Sparse $\ell_0 - \ell_1$ image reconstruction

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Projet MORPHEME - UCA, CNRS, INRIA -

M2 MScDAI

Outline of the talk

1. Introduction and examples
2. ℓ_1 promotes sparsity
3. Algorithms for $\ell_2 - \ell_1$ optimization
4. Algorithms for $\ell_2 - \ell_0$ optimization
5. Results on super-resolution Microscopy by Single Molecule Localization.

1.Introduction

Many signal processing areas are concerned with

- Linear observation : $A\mathrm{x} = d$
  - d : observed data, vector in $\mathbb{R}^M$
  - x unknown data to be estimated in $\mathbb{R}^N$
  - A observation matrix, $M \times N$ matrix.

  where we have **few observations** for a **large explicative unknown variables** x $M << N$
  The system is undertermined, A is ill-conditioned, observations are noisy
- Least square solution $\hat{\mathrm{x}} = \arg \min_{\mathrm{x} \in \mathbb{R}^N} \|A\mathrm{x} - d\|_2^2$
  ($\|\mathrm{x}\|_2^2 = \|\mathrm{x}\|^2 = \sum_{i=1}^N x_i^2$)
- Regularization: **sparse** signal hypothesis modeled by considering ℓ_1-norm or ℓ_0 semi-norm constraints:

$$\|\mathrm{x}\|_1 \leq K \quad \text{where} \quad \|\mathrm{x}\|_1 = \sum_{i=1}^N |\mathrm{x}_i|$$

$$\|\mathrm{x}\|_0 \leq K \quad \text{where} \quad \|\mathrm{x}\|_0 = \#\{\mathrm{x}_i,\ i = 1, \dots, N : \mathrm{x}_i \neq 0\}$$

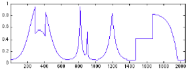
NB: ℓ_0-norm **is NOT** a norm as $\|\lambda\mathrm{x}\|_0 = \|\mathrm{x}\|_0 \neq \lambda\|\mathrm{x}\|_0$.

1.0 Dictionary representation in image processing

- Image are non-stationary, they exhibit smooth areas, oscillations, edges, textures,...

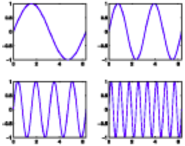
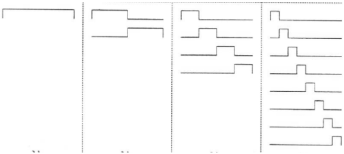
Let's $d \in \mathbb{R}^M$ be a patch of an image or a signal:



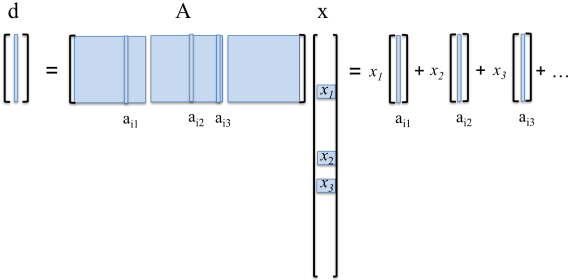


- Each part is represented by given waveforms which best match the image structure, for example Basis B_i as Haar, smooth wavelets, sine/cosine transform,...

Let's $A = [\mathrm{a}_1, ..., \mathrm{a}_N] \in \mathbb{R}^{M \times N}$ be a set of basis vectors, or normalized vectors

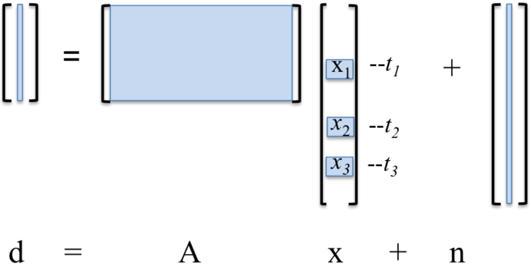




- Such A is a redundant **dictionary** (succession of representative waveforms, possibly a succession of bases)
- The dictionary A is adapted to the signal d if d can be represented by a **few** number of vectors of the dictionary A, that is $d \approx A\mathrm{x}$ with x is a **sparse** vector, that is $\|\mathrm{x}\|_0 \leq K$, where $K << N$.



- signal is a sum of pulses, spikes, modeled by a sum of Dirac $\sum_{r=1}^{K} x_r \delta_{t_r}$.
- acquisition system, channel, is modeled as a linear system, e.g. convolution by a Gaussian function: $d(.) = h * \sum_{r=1}^{K} x_r \delta_{t_r} = \sum_{r=1}^{K} x_r h(. - t_r)$.

By assuming the Dirac locations t_r are on a regular grid indexed by $i = 1, \ldots N$

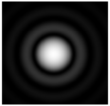


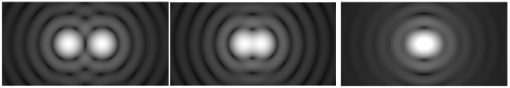
- 1D example: Channel estimation in communications, ...
- 2D example: Single Molecule Localization in super-resolution microscopy , ...

2D example in Super-resolution microscopy: SMLM (continued)

Conventional fluorescence microscopy limits

- physical diffraction limit of optical systems
- Airy patch = impulse response of the microscope (PSF: *Point Spread Function*)
- overlapping patches limit at ≈200nm the distance between two molecules to be resolved (Rayleigh limit)





Super-resolution by single molecule localization

- **Photo-activable molecules**: PALM *Photo Activated Localisation Microscopy* ([Betzig & al 06, Hess & al, 2006]) et STORM *STochastic Optical Reconstruction Microscopy* ([Rust & al, 2006])
- Sequentially activate and image a small random set of fluorescent molecules.

2D example in Super-resolution microscopy: SMLM (continued)

- activation
- imaging
- localization
- assembling

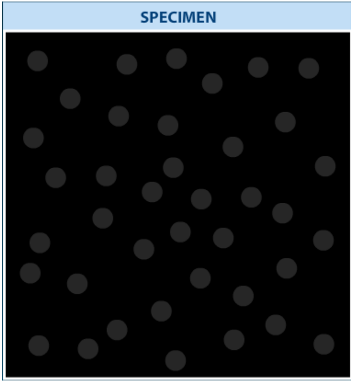


Figure: PALM microscopy principle. From Zeiss tutorials [http://zeiss-campus.magnet.fsu.edu/tutorials/index.html]

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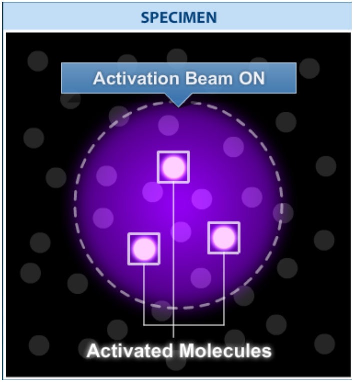

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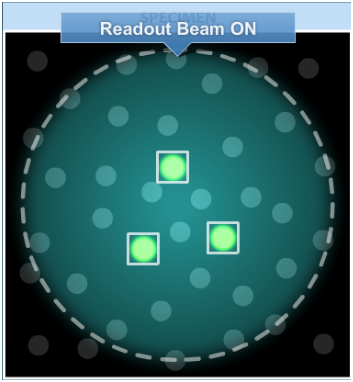

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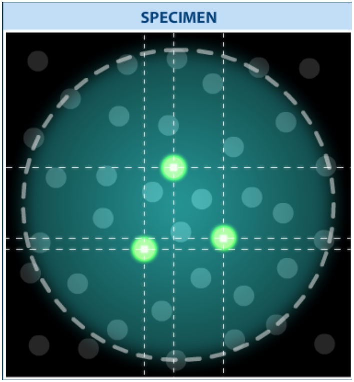

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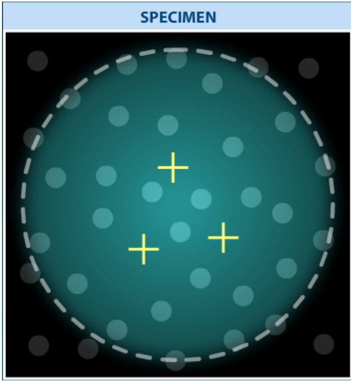

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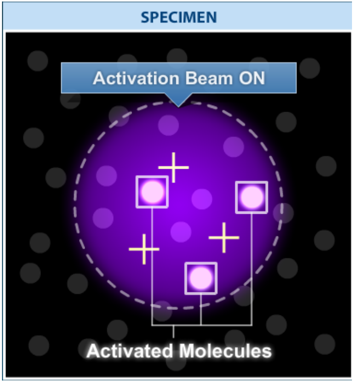

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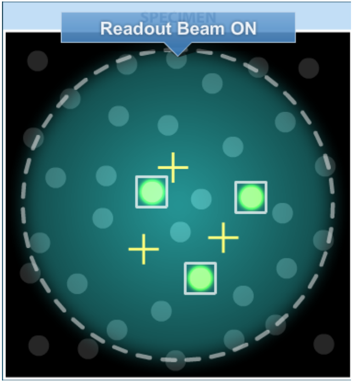

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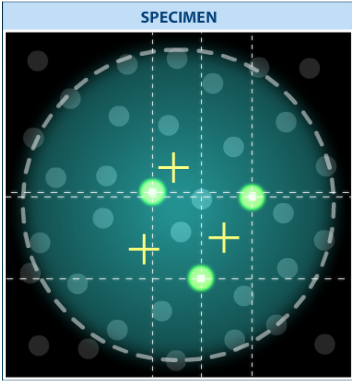

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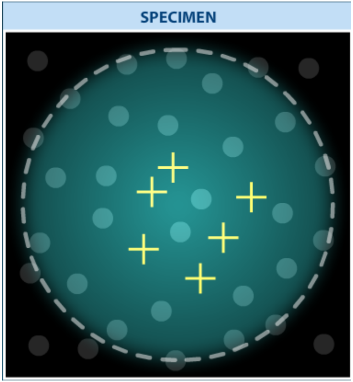

