Complexe getallen

Definitie complex getal:
$a + bj$ met $j^2 = -1$

Bewerkingen:
1. optellen
$(a + bj) + (c + dj) = (a + c) + (b + d)j$

2. aftrekken
$(a + bj) - (c + dj) = (a - c) + (b - d)j$

Vermenigvuldigen en delen

3. vermenigvuldigen

$(a + bj) \cdot (c + dj) = (ac - bd) + (ad + bc)j$

4. delen

$\frac{a+bj}{c+dj} = \frac{(a+bj)(c-dj)}{(c+dj)(c-dj)} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}j$

Opgaven

Bereken $(4 - 3j) + (2 + 6j)$

$(4 - 3j) + (2 + 6j) = 6 + 3j$

Bereken $(1+j)(2+j)$

$(1+j)(2+j) = 2 + j + 2j + j^2 = 1 + 3j$

Bereken $\frac{1}{1+j}$

$\frac{1}{1+j} = \frac{1-j}{(1+j)(1-j)} = \frac{1-j}{1-j^2} = \frac{1-j}{2} = 0.5 - 0.5j$

Reëel en imaginair

$$z = a + bj$$

Reële deel: $Re(z) = a$

Imaginaire deel: $Im(z) = b$

Opgaven

$\alpha = 1 - j$. Bereken:

a. $Re(\alpha)$, $Re(\alpha^2)$

b. $Im(\alpha)$, $Im(\alpha^2)$

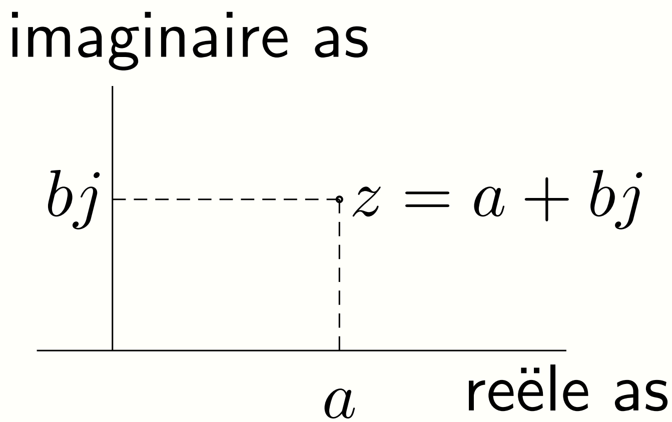
Antwoord

$\alpha = 1 - j$ en
$\alpha^2 = (1-j)(1-j) = 1 - j - j + j^2 = 1 - 2j - 1 = -2j$

a. $Re(\alpha) = 1$ en $Re(\alpha^2) = 0$

b. $Im(\alpha) = -1$, $Im(\alpha^2) = -2$

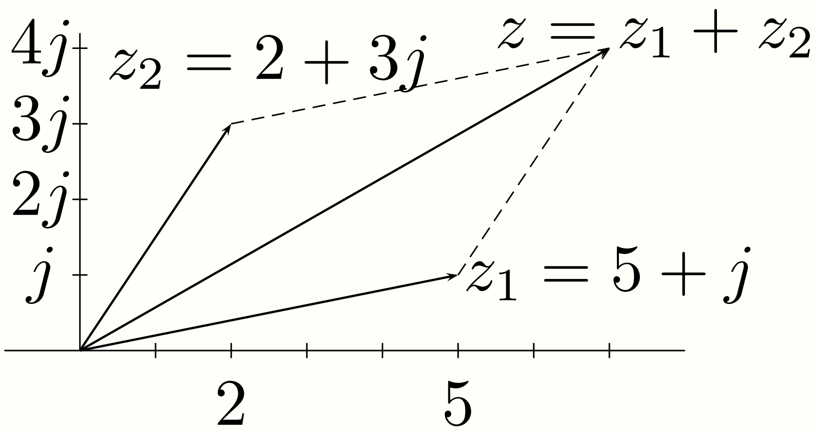
Complexe vlak



Vectoren: $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$

$\mathbb{C}$: $\quad (a+bj) + (c+dj) = (a+c) + (b+d)j$

Complexe vlak



Voor optellen en aftrekken: $\mathbb{C}$ als vectoren

Absolute waarde van $z = a + bj$: $|z| = \sqrt{a^2 + b^2}$

Absolute waarde

Absolute waarde van $z = a + bj$: $|z| = \sqrt{a^2 + b^2}$

Voorbeeld

$|3 + 4j| = \sqrt{3^2 + 4^2} = 5$

Opgaven

Bereken de absolute waarde van:

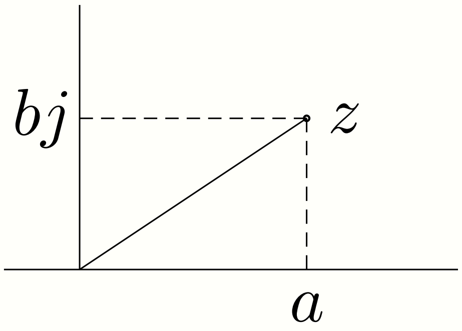
$3 + 2j$

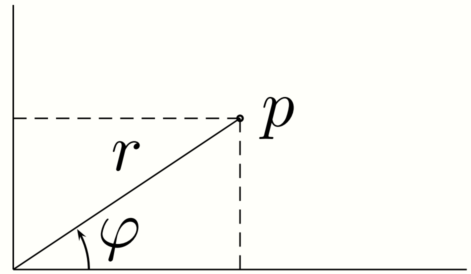
$-j$

Antwoord

$$abs(3+2j) = \sqrt{9+4} = \sqrt{13} \approx 3.61$$

$$abs(-j) = \sqrt{1} = 1$$





$$a + bj \qquad \stackrel{?}{\longrightarrow} \qquad r, \varphi$$

$$r = |z| = \sqrt{a^2 + b^2}$$

$$\varphi = arg(z) : \tan \varphi = b/a$$

$$a + bj \qquad \stackrel{?}{\longleftarrow} \qquad r, \varphi$$

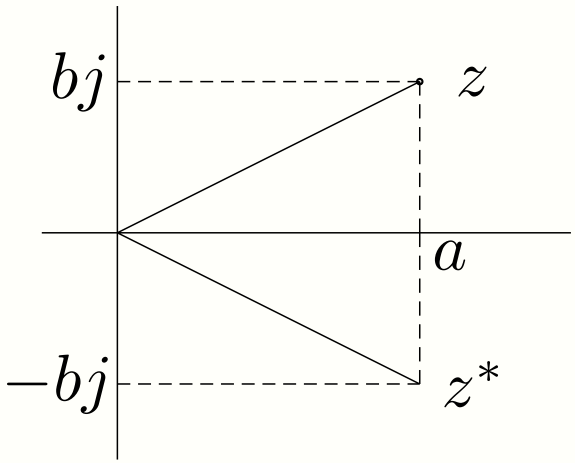
$$a = r \cos \varphi$$

$$b = r \sin \varphi$$

$$z = r(\cos \varphi + j \sin \varphi)$$

Complex geconjugeerde

$$z^* = a - bj$$



Complex geconjugeerde

Opgave

Bereken de absolute waarde en het argument van:

$z = 2 + 2j$

Antwoord

$abs(2 + 2j) = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$

$arg(2 + 2j) = \arctan \frac{2}{2} = \arctan(1) = \frac{\pi}{4}$

Complex geconjugeerde

Opgave

Schrijf in de vorm $z = a + bj$

$|z| = 1$, $arg(z) = \frac{\pi}{2}$

Antwoord

$$Re(z) = \cos(\frac{\pi}{2}) = 0$$

$$Im(z) = \sin(\frac{\pi}{2}) = 1$$

$$z = j$$

Vermenigvuldigen

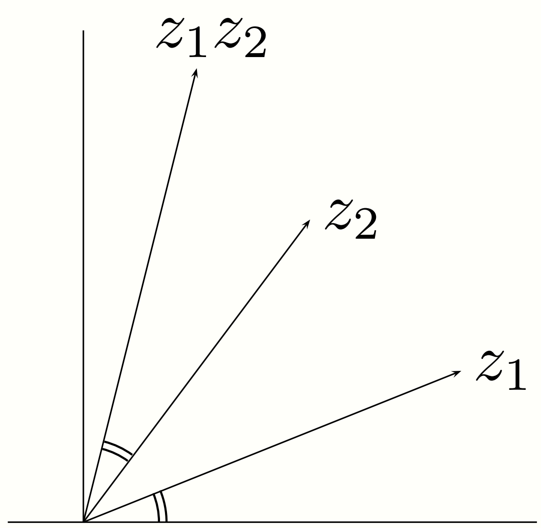
$$z_1 = r_1(\cos\varphi_1 + j\sin\varphi_1)$$
$$z_2 = r_2(\cos\varphi_2 + j\sin\varphi_2)$$

$$z_1 \cdot z_2 = r_1 r_2(\cos\varphi_1 + j\sin\varphi_1)(\cos\varphi_2 + j\sin\varphi_2) =$$

$$r_1 r_2\{(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2) + \\ j(\cos\varphi_1\sin\varphi_2 + \cos\varphi_2\sin\varphi_1)\} =$$

$$r_1 r_2(\cos(\varphi_1 + \varphi_2) + j(\sin\varphi_1 + \varphi_2))$$

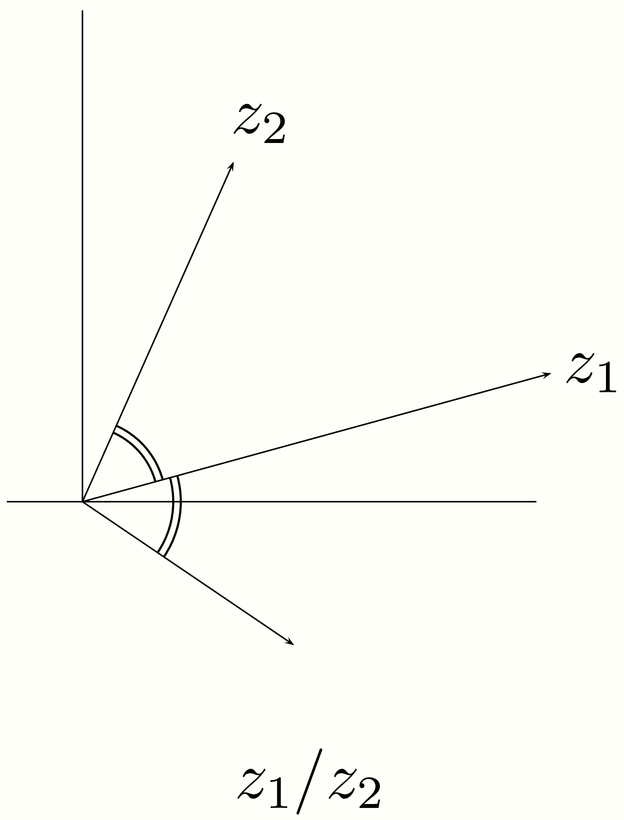
Vermenigvuldigen



$$|z_1z_2| = r_1r_2 = |z_1||z_2|$$
$$arg(z_1z_2) = arg(z_1) + arg(z_2)$$

Delen

$|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$ en $arg(\frac{z_1}{z_2}) = arg(z_1) - arg(z_2)$



