Ex. 2, Chapter 7

$X_1, \dots, X_n \overset{i.i.d.}{\sim} Ber(p)$

$Y_1, \dots, Y_m \overset{i.i.d.}{\sim} Ber(q)$

i) Provide a plug-in estimator of p and an approximate 90% CI for p.

$p = E(X_i) \quad \forall i \in \{1, \dots, n\}$

$\hat{p} = E(Z)$ with $\underbrace{\mathbb{P}\{Z = x_i \mid x_1, \dots, x_n\}}_{= \frac{1}{n}}$

$= \sum_{i=1}^{n} x_i \mathbb{P}\{Z = x_i \mid x_{1:n}\}$

$= \frac{1}{n} \sum_{i=1}^{n} x_i = \overline{X}$ plug-in estimator

$CI = [\overline{X} - 1.64\,\hat{se}(\overline{X}),\ \overline{X} + 1.64\,\hat{se}(\overline{X})]$

with $Var(\overline{X}) = \frac{Var(X_i)}{n} = \frac{p(1-p)}{n}$

$\Rightarrow se(\overline{X}) = \left(\frac{p(1-p)}{n}\right)^{1/2}$

$\hat{se}(\overline{X}) = \left(\frac{\hat{p}(1-\hat{p})}{n}\right)^{1/2}$

$$\boxed{\mathbb{P}\{p \in CI\} \xrightarrow[n \to +\infty]{} 90\%}$$

ii) $p - q = \mathbb{E}(x_1) - \mathbb{E}(y_1)$

$$\hat{p} - \hat{q} = \overline{x} - \overline{y}$$

$$CI = \left[(\overline{x} - \overline{y}) \pm 1.64\, \hat{se}(\overline{x} - \overline{y}) \right]$$

Since $\overline{X} \perp \overline{Y} \Rightarrow Var(\overline{x} - \overline{y}) =$

$$= Var(\overline{x}) + Var(\overline{y}) - 2 \underbrace{Cov(\overline{x}, \overline{y})}_{=0}$$ because of indep.

$$= \frac{p(1-p)}{n} + \frac{q(1-q)}{m}$$

$$\hat{Var}(\overline{x} - \overline{y}) = \frac{\hat{p}(1-\hat{p})}{n} + \frac{\hat{q}(1-\hat{q})}{m}$$

$$\hat{se}(\overline{x} - \overline{y}) = \sqrt{\hat{Var}(\overline{x} - \overline{y})}$$

$$\mathbb{P}\{p - q \in CI\} \rightarrow 90\%$$

$\min\{n, m\} \rightarrow +\infty$

Skewness of $Y_i = e^{X_i}$ with $X_i \sim \mathcal{N}(0,\sigma^2)$

$$Sk(Y_i) = \left(e^{\sigma^2}+2\right)\left[\left(e^{\sigma^2}-1\right)\right]^{1/2}$$

* Invariance of ML estimator

$X_1, \dots, X_n \sim F(\theta)$, $\hat{\theta}_{ML}$ is the ML of θ

Now if $\tau = g(\theta) \Rightarrow \hat{\tau}_{ML} = g(\hat{\theta}_{ML})$

$$\Rightarrow * \ \widehat{Sk}_{ML}(Y_i) = \left(e^{\hat{\sigma}^2_{ML}}+2\right)\left[e^{\hat{\sigma}^2_{ML}}-1\right]^{1/2}$$

$$* \ \widehat{Sk}_{Pin}(Y) = \text{empirical skewness}$$

with $\hat{\sigma}^2_{ML} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$

Method of Moments

$X_1, \dots, X_N \overset{i.i.d}{\sim} F(\Theta) \qquad \theta \in \Theta \subseteq \mathbb{R}^{k}$

