COM303: Digital Signal Processing

Lecture 5: The Discrete Fourier Transform

Overview

- the Fourier basis for $\mathbb{C}^N$ (recap)
- the DFT: definition and examples
- interpreting a DFT plot

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- the DFT: definition and examples
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The canonical (time) basis for $\mathbb{C}^N$

- in "signal" notation: $\delta_k[n] = \delta[n-k], \qquad n, k = 0, 1, \ldots, N-1$
- in vector notation: $\{\boldsymbol{\delta}^{(k)}\}_{k=0,1,\ldots,N-1}$ with $\delta_n^{(k)} = \delta[n-k]$

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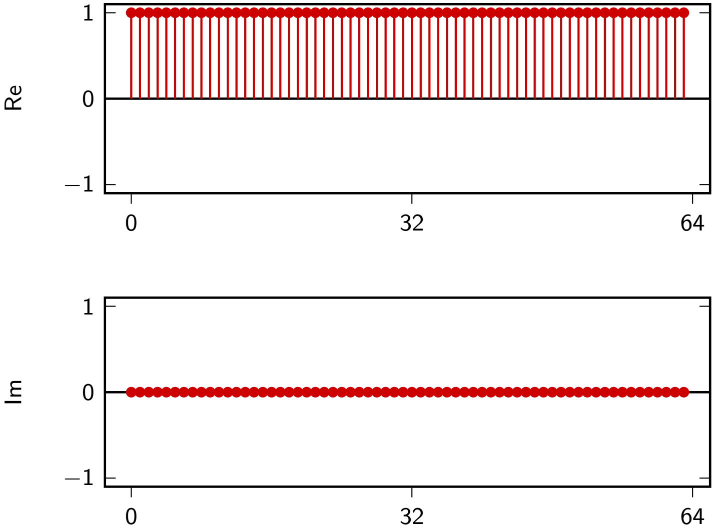
The Fourier (frequency) basis for $\mathbb{C}^N$

- in "signal" notation: $w_k[n] = e^{j\frac{2\pi}{N}nk}, \qquad n, k = 0, 1, \dots, N-1$
- in vector notation: $\{\mathbf{w}^{(k)}\}_{k=0,1,\dots,N-1}$ with $w_n^{(k)} = e^{j\frac{2\pi}{N}nk}$

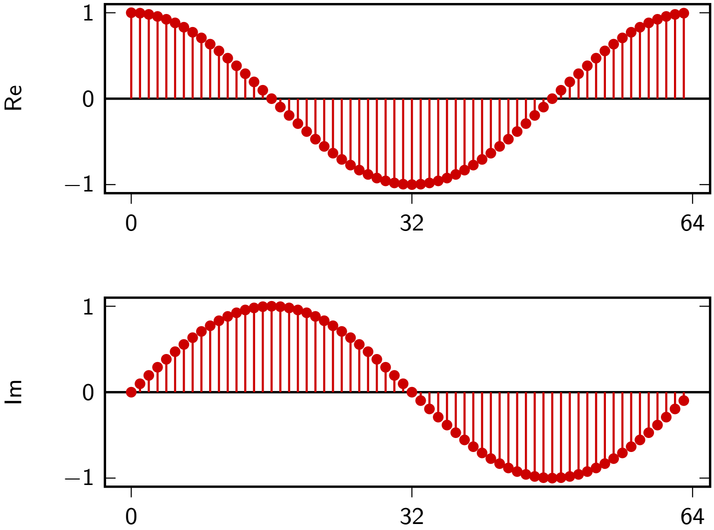
The Fourier (frequency) basis for $\mathbb{C}^N$

- in "signal" notation: $w_k[n] = e^{j\frac{2\pi}{N}nk}, \qquad n, k = 0, 1, \ldots, N-1$
- in vector notation: $\{\mathbf{w}^{(k)}\}_{k=0,1,\ldots,N-1}$ with $w_n^{(k)} = e^{j\frac{2\pi}{N}nk}$

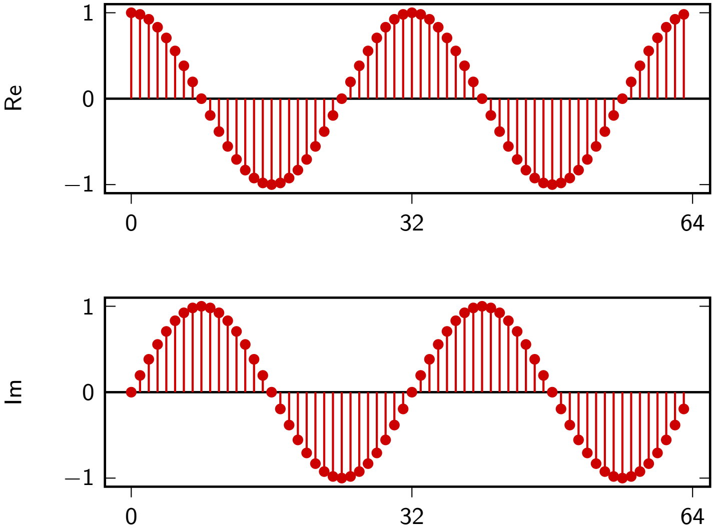
Basis vector $\mathbf{w}^{(0)} \in \mathbb{C}^{64}$ $\omega = (2\pi/64) \cdot 0$



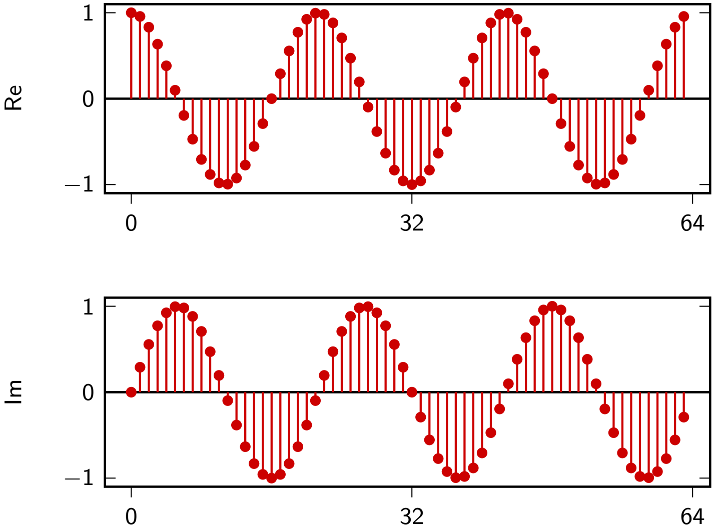
Basis vector $\mathbf{w}^{(1)} \in \mathbb{C}^{64}$ $\omega = (2\pi/64) \cdot 1$



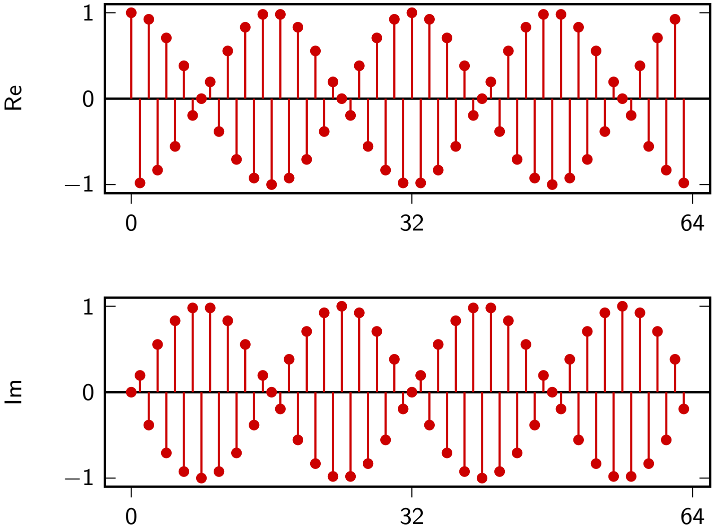
Basis vector $\mathbf{w}^{(2)} \in \mathbb{C}^{64}$ $\quad \omega = (2\pi/64) \cdot 2$



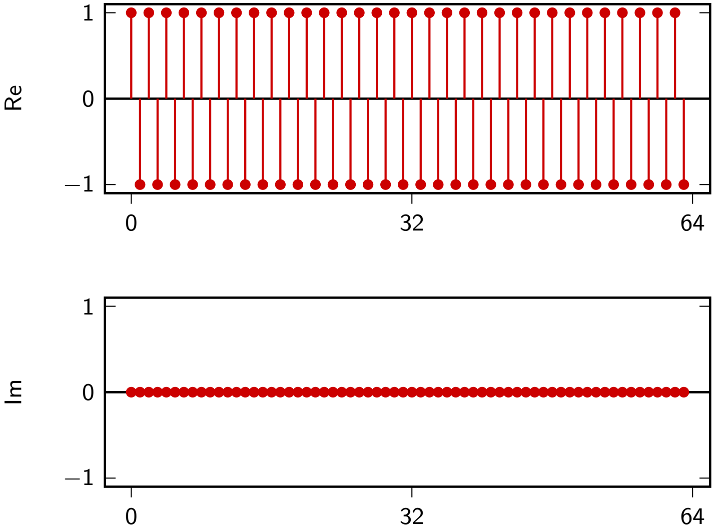
Basis vector $\mathbf{w}^{(3)} \in \mathbb{C}^{64}$ $\quad \omega = (2\pi/64) \cdot 3$



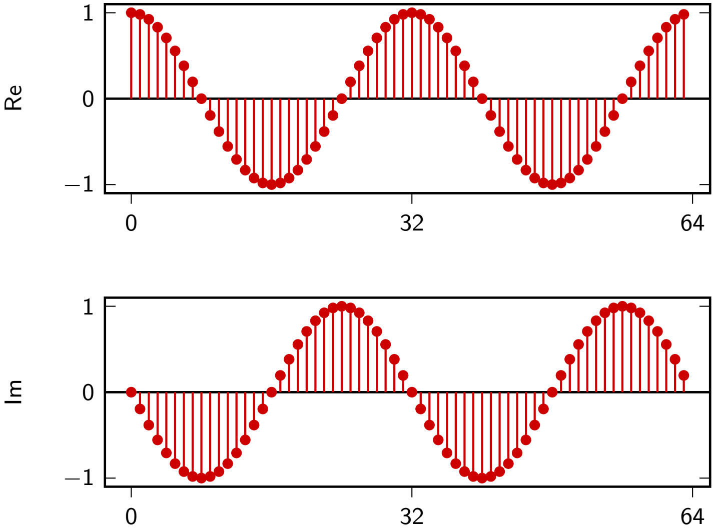
Basis vector $\mathbf{w}^{(30)} \in \mathbb{C}^{64}$ $\qquad \omega = (2\pi/64) \cdot 30$



