Inverse Problems in Image Processing

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Exercise 1

Given $y \in \mathbb{R}^n$ and a linear operator $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, compute the **gradient** of the n-dimensional function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined as

$$f(x) = \frac{1}{2}\|Ax - y\|_2^2 \tag{1}$$

Exercise 2

Compute the **Lipschitz constant** of the gradient of f defined in (1).

Exercise 3

Consider the 1-dimensional function $f(x) = |x|$, $x \in \mathbb{R}$. Compute the **subdifferential** of f for all $x \in \mathbb{R}$.

Properties of subdifferential calculus

Proposition 1. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be two proper convex functions.*

1. *$\forall x \in \mathbb{R}^n$ there holds*
$$\partial f(x) + \partial g(x) \subset \partial(f + g)(x). \tag{2}$$
2. *Moreover, if $\operatorname{int}(\operatorname{dom} f) \cap \operatorname{int}(\operatorname{dom} g) \neq \emptyset$, then $\forall x \in \mathbb{R}^n$*
$$\partial f(x) + \partial g(x) = \partial(f + g)(x). \tag{3}$$

Proof. First of all, we note that $\operatorname{dom}(f + g) = \operatorname{dom} f \cap \operatorname{dom} g$. Let $u \in \partial f(x)$ and $v \in \partial g(x)$. Let $z \in \mathbb{R}^n$. Then, we have

$$f(z) \geq f(x) + \langle u, z - x\rangle$$
$$g(z) \geq g(x) + \langle v, z - x\rangle$$

Summing the two inequalities above, we obtain

$$f(z) + g(z) \geq f(x) + g(x) + \langle u + v, z - x\rangle.$$

Hence $u + v \in \partial(f + g)(x)$. □

Proposition 2. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $\lambda > 0$. Then $\forall x \in \mathbb{R}^n$*

$$\lambda \partial f(x) = \partial(\lambda f)(x). \tag{4}$$

Proof. Let $x \in \mathbb{R}^n$ and $p \in \partial f(x)$. Then, by definition of subgradient for any $z \in \mathbb{R}^n$ we have $f(z) \geq f(x) + \langle p, z - x\rangle$. Hence, $\lambda f(z) \geq \lambda f(x) + \langle \lambda p, z - x\rangle$ and thus $\lambda p \in \partial(\lambda f)(x) \Longrightarrow \lambda \partial f(x) \subseteq \partial(\lambda f)(x)$. On the contrary, let $\tilde{p} \in \partial(\lambda f)(x)$. By definition, we have for any $z \in \mathbb{R}^n$ that $\lambda f(z) \geq \lambda f(x) + \langle \tilde{p}, z - x\rangle$. Dividing by $\lambda > 0$: $f(z) \geq f(x) + \langle \frac{\tilde{p}}{\lambda}, z - x\rangle$. It follows that $\frac{\tilde{p}}{\lambda} \in \partial f(x) \Longrightarrow \tilde{p} \in \lambda \partial f(x) \Longrightarrow \partial(\lambda f)(x) \subseteq \lambda \partial f(x)$. □

Proposition 3. ***Separability.*** *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ a convex separable function i.e.*

$$f(x) = \sum_{i=1}^{n} f_i(x_i) \tag{5}$$

where $f_i : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is proper and convex for any $i = 1, \ldots, n$. Then, the subdifferential of f can be obtained as

$$\begin{aligned}\partial f(x) =& \partial f_1(x_1) \times \partial f_2(x_2) \times \cdots \times \partial f_n(x_n) \\ =& \{(p_1, p_2, \ldots, p_n) \text{ s.t. } p_1 \in \partial f_1(x_1),\ p_2 \in \partial f_2(x_2),\ \ldots,\ p_n \in \partial f_n(x_n)\}.\end{aligned} \tag{6}$$

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Exercise 4

Given $y \in \mathbb{R}^n$ and a linear operator $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, use the above propositions to compute the **subdifferential** of the n-dimensional function

$$F(x) = \frac{1}{2}\|Ax - y\|_2^2 + \lambda\|x\|_1, \ \ \lambda > 0. \tag{7}$$