COM303: Digital Signal Processing

Lecture 2: Discrete-Time Signals

Module Overview:

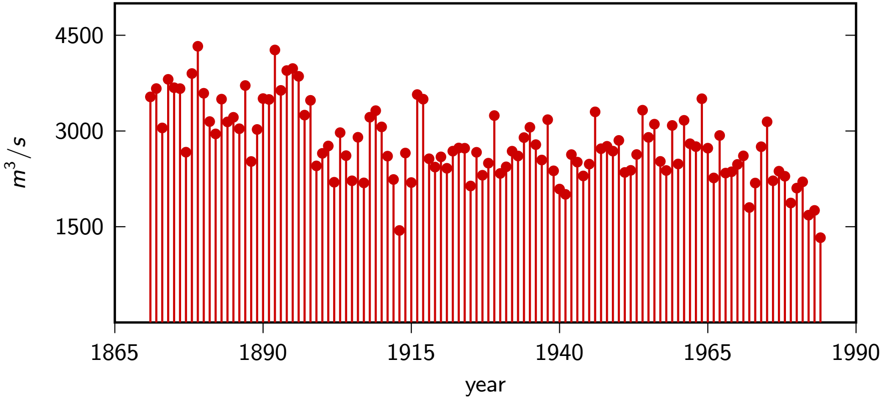
- discrete-time signals
- elementary signal operations
- the Karplus-Strong algorithm

Discrete-time signals have a long tradition...

Meteorology (limnology): the floods of the Nile

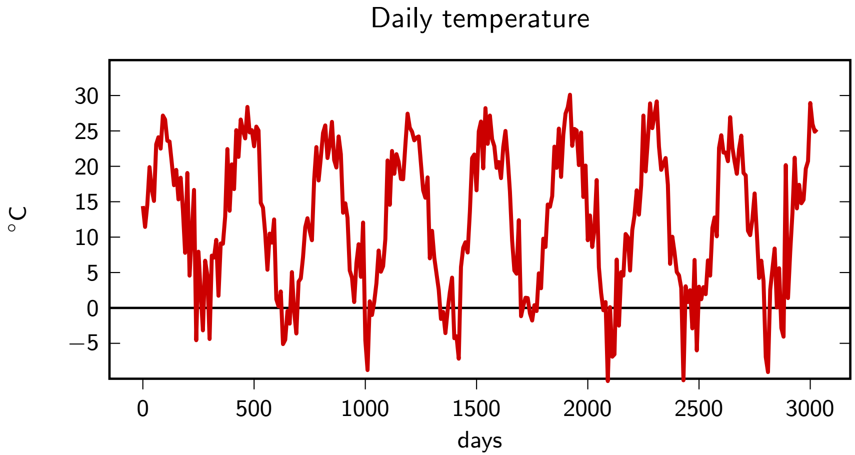
Representations of flood data: circa 2500 BC

Discrete-time signals have a long tradition...



Representations of flood data: circa AD 2000

Probably your first scientific experiment...



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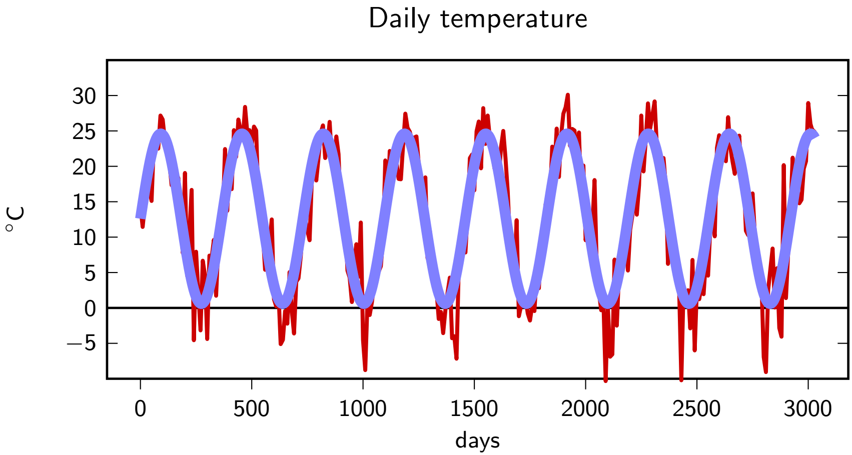
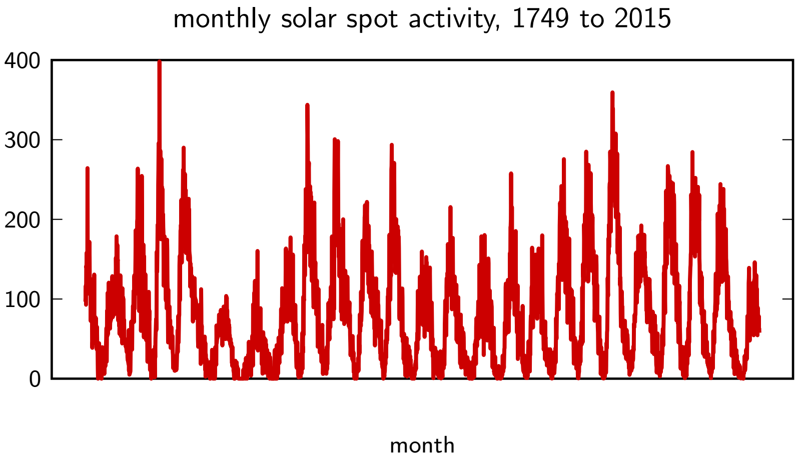
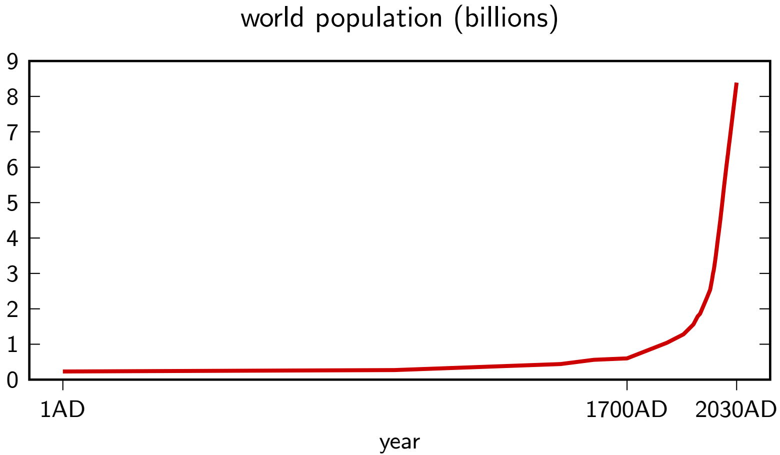


Astronomy

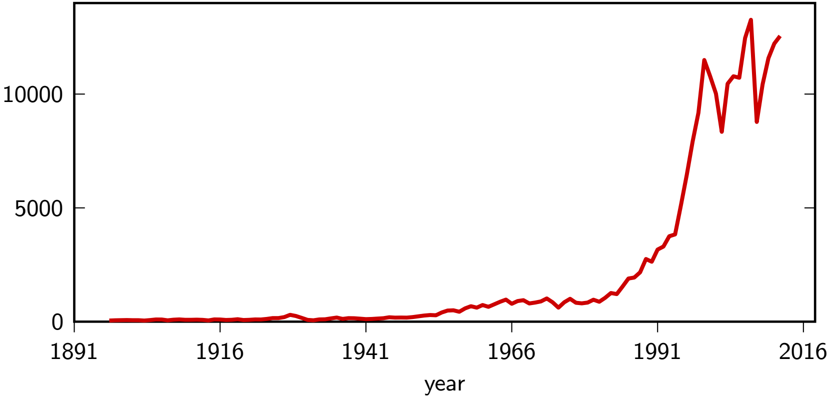

monthly solar spot activity, 1749 to 2015
400
300
200
100
0
month

History and sociology



Economics

a purely man-made signal: the Dow Jones industrial average



More formally...

