COM303: Digital Signal Processing

Lecture 7: The DTFT Formalism

Overview:

- the DTFT of non square-summable sequences
- relationships between transforms
- modulation

the DTFT formalism for non ℓ_2 sequences

The path to the DTFT

- DFT: simple change of basis in $\mathbb{C}^N$
- DFS: same, with N-periodicity explicit

Review: DFT

$$
\begin{tikzcd}[column sep=huge]
\mathbb{C}^N \arrow[r, bend left, "X[k] = \langle e^{j\frac{2\pi}{N}nk}, x[n] \rangle"] \arrow[r, phantom, "\text{basis: } \{e^{j\frac{2\pi}{N}nk}\}_k"] & \mathbb{C}^N \arrow[l, bend left, "x[n] = (1/N)\sum X[k]\, e^{j\frac{2\pi}{N}nk}"]
\end{tikzcd}
$$

Review: DFS

$$\tilde{X}[k] = \langle e^{j\frac{2\pi}{N}nk}, \tilde{x}[n]\rangle$$

$$\begin{tikzcd}
\tilde{\mathbb{C}}^N \arrow[r, bend left] & \tilde{\mathbb{C}}^N \arrow[l, bend left]
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basis: $\{e^{j\frac{2\pi}{N}nk}\}_k$

$$\tilde{x}[n] = (1/N)\sum \tilde{X}[k]\, e^{j\frac{2\pi}{N}nk}$$

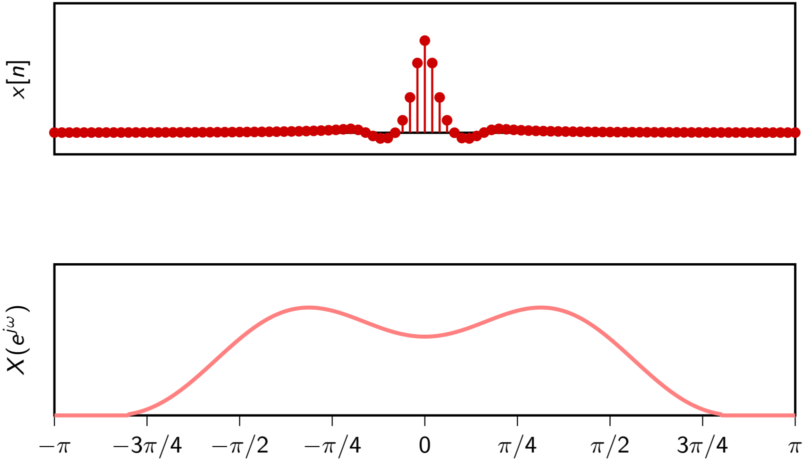
The path to the DTFT

- DFT: simple change of basis in $\mathbb{C}^N$
- DFS: same, with N-periodicity explicit
- when $N \rightarrow \infty$:
  - DFT (informally): $\displaystyle\sum_n x[n]e^{j\frac{2\pi}{N}kn} \rightarrow \sum_n x[n]e^{j\omega n}$
  - DFS (formally): $\displaystyle\sum_{n=0}^{N-1} \tilde{x}_N[n]e^{j\frac{2\pi}{N}kn} = \sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}\Big|_{\omega=\frac{2\pi}{N}k}$

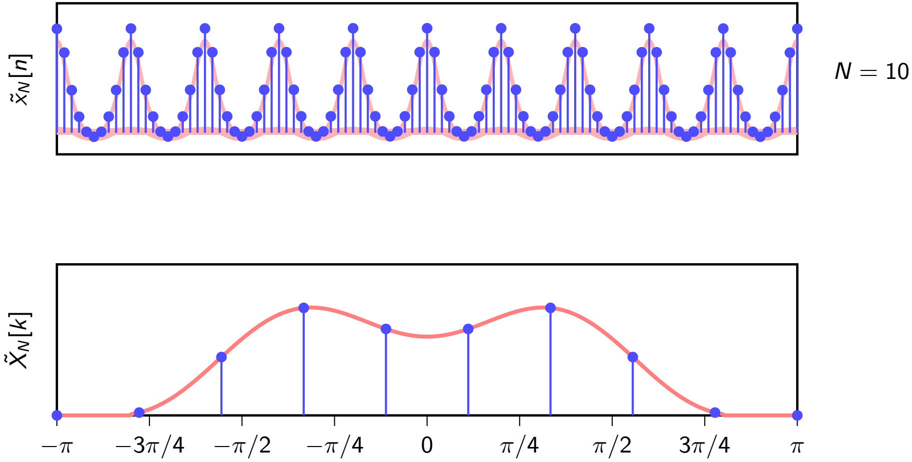
Discrete-Time Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

From DFS to DTFT



From DFS to DTFT



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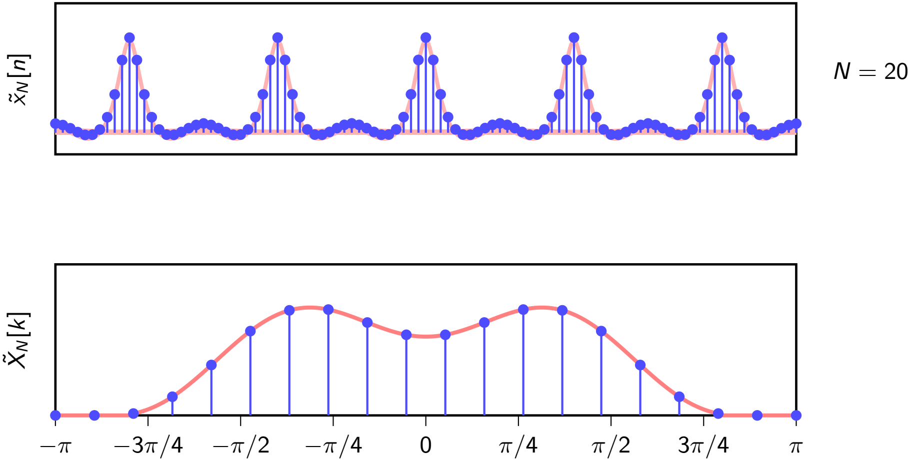

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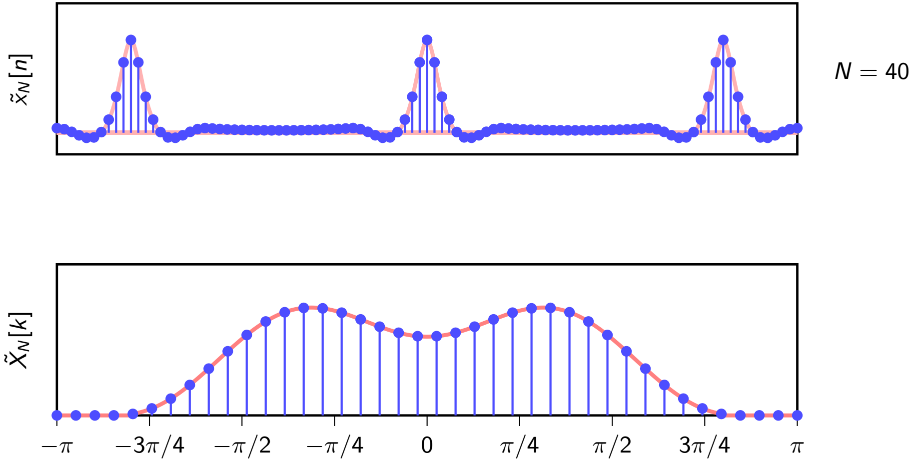

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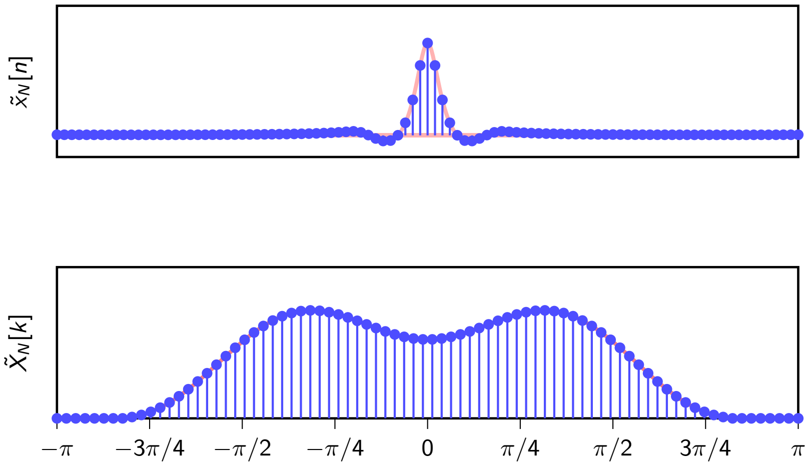

$N = 80$

Discrete-Time Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \qquad \omega \in [-\pi, \pi]$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega, \qquad n \in \mathbb{Z}$$

DTFT

$$\begin{tikzcd}
\ell_2(\mathbb{Z}) \arrow[rr, bend left, "X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}"] & \text{``basis'':}\ \{e^{j\omega n}\}_{\omega} & L_2([-\pi, \pi]) \arrow[ll, bend left, "x[n] = (1/2\pi) \int X(e^{j\omega}) e^{j\omega n} d\omega"]
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Inverse DTFT as a basis expansion

- DTFT is an (invertible) mapping from $\ell_2(\mathbb{Z})$ to $L_2([-\pi, \pi])$
- $L_2([-\pi, \pi])$ is a Hilbert space
- the set of 2π-periodic functions $\{e^{-j\omega n}\}_n$ is an orthogonal basis for $L_2([-\pi, \pi])$:

$$\langle e^{-j\omega n}, e^{-j\omega m})\rangle = 2\pi\delta[n-m]$$

- the inverse DTFT is a basis expansion; the analysis coefficients are the time-domain values:

$$x[n] \propto \langle e^{-j\omega n}, X(e^{j\omega n})\rangle$$

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DTFT as a formal basis expansion

- DTFT can be seen as an inner product

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = \langle e^{j\omega n}, x[n] \rangle$$

- the set $\{e^{j\omega n}\}_{\omega}$ is not countable
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DTFT as basis expansion

Some things are OK:

- $\text{DFT}\{\delta[n]\} = 1$
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DTFT as basis expansion

Some things aren't:

- $\mathrm{DFT}\{1\} = \delta[n]$
- $\mathrm{DTFT}\{1\} = \sum_{n=-\infty}^{\infty} e^{-j\omega n} = ?$

- problem: too many interesting sequences are *not* square summable!

