Introduction to Information Theory

Joris LIMONIER

January 30, 2023

Contents

1 Exercise sheet

1.1 Exercise 1

Given a game of 52 cards, we draws cards with replacement.

Probability to get a queen   There are 4 queens in the deck, so the probability to get a queen is:
$$\mathbb{P}(queen) = 4/52.$$

Probability to get a heart   There are 13 hearts in the deck, so the probability to get a heart is:
$$\mathbb{P}(heart) = 13/52.$$

Probability to get the queen of hearts or the ace of spades   There is one queen of heart and one ace of spades, which are disjoint events, so the probability to get one of them is:

$$
\begin{aligned}
\mathbb{P}(\text{QH or AS}) &= \mathbb{P}(QH) + \mathbb{P}(AS) \\
&= 1/52 + 1/52 \\
&= 2/52
\end{aligned}
$$

Probability to get a queen or a spade   There are 4 queens and 13 spades, which are not disjoint events, so the probability to get one of them is:

$$
\begin{aligned}
\mathbb{P}(\text{Q or S}) &= \mathbb{P}(Q) + \mathbb{P}(S) - \mathbb{P}(Q \cap S) \\
&= 4/52 + 13/52 - 1/52 \\
&= 16/52
\end{aligned}
$$

Probability to get neither a queen, nor a spade   This is the complement of the previous event, so the probability to get neither a queen, nor a spade is:

$$\mathbb{P}(\text{not Q and not S}) = 1 - \mathbb{P}(\text{Q or S})$$
$$= 1 - 16/52$$
$$= 36/52$$

1.2   Exercise 2

We draw one ball from an urn with 5 white balls, 4 red balls and 2 black balls.

Probability to draw white   There are 11 balls in total and 5 white balls, so the probability to draw a white ball is:

$$\mathbb{P}(W) = 5/11$$

Probability to draw not white   This is the complement of the previous event, so the probability to draw not white is:

$$\mathbb{P}(\neg W) = 1 - \mathbb{P}(W)$$
$$= 6/11$$

Probability to draw white or red   These events are disoint, so the probability to draw a white or a red ball is:

$$\mathbb{P}(W \cup R) = \mathbb{P}(W) + \mathbb{P}(R)$$
$$= 5/11 + 4/11$$
$$= 9/11$$

2 Exercise sheet Quantitative Measure of Information (part 1)

2.1 Exercise 6

Given

$$\mathbb{P}(X = 0) = \frac{1}{4}$$
$$\mathbb{P}(X = 1) = \frac{3}{4}$$
$$p_0 = 10^{-1}$$

where p_0 is the probability of incorrect transmission of a bit.

Compute $H(X)$

$$H(X) = -\sum_{i=1}^{m} \mathbb{P}(X = x_i) \log_2 \mathbb{P}(X = x_i)$$
$$= -\frac{1}{4} \log_2 \frac{1}{4} - \frac{3}{4} \log_2 \frac{3}{4}$$
$$\approx 0.811 \text{ sh / state of X}$$

Compute $H(Y)$ We first compute the probability distribution of Y:

$$
\begin{aligned}
\mathbb{P}(Y = 1) &= \mathbb{P}(Y = 1, X = 0) + \mathbb{P}(Y = 1, X = 1) \\
&= \mathbb{P}(Y = 1 \mid X = 0)\mathbb{P}(X = 0) + \mathbb{P}(Y = 1 \mid X = 1)\mathbb{P}(X = 1) \\
&= p_0 \frac{1}{4} + (1 - p_0)\frac{3}{4} \\
&= 10^{-1}\frac{1}{4} + (1 - 10^{-1})\frac{3}{4} \\
&= \frac{1}{10}\frac{1}{4} + \frac{9}{10}\frac{3}{4} \\
&= \frac{1}{40} + \frac{27}{40} \\
&= \frac{28}{40}
\end{aligned}
$$

and on the other hand:

$$
\begin{aligned}
\mathbb{P}(Y = 0) &= \mathbb{P}(Y = 0, X = 0) + \mathbb{P}(Y = 0, X = 1) \\
&= \mathbb{P}(Y = 0 \mid X = 0)\mathbb{P}(X = 0) + \mathbb{P}(Y = 0 \mid X = 1)\mathbb{P}(X = 1) \\
&= (1 - p_0)\frac{1}{4} + p_0 \frac{3}{4} \\
&= \frac{9}{10}\frac{1}{4} + \frac{1}{10}\frac{3}{4} \\
&= \frac{9}{40} + \frac{3}{40} \\
&= \frac{12}{40}
\end{aligned}
$$

$$
\begin{aligned}
H(Y) &= -\sum_{i=1}^{m} \mathbb{P}(Y = y_i)\log_2 \mathbb{P}(Y = y_i) \\
&= -\frac{28}{40}\log_2 \frac{28}{40} - \frac{12}{40}\log_2 \frac{12}{40} \\
&= -\frac{28}{40}\log_2 \frac{28}{40} - \frac{12}{40}\log_2 \frac{12}{40} \\
&\approx 0.881 \text{ sh / state of Y}
\end{aligned}
$$

So we have:
$$
\begin{cases}
\mathbb{P}(X = 0, Y = 0) = \frac{9}{40} \\
\mathbb{P}(X = 0, Y = 1) = \frac{1}{40} \\
\mathbb{P}(X = 1, Y = 0) = \frac{3}{40} \\
\mathbb{P}(X = 1, Y = 1) = \frac{27}{40}
\end{cases}
$$

Compute $H(X, Y)$

$$H(X, Y) = -\sum_{i=1}^{n}\sum_{j=1}^{m} \mathbb{P}(X = x_i, Y = y_j) \log \mathbb{P}(X = x_i, Y = y_j)$$

$$\begin{aligned}
&= -\mathbb{P}(X = 0, Y = 0) \log \mathbb{P}(X = 0, Y = 0) \\
&\quad - \mathbb{P}(X = 0, Y = 1) \log \mathbb{P}(X = 0, Y = 1) \\
&\quad - \mathbb{P}(X = 1, Y = 0) \log \mathbb{P}(X = 1, Y = 0) \\
&\quad - \mathbb{P}(X = 1, Y = 1) \log \mathbb{P}(X = 1, Y = 1) \\
&= -\frac{9}{40}\log\frac{9}{40} - \frac{1}{40}\log\frac{1}{40} - \frac{3}{40}\log\frac{3}{40} - \frac{27}{40}\log\frac{27}{40} \\
&\approx 1.28
\end{aligned}$$

Compute $H(Y \mid X)$

$$\begin{aligned}
H(Y \mid X) &= H(X, Y) - H(Y) \\
&= 1.28 - 0.88 \\
&= 0.4 \text{ sh / state of X}
\end{aligned}$$

Compute $H(X \mid Y)$

$$\begin{aligned}
H(X \mid Y) &= H(X, Y) - H(X) \\
&= 1.28 - 0.81 \\
&= 0.47 \text{ sh / state of Y}
\end{aligned}$$

Compute $I(X, Y)$

$$\begin{aligned}
I(X, Y) &= H(X) - H(X \mid Y) \\
&= 0.81 - 0.40 \\
&= 0.41 \text{ sh / state of (X, Y)}
\end{aligned}$$

2.2 Problem 2

Consider a twin-pan balance and $c = 9$ coins. We know that one of these coins is fake. The problem is to find the fake coin given that it only differs from the other 8 coins by its weight.

