Bachelor Degree in Informatics Engineering

Barcelona School of Informatics

Mathematics 1

Part II: Linear Algebra

Exercises and problems

Academic Year 2019-2020(1)

Departament de Matemàtiques

Universitat Politècnica de Catalunya

The problems of this collection were initially gathered by Anna de Mier and Montserrat Maureso. Many of them were taken from the problem sets of several courses taught over the years by the members of the Departament de Matemàtica Aplicada 2. Other exercises came from the bibliography of the course or from other texts, and some of them were new. Since Mathematics 1 was first taught in 2010 several problems have been modified or rewritten by the professors involved in the teaching of the course.

We would like to acknowledge the assistance of the scholar Gabriel Bernardino in the writing of the solutions.

Translation by Josep Elgueta and the scholar Bernat Coma.

This problem list has been revised during the academic year 2018/19.

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Matrices, systems of linear equations and determinants

Unless otherwise indicated, we always work in the field $\mathbb{R}$ of real numbers.

5.1 Matrix algebra

5.1 Given the matrices

$$A = \begin{pmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 3 \\ -1 & 0 & 2 \\ 4 & 5 & -1 \end{pmatrix}$$

compute: 1) $3A$; 2) $3A - B$; 3) AB; 4) BA; 5) $C(3A - 2B)$.

5.2 Compute the products $(1\ 2\ -3)\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}(1\ 2\ -3)$.

5.3 Let A and B be matrices such that AB is a square matrix. Show that the product BA is well defined.

5.4 For the following matrices A and B, give the elements c_{13} and c_{22} of the matrix $C = AB$ without computing all the elements of C.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -3 & 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{pmatrix}.$$

5.5 A company makes bags and suitcases in two different factories. The table below gives the cost, in thousands of euros, of manufacturing each product in each factory:

	Factory 1	Factory 2
Bags	135	150
Suitcases	627	681

Answer the following questions using matrix operations.

1) Knowing that the personnel costs represent 2/3 of the total cost, find the matrix that gives the personnel cost of each product in each factory.
2) Find the matrix that gives the cost of the material of each product in each factory, assuming that, in addition to the personnel and material costs, there is a cost of 20.000 euros for each product in each factory.

5.6 In this exercise we want to find a formula giving the successive powers of a diagonal matrix.

a) Compute A^2, A^3 and A^5 if
$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$
b) Compute A^{32}.
c) Let D be a diagonal $n \times n$ matrix whose diagonal entries are $\lambda_1, \lambda_2, \dots, \lambda_n$. Make a guess about D^r, for $r \in \mathbb{Z}$, $r \geq 1$, and prove it by induction.

5.7 Let $A = \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$. Compute $(AB)^t$ and $B^t A^t$. Observe that AB can be a non-symmetric matrix even when both A and B are symmetric.

5.8 Give an example of two 2×2 matrices A and B such that $(AB)^t \neq A^t B^t$.

5.9 Let I be the identity matrix and $\boldsymbol{O}$ the null matrix of $\mathcal{M}_{2\times 2}(\mathbb{R})$. Find matrices $A, B, C, D, E \in \mathcal{M}_{2\times 2}(\mathbb{R})$ such that

1) $A^2 = I$ and $A \neq I$;
2) $B^2 = \boldsymbol{O}$ and $B \neq \boldsymbol{O}$;
3) $C^2 = C$ and $C \neq I, \boldsymbol{O}$;
4) $DE = \boldsymbol{O}$ but $E \neq D$ and $ED \neq \boldsymbol{O}$.

5.10 Let A and B be two symmetric matrices of the same type. Prove that AB is a symmetric matrix if and only if A and B commute.

5.11 Do the following equalities hold for all matrices $A, B \in \mathcal{M}_n(\mathbb{R})$? If not, give a condition on A and B ensuring that the equations hold.

1) $(A + B)^2 = A^2 + B^2 + 2AB$;
2) $(A - B)(A + B) = A^2 - B^2$.

5.12 Let A and B be square matrices of the same type. A is said to be *similar* to B if there exists an invertible matrix P such that $B = P^{-1}AP$. If A is similar to B, prove:

1) B is similar to A. In general we say that A and B are similar.
2) The relation *"being similar"* is an equivalence relation.
3) A is invertible if and only if B is invertible.
4) A^t is similar to B^t.
5) If $A^n = \boldsymbol{O}$ and C is an invertible matrix of the same type as A, then $(C^{-1}AC)^n = \boldsymbol{O}$.

5.13 Find a matrix in row echelon form equivalent to each of the following matrices. Give the rank of the matrix in each case.

1) $\begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 1 & 3 \\ -1 & 2 & 0 & 0 \end{pmatrix}$ 2) $\begin{pmatrix} -3 & 1 \\ 2 & 0 \\ 6 & 4 \end{pmatrix}$ 3) $\begin{pmatrix} 5 & 11 & 6 \\ 2 & 1 & 4 \\ 3 & -2 & 8 \\ 0 & 0 & 4 \end{pmatrix}$ 4) $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 & 3 \\ -2 & -1 & 0 & 1 & 2 \\ -3 & -2 & -1 & 0 & 1 \end{pmatrix}$

5.14 Find the inverse of the following elementary matrices.

1) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ 3) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ 5) $\begin{pmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $k \neq 0$

2) $\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$ 4) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$

5.15 Using the Gauss-Jordan method, find, if it exists, the inverse of each of the following matrices.

1) $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ 2) $\begin{pmatrix} 3 & 1 & 5 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$ 3) $\begin{pmatrix} -2 & 3 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 2 \\ 3 & 1 & -2 & -4 \end{pmatrix}$

4) $A = (a_{i,j})_{4\times 4}$, such that $a_{i,j} = 1$ if $|i - j| \leq 1$, and $a_{i,j} = 0$ otherwise.
5) $A = (a_{i,j})_{4\times 4}$, such that $a_{i,j} = 2^{j-1}$ if $i \geq j$, and $a_{i,j} = 0$ otherwise.
6) $A = (a_{i,j})_{4\times 4}$, such that $a_{i,i} = k$, $a_{i,j} = 1$ if $i - j = 1$, and $a_{i,j} = 0$ otherwise.

5.2 Systems of linear equations

5.16 Which of the following equations are linear in x, y and z?

1) $x + 3xy + 2z = 2$;
2) $y + x + \sqrt{2}z = e^2$;
3) $x - 4y + 3z^{1/2} = 0$;
4) $y = z \sin \frac{\pi}{4} - 2y + 3$;
5) $z + x - y^{-1} + 4 = 0$;
6) $x = z$.

5.17 Find a system of linear equations for each of the following augmented matrices.

1) $\begin{pmatrix} 1 & 0 & 3 & 2 \\ 2 & 1 & 1 & 3 \\ 0 & -1 & 2 & 4 \end{pmatrix}$

2) $\begin{pmatrix} -1 & 5 & -2 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

3) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1/3 & 1/4 & 1/5 & 1/2 & 1 \end{pmatrix}$

4) $\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 1 & 4 \end{pmatrix}$

5.18 Answer the following questions. Justify your answers.

1) What is the rank of the system matrix of a determined consistent system with 5 equations and 4 unknowns? What about if the system is underdetermined?
2) How many equations are needed at least to have an underdetermined consistent system with 2 degrees of freedom and rank 3? How many unknowns would this system have?
3) Is it possible to have a determined consistent system with 7 equations and 10 unknowns?
4) Can a system with fewer equations than unknowns be inconsistent?
5) Give a determined consistent system, an underdetermined consistent system and an inconsistent system, each with 3 unknowns and 4 equations.

5.19 Solve the following systems of linear equations with coefficients in $\mathbb{Z}_2$. Use Gaussian elimination and give the solution in parametric form.

1) $\left\{ \begin{array}{rcl} x+y & = & 1 \\ x+z & = & 0 \\ x+y+z & = & 1 \end{array} \right.$
2) $\left\{ \begin{array}{rcl} x+y & = & 1 \\ y+z & = & 1 \\ x+z & = & 1 \end{array} \right.$
3) $\left\{ \begin{array}{rcl} x+y & = & 0 \\ y+z & = & 0 \\ x+z & = & 0 \end{array} \right.$

5.20 Solve the following systems of linear equations. Use Gaussian elimination and give the solution in parametric form.

1) $\begin{cases} x+y+2z &= 8 \\ -x-2y+3z &= 1 \\ 3x-7y+4z &= 10 \\ 3y-2z &= -1 \end{cases}$

2) $\begin{cases} x-y+2z &= 3 \\ 2x-2y+5z &= 4 \\ x+2y-z &= -3 \\ 2y+2z &= 1 \end{cases}$

3) $\begin{cases} x-y+2z-w &= -1 \\ 2x+y-2z-2w &= -2 \\ -x+2y-4z+w &= 1 \\ 3x-3w &= -3 \end{cases}$

4) $\begin{cases} x_1+3x_2-2x_3+2x_5 &= 0 \\ 2x_1+6x_2-5x_3-2x_4+4x_5-3x_6 &= -1 \\ 5x_3+10x_4+15x_6 &= 5 \\ 2x_1+6x_2+8x_4+4x_5+18x_6 &= 6 \end{cases}$

5.21 Solve the following homogeneous systems of linear equations. Use Gaussian elimination and give the solution in parametric form.