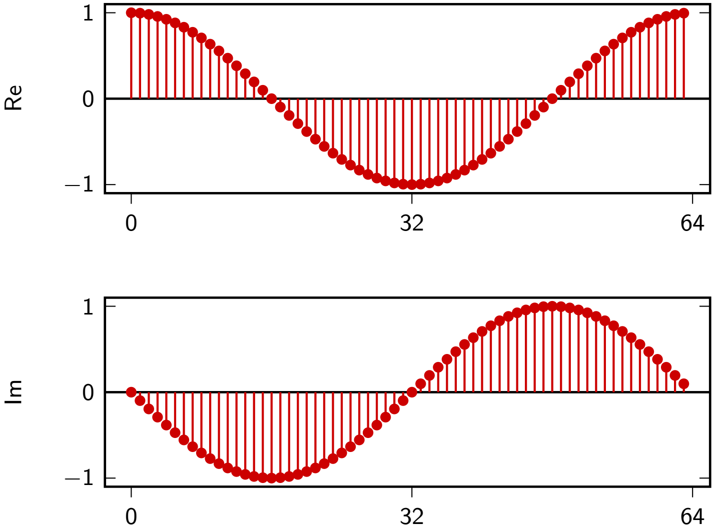
Basis vector $\mathbf{w}^{(32)} \in \mathbb{C}^{64}$ $\quad\quad \omega = (2\pi/64) \cdot 32$



Basis vector $\mathbf{w}^{(62)} \in \mathbb{C}^{64}$ $\qquad \omega = (2\pi/64) \cdot 62$



Basis vector $\mathbf{w}^{(63)} \in \mathbb{C}^{64}$ $\qquad \omega = (2\pi/64) \cdot 63$



Proof of orthogonality

$$
\begin{aligned}
\langle \mathbf{w}^{(k)}, \mathbf{w}^{(h)} \rangle &= \sum_{n=0}^{N-1} (e^{j\frac{2\pi}{N}nk})^* \, e^{j\frac{2\pi}{N}nh} \\
&= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(h-k)n} \\
&= \begin{cases} N & \text{for } h = k \\ \dfrac{1 - e^{j2\pi(h-k)}}{1 - e^{j\frac{2\pi}{N}(h-k)}} = 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

The Fourier Basis for $\mathbb{C}^N$

- N orthogonal vectors $\longrightarrow$ basis for $\mathbb{C}^N$
- vectors are not ortho*normal*. Normalization factor would be $1/\sqrt{N}$
- will keep normalization factor explicit in DFT formulas

Basis expansion

Analysis formula:

$$X_k = \langle \mathbf{w}^{(k)}, \mathbf{x} \rangle$$

Synthesis formula:

$$\mathbf{x} = \frac{1}{N} \sum_{k=0}^{N-1} X_k \mathbf{w}^{(k)}$$

Basis expansion (signal notation)

Analysis formula:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}, \qquad k = 0, 1, \dots, N-1$$

N-point signal in the *frequency domain*

Synthesis formula:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk}, \qquad n = 0, 1, \dots, N-1$$

N-point signal in the *"time" domain*

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Analysis formula:

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N-point signal in the *"time" domain*

Change of basis in matrix form

Define $W_N = e^{-j\frac{2\pi}{N}}$
(or simply W when N is evident from the context)

Change of basis matrix $\mathbf{W}$ with $\mathbf{W}[n, m] = W_N^{nm}$:

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & W^1 & W^2 & W^3 & \dots & W^{N-1} \\ 1 & W^2 & W^4 & W^6 & \dots & W^{2(N-1)} \\ & & & \dots & & \\ 1 & W^{N-1} & W^{2(N-1)} & W^{3(N-1)} & \dots & W^{(N-1)^2} \end{bmatrix}$$

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Change of basis in matrix form

Analysis formula:

$$\mathbf{X} = \mathbf{W}\mathbf{x}$$

Synthesis formula:

$$\mathbf{x} = \frac{1}{N}\mathbf{W}^H\mathbf{X}$$

DFT Matrix

$$W_N^m = W_N^{(m \mod N)}$$

e.g. $W_8^{11} = W_8^3$

DFT Matrix

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Small DFT matrices: $N = 2, 3$

$$W_2 = e^{-j\frac{2\pi}{2}} = -1$$

$$\mathbf{W}_2 = \begin{bmatrix} 1 & 1 \\ 1 & W \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$W_3 = e^{-j\frac{2\pi}{3}} = -(1 + j\sqrt{3})/2$$

$$\mathbf{W}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W & W^2 \\ 1 & W^2 & W^4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W & W^2 \\ 1 & W^2 & W \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -(1 + j\sqrt{3})/2 & -(1 - j\sqrt{3})/2 \\ 1 & -(1 - j\sqrt{3})/2 & (1 - j\sqrt{3})/2 \end{bmatrix}$$

Small DFT matrices: $N = 4$

$$W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}} = -j$$

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 \\ 1 & W^2 & W^4 & W^6 \\ 1 & W^3 & W^6 & W^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 \\ 1 & W^2 & 1 & W^2 \\ 1 & W^3 & W^2 & W \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Small DFT matrices: $N = 5$

$$\mathbf{W}_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 \\ 1 & W^2 & W^4 & W^6 & W^8 \\ 1 & W^3 & W^6 & W^9 & W^{12} \\ 1 & W^4 & W^8 & W^{12} & W^{16} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 \\ 1 & W^2 & W^4 & W & W^3 \\ 1 & W^3 & W & W^4 & W^2 \\ 1 & W^4 & W^3 & W^2 & W \end{bmatrix}$$

Small DFT matrices: $N = 6$

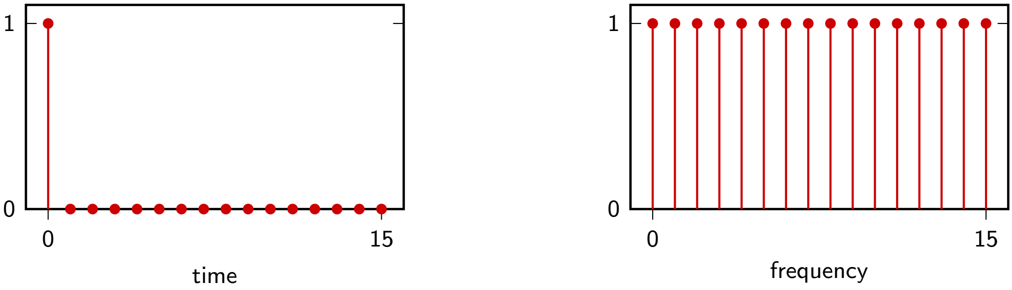
$$\mathbf{W}_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 & W^5 \\ 1 & W^2 & W^4 & W^6 & W^8 & W^{10} \\ 1 & W^3 & W^6 & W^9 & W^{12} & W^{15} \\ 1 & W^4 & W^8 & W^{12} & W^{16} & W^{20} \\ 1 & W^5 & W^{10} & W^{15} & W^{20} & W^{25} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 & W^5 \\ 1 & W^2 & W^4 & 1 & W^2 & W^4 \\ 1 & W^3 & 1 & W^3 & 1 & W^3 \\ 1 & W^4 & W^2 & 1 & W^4 & W^2 \\ 1 & W^5 & W^4 & W^3 & W^2 & W \end{bmatrix}$$

DFT is obviously linear

$$\mathrm{DFT}\left\{\alpha\, x[n] + \beta\, y[n]\right\} = \alpha\, \mathrm{DFT}\left\{x[n]\right\} + \beta\, \mathrm{DFT}\left\{y[n]\right\}$$

DFT of $\delta[n] \in \mathbb{C}^N$

$$\sum_{n=0}^{N-1} \delta[n]\, e^{-j\frac{2\pi}{N}nk} = 1 \quad \forall k$$



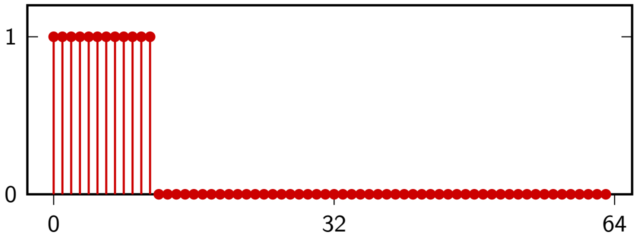
DFT of $\boldsymbol{\delta}^{(m)} = \delta[n-m] \in \mathbb{C}^N$

$$\sum_{n=0}^{N-1} \delta[n-m]\, e^{-j\frac{2\pi}{N}nk} = e^{-j\frac{2\pi}{N}mk}$$

$$\mathrm{DFT}\left\{\boldsymbol{\delta}^{(m)}\right\} = \left(\mathbf{w}^{(m)}\right)^*$$

DFT of length-M step in $\mathbb{C}^N$

$$x[n] = \sum_{h=0}^{M-1} \delta[n-m], \qquad n = 0, 1, \ldots, N-1$$



DFT of length-M step in $\mathbb{C}^N$

$$X[k] = \sum_{m=0}^{M-1} \text{DFT}\left\{\delta[n-m]\right\} = \sum_{m=0}^{M-1} e^{-j\frac{2\pi}{N}mk}$$

$$= \frac{1-e^{-j\frac{2\pi}{N}kM}}{1-e^{-j\frac{2\pi}{N}k}}$$

$$= \frac{e^{-j\frac{\pi}{N}kM}\left[e^{j\frac{\pi}{N}kM}-e^{-j\frac{\pi}{N}kM}\right]}{e^{-j\frac{\pi}{N}k}\left[e^{j\frac{\pi}{N}k}-e^{-j\frac{\pi}{N}k}\right]}$$

$$= \frac{\sin\left(\frac{\pi}{N}Mk\right)}{\sin\left(\frac{\pi}{N}k\right)} e^{-j\frac{\pi}{N}(M-1)k}$$

DFT of length-M step in $\mathbb{C}^N$

$$X[k] = \sum_{m=0}^{M-1} \text{DFT}\left\{\delta[n-m]\right\} = \sum_{m=0}^{M-1} e^{-j\frac{2\pi}{N}mk}$$

$$= \frac{1 - e^{-j\frac{2\pi}{N}kM}}{1 - e^{-j\frac{2\pi}{N}k}}$$

$$= \frac{e^{-j\frac{\pi}{N}kM}\left[e^{j\frac{\pi}{N}kM} - e^{-j\frac{\pi}{N}kM}\right]}{e^{-j\frac{\pi}{N}k}\left[e^{j\frac{\pi}{N}k} - e^{-j\frac{\pi}{N}k}\right]}$$

$$= \frac{\sin\left(\frac{\pi}{N}Mk\right)}{\sin\left(\frac{\pi}{N}k\right)} e^{-j\frac{\pi}{N}(M-1)k}$$