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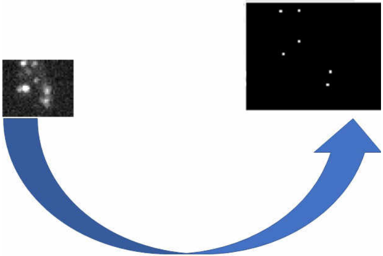
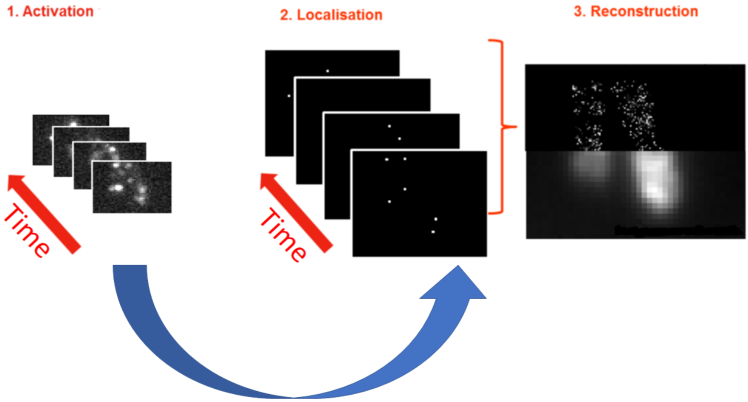


$$\hat{x} \in \arg\min_{x} \frac{1}{2}\|Ax - d\|_2^2 + \lambda \|x\|_0$$

2D example in Super-resolution microscopy: SMLM (continued)



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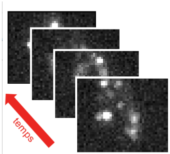
2D example in Super-resolution microscopy: SMLM (continued)

Limitations: number of acquisition needed to obtain the super-resolved image

- cost time and memory
- temporal resolution restricted (motion)

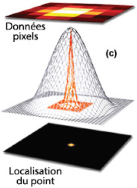


→ **Increase molecule density**

- Localization more difficult due to more overlapping

Localization algorithms

- Challenge ISBI 2013 [Sage et al 15]
- PSF fitting, and derived methods for high density molecule localization (e.g. DAOSTORM, [Holden & al 11]).



- **Deconvolution and reconstruction on a finer grid** (e.g. FALCON, [Min & al, 2014])

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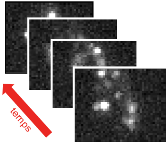

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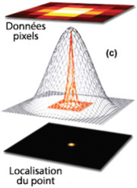

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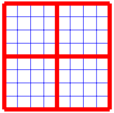
2D example in Super-resolution microscopy: SMLM (continued)

Image formation model PALM / STORM

$\mathbf{d} \in \mathbb{R}^{M \times M}$ one acquisition.
$\mathbf{X} \in \mathbb{R}^{ML \times ML}$ an image where each pixel of $\mathbf{d}$ is divided in $\mathbf{L}\times\mathbf{L}$ pixels.



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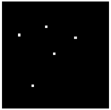

$* \quad \xrightarrow{H(\cdot)} \quad \xrightarrow{M_L(\cdot)} \quad \xrightarrow{+\eta}$

X PSF H(X) $M_L(H(X))$ d

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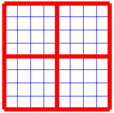

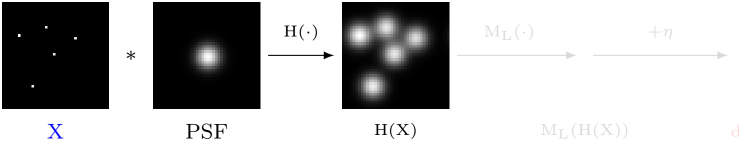

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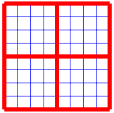

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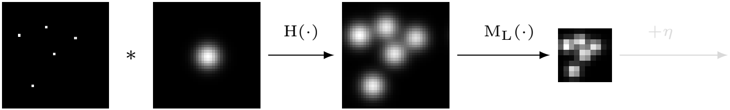

X | PSF | H(X) | $M_L(H(X))$ | d

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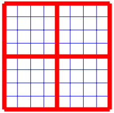

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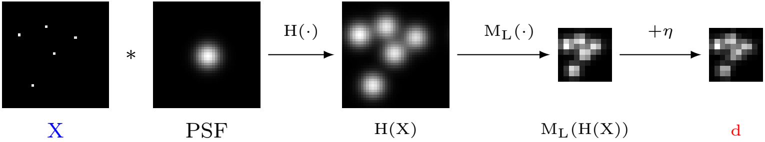

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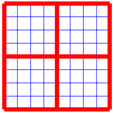

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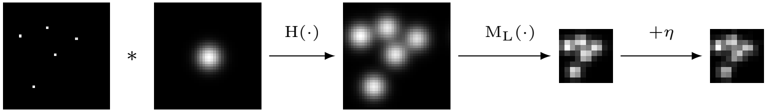

Model

$$\mathrm{d} = \mathrm{M_L}(\mathrm{H}(\mathrm{X})) + \eta,$$

Image formation model PALM / STORM

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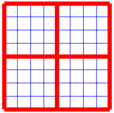

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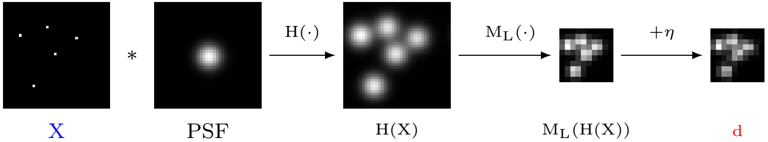


Problem $\ell_2 - \ell_0$

$$\hat{\mathrm{X}} \in \arg\min_{\mathrm{X}} \frac{1}{2}\|\mathrm{d} - \mathrm{M_L}(\mathrm{H}(\mathrm{X}))\|_2^2 + \lambda\|X\|_0$$

$\|X\|_0 = \#\{X_i / X_i \neq 0\}$ is the number of non zero components of X.

1.3 ℓ_2-ℓ_0 optimization problems

Noisy problem: two constrained forms ($\epsilon > 0$, $K > 0$)

$$\hat{\mathrm{x}} = \arg\min_{\mathrm{x}\in\mathbb{R}^N} \frac{1}{2}\|A\mathrm{x} - d\|_2^2 \ \text{ subject to } \ \|\mathrm{x}\|_0 \leq K$$

$$\hat{\mathrm{x}} = \arg\min_{\mathrm{x}\in\mathbb{R}^N} \|\mathrm{x}\|_0 \ \text{ subject to } \ \|A\mathrm{x} - d\|_2^2 \leq \epsilon$$

Noisy problem : penalized form ($\lambda > 0$)

$$\hat{\mathrm{x}} = \arg\min_{\mathrm{x}\in\mathbb{R}^N} \mathrm{G}_{\ell_0}(\mathrm{x}) := \frac{1}{2}\|A\mathrm{x} - d\|_2^2 + \lambda\|\mathrm{x}\|_0$$

$$A \in \mathbb{R}^{M\times N} \text{ with } M \ll N$$

- Non equivalent formulations
- Existence of an optimal solution and relationships between optimal solutions in [Nikolova 16]
- Intensive work in signal and image processing, and in statistics.
- **non-continuous**, **non-convex** and **NP-hard** optimization problem. [Natarajan 95] [Davis & al 97]. Rouhgly speaking, *a solution cannot be verified in polynomial time w.r.t the dimension of the problem*

Replacing ℓ_0-semi-norm by ℓ_1-norm

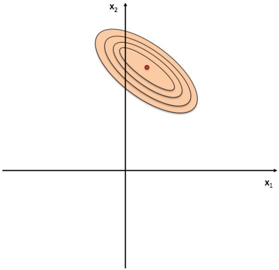
$$\hat{\mathrm{x}} = \arg\min_{\mathrm{x}\in\mathbb{R}^N} \|A\mathrm{x} - d\|_2^2 \ \text{ subject to } \ \|\mathrm{x}\|_1 \leq K$$

$$\hat{\mathrm{x}} = \arg\min_{\mathrm{x}\in\mathbb{R}^N} \|A\mathrm{x} - d\|_2^2 + \lambda\|\mathrm{x}\|_1$$

with $\|\mathrm{x}\|_1 = \sum_{i=1}^N |x_i|$.

- gives easier optimization problems: convex and continuous (but non smooth)
- Different (contraint/penalized) formulations are equivalent
- ℓ_1-norm promotes sparsity
- They are known as **Basis Pursuit De-Noising** (BPDN) [Chen et al 98], or **LASSO** [Tibshirani 96] problems.

2. ℓ_1 -norm promotes sparsity



Level lines of $\|Ax - d\|_2^2$.

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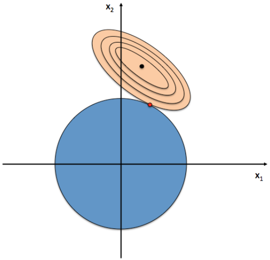
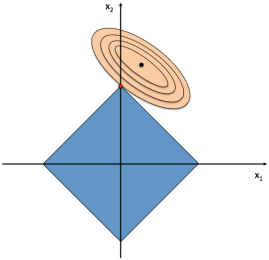


Level lines of $\|A\mathrm{x} - d\|_2^2$ with the ℓ_2 constraint $\|\mathrm{x}\|_2 \leq K$.

2. ℓ_1 -norm promotes sparsity



Level lines of $\|Ax - d\|_2^2$ with the ℓ_1 constraint $\|x\|_1 \leq K$.

