Introduction to Information Theory

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1 Exercise sheet

1.1 Exercise 1

Given a game of 52 cards, we draws cards with replacement.

Probability to get a queen There are 4 queens in the deck, so the probability to get a queen is:

$$\mathbb{P}(queen) = 4/52.$$

Probability to get a heart There are 13 hearts in the deck, so the probability to get a heart is:

$$\mathbb{P}(heart) = 13/52.$$

Probability to get the queen of hearts or the ace of spades There is one queen of heart and one ace of spades, which are disjoint events, so the probability to get one of them is:

$$\begin{aligned}\mathbb{P}(\text{QH or AS}) &= \mathbb{P}(QH) + \mathbb{P}(AS)\\ &= 1/52 + 1/52\\ &= 2/52\end{aligned}$$

Probability to get a queen or a spade There are 4 queens and 13 spades, which are not disjoint events, so the probability to get one of them is:

$$\begin{aligned}\mathbb{P}(\text{Q or S}) &= \mathbb{P}(Q) + \mathbb{P}(S) - \mathbb{P}(Q \cap S)\\ &= 4/52 + 13/52 - 1/52\\ &= 16/52\end{aligned}$$

Probability to get neither a queen, nor a spade This is the complement of the previous event, so the probability to get neither a queen, nor a spade is:

$$\begin{aligned}\mathbb{P}(\text{not Q and not S}) &= 1 - \mathbb{P}(\text{Q or S})\\ &= 1 - 16/52\\ &= 36/52\end{aligned}$$

1.2 Exercise 2

We draw one ball from an urn with 5 white balls, 4 red balls and 2 black balls.

Probability to draw white There are 11 balls in total and 5 white balls, so the probability to draw a white ball is:

$$\mathbb{P}(W) = 5/11$$

Probability to draw not white This is the complement of the previous event, so the probability to draw not white is:

$$\begin{aligned}\mathbb{P}(\neg W) &= 1 - \mathbb{P}(W) \\ &= 6/11\end{aligned}$$

Probability to draw white or red These events are disoint, so the probability to draw a white or a red ball is:

$$\begin{aligned}\mathbb{P}(W \cup R) &= \mathbb{P}(W) + \mathbb{P}(R) \\ &= 5/11 + 4/11 \\ &= 9/11\end{aligned}$$

1.3 Exercise 3

1.4 Exercise 4

1.5 Exercise 5

1.6 Exercise 6

1.7 Exercise 7

2 Exercise sheet Quantitative Measure of Information (part 1)

2.1 Exercise 6

Given

$$\begin{aligned}\mathbb{P}(X = 0) &= \frac{1}{4} \\ \mathbb{P}(X = 1) &= \frac{3}{4} \\ p_0 &= 10^{-1}\end{aligned}$$

where p_0 is the probability of incorrect transmission of a bit.

Compute $H(X)$

$$\begin{aligned}
H(X) &= -\sum_{i=1}^{m} \mathbb{P}(X = x_i) \log_2 \mathbb{P}(X = x_i) \\
&= -\frac{1}{4} \log_2 \frac{1}{4} - \frac{3}{4} \log_2 \frac{3}{4} \\
&\approx 0.811 \text{ sh / state of X}
\end{aligned}$$

Compute $H(Y)$ We first compute the probability distribution of Y:

$$\begin{aligned}
\mathbb{P}(Y = 1) &= \mathbb{P}(Y = 1, X = 0) + \mathbb{P}(Y = 1, X = 1) \\
&= \mathbb{P}(Y = 1 \mid X = 0)\mathbb{P}(X = 0) + \mathbb{P}(Y = 1 \mid X = 1)\mathbb{P}(X = 1) \\
&= p_0 \frac{1}{4} + (1 - p_0) \frac{3}{4} \\
&= 10^{-1} \frac{1}{4} + (1 - 10^{-1}) \frac{3}{4} \\
&= \frac{1}{10} \frac{1}{4} + \frac{9}{10} \frac{3}{4} \\
&= \frac{1}{40} + \frac{27}{40} \\
&= \frac{28}{40}
\end{aligned}$$

and on the other hand:

$$\begin{aligned}
\mathbb{P}(Y = 0) &= \mathbb{P}(Y = 0, X = 0) + \mathbb{P}(Y = 0, X = 1) \\
&= \mathbb{P}(Y = 0 \mid X = 0)\mathbb{P}(X = 0) + \mathbb{P}(Y = 0 \mid X = 1)\mathbb{P}(X = 1) \\
&= (1 - p_0) \frac{1}{4} + p_0 \frac{3}{4} \\
&= \frac{9}{10} \frac{1}{4} + \frac{1}{10} \frac{3}{4} \\
&= \frac{9}{40} + \frac{3}{40} \\
&= \frac{12}{40}
\end{aligned}$$

$$\begin{aligned}
H(Y) &= -\sum_{i=1}^{m} \mathbb{P}(Y = y_i) \log_2 \mathbb{P}(Y = y_i) \\
&= -\frac{28}{40} \log_2 \frac{28}{40} - \frac{12}{40} \log_2 \frac{12}{40} \\
&= -\frac{28}{40} \log_2 \frac{28}{40} - \frac{12}{40} \log_2 \frac{12}{40} \\
&\approx 0.881 \text{ sh / state of Y}
\end{aligned}$$

