Statistical inference theory: Exercises

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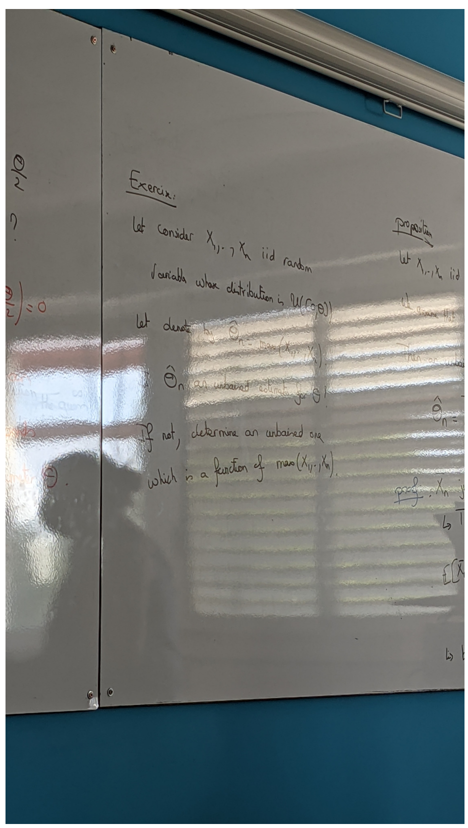

1.1 Exercise

Let $X_1, \dots, X_n$ be i.i.d. random variables with distribution $\mathcal{U}(0, \theta)$.
Let $\hat{\theta}_n = \max(X_1, \dots, X_n)$.

Is $\hat{\theta}_n$ an unbiased estimator for θ ?

$\hat{\theta}_n$ is an estimator since it does not depend on any parameter.
Now, $\hat{\theta}_n$ is unbiased if $\mathbb{E}[\hat{\theta}_n] = \theta$. We compute $F_{\hat{\theta}_n}$, the CDF of $\hat{\theta}_n$:

$$\begin{aligned} F_{\hat{\theta}_n}(x) &= \mathbb{P}(X_1 \leq x, \dots, X_n \leq x) \\ &= \mathbb{P}(X_1 \leq x) \dots \mathbb{P}(X_n \leq x) \\ &= [\mathbb{P}(X_1 \leq x)]^n \\ &= \begin{cases} 0 & x < 0 \\ \left[\frac{x}{\theta}\right]^n & 0 \leq x \leq \theta \\ 1 & x > \theta \end{cases} \end{aligned}$$

Then $f_{\hat{\theta}_n}$, the CDF of $\hat{\theta}_n$ is given by:

$$\begin{aligned} f_{\hat{\theta}_n}(x) &= \frac{d}{dx} F_{\hat{\theta}_n}(x) \\ &= \begin{cases} \frac{nx^{n-1}}{\theta^n} & x \in [0, \theta] \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Hence the expected value is given by:

$$\begin{aligned} \mathbb{E}[\hat{\theta}_n] &= \int_{-\infty}^{+\infty} t f(t) dt \\ &= \frac{n}{\theta^n} \int_0^\theta t^n dt \\ &= \frac{n}{\theta^n (n+1)} \left[t^{n+1}\right]_0^\theta \\ &= \frac{n}{\theta^n (n+1)} \left[\theta^{n+1} - 0\right] \\ &= \frac{n\theta}{n+1} \end{aligned}$$

Since $\mathbb{E}[\hat{\theta}_n] = \frac{n\theta}{n+1} \neq \theta$, we have that $\hat{\theta}_n$ is not an unbiased estimator for θ.

If not, determine an unbiased one which is a funtion of $\max(X_1, \ldots, X_n)$
Method 1:

$$\mathbb{E}[\hat{\theta}_n] = \frac{n\theta}{n+1}$$

$$\implies \mathbb{E}[\hat{\theta}_n]\frac{n+1}{n} = \theta$$

$$\implies \mathbb{E}\left[\frac{n+1}{n}\hat{\theta}_n\right] = \theta$$

So $\overline{\theta} := \frac{n+1}{n}\hat{\theta}_n$ is an unbiased estiamtor for θ.
Method 2: We consider $\bar{\theta} := \frac{n+1}{n}\hat{\theta} = \frac{n+1}{n}\max(X_1, \ldots, X_n)$. Then $F_{\hat{\theta}_n}$, the CDF of $\hat{\theta}_n$ is given by:

$$\begin{aligned}
F_{\bar{\theta}_n}(x) &= \mathbb{P}\left(\bar{\theta} \leq x\right) \\
&= \mathbb{P}\left(\frac{n+1}{n}\hat{\theta} \leq x\right) \\
&= \mathbb{P}\left(X_1 \leq \frac{n}{n+1}x, \ldots, X_n \leq \frac{n}{n+1}x\right) \\
&= \mathbb{P}\left(X_1 \leq \frac{n}{n+1}x\right) \ldots \mathbb{P}\left(X_n \leq \frac{n}{n+1}x\right) \\
&= \left[\mathbb{P}\left(X_1 \leq \frac{n}{n+1}x\right)\right]^n \\
&= \begin{cases} 0 & x < 0 \\ \left[\frac{n}{\theta(n+1)}x\right]^n & 0 \leq x \leq \frac{\theta(n+1)}{n} \\ 1 & x > \frac{\theta(n+1)}{n} \end{cases}
\end{aligned}$$

Then $f_{\hat{\theta}_n}$, the CDF of $\hat{\theta}_n$ is given by:

$$\begin{aligned}
f_{\hat{\theta}_n}(x) &= \frac{d}{dx}F_{\hat{\theta}_n}(x) \\
&= \begin{cases} \left[\frac{n}{\theta(n+1)}\right]^n nx^{n-1} & x \in [0, \frac{\theta(n+1)}{n}] \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{n^{n+1}}{\theta^n(n+1)^n}x^{n-1} & x \in [0, \frac{\theta(n+1)}{n}] \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Hence the expected value is given by:

$$
\begin{aligned}
\mathbb{E}[\hat{\theta}_n] &= \int_{-\infty}^{+\infty} tf(t)dt \\
&= \frac{n^{n+1}}{\theta^n(n+1)^n} \int_0^{\frac{\theta(n+1)}{n}} t^n dt \\
&= \frac{n^{n+1}}{\theta^n(n+1)^n} \left[\frac{t^{n+1}}{n+1}\right]_0^{\frac{\theta(n+1)}{n}} \\
&= \frac{n^{n+1}}{\theta^n(n+1)^{n+1}} \left[t^{n+1}\right]_0^{\frac{\theta(n+1)}{n}} \\
&= \frac{n^{n+1}}{\theta^n(n+1)^{n+1}} \left[\left(\frac{\theta(n+1)}{n}\right)^{n+1} - 0\right] \\
&= \theta
\end{aligned}
$$

Now $\mathbb{E}[\hat{\theta}_n] = \theta$, therefore $\hat{\theta}_n$ is an unbiased estimator for θ.