Inverse Problems in images by variational methods

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Image Construction

$$g = H(u) \text{ ☺ } n$$

- g : observation = Physical quantities : optics, radar, laser, IR, magnetic field, X rays, ultrasons, …
- H : operator which links the observation to the quantity we are looking for, we want to image u, through the measuring instrument and possibly a reconstruction process.
- u : image we want to obtain
- n : random part in the observation process (noise)

Observation model: Gaussian noise case

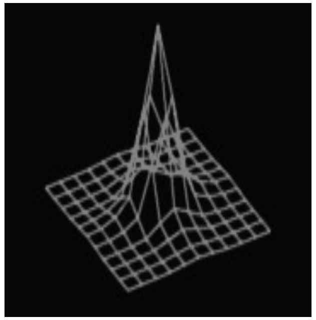
Observed images are degraded :

$$g = h * u + n$$

Noise
white Gaussian

Observed Image

Convolution kernel (PSF)

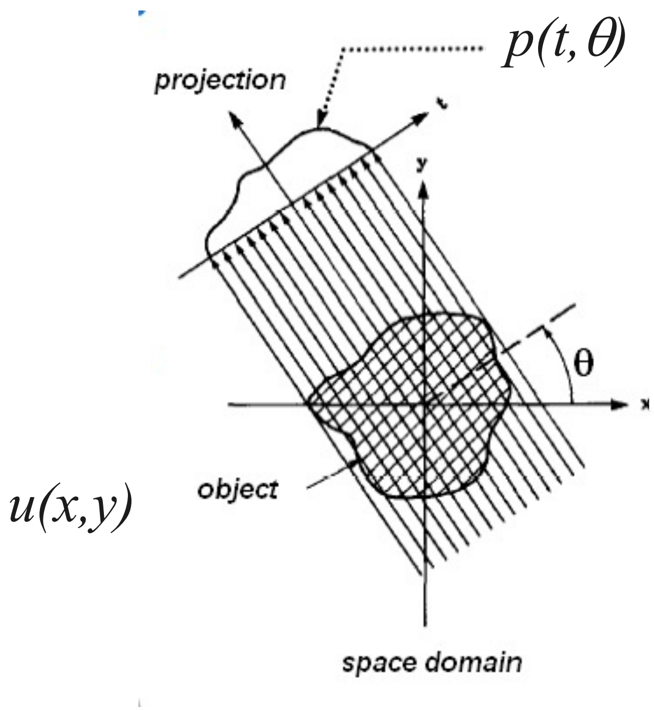


Original Image

- Restoration : retrieve u from g
- Inverse $g = h * u + n$ is an ill posed problem

Examples

- Linéaires : $H(u)= H.u$
  - Tomographic Reconstruction, in medical imaging or geosciences,….
  - Reconstruct the volume of an object (the human body in the case of medical imaging), based on a series of measurements made outside the object.
    Example of X-ray tomography

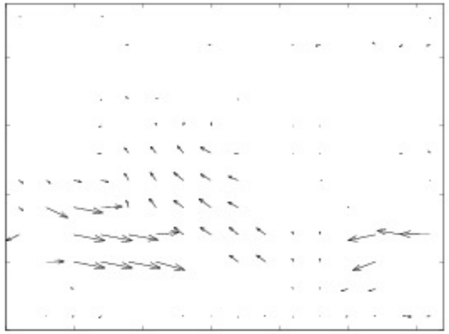


H= Radon Matrix = 1D projection according to the angle θ

Examples

- Non Linear example :

  - Optical flow
    we observe $I(x,t)$ and $I(x,t+1)$ and we're looking for the apparent movement $v(x,t)$ such that $I(x+v,t) = I(x,t+1)$, or
  - $I(x+v(x,t),t) - I(x,t+1)=0$



Linearization by derivation: optical flow equation

$$\nabla I\left(x,t+1\right).v\left(x,t\right)+I_t\left(x,t\right)=0$$

  - Microwave reverse diffraction, non-destructive control: Maxwell equations …

- **Direct Problem** $g = H(u)$ ☺ n

  It is the equation of image construction, the mathematical modeling of the physical phenomena of acquisition.

It defines g from H, u and n.

- **Inverse Problem**

  From the observed data g, and modeling of the direct problem, find u assuming H known and the law parameters of n known.

  → H could be partially only known,

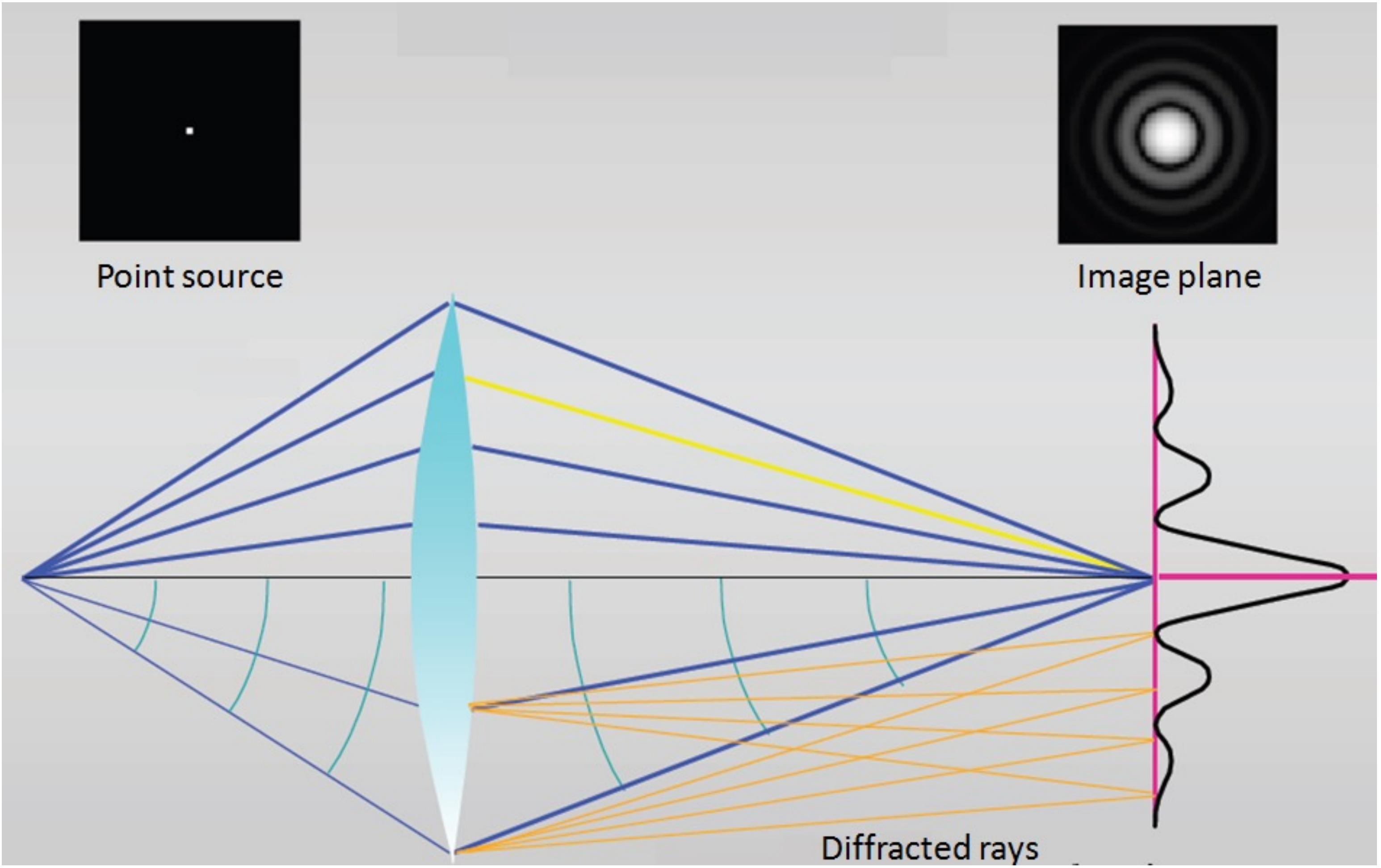
  → Noise parameters must be estimated before or in the same time.