Let's look at the **penalized** form in the 1-dimensional case: we want to compute

$$\arg\min_{x\in\mathbb{R}} \left\{ g(x) := \frac{1}{2}(x-d)^2 + \lambda|x| \right\}$$

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- if $x \geq 0$ then $g(x) = \frac{1}{2}(x-d)^2 + \lambda x$
- The minimum is reached at $\hat{x} = d - \lambda$, if $d \geq \lambda$
- if $d < \lambda$ and $x \geq 0$ the minimum is reached in $\hat{x} = 0$

- if $x \leq 0$ then $g(x) = \frac{1}{2}(x-d)^2 - \lambda x$
- The minimum is reached at $\hat{x} = d + \lambda$, if $d \leq -\lambda$
- if $d > -\lambda$ and $x \leq 0$ the minimum is reached in $\hat{x} = 0$

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if $d \geq \lambda$ then $\hat{x} = d - \lambda$

- if $x \leq 0$ then $g(x) = \frac{1}{2}(x-d)^2 - \lambda x$
- The minimum is reached at $\hat{x} = d + \lambda$, if $d \leq -\lambda$
- if $d > -\lambda$ and $x \leq 0$ the minimum is reached in $\hat{x} = 0$

if $d \leq -\lambda$ then $\hat{x} = d + \lambda$

and if $-\lambda \leq d \leq \lambda$ then $\hat{x} = 0$

The solution is given by the **Soft Threshold** function.

2. ℓ_1 -norm promotes sparsity

Let's look at the **penalized** form in the 1-dimensional case: we want to compute

$$\arg\min_{x \in \mathbb{R}} \left\{ g(x) := \frac{1}{2}(x-d)^2 + \lambda|x| \right\}$$

- if $x \geq 0$ then $g(x) = \frac{1}{2}(x-d)^2 + \lambda x$
- The minimum is reached at $\hat{x} = d - \lambda$, if $d \geq \lambda$
- if $d < \lambda$ and $x \geq 0$ the minimum is reached in $\hat{x} = 0$

- if $x \leq 0$ then $g(x) = \frac{1}{2}(x-d)^2 - \lambda x$
- The minimum is reached at $\hat{x} = d + \lambda$, if $d \leq -\lambda$
- if $d > -\lambda$ and $x \leq 0$ the minimum is reached in $\hat{x} = 0$

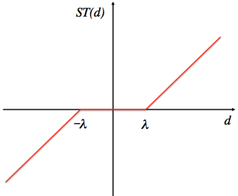
if $d \geq \lambda$ then $\hat{x} = d - \lambda$

if $d \leq -\lambda$ then $\hat{x} = d + \lambda$

and if $-\lambda \leq d \leq \lambda$ then $\hat{x} = 0$

The solution is given by the **Soft Threshold** function.

$$\hat{x}(d) = ST_\lambda(d) = \begin{cases} d - \lambda & \text{if } d > \lambda \\ d + \lambda & \text{if } d < -\lambda \\ 0 & \text{if } |d| \leq \lambda \end{cases}$$



In the 1-dimensional case, the solution of

$$\arg\min_{x\in\mathbb{R}} \left\{ \frac{1}{2}(d-x)^2 + \lambda|x| \right\}.$$

is reached in

$$\hat{x}(d) = ST_\lambda(d) = \begin{cases} d - \text{sign}(d)\lambda & \text{if } |d| > \lambda \\ 0 & \text{if } |d| \leq \lambda \end{cases} \tag{1}$$

which is the soft-thresholding (ST) function. Then we have that $\hat{x} = ST_\lambda(d)$ and $\hat{x} = 0$ for all $|d| \leq \lambda$.

Remark: if we use the ℓ_2-norm the problem is $\arg\min_{x\in\mathbb{R}} \left\{ \frac{1}{2}(d-x)^2 + \lambda x^2 \right\}$. The solution is $\hat{x} = \frac{d}{1+2\lambda}$ which is different from 0 as soon as $d \neq 0$.

Algorithms for ℓ_2 - ℓ_1 optimization

1. Convex non smooth optimization
2. Forward-Backward Splitting (FBS) algorithm for ℓ_1: IST, and Fast version (FISTA,...)
3. ADMM / Split Bregman Algorithm
4. ...

The optimization problem is

$$\arg\min_{\mathrm{x}\in\mathbb{R}^N} \frac{1}{2}\|A\mathrm{x}-d\|_2^2+\lambda\|\mathrm{x}\|_1$$

But $\|\mathrm{x}\|_1=\sum_{i=1}^N |x_i|$ is convex but **non differentiable** in all x (x such that $\exists i, x_i=0$).
An algorithm adapted to the minimization of this $\ell_2-\ell_1$ non smooth function is the **Forward-Backward Splitting** Algorithm.
Let consider the optimization problem

$$\arg\min_{x\in\mathbb{R}} \{f(\mathrm{x})+g(\mathrm{x})\}$$

where f is convex, differentiable and g is continuous, convex, non differentiable but such that its proximal has an explicit form.
Definition Proximal of g :

$$\mathrm{prox}_g(\mathrm{y})=\arg\min_{\mathrm{x}\in\mathbb{R}^N}\left\{\frac{1}{2}\|\mathrm{x}-\mathrm{y}\|^2+g(\mathrm{x})\right\}$$

Example $g(.)=\lambda\|.\|_1$, then

$$\mathrm{prox}_{\|.\|_1}(\mathrm{y})=ST_\lambda(y).$$

Forward-Backward Algorithm

Optimization problem

$$\arg\min_{x\in\mathbb{R}} \{f(\mathrm{x}) + g(\mathrm{x})\}$$

$f : \mathbb{R}^N \to \mathbb{R}$ convex, differentiable, L-gradient Lipschitz;
$g : \mathbb{R}^N \to \mathbb{R}$ continuous, non differentiable, with explicit proximal.

Forwars-Backward Splitting (FBS) Algorithm

Data: $\mathrm{x}^0, 0 < \gamma < \frac{1}{L}, TOL$

$k = 0$, $\mathrm{x}^1 = \mathrm{prox}_{\gamma g}\left(\mathrm{x}^0 - \gamma\nabla f(\mathrm{x}^0)\right)$

while $\left(\frac{\|\mathrm{x}^{k+1}-\mathrm{x}^k\|}{\|\mathrm{x}^k\|} > TOL\right)$ **do**

  $\mathrm{x}^{k+1} = \mathrm{prox}_{\gamma g}\left(\mathrm{x}^k - \gamma\nabla f(\mathrm{x}^k)\right)$

  $k = k + 1$

end

- The FBS algorithm converges to a minimizer of $f + g$ if f and g are convex functions [Combettes and Wajs 05], and to a stationary point for non convex functions [Attouch et al 13].
- Very easy to use and program on large scale data

Penalized form

$$\hat{\mathrm{x}} = \arg \min_{\mathrm{x} \in \mathbb{R}^N} \frac{1}{2} \|A\mathrm{x} - d\|_2^2 + \lambda \|\mathrm{x}\|_1$$

- $\frac{1}{2}\|A\mathrm{x} - d\|_2^2$ is L-gradient Lipschitz ($L = \|A\|^2$)
- Proximal of $\|.\|_1$ has explicit expression, this is the Soft Threshold:

$$\mathrm{prox}_{\gamma\lambda\|.\|_1}(\mathrm{y}) = ST_{\gamma\lambda}(\mathrm{y})$$

Iterative Soft Thresholding

(IST): Forward-Backward Splitting (FBS) algorithm

$$\mathrm{x}^{k+1} = ST_{\gamma\lambda}\left(\mathrm{x}^k - \gamma A^t \left(A\mathrm{x}^k - d\right)\right)$$

$\gamma < \frac{2}{L}$ is the gradient step.

4.0 ℓ_1 or ℓ_0 minimization

Let consider the penalized and constrained ℓ_1 problems ($\lambda > 0$)

$$\arg\min_{\mathrm{x}\in\mathbb{R}^N} \frac{1}{2}\|A\mathrm{x} - d\|_2^2 + \lambda\|\mathrm{x}\|_1 \tag{2}$$

$$\arg\min_{\mathrm{x}\in\mathbb{R}^N} \frac{1}{2}\|A\mathrm{x} - d\|_2^2 \ \text{ subject to } \ \|\mathrm{x}\|_1 \leq K \tag{3}$$

and the penalized and constrained ℓ_0 problems ($\lambda > 0$)

$$\arg\min_{\mathrm{x}\in\mathbb{R}^N} \frac{1}{2}\|A\mathrm{x} - d\|_2^2 + \lambda\|\mathrm{x}\|_0 \tag{4}$$

$$\arg\min_{\mathrm{x}\in\mathbb{R}^N} \frac{1}{2}\|A\mathrm{x} - d\|_2^2 \ \text{ subject to } \ \|\mathrm{x}\|_0 \leq K \tag{5}$$

$$A \in \mathbb{R}^{M\times N} \text{ with } M \ll N$$

- Problems (2) and (3) are continuous and convex.
- Problems (2) and (3) are "equivalent".
- Problems (4) and (5) are **non-continuous**, **non-convex** and **NP-hard** optimization problems [Natarajan 95] [Davis & al 97].
- Problems (4) and (5) are not equivalent.

ℓ_2-ℓ_0 Optimization

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1. Iterative Hard Thresholding,
2. Continuous relaxation,
3. Greedy algorithms,
4. Exact reformulation.

4.1 FBS = IHT Algorithm

Penalized form

$$\hat{x} = \arg\min_{x \in \mathbb{R}^N} \frac{1}{2}\|Ax - d\|_2^2 + \lambda \|x\|_0$$

- $\frac{1}{2}\|Ax - d\|_2^2$ is L-gradient Lipschitz ($L = \|A\|^2$)
- Proximal of $\|.\|_0$ has explicit expression, this is the Hard Threshold

Iterative Hard Thresholding

(IHT): Forward-Backward Splitting (FBS) algorithm

$$x^{k+1} = \text{prox}_{\gamma\lambda\|.\|_0}\left(x^k - \gamma A^t\left(Ax^k - d\right)\right)$$

$\gamma < \frac{1}{L}$ is the gradient step.