discrete-time signal: a sequence of **complex** numbers

- one dimension (for now)
- notation: $x[n]$
- two-sided sequences: $x : \mathbb{Z} \to \mathbb{C}$
- n is *a-dimensional* "time"
- analysis: periodic measurement
- synthesis: stream of generated samples

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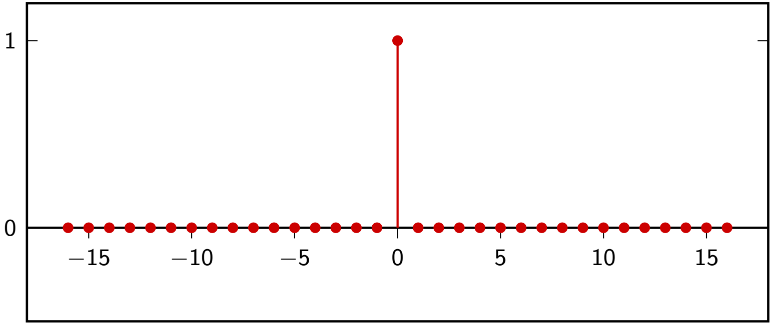
The delta signal

$$x[n] = \delta[n]$$



How do you synchronize audio and video...

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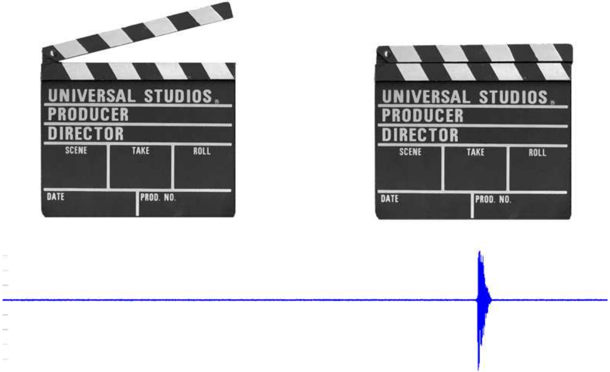
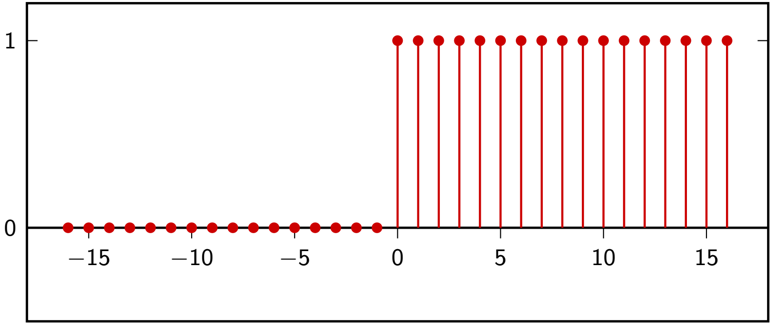


The unit step

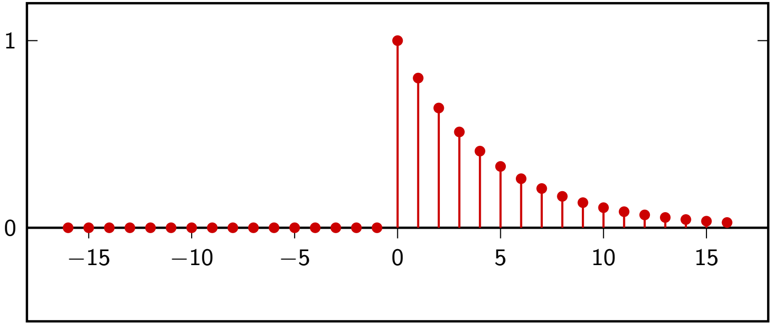
$$x[n] = u[n]$$



The Frankenstein switch...

The exponential decay

$$x[n] = |a|^n \, u[n], \quad |a| < 1$$



How fast does your coffee get cold...

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Newton's law of cooling:

$$\frac{dT}{dt} = -c(T - T_{\text{env}})$$

$$T(t) = T_{\text{env}} + (T_0 - T_{\text{env}})e^{-ct}$$

In practice:

- must have convection only
- must have large conductivity

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In practice:

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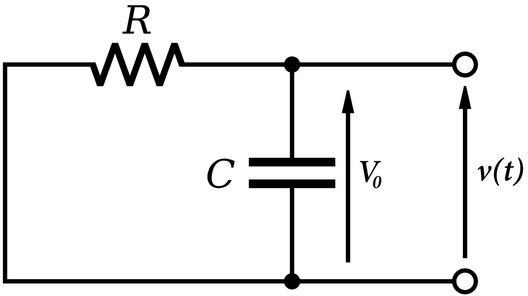
- must have large conductivity

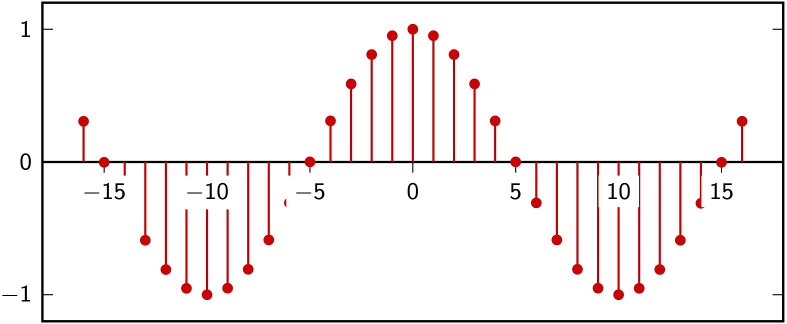
Also, how fast your capacitor discharges



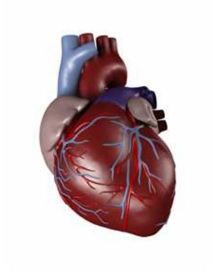
$$v(t) = V_0 \, e^{-\frac{t}{RC}}$$

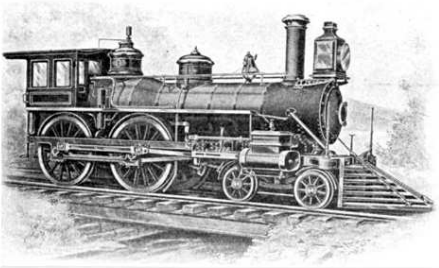
The sinusoid

$$x[n] = \sin(\omega_0 n + \theta)$$



Oscillations are everywhere!