The Dirac delta

The Dirac delta functional

the functional $\delta(t)$ is defined by its "sifting" property:

$$\int_{-\infty}^{\infty} \delta(t-s)f(t)dt = f(s)$$

$$\text{for all } f(t), \quad t \in \mathbb{R}.$$

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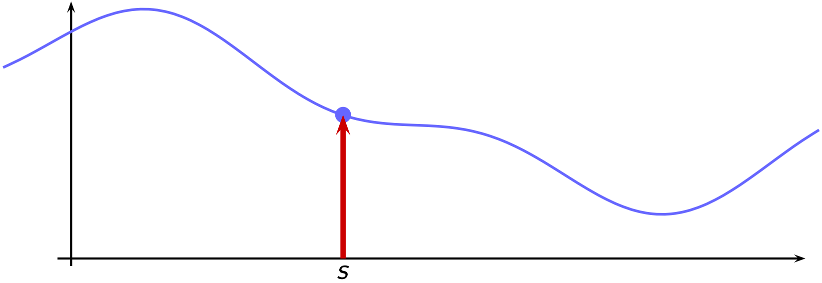
$$\int_{s-\delta}^{s+\delta} \delta(t-s)f(t)dt = f(s) \qquad \forall \delta \in \mathbb{R}^+$$

for all $f(t), \quad t \in \mathbb{R}.$

The Dirac delta functional



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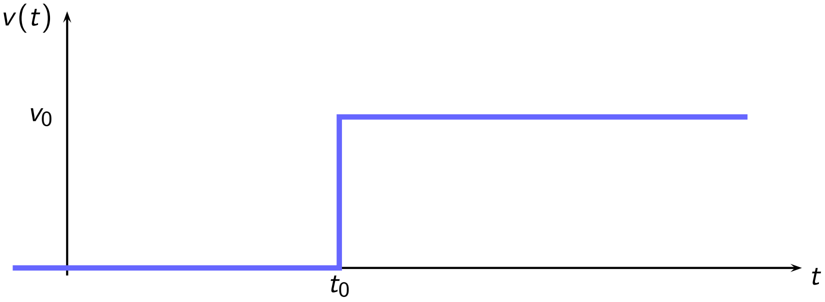
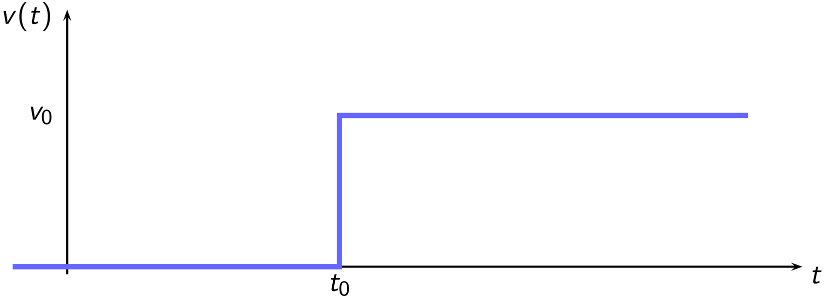
The Dirac delta functional in physics



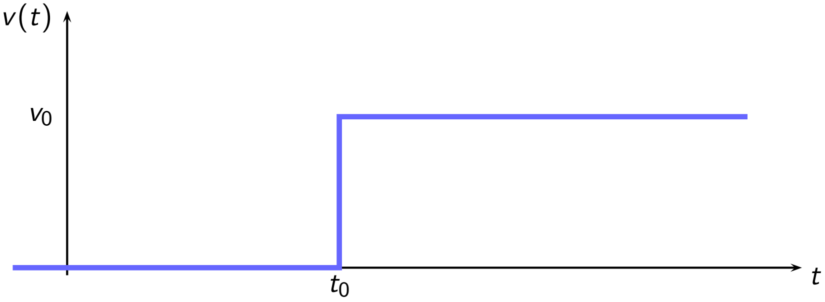
$$F(t) = ma(t) = m\frac{\partial v(t)}{\partial t}$$

The Dirac delta functional in physics



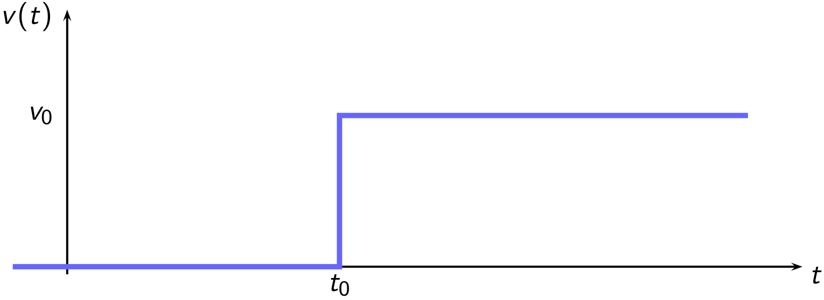
$$a(t_0) = \infty?$$

The Dirac delta functional in physics



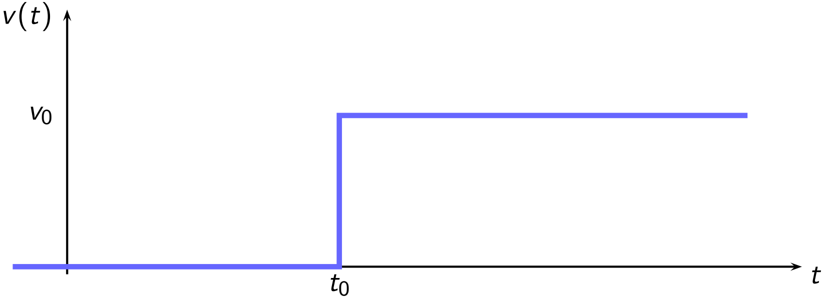
from the other side: $v(t) = \int_{-\infty}^{t} a(\tau) d\tau = \begin{cases} 0 & \text{for } t < t_0 \\ v_0 & \text{for } t > t_0 \end{cases}$

The Dirac delta functional in physics



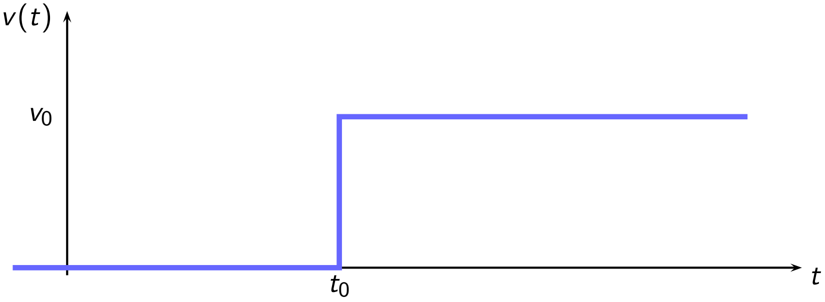
from the other side: $v(t) = \int_{-\infty}^{t} v_0 \delta(\tau - t_0) d\tau$

The Dirac delta functional in physics



$$a(t) = v_0 \delta(t - t_0)$$

The Dirac delta functional in physics



$$F(t) \propto \delta(t - t_0) \approx \begin{cases} \infty & \text{for } t = t_0 \\ 0 & \text{otherwise} \end{cases}$$

Intuition

consider a family of *localizing* functions $r_k(t)$ with $k \in \mathbb{N}$ and $t \in \mathbb{R}$ where:

- support inversely proportional to k
- constant area

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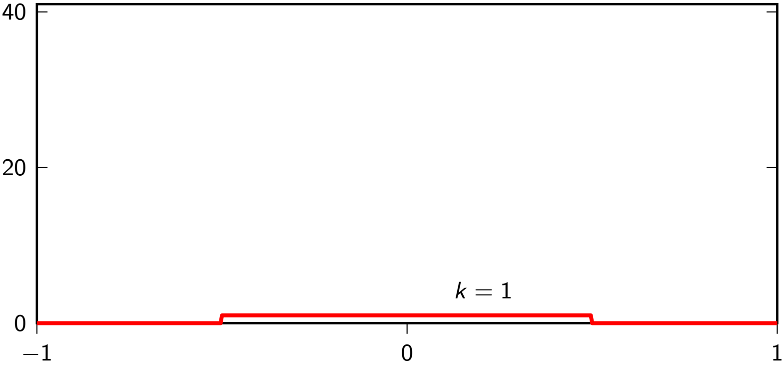
Intuition

$$\mathrm{rect}(t) = \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

we can build a localizing family as $r_k(t) = k\,\mathrm{rect}(kt)$:

- nonzero over $[-1/2k, 1/2k]$, i.e. support is $1/k$
- area is 1

The family $r_k(t) = k\,\mathrm{rect}(kt)$



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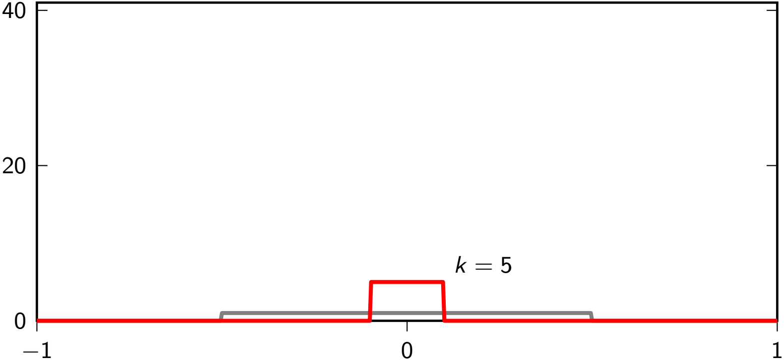

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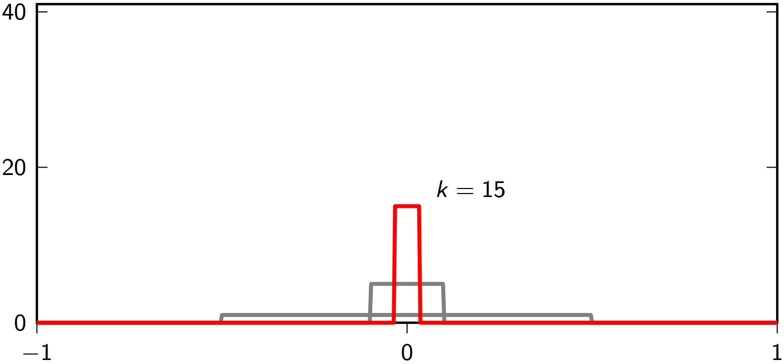

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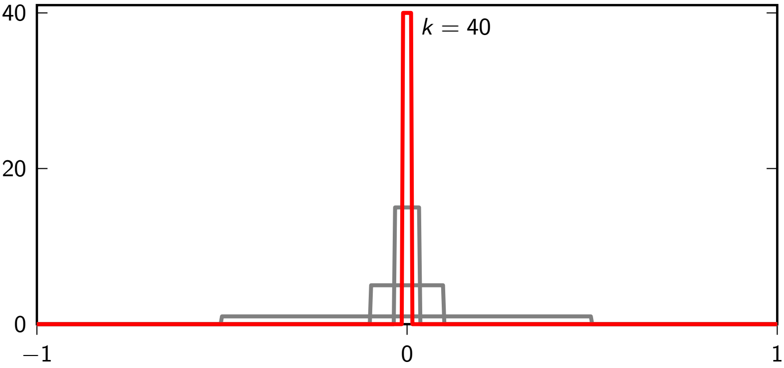
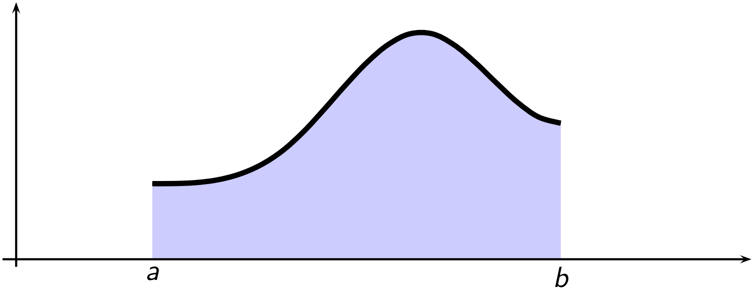


Remember the Mean Value Theorem?