1) $\begin{cases} 2x+2y+2z &= 0 \\ -2x+5y+2z &= 0 \\ -7x+7y+z &= 0 \end{cases}$

2) $\begin{cases} 2x-4y+z+w &= 0 \\ x-5y+2z &= 0 \\ -2y-2z-w &= 0 \\ x+3y+w &= 0 \\ x-2y-z+w &= 0 \end{cases}$

3) $\begin{cases} x_2-3x_3+x_4 &= 0 \\ x_1+x_2-x_3+4x_4 &= 0 \\ -2x_1-2x_2+2x_3-8x_4 &= 0 \end{cases}$

4) $\begin{cases} 2x_1+2x_2-x_3+x_5 &= 0 \\ -x_1-x_2+2x_3-3x_4+x_5 &= 0 \\ x_1+x_2+x_3+2x_5 &= 0 \\ 2x_3+2x_4+2x_5 &= 0 \end{cases}$

5.22 Discus the following systems of linear equations according to the values of the parameters (assumed to be real).

1) $\begin{cases} x+y+2z &= a \\ x+z &= b \\ 2x+y+3z &= c \end{cases}$

2) $\begin{cases} bx+y+z &= b^2 \\ x-y+z &= 1 \\ 3x-y-z &= 1 \\ 6x-y+z &= 3b \end{cases}$

3) $\begin{cases} ax+y-z+t-u &= 0 \\ x+ay+z-t+u &= 0 \\ -x+y+az+t-u &= 0 \\ x-y+z+at+u &= 0 \\ -x+y-z+t+au &= 0 \end{cases}$

4) $\begin{cases} x+2y-3z &= 4 \\ 3x-y+5z &= 2 \\ 4x+y+(a^2-14)z &= a+2 \end{cases}$

5) $\begin{cases} x+y &= k \\ ax+by &= k^2 \\ a^2x+b^2y &= k^3 \end{cases}$

5.3 Determinants

5.23 Assuming that $\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = 5$, compute the following determinants..

1) $\begin{vmatrix} e & f & g & h \\ i & j & k & l \\ a & b & c & d \\ m & n & o & p \end{vmatrix}$

2) $\begin{vmatrix} -a & -b & -c & -d \\ 2e & 2f & 2g & 2h \\ i & j & k & l \\ m & n & o & p \end{vmatrix}$

3) $\begin{vmatrix} a+e & b+f & c+g & d+h \\ e & f & g & h \\ m & n & o & p \\ i & j & k & l \end{vmatrix}$

4) $\begin{vmatrix} a & b & c & d \\ e-3a & f-3b & g-3c & h-3d \\ i & j & k & l \\ 4m & 4n & 4o & 4p \end{vmatrix}$

5.24 For which values of λ is the determinant of the following matrices equal to 0?

1) $\begin{pmatrix} \lambda-1 & -2 \\ 1 & \lambda-4 \end{pmatrix}$

2) $\begin{pmatrix} \lambda-6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda-4 \end{pmatrix}$

5.25 Compute the following determinants.

1) $\begin{vmatrix} 5 & 15 \\ 10 & -20 \end{vmatrix}$

2) $\begin{vmatrix} 2 & 1 & 2 \\ 0 & 3 & -1 \\ 4 & 1 & 1 \end{vmatrix}$

3) $\begin{vmatrix} 3 & -1 & 5 \\ -1 & 2 & 1 \\ -2 & 4 & 3 \end{vmatrix}$

4) $\begin{vmatrix} 4 & 16 & 0 \\ 12 & -8 & 8 \\ 16 & 20 & -4 \end{vmatrix}$

5) $\begin{vmatrix} 0 & 2 & 1 \\ -1 & 3 & 0 \\ 2 & 4 & 3 \end{vmatrix}$

6) $\begin{vmatrix} -1 & 2 & 1 & 2 \\ 1 & 2 & 4 & 1 \\ 2 & 0 & -1 & 3 \\ 3 & 2 & -1 & 0 \end{vmatrix}$

7) $\begin{vmatrix} 0 & -3 & 8 & 2 \\ 8 & 1 & -1 & 6 \\ -4 & 6 & 0 & 9 \\ -7 & 0 & 0 & 14 \end{vmatrix}$

8) $\begin{vmatrix} 1 & -1 & 8 & 4 & 2 \\ 2 & 6 & 0 & -4 & 3 \\ 2 & 0 & 2 & 6 & 2 \\ 0 & 2 & 8 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 \end{vmatrix}$

5.26 Let A and B be 3×3 matrices such that $\det(A) = 10$ and $\det(B) = 12$. Compute

1) $\det(AB)$, 2) $\det(A^4)$, 3) $\det(2B)$, 4) $\det(A^t)$, 5) $\det(A^{-1})$.

5.27 Prove that

$$\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} = (a+3)(a-1)^3.$$

6
Vector spaces

A *vector space over a field* $\mathbb{K}$ consists of

1) a non-empty set E,
2) a binary operation $E \times E \to E$ called *addition* and denoted $+$, and
3) a map $\mathbb{K} \times E \to E$ called *scalar multiplication* and denoted $\cdot$,

satisfying the following eight properties for each $u, v, w \in E$ and each $\lambda, \mu \in \mathbb{K}$:

e1) $u + (v + w) = (u + v) + w$ *(associative law)*;

e2) $u + v = v + u$ *(commutative law)*;

e3) there exists a unique element $0_E \in E$ such that $u + 0_E = u$ *(zero vector)*;

e4) for each $u \in E$ there exists a unique $u' \in E$ such that $u + u' = 0_E$ *(additive inverse)*;

e5) $\lambda(\mu u) = (\lambda\mu)u$;

e6) $\lambda(u + v) = \lambda u + \lambda v$;

e7) $(\lambda + \mu)u = \lambda u + \mu u$;

e8) $1u = u$, where 1 denotes the multiplicative identity of $\mathbb{K}$.

(Note: in most cases, the field $\mathbb{K}$ will be $\mathbb{R}$; other fields that may appear are $\mathbb{Q}$ and $\mathbb{Z}_p$.)

Exercises

6.1 Let $u = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, $v = \begin{pmatrix} 4 \\ 0 \\ -8 \end{pmatrix}$ and $w = \begin{pmatrix} 6 \\ -1 \\ -4 \end{pmatrix}$ be vectors of $\mathbb{R}^3$. Compute

1) $u - v$; 2) $5v + 3w$; 3) $5(v + 3w)$; 4) $(2w - u) - 3(2v + u)$.

6.2 Draw the following vectors of $\mathbb{R}^2$.

$$1)\ v_1 = \begin{pmatrix} 2 \\ 6 \end{pmatrix}; \quad 2)\ v_2 = \begin{pmatrix} -4 \\ -8 \end{pmatrix}; \quad 3)\ v_3 = \begin{pmatrix} -1 \\ 5 \end{pmatrix}; \quad 4)\ v_4 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

6.3 For the vectors in the previous exercise, compute graphically the vectors $v_1 + v_2$, $v_1 - v_3$ and $v_2 - v_4$ and check your answers algebraically.

6.4 Let u, v, w be elements of a vector space and let α, β, γ be elements of the corresponding field of scalars, with $\alpha \neq 0$. Assuming that the relation $\alpha u + \beta v + \gamma w = 0$ holds, write the vectors u, $u - v$ and $u + \alpha^{-1}\beta v$ in terms of v and w.

6.5 Let $P(\mathbb{R})_e$ be the set of all polynomials with coefficients in $\mathbb{R}$ and with only even powers of x. Determine if $P(\mathbb{R})_e$ is a vector space with the usual addition and scalar multiplication of polynomials. (Assume that the polynomial 0 has degree 0.)

6.6 Let $\mathcal{F}(\mathbb{R})$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Given two functions $f, g \in \mathcal{F}(\mathbb{R})$ and $\lambda \in \mathbb{R}$, let us define the functions $f + g$ and λf by

$$\begin{aligned} (f+g)(x) &= f(x) + g(x), \\ (\lambda f)(x) &= \lambda f(x). \end{aligned}$$

Prove that $\mathcal{F}(\mathbb{R})$ is a $\mathbb{R}$-vector space with these operations.