Four signal classes

- finite-length
- infinite-length
- periodic
- finite-support

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Finite-length signals

- sequence notation: $x[n], \quad n = 0, 1, \ldots, N-1$
- vector notation: $\mathbf{x} = \begin{bmatrix} x_0 & x_1 & \ldots & x_{N-1} \end{bmatrix}^T$
- practical entities, good for numerical packages (e.g.numpy)

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Periodic signals

- N-periodic sequence: $\tilde{x}[n] = \tilde{x}[n + kN], \quad n, k, N \in \mathbb{Z}$
- $\tilde{\mathbf{x}} = \begin{bmatrix} \ldots & x_{N-2} & x_{N-1} & x_0 & x_1 & \ldots & x_{N-1} & x_0 & x_1 & \ldots \end{bmatrix}^T$
- infinite length but same amount of information as length-N
- "natural" bridge between finite and infinite lengths

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Finite-support signals

- Finite-support sequence:

$$\bar{x}[n] = \begin{cases} x[n] & \text{if } 0 \leq n < N \\ 0 & \text{otherwise} \end{cases} \qquad n \in \mathbb{Z}$$

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Elementary operators

- scaling:

$$y[n] = \alpha x[n]$$

- sum:

$$y[n] = x[n] + z[n]$$

- product:

$$y[n] = x[n] \cdot z[n]$$

- shift by k (delay):

$$y[n] = x[n-k]$$

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Shift of a finite-length: finite-support

$$[x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7]$$

Shift of a finite-length: finite-support

$$x[n]$$

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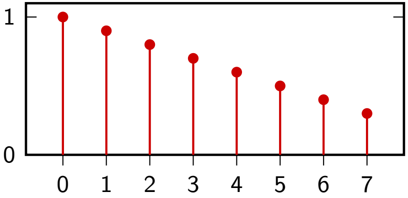

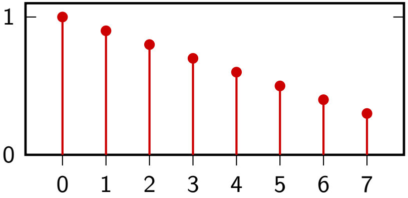

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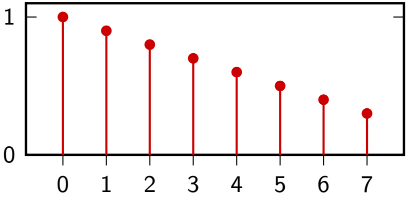
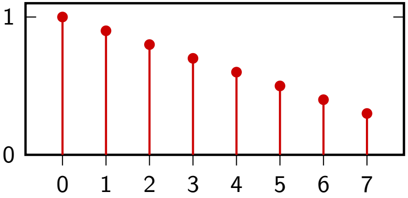
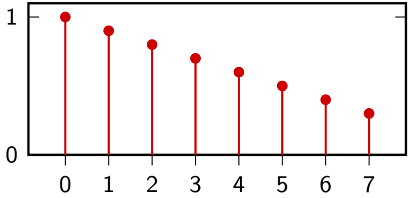


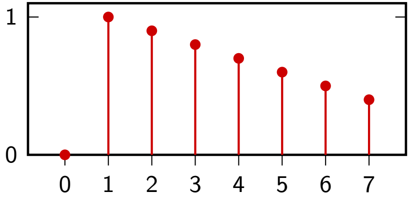


Shift of a finite-length: finite-support

$$\bar{x}[n-1]$$

$$\ldots \quad 0 \quad 0 \quad 0 \quad \boxed{0 \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6} \quad x_7 \quad 0 \quad 0 \quad \ldots$$





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$$\bar{x}[n-2]$$

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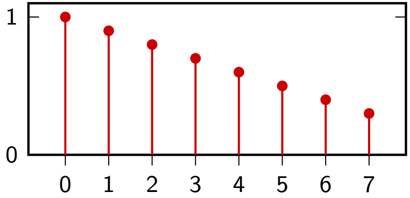

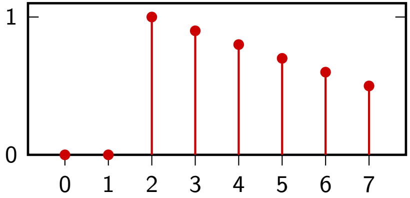

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$$\bar{x}[n-3]$$

$$\ldots \quad 0 \quad 0 \quad 0 \quad \boxed{0 \quad 0 \quad 0 \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4} \quad x_5 \quad x_6 \quad x_7 \quad \ldots$$

Shift of a finite-length: finite-support

$$\bar{x}[n-4]$$

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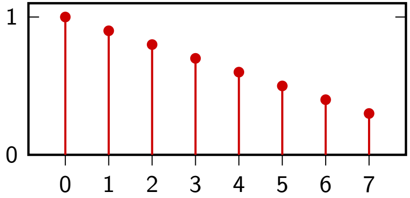
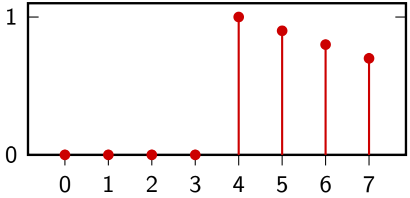
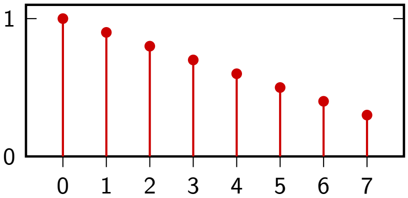
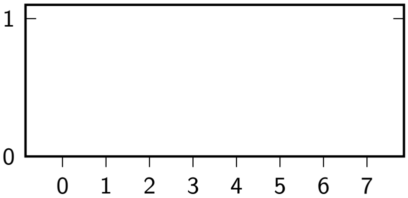




Shift of a finite-length: periodic extension

$$[x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7]$$





Shift of a finite-length: periodic extension

$x[n]$

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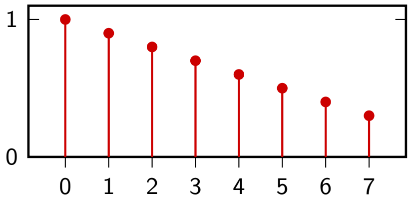
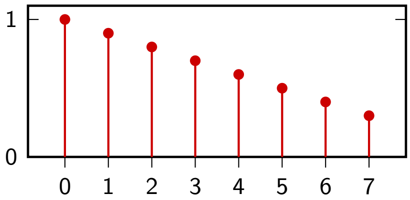
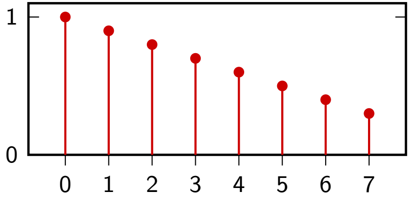


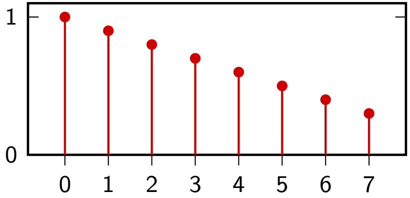


Shift of a finite-length: periodic extension

$$\tilde{x}[n]$$

$$\ldots \quad x_5 \quad x_6 \quad x_7 \quad \boxed{x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7} \quad x_0 \quad x_1 \quad x_2 \quad \ldots$$

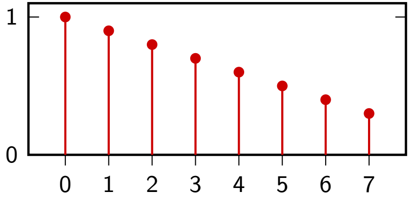


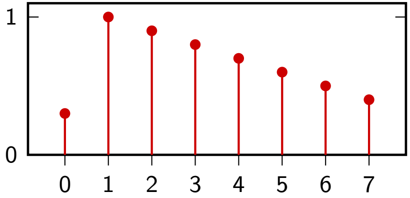


Shift of a finite-length: periodic extension

$$\tilde{x}[n-1]$$

$$\ldots \quad x_4 \quad x_5 \quad x_6 \quad \boxed{x_7 \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6} \quad x_7 \quad x_0 \quad x_1 \quad \ldots$$