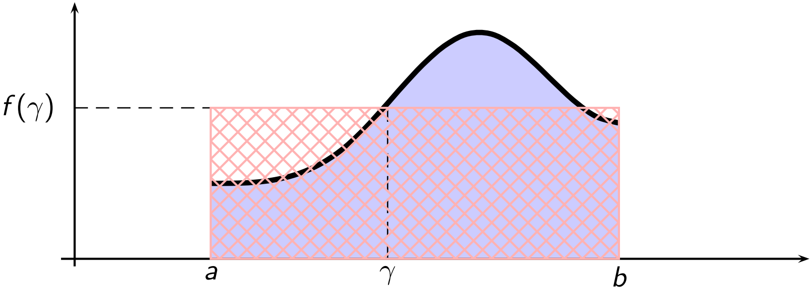
for any continuous function over the interval $[a, b]$ there exists $\gamma \in [a, b]$ s.t.

$$\int_a^b f(t)dt = (b - a)\, f(\gamma)$$

The Mean Value Theorem



The Mean Value Theorem



Extracting a point value

for our family of localizing functions:

$$\int_{-\infty}^{\infty} r_k(t)f(t)dt = k\int_{-1/2k}^{1/2k} f(t)dt$$
$$= f(\gamma)|_{\gamma\in[-1/2k,1/2k]}$$

and so:

$$\lim_{k\to\infty}\int_{-\infty}^{\infty} r_k(t)f(t)dt = f(0)$$

The Dirac delta functional

The delta functional is a shorthand. Instead of writing

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} r_k(t-s) f(t) dt$$

we write

$$\int_{-\infty}^{\infty} \delta(t-s) f(t) dt.$$

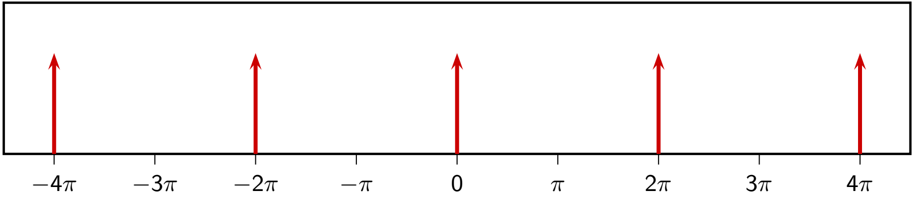
as if $\lim_{k \to \infty} r_k(t) = \delta(t)$,

The "pulse train"

$$\tilde{\delta}(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

just a technicality to use the Dirac delta in the space of 2π-periodic functions

Graphical representation



Now let the show begin!

$$\text{IDTFT}\left\{\tilde{\delta}(\omega)\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{\delta}(\omega)e^{j\omega n}d\omega$$

$$= \int_{-\pi}^{\pi}\delta(\omega)e^{j\omega n}d\omega$$

$$= e^{j\omega n}\big|_{\omega=0}$$

$$= 1$$

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In other words

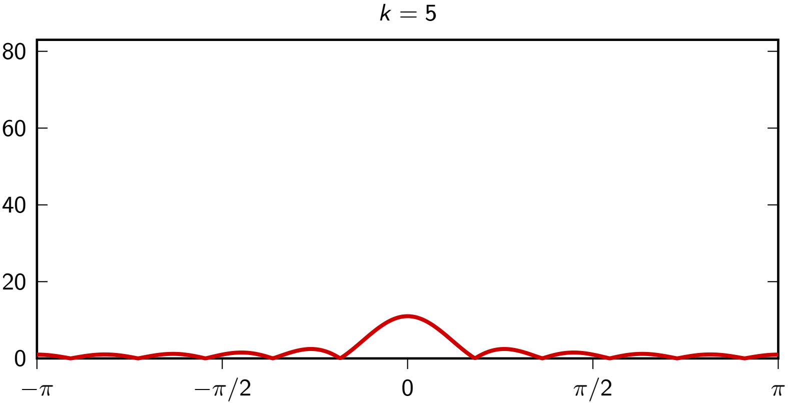
$$\mathsf{DTFT}\left\{1\right\} = \tilde{\delta}(\omega)$$

Does it make sense?

Partial DTFT sum:

$$S_k(\omega) = \sum_{n=-k}^{k} e^{-j\omega n}$$

Plotting $|S_k(\omega)|$



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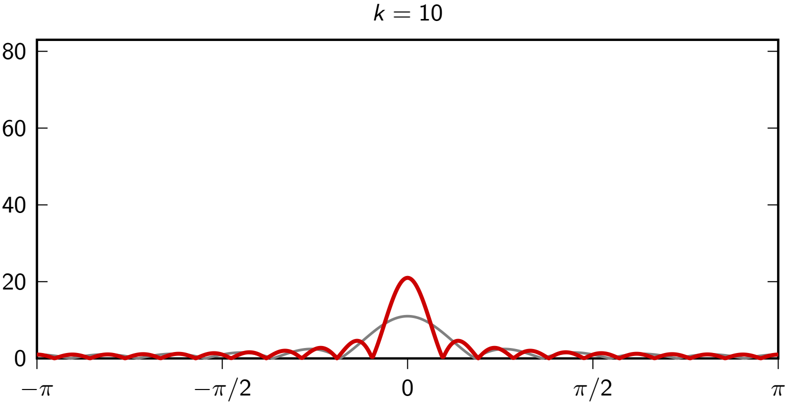

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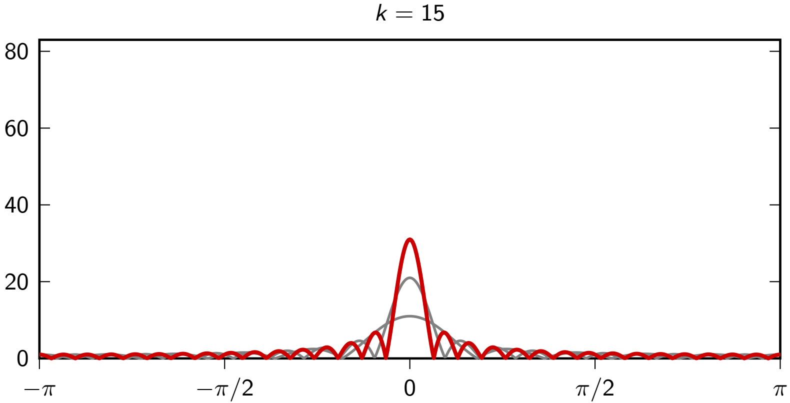

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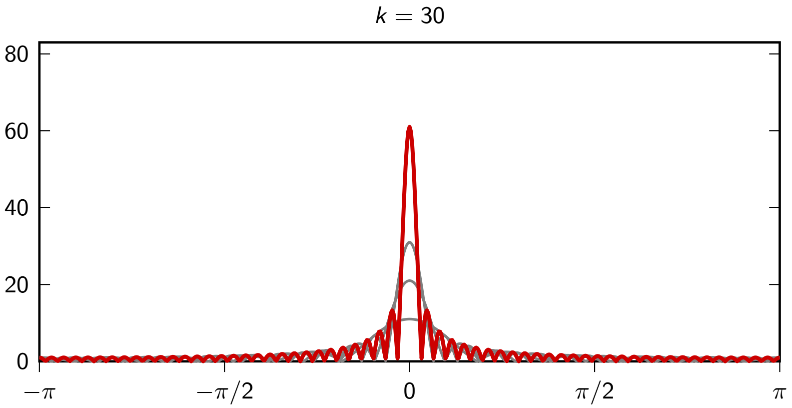

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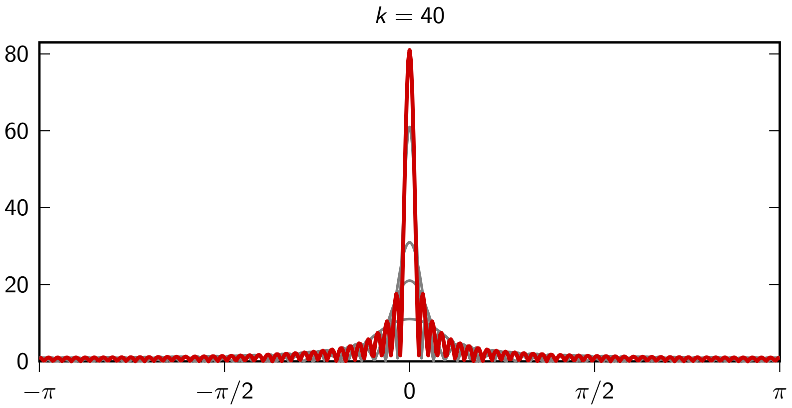

Does it make sense?

Partial DTFT sums look like a family of localizing functions:

$$S_k(\omega) \to \tilde{\delta}(\omega)$$

Using the same technique

$$\text{IDTFT}\left\{\tilde{\delta}(\omega - \omega_0)\right\} = e^{j\omega_0 n}$$

So:

- $\text{DTFT}\{1\} = \tilde{\delta}(\omega)$
- $\text{DTFT}\left\{e^{j\omega_0 n}\right\} = \tilde{\delta}(\omega - \omega_0)$
- $\text{DTFT}\{\cos \omega_0 n\} = [\tilde{\delta}(\omega - \omega_0) + \tilde{\delta}(\omega + \omega_0)]/2$
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Warning: use with caution!

- Dirac delta in the DTFT $\Rightarrow$ signal is NOT finite-energy (eg. periodic, constant etc)
- signal must still be a power signal (finite energy over finite sections)
- Dirac deltas make sense only if integrals are involved; where are the integrals?
  - $\bar{w}[n] = 1$ only for $M \leq n \leq N \Rightarrow \text{DTFT}\left\{x[n]\bar{w}[n]\right\} = \int_{-\pi}^{\pi} X(e^{j(\sigma-\omega)})W(e^{j\sigma})d\sigma$: OK

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relationships between transforms

Overview:

- DFT, DFS, DTFT
- DTFT of periodic sequences
- DTFT of finite-support sequences
- Zero padding

Transforms

- DFT, DFS: change of basis in $\mathbb{C}^N$
- DTFT: "formal" change of basis in $\ell_2(\mathbb{Z})$
- basis vectors are "building blocks" for any signal

- DFT: numerical algorithm (computable)
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- basis vectors are "building blocks" for any signal

- DFT: numerical algorithm (computable)
- DTFT: mathematical tool (proofs)

Transforms

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- DTFT: mathematical tool (proofs)

Embedding finite-length signals

- N-tap signal $x[n]$
- natural spectral representation: DFT $X[k]$
- two ways to embed $x[n]$ into an infinite sequence:
  - periodic extension: $\tilde{x}[n] = x[n \mod N]$
  - finite-support extension: $\bar{x}[n] = \begin{cases} x[n] & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$
- how does $X[k]$ relate to the DTFT of the embedded signals?

