

# Introduction to Information Theory

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## 1 Exercise sheet

### 1.1 Exercise 1

Given a game of 52 cards, we draw cards with replacement.

**Probability to get a queen** There are 4 queens in the deck, so the probability to get a queen is:

$$\mathbb{P}(\text{queen}) = 4/52.$$

**Probability to get a heart** There are 13 hearts in the deck, so the probability to get a heart is:

$$\mathbb{P}(\text{heart}) = 13/52.$$

**Probability to get the queen of hearts or the ace of spades** There is one queen of heart and one ace of spades, which are disjoint events, so the probability to get one of them is:

$$\begin{aligned}\mathbb{P}(\text{QH or AS}) &= \mathbb{P}(\text{QH}) + \mathbb{P}(\text{AS}) \\ &= 1/52 + 1/52 \\ &= 2/52\end{aligned}$$

**Probability to get a queen or a spade** There are 4 queens and 13 spades, which are not disjoint events, so the probability to get one of them is:

$$\begin{aligned}\mathbb{P}(\text{Q or S}) &= \mathbb{P}(\text{Q}) + \mathbb{P}(\text{S}) - \mathbb{P}(\text{Q} \cap \text{S}) \\ &= 4/52 + 13/52 - 1/52 \\ &= 16/52\end{aligned}$$

**Probability to get neither a queen, nor a spade** This is the complement of the previous event, so the probability to get neither a queen, nor a spade is:

$$\begin{aligned}\mathbb{P}(\text{not Q and not S}) &= 1 - \mathbb{P}(\text{Q or S}) \\ &= 1 - 16/52 \\ &= 36/52\end{aligned}$$

## 1.2 Exercise 2

We draw one ball from an urn with 5 white balls, 4 red balls and 2 black balls.

**Probability to draw white** There are 11 balls in total and 5 white balls, so the probability to draw a white ball is:

$$\mathbb{P}(W) = 5/11$$

**Probability to draw not white** This is the complement of the previous event, so the probability to draw not white is:

$$\begin{aligned}\mathbb{P}(\neg W) &= 1 - \mathbb{P}(W) \\ &= 6/11\end{aligned}$$

**Probability to draw white or red** These events are disjoint, so the probability to draw a white or a red ball is:

$$\begin{aligned}\mathbb{P}(W \cup R) &= \mathbb{P}(W) + \mathbb{P}(R) \\ &= 5/11 + 4/11 \\ &= 9/11\end{aligned}$$

### 1.3 Exercise 3

### 1.4 Exercise 4

### 1.5 Exercise 5

### 1.6 Exercise 6

### 1.7 Exercise 7

## 2 Exercise sheet Quantitative Measure of Information (part 1)

### 2.1 Exercise 6

Given

$$\begin{aligned}\mathbb{P}(X = 0) &= \frac{1}{4} \\ \mathbb{P}(X = 1) &= \frac{3}{4} \\ p_0 &= 10^{-1}\end{aligned}$$

where  $p_0$  is the probability of incorrect transmission of a bit.

**Compute  $H(X)$**

$$\begin{aligned}H(X) &= - \sum_{i=1}^m \mathbb{P}(X = x_i) \log_2 \mathbb{P}(X = x_i) \\ &= -\frac{1}{4} \log_2 \frac{1}{4} - \frac{3}{4} \log_2 \frac{3}{4} \\ &\approx 0.811 \text{ sh / state of X}\end{aligned}$$

**Compute  $H(Y)$**  We first compute the probability distribution of  $Y$ :

$$\begin{aligned}
\mathbb{P}(Y = 1) &= \mathbb{P}(Y = 1, X = 0) + \mathbb{P}(Y = 1, X = 1) \\
&= \mathbb{P}(Y = 1 \mid X = 0)\mathbb{P}(X = 0) + \mathbb{P}(Y = 1 \mid X = 1)\mathbb{P}(X = 1) \\
&= p_0 \frac{1}{4} + (1 - p_0) \frac{3}{4} \\
&= 10^{-1} \frac{1}{4} + (1 - 10^{-1}) \frac{3}{4} \\
&= \frac{1}{10} \frac{1}{4} + \frac{9}{10} \frac{3}{4} \\
&= \frac{1}{40} + \frac{27}{40} \\
&= \frac{28}{40}
\end{aligned}$$