Computation of $\text{prox}_{\gamma\lambda\|.\|_0}$:

$$\text{prox}_{\gamma\lambda\|.\|_0}(y) = \arg\min_{x \in \mathbb{R}^N} \left\{\frac{1}{2}\|x - y\|^2 + \gamma\lambda\|x\|\|_0\right\}$$

$$\frac{1}{2}(x - y)^2 + \gamma\lambda\|x\|\|_0 = \sum_{i=1}^{N}(x_i - y_i)^2 + \gamma\lambda|x_i|_0$$

where $|u|_0 = 1$ if $u \neq 0$, 0 elsewhere.
Then it is sufficient to compute in 1D $\arg\min_{u \in \mathbb{R}} \left\{g(u) := \frac{1}{2}(u - y)^2 + \gamma\lambda|u|_0\right\}$

Computation of $\arg\min_{u\in\mathbb{R}} \left\{ g(u) := \frac{1}{2}(u-y)^2 + \gamma\lambda|u|_0 \right\}$

- if $u = 0$ then $g(0) = \frac{1}{2}(y)^2$
- The minimum could be reached at $\hat{u} = 0$, the value is $g(\hat{u}) = \frac{1}{2}(y)^2$

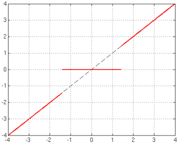
- if $u \neq 0$ then $g(u) = \frac{1}{2}(u-y)^2 + \lambda$
- The minimum is reached at $\hat{u} = y$ and the value is $g(\hat{u}) = \lambda$

if $|y| \leq \sqrt{2\lambda}$ then $\hat{u} = 0$

if $|y| \geq \sqrt{2\lambda}$ then $\hat{u} = y$

The solution is given by the Hard Threshold function

$$\hat{u} = \begin{cases} y & \text{if } |y| > \sqrt{2\lambda}, \\ 0 & \text{if } |y| \leq \sqrt{2\lambda}. \end{cases}$$



Find the solution of the optimal problem

$$\hat{\mathrm{x}} = \arg\min_{\mathrm{x}\in\mathbb{R}^N} \frac{1}{2}\|A\mathrm{x} - d\|_2^2 + \lambda\|\mathrm{x}\|_0$$

by Forward Backward Splitting algorithm (Iterative Hard Thresholding)

$$\mathrm{x}^{k+1} = \mathrm{prox}_{\gamma\lambda\|.\|_0}\left(\mathrm{x}^k - \gamma A^t\left(A\mathrm{x}^k - d\right)\right)$$

- IHT algorithm converges to a critical point [Blumensath and Davies 08, Attouch et al 13].
- **Initialization** point is important, for example initialize with the solution with the ℓ_1-norm problem: $\arg\min_{\mathrm{x}\in\mathbb{R}^N}\left\{\frac{1}{2}\|A\mathrm{x} - \mathrm{y}\|^2 + \gamma\lambda\|\mathrm{x}\|_1\right\}$. It is not guaranty that this solution is sparse.

Continuous separable relaxation (convex and non-convex)

$$\frac{1}{2}\|Ax - d\|_2^2 + \lambda\|\mathrm{x}\|_0 \quad \rightarrow \quad \frac{1}{2}\|Ax - d\|_2^2 + \lambda\sum_{i\in\mathbb{I}_N} \phi(\mathrm{x}_i)$$

Continuous approximation of the ℓ_0-norm function:

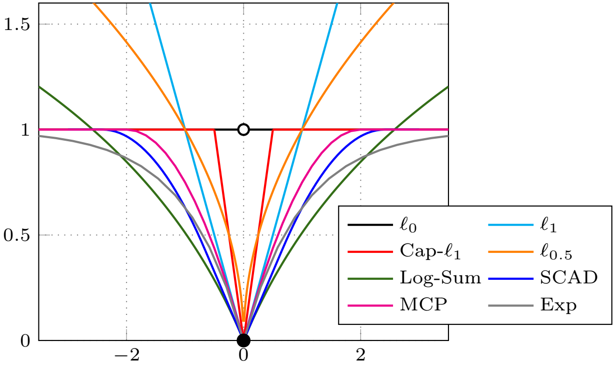
- ℓ_1-norm: Lasso [Tibshirani 96] ; Basic Pursuit [Chen et al 98] ; Compressed Sensing [Donoho 06, Candès et al 06])
- Adaptive Lasso [Zou 06] ;
- Nonnegative Garrote [Breiman 95] ;
- Exponential approximation [Mangasarian 96] ;
- Log-Sum Penalty [Candès et al 08] ;
- Smoothly Clipped Absolute Deviation (SCAD) [Fan and Li 01] ;
- Minimax Concave Penalty (MCP) [Zhang 10] ;
- ℓ_p-norms $0 < p < 1$ [Chartrand 07, Foucart and Lai 09] ;
- Smoothed ℓ_0-norm Penalty (SL0) [Mohimani et al 09] ;

Are they *good* approximations?
Which one to use?

Continuous separable relaxation (convex and non-convex)

$$\frac{1}{2}\|A\mathrm{x} - d\|_2^2 + \lambda\|\mathrm{x}\|_0 \quad \to \quad \frac{1}{2}\|A\mathrm{x} - d\|_2^2 + \lambda\sum_{i \in \mathbb{I}_N} \phi(\mathrm{x}_i)$$

Continuous approximation of the ℓ_0-norm function:



Are they *good* approximations?
Which one to use?

Example of continuous approximation functions of the ℓ_0-norm:

- ℓ_1-norm: $\phi(t) = |t|$
- Log-Sum Penalty [Candès et al 08] $\phi_{Log}(\theta; t) := \log(1 + |t|\theta)$, with $\theta \in \mathbb{R}^{\star}_{+}$.
- Minimax Concave Penalty (MCP) [Zhang 10]
  $\phi_{MCP}(\gamma, \lambda; t) = \lambda \left(\frac{\gamma\lambda}{2} \mathbb{1}_{\{|t|>\gamma\lambda\}} + \left(|t| - \frac{t^2}{2\gamma\lambda} \right) \mathbb{1}_{\{|t|\leq\gamma\lambda\}} \right)$
  with $\mathbb{1}_{\{x\in C\}} = 1$ if $x \in C$ and 0 otherwise.

$$\mathrm{G}_{\ell_0}(x) := \frac{1}{2}\|A\mathrm{x} - d\|_2^2 + \lambda\|\mathrm{x}\|_0 \quad \rightarrow \quad \tilde{\mathrm{G}}(x) := \frac{1}{2}\|A\mathrm{x} - d\|_2^2 + \sum_{i=1}^{N} \phi(\mathrm{x}_i)$$

Definition of a *good* continuous approximation

- $\mathrm{G}_{\ell_0}(x)$ and $\tilde{\mathrm{G}}(x)$ have **same global** minimizers

$$\arg\min_{\mathrm{x}\in\mathbb{R}^N} \tilde{\mathrm{G}}(\mathrm{x}) = \arg\min_{\mathrm{x}\in\mathbb{R}^N} \mathrm{G}_{\ell_0}(\mathrm{x}) \tag{P1}$$

- $\tilde{\mathrm{G}}(x)$ has **less local** minimizers than $\mathrm{G}_{\ell_0}(x)$

$$\hat{\mathrm{x}} \text{ minimiseur de } \tilde{\mathrm{G}} \implies \hat{\mathrm{x}} \text{ minimiseur de } \mathrm{G}_{\ell_0} \tag{P2}$$

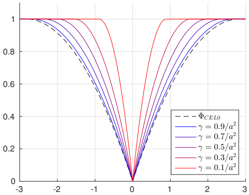
ϕ depends on $\|a_i\|$ and λ when applied on x_i:

$$\tilde{\mathrm{G}}(x) := \frac{1}{2}\|A\mathrm{x} - d\|_2^2 + \sum_{i \in \mathbb{I}_N} \phi(\|a_i\|, \lambda, \mathrm{x}_i)$$

The one which removes the most of local minimizers is $\phi_{MCP}(\frac{1}{\|a_i\|}, \lambda, t)$ that we call $\phi_{\texttt{CEL0}}$:

$$\phi_{\texttt{CEL0}}(\|a_i\|, \lambda, x) = \lambda - \frac{\|a_i\|^2}{2}\left(|x| - \frac{\sqrt{2\lambda}}{\|a_i\|}\right)^2 \mathbb{1}_{\left\{|x| \leq \frac{\sqrt{2\lambda}}{\|a_i\|}\right\}}$$

where $\mathbb{1}_{\{\mathrm{x} \in D\}} = 1$ if $\mathrm{x} \in D$; 0 otherwise.



$\phi_{MCP}(\gamma, \lambda, t)$

The $\ell_2 - \ell_0$ and ℓ_2- CEL0 functionals :

$$\mathrm{G}_{\ell_0}(\mathrm{x}) := \frac{1}{2}\|A\mathrm{x} - d\|^2 + \lambda\|\mathrm{x}\|_0$$

$$\mathrm{G}_{\text{CEL0}}(\mathrm{x}) = \frac{1}{2}\|A\mathrm{x} - d\|^2 + \sum_{i \in \mathbb{I}_N} \phi_{\text{CEL0}}(\|a_i\|, \lambda, \mathrm{x}_i)$$

where $\phi_{\text{CEL0}}(\|a_i\|, \lambda, x) = \lambda - \frac{\|a_i\|^2}{2}\left(|x| - \frac{\sqrt{2\lambda}}{\|a_i\|}\right)^2 \mathbb{1}_{\left\{|x| \le \frac{\sqrt{2\lambda}}{\|a_i\|}\right\}}$

Properties of $\mathrm{G}_{\text{CEL0}}(\mathrm{x})$

- **Limit inf** of the functions satisfying (P1) and (P2) ; the one which removes the most of local minimizers
- **Continuity**
- **Non convex** in the general case (for any A)
- but **convexity** with respect to each **component**

Nonsmooth nonconvex algorithms

The **continuity of** $\mathrm{G}_{\texttt{CEL0}}$ allows to use recent *nonsmooth nonconvex* algorithms to minimize (indirectly) G_{ℓ_0},

- *Difference of Convex* (DC) functions programming [Gasso et al 09]
- *Majorization-Minimization*(MM) algorithms (*e.g.* Iteratively Reweighted ℓ_1 (IRL1) [Ochs et al 2015])
- *Forward-Backward splitting* (GIST [Gong et al 13], [Attouch et al 13])

Forward-Backward Splitting Algorithm

$$\mathrm{x}^{k+1} \in \mathrm{prox}_{\gamma\Phi_{\mathtt{CEL0}}(\cdot)}\left(\mathrm{x}^k - \gamma^k A^T(A\mathrm{x}^k - d)\right),$$

where $0 < \gamma < \frac{1}{\|A\|^2}$ and

$$\mathrm{prox}_{\gamma\phi_{\mathtt{CEL0}}(a,\lambda;\cdot)}(u) = \begin{cases} \mathrm{sign}(u)\min\left(|u|, (|u| - \sqrt{2\lambda}\gamma a)_+/(1 - a^2\gamma)\right) & \text{if } a^2\gamma < 1 \\ u\mathbb{1}_{\{|u|>\sqrt{2\gamma\lambda}\}} + \{0, u\}\mathbb{1}_{\{|u|=\sqrt{2\gamma\lambda}\}} & \text{if } a^2\gamma \geq 1 \end{cases}$$

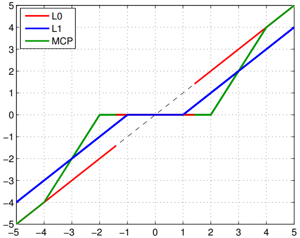


Figure: Proximal operators. Red: ℓ_0, Blue: ℓ_1, Green: $\Phi_{\mathtt{CEL0}}$ (depends on $a = \|a_i\|$ at component $u = \mathrm{x}_i$).

