COM-303 - Signal Processing for Communications
Final Exam

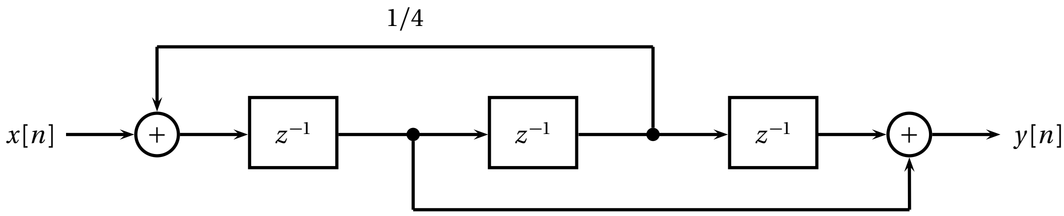
July 2, 2015, 08:15 to 11:15

Verify that this exam has YOUR last name on top

DO NOT OPEN THE EXAM UNTIL INSTRUCTED TO DO SO

- **Write your name** on the top left corner of **ALL the sheets you turn in**.
- There are 4 problems for a total of 100 points; the number of points is indicated for each problem.
- Please **write your derivations clearly!**
- You can have two A4 sheets of *handwritten* notes (front and back). Please **no photocopies, no books and no electronic devices**. Turn off your phone and store it in your bag.
- When you are done, simply leave your solution on your desk **with this page on top** and exit the classroom.

Exercise 1. (25 points)
Consider the causal system implemented by the following block diagram:



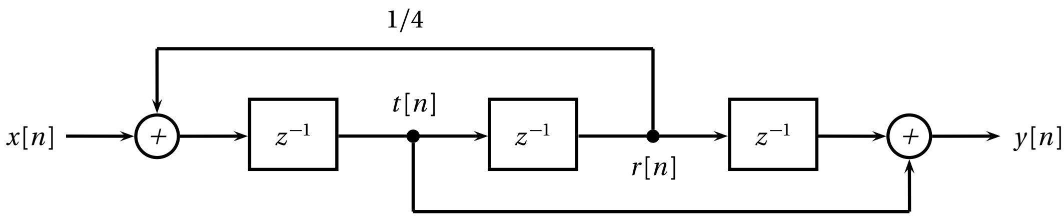
(a) Compute the system's transfer function $H(z)$

(b) Plot the system's poles and zeros on the complex plane

(c) Sketch the magnitude of the system's frequency response $|H(e^{j\omega})|$

(d) Draw another block diagram that implements the same transfer function $H(z)$ as a cascade of a second-order direct form II structure and a simple delay.

First set two auxiliary variables $t[n]$ and $r[n]$ like so:



(a) From simple inspection:

$$y[n] = t[n] + r[n-1]$$
$$r[n] = t[n-1]$$
$$t[n] = x[n-1] + (1/4)r[n-1]$$

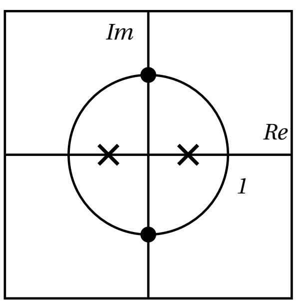
which, in the z-domain, becomes

$$Y(z) = T(z) + z^{-1}R(z)$$
$$R(z) = z^{-1}T(z)$$
$$T(z) = z^{-1}X(z) + (1/4)z^{-1}R(z)$$