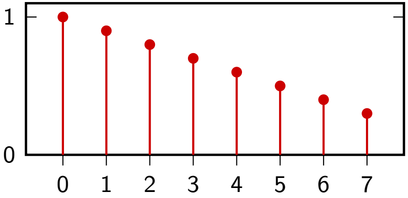


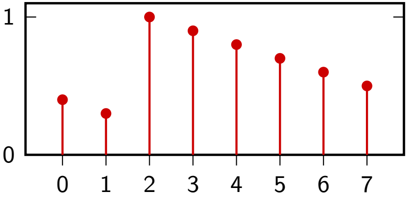


Shift of a finite-length: periodic extension

$$\tilde{x}[n-2]$$

$$\ldots \; x_3 \; x_4 \; x_5 \; \boxed{x_6 \; x_7 \; x_0 \; x_1 \; x_2 \; x_3 \; x_4 \; x_5} \; x_6 \; x_7 \; x_0 \; \ldots$$

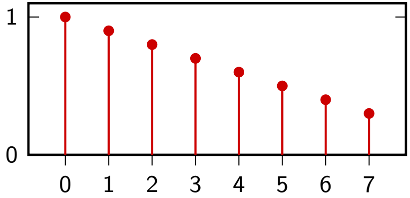


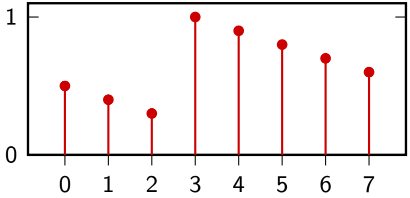


Shift of a finite-length: periodic extension

$$\tilde{x}[n-3]$$

$$\ldots \; x_2 \; x_3 \; x_4 \; \boxed{x_5 \; x_6 \; x_7 \; x_0 \; x_1 \; x_2 \; x_3 \; x_4} \; x_5 \; x_6 \; x_7 \; \ldots$$

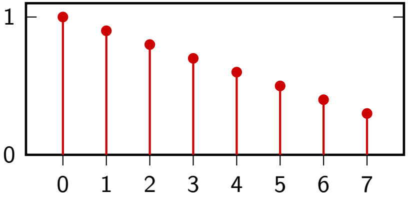


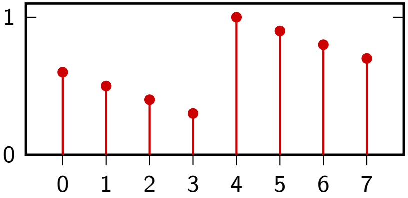


Shift of a finite-length: periodic extension

$$\tilde{x}[n-4]$$

$$\ldots\ x_1\ \ x_2\ \ x_3\ \ \boxed{x_4\ \ x_5\ \ x_6\ \ x_7\ \ x_0\ \ x_1\ \ x_2\ \ x_3}\ \ x_4\ \ x_5\ \ x_6\ \ \ldots$$





Energy and power

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

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Energy and power: periodic signals

$$E_{\tilde{x}} = \infty$$

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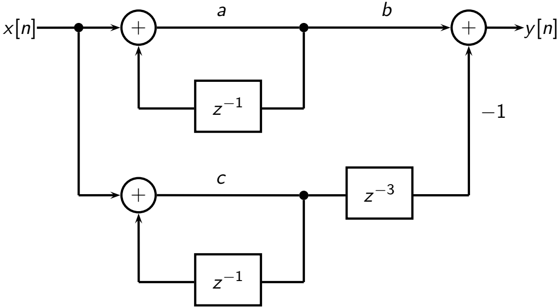
Let’s play with LEGO!

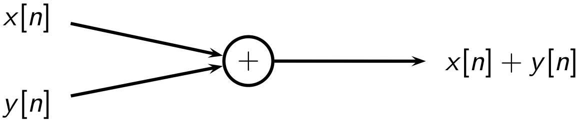
Overview:

- DSP as Lego: The fundamental building blocks
- Averages and moving averages
- Recursion: Revisiting your bank account
- Building a simple recursive synthesizer
- Examples of sounds

DSP as Lego



Building Blocks: Adder



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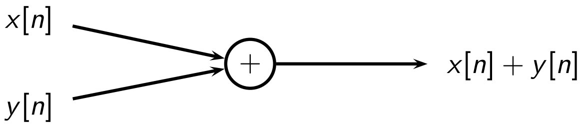
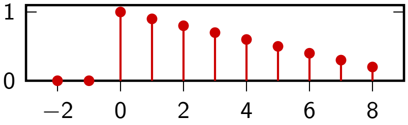


$x[n] = 1 - n/10$



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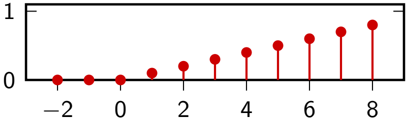

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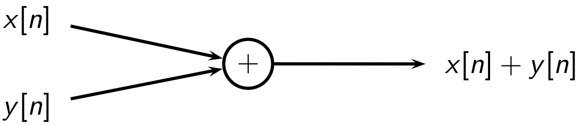

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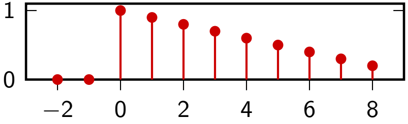

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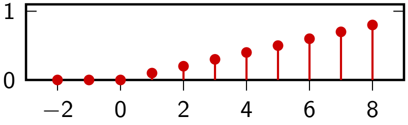

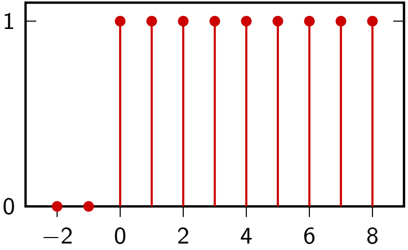

Building Blocks: Multiplier

$$x[n] \xrightarrow{\quad\alpha\quad} \alpha x[n]$$

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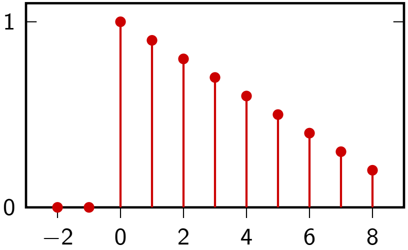

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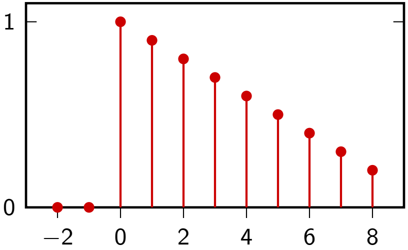
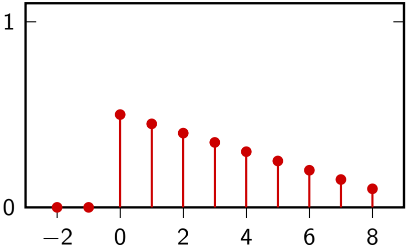


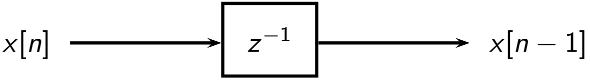


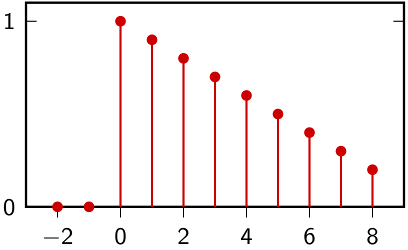
$\alpha = 0.5$

Building Blocks: Unit Delay

$$x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow x[n-1]$$

Building Blocks: Unit Delay





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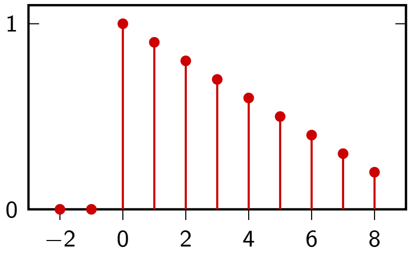
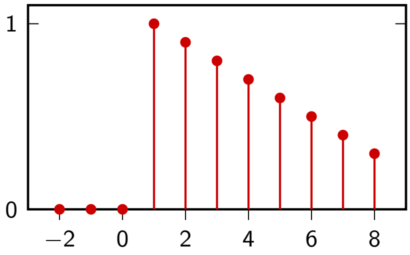
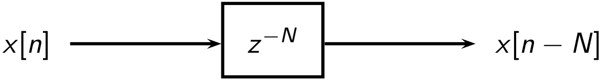


