Probabilities refresher: Exam

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1 Exercise 1

1.1 Question 1

Let F_X be the distribution function of X. From the plot, we have that:

$$F_X(x) := \begin{cases} 0 & \text{if } x \in]-\infty, -3[\\ 0.05 & \text{if } x \in [-3, 1[\\ 0.4 & \text{if } x \in [1, 2[\\ 0.5 & \text{if } x \in [2, 3[\\ 0.65 & \text{if } x \in [3, 4[\\ 0.9 & \text{if } x \in [4, 7[\\ 1 & \text{if } x \in [7, +\infty[\end{cases}$$

1.2 Question 2

By definition, we know that the variance of X is defined as

$$Var(X) := \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

Let us compute the components of that equation. First we deduce by the distribution function that the mass function is given as:

k	-3	1	2	3	4	7
$\mathbb{P}(X = k)$	0.05	0.35	0.1	0.15	0.25	0.1

and the sample space Ω is given by $\Omega := \{-3, 1, 2, 3, 4, 7\}$. Therefore the expected value of X is given by:

$$\begin{aligned}
\mathbb{E}[X] &:= \sum_{k \in \Omega} k \mathbb{P}(X = k) \\
&= -3 \times 0.05 + 1 \times 0.35 + 2 \times 0.1 \\
&\quad + 3 \times 0.15 + 4 \times 0.25 + 7 \times 0.1 \\
&= -0.15 + 0.35 + 0.2 + 0.45 + 1 + 0.7 \\
&= 2.55
\end{aligned}$$

Similarly, the expected value of X^2 is given by:

$$\begin{aligned}
\mathbb{E}[X^2] &:= \sum_{k \in \Omega} k^2 \mathbb{P}(X = k) \\
&= 9 \times 0.05 + 1 \times 0.35 + 4 \times 0.1 \\
&\quad + 9 \times 0.15 + 16 \times 0.25 + 49 \times 0.1 \\
&= 0.45 + 0.35 + 0.4 + 1.35 + 4 + 4.9 \\
&= 11.45
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
Var(X) &:= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= 11.45 - 2.55^2 \\
&= 11.45 - 6.5025 \\
&= 4.9475
\end{aligned}$$

2 Exercise 2

2.1 Question 1

We will use two pieces of knowledge. First, the probabilities over the sample space sum up to 1. Second, that the expectation of X is 1.8.
Let Ω be the sample space, with $\Omega := \{-1, 2, 3, 4, 5\}$. We have on the one hand:

$$\begin{aligned}
&\sum_{k \in \Omega} \mathbb{P}(X = k) = 1 \\
&\implies a + 0.1 + 0.1 + b + 0.1 = 1 \\
&\implies a + b + 0.3 = 1 \\
&\implies a + b = 0.7 \qquad (1)
\end{aligned}$$

We have on the other hand:

$$\begin{aligned}
&\mathbb{E}[X] = 1.8 \\
&\implies \sum_{k\in\Omega} k\mathbb{P}(X = k) = 1.8 \\
&\implies -1 \times a + 2 \times 0.1 + 3 \times 0.1 + 4 \times b + 5 \times 0.1 = 1.8 \\
&\implies -a + 0.2 + 0.3 + 4b + 0.5 = 1.8 \\
&\implies -a + 4b = 0.8 \qquad (2)
\end{aligned}$$

Thus by combining (1) and (2), we obtain a system with two equations and two unknowns:

$$\begin{aligned}
&\begin{cases} a + b = 0.7 \\ -a + 4b = 0.8 \end{cases} \\
\implies &\begin{cases} a + b = 0.7 \\ 5b = 1.5 \end{cases} \\
\implies &\begin{cases} a + b = 0.7 \\ b = 0.3 \end{cases} \\
\implies &\begin{cases} a = 0.4 \\ b = 0.3 \end{cases}
\end{aligned}$$

2.2 Question 2

Now from the table and question 1, it follows that the distribution function is given by:

$$\begin{cases}
0 & \text{if } x \in]-\infty, -1[\\
0.4 & \text{if } x \in [-1, 2[\\
0.5 & \text{if } x \in [2, 3[\\
0.6 & \text{if } x \in [3, 4[\\
0.9 & \text{if } x \in [4, 5[\\
1 & \text{if } x \in [5, +\infty[
\end{cases}$$

2.3 Question 3

By simply getting the desired informqtion from the table, we get that:

$$\begin{aligned}
\mathbb{P}(X \in \{-1, 4, 5\}) &= \sum_{k\in\{-1,4,5\}} \mathbb{P}(X = k) \\
&= a + b + 0.1 \\
&= 0.4 + 0.3 + 0.1 \\
&= 0.8
\end{aligned}$$

