COM303: Digital Signal Processing

Lecture 20: Image Processing

overview

- Introduction to Images and Image Processing
- Affine Transforms
- 2D Fourier Analysis
- Image Filters

Overview:

- Images as multidimensional digital signals
- 2D signal representations
- Basic signals and operators

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In the old, non-PC days...

Please meet ...

Digital images

- two-dimensional signal $x[n_1, n_2]$, $n_1, n_2 \in \mathbb{Z}$
- indices locate a point on a grid $\rightarrow$ pixel
- grid is usually regularly spaced
- values $x[n_1, n_2]$ refer to the pixel's appearance

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Digital images: grayscale vs color

- grayscale images: scalar pixel values
- color images: multidimensional pixel values in a color space (RGB, HSV, YUV, etc)
- we can consider the single components separately:

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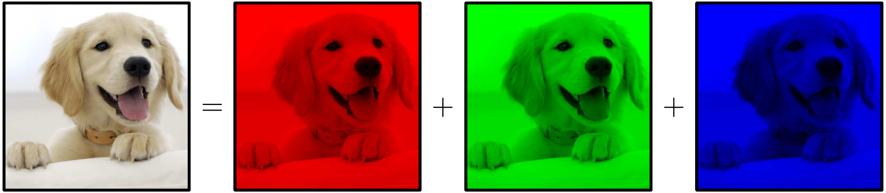

Image processing

From one to two dimensions...

- something still works
- something breaks down
- something is new

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A thought experiment

- consider all possible 256×256, 8bpp "images"
- each image is 524,288 bits
- total number of possible images: $2^{524,288} \approx 10^{157,826}$
- number of atoms in the universe: 10^{82}
- how many "images" are proper images?

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Image processing

images are very specialized signals,
"designed" for a very specific processing system: the human brain!

visual semantics is extremely hard to deal with

Image processing

What works:

- the maths: linearity, space-invariance, convolution
- Fourier transform formulas
- interpolation, sampling, quantization

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What breaks down:

- Fourier analysis less relevant
- filter design hard, IIRs rare
- linear, space-invariant operators only mildly useful because of their isotropy

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What's new:

- new manipulations: affine transforms
- images are finite-support signals
- images are usually available in their entirety $\rightarrow$ causality mostly irrelevant

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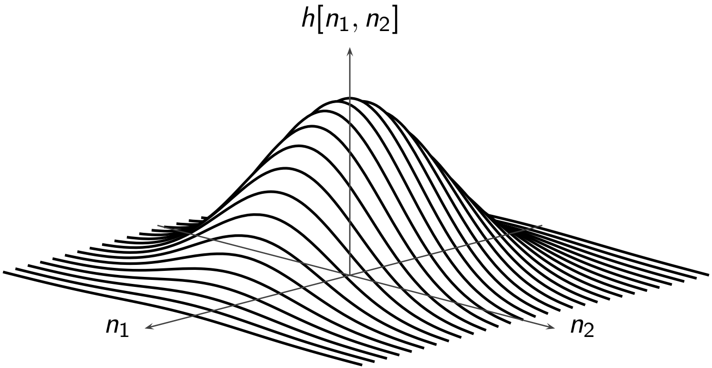
2D signal processing: the basics

A two-dimensional discrete-space signal:

$$x[n_1, n_2], \qquad n_1, n_2 \in \mathbb{Z}$$

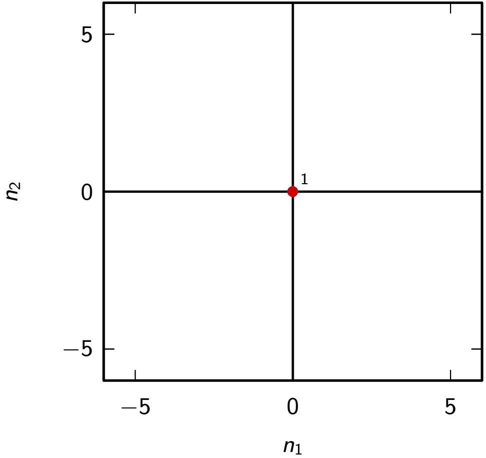
2D signals: Cartesian representation



2D signals: support representation

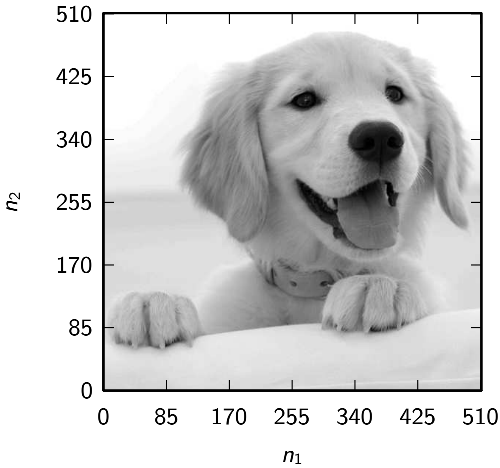
- just show coordinates of nonzero samples
- amplitude may be written along explicitly
- example:

$$\delta[n_1, n_2] = \begin{cases} 1 & \text{if } n_1 = n_2 = 0 \\ 0 & \text{otherwise.} \end{cases}$$



2D signals: image representation

- medium has a certain dynamic range (paper, screen)
- image values are quantized (usually to 8 bits, or 256 levels)
- the eye does the interpolation in space provided the pixel density is high enough



About dynamic ranges...

Images:

- human eye: 120dB
- prints: 12dB to 36dB
- LCD: 60dB
- digital cinema: 90dB

Sounds:

- human ear: 140dB
- speech: 40dB
- vinyl, tape: 50dB
- CD: 96dB

Why 2D?

- images could be unrolled (printers, fax)
- but what about spatial correlation?

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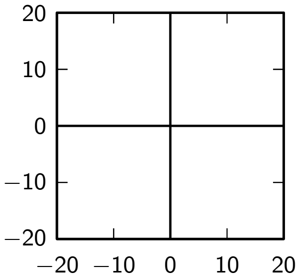
2D vs raster scan



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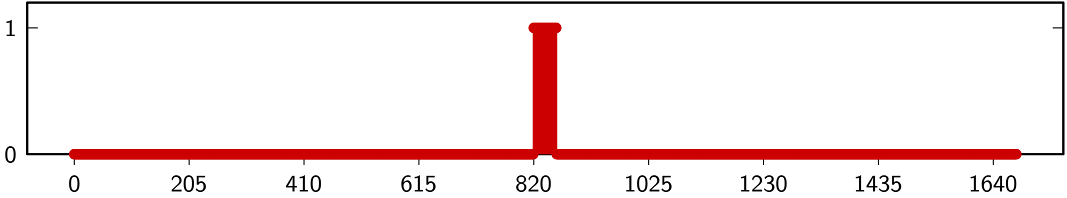

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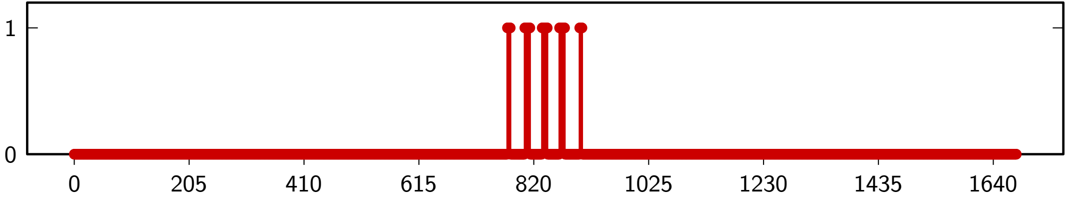

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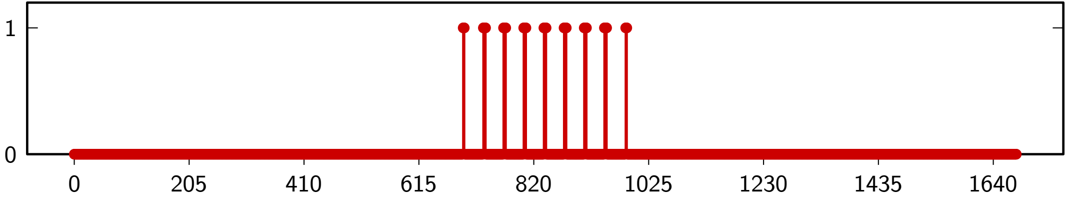