6.7 Which of the following sets are vector subspaces over $\mathbb{R}$? (Justify your answers.)

$$E_1 = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x + \pi y = 0\}, \quad E_5 = \{\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 : x + y + z + t = 0, x - t = 0\},$$

$$E_2 = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + z = \pi\}, \quad E_6 = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x^2 + 2xy + y^2 = 0\},$$

$$E_3 = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : xy = 0\}, \quad E_7 = \{\begin{pmatrix} a+b \\ a-2b \\ c \\ 2a+c \end{pmatrix} \in \mathbb{R}^4 : a, b, c \in \mathbb{R}\},$$

$$E_4 = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x \in \mathbb{Q}\}, \quad E_8 = \{\begin{pmatrix} a^2 \\ a \\ b+a \\ 2+a \end{pmatrix} \in \mathbb{R}^4 : a, b \in \mathbb{R}\}.$$

6.8 Let $P(\mathbb{R})$ be the vector space of all polynomials in x with coefficients in $\mathbb{R}$. Which of the following subsets are vector subspaces of $P(\mathbb{R})$? (Justify your answers.)

$$\begin{aligned}
F_1 &= \{a_3x^3 + a_2x^2 + a_1x + a_0 \in P(\mathbb{R}) : a_2 = a_0\} \\
F_2 &= \{p(x) \in P(\mathbb{R}) : p(x) \text{ has degree } 3\} \\
F_3 &= \{p(x) \in P(\mathbb{R}) : p(x) \text{ has even degree}\} \\
F_4 &= \{p(x) \in P(\mathbb{R}) : p(1) = 0\} \\
F_5 &= \{p(x) \in P(\mathbb{R}) : p(0) = 1\} \\
F_6 &= \{p(x) \in P(\mathbb{R}) : p'(5) = 0\}
\end{aligned}$$

6.9 Let $\mathcal{M}_{n\times m}(\mathbb{R})$ be the vector space of all $n \times m$ matrices with coefficients in $\mathbb{R}$. Which of the following subsets are vector subspaces of $\mathcal{M}_{n\times m}(\mathbb{R})$? (Justify your answers.)

$$\begin{aligned}
M_1 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\} \\
M_2 &= \{A \in \mathcal{M}_{n\times m}(\mathbb{R}) : A = A^t\} \\
M_3 &= \{A \in \mathcal{M}_{n\times m}(\mathbb{R}) : a_{1\,i} = 0 \ \forall i \in [m]\} \\
M_4 &= \{A \in \mathcal{M}_{n\times m}(\mathbb{R}) : a_{1\,i} = 1 \ \forall i \in [m]\} \\
M_5 &= \{A \in \mathcal{M}_{n\times m}(\mathbb{R}) : AB = 0\} \text{ (where } B \text{ is a fixed matrix)}
\end{aligned}$$

6.10 Let $T \subset \mathbb{R}^4$. Prove that the vector u below can be written as a linear combination of the vectors in T in at least two different ways.

$$T = \{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \}, \quad u = \begin{pmatrix} 0 \\ 3 \\ 5 \\ 1 \end{pmatrix}$$

6.11 For which values of the parameter a can the vector $u \in \mathbb{R}^3$ be written as a linear combination of the vectors in T?

$$u = \begin{pmatrix} 1 \\ 5 \\ a \end{pmatrix}, \quad T = \{ \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 7 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \}$$

6.12 Give the values of the parameters a and b for which the matrix $\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 1 & 4 \\ -5 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$.

6.13 Given the vectors $u = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ of $\mathbb{R}^3$, find a condition on the components of a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ which ensures that this vector is in the subspace spanned by $\{u, v\}$.

6.14 Give the form of a generic polynomial in $P(\mathbb{R})$ that belongs to the vector subspace spanned by the set $\{1+x, x^2\}$.

6.15 Let $F = \langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \rangle$ and $G = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \rangle$ be subspaces of $\mathbb{R}^3$.

1) Prove that $F = G$.

2) Let $e = \begin{pmatrix} 9 \\ \sqrt{2}-1 \\ 1-\sqrt{2} \end{pmatrix}$. Prove that $e \in F$ and express it as a linear combination of the two families of vectors that span F.

6.16 Determine whether the following sets of vectors are linearly independent in the respective vector spaces.

1) $\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 8 \end{pmatrix} \}$ in $\mathbb{R}^3$;

2) $\{ \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} \}$ in $\mathbb{R}^3$;

3) $\{ \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 8 \\ 1 \\ 3 \end{pmatrix} \}$ in $\mathbb{R}^3$;

4) $\{ \begin{pmatrix} 4 \\ -5 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 6 \\ -3 \\ 3 \\ 9 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 5 \\ 6 \end{pmatrix} \}$ in $\mathbb{R}^4$;

5) $\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \\ 11 \\ 12 \end{pmatrix} \}$ in $\mathbb{R}^5$.

6.17 Let us consider the vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ b \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 5 \\ a \\ -4 \end{pmatrix}$ in the vector space $\mathbb{R}^4$. Determine a and b so that they are linearly dependent vectors, and in that case express the vector $0_{\mathbb{R}^4}$ as a non-trivial linear combination of them.

6.18 Let E be a $\mathbb{R}$-vector space and let u, v, w be any three vectors in E. Prove that the set $\{u-v, v-w, w-u\}$ is linearly dependent.

6.19 Prove that the following matrices A, B and C are linearly independent as vectors in $\mathcal{M}_{2\times 3}(\mathbb{R})$.

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} -1 & 1 & -2 \\ 3 & -2 & 0 \end{pmatrix}.$$

Prove that for any λ the matrix

$$\begin{pmatrix} \lambda & 2 & -4 \\ 2-\lambda & 2 & 2 \end{pmatrix}$$

is a linear combination of A and B.

6.20 Prove that the polynomials $x^2 + 2x - 1$, $x^2 + 1$ and $x^2 + x$ are linearly dependent in $P(\mathbb{R})$.

6.21 If $\{e_1, e_2, \ldots, e_r\}$ is a family of linearly dependent vectors in some vector space, is it true that any vector e_i can be written as a linear combination of the other vectors? Prove it or give a counterexample.

6.22 Determine whether the following statements about a family of vectors in a vector space E are true, proving them when they are true and giving a counterexample otherwise.

1) If $\{e_1, \ldots, e_r\}$ are linearly independent and $v \neq e_i$ for each i, then the set $\{e_1, \ldots, e_r, v\}$ is linearly independent.
2) If $\{e_1, \ldots, e_r\}$ are linearly independent and $v \notin \langle e_1, \ldots, e_r \rangle$, then $\{e_1, \ldots, e_r, v\}$ is linearly independent.
3) If $\{e_1, \ldots, e_r\}$ spans E and $v \neq e_i$ for each i, then $\{e_1, \ldots, e_r, v\}$ also spans E.
4) If $\{e_1, \ldots, e_r\}$ spans E and $e_r \in \langle e_1, \ldots, e_{r-1} \rangle$, then $\{e_1, \ldots, e_{r-1}\}$ also spans E.
5) Every set with only one vector is linearly independent.

6.23 Let us consider the set of vectors $\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\4\end{pmatrix}, \begin{pmatrix}0\\0\\0\\2\end{pmatrix}\}$.

1) Prove that they are a basis of $\mathbb{R}^4$.
2) Give the coordinates of the vector $\begin{pmatrix}1\\0\\2\\-3\end{pmatrix}$ in this basis.
3) Give the coordinates of an arbitrary vector $\begin{pmatrix}x\\y\\z\\t\end{pmatrix}$ in this basis.

6.24 Let $B = \left\{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}1 & 2\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right\}$. Prove that B is a basis of $\mathcal{M}_2(\mathbb{R})$.

Give the coordinates of $A = \begin{pmatrix}1 & 3\\3 & 1\end{pmatrix}$ in this basis.

6.25 Let $P_3(\mathbb{R})$ be the vector space of all polynomials of degree at most 3. Prove that the polynomials $1 + x, -1 + x, 1 + x^2$ and $1 - x + x^3$ form a basis of $P_3(\mathbb{R})$ and give the coordinates of the polynomial $x^3 + 3x^2 + 6x - 5$ in this basis.

6.26 Let $F = \langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \rangle$ in $\mathbb{R}^3$. Give a basis of F and find the condition (in terms of an homogeneous system of linear equations) that the components of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ must satisfy for this vector to belong to F.

6.27 Consider the following subspaces of $\mathbb{R}^4$.

$$F = \langle \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \rangle, \qquad G = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \rangle.$$

Prove that $F = G$ and that the respective spanning sets are bases. Do the vectors $\begin{pmatrix} \sqrt{3} \\ \sqrt{2}-1 \\ 1-\sqrt{2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ belong to F? If so, give their coordinates in the two bases.

6.28 Let $\{v_1, v_2, v_3\}$ be a basis of a vector space E. Prove that the set $\{v_1 + 2v_2, 2v_2 + 3v_3, 3v_3 + v_1\}$ is also a basis of E.

6.29 Give a basis of the subspace E of $\mathbb{R}^5$ and complete it to a basis of $\mathbb{R}^5$ for

$$E = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \in \mathbb{R}^5 : x_3 = x_1 + x_2 - x_4,\ x_5 = x_2 - x_1 \}.$$

6.30 Let us consider the vectors in $\mathcal{M}_2(\mathbb{R})$

$$\begin{pmatrix} 1 & 4 \\ -1 & 10 \end{pmatrix}, \begin{pmatrix} 6 & 10 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}.$$

Prove that they are linearly independent and give a vector that together with them forms a basis of $\mathcal{M}_2(\mathbb{R})$.

6.31 For which values of λ the vectors $\begin{pmatrix} \lambda \\ 0 \\ 1 \\ \lambda \end{pmatrix}, \begin{pmatrix} \lambda \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \lambda \\ \lambda \end{pmatrix}$ span a vector subspace of $\mathbb{R}^4$ of dimension 2?

6.32 Consider the subspace $F_a = \langle \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & a \\ 0 & -1 \end{pmatrix} \rangle$ of $\mathcal{M}_2(\mathbb{R})$, with $a \in \mathbb{R}$.

