Exam (A)

1) $X_1, \dots, X_N \overset{i.i.d.}{\sim} F$

$$\hat{F}_N(u) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{]-\infty, u]}(X_i)$$

a) Show $\hat{F}_N(u)$ is an unbiased estimator of $F(u)$.

$$\mathbb{E}(\hat{F}_N(u)) = \frac{1}{N} \sum_{i=1}^{N} \underbrace{\mathbb{E}\,\mathbb{1}_{]-\infty,u]}(X_i)}_{F(u)}$$

$$= \frac{1}{\cancel{N}} \cdot \cancel{N} F(u) \checkmark$$

b) $p_{\bar{u}} \triangleq \mathbb{P}\{X_i \geq \bar{u}\}$. Obtain the plug-in estimator of $p_{\bar{u}}$

$$p_{\bar{u}} = 1 - \mathbb{P}\{X_i \leq \bar{u}\} = 1 - F(u)$$

$$\hat{p}_{\bar{u}} := 1 - \hat{F}_N(u)$$

c) Using CLT find an asy 95% CI for $p_{\bar{u}}$

Recall that $\hat{F}_N(u) \xrightarrow{d} \mathcal{N}(F(u), Var(\hat{F}_N(u)))$

$$\mathrm{Var}(\hat{F}_N(u)) \overset{\perp}{=} \frac{1}{N^2}\sum_{i=1}^{N} \mathrm{Var}\, \mathbb{1}_{]-\infty,u]}(X_i)$$

$$= \frac{1}{N^{\cancel{2}}} \cancel{\sum_{i=1}^{N}} F(u)(1-F(u))$$

$$= \frac{1}{N} F(u)(1-F(u))$$

$$\Rightarrow \quad \hat{F}_N(u) - F(u) \sim \mathcal{N}\left(0, \frac{1}{N} \hat{F}_N(u)(1-\hat{F}_N(u))\right)$$

$$\left[\hat{F}_N(u) \pm 1.96 \cdot \frac{1}{\sqrt{N}} \sqrt{\hat{F}_N(u)(1-\hat{F}_N(u))}\right]$$

is an asy. 95 % CI for $\underbrace{F(u)}_{1-p_u}$

$$\Rightarrow p_u = 1 - F(u) \Rightarrow$$

$$\left[(1-\hat{F}_N(u)) \pm 1.96 \sqrt{\frac{1}{N} \hat{F}_N(u)(1-\hat{F}_N(u))}\right]$$

is a 95 % asy. CI for

p_u $\square$

d) A bootstrap pivot CI for $p_{\bar{u}}$

Given $X_1, .., X_N$

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B = 1000
store_e_cdf = vector(length = B)
for (b in 1:B){
      obs = sample(X1.., XN, replace=T)
      e_cdf = mean(obs >= u)
      store_e_cdf = c(store_e_cdf, e_cdf)
}
lower_q = quantile(store_e_cdf, α/2)
higher_q = quantile(store_e_cdf, 1 - α/2)
lower_bound = 2·F̂N(u) - higher_q
upper_bound = 2 * F̂N(u) - lower_q
return c(lower_bound, upper_bound)
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$(\hat{p}^b_{\bar{u}} =)$

Ex. 2

$$X_1, \dots, X_N \overset{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$$

$$f_\theta(x) = \theta e^{-\theta x} \mathbb{1}_{[0,+\infty[}(x) \qquad \theta > 0$$

a) Compute $\hat{\theta}_{MM}$

$$\boxed{\begin{aligned} \hat{\mu}_1 &= \\ \mu_1 &= \end{aligned}} \begin{aligned} &\frac{1}{N}\sum_{i=1}^{N} X_i = \overline{X} \\ &\mathbb{E}(X_i) = \int_{-\infty}^{+\infty} x\, \theta e^{-\theta x} \mathbb{1}_{[0,+\infty[}(x)\, dx \end{aligned}$$

$$= \int_0^{+\infty} \underset{f}{x} \underbrace{\theta e^{-\theta x}}_{g'} dx = \int_0^{+\infty} \theta x e^{-\theta x} dx$$

$$= \left[f g \right]_0^{+\infty} - \int_0^{+\infty} f' g \, dx$$

$$= \underbrace{x(-e^{-\theta x}) \Big|_0^{+\infty}}_{0} + \underbrace{\int_0^{+\infty} e^{-\theta x} dx}_{-\frac{1}{\theta} e^{-\theta x} \Big|_0^{+\infty}} =$$

$$= 0 - \left(-\frac{1}{\theta}\right) = \frac{1}{\theta}$$

$$\Rightarrow \frac{1}{\hat{\theta}_{MM}} = \overline{X} \Rightarrow \hat{\theta}_{MM} = \frac{1}{\overline{X}}$$