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DTFT of periodic signals

$$\tilde{x}[n] = x[n \mod N]$$

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \tilde{x}[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\omega n}$$

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\end{aligned}$$

We've seen this before

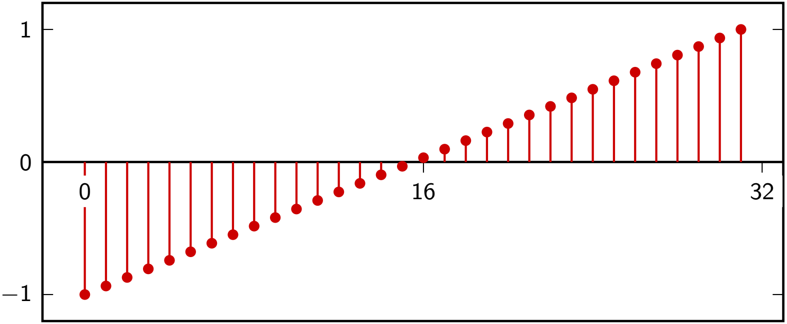
$$\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{N}nk} e^{-j\omega n} = \text{DTFT}\left\{e^{j\frac{2\pi}{N}nk}\right\}$$

$$= \tilde{\delta}(\omega - \frac{2\pi}{N}k)$$

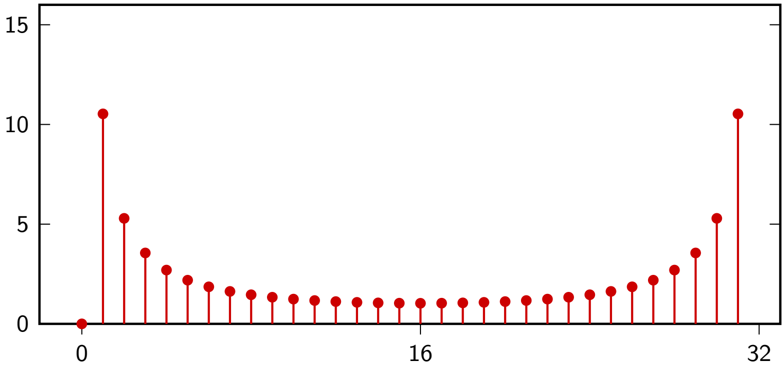
DTFT of periodic signals

$$\tilde{X}(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\tilde{\delta}(\omega - \frac{2\pi}{N}k)$$

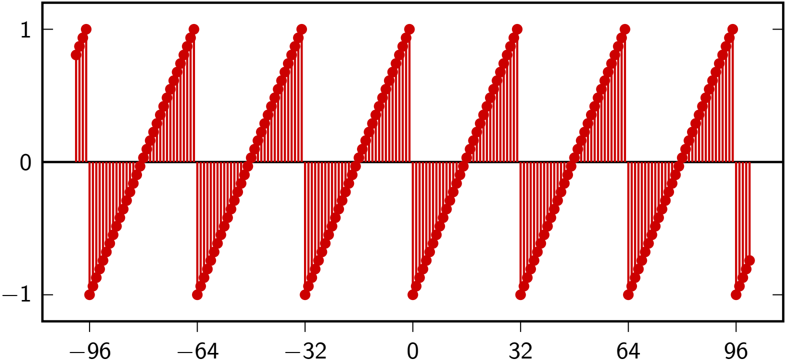
32-tap sawtooth



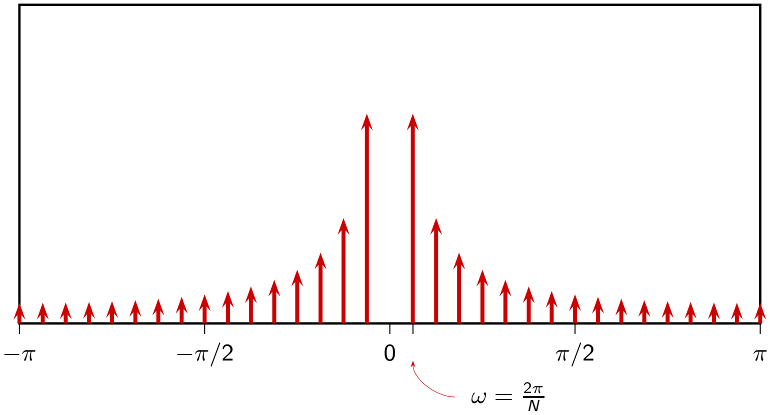
DFT of 32-tap sawtooth



32-periodic sawtooth



DTFT of periodic extension



DTFT of finite-support signals

$$\bar{x}[n] = \begin{cases} x[n] & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

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\bar{X}(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \bar{x}[n] e^{-j\omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \\
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&= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \left(\sum_{n=0}^{N-1} e^{-j(\omega - \frac{2\pi}{N}k)n} \right)
\end{aligned}$$

DTFT of finite-support signals

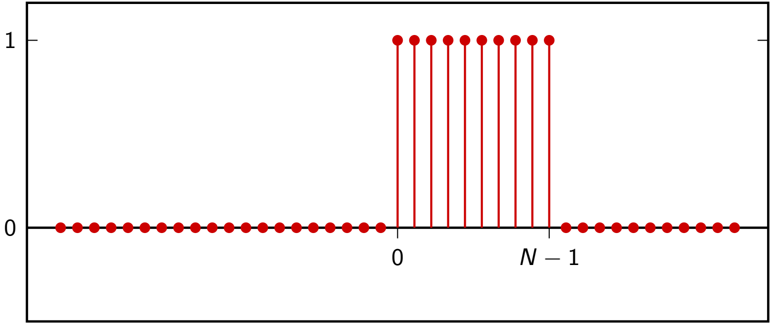
$$\sum_{n=0}^{N-1} e^{-j(\omega - \frac{2\pi}{N}k)n} = \bar{R}(e^{j(\omega - \frac{2\pi}{N}k)})$$

where $\bar{R}(e^{j\omega})$ is the DTFT of $\bar{r}[n]$, the rectangular signal:

$$\bar{r}[n] = \begin{cases} 1 & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

Rectangular step signal

$$\bar{r}[n] = \begin{cases} 1 & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$



DTFT of rectangular step signal

$$
\begin{aligned}
\bar{R}(e^{j\omega}) &= \sum_{n=0}^{N-1} e^{-j\omega n} \\
&= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \\
&= \frac{e^{-j\frac{\omega N}{2}}\left[e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}}\right]}{e^{-j\frac{\omega}{2}}\left[e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right]} \\
&= \frac{\sin\left(\frac{\omega}{2}N\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\frac{\omega}{2}(N-1)}
\end{aligned}
$$

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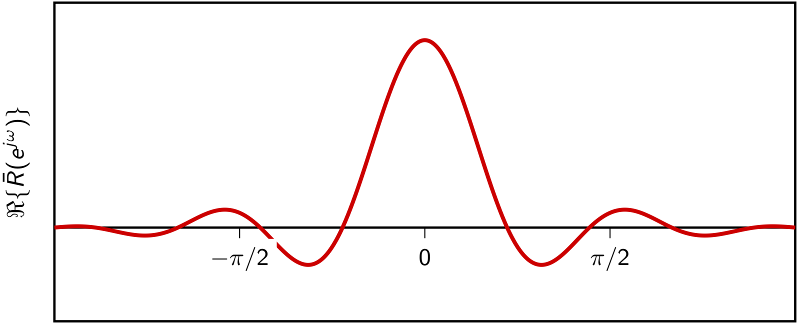
&= \frac{\sin\left(\frac{\omega}{2} N\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\frac{\omega}{2}(N-1)}
\end{aligned}
$$

DTFT of interval signal ($N = 9$)



note that $\bar{R}(e^{j\omega}) = 0$ for $\omega = 2\pi k/N$, $k \in \mathbb{Z}/\{0\}$

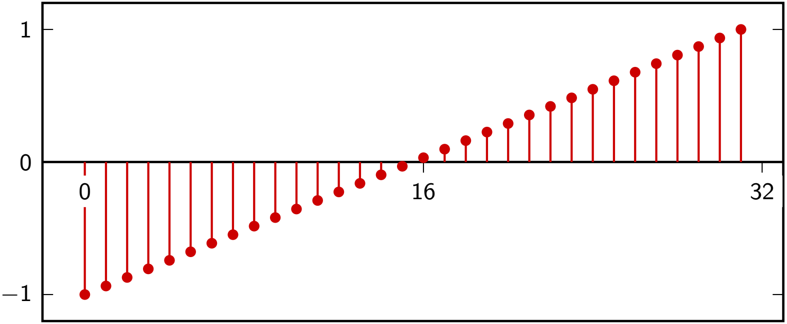
DTFT of finite-support signals

define $\Lambda(\omega) = \frac{1}{N}\bar{R}(e^{j\omega})$

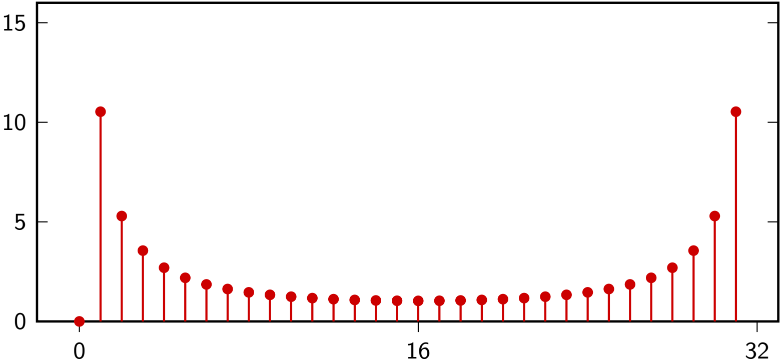
$$\bar{X}(e^{j\omega}) = \sum_{k=0}^{N-1} X[k]\Lambda(\omega - \frac{2\pi}{N}k)$$