1. Number of states and entropy   The number of states is 18. Each coin may be lighter or heavier than the other one it is checked against.
The entropy is:

$$H(S) = \log_2 18 \approx 4.17 \text{ sh}$$

2. Each weighting operation may give one of the following results: left is heavier, right is heavier or equal weight.

The maximum entropy that can be achieved is:

$$H_{max}(S) = \log_2 3 \approx 1.58 \text{ sh}$$

so the average number of weighting operations is equal to:

$$\bar{n}_{min} = \frac{\log_2(18)}{\log_2(3)}$$

$$\approx 2.63$$

3.

$$\mathbb{P}_e = \frac{C_9^{2n}}{C_8^{2n}}$$

$$= \frac{\frac{8!}{2n!(8-2n)!}}{\frac{9!}{2n!(9-2n)!}}$$

$$= 1 - \frac{2}{9}n$$

$$\mathbb{P}_l = \mathbb{P}_r = \frac{1}{2}(1 - \mathbb{P}_e)$$

$$= \frac{1}{9}n$$

We want $\mathbb{P}_l = \mathbb{P}_r = \mathbb{P}_e$, to get maximum information. This is achieved for $n = 3$, that is 3 coins on each side.

3 Exercise sheet Quantitative Measure of Information (part 2)

3.1 Exercise

Y is uniformly distributed

$$\mathbb{P}(Y = y_1) = \frac{1}{24} + \frac{1}{6} + \frac{1}{24}$$

$$= \frac{6}{24}$$

$$\mathbb{P}(Y = y_2) = \frac{1}{12} + \frac{1}{8} + \frac{1}{24}$$
$$= \frac{6}{24}$$

$$\mathbb{P}(Y = y_3) = \frac{1}{6} + \frac{1}{24} + \frac{1}{24}$$
$$= \frac{6}{24}$$

$$\mathbb{P}(Y = y_4) = \frac{1}{6} + \frac{1}{6} + \frac{1}{24}$$
$$= \frac{6}{24}$$

Y is indeed uniformly distributed.

Compute $I(X, Y)$

$$I(X, Y) = H(Y) - H(Y \mid X)$$
$$= \log_2 4 - H(Y \mid X)$$

The distribution of X is given by:
$$\begin{cases} \mathbb{P}(X = x_1) = \frac{1}{3} \\ \mathbb{P}(X = x_2) = \frac{1}{2} \\ \mathbb{P}(X = x_3) = \frac{1}{6} \end{cases}$$

$$H(Y \mid X) = \sum_{i=1}^{3} H(Y \mid X = x_i) \log \mathbb{P}(X = x_i)$$

3.2 Problem

3.2.1 1.

$$\mathbb{P}(M = 0) = \mathbb{P}(M = 0, T = 0) + \mathbb{P}(M = 0, T = 1)$$
$$= 1/8 + 3/16$$
$$= 5/16$$

$$\mathbb{P}(M=1) = \mathbb{P}(M=1, T=0) + \mathbb{P}(M=1, T=1)$$
$$= 1/16 + 5/8$$
$$= 11/16$$

$$\mathbb{P}(T=0) = \mathbb{P}(T=0, M=0) + \mathbb{P}(T=0, M=1)$$
$$= 1/8 + 1/16$$
$$= 3/16$$

$$\mathbb{P}(T=1) = \mathbb{P}(T=1, M=0) + \mathbb{P}(T=1, M=1)$$
$$= 3/16 + 5/8$$
$$= 13/16$$

3.2.2   2.

$$\mathbb{P}(M=0, T=1) + \mathbb{P}(M=1, T=0) = 3/16 + 1/16 = 4/16 = 1/4$$

3.2.3   3.

$$1 - \mathbb{P}(T=1) = 1 - 13/16 = 3/16 < 1/4$$

Indeed.