Image Restoration: deblurring, denoising

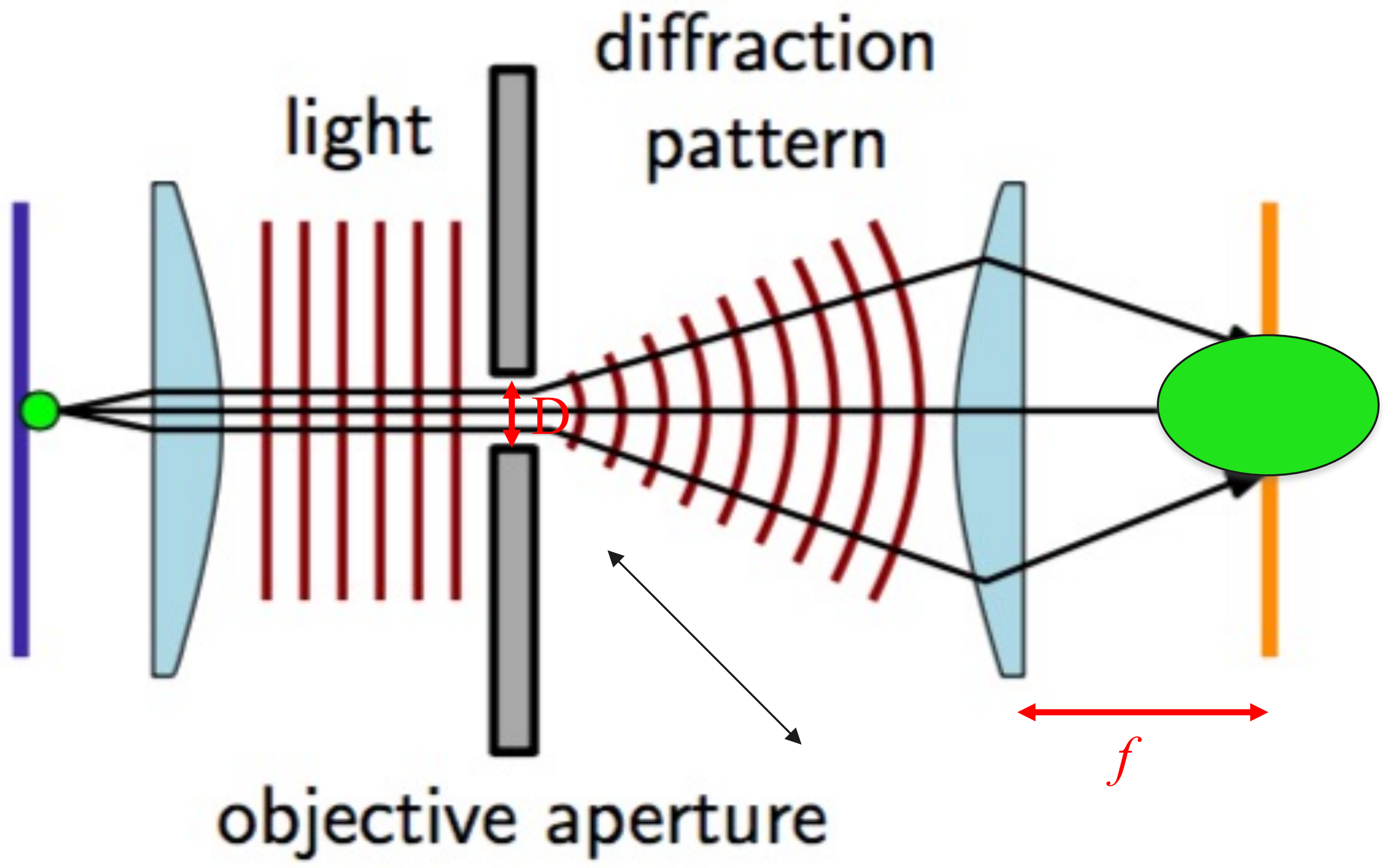
- Optical images have limited resolution due to diffraction limit
- Model the diffraction effect on images : blur is modeled by convolution
- Simple Model:
  - Convolution is linear
  - But difficult to inverse: ill-posed problem.

Diffraction

The image of a point source is the Airy patch

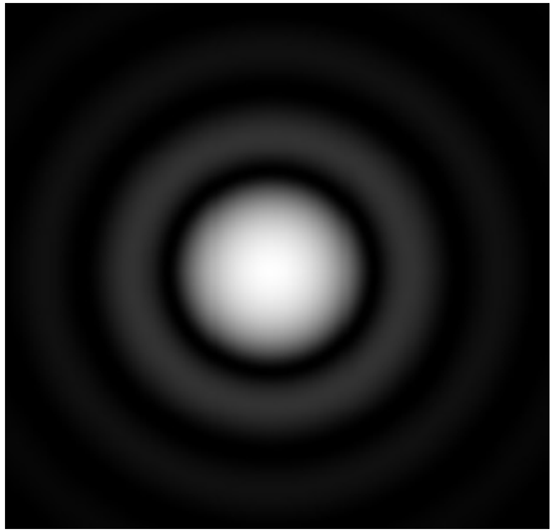


Diffraction



Optical system blurring : diffraction (continuing)

- Objective aperture D, λ is the wavelength (in visible light $\lambda = 0.6 \mu m$), f is the focal length

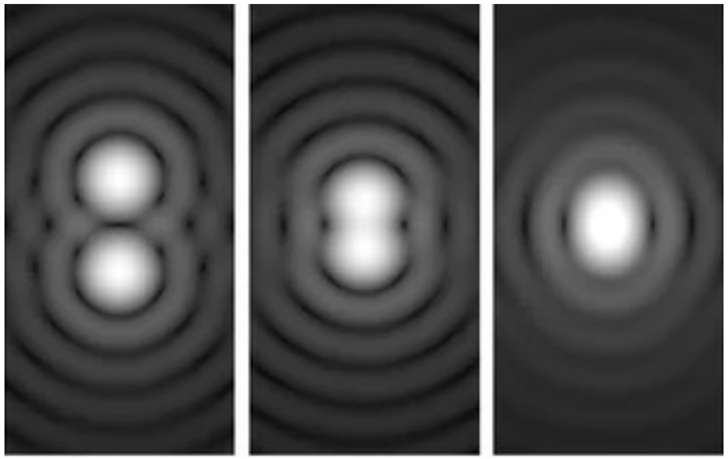


- The image of a point is the Airy patch → bright circular spot with attenuated annulus.

The radius of the Airy patch brings an idea of the dimension of the smallest details which can be view with an ideal optic. The radius is given by

$$r_A = 1.22 \frac{\lambda f}{D}$$

It also gives information on the resolution of the image. If two points are closer than r they cannot be resolved.



PSF

- All image formation system is not perfect and introduces blur in the observed image.
- The degree of spreading (blurring) of a single point like (Sub Resolution) object is a measure for the quality of an optical system. The 2D or 3D blurry image of such a single point light source is called the **Point Spread Function** (PSF).
- In general, the blurring is largely due to diffraction limited imaging by the instrument (in x,y directions)
- It could also be due to out of focus.
- The convolution is the mathematical model which explains the formation of an image that is degraded by blurring.

Spatial Dispersion: IR or PSF

We call impulse response (IR) or Point Spread Function (PSF), the imaging system response to a punctual intensity distribution (Dirac distribution). The PSF function is expressed in the continuous setting image coordinates. It is normalized and positive; It is usually denoted by h variable.

$$PSF \quad h : \Omega \subset R^2 \rightarrow R^+$$

$$x \rightarrow h(x) \quad \text{and} \quad \int_\Omega h(x)dx = 1$$

In the most general case, h is depending on the position and the spatial intensity distribution of the scene.