Forward-Backward Splitting Algorithm

$$\mathrm{x}^{k+1} \in \mathrm{prox}_{\gamma \Phi_{\mathtt{CEL0}}(\cdot)} \left(\mathrm{x}^k - \gamma^k A^T (A\mathrm{x}^k - d) \right),$$

where $0 < \gamma < \frac{1}{\|A\|^2}$ and

$$\mathrm{prox}_{\gamma \phi_{\mathtt{CEL0}}(a,\lambda;\cdot)}(u) = \begin{cases} \mathrm{sign}(u) \min \left(|u|, (|u| - \sqrt{2\lambda}\gamma a)_+ / (1 - a^2\gamma) \right) & \text{if } \ a^2\gamma < 1 \\ u \mathbb{1}_{\{|u| > \sqrt{2\gamma\lambda}\}} + \{0, u\} \mathbb{1}_{\{|u| = \sqrt{2\gamma\lambda}\}} & \text{if } \ a^2\gamma \geq 1 \end{cases}$$

- Convergence to a critical point under Kurdyka-Lojaseiwicz (KL) property [Attouch et al 13].
- Accelerated algorithm in the non convex case [Li Lin 15]

5.1 Results, ISBI challenge 2013, simulated dataset

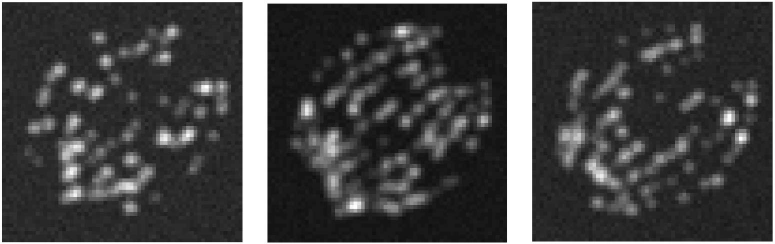


Figure: Simulated images (among the 361 simulated high density images for this sample). Data from IEEE ISBI Challenge 2013.
http://bigwww.epfl.ch/smlm/datasets/index.html

8 simulated tubes of 30nm diameter
Camera of 64×64 pixels of size 100nm.
Gaussian PSF, FWHM = 258.21 nm (full width at half maximum)
80932 molecules activated on 361 frames.

5.1 Results, ISBI challenge 2013, simulated dataset

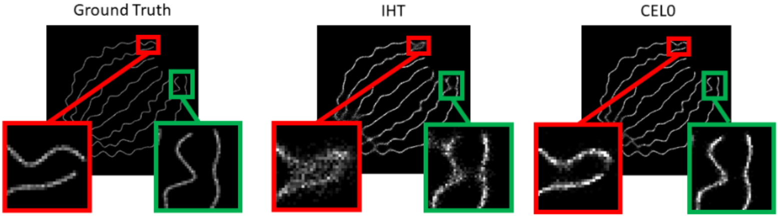
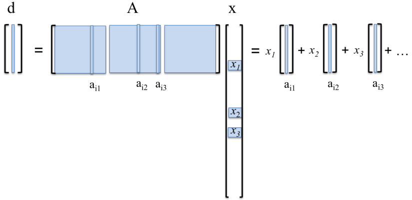


Figure: Reconstruction from simulated data set, reduction ratio $L = 4$.

Greedy algorithms, *Matching Pursuit (MP)* [Mallat et al 93], *Orthogonal MP* [Pati et al 93], *Orthogonal Least Squares (OLS)* [Chen et al 89], *Bayesian OMP* [Herzet et al 10], *Single Best Replacement* [Soussen et al 11] and further variants.

Matching Pursuit:

d is the signal we want to represent with the a limited number $K << N$ of waveforms or atoms of dictionary A, one atom is one column of A, *i.e.* $A_{.,i} = \mathrm{a}_i$, $i = 1, ..N$.



For that we have to solve

$$\hat{\mathrm{x}} = \arg\min_{\mathrm{x}\in\mathbb{R}^N} \|A\mathrm{x} - d\|_2^2 \text{ subject to } \|\mathrm{x}\|_0 \leq K.$$

$$(\text{ or } \hat{\mathrm{x}} = \arg\min_{\mathrm{x}\in\mathbb{R}^N} \|\mathrm{x}\|_0 \text{ subject to } \|A\mathrm{x} - d\|_2^2 \leq \epsilon)$$

Matching Pursuit algorithm add one component at a time.

Matching Pursuit principle

It is assumed without loss of generality that A has unit norm columns, $\|A_{.,i}\| = \|a_i\| = 1$.

The **first component** $i^1 \in \{1, ..., N\}$ will be such that the **correlation** between d and atom i is maximum: $i^1 = \arg \max_{j \in \{1,..,N\}} |\langle a_j, d\rangle|$.

Then the **optimal solution** is $\mathrm{x}^1 = (0, 0, .., \langle a_{i^1}, d\rangle, 0, .., 0)$, where the non null component is at index i^1, which is written as $\mathrm{x}^1 = \langle a_{i^1}, d\rangle . e_{i^1}$, $e_i \in \mathbb{R}^N$, $i \in \{1, .., N\}$ is the canonical basis in $\mathbb{R}^N$.

The criterion is $\|A.\mathrm{x}^1 - d\|^2 = \|d\|^2 - (\langle a_{i^1}, d\rangle)^2$.

The **residual** is $r = d - A.\mathrm{x}^1 = d - \langle a_{i^1}, d\rangle a_{i^1}$, and the process is repeated.

Matching Pursuit Algorithm
Input: A (with unit norm column), d, K.

Initialize: $r^0 = d, \sigma^0 = \varnothing, (x^0 = 0)$.

Repeat, while $\#\sigma^k \leq K$: (or while $\|r^k\| > \epsilon$)

$$\begin{array}{rcl} i^k & = & \arg \max_{j \in \{1,..,N\}} |\langle r^k, a_j \rangle| \\ \sigma^{k+1} & = & \sigma^k \cup \{i^k\} \\ r^{k+1} & = & r^k - \langle r^k, a_{i^k} \rangle . a_{i^k} \end{array} \tag{6}$$

σ^k is the support of the current solution x^k, that is the indexes of the non-zero components. $\#\sigma^k$ is the cardinal of σ^k. The initial value of $\#\sigma^0$ is 0 and it increases by 1 at each iteration.
The optimal solution at current iteration is $x^{k+1} = x^k + \langle r^k, a_{i^k} \rangle . e_{i^k}$.

- The residual $\|r^k\|$ converges exponentially to 0 [Mallat et al 93].
- Sub-optimal solution: retro-project the residual onto $Span\{(a_i)_{i \in \sigma^K}\}$ reduce the approximation error ($\|A.\mathrm{x}^K - d\|^2$).

Orthogonal Matching Pursuit [Pati et al 93, Tropp 04]: at each iteration, optimally estimate the intensities with the current support of the solution fixed, by $\mathrm{x}^{k+1} = \arg\min_{\{\mathrm{x}/\sigma_{\mathrm{x}} \subset \sigma^{k+1}\}} \|A\mathrm{x} - d\|^2$.

Orthogonal Matching Pursuit (OMP) Algorithm Input: A (with unit

norm column), d, K.

Initialize: $r^0 = d, \sigma^0 = \varnothing$

Repeat, while $\#\sigma^k \leq K$:

$$
\begin{aligned}
i^k &= \arg\max_{j \notin \sigma^k} |\langle r^k, a_j \rangle| \\
\sigma^{k+1} &= \sigma^k \cup \{i^k\} \\
\mathrm{x}^{k+1} &= \arg\min_{\{\mathrm{x}/\sigma_{\mathrm{x}} \subset \sigma^{k+1}\}} \|A\mathrm{x} - d\|^2 \\
r^{k+1} &= d - A\mathrm{x}^{k+1}
\end{aligned}
$$

- Convergence in N iterations at most (at each iteration a **new** component is selected),
- Exact sparse recovery results (under conditions on A) [Tropp 04].

Further algorithms:

At each iteration, several strategies for one component to be

- added,
- removed,
- replaced.

Orthogonal Least Squares (OLS) [Chen et al 89], *Bayesian OMP* [Herzet et al 10], *Single Best Replacement* [Soussen et al 11] and further variants [Jain & al 11, Soussen et al 15]...

The more complex is the strategy, the best is the solution and the longest is the computing time.