1) Find the value of a for which F_a is of dimension 2.
2) Let $a = a_0$ be the value of a from a). Find the conditions, in terms of an homogeneous system of equations in x, y, z, t, so that the matrix $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ belongs to F_{a_0}.
3) Argue that $B = \{ \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \}$ and $B' = \{ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \}$ are bases of F_{a_0}.

6.33 In each case, give a basis and the dimension of the spaces E, F and $E \cap F$:

1) $E = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : 2x = 2y = z \}$ and $F = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + y = z, 3x + y + z = 0 \}$ as subspaces of $\mathbb{R}^3$.
2) $E = \langle \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \rangle$ and $F = \langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} \rangle$ as subspaces of $\mathbb{R}^3$.
3) $E = \{ \begin{pmatrix} a \\ a+3b \\ 2a-b \\ c \end{pmatrix} : a, b, c \in \mathbb{R} \}$ and $F = \{ \begin{pmatrix} -2a \\ b \\ 0 \\ 3b \end{pmatrix} : a, b \in \mathbb{R} \}$ as subspaces of $\mathbb{R}^4$.
4) $E = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}) : a = b = c \right\}$ and $F = \langle \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \rangle$ as subspaces of $\mathcal{M}_2(\mathbb{R})$.

6.34 For each of the subspaces E in the previous exercise (exercise 6.33), complete the respective bases to a basis of the whole vector space.

6.35 Consider the basis $B = \{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \}$ of $\mathbb{R}^3$.

1) Give the change-of-basis matrix P_B^C from the canonical basis of $\mathbb{R}^3$ to B.
2) Let now $B' = \{ \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \}$ be another basis of $\mathbb{R}^3$. Give the change-of-basis matrix $P_B^{B'}$ from basis B' to B.

6.36 Consider the vector space $P_2(\mathbb{R})$ of polynomials of degree at most 2.

1) Prove that $B = \{-1 + 2x + 3x^2, x - x^2, x - 2x^2\}$ is a basis of $P_2(\mathbb{R})$ and compute the change-of-basis matrix from canonical basis to B.
2) Find the coordinates of $p(x) = 3 - x + 2x^2$ in basis B.

6.37 Let $B = \{\begin{pmatrix}1\\5\\6\end{pmatrix}, \begin{pmatrix}-2\\-5\\3\end{pmatrix}, \begin{pmatrix}1\\4\\-1\end{pmatrix}\}$ and $B' = \{\begin{pmatrix}1\\3\\2\end{pmatrix}, \begin{pmatrix}-1\\-2\\5\end{pmatrix}, \begin{pmatrix}0\\2\\4\end{pmatrix}\}$ be bases of $\mathbb{R}^3$.

1) Prove that they are indeed bases.
2) Give the change-of-basis matrix from B to B' ($P_{B'}^{B}$) and the change-of-basis matrix from B' to B ($P_{B}^{B'}$).
3) Compute the coordinates in both bases B and B' of the vector that in the canonical basis has coordinates $\begin{pmatrix}2\\5\\2\end{pmatrix}$.

6.38 Let

$$\begin{aligned} B &= \left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}, \\ B' &= \left\{\begin{pmatrix}1&1\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\1&1\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\} \end{aligned}$$

be two bases of $\mathcal{M}_2(\mathbb{R})$. Give the change-of-basis matrices $P_{B'}^{B}$ and $P_{B}^{B'}$.

6.39 Given B and B' below, prove that they are basis of $\mathbb{R}^3$.

$$B = \{\begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}\}, \qquad B' = \{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\}.$$

Let u be a vector of $\mathbb{R}^3$ whose coordinates in the basis B and B' are $u_B = \begin{pmatrix}x\\y\\z\end{pmatrix}$ and $u_{B'} = \begin{pmatrix}x'\\y'\\z'\end{pmatrix}$, respectively. Express x, y and z in terms of x', y' and z', and conversely.

6.40 Let $B = \{p_1(x), p_2(x), p_3(x)\}$ be a basis of $P_2(\mathbb{R})$, the space of all polynomials of degree ≤ 2. Let $u(x) = x^2 + x + 2$, $v(x) = 2x^2 + 3$ and $w(x) = x^2 + x$. If the coordinates of $u(x)$, $v(x)$ and $w(x)$ in the basis B are

$$u(x)_B = \begin{pmatrix}2\\1\\0\end{pmatrix}, v(x)_B = \begin{pmatrix}2\\0\\2\end{pmatrix}, w(x)_B = \begin{pmatrix}1\\1\\-2\end{pmatrix},$$

respectively, give the coordinates of the vectors in B in the canonical basis $\{x^2, x, 1\}$.

7
Linear maps

Exercises

7.1 Determine which of the following maps are linear:

1) $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f\begin{pmatrix} x \\ y \end{pmatrix} = x + y$;

2) $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f\begin{pmatrix} x \\ y \end{pmatrix} = x^2y^2$;

3) $f\colon \mathbb{R}^3 \to \mathbb{R}^3$, $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+7 \\ 2y \\ x+y+z \end{pmatrix}$;

4) $f\colon \mathbb{R}^2 \to \mathbb{R}^3$, $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y-x \\ x+y \end{pmatrix}$;

5) $f\colon \mathbb{R}^3 \to \mathbb{R}^3$, $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} xy \\ z \\ x \end{pmatrix}$.

7.2 Determine which of the following maps $f\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$ are linear:

1) $f(a_0 + a_1x + a_2x^2) = 0$;

2) $f(a_0 + a_1x + a_2x^2) = a_0 + (a_1 + a_2)x + (2a_0 - 3a_1)x^2$;

3) $f(a_0 + a_1x + a_2x^2) = a_0 + a_1(1 + x) + a_2(1 + x)^2$.

7.3 Determine which of the following maps are linear:

1) $f\colon \mathcal{M}_2(\mathbb{R}) \to \mathbb{R}$ defined by $f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = a + d$;

2) $f\colon \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_{2\times 3}(\mathbb{R})$ defined by $f(A) = AB$, with $B \in \mathcal{M}_{2\times 3}(\mathbb{R})$ a fixed matrix;

3) $f\colon \mathcal{M}_n(\mathbb{Z}_2) \to \mathbb{Z}_2$, defined by $f(A) = \det(A)$.

7.4 Let $f\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear map defined by: $f(1) = 1 + x$, $f(x) = 3 - x^2$ and $f(x^2) = 4 + 2x - 3x^2$. What is the image of the polynomial $a_0 + a_1x + a_2x^2$? Compute $f(2 - 2x + 3x^2)$.

7.5 In each case, determine if there exists an endomorphism $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(u_i) = v_i,\ i = 1,2,3,$:

1) $u_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $u_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}$, $u_3 = \begin{pmatrix}-1\\1\\0\end{pmatrix}$ and $v_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $v_2 = \begin{pmatrix}3\\2\\1\end{pmatrix}$, $v_3 = \begin{pmatrix}8\\4\\0\end{pmatrix}$;

2) $u_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $u_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}$, $u_3 = \begin{pmatrix}-1\\2\\3\end{pmatrix}$ and $v_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $v_2 = \begin{pmatrix}3\\2\\1\end{pmatrix}$, $v_3 = \begin{pmatrix}8\\4\\0\end{pmatrix}$;

3) $u_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $u_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}$, $u_3 = \begin{pmatrix}-1\\2\\3\end{pmatrix}$ and $v_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $v_2 = \begin{pmatrix}3\\2\\1\end{pmatrix}$, $v_3 = \begin{pmatrix}0\\1\\2\end{pmatrix}$.

7.6 Let E and F be two vector spaces, $f\colon E \to F$ a linear map, and $v_1, v_2, \ldots, v_n$ vectors of E. For each of the following statements, prove it if it is true and give a counterexample if it is false.

1) If $v_1, v_2, \ldots, v_n$ are linearly independent, then $f(v_1), f(v_2), \ldots, f(v_n)$ are linearly independent.
2) If $f(v_1), f(v_2), \ldots, f(v_n)$ are linearly independent, then $v_1, v_2, \ldots, v_n$ are linearly independent.
3) If $v_1, v_2, \ldots, v_n$ is a spanning set of E, then $f(v_1), f(v_2), \ldots, f(v_n)$ is a spanning set of F.
4) If $f(v_1), f(v_2), \ldots, f(v_n)$ a spanning set of F, then $v_1, v_2, \ldots, v_n$ is a spanning set of E.
5) If $v_1, v_2, \ldots, v_n$ is a spanning set of E, then $f(v_1), f(v_2), \ldots, f(v_n)$ is a spanning set of $\operatorname{Im} f$.

7.7 For the following subspaces E and F of $\mathbb{R}^4$, determine if there exists a linear map $f\colon \mathbb{R}^4 \to \mathbb{R}^4$ such that $f(u) = 0$ for each $u \in E$ and $f(v) = v$ for each $v \in F$.

1) $E = \langle \begin{pmatrix}1\\2\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\2\end{pmatrix} \rangle$ and $F = \langle \begin{pmatrix}2\\2\\2\\3\end{pmatrix}, \begin{pmatrix}1\\1\\0\\1\end{pmatrix} \rangle$.