DFT of length-M step in $\mathbb{C}^N$

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DFT of length-M step in $\mathbb{C}^N$

$$
\begin{aligned}
X[k] &= \sum_{m=0}^{M-1} \mathrm{DFT}\left\{\delta[n-m]\right\} = \sum_{m=0}^{M-1} e^{-j\frac{2\pi}{N}mk} \\
&= \frac{1-e^{-j\frac{2\pi}{N}kM}}{1-e^{-j\frac{2\pi}{N}k}} \\
&= \frac{e^{-j\frac{\pi}{N}kM}\left[e^{j\frac{\pi}{N}kM}-e^{-j\frac{\pi}{N}kM}\right]}{e^{-j\frac{\pi}{N}k}\left[e^{j\frac{\pi}{N}k}-e^{-j\frac{\pi}{N}k}\right]} \\
&= \frac{\sin\left(\frac{\pi}{N}Mk\right)}{\sin\left(\frac{\pi}{N}k\right)} e^{-j\frac{\pi}{N}(M-1)k}
\end{aligned}
$$

DFT of length-M step in $\mathbb{C}^N$

$$X[k] = \frac{\sin\left(\frac{\pi}{N}Mk\right)}{\sin\left(\frac{\pi}{N}k\right)} e^{-j\frac{\pi}{N}(M-1)k}$$

- $X[0] = M$, from the definition of the sum
- $X[k] = 0$ if Mk/N integer $(0 \leq k < N)$
- $\angle X[k]$ linear in k (except at sign changes for the real part)

DFT of length-M step in $\mathbb{C}^N$

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- $X[0] = M$, from the definition of the sum
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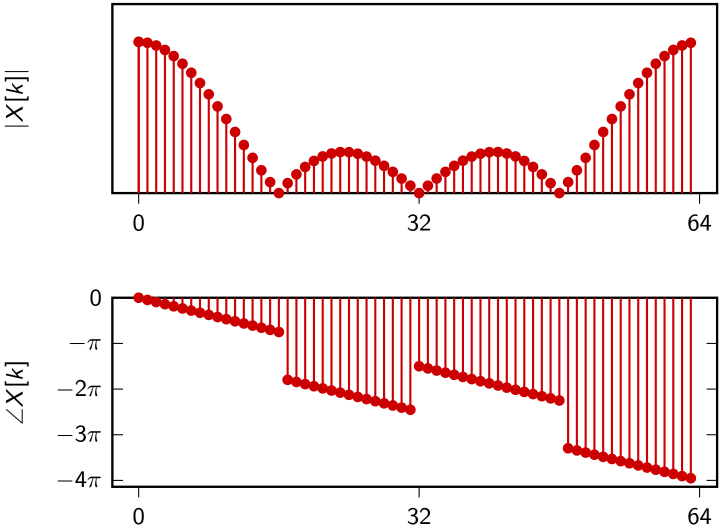
- $\angle X[k]$ linear in k (except at sign changes for the real part)

DFT of length-4 step in $\mathbb{C}^{64}$

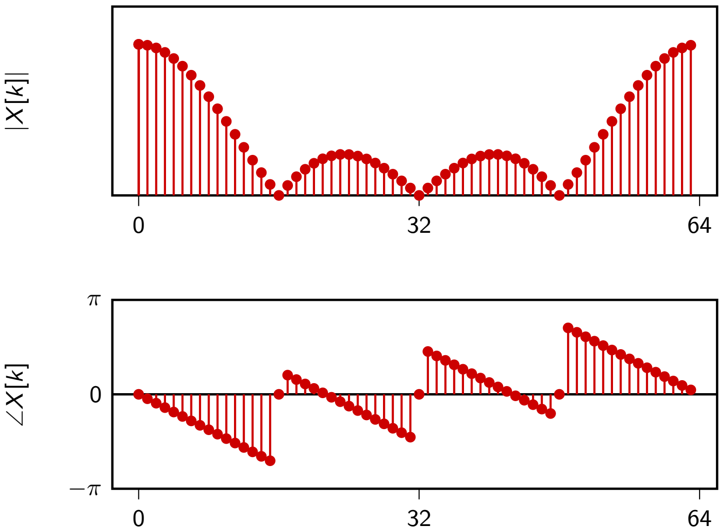


Wrapping the phase

Often the phase is displayed "wrapped" over the $[-\pi, \pi]$ interval.

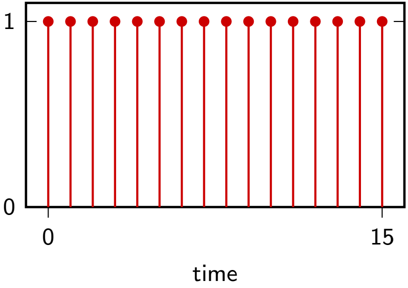
- most numerical packages return wrapped phase
- phase can be unwrapped by adding multiples of 2π

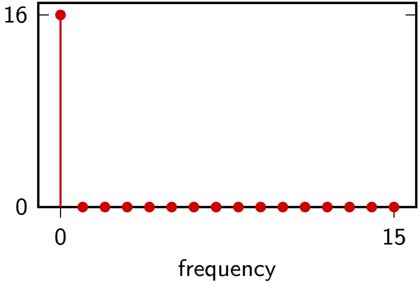
DFT of length-4 step in $\mathbb{C}^{64}$ (phase wrapped)



DFT of $x[n] = 1, \quad x[n] \in \mathbb{C}^N$

$$X[k] = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}nk} = N\delta[k]$$





DFT of harmonic oscillations

$$\mathrm{DFT}\left\{e^{j\frac{2\pi}{N}mn}\right\} = \langle \mathbf{w}^{(k)}, \mathbf{w}^{(m)} \rangle = N\delta[m-k]$$

DFT of $x[n] = 3\cos(2\pi/16\, n)$, $\quad x[n] \in \mathbb{C}^{64}$

$$
\begin{aligned}
x[n] &= 3\cos\left(\frac{2\pi}{16}\, n\right) \\
&= 3\cos\left(\frac{2\pi}{64} 4\, n\right) \\
&= \frac{3}{2}\left[e^{j\frac{2\pi}{64}4n} + e^{-j\frac{2\pi}{64}4n}\right] \\
&= \frac{3}{2}\left[e^{j\frac{2\pi}{64}4n} + e^{j\frac{2\pi}{64}60n}\right] \\
&= \frac{3}{2}(w_4[n] + w_{60}[n])
\end{aligned}
$$

DFT of $x[n] = 3\cos(2\pi/16\, n), \quad x[n] \in \mathbb{C}^{64}$

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DFT of $x[n] = 3\cos(2\pi/16\, n)$, $\quad x[n] \in \mathbb{C}^{64}$

$$
\begin{aligned}
X[k] &= \langle w_k[n], x[n] \rangle \\
&= \langle w_k[n], \frac{3}{2}(w_4[n] + w_{60}[n]) \rangle \\
&= \frac{3}{2}\langle w_k[n], w_4[n] \rangle + \frac{3}{2}\langle w_k[n], w_{60}[n] \rangle \\
&= \begin{cases} 96 & \text{for } k = 4,\ 60 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

DFT of $x[n] = 3\cos(2\pi/16\, n)$, $\quad x[n] \in \mathbb{C}^{64}$

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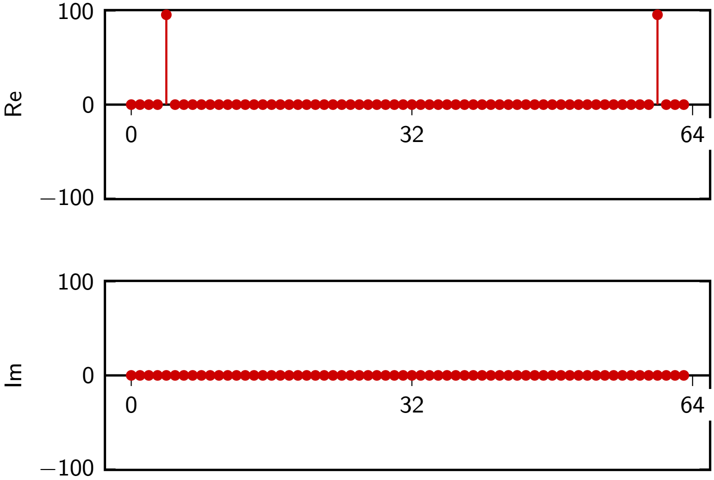
DFT of $x[n] = 3\cos(2\pi/16\, n)$, $\quad x[n] \in \mathbb{C}^{64}$

$$\begin{aligned} X[k] &= \langle w_k[n], x[n] \rangle \\ &= \langle w_k[n], \frac{3}{2}(w_4[n] + w_{60}[n]) \rangle \\ &= \frac{3}{2}\langle w_k[n], w_4[n] \rangle + \frac{3}{2}\langle w_k[n], w_{60}[n] \rangle \\ &= \begin{cases} 96 & \text{for } k = 4,\ 60 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

DFT of $x[n] = 3\cos(2\pi/16\, n)$, $\quad x[n] \in \mathbb{C}^{64}$

$$
\begin{aligned}
X[k] &= \langle w_k[n], x[n] \rangle \\
&= \langle w_k[n], \frac{3}{2}(w_4[n] + w_{60}[n]) \rangle \\
&= \frac{3}{2}\langle w_k[n], w_4[n] \rangle + \frac{3}{2}\langle w_k[n], w_{60}[n] \rangle \\
&= \begin{cases} 96 & \text{for } k = 4,\ 60 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

DFT of $x[n] = 3\cos(2\pi/16\, n)$, $\quad x[n] \in \mathbb{C}^{64}$



DFT of $x[n] = 3\cos(2\pi/16\, n + \pi/3), \quad x[n] \in \mathbb{C}^{64}$

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\begin{aligned}
x[n] &= 3\cos\left(\frac{2\pi}{16}\, n + \frac{\pi}{3}\right) \\
&= 3\cos\left(\frac{2\pi}{64}\, 4\, n + \frac{\pi}{3}\right) \\
&= \frac{3}{2}\left[e^{j\frac{2\pi}{64}4n} e^{j\frac{\pi}{3}} + e^{-j\frac{2\pi}{64}4n} e^{-j\frac{\pi}{3}}\right] \\
&= \frac{3}{2}\left(e^{j\frac{\pi}{3}}\, w_4[n] + e^{-j\frac{\pi}{3}}\, w_{60}[n]\right)
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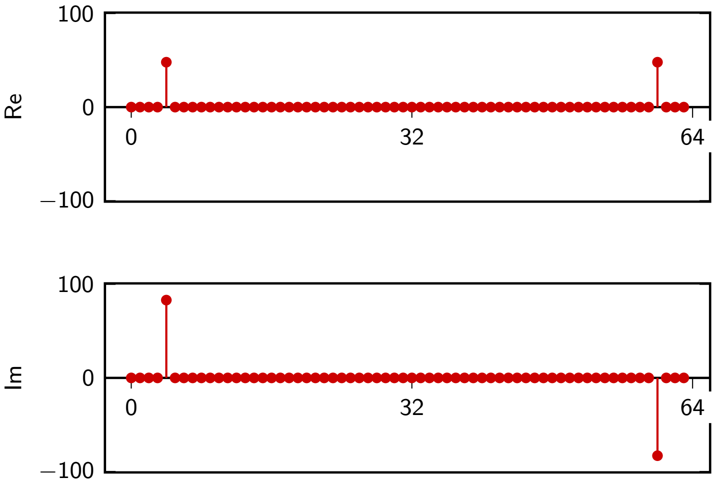

