1 ELECTRONICS FOR MEASUREMENT AND CONTOL IN ENERGY CONVERSION SYSTEMS (18.09.01)

Measurement and control are among of the most important area of electrical engineering because all engineering disciplines require the ability to perform measurement and eventually control of some kind.

In this introductory chapter we examine the architecture of typical measurement and control system, identifying the system components and introducing the basic principles, terminology and characteristics that are describing the operation of these systems. We also discuss how noise, calibration errors, sensor dynamics and non linearity can introduce errors and affect the accuracy, precision and resolution of measurements.

Measurement systems are often used to measure some physical quantities such as temperature, pressure, mechanical motion or electrical quantities like voltage or current. However they can also be designed to locate things or events, or they can be used to count things such as red blood cells or cars passing a checkpoint.

A measurement system is often an important part of a control system. In this text we will therefore use the term instrumentation system to include the components for process actuation in addition to the measurement and display components in a control system.

The term *Industrial Electronics* can also be used to describe the topics covered in this course, because we will limit our treatment to electronic component for signal processing in electrical energy conversion systems.

The field of instrumentation, measurement and control are rapidly changing and new standards, sensors and measurement systems are continually being introduced in the scientific literature and in the marked place. No text can cover the complete field.

1.1 Control System system architecture

A *measurement system* consists of three essential elements: a sensor, signal conditioning circuits, and data storage or display device. In the general form an *instrumentation system* will also include control elements consisting of process actuator devices and operator input devices. The operator input, storage, display and control devices are often based on digital computers.

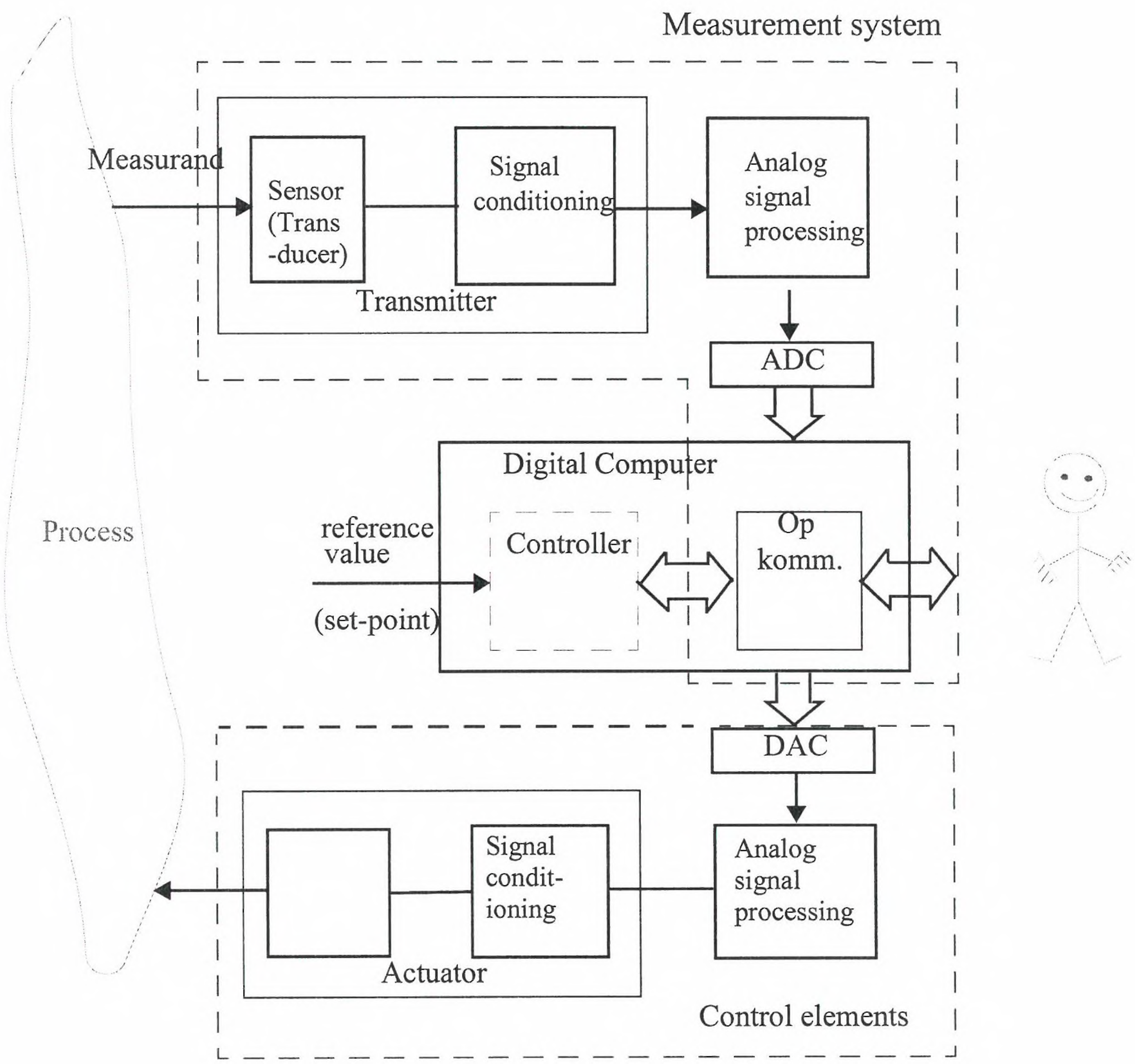


Figure 1.1 Control and measurement system

Figure 1.1 illustrates the block diagram of a general control and measurement system. The essential elements of the system are defined in terms of separate functional blocks.

Sensors or *transducers* are devices that convert one type of physical phenomenon, such as temperature, strain, pressure, or light, into another. The most common transducers convert physical quantities to electrical quantities, such as voltage or resistance. A broad range of sensors exists to measure virtually all physical phenomena.

The input to the sensor, called the *measurand* is converted to an output *measured signal* represented in some form ready for further signal processing in the measurement system. This *conditioned signal* is usually in the form of an analog voltage or current signal. The purpose of the signal conditioning sub-system may be to amplify, give a low or matched output impedance and to improve the signal-to-noise ratio of the analog conditioned signal. Devices carrying out the combined sensing and signal conversion operation are called *measurement transmitters.*

The conditioned signal from the transmitter can be distributed to various display and recording devices, such as analog or digital oscilloscope or a strip-chart recorder. Today the measured analog signal are most often converted to an equivalent digital representation, to exploit the capabilities of a computer or microprocessor in processing the signal. In this case the analog signal must be low-pass filtered, to prevent aliasing, and then periodically sampled and converted to a digital word by an *analog to digital converter* (ADC). The data converted from analog to digital form remain in digital form for ease of storage, digital filtering and further signal processing.

The final elements that eventually are included in the instrumentation system are the devices that direct influences the process. These elements takes the controller outputs and transform these signals into some proportional operation performed on the process. These elements are referred to as the *control elements*.

Often an intermediate operation is required between the controller output and the final control element. The device carrying out this operation is referred to as an *actuator* because it uses the control signal to actuate the final control element. The *actuator* device translates the small energy signal from the controller into a larger energy action on the process.

In a digital processor based instrumentation or control system the control elements includes devices for *digital to analog conversion* (DAC) and most often some analog or digital signal processing for signal conditioning.

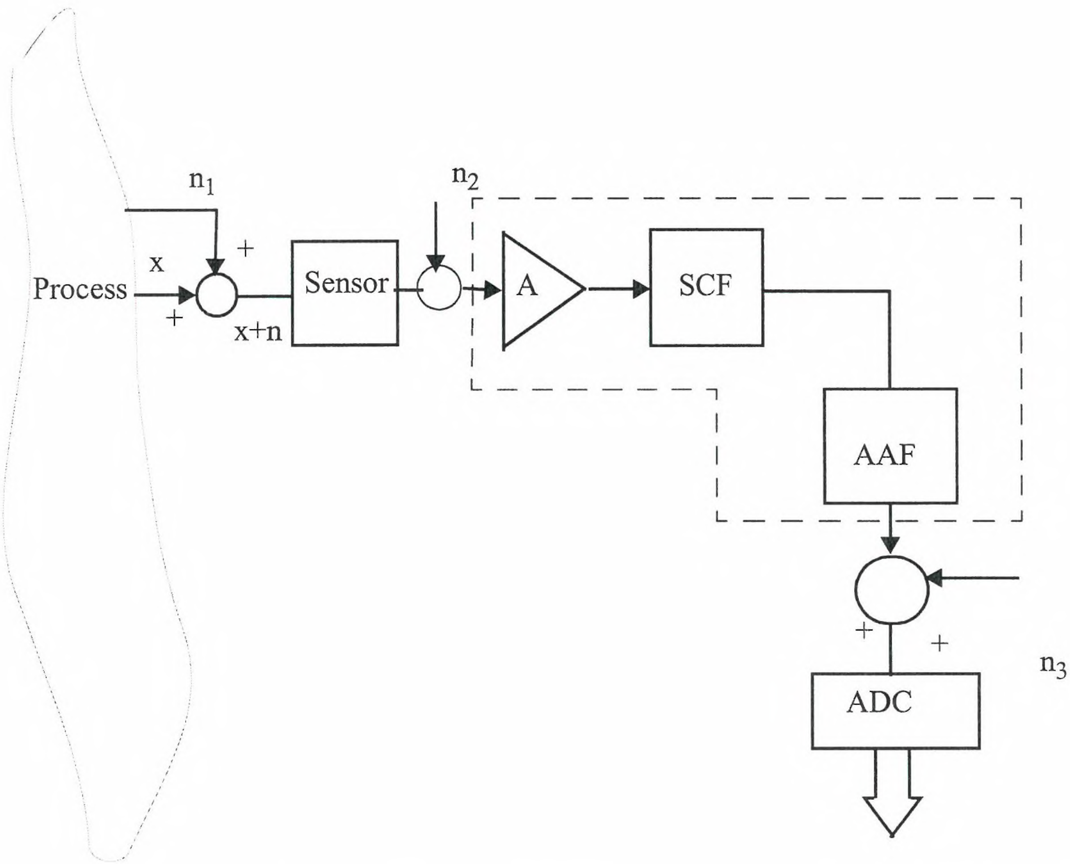


Figure 1.2 A generalized digital measurement system - SCF: Signal Condition Filter - AAF: Anti Aliasing Filter

In Figure 1.2 the measurement part of Figure 1.1 is redrawn to highlight some more details. Note that there are three major sources of noise in the digital measurement system, these are:

Environmental noise (n1) This is noise from the process or the outside of the sensor system which will be added to the measurand variable.

Noise associated with the electronic signal conditioning circuits (n2) referred to its input.

Quantization noise (n3) is noise generated in the analog to digital conversion process, modelled as an analog input noise to the ADC.

These noise sources adds up and generally set a limit to the measurement system resolution and its accuracy.

1.2 Representation of signals

All signals can be mathematically described using two basic methods: the *time domain* and the *frequency domain.*

The *time domain* is the form of representation that most people are familiar with. This method shows variations of a signal with time. A pure sine wave is shown on the left side of Figure 1.3. The signal has a *frequency* (the number of times that the signal repeats itself in a second) and a *period* (the time duration of one complete signal cycle).

Frequency and period are not different quantities, but different methods of describing the time measurement of the same signal. The diagram shows two waveforms. The lower one has a higher frequency and a shorter period. Both signals have the same amplitude.

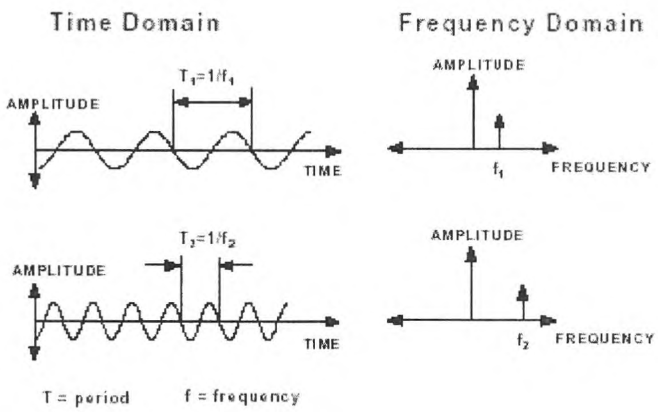


Figure 1.3 Signal measured in both time and frequency domain

The *frequency domain* is very important in signal processing. Instead of showing the variation of a signal with respect to time, we show the variation of the signal with respect to frequency.

The right side of Figure 1.3 shows a frequency domain display. In the top graph, we see a single line representing a single pure frequency (f_1). In the lower graph, we see a higher-frequency signal (f_2). This line is further to the right on the frequency axis.

Real signals are typically more complicated than a simple sine wave. They consist of many frequencies of differing amplitudes that combine to a composite signal.

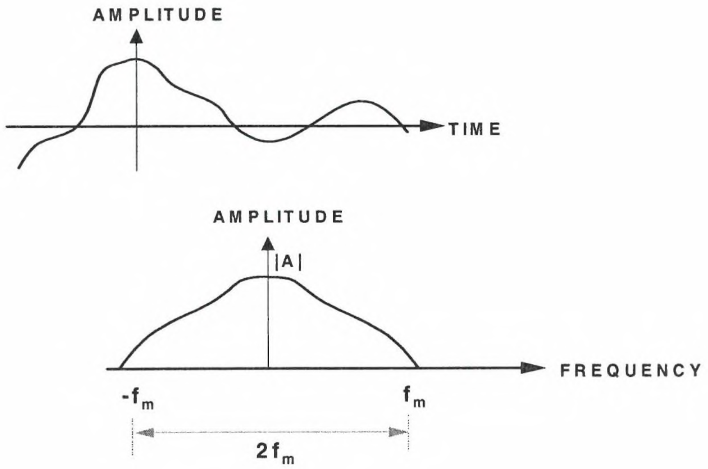


Figure 1.4 Real signals are combination of many frequencies

The *bandwidth* of a signal is the difference between the maximum and minimum frequencies above a certain amplitude (in the above case it is fm). Knowing the bandwidth of a signal is very useful, as without this information we might either design equipment which cannot process all of a signal, or is over specified and costs too much for the required purpose.

In Figure 1.4 we see instead of a single line of a set frequency and amplitude, we see a spread of frequencies and amplitudes, which illustrate the highest and lowest frequencies that exist in the signal. The bandwidth of a signal tells us nothing about the actual frequencies of the signal. For example, we could have two signals, both with a bandwidth of 10kHz. One signal could have a signal ranging from 5kHz to 15kHz, while the other might have signals ranging from 500kHz to 510kHz.

The shape of the signal in the lower part of Figure 1.4 shows the distribution of signal energy with frequency or the *spectrum* of the signal. In this case a high proportion of low-frequency energy is present. Note that the above diagram is also an *even* function, showing both positive and negative frequencies. This common representation of a signal is used because the frequency component of a signal can be expressed as the sum of two exponential.

A signal whose amplitude is time dependent is called a dynamic signal. The magnitude of a dynamic signal can be described as a function of time. If the function is defined for all points in time over a certain finite time interval, we refer to this signal as a *continuous-time signal.* If the amplitude of the signal, and thus the function value, can assume any value within a certain amplitude interval, this signal is called a *continuous-amplitude signal.*

Virtually all measurable variables of physical processes are analog in nature. We therefore define an *analog* measurement signal as a signal whose value (amplitude) is defined everywhere on a specific time interval, and whose value may assume all values between specific lower and upper limits. An *analog signal* is therefore a signal which is continuous in the time domain as well as in the amplitude.

There are also *discrete-time signals*. The amplitude of such signals is only defined or known at certain discrete point in time. We can consider a discrete-time signal as the result of *sampling* of a continuous-time signal.

Signals that are both discrete in amplitude and discrete in time will be referred to as *digital signals*.

Digital computers can not manipulate analog signals directly; they can only process and produce digital signals in form of numbers. To go from analog to digital signals, and vice versa, we need a type of processing called signal conversion. Analog to digital conversion (ADC) and digital to analog conversion (DAC) respectively.

1.2.1 Analog data representation for signal transmission.

For instrumentation systems part of the specification is the range of the variables involved. For example, if a system is to measure temperature, there will be a range of temperature specified, say 0 degC to 100 degC. Similar if the system is to output an actuating signal to continuous valve, the signal will be designed to cover the range from fully closed to fully open, with all the various valve settings in between. We have to decide upon how to represent these signals as analog quantities when designing our instrumentation system. For electrical systems we usually use a range of voltage or current for the analog signal variables.

When the information in the signal variables are to be transmitted over some distance, such as to and from the control computer and the plant or between the measurement transmitter and the signal processing components, one have to concern about the representation form of the signal. When the information is already represented in a digital form it is usually preferable to transmit data in some form of digital codes rather than as analog voltage or current signals.The main reason for this choice of digital over analog is that digital data is less sensitive to noise and interference than analog signals. More will be said about digital data transmission in later chapters. Here we make some comment on analog data representation.

Two analog standards are in common use as means of representation the range of variables in electrical instrumentation systems:

For *voltage signal* representation the most common range are defined in the IEC 381 standard to be:

0V to +5V

0V to +10V

-10V to +10V

Current signal are the most used representation for analog signal transmission over some distance. The International standard values according to IEC 381 is the range of 4 to 20 mA. Thus, in the preceding temperature example 0 degC might be represented by 4 mA and 100 degC by 20 mA.

Current is used instead of voltage when analog signals are to be transmitted over some distance because the system is less dependent on line resistance and the receiver load. In the receiving end the incoming current are usually converted to a voltage signal for further signal processing.

Voltage is not used for transmission because of its susceptibility to change of resistance in the line. However voltage signals are more easy to used in parallel connection of more signal processing units like amplifiers and filters.

1.3 Sensors

The term *sensor* and *transducer* will be used interchange in this text to describe the same thing, although they sometimes may have different meaning, and are sometimes confused. A transducer is any device that transforms one form of energy to another. Hence a sensor is usually a transducer, but not all transducers are used as sensors.

There is no standard classification of sensors. They may be grouped according to their physical characteristics (e.g., optical sensors, electronic sensors, resistive sensors) or by the physical variable measured by the sensor (e.g., position, pressure, temperature). In this text we will present and group the sensors according to the physical variable sensed. Table 1.1, "Sensor Classification," on page 9 lists the most interesting categories of sensors for engineering measurement

Table 1.1 Sensor Classification

Sensed variable	Sensors	Note
Motion	Resistive potentiometers Optical encoders Strain gauges Differential transformers Variable reluctance sensors	
Temperature	Thermocouple Thermistor Resistance thermometer (RTD)	

Many sensors provide the rate-limiting element in an instrumentation system, because the response of the sensor may be slower than the other elements in the system. As a result of the sensor dynamics, the measurement system will have a settling time before a stable measurement can be made when there is a step input of the measurand.

A sensor is characterised by a set of specifications that indicate its overall effectiveness in measuring the physical variable. The following definitions are used to characterise the operating performance of a specific sensor.

- *Span*: Linear operating range.
- *Linearity*: Conformity to an ideal linear calibration curve, usually given in percent of reading or of full-scale reading.
- *Accuracy*: Conformity of the measurement to the true value, usually given as percent of full-scale value.
- *Error*: Difference between measurement and the true value, usually given as percent of full-scale value.
- *Resolution*: The smallest measurable increment

Selection of an appropriate sensor for a given application is the first and perhaps the most important step in obtaining accurate results in a measurement system. Before a sensor can be selected some elementary questions should be asked:

- What is the physical quantity (*type* and *range* of the measurand) to be measured?
- Which sensor principle can best be used to get the measurand? (Input output characteristic of the transducer compatible with the measurement system).
- What accuracy is required for the measurement?

The accuracy requirements of the total measurement system determine the degree to which individual factors contribution to accuracy must be considered. Important factors are:

- *Ambient conditions*: non linearity effects, hysteresis effects, frequency response, resolution.
- *Environmental conditions*: Temperature effects, acceleration, shock and vibrations.