ex: i) $X_1, \dots, X_N \sim Ber(p) \Rightarrow \theta = p, \quad k = 1$

ii) $X_1, \dots, X_N \sim \mathcal{N}(\mu, \sigma^2) \qquad \Theta = [0, 1]$

$\theta = (\mu, \sigma), \quad k = 2$

$\Theta =]-\infty, +\infty[\times]0, +\infty[$

Theoretical moment on order r for X_i

$$\alpha_r = \mathbb{E}(X_i^r) = \int x^r f(x)\,dx$$

$$\left(\sum_{j=1}^{r} x_j \, \mathbb{P}\{X_i = x_j\} \right)$$

Empirical moment of order r

$$\hat{\alpha}_r = \frac{1}{N} \sum_{i=1}^{N} X_i^r$$

you set $\hat{\theta}_{MM}$ such that

$$\alpha_1(\hat{\theta}_{MM}) = \hat{\alpha}_1$$

$$\alpha_2(\hat{\theta}_{MM}) = \hat{\alpha}_2$$

$$\vdots$$

$$\alpha_k(\hat{\theta}_{MM}) = \hat{\alpha}_k$$

Ex: $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$

$$\alpha_1 = \mathbb{E}(X_i) = \underline{\mu} \quad ; \quad \alpha_2 = \mathbb{E}(X_i^2) = \underline{\mu^2 + \sigma^2}$$

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i = \overline{X}$$

$$\hat{\mu}^2 + \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} X_i^2$$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n} \sum_{i=1}^{n} X_i \right)^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2$$

Maximum likelihood estimator

$$\mathcal{L}(\theta) = f(X_1, \dots, X_n | \theta) \stackrel{!}{=} \prod_{i=1}^{n} f(X_i | \theta)$$

$$\hat{\theta}_{ML} = \underset{\Theta}{\arg\max}\, \mathcal{L}(\theta) = \underset{\Theta}{\arg\max} \prod_i f(X_i | \theta)$$

which is the same as

$$\hat{\theta}_{ML} = \underset{\Theta}{\arg\max}\, \underbrace{\log \mathcal{L}(\theta)}_{\sum_{i=1}^{n} \log f(X_i | \theta)}$$

i) Bernoulli distributed data:

$X_1, \dots, X_N \sim Ber(p)$, find $\hat{p}_{ML}$

first we need $p(x_i) = p^{x_i}(1-p)^{1-x_i}$

(equivalent to say $x_i = 1$ with p and $x_i = 0$ with $1-p$)

$$\log p(x_i) = x_i \log p + (1-x_i)\log(1-p)$$

$$\log \mathcal{L}(p) = \sum_{i=1}^{n} (x_i \log p) + (1-x_i) \log(1-p)$$

$$= n\bar{X} \log p + (n - n\bar{X}) \log(1-p)$$

this is the function to maximise w.r. to p.

$$\frac{d}{dp} \log \mathcal{L}(p) = \frac{n\bar{X}}{p} - \frac{(n - \bar{X}n)}{(1-p)} = 0$$

$$(1-p)n\bar{X} - p(n - n\bar{X}) = 0$$

$$\cancel{n\bar{X}} - \cancel{pn\bar{X}} - pn + \cancel{pn\bar{X}} = 0$$

$$\hat{p}_{ML} = \bar{X}$$

ii) $X_1, \dots, X_N \overset{i.i.d}{\sim} \mathcal{U}([0,\theta])$ $\qquad$ θ unknown

Compute $\hat{\theta}_{ML}$

$$f(x_i|\theta) = \frac{1}{\theta} \mathbb{1}_{[0,\theta]}(x_i),$$

The joint density of $X_1, \dots, X_N$ (i.e. the likelihood) is

$$\mathcal{L}(\theta) = \prod_{i=1}^{N} f(x_i|\theta) = \frac{1}{\theta^N} \prod_{i=1}^{N} \mathbb{1}_{[0,\theta]}(x_i)$$

if $\theta < \max\{x_1, \dots, x_N\} \Rightarrow \mathcal{L}(\theta) = 0$

$\Rightarrow$ we set $\theta \geq \max\{x_1, \dots, x_N\}$

Since we want to maximize $\mathcal{L}(\theta)$

if $\theta \geq \max\{x_1, \dots, x_N\} \Rightarrow \mathcal{L}(\theta) = \frac{1}{\theta^N}$

$\Rightarrow \log \mathcal{L}(\theta) = -N \log \theta$

$\frac{d}{d\theta} \log \mathcal{L}(\theta) = -\frac{N}{\theta} < 0$ on $\Theta = [\max\{x_i\}, +\infty[$

$\Rightarrow \hat{\theta}_{ML} = \max\{x_1, \dots, x_N\}$