b) Compute $\hat{\theta}_{ML}$:

$$\log \prod_{i=1}^{N} f_\theta(x_i)$$

$$\log \mathcal{L}(\theta) \doteq \sum_{i=1}^{N} \log f_\theta(x_i) = \sum_{i=1}^{N} \log \theta e^{-\theta x_i} \qquad x_i > 0$$

$$= \sum_{i=1}^{N} \log \theta - \sum_{i=1}^{N} \theta x_i$$

$$\frac{d}{d\theta} \log \mathcal{L}(\theta) = \frac{N}{\theta} - \bar{X} N \stackrel{!}{=} 0$$

$$\Rightarrow \hat{\theta}_{ML} = \frac{1}{\bar{X}}$$

c) Compute the Fisher information $I_N(\theta)$, and find a limit distribution for $\hat{\theta}_{ML}$

$$\hat{\theta}_{ML} - \theta \xrightarrow{d} \mathcal{N}(0, se^2(\hat{\theta}_{ML}))$$

where $se(\hat{\theta}_{ML}) \underset{N \to +\infty}{\longrightarrow} \sqrt{\frac{1}{I_N(\theta)}}$

$I_N(\theta) = N I_1(\theta)$ because of $\perp$

and $I_1(\theta) = -\mathbb{E}\left[\frac{d^2}{d\theta^2} \log f_\theta(x_1)\right]$ -

$$\frac{d^2}{d\theta^2} \log \theta e^{-\theta x_1} = \frac{d^2}{d\theta^2} (\log \theta - \theta x_1) =$$

$$= \frac{d}{d\theta}\left(\frac{1}{\theta} - x_1\right) = -\frac{1}{\theta^2}$$

$$\Rightarrow I_N(\theta) = N I_1(\theta) = \frac{N}{\theta^2}$$

$$\Rightarrow \hat{se}(\hat{\theta}_{ML}) = \frac{\hat{\theta}_{ML}}{\sqrt{N}}$$

$$\Rightarrow \hat{\theta}_{ML} \xrightarrow{d} \mathcal{N}\left(\theta, \frac{\hat{\theta}_{ML}^2}{N}\right)$$

Delta Method

$X_1, \dots, X_N \overset{i.i.d.}{\sim} F(\theta))$

you focus on $\tau = g(\theta)$ where g is a <u>differentiable</u> function.

By equivariance you know

$$\hat{\tau}_{ML} = g(\hat{\theta}_{ML})$$

How to compute a conf. interval for τ?

$$\hat{\tau}_{ML} - \tau = g(\hat{\theta}_{ML}) - g(\theta) = g'(\theta)(\hat{\theta}_{ML} - \theta) + o(|\hat{\theta}_{ML} - \theta|)$$

$$\sim g'(\theta)\underbrace{(\hat{\theta}_{ML} - \theta)}_{\xrightarrow[CLT]{d}\ N(0, \frac{1}{I_N(\theta)})}$$

$\mathrm{Var}(\hat{\theta}_{ML})$

$\downarrow$

$$\Rightarrow \hat{\tau}_{ML} - \tau \xrightarrow{d} N\left(0, \frac{g'^2(\theta)}{I_N(\theta)}\right)$$

$$\Rightarrow \left[\hat{\tau}_{ML} \pm z_{\alpha/2} \, |g'(\theta)| / \sqrt{I_N(\theta)}\right]$$

is an asy. $1-\alpha$ CI for τ —

Problem: $X_1, \dots, X_N \overset{i.i.d.}{\sim} \mathcal{N}(\mu, \sigma^2)$ with μ given

i) Compute $\hat{\sigma}_{ML}$ and estimate $se(\hat{\sigma}_{ML})$

ii) Compute a 95% CI (approximate) for $\log \sigma$

i) $$\log \mathcal{L}(\sigma) = \sum_{i=1}^{N} \log\left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}\right)$$

$$= -N\log\sigma - \frac{1}{2}\left(\sum_{i=1}^{N}(x_i-\mu)^2\right)\cdot\frac{1}{\sigma^2} + cst$$

$$\Rightarrow \frac{\partial}{\partial\sigma}\log\mathcal{L}(\sigma) = -\frac{N}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{N}(x_i-\mu)^2 \overset{!}{=} 0$$

$$\Rightarrow \frac{1}{\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2 = N$$

$$\Rightarrow \hat{\sigma}^2_{ML} = \frac{1}{N}\sum_i (x_i-\mu)^2$$

$$\Rightarrow \hat{\sigma}_{ML} = \sqrt{\frac{1}{N}\sum_i (x_i-\mu)^2}$$

and $$\hat{se}(\hat{\sigma}_{ML}) = \frac{\hat{\sigma}_{ML}}{\sqrt{2N}}$$

ii) $\tau = \log \sigma \Rightarrow g(\cdot) = \log(\cdot)$.

$$\left[\hat{\tau}_{ML} \pm z_{\frac{\alpha}{2}}\, g'(\theta)\, se\, \hat{\theta}_{ML}\right]$$ an

asymptotic $1-\alpha$ CI for τ

$$\rightarrow \left[\log \hat{\sigma}_{ML} \pm z_{\frac{\alpha}{2}}\, \frac{1}{\cancel{\sigma}} \cdot \frac{\cancel{\sigma}}{\sqrt{2N}}\right]$$

So far we considered $\theta \in \mathbb{R}$
but θ might be multidim.