2) $E = \langle \begin{pmatrix}1\\0\\-1\\3\end{pmatrix}, \begin{pmatrix}4\\1\\0\\2\end{pmatrix} \rangle$ and $F = \langle \begin{pmatrix}1\\0\\0\\2\end{pmatrix}, \begin{pmatrix}1\\1\\1\\-5\end{pmatrix} \rangle$.

7.8 For each of the following linear maps, give the associated matrix in canonical basis and compute the dimension of the kernel and of the image:

1) $f\colon \mathbb{R} \to \mathbb{R}$, on $f(x) = 3x$;
2) $f\colon \mathbb{R}^2 \to \mathbb{R}^3$, on $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ x-y \end{pmatrix}$;
3) $f\colon \mathbb{R}^3 \to \mathbb{R}^3$, on $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ y+z \\ z \end{pmatrix}$;
4) $f\colon \mathcal{M}_2(\mathbb{R}) \to \mathbb{R}^3$, on $f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} b+c \\ c+d \\ 2a-b+c-d \end{pmatrix}$;
5) $f\colon P_2(\mathbb{R}) \to P_3(\mathbb{R})$, on $f(a_0 + a_1x + a_2x^2) = (2a_1 - a_0) + (2a_1 - a_2)x + (3a_2 - 2a_1 + a_0)x^2 + (a_0 + a_1 + a_2)x^3$.

7.9 Let f be an endomorphism of $\mathbb{R}^3$ with associated matrix

$$\begin{pmatrix} (m-2) & 2 & -1 \\ 2 & m & 2 \\ 2m & 2(m+1) & m+1 \end{pmatrix}.$$

Find the dimension of the image according to the values of m.

7.10 Let E be an $\mathbb{R}$-vector space and $B = \{u, v, w, t\}$ a basis of E. Let f be an endomorphism of E such that

$$f(u) = u + 2w,\ f(v) = v + w,\ f(w) = 2u + v + w,\ f(t) = 2u + 2v + 4w.$$

Find the matrix of f in the basis B, and find a basis and the dimension of the image of f.

7.11 Find the kernel of the linear map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ y-z \\ z-x \end{pmatrix}$, compute $f\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and the preimages of the vectors $\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$.

7.12 Determine whether the following linear maps are bijective or not using the information given:

1) $f\colon \mathbb{R}^n \to \mathbb{R}^n$, with $\mathrm{Ker} f = \{0_{\mathbb{R}^n}\}$;
2) $f\colon \mathbb{R}^n \to \mathbb{R}^n$, with $\dim(\mathrm{Im}\, f) = n-1$;
3) $f\colon \mathbb{R}^m \to \mathbb{R}^n$, with $n < m$;
4) $f\colon \mathbb{R}^n \to \mathbb{R}^n$, with $\mathrm{Im}\, f = \mathbb{R}^n$.

7.13 For each of the following linear maps, give the associated matrix in the canonical bases; give the dimension and a basis of the kernel and of the image of the map; determine if the map is injective, surjective, bijective or none of them; and find the inverse map, if it exists:

1) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = ax$, for given $a \in \mathbb{R}$;s

2) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+3y \\ 2x+7y \end{pmatrix}$;

3) $f\colon \mathbb{R}^4 \to \mathbb{R}^3$, $f\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x-y+z+2t \\ y-z+t \\ x-2y+2z \end{pmatrix}$;

4) $f\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $f(a_0 + a_1x + a_2x^2) = (a_0 - a_1) + (a_1 - a_2)x + (a_2 - a_0)x^2$;

5) $f\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $f(a_0 + a_1x + a_2x^2) = 3a_0 + (a_0 - a_1)x + (2a_0 + a_1 + a_2)x^2$;

6) $f\colon \mathcal{M}_2(\mathbb{R}) \to \mathbb{R}^2$, $f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a+d \\ b+c \end{pmatrix}$;

7) $f\colon \mathbb{R}^3 \to \mathcal{M}_2(\mathbb{R})$, $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y & y-z \\ z-y & x-z \end{pmatrix}$.

7.14 Let B be an invertible matrix $n \times n$. Prove that the map $f : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ defined by $f(A) = AB$ is a bijective endomorphism.

7.15 For the following linear maps f_1 and f_2, determine whether the composition function $f = f_2 \circ f_1$ is injective, surjective or bijective.

1) $f_1\colon \mathbb{R}^3 \to \mathbb{R}^3$ and $f_2\colon \mathbb{R}^3 \to \mathbb{R}^2$, where $f_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ x+y \\ x+2z-y \end{pmatrix}$ and $f_2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x-3z \\ y+4z \end{pmatrix}$;

2) $f_1\colon P_3(\mathbb{R}) \to P_2(\mathbb{R})$ and $f_2\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$, where $f_1(a_0 + a_1x + a_2x^2 + a_3x^3) = a_2 + a_3x + a_0x^2$ and $f_2(a_0 + a_1x + a_2x^2) = (a_1 + a_2) + (a_0 + a_2)x + (a_0 + a_1)x^2$;

3) $f_1\colon \mathbb{R}^2 \to \mathbb{R}^3$ and $f_2\colon \mathbb{R}^3 \to \mathbb{R}^3$, where $f_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x+y \\ y \end{pmatrix}$ and $f_2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+z \\ x+y+z \\ y+x \end{pmatrix}$.

7.16 Give the matrices of the following linear maps in the canonical basis:

1) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $f\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$;

2) $f\colon \mathbb{R}^2 \to \mathbb{R}^4$ such that $f\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 3 \end{pmatrix}$ and $f\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 3 \\ 1 \end{pmatrix}$;

3) $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that $f\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $f\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $f\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

7.17 Let f be the endomorphism of $P_2(\mathbb{R})$ given by $f(a_0 + a_1x + a_2x^2) = 3a_0 + (a_0 - a_1)x + (2a_0 + a_1 + a_2)x^2$. Give the matrix of f in the basis $B = \{1 + x^2, -1 + 2x + x^2, 2 + x + x^2\}$.

7.18 Let $B_E = \{e_1, e_2, e_3\}$ be a basis of an $\mathbb{R}$-vector space E and $B_F = \{v_1, v_2\}$ a basis of an $\mathbb{R}$-vector space F. Let us consider the linear map $f\colon E \to F$ defined by

$$f(xe_1 + ye_2 + ze_3) = (x - 2z)v_1 + (y + z)v_2.$$

Find the matrix of f in the indicated bases:

1) $B_E = \{e_1, e_2, e_3\}$ and $B_F = \{v_1, v_2\}$.
2) $B_E = \{e_1, e_2, e_3\}$ and $B'_F = \{2v_1, 2v_2\}$.
3) $B'_E = \{e_1 + e_2 + e_3, 2e_1 + 2e_3, 3e_3\}$ and $B_F = \{v_1, v_2\}$.
4) $B'_E = \{e_1 + e_2 + e_3, 2e_1 + 2e_3, 3e_3\}$ and $B'_F = \{2v_1, 2v_2\}$.

7.19 Let us consider the endomorphism $f_N\colon \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ defined by $f_N(A) = NA$ where

$$N = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

1) Find the matrix of f_N in the canonical basis of $\mathcal{M}_2(\mathbb{R})$.
2) Compute $\mathrm{Ker} f_N$ and $\mathrm{Im}\, f_N$.
3) Find the matrix of f_N in the basis

$$B = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

7.20 Let $M = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ be the matrix of an endomorphism f of $\mathbb{R}^3$ in canonical basis.

1) Find the subspaces $\mathrm{Ker} f$ and $\mathrm{Im} f$.
2) Find a basis B of $\mathbb{R}^3$ such that the matrix of f in basis B is $M_B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

7.21 Let $u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ be vectors of $\mathbb{R}^4$ and let E be the subspace spanned by $B_E = \{u_1, u_2\}$. Let $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ be vectors of $\mathbb{R}^3$ and F the subspace spanned by $B_F = \{v_1, v_2\}$. Let $f\colon E \to F$ be given by $f(x_1u_1 + x_2u_2) = (x_1 - x_2)v_1 + (x_1 + x_2)v_2$.

1) Find the matrix of f in the bases B_E and B_F.

2) Is f injective? Is it surjective?

3) Let $B'_E = \{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 0 \end{pmatrix} \}$ and $B'_F = \{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \}$. Prove that they are bases of E and F, respectively, and give the matrix of f in these new bases.

7.22 Let us consider the linear maps associated to the following matrices:

1) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

2) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

3) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

4) $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

5) $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

6) $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

Observe that the result of applying to a vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ is, respectively,

1) the reflection through the axis OX;

2) the reflection through the axis OY;

3) the orthogonal projection onto the axis OX;

4) the orthogonal projection onto the axis OY;

5) scaling by a factor of k;

6) the counterclockwise rotation of angle α.

7.23 Give the matrix of the following compositions of linear maps of $\mathbb{R}^2$:

1) a counterclockwise rotation of 30^o, followed by a reflection through the axis OY;

2) an orthogonal projection onto the y axis, followed by a contraction of factor $k = 1/2$;

3) a scaling of factor $k = 2$, followed by a counterclockwise rotation of 45^o, followed by a reflection through the axis OY.