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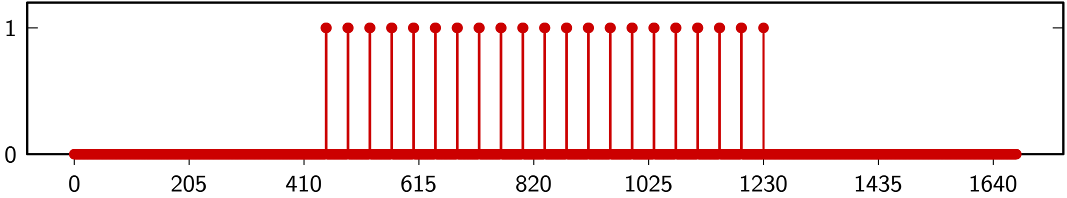

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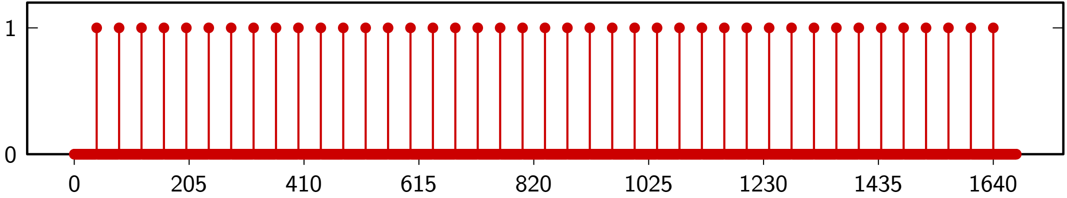

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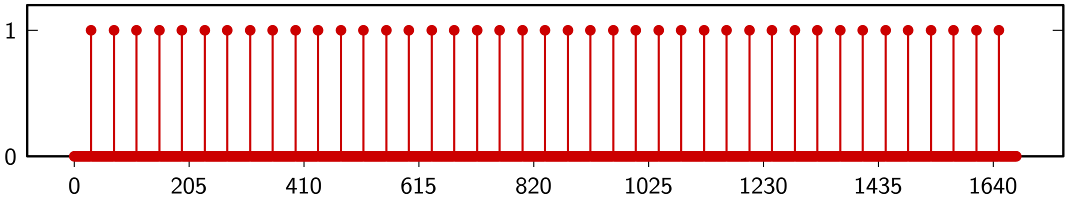

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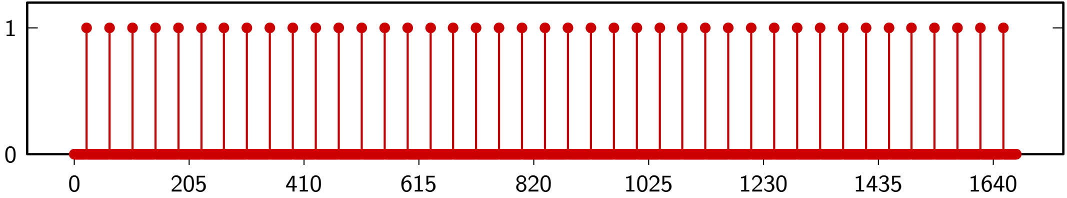

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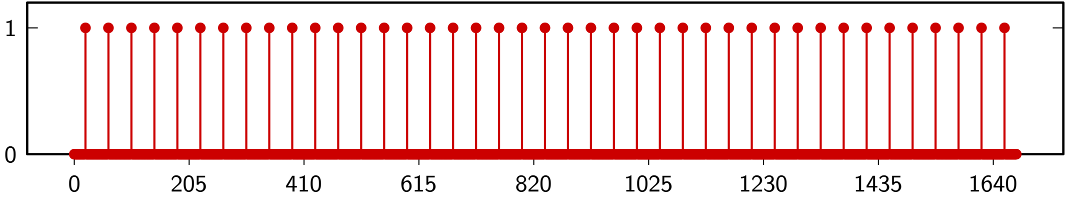

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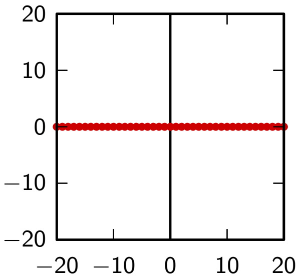

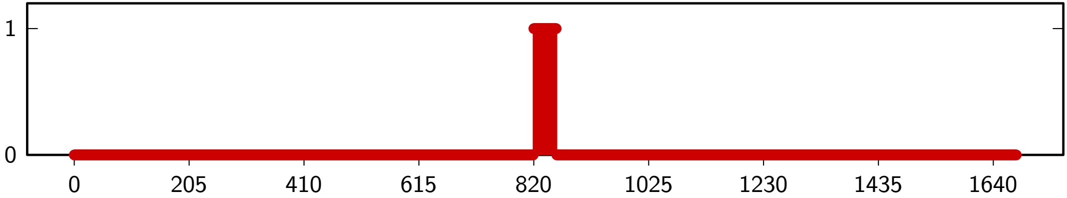

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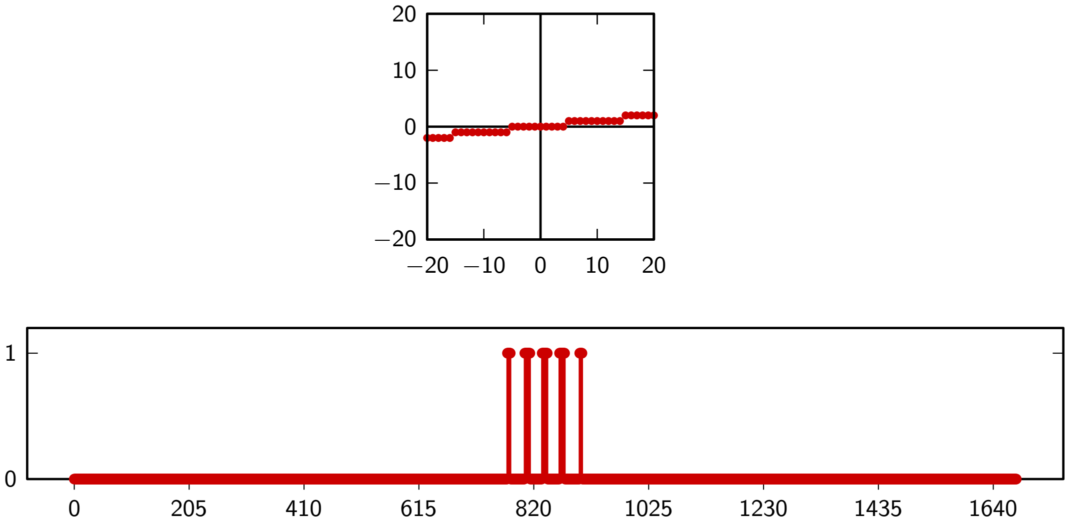

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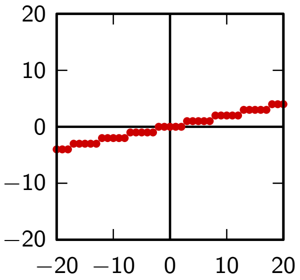

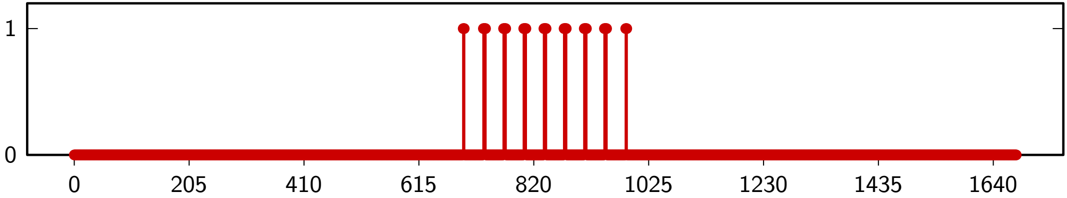

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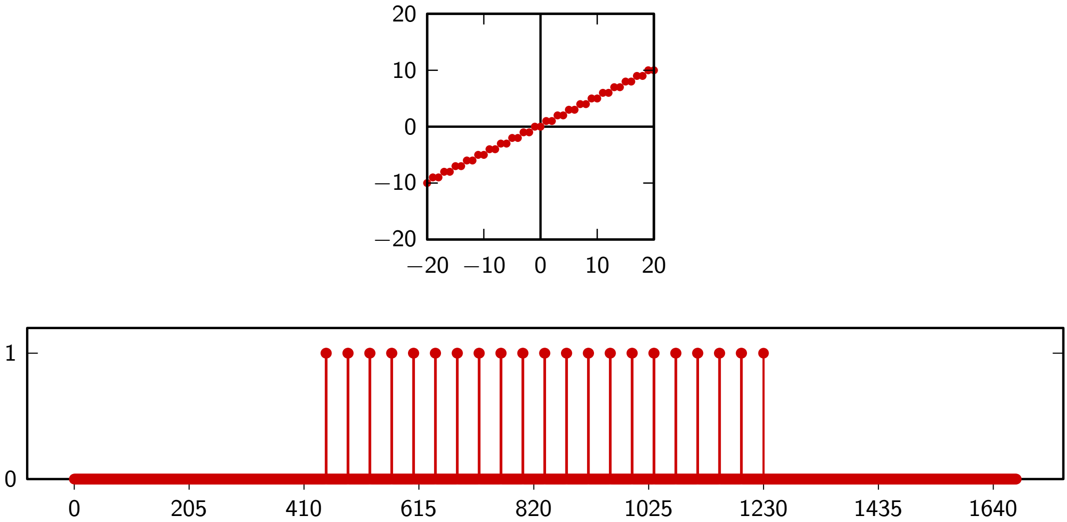