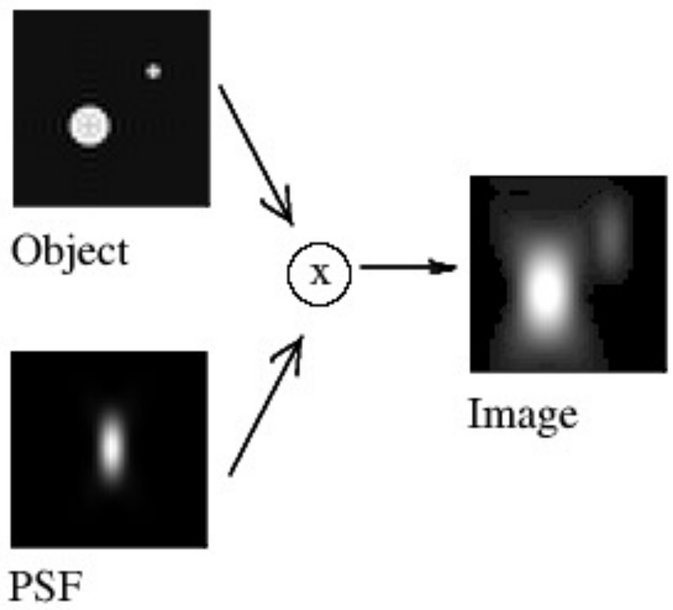
This is the PSF: Point Spread Function

Optical system blurring

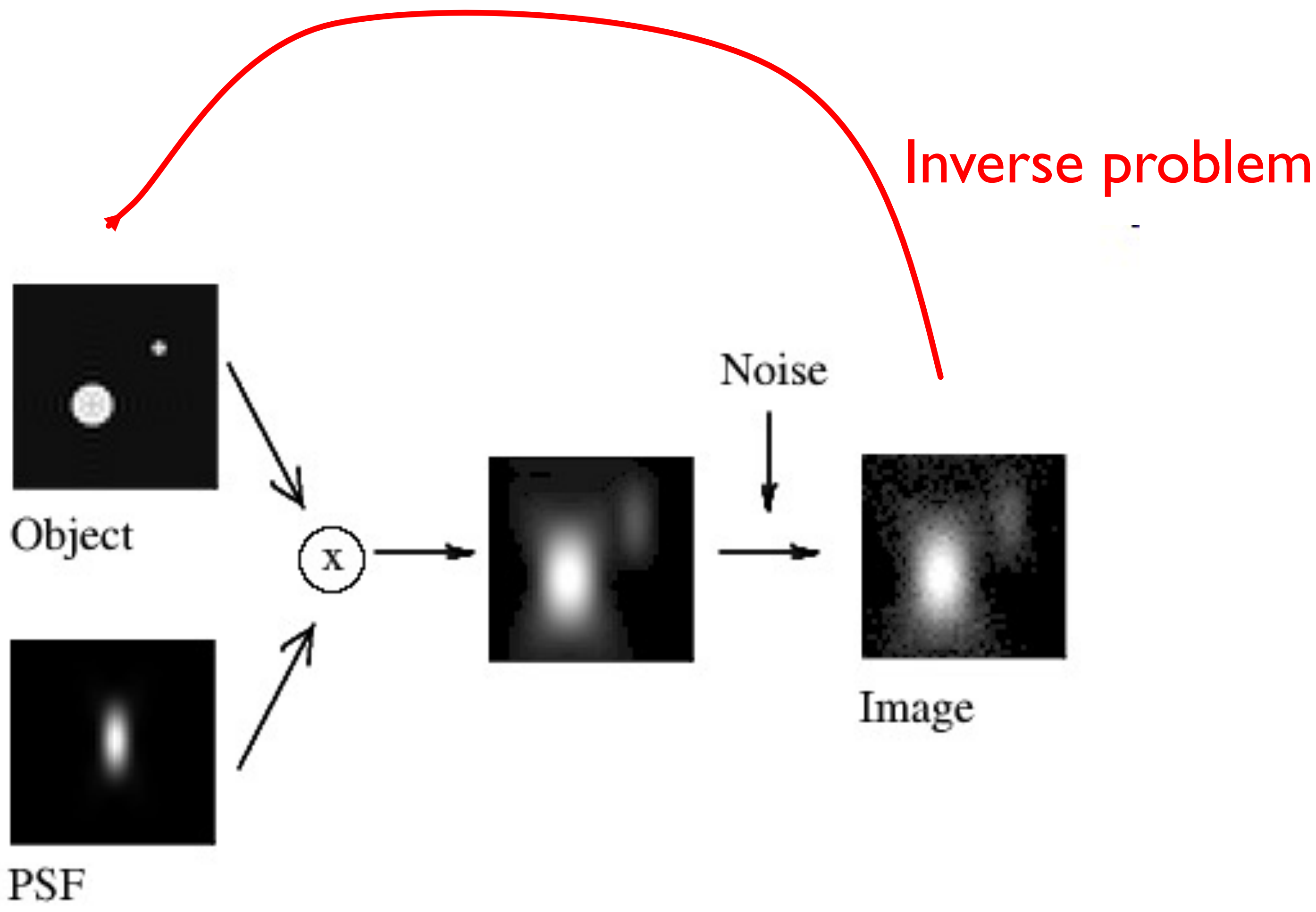
- Diffraction is mathematically described by a convolution equation of the form $$g_{x,y} = \left(h * u\right)_{x,y}$$
- where the image g arises from the convolution of the real light sources u (the specimen) and the PSF h. The convolution operator $*$ implies an integral all over the space: $$g_{x,y} = \int_{\Omega} h(x-s, y-t)\, u(s,t)\, ds\, dt$$
- **Interpretation** You can interpret the convolution equation as follows: the recorded intensity in a voxel located at point (x,y) of the image g arises from the contributions of all points of the specimen u, their real intensities weighted by the PSF h depending on the distance to the considered point.

Optical system blurring

- The image formation model is a convolution btw the object and the PSF. It gives for example:



Restoration : inversion of the image formation model (blur + noise)



Notations, assumptions

$\Omega \subset \mathrm{R}^2$ Open bounded subset

Continuous variables: $u(x)$

$$u: \quad \Omega \to \mathrm{R}$$

$$x \to u(x)$$ Grey level at point $x=(x_1,x_2)$

$\Omega \subset \mathrm{N}^2$ Bounded subset of discrete points

Discrete variables : pixel i,j

$$\mathrm{u}_{i,j} = \mathrm{u}(i\Delta x, j\Delta y), i, j = 0 \ldots N$$

g :observed image, degraded from u

Discrete Fourier Transform (recall)

- let u be a discrete signal of finite support :

  $u_0, u_1, \ldots\ldots u_{N-1}$

- Its Dicrete Fourier Transform is

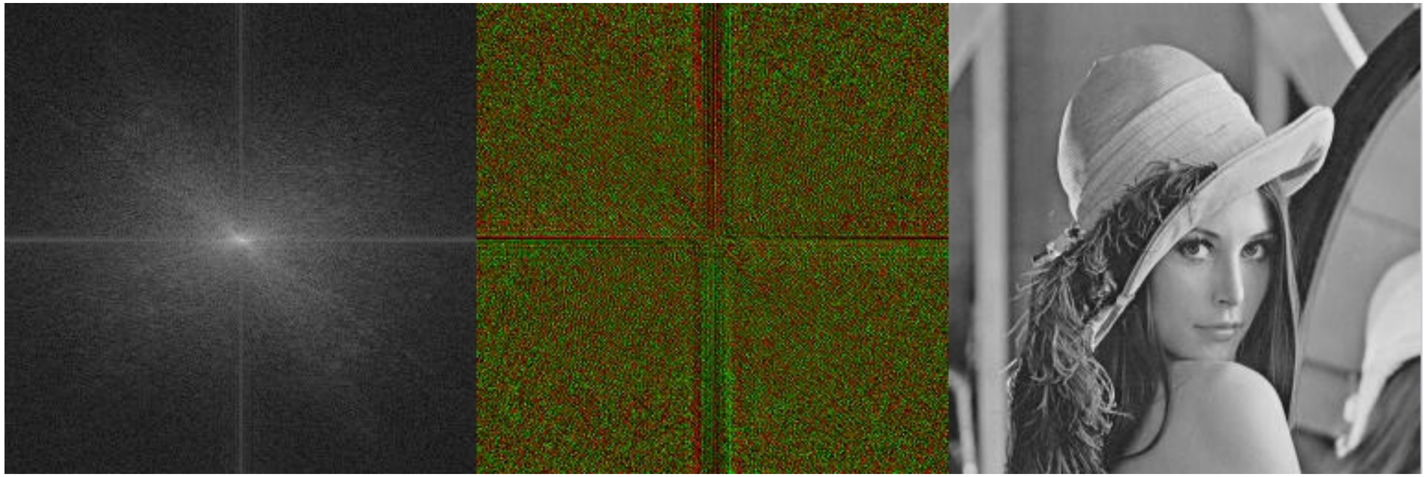
$$\hat{u}_k = \sum_{n=0}^{N-1} u_n \exp\left(\frac{-2i\pi kn}{N}\right)$$