7.24 Let $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be linear maps. Determine if $f_1 \circ f_2 = f_2 \circ f_1$ when:

1) f_1 and f_2 are the orthogonal projections onto the axes OY and OX, respectively;

2) f_1 is the counterclockwise rotation of angle θ_1 and f_2 is the counterclockwise rotation of angle θ_2;

3) f_1 is the reflection through the x axis and f_2 is the reflection through y axis;

4) f_1 is the orthogonal projection on the y axis and f_2 is the counterclockwise rotation of angle θ.

7.25 Consider the linear maps associated to the matrices:

1) $\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$ 4) $\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$ 7) $\begin{pmatrix}\cos\alpha&-\sin\alpha&0\\\sin\alpha&\cos\alpha&0\\0&0&1\end{pmatrix}$

2) $\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$ 5) $\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}$ 8) $\begin{pmatrix}\sin\alpha&0&\cos\alpha\\0&1&0\\\cos\alpha&0&-\sin\alpha\end{pmatrix}$

3) $\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ 6) $\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}$ 9) $\begin{pmatrix}1&0&0\\0&\cos\alpha&-\sin\alpha\\0&\sin\alpha&\cos\alpha\end{pmatrix}$

Observe that the result of applyinng them to a vector $\begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3$ is, respectively,

1) the reflection through the plane $z = 0$;
2) the reflection through the plane $y = 0$;
3) the reflection through the plane $x = 0$;
4) the orthogonal projection onto the plane $z = 0$;
5) the orthogonal projection onto the plane $y = 0$;
6) the orthogonal projection onto the plane $x = 0$;
7) a counterclockwise rotation of angle α through the axis OZ, when looking at the plane $z = 0$ from the semiplane $z > 0$;
8) a counterclockwise rotation of angle α through the axis OY, when looking at the plane $y = 0$ from the semiplane $y > 0$;
9) a counterclockwise rotation of angle α through the axis OX, when looking at the plane $x = 0$ from the semiplane $x > 0$.

7.26 Give the matrix of the following compositions of linear maps of $\mathbb{R}^3$:

1) a reflection through the plane $x = 0$, followed by an orthogonal projection onto the plane $y = 0$;

2) a counterclockwise rotation of 45^o around the axis OY, followed by a scaling of factor $k = \sqrt{2}$;

3) a counterclockwise rotation of 30^o around the axis OX, followed by a counterclockwise rotation of 30^o around the axis $= Z$, followed by a scaling of factor $k = 1/3$.

7.27 Let $f_1 : \mathbb{R}^3 \to \mathbb{R}^3$ and $f_2 : \mathbb{R}^3 \to \mathbb{R}^3$ be linear maps. Determine if $f_1 \circ f_2 = f_2 \circ f_1$ when:

1) f_1 is a scaling of factor k and f_2 is a counterclockwise rotation around the z axis of angle θ;

2) f_1 is a counterclockwise rotation around the axis $= X$ of angle θ_1 and f_2 is counterclockwise rotation around the axis OZ of angle θ_2.

8
Diagonalization

Exercises

8.1 Compute the characteristic polynomial, the eigenvalues and the eigenspaces of the following matrices. In each case, determine if they are diagonalizable and give, when it exists, a basis in which the associated matrix is diagonal.

1) $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$,

2) $\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$,

3) $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix}$,

4) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$,

5) $\begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$,

6) $\begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$,

7) $\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$,

8) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 12 & -13 & 6 \end{pmatrix}$,

9) $\begin{pmatrix} -2 & 0 & 0 & 4 \\ 2 & 1 & 0 & 2 \\ 3 & 0 & -1 & 3 \\ 4 & 0 & 0 & -2 \end{pmatrix}$.

8.2 Let J be the matrix of $\mathcal{M}_5(\mathbb{R})$ with all entries equal to 1. Find a basis of $\mathbb{R}^5$ consisting of an eigenvector of J of eigenvalue 5 and 4 eigenvectors of eigenvalue 0.

8.3 Find the eigenvalues and eigenvectors of the following endomorphisms. If they are diagonalizable, give a basis in which they diagonalize and the associated diagonal matrix.

1) $f\colon \mathbb{R}^3 \to \mathbb{R}^3$, $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + 4z \\ 3x - 4y + 12z \\ x - 2y + 5z \end{pmatrix}$.

2) $f\colon P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$,
$$f(a + bx + cx^2) = (5a + 6b + 2c) - (b + 8c)x + (a - 2c)x^2.$$

3) $f\colon P_3(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$,
$$f(a + bx + cx^2 + dx^3) = (a + b + c + d) + 2(b + c + d)x + 3(c + d)x^2 + 4dx^3.$$

8.4 Find the eigenvalues and eigenvectors of the endomorphism $f\colon \mathcal{M}_2(\mathbb{R}) \longrightarrow \mathcal{M}_2(\mathbb{R})$ defined by

$$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2c & a+c \\ b-2c & d \end{pmatrix}.$$

8.5 Discuss whether the following matrices diagonalize over $\mathbb{R}$ according to the values of the parameters:

1) $\begin{pmatrix} a & b \\ -b & 0 \end{pmatrix}$

2) $\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 2 \end{pmatrix}$,

3) $\begin{pmatrix} a & b & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

4) $\begin{pmatrix} c & 2a & 0 \\ b & 0 & a \\ 0 & 2b & -c \end{pmatrix}$,

5) $\begin{pmatrix} 1 & a & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$,

6) $\begin{pmatrix} 2a-1 & 1-a & 1-a \\ a-1 & 1 & 1-a \\ a-1 & 1-a & 1 \end{pmatrix}$,

7) $\begin{pmatrix} 1 & 0 & 0 \\ b & a & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

8.6 Let f be an endomorphism of a $\mathbb{K}$-vector space E and let $u \in E$ be an eigenvector of f of eigenvalue $\lambda \in \mathbb{K}$. Prove that:

1) $-u$ is an eigenvector of f of eigenvalue λ;

2) u is an eigenvector of f^2 of eigenvalue λ^2.

8.7 Let E be an $\mathbb{R}$-vector space and f an endomorphism of E. Prove that f is bijective if and only if 0 is not an eigenvalue of f.

8.8 Prove that if $A \in \mathcal{M}_n(\mathbb{R})$ is an upper triangular matrix, with the elements in the diagonal pairwise different, then A is diagonalizable.

8.9 Determine if there exists an endomorphism $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ satisfying the conditions specified below. If it exists, give it explicitly, compute its characteristic polynomial and determine if it is diagonalizable or not.

1) $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ are eigenvectors of eigenvalue 1 and $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is an eigenvector of eigenvalue 0.

2) $f^{-1}(0) = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid 5x+y-2z=0\}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of eigenvalue $-1/2$.

3) $f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $f\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

4) $f^{-1}(0) = \langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$ and $F = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 | \, x + y = z\}$ is the eigenspace of the eigenvalue 2.

8.10 Consider the endomorphism f of $\mathbb{R}^3$ defined by $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax - y \\ x + y + z \\ 2z \end{pmatrix}$, where a is a real parameter.

1) Give the dimension of $\operatorname{Im} f$ according to the values of $a \in \mathbb{R}$.
2) Is f diagonalizable when $a = 3$?
3) Give conditions on a such that all eigenvalues of f are real.

8.11 Let $A \in \mathcal{M}_n(\mathbb{R})$.

1) How are the eigenvalues of A and A^k related? And the eigenvectors?
2) Prove that if the matrix A can be written as $A = PDP^{-1}$, where P is an invertible matrix, then $A^k = PD^kP^{-1}$.
3) Using the previous result, compute

$$\text{i)} \begin{pmatrix} 17 & -6 \\ 35 & -12 \end{pmatrix}^{100}, \quad \text{ii)} \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}^{2001}, \quad \text{iii)} \begin{pmatrix} 2 & 0 & 0 & 9 \\ 5 & 13 & 0 & 5 \\ 7 & 0 & -1 & 7 \\ 9 & 0 & 0 & 2 \end{pmatrix}^{70}.$$

8.12 An UFO leaves a planet in which the vectors v_1, v_2, v_3 have their origin. These vectors are used as a basis of a coordinate system of the universe ($\mathbb{R}^3$). After arriving at the point $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ the spaceship is pushed by a strange force such that every day it is moved from the point v to the point Av, where

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

1) What will be the position of the spaceship after 10 days?
2) Will the spaceship arrive some day to Earth, which is located in the point $\begin{pmatrix} -4098 \\ 2049 \\ 4149 \end{pmatrix}$?

Review exercises

Matrices, systems of linear equations and determinants

B.1 Given two diagonal matrices A and B of the same type, prove that $AB = BA$.

B.2 Let A and B be two upper (lower) triangular matrices of the same type. Prove that AB is an upper (lower) triangular matrix.

B.3 Prove that if A is an $m \times n$ matrix, then AA^t and A^tA are symmetric matrices.

B.4 Let A be a symmetric $n \times n$ matrix such that $A^2 = A$, and let us assume that for any i we have $a_{ii} = 0$. Prove that all entries in the ith row and in the ith column of A are zero.

B.5 Find the set of matrices $A \in \mathcal{M}_2(\mathbb{R})$ such that $A^2 = I$.

B.6 Let A, B and C be matrices. Prove that

1) if A is row equivalent to B, then B is row equivalent to A;
2) if A is row equivalent to B and B is row equivalent to C, then A is row-equivalent to C.