smooth interpolation of DFT values

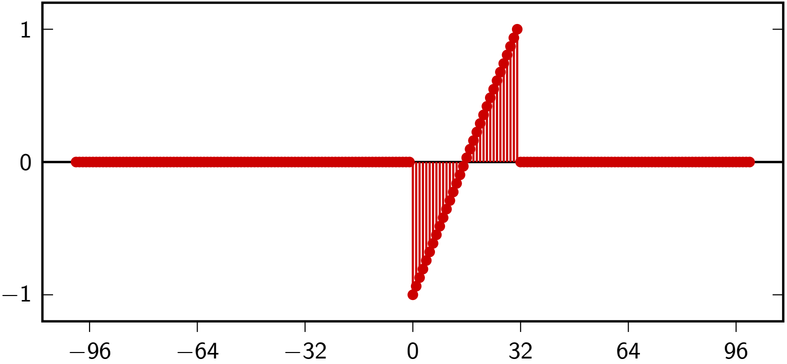
32-tap sawtooth



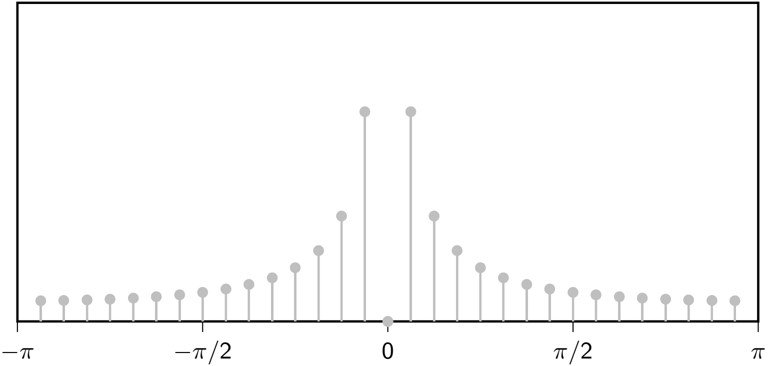
DFT of 32-tap sawtooth



Sawtooth: finite support extension



DTFT of finite support extension (sketch)



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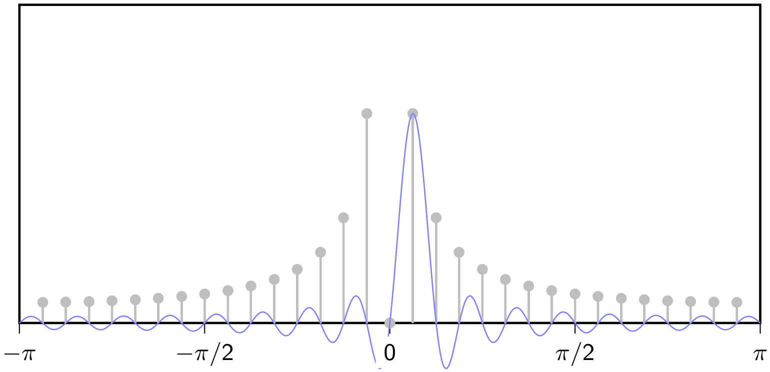

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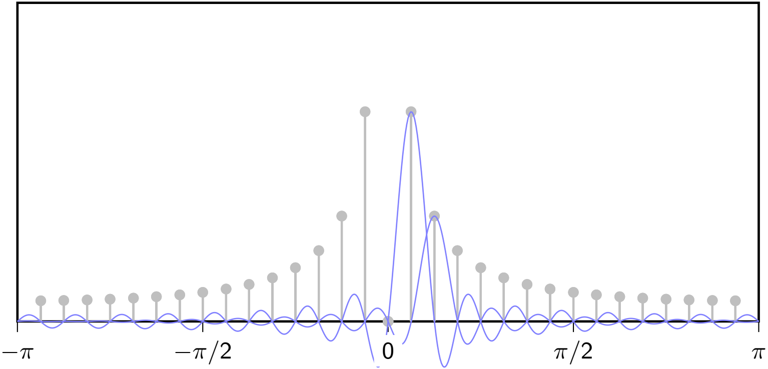

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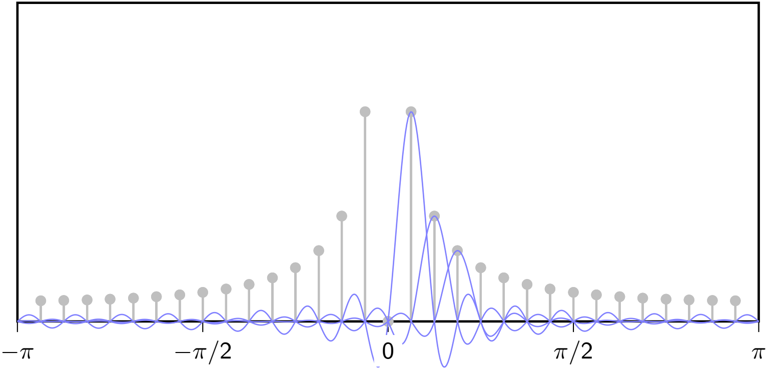

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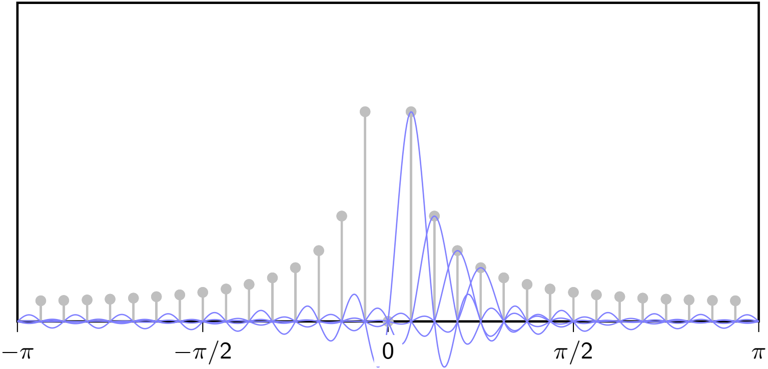

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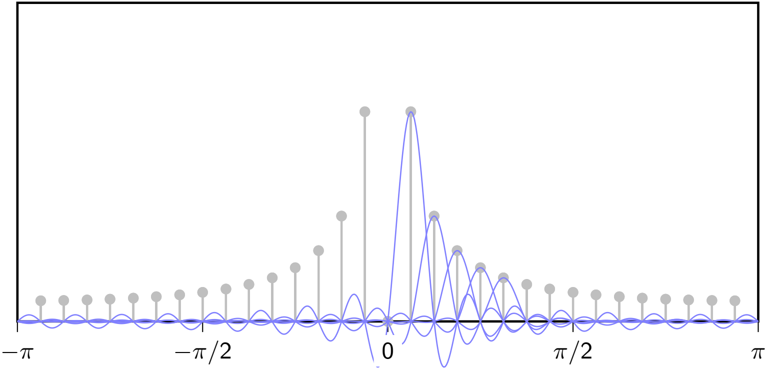
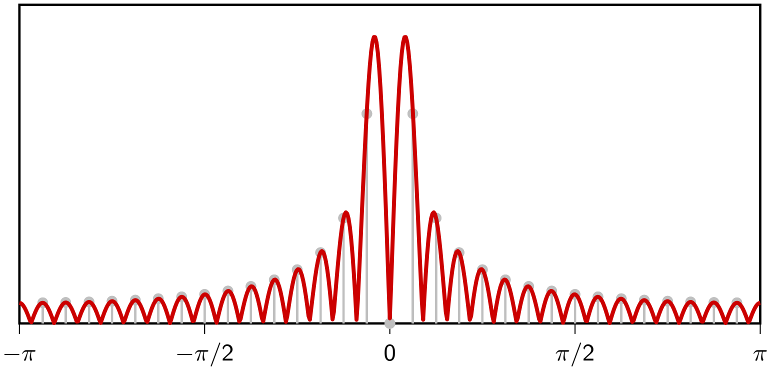
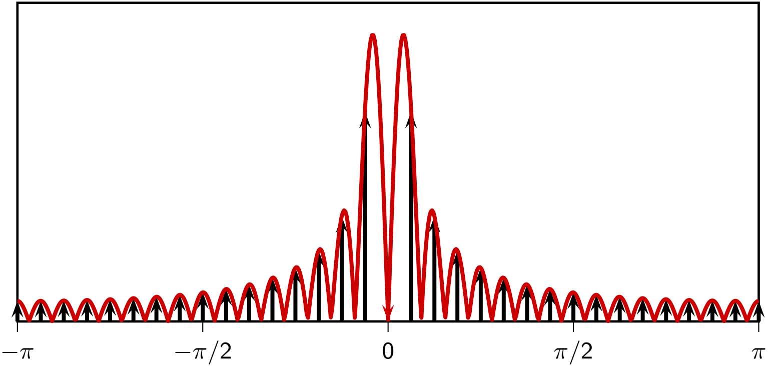


DTFT of finite support extension



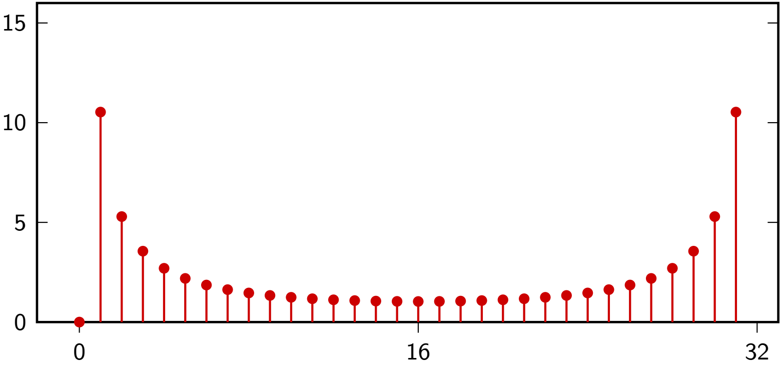
As a comparison...



