

Statistical inference practice

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1 Homework for October 8, 2021

1.1 Chapter 6 - Exercise 2

Let X_1, \dots, X_n be i.i.d. random variables with distribution $\mathcal{U}(0, \theta)$.

Let $\hat{\theta}_n = \max(X_1, \dots, X_n)$.

What is the bias of $\hat{\theta}_n$?

$\hat{\theta}_n$ is unbiased if $\mathbb{E}[\hat{\theta}_n] = \theta$.

We compute $F_{\hat{\theta}_n}$, the CDF of $\hat{\theta}_n$:

$$\begin{aligned}
F_{\hat{\theta}_n}(x) &= \mathbb{P}(\hat{\theta} \leq x) \\
&= \mathbb{P}(X_1 \leq x, \dots, X_n \leq x) \\
&= \mathbb{P}(X_1 \leq x) \dots \mathbb{P}(X_n \leq x) && \text{(independence)} \\
&= [\mathbb{P}(X_1 \leq x)]^n && \text{(identity of distribution)} \\
&= \begin{cases} 0 & x < 0 \\ \left[\frac{x}{\theta}\right]^n & 0 \leq x \leq \theta \\ 1 & x > \theta \end{cases}
\end{aligned}$$

Then $f_{\hat{\theta}_n}$, the PDF of $\hat{\theta}_n$ is given by:

$$\begin{aligned}
f_{\hat{\theta}_n}(x) &= \frac{d}{dx} F_{\hat{\theta}_n}(x) \\
&= \begin{cases} \frac{nx^{n-1}}{\theta^n} & x \in [0, \theta] \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Hence the expected value is given by:

$$\begin{aligned}
\mathbb{E}[\hat{\theta}_n] &= \int_{-\infty}^{+\infty} tf(t)dt \\
&= \frac{n}{\theta^n} \int_0^\theta t^n dt \\
&= \frac{n}{\theta^n(n+1)} [t^{n+1}]_0^\theta \\
&= \frac{n}{\theta^n(n+1)} [\theta^{n+1} - 0] \\
&= \frac{n\theta}{n+1}
\end{aligned}$$

Since $\mathbb{E}[\hat{\theta}_n] = n\theta/(n+1) \neq \theta$, we have that $\hat{\theta}_n$ is not an unbiased estimator for θ . However, $\mathbb{E}[\hat{\theta}_n] \xrightarrow{n \rightarrow \infty} \theta$, therefore $\hat{\theta}_n$ is asymptotically unbiased.

What is SE , the standard error of $\hat{\theta}_n$?

$$\begin{aligned}
SE &= SE(\hat{\theta}_n) \\
&= \sqrt{Var(\hat{\theta}_n)} \\
&= \sqrt{\mathbb{E}[\hat{\theta}_n^2] - \mathbb{E}[\hat{\theta}_n]^2} \tag{1}
\end{aligned}$$

We need to find $\mathbb{E}[\hat{\theta}_n^2]$.

$$\begin{aligned}
\mathbb{E}[\hat{\theta}_n^2] &:= \int_{-\infty}^{+\infty} t^2 f(t) dt \\
&= \frac{n}{\theta^n} \int_0^\theta t^{n+1} dt \\
&= \frac{n}{\theta^n(n+2)} [t^{n+2}]_0^\theta \\
&= \frac{n}{\theta^n(n+2)} [\theta^{n+2} - 0] \\
&= \frac{n\theta^2}{n+2}
\end{aligned}$$

Then (1) becomes:

$$\begin{aligned}
SE &= \sqrt{\mathbb{E}[\hat{\theta}_n^2] - \mathbb{E}[\hat{\theta}_n]^2} \\
&= \sqrt{\frac{n\theta^2}{n+2} - \left[\frac{n\theta}{n+1}\right]^2} \\
&= \sqrt{n\theta^2 \left[\frac{1}{n+2} - \frac{n}{(n+1)^2}\right]} \\
&= \sqrt{n\theta^2 \left[\frac{1}{n+2} - \frac{n}{n^2+2n+1}\right]} \\
&= \sqrt{n\theta^2 \left[\frac{n^2+2n+1}{(n+2)(n^2+2n+1)} - \frac{n(n+2)}{(n^2+2n+1)(n+2)}\right]} \\
&= \sqrt{n\theta^2 \frac{n^2+2n+1-n(n+2)}{(n^2+2n+1)(n+2)}} \\
&= \sqrt{\frac{n\theta^2}{(n^2+2n+1)(n+2)}} \\
&= \frac{\theta}{n+1} \sqrt{\frac{n}{n+2}}
\end{aligned}$$

What is MSE , the Mean-Square Error of $\hat{\theta}_n$?