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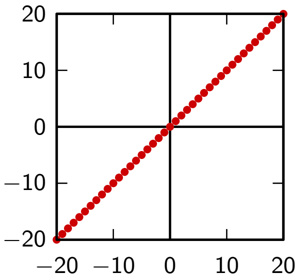

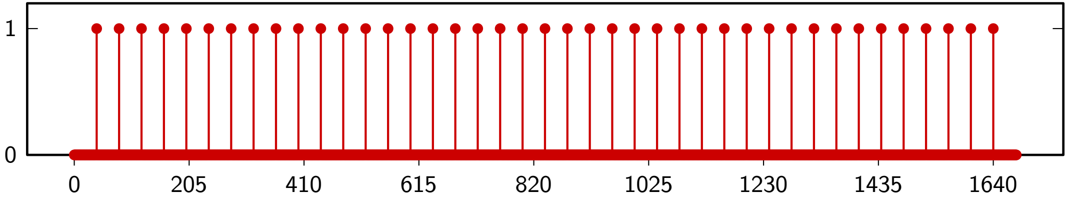

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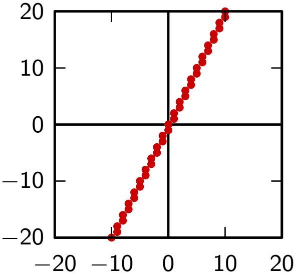

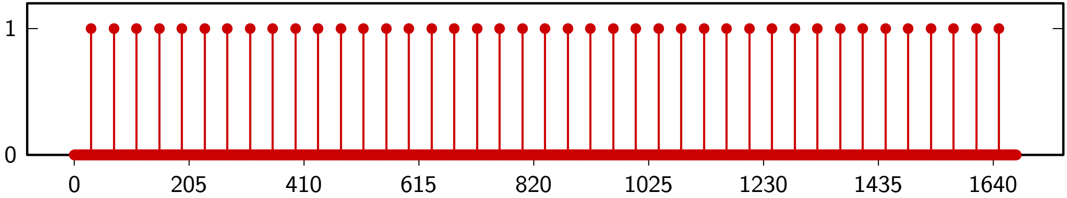

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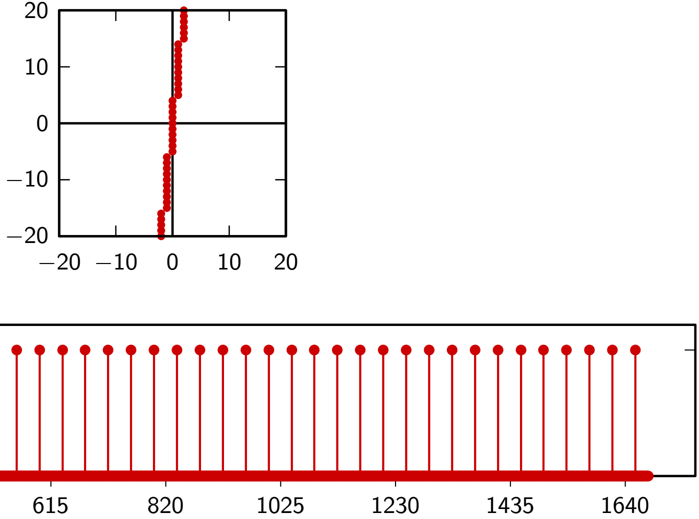

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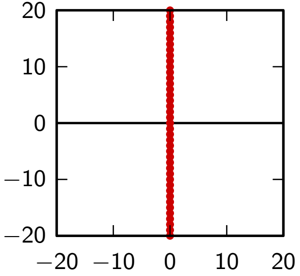
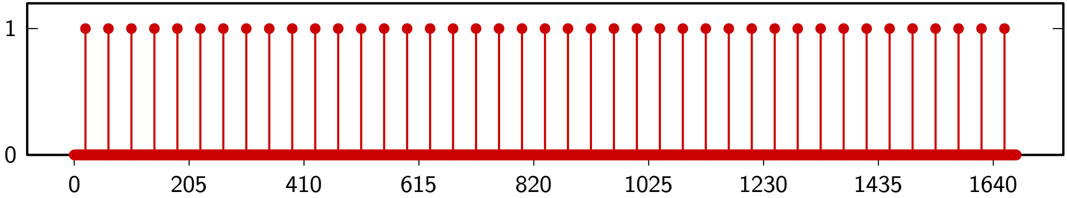
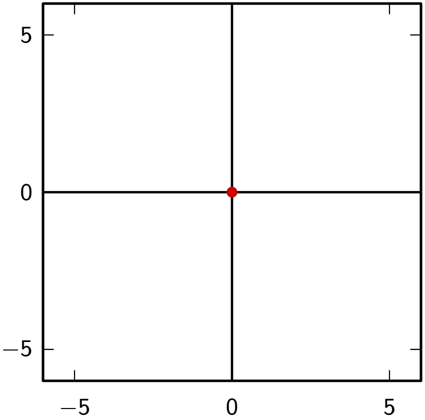




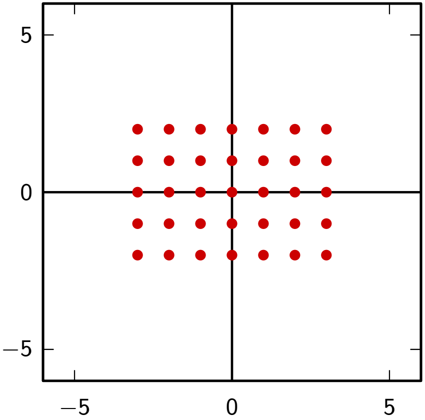
Basic 2D signals: the impulse

$$\delta[n_1, n_2] = \begin{cases} 1 & \text{if } n_1 = n_2 = 0 \\ 0 & \text{otherwise.} \end{cases}$$



Basic 2D signals: the rect

$$\text{rect}\left(\frac{n_1}{2N_1}, \frac{n_2}{2N_2}\right) = \begin{cases} 1 & \text{if } |n_1| < N_1 \\ & \text{and } |n_2| < N_2 \\ \\ 0 & \text{otherwise;} \end{cases}$$



Separability

$$x[n_1, n_2] = x_1[n_1]x_2[n_2]$$

Separable signals

$$\delta[n_1, n_2] = \delta[n_1]\delta[n_2]$$

$$\mathrm{rect}\left(\frac{n_1}{2N_1}, \frac{n_2}{2N_2}\right) = \mathrm{rect}\left(\frac{n_1}{2N_1}\right)\mathrm{rect}\left(\frac{n_2}{2N_2}\right).$$

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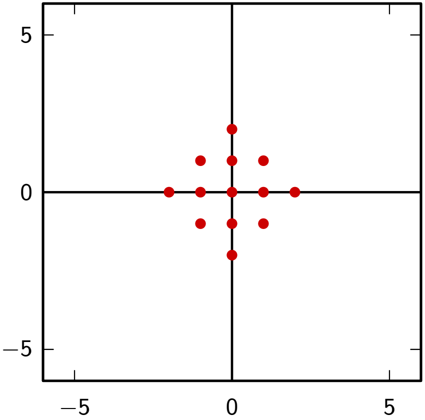
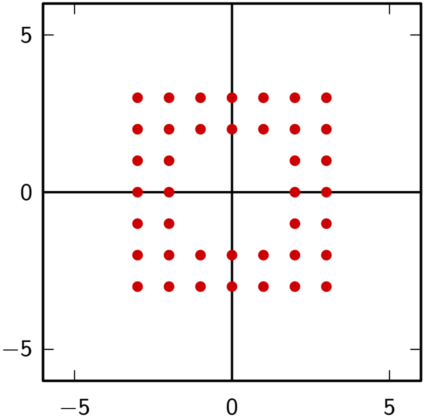
Nonseparable signal

$$x[n_1, n_2] = \begin{cases} 1 & \text{if } |n_1| + |n_2| < N \\ 0 & \text{otherwise} \end{cases}$$