About zero-padding

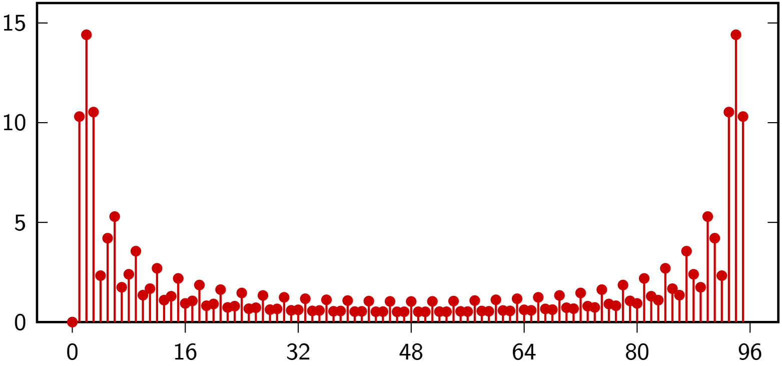
When computing the DFT numerically
one may “pad” the data vector with zeros to obtain “nicer” plots

DFT of 32-tap sawtooth



$$\mathbf{x} = [x_0 \; x_1 \; \ldots \; x_{31}]$$

DFT of 32-tap sawtooth, zero-padded to 96 points



$$\mathbf{x} = [x_0 \; x_1 \; \ldots \; x_{31} \; 0 \; \ldots \; 0]$$

About zero-padding

$$x_M[n] = \begin{cases} x[n] & \text{for } 0 \leq n < N \\ 0 & \text{for } N \leq n < M \end{cases}$$

About zero-padding

$$X_M[h] = \sum_{n=0}^{M-1} x'[n] e^{-j\frac{2\pi}{M}nh} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{M}nh}$$

$$= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\frac{2\pi}{M}nh}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \left(\sum_{n=0}^{N-1} e^{-j(\frac{2\pi}{M}h - \frac{2\pi}{N}k)n} \right)$$

$$= \bar{X}(e^{j\omega})\big|_{\omega=\frac{2\pi}{M}h}$$

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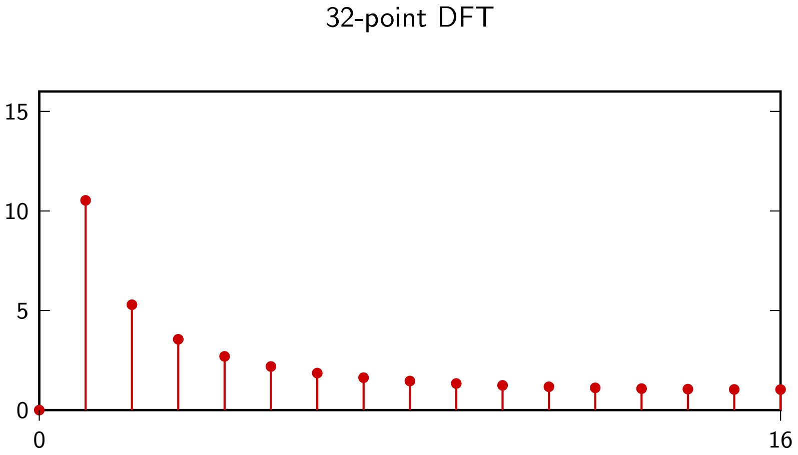
&= \bar{X}(e^{j\omega})\big|_{\omega=\frac{2\pi}{M}h}
\end{aligned}
$$

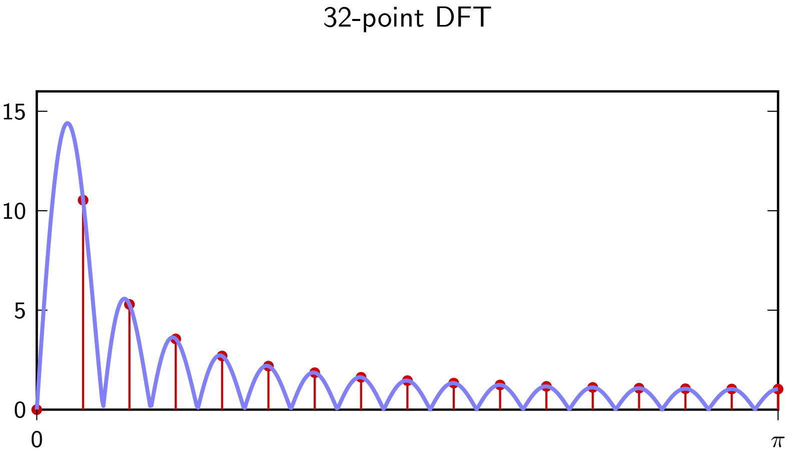
About zero-padding

- zero padding does not add information
- a zero-padded DFT is simply a sampled DTFT of the finite-support extension

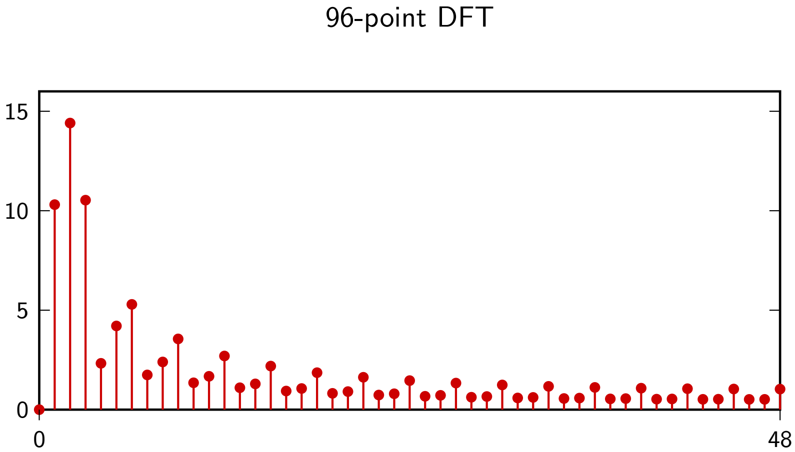
DFT of 32-tap sawtooth, zero-padded



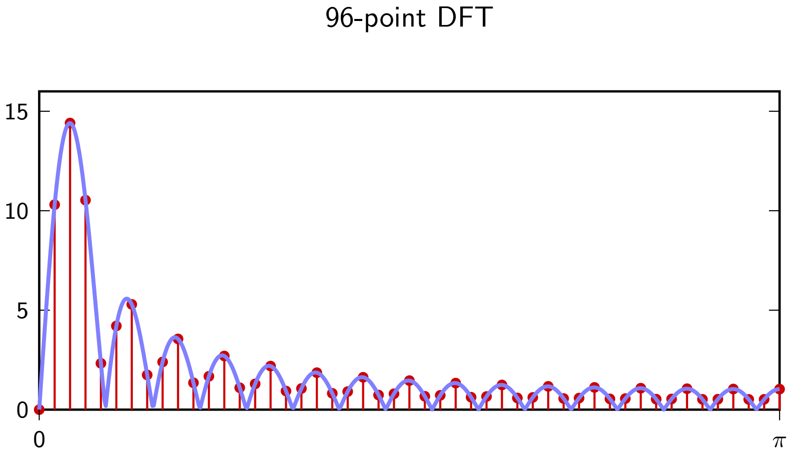
DFT of 32-tap sawtooth, zero-padded



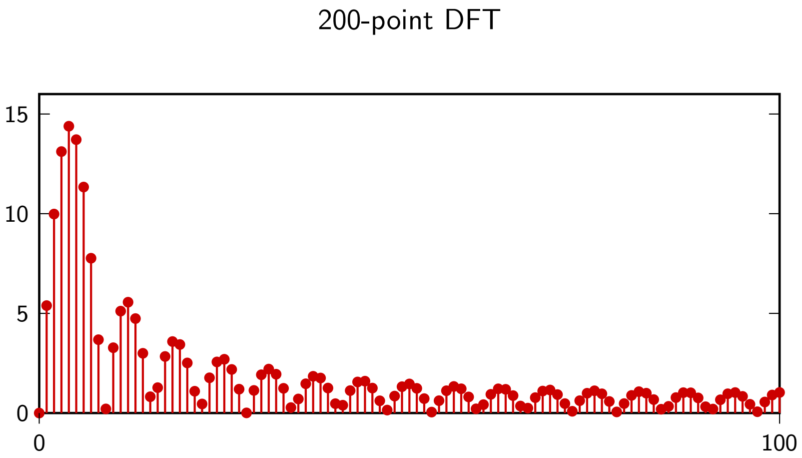
DFT of 32-tap sawtooth, zero-padded



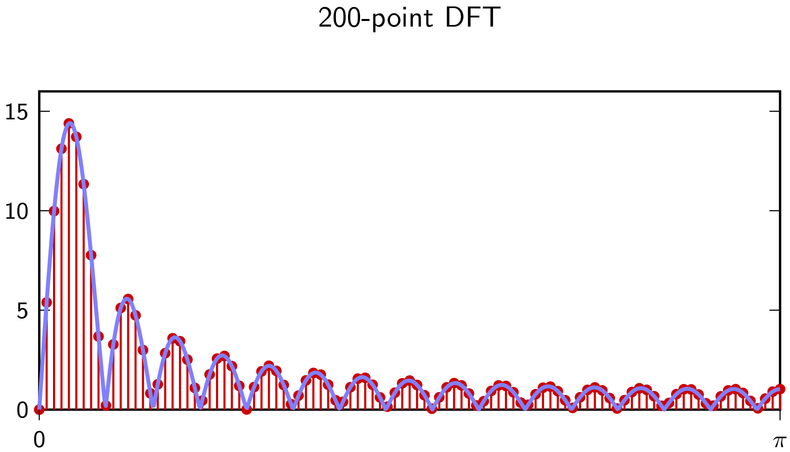
DFT of 32-tap sawtooth, zero-padded



DFT of 32-tap sawtooth, zero-padded



DFT of 32-tap sawtooth, zero-padded



modulation

Overview:

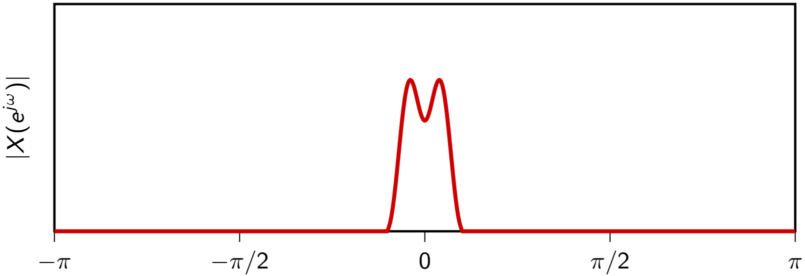
- Lowpass, highpass and bandpass signals
- Sinusoidal modulation
- Tuning a guitar

Classifying signals in frequency

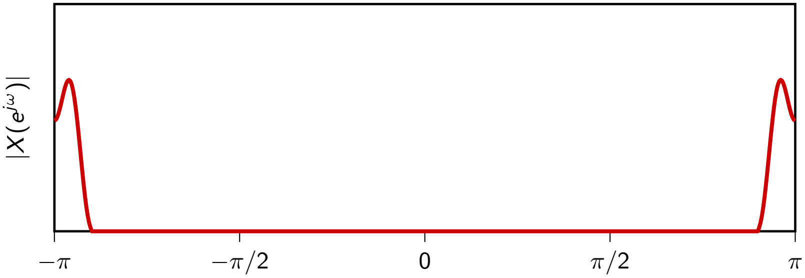
Three broad categories according to where most of the spectral energy resides:

- lowpass signals (also known as "baseband" signals)
- highpass signals
- bandpass signals

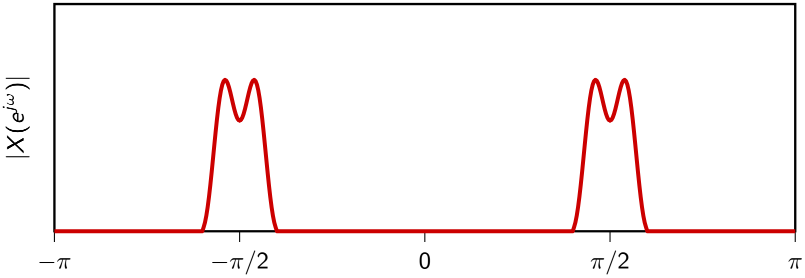
Lowpass example



Highpass example



Bandpass example



Sinusoidal modulation

$$\text{DTFT}\left\{x[n]\cos(\omega_c n)\right\} = \text{DTFT}\left\{\frac{1}{2}e^{j\omega_c n}x[n] + \frac{1}{2}e^{-j\omega_c n}x[n]\right\}$$

$$= \frac{1}{2}\left[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})\right]$$