The MSE is given by:

$$\begin{aligned}
 MSE &:= bias^2(\hat{\theta}_n) + Var(\hat{\theta}_n) \\
 &= \left[\mathbb{E}[\hat{\theta}] - \theta \right]^2 + \frac{n\theta^2}{(n+1)^2(n+2)} \\
 &= \left[\frac{n\theta}{n+1} - \theta \right]^2 + \frac{n\theta^2}{(n+1)^2(n+2)} \\
 &= \left[\frac{-\theta}{n+1} \right]^2 + \frac{n\theta^2}{(n+1)^2(n+2)} \\
 &= \frac{\theta^2}{(n+1)^2} \left[1 + \frac{n}{n+2} \right] \\
 &= \frac{\theta^2}{(n+1)^2} \frac{2n+2}{n+2} \\
 &= \frac{2\theta^2}{(n+1)(n+2)}
 \end{aligned}$$

1.2 Chapter 6 - Exercise 3

Let X_1, \dots, X_n be i.i.d. random variables with distribution $\mathcal{U}(0, \theta)$.

Let $\hat{\theta}_n := 2\bar{X}_n$.

What is the bias of $\hat{\theta}_n$?

$\hat{\theta}_n$ is unbiased if $\mathbb{E}[\hat{\theta}_n] = \theta$.

We compute $\mathbb{E}[\hat{\theta}_n]$:

$$\begin{aligned}
 \mathbb{E}[\hat{\theta}_n] &= \mathbb{E}[2\bar{X}_n] \\
 &= \mathbb{E}\left[2 \frac{X_1 + \dots + X_n}{n}\right] \\
 &= \frac{2}{n} \mathbb{E}[X_1 + \dots + X_n] \\
 &= \frac{2}{n} \mathbb{E}[X_1] + \dots + \mathbb{E}[X_n] \\
 &= 2\mathbb{E}[X_1] \\
 &= \theta
 \end{aligned}$$

Therefore $\hat{\theta}$ is unbiased.

What is SE , the standard error of $\hat{\theta}_n$?

$$\begin{aligned}
 SE &= SE(\hat{\theta}_n) \\
 &= \sqrt{Var(\hat{\theta}_n)} \\
 &= \sqrt{Var(2\bar{X}_n)} \\
 &= \frac{2}{n} \sqrt{Var(X_1 + \dots + X_n)} \\
 &= \frac{2}{n} \sqrt{Var(X_1) + \dots + Var(X_n)} \quad (\text{The } X_i \text{ are i.i.d.}) \\
 &= \frac{2}{n} \sqrt{\frac{n\theta^2}{12}} \\
 &= \frac{2\theta}{2\sqrt{3n}} \\
 &= \frac{\theta}{\sqrt{3n}}
 \end{aligned}$$

What is MSE , the Mean-Square Error of $\hat{\theta}_n$?

The MSE is given by:

$$\begin{aligned}
 MSE &:= bias^2(\hat{\theta}_n) + Var(\hat{\theta}_n) \\
 &= \underbrace{\left[\mathbb{E}[\hat{\theta}] - \theta \right]^2}_{=0} + \frac{\theta^2}{3n} \\
 &= \frac{\theta^2}{3n}
 \end{aligned}$$

2 Homework for October 20, 2021

2.1 Chapter 7 - Exercise 2

Let $X_1, \dots, X_n \sim \text{Bernoulli}(p)$ and let $Y_1, \dots, Y_m \sim \text{Bernoulli}(q)$.