So we have:

$$\begin{cases} \mathbb{P}(X=0, Y=0) = \frac{9}{40} \\ \mathbb{P}(X=0, Y=1) = \frac{1}{40} \\ \mathbb{P}(X=1, Y=0) = \frac{3}{40} \\ \mathbb{P}(X=1, Y=1) = \frac{27}{40} \end{cases}$$

Compute $H(X, Y)$

$$\begin{aligned} H(X,Y) = &-\sum_{i=1}^{n}\sum_{j=1}^{m} \mathbb{P}(X=x_i, Y=y_j) \log \mathbb{P}(X=x_i, Y=y_j) \\ &= -\mathbb{P}(X=0, Y=0) \log \mathbb{P}(X=0, Y=0) \\ &- \mathbb{P}(X=0, Y=1) \log \mathbb{P}(X=0, Y=1) \\ &- \mathbb{P}(X=1, Y=0) \log \mathbb{P}(X=1, Y=0) \\ &- \mathbb{P}(X=1, Y=1) \log \mathbb{P}(X=1, Y=1) \\ &= -\frac{9}{40}\log\frac{9}{40} - \frac{1}{40}\log\frac{1}{40} - \frac{3}{40}\log\frac{3}{40} - \frac{27}{40}\log\frac{27}{40} \\ &\approx 1.28 \end{aligned}$$

Compute $H(Y \mid X)$

$$\begin{aligned} H(Y \mid X) &= H(X,Y) - H(Y) \\ &= 1.28 - 0.88 \\ &= 0.4 \text{ sh / state of X} \end{aligned}$$

Compute $H(X \mid Y)$

$$\begin{aligned} H(X \mid Y) &= H(X,Y) - H(X) \\ &= 1.28 - 0.81 \\ &= 0.47 \text{ sh / state of Y} \end{aligned}$$

Compute $I(X, Y)$

$$\begin{aligned} I(X,Y) &= H(X) - H(X \mid Y) \\ &= 0.81 - 0.40 \\ &= 0.41 \text{ sh / state of (X, Y)} \end{aligned}$$

2.2 Problem 2

Consider a twin-pan balance and $c = 9$ coins. We know that one of these coins is fake. The problem is to find the fake coin given that it only differs from the other 8 coins by its weight.

1. Number of states and entropy The number of states is 18. Each coin may be lighter or heavier than the other one it is checked against.
The entropy is:

$$H(S) = \log_2 18 \approx 4.17 \text{ sh}$$

2. Each weighting operation may give one of the following results: left is heavier, right is heavier or equal weight.
The maximum entropy that can be achieved is:

$$H_{max}(S) = \log_2 3 \approx 1.58 \text{ sh}$$

so the average number of weighting operations is equal to:

$$\begin{aligned}\bar{n}_{min} &= \frac{\log_2(18)}{\log_2(3)} \\ &\approx 2.63\end{aligned}$$

3.

$$\begin{aligned}\mathbb{P}_e &= \frac{C_9^{2n}}{C_8^{2n}} \\ &= \frac{\frac{8!}{2n!(8-2n)!}}{\frac{9!}{2n!(9-2n)!}} \\ &= 1 - \frac{2}{9}n\end{aligned}$$

$$\begin{aligned}\mathbb{P}_l = \mathbb{P}_r &= \frac{1}{2}(1 - \mathbb{P}_e) \\ &= \frac{1}{9}n\end{aligned}$$

We want $\mathbb{P}_l = \mathbb{P}_r = \mathbb{P}_e$, to get maximum information. This is achieved for $n = 3$, that is 3 coins on each side.

3 Exercise sheet Quantitative Measure of Information (part 2)

3.1 Exercise

Y is uniformly distributed

$$\begin{aligned}\mathbb{P}(Y = y_1) &= \frac{1}{24} + \frac{1}{6} + \frac{1}{24} \\ &= \frac{6}{24}\end{aligned}$$

$$\begin{aligned}\mathbb{P}(Y = y_2) &= \frac{1}{12} + \frac{1}{8} + \frac{1}{24} \\ &= \frac{6}{24}\end{aligned}$$

$$\begin{aligned}\mathbb{P}(Y = y_3) &= \frac{1}{6} + \frac{1}{24} + \frac{1}{24} \\ &= \frac{6}{24}\end{aligned}$$

$$\begin{aligned}\mathbb{P}(Y = y_4) &= \frac{1}{6} + \frac{1}{6} + \frac{1}{24} \\ &= \frac{6}{24}\end{aligned}$$

Y is indeed uniformly distributed.

