FACULTY OF SCIENCE,
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Thesis for the Bachelor of Mathematics

High Dimensional Regression Models

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Chapter 1

Introduction

1.1 The first section

1.1.1 The first subsection

$$\mathbf{Y} = \mathbf{X}\beta + \boldsymbol{\varepsilon} \tag{1.1}$$

We define $\hat{\beta}$ as follows

$$\hat{\beta} := \arg\min_{\beta} \left\{ \frac{\|\mathbf{Y} - \mathbf{X}\beta\|_2^2}{n} + \lambda\|\beta\|_1 \right\} \tag{1.2}$$

Lemma 1.1 (Basic Inequality)**.**

$$\frac{\|\mathbf{X}(\hat{\beta} - \beta^0)\|_2^2}{n} + \lambda\|\hat{\beta}\|_1 \leq 2\frac{\varepsilon^T\mathbf{X}(\hat{\beta} - \beta^0)}{n} + \lambda\|\beta^0\|_1$$

Proof. By definition of $\hat{\beta}$, we have that

$$\forall\beta \quad \frac{\|\mathbf{Y} - \mathbf{X}\hat{\beta}\|_2^2}{n} + \lambda\|\hat{\beta}\|_1 \leq \frac{\|\mathbf{Y} - \mathbf{X}\beta\|_2^2}{n} + \lambda\|\beta\|_1$$

In particular for $\beta = \beta^0$ we have

$$\frac{\|\mathbf{Y} - \mathbf{X}\hat{\beta}\|_2^2}{n} + \lambda\|\hat{\beta}\|_1 \leq \frac{\|\mathbf{Y} - \mathbf{X}\beta^0\|_2^2}{n} + \lambda\|\beta^0\|_1$$

We now replace $\mathbf{Y}$ using equation (1.1).

$$
\begin{aligned}
& \frac{\|\mathbf{Y}-\mathbf{X}\hat{\beta}\|_2^2}{n} + \lambda\|\hat{\beta}\|_1 \leq \frac{\|\mathbf{Y}-\mathbf{X}\beta^0\|_2^2}{n} + \lambda\|\beta^0\|_1 \\
\implies & \frac{\|(\mathbf{X}\beta^0+\boldsymbol{\varepsilon})-\mathbf{X}\hat{\beta}\|_2^2}{n} + \lambda\|\hat{\beta}\|_1 \leq \frac{\|(\mathbf{X}\beta^0+\boldsymbol{\varepsilon})-\mathbf{X}\beta^0\|_2^2}{n} + \lambda\|\beta^0\|_1 \\
\implies & \frac{\|\mathbf{X}(\beta^0-\hat{\beta})+\boldsymbol{\varepsilon}\|_2^2}{n} + \lambda\|\hat{\beta}\|_1 \leq \frac{\|\mathbf{X}(\beta^0-\beta^0)+\boldsymbol{\varepsilon}\|_2^2}{n} + \lambda\|\beta^0\|_1 \\
\implies & \frac{\langle \mathbf{X}(\beta^0-\hat{\beta})+\boldsymbol{\varepsilon}, \mathbf{X}(\beta^0-\hat{\beta})+\boldsymbol{\varepsilon}\rangle}{n} + \lambda\|\hat{\beta}\|_1 \leq \frac{\|\boldsymbol{\varepsilon}\|_2^2}{n} + \lambda\|\beta^0\|_1 \\
\implies & \frac{\|\mathbf{X}(\beta^0-\hat{\beta})\|_2^2 + \|\boldsymbol{\varepsilon}\|_2^2 + 2\langle \mathbf{X}(\beta^0-\hat{\beta}), \boldsymbol{\varepsilon}\rangle}{n} + \lambda\|\hat{\beta}\|_1 \leq \frac{\|\boldsymbol{\varepsilon}\|_2^2}{n} + \lambda\|\beta^0\|_1 \\
\implies & \frac{\|\mathbf{X}(\hat{\beta}-\beta^0)\|_2^2}{n} + \lambda\|\hat{\beta}\|_1 \leq \frac{2\langle \mathbf{X}(\hat{\beta}-\beta^0), \boldsymbol{\varepsilon}\rangle}{n} + \lambda\|\beta^0\|_1 \\
\implies & \frac{\|\mathbf{X}(\hat{\beta}-\beta^0)\|_2^2}{n} + \lambda\|\hat{\beta}\|_1 \leq \frac{2\boldsymbol{\varepsilon}^T\mathbf{X}(\hat{\beta}-\beta^0)}{n} + \lambda\|\beta^0\|_1
\end{aligned}
$$

□

Let

$$\mathscr{T} := \left\{ \max_{1\leq j\leq p} 2\frac{\left|\varepsilon^T\mathbf{X}^{(j)}\right|}{n} \leq \lambda_0 \right\}$$

Lemma 1.2 (Lemma 6.2.). *For all $t>0$ and*

$$\lambda_0 := 2\sigma\sqrt{\frac{t^2+2\log p}{n}}$$

we have

$$\mathbb{P}(\mathscr{T}) \geq 1 - 2\exp\left[-t^2/2\right]$$

Proof. We define

$$V_j := \frac{\varepsilon^T\mathbf{X}^{(j)}}{\sqrt{n\sigma^2}}$$

Then we have

$$\begin{aligned}
\mathbb{P}(\mathscr{T}) &= \mathbb{P}\left(\left\{\max_{1\le j\le p} 2\frac{\left|\varepsilon^T \mathbf{X}^{(j)}\right|}{n} \le 2\sigma\sqrt{\frac{t^2+2\log p}{n}}\right\}\right)\\
&= \mathbb{P}\left(\left\{\max_{1\le j\le p} \frac{\varepsilon^T \mathbf{X}^{(j)}}{\sqrt{n\sigma^2}} \le \sqrt{t^2+2\log p}\right\}\right)\\
&= \mathbb{P}\left(\left\{\max_{1\le j\le p} |V_j| \le \sqrt{t^2+2\log p}\right\}\right)\\
&= 1-\mathbb{P}\left(\left\{\max_{1\le j\le p} |V_j| > \sqrt{t^2+2\log p}\right\}\right)
\end{aligned}$$

$\square$

To be added

- how to get $\hat{b}$ on page 101.
- where the χ^2 distribution comes from in page 101