Sensor characteristics define many of the signal conditioning requirements of a measurement or instrumentation system. Table 1 summarizes the basic characteristics and signal conditioning requirements of some common sensors.

Table 1.2 Electrical Characteristics and basic Signal Conditioning Requirements of some common Sensors

Sensor	Electrical Characteristics	Signal Conditioning Requirements
Thermocouple	Low-voltage output Low sensitivity Nonlinear output	Reference temperature sensor (for cold-junction compensation) High amplification Linearization
RTD	Low resistance (100 ¾ typical) Low sensitivity Nonlinear output	Current excitation Four-wire/three-wire configuration Linearization
Strain gauge	Low-resistance device Low sensitivity Nonlinear output	Voltage or current excitation Bridge completion Linearization
Current-output device	Current loop output (4-20 mA typical)	Precision resistor
Thermistor	Resistive device High resistance and sensitivity Very nonlinear output	Current excitation or voltage excitation with reference resistor Linearization
Integrated circuit (IC) temperature sensor	High-level voltage or current output Linear output	Power source Moderate gain

1.4 Signal Conditioning Fundamentals for Measurement and Control Systems

Often, sensor outputs need to be conditioned before further processing can take place. The most common signal conditioning circuits are amplifiers and filters.

Proper wiring, grounding and shielding techniques are required to minimize undesired interference and noise.

If the conditioned sensor signal are to be stored in a digital form by a computer, it is necessary to perform an analog to digital conversion.

Once the digital data corresponding to the measured variable is available, the need for digital data transmission may arise. Standard transmission formats exists.

1.5 General Signal Conditioning Functions

Regardless of the types of sensors or transducers you are using, proper signal conditioning circuits can improve the quality and performance of our system. Signal conditioning is an important component of a computer based measurement system.

Signal conditioning must be used before we can connect sensors such as thermocouples, RTDs, strain gauges, and current-output devices to digital computer analog input channels. No matter what sensors we are using, signal conditioning can improve the accuracy, effectiveness, and safety of your measurements because of capabilities such as amplification, isolation, and filtering.

1.5.1 Amplification

Unwanted noise can reduce the measurement accuracy of a system. The effects of system noise on your measurements can be extreme if you are not careful. Signal conditioning circuitry with amplification, which applies gain outside the analog input port and near the signal source, can increase measurement resolution and effectively reduce the effects of noise. An amplifier, whether located directly at the analog input port or in external signal conditioners, can apply gain to the small signal before the ADC converts the signal to a digital value. Boosting the input signal uses as much of the ADC input range as possible.

However, many transducers produce voltage output signals on the order of mV or even micro volts. Amplifying these low-level analog signals directly at the analog input port also amplifies any noise picked up from the signal lead wires or from within the computer chassis. When the input signal is as small as micro volts, this noise can drown out the signal itself, leading to meaningless data.

A simple method for reducing the effects of system noise on your signal is to amplify the signal as close to the source as possible, which boosts the analog signal above the noise level before noise in the lead wires or computer chassis can corrupt the signal.

For example, a J-type thermocouple outputs a very low-level voltage signal that varies by about 50 μV/°C. Suppose that the thermocouple leads must travel 10 m through an electrically noisy plant environment to the computer input port. If the various noise sources in the environment couple 200 μV onto the thermocouple leads, you obtain a noisy temperature reading with about 4 degC of noise. However, amplifying the signal close to the thermocouple before noise corrupts the signal alleviates this problem. Amplifying the signal with a gain of 500 with a signal conditioner placed near the thermocouple produces a thermocouple signal that varies by about 25 mV/°C. As this high-level signal travels the same 10 m, the 200 μV of noise coupled onto this signal after amplification has much less of an effect on the final measurement, adding only a fraction of a degC of noise to the measured temperature reading.

1.5.2 Filtering and Averaging

You can also use filters to reject unwanted noise within a certain frequency range. Many systems will exhibit 50 Hz periodic noise components from sources such as power supplies or machinery. Low pass filters on our signal conditioning circuitry can eliminate unwanted high-frequency components. However, we must be sure to select the filter bandwidth carefully so that we do not affect the time response of our signals.

Although many signal conditioners include low pass noise filters to remove unwanted noise, an extra precaution is to use software averaging to remove additional noise. *Software averaging* is a simple and effective technique of digitally filtering. For every data point you need, the measurement system acquires and averages many voltage readings.

For example, a common approach is to acquire 100 points and average those points for each measurement you need. For slower applications in which you can over sample in this way, averaging is a very effective noise filtering technique.

1.5.3 Isolation

Improper grounding of the measurement system is the most common cause of measurement problems and damaged electronic boards. Isolated signal conditioners can prevent most of these problems by passing the signal from its source to the measurement device without a galvanic or physical connection. Isolation breaks ground loops, rejects high common-mode voltages, and protects expensive instrumentation equipment.

Common methods for circuit isolation include using optical, magnetic, or capacitive isolators. Magnetic and capacitive isolators modulate the signal to convert it from a voltage to a frequency. The frequency can then be transmitted across a transformer or capacitor without a direct physical connection before being converted back to a voltage value.

When you connect your sensor or equipment ground to your measurement system, you will see any potential difference in the grounds on both inputs to your measurement system. This voltage is referred to as *common-mode voltage*. If you are using a single-ended measurement system, as shown in Figure 1.5 the measured voltage includes the voltage from the desired signal, VS, as well as this common-mode voltage from the additional ground currents in the system, VG.

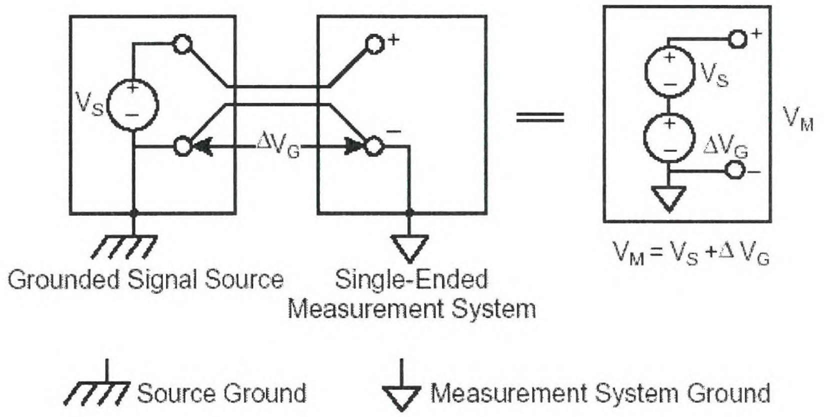


Figure 1.5 Single-Ended Measurement System

If you are using differential inputs, you can reject some of this common-mode voltage. However, larger ground potential differences, or ground loops, will damage unprotected devices. If you cannot remove the ground references, use isolating signal conditioners that break these ground loops and reject very large common-mode voltages.

Isolators also provide an important safety function by protecting against high-voltage surges from sources like power lines, lightning, or high-voltage equipment. When dealing with high voltages, a surge can damage the equipment or even harm equipment operators. By breaking the galvanic connection, isolated signal conditioners produce an effective barrier between the measurement system and these high-voltage surges.

1.5.4 Multiplexing

Signal multiplexers can cost-effectively expand the input/output (I/O) capabilities of the measurement system. The typical plug-in analog measurement board for PC has often 8 to 16 analog inputs and 8 to 24 digital I/O lines.

External multiplexers can increase the I/O capacity of a plug-in board to hundreds and even thousands of channels.

Analog input multiplexers use solid-state or relay switches to sequential switch, or *scan*, multiple analog input signals onto a single channel.

1.5.5 Digital I/O signal conditioning

Digital On/Off signals can also require signal conditioning peripherals. Usually, you should not directly connect digital signals used in industrial environments to a computer board without some type of isolation because of the possibility of large voltage spikes or large common-mode voltages. It is often required to have some signal conditioning modules to optically isolate the digital I/O signals in order to remove these problems. Digital I/O signals are used to drive actuator devices such as power transistors, electromechanical or solid-state relays. These devices are used to switch loads such as solenoids, lights, electrical motors, or to control power electronic converters. You can also use solid-state relays or opto couplers to sense high-voltage signals and convert them to digital I/O signals.

1.6 Definitions of terms and expressions used in specification of instrumentation devices

The *specification* of a device is a description of its characteristics, construction and performance and any other information relevant to its use. The specification for a specific device is given in the manufacturers *data-sheet* for the actual device. Several sample specifications illustrating typical measurement devices will be given in the course material. This section presents definitions of some of the common terms and expressions used to characterise sensors and measurement systems.

The terms and characteristics that follow can be applied to the whole measurement system and to all components within a measurement system, including the sensor and signal condition circuits.

The most important quantity in a measurement system is the measurement error.

Error is the difference between a *measured value* (measured indication) and the *true value* (actual value).

True value: a theoretical concept, the real value of a physical property that the measurement attempt to determine

Systematic (bias) error: The deviation of the measurement from the true value resulting from inherent mistakes in the measurement methods. This type of error tends to stay constant from trial to trial.

Random (precision) error: The deviation of the measurement from the true value resulting from the finite precision of the measurement method

Reduction of errors: Systematic error can be reduced by using more measurement methods. Random errors can be reduced by taking more trials.

Accuracy: How far are the measurements from the true value

Precision: How small an increment that an measurement is made; number of significant figures

1.7 Summary and additional readings.

This chapter has introduced you to the field of measurement, control and instrumentation systems. It has also defined some basic terminology and characteristics used when discussing such systems. We have discussed sensors and actuator technology and given a short introduction to the signal processing involved when using these in a measurement and control.

In the following chapters we will look in details at analog and digital signal processing and at a selection of different type of sensors. You will meet many more definitions and concepts in the following parts of the course but it may at times be useful to refer back to this chapter and the chapters of the addition reading

3 MATHEMATICAL MODELLING OF DIGITAL CONTROL SYSTEMS FOR ELECTRONIC ENERGY CONVERTERS (30.10.01)

A digital computer based controller is concerned with processing of number sequences. These sequences are discrete time signals generated by sampling a continuous signal at constant time intervals. In order to analyse and design such a control system it is necessary to have a mathematical model of these signals, the digital control system components, and the controlled power electronics plant.

This chapter contains a review of the theoretical aspects concerning discrete and sampled system and signals. No attempt is made to present the complete theory in this report. Rather, attention is focused on a limited number of topics which can serve as an adequate background necessary for the understanding and guiding of practical design of a digital control system in power electronics.

The introduction of the discrete time concept is done as a logical and gradual extension of the classical philosophy for continuous systems. The presentation is based both on the input-output model approach for mono-variable systems, where the transform technique is used, and the state space representation.

3.1 System overview, terminology, and basic assumptions

This section serves two purposes. The first is to give an overview of a power electronic system controlled by a digital computer. The second is to identify system characteristics and to introduce terminology related to signals and systems. The discussion will be done with reference to Figure 3.1, which is a revision of figure 1.1 where the basic element and signals in the closed loop control system are highlighted.

The part of the system that contains the manipulated and measured variables is called the process or plant. In this text we are dealing with processes in connection with power electronics.

The process/plant in a power electronic system is most often some kind of energy conversion equipment. Depending on the actual application, the manipulated variables may be rotating speed, current, voltage, temperature, etc. In some applications it is convenient to include the electrical power converter as part of the process while in other applications it can be regarded as a separate unit.

The measured output from the process $y(t)$ is a continuous time signal. A computer is able to work with digital coded information only. The output from the process must therefore be converted into digital form by some kind of analog to digital (A/D) converter. The digital output signal from the A/D converter is discrete in both time and amplitude level. The discretizing in time is called sampling. The discretizing in amplitude level is called quantizing.

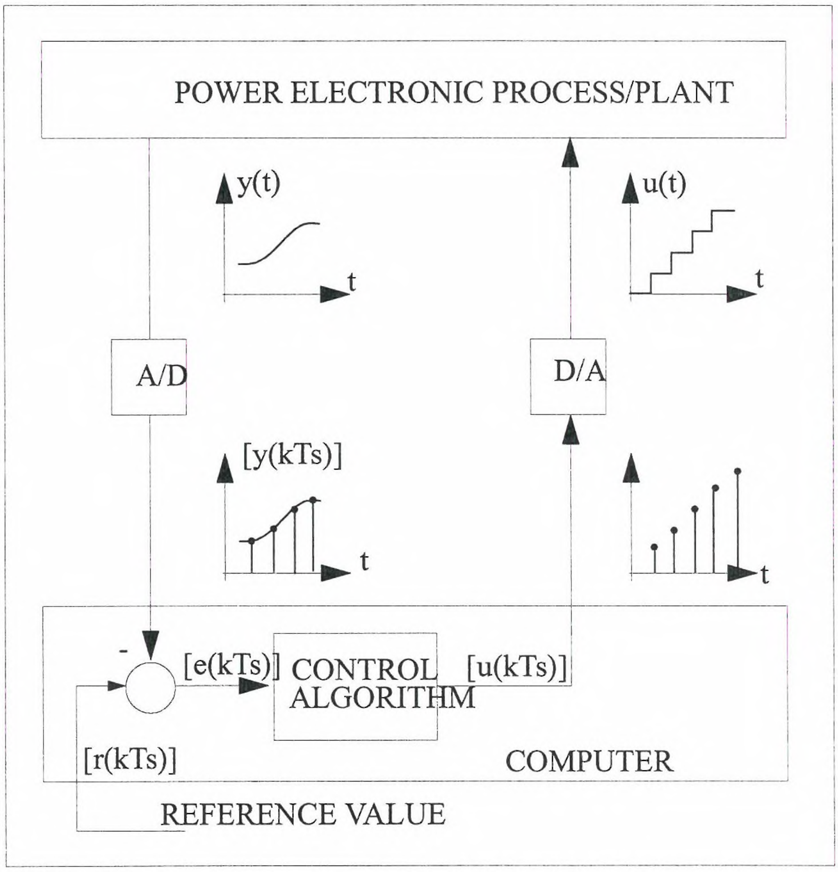


Figure 3.1 Schematic diagram illustrating the interface signal between the system elements.

The computer interprets the converted signal $y(kT_s)$ as a sequence of numbers. These discrete signals are manipulated by the computer algorithm which solves a difference equation implementing the desired control law. This algorithm generates a new sequence of numbers $u(kT_s)$. This sequence is passed on to the process through some kind of a reconstruction device represented here by a digital to analog (D/A) converter. The D/A converter must produce a continuous-time signal. This is normally done by keeping the control signal constant between the conversions (zero order hold). Note that the system runs open loop between the sampling instants.

Ideally the A/D and D/A conversions are done at the sampling instants $t = kT_S$ where $k = 0,1,2....$ The time interval between these instants are usually constant and are denoted by the sampling period T_S. The inverse of T_S is the sampling frequency

$$f_s = 1/T_s \tag{3.1}$$

The execution of the control program may be started by a clock which gives an interrupt signal to the computer at each sampling instant.

The events that takes place in the system are illustrated in figure (3.2) and and figure (3.3).

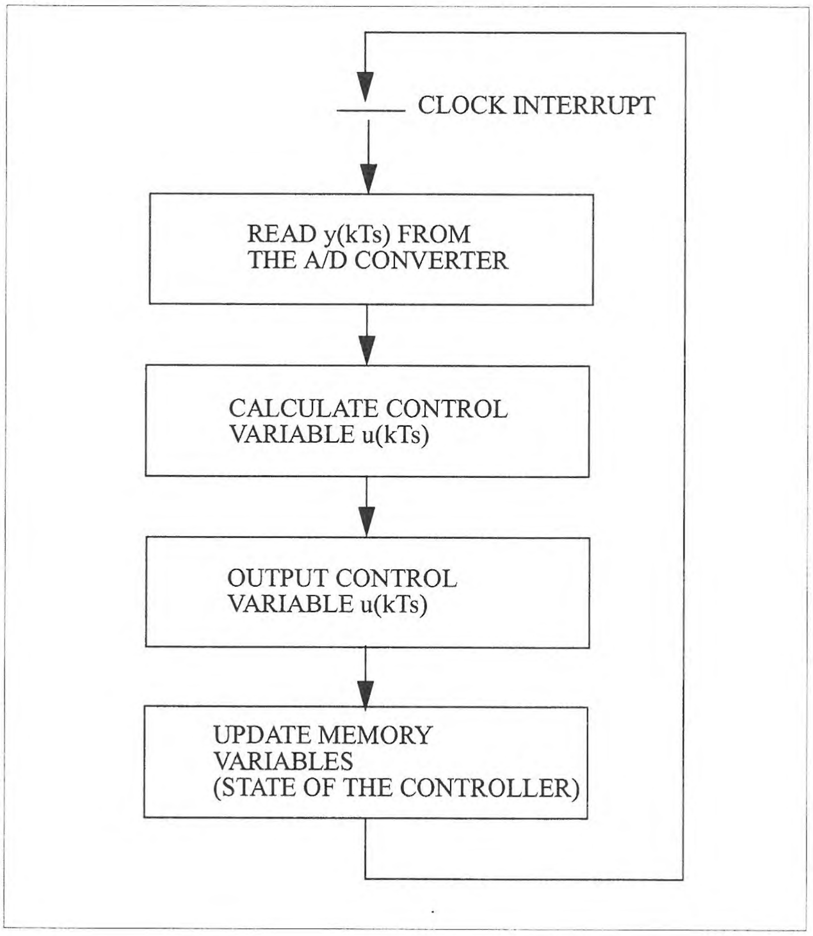
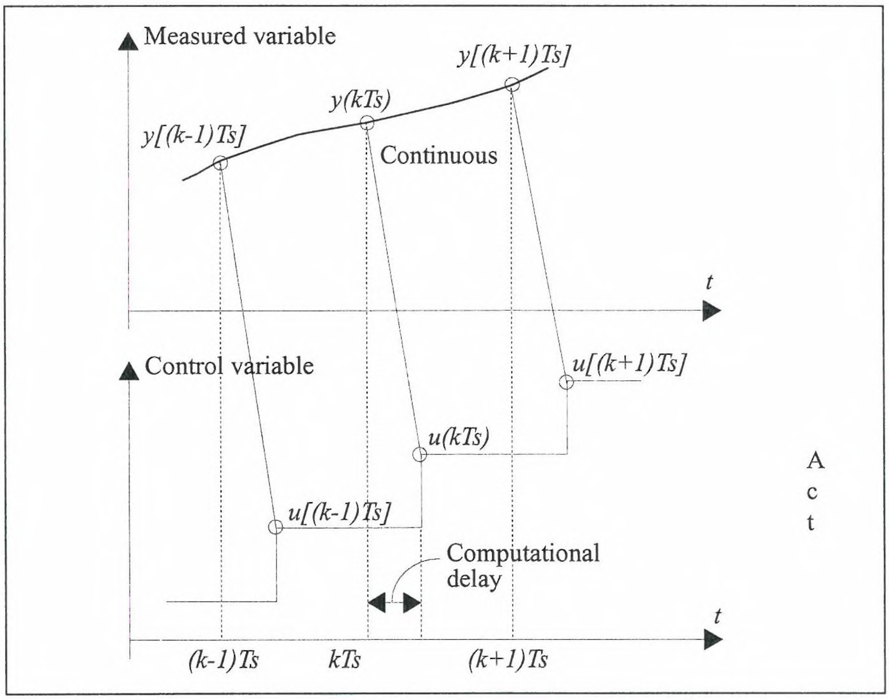


Figure 3.2 Graphical illustration of the events that take place in a program that represents a digital controller.

Since the computer is performing the tasks in sequence, there will be always a time delay due to the computing time. The A/D and D/A conversion also takes time. These time delays must often be taken into account when designing the control system. We will show later how this can be done.



A
c
t

Figure 3.3 Synchronizing input and output. $y(kT_S)$ and $u(kT_S)$ are numbers stored in the computer.