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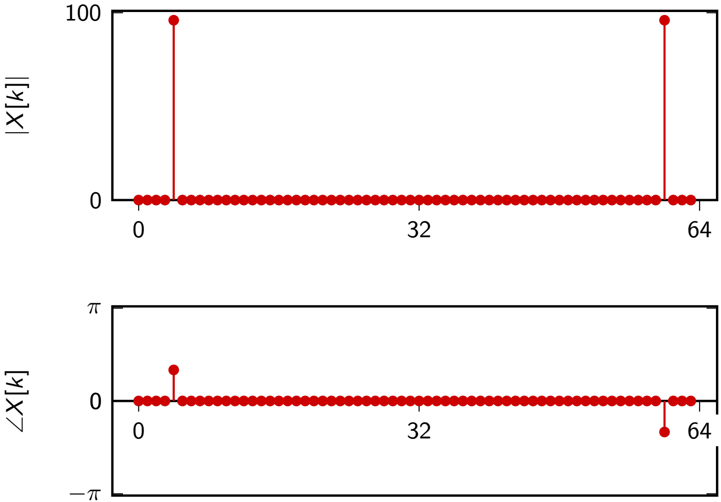


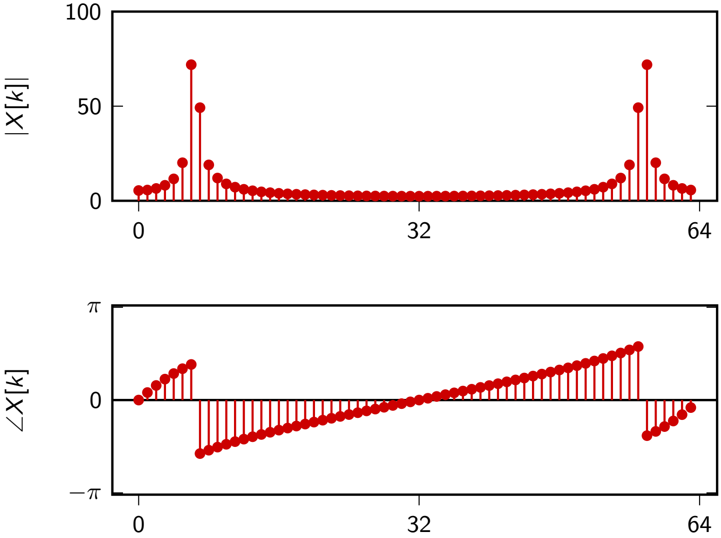
DFT of $x[n] = 3\cos(2\pi/10\, n)$, $\quad x[n] \in \mathbb{C}^{64}$

$$\frac{2\pi}{64}\,6 < \frac{2\pi}{10} < \frac{2\pi}{64}\,7$$

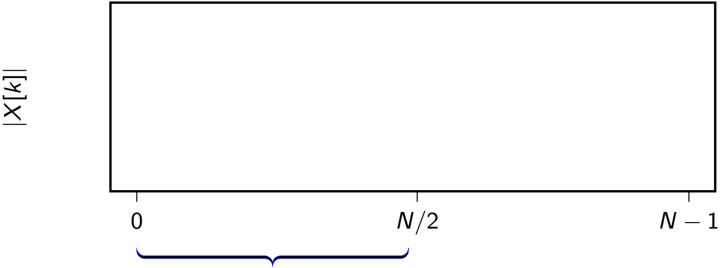
The DFT is an algorithm!

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}, \qquad k = 0, 1, \ldots, N-1$$

DFT of $x[n] = 3\cos(2\pi/10\, n)$, $\quad x[n] \in \mathbb{C}^{64}$

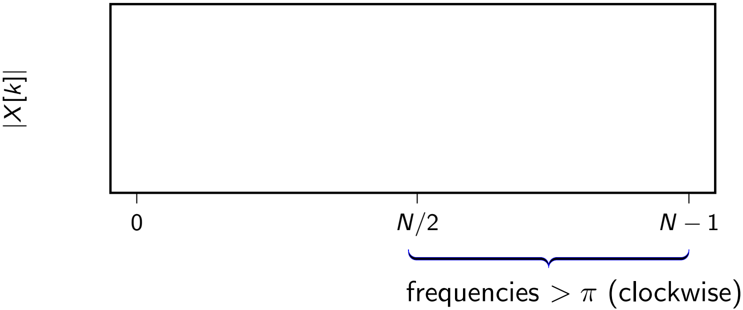


Interpreting a DFT plot



frequencies $< \pi$ (counterclockwise)

Interpreting a DFT plot



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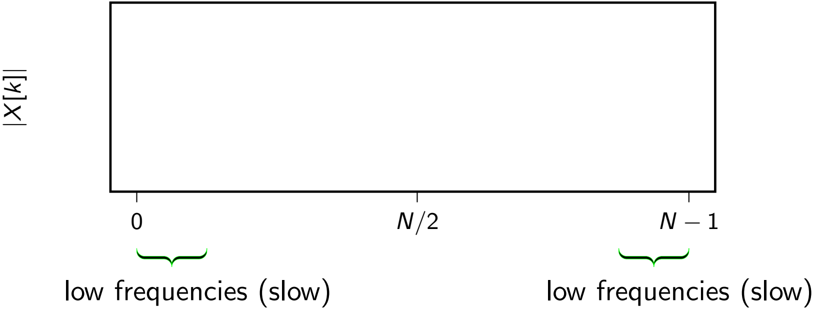

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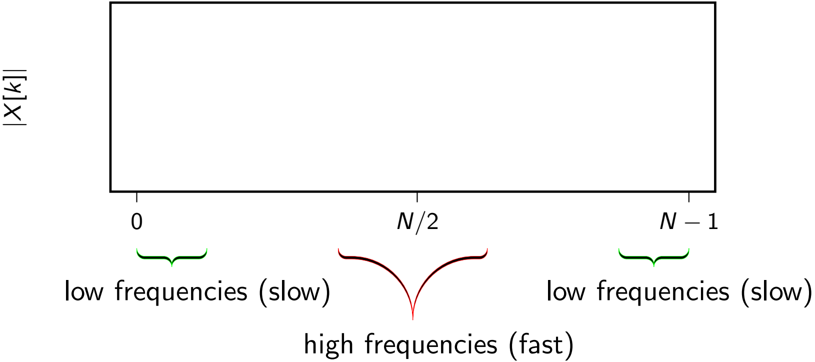
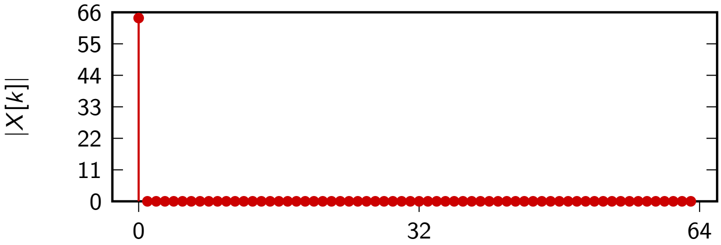


Interpreting a DFT plot

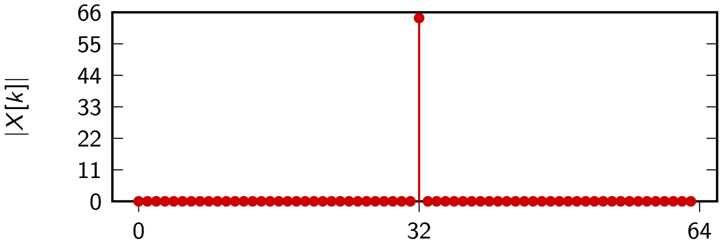
$x[n] = 1$ (slowest signal)



only lowest frequency

Interpreting a DFT plot

$$x[n] = \cos \pi n = (-1)^n \text{ (fastest signal)}$$



only highest frequency

Energy distribution

Recall Parseval's Theorem: $\|\mathbf{x}\|^2 = \sum |\alpha_k|^2$

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

square magnitude of k-th DFT coefficient
proportional to signal's energy at frequency $\omega = (2\pi/N)k$

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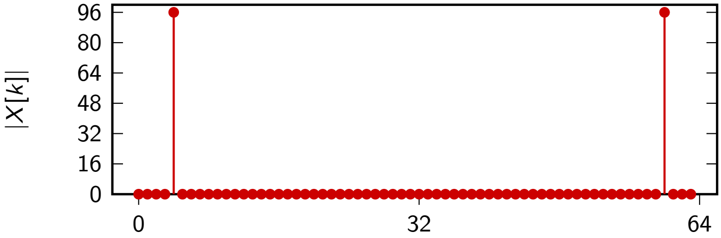
Interpreting a DFT plot

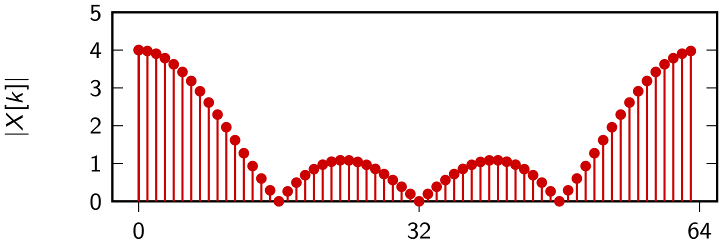
$$x[n] = 3\cos(2\pi/16\, n) \text{ (sinusoid)}$$



energy concentrated on single frequency
(counterclockwise and clockwise combine to give real signal)