Nonseparable signal

$$x[n_1, n_2] = \mathrm{rect}\left(\frac{n_1}{2N_1}, \frac{n_2}{2N_2}\right) - \mathrm{rect}\left(\frac{n_1}{2M_1}, \frac{n_2}{2M_2}\right)$$



2D convolution

$$x[n_1, n_2] * h[n_1, n_2] = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x[k_1, k_2] h[n_1 - k_1, n_2 - k_2]$$

2D convolution for separable signals

If $h[n_1, n_2] = h_1[n_1]h_2[n_2]$:

$$x[n_1, n_2] * h[n_1, n_2] = \sum_{k_1=-\infty}^{\infty} h_1[n_1 - k_1] \sum_{k_2=-\infty}^{\infty} x[k_1, k_2]h_2[n_2 - k_2]$$

$$= h_1[n_1] * (h_2[n_2] * x[n_1, n_2]).$$

2D convolution for separable signals

If $h[n_1, n_2]$ is an $M_1 \times M_2$ finite-support signal:

- non-separable convolution: $M_1 M_2$ operations per output sample
- separable convolution: $M_1 + M_2$ operations per output sample!

affine transforms

Affine transforms

mapping $\mathbb{R}^2 \to \mathbb{R}^2$ that reshapes the coordinate system (in continuous space):

$$\begin{bmatrix} t_1' \\ t_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} - \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$\begin{bmatrix} t_1' \\ t_2' \end{bmatrix} = \mathbf{A} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} - \mathbf{d}$$

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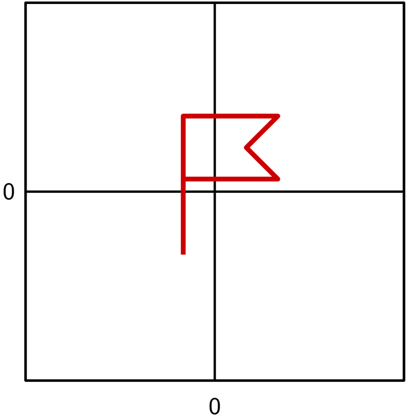
Translation

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$



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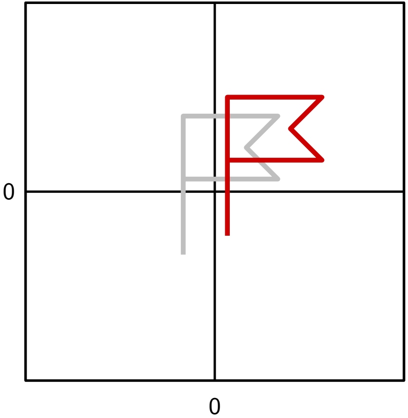
$d_1 = -0.7, d_2 = -0.3$



Scaling

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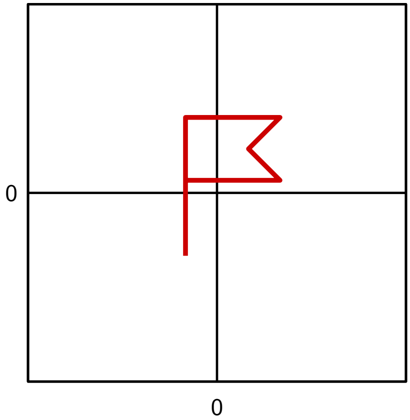

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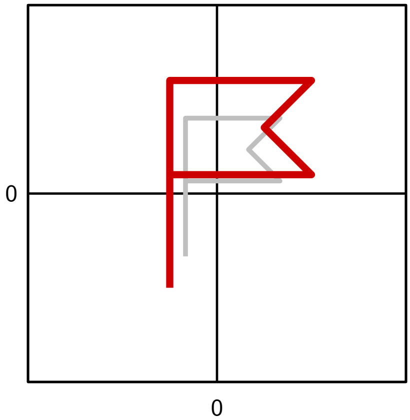
if $a_1 = a_2$ the *aspect ratio* is preserved

$a_1 = a_2 = 1.5$



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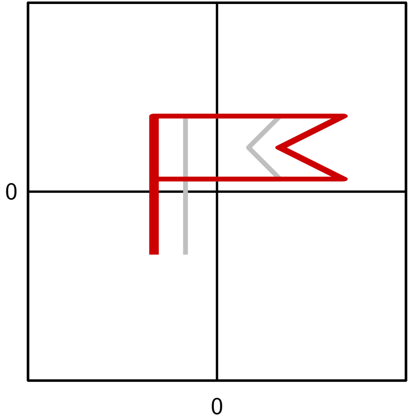
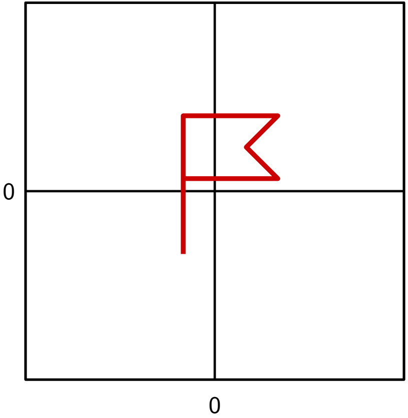
$a_1 = 2, a_2 = 1$



Rotation

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{d} = 0$$



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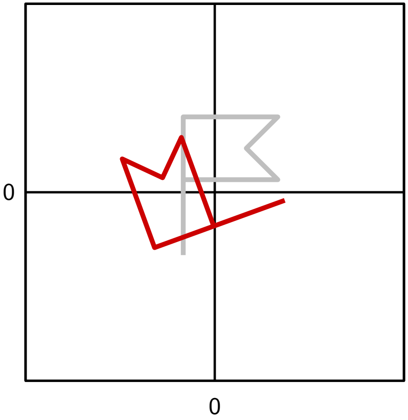
$\theta = -0.6\pi$



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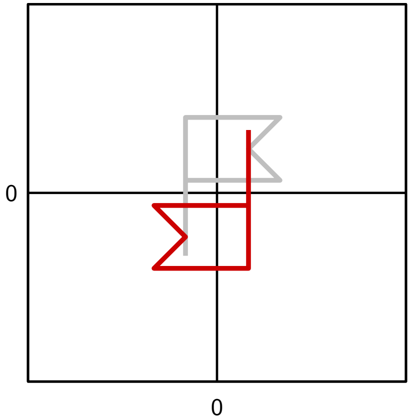
$\theta = \pi$



Flips

Horizontal:

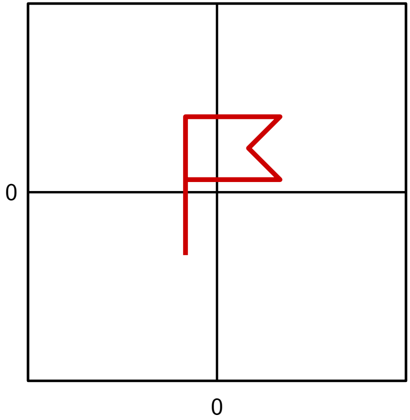
$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{d} = 0$$

Vertical:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\mathbf{d} = 0$$



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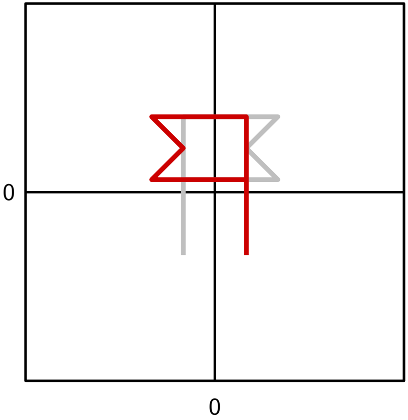


Shear

Horizontal:

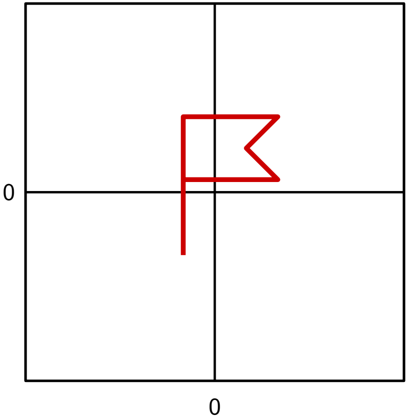
$$\mathbf{A} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{d} = 0$$

Vertical:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

$$\mathbf{d} = 0$$



Shear

Horizontal:

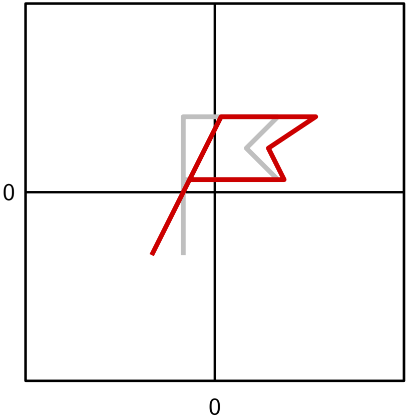
$$\mathbf{A} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{d} = 0$$