In order to be concise, let us give the applied definition of the various type of signals and systems that will be used in the following sections:

Continuous-time signals (CT-signals) are defined over a continuous span of time. The amplitudes of these signals range either over a continuous range of values or a finite number of possible values. Sometimes one will use the term analog signals to denote CT-signals.

Discrete-time signals or sequences (DT-signals) are defined over only a particular set of discrete values of time, which means that such signals can be represented as sequences of numbers. A sequence of numbers, x, is denoted as

$$x = \{x(kT_S)\}, k = 0, 1, 2... \tag{3.2}$$

Although this implies that $x(kT_S)$ is actually the k'th member, it is convenient to denote the sequence itself by $x(kT_S)$. The short form of notation

$$x(kT_s) = x(k) = x_k \tag{3.3}$$

will also be used interchangeably throughout the text.

Sampled signals represent discretized version of CT-signals. They are a special case of DT-signals which are pulse amplitude modulated and denoted as

$$x^*(t) = \sum_{k=0}^{n} x(kT_s)\delta(t - kT_s)$$

Digital signals are signals where the information is in some kind of coded form. They are quantized in amplitude and discrete in time. Thus, they are a special case of DT-signals.

Systems are classified by the same criteria as signals.

Continuous (or analog) systems are systems where both input and output signals are CT-signals.

Discrete-time systems are systems whose input and output are DT-signals.

Sampled-data systems contain both discrete- and continuous time signals.

Digital Control system is a system that contains both digital and continuous time signals.

From the previous discussion we have seen that a power electronic system that contains a computer for control, operates on signals that are of digital and continuous nature. It may therefore be characterized as a digital control system.

If we assume that the computer and the signal converters have a sufficiently large word length, we may neglect the effect of amplitude quantizing. With this approximation our system can be said to be a sampled system consisting of a purely discrete component, (the computer algorithm), and a continuous system component, (the process or plant). Between these two sub-systems there must be some kind of signal conversion component, logically represented by the A/D and the D/A converters.

In order to analyse this system, we must have a mathematical representation of the various system elements. The aim of this chapter is to appropriately model each element in order to connect these in an overall representation of the complete system.

We will assume that the system components can be modelled as linear and time invariant components. Components in power electronic systems are, as will be shown, not all linear. A first approximation of the system's behaviour may be considered essentially linear, or at least having a linear working domain. Because of this approximation it is necessary to do computer simulations of the designed system to check the behaviour outside the linear domain.

3.2 Modelling of discrete time systems - The computer control algorithm represented as a difference equation

Assume that the input to the digital processor up to the time $t=kT_s$ has been $e(0)$, $e(T_s)$, $e(2T_s)$, ..., $e(kT_s)$ and the output sequence prior to that time was $u(0)$, $u(T_s)$, $u(2T_s)$, ..., $u((k-1)T_s)$. The next output at $t=kT_s$ is written as:

$$u_k = f(e_k, e_{k-1}, e_{k-2}, \ldots e_{k-n}; u_{k-1}, u_{k-2} \ldots u_{k-n})$$

We will assume that the computer algorithm performs a linear combination of the input and the past control output. Thus we write:

$$u_k = g_0 e_k + g_1 e_{k-1} + \ldots + g_n e_{k-n} - f_1 u_{k-1} - \ldots - f_n u_{k-n} \tag{3.4}$$

This is a linear difference equation. If the coefficients are constant it is said to be time invariant. The order of the equation is n if the signals from only the last n sampling instants enter the equation. If not all f_i are zero the equation is said to be recursive because it specifies a recursive procedure for determining the output in terms of the inputs and previous outputs. If all f_i are zero the equation is said to be non-recursive.

The algorithm of the general form (3.4) can adequately perform most control tasks in power electronic systems. The aim of the control system design procedure is to select the order of the equation, the sampling rate, and give values to g_i and f_i. This must be done in such a way that the overall system attains the desired dynamic properties.

3.2.1 A discrete PI-control algorithm

One of the most common control algorithms in power electronic systems is the proportional-integral (PI) control action. As an example of the origins of a difference equation we will consider a discrete approximation to the continuous PI-control law. Suppose we have a continuous input signal, $e(t)$, of which a segment is sketched in figure

(3.4), and we wish to compute an approximation to the continuous PI control law given by

$$u(t) = K_p\left(e(t) + \frac{1}{T_i}\int_{-\infty}^{t} e(\tau)d\tau\right) \tag{3.5}$$

with the transfer function

$$h_r(s) = K_p\frac{1+sT_i}{sT_i} \tag{3.6}$$

Using only the discrete values *e(0),......,e(k-1), e(k)* the output of the discrete PI controller can be written as

$$u(k) = K_p\left(e(k) + \frac{1}{T_i}I(k)\right) \tag{3.7}$$

where $I(k)$ is the approximation to the integral up to the time $t = kT_s$. We will assume that we have an approximation of the integral up to the time $t=(k-1)T_s$ and we call it $I(k-1)$ the integral then can be written

$$I(k) = I(k-1) + \Delta I(k) \tag{3.8}$$

The problem is to obtain $I(k)$ from this information. By interpreting the integral as the area under the curve $e(t)$ the problem reduces to finding an approximation to the area under the curve between $(k-1)T_s$ and kT_s. See figure (3.4). By the trapezoid approximation we can write:

$$\Delta I(k) = \frac{e(kT_s) + e((k-1)T_s)}{2}\cdot T_s \tag{3.9}$$

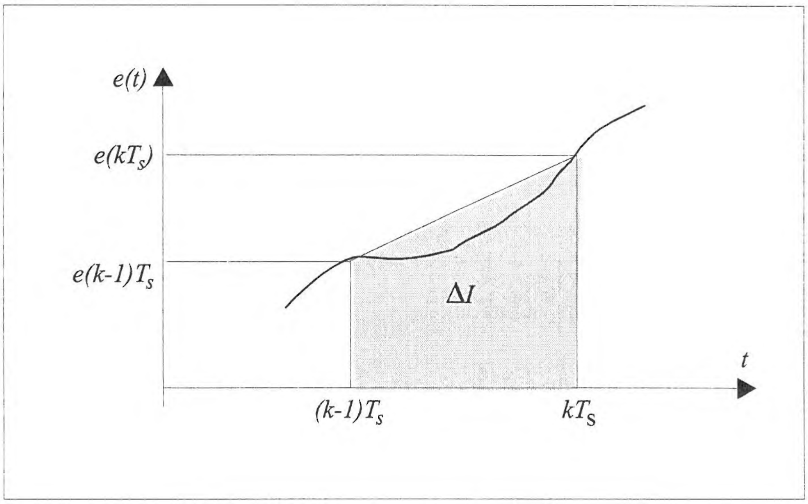


Figure 3.4 Trapezoid approximation of the integral.

By observing that

$$\frac{K_P}{T_i}(I(k-1)) = u(k-1) - K_p e(k-1) \tag{3.10}$$

and combining equations (3.7) through (3.10) we find

$$u(k) = u(k-1) - K_p e(k-1) + \frac{K_p}{T_i} \cdot \frac{T_s}{2}(e(k) + e(k+1)) + K_p e(k) \tag{3.11}$$

By collecting terms, this recursive differential equation describing a discrete PI control algorithm can be expressed as

$$u(k) = u(k-1) + g_0 e(k) + g_1 e(k-1) \tag{3.12}$$

where the coefficients are

$$g_0 = K_p\left(1 + \frac{T_S}{2T_i}\right) \tag{3.13}$$

$$g_1 = -K_p\left(1 - \frac{T_s}{2T_i}\right) \tag{3.14}$$

If we approximate the area under the curve *e(t)* in the time interval from $t=(k-1)T_s$ to $t=kT_s$ by the rectangle of height *e(k-1)* the resulting formula for the integral is called the Forward Rectangular Rule of integration and is given by

$$I(k) = I(k-1) + T_s e(k-1) \tag{3.15}$$

A third possible integration method is the Backward Rectangular Rule, given by

$$I(k) = I(k-1) + T_s e(k) \tag{3.16}$$

A block diagram representation of equation (3.12) is shown in Figure 3.5. Note that the data storage is represented by a time delay which delays the input data for one sampling period.

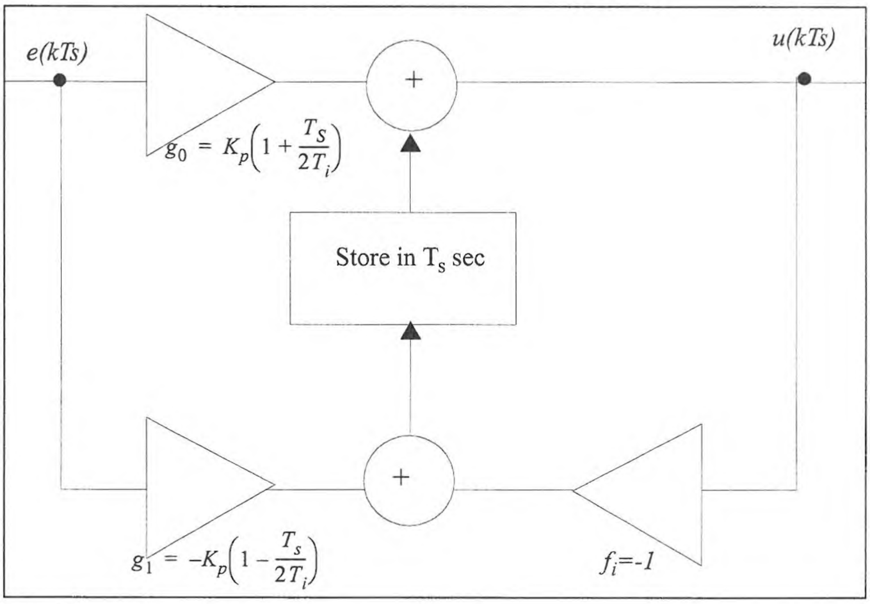


Figure 3.5 Block diagram representation of equation (3.12)

Recall that in a block diagram representation of CT-systems the basic element is the integrator. In DT-systems the basic element is the time delay (or storage) of T_S seconds.

3.2.2 Discretizing of a continuous system

We have seen that the computer control algorithm can be represented as a difference equation. As another example of the origins of a difference equation we will consider the situation where we have a time-discrete input control of a continuous plant. Suppose we have a continuous input signal, $u(t)$, and the output signal from the CT-plant, $y(t)$ as shown in Figure (3.6).

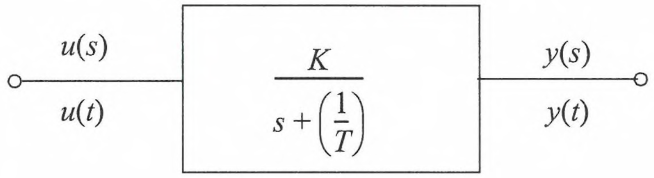


Figure 3.6 Block diagram representation of a first order plant.

The system differential equation is given by

$$\frac{d}{dt}y(t) + \left(\frac{1}{T}\right)y(t) = Ku(t) \tag{3.17}$$

The solution to this equation is known to be

$$y(t) = e^{-\left(\frac{1}{T}\right)(t-t_0)}y(t_0) + \int_{t_0}^{t} e^{-\left(\frac{1}{T}\right)(t-\tau)}Ku(\tau)d\tau \tag{3.18}$$

We will now assume that the input signal to the plant is generated by a computer which work with a sampling period of T_s seconds. We will further assume that the computer has a A/D converter which holds the value $u(t)$ constant between the sampling instants. The output signal $y(t)$ is sampled at the same instants as placement of a new input takes place. The situation is illustrated in Figure (3.7)

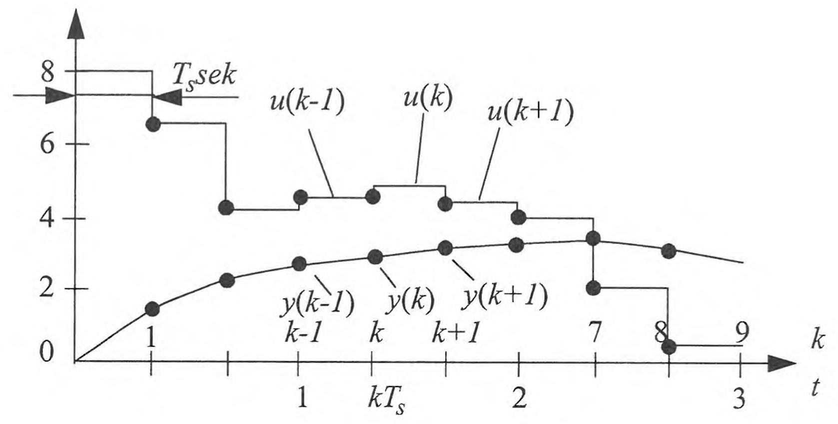


Figure 3.7 Input output signals in a continuous plant.

If we are not interested in the variables between the sampling instants but we want to focus on the values at the sampling instants we may define the following time discrete variables which are marked in the figure.

$$y(k) = y(kT_s) \quad ; k = 0, 1, 2, 3, ...$$

$$u(k) = u(t) \quad ; kT_s \le t < (k+1)T_s$$

By inserting $t_0 = kT_s$ and $t = (k+1)T_s$ in equation (3.18) we find:

$$y(k+1) = ay(k) + bu(k) \tag{3.19}$$

Where

$$a = e^{\left(-\left(\frac{1}{T}\right)\right)T_s} \tag{3.20}$$

$$b = K\int_0^{T_s} e^{-\left(\frac{1}{T}\right)\tau} d\tau = \frac{K\left(1 - e^{-\left(\frac{1}{T}\right)T_s}\right)}{\frac{1}{T}} \tag{3.21}$$

The difference equation (3.19) represents the result from discretizing of the continuous plant given by (3.17). The discretizing was carried out based on constant sampling period T_s and constant input $u(t)$ during the sampling interval.

One particular simple approximate method for discretizing differential equations which work fine for short sample intervals, is the Eulers method (also called the forward rectangular rule).

$$\frac{dy}{dt} \cong \frac{y(k+1) - y(k)}{T_s} \tag{3.22}$$

The approximation given in equation (3.22) will be used in place of all derivatives that appear in the differential equation.

Applying Eulers method to equation (3.17) we arrive to a difference equation given by (3.19) where the constant coefficients are given by:

$$a = 1 - \left(\frac{1}{T}\right)T_s \tag{3.23}$$

$$b = KT_s \tag{3.24}$$

By comparing (3.23) with (3.20) and (3.24) with (3.21), we see that the coefficients calculated by the approximate method are similar to the result we get if we take only the first two terms in the series expansion of the exponential function of the exact solution.

Example 3.1

Given a first order low-pass analog passive filter with low frequency gain 1 and time constant 1s. The filter topology and the transfer function is given in figure (3.8)

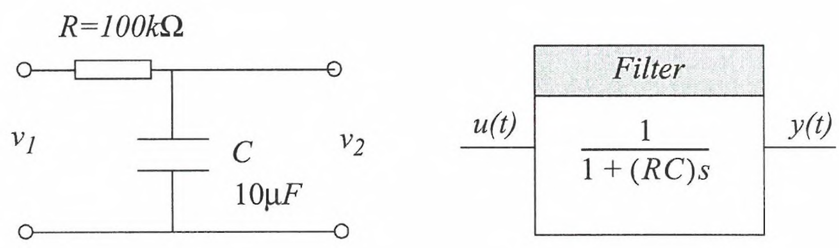


Figure 3.8 Analog low-pass filter

The output signal is sampled 3 times in a second. $\left(T_s = \frac{1}{3}\right)$. *The input signal is constant during the sampling interval* T_S. *Find the difference equation describing the filter.*

3.2.3 Difference equations

We have seen that discrete systems can be modelled in the time domain by difference equations. We have seen that the difference equation can be evaluated directly by a computer and that they can also represent discrete models of physical processes defined at the sampling instants. If the system is linear and time invariant the equation will be a *constant coefficients difference equation* (CCDE).

A general linear difference equation of order *n* with constant coefficients can be written as:

$$y(k) + a_1 y(k-1) + \ldots + a_n y(k-n) = b_1 u(k-1) + \ldots + b_n u(k-n) \quad (3.25)$$

For solving linear time-invariant difference equation there are different techniques that can be used. One approach consists of finding the complementary and the particular parts of the solution, in a manner similar to that used in the classical solution of linear differential equations. We will not take that approach here, but use a direct method which is a sequential procedure simular to the method used in the digital computer solution of difference equations. The method will be illustrated by examples.

To solve a specific CCDE we need a starting time (value of k) and a number of initial values depending on the order of the equation. The initial conditions represents the state of the system characterized by the computer memory at that time. For a physical process (for example a power electronics plant) the initial state may represent the energy stored in the system at starting time.

Example 3.2 First order difference equation

It is desired to find the unit step response y(k) for the differenc equation

$$y(k) = \frac{1}{2} y(k-1) + u(k-1) \quad \text{for } k \geq 0$$

Compared with the general equation (equation 3.25) we see that a_1*=-0.5 and* b_1*=1.*

The y(k) can be determined by solving the difference equation first for k=0, then for k=1, k=2 and so on. Thus

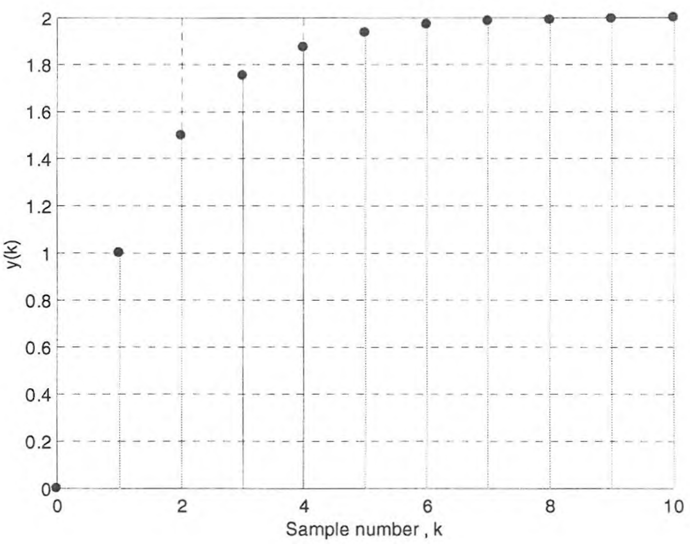
Table 3.1

k	*u(k-1)*	*0.5y(k-1)*	*y(k)*
0	0	0	0
1	1	0	1
2	1	0.5	1.5
3	1	0.75	1.75
4	1	0.875	1.875
5	1	0.975	1.9375

Note the sequential nature of the solution process. Continuing this procedure we can find y(k) for any value of k. This technique is not practical except when implemented on a digital computer. For this example a matlab script which solve the equation is :

```
k=0:10; % 11 samples
u=ones(size(k)); % step input
bcoeff=[0 1];
acoeff=[1 -0.5];
y=filter(bcoeff,acoeff,u);
[k'y'] % display values of k and y
stem(k,y,); % plot responce
xlabel('Sample number, k');
ylabel('y(k)');
grid;
```

Plot from the above code is:



Example 3.3 Second order difference equation

Consider the difference equation

$$y(k) - y(k-1) + y(k-2) = u(k-2) \tag{3.26}$$

Calculate the impulse response ($u(0)=1$; $u(k)=0$ for $k \neq 0$)

Table 3.2

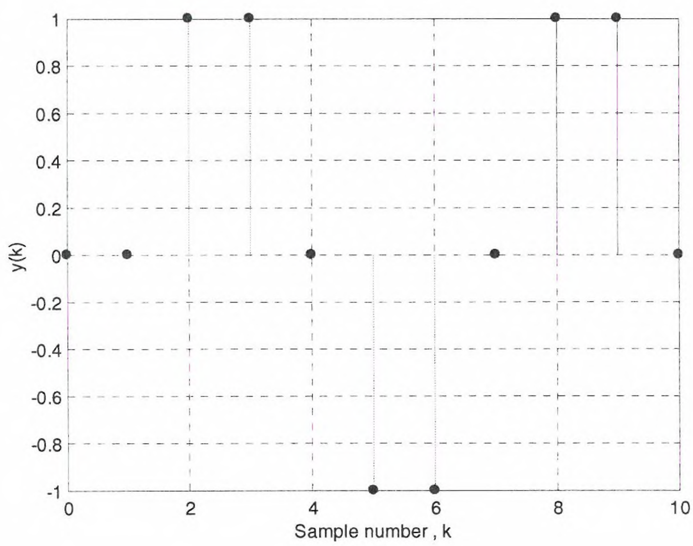
k	$u(k-2)$	$y(k-1)-y(k-2)$	$y(k)$
0	0	0	0
1	0	0	0
2	1	0	1
3	0	1	1
4	0	0	0
5	0	-1	-1
6	0	-1	-1

A matlab script for the solution is given below

```
k=0:10;
u=zeros(size(k));
```

```
u(1)=1;
bcoeff=[0 0 1];
acoeff=[1 -1 1];
y=filter(bcoeff,acoeff,u);
[k'y'] % display values of k and y
stem(k,y,); % plot responce
xlabel('Sample number, k');
ylabel('y(k)');
grid;
```

Plot from the above code is:



We see from the last examples that the solution to difference equations, simular to differential equations, can have stable response and approach a specific limit (the limit is 2 in example (3.2)) or be oscillating with a constant amplitude. In some case the response can be growing without bound. If the response of a dynamic system to any finite initial conditions can grow without bound, we call the system *unstable*. We would like to be able to examine equations like (3.25) and, without having to solve them explicitly, see if the response will be stable and even understand the general shape of the solution.