Interpreting a DFT plot

$$x[n] = u[n] - u[n-4] \text{ (step)}$$

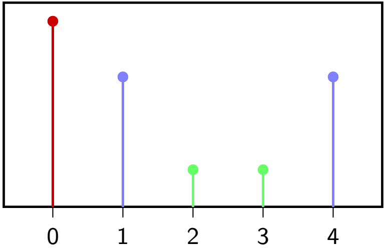


energy mostly in low frequencies

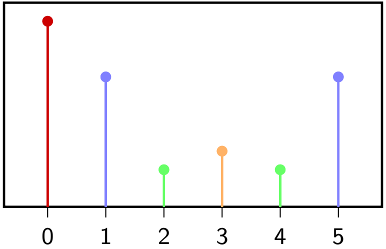
DFT of real signals

For real signals the DFT is “symmetric” in magnitude:

$$|X[k]| = |X[N-k]| \text{ for } k = 1, 2, \ldots, \lfloor N/2 \rfloor$$



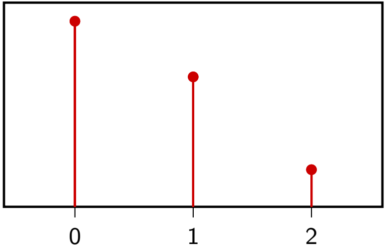
$N = 5$, odd length



$N = 6$, even length

DFT of real signals

For real signals, magnitude plots need only $\lfloor N/2 \rfloor + 1$ points



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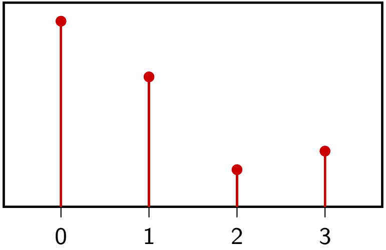


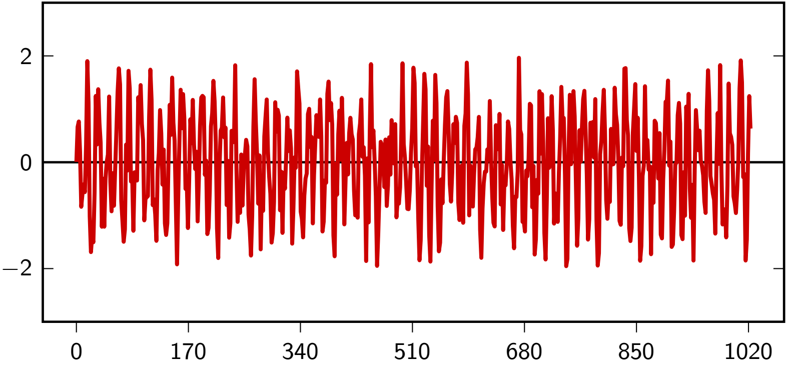
$N = 6$, even length

the DFT as an analysis tool

Overview

- DFT analysis examples
- Labeling the DFT axes

Mystery signal revisited



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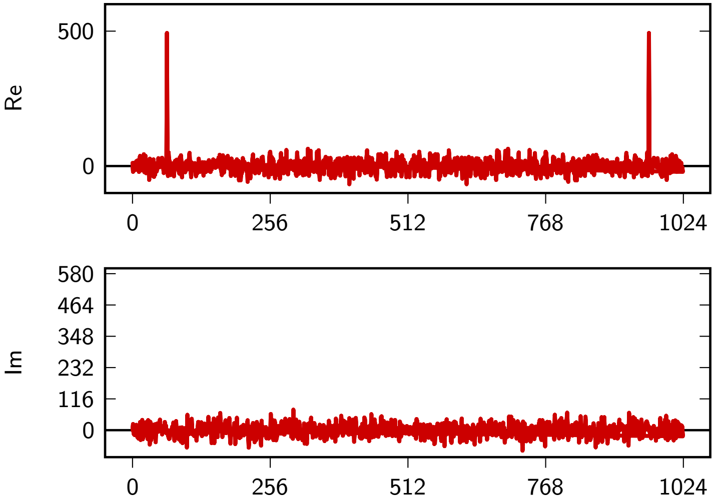

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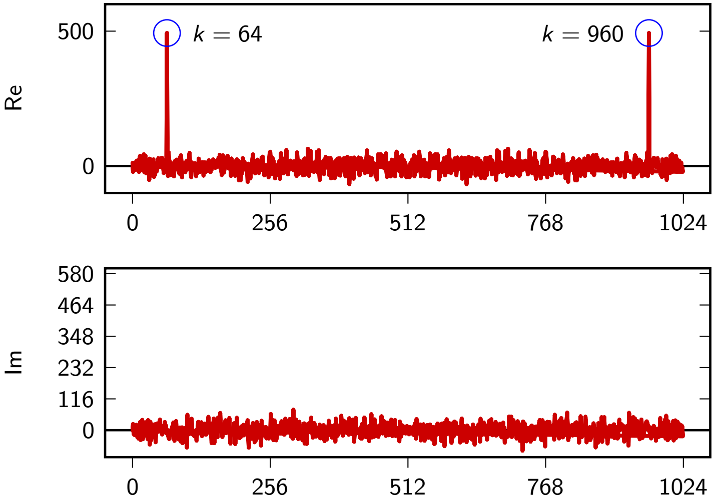

Mystery signal revisited

$$x[n] = \cos(\omega n + \phi) + \eta[n]$$

with

$$\begin{aligned} \phi &= 0 \\ \omega &= \frac{2\pi}{1024}64 \end{aligned}$$

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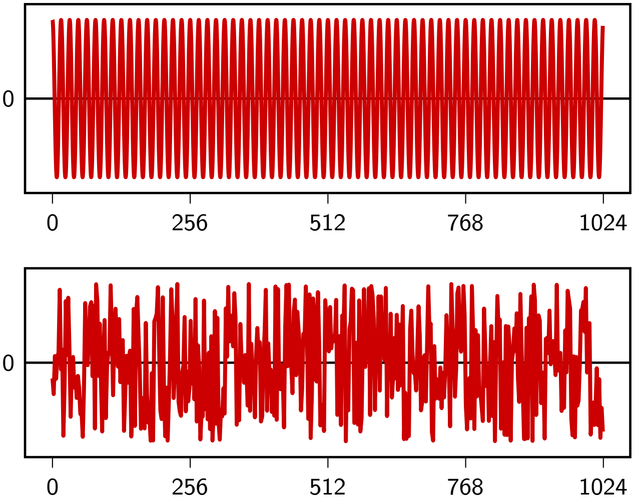
Mystery signal unveiled



Solar spots

- sunspot number: $s = 10 \times \#\text{ of clusters} + \#\text{ of spots}$
- data set from 1749 to 2003, 2904 months

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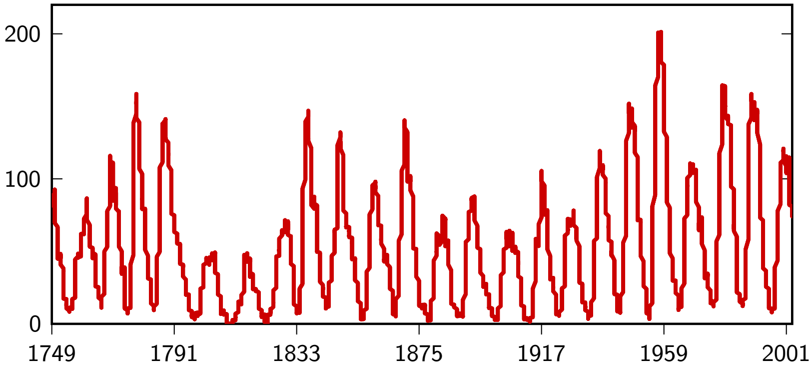

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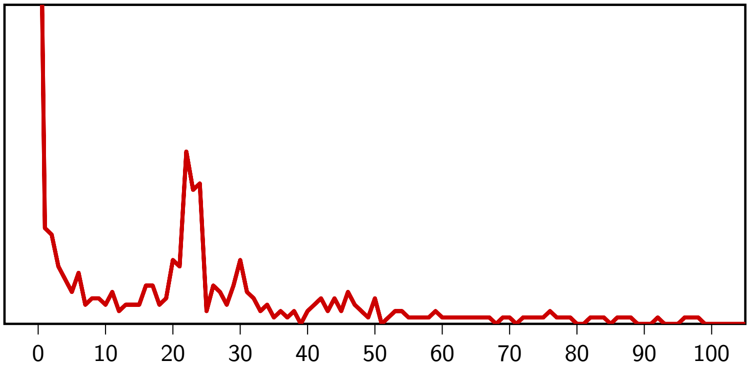

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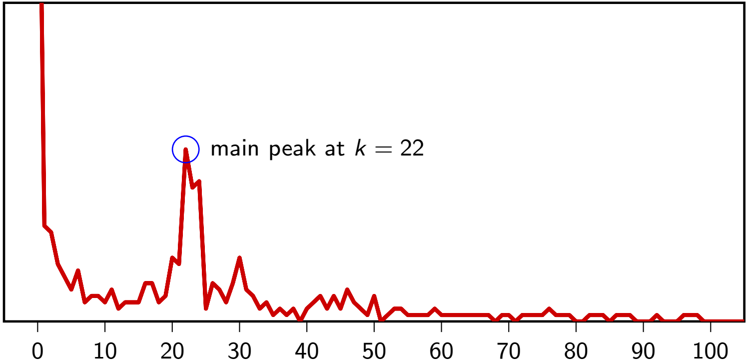

Solar spots

- DFT main peak for $k = 22$
- 22 cycles over 2904 months
- period: $\frac{2904}{22} \approx 11$ years

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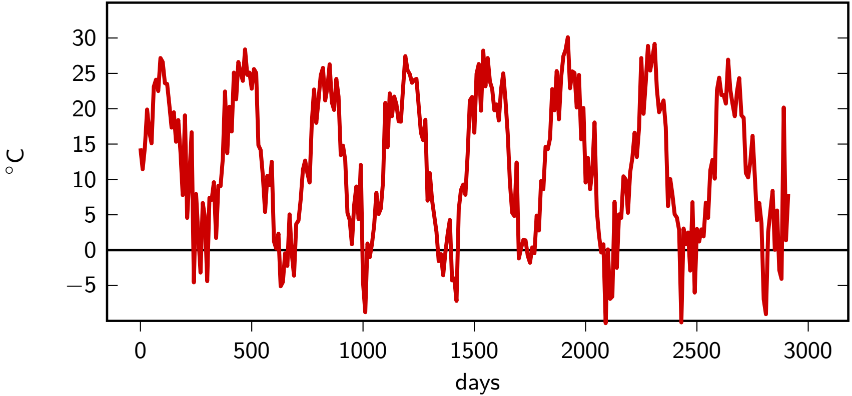
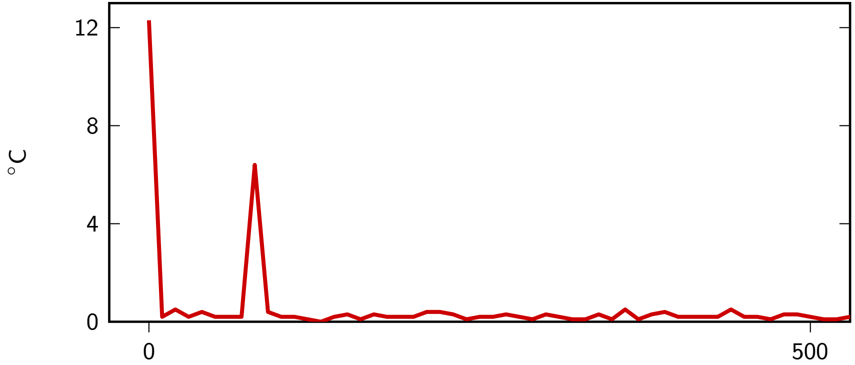
Daily temperature (2920 days)