- usually $x[n]$ baseband
- ω_c is the *carrier* frequency

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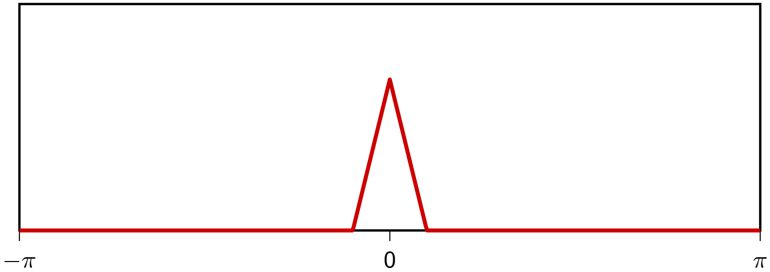
Example



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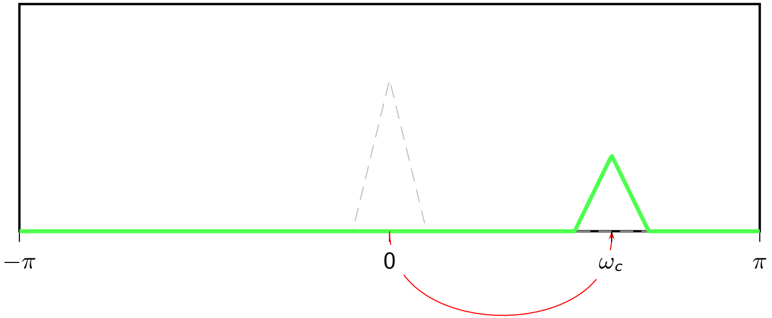

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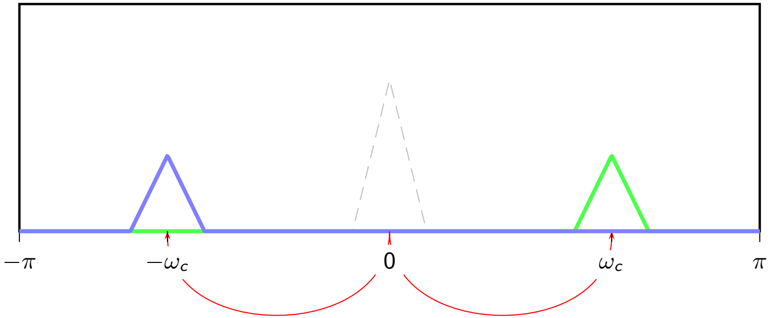

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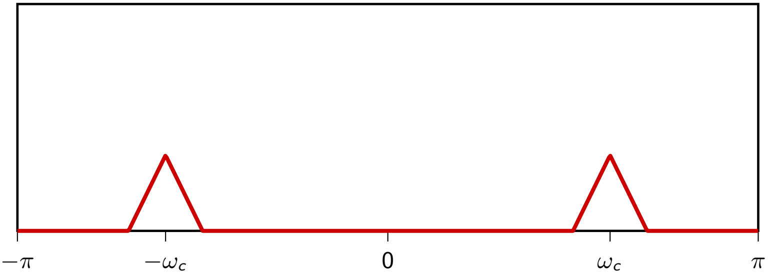
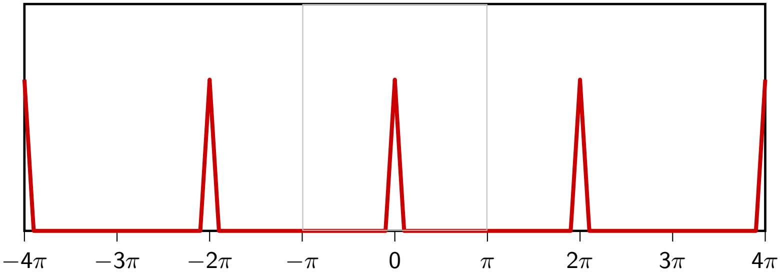


Again, explicitly showing the periodicity of the spectrum



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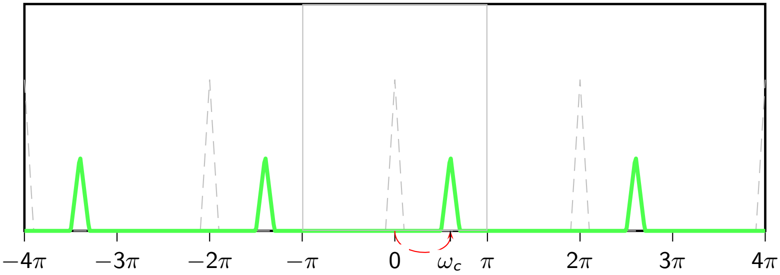

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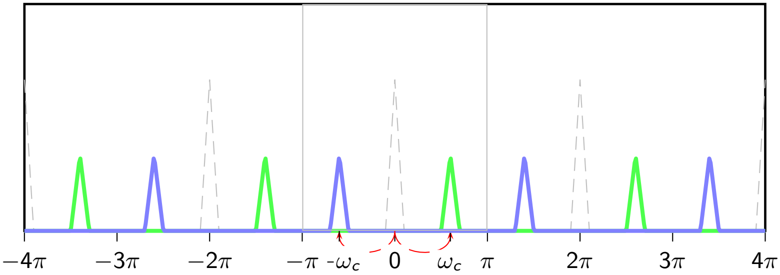

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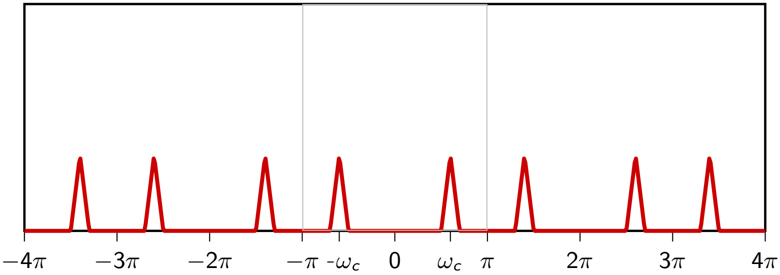
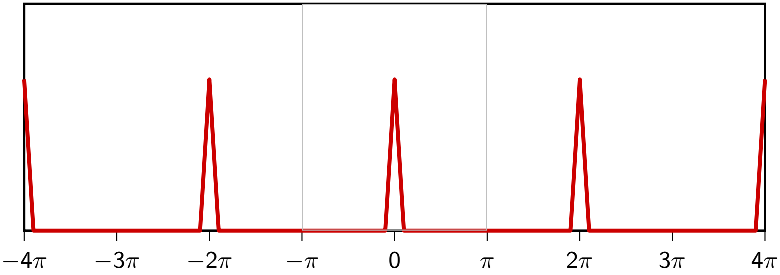


Careful when the modulation frequency is too large!



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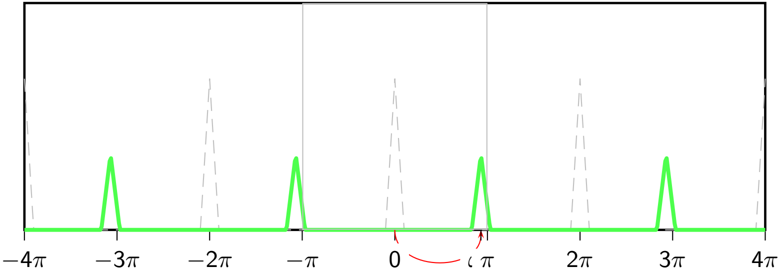

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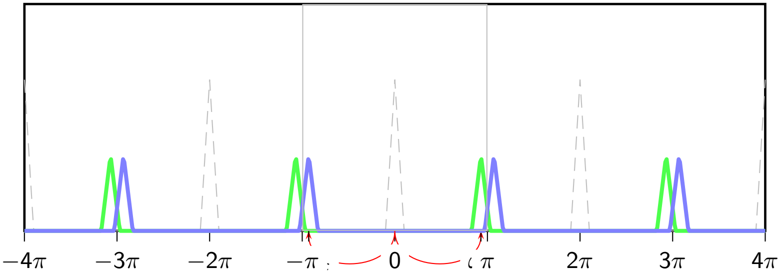

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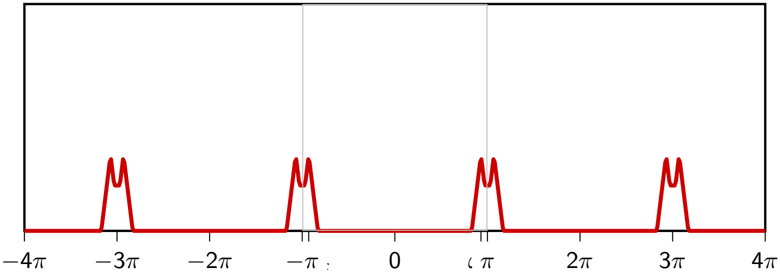

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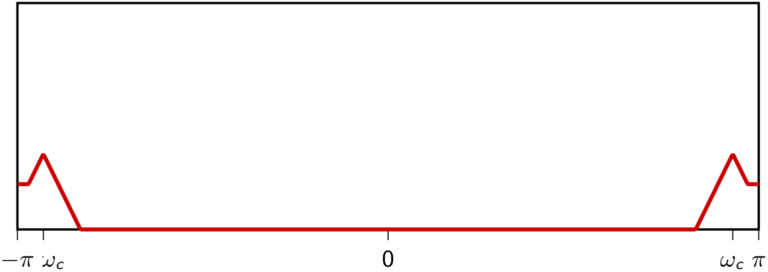

Sinusoidal modulation: applications

- voice and music are lowpass signals
- radio channels are bandpass, in much higher frequencies
- modulation brings the baseband signal in the transmission band
- demodulation at the receiver brings it back

Sinusoidal demodulation

just multiply the received signal by the carrier again

$$y[n] = x[n]\cos(\omega_c n) \qquad Y(e^{j\omega}) = \frac{1}{2}\left[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})\right]$$

$$\begin{aligned}
\text{DTFT}\left\{y[n]\cdot 2\cos(\omega_c n)\right\} &= Y(e^{j(\omega-\omega_c)}) + Y(e^{j(\omega+\omega_c)}) \\
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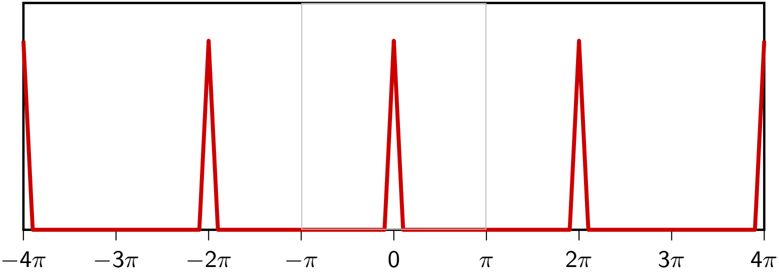
Demodulation in the frequency domain