- Find the plug-in estimator and estimated standard error for p .
- Find an approximate 90 percent confidence interval for p .
- Find the plug-in estimator and estimated standard error for $p - q$.
- Find an approximate 90 percent confidence interval for $p - q$.

Find the plug-in estimator and estimated standard error for p .

Let ϕ be the plug-in estimator for p , it is given by:

$$\phi = \mathbb{E}[X_i], \quad i = 1, \dots, n$$

and

$$\hat{\phi} = \mathbb{E}[Z], \quad \text{with } \mathbb{P}(Z = X_i \mid X_1, \dots, X_n) = \frac{1}{n}$$

$$\begin{aligned} \hat{\phi} &= \mathbb{E}[Z] \\ &= \sum_{i=1}^n X_i \mathbb{P}(Z = X_i) \\ &= \sum_{i=1}^n \frac{1}{n} X_i \\ &= \bar{X} \end{aligned}$$

and the standard error se is given by:

$$\begin{aligned} se(\phi) &= \sqrt{Var(\phi)} \\ &= \sqrt{Var(\bar{X})} \\ &= \sqrt{\frac{p(1-p)}{n}} \end{aligned}$$

Find an approximate 90 percent confidence interval for p .

We know that 90% (*i.e.* $\alpha = 0.05$) confidence intervals are of the following form:

$$\begin{aligned} &\bar{X} \pm z_{\alpha/2} se(p_{pin}) \\ &= \bar{X} \pm 1.645 \sqrt{\frac{1}{n} \left[\sum_{i=1}^n X_i^2 \right] - \bar{X}^2} \end{aligned}$$

Find the plug-in estimator and estimated standard error for $p - q$.

Let Π be the plug-in estimator for $p - q$ and $\chi \in \{X_1, \dots, X_n, Y_1, \dots, Y_m\}$

$$\begin{aligned}\mathbb{P}(\Pi = \chi) &= \frac{1}{\#\{X_1, \dots, X_n\} + \#\{Y_1, \dots, Y_m\}} \\ &= \frac{1}{m+n}\end{aligned}$$

$$\begin{aligned}se(\Pi) &= \sqrt{Var(\Pi)} \\ &= \sqrt{\mathbb{E}[\Pi^2] - \mathbb{E}[\Pi]^2} \\ &= \sqrt{\left[\sum_{i=1}^n \chi^2 \mathbb{P}\{\Pi = \chi\} \right] - \left[\sum_{i=1}^n \chi \mathbb{P}\{\Pi = \chi\} \right]^2} \\ &= \sqrt{\left[\sum_{i=1}^n \chi^2 \frac{1}{m+n} \right] - \left[\sum_{i=1}^n \chi \frac{1}{m+n} \right]^2} \\ &= \sqrt{\frac{1}{m+n} \left[\sum_{i=1}^n \chi^2 \right] - \bar{\chi}^2}\end{aligned}$$

Find an approximate 90 percent confidence interval for $p - q$.

We know that 90% (*i.e.* $\alpha = 0.05$) confidence intervals are of the following form:

$$\begin{aligned}\bar{\chi} \pm z_{\alpha/2} se(p_{pin}) \\ = \bar{\chi} \pm 1.645 \sqrt{\frac{1}{m+n} \left[\sum_{i=1}^n \chi^2 \right] - \bar{\chi}^2}\end{aligned}$$

2.2 Chapter 7 - Exercise 5

Let x and y be two distinct points. Find $\text{Cov}(\hat{F}_n(x), \hat{F}_n(y))$.

2.3 Chapter 7 - Exercise 6

3 Homework for October 29

3.1 Custom exercise

Let $N = 50$, Y_1, \dots, Y_n are i.i.d. $\mathcal{N}(0, 1)$. Let $X_i = e^{Y_i}$.

Let $\theta = \text{skewness}(X) = (e + 2)\sqrt{e - 1}$ (X is log normal distributed)

Compute the 3 types of normal confidence intervals for θ .

Repeat the experiment to check how often θ belongs to the confidence intervals.