B.7 Prove that the following system of linear equations is consistent but underdetermined with two degrees of freedom.

$$\left\{\begin{array}{rcl} 4x - y + z + 2t + 2u & = & 1 \\ y + z - 2u & = & 1 \\ 2x + z + t & = & 1 \\ x - y + t + 2u & = & 0 \\ 5x + y + 3z + 2t - 2u & = & 3 \end{array}\right.$$

Answer the following questions.

1) What is the maximum number of independent equations?
2) Is the second equation a linear combination of the others? Answer the same question for the fourth equation.

3) Is there any solution of the system such that $x = 2\pi$? Answer the same question for $y = 2\sqrt{3}$.

4) Is there any solution of the system such that $x = 0$ and $y = 2\sqrt{3}$? The same for $z = 2\pi$ and $t = 2\sqrt{3}$.

B.8 For any matrix A let B be the matrix obtained by multiplying one row of A by a constant c. Prove that $\det B = c \det A$.

B.9 Solve the following systems of linear equations. Use Gausssian elimination and give the solution in parametric form.

$$1)\ \left\{\begin{array}{rcr} x+y &=& 2\\ x+3y &=& 0\\ 2x+4y &=& 2\\ 2x+3y &=& -3\end{array}\right.$$

$$3)\ \left\{\begin{array}{rcr} x-y+2z &=& 7\\ 2x-2y+2z-4w &=& 12\\ -x+y-z+2w &=& -4\\ -3x+y-8z-10w &=& -29\end{array}\right.$$

$$2)\ \left\{\begin{array}{rcr} -x+y+2z &=& 1\\ 2x+3y+z &=& -2\\ 5x+4y+2z &=& 4\end{array}\right.$$

$$4)\ \left\{\begin{array}{rcr} x+6y-z-4w &=& 0\\ -2x-12y+5z+17w &=& 0\\ 3x+18y-z-6w &=& 0\\ 5x+30y-6z-23w &=& 0\end{array}\right.$$

B.10 Discuss the following systems according to the values of the parameters (assumed to be real).

$$1)\ \left\{\begin{array}{rcl} x+y+mz &=& m\\ x+my+z &=& m\\ mx+y+z &=& m\end{array}\right.$$

$$3)\ \left\{\begin{array}{rcl} ax+3y &=& 2\\ 3x+2y &=& a\\ 2x+ay &=& 3\end{array}\right.$$

$$2)\ \left\{\begin{array}{rcc} (a-1)x-ay &=& 2\\ 6ax-(a-2)y &=& 5-a\end{array}\right.$$

$$4)\ \left\{\begin{array}{rcl} ax+2y+3z+u &=& 6\\ x+3y-z+2u &=& b\\ 3x-ay+z &=& 2\\ 5x+4y+3z+3u &=& 9\end{array}\right.$$

Vector spaces

B.11 Let $\mathcal{F}(\mathbb{R})$ be the vector space defined in the exercise 6.6. Determine which of the following subsets are vector subspaces of $\mathcal{F}(\mathbb{R})$. (Justify your answers.)

$$\begin{array}{rcl} F_1 &=& \{f \in \mathcal{F}(\mathbb{R}) : f(1) = f(2)\}\\ F_2 &=& \{f \in \mathcal{F}(\mathbb{R}) : f(1) = f(2)+1\}\\ F_3 &=& \{f \in \mathcal{F}(\mathbb{R}) : f(1) = 0\}\\ F_4 &=& \{f \in \mathcal{F}(\mathbb{R}) : f(-x) = f(x)\ \forall x \in \mathbb{R}\}\\ F_5 &=& \{f \in \mathcal{F}(\mathbb{R}) : f \text{ is continuous}\}\end{array}$$

B.12 Prove that the set of solutions of a system of linear equations with n unknowns is a vector subspace of $\mathbb{R}^n$ if and only if the system is homogeneous.

B.13 Let us consider the subspaces of $\mathbb{R}^3$

$$F = \langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} \rangle \text{ and } G = \langle \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ a \end{pmatrix} \rangle.$$

Determine the values of a such that $F = G$.

B.14 Let E be an $\mathbb{R}$-vector space and u, v, w three vectors such that $2u+2v-w = -4u-5v+w$. Prove that $\{u, v, w\}$ is a linearly dependent set.

B.15 Let E be a vector space and v_1, v_2, v_3 vectors of E. Prove that the following statements are equivalent.

a) $\{v_1, v_2, v_3\}$ is linearly independent.

b) $\{v_1 + v_2, v_2, v_2 + v_3\}$ is linearly independent.

c) $\{v_1 + v_2, v_1 + v_3, v_2 + v_3\}$ is linearly independent.

B.16 Prove that the set $\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\}$ is a basis of $\mathbb{R}^4$. Write down the vector $\begin{pmatrix} -3 \\ 7 \\ 6 \\ -5 \end{pmatrix}$ as a linear combination of the vectors in this basis.

B.17 Let F be the subspace of $\mathbb{R}^4$ given by

$$F = \langle \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 9 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 3 \\ 42 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ -1 \\ 37 \end{pmatrix} \rangle.$$

1) Find a basis of F.

2) Prove that $e = \begin{pmatrix} 9 \\ 7 \\ 2 \\ 79 \end{pmatrix} \in F$ and compute the coordinates of e in the basis of the previous question.

3) Find a basis of F that contains e.

B.18 Let $\{u, v\}$ be a basis of $\mathbb{R}^2$. Prove that the set $\{\alpha u+\beta v : \alpha+\beta = 0\}$ is a vector subspace. Describe this subspace geometrically and find a basis of it.

B.19 Let E be a vector subspace. Prove:

1) If $\{e_1, \ldots, e_n\}$ is a spanning set of E such that when one removes any of the e_i it no longer spans E, then $\{e_1, \ldots, e_n\}$ is a basis of E.

2) If $\{e_1, \ldots, e_n\}$ is a linearly independent set of E such that it becomes linearly dependent after adding any new vector, then $\{e_1, \ldots, e_n\}$ is a basis of E.

B.20 Find a basis and the dimension of the following subspaces.

1) $E_1 = \{\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 : x = y = 2z\}$.

2) $E_2 = \{\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 : x = y - 3z, z = t\}$.

3) $E_3 = \langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \rangle \subseteq (\mathbb{R})^3$.

4) $E_4 = \langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \rangle \subseteq (\mathbb{R})^3$.

5) $E_1 \cap E_2$.

6) $E_3 \cap E_4$.

B.21 Let E_1, E_2, E_3 and E_4 be the subspaces in the previous exercise. For each of them, complete the basis found to a basis of the whole space.

B.22 Let $F = \{\begin{pmatrix} a \\ 2a \\ 2b - a \end{pmatrix} : a, b \in \mathbb{Q}\}$. Prove that it is a vector subspace of $\mathbb{Q}^3$. Find a basis and its dimension.

Answer the same questions if we take $\mathbb{Z}_2$ instead of $\mathbb{Q}$ as the field of scalars.

B.23 A matrix $M \in \mathcal{M}_n(\mathbb{R})$ is called magic if the sum of the elements in each row, column and in the main diagonal is the same in all cases. Prove that the set of magic matrices is a subspace of $\mathcal{M}_n(\mathbb{R})$. For $n = 2, 3$ find a basis of this subspace and its dimension.

B.24 Let $B = \{\begin{pmatrix}-3\\0\\3\end{pmatrix}, \begin{pmatrix}-3\\2\\-1\end{pmatrix}, \begin{pmatrix}1\\6\\-1\end{pmatrix}\}$ and $B' = \{\begin{pmatrix}-6\\-6\\0\end{pmatrix}, \begin{pmatrix}-2\\-6\\4\end{pmatrix}, \begin{pmatrix}-2\\-3\\7\end{pmatrix}\}$ be bases of $\mathbb{R}^3$.

1) Prove that they are indeed bases.

2) Give the change-of-basis matrix from the basis B to the basis B' ($P_{B'}^{B}$) and the change-of-basis matrix from B' to B ($P_{B}^{B'}$).

3) Compute the coordinates in the bases B and B' of the vector v whose coordinates in the canonical basis are $\begin{pmatrix}-5\\8\\-5\end{pmatrix}$.

B.25 Let E be a vector space over $\mathbb{R}$ of dimension 3 and let $\{e_1, e_2, e_3\}$ be a basis of E. The vector v has coordinates $\begin{pmatrix}1\\0\\-1\end{pmatrix}$ in this basis. Compute the coordinates of v in the basis:

$$\begin{aligned} u_1 &= e_1 + e_2 + e_3 \\ u_2 &= e_1 - e_2 \\ u_3 &= e_2 - e_3 \end{aligned}$$

(You do not need to prove that it is a basis.)

B.26 Let

$$\begin{aligned} B &= \left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}, \\ B' &= \left\{\begin{pmatrix}1&3\\2&-1\end{pmatrix}, \begin{pmatrix}-2&-5\\-5&4\end{pmatrix}, \begin{pmatrix}-1&-2\\-2&4\end{pmatrix}, \begin{pmatrix}-2&-3\\-5&11\end{pmatrix}\right\} \end{aligned}$$

be two bases of $\mathcal{M}_2(\mathbb{R})$. Give the change-of-basis matrices $P_{B'}^{B}$ and $P_{B}^{B'}$.

