MSc. Data Science & Artificial Intelligence

Inverse Problems in Image Processing

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Assignment 1

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1 Exercise 1

$$\mathrm{prox}_{\tau f}(x) = \arg\min_u \frac{1}{2\tau}\|u-x\|^2 + f(u) \tag{1}$$

$$\mathrm{prox}_{\tau|\cdot|}(x) = \arg\min_u \frac{1}{2\tau}\|u-x\|^2 + |u| \tag{2}$$

$$\tag{3}$$

Let $h(x) = \frac{1}{2\tau}\|u-x\|^2 + |u|$. Then

$$\begin{aligned}\frac{\partial}{\partial u}h(x) &= \frac{\partial}{\partial u}\frac{1}{2\tau}\|u-x\|^2 + |u| \\ &= \begin{cases}\frac{1}{\tau}(u-x)-1, & u<0\\ 0, & u<0\\ \frac{1}{\tau}(u-x)+1, & u>0\end{cases}\end{aligned}$$

$u > 0$

$$\begin{aligned}&\frac{1}{\tau}(u-x)+1 = 0\\ \implies & u = x-\tau\end{aligned}$$

$u < 0$

$$\begin{aligned}&\frac{1}{\tau}(u-x)-1 = 0\\ \implies & u = x+\tau\end{aligned}$$

$u = 0 \quad \partial h(u) = [-1, 1]$ Therefore

$$\mathrm{prox}_{\tau|\cdot|}(x) = \begin{cases} x-\tau, & x<-\tau\\ 0, & -\tau\le x\le\tau\\ x+\tau, & x>\tau\end{cases}$$

2 Exercise 2

$$f(x) = |x|_0 = \begin{cases}0, & x=0\\ 1, & x\neq 0\end{cases} \tag{4}$$

$$\mathrm{prox}_{\tau|\cdot|_0}(x) = \arg\min_u \frac{1}{2\tau}\|u-x\|^2 + |u|_0 \tag{5}$$

$$\mathrm{prox}_{\tau|\cdot|_0}(x) = \begin{cases}0, & x=0\\ x, & x\neq 0\end{cases} \tag{6}$$

$$h'(x) = \begin{cases}0, & x\neq 0\\ 1, & x=0\end{cases}$$

So it is better to choose $u=0$ when $\frac{x^2}{2\tau} < 1$, else, set $u = x$.

3 Exercise 3

$$\delta_{\mathbb{R}_+^n}(x) = \begin{cases} \infty, & x \notin \mathbb{R}_+^n \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

$$\text{prox}_{\tau|\cdot|_1 + \delta_{\mathbb{R}_+^n}(\cdot)}(x) = \arg\min_u \frac{1}{2\tau}\|u - x\|^2 + |u|_1 + \delta_{\mathbb{R}_+^n}(u) \tag{8}$$

$$\text{prox}_{\tau|\cdot|_1 + \delta_{\mathbb{R}_+^n}(\cdot)}(x) = \arg\min_u \frac{1}{2\tau}\|u - x\|^2 + |u|_1 + \delta_{\mathbb{R}_+^n}(u) \tag{9}$$

Given by prof:

$$\text{prox}(x) = \max(\text{prox}_{\tau|\cdot|_1}(x), 0) \tag{10}$$

4 Exercise 4

Compute $\text{prox}_f(x)$.