SOLUTION OF CCDE

The classical methods for solving difference equation is similar to the methods of differential equations. These methods require the prior determination of the homogeneous solution. The homogeneous difference equation give us the characteristic equation of the corresponding difference equation.

The solution to the characteristic equations describes the natural behaviour of the solution and predicts the stability of the solution to the difference equation.

Given the CCDE

$$\begin{aligned} y(k) &= -a_1 y(k-1) - a_2 y(k-2) - \ldots - a_n y(k-n) \\ &\quad + b_0 u(k) + \ldots + b_m u(k-m) \end{aligned} \tag{3.27}$$

The homogeneous equation is

$$y(k) + a_1 y(k-1) + a_2 y(k-2) + \ldots + a_n y(k-n) = 0 \tag{3.28}$$

Assuming $y(k) = A\lambda^k$ we get the equation

$$A\lambda^k + a_1 A\lambda^{k-1} + \ldots + a_n A\lambda^{k-n} = 0 \tag{3.29}$$

$$A\lambda^k(\lambda^n + a_1\lambda^{n-1} + \ldots a_n\lambda^0) = 0 \tag{3.30}$$

The characteristic equation is:

$$\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0 \tag{3.31}$$

Note that λ^k plays the same role in DE as $e^{\lambda t}$ in LTI differential equations.

The geneal solution to the CCDE is a linear combination of solutions, based on the roots of the characteristic equation.

For example

$$y(k) = 0{,}9y(k-1) - 0{,}2y(k-2) + b_0 u(k) \tag{3.32}$$

The homogeneous equation

$$\lambda^2 - 0{,}9\lambda + 0{,}2 = 0 \tag{3.33}$$

The roots are

$\lambda_1 = 0,5$ and $\lambda_2 = 0,4$

and the solution is

$$y(k) = A_1(0,5)^k + A_2(0,4)^k \tag{3.34}$$

Since both roots are inside the unit circle, equation (3.32) is stable. The value of the roots describe the natural behaviour of the solution $y(k)$.

Assuming that one of the roots is real.

$$y(k) = A\lambda^k \tag{3.35}$$

The numerical value of $y(k)$ is summarized in table 3.3.

Table 3.3

λ	value of $y(k)$ for $k = 0, 1, 2\ldots$
$\lambda > 1$	increasing
$\lambda = 1$	constant
$0 < \lambda < 1$	decreasing
$-1 < \lambda < 0$	decreasing, alternating sign
$\lambda = -1$	oscillating between +A and -A
$\lambda < -1$	increasing, alternating sign

Complex or imaginary roots always occur in conjugate pairs and give solution of the form:

$$y(k) = A_1(\lambda)^k + A_2(\lambda')^k \tag{3.36}$$

where

$\lambda = a + jb$

$\lambda' = a - jb$

Multiple real roots generate behaviour which consists of the term $k\lambda^k$. In general, if the repeated roots are indicated by the factor $(\lambda - r_k)^q$, terms of the form:

$$A_1 k^{q-1}(r_k)^k + A_2 k^{q-2}(r_k)^k + \ldots + A_q(r_k)^k$$

will appear in the solution.

A practical method of finding the particular solution to the difference equation is to use the z-transform approach which will be presented later.

3.3 Z-transform representation of discrete systems.

So far we have treated the DT-system in the time domain. We will now introduce the Z-transform which is a convenient tool when dealing with problems of a discrete nature. The role of the Z-transform (ZT) in discrete systems is similar to that of the Laplace transform in continuous systems.

3.3.1 Definition of the Z-transform.

The Z-transform maps a time sequence into the complex z-plane.

Given a time sequence $\{x(kT_s)\}$. The basic one sided Z-transform is defined as [15]:

$$x(z) = Z\{x(kT_s)\} = \sum_{k=0}^{\infty} x(kT_s)z^{-k} \tag{3.37}$$

$x(kT_s) = 0$ for $k < 0$.

That is a causal sequence. Since in practical situations the sequence is causal, this one-sided ZT will be satisfactory for most engineering applications. To assure that the ZT exists we assume that $\lim_{k \to \infty} \sum x(kT_s)z^{-k}$ exists in some region in the complex z-plane.

A fundamental property of the ZT is that there is a one-to-one correspondence between the sequence $\{x(kT_s)\}$, and its ZT. That is, given $x(z)$ we can recover $\{x(kT_s)\}$ via the inverse Z-transform (IZT) which we denote as:

$$\{x(kT_s)\} = Z^{-1}\{x(z)\} = \frac{1}{2\pi i}\int_C z^{k-1}x(z)dz \tag{3.38}$$

where the integral is taken along a closed path, C, in the complex z-plane which must contain the origin. FT and IFT form a transform pair.

If the sequence $x(kT_s)$ is the result of sampling a CT function $x(t)$ we must observe that there is no a unique correspondence between $x(x)$ and $x(t)$. This occurs because many CT-signals can give the same $\{x(kT_s)\}$ and therefore $x(x)$. The ZT represents a sampled CT-signals $x(t)$, with samples $x(kT_s)$, at the sampling instants only.

From the definition of the ZT, equation (3.37), many useful properties of the ZT can be derived. The most importants ones are summarized in appendix A.

3.3.2 The Z-transform of some elementary time functions/sequences.

For larger reference we will in this section calculate the ZT for some elementary time sequences. The calculations will be based on the ZT-definition.

(a) The unit impulse.

$$\delta(k) = \begin{cases} 1 & \text{for k=0} \\ 0 & \text{for k=1,2,...,}\infty \end{cases}$$

$$Z[\delta(k)] = \sum_{k=0}^{\infty} 1z^{-k} = 1$$

(b) The unit step.

$$u(k) = 1 \qquad for \qquad 0 \le k \le \infty$$

$$Z[u(k)] = 1 + z^{-1} + z^{-2} + \dots$$

$$= \frac{1}{1-z^{-1}} = \frac{z}{z-1} \qquad |z| > 1$$

(c) The discrete ramp function = $f(kT_s) = kT_s$

$$F(z) = \sum_{k=0}^{\infty} (kT_s)z^{-k} = T_s z^{-1} + 2T_s z^{-2} + 3T_s z^{-3} + \tag{3.39}$$

To express $F(z)$ in closed form, we first multiply both sides by z^{-1}, resulting in

$$z^{-1}F(z) = T_s z^{-2} + 2T_s z^{-3} + \ldots \tag{3.40}$$

Subtracting the last equation from the first we get

$$\begin{aligned}(1-z^{-1})F(z) &= T_s z^{-1} + T_s z^{-2} + T_s z^{-3} + \ldots \\ &= T_s z^{-1}(1 + z^{-1} + z^{-2} + \ldots) \\ &= T_s z^{-1}\left(\frac{1}{1-z^{-1}}\right)\end{aligned} \tag{3.41}$$

$$F(z) = \frac{T_s z^{-1}}{(1-z^{-1})^2} = \frac{T_s z}{(z-1)^2} \tag{3.42}$$

(d) The discrete exponential function

$x(k) = a^k$

$$\begin{aligned}Z[x(k)] = Z[a^k] &= \sum_{k=0}^{\infty} a^k \cdot z^k \\ &= 1 + \left(\frac{z}{a}\right)^{-1} + \left(\frac{z}{a}\right)^{-2} + \ldots \\ &= \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad |z| > a\end{aligned} \tag{3.43}$$

Another form of the discrete exponential function is:

$x(k) = e^{-akT_s}$ where a is a real constant.

$$x(z) = Z[e^{-akT_s}] = \sum_{k=0}^{\infty} e^{-akT_s} z^{-k} \tag{3.44}$$

$$= 1 + e^{-aT_s} z^{-1} + (e^{-2aT_s} z^{-2}) + \dots$$

$$= 1 + e^{-aT_s} z^{-1} + (e^{-aT_s} z^{-1})^2 + \dots$$

This infinite series converges for all values of z that satisfy

$|e^{-aT_s} z^{-1}| < 1$

$$= \frac{1}{1 - e^{-aT_s} z^{-1}} = \frac{z}{z - e^{-aT_s}}$$

for $|e^{-aT_s} z^{-1}| < 1$ $\quad or \quad$ $|z^{-1}| < e^{aT_s}$

(e) The discrete sinusoidal function

$f(kT_s) = \sin(\omega k T_s)$

$$F(z) = \sum_{k=0}^{\infty} (\sin \omega k T_s) z^{-k} \tag{3.45}$$

It is convenient to express $\sin \omega k T_s$ in the exponential form

$$F(z) = \sum_{k=0}^{\infty} \left(\frac{e^{j\omega k T_s} - e^{-j\omega k T_s}}{2j} \right) z^{-k} \tag{3.46}$$

Since the z transform of the exponential function is:

$$Z\left(e^{-akT_s}\right) = \frac{1}{1 - e^{-aT_s} z^{-1}}$$

then we get:

$$F(z) = \frac{1}{2j}\left[\frac{1}{1-e^{j\omega T_s}z^{-1}} - \frac{1}{1-e^{-j\omega T_s}z^{-1}}\right]$$

$$= \frac{1}{2j}\frac{\left(e^{j\omega T_s} - e^{-j\omega T_s}\right)z^{-1}}{1-\left(e^{j\omega T_s}+e^{-j\omega T_s}\right)z^{-1}+z^{-2}}$$

$$= \frac{z^{-1}\sin\omega T_s}{1-2z^{-1}\cos\omega T_s + z^{-2}}$$

Therefore:

$$F(z) = \frac{z\sin\omega T_s}{z^2 - 2z\cos\omega T_s + 1} \qquad \text{for } |z| > 1 \tag{3.47}$$

.

.

.

3.3.3 The transfer function for discrete time systems.

We know that for CT-systems the Laplace transform (LT) enables us to represent these systems in terms of transfer functions. We will now show that difference equations and the ZT enable us to do so for DT-systems.

We have seen that the computer algorithm in Figure 3.1 represents a DT-system expressed by the following input-output relationship:

$$u_k = g_0 e_k + g_1 e_{k-1} + \ldots + g_n e_{k-n} + -f_1 u_{k-1} (-f_n) u_{k-n} \tag{3.48}$$

Assume that the system is initially at rest, that is, all initial conditions are zero.

If we multiply (3.48) by z^k and sum over k we get:

$$\sum_{k=0}^{\infty} u_k z^{-k} = g_0 \sum_{k=0}^{\infty} e_k z^{-k} + g_1 \sum_{k=0}^{\infty} e_{k-1} z^{-k} + \ldots + g_n \sum_{k=0}^{\infty} e_{k-n} z^{-k}$$

$$-f_1 \sum_{k=0}^{\infty} u_{k-1} z^{-k} - f_2 \sum_{k=0}^{\infty} u_{k-2} z^{-k} - \ldots - f_n \sum_{k=0}^{\infty} u_{k-n} z^{-k}$$

For a general term which is of the form:

$$\alpha_j \sum_{k=0}^{\infty} k_{k-j} z^{-k} \tag{3.49}$$

Using the definition of the ZT which defines $x_i = 0$ for $i < 0$

$\sum_{i=0}^{\infty} x(iT_s) z^{-i} = x(z)$ and letting k-j = i we obtain

$$\alpha_j z^{-j} \sum_{i=-j}^{\infty} x_i x^{-i} = \alpha_i z^{-j} \sum_{i=0}^{\infty} x(iT_s) z^{-i} = \alpha_j z^{-j} x(z) \tag{3.50}$$

Thus (3.48) is now transformed to an algebraic equation in the z-domain

$$u(z) = g_0 e(z) + \ldots + g_n z^{-n} e(z) - f_1 z^{-1} u(z) - \ldots - f_n z^{-n} u(z) \tag{3.51}$$

Rearranging the terms gives:

$$u(z) = \frac{g_0 + g_1 z^{-1} + \ldots + g_2 z^{-n}}{1 + f_1 z^{-1} + \ldots f_n z^{-n}} e(z) \tag{3.52}$$

This result could also been obtained directly by the backward shift theorem.

The ratio

$$\frac{u(z)}{e(z)} = h_R(z) = \frac{g_0 + g_1 z^{-1} + \ldots + g_n z^{-n}}{1 + f_1 z^{-1} + \ldots + f_n z^{-n}} \tag{3.53}$$

is defined as the transfer function $h_R(z)$ which also can be written as

$$h_R(z) = \frac{g_0 z^n + g_1 z^{n-1} + \ldots g_n}{z^n + f_1 z^{n-1} + \ldots f_n} \tag{3.54}$$

The causality condition, or practical realization of equation (3.54) requires that no future values can be used for calculation of *uk*. This is reflected in the ZT function (3.54) by the fact that the order of the numerator must be less than or equal to the order of denominator.

Factoring the polynomials in the numerator and denominator of $h_R(z)$, we obtain

$$h_R(z) = \frac{g_0(z-z_1)(z-z_2)\ldots(z-z_n)}{(z-p_1)(z-p_2)\ldots(z-p_n)} \tag{3.55}$$

where z_i and p_i denote the ith zero and pole, respectively, of $h_R(z)$.

The impulse response of a DT-system is the output that results when the input is the unit impulse sequence $\{\delta(k)\} = 1$ for $k = 0$ and zero elsewhere.

The ZT of the unit impulse is:

$$Z[\delta(k)] = \sum_{k=0}^{\infty} 1z^{-k} = 1 \tag{3.56}$$

The corresponding output is:

$$y(z) = h(z) * 1 = h(z)$$

Upon inverting the z-transform the result is

$$h(k) = Z^{-1}\, h(z)$$

Thus the inverse transform of the transfer function is the impulse response sequence.

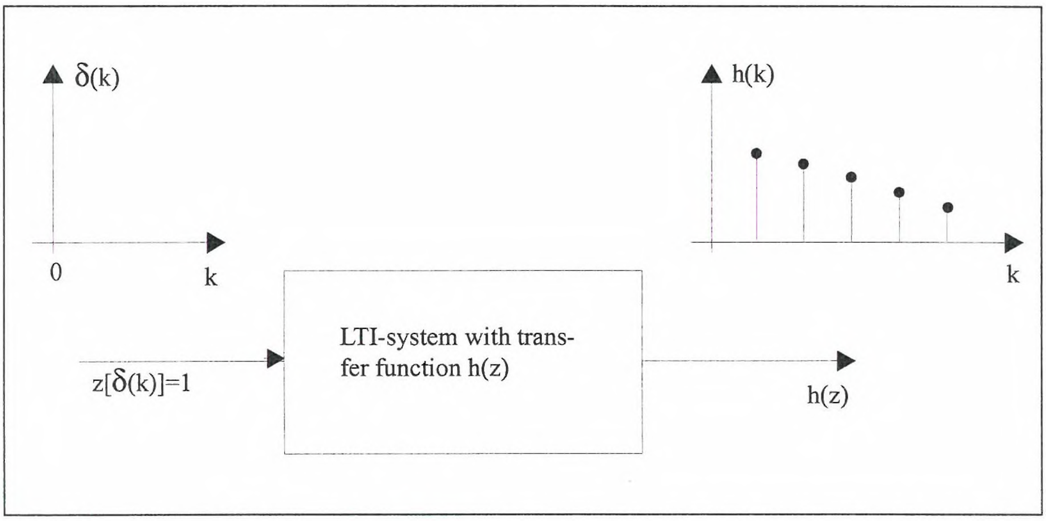


Figure 3.9 The impulse response of a discrete time invariant system

3.4 Representation of the conversion between discrete and continuous signals

Special electronic devices are necessary to make the digital controller able to exchange information with the CT-system which is to be controlled. These devices are logically represented in Figure 3.1 as D/A and A/D converters. The aim of this section is to present a mathematical representation of these conversion processes.

The D/A converter acts on the DT-sequence from the computer.

This signal, u_k, represented as a numerical content of some register in the processor, is converted to a CT-signal u(t) by keeping the amplitude constant between the sampling instants.

That is to say:

$$u(t) = u(kT_s) \qquad kT_s \le t \le (k+1)T_s$$

From a functional viewpoint it may be regarded as a device which consists of a decoder and a zero order hold unit as shown in Figure 3.10. The decoder acts simply as a constant gain.

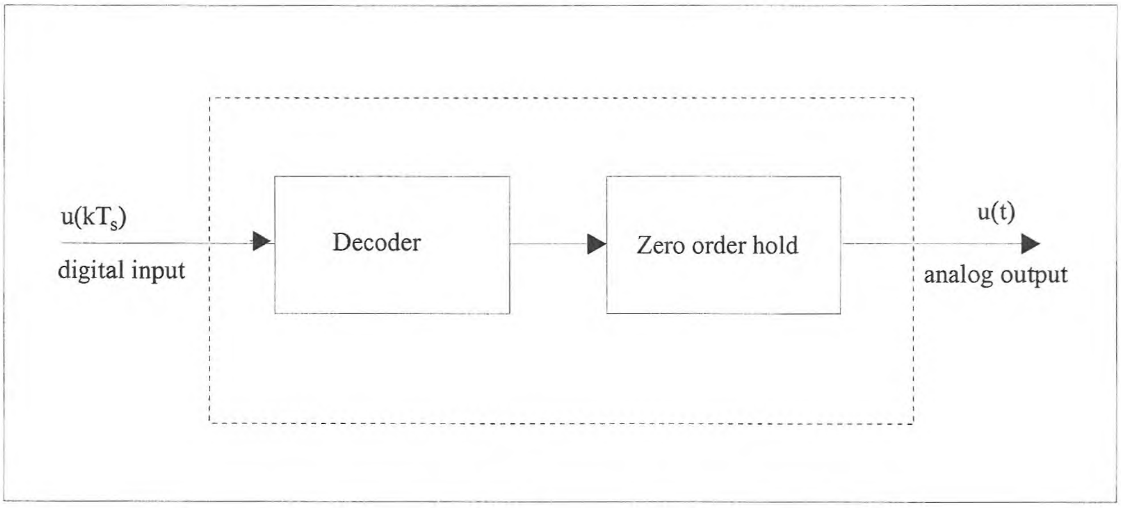


Figure 3.10 Functional block diagram representation of D/A conversion

The A/D conversion process is the transformation of information contained in a CT-signal into a digital-coded word. This involves the following sequential operations: sample, quantizing, and encoding.

