MSc. Data Science & Artificial Intelligence

Inverse Problems in Image Processing

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Assignment 1

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1 Tutorial 5

1.1 Exercise 1

The proximal operator of τf is defined as:

$$\text{prox}_{\tau f}(x) = \arg\min_{u \in \mathbb{R}} \frac{1}{2\tau}\|u - x\|^2 + f(u) \tag{1}$$

which we can apply to the ℓ_1 norm to get:

$$\text{prox}_{\tau|\cdot|}(x) = \arg\min_{u \in \mathbb{R}} \frac{1}{2\tau}\|u - x\|^2 + |u| \tag{2}$$

Let $h(u) = \frac{1}{2\tau}\|u - x\|^2 + |u|$, then

$$\begin{aligned}
\frac{\partial}{\partial u} h(u) &= \frac{\partial}{\partial u}\left[\frac{1}{2\tau}\|u - x\|^2 + |u|\right] \\
&= \begin{cases} \frac{1}{\tau}(u - x) - 1, & u < 0 \\ 0, & u = 0 \\ \frac{1}{\tau}(u - x) + 1, & u > 0 \end{cases}
\end{aligned}$$

Case $u > 0$.

$$\begin{aligned}
&\frac{\partial}{\partial u} h(u) = 0 \\
\implies &\frac{1}{\tau}(u - x) + 1 = 0 \\
\implies &u = x - \tau
\end{aligned}$$

Case $u < 0$.

$$\begin{aligned}
&\frac{\partial}{\partial u} h(u) = 0 \\
\implies &\frac{1}{\tau}(u - x) - 1 = 0 \\
\implies &u = x + \tau
\end{aligned}$$

Case $u = 0$. In this case, we cannot compute the derivative as the function is non-differentiable in $u = 0$. We have however that the subdifferential of h is given by:

$$\partial h(u) = [-1, 1]$$

In particular, we have by the optimality condition that:

$$0 \in \partial h(u^*) \iff u^* \in \arg\min_{u \in \mathbb{R}} f(u) \tag{3}$$

As a result, we have that the proximal operator of $\tau|\cdot|$ is given by:

$$\text{prox}_{\tau|\cdot|}(x) = \begin{cases} x - \tau, & x < -\tau \\ 0, & -\tau \le x \le \tau \\ x + \tau, & x > \tau \end{cases}$$

We plot this proximal operator in the companion notebook.

1.2 Exercise 2

We define f as the ℓ_0 norm:

$$f(x) = |x|_0 = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0 \end{cases} \tag{4}$$

The proximal operator of τf is defined as:

$$\operatorname{prox}_{\tau|\cdot|_0}(x) = \arg\min_{u \in \mathbb{R}} \frac{1}{2\tau}\|u - x\|^2 + |u|_0 \tag{5}$$

Let $h(u)$ be given by:

$$h(u) := \frac{1}{2\tau}\|u - x\|^2 + |u|_0$$

then

$$h'(x) = \begin{cases} \frac{1}{2\tau}(u - x) + 0, & x \neq 0 \\ \frac{1}{2\tau}(u - x) + 1, & x = 0 \end{cases}$$

So it is better to choose $u = 0$ when $\frac{x^2}{2\tau} < 1$, else, set $u = x$.

1.3 Exercise 3

$$\delta_{\mathbb{R}^n_+}(x) = \begin{cases} \infty, & x \notin \mathbb{R}^n_+ \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

$$\operatorname{prox}_{\tau|\cdot|_1 + \delta_{\mathbb{R}^n_+}(\cdot)}(x) = \arg\min_{u} \frac{1}{2\tau}\|u - x\|^2 + |u|_1 + \delta_{\mathbb{R}^n_+}(u) \tag{7}$$

$$\operatorname{prox}_{\tau|\cdot|_1 + \delta_{\mathbb{R}^n_+}(\cdot)}(x) = \arg\min_{u} \frac{1}{2\tau}\|u - x\|^2 + |u|_1 + \delta_{\mathbb{R}^n_+}(u) \tag{8}$$

Given by prof:

$$\operatorname{prox}(x) = \max(\operatorname{prox}_{\tau|\cdot|_1}(x), 0) \tag{9}$$

1.4 Exercise 4

Compute $\operatorname{prox}_f(x)$.

2 Tutorial 6

2.1 Question 7