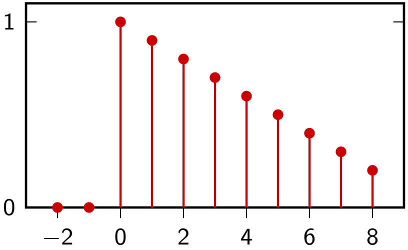


Building Blocks: Arbitrary Delay

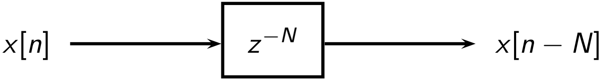
$$x[n] \longrightarrow \boxed{z^{-N}} \longrightarrow x[n-N]$$

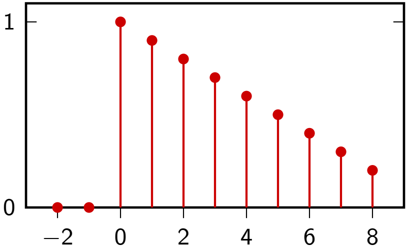
Building Blocks: Arbitrary Delay

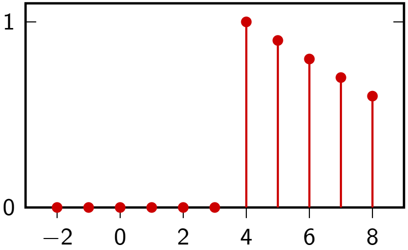




Building Blocks: Arbitrary Delay







$N = 4$

The 2-point Moving Average

- simple average:

$$m = \frac{a+b}{2}$$

- moving average: take a "local" average

$$y[n] = \frac{x[n] + x[n-1]}{2}$$

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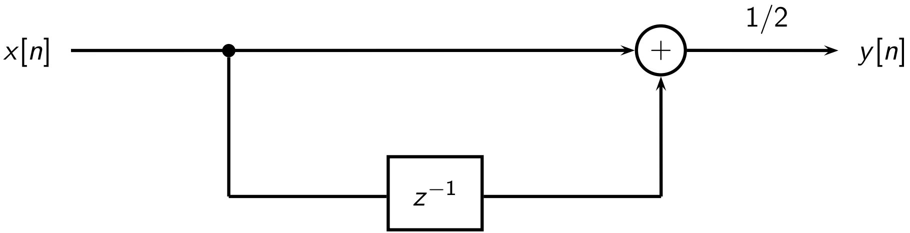
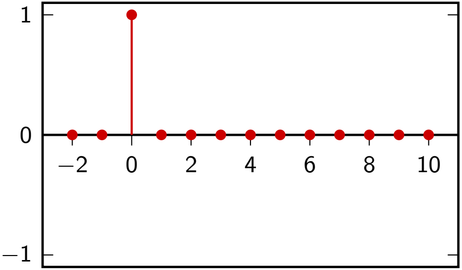
The 2-point Moving Average Using Lego



Let's average...

$x[n] = \delta[n]$



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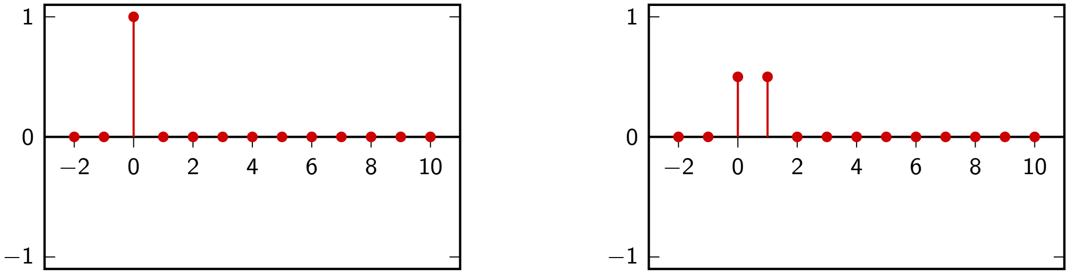
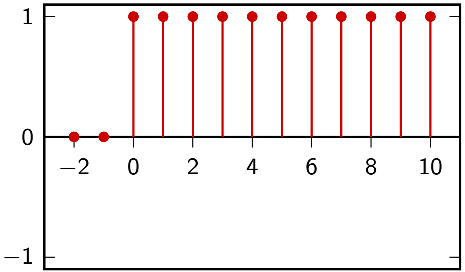


Let's average...

$x[n] = u[n]$



Let's average...

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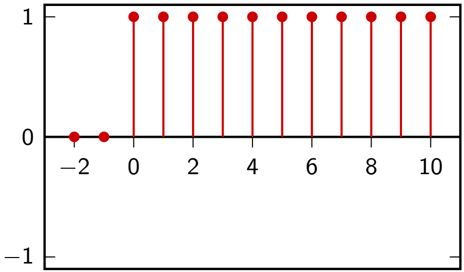
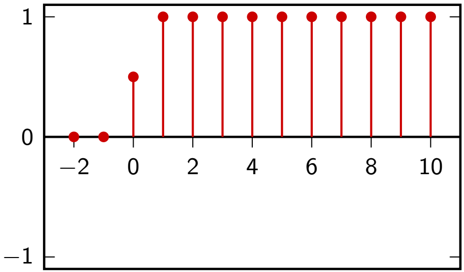
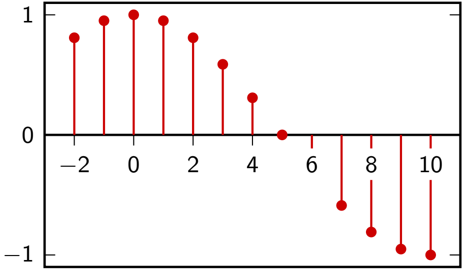




Let's average...

$$x[n] = \cos(\omega n), \quad \omega = \pi/10$$



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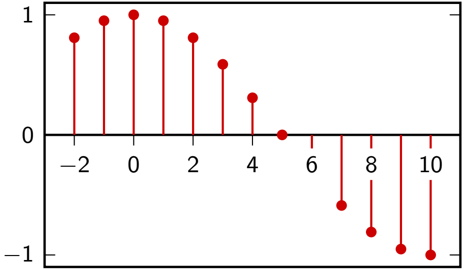
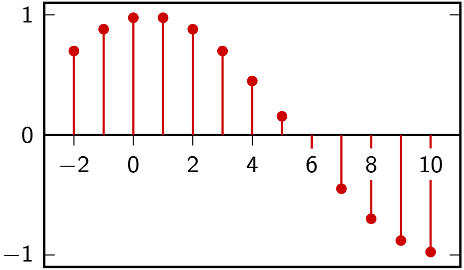




Let's average...

$$\cos(A) + \cos(B) = 2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2})$$

$$\begin{aligned}
\frac{\cos(\omega n) + \cos(\omega(n-1))}{2} &= \cos\frac{\omega n + \omega n - \omega}{2} \cos\frac{\omega n - \omega n + \omega}{2} \\
&= \cos(\omega/2)\cos(\omega n - \omega/2)
\end{aligned}$$