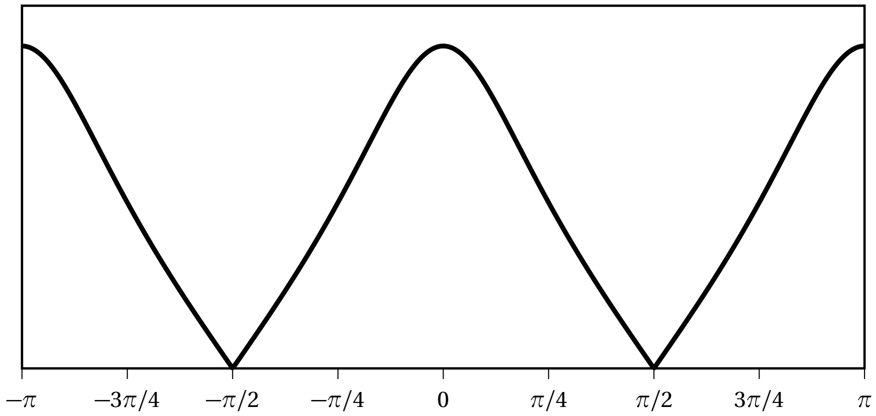
Solving for $X(z)$ and $Y(z)$ yields

$$H(z) = \frac{X(z)}{Y(z)} = z^{-1}\frac{1+z^{-2}}{1-(1/4)z^{-2}}$$

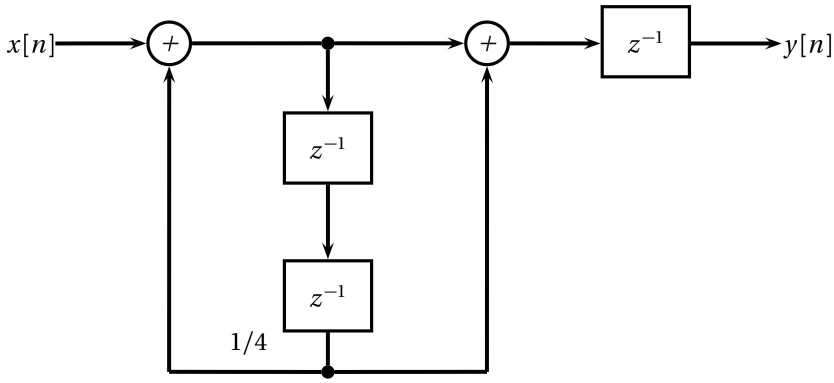
(b) the roots of the numerator are $\pm j$ and those of the denominator are $\pm 1/2$ therefore



(c) The filter is an approximate stopband

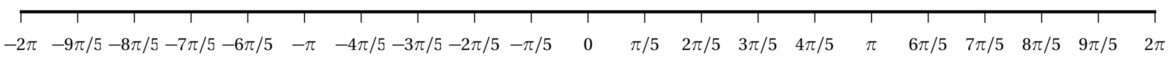


(d)



Exercise 2. (25 points)
Plot the DTFT of the signal $x[n] = \mathrm{sinc}(5n/2)$.

Hint: you can either work mostly in the time domain using simple trigonometry (but careful with the value of $x[0]$*) or you can work mostly in the frequency domain by considering* $x[n]$ *as a continuous-time sinc sampled with period* $T_s = 5/2$*; in this case the grid below can be of help.*



The standard derivation of the rect-sinc DTFT pair starts from a rect in frequency. Since the cutoff of the rect must be less than π*, the inverse DTFT returns a signal of the form* $\mathrm{sinc}(\alpha n)$ *with* $\alpha < 1$ *and therefore we cannot use the rect-sinc pair formula if, like in this case,* $\alpha > 1$.

Working in the time domain: *by exploting the* 2π*-periodicity of the sine, we have for* $n \neq 0$*:*

$$\mathrm{sinc}(5n/2) = \frac{\sin 5\pi n/2}{5\pi n/2} = \frac{\sin(2\pi n + \pi n/2)}{5\pi n/2} = (1/5)\frac{\sin \pi n/2}{\pi n/2} = (1/5)\mathrm{sinc}(n/2)$$

For $n = 0$, $x[n] = 1$ *while the above expression is equal to* $1/5$. *Therefore we can write*

$$x[n] = (1/5)\mathrm{sinc}(n/2) + (4/5)\delta[n]$$

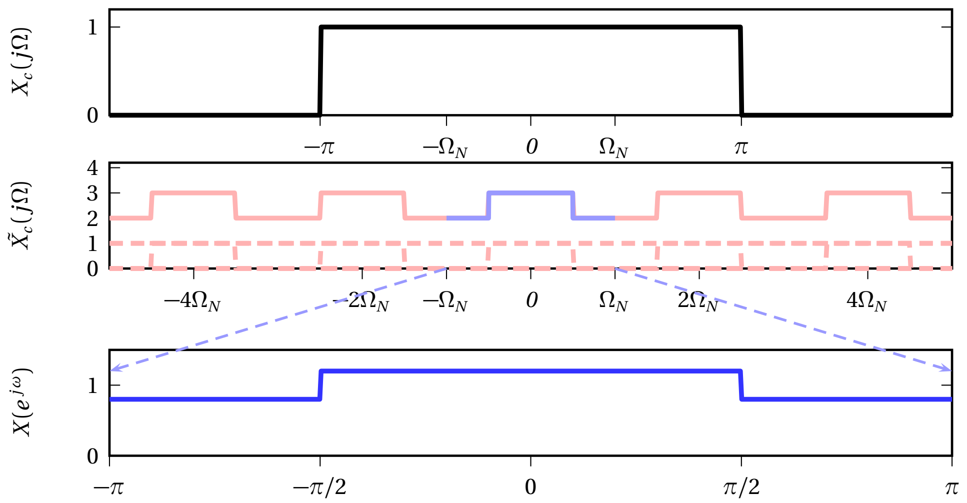
We can now use the standard sinc-rect transform pair formula to obtain