Vertical:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

$$\mathbf{d} = 0$$



Affine transforms in discrete-space

$$\begin{bmatrix} t_1' \\ t_2' \end{bmatrix} = \mathbf{A} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} - \mathbf{d} \quad \in \mathbb{R}^2 \neq \mathbb{Z}^2$$

Solution for images

- take each *output point* $y[m_1, m_2]$
- apply the *inverse* transform to $[m_1, m_2]$ and find the *source* point's coordinates:

$$\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} m_1 + d_1 \\ m_2 + d_2 \end{bmatrix};$$

- if source point not on source grid, write

$$(t_1, t_2) = (\eta_1 + \tau_1, \eta_2 + \tau_2), \qquad \eta_{1,2} \in \mathbb{Z}, \quad 0 \leq \tau_{1,2} < 1$$

  and interpolate from the surrounding original grid points

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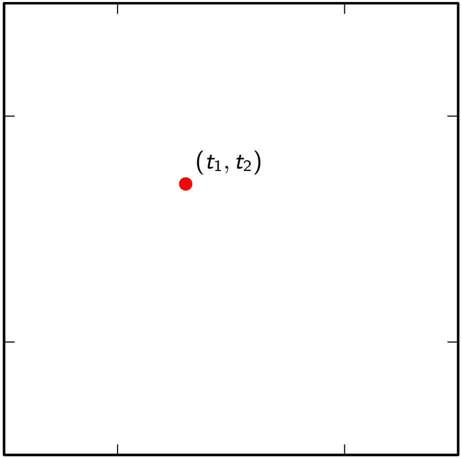
Bilinear Interpolation



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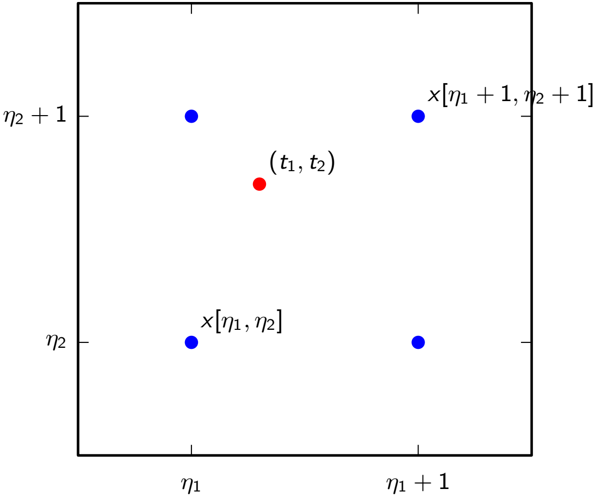

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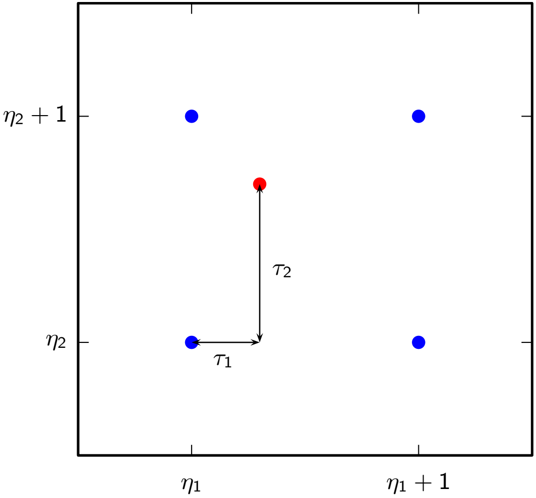

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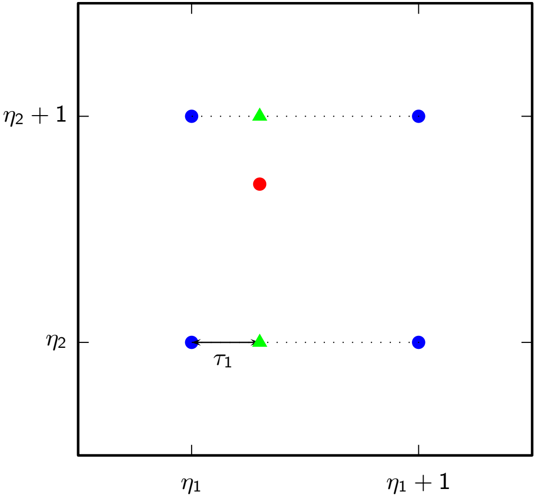

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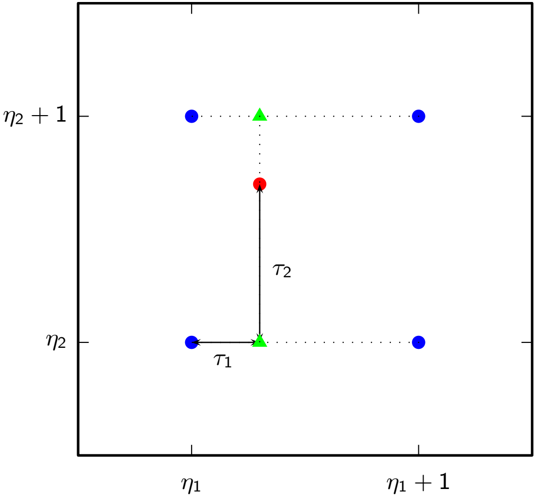

Bilinear Interpolation

If we use a first-order interpolator:

$$
\begin{aligned}
y[m_1, m_2] = {} & (1-\tau_1)(1-\tau_2)x[\eta_1, \eta_2] + \tau_1(1-\tau_2)x[\eta_1+1, \eta_2] \\
& + (1-\tau_1)\tau_2 x[\eta_1, \eta_2+1] + \tau_1\tau_2 x[\eta_1+1, \eta_2+1]
\end{aligned}
$$

Shearing

2D Fourier Analysis

2D DFT

$$X[k_1, k_2] = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[n_1, n_2] e^{-j\frac{2\pi}{N_1}n_1 k_1} e^{-j\frac{2\pi}{N_2}n_2 k_2}$$

$$x[n_1, n_2] = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} X[k_1, k_2] e^{j\frac{2\pi}{N_1}n_1 k_1} e^{j\frac{2\pi}{N_2}n_2 k_2}$$

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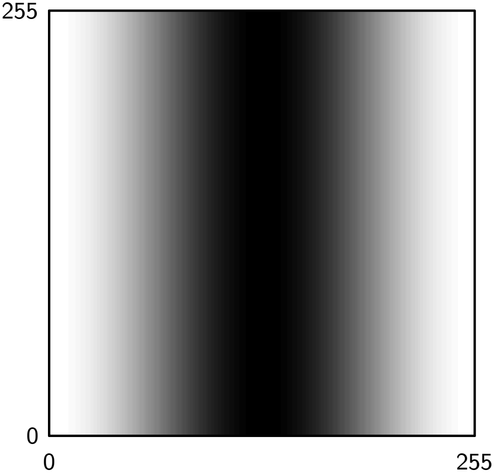
2D-DFT Basis Vectors

There are $N_1 N_2$ orthogonal basis vectors for an $N_1 \times N_2$ image:

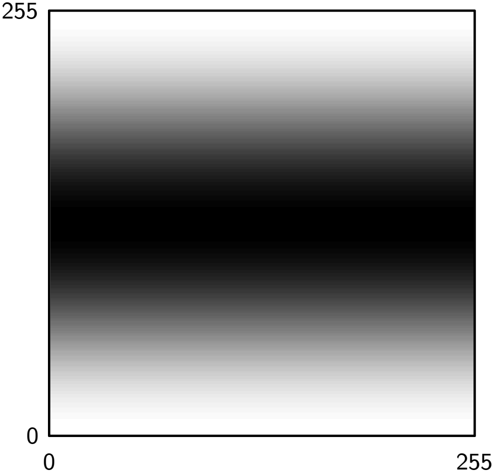
$$w_{k_1,k_2}[n_1, n_2] = e^{j\frac{2\pi}{N_1}n_1 k_1} e^{j\frac{2\pi}{N_2}n_2 k_2}$$

for $n_1, k_1 = 0, 1, \ldots, N_1 - 1$ and $n_2, k_2 = 0, 1, \ldots, N_2 - 1$

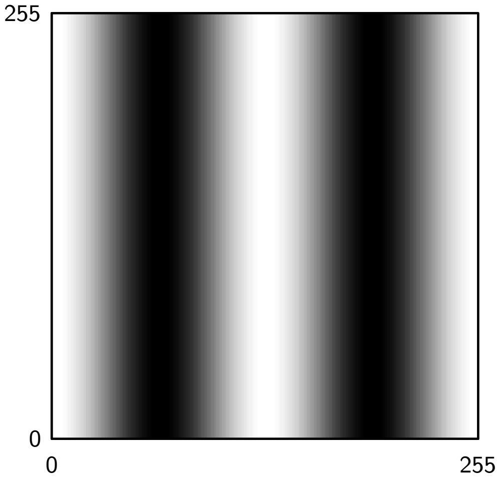
2D-DFT basis vectors for $k_1 = 1, k_2 = 0$ (real part)