Don't remember the trigonometry? No problem

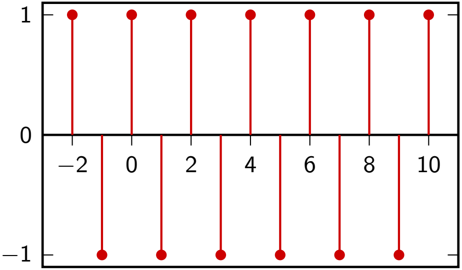
$$\begin{aligned}
\frac{\cos(\omega n) + \cos(\omega(n-1))}{2} &= \frac{1}{2}\,\mathrm{Re}\{e^{j\omega n} + e^{j\omega(n+1)}\} \\
&= \frac{1}{2}\,\mathrm{Re}\{e^{j\omega(n+\frac{1}{2})}(e^{-j\omega/2} + e^{j\omega/2})\} \\
&= \cos(\omega/2)\,\mathrm{Re}\{e^{j\omega(n+\frac{1}{2})}\} \\
&= \cos(\omega/2)\cos(\omega n - \omega/2)
\end{aligned}$$

Useful thing to remember

any linear combination of trigonometric functions at frequency ω_0 produces a sinusoid at the same frequency; linear combinations only alter the magnitude and phase

Let's average...

$x[n] = (-1)^n$



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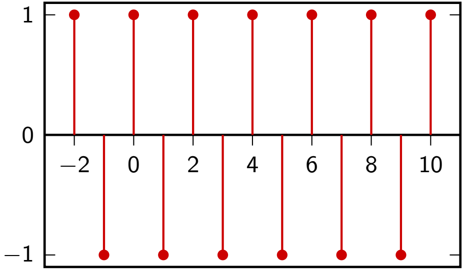
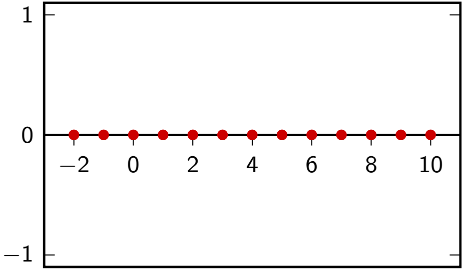
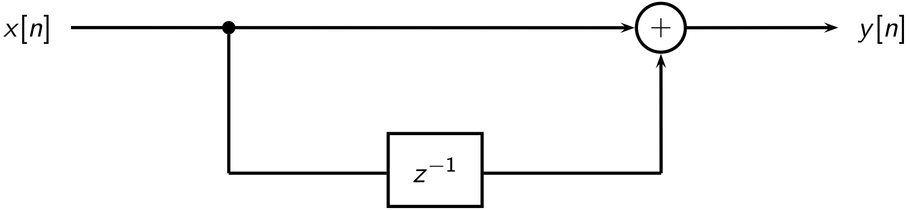




What if we reverse the loop?



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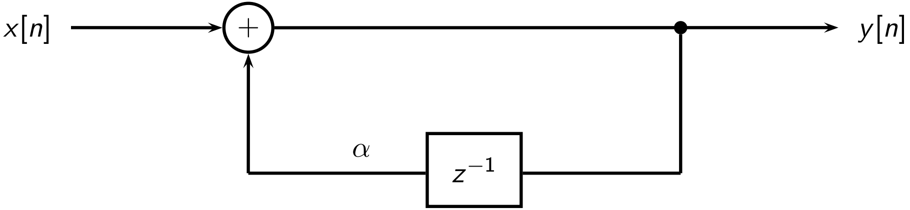

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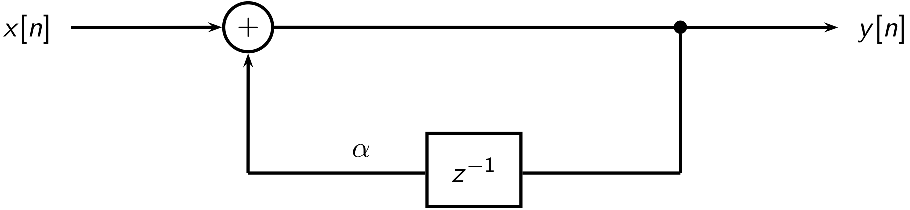
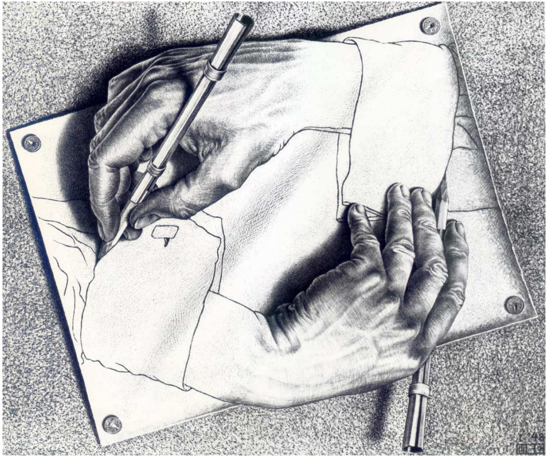


$$y[n] = x[n] + \alpha y[n-1]$$

A powerful concept: recursion



How we solve the chicken-and-egg problem

Zero Initial Conditions

- set a start time (usually $n_0 = 0$)
- assume input and output are zero for *all time* before n_0

A simple model for banking

A simple equation to describe compound interest:

- constant interest/borrowing rate of 5% per year
- interest accrues on Dec 31
- deposits/withdrawals during year n: $x[n]$
- balance at year n:

$$y[n] = 1.05\, y[n-1] + x[n]$$

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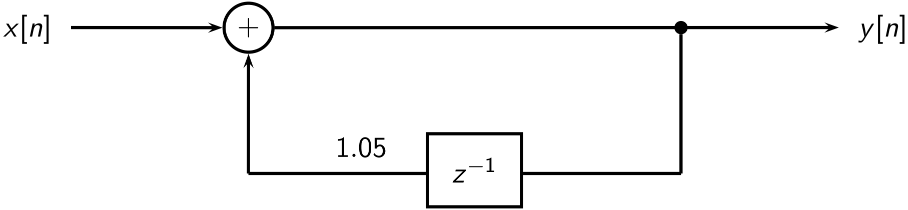
Accumulation of interest: first-order recursion

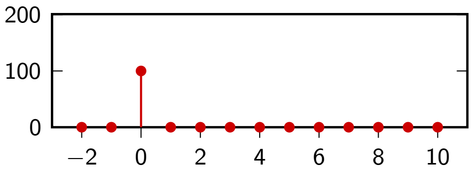


$$y[n] = 1.05\, y[n-1] + x[n]$$

Example: the one-time investment

$x[n] = 100\,\delta[n]$

- $y[0] = 100$
- $y[1] = 105$
- $y[2] = 110.25,\ y[3] = 115.7625$ etc.
- In general: $y[n] = (1.05)^n 100\, u[n]$



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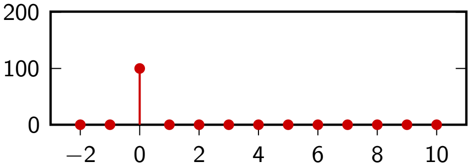

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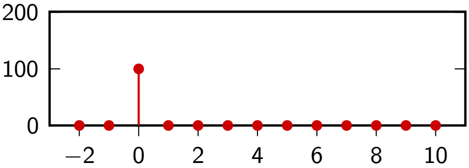

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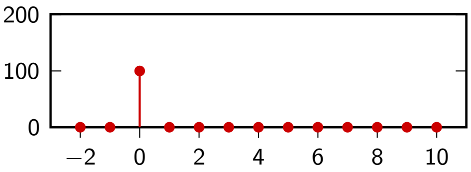