3.2.4   4.

$$I(E,T) = \sum_{i=0,1} \sum_{j=0,1} \mathbb{P}(E=i, T=j) \log \frac{\mathbb{P}(E=i, T=j)}{\mathbb{P}(E=i)\mathbb{P}(T=j)}$$
$$= \sum_{j=0,1} \mathbb{P}(E=1, T=j) \log \frac{\mathbb{P}(E=1, T=j)}{\mathbb{P}(E=1)\mathbb{P}(T=j)}$$
$$= \mathbb{P}(E=1, T=0) \log \frac{\mathbb{P}(E=1, T=0)}{\mathbb{P}(E=1)\mathbb{P}(T=0)} +$$
$$\mathbb{P}(E=1, T=1) \log \frac{\mathbb{P}(E=1, T=1)}{\mathbb{P}(E=1)\mathbb{P}(T=1)}$$
$$= \mathbb{P}(T=0) \log \frac{\mathbb{P}(T=0)}{1 * \mathbb{P}(T=0)} + \mathbb{P}(T=1) \log \frac{\mathbb{P}(T=1)}{1 * \mathbb{P}(T=1)}$$
$$= 0$$

3.2.5   5.

$$I(M,T) = \sum_{i=0,1} \sum_{j=0,1} \mathbb{P}(M = i, T = j) \log \frac{\mathbb{P}(M = i, T = j)}{\mathbb{P}(M = i)\mathbb{P}(T = j)}$$

$$= \frac{1}{8} \log \frac{\frac{1}{8}}{\frac{5}{16}\frac{3}{16}} + \frac{3}{16} \log \frac{\frac{3}{16}}{\frac{5}{16}\frac{13}{16}} + \frac{1}{16} \log \frac{\frac{1}{16}}{\frac{11}{16}\frac{3}{16}} + \frac{1}{16} \log \frac{\frac{5}{8}}{\frac{11}{16}\frac{13}{16}}$$

$$= \frac{1}{16} \left(2 \log \frac{2}{\frac{5}{16} * 3} + 3 \log \frac{3}{\frac{5}{16} * 13} + 1 \log \frac{1}{\frac{11}{16} * 3} + 1 \log \frac{10}{\frac{11}{16} * 13} \right)$$

$$= \frac{1}{16} \left(2 \log \frac{16 * 2}{5 * 3} + 3 \log \frac{16 * 3}{5 * 13} + 1 \log \frac{16 * 1}{11 * 3} + 1 \log \frac{16 * 10}{11 * 13} \right)$$

3.2.6   6.

3.2.7   7.

$$\mathbb{P}(E = 0) = \mathbb{P}(E = 0, T = 0) + \mathbb{P}(E = 0, T = 1)$$
$$= 3/512 + 13/512$$
$$= 16/512$$

$$\mathbb{P}(E = 1) = \mathbb{P}(E = 1, T = 0) + \mathbb{P}(E = 1, T = 1)$$
$$= 93/512 + 403/512$$
$$= 496/512$$

$$\mathbb{P}(T = 0) = \mathbb{P}(T = 0, E = 0) + \mathbb{P}(T = 0, E = 1)$$
$$= 3/512 + 93/512$$
$$= 96/512$$
$$= 3/16$$

$$\mathbb{P}(T = 1) = \mathbb{P}(T = 1, E = 0) + \mathbb{P}(T = 1, E = 1)$$
$$= 13/512 + 403/512$$
$$= 416/512$$
$$= 13/16$$

$$\mathbb{P}_{error}(E) = \mathbb{P}(E = 0, T = 1) + \mathbb{P}(E = 1, T = 0)$$
$$= 13/512 + 93/512$$
$$= 106/512 \approx 0.21$$
$$< \mathbb{P}_{error}(M)$$

3.2.8   8.

$$\mathbb{P}(E = 0)\mathbb{P}(T = 0) = 16/512 * 3/16$$
$$= 3/512$$
$$= \mathbb{P}(E = 0, T = 0)$$

$$\mathbb{P}(E = 0)\mathbb{P}(T = 1) = 16/512 * 416/512$$
$$= 13/512$$
$$= \mathbb{P}(E = 0, T = 1)$$

$$\mathbb{P}(E = 1)\mathbb{P}(T = 0) = 496/512 * 3/16$$
$$= 93/512$$
$$= \mathbb{P}(E = 1, T = 0)$$

$$\mathbb{P}(E = 1)\mathbb{P}(T = 1) = 496/512 * 13/16$$
$$= 403/512$$
$$= \mathbb{P}(E = 1, T = 1)$$

E and T are independent.