Exact reformulation

- *Class of continuous nonconvex penalties* $\rightarrow$ asymptotic connections with the ℓ_2-ℓ_0 criteria [Chouzenoux et al 13]
- *Reformulation using Difference of Convex functions* $\rightarrow$ asymptotic or local minimizer results [Le Thi et al 14, Le Thi et al 15]
- *Equivalence of ℓ_0- and ℓ_p-norm $(0 < p \leq 1)$ minimization under linear equalities or inequalities (e.g. exact reconstruction problem)* [Fung and Mangasarian 11]
- *Reformulation and optimization through Mixed-Integer Programs (MIPs)* $\rightarrow$ global optimum for problems of reasonable size (a few hundred variables) [Bourguignon et al 15]
- **Exact reformulation** ([Bi et al 14, Yuan & Ghanem 16, Liu et al 18], ,...)

Lemma 1 [Liu et al 18, Yuan & Ghanem 16]

$$\|\mathrm{x}\|_0 = \min_{-1 \leq \mathrm{u} \leq 1} \|\mathrm{u}\|_1 \text{ s.t } \|\mathrm{x}\|_1 = < \mathrm{u}, \mathrm{x} >$$

Exact reformulation for the $\ell_2 - \ell_0$ penalized problem

Initial problem:

$$\min_{\mathrm{x}} \frac{1}{2}\|A\mathrm{x} - d\|_2^2 + \lambda \|\mathrm{x}\|_0$$

Penalized reformulation:

$$\min_{\mathrm{x},\mathrm{u}} G_\rho(\mathrm{x},\mathrm{u}) := \frac{1}{2}\|A\mathrm{x} - d\|^2 + \iota_{\{-1 \leq \cdot \leq 1\}}(\mathrm{u}) + \lambda \|u\|_1 + \rho(\|\mathrm{x}\|_1 - < \mathrm{x}, \mathrm{u} >)$$

with $\iota_{\{\mathrm{x} \in D\}}(\mathrm{x}) = 0$ if $\mathrm{x} \in D$, $+\infty$ otherwise.

Theorem [Bechensteen,et al.]

If $\rho > \sigma_{max}(A)\|d\|_2$, and A is of full rank. Then:

1. If (x_ρ, u_ρ) is a local (respectively global) minimizer of G_ρ, then x_ρ is a local (respectively global) minimizer of the initial problem.
2. If $\hat{x}$ is a global minimizer of the initial problem, then $(\hat{x}, \hat{u})$ is a global minimizer of G_ρ with $\hat{u}$ associated with Lemma 1.

Lemma 1 [Liu et al 18, Yuan & Ghanem 16]

$$\|\mathrm{x}\|_0 = \min_{-1\leq \mathrm{u} \leq 1} \|\mathrm{u}\|_1 \text{ s.t } \|\mathrm{x}\|_1 =< \mathrm{u}, \mathrm{x} >$$

Exact reformulation for the $\ell_2 - \ell_0$ constrained problem

Initial problem:

$$\min_{\mathrm{x}} \frac{1}{2}\|A\mathrm{x} - d\|_2^2 + \iota_{\{\|\cdot\|_0 \leq K\}}(\mathrm{x})$$

Constrained reformulation:

$$\min_{\mathrm{x},\mathrm{u}} G_\rho(\mathrm{x}, \mathrm{u}) := \frac{1}{2}\|A\mathrm{x} - d\|^2 + \iota_{\{\cdot \geq 0\}}(\mathrm{x}) + \iota_{\{-1 \leq \cdot \leq 1\}}(\mathrm{u}) + \iota_{\{\|\cdot\|_1 \leq K\}}(u) + \rho(\|\mathrm{x}\|_1 - < \mathrm{x}, \mathrm{u} >)$$

Theorem [Bechensteen,et al.]

If $\rho > \sigma_{max}(A)\|d\|_2$, and A is of full rank. Then:

1. If (x_ρ, u_ρ) is a local (respectively global) minimizer of G_ρ, then x_ρ is a local (respectively global) minimizer of the initial problem.
2. If $\hat{x}$ is a global minimizer of the initial problem, then $(\hat{x}, \hat{u})$ is a global minimizer of G_ρ with $\hat{u}$ associated with Lemma 1.

Why minimize the constrained or penalized reformulation instead of their initial formulation?

Constrained reformulation:

$$\min_{x,u} \frac{1}{2}\|Ax - d\|^2 + \iota_{\{\cdot \geq 0\}}(x) + \iota_{\{-1 \leq \cdot \leq 1\}}(u) + \iota_{\{\|\cdot\|_1 \leq K\}}(u) + \rho(\|x\|_1 - < x, u >)$$

Penalized reformulation:

$$\min_{x,u} \frac{1}{2}\|Ax - d\|^2 + \iota_{\{\cdot \geq 0\}}(x) + \iota_{\{-1 \leq \cdot \leq 1\}}(u) + \lambda\|u\|_1 + \rho(\|x\|_1 - < x, u >)$$

- Biconvex
- Non-convexity linked to the coupling term $< x, u >$
- Minimizing the reformulation is equivalent to minimize the initial problem regarding local and global minimizers

We add a positivity constraint on x and we finally define

$$G_\rho(\mathrm{x},\mathrm{u}) = \frac{1}{2}\|A\mathrm{x}-d\|^2 + \iota_{\{\cdot\geq 0\}}(\mathrm{x}) + \rho\|\mathrm{x}\|_1 + \iota_{\{\|\cdot\|_1\leq K\}}(\mathrm{u}) + \iota_{\{-1\leq\cdot\leq 1\}}(\mathrm{u}) - \rho < \mathrm{x},\mathrm{u} >$$

The global optimization scheme is (continuation method)

Initialize: $\rho^0 > 0, n = 0$

Repeat: Solve the problem G_{ρ^n}:

$$\{\mathrm{x}^{n+1},\mathrm{u}^{n+1}\} = \arg\min_{\mathrm{x},\mathrm{u}} G_{\rho^n}(\mathrm{x},\mathrm{u})$$

Update: $\rho^{n+1} = \alpha\rho^n$, $\alpha > 1$

Until: $\rho^{n+1} > \sigma_{max}(A)\|d\|_2$

$$G_{\rho^n}(\mathrm{x},\mathrm{u}) = \frac{1}{2}\|A\mathrm{x}-d\|^2 + \iota_{\{\cdot\geq 0\}}(\mathrm{x}) + \rho^n\|\mathrm{x}\|_1 + \iota_{\{\|\cdot\|_1\leq K\}}(\mathrm{u}) + \iota_{\{-1\leq\cdot\leq 1\}}(\mathrm{u}) - \rho^n < \mathrm{x},\mathrm{u} >$$

At fixed ρ^n we apply the Proximal Alternate Minimization (PAM) algorithm [Attouch & al 10]

Initialize: $\mathrm{u}^0 = 0 \in \mathbb{R}^M$

Repeat: $\arg\min G_{\rho^n}$ using alternate minimizations

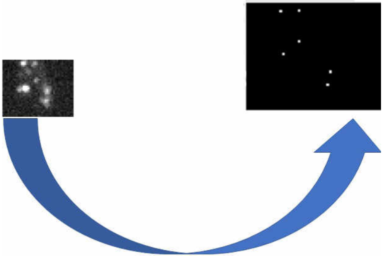
- $\{\mathrm{x}^{n+1}\} = \arg\min_{\mathrm{x}} G_{\rho^n}(\mathrm{x},\mathrm{u}^n) + \frac{1}{2c^n}\|\mathrm{x}-\mathrm{x}^n\|^2$
  $\rightarrow$ FISTA Algorithm [Beck et al 09]

- $\{\mathrm{u}^{n+1}\} = \arg\min_{\mathrm{u}} G_{\rho^n}(\mathrm{x}^{n+1},\mathrm{u}) + \frac{1}{2d^n}\|\mathrm{u}-\mathrm{u}^n\|^2$
  $\rightarrow$ Algorithm [Stefanov, 2004]

Until: convergence

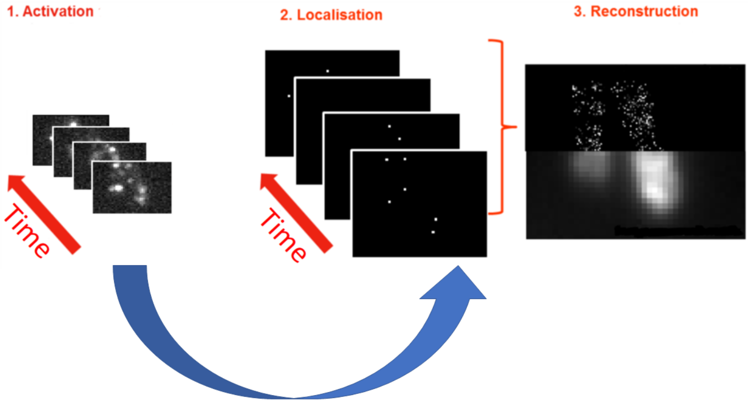
Convergence of the algorithm towards a critical point of G_{ρ^n} for c^n and d^n such that $0 < r_- < c^n, d^n < r_+$ and under KL condition on G_{ρ^n} and assuming that x_n and u_n are bounded [Attouch & al 10].

5. Results: Single-Molecule Localization Microscopy



$$\hat{x} \in \arg\min_{x} \frac{1}{2}\|Ax - d\|_2^2 + \iota_{\{\cdot \geq 0\}}(x) + R(x)$$

5. Results: Single-Molecule Localization Microscopy



$$\hat{x} \in \arg\min_{x} \frac{1}{2}\|Ax - d\|_2^2 + \iota_{\{\cdot \geq 0\}}(x) + R(x)$$

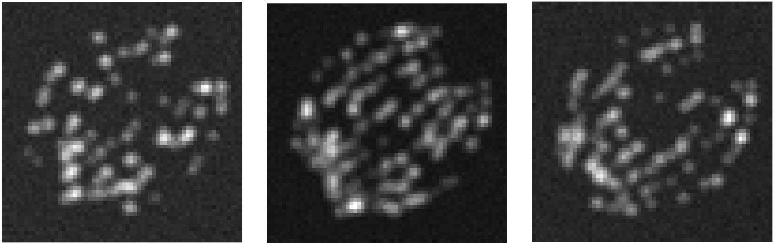


Figure: Simulated images (among the 361 simulated high density images for this sample). Data from IEEE ISBI Challenge 2013.
http://bigwww.epfl.ch/smlm/datasets/index.html

8 simulated tubes of 30nm diameter
Camera of 64×64 pixels of size 100nm.
Gaussian PSF, FWHM = 258.21 nm (full width at half maximum)
80932 molecules activated on 361 frames.