3 Exercise 3

3.1 Question 1

From the plot, we have that:

$$f_X(x) = \begin{cases} a & [-1, 1[\\ -a(x-5) & [3, 5[\\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Now we know the following:

$$\begin{aligned}
&\int_{-\infty}^{\infty} f_X(x)dx = 1 \\
\implies &\int_{-1}^{1} f_X(x)dx + \int_{3}^{5} f_X(x)dx = 1 \\
\implies &\int_{-1}^{1} a\ dx + \int_{3}^{5} -a(x-5)\ dx = 1 \\
\implies &[ax]_{-1}^{1} - a\left[\frac{x^2}{2} - 5x\right]_3^5 = 1 \\
\implies &a\,[1+1] - a\left[-\frac{25}{2} + \frac{21}{2}\right] = 1 \\
\implies &2a + 2a = 1 \\
\implies &a = \frac{1}{4}
\end{aligned}$$

3.2 Question 2

As in previous exercises, we first compute the expected value of X and the expected value of X^2.

$$
\begin{aligned}
\mathbb{E}[X] &:= \int_{-\infty}^{+\infty} x f_X(x)\ dx \\
&= \int_{-1}^{1} ax\ dx - \int_{3}^{5} ax(x-5)\ dx \\
&= a\int_{-1}^{1} x\ dx - a\int_{3}^{5} x^2 - 5x\ dx \\
&= \underbrace{a\left[\frac{x^2}{2}\right]_{-1}^{1}}_{=0} - a\left[\frac{x^3}{3} - 5\frac{x^2}{2}\right]_{3}^{5} \\
&= -a\left[\left(\frac{125}{3} - 5\frac{25}{2}\right) - \left(\frac{27}{3} - 5\frac{9}{2}\right)\right] \\
&= -a\left[-\frac{125}{6} - \left(9 - \frac{45}{2}\right)\right] \\
&= -a\left[-\frac{125}{6} + \frac{27}{2}\right] \\
&= -a\left[-\frac{125}{6} + \frac{81}{6}\right] \\
&= -a\left[-\frac{44}{6}\right] \\
&= \frac{22a}{3} \\
&= \frac{11}{6}
\end{aligned}
$$

Then:

$$\begin{aligned}
\mathbb{E}[X^2] &:= \int_{-\infty}^{+\infty} x^2 f_X(x)\ dx \\
&= \int_{-1}^{1} ax^2\ dx - \int_{3}^{5} ax^2(x-5)\ dx \\
&= a\int_{-1}^{1} x^2\ dx - a\int_{3}^{5} x^3 - 5x^2\ dx \\
&= a\left[\frac{x^3}{3}\right]_{-1}^{1} - a\left[\frac{x^4}{4} - 5\frac{x^3}{3}\right]_{3}^{5} \\
&= a\frac{2}{3} - a\left[\left(\frac{5^4}{4} - 5\frac{5^3}{3}\right) - \left(\frac{3^4}{4} - 5\frac{3^3}{3}\right)\right] \\
&= a\frac{2}{3} - a\left[-\frac{5^4}{12} - \left(\frac{81}{4} - \frac{135}{3}\right)\right] \\
&= a\frac{2}{3} - a\left[-\frac{625}{12} - \frac{243}{12} + \frac{540}{12}\right] \\
&= a\frac{8}{12} + a\frac{328}{12} \\
&= \frac{82a}{3} \\
&= \frac{41}{6}
\end{aligned}$$

Thus we get that the variance of X $Var(X)$ is given by:

$$\begin{aligned}
Var(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \frac{41}{6} - \left[\frac{11}{6}\right]^2 \\
&= \frac{246}{36} - \frac{121}{36} \\
&= \frac{125}{36}
\end{aligned}$$

3.3 Question 3

We know that the CDF (Cumulative Distribution Function) is defined as follows:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\ dt$$

Then, reusing the definition of the PDF (Probability Density Function) from (3), we get that:

- If $x \in]-\infty, -1[$ then it is clear that $F_X(x) = 0$.
- If $x \in [-1, 1[$

$$
\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} f_X(t)\ dt \\
&= \underbrace{\int_{-\infty}^{-1} f_X(t)\ dt}_{=0} + \int_{-1}^{x} f_X(t)\ dt \\
&= \int_{-1}^{x} a\ dt \\
&= [at]_{-1}^{x} \\
&= a(x+1) \\
&= \frac{x+1}{4}
\end{aligned}
$$

- If $x \in [1, 3[$

$$
\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} f_X(t)\ dt \\
&= \underbrace{\int_{-\infty}^{1} f_X(t)\ dt}_{=F_X(1)} + \int_{1}^{x} \underbrace{f_X(t)}_{=0}\ dt \\
&= a(1+1) \\
&= 2a \\
&= \frac{1}{2}
\end{aligned}
$$

- If $x \in [3, 5[$

$$\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} f_X(t)\ dt \\
&= \underbrace{\int_{-\infty}^{3} f_X(t)\ dt}_{=F_X(3)} + \int_{3}^{x} f_X(t)\ dt \\
&= 2a + \int_{3}^{x} -a(t-5)\ dt \\
&= 2a - a\left[\frac{t^2}{2} - 5t\right]_3^x \\
&= 2a - a\left[\left(\frac{x^2}{2} - 5x\right) - \left(\frac{3^2}{2} - 5\times 3\right)\right] \\
&= 2a - a\left[\frac{x^2}{2} - 5x + \frac{21}{2}\right] \\
&= a\left[-\frac{x^2}{2} + 5x - \frac{17}{2}\right] \\
&= -\frac{x^2}{8} + \frac{5x}{4} - \frac{17}{8}
\end{aligned}$$

(Here, we can easily check that when setting $x = 3$ and $x = 5$, we obtain respectively $F_X(x) = \frac{1}{4}$ and $F_X(x) = 1$, which is coherent with the plot.)