Delen

Voorbeeld 1

$z_1 = 6(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2})$
$z_2 = 3(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3})$

$$z_1 z_2 = 18(\cos\frac{5\pi}{6} + j\sin\frac{5\pi}{6})$$

$$\frac{z_1}{z_2} = 2(\cos\frac{\pi}{6} + j\sin\frac{\pi}{6})$$

Delen

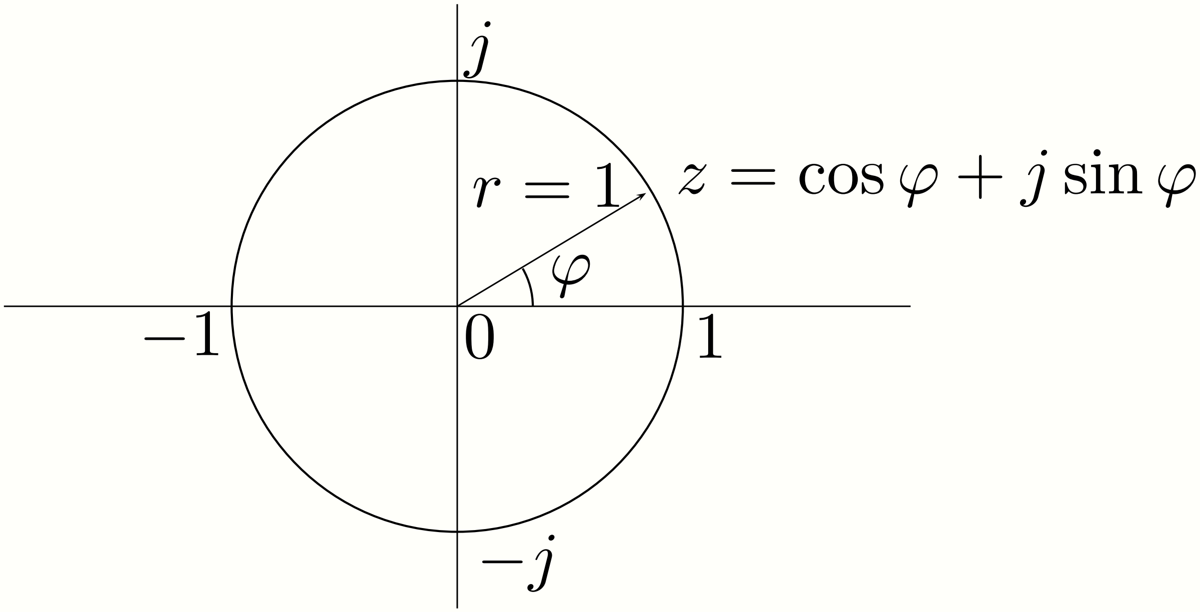
Voorbeeld 2

jz?