and on the other hand:

$$\begin{aligned}
\mathbb{P}(Y = 0) &= \mathbb{P}(Y = 0, X = 0) + \mathbb{P}(Y = 0, X = 1) \\
&= \mathbb{P}(Y = 0 \mid X = 0)\mathbb{P}(X = 0) + \mathbb{P}(Y = 0 \mid X = 1)\mathbb{P}(X = 1) \\
&= (1 - p_0) \frac{1}{4} + p_0 \frac{3}{4} \\
&= \frac{9}{10} \frac{1}{4} + \frac{1}{10} \frac{3}{4} \\
&= \frac{9}{40} + \frac{3}{40} \\
&= \frac{12}{40}
\end{aligned}$$

$$\begin{aligned}
H(Y) &= - \sum_{i=1}^m \mathbb{P}(Y = y_i) \log_2 \mathbb{P}(Y = y_i) \\
&= - \frac{28}{40} \log_2 \frac{28}{40} - \frac{12}{40} \log_2 \frac{12}{40} \\
&= - \frac{28}{40} \log_2 \frac{28}{40} - \frac{12}{40} \log_2 \frac{12}{40} \\
&\approx 0.881 \text{ sh / state of } Y
\end{aligned}$$

So we have:

$$\begin{cases} \mathbb{P}(X = 0, Y = 0) = \frac{9}{40} \\ \mathbb{P}(X = 0, Y = 1) = \frac{1}{40} \\ \mathbb{P}(X = 1, Y = 0) = \frac{3}{40} \\ \mathbb{P}(X = 1, Y = 1) = \frac{27}{40} \end{cases}$$

**Compute  $H(X, Y)$**

$$\begin{aligned}
 H(X, Y) &= - \sum_{i=1}^n \sum_{j=1}^m \mathbb{P}(X = x_i, Y = y_j) \log \mathbb{P}(X = x_i, Y = y_j) \\
 &= -\mathbb{P}(X = 0, Y = 0) \log \mathbb{P}(X = 0, Y = 0) \\
 &\quad - \mathbb{P}(X = 0, Y = 1) \log \mathbb{P}(X = 0, Y = 1) \\
 &\quad - \mathbb{P}(X = 1, Y = 0) \log \mathbb{P}(X = 1, Y = 0) \\
 &\quad - \mathbb{P}(X = 1, Y = 1) \log \mathbb{P}(X = 1, Y = 1) \\
 &= -\frac{9}{40} \log \frac{9}{40} - \frac{1}{40} \log \frac{1}{40} - \frac{3}{40} \log \frac{3}{40} - \frac{27}{40} \log \frac{27}{40} \\
 &\approx 1.28
 \end{aligned}$$

**Compute  $H(Y | X)$**

$$\begin{aligned}
 H(Y | X) &= H(X, Y) - H(X) \\
 &= 1.28 - 0.88 \\
 &= 0.4 \text{ sh / state of X}
 \end{aligned}$$

**Compute  $H(X | Y)$**

$$\begin{aligned}
 H(X | Y) &= H(X, Y) - H(Y) \\
 &= 1.28 - 0.81 \\
 &= 0.47 \text{ sh / state of Y}
 \end{aligned}$$

**Compute  $I(X, Y)$**

$$\begin{aligned}
 I(X, Y) &= H(X) - H(X | Y) \\
 &= 0.81 - 0.40 \\
 &= 0.41 \text{ sh / state of (X, Y)}
 \end{aligned}$$