- The Inverse Discrete Fourier transform is

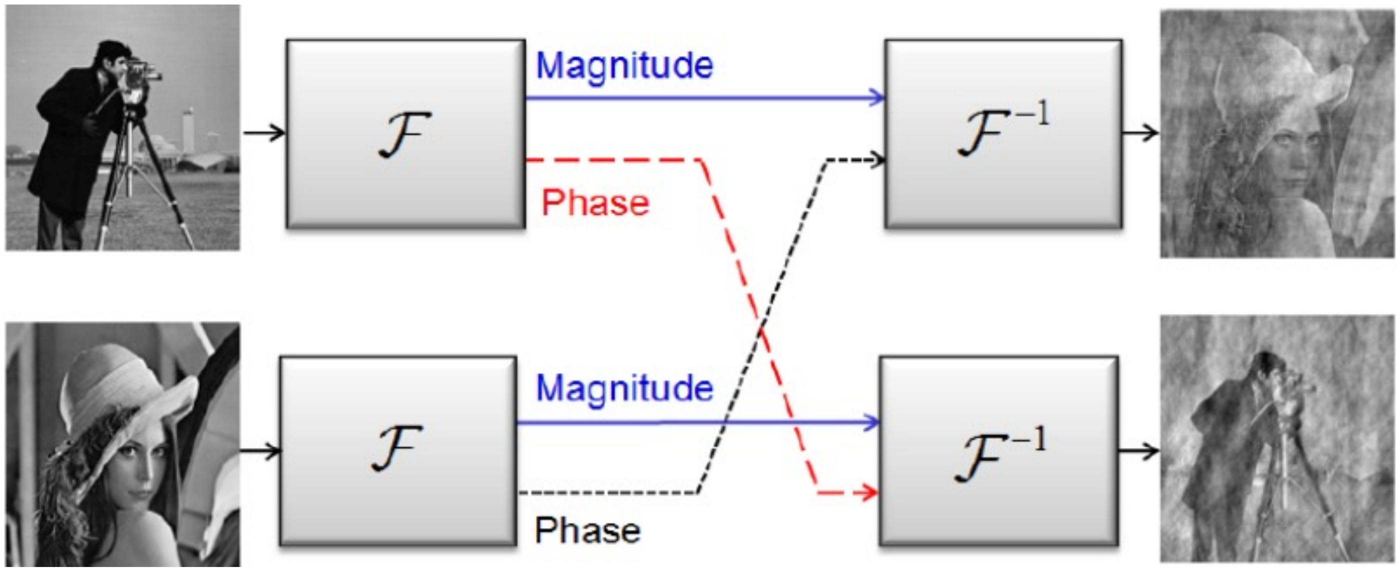
$$u_n = \frac{1}{N} \sum_{k=0}^{N-1} \hat{u}_n \exp\left(\frac{2i\pi kn}{N}\right)$$

- Fast algorithm : FFT

Discrete Fourier Transform: magnitude/phase



magnitude phase



Continuous Fourier Transform (recall)

- Let u be a function from $\mathrm{R}^{\mathrm{d}} \rightarrow \mathrm{R}$ (for 2D images, d=2). We assume that $u \in L^1(\mathrm{R}^{\mathrm{d}})$, i.e.

$$\int_{R^d} |u(x)|\, dx < \infty$$

- The Fourier Transform (F) of $u \in L^1(\mathrm{R}^{\mathrm{d}})$, is the continuous function defined by

$$F(u) = \hat{u} \quad \text{and} \quad \forall \zeta \in R^d,\ \hat{u}(\zeta) = \int_{R^d} u(x) \exp\left(-i \langle \zeta, x \rangle\right) dx \qquad (1)$$

where $< \zeta, x >$ is the standard real scalar product $\langle \zeta, x \rangle = \sum_{i=1}^{d} \zeta_i . x_i$

$\hat{u}$ is continuous and $\hat{u}(\zeta) \rightarrow 0$ when $|\zeta| \rightarrow +\infty$

When $\hat{u} \in L^1$, we can retrieve the initial function u with the inverse Fourier transform

$$u(x) = \frac{1}{(2\pi)^d} \int_{R^d} \hat{u}(\zeta) \exp\left(i \langle \zeta, x \rangle\right) d\zeta \qquad (2)$$

Equations (1) and (2) are written in the L^1 sense, that is u (or $\hat{u}$) equals a.e. the continuous function defined by the right-hand side term.

Fourier Transform (recall)

- If $u \in S$ where S is the Schwartz space of functions $u \in C^{\infty}$ quickly decreasing that is $x^{\alpha}\partial^{\beta} u(x) \to 0$ when $|x| \to \infty\ \forall(\alpha,\beta) \in N^2$, then $\hat{u} \in S$ too.

The Fourrier transform $F : u \to \hat{u}$ is an isomorphism of S and can be continuously extended to an isomorphism on L^2.

- Parseval : $\boxed{\|u\|_2 = \|\hat{u}\|_2}$

- Some properties: let u and v two functions in S, we have

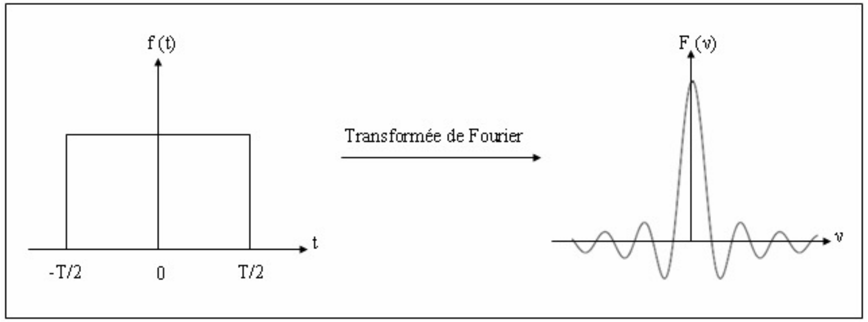
$$\boxed{\widehat{u * v} = \hat{u}.\hat{v}} \quad \text{and} \quad \widehat{u.v} = \frac{1}{(2\pi)^d}\hat{u} * \hat{v} \qquad (3)$$