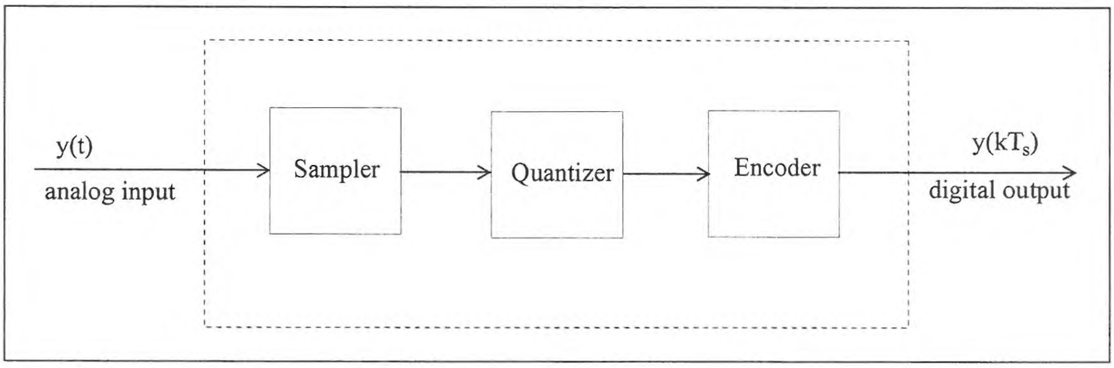


Figure 3.11 Functional block diagram representation of A/D conversion

If we assume that the resolution of the A/D conversion is very high, quantizing can be neglected. The encoder also acts simply as a constant gain, normally a unity gain. Thus for analytical purposes the block diagram can be reduced to a sampler.

We are now going to establish a mathematical model for the basic elements sampler and hold element.

The sampler can be considered as a switch, which is closed in a very short time interval, of equal length, at every sampling instant. The pulses thus generated will have a strength or area proportional to the magnitude of the input signal at the sampling instants.

We can model this process by a mathematical idealization where the CT signal, x(t), is amplitude modulated by the impulse train

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \tag{3.57}$$

as shown in Figure 3.12.

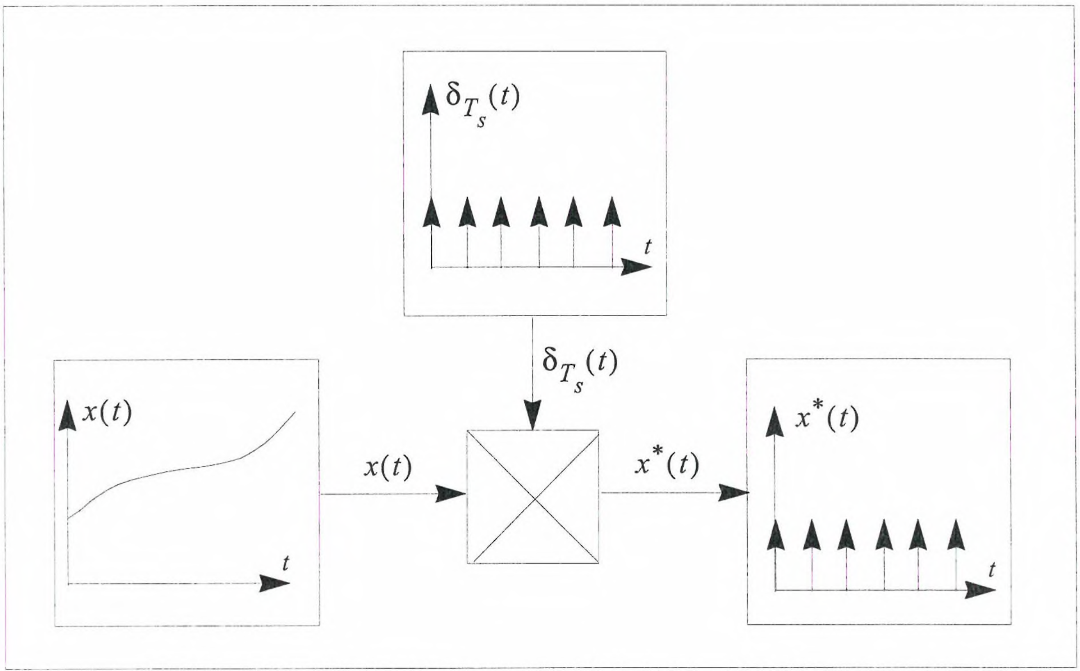


Figure 3.12 Representation of ideal sampling

The output from this ideal sampler is a train of impulses with strengths equal to $x(kT_S)$:

$$x^*(t) = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_s) = \sum_{k=-\infty}^{\infty} x(kT_s)\delta(t - kT_s) \tag{3.58}$$

The symbol (*) is a common way of denoting a signal sampled by ideal impulse sampling. Later in the text, a switch with the sampling period indicated below, will be used to represent an ideal sampler.

Since the train of impulses is a periodic function with period T_S and angular frequency $\omega_s = \frac{2\pi}{T_s}$ the function can be represented by the complex Fourier series

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_s t} \tag{3.59}$$

where the complex Fourier coefficient is given by

$$c_n = \frac{1}{T_s} \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta_T(t) e^{-jn\omega_s t} dt = \frac{1}{T_s} \tag{3.60}$$

Thus the infinite train of impulses may be written as:

$$\delta_T(t) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \tag{3.61}$$

If we substitute (3.61) for the impulse train in (3.58), for the sampled signal we get:

$$x^*(t) = \frac{x(t)}{T_s} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \tag{3.62}$$

From the definition of the Laplace transform we can now calculate the LT of the signal x*(t)

$$x^*(s) = \int_0^{\infty} x^*(t) e^{-st} dt = \frac{1}{T_s} \int_0^{\infty} x(t) \sum_{n=-\infty}^{\infty} e^{jn\omega_s t} e^{-st} dt \tag{3.63}$$

which is equal to

$$x^*(s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} x(s - jn\omega_s) \tag{3.64}$$

where x(s) is the LT of the CT signal x(t).

Another expression for the LT of the sampled signal x*(t) can be derived directly from (3.58)

$$x^*(s) = \int_0^{\infty} \sum_{k=-\infty}^{\infty} x(kT_s)\delta(t - kT_s)e^{-st}dt$$

$$x^*(s) = \sum_{k=0}^{\infty} x(kT_s)e^{-skT_s} \tag{3.65}$$

The expressions (3.64) and (3.65) have many interesting implications. We will return to eq. (3.65) in a later section. One of the most interesting implications of (3.64) is the important sampling theorem which we are now going to discuss.

The operation of sampling a CT signal gives a sampled frequency domain function which can be evaluated by simply evaluating (3.64) on the s=jω axis. This gives:

$$x^*(j\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} x[j(\omega - n\omega_s)] \tag{3.66}$$

where x(jω) is the Fourier transform of the unsampled function x(t).

The equation (3.66) makes the connection between the FT of x(t) and the FT of x*(t) very clear. Consider a band limited signal x(t) which has a amplitude spectrum |x(jω)| and a cut off frequency ω_0 as shown in Figure 3.13.

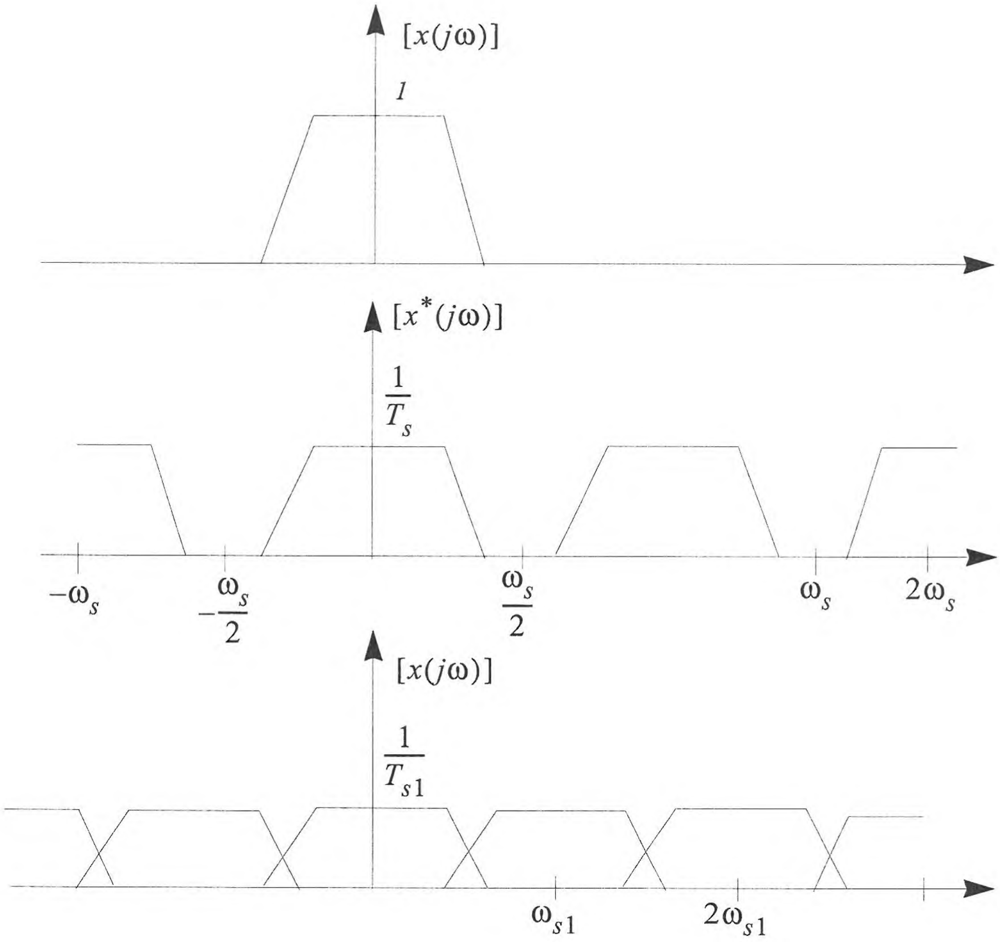


Figure 3.13 The effect of sampling on the amplitude spectrum of a band limited signal.

As we see from the figure and equation (3.66) the amplitude spectrum of x(t) is repeated with period ω_s along the ω-axis. When the sampling frequency $\omega_s > 2\omega_0$ the spectrum $|x^*(j\omega)|$ will be periodic with no overlap. The sampler acts as a harmonic generator.

If we lower the sampling frequency so that $\omega_s < 2\omega_0$ we get overlap between the base band and the frequency shifted bands. This is called aliasing. The higher frequencies are folded back into the low frequencies. Information in the CT signal spectrum is now lost. It is not possible to reconstruct the original frequency spectrum. Figure 3.14 clearly shows what happens when a single frequency signal is sampled at a rate which is too low (i.e. aliasing occurs).

We will recall here that the derivation of the frequency spectrum $|x^*(j\omega)|$ was based on ideal sampling. Non-ideal sampling (i.e. finite-width pulse amplitude sampling) will give the same frequency distribution of $|X^*(j\omega)|$, but the amplitudes will have a factor $\frac{p}{T_s}\left|\frac{\sin(n\omega_s p/2)}{(n\omega_s p/2)}\right|$ instead of $\frac{1}{T_s}$ where p is the pulse width. [16].

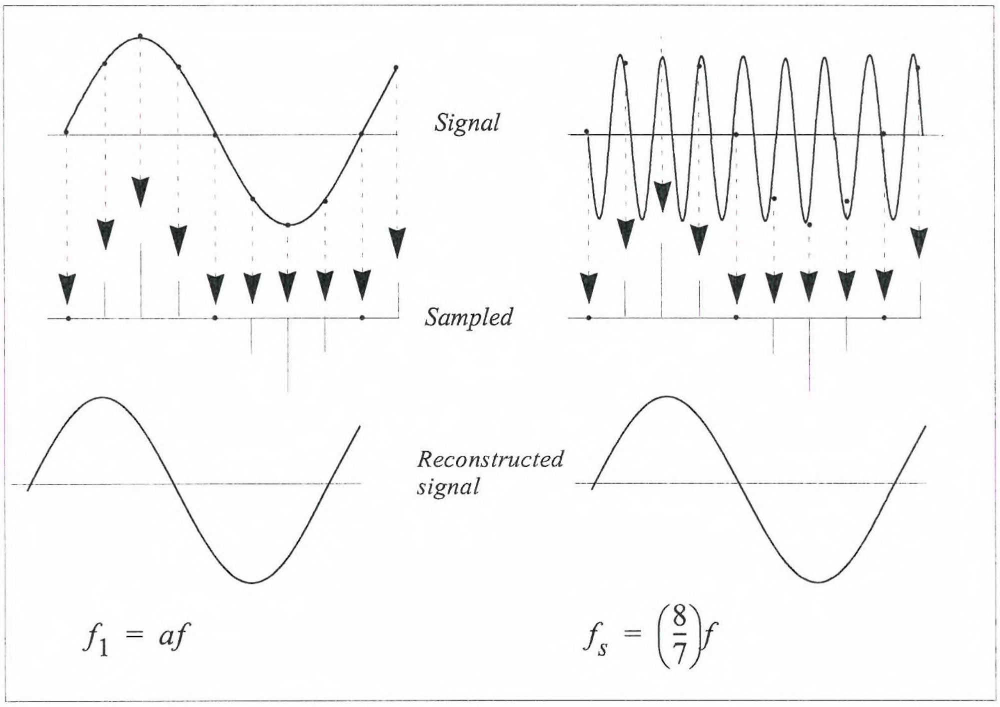


Figure 3.14 The higher frequency signal is folded down to a low frequency signal due to sampling

We can now conclude with the Shannon's Sampling Theorem which states: If a signal contains no frequency higher than ω_0 radians per second, it is completely characterized by the values of the signal sampled with a frequency $\omega_s > 2\omega_0$. The frequency $\frac{\omega_s}{2} = \frac{\pi}{T_s}$ is often called the Nyquist frequency.

In practice however, stability of the closed-loop system and other practical considerations may make it necessary to sample at a rate higher than this theoretical minimum.

Strictly speaking, a band-limited signal rarely exists in a physically control system. Only approximate band limited signals are found. Therefore, in practice when sampling a CT signal one must:

1. Choose a sampling frequency $\omega_s > 2\omega_b$ where ω_b is the highest frequency of interest in the CT signal.
2. Implement a low pass analog pre-sampling filter with cross-over frequency, ω_f where $\omega_b < \omega_f < \frac{\omega_s}{2}$

The highest frequency of interest is related to the bandwidth of the close-loop system. The selection of sampling rate can then be based on bandwidth, or equivalently, on the rise time of the closed loop system. A rule of thumb often used is to choose the sampling frequency about 10 times the bandwidth, or 3-4 samples during the rise time [17].

Because of the high frequency components inherently present, it is not desirable to apply the pulse sampled or discrete signal directly to an analog system. An equivalent time domain explanation is that these short pulses are not able to control the continuous plant without some sort of a reconstruction device. It is the strength of these pulses that are of interest. In a practical situation the sampled signal must therefore be followed by some sort of data hold, most often a zero-order hold, which is integrating each pulse.

The purpose of the zero order hold reconstruction filter is to hold the $u(kT_s)$ value until the succeeding sampling instant, as illustrated in Figure 3.15.

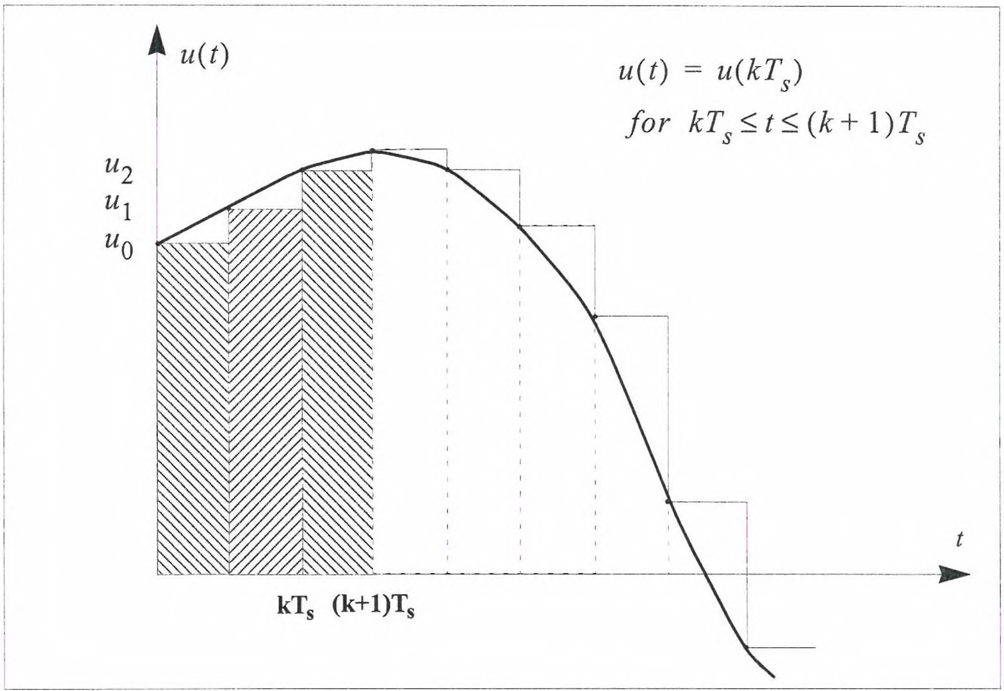


Figure 3.15 Reconstruction with a zero order hold

The impulse response of the zero order hold filter can be written as:

$$h(t) = u_1(t) - u_1(t - T_s) \tag{3.67}$$

where u(t) denotes a unit step function.

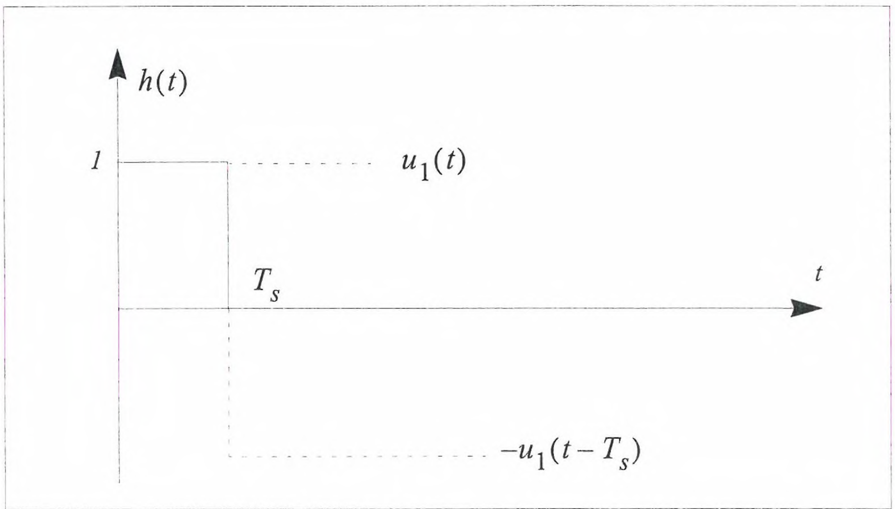


Figure 3.16 The impulse response of zero order hold.

For the LT we have:

$$h_h(s) = \frac{1}{s} + \frac{-e^{-sT_s}}{s} = \frac{1-e^{-sT_s}}{s} \tag{3.68}$$

And the frequency response is:

$$\begin{aligned} h_h(j\omega) &= \frac{1-e^{-j\omega T_s}}{j\omega} = e^{-j\omega\frac{T_s}{2}}\left\{\frac{e^{j\omega\frac{T_s}{2}} - e^{-j\omega\frac{T_s}{2}}}{2j}\right\}\frac{2j}{j\omega} \\ &= T_s e^{-j\omega\frac{T_s}{2}} \frac{\sin\frac{\omega T_s}{2}}{\frac{\omega T_s}{2}} \end{aligned} \tag{3.69}$$

The amplitudes and phase plots are shown in Figure 3.17.