$$X(e^{j\omega}) = 4/5 + (2/5)\mathrm{rect}(\omega/\pi)$$

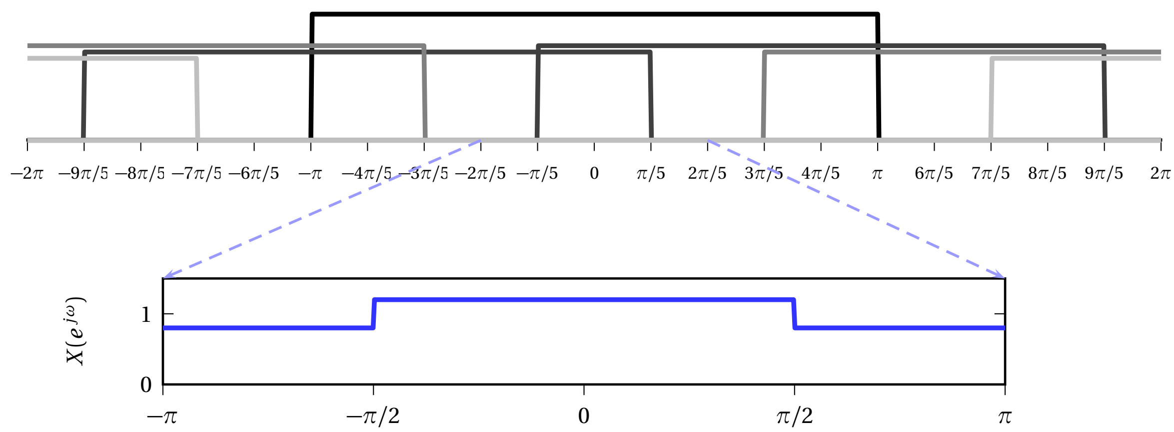
Working in the frequency domain: *Consider the continuous-time sinc function* $x(t) = \mathrm{sinc}(t)$*; its Fourier transform is* $\Phi(j\Omega) = \mathrm{rect}(\Omega/2\pi)$*, bandlimited to* $\Omega_0 = \pi$*. Clearly* $x[n] = x(nT)$ *for* $T_s = 5/2$*; however,* $5/2$ *is larger than the maximum alias-free sampling period for* $x(t)$*, which is* $T_{\max} = \pi/\Omega_0 = 1$ *so we will have aliasing. The resulting DTFT will be*

$$X(e^{j\omega}) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} \Phi\left(\frac{j\omega}{\pi}\Omega_N + 2jk\Omega_N\right)$$

with $\Omega_N = \pi/T_s = 2\pi/5$*. We can determine the shape of the DTFT graphically:*



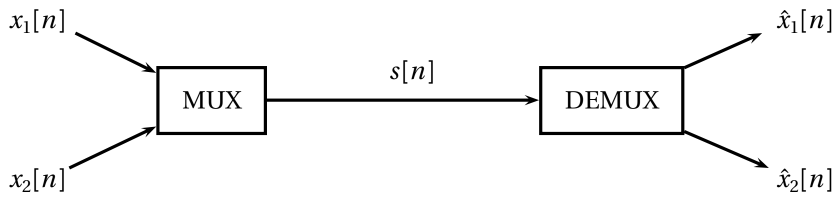
The overlaps are illustrated in detail by this figure, using an artificial different height for each spectral copy:



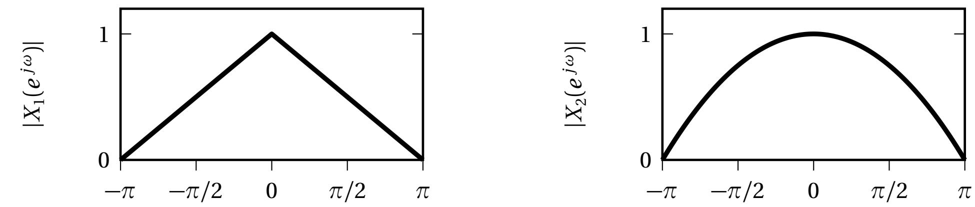
so that

$$X(e^{j\omega}) = \begin{cases} 6/5 & \text{for } |\omega| < \pi/2 \\ 4/5 & \text{otherwise} \end{cases} \qquad \text{extended by } 2\pi\text{-periodicity}$$

Exercise 3. (26 points)
In communication systems, it often happens that several data streams must share a single communication channel; in these cases the data streams are *multiplexed* and transmitted together and are de-multiplexed at the receiving end:



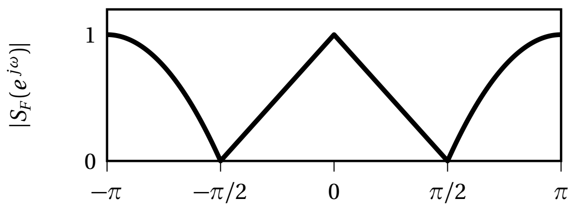
The two simplest multiplexing schemes are time-division multiplexing (TDM) and frequency-division multiplexing (FDM). Consider the simple case of two discrete-time data sequences $x_1[n]$ and $x_2[n]$, whose magnitude spectra are sketched here:



Using 2-TDM (i.e. time-division multiplexing for two streams), the transmitted sequence $s_T[n]$ is obtained by interleaving the original sequences:

$$s_T[n] = \begin{cases} x_1[n/2] & \text{for } n \text{ even} \\ x_2[(n-1)/2] & \text{for } n \text{ odd} \end{cases}$$

In 2-FDM, on the other hand, the two sequences are combined into a sequence $s_F[n]$ so that $S_F(e^{j\omega})$ looks like this:

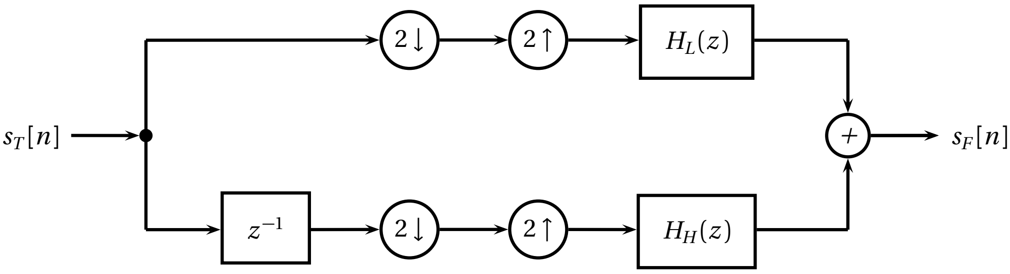


(a) Assume you receive a 2-TDM signal and you need to retransmit it to a 2-FDM receiver. Draw the block diagram of a system that takes $s_T[n]$ as the input and converts it into $s_F[n]$ as the output. There should be no data loss and the data rate (i.e. number of samples per second) should remain the same between the input and the output.

(b) Design an 2-FDM de-multiplexer (i.e. a system that takes $s_F[n]$ as the input and produces the original components $x_1[n]$ and $x_2[n]$).

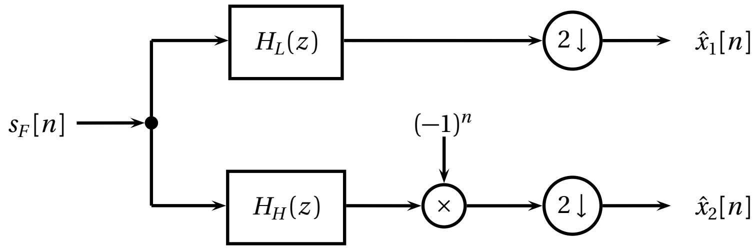
Please be extremely precise in your block diagrams.

(a)

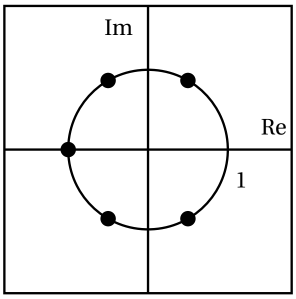


where $H_L(z)$ and $H_H(z)$ are ideal low- and high-pass filters with cutoff frequency $\pi/2$.