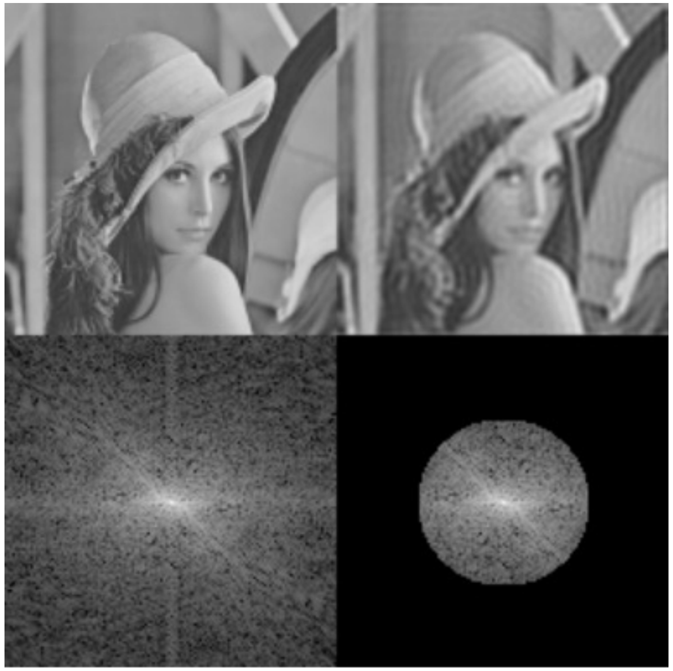
NB : making the change of variable $\zeta = 2\pi f$ we have:

$$u(x) = \int_{R^d} \hat{u}(f)\exp(2i\pi\langle f, x\rangle)\,df$$

and $\boxed{\widehat{u.v} = \hat{u} * \hat{v}}$

Fourier Transform and convolution





spatial coordinates:
convolution by a sinus cardinal

frequency coordinates:
cut-off high frequencies

Spatial Dispersion: FTM

Under assumption of blur kernel is stationary and independent of the scene:

Blur = convolution

$$g(x,y) = (h * u)_{x,y} = \int_{\Omega} h(x-s, y-t)\,u(s,t)\,ds\,dt$$

In Fourier space

$$F(g)_{u,v} = F(h)_{u,v} . F(u)_{u,v}$$

From the scene to the image on the sensor:

Optical system, sensor integration

Discrete Convolution

- Let U a discrete signal (finite length) U(0)…U(N-1)
- Let h be the discrete PSF

$$g(k) = \sum_{n=0}^{N-1} h(n)u(k-n) = \sum_{n=0}^{N-1} h(k-n)u(n)$$

- With centered PSF h(-K)….h(K) $\quad g(n) = \sum_{k=-K}^{K} h(k)u(n-k)$

- We have for the PSF $\quad \sum_{k=-K}^{K} h(k) = 1 \qquad h(k) \geq 0 \quad \forall k$

- Matrix/vector writting $\quad \mathrm{g} = \mathrm{Hu}$

  H is a band Toeplitz matrix if boundary conditions are zero

  H is a circulante matrix if boundary conditions are periodic.

  In the circulante case H can be diagonalized by DFT.

Sensor

- **Integration**
  - Each sensor is an integrator. If we have a matrix of sensors, each one is modeled by a rectangular cell of size p_x x p_y : photosensible area, which are distributed in a grid p_{ex} , p_{ey}: pixel step size.

$$u_{k,l} = \int_{[-p_x,p_x]\times[-p_y,p_y]} u\left(kp_{ex} + x, lp_{ey} + y\right) dxdy$$

  - It is also a convolution, the following PSF:

$$PSF_{int} = \sum_{k,l} 1_{[-p_x,p_x]\times[-p_y,p_y]} \delta_{kp_{ex},lp_{ey}} \quad \text{et} \quad u_{k,l} = \left(PSF_{int} * u\right)\left(kp_{ex}, lp_{ey}\right)$$

  - Associated FTM is

$$\left(FTM_{int}\right)_{u,v} = \frac{\sin\left(\pi u \frac{p_x}{p_{ex}}\right)}{\pi u \frac{p_x}{p_{ex}}} \frac{\sin\left(\pi u \frac{p_y}{p_{ey}}\right)}{\pi u \frac{p_y}{p_{ey}}}$$

Spatial Dispersion

- PSF is (at least)

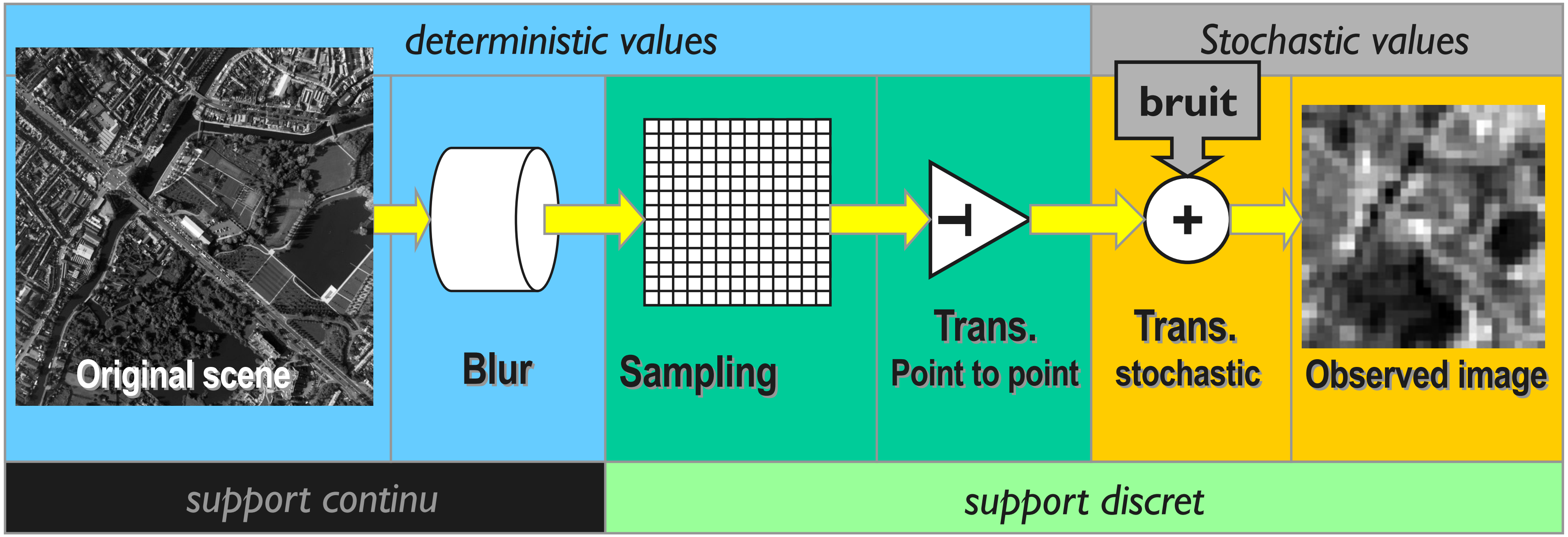
$$PSF = PSF_{optics} * PSF_{sensors}$$

Which correponds to the FTM

$$FTM = FTM_{optics} \,.\, FTM_{sensor}$$

- We use simplified models with few parameters, and usually PSF is approximated by a Gaussian function.

Observation model for satellite imaging



- Physical degradation (diffraction blur) is applied in the continuous setting, then the sensor integration makes the variables discrete. The true model is discrete/continuous:

$$g_{i,j} = (h_s * h_b * u)_{i,j} + n_{i,j}$$

$$u: \quad \Omega \subset R^2 \to \mathrm{R}$$
$$(x_1, x_2) \to u(x_1, x_2)$$

$$h_b: \quad \Omega \subset R^2 \to \mathrm{R}$$
$$(x_1, x_2) \to u(x_1, x_2)$$

$$h_s: \quad \Omega \subset R^2 \to \mathrm{R}$$
$$(x_1, x_2) \to h_s(x_1, x_2) = \sum_{k,l \in Z^2} \delta_{kp_{ex}, lp_{ey}} . 1_{[-p_x, p_x] \times [-p_y, p_y]}(x_1, x_2)$$

- Let X be a random sequence. If X models an image it is a random field $X = X_{i,j}$ $i,j=1,\ldots N$ Each $X_{i,j}$ is a random variable, which is characterized by its density probability, continuous or discrete, denoted $p_X(x,i,j)$
- Stationary assumption: the density probality is the same for all pixels: $p_X(x,i,j) = p_X(x)$ for all (i,j)
- Under stationary assumption, the mean and the variance are given by

$$m_X = \int_{x \in \mathrm{R}} x \, p_X(x) \, dx$$

$$\sigma_X^2 = \int_{x \in \mathrm{R}} \left(x - m_X\right)^2 p_X(x) \, dx$$

Random sequences (stationay assumption)

