No, More Heads Are Not Better!

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Some math details

Condorcet's Jury Theorem

For any dichotomous/binary choice by a group:

probability p that an individual votes correctly > 0.5

(p also denoted competence), and votes are independent/uncorrelated (with uniform probabilities),

the probability of a correct majority approaches 1 as the number of voters increases

Condorcet's Jury Theorem

- In order to avoid the problem of tied votes, we stick to odd numbers of voters
- Any odd number n of voters: a row of bits: 0010101...011101
- The probability that a single bit will be 1 is p.
- The probability that a single bit will be 0 is $1 - p$.
- The probability of a single bit combination is the product of all single bit probabilities.

Condorcet’s Jury Theorem

- The probability of a single bit combination of k 1s (correct) and $(n-k)$ 0s (incorrect):

$$p^k(1-p)^{(n-k)}$$

- We then need to calculate the probability for all combinations with a given number of 1s and 0s.
- Number of combinations with k 1s and $(n-k)$ 0s:

$$\binom{n}{k} \quad \left[= \frac{n!}{k!(n-k)!}\right]$$

Condorcet's Jury Theorem

- The probability of all bit combination of k 1s (correct) and $(n - k)$ 0s (incorrect):

$$\binom{n}{k} p^k (1-p)^{(n-k)}$$

Condorcet’s Jury Theorem

- We then need the probability for all bit combinations where 1s outnumber the 0s.
- Correct majority decision is where $k \geq \frac{n-1}{2}$.
- Sum of all probabilities for correct majority decision:

$$\sum_{k=\frac{n-1}{2}}^{n} \binom{n}{k} p^{k} (1-p)^{(n-k)}$$

Individual Incentive?

- Any odd number n of voters: a row of bits
- Number of odd combinations: 2^n
- Number of other voters (even): 2^{n-1} = 2^m
- random voting model
  - equal probability
  - statistically independent
- Probability of number of 1s equaling number of 0s:

$$\frac{number\ of\ combinations\ where\ number\ of\ 1s = number\ of\ 0s}{total\ number\ of\ combinations\ (2^m)}$$

Individual Incentive?

- Number of combinations with equal number of 0s and 1s of all other voters:

$$\binom{m}{\frac{m}{2}} \quad \left[= \frac{m!}{\frac{m}{2}!\left(m-\frac{m}{2}\right)!}\right] \quad [= \frac{m!}{\frac{m}{2}!\frac{m}{2}!}]$$

- Probability of a tie to break:

$$\frac{\frac{m!}{\frac{m}{2}!\frac{m}{2}!}}{2^m} = 2^{-m} \ \frac{m!}{\frac{m}{2}!\frac{m}{2}!}$$

Stirling's Formula

- An approximation of factorial:

$$x! \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x$$

- Optional step(s) is to prove Stirling's formula. In order to do this check out any advanced mathematics book, e.g., *Advanced Engineering Mathematics* by Erwin Kreyszig.
- We simplify the formula by inserting the approximation for factorial.

Penrose Square Root Law

Probability of tie-breaking for odd number of voters n:

$$\sqrt{\frac{2}{\pi}}\sqrt{\frac{1}{n-1}}$$