Linear maps

B.27 Let $B = \{v_1, v_2, v_3\}$ be a basis of $\mathbb{R}^3$, with $v_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $v_2 = \begin{pmatrix}2\\5\\3\end{pmatrix}$ and $v_3 = \begin{pmatrix}1\\0\\10\end{pmatrix}$. Is there a linear map $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that $f(v_1) = \begin{pmatrix}1\\0\end{pmatrix}$, $f(v_2) = \begin{pmatrix}1\\0\end{pmatrix}$, and $f(v_3) = \begin{pmatrix}0\\1\end{pmatrix}$? If so, give the image of the vector $\begin{pmatrix}x\\y\\z\end{pmatrix}$ by this map f.

B.28 Determine which of the following vector spaces are isomorphic to $\mathbb{R}^6$: $\mathcal{M}_{2\times 3}(\mathbb{R})$, $\mathbb{R}_6[x]$, $\mathcal{M}_{6\times 1}(\mathbb{R})$, $W = \{(x_1, x_2, x_3, 0, x_5, x_6, x_7) : x_i \in \mathbb{R}\}$.

B.29 Let $f : \mathcal{M}_3(\mathbb{R}) \to \mathcal{M}_3(\mathbb{R})$ be defined by $f(A) = A - A^t$. Compute the preimage of the zero vector and its dimension.

B.30 Let $B = \{u, v, w\}$ be a basis of a $\mathbb{K}$-vector space E, and let f be an endomorphism of E such that:

$$f(u) = u + v, \ f(w) = u, \ \mathrm{Ker} f = \langle u + v \rangle.$$

Give a basis and the dimension of the subspaces $\mathrm{Im} f$ and $\mathrm{Im} f^2$

B.31 Let f be an endomorphism of $\mathbb{R}^3$ such that $f\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$ and $f^{-1}(0) = \{\begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 : x - 3y + 2z = 0\}$.

1) Give the matrix of f in the canonical basis.
2) Compute the dimension of $f^{-1}(0)$ and of the image of f.
3) Give an explicit expression for the image of an arbitrary vector.
4) Determine if it is injective, surjective, bijective or none of them.

B.32 Let f_r be the endomorphisms of $\mathbb{R}^3$ defined by

$$f_r\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2x + y + 2z\\x + y\\x + 2y + rz\end{pmatrix}, \quad r \in \mathbb{R}.$$

1) Find the value of r such that the dimension of $\mathrm{Im}\, f_r$ is the lowest possible one.
2) For the values of r obtained in the previous item, give a basis and the dimension of the preimage of the zero vector by f_r.
3) Given $v = \begin{pmatrix}3\\2\\t\end{pmatrix}$, does it exist any t such that $f_r^{-1}(0) = \langle v \rangle$?
4) If $w = \begin{pmatrix}1\\2\\1\end{pmatrix}$, compute $f_r(w)$ and $f_r^{-1}(f_r(w))$.

B.33 Let E and F be two $\mathbb{K}$-vector spaces and $f \colon E \to F$ a linear map. Let v_1, v_2 be two vectors of E. Prove the following assertions:

1) If v_1 and v_2 are linearly dependent, then $f(v_1)$ and $f(v_2)$ are linearly dependent.
2) If $f(v_1)$ and $f(v_2)$ are linearly independent, then v_1 and v_2 are linearly independent.
3) Assuming f is injective, if $f(v_1)$ and $f(v_2)$ are linearly dependent, then v_1 and v_2 are linearly dependent.

B.34 Let E be a $\mathbb{R}$-vector space and $v_1, v_2, \ldots, v_n$ vectors of E. Let us also consider the map $f\colon \mathbb{R}^n \to E$ given by $f(x_1, x_2, \ldots, x_n)^t = x_1v_1 + x_2v_2 + \cdots + x_nv_n$. Prove the following assertions.

1) f is injective if and only if the vectors $v_1, v_2, \ldots, v_n$ are linearly independent.
2) f is surjective, if and only if, the vectors $v_1, v_2, \ldots, v_n$ span E.
3) f is bijective, if and only if, the vectors $v_1, v_2, \ldots, v_n$ are a basis of E.

B.35 Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ be the matrix of $f : \mathbb{R}^3 \to \mathbb{R}^2$ in the bases $B_1 = \{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \}$ of $\mathbb{R}^3$ and $B_2 = \{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}$ of $\mathbb{R}^2$.

1) Give the matrix of f in the canonical bases of $\mathbb{R}^3$ and $\mathbb{R}^2$.
2) Find the matrix of f in the basis $B'_1 = \{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \}$ of $\mathbb{R}^3$ and $B'_2 = \{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}$ of $\mathbb{R}^2$.
3) Let $v \in \mathbb{R}^3$ be the coordinates $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ in the basis B_1. Give the coordinates of $f(v)$ in the basis B'_2.

B.36 Let f be an endomorphism of $\mathbb{R}^3$ whose matrix in the basis $u_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, u_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is given by

$$A = \begin{pmatrix} 5 & 1 & 3 \\ -5 & -1 & -3 \\ -8 & -1 & -5 \end{pmatrix}.$$

1) Find the matrix of f in the canonical basis.
2) Determine if f is injective, surjective or bijective.
3) Find the preimage of the vector $w = au_1 + bu_2 - bu_3$ according to the values of a and b. Give the result in the canonical basis.

B.37 Let $B = \{e_1, e_2, e_3, e_4\}$ be a basis of a $\mathbb{R}$-vector space E, and let $f : E \to E$ be the linear map defined by

$$f(x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4) = (x_1 + x_2)e_1 + (x_3 + x_4)e_2.$$

1) Find the matrix of f in the basis B.
2) Prove that $f \neq f^2$ and $f^2 = f^3 \neq 0$.
3) Find a basis and the dimension of the subspaces $\mathrm{Im} f, \mathrm{Im} f^2$.
4) Prove that the vectors $e_4, e_1 + e_2, e_2 + e_4, e_1 + e_2 + e_3$ form a basis of E and write the matrix of f in this basis.

B.38 Give the matrix of the following compositions of the linear maps of $\mathbb{R}^2$:

1) a reflection through the x axis, followed by a dilation of factor $k = 3$;
2) a counterclockwise rotation of 60^o, followed by an orthogonal projection onto the x axis, followed by a central symmetry.

B.39 Give the matrix of the following compositions of linear maps of $\mathbb{R}^3$:

1) an orthogonal projection onto the xy plane, followed by a reflection through the yz plane, followed by a contraction of factor $k = 3$;
2) a reflection through the xy plane, followed by a reflection through the xz plane, followed by a counterclockwise rotation of 60^o around the z axis.

Diagonalization

B.40 Let $f\colon E \to E$ be a bijective endomorphism. Prove that f diagonalizes in the basis B if and only if f^{-1} diagonalizes in the basis B. What is the relationship between the eigenvalues of f and those of f^{-1}?

B.41 Let $A \in \mathcal{M}_n(\mathbb{R})$ be a matrix such that $A^2 = A$, and let $B \in \mathcal{M}_n(\mathbb{R})$ be any matrix. Prove that any eigenvalue of AB is also an eigenvalue of ABA.

B.42 For each of the following endomorphisms, find the characteristic polynomial, the eigenvalues and the eigenspaces. Determine if they are diagonalizable.

1) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x + 3y \end{pmatrix}$.
2) $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 3y \\ -z \end{pmatrix}$.
3) $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 3y \\ 0 \end{pmatrix}$.
4) $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ -8x - 2y + 36z \\ -2x + 4z \end{pmatrix}$.

5) $f:\mathbb{R}^3 \to \mathbb{R}^3,\ f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+z \\ 3x+2y+3z \\ x+z \end{pmatrix}$.

6) $f:\mathbb{R}^4 \to \mathbb{R}^4,\ f\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x \\ 2x+5y+6z+7t \\ 3x+8z+9t \\ 4x+10t \end{pmatrix}$.

7) $f:\mathbb{R}^n \to \mathbb{R}^n,\ f\begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1+x_2+\cdots+x_n \\ x_2+\cdots+x_n \\ \cdots \\ x_n \end{pmatrix}$.

B.43 Let us consider the endomorphism f of $\mathbb{R}^4$ whose matrix in canonical basis is

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & -2 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 \end{pmatrix}.$$

1) Compute the characteristic polynomial and the eigenvalues.
2) Find a basis of $\mathbb{R}^4$ given by eigenvectors of f.
3) Give the change-of-basis matrix P from the basis obtained in 2) to the canonical basis. Compute P^{-1}.
4) Write down the diagonal matrix of f in the basis obtained in 2) and prove that it is equal to $P^{-1}AP$.

B.44 Let $C \in \mathcal{M}_n(\mathbb{R})$ be a non-null matrix and let $f_C\colon \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ be the map defined by $f_C(A) = CA - AC$, for each matrix $A \in \mathcal{M}_n(\mathbb{R})$.

1) Prove that f_C is not injective.
2) If $C = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$, for which values of the real parameters a, c and d the endomorphism f_C diagonalizes?