### 3 Exercise sheet Quantitative Measure of Information (part 2)

#### 3.1 Exercise

$Y$  is uniformly distributed

$$\begin{aligned}
 \mathbb{P}(Y = y_1) &= \frac{1}{24} + \frac{1}{6} + \frac{1}{24} \\
 &= \frac{6}{24}
 \end{aligned}$$

$$\begin{aligned}\mathbb{P}(Y = y_2) &= \frac{1}{12} + \frac{1}{8} + \frac{1}{24} \\ &= \frac{6}{24}\end{aligned}$$

$$\begin{aligned}\mathbb{P}(Y = y_3) &= \frac{1}{6} + \frac{1}{24} + \frac{1}{24} \\ &= \frac{6}{24}\end{aligned}$$

$$\begin{aligned}\mathbb{P}(Y = y_4) &= \frac{1}{6} + \frac{1}{6} + \frac{1}{24} \\ &= \frac{6}{24}\end{aligned}$$

$Y$  is indeed uniformly distributed.

**Compute  $I(X, Y)$**

$$\begin{aligned}I(X, Y) &= H(Y) - H(Y \mid X) \\ &= \log_2 4 - H(Y \mid X)\end{aligned}$$

The distribution of  $X$  is given by:

$$\begin{cases} \mathbb{P}(X = x_1) = \frac{1}{3} \\ \mathbb{P}(X = x_2) = \frac{1}{2} \\ \mathbb{P}(X = x_3) = \frac{1}{6} \end{cases}$$

$$H(Y \mid X) = \sum_{i=1}^3 H(Y \mid X = x_i) \log \mathbb{P}(X = x_i)$$

## 3.2 Problem

1.

$$\begin{aligned}\mathbb{P}(M = 0) &= \mathbb{P}(M = 0, T = 0) + \mathbb{P}(M = 0, T = 1) \\ &= 1/8 + 3/16 \\ &= 5/16\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(M = 1) &= \mathbb{P}(M = 1, T = 0) + \mathbb{P}(M = 1, T = 1) \\
&= 1/16 + 5/8 \\
&= 11/16
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(T = 0) &= \mathbb{P}(T = 0, M = 0) + \mathbb{P}(T = 0, M = 1) \\
&= 1/8 + 1/16 \\
&= 3/16
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(T = 1) &= \mathbb{P}(T = 1, M = 0) + \mathbb{P}(T = 1, M = 1) \\
&= 3/16 + 5/8 \\
&= 13/16
\end{aligned}$$

**2.**

$$\mathbb{P}(M = 0, T = 1) + \mathbb{P}(M = 1, T = 0) = 3/16 + 1/16 = 4/16 = 1/4$$

**3.**

$$1 - \mathbb{P}(T = 1) = 1 - 13/16 = 3/16 < 1/4$$

Indeed.

**4.**

$$\begin{aligned}
I(E, T) &= \sum_{i=0,1} \sum_{j=0,1} \mathbb{P}(E = i, T = j) \log \frac{\mathbb{P}(E = i, T = j)}{\mathbb{P}(E = i)\mathbb{P}(T = j)} \\
&= \sum_{j=0,1} \mathbb{P}(E = 1, T = j) \log \frac{\mathbb{P}(E = 1, T = j)}{\mathbb{P}(E = 1)\mathbb{P}(T = j)} \\
&= \mathbb{P}(E = 1, T = 0) \log \frac{\mathbb{P}(E = 1, T = 0)}{\mathbb{P}(E = 1)\mathbb{P}(T = 0)} + \\
&\quad \mathbb{P}(E = 1, T = 1) \log \frac{\mathbb{P}(E = 1, T = 1)}{\mathbb{P}(E = 1)\mathbb{P}(T = 1)} \\
&= \mathbb{P}(T = 0) \log \frac{\mathbb{P}(T = 0)}{1 * \mathbb{P}(T = 0)} + \mathbb{P}(T = 1) \log \frac{\mathbb{P}(T = 1)}{1 * \mathbb{P}(T = 1)} \\
&= 0
\end{aligned}$$

5.