- If $x \in [5, +\infty[$ then it is clear that $F_X(x) = 1$.

3.4 Question 4

$$\begin{aligned}
&\mathbb{P}(X \in [-0.1, 0.7] \cup [3.5, 7]) \\
&= \mathbb{P}(X \in [-0.1, 0.7]) + \mathbb{P}(X \in [3.5, 7]) - \underbrace{\mathbb{P}(X \in [-0.1, 0.7] \cap [3.5, 7])}_{=0} \\
&= [F_X(0.7) - F_X(-0.1)] + [F_X(7) - F_X(3.5)] \\
&= [a(0.7+1) - a(-0.1+1)] + \left[1 - a\left(-\frac{3.5^2}{2} + 5\times 3.5 - \frac{17}{2}\right)\right] \\
&= \frac{8}{10}a + 1 - a\left(-\frac{49}{8} + \frac{35}{2} - \frac{17}{2}\right) \\
&= \frac{8}{10}a + 1 - \frac{23}{8}a \\
&= 1 - \frac{83}{40}a \\
&= \frac{77}{160}
\end{aligned}$$

4 Exercise 4

Let X be $\mathcal{N}(10, 100)$. We center and scale X in order to get a $\mathcal{N}(0, 1)$. That is, we consider

$$Y := \frac{X - \mathbb{E}[X]}{\sqrt{Var(X)}} = \frac{X - 10}{10}$$

Then we have that $X = 10Y + 10$, therefore $\forall a, b \in \mathbb{R}$ such that $a \leq b$:

$$\begin{aligned}
\mathbb{P}(a \leq X \leq b) &= \mathbb{P}(a \leq 10Y + 10 \leq b) \\
&= \mathbb{P}(a - 10 \leq 10Y \leq b - 10) \\
&= \mathbb{P}(\frac{a - 10}{10} \leq Y \leq \frac{b - 10}{10})
\end{aligned}$$

Let Φ be the CDF of $\mathcal{N}(0, 1)$.

4.1 Question 1

$$\begin{aligned}
\mathbb{P}(X \in [12.5, 23.1]) &= \mathbb{P}(\frac{12.5 - 10}{10} \leq Y \leq \frac{23.1 - 10}{10}) \\
&= \mathbb{P}(0.25 \leq Y \leq 1.31) \\
&= \Phi(1.31) - \Phi(0.25) \\
&= 0.9049 - 0.5987 \\
&= 0.3062
\end{aligned}$$

4.2 Question 2

$$\begin{aligned}
\mathbb{P}(X \in [1.6, 6.9]) &= \mathbb{P}(\frac{1.6 - 10}{10} \leq Y \leq \frac{6.9 - 10}{10}) \\
&= \mathbb{P}(-0.84 \leq Y \leq -0.31) \\
&= \Phi(-0.31) - \Phi(-0.84) \\
&= [1 - \Phi(0.31)] - [1 - \Phi(0.84)] \\
&= -\Phi(0.31) + \Phi(0.84) \\
&= -0.5517 + 0.7995 \\
&= 0.2478
\end{aligned}$$

4.3 Question 3

On the one hand:

$$\begin{aligned}
\mathbb{P}(X \in [8.9, 9.5]) &= \mathbb{P}(\frac{8.9-10}{10} \leq Y \leq \frac{9.5-10}{10}) \\
&= \mathbb{P}(-0.11 \leq Y \leq -0.05) \\
&= \Phi(-0.05) - \Phi(-0.11) \\
&= [1 - \Phi(0.05)] - [1 - \Phi(0.11)] \\
&= -\Phi(0.05) + \Phi(0.11) \\
&= -0.5199 + 0.5438 \\
&= 0.0239
\end{aligned}$$

On the other hand:

$$\begin{aligned}
\mathbb{P}(X \in [22.4, 43.2]) &= \mathbb{P}(\frac{22.4-10}{10} \leq Y \leq \frac{43.2-10}{10}) \\
&= \mathbb{P}(1.24 \leq Y \leq 3.32) \\
&= \Phi(3.32) - \Phi(1.24) \\
&= 0.99955 - 0.8925 \\
&= 0.10705
\end{aligned}$$

Therefore we get for the union:

$$\begin{aligned}
\mathbb{P}(X \in [8.9, 9.5] \cup [22.4, 43.2]) &= \mathbb{P}(X \in [8.9, 9.5]) + \mathbb{P}(X \in [22.4, 43.2]) \\
&\quad - \underbrace{\mathbb{P}(X \in [8.9, 9.5] \cap [22.4, 43.2])}_{=0} \\
&= 0.0239 + 0.10705 \\
&= 0.13095
\end{aligned}$$