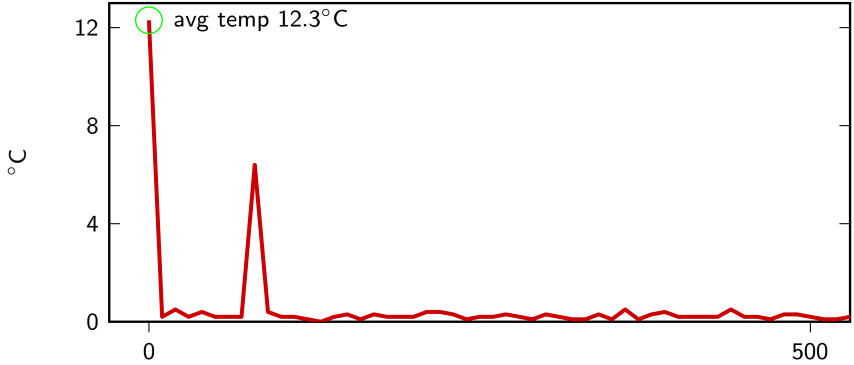
Daily temperature: DFT



first few hundred DFT coefficients

(in magnitude and normalized by the length of the temperature vector)

Daily temperature: DFT



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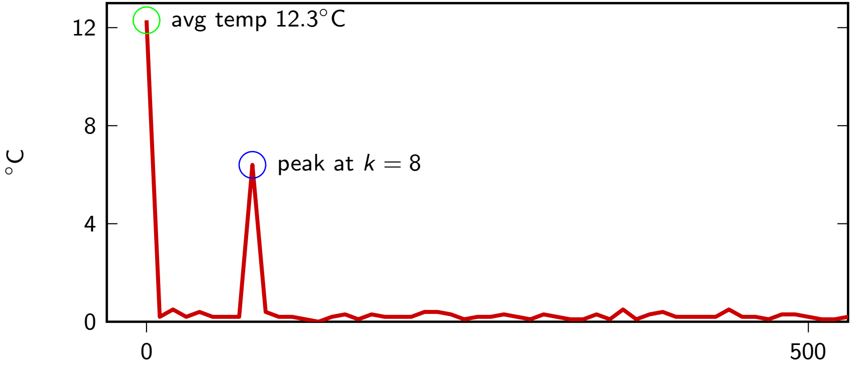

first few hundred DFT coefficients
(in magnitude and normalized by the length of the temperature vector)

Daily temperature

- average value (0-th DFT coefficient): $12.3^\circ\mathrm{C}$
- DFT main peak for $k = 8$, value $6.4^\circ\mathrm{C}$
- 8 cycles over 2920 days
- period: $\dfrac{2920}{8} = 365$ days
- temperature excursion: $12.3^\circ\mathrm{C} \pm 12.8^\circ\mathrm{C}$

Daily temperature

In case you're wondering why $\pm 12.8^\circ$:

$$\mathrm{DFT}\left\{A\cos\left(\frac{2\pi}{N}Mn\right)\right\}[k] = \begin{cases} \frac{A}{2}N & \text{for } k = M, N-M \\ 0 & \text{otherwise} \end{cases}$$

Labeling the frequency axis

If we know the "clock" of the system T_s (or its frequency F_s)

- fastest (positive) frequency is $\omega = \pi$
- sinusoid at $\omega = \pi$ needs two samples to do a full revolution
- time between samples: $T_s = 1/F_s$ seconds
- real-world period for fastest sinusoid: $2T_s$ seconds
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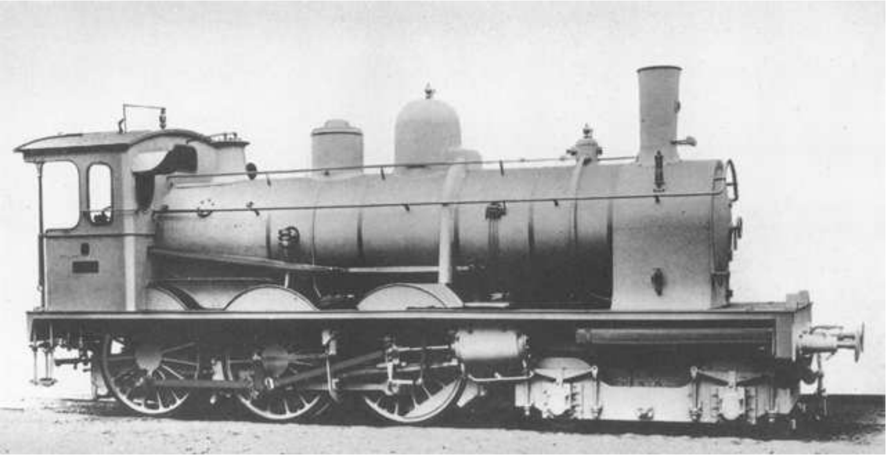
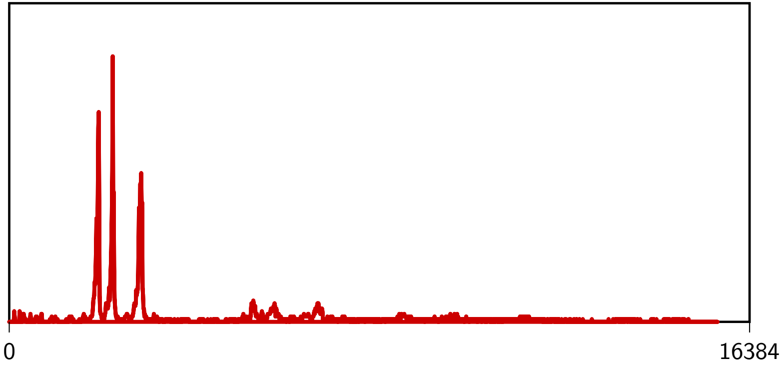
Example: train whistle



Play

Example: train whistle

32768 samples (the "clock" of the system $F_s = 8000\text{Hz}$)



Example: train whistle

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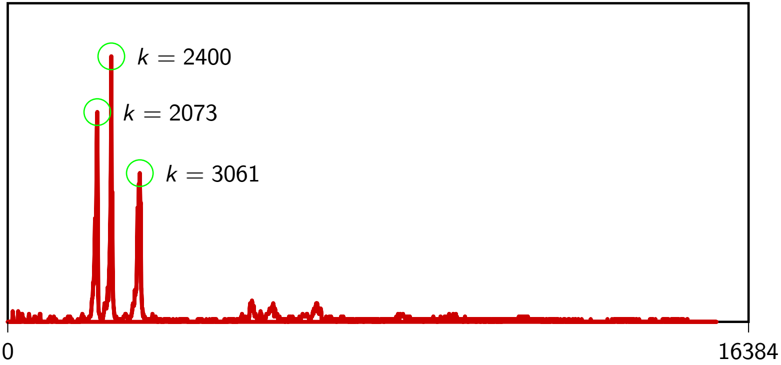

Example: train whistle

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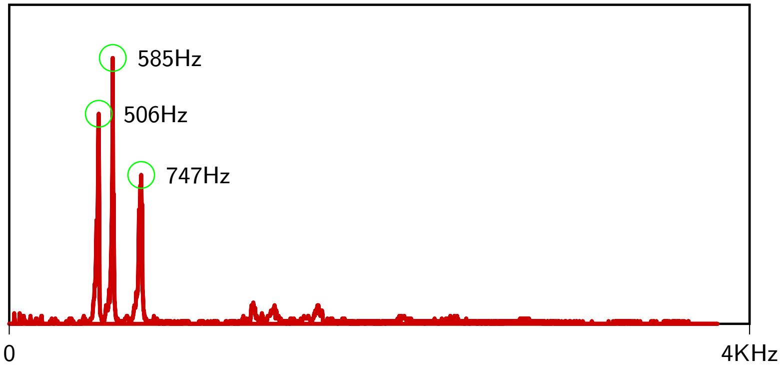
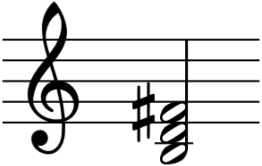


Example: train whistle

if we look up the frequencies:



B minor chord

the DFT as a synthesis tool

DFT formulas

Analysis formula:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}, \qquad k = 0, 1, \ldots, N-1$$

N-point signal in the *frequency domain*

Synthesis formula:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk}, \qquad n = 0, 1, \ldots, N-1$$

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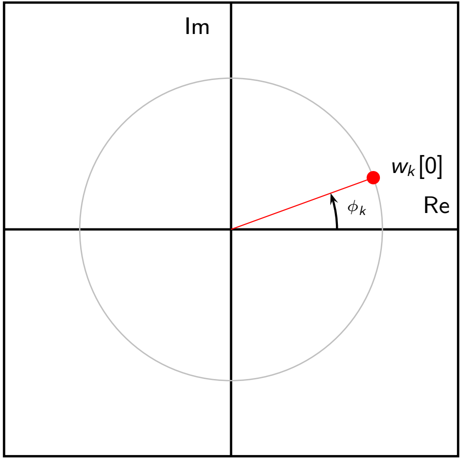
Synthesis: the sinusoidal generator

$$w_k[n] = e^{j(\frac{2\pi}{N}kn+\phi_k)}$$



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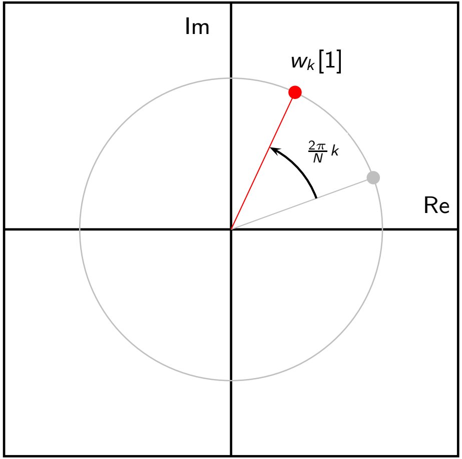