$$|jz| = |j| \cdot |z| = |z|$$

$$arg(jz) = arg(j) + arg(z) = \frac{\pi}{2} + arg(z)$$

Eenheidscirkel

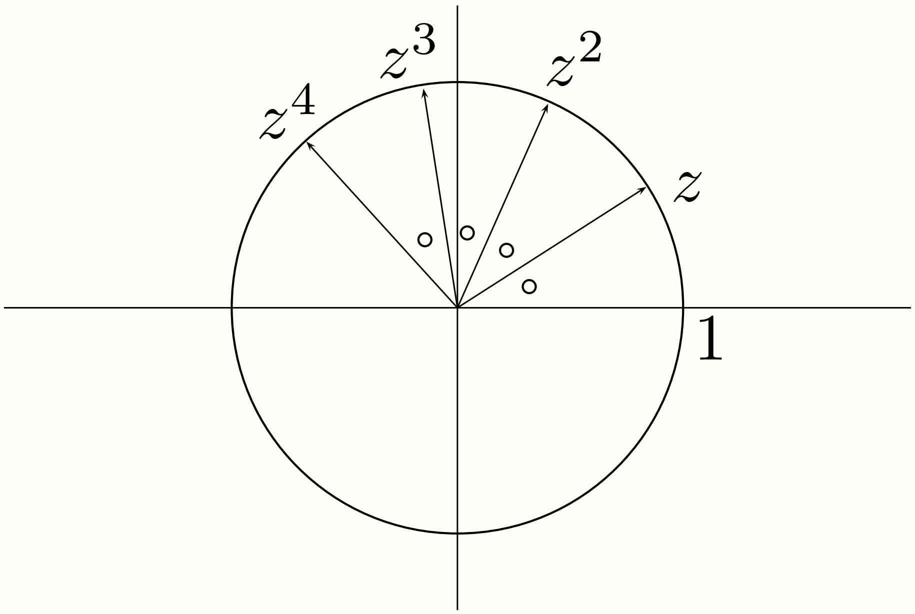


$z_1 = \cos\varphi_1 + j\sin\varphi_1$ en $z_2 = \cos\varphi_2 + j\sin\varphi_2$

$z_1 \cdot z_2 = \cos(\varphi_1 + \varphi_2) + j\sin(\varphi_1 + \varphi_2)$

Eenheidscirkel

$$z = \cos\varphi + j\sin\varphi$$
$$z^n = (\cos\varphi + j\sin\varphi)^n = \cos n\varphi + j\sin n\varphi$$



Samenvatting

$$\left.\begin{array}{c} + \\ - \end{array}\right\} \Rightarrow z = a + bj \quad \text{als vectoren}$$

$$\left.\begin{array}{c} * \\ : \end{array}\right\} \Rightarrow z = r(\cos\varphi + j\sin\varphi)$$

absolute waarden $*$ of :

argumenten $+$ of $-$

Opgaven

$z = 1 + j$

Teken z, jz en $(j-1)z$

Opgaven

$z = \frac{1+2j}{1-j}$

Bereken op twee manieren $|z|$ en $arg(z)$:

a. door $z = a + jb$ te schrijven

b. regel van delen

Antwoord

$z = \frac{1+2j}{1-j}$, Bereken op twee manieren $|z|$ en $arg(z)$:

a. door $z = a + jb$ te schrijven

$z = \frac{(1+2j)(1+j)}{(1-j)(1+j)} = \frac{1+2j+j-2}{1+1} = \frac{-1+3j}{2} = -0.5 + 1.5j$

$|z| = \sqrt{0.5^2 + 1.5^2} = 0.5\sqrt{10} = 1.58$,

$arg(z) = \arctan(\frac{1.5}{-0.5}) = \arctan(-3) = 0.6\pi$

b. regel van delen

$|\frac{1+2j}{1-j}| = \frac{|1+2j|}{|1-j|} = \frac{\sqrt{1+4}}{\sqrt{1+1}} = \sqrt{\frac{5}{2}} = \sqrt{2.5}$

$arg(\frac{1+2j}{1-j}) = arg(1+2j) - arg(1-j) = \arctan(\frac{2}{1}) -$

$arctan(\frac{1}{1}) = \arctan(2) - \arctan(-1) = 0.35\pi + 0.25\pi = 0.6\pi$

Complexe e-macht

Formule van Euler

$$e^{j\varphi} = \cos\varphi + j\sin\varphi$$

$e^{j\varphi_1} \cdot e^{j\varphi_2} = e^{j(\varphi_1+\varphi_2)}$
$e^{j\cdot 0} = e^0 = 1$
$(e^{j\varphi})^n = e^{jn\varphi}$
$(e^{j\varphi})' = je^{j\varphi}$

Dus nu: $z = r(\cos\varphi + j\sin\varphi) = re^{j\varphi}$

Voorbeelden

$e^{j\pi} = e^{-j\pi} = -1, \qquad e^{j2\pi} = e^0 = 1$

$e^{-j\frac{1}{2}\pi} = e^{j\frac{3}{2}\pi} = -j$

Wat is e^z voor $z = a + bj$?