$\mathrm{DTFT}\{x[n]\}$



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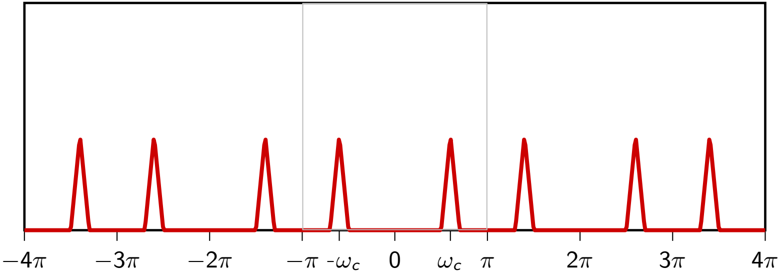

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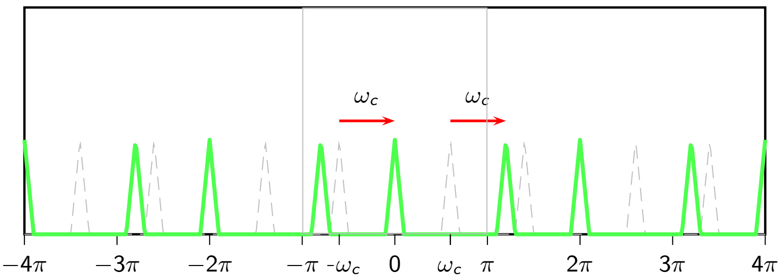

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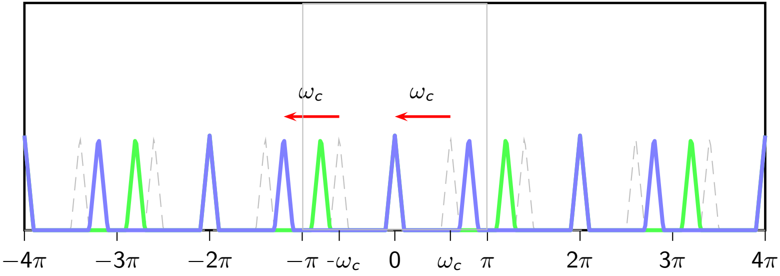

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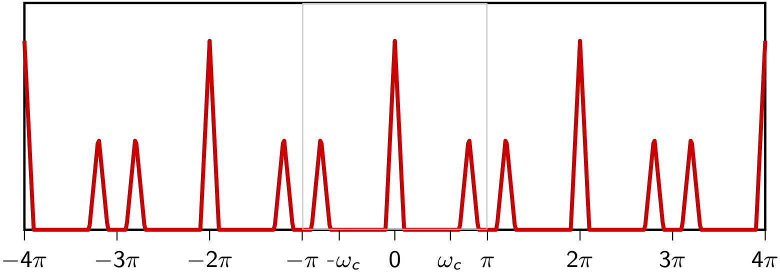

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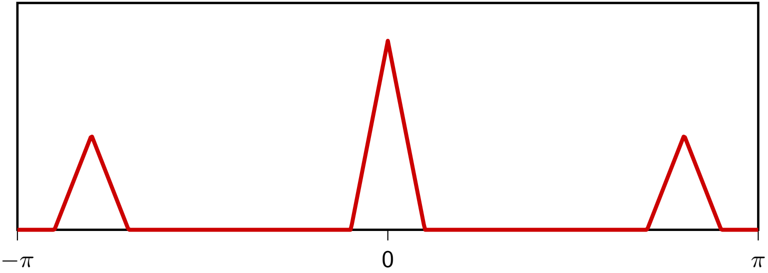

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- we recovered the baseband signal exactly...
- but we have some spurious high-frequency components
- in the next lectures we will learn how to get rid of them!

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Another application: tuning a guitar

Problem (abstraction):

- reference sinusoid at frequency ω_0
- tunable sinusoid of frequency ω
- make $\omega = \omega_0$ "by ear"

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The procedure

1. bring ω close to ω_0 (easy)
2. when $\omega \approx \omega_0$ play both sinusoids together
3. trigonometry comes to the rescue:

$$
\begin{aligned}
x[n] &= \cos(\omega_0 n) + \cos(\omega n) \\
&= 2\cos\left(\frac{\omega_0 + \omega}{2} n\right) \cos\left(\frac{\omega_0 - \omega}{2} n\right) \\
&\approx 2\cos(\Delta_\omega n)\cos(\omega_0 n)
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Let's see what's happening

$$x[n] \approx 2\cos(\Delta_\omega n)\cos(\omega_0 n)$$

- "error" signal
- modulation at ω_0
- when $\omega \approx \omega_0$, the error signal is too low to be heard; modulation brings it up to hearing range and we perceive it as amplitude oscillations of the carrier frequency

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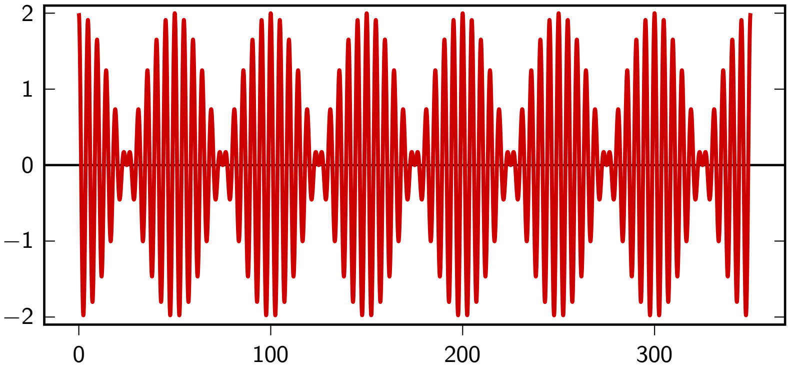
In the time domain...



$$\omega_0 = 2\pi \cdot 0.2, \quad \omega = 2\pi \cdot 0.22, \quad \Delta_\omega = 2\pi \cdot 0.0100$$

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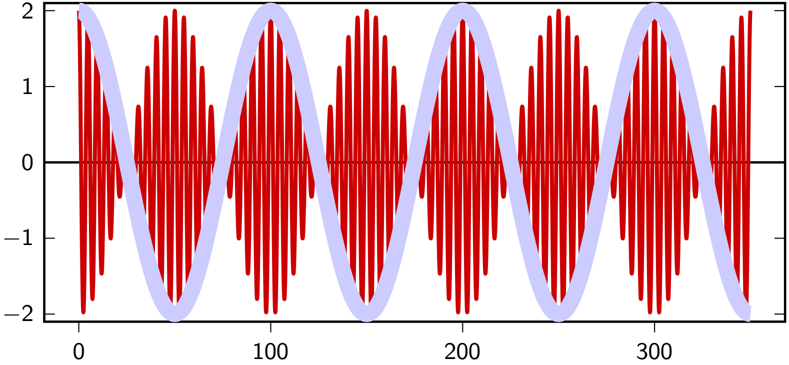

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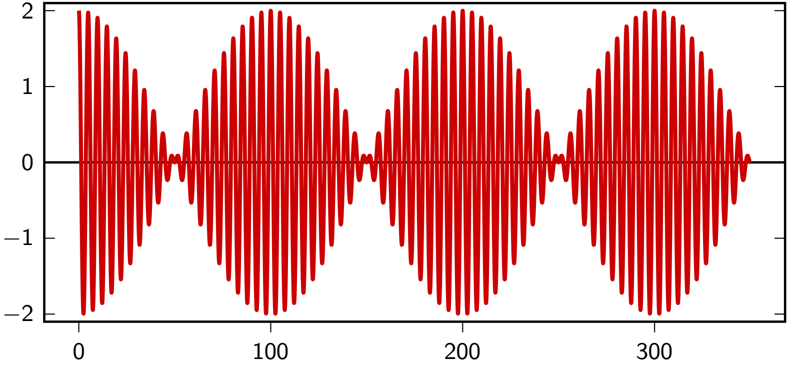

$$\omega_0 = 2\pi \cdot 0.2, \quad \omega = 2\pi \cdot 0.21, \quad \Delta_\omega = 2\pi \cdot 0.0050$$

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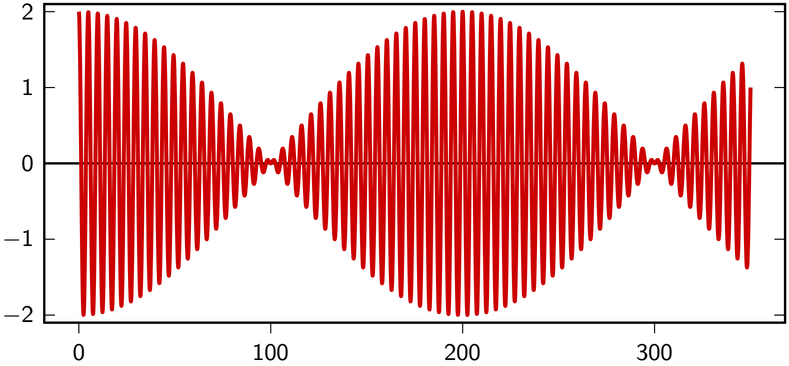

$$\omega_0 = 2\pi \cdot 0.2, \quad \omega = 2\pi \cdot 0.205, \quad \Delta_\omega = 2\pi \cdot 0.0025$$

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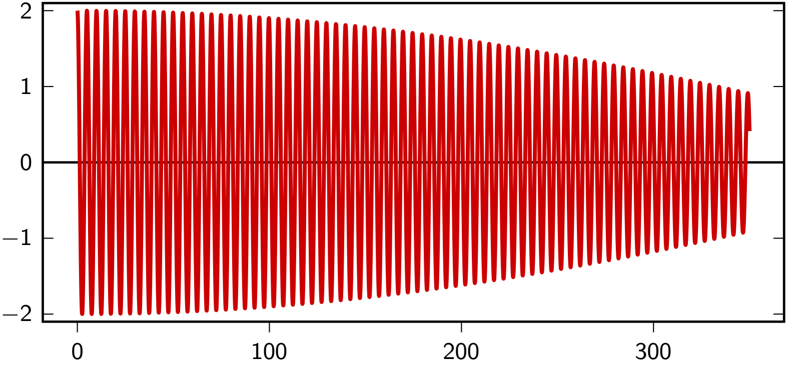

$$\omega_0 = 2\pi \cdot 0.2, \quad \omega = 2\pi \cdot 0.201, \quad \Delta_\omega = 2\pi \cdot 0.0005$$

demonstration