Compute $I(X, Y)$

$$\begin{aligned}I(X, Y) &= H(Y) - H(Y \mid X) \\ &= \log_2 4 - H(Y \mid X)\end{aligned}$$

The distribution of X is given by:

$$\begin{cases}\mathbb{P}(X = x_1) = \frac{1}{3} \\ \mathbb{P}(X = x_2) = \frac{1}{2} \\ \mathbb{P}(X = x_3) = \frac{1}{6}\end{cases}$$

$$H(Y \mid X) = \sum_{i=1}^{3} H(Y \mid X = x_i) \log \mathbb{P}(X = x_i)$$

3.2 Problem

3.2.1 1.

$$\begin{aligned}
\mathbb{P}(M = 0) &= \mathbb{P}(M = 0, T = 0) + \mathbb{P}(M = 0, T = 1) \\
&= 1/8 + 3/16 \\
&= 5/16
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(M = 1) &= \mathbb{P}(M = 1, T = 0) + \mathbb{P}(M = 1, T = 1) \\
&= 1/16 + 5/8 \\
&= 11/16
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(T = 0) &= \mathbb{P}(T = 0, M = 0) + \mathbb{P}(T = 0, M = 1) \\
&= 1/8 + 1/16 \\
&= 3/16
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(T = 1) &= \mathbb{P}(T = 1, M = 0) + \mathbb{P}(T = 1, M = 1) \\
&= 3/16 + 5/8 \\
&= 13/16
\end{aligned}$$

3.2.2 2.

$$\mathbb{P}(M = 0, T = 1) + \mathbb{P}(M = 1, T = 0) = 3/16 + 1/16 = 4/16 = 1/4$$

3.2.3 3.

$$1 - \mathbb{P}(T = 1) = 1 - 13/16 = 3/16 < 1/4$$

Indeed.

3.2.4 4.

$$
\begin{aligned}
I(E,T) &= \sum_{i=0,1}\sum_{j=0,1} \mathbb{P}(E=i,T=j)\log\frac{\mathbb{P}(E=i,T=j)}{\mathbb{P}(E=i)\mathbb{P}(T=j)} \\
&= \sum_{j=0,1} \mathbb{P}(E=1,T=j)\log\frac{\mathbb{P}(E=1,T=j)}{\mathbb{P}(E=1)\mathbb{P}(T=j)} \\
&= \mathbb{P}(E=1,T=0)\log\frac{\mathbb{P}(E=1,T=0)}{\mathbb{P}(E=1)\mathbb{P}(T=0)} + \\
&\mathbb{P}(E=1,T=1)\log\frac{\mathbb{P}(E=1,T=1)}{\mathbb{P}(E=1)\mathbb{P}(T=1)} \\
&= \mathbb{P}(T=0)\log\frac{\mathbb{P}(T=0)}{1*\mathbb{P}(T=0)} + \mathbb{P}(T=1)\log\frac{\mathbb{P}(T=1)}{1*\mathbb{P}(T=1)} \\
&= 0
\end{aligned}
$$

3.2.5 5.

$$
\begin{aligned}
I(M,T) &= \sum_{i=0,1}\sum_{j=0,1} \mathbb{P}(M=i,T=j)\log\frac{\mathbb{P}(M=i,T=j)}{\mathbb{P}(M=i)\mathbb{P}(T=j)} \\
&= \frac{1}{8}\log\frac{\frac{1}{8}}{\frac{5}{16}\frac{3}{16}} + \frac{3}{16}\log\frac{\frac{3}{16}}{\frac{5}{16}\frac{13}{16}} + \frac{1}{16}\log\frac{\frac{1}{16}}{\frac{11}{16}\frac{3}{16}} + \frac{1}{16}\log\frac{\frac{5}{8}}{\frac{11}{16}\frac{13}{16}} \\
&= \frac{1}{16}\left(2\log\frac{2}{\frac{5}{16}*3} + 3\log\frac{3}{\frac{5}{16}*13} + 1\log\frac{1}{\frac{11}{16}*3} + 1\log\frac{10}{\frac{11}{16}*13}\right) \\
&= \frac{1}{16}\left(2\log\frac{16*2}{5*3} + 3\log\frac{16*3}{5*13} + 1\log\frac{16*1}{11*3} + 1\log\frac{16*10}{11*13}\right)
\end{aligned}
$$

3.2.6 6.

