MSc. Data Science & Artificial Intelligence

Inverse Problems in Image Processing

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Assignment 1

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1 Exercise 1

$$\text{prox}_{\tau f}(x) = \arg\min_u \frac{1}{2\tau}\|u - x\|^2 + f(u) \tag{1}$$

$$\text{prox}_{\tau|\cdot|}(x) = \arg\min_u \frac{1}{2\tau}\|u - x\|^2 + |u| \tag{2}$$

$$\tag{3}$$

Let $h(x) = \frac{1}{2\tau}\|u - x\|^2 + |u|$. Then

$$\frac{\partial}{\partial u}h(x) = \frac{\partial}{\partial u}\frac{1}{2\tau}\|u - x\|^2 + |u|$$

$$= \begin{cases} \frac{1}{\tau}(u - x) - 1, & u < 0 \\ 0, & u = 0 \\ \frac{1}{\tau}(u - x) + 1, & u > 0 \end{cases}$$

Case $u > 0$.

$$\frac{\partial}{\partial u}h(x) = 0$$

$$\implies \frac{1}{\tau}(u - x) + 1 = 0$$

$$\implies u = x - \tau$$

Case $u < 0$.

$$\frac{\partial}{\partial u}h(x) = 0$$

$$\implies \frac{1}{\tau}(u - x) - 1 = 0$$

$$\implies u = x + \tau$$

Case $u = 0$. $\quad \partial h(u) = [-1, 1]$ Therefore

$$\text{prox}_{\tau|\cdot|}(x) = \begin{cases} x - \tau, & x < -\tau \\ 0, & -\tau \le x \le \tau \\ x + \tau, & x > \tau \end{cases}$$

2 Exercise 2

$$f(x) = |x|_0 = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0 \end{cases} \tag{4}$$

$$\text{prox}_{\tau|\cdot|_0}(x) = \arg\min_u \frac{1}{2\tau}\|u - x\|^2 + |u|_0 \tag{5}$$

$$\text{prox}_{\tau|\cdot|_0}(x) = \begin{cases} 0, & x = 0 \\ x, & x \neq 0 \end{cases} \tag{6}$$

$$h'(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

So it is better to choose $u = 0$ when $\frac{x^2}{2\tau} < 1$, else, set $u = x$.

3 Exercise 3

$$\delta_{\mathbb{R}_+^n}(x) = \begin{cases} \infty, & x \notin \mathbb{R}_+^n \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

$$\text{prox}_{\tau|\cdot|_1 + \delta_{\mathbb{R}_+^n}(\cdot)}(x) = \arg\min_u \frac{1}{2\tau}\|u - x\|^2 + |u|_1 + \delta_{\mathbb{R}_+^n}(u) \tag{8}$$

$$\text{prox}_{\tau|\cdot|_1 + \delta_{\mathbb{R}_+^n}(\cdot)}(x) = \arg\min_u \frac{1}{2\tau}\|u - x\|^2 + |u|_1 + \delta_{\mathbb{R}_+^n}(u) \tag{9}$$

Given by prof:

$$\text{prox}(x) = \max(\text{prox}_{\tau|\cdot|_1}(x), 0) \tag{10}$$

4 Exercise 4

Compute $\text{prox}_f(x)$.