Ex: $X_1, \dots, X_N \sim \mathcal{N}(\mu, \sigma^2)$
$$\Rightarrow \theta = (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}^+$$

$$\hat{\theta}_{ML} = (\hat{\mu}_{ML}, \hat{\sigma}_{ML})$$

Th: $\hat{\theta}_{ML} - \theta \xrightarrow{d} \mathcal{N}(0, I_N^{-1}(\theta))$

with $I_N(\theta) = -E[H \log L(\theta)]$

$$\log \mathcal{L}(\mu,\sigma) = \sum_{i=1}^{N}\left[-\log\sigma - \frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right] + \text{cost}$$

$$= -N\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2$$

in order to compute $\hat{\mu}_{ML}$, $\hat{\sigma}_{ML}$
we need to solve

$$\begin{cases} \frac{\partial}{\partial\mu}\log\mathcal{L}(\mu,\sigma) = 0 \\ \frac{\partial}{\partial\sigma}\log\mathcal{L}(\mu,\sigma) = 0 \end{cases} \qquad \nabla\log\mathcal{L}(\mu,\sigma) = 0$$

The solution is $\hat{\mu}_{ML} = \overline{X}$
and $\hat{\sigma}_{ML} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i-\overline{X})^2}$

Recall: $(\hat{\mu}_{ML}, \hat{\sigma}_{ML}) \sim \mathcal{N}((\mu,\sigma), I_N^{-1}(\mu,\sigma))$

$$I_N(\mu,\sigma) \stackrel{1}{=} N I_1(\mu,\sigma) = -\mathbb{E}_{\mu,\sigma}\left[H\log\mathcal{L}(\mu,\sigma)\right]$$

$$\log\mathcal{L}(\mu,\sigma)_{\mu\mu} = \frac{\partial^2}{\partial\mu^2}\left(-N\log\sigma - \frac{1}{2\sigma^2}\sum_i(x_i-\mu)^2\right)$$

$$= \frac{\partial}{\partial\mu}\left(\frac{1}{2\sigma^2}\cdot 2\sum_i(x_i-\mu)\right)$$

$$= -\frac{N}{\sigma^2}$$

$$\log_{\sigma\sigma} \mathcal{L}(\mu,\sigma) = \frac{\partial^2}{\partial\sigma^2}\left[-N\log\sigma - \frac{1}{2\sigma^2}\sum_i (x_i-\mu)^2\right]$$

$$= \frac{\partial}{\partial\sigma}\left[-\frac{N}{\sigma} + \frac{1}{\sigma^3}\sum_i (x_i-\mu)^2\right]$$

$$= \frac{N}{\sigma^2} - \frac{3}{\sigma^4}\sum_i (x_i-\mu)^2$$

$$\log_{\sigma\mu} \mathcal{L}(\mu,\sigma) = \frac{\partial}{\partial\mu}\left[-\frac{N}{\sigma} + \frac{1}{\sigma^3}\sum_i (x_i-\mu)^2\right]$$

$$= -\frac{2}{\sigma^3}\sum_i (x_i-\mu)$$

$$\Rightarrow H_{\mu,\sigma}\log\mathcal{L}(\mu,\sigma) = \begin{pmatrix} -\frac{N}{\sigma^2} & -\frac{2}{\sigma^3}\sum_i (x_i-\mu) \\ -\frac{2}{\sigma^3}\sum_i (x_i-\mu) & \frac{N}{\sigma^2} - \frac{3}{\sigma^4}\sum_i (x_i-\mu)^2 \end{pmatrix}$$

$$I_N(\mu,\sigma) = -\mathbb{E}\, H_{\mu,\sigma}\log\mathcal{L}(\mu,\sigma) = \begin{pmatrix} -\frac{N}{\sigma^2} & 0 \\ 0 & -\frac{2N}{\sigma^2} \end{pmatrix}$$

$$\Rightarrow (\hat{\mu}_{ML}, \hat{\sigma}_{ML}) \sim \mathcal{N}\left((\mu,\sigma), \begin{pmatrix} \frac{\sigma^2}{N} & 0 \\ 0 & \frac{\sigma^2}{2N} \end{pmatrix}\right)$$