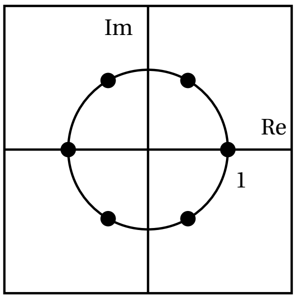
(b)



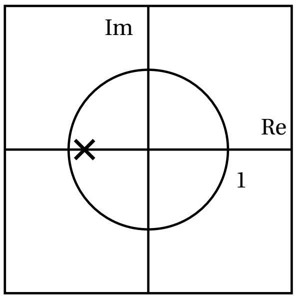
Exercise 4. (24 points)
Associate each pole-zero plot to the corresponding impulse response.



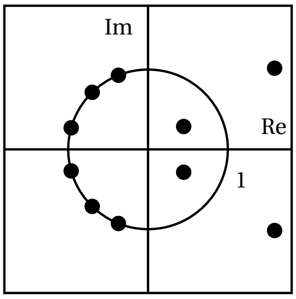
(a)



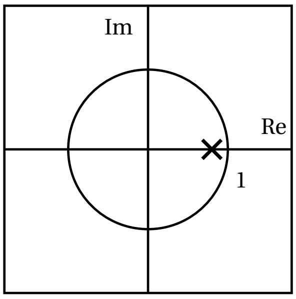
(b)



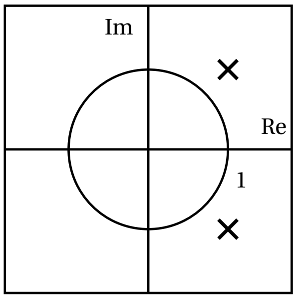
(c)



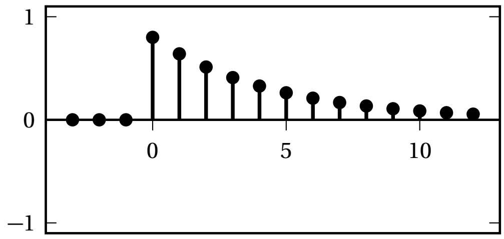
(d)



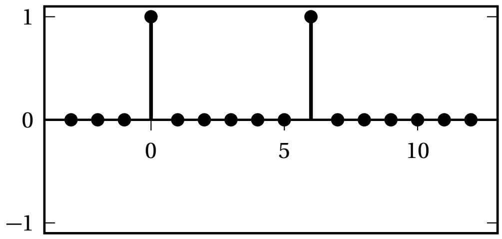
(e)



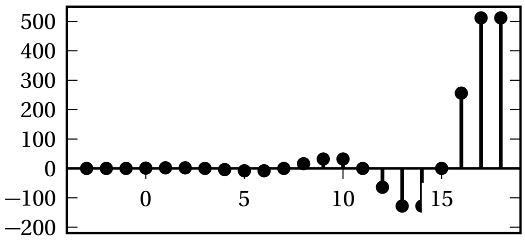
(f)



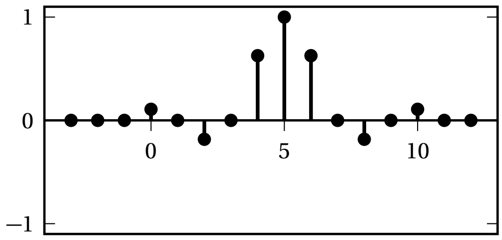
(1)



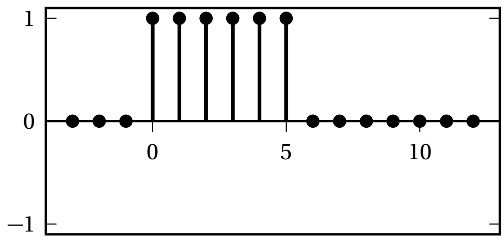
(2)



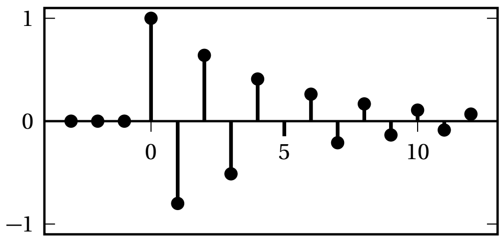
(3)



(4)



(5)



(6)

(a) - (5)
(b) - (2)
(c) - (6)
(d) - (4)
(e) - (1)
(f) - (3)