Exercise 3

Chapter 5

1. Prove that Equation (5.2) suffices for showing that $\mathbb{P}[L_{\mathcal{D}}(A(S)) \geq 1/8] \geq 1/7$. *Hint:* Let θ be a random variable that receives values in $[0,1]$ and whose expectation satisfies $\mathbb{E}[\theta] \geq 1/4$. Use Lemma B.1 to show that $\mathbb{P}[\theta \geq 1/8] \geq 1/7$.

Chapter 6

2. Given some finite domain set, $\mathcal{X}$, and a number $k \leq |\mathcal{X}|$, figure out the VC-dimension of each of the following classes (and prove your claims):
   1. $\mathcal{H}_{=k}^{\mathcal{X}} = \{h \in \{0,1\}^{\mathcal{X}} : |\{x : h(x) = 1\}| = k\}$. That is, the set of all functions that assign the value 1 to exactly k elements of $\mathcal{X}$.

   2. $\mathcal{H}_{at-most-k} = \{h \in \{0,1\}^{\mathcal{X}} : |\{x : h(x) = 1\}| \leq k \text{ or } |\{x : h(x) = 0\}| \leq k\}$.

5. **VC-dimension of axis aligned rectangles in $\mathbb{R}^d$:** Let $\mathcal{H}_{\text{rec}}^d$ be the class of axis aligned rectangles in $\mathbb{R}^d$. We have already seen that $\text{VCdim}(\mathcal{H}_{\text{rec}}^2) = 4$. Prove that in general, $\text{VCdim}(\mathcal{H}_{\text{rec}}^d) = 2d$.

8. (*) It is often the case that the VC-dimension of a hypothesis class equals (or can be bounded above by) the number of parameters one needs to set in order to define each hypothesis in the class. For instance, if $\mathcal{H}$ is the class of axis aligned rectangles in $\mathbb{R}^d$, then $\text{VCdim}(\mathcal{H}) = 2d$, which is equal to the number of parameters used to define a rectangle in $\mathbb{R}^d$. Here is an example that shows that this is not always the case. We will see that a hypothesis class might be very complex and even not learnable, although it has a small number of parameters.

   Consider the domain $\mathcal{X} = \mathbb{R}$, and the hypothesis class

$$\mathcal{H} = \{x \mapsto \lceil \sin(\theta x) \rceil : \theta \in \mathbb{R}\}$$

   (here, we take $\lceil -1 \rceil = 0$). Prove that $\text{VCdim}(\mathcal{H}) = \infty$.

   Hint: There is more than one way to prove the required result. One option is by applying the following lemma: If $0.x_1x_2x_3\ldots$, is the binary expansion of $x \in (0,1)$, then for any natural number m, $\lceil \sin(2^m \pi x) \rceil = (1 - x_m)$, provided that $\exists k \geq m$ s.t. $x_k = 1$.

9. Let $\mathcal{H}$ be the class of signed intervals, that is, $\mathcal{H} = \{h_{a,b,s} : a \le b, s \in \{-1, 1\}\}$ where

$$h_{a,b,s}(x) = \begin{cases} s & \text{if } x \in [a,b] \\ -s & \text{if } x \notin [a,b] \end{cases}$$

Calculate VCdim($\mathcal{H}$).

Chapter 7

3. - Consider a hypothesis class $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$, where for every $n \in \mathbb{N}$, $\mathcal{H}_n$ is finite. Find a weighting function $w : \mathcal{H} \to [0, 1]$ such that $\sum_{h \in \mathcal{H}} w(h) \le 1$ and so that for all $h \in \mathcal{H}$, $w(h)$ is determined by $n(h) = \min\{n : h \in \mathcal{H}_n\}$ and by $|\mathcal{H}_{n(h)}|$.
   - (*) Define such a function w when for all n $\mathcal{H}_n$ is countable (possibly infinite).