3.2.9   9.

From Shannon's first therorem:

$$\bar{n}_{min}(T) = \frac{H(T)}{\log_2 q}$$
$$= H(T)$$
$$= 0.70 \text{ bit/realization of } T$$

3.2.10   11.

$$\bar{n}_{min}(T, M) = H(T, M)$$
$$= 1.50 \text{ sh/realization of } (M, T)$$

4 Discrete source coding

4.1 Exercise 4

- We encore the following commands:

- Raise the stylus (RS)

- Press the stylus (PS)

- move the stylus left (–X)

- move the stylus right (+X)

- move the stylus up (+Y)

- move the stylus down (–Y)

We have:

$$\mathbb{P}_{RS} = \mathbb{P}_{PS} = \mathbb{P}_{-X} = 0.1, \quad \mathbb{P}_{+X} = 0.3, \quad \mathbb{P}_{+Y} = \mathbb{P}_{-Y} = 0.2$$

4.2 Problem 2

4.2.1 1.

Let

$$\Pi := \begin{bmatrix} 1-q & q \\ p & 1-p \end{bmatrix}$$

We want $\lambda\Pi = \lambda$, that is:

$$\lambda\Pi = \lambda \implies \begin{cases} \lambda_1(1-q) + \lambda_2 p = \lambda_1 \\ \lambda_1 q + \lambda_2(1-p) = \lambda_1 \end{cases}$$

$$\implies \begin{cases} \lambda_1 = \frac{p}{p+q} = \frac{1}{3} \\ \lambda_2 = \frac{q}{p+q} = \frac{2}{3} \end{cases}$$

4.2.2 2.

4.2.3 3.

5 In-class exercises 2023-01-23

Consider the following matrix:

$$\Pi = \begin{bmatrix} 0 & 0.7 & 0.3 \\ 0.5 & 0 & 0.5 \\ 0 & 0.8 & 0.2 \end{bmatrix}$$

and:
$$p_\infty = \begin{bmatrix} x & y & z \end{bmatrix}$$

Therefore:

$$p_\infty \Pi = p_\infty$$
$$\implies \det(p_\infty(\Pi - Id)) = 0$$
$$\implies \det\left[p_\infty(\Pi - Id)\right] = 0$$
$$\implies \det p_\infty \begin{bmatrix} -1 & 0.7 & 0.3 \\ 0.5 & -1 & 0.5 \\ 0 & 0.8 & -0.8 \end{bmatrix} = 0$$

6 In-class exercises 2023-01-30

6.1 Lecture notes exercise

$$H(X) = -\sum_{i=1}^{2} \mathbb{P}(X = x_i) \log \mathbb{P}(X = x_i)$$
$$= -2 * \frac{1}{2} \log \frac{1}{2}$$
$$= 1$$

$$\mathbb{P}(Y = 0) = \mathbb{P}(Y = 0, X = 0) + \mathbb{P}(Y = 0, X = 1)$$
$$= \mathbb{P}(Y = 0 \mid X = 0)\mathbb{P}(X = 0) + \mathbb{P}(Y = 0 \mid X = 1)\mathbb{P}(X = 1)$$
$$= (1 - p)\frac{1}{2} + p\frac{1}{2}$$
$$= \frac{1}{2}$$

$$\mathbb{P}(Y = 1) = 1 - \mathbb{P}(Y = 0)$$
$$= \frac{1}{2}$$

$$\mathbb{P}(X = 0 \mid Y = 0) = \frac{\mathbb{P}(Y = 0 \mid X = 0)\mathbb{P}(X = 0)}{\mathbb{P}(Y = 0)}$$
$$= \frac{(1 - p)1/2}{1/2}$$
$$= 1 - p$$

$$\mathbb{P}(X = 1 \mid Y = 0) = \frac{\mathbb{P}(Y = 0 \mid X = 1)\mathbb{P}(X = 1)}{\mathbb{P}(Y = 0)}$$
$$= \frac{p1/2}{1/2}$$
$$= p$$