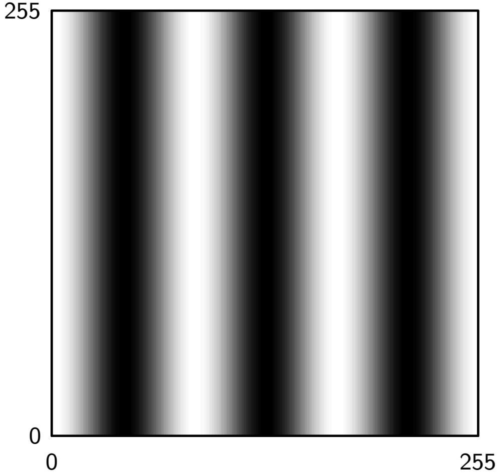
2D-DFT basis vectors for $k_1 = 0, k_2 = 1$ (real part)



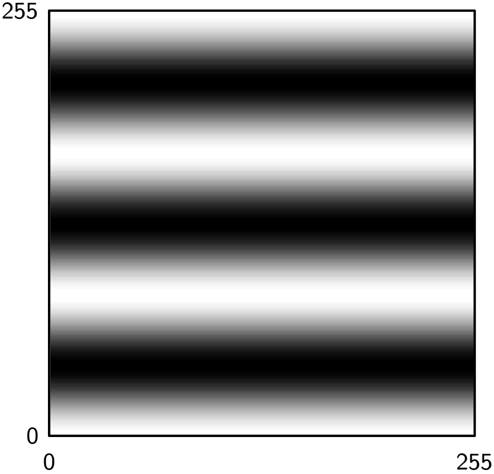
2D-DFT basis vectors for $k_1 = 2, k_2 = 0$ (real part)



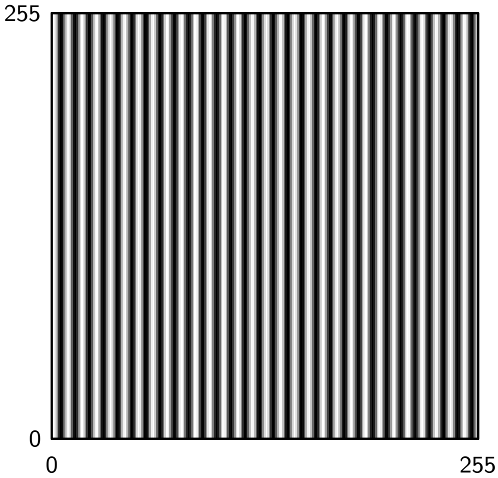
2D-DFT basis vectors for $k_1 = 3, k_2 = 0$ (real part)



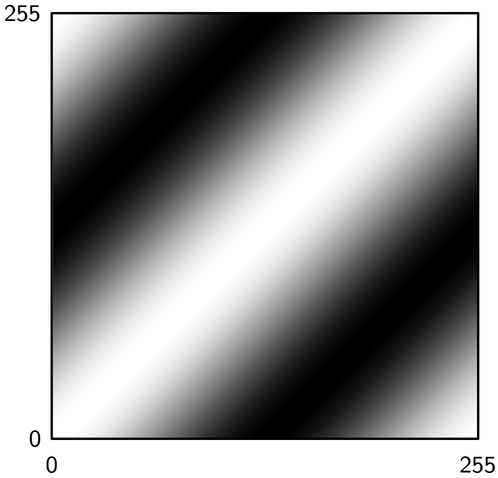
2D-DFT basis vectors for $k_1 = 0, k_2 = 3$ (real part)



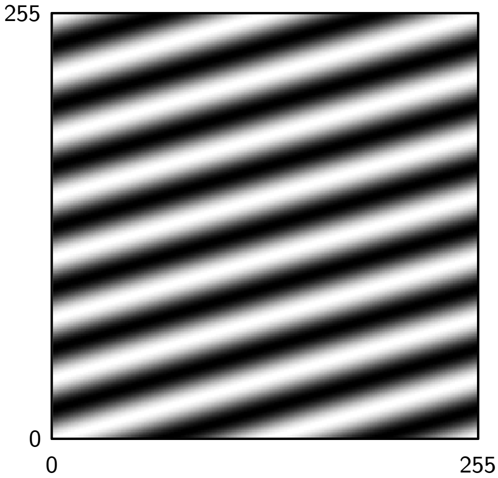
2D-DFT basis vectors for $k_1 = 30, k_2 = 0$ (real part)



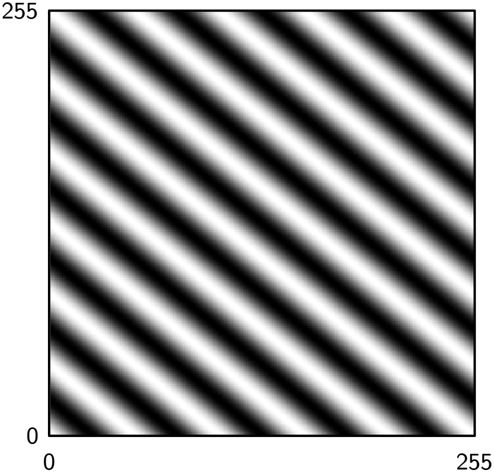
2D-DFT basis vectors for $k_1 = 1, k_2 = 1$ (real part)



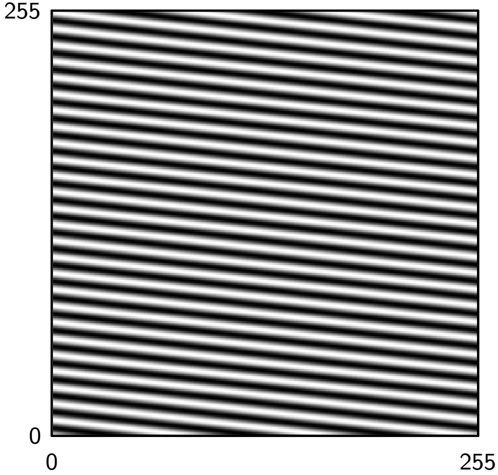
2D-DFT basis vectors for $k_1 = 2, k_2 = 7$ (real part)



2D-DFT basis vectors for $k_1 = 5, k_2 = 250$ (real part)



2D-DFT basis vectors for $k_1 = 3, k_2 = 230$ (real part)



2D DFT

2D-DFT basis functions are separable, and so is the 2D-DFT:

$$X[k_1, k_2] = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[n_1, n_2] e^{-j\frac{2\pi}{N_1} n_1 k_1} e^{-j\frac{2\pi}{N_2} n_2 k_2}$$

- 1D-DFT along n_2 (the columns)
- 1D-DFT along n_1 (the rows)

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2D DFT in matrix form

- finite-support 2D signal can be written as a matrix $\mathbf{x}$
- $N_1 \times N_2$ image is an $N_2 \times N_1$ matrix (n_1 spans the columns, n_2 spans the rows)
- recall also the $N \times N$ DFT matrix:

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & W_N^1 & W_N^2 & W^3 & \ldots & W_N^{N-1} \\ 1 & W_N^2 & W^4 & W_N^6 & \ldots & W_N^{2(N-1)} \\ & & & \ldots & & \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & W_N^{3(N-1)} & \ldots & W_N^{(N-1)^2} \end{bmatrix}$$

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$$\mathbf{V} \in \mathbb{C}^{N_2 \times N_1}$$

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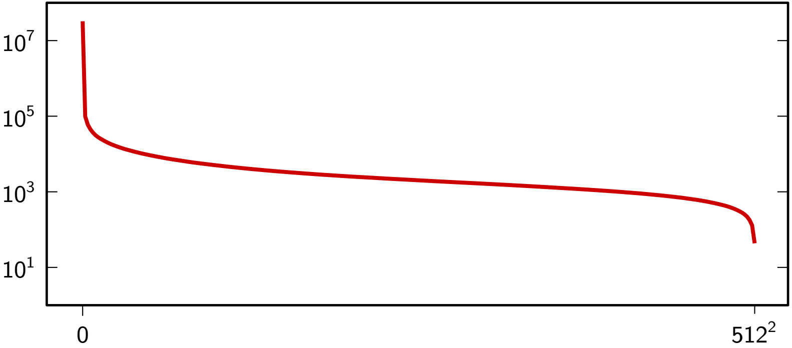
$$\mathbf{X} = \mathbf{W}_{N_2}\,\mathbf{x}\,\mathbf{W}_{N_1}$$

How does a 2D-DFT look like?