4 Homework for November 5, 2021

4.1 Chapter 9 - Exercise 1

Let $X_1, \dots, X_n \sim \text{Gamma}(\alpha, \beta)$. Find the method of moments estimator for α and β . From tables, we have that:

$$\begin{cases} \mathbb{E}[X_i] = \frac{\alpha}{\beta} \\ \text{Var}(X_i) = \frac{\alpha}{\beta^2} \end{cases}$$

Equating with empirical expected value and empirical variance respectively.

$$\begin{aligned} & \begin{cases} \frac{1}{n} \sum_{i=1}^n X_i = \mathbb{E}[X_i] \\ \frac{1}{n} \sum_{i=1}^n (\bar{X} - X_i)^2 = \text{Var}(X_i) \end{cases} \\ \implies & \begin{cases} \frac{1}{n} \sum_{i=1}^n X_i = \frac{\hat{\alpha}}{\hat{\beta}} \\ \frac{1}{n} \sum_{i=1}^n (\bar{X} - X_i)^2 = \frac{\hat{\alpha}}{\hat{\beta}^2} \end{cases} \\ \implies & \begin{cases} \hat{\alpha} = \frac{\hat{\beta}}{n} \sum_{i=1}^n X_i \\ \frac{1}{n} \sum_{i=1}^n (\bar{X} - X_i)^2 = \frac{\hat{\alpha}}{\hat{\beta}^2} \end{cases} \\ \implies & \begin{cases} \hat{\alpha} = \frac{\hat{\beta}}{n} \sum_{i=1}^n X_i \\ \frac{1}{n} \sum_{i=1}^n (\bar{X} - X_i)^2 = \frac{1}{\hat{\beta}^2} \frac{\hat{\beta}}{n} \sum_{i=1}^n X_i \end{cases} \\ \implies & \begin{cases} \hat{\alpha} = \frac{\hat{\beta}}{n} \sum_{i=1}^n X_i \\ \frac{1}{n} \sum_{i=1}^n (\bar{X} - X_i)^2 = \frac{1}{\hat{\beta} n} \sum_{i=1}^n X_i \end{cases} \\ \implies & \begin{cases} \hat{\alpha} = \frac{\hat{\beta}}{n} \sum_{i=1}^n X_i \\ \hat{\beta} = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n (\bar{X} - X_i)^2} \end{cases} \\ \implies & \begin{cases} \hat{\alpha} = \frac{1}{n} \frac{[\sum_{i=1}^n X_i]^2}{\sum_{i=1}^n (\bar{X} - X_i)^2} \\ \hat{\beta} = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n (\bar{X} - X_i)^2} \end{cases} \end{aligned}$$

4.2 Chapter 9 - Exercise 2

Let $X_1, \dots, X_n \sim \text{Uniform}(a, b)$ where a and b are unknown parameters and $a < b$.

- (a) Find the method of moments estimators for a and b .

- (b) Find the MLE \hat{a} and \hat{b} .
- (c) Let $\tau = \int x dF(x)$. Find the MLE of τ .
- (d) Let $\hat{\tau}$ be the MLE of τ . Let $\tilde{\tau}$ be the nonparametric plug-in estimator of $\tau = \int x dF(x)$. Suppose that $a = 1, b = 3$, and $n = 10$. Find the MSE of $\hat{\tau}$ by simulation. Find the MSE of $\tilde{\tau}$ analytically. Compare.

(a) Find the method of moments estimators for a and b .

We compute the first order of moments:

$$\begin{aligned}
 \mathbb{E}[X_1] &= \int_a^b x f(x) dx \\
 &= \frac{1}{b-a} \int_a^b x dx \\
 &= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b \\
 &= \frac{1}{b-a} \frac{b^2 - a^2}{2} \\
 &= \frac{a+b}{2}
 \end{aligned} \tag{2}$$

Now compute the second order of moments:

$$\begin{aligned}
 \mathbb{E}[X_1^2] &= \int_a^b x^2 f(x) dx \\
 &= \int_a^b x^2 \frac{1}{b-a} dx \\
 &= \frac{1}{b-a} \left[\frac{x^3}{3} \right]_a^b \\
 &= \frac{1}{b-a} \left[\frac{b^3 - a^3}{3} \right] \\
 &= \frac{1}{b-a} \left[\frac{(b-a)(a^2 + ab + b^2)}{3} \right] \\
 &= \frac{a^2 + ab + b^2}{3}
 \end{aligned} \tag{3}$$