$$\mathbb{P}(X = 0 \mid Y = 1) = \frac{\mathbb{P}(Y = 1 \mid X = 0)\mathbb{P}(X = 0)}{\mathbb{P}(Y = 1)}$$
$$= p$$

$$\mathbb{P}(X = 1 \mid Y = 1) = \frac{\mathbb{P}(Y = 0 \mid X = 1)\mathbb{P}(X = 1)}{\mathbb{P}(Y = 1)}$$
$$= 1 - p$$

$$H(X \mid Y) = \frac{1}{2}H(p, 1 - p) + \frac{1}{2}H(p, 1 - p)$$
$$= H(p, 1 - p)$$

$$I(X, Y) = H(X) - H(X \mid Y)$$
$$= 1 - H(p, 1 - p)$$

6.2   Previous exam - exercise 2

6.2.1   1.

$$H(X) = h(\beta)$$

6.2.2   2.

$$\mathbb{P}(X = 0, Y = 0) = \mathbb{P}(Y = 0 \mid X = 0)\mathbb{P}(X = 0) = q\beta$$
$$\mathbb{P}(X = 0, Y = 1) = \mathbb{P}(Y = 1 \mid X = 0)\mathbb{P}(X = 0) = (1 - q)\beta$$
$$\mathbb{P}(X = 1, Y = 0) = \mathbb{P}(Y = 0 \mid X = 1)\mathbb{P}(X = 1) = 0$$
$$\mathbb{P}(X = 1, Y = 1) = \mathbb{P}(Y = 1 \mid X = 1)\mathbb{P}(X = 1) = (1 - \beta)$$

6.2.3 3.

$$\mathbb{P}(Y = 0) = \mathbb{P}(Y = 0, X = 0) + \mathbb{P}(Y = 0, X = 1)$$
$$= q\beta + 0$$
$$= q\beta$$

$$\mathbb{P}(Y = 1) = \mathbb{P}(Y = 1, X = 0) + \mathbb{P}(Y = 1, X = 1)$$
$$= (1 - q)\beta$$

6.2.4 4.

$$H(Y \mid X = 0) = h(q)$$

$$H(Y \mid X = 1) = 0$$

$$H(Y) = H(Y \mid X = 0)\mathbb{P}(X = 0) + H(Y \mid X = 1)\mathbb{P}(X = 1)$$
$$= h(q)\beta + 0(1 - \beta)$$
$$= h(q\beta)$$

6.2.5 5.

$$H(Y \mid X) = H(Y \mid X = 0)\mathbb{P}(X = 0) + H(Y \mid X = 1)\mathbb{P}(X = 1)$$
$$= h(q)\beta + 0(1 - \beta)$$
$$= h(q)\beta$$

$$I(X, Y) = H(Y) - H(Y \mid X)$$
$$= \beta h(q) - h(q)\beta$$

6.2.6   6.

$$f(x) = h(qx) - xh(q)$$
$$\Longrightarrow f'(x) = h'(qx) - h(q)$$
$$\Longrightarrow f'(x) = (-qx \log qx - (1 - qx) \log(1 - qx))' - h(q)$$
$$\Longrightarrow f'(x) = (-q \log qx - qx\frac{q}{qx} + q \log(1 - qx) - (1 - qx)\frac{-q}{1 - qx} - h(q)$$

$$f'(x) = 0$$
$$\Longrightarrow -qx \log qx - (1 - qx) \log(1 - qx) - h(q) = 0$$
$$\Longrightarrow -qx \log qx - (1 - qx) \log(1 - qx) = h(q)$$
$$\Longrightarrow -x \log qx - \frac{(1 - qx)}{q} \log(1 - qx) = \frac{h(q)}{q}$$
$$\Longrightarrow -x \log qx - \frac{(1 - qx)}{q} \log(1 - qx) = \frac{h(q)}{q}$$