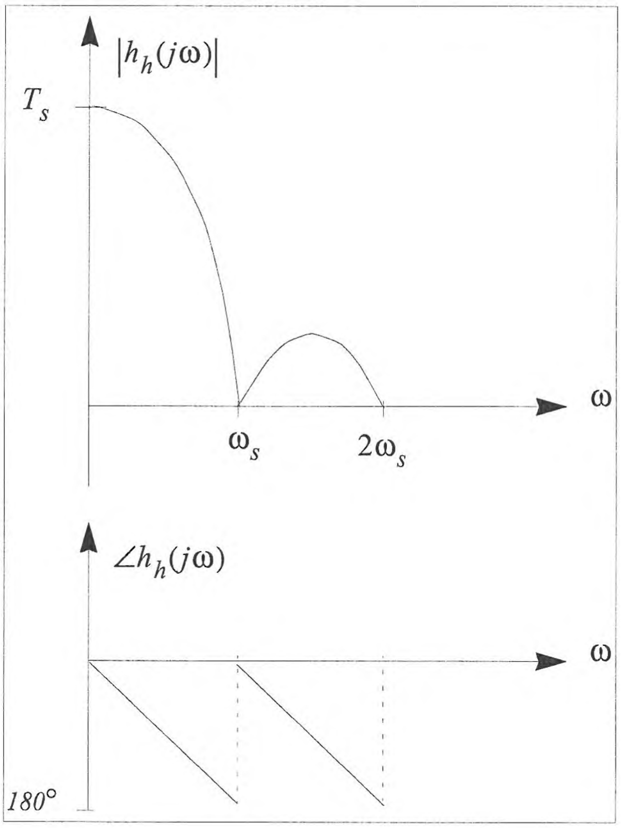
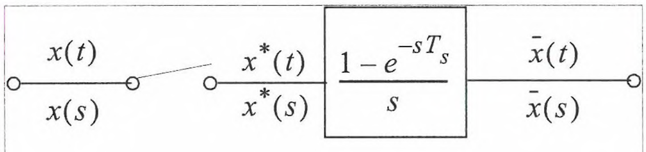


Figure 3.17 Frequency response of the zero order hold

We observe that the magnitude becomes zero at the frequency equal to the sampling frequency and at integral multiples of the sampling frequency. Notice that the steady-state gain of the zero order hold is $h_h(s=0)=T_s$. We have seen, eq. (3.64), that the ideal sampling process has a gain $\frac{1}{T_s}$. Thus the combination of a sampler and a zero order hold has unit steady state gain. We also observe that for very fast sampling, $\omega_s \gg \frac{\pi}{T_s}$, a series connection of a sampler and zero order hold element act as a CT system with unit transfer function.

The ideal sampler is not a physical device but a mathematical approximation. In a physical situation the sampler is always followed by a hold element. Thus the combination of the ideal sampler and the zero order hold element accurately models a physical transfer operation. See Figure 3.18.



$$\bar{x}(s) = [\Sigma x(kT_s)e^{-kT_s s}]\left[\frac{1-e^{-sT_s}}{s}\right]$$

Figure 3.18 Representation of sampler and data hold

As can be seen from equation (3.65) sampling is a linear but time-variant operation. Many different input signals can result in the same output signal.

Thus it is not possible to find a transfer function for the ideal sampler. These properties complicate the analysis of a discrete control system, especially if we are interesting in the exact behaviour between sampling instants.

We have now found the LT, $x^*(s)$, of a sampled signal $x^*(t)$. In section 3.3 we introduced the ZT for a discrete time sequence $\{x(kT_s)\}$.

We may now ask tourselves: Is there a connection between these two transforms?

If we can establish such a relationship then the DT and the CT system models can be linked together. The CT-system part may then be modelled as a discretized or sampled system. In this way we may consider the complete Control System as discrete, and use the ZT representation.

The similarities between the transforms X(z) and x*(s) is obvious. In fact, if we assume the number sequence $\{x(kT_S)\}$ is obtained from sampling a time function x(t) and $e^{sT_s} = z$ in (3.65), then it becomes the z-transform. In this case we have

$$x(z) = x^*(s)\Big|_{e^{sT_s} = z} \tag{3.70}$$

We will use the following change in variable:

$$e^{sT_s} = z \qquad , s = \frac{1}{T_s} lnz \tag{3.71}$$

In general, the ZT instead of the LT* in our analysis of digital control system will be used.

One advantage of this approach is that according to equation (3.64) x*(s) has an infinity number of poles and zeroes in the s-plane. However, x(z) has a limited number of poles and zeroes. In this way analysis or design procedures that utilize a pole-zero approach are greatly simplified through the use of the ZT.

3.5 MATHEMATICALREPRESENTATION OF THE CONTINUOUS PLANT BY DISCRETIZATION

Now that we have described the conversion of signals between the computer and the CT-plant we turn to the fundamental problem of finding a discrete time equivalent representation of the CT-plant.

This representation will give the relation between the output sequence of the plant {y(kT} and the input {u(kT)}. Using this model representation the system variables are considered only at the sampling instants because the system is the time invariant.

We will consider the situation when a sample and hold device is connected to a LTI-system with the transfer function $h_p(s)$.

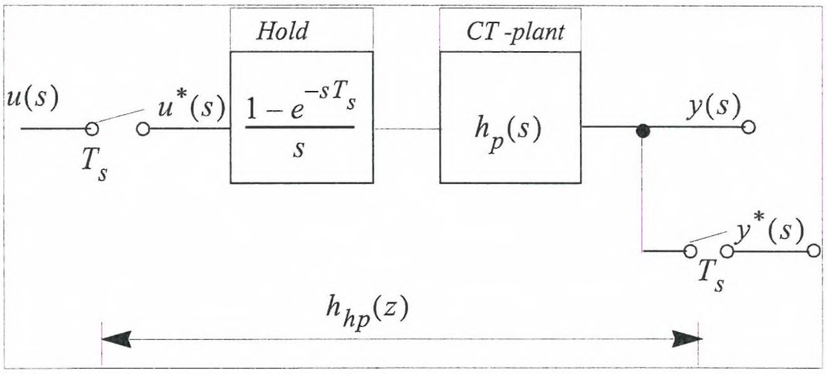


Figure 3.19 Forward loop for CT-process driven through zero order hold.

If we let $h_{hp}(s)$ be the transfer function of the zero-order hold and the process, i.e.

$$h_{hp}(s) = \frac{1}{s}\left(1 - e^{-sT_s}\right)h_p(s) \tag{3.72}$$

The LT of the sampled input signal u*(t) is given by

$$u^*(s) = \sum_{k=0}^{\infty} u(kT_s)e^{-skT_s}$$

The LT of the CT-output signal from the process is:

$$y(s) = h_{hp}(s)\sum_{k=0}^{\infty} u(kT_s)e^{-skT_s}$$

$$y(s) = h_{hp}(s)u^*(s) \tag{3.73}$$

Thus we have calculated the LT of the CT-output signal. As (3.73) shows, it is not possible to factor out the LT of the CT-signal u(t). So, we cannot find a transfer function between the input u(t) and the output y(t). This is because the system is not time invariant.

If we are interested in the output signal only at the sampling instants then we are able to define a transfer function. By adding a fictive synchronous sampler to the system as shown in the figure, the input output relation is then given by:

$y^*(s) = [h_{hp}(s)\ u^*(s)]^*$

From (3.64) we know

$$y^*(s) = \frac{1}{T_s}\sum_n y(s + jn\omega_s) = \frac{1}{T_s}\sum_n h_{hp}(s + jn\omega_s)u^*(s + jn\omega_s)$$

Because $u^*(s+jn\omega_s) = u^*(s)$ we can write

$$y^*(s) = u^*(s)\frac{1}{T_s}\sum_n h_{hp}(s + jn\omega_s)$$

$$y^*(s) = u^*(s)h_{hp}{}^*(s) \tag{3.74}$$

By using the relationship between x*(s) and x(z) equation we may write

$$y(z) = u(z)\ h_{hp}(z)$$

and

$$\frac{y(z)}{u(z)} = h_{hp}(z) \tag{3.75}$$

We have now seen that when sampling a CT-plant which is driven by a zero order hold reconstruction device, the discretized plant can be represented by the pulse transfer function $h_{hp}(z)$. This is a very important relation for the digital control of a CT plant.

3.6 COMPUTING THE PULSE TRANSFER FUNCTION FROM THE CT-TRANSFER FUNCTION $h_p(s)$.

The pulse transfer function, $h_{hp}(z)$, can be derived directly from the CT transfer function $h_p(s)$ by the following argumentation:

The signal at the output of the zero order hold could be decomposed into a series of gate functions as indicated in Figure 3.15. Each of these gate function could be further decomposed into step function as shown in Figure 3.16. The continuous time response to the first step of magnitude u_0 is

$$y(t) = u_0 L^{-1}\left[\frac{h_p(s)}{s}\right]$$

where L^{-1} is inverse Laplace transform. The ZT of the output sampled sequence y(k) in Figure 3.16 is given by:

$$y(z) = u_0 Z\left[L^{-1}\left[\frac{hp(s)}{s}\right]\right]$$

the response due to the negative step increment is similar except for the delay of one sampling period. The total ZT due to the first gate function is:

$$y(z) = u_0\left(Z\left[L^{-1}\left[\frac{h_p(s)}{s}\right]\right] - z^{-1}Z\left[L^{-1}\left[\frac{h_p(s)}{s}\right]\right]\right)$$

where z^{-1} represents the delay of one sample period. Extending this procedure to the entire series of gate functions representing u(t), the ZT of the sampled output y(kT) is:

$$y(z) = \left[\sum_{k=0}^{\infty} u(kT)z^{-k}\right](1 - z^{-1})Z\left[L^{-1}\left[\frac{h_p(s)}{s}\right]\right]$$

We recognize the first term as the ZT of the {u(kT)} sequence, so the DT transfer function between input and output, the pulse transfer function, is:

$$h_{hp}(z) = \frac{y(z)}{u(z)} = (1 - z^{-1})Z\left[L^{-1}\left[\frac{h_p(s)}{s}\right]\right] \tag{3.76}$$

We can summarize the method for obtaining the Pulse transfer function by the following steps:

1. Determine the time function corresponding to $\frac{h_p(s)}{s}$
2. Determine the corresponding ZT (Usually from a table which can be found in many text-books [16].)
3. Multiply by $(1 - z^{-1})$ to get the pulse transfer function including the zero order hold.

Another method for calculating an approximate value of the pulse transfer function $h_{hp}(z)$ is to use the so-called Tustin transformation

$$s = \frac{2}{T_s}\frac{z-1}{z+1}$$

An evaluation of this approximation is given in [18].

3.7 MATHEMATICAL REPRESENTATION OF A DC-MACHINE

As we have seen, the starting point for determining the discrete pulse transfer function for a continuous plant is a model representation of the plant given by the state space equations or the CT transfer function of the plant.

As the DC-machine will be used in the case studies that follow, the dynamic equations of this machine will be reviewed in this section. The power converter feeding the machine which is assumed to be a thyristor or transistor converter is also a part of the CT plant. The mathematical representation of the converter is discussed in the next section.

The dynamic state equations and the CT transfer function block diagram representation of the DC machine is shown in Figure 3.20. The model is linearized and no connection between the armature and field windings is assumed.

In the block diagram and equations, all voltages, currents, and torque are referenced to their rated values. The speed *n* is referenced to the value that appears when an unloaded machine is fed with rated voltage. Thus, the electro-mechanical system is characterized by the gain and the time constant of the armature circuit and by the integration time of the mechanical inertia:

$$K_a = \frac{U_{an}}{I_{an}\sum R_a} = \frac{I_{ak}}{I_{an}} = \frac{U_{an}I_{an}}{I_{an}^2\sum R_a}$$

is the ratio of nominal armature voltage and voltage drop across the armature resistance carrying rated current I_{an}. This is equal to the ratio of the current I_{ak} and I_{an}. I_{ak} is the current we get when the armature voltage is equal to U_{an} at zero speed. K_a can also be expressed as the ratio of nominal input armature terminal power and armature power loss. Typical values of K_a are in the range 8 to 15.

$$T_r = \frac{2\pi n_0 J}{M_n}$$

is the mechanical run up time of the machine. It is the time the machine will use to reach the unloaded or rated speed n_0 when it starts at zero speed and the torque is equal to the rated value M_n. J is the moment of inertia and includes all masses involved in the turning motion.

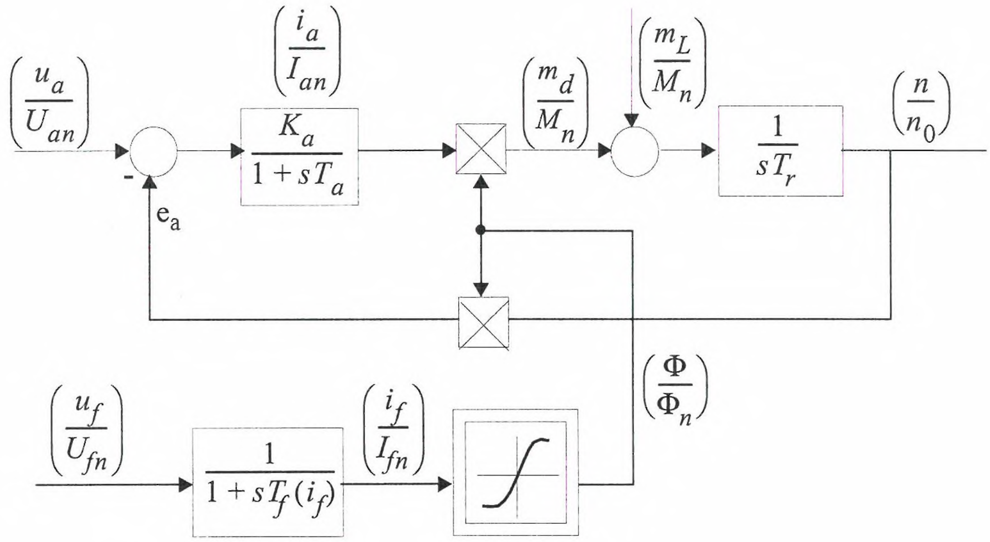


Figure 3.20 Mathematical representation of a DC-machine

$$T_m = \frac{T_r}{K_a\left(\frac{\phi}{\phi_n}\right)^2}$$

Sometimes T_m will be used instead of T_r. It is called the mechanical time constant because of its physical background: when the electrical time constant T_a is neglected and the machine thereby becomes a first order system, the time constant T_m characterizes all transient behaviour of the machine.

The given block diagram representation of the armature circuit of the DC machine is in accordance with the representation given in most text books on electrical machines [19], [20] and [21].

The discussion which follows will be concerned with the field circuits.

The effect of a rapid changing excitation voltage, which produces eddy currents in the iron laminations is not reflected in the diagram. The effects of eddy currents are often accounted for by means of a resistance R_ω in parallel with the inductance L_e of the field circuits [20]. In this case the corresponding transfer function also includes a differentiation term giving a phase advance:

$$\frac{\left(\frac{i_f}{I_{fn}}\right)}{\left(\frac{u_f}{U_{fn}}\right)} = \frac{1 + s\frac{L_e}{R_\omega}}{1 + sL_e\frac{R_e + R_\omega}{R_e R_\omega}} \tag{3.77}$$

Due to the saturation of the iron, the magnetic flux in the machine is a non-linear function of the excitation current. The slope at each point of the magnetization curve is the effective amplification in the field circuit. This slope is also a measure of the permeability of the iron and also the winding inductance. The following relationship is obtained for any point on the magnetization characteristic:

$$\frac{\Delta\phi}{\Delta i_e} = KL_e \tag{3.78}$$

Since the winding resistance is almost constant, the ratio of the amplification factor and the time constant is approximately constant over the operating range:

$$\frac{T_e}{\frac{\Delta\phi}{\Delta i_e}} = \frac{k}{R_e} = \text{constant} \tag{3.79}$$

For the purpose of an excitation current controller design the mean value of the excitation time constant is usually adequate. As will be shown later, the correct controller coefficients can be chosen on the basis of the relation between loop amplification and time constant (symmetrical optimum). Thus, in these cases the variation of the time constant has no influence.

3.8 MATHEMATICAL REPRESENTATION OF THE POWER CONVERTER

We will now face the problem of developing a mathematical model for a power converter. For the purpose of closed loop control this model should be linear and represent the converter as a CT transfer function. The input will be a coded digital word representing the wanted mean value of the voltage at the output terminal of the converter.

Let us first consider a DC chopper converter. The input reference to the chopper is the sequence $\{u(kT_s)\}$. The mean value of the output voltage of the chopper is a CT voltage with the mean value

$$u_d(t) = U_b \frac{t_{on}(nT_{ch})}{T_{ch}} \qquad nT_{ch} \le t < (n+1)T_{ch} \tag{3.80}$$

where U_b is the DC-supply voltage, t_{on} is the on-time of the switching element and T_{ch} is the chopper period.

The system is normally designed so that t_{on} is proportional to the last input reference value, at the start of the chopper period. Assume that the start of chopper period is synchronized to the sampling interval. Thus:

$$\frac{t_{on}(t)}{T_{ch}} = K_{ch}\frac{u(kT_s)}{u_{cmax}} \qquad for\ \ kT_s \le t < (k+1)T_s$$

For the mean value of output voltage we have:

$$\frac{u_d(t)}{U_b} = K_{ch}\frac{u(kT_s)}{u_{cmax}} \qquad for\ \ kT_s \le t < (k+1)T_s \tag{3.81}$$

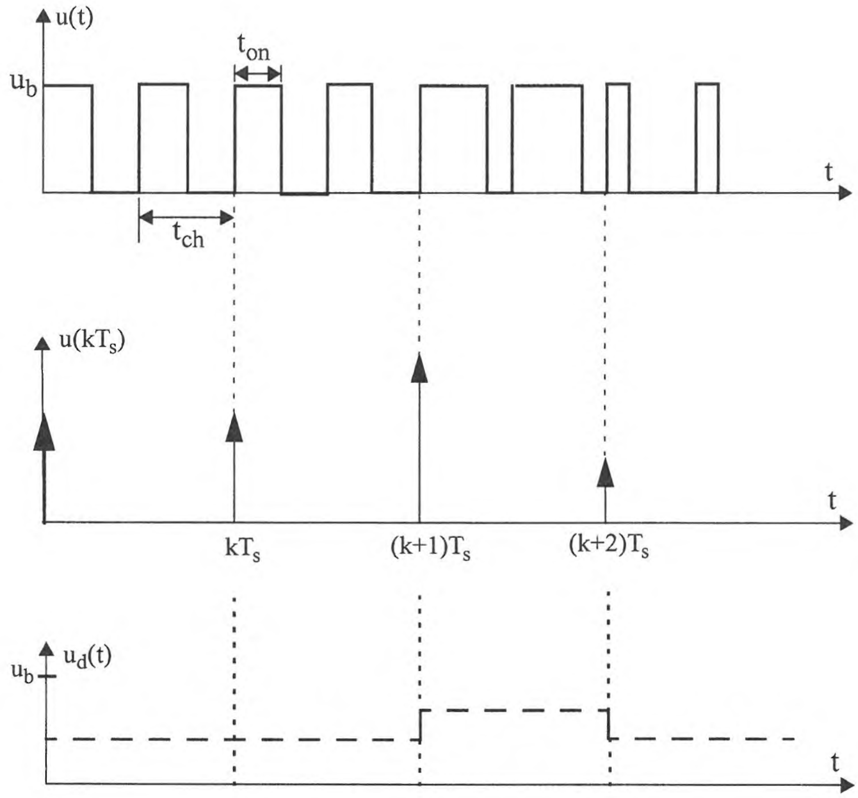


Figure 3.21 DC-chopper operation

The chopper will be latched to this value until the next sampling interval, as illustrated in Figure 3.21. We see that the PWM chopper acts as a zero order hold. As described in section 3.4 a dynamic model can then be developed as for the ZOH element.