Now on the one hand we equate (2) with $\hat{\mu}_1 := \frac{1}{n} \sum_{i=1}^n X_i$. On the other hand, we equate (3) with $\hat{\mu}_2 := \frac{1}{n} \sum_{i=1}^n X_i^2$. Therefore we get a system of equations:

$$\begin{cases} \hat{\mu}_1 = \frac{a+b}{2} \\ \hat{\mu}_2 = \frac{a^2+ab+b^2}{3} \end{cases}$$

$$\begin{cases} b = 2\hat{\mu}_1 - a \\ 3\hat{\mu}_2 = a^2 + a[2\hat{\mu}_1 - a] + [2\hat{\mu}_1 - a]^2 \end{cases}$$

$$\begin{cases} b = 2\hat{\mu}_1 - a \\ 3\hat{\mu}_2 = a^2 + 2\hat{\mu}_1 a - a^2 + 4\hat{\mu}_1^2 - 4\hat{\mu}_1 a + a^2 \end{cases}$$

$$\begin{cases} b = 2\hat{\mu}_1 - a \\ a^2 - 2\hat{\mu}_1 a + (4\hat{\mu}_1^2 - 3\hat{\mu}_2) = 0 \end{cases} \quad (4)$$

Then, the second equation of (4) yields:

$$\begin{aligned} a_1 &= \frac{2\hat{\mu}_1 - \sqrt{(2\hat{\mu}_1)^2 - 4(4\hat{\mu}_1^2 - 3\hat{\mu}_2)}}{2} \\ &= \hat{\mu}_1 - \sqrt{\hat{\mu}_1^2 - 4\hat{\mu}_1^2 + 3\hat{\mu}_2} \\ &= \hat{\mu}_1 - \sqrt{3(\hat{\mu}_2 - \hat{\mu}_1^2)} \end{aligned}$$

and

$$\begin{aligned} a_1 &= \frac{2\hat{\mu}_1 + \sqrt{(2\hat{\mu}_1)^2 - 4(4\hat{\mu}_1^2 - 3\hat{\mu}_2)}}{2} \\ &= \hat{\mu}_1 + \sqrt{\hat{\mu}_1^2 - 4\hat{\mu}_1^2 + 3\hat{\mu}_2} \\ &= \hat{\mu}_1 + \sqrt{3(\hat{\mu}_2 - \hat{\mu}_1^2)} \end{aligned}$$

Let b_1, b_2 be associated with a_1, a_2 respectively. Then the first equation of (4) becomes:

$$\begin{cases} b_1 := 2\hat{\mu}_1 - \left(\hat{\mu}_1 - \sqrt{3(\hat{\mu}_2 - \hat{\mu}_1^2)} \right) \\ b_2 := 2\hat{\mu}_1 - \left(\hat{\mu}_1 + \sqrt{3(\hat{\mu}_2 - \hat{\mu}_1^2)} \right) \end{cases}$$

$$\begin{cases} b_1 := \hat{\mu}_1 + \sqrt{3(\hat{\mu}_2 - \hat{\mu}_1^2)} \\ b_2 := \hat{\mu}_1 - \sqrt{3(\hat{\mu}_2 - \hat{\mu}_1^2)} \end{cases}$$

Since $b_2 > a_2$, which is impossible by the exercise, we have that the method of moment estimators are:

$$\begin{cases} a = \hat{\mu}_1 - \sqrt{3(\hat{\mu}_2 - \hat{\mu}_1^2)} \\ b = \hat{\mu}_1 + \sqrt{3(\hat{\mu}_2 - \hat{\mu}_1^2)} \end{cases}$$

(b) Find the MLE \hat{a} and \hat{b} .