- The correlation function or autocorrelation needs the joint probability densities

$$R_X(k,l) = E\left[X(i,j)X^*(i+k,j+l)\right] =$$
$$= \int_{x_1}\int_{x_2} x_1 x_2^* \, p_{XX}(x_1,x_2,k,l)\,dx_1\,dx_2, \quad \forall i,j$$

- The covariance or autocovariance is defined by

$$C_X(k,l) = E\left[\left[X(i,j)-m_X\right]\left[X(i+k,j+l)-m_X\right]^*\right] \quad \forall i,j$$
$$= \int_{x_1\in\mathrm{R}}\int_{x_2}\left[x_1-m_X\right]\left[x_2-m_X\right]^* p_X(x_1,x_2,k,l)\,dx_1\,dx_2$$

Let remark that $\sigma_X^2 = C_X(0,0)$

- Ergodicity : allows to identify mathematical expectations (over sets) with infinite spatial means. For a stationary random sequence, it means that

$$R_X(k,l) = \lim_{N \to +\infty} \frac{1}{N^2} \sum_{i,j=0}^{N} x(i+k, j+l)\, x^*(i,j)$$

We also have

$$m_X = \lim_{N \to +\infty} \frac{1}{N^2} \sum_{i,j=0}^{N} x(i,j)$$

Random sequences

- Correlation matrix of a real signal $X^T = (X_1, \dots X_N)$
  1D indexes, for 2D images, rank the 2D indexes in a 1D vector by lexicographic ordering (line by line)

$$E\left(X\, X^* \right) = \begin{pmatrix} E(X_1 X_1) & E(X_1 X_2) & . & . & E(X_1 X_N) \\ E(X_1 X_2) & E(X_2 X_2) & & & \\ & & . & & \\ & & & . & \\ E(X_1 X_N) & & & & E(X_N X_N) \end{pmatrix}$$

- In the stationay case:

$$E\left(X\, X^T \right) = \begin{pmatrix} R_X(0) & R_X(1) & . & . & R_X(N-1) \\ R_X(1) & R_X(0) & & & \\ & & . & & \\ & & & . & \\ R_X(N-1) & & & & R_X(0) \end{pmatrix}$$

They are symmetric Toeplitz matrices

Gaussian White Noise

- Let n be the noise. It is a multidimensional variable on a field of pixels (i,j) $i,j=1,\dots N$.

- If n is a white noise, the random variables $n(i,j)$ are mutually independent, so uncorrelated variables. Then the autocorrelation matrix is diagonal. Moreover if we assume a stationary noise then the autocorrelation matrix written as

$$E\left(nn^t\right) = \sigma^2 Id_N$$

- If the noise is a white Gaussian noise with 0 mean, the joint density probabilities of all pixels is of Gaussian law $N(0_N, \sigma^2 Id_N)$ given by

$$P_n\left(n\right) = \frac{1}{\left[2\pi\right]^{\frac{1}{2}} \sigma^n} \exp - \frac{\left(n\right)^t \left(n\right)}{2\sigma^2} = \frac{1}{\left[2\pi\right]^{\frac{1}{2}} \sigma^n} \exp - \frac{\left\|n\right\|^2}{2\sigma^2}$$

Noise

Several sources of noise

- Quantum noise: electron accumulation, photon count, Poisson statistic.
- Thermal noise and acquisition noise: Gaussian statistic.
- Quantification noise: uniform, small variance wrt other noise sources.
- Compression noise : colored, correlated, non stationary. Difficult to take into account, considered as Gaussian noise in first approximation
- Transmission noise : loss of bits… Difficult to take into account.

Assumptions (realistic) : independence of noises between themselves and independence between pixels (white noise) and stationarity of the distribution (same law in each pixel).

Poisson Noise + Gaussian Noise → approximation by white additive Gaussisn noise with zero mean and variance which depends on the intensityt de variance $u_{i,j}$ at pixel *(i,j)*

$$P(n/u) = \prod_{i,j} \mathcal{N}_2\left(0, \left(A + B u_{i,j}\right) Id\right)$$

Noise with stationary law and non stationary variance.

At high count rate (real optical scene, long time exposure), Poisson law tends to Gaussian law. The noise is then white Gaussian $\mathcal{N}(0, \sigma^2 Id)$

Gaussian noise assumption

The noise is additive : $g=h*u+n$

In each pixel i, n_i is a random variable with a Gaussian distribution :

$$P\left(n_i=\alpha_i\right)=\frac{1}{Z}\exp-\frac{\alpha_i^2}{2\sigma^2}$$

The random variable N^3-dimensionnel $n=(n_1,n_2,,,,n_{N3})$ has a Gaussian distribution with zero mean parameter and variance σ^2 with independence between pixels:

$$P\left(n_1=\alpha_1,n_2=\alpha_2,\ldots,n_{N^3}=\alpha_{N^3}\right)=\prod_{i=1}^{N^3}P\left(n_i=\alpha_i\right)$$

$$P\left(n_1=\alpha_1,n_2=\alpha_2,\ldots,n_{N^3}=\alpha_{N^3}\right)=\frac{1}{Z'}\exp-\frac{\sum_{i=1}^{N^3}\alpha_i^2}{2\sigma^2}=\frac{1}{Z'}\exp-\frac{\|\alpha\|^2}{2\sigma^2}$$

Gaussian noise assumption

With the model $g=h*u+n$ the probability of observing g if I know that u is the result of the convolution of h with the specimen u is the likelihood:

$$P\left(g/\left(h*u\right)\right)=P\left(g-\left(h*u\right)=n/\left(h*u\right)\right)=P\left(g-\left(h*u\right)=n\right)$$

$$P\left(g/\left(h*u\right)\right)=\frac{1}{Z'}\exp-\frac{\sum_{i=1}^{N^3}\left[\left(h*u\right)-g\right]_i^2}{2\sigma^2}=\frac{1}{Z'}\exp-\frac{\left\|g-h*u\right\|^2}{2\sigma^2}$$

The Maximum Likelihood estimator of u is $\max_u P\left(g/\left(h*u\right)\right)$

Which is equivalent to $\min_u \left\|g-h*u\right\|^2$

Poisson density

The scalar random variable Y has a Poisson distribution with λ parameter

$$Y \propto \mathcal{P}(\lambda) \quad \Leftrightarrow \quad P(Y = y / \lambda) = \frac{\lambda^{y} e^{-\lambda}}{y!}$$

This models the photon shot noise, which is a count measure, modeled by Poisson law.

The λ parameter is the mean and the variance.

For high count (high λ parameter), the Poisson law is well approximated by a Gaussian law