If the sampling instant is not synchronized to the moment for switching of the chopping element, we will get a dead time with the average value equal to $\frac{T_{ch}}{2}$. The complete transfer function of the chopper is then:

$$h_{ch}(s) = K_{ch}\left[\frac{1-e^{-sT_s}}{s}\right]e^{-s\frac{T_{ch}}{2}} \tag{3.82}$$

Similar development can be carried out for a six pulse thyristor bridge. The gain of the bridge $\frac{du_d}{d\alpha}$ is not linear, but depends on the actual firing delay α. It will be shown in detail in chapter 7 how a gate firing algorithm can be designed to compensate for this non-linearity. The stationary gain, K_{br}, of the bridge can then be considered constant.

To establish the dynamic part of the transfer function the following argument will be used:

Assume that the sampling period of the input sequence is $T_s = T/6$ where T is the period of the power line voltages. The bridge will receive a new firing reference, α, every 1/6 of the power line period. Between the sampling instants the mean value of the output voltage, $u_d(t)$, will be kept constant according to the firing delay given at the last sampling instant. This is illustrated in Figure 3.22.

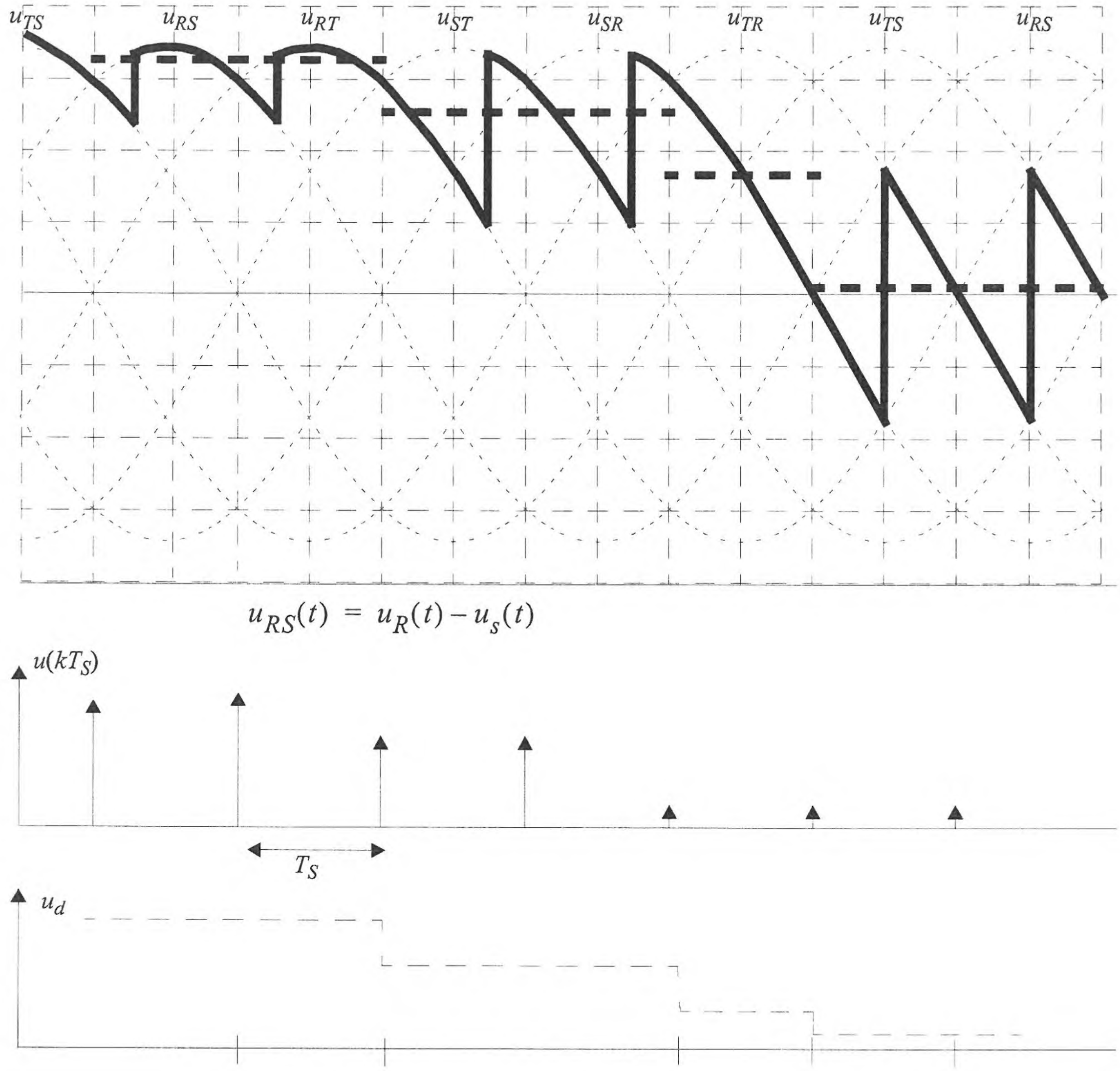


Figure 3.22 Thyristor bridge waveforms.

The thyristor bridge rectifier is inherently a sample and hold element in the control loop. It may, therefore be modelled as a ZOH element.

The thyristors are not necessarily fired at the sampling instant, but the firing is delayed according to the actual α. This can be modelled by a dead time with mean value $T_t = T/12 = \frac{T_s}{2}$ Thus the complete transfer function for the rectifier bridge is:

$$h_{br}(s) = K_{br}\left[\frac{1-e^{-sT_s}}{s}\right]e^{-s\frac{T_s}{2}} \tag{3.83}$$

3.9 PULSE TRANSFER FUNCTION FOR SYSTEMS WITH DEAD TIME

The method to determine the pulse transfer function of discrete or discretized systems so far does not apply to systems containing dead time.

As we have seen in previous sections dead times are encountered in digital power electronic control systems due to:

- computation time of the digital control algorithm
- dead time in the power electronic converter

In order to analyse system with dead time, a modification of the ZT is necessary. A continuous system containing a dead time element can be written as:

$$h_p(s) = h(s)*e^{-sT_d} \tag{3.84}$$

where h(s) is a rational function. In general, the dead time T_d can be expressed by an integer multiple of the sampling period T_s and a difference term εT_s:

$$T_d = mT_s - \varepsilon T_s = (m-\varepsilon)T_s \tag{3.85}$$

where m = 1, 2, 3.... and $0 < \varepsilon \le 1$ as illustrated in Figure 3.23.

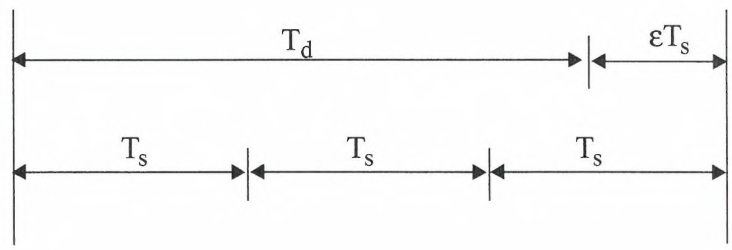


Figure 3.23 Expression of dead time

If we include the hold element, the CT transfer function is:

$$h_{hp}(s) = \frac{1-e^{-sT_s}}{s}e^{-sT_d}h(s) \tag{3.86}$$

Let the step response for the rational part be

$$v(t) = L^{-1}\left[\frac{h(s)}{s}\right]$$

Then we may write:

$$\begin{aligned} h_{hp}(t) &= L^{-1}\left[e^{-sT_d}\left(\frac{h(s)}{s}\right)-e^{-s(T_s+T_d)}\left(\frac{h(s)}{s}\right)\right] \\ &= v(t-T_d)-v(t-T_d-T_s) \end{aligned}$$

Using equation (3.85) and the shift theorem for ZT we get

$$\begin{aligned} h_{hp}(z) &= (1-z^{-1})Z[v(kT_s-mT_s+\varepsilon T_s)] \\ &= (1-z^{-1})z^{-m}Z[v(kT_s+\varepsilon T_s)] \\ &= (1-z^{-1})z^{-m}Z\left[L^{-1}\left[\frac{h(s)}{s}\right]_{t=kT_s+\varepsilon T_s}\right] \end{aligned} \tag{3.87}$$

This z-transform will be denoted by

$$h(z,\varepsilon) = Z_{\varepsilon}\left[\frac{h(s)}{s}\right] \tag{3.88}$$

This new transform now introduced for the sequence $\{v(kT_s+\varepsilon T_s)\}$ will be called the advanced ZT. In general we write

$$x(z,\varepsilon) = Z_{\varepsilon}\{x(t)\} = \sum_{k=0}^{\infty} x(kT_s+\varepsilon T_s)z^{-k} \tag{3.89}$$

Tables of this transform can be found in [22]. It is not necessary to consider this transform as a new transform since all the rules of the ordinary ZT also apply to this transform.

In most literature on sampled data systems the modified ZT is introduced and denoted as:

$$X(z, m) = Z_m\{x(t)\} = z^{-1}\sum_{k=0}^{\infty} x(kT_s + mT_s)z^{-k} \tag{3.90}$$

for $0 \le m \le 1$

Note that m has another meaning than used earlier. Here m is a nymber between 0 and 1. Tables of the modified ZT can be found in [23] and [24].

The advanced ZT can be calculated from the modified ZT by the substitution m=1+ε or equivalently, multiplication by the factor z^{-1} and use of the substitution m=ε.

For ε=0 or m=1 both transforms become the ordinary ZT.

For later reference we will calculate the pulse transfer function for the plant:

$$h_{hp}(s) = \frac{1-e^{-sT_d}}{s}K\frac{e^{-sT_d}}{1+sT} \tag{3.91}$$

where $T_d = (m-\varepsilon)T_s$ $m = 1, 2, \ldots$, and $0 < \varepsilon \leq 1$.

This CT transfer function can represent a power converter feeding a inductive load with time constant T (example, field circuit of a DC motor). For the pulse transfer function we have

$$h_{hp}(z) = (1-z^{-1})z^{-m}Z_{\varepsilon}\left[L^{-1}\left[\frac{K}{s(1+sT)}\right]\right]$$

From a table giving the advanced-ZT or from a table giving the modified-ZT we find:

$$h_{hp}(z) = \frac{z-1}{z}z^{-m}K\frac{z\left(\left(1-e^{-\varepsilon\frac{T_s}{T}}\right)z+\left(e^{-\varepsilon\frac{T_s}{T}}-e^{-\varepsilon\frac{T_s}{T}}\right)\right)}{(z-1)\left(z-e^{-\frac{T_s}{T}}\right)}$$

Thus we have:

$$h_{hp}(z) = z^{-m}K\frac{\left(1-e^{-\varepsilon\frac{T_s}{T}}\right)z+\left(e^{-\varepsilon\frac{T_s}{T}}-e^{-\varepsilon\frac{T_s}{T}}\right)}{\left(z-e^{-\frac{T_s}{T}}\right)} \tag{3.92}$$

3.10 MATHEMATICAL MODEL OF THE COMPLETE CLOSED LOOP SYSTEM INCLUDING THE DISCRETIZED CT-PLANT

We have now established the necessary theoretical tool to represent a complete model for the closed loop digital system. This model will be used for analysis and design of the digital control system.

To assemble the different system elements into the complete model the following systematic approach will be given. Again Figure 3.1 is used as a starting point.

From the physical representation, we identify the elements that represent the signal conversion between the analog and digital parts of the system. These elements have been logically described as AD- and DA-converters.

- The AD converters are represented by ideal samplers.
- The DA converters are represented by ideal samplers followed by zero-order-hold elements.
- It is assumed that the samplers are perfectly synchronized.

We then get the block diagram representation given in Figure 3.24.

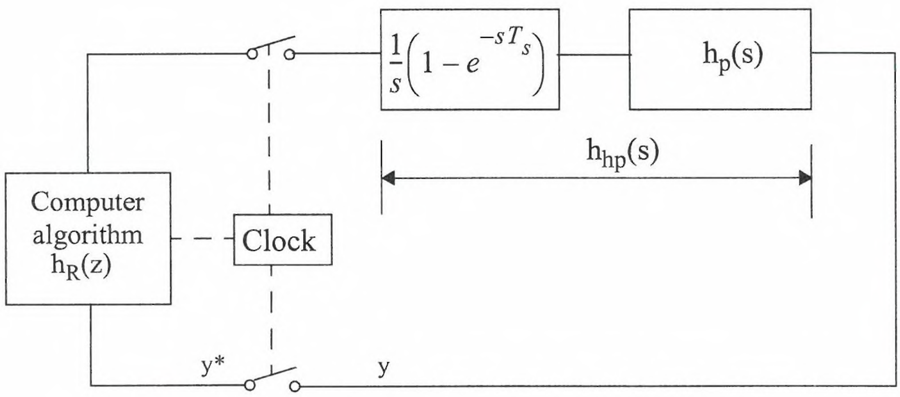


Figure 3.24 Model of the system. The computer algorithm must be described by impulse modulated signal.

- In this model the computer must be represented as a system component that transforms an impulse modulated signal to another impulse modulated signal.
- The analog parts are the zero order hold and the process.

The transfer function of the analog part is

$$h_{hp}(s) = \left(\frac{1 - e^{-sT_s}}{s}\right) h_p(s)$$

The LT y(s) of the output y(t) is

$$y(s) = h_{hp}(s)*u(s)*$$

The sampled output has the LT

$$y*(s) = [h_{hp}(s)u*(s)]* = h*_{hp}(s)\, u*(s)$$

If we represent the sampled signals as sequences we get a block diagram of the system as shown in Figure 3.25.

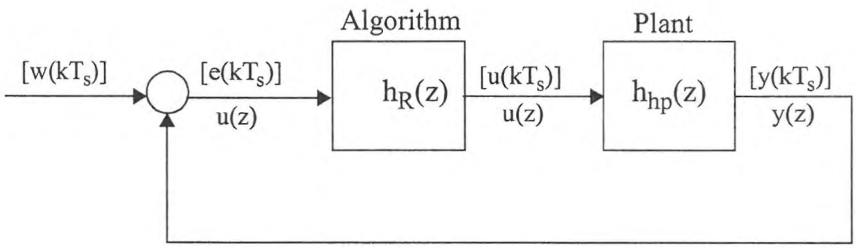


Figure 3.25 Model of the system, represented by pulse transfer functions and sequences.

- In this model the calculations in the computer are modelled by the Pulse transfer function $h_R(z)$.
- The CT part is represented by the Pulse transfer function

$$h_{hp}(z) = h_{hp}*(s)\big|_{e^{sT_s} = z}$$

This block diagram gives the properties of the system as seen by the computer.

It is an input-output representation that gives the relationship between the system variables at the sampling instants only.

Based on this representation we are able to find the transfer function from the input *w(k)*, and the output y(k), and the characteristic equation of the system, valid at the sampling instants.

At the summation point we have

$$e(z) = w(z) - y(z)$$

Substituting the equation $u(z) = h_R(z)e(z)$ for the controller, and $y(z) = h_{hp}(z)u(z)$ for the plant we get:

$$m(z) = \frac{y(z)}{w(z)} = \frac{h_R(z)h_{hp}(z)}{1 + h_R(z)h_{hp}(z)} \tag{3.93}$$

The characteristic equation of the system is defined as:

$$1 + h_0(z) = 0 \tag{3.94}$$

where $h_0(z)$ is the open loop transfer function. In this case we have

$$h_0(z) = h_R(z)h_{hp}(z) \tag{3.95}$$

When the system contains cascaded elements, care must be taken when deriving DT transfer function for the complete system. It is not always possible to write a transfer function between variables [25].

The following rules may be helpful:

- The ZT of two CT elements separated by a sampler is the product of the two Z-transforms.

$$Z[h_1(s)h_2(s)] = Z[h_1h_2(s)] \tag{3.96}$$

- In general no transfer function can be found for a system if the input is applied to a CT element before being sampled.

$$Z(h_1h_2) \neq (h_1(z)*h_2(z)) \tag{3.97}$$

In complex situations with feedback and many samplers the algebraic manipulations can be tedious. In such situations the signal flow graph method [23] may be used.

3.11 TRANSIENT BEHAVIOUR OF THE CLOSED LOOP SYSTEM

The roots of the characteristic equation (3.94) are the poles of the closed loop transfer function.The location of the roots of the characteristic equation in the z-plane determine the dynamic characteristic of the closed loop system. Thus the controller $h_R(z)$ must be designed so that the roots receive preferable locations.

For the output of the system we may write

$$y(z) = \frac{h_0(z)w(z)}{1+h_0(z)} = \frac{\prod^{m}(z-z_i)}{\prod^{n}(z-p_i)}\omega(z) \tag{3.98}$$

If the input sequence w_k is the unit impulse at k=0 the output y(z) can be expressed by using a partial fraction expansion:

$$y(z) = \frac{k_1 z}{z-p_1} + \ldots \frac{k_n z}{z-p_n}$$

The system is stable if the output sequence {y(k)} approaches zero as time increases. By taking the IZT of y(z) we find:

$$\{y(k)\} = Z^{-1}\left(\sum_{i=1}^{n}\frac{k_i z}{z-p_i}\right) = \sum_{i=1}^{n} k_i(p_i)^k \tag{3.99}$$

for $|p_i| < 1$ we see that y(k) converges to zero.

Thus, the system is stable if all the roots of the characteristic equation

$$1 + h_0(z) = 0$$

lie inside the unit circle in the z-plane. The output sequence {y(k)} will then be bounded if the input is a bounded signal. We see that the unit circle in the complex z-plane is the stability boundary, similar to the imaginary axis of the s-plane for CT systems.

Valuable insight into the dynamic behaviour of a discrete or discretized system can be gained by studying the locations of the system poles in the z-plane. As can be seen from equation (3.99) each pole in the transfer function contribute separate dynamic modes to the resulting response sequence at the sampling instants.

If we have a real pole $p_i < -1$ the associated time sequence will oscillate and increase in an oscillatory fashion. If $0 < p_i < 1$ it will decay in an exponential manner as k become large. If $p_i > 1$, the associated sequence will grow exponential without bound.

If the characteristic equation has complex roots, p and p* similar results can be derived. In both cases the decay for $|p| < 1$ will be faster if the poles lies close to zero.

A summary of various pole locations in the z-plane and the type of response sequences they represent is given in Figure 3.26.

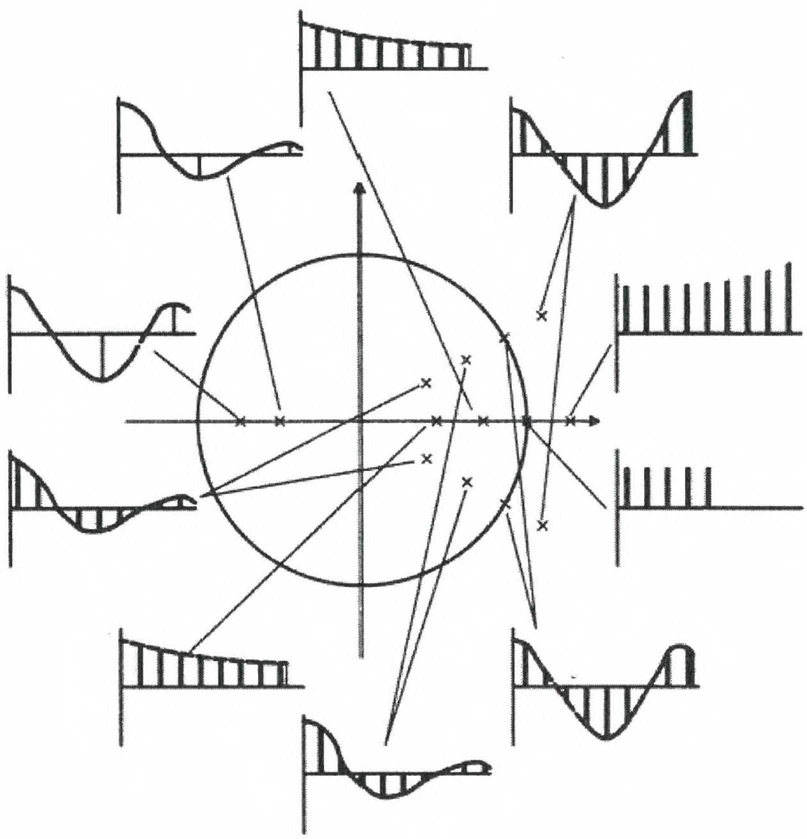


Figure 3.26 Pole locations and associated response.