- try to show the magnitude as an image
- problem: the range is too big for the grayscale range of paper or screen
- try to normalize: $|X'[n_1, n_2]| = |X[n_1, n_2]| / \max |X[n_1, n_2]|$
- but it doesn't work...

DFT coefficients sorted by magnitude



Dealing with HDR images

if the image is high dynamic range we need to compress the levels

- remove flagrant outliers (e.g. $X[0,0] = \sum\sum x[n_1, n_2]$)
- use a nonlinear mapping: e.g. $y = x^{1/3}$ after normalization ($x \leq 1$)

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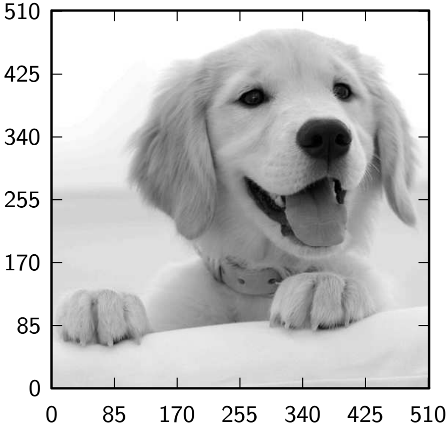
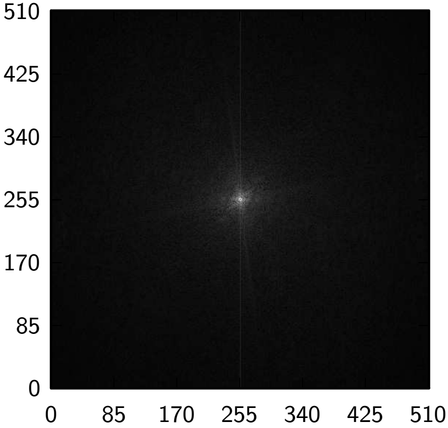
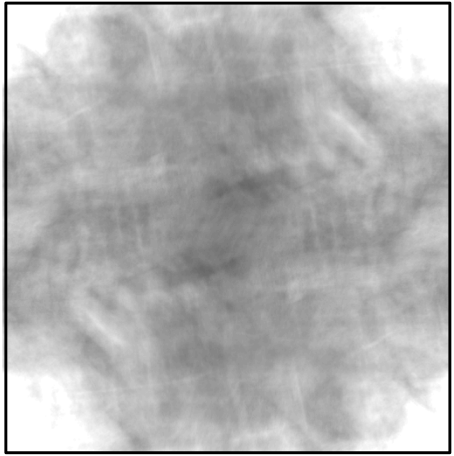




DFT magnitude doesn’t carry much information



DFT phase, on the other hand...

Image frequency analysis

- most of the information is contained in image's *edges*
- edges are points of abrupt change in signal's values
- edges are a "space-domain" feature $\rightarrow$ not captured by DFT's magnitude
- phase alignment is important for reproducing edges

image filtering

Overview:

- Filters for image processing
- Classification
- Examples

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Analogies with 1D filters

- linearity
- *space* invariance
- impulse response
- frequency response
- stability
- 2D CCDE

The problem with LSI operators

- interesting images contain lots of *semantics*: different information in different areas
- space-invariant filters process everything in the same way
- but we should process things differently
  - edges
  - gradients
  - textures
  - ...

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- IIR, FIR
- causal or noncausal
- highpass, lowpass, ...
  - lowpass $\rightarrow$ image smoothing
  - highpass $\rightarrow$ enhancement, edge detection

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- nonlinear phase (edges!)
- border effects
- stability: the fundamental theorem of algebra doesn't hold in multiple dimensions!
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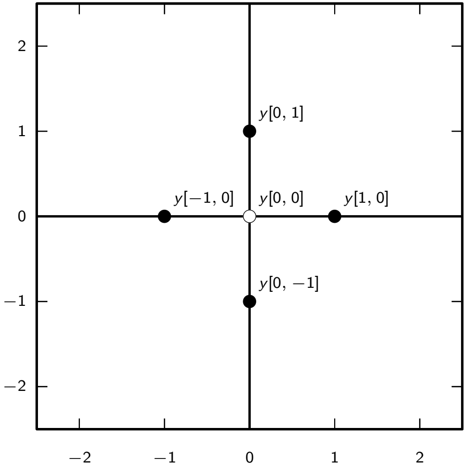
A noncomputable CCDE

$$y[n_1, n_2] = a_0 y[n_1 + 1, n_2] + a_1 y[n_1, n_2 - 1] + a_2 y[n_1 - 1, n_2] + a_3 y[n_1, n_2 + 1] + x[n_1, n_2];$$



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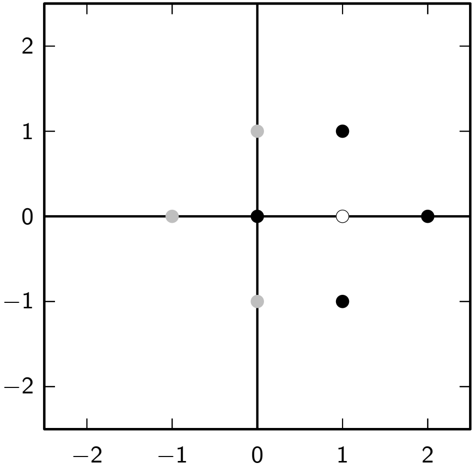

Practical FIR filters

- generally zero centered (causality not an issue) $\Rightarrow$ odd number of taps in both directions
- per-sample complexity:
  - $M_1 M_2$ for nonseparable impulse responses
  - $M_1 + M_2$ for separable impulse responses
- obviously always stable

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- obviously always stable

Practical FIR filters

- generally zero centered (causality not an issue) $\Rightarrow$ odd number of taps in both directions
- per-sample complexity:
  - $M_1 M_2$ for nonseparable impulse responses
  - $M_1 + M_2$ for separable impulse responses
- obviously always stable

Moving Average

$$y[n_1, n_2] = \frac{1}{(2N+1)^2} \sum_{k_1=-N}^{N} \sum_{k_2=-N}^{N} x[n_1 - k_1, n_2 - k_2]$$

$$h[n_1, n_2] = \frac{1}{(2N+1)^2} \text{rect}\left(\frac{n_1}{2N}, \frac{n_2}{2N}\right)$$

Moving Average

$$y[n_1, n_2] = \frac{1}{(2N+1)^2} \sum_{k_1=-N}^{N} \sum_{k_2=-N}^{N} x[n_1 - k_1, n_2 - k_2]$$

$$h[n_1, n_2] = \frac{1}{(2N+1)^2} \mathrm{rect}\left(\frac{n_1}{2N}, \frac{n_2}{2N}\right)$$

Moving Average

$$h[n_1, n_2] = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Moving Average

original

11×11 MA

Moving Average

original

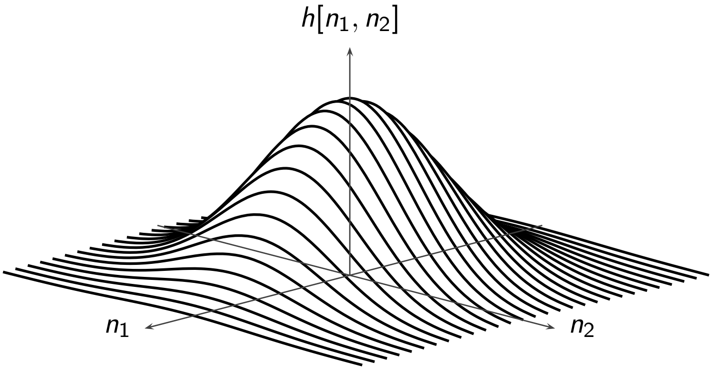
51×51 MA

Gaussian Blur

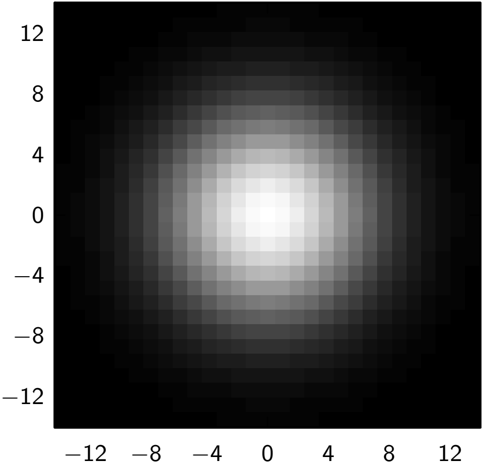
$$h[n_1, n_2] = \frac{1}{2\pi\sigma^2} e^{-\frac{n_1^2 + n_2^2}{2\sigma^2}}, \qquad |n_1, n_2| < N$$