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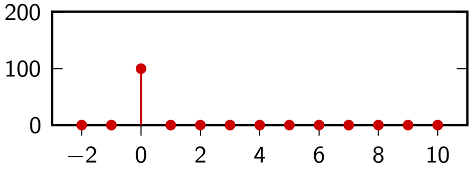
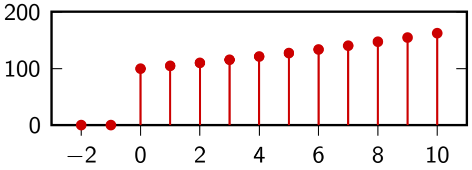
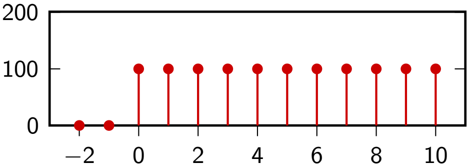




Example: the saver

$x[n] = 100\, u[n]$

- $y[0] = 100$
- $y[1] = 205$
- $y[2] = 315.25, \; y[3] = 431.0125$ etc.
- In general: $y[n] = 2000\,((1.05)^{n+1} - 1)\, u[n]$



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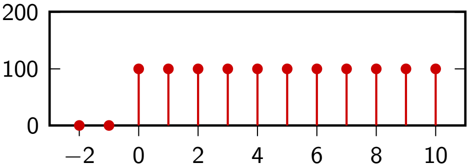

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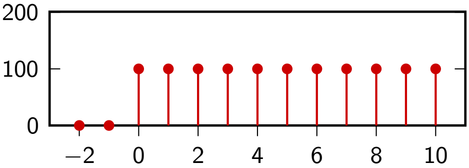

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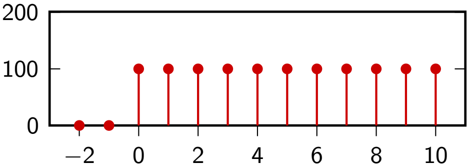
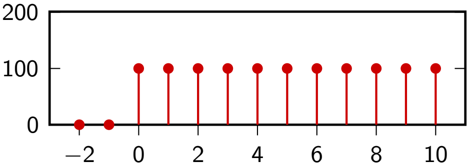
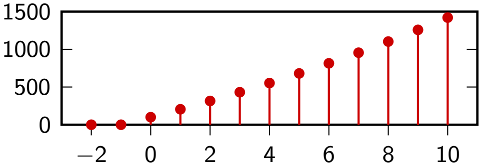


Example: the saver

$x[n] = 100\, u[n]$

- $y[0] = 100$
- $y[1] = 205$
- $y[2] = 315.25$, $y[3] = 431.0125$ etc.
- In general: $y[n] = 2000\left((1.05)^{n+1} - 1\right) u[n]$

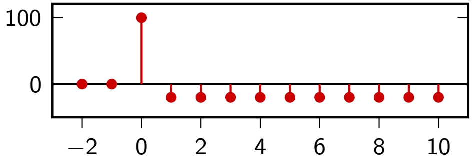




Example: living off the interest

$x[n] = 100\,\delta[n] - 5\,u[n-1]$

- $y[0] = 100$
- $y[1] = 100$
- $y[2] = 100$, $y[3] = 100$ etc.
- In general: $y[n] = 100\,u[n]$



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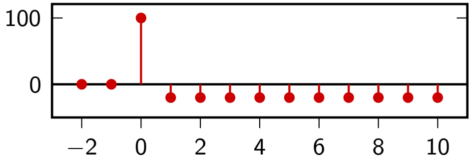

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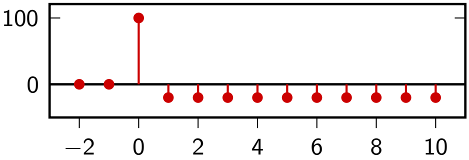

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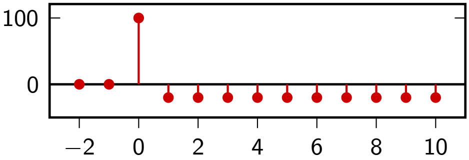

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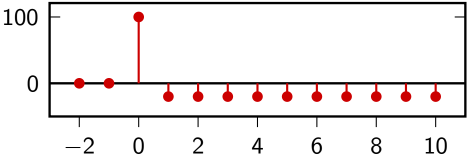
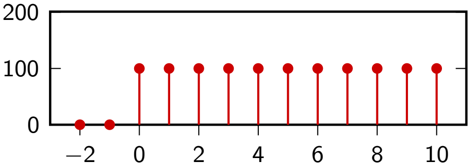
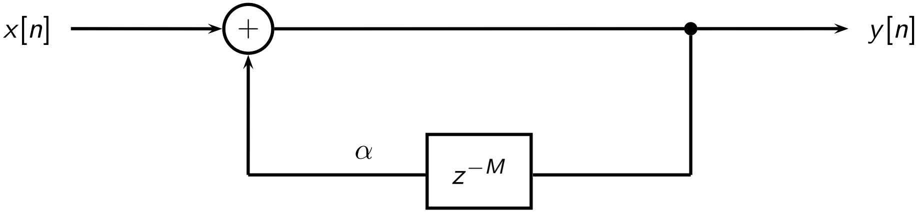


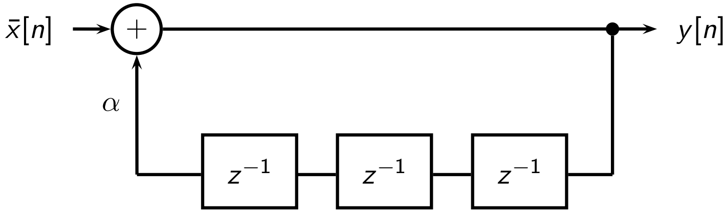


An interesting generalization



$$y[n] = \alpha\, y[n-M] + x[n]$$

Creating loops

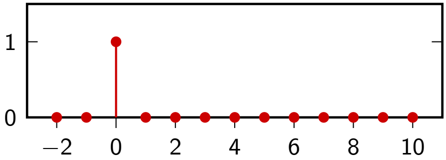


$$y[n] = \alpha y[n-3] + \bar{x}[n]$$

Example

$M = 3,\ \alpha = 0.7,\ x[n] = \delta[n]$

- $y[0] = 1,\ y[1] = 0,\ y[2] = 0$
- $y[3] = 0.7,\ y[4] = 0,\ y[5] = 0$
- $y[6] = 0.7^2,\ y[7] = 0,\ y[8] = 0$, etc.



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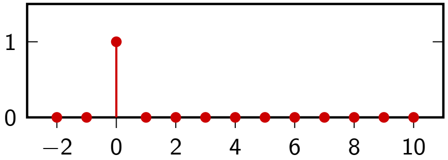

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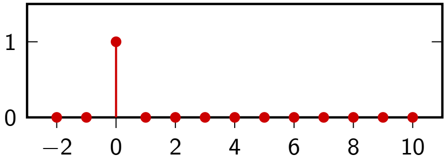

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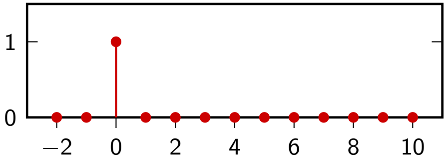
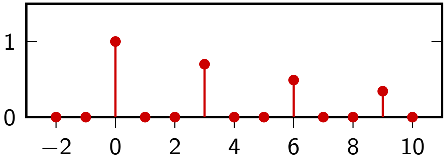
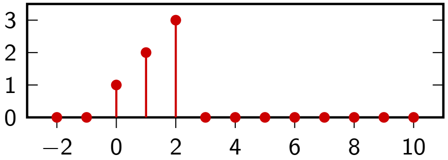