For subsequent reference and to illustrate the methods presented we will investigate the digital control system shown in Figure 3.27. The block diagram shows a microprocessor controlled power converter driving a passive inductive load with time constant T. Assume a current controller with proportional control law and negligible dead time of the converter.

The pulse transfer function of the power converter, including the load, is given according to equation (3.76) and (3.82).

$$h_{hp}(z) = (1 - z^{-1})Z\left\{L^{-1}\left[\frac{K}{s(1 + sT)}\right]\right\} \tag{3.100}$$

From a ZT table we get:

$$h_{hp}(z) = K\frac{z^{-1}\left(1 - e^{-\frac{T_s}{T}}\right)}{1 - e^{-\frac{T_s}{T}}z^{-1}} \tag{3.101}$$

This pulse transfer function could also been obtained by letting m=1 and ε=1 in equation (4.49).

Choosing $T_s/T=1/3$ we get:

$$h_{hp}(z) = K\frac{0{,}2835z^{-1}}{1 - 0{,}7165z^{-1}}$$

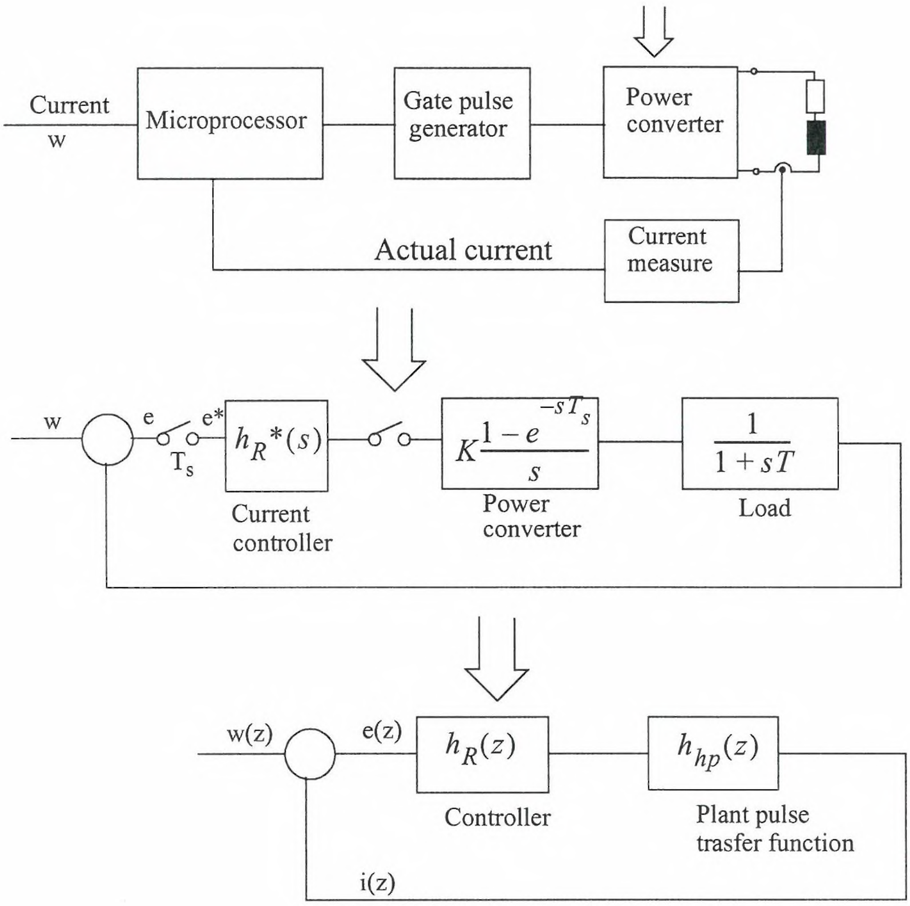


Figure 3.27 Block diagram of a microprocessor based current control system

The characteristic equation of the system is then:

$$1 + h_0(z) = 1 + h_R(z)h_{hp}(z) = 0$$

$$1 + \frac{K_pK(0{,}2835)z^{-1}}{1 - 0{,}7165*z^{-1}} = 0$$

The root of this equation is

$$p = 0{,}7165 - 0{,}2835K_0$$

where K_0 = KpK is the total loop gain.

As K_0 increases from zero, the system pole p moves to left from the location z=0.7165 for K_0=0 and eventually leaves the unit circle as illustrated in Figure 3.28. The value of the gain K_{crit} for which the system become unstable is of interest. The system is marginal stable for p=-1. We have K_{ocrit} = 6.05.

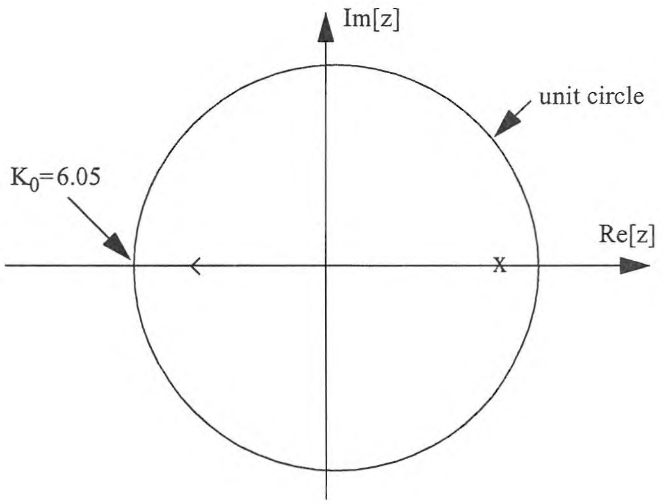


Figure 3.28 Root locus of the system

Important properties of a digital control system can be seen from this simple example:

- Note that the pole at s= - 1/T of the CT plant is transformed to a pole $z = e^{-\frac{T_s}{T}}$ in the discretized plant transfer function. The relation $z_p = e^{s_p T_s}$ between s-domain poles and poles of the pulse transfer function in the z-plane is generally valid. This sort of relationship does not necessarily hold for the zeros.
- The z domain poles location are in addition to the original s-domain location also dependent of the sampling interval T_s poles close to z=1 correspond to fast sampling or a short time constant for the plant.
- Another interesting observation is that though this system, can be unstable, the same plant under action of a CT control is always stable. The reason for this is the basic nature of a digital control sys-

tem. The digital controller receives only samples of the error signal at discrete time instances. Thus the control action must be performed based on a limited amount of information while for a CT control system an infinite amount of information about the error signal is available. As the DT system runs open loop between sampling instants, when the controller does take action it overcompensates by generating too high gain.

- Higher gain in the loop can be tolerated by increasing the sampling frequency before the system becomes unstable.

The relation given in equation (3.71) is a transformation between the s-plane pole locations and between the z-plane locations. To gain further insight into the characteristics of pole locations in the z-plane, several mappings will be considered.

The complex variable s may be written as s=α+jω. Hence, according to (3.71) we have:

$$z = e^{(\alpha + j\omega)T_s} = e^{\alpha T_s} e^{j\omega T_s} = e^{\alpha T_s} e^{(j\omega T_s + n2\pi)} \tag{3.102}$$

From equation (3.64) we see that the poles on the s-plane, whose frequencies differ in integral multiple of the sampling frequency, are transformed to the same position on the z-plane. So studying the so-called primary strip in the s-plane (n=0) is sufficient.

The mapping of the left half plane portion of the primary strip is mapped into the interior of the unit circle (since α<0, magnitude is less than 1). The imaginary axis in the s-plane (α=0) corresponds to |z|=1.

The right half portion maps into the exterior of the unit circle. (See Figure 3.29). This result is in agreement with the stability boundaries of the two planes. Figure 3.29 also shows how poles with constant damping, constant frequency, and constant relative damping factor are mapped into the z-plane.

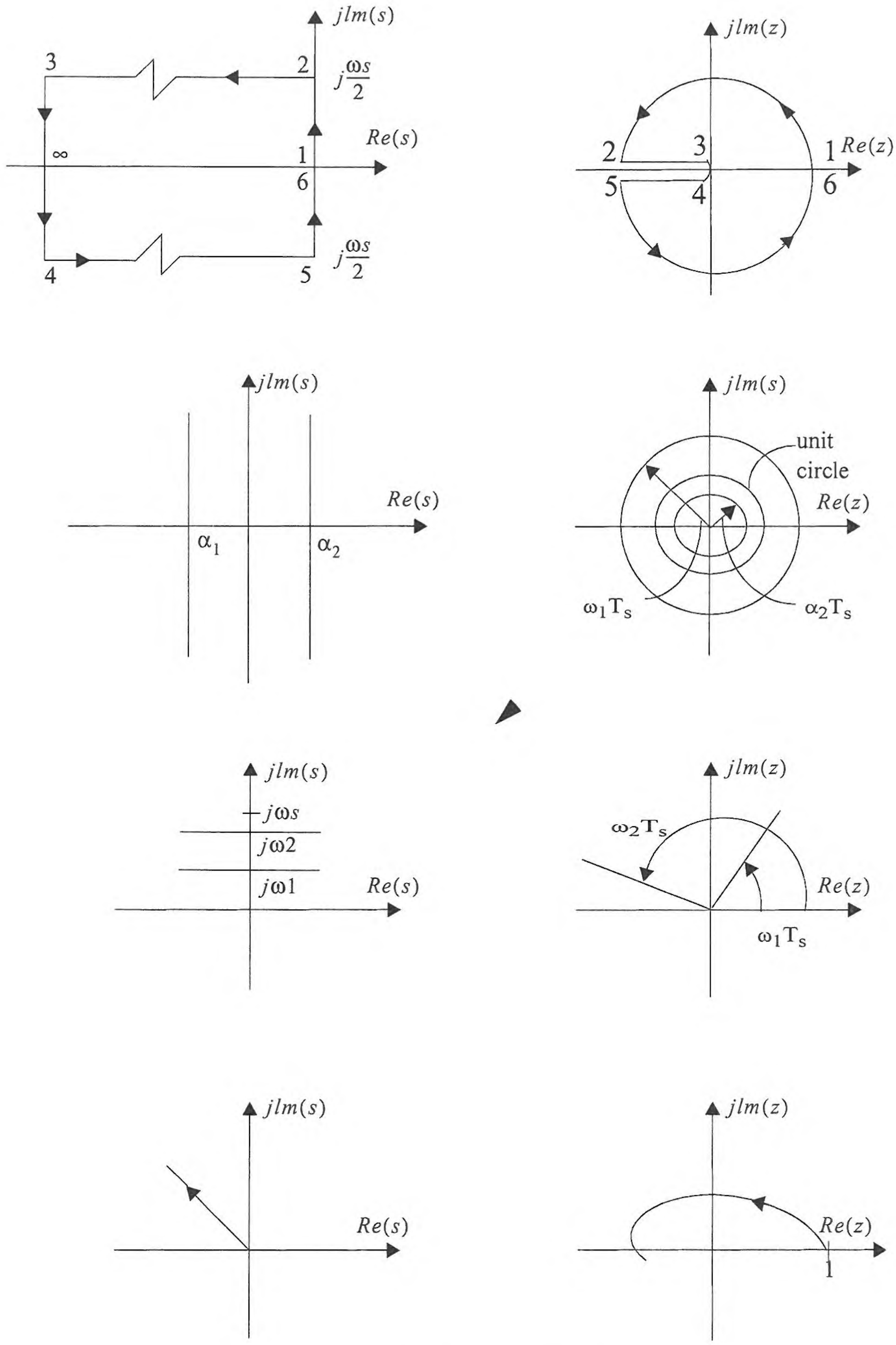


Figure 3.29 Mapping of s-plane into the z-plane

Complex poles on the s-plane can be written in the standard form:

$$s_{1,2} = \alpha_{1,2} + j\omega_{1,2} = -\xi\omega_0 \pm j\omega_0\sqrt{1-\xi^2} \tag{3.103}$$

The constant relative damping factor (ξ = constant) path on the z-plane is a family of logarithmic spirals, except for $\xi = 0$ and $\xi = 1$. This can be seen as follows: From equation (3.103), if we let ω_d be defined as $\omega_d = \omega_0\sqrt{1-\xi^2}$, then in the z-plane, the line with constant ξ becomes:

$$\begin{aligned} z &= e^{sT_s} = \exp(-\xi\omega_0 T_s + j\omega_d T_s) \\ &= \exp\left(-\frac{2\pi\xi}{\sqrt{1-\xi^2}}\frac{\omega_d}{\omega_s} + j2\pi\frac{\omega_d}{\omega_s}\right) \end{aligned}$$

Hence,

$$|z| = \exp\left(-\frac{2\pi\xi}{\sqrt{1-\xi^2}}\frac{\omega_d}{\omega_s}\right)$$

$$\angle z = 2\pi\left(\frac{\omega_d}{\omega_s}\right)$$

This implies that magnitude of z decreases and the angle of z increases as ω_d increases. The locus becomes a logarithmic spiral in the z-plane.The path for $\xi = 0.5$ and $\omega_d < \frac{\omega_s}{2}$ is shown in Figure 3.26 where ξ is the relative damping factor equal to $\xi = -\frac{\alpha}{\omega_0} = \sin\varphi$ and ω_0 is the undamped resonance frequency given by

$$\omega_0 = \sqrt{\omega_d^2 + \alpha^2} = \frac{\omega_d}{\cos\varphi}$$

3.12 STEADY STATE BEHAVIOUR OF THE CLOSED LOOP SYSTEM

The steady state performance of the digital control system can be determined through use of the final value theorem of the ZT.

The steady state error at the sampling instants is defined as:

$$e_{ss} = \lim_{k \to \infty} e(kT) = \lim_{z \to 1} (1 - z^{-1}) e(z) \tag{3.104}$$

where we assumed that $(1 - z^{-1})\, e(z)$ does not have any pole on/or outside the unit circle. For the system given in Figure 3.25 the ZT of the error signal is:

$$e(z) = \frac{w(z)}{1 + h_R(z) h_{hp}(z)} = \frac{w(z)}{1 + h_0(z)} \tag{3.105}$$

Thus we have

$$e_{ss} = \lim_{k \to \infty} e(kT) = \lim_{z \to 1} (1 - z^{-1}) \frac{w(z)}{1 + h_0(z)} \tag{3.106}$$

This equation shows that the steady state error depends on the reference input w(z), as well as the open loop transfer function $h_0(z)$. In the following we will calculate the steady state error for three basic type of input signals: step sequence, ramp sequence and parabolic sequence. The results are presented in table 4.1.

The limit as $z \to 1$ of the open loop transfer function $h_0(z)$ can always be expressed as:

$$\lim_{z \to 1} h_0(z) = \lim_{z \to 1} \frac{a_0 \prod^{m} (z - z_i)}{(z - 1) \prod^{p} (z - p_i)} = \lim_{z \to 1} \frac{K_{dc}}{(z - 1)^N} \qquad p_i \neq 1$$

where $K_{dc} = \left. \dfrac{a_0 \prod^{m} (z - z_i)}{\prod^{p} (z - p_i)} \right|_{z=1}$ is the open loop dc gain when all poles of z = 1 are removed

Table 3.4 Steady state error of a plant with N poles of $h_0(z)$ at z = 1

Input sequence {w(kT)}	w(z)	Steady state error $e_{ss} = \lim_{k \to \infty} e(kT)$	Number of poles at z = 1 in order to get $e_{ss} = 0$
unit step	$\dfrac{z}{z-1}$	$\dfrac{1}{1 + \lim_{z \to 1} h_0(z)} = \left(\dfrac{1}{1 + K_{dc}}\right)(z-1)^N \Big\|_{z=1}$	$N \geq 1$
ramp $\{kT_s\}$	$\dfrac{T_s z}{(z-1)^2}$	$\dfrac{1}{1 + \lim_{z \to 1} (z-1) h_0(z)} = \left(\dfrac{T_s}{K_{dc}}\right)(z-1)^{N-1} \Big\|_{z=1}$	$N \geq 2$
parabolic $\left\{\dfrac{(kT_s)^2}{2}\right\}$	$\dfrac{T_s^2 z(z+1)}{2z(z-1)^3}$	$\dfrac{1}{\frac{1}{T_s^2} \lim_{z \to 1} (z-1)^2 h_0(z)} = \left(\dfrac{T_s^2}{K_{dc}}\right)(z-1)^{N-2} \Big\|_{z=1}$	$N \geq 3$

The above calculations illustrate that, in general, increased system gain and/or addition of poles at z = 1 in the open loop transfer function $h_0(z)$ decreases the steady state error. It will also be noted that the designer must find a compromise between small steady state error and system stability because in general, both large system gain and poles of $h_0(z)$ at z = 1 have destabilizing effects on the system.

4 PRACTICAL METHODS FOR DIGITAL CONTROLLER DESIGN (12.11.01)

In chapter 3 several aspects of modelling the system elements and control system analysis have been discussed. We will now put this knowledge together and discuss several design procedures.

We will focus on methods for design of controllers to be implemented in standard microprocessors, where single input single output systems are considered.

Based on the specifications given for control of power electronic systems it is found that design methods based on a bilinear transformation to a fictive frequency plane (q-plane) is adequate and easy to use. This method allows most of the experience from CT-system design to be carried over to DT-systems.

In the concluding sections explicit formulas for selecting optimal controller parameters are derived.

4.1 GENERAL REMARKS ON DIFFERENT DESIGN APPROACHES

There is no general design procedure for all power electronic control system. The designer may tailor a procedure for his or her particular problem depending on the given requirement specification and physical constraints, such as sampling time, actuator limitation, etc.

The following steps must be performed for all procedures:

- Establish a *mathematical model of the plan*t. This is done either through identification methods or from physical balance equations.
- Select appropriate *sensor and actuator* elements.
- Based on the preceding steps and the requirement specifications select the *controller structure.*
- Select a *design method to determine the parameters* of the controller so that the requirements on the control system behaviour are satisfied.

The first two steps of the procedure were covered in the previous chapters. Before we proceed with the others, we will put a few comments on the first step:

The established model of the system must describe the important physical dynamic behaviour, particularly for the low frequency behaviour inside the design bandwidth. If only the input/output behaviour is of interest, the CT model should be simplified as far as possible before it is converted to a DT model. Some helpful rules are found in [18] and [ ].

For a linear CT plant models with transfer functions of the form

$$h_p(s) = \frac{y(s)}{u(s)} = \frac{\prod_{\beta=1}^{m}(1+T_\beta s)}{(1+2\xi\omega_0 s+\omega_0^2 s)\prod_{\alpha=1}^{m-2}(1+T_\alpha s)} \tag{4.1}$$

the dynamic behaviour is approximately the same in both open and closed loop if the generalized sum of the time constants

$$T_\Sigma = 2\xi\omega_0 + \sum_{\alpha=1}^{m-2} T_\alpha - \prod_{\beta=1}^{m} T_\beta \tag{4.2}$$

remains constant.

A small time constants T_i should be replaced to be a dead time before discretizing:

$$T_\alpha = \sum_{i=1}^{l} T_i \tag{4.3}$$

Poles and zeros which are approximately equal should be cancelled, considering equation (4.2).

After calculation of the plant pole and zeros we can decide upon the activities necessary to improve the dynamic behaviour in order to satisfy