Inverse Problems in Image Processing

Faisal Jayousi

TD 5: 8th February 2023

Exercise 1: soft-thresholding

Consider the 1-dimensional function $f(x) = |x|$, $x \in \mathbb{R}$. Given a positive scalar $\tau > 0$, compute the **proximal operator** of τf for all $x \in \mathbb{R}$, i.e.

$$\operatorname{prox}_{\tau f}(x) = \underset{u \in \mathbb{R}}{\operatorname{argmin}} \frac{1}{2\tau}(u - x)^2 + f(u) \tag{1}$$

$$\operatorname{prox}_{\tau |\cdot|}(x) = \underset{u \in \mathbb{R}}{\operatorname{argmin}} \frac{1}{2\tau}(u - x)^2 + |u| \tag{2}$$

and **plot** it as a function of x.

Exercise 2: hard-thresholding

Consider the 1-dimensional function

$$f(x) = |x|_0 = \begin{cases} 1 & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}. \tag{3}$$

Given a positive scalar $\tau > 0$, compute the **proximal operator** of τf for all $x \in \mathbb{R}$, i.e.

$$\operatorname{prox}_{\tau |\cdot|_0}(x) = \underset{u \in \mathbb{R}}{\operatorname{argmin}} \frac{1}{2\tau}(u - x)^2 + |u|_0 \tag{4}$$

and **plot** it as a function of x.

Separability

Proposition 1. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ a convex separable function i.e.*

$$f(x) = \sum_{i=1}^{n} f_i(x_i) \tag{5}$$

where $f_i : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is proper and convex for any $i = 1, \dots, n$. Then, the proximal operator of f is equivalent to

$$\operatorname{prox}_f(x) = \big(\operatorname{prox}_{f_1}(x_1), \cdots, \operatorname{prox}_{f_n}(x_n)\big). \tag{6}$$

Exercise 3: non-negativity constraints

Given the **indicator function** $\delta_{\mathbb{R}^n_+}$, compute its **proximal operator** $\operatorname{prox}_{\delta_{\mathbb{R}^n_+}}(x)$ for any $x \in \mathbb{R}^n$, taking into account the separability (see Proposition 3 in the following).

Compute the proximal operator of $\tau\|\cdot\|_1 + \delta_{\mathbb{R}^n_+}(\cdot)$.

Properites of proximal operators

Proposition 2. *Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper, and let $\lambda \neq 0$. Define $f(x) = \lambda g(x/\lambda)$. Then*

$$\operatorname{prox}_f(x) = \lambda \operatorname{prox}_{g/\lambda}(x/\lambda). \tag{7}$$

Proof. Note that

$$\operatorname{prox}_f(x) = \underset{u \in \mathbb{R}^n}{\operatorname{argmin}} \left\{ f(u) + \frac{1}{2}\|u - x\|^2 \right\} = \underset{u \in \mathbb{R}^n}{\operatorname{argmin}} \left\{ \lambda g\left(\frac{u}{\lambda}\right) + \frac{1}{2}\|u - x\|^2 \right\}.$$

Making the change of variables $z = \frac{u}{\lambda}$, we can continue to write

$$\begin{aligned}
\operatorname{prox}_f(x) &= \lambda \operatorname*{argmin}_{z\in\mathbb{R}^n} \left\{ \lambda g(z) + \frac{1}{2}\|\lambda z - x\|^2 \right\} \\
&= \lambda \operatorname*{argmin}_{z\in\mathbb{R}^n} \left\{ \lambda^2 \left[\frac{g(z)}{\lambda} + \frac{1}{2}\left\| z - \frac{x}{\lambda} \right\|^2 \right] \right\} \\
&= \lambda \operatorname*{argmin}_{z\in\mathbb{R}^n} \left\{ \frac{g(z)}{\lambda} + \frac{1}{2}\left\| z - \frac{x}{\lambda} \right\|^2 \right\} \\
&= \lambda \operatorname{prox}_{g/\lambda}(x/\lambda).
\end{aligned}$$

□

Proposition 3. *Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper, and let $f(x) = g(x) + \frac{c}{2}\|x\|^2$ with $c > 0$. Then*

$$\operatorname{prox}_f(x) = \operatorname{prox}_{\frac{1}{c+1}g}\left(\frac{x}{c+1}\right) \tag{8}$$

Proof. Follows by the following simple computation:

$$\begin{aligned}
\operatorname{prox}_f(x) &= \operatorname*{argmin}_{u\in\mathbb{R}^n} \left\{ f(u) + \frac{1}{2}\|u - x\|^2 \right\} \\
&= \operatorname*{argmin}_{u\in\mathbb{R}^n} \left\{ g(u) + \frac{c}{2}\|u\|^2 + \frac{1}{2}\|u - x\|^2 \right\} \\
&= \operatorname*{argmin}_{u\in\mathbb{R}^n} \left\{ g(u) + \frac{c+1}{2}\left\| u - \left(\frac{x}{c+1}\right) \right\|^2 \right\} \\
&= \operatorname{prox}_{\frac{1}{c+1}g}\left(\frac{x}{c+1}\right).
\end{aligned}$$

□

Exercise 4

Compute the **proximal operator** of the **elastic net** functional

$$f(x) = \|x\|_1 + \lambda/2\|x\|_2^2 \tag{9}$$

using the properties above.