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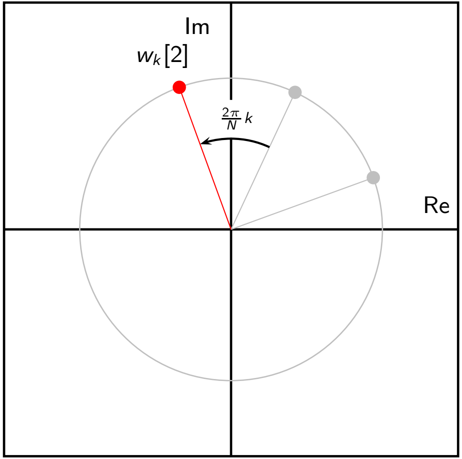

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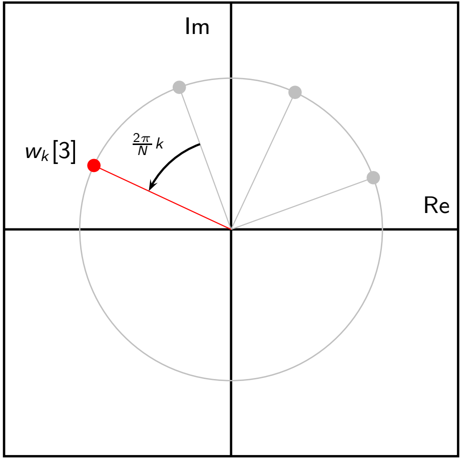

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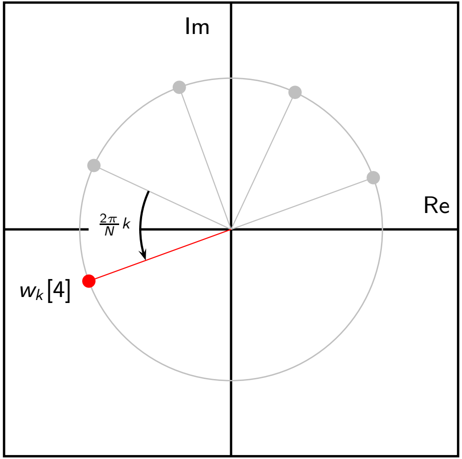

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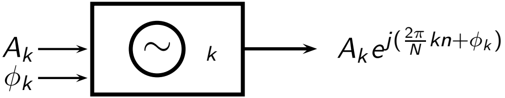
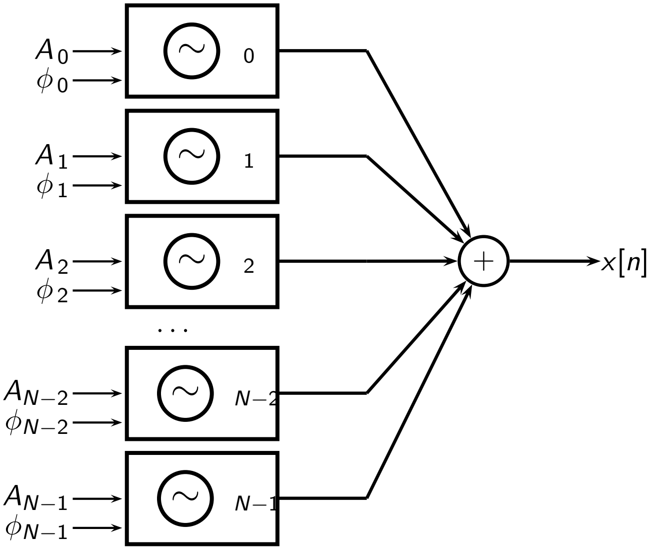


DFT synthesis formula



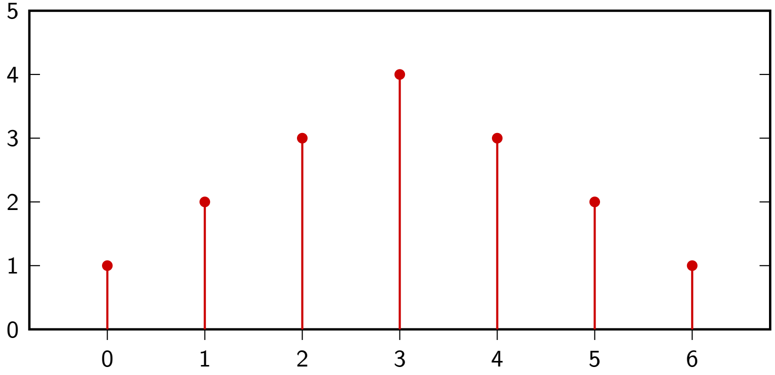
Initializing the machine

$$A_k = |X[k]|/N$$

$$\phi_k = \angle X[k]$$

Example

$$\mathbf{x} = [1\ 2\ 3\ 4\ 3\ 2\ 1]^T$$



Example

k	A_k	ϕ_k
0	2.2857	0.0000
1	0.7213	-2.6928
2	0.0440	0.8976
3	0.0919	-1.7952
4	0.0919	1.7952
5	0.0440	-0.8976
6	0.7213	2.6928

Example

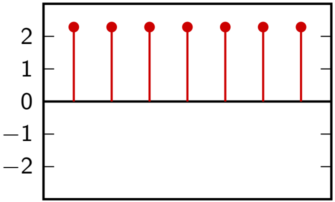
$k = 0$

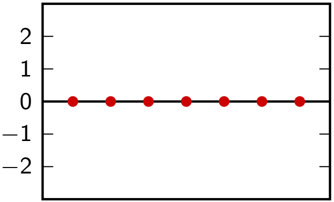
Re

Im

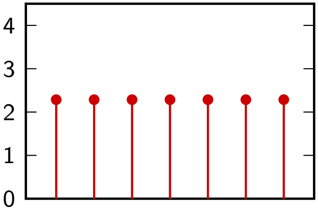
$A_k e^{j(\frac{2\pi}{N}kn+\phi_k)}$

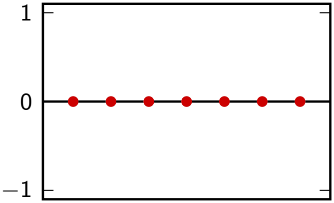
$A_0 = 2.28, \phi_0 = 0$





$$\sum_{k=0}^{0} A_k e^{j(\frac{2\pi}{N}kn+\phi_k)}$$





Example

$k = 1$

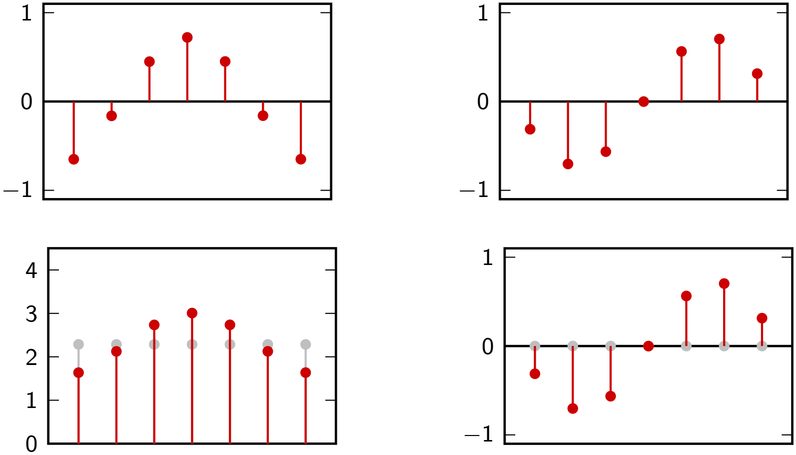
Re

Im

$A_k e^{j(\frac{2\pi}{N} kn + \phi_k)}$

$A_1 = 0.72, \phi_1 = -2.69$

$\sum_{k=0}^{1} A_k e^{j(\frac{2\pi}{N} kn + \phi_k)}$



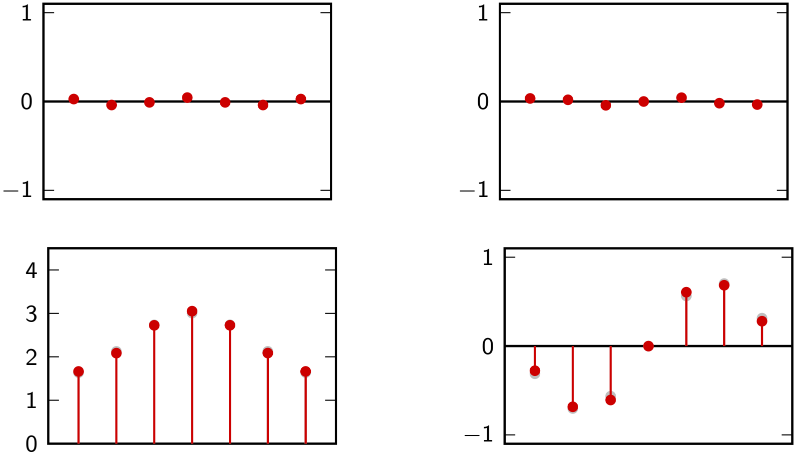
Example

$k = 2$

Re Im

$A_k e^{j(\frac{2\pi}{N}kn+\phi_k)}$

$A_2 = 0.04, \phi_2 = 0.89$



$$\sum_{k=0}^{2} A_k e^{j(\frac{2\pi}{N}kn+\phi_k)}$$

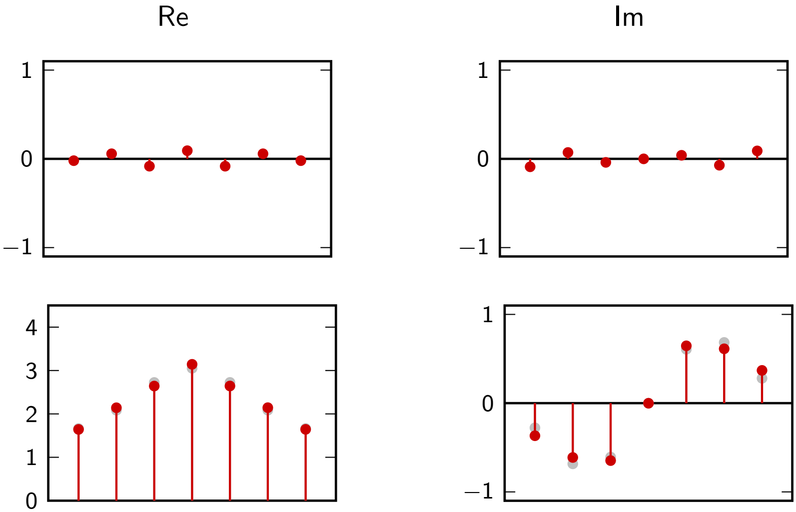
Example

$k = 3$

$$A_k e^{j(\frac{2\pi}{N}kn+\phi_k)}$$

$$A_3 = 0.09, \phi_3 = -1.79$$

$$\sum_{k=0}^{3} A_k e^{j(\frac{2\pi}{N}kn+\phi_k)}$$



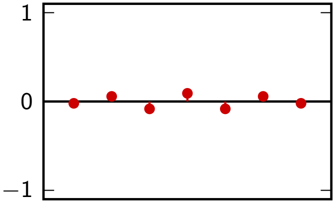
Example

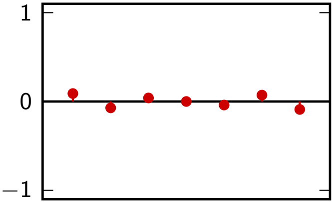
$k = 4$

Re

Im

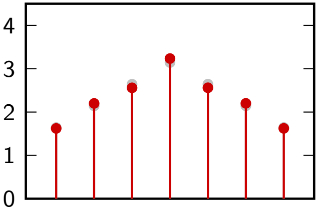
$A_k e^{j\left(\frac{2\pi}{N}kn+\phi_k\right)}$