Let $\theta := (a, b) \in \Theta \subseteq \mathbb{R}^2$. We define \mathcal{L} the likelihood function as follows:

$$\begin{aligned}\mathcal{L}(\theta) &= f(X_1, \dots, X_n \mid \theta) \\ &= \prod_{i=1}^n f(X_i \mid \theta)\end{aligned}$$

Now we want to find $\hat{\theta} := (\hat{a}, \hat{b})$ the argument maximizing the likelihood function

$$\begin{aligned}\hat{\theta} &:= (\hat{a}, \hat{b}) \\ &:= \arg \max_{\Theta} \mathcal{L}(\theta) \\ &= \arg \max_{\Theta} \log \mathcal{L}(\theta)\end{aligned}$$

therefore we have

$$\begin{aligned}\log \mathcal{L}(\theta) &= \log \prod_{i=1}^n f(X_i \mid \theta) \\ &= \sum_{i=1}^n \log f(X_i \mid (a, b)) \\ &= \sum_{i=1}^n \log \frac{1}{b-a} \\ &= -n \log(b-a)\end{aligned}$$

hence

$$\begin{cases} \frac{\partial}{\partial a} \log \mathcal{L}(\theta) = \frac{n}{b-a} \\ \frac{\partial}{\partial b} \log \mathcal{L}(\theta) = -\frac{n}{b-a} \end{cases}$$

Now we have that

$$\begin{aligned}&\begin{cases} \frac{\partial}{\partial a} \log \mathcal{L}(\theta) > 0 \\ \frac{\partial}{\partial b} \log \mathcal{L}(\theta) < 0 \end{cases} \\ \implies &\begin{cases} \log \mathcal{L}(\theta) \text{ is increasing with respect to } a \\ \log \mathcal{L}(\theta) \text{ is decreasing with respect to } b \end{cases} \\ \implies &\begin{cases} \hat{a} = \min \{X_1, \dots, X_n\} \\ \hat{b} = \max \{X_1, \dots, X_n\} \end{cases}\end{aligned}$$

(c) Let $\tau = \int x dF(x)$. Find the MLE of τ .

Let $\theta := \tau \in \Theta \subseteq \mathbb{R}$. We have the following:

$$\begin{aligned}\tau &= \int x dF(x) \\ &= \int x \frac{dF(x)}{dx} dx \\ &= \int x f(x) dx \\ &= \mathbb{E}[X_1] \\ &= \frac{a+b}{2}\end{aligned}$$

Hence

$$\hat{\tau} = \frac{\hat{a} + \hat{b}}{2}$$

(d) Let $\hat{\tau}$ be the MLE of τ . Let $\tilde{\tau}$ be the nonparametric plug-in estimator of $\tau = \int x dF(x)$. Suppose that $a = 1, b = 3$, and $n = 10$. Find the MSE of $\hat{\tau}$ by simulation. Find the MSE of $\tilde{\tau}$ analytically. Compare.

The MSE is defined by:

$$\begin{aligned}
MSE(\tilde{\tau}) &:= Var(\tilde{\tau}) + bias^2(\tilde{\tau}) \\
&= \mathbb{E}[\tilde{\tau}^2] - \mathbb{E}[\tilde{\tau}]^2 + [\mathbb{E}[\tilde{\tau}] - \tilde{\tau}]^2 \\
&= \mathbb{E}[\bar{X}^2] - \mathbb{E}[\bar{X}]^2 + \underbrace{[\mathbb{E}[\bar{X}] - \bar{X}]^2}_{=0} \\
&= \frac{1}{n} \left[\frac{a^2 + ab + b^2}{3} - \left[\frac{a+b}{2} \right]^2 \right] \\
&= \frac{1}{n} \left[\frac{a^2 + ab + b^2}{3} - \frac{a^2 + 2ab + b^2}{4} \right] \\
&= \frac{1}{n} \left[\frac{a^2 + ab + b^2 - 3a^2 - 6ab - 3b^2}{12} \right] \\
&= \frac{1}{n} \left[\frac{4a^2 + 4ab + 4b^2 - 3a^2 - 6ab - 3b^2}{12} \right] \\
&= \frac{1}{n} \left[\frac{a^2 - 2ab + b^2}{12} \right] \\
&= \frac{1}{n} \left[\frac{(a-b)^2}{12} \right] \\
&= \frac{(a-b)^2}{12n}
\end{aligned}$$