$$\begin{aligned}
I(M, T) &= \sum_{i=0,1} \sum_{j=0,1} \mathbb{P}(M = i, T = j) \log \frac{\mathbb{P}(M = i, T = j)}{\mathbb{P}(M = i)\mathbb{P}(T = j)} \\
&= \frac{1}{8} \log \frac{\frac{1}{8}}{\frac{5}{16} \frac{3}{16}} + \frac{3}{16} \log \frac{\frac{3}{16}}{\frac{5}{16} \frac{13}{16}} + \frac{1}{16} \log \frac{\frac{1}{16}}{\frac{11}{16} \frac{3}{16}} + \frac{1}{16} \log \frac{\frac{5}{8}}{\frac{11}{16} \frac{13}{16}} \\
&= \frac{1}{16} \left( 2 \log \frac{2}{\frac{5}{16} * 3} + 3 \log \frac{3}{\frac{5}{16} * 13} + 1 \log \frac{1}{\frac{11}{16} * 3} + 1 \log \frac{10}{\frac{11}{16} * 13} \right) \\
&= \frac{1}{16} \left( 2 \log \frac{16 * 2}{5 * 3} + 3 \log \frac{16 * 3}{5 * 13} + 1 \log \frac{16 * 1}{11 * 3} + 1 \log \frac{16 * 10}{11 * 13} \right)
\end{aligned}$$

### 3.3 Problem 2

Consider a twin-pan balance and  $c = 9$  coins. We know that one of these coins is fake. The problem is to find the fake coin given that it only differs from the other 8 coins by its weight.

**1. Number of states and entropy** The number of states is 18. Each coin may be lighter or heavier than the other one it is checked against. The entropy is:

$$H(S) = \log_2 18 \approx 4.17 \text{ sh}$$

**2.** Each weighting operation may give one of the following results: left is heavier, right is heavier or equal weight.

The maximum entropy that can be achieved is:

$$H_{max}(S) = \log_2 3 \approx 1.58 \text{ sh}$$

so the average number of weighting operations is equal to:

$$\begin{aligned}
\bar{n}_{min} &= \frac{\log_2(18)}{\log_2(3)} \\
&\approx 2.63
\end{aligned}$$



3.

$$\begin{aligned}\mathbb{P}_e &= \frac{C_9^{2n}}{C_8^{2n}} \\ &= \frac{\frac{8!}{2n!(8-2n)!}}{\frac{9!}{2n!(9-2n)!}} \\ &= 1 - \frac{2}{9}n\end{aligned}$$

$$\begin{aligned}\mathbb{P}_l = \mathbb{P}_r &= \frac{1}{2}(1 - \mathbb{P}_e) \\ &= \frac{1}{9}n\end{aligned}$$

We want  $\mathbb{P}_l = \mathbb{P}_r = \mathbb{P}_e$ , to get maximum information. This is achieved for  $n = 3$ , that is 3 coins on each side.

## 4 Discrete source coding

### 4.1 Exercise 4

- We encode the following commands:
- Raise the stylus (RS)
- Press the stylus (PS)
- move the stylus left (-X)
- move the stylus right (+X)
- move the stylus up (+Y)
- move the stylus down (-Y)

We have:

$$\mathbb{P}_{RS} = \mathbb{P}_{PS} = \mathbb{P}_{-X} = 0.1, \quad \mathbb{P}_{+X} = 0.3, \quad \mathbb{P}_{+Y} = \mathbb{P}_{-Y} = 0.2$$

## 5 In-class exercises 2023-01-23

Consider the following matrix:

$$\Pi = \begin{bmatrix} 0 & 0.7 & 0.3 \\ 0.5 & 0 & 0.5 \\ 0 & 0.8 & 0.2 \end{bmatrix}$$

and:

$$p_\infty = \begin{bmatrix} x & y & z \end{bmatrix}$$

Therefore:

$$\begin{aligned} p_\infty \Pi &= p_\infty \\ \implies \det(p_\infty(\Pi - Id)) &= 0 \\ \implies \det[p_\infty(\Pi - Id)] &= 0 \\ \implies \det p_\infty \begin{bmatrix} -1 & 0.7 & 0.3 \\ 0.5 & -1 & 0.5 \\ 0 & 0.8 & -0.8 \end{bmatrix} &= 0 \end{aligned}$$