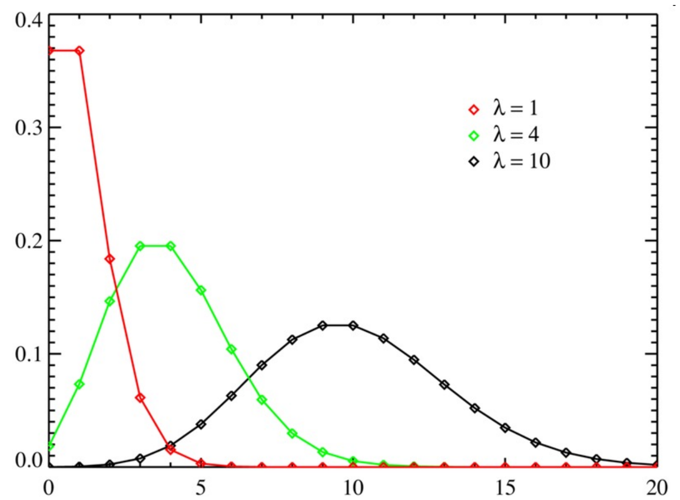
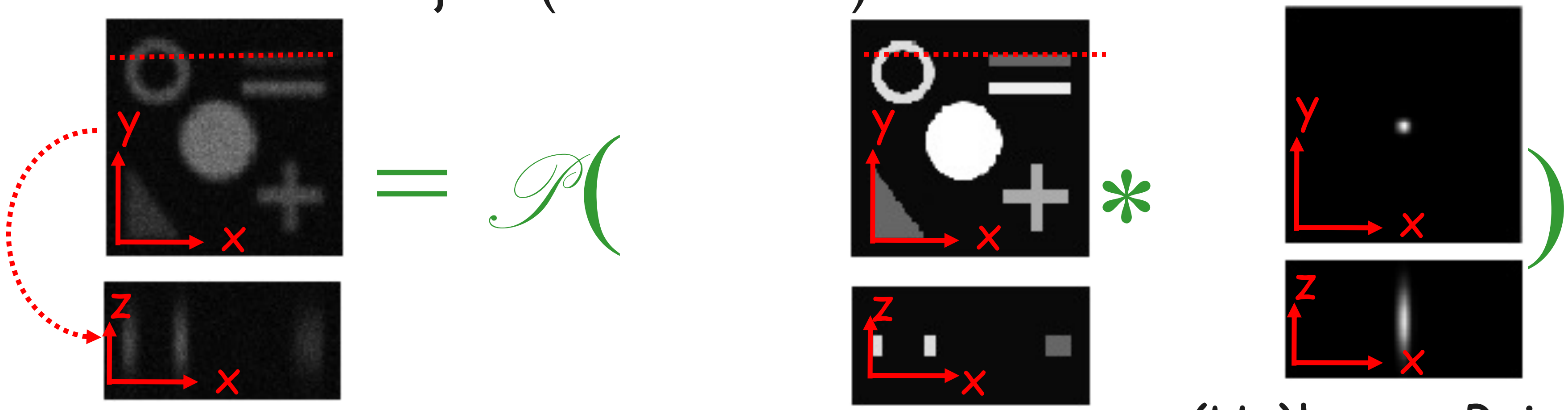


Image observation model (Poisson noise)

Simulated 3D object (128x128x64)



Observed image

Unknown true object

(Un)known Point Spread Function

$$g = \mathscr{P}(u * h)$$

- The image is blurred: the degradation is given by the Point Spread Function h.
- The image is noisy: the noise is usually photon noise, a term that refers to the inherent natural variation of the incident photon flux.

➤ Restoration Goal: Given the observation i, recover the object o

Poisson density

The random variable Y has a Poisson distribution with λ parameter which models the photon count noise :

$$Y \propto \mathsf{P}(\alpha) \quad \Leftrightarrow \quad P\left(Y = y / \alpha\right) = \frac{\alpha^{y} e^{-\alpha}}{y!}$$

Writing $g = \mathscr{P}(u*h)$ means that g has a Poisson distribution with parameter $u*h$. More precisely, in each pixel i, g_i has a Poisson distribution with parameter $(u*h)_i$

$$P\left(g_i / \left[u * h\right]_i\right) = \frac{[h*u]_i^{g_i} e^{-[h*u]_i}}{g_i\,!}$$

Poisson noise assumption

The Maximum Likelihood estimator of u is $\max_u P\left(g/\left(h*u\right)\right)$

In the Poisson noise case, due to independence between pixels:

$$P(g/u,h)=\prod_i \frac{[h*u]_i^{g_i}\, e^{-[h*u]_i}}{g_i!}$$

Which is equivalent to the minimization problem of
– log(P(g/u,h))

$$\boxed{\min_u \sum_i [(h*u)_i - g_i.\log(h*u)_i]}$$

Observation model: Gaussian noise case

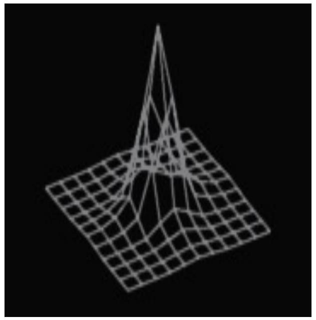
Observed images are degraded :

$$g = h * u + n$$

Noise
white Gaussian

Observed Image

Convolution kernel (PSF)



Original Image

- Restoration : retrieve u from g
- Inverse $g = h * u + n$ is an ill posed problem

Image Restoration

- Retrieve u, from the observed image $g = Hu + n$
- We assume that
  - Operator H is known,
  - Statistics (pdf, mean, standard deviation…) of the noise n are known

- Restoration = deconvolution problem.
- if $H = id$: denoising

- What is the difficulty of this inverse problem?

Difficulty of the inversion

- Assume that we are in the discrete setting, and that u have N points and u_0 have M points.
  - if $M>N$ we observe more points than the number of point we want to compute. The problem is over-determinate. The equations can be compatible or not. In any cases, select N equations among M or compute the least solution:

$$\underset{u}{Min}\left\| Hu - g \right\|^2$$

  - If $M<N$ we observe less points than the number of point we want to compute. The problem is under-determinate. Add constraints on the solution to select one solution among the set of solution.
  - If $M=N$ the H matrix is square (NxN). Then H can be invertible or not. If it is not invertible, with almost one null singular value, then the problem is again under-determinate. Add constraints on the solution to select one solution among the set of solution.

Difficulty of the inversion

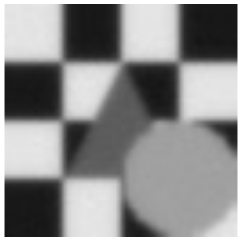
- Assume the matrix H is square ($M=N$) and invertible. Then we can compute

$$H^{-1}g = H^{-1}\left(Hu + n\right)$$

which gives the inverse solution

$$\hat{u} = u + H^{-1}n$$

Example



Blurred and noisy image u_0

$\longrightarrow$

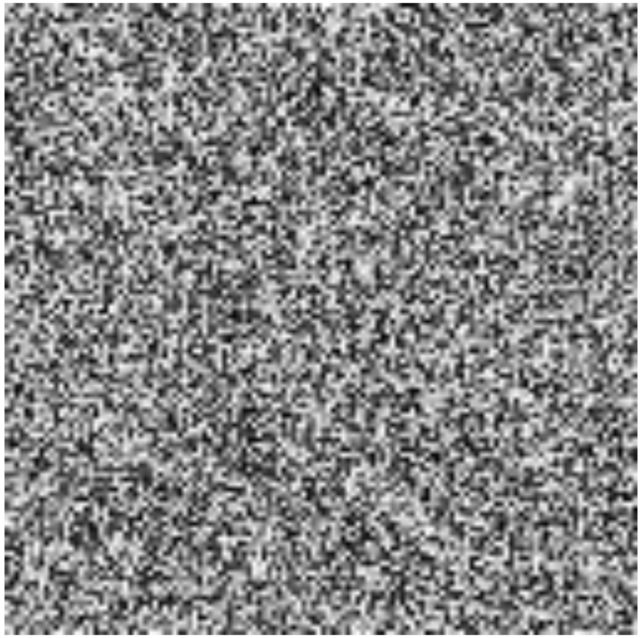
Inverse solution

$\neq$

Original image

Inversion

Blurred and noisy image. Image given the space french agency which simulate the optics of the french satelitte SPOT5. Resolution 2,5m.
@CNES



Inverse solution

Inverse solution

- We have assumed that H is invertible in the mathematical sense, which means that the solution of an equation

$$v=Hu$$

  - Exists
  - Is unique
- We need one condition more to obtain an acceptable solution: the stability of the solution wrt the data, which means that u depends continuously on v, that is

for any sequence u_n such that $Hu_n \xrightarrow[n\to+\infty]{} Hu$

Then $u_n \xrightarrow[n\to+\infty]{} u$

Well-posed problem Hadamard 1923

Consider the equation

$$v = Hu \quad (1)$$

where u and $v : \Omega \subset R^2 \to R$ and $H : L^2(\Omega) \to L^2(\Omega)$

The inverse problem consists in finding u form a given v. This inverse problem is well-posed if the three following conditions are satisfied:

- Existence: for any v, we can find a u such that (1) is satisfied,
- Uniqueness: the solution *u* is unique,
- Stability: for any sequence u_n such that $\lim_{n \to +\infty} Hu_n = Hu$

Then $\lim_{n \to +\infty} u_n = u$

Deconvolution: an ill-posed problem

- Convolution: $h(x,y)=h(y-x)$
- Riemann-Lebesgue Lemma

If $h \in L^2(\Omega)$ then it can be shown that

$$\lim_{\alpha \to +\infty} \int_\Omega h(y)\sin(\alpha y)dy = 0$$

Then we have $$\lim_{\alpha \to +\infty} \int_\Omega h(x-y)\left[u(y)+\sin(\alpha y)\right]dy = v(x)$$

Any high frequency signal added to u leaves the integral unchanged.