5.1 Results, ISBI challenge 2013, simulated dataset

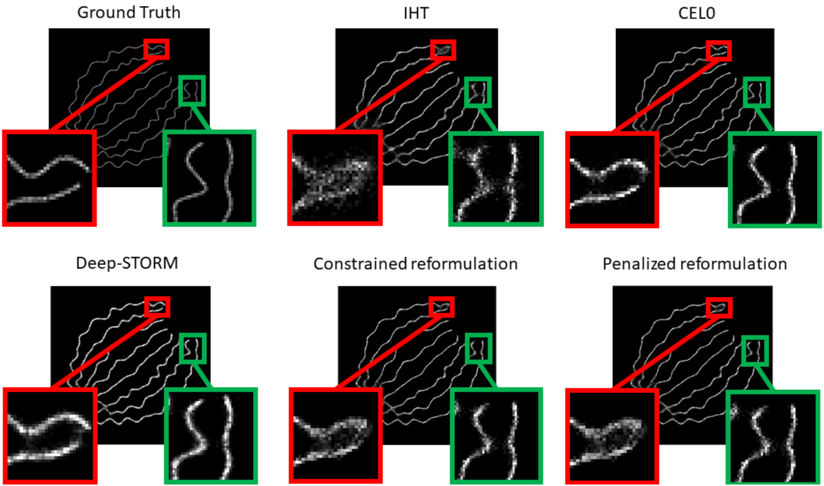
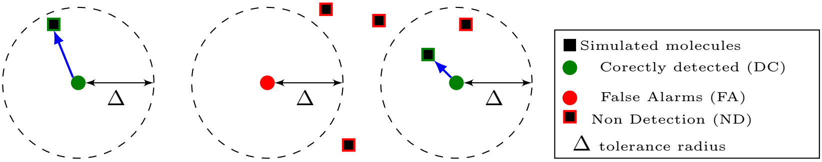


Figure: Reconstruction from simulated data set, reduction ratio $L = 4$.

5.1 Results, ISBI challenge 2013, simulated dataset

Jaccard index calculus



$$\text{Jaccard index} = \frac{\text{DC}}{\text{DC} + \text{FA} + \text{ND}}$$

Jaccard index results

	Jaccard index (%)			
Method - Tolerance (nm)	50	100	150	200
IHT	20.1	35.9	40.4	41.3
CEL0	**29.3**	**41.3**	42.4	42.6
Constrained reformulation	25.2	40.0	**43.2**	**43.9**
Penalized reformulation	25.0	39.3	42.2	42.8
Deep-STORM	×	×	×	×

Table: The jaccard index obtained and the tolerance

5.2 Results, ISBI challenge 2013, Real dataset

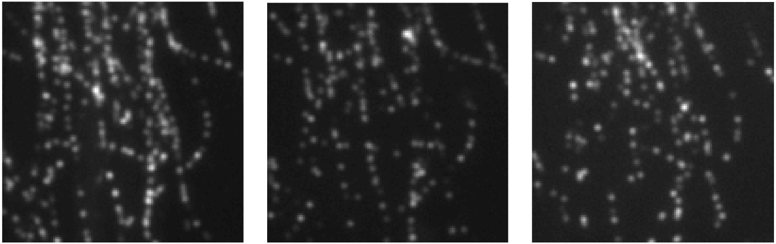


Figure: Real images (among the 500 real high density images for this sample). Data from IEEE ISBI Challenge 2013.
http://bigwww.epfl.ch/smlm/datasets/index.html

Camera of 128×128 pixels of size 100nm.
Gaussian PSF, FWHM = 358.1 nm (full width at half maximum)

5.2 Results, ISBI challenge 2013, Real dataset

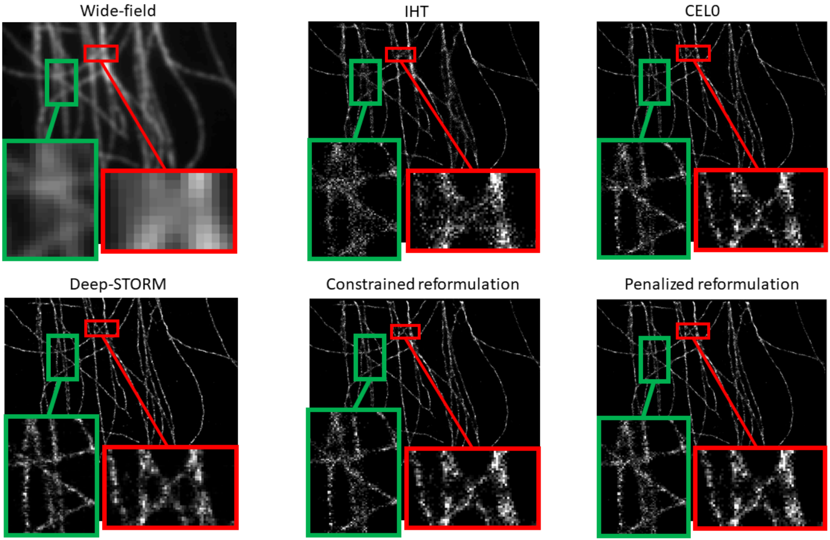


Figure: Reconstruction from the real data set, reduction ratio $L = 4$.

References I

Hedy Attouch, Jérôme Bolte, Patrick Redont and Antoine Soubeyran, *Proximal alternating minimization and projection methods for nonconvex problems. An approach based on the Kurdyka-Lojasiewicz inequality*, Mathematics of operation research, 35(2), (2010), pp. 438–457.

Hedy Attouch, Jérôme Bolte, and Benar Fux Svaiter, *Convergence of descent methods for semi-algebraic and tame problems: proximal algorithms, forward–backward splitting, and regularized gauss–seidel methods*, Mathematical Programming, 137 (2013), pp. 91–129.

Arne Bechensteen, Laure Blanc-Féraud and Gilles Aubert, *Reformulation of l_2-l_0 constrained criterion for SMLM*, preprint, 2018.

Amir Beck and Marc Teboulle, *A fast iterative shrinkage-thresholding algorithm for linear inverse problems*, SIAM Journal on Imaging Sciences, 2 (2009), pp. 183–202.

S. Bi, X. Liu, and S. Pan, *Exact penalty decomposition method for zero-norm minimization based on MPEC formulation*, SIAM Journal on Scientific Computing, 36(4) (2014).

Thomas Blumensath and Mike E Davies, *Iterative thresholding for sparse approximations*, Journal of Fourier Analysis and Applications, 14 (2008), pp. 629–654.

J. Bioucas-Dias and M. Figueiredo, *A new TwIST: Two-step iterative shrinkage/thresholding algorithms for image restoration* , IEEE Transactions on Image processing 16 (12), (2007), pp.2992-3004

Léon Bottou, Frank E. Curtis and Jorge Nocedal, *Optimization Methods for Large-Scale Machine Learning*, SIAM Review, 60(2), (2018), pp.223-311.

S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, *Distributed optimization and statistical learning via the alternating direction method of multipliers. Foundations andTrends in Machine Learning*, 3(1), (2011), pp.1-122.

References II

Sébastien Bourguignon, Jordan Ninin, Hervé Carfantan and Marcel Monceau, *Optimisation exacte de critères parcimonieux en norme ℓ_0 par programmation mixte en nombres entiers*, Colloque GRETSI 2015.

Leo Breiman, *Better subset regression using the nonnegative garrote*, Technometrics, 37 (1995), pp. 373–384.

Betzig, Eric and Patterson, George H and Sougrat, Rachid and Lindwasser, O Wolf and Olenych, Scott and Bonifacino, Juan S and Davidson, Michael W and Lippincott-Schwartz, Jennifer and Hess, Harald F, *Imaging intracellular fluorescent proteins at nanometer resolution*, Science, 5793 (2006), pp. 1642-1645.

L. Bregman, *A Relaxation Method of Finding a Common Point of Convex Sets and its Application to Problems of Optimization*, Dokl. Akad. Nauk SSSR, 171(5), (1966) pp. 1019-1022. (English translation: Soviet Math. Dokl., v. 7, 1966, p.p. 1578-1581).

Emmanuel J Candès, Justin Romberg, and Terence Tao, *Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information*, Information Theory, IEEE Transactions on, 52 (2006), pp. 489–509.

Emmanuel J Candès, Michael B Wakin, and Stephen P Boyd, *Enhancing sparsity by reweighted ℓ_1 minimization*, Journal of Fourier analysis and applications, 14 (2008), pp. 877–905.

Emmanuel J Candès, and Michael B Wakin, *An introduction to compressive sampling*, IEEE Signal Processing Magazine, 25(2), (2008), pp. 21–30.

P. Catala, V. Duval and G. Peyre, *A low - rank approach to off - the - grid sparse deconvolution*, NCMIP conf 2017, arXiv 2019.

Rick Chartrand, *Exact reconstruction of sparse signals via nonconvex minimization*, Signal Processing Letters, IEEE, 14 (2007), pp. 707–710.

References III

S. Chen, S. Billings and W. Luo, *Orhogonal least squares methods and their application to non-linear system identification*, International journal of Control, 50(5) (1989), pp. 1873–1896.

Scott Shaobing Chen, David L Donoho, and Michael A Saunders, *Atomic decomposition by Basis Pursuit*, SIAM journal on scientific computing, 20 (1998), pp. 33–61.

Emilie Chouzenoux, Anna Jezierska, Jean-Christophe Pesquet and Hugues Talbot, *A majorize-minimize subspace approach for \ell_2-\ell_0 image regularization*, SIAM Journal on Imaging Sciences, 6 (2013), pp.563–591.

Frank H Clarke, *Optimization and nonsmooth analysis*, vol. 5, Siam, 1990.