$e^z = e^{a+bj} = e^a \cdot e^{bj} = e^a(\cos b + j \sin b)$

Opgaven

Schrijf het volgende getal in de vorm $re^{j\varphi}$:

$2 + j$

Antwoord

Schrijf het volgende getal in de vorm $re^{j\varphi}$:

$2 + j$

$r = |z| = \sqrt{5},$

$\varphi = arg(z) = \arctan(\frac{1}{2}) = \arctan(0.5) = 0.15\pi$

$z = \sqrt{5}e^{j0.15\pi}$

Inverse Euler formule

$$
\begin{aligned}
e^{j\varphi} &= \cos\varphi + j\sin\varphi \\
e^{-j\varphi} &= \cos(-\varphi) + j\sin(-\varphi) = \\
&\cos\varphi - j\sin\varphi
\end{aligned}
$$

$$\overline{\hspace{10cm}}+$$

$$e^{j\varphi} + e^{-j\varphi} = 2\cos\varphi \Rightarrow$$

Inverse formule van Euler

$$\cos\varphi = \frac{e^{j\varphi} + e^{-j\varphi}}{2}$$

Inverse Euler formule

$$
\begin{aligned}
e^{j\varphi} &= \cos\varphi + j\sin\varphi \\
e^{-j\varphi} &= \cos\varphi - j\sin\varphi
\end{aligned}
$$

$$\overline{\hspace{10cm}} -$$

$$e^{j\varphi} - e^{-j\varphi} = 2j\sin\varphi \Rightarrow$$

Inverse formule van Euler

$$\sin\varphi = \frac{e^{j\varphi} - e^{-j\varphi}}{2j}$$

Opgave Inverse Euler formule

Opgaven

Vereenvoudig de volgende som van complexe e-machten:

$(1+j)e^{-j\varphi} + (1-j)e^{j\varphi}$

Antwoord

$(1+j)e^{-j\varphi} + (1-j)e^{j\varphi}$

$e^{-j\varphi} + e^{j\varphi} + je^{-j\varphi} - je^{j\varphi} =$

$\frac{2(e^{j\varphi}+e^{-j\varphi})}{2} - j(e^{j\varphi} - e^{-j\varphi}) =$

$2\cos\varphi + \frac{2(e^{j\varphi}-e^{-j\varphi})}{2j} = 2\cos\varphi + 2\sin\varphi$

Bewerkingen met $re^{j\varphi}$

Vermenigvuldigen

$$z_1 \times z_2 = r_1 e^{j\varphi_1} \times r_2 e^{j\varphi_2} = (r_1 r_2) e^{j(\varphi_1 + \varphi_2)}$$

Delen

$$z_1 : z_2 = \frac{r_1 e^{j\varphi_1}}{r_2 e^{j\varphi_2}} = \frac{r_1}{r_2} e^{j(\varphi_1 - \varphi_2)}$$

Geconjugeerde

$$z^* = (re^{j\varphi})^* = re^{-j\varphi}$$

Vergelijkingen

Los op: $z^n = \alpha$ (α is gegeven complex getal)

$$|z^n| = |z|^n = |\alpha| \Rightarrow |z| = \sqrt[n]{|\alpha|}$$

$$arg(z^n) = n \cdot arg(z) = arg(\alpha) + k \cdot 2\pi \Rightarrow$$

$$arg(z) = \frac{1}{n} arg(\alpha) + \frac{k \cdot 2\pi}{n},\ k = 0, 1, \ldots, n-1$$

Vergelijkingen

Voorbeeld

Los op: $z^4 = 1$

$$|z| = \sqrt[n]{|\alpha|} \text{ en } arg(z) = \frac{1}{n}arg(\alpha) + \frac{k\cdot 2\pi}{n}, \qquad k = 0, 1, \ldots, n-1$$

$$|z^4| = |z|^4 = |1| = 1 \Rightarrow |z| = 1$$

$$arg(z^4) = 4 \cdot arg(z) = arg(1) + k \cdot 2\pi = k \cdot 2\pi \Rightarrow$$

$$arg(z) = \frac{1}{4}k \cdot 2\pi = \frac{\pi}{2}k, k = 0, \ldots, 3$$

Oplossingen van $z^4 = 1$

$|z| = 1$ en $arg(z) = \frac{\pi}{2}k, \qquad k = 0, \ldots, 3$