The continuous problem is per se ill-posed.

Fourier analysis

- Back to the discrete problem.
- Circular discrete convolution (convolution with periodic boundary conditions) is a simple product in the Fourier plane.

$$\left(g\right)_{i,j} = \left(h * u\right)_{i,j} + \left(n\right)_{i,j}$$

$$\rightarrow \quad F(g)_{k,l} = F(h)_{k,l} . F(u)_{k,l} + F(n)_{k,l}$$

- Matrix-vector form: under periodic boundary conditions, the matrix H is circular block circular, so eigen vectors are the basis vectors of the 2D Fourier transform

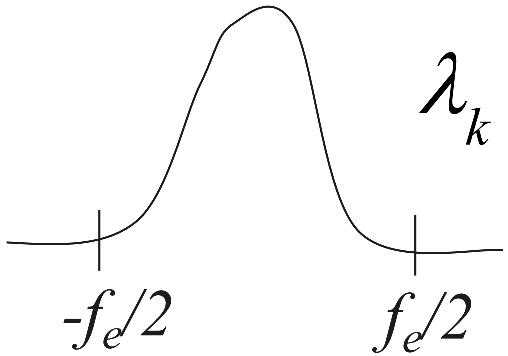
$$g = H u + n$$

$$\rightarrow \quad F(g) = diag\left\{\lambda_{k,l}\right\} . F(u) + F(n)$$

- The eigen values $\lambda_{k,l}$ are the coefficients of the 2D Fourier transform of the kernel h. So it is the MTF.

Fourier Analysis

- As h models a low frequency filter (blur), then the eigen values $\lambda_{k,l}$ of H matrix corresponding to high frequencies are small, may be zero.
- The PSF corresponding to blur attenuate high frequencies of the image (think to the Gaussian blur model)



- We want to inverse $F(u_0) = diag\{\lambda_{k,l}\}.F(u) + F(n)$
- If $\exists$ (k,l), $\lambda_{k,l}$ =0, the problem in $F(u)$ has an infinity of solutions
- If $\forall (k,l)$, $\lambda_{k,l} \neq 0$, the problem in $F(u)$ has a unique solution, but instable.

$$F(u)_{k,l} = \frac{F(g)_{k,l}}{\lambda_{k,l}} + \frac{F(n)_{k,l}}{\lambda_{k,l}}$$

Matrix Conditioning

- Let consider the matrix vector equation $v=Hu$
- Existence and uniqueness are ensured as soon as H is a square non singular matrix.
- Stability is measured by the condition number of the matrix.
- Definition : the condition number is defined, when H is regular, by

$$Cond(H) = \|H\|.\|H^{-1}\|$$

where $\|H\|$ is the matrix norm induced by the vector norm on R^n: $\|H\| = \sup_{x\neq 0}\left\{\frac{\|Hx\|}{\|x\|}\right\}$

- Properties

$$Cond(H) \geq 1, \qquad Cond(H) = Cond(H^{-1}),$$

$$Cond(I) = 1, \qquad Cond(\lambda H) = Cond(H) \quad \text{for } \lambda \neq 0,$$

with the Euclidian norm, $Cond(H) = \frac{|\mu_{\max}|}{|\mu_{\min}|}$ μ_i : singular values of H

if H is normal $Cond(H) = \frac{|\lambda_{\max}|}{|\lambda_{\min}|}$ λ_i : eigen values of H

What *Cond(H)* measures?

- Let consider the matrix vector equation $v=Hu$, and let δv be a pertubation on v. δv leads to a perturbation δu in u such that

$$v+\delta v=H(u+\delta u)$$

We have $\delta v=H\delta u$ so $\delta u=H^{-1}\delta v$ and we can deduce

$$\|\delta u\|\leq\|H^{-1}\|.\|\delta v\|$$

But we also have $\|v\|\leq\|H\|.\|u\|$. Then for non null vectors u,v we have

$$\frac{\|\delta u\|}{\|u\|}\leq\|H\|.\|H^{-1}\|\frac{\|\delta v\|}{\|v\|}$$ also wirtten as $$\boxed{\frac{\|\delta u\|}{\|u\|}\leq Cond(H)\frac{\|\delta v\|}{\|v\|}}$$

Then a small condition number (near 1) will ensure stability because a small relative perturbation on the observed data v will produce a small relative perturbation on the solution u

Least square solution

- The least square solution is given by the resolution of the optimisation problem

$$\underset{u\in L^2(\Omega)}{Inf} \int_{\Omega} \left| g - Hu \right|^2 dx$$

If the operator H is such that Ker(H)={0}, then it exists a unique solution, given by the Euler equation

$$\begin{cases} H^*\left(Hu - g\right) = 0 \\ \left.\dfrac{\partial u}{\partial \mathrm{N}}\right|_{\partial\Omega} = 0 \end{cases}$$

H is a linear operator,
H^* is its adjoint

The solution can be computed by solving the associated dynamical system, where u is now depending on time t (equivalent to gradient descent with fixed iteration step)

$$\begin{cases} \dfrac{\partial u}{\partial t} = H^*\left(g - Hu\right) \\ \left.\dfrac{\partial u}{\partial \mathrm{N}}\right|_{\partial\Omega} = 0 , \quad u\left(x, t = 0\right) = g\left(x\right) \end{cases}$$

- Of course if the inverse solution is unstable, so is the least square solution.

Least square solution

- In discrete variables: $$\underset{u}{Min}\left\|Hu-g\right\|^2 = \underset{u}{Min}\sum_{i,j=1}^{N}\left(Hu-g\right)_{i,j}^2$$

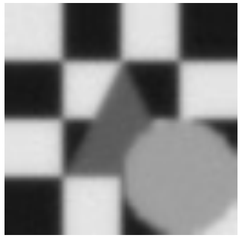
If the matrix H has non null singular values then the problem has a unique solution, given by the resolution of the linear system

$$\left(Hu-g\right)=0 \quad \text{and the solution is} \quad \hat{u}=\left(H^*H\right)^{-1}H^*g$$

- Boundary conditions are included in the construction of the matrix H
- Of course if H is ill-conditioned, so is H^*H
- In the frequency domain, MTF is the Modulation Transfer Function $MTF = F(h)$ if H admits an inverse then it has non null eigen values, so $(MTF)_{k,l}\neq 0$, and the least square solution is given by

$$F\left(\hat{u}\right)=\frac{MTF^*}{\left|MTF\right|^2}F\left(g\right)$$

Least square solution



Blurred and noisy image u_0

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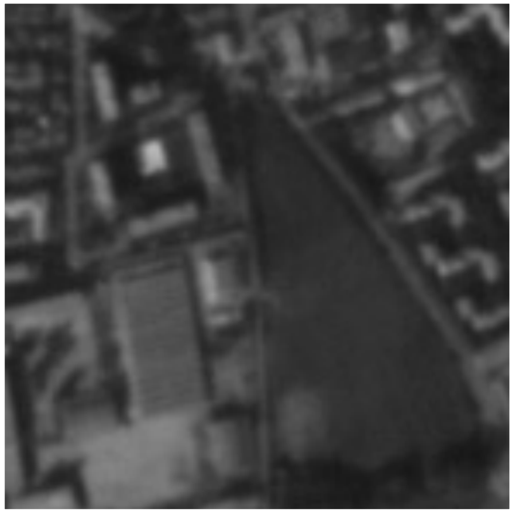
Least square solution

$\neq$

Original image



$\longrightarrow$