Example

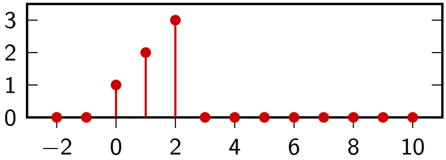
$$M = 3,\ \alpha = 1,\ x[n] = \delta[n] + 2\,\delta[n-1] + 3\,\delta[n-2]$$



Example

$M = 3,\ \alpha = 1,\ x[n] = \delta[n] + 2\,\delta[n-1] + 3\,\delta[n-2]$

- $y[0] = 1,\ y[1] = 2,\ y[2] = 3$



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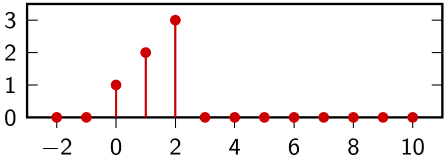

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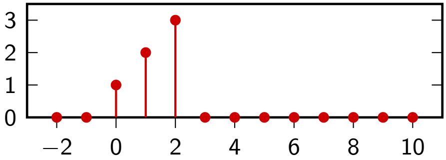

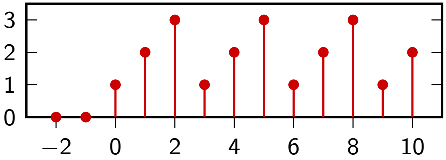

We can make music with that!

- build a recursion loop with a delay of M
- choose a signal $\bar{x}[n]$ that is nonzero only for $0 \leq n < M$
- choose a decay factor
- input $\bar{x}[n]$ to the system
- play the output

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How do we “play” it, really?

- M-tap delay $\longrightarrow$ M-sample “periodicity”
- associate time T to sample interval
- periodic signal of frequency

$$f = \frac{1}{MT}\text{Hz}$$

- example: $T = 22.7\mu s$, $M = 100$

$$f \approx 440\text{Hz}$$

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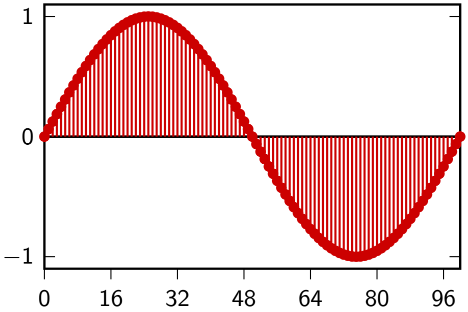
Playing a sine wave

$M = 100$, $\alpha = 1$, $\bar{x}[n] = \sin(2\pi\, n/100)$ for $0 \leq n < 100$ and zero elsewhere



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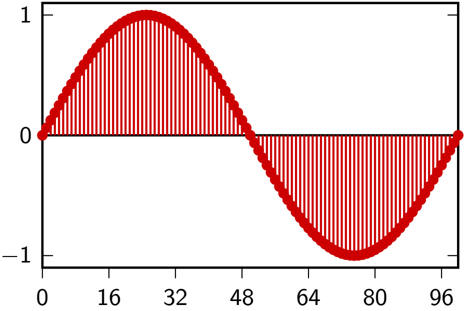

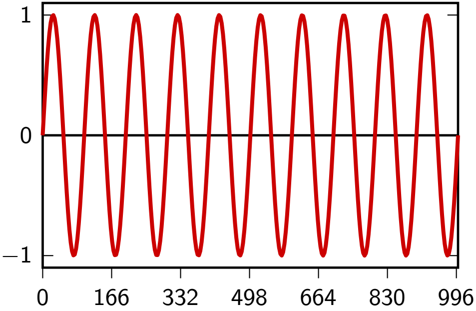

Play

Introducing some realism

- M controls frequency (pitch)
- α controls envelope (decay)
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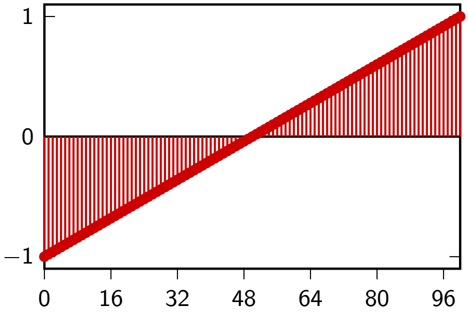
A proto-violin

$M = 100$, $\alpha = 0.95$, $\bar{x}[n]$: zero-mean sawtooth wave between 0 and 99, zero elsewhere



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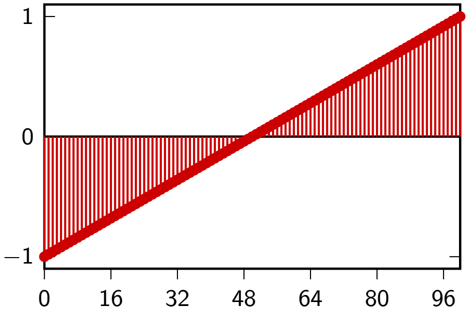
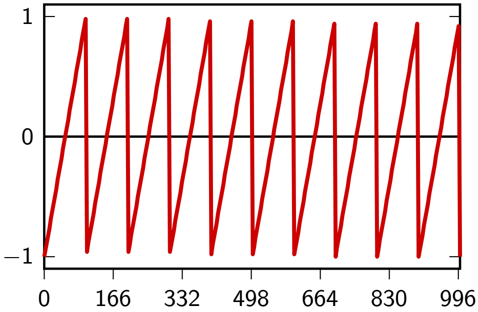
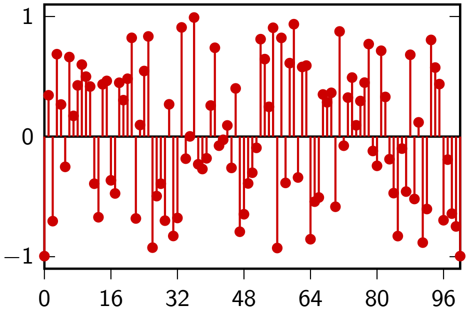




Play

The Karplus-Strong Algorithm

$M = 100$, $\alpha = 0.9$, $\bar{x}[n]$: 100 random values between 0 and 99, zero elsewhere



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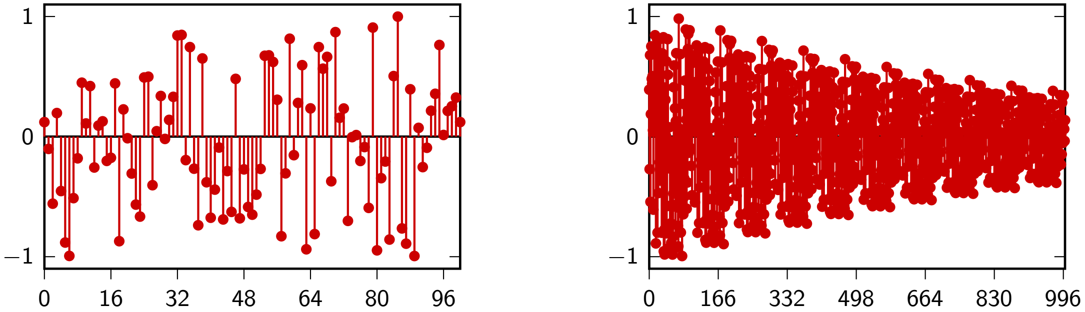

Play

Recap

- We have seen basic elements:
  - adders
  - multipliers
  - delays
- We have seen two systems
  - moving averages
  - recursive systems
- We were able to build simple systems with interesting properties
- to understand all of this in more details we need a mathematical framework!