3.2.7 7.

$$
\begin{aligned}
\mathbb{P}(E=0) &= \mathbb{P}(E=0,T=0) + \mathbb{P}(E=0,T=1) \\
&= 3/512 + 13/512 \\
&= 16/512
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}(E=1) &= \mathbb{P}(E=1,T=0) + \mathbb{P}(E=1,T=1) \\
&= 93/512 + 403/512 \\
&= 496/512
\end{aligned}
$$

$$\begin{aligned}\mathbb{P}(T=0) &= \mathbb{P}(T=0,E=0)+\mathbb{P}(T=0,E=1)\\ &= 3/512+93/512\\ &= 96/512\\ &= 3/16\end{aligned}$$

$$\begin{aligned}\mathbb{P}(T=1) &= \mathbb{P}(T=1,E=0)+\mathbb{P}(T=1,E=1)\\ &= 13/512+403/512\\ &= 416/512\\ &= 13/16\end{aligned}$$

$$\begin{aligned}\mathbb{P}_{error}(E) &= \mathbb{P}(E=0,T=1)+\mathbb{P}(E=1,T=0)\\ &= 13/512+93/512\\ &= 106/512 \approx 0.21\\ &< \mathbb{P}_{error}(M)\end{aligned}$$

3.2.8 8.

$$\begin{aligned}\mathbb{P}(E=0)\mathbb{P}(T=0) &= 16/512 * 3/16\\ &= 3/512\\ &= \mathbb{P}(E=0,T=0)\end{aligned}$$

$$\begin{aligned}\mathbb{P}(E=0)\mathbb{P}(T=1) &= 16/512 * 416/512\\ &= 13/512\\ &= \mathbb{P}(E=0,T=1)\end{aligned}$$

$$\begin{aligned}\mathbb{P}(E=1)\mathbb{P}(T=0) &= 496/512 * 3/16\\ &= 93/512\\ &= \mathbb{P}(E=1,T=0)\end{aligned}$$

$$\begin{aligned}\mathbb{P}(E=1)\mathbb{P}(T=1) &= 496/512 * 13/16\\ &= 403/512\\ &= \mathbb{P}(E=1,T=1)\end{aligned}$$

E and T are independent.

3.2.9 9.

From Shannon's first therorem:

$$\begin{aligned}\bar{n}_{min}(T) &= \frac{H(T)}{\log_2 q} \\ &= H(T) \\ &= 0.70 \text{ bit/realization of } T\end{aligned}$$

3.2.10 11.

$$\begin{aligned}\bar{n}_{min}(T, M) &= H(T, M) \\ &= 1.50 \text{ sh/realization of } (M, T)\end{aligned}$$

4 Discrete source coding

4.1 Exercise 4

- We encore the following commands:
- Raise the stylus (RS)
- Press the stylus (PS)
- move the stylus left (–X)
- move the stylus right (+X)
- move the stylus up (+Y)
- move the stylus down (–Y)

We have:

$$\mathbb{P}_{RS} = \mathbb{P}_{PS} = \mathbb{P}_{-X} = 0.1, \quad \mathbb{P}_{+X} = 0.3, \quad \mathbb{P}_{+Y} = \mathbb{P}_{-Y} = 0.2$$

4.2 Problem 2

4.2.1 1.

Let

$$\Pi := \begin{bmatrix} 1-q & q \\ p & 1-p \end{bmatrix}$$

We want $\lambda\Pi = \lambda$, that is:

$$\lambda\Pi = \lambda \implies \begin{cases} \lambda_1(1-q) + \lambda_2 p = \lambda_1 \\ \lambda_1 q + \lambda_2(1-p) = \lambda_1 \end{cases}$$
$$\implies \begin{cases} \lambda_1 = \frac{p}{p+q} = \frac{1}{3} \\ \lambda_2 = \frac{q}{p+q} = \frac{2}{3} \end{cases}$$

4.2.2 2.

4.2.3 3.

5 In-class exercises 2023-01-23

Consider the following matrix:

$$\Pi = \begin{bmatrix} 0 & 0.7 & 0.3 \\ 0.5 & 0 & 0.5 \\ 0 & 0.8 & 0.2 \end{bmatrix}$$

and:

$$p_\infty = \begin{bmatrix} x & y & z \end{bmatrix}$$

Therefore:

$$\begin{aligned} & p_\infty \Pi = p_\infty \\ \implies & \det(p_\infty(\Pi - Id)) = 0 \\ \implies & \det\left[p_\infty(\Pi - Id)\right] = 0 \\ \implies & \det p_\infty \begin{bmatrix} -1 & 0.7 & 0.3 \\ 0.5 & -1 & 0.5 \\ 0 & 0.8 & -0.8 \end{bmatrix} = 0 \end{aligned}$$