Algebra refreshers
Exercises

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5 Matrices, systems of linear equations and determinants

5.1 Matrix algebra

5.1.1

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \\ \implies 3A &= \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} \end{aligned}$$

5.1.2

$$\begin{aligned} 3A - B &= \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix} \end{aligned}$$

5.1.3

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ -6 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -8 & 6 \\ 0 & 0 & 0 \\ 1 & -4 & 3 \end{bmatrix} + \begin{bmatrix} -4 & 12 & 8 \\ 1 & -3 & -2 \\ -2 & 6 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 4 & 14 \\ -5 & -3 & -2 \\ 3 & 2 & 7 \end{bmatrix} \end{aligned}$$

5.1.4

$$\begin{aligned}
BA &= \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 4 & 8 \\ 1 & 2 & 4 \\ -1 & -2 & -4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 12 & 0 & 4 \\ -9 & 0 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 6 & 3 & 6 \\ 4 & 2 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 4 & 8 \\ 19 & 5 & 14 \\ -6 & 0 & -3 \end{bmatrix}
\end{aligned}$$

5.1.5

$$\begin{aligned}
C(3A-2B) &= C\left(3\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} - 2\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}\right) \\
&= C\left(\begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 2 & -8 & 6 \\ -2 & 6 & 4 \end{bmatrix}\right) \\
&= \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 3 \\ -1 & 0 & 2 \\ 4 & 5 & -1 \end{bmatrix} \begin{bmatrix} -1 & 6 & 12 \\ -11 & 8 & -9 \\ 8 & -3 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 3 \\ -1 & 0 & 2 \\ 4 & 5 & -1 \end{bmatrix} \begin{bmatrix} -1 & 6 & 12 \\ -11 & 8 & -9 \\ 8 & -3 & 2 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 12 & 24 \\ -1 & 6 & 12 \\ 1 & -6 & -12 \\ -4 & 24 & 48 \end{bmatrix} + \begin{bmatrix} -11 & 8 & -9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -55 & 40 & -45 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 24 & -9 & 6 \\ 16 & -6 & 4 \\ -8 & 3 & -2 \end{bmatrix} \\
&= \begin{bmatrix} -13 & 20 & 15 \\ 23 & -3 & 18 \\ 17 & -12 & -8 \\ -67 & 67 & 1 \end{bmatrix}
\end{aligned}$$

5.2

$$\begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = 2 + 2 - 15 = -11$$

$$\begin{bmatrix}2\\1\\5\end{bmatrix}\begin{bmatrix}1 & 2 & -3\end{bmatrix} = \begin{bmatrix}2 & 4 & -6\\1 & 2 & -3\\5 & 10 & -15\end{bmatrix}$$

5.3

Let $m, n, p \in \mathbb{N}$ such that $A \in \mathcal{M}_{m,n}(\mathbb{R})$ and $B \in \mathcal{M}_{n,p}(\mathbb{R})$, then $AB \in \mathcal{M}_{m,p}(\mathbb{R})$. Since AB is squared, we have that $m = p$. Thus $B \in \mathcal{M}_{n,p} = \mathcal{M}_{n,m}$ so $BA \in \mathcal{M}_{n,n}$ is well defined.

5.4

$c_{13} = 1 \times 0 + 2 \times 3 + 1 \times 2 = 8$
$c_{22} = -3 \times 0 + 0 \times -4 + (-1) \times 3 = -3$

5.5

5.6

(a) and (b)

$$A = \begin{bmatrix}2 & 0 & 0\\0 & -1 & 0\\0 & 0 & 3\end{bmatrix}$$
$$A^n = \begin{bmatrix}2^n & 0 & 0\\0 & (-1)^n & 0\\0 & 0 & 3^n\end{bmatrix}$$

(c) Initialization: let $k = 0$. $D^0 = I \implies \forall 1 \le i \le r, D_{ii} = 1 = \lambda^0$

5.7

$$A = \begin{bmatrix}1 & -2\\-2 & 3\end{bmatrix} \qquad B = \begin{bmatrix}-2 & 1\\1 & 1\end{bmatrix}$$

$$\begin{aligned}
AB &= \begin{bmatrix}1 & -2\\-2 & 3\end{bmatrix}\begin{bmatrix}-2 & 1\\1 & 1\end{bmatrix}\\
\implies AB &= \begin{bmatrix}-2 & 1\\4 & -2\end{bmatrix} + \begin{bmatrix}-2 & 3\\-2 & 3\end{bmatrix}\\
\implies AB &= \begin{bmatrix}-4 & 4\\2 & 1\end{bmatrix}\\
\implies (AB)^t &= \begin{bmatrix}-4 & 2\\4 & 1\end{bmatrix}
\end{aligned}$$

6 Exercises from lecture notes

6.1 Exercise p. 20.

6.1.1 Prove that $v - P_U(v) \in U^\perp$

$$
\begin{aligned}
v - P_U(v) &= \underbrace{u}_{=P_U(v)} + u^\perp - P_U(v) \\
&= P_U(v) + u^\perp - P_U(v) \\
&= u^\perp \\
&\in U^\perp
\end{aligned}
$$

6.1.2 Exercise

Prove that, given $v \in V$ and $u \in U \subset V$, then

$$P_U(v) = \arg\min_{u \in U} \|v - u\|^2$$

In other words, prove that the orthogonal projection of v on U is the nearest point of U to v (Hint: use that $v - u = v - P_U(v) + P_U(v) - u$).

7 Extra exercises

7.1 Exercise 1

$$\frac{\partial}{\partial x}\left[(x^2+y^2)^{1/2}\right] = \frac{x}{(x^2+y^2)^{1/2}}$$

7.2 Exercise 2

$$\frac{\partial}{\partial x}\left[\log(x^2+y^2)\right] = \frac{2x}{x^2+y^2}$$