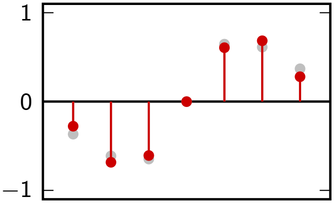




$A_4 = 0.09, \phi_4 = 1.79$

$$\sum_{k=0}^{4} A_k e^{j\left(\frac{2\pi}{N}kn+\phi_k\right)}$$





Example

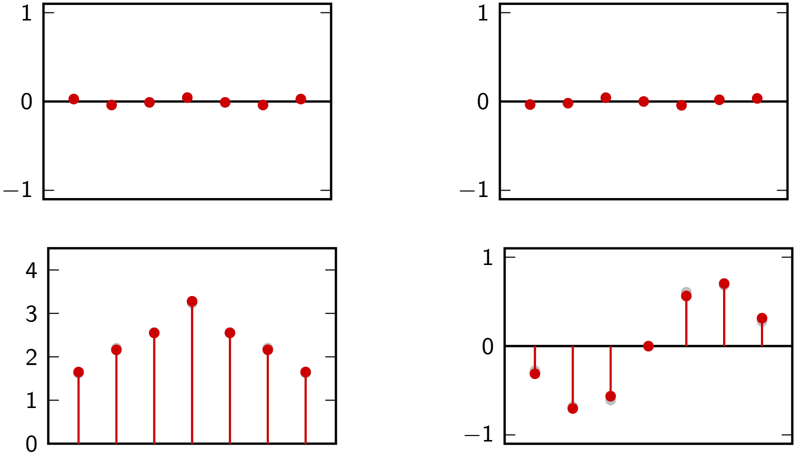
$k = 5$

Re Im

$$A_k e^{j\left(\frac{2\pi}{N}kn+\phi_k\right)}$$

$$A_5 = 0.04, \phi_5 = -0.89$$

$$\sum_{k=0}^{5} A_k e^{j\left(\frac{2\pi}{N}kn+\phi_k\right)}$$



Example

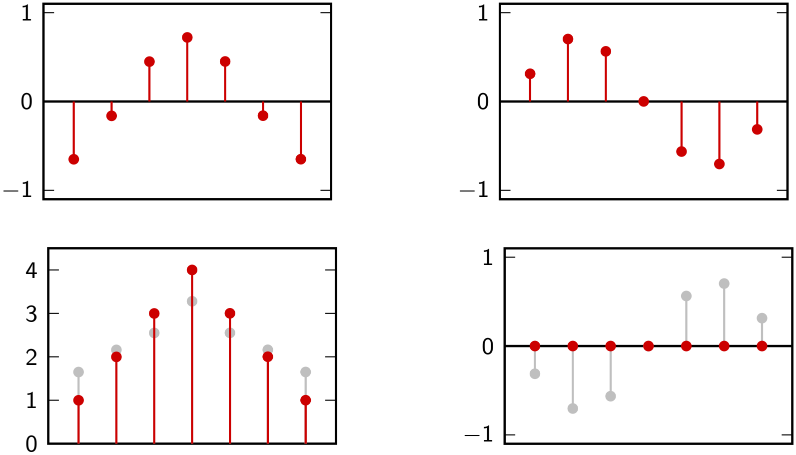
$k = 6$

$A_k e^{j(\frac{2\pi}{N} kn + \phi_k)}$

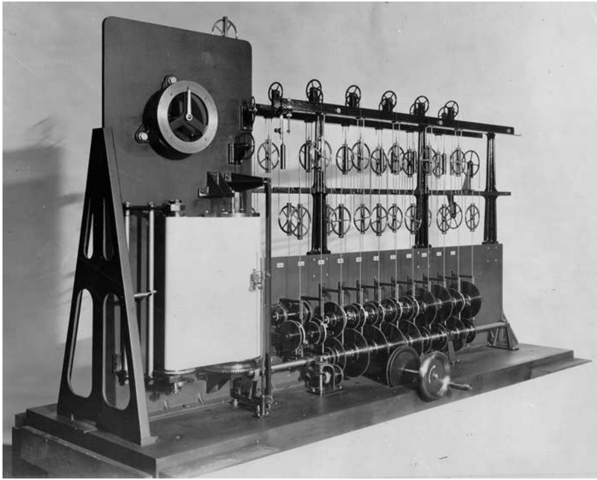
$A_6 = 0.72, \phi_6 = 2.69$

$$\sum_{k=0}^{6} A_k e^{j(\frac{2\pi}{N} kn + \phi_k)}$$

Re Im



The machine before DSP



tide-predicting machine (originally invented by Lord Kelvin)

Wonderful website

http://jackschaedler.github.io/circles-sines-signals

Running the machine too long...

$$x[n+N] = x[n]$$

output signal is N-periodic!

Inherent periodicities in the DFT

the synthesis formula:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk}, \qquad n = 0, 1, \ldots, N-1$$

produces an N-point signal in the time domain

the analysis formula:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}, \qquad k = 0, 1, \ldots, N-1$$

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Discrete Fourier Series (DFS)

$$\text{DFS} = \text{DFT with periodicity explicit}$$

- the DFS maps an N-periodic signal onto an N-periodic sequence of Fourier coefficients
- the inverse DFS maps an N-periodic sequence of Fourier coefficients a set onto an N-periodic signal
- the DFS of an N-periodic signal is mathematically equivalent to the DFT of one period

Finite-length time shifts revisited

The DFS helps us understand how to define time shifts for finite-length signals.

For an N- periodic sequence $\tilde{x}[n]$:

- $\tilde{x}[n-M]$ is well-defined for all $M \in \mathbb{N}$
- $\text{DFS}\left\{\tilde{x}[n-M]\right\} = e^{-j\frac{2\pi}{N}Mk}\tilde{X}[k]$ (easy derivation)
- $\text{IDFS}\left\{\tilde{X}[k]\right\} = \tilde{x}[n-M]$

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- $\text{IDFS}\left\{\boxed{e^{-j\frac{2\pi}{N}Mk}}\ \tilde{X}[k]\right\} = \tilde{x}[n-M]$

a delay in time becomes a *linear phase* factor in frequency

Finite-length time shifts revisited

For an N-point signal $x[n]$:

- $x[n - M]$ is *not* well-defined
- what is $\text{IDFT}\left\{e^{-j\frac{2\pi}{N}Mk}\, X[k]\right\}$?

Finite-length time shifts revisited

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- $x[n - M]$ is *not* well-defined
- what is $\text{IDFT}\left\{e^{-j\frac{2\pi}{N}Mk}\;X[k]\right\}$?

Finite-length time shifts revisited

$$\mathrm{IDFT}\left\{e^{-j\frac{2\pi}{N}Mk}\,X[k]\right\}[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{-j\frac{2\pi}{N}Mk}\,e^{j\frac{2\pi}{N}nk}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1} x[m]\,e^{-j\frac{2\pi}{N}mk}\right)\,e^{-j\frac{2\pi}{N}Mk}\,e^{j\frac{2\pi}{N}nk}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x[m]\sum_{k=0}^{N-1}\,e^{j\frac{2\pi}{N}(n-M-m)k}$$

Finite-length time shifts revisited

$$\begin{aligned}
\mathrm{IDFT}\left\{e^{-j\frac{2\pi}{N}Mk}\,X[k]\right\}[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{-j\frac{2\pi}{N}Mk}\,e^{j\frac{2\pi}{N}nk} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1} x[m]\,e^{-j\frac{2\pi}{N}mk}\right)\,e^{-j\frac{2\pi}{N}Mk}\,e^{j\frac{2\pi}{N}nk} \\
&= \frac{1}{N}\sum_{m=0}^{N-1} x[m]\sum_{k=0}^{N-1}\,e^{j\frac{2\pi}{N}(n-M-m)k}
\end{aligned}$$

Finite-length time shifts revisited

$$\begin{aligned}
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&= \frac{1}{N}\sum_{m=0}^{N-1} x[m]\sum_{k=0}^{N-1}\,e^{j\frac{2\pi}{N}(n-M-m)k}
\end{aligned}$$

We've seen something like this before...

$$\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}nk} = \begin{cases} N & \text{if } k \text{ multiple of } N \\ 0 & \text{otherwise} \end{cases}$$

(remember the orthogonality proof for the DFT basis)

Finite-length time shifts revisited

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}((n-M)-m)k} = \begin{cases} N & \text{for } ((n-M)-m) \text{ multiple of } N \\ 0 & \text{otherwise} \end{cases}$$

is there an m such that $0 \leq m < N$ and $((n-M)-m)$ multiple of N?

Finite-length time shifts revisited

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Modulo operator

given $C \in \mathbb{N}$, find m such that $0 \leq m < N$ and $C - m$ is a multiple of N

any integer C can be written as $C = pN + (C \mod N), \quad p \in \mathbb{N}$:

- $0 \leq (C \mod N) < N$
- $C - (C \mod N)$ is a multiple of N

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Finite-length time shifts revisited

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-M-m)k} = \begin{cases} N & \text{for } m = (n-M) \mod N \\ 0 & \text{otherwise} \end{cases}$$

$$\text{IDFT}\left\{e^{-j\frac{2\pi}{N}Mk}X[k]\right\}[n] = \frac{1}{N}\sum_{m=0}^{N-1} x[m] \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-M-m)k}$$

$$= x[(n-M) \mod N]$$

shifts for finite-length signals are "naturally" circular

Finite-length time shifts revisited

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