Patrick Combettes and V. Wajs, *Signal recovery by proximal forward-backward splitting*, Multiscale Modeling and Simulation, 4(4), (2005), pp. 1168-1200.

Ingrid Daubechies, Michel Defrise, and Christine De Mol, *An iterative thresholding algorithm for linear inverse problems with a sparsity constraint*, Communications on Pure and Applied Mathematics, 57 (2004), pp. 1413–1457.

Geoff Davis, Stéphane Mallat and Marco Avellaneda *Adaptive greedy approximations*, Constructive approximation, 13 (1997), pp. 57–98.

David L Donoho and Micael Elad, *Optimally sparse representation in general (nonorthogonal) dictionaries via ℓ_1 minimization*, in Proceedings of the National Academy of Sciences 100(5) (2003), pp. 72–76.

David L Donoho, *For most large underdetermined systems of linear equations the minimal ℓ_1-norm solution is also the sparsest solution*, Communications on Pure and Applied Mathematics, 59 (2006), pp. 797–829.

References IV

David L Donoho, *Compressed sensing*, IEEE Transactions on Information Theory. 52 (4), (2006), pp. 1289?1306.

M. Elad, B. Matalon, and M. Zibulevsky, *Subspace optimization methods for linear least squares with non-quadratic regularization*, Appl. Comput. Harmon. Anal., 23 (2007), pp. 346-367.

Jon than Eckstein, *Augmented Lagrangian and Alternating Direction Methods for Convex Optimization: A tutorial and some illustrative computational results*, Rutcor Resaerch Report, RRR 32-2012 (2012).

Jianqing Fan and Runze Li, *Variable selection via nonconcave penalized likelihood and its oracle properties*, Journal of the American Statistical Association, 96 (2001), pp. 1348–1360.

Simon Foucart and Ming-Jun Lai, *Sparsest solutions of underdetermined linear systems via ℓ_q-minimization for $0 < q \leq 1$*, Applied and Computational Harmonic Analysis, 26 (2009), pp. 395–407.

G.M. Fung and O.L. Mangasarian, *Equivalence of minimal ℓ_0- and ℓ_p-norm solutions of linear equalities, inequalities and linear programs for sufficiently small p*, Journal of optimization theory and applications, 151 (2011), pp. 1–10.

Gilles Gasso, Alain Rakotomamonjy, and Stéphane Canu, *Recovering sparse signals with a certain family of nonconvex penalties and DC programming*, Signal Processing, IEEE Transactions on, 57 (2009), pp. 4686–4698.

R. Glowinski and A. Marroco, *Sur l'approximation par éléments finis d'ordre un et la résolution, par pénalisation-dualité, d'une classe de problèmes de Dirichlet non linéaires*, Revue Française d'Automatique, Informatique, et Recherche Opérationelle, 9, (1975) pp.41-76.

References V

T. Goldstein and S. Osher *The split Bregman method for ℓ_1-regularization problems* , SIAM Journal of Imaging Sciences, 2(2) (2009), pp. 323-343.

Pinghua Gong, Changshui Zhang, Zhaosong Lu, Jianhua Huang, and Jieping Ye, *A General Iterative Shrinkage and Thresholding Algorithm for Non-convex Regularized Optimization Problems*, in Proceedings of The 30th International Conference on Machine Learning (2013), pp. 37–45.

Remi Gribonval and Morten Nielsen, *Sparse representation in unions of bases*, IEEE Transactions on Information Theory, 49(12), (2003), pp. 73–76.

Hess, Samuel T and Girirajan, Thanu PK and Mason, Michael D *Ultra-high resolution imaging by fluorescence photoactivation localization microscopy* Biophysical journal, 11 (2006), Elsevier, pp. 4258-4272.

Cédric Herzet and Angélique Drémeau, *Bayesian Pursuit Algorithms* , in Proceedings of European Signal Processing conference (EUSIPCO), Aalborg, Danemark, August 2010.

Seamus J Holden, Stephan Uphoff and Achillefs N Kapanidis *DAOSTORM: an algorithm for high-density super-resolution microscopy* Nature Methods, 8 (2011), pp. 279–280.

P. Jain, A. Tewari and I.S.Dhillon *Orthogonal Matching Pursuit with Replacement* Advanced in Neural Information Processing Systems, 24 (2011), pp. 1215–1223.

Hoai An Le Thi, Hoai Minh Le and Tao Pham Dinh, *Feature selection in machine learning: an exact penalty approach using a Difference of Convex function Algorithm*, Machine Learning, 2014, pp. 1–24.

Hoai An Le Thi, Tao Pham Dinh, Hoai Minh Le and Xuan Thanh Vo, *DC approximation approaches for sparse optimization*, European Journal of Operational Research, 244 (2015), pp. 26–46.

References VI

Huan Li and Zhouchen Lin, *Accelerated Proximal Gradient Methods for Nonconvex Programming*, Part of: Advances in Neural Information Processing Systems 28 (NIPS 2015), pp. 679?704.

Yulan Liu, Shujun. Bi, and Shoahua Pan, *Equivalent Lipschitz surrogates for zero-norm and rank optimization problems.*, Journal of Global Optimization, 72 (4), (2018), pp. 679?704.

Stéphane G Mallat and Zhifeng Zhang, *Matching pursuits with time-frequency dictionaries*, Signal Processing, IEEE Transactions on, 41 (1993), pp. 3397–3415.

OL Mangasarian *Machine learning via polyhedral concave minimization* Applied Mathematics and Parallel Computing (1996), pp. 175–188.

Min, J. and Vonesch, C. and Kirshner, H. and Carlini, L. and Olivier, N. and Holden, S. and Unser, M. *FALCON: fast and unbiased reconstruction of high-density super-resolution microscopy data.* Scientific Reports, 4 (2014), pp. 4577.

Hosein Mohimani, Massoud Babaie-Zadeh and Christian Jutten, *A fast approach for overcomplete sparse decomposition based on smoothed ℓ_0 norm*, Signal Processing, IEEE Transactions on, 57 (2008), pp. 289–301.

Balas Kausik Natarajan *Sparse approximate solutions to linear systems*, SIAM journal on computing, 24 (1995), pp. 227–234.

Yuri Nesterov *Gradient methods for minimizing composite functions*, Mathematical Programming, 140(1), (2013), pp 125?161

Mila Nikolova, *Description of the minimizers of least squares regularized with ℓ_0-norm. Uniqueness of the global minimizer*, SIAM Journal on Imaging Sciences, 6 (2013), pp. 904–937.

References VII

Mila Nikolova, *Relationship between the optimal solutions of least squares regularized with L0-norm and constrained by k-sparsity*, Appl. Comput. Harmon. Anal., 41(1), (2016), pp. 237–265.

Jorge Nocedal and Stephen J. Wright, *Numerical Optimization* , Springer, 2 edition (2006).

P. Ochs, A. Dosovitskiy, T. Brox, and T. Pock, *An iteratively reweighted Algorithm for Non-smooth Non-convex Optimization in Computer Vision*, SIAM Journal on Imaging Sciences, 8(1), 2015.

Yagyensh Chandra Pati, Ramin Rezaiifar, and PS Krishnaprasad, *Orthogonal matching pursuit: Recursive function approximation with applications to wavelet decomposition*, in Signals, Systems and Computers, 1993. 1993 Conference Record of The Twenty-Seventh Asilomar Conference on, IEEE, 1993, pp. 40–44.

R. T. Rockafellar, *Convex analysis* Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, (1997). Reprint of the 1970 original, Princeton Paperbacks.

Rust, Michael J and Bates, Mark and Zhuang, Xiaowei *Sub-diffraction-limit imaging by stochastic optical reconstruction microscopy (STORM)*, Nature methods, 10 (2006), pp. 793-796.

Daniel Sage, Hagai Kirshner, Thomas Pengo, Nico Stuurman, Junhong Min, Suliana Manley and Michael Unser, *Quantitative evaluation of software packages for single-molecule localization microscopy*, in Nature methods, 2015, pp. 717–724.

Stefan M. Stefanov *Convex quadratic minimization subject to a linear constraint and box constraints* Applied Mathematics Research Express, 1 (2004), pp. 17-42.

Emmanuel Soubies, Laure Blanc-Féraud and Gilles Aubert *A Continuous Exact l0 Penalty (CEL0) for Least Squares Regularized Problem*, SIAM Journal on Imaging Sciences, 8(3), 2015.

References VIII

Emmanuel Soubies, Laure Blanc-Féraud and Gilles Aubert *A Unified View of Exact Continuous Penalties for l2-l0 Minimization*, SIAM Journal of Optimization, 27(3), 2017.

Charles Soussen, Jérôme Idier, David Brie, and Junbo Duan, *From Bernoulli–Gaussian deconvolution to sparse signal restoration*, Signal Processing, IEEE Transactions on, 59 (2011), pp. 4572–4584.

Charles Soussen, Jérôme Idier, Junbo Duan and David Brie, *Homotopy Based Algorithms for ℓ_0-Regularized Least-Squares*, Signal Processing, IEEE Transactions on, 63(13) (2015), pp. 3301–3316.

R. Tibshirani, *Regression shrinkage and selection via the lasso*, Journal of the Royal Statistical Society, 46 (1996), pp. 431–439.

Joel A Tropp, *Greed is good: Algorithmic results for sparse approximation*, Information Theory, IEEE Transactions on, 50 (2004), pp. 2231–2242.

Ganzhao Yuan and Bernard Ghanem, *Sparsity Constrained Minimization via Mathematical Programming with Equilibrium Constraints*, arXiv:1608.04430 (2016).

Cun-Hui Zhang, *Multi-stage convex relaxation for learning with sparse regularization*, Advances in Neural Information Processing Systems, (2009), pp. 1929–1936.

Cun-Hui Zhang, *Nearly unbiased variable selection under minimax concave penalty*, The Annals of Statistics, (2010), pp. 894–942.

Hui Zou, *The adaptive lasso and its oracle properties*, Journal of the American statistical association, 101 (2006), pp. 1418–1429.