with $N \approx 3\sigma$

Gaussian Blur



Gaussian Blur



$\sigma = 5, N = 14$

Gaussian Blur

original

$\sigma = 1.8, 11 \times 11$ blur

Gaussian Blur

original

$\sigma = 8.7, 51 \times 51$ blur

Gaussian blur more "photographic" than moving average

11×11 MA

$\sigma = 1.8, 11 \times 11$ blur

Gaussian blur more "photographic" than moving average

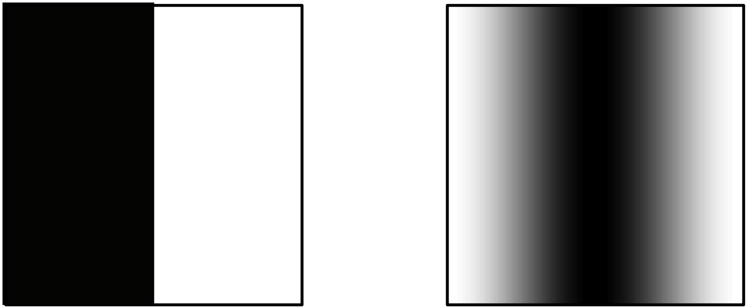
51×51 MA

$\sigma = 8.7, 51 \times 51$ blur

Filters for edge detection

What is an edge? Very complicated but, simplifying:

- points of "discontinuity" in intensity
- points of inflection in intensity



Sobel filter

Goal: find points where derivative is large.

$$\nabla f(t_1, t_2) = \begin{bmatrix} \dfrac{\partial f}{\partial t_1} & \dfrac{\partial f}{\partial t_2} \end{bmatrix}^T$$

Sobel filter

approximating the first derivative on the discrete grid in a circularly symmetric way:

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x & x_5 \\ x_6 & x_7 & x_8 \end{bmatrix}$$

$$\mathbf{x}' \approx \sum_{i=1}^{8} \frac{x - x_i}{d(x, x_i)} \mathbf{x}_i$$

Sobel filter

approximating the first derivative on the discrete grid in a circularly symmetric way:

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x & x_5 \\ x_6 & x_7 & x_8 \end{bmatrix}$$

$$\mathbf{x}' \approx \sum_{i=1}^{8} \frac{x - x_i}{d(x, x_i)} \mathbf{x}_i$$

Sobel filter

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x & x_5 \\ x_6 & x_7 & x_8 \end{bmatrix}$$

$$d(x, x_i) = \begin{cases} 2 & \text{for } i = 2, 4, 5, 7 \\ 4 & \text{for } i = 1, 3, 6, 8 \end{cases}$$

$$\mathbf{x}_1 = \begin{bmatrix} -1 & 1 \end{bmatrix}^T, \quad \mathbf{x}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T, \quad \dots, \quad \mathbf{x}_8 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$$

Sobel filter

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x & x_5 \\ x_6 & x_7 & x_8 \end{bmatrix}$$

$$4\nabla x \approx \begin{bmatrix} x_3 - x_1 + 2(x_4 - x_8) + x_5 - x_7 \\ x_7 - x_1 + 2(x_6 - x_2) + x_5 - x_3 \end{bmatrix}$$

Sobel filter

approximation of the first derivative in the horizontal direction:

$$s_h[n_1, n_2] = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

approximation of the first derivative in the vertical direction:

$$s_v[n_1, n_2] = \begin{bmatrix} -1 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Sobel filter

approximation of the first derivative in the horizontal direction:

$$s_h[n_1, n_2] = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

approximation of the first derivative in the vertical direction:

$$s_v[n_1, n_2] = \begin{bmatrix} -1 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

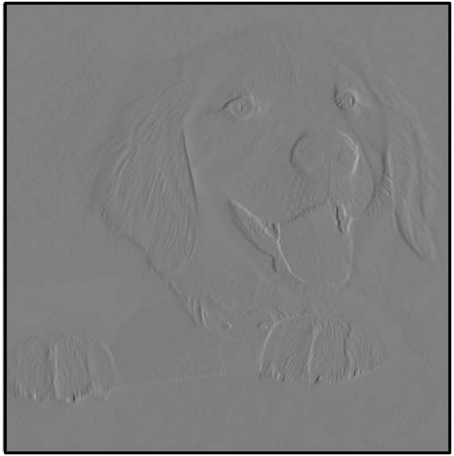
Sobel filter

filter is separable, e.g.:

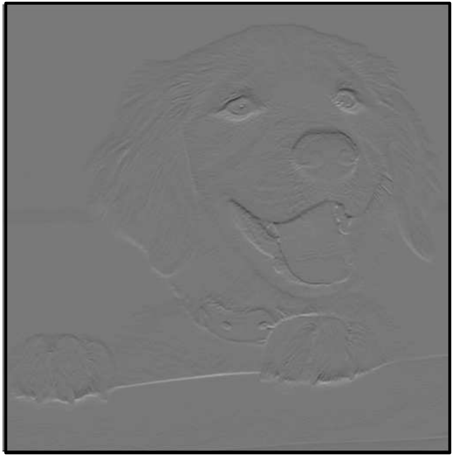
$$s_h[n_1, n_2] = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$$

horizontal gradient = vertical averaging followed by horizontal differentiation

Sobel filter



horizontal Sobel filter



vertical Sobel filter

Sobel operator

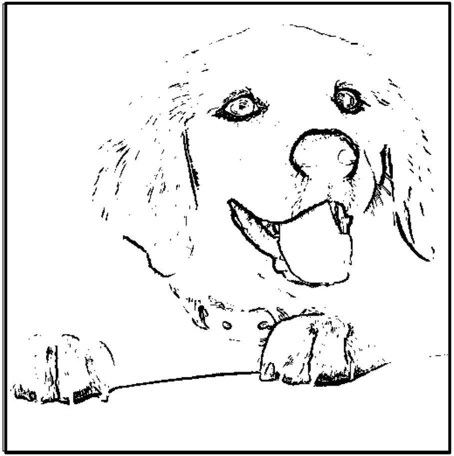
approximation for the square magnitude of the gradient:

$$|\nabla x[n_1, n_2]|^2 = |s_h[n_1, n_2] * x[n_1, n_2]|^2 + |s_v[n_1, n_2] * x[n_1, n_2]|^2$$

("operator" because it's nonlinear)

Gradient approximation for edge detection

Sobel operator



thresholeded Sobel operator

Laplacian operator

Laplacian of a function in continuous-space:

$$\Delta f(t_1, t_2) = \frac{\partial^2 f}{\partial t_1^2} + \frac{\partial^2 f}{\partial t_2^2}$$

Laplacian operator

approximating the Laplacian; start with a Taylor expansion

$$f(t+\tau) = \sum_{n=0}^{\infty} \frac{f^{(n)}(t)}{n!} \tau^n$$

and compute the expansion in $(t+\tau)$ and $(t-\tau)$:

$$f(t+\tau) = f(t) + f'(t)\tau + \frac{1}{2} f''(t)\tau^2$$

$$f(t-\tau) = f(t) - f'(t)\tau + \frac{1}{2} f''(t)\tau^2$$

Laplacian operator

by rearranging terms:

$$f''(t) = \frac{1}{\tau^2}(f(t-\tau) - 2f(t) + f(t+\tau))$$

which, on the discrete grid, is the FIR $h[n] = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$

Laplacian

summing the horizontal and vertical components:

$$h[n_1, n_2] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

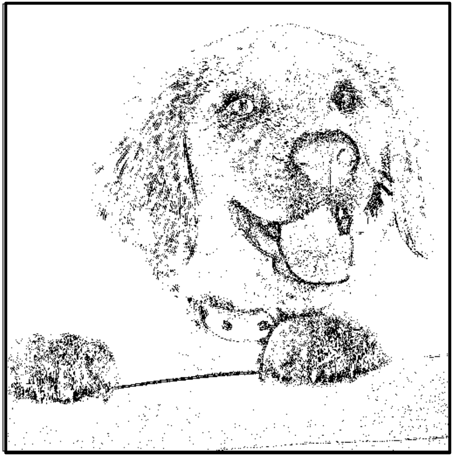
Laplacian

If we use the diagonals too:

$$h[n_1, n_2] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Laplacian for Edge Detection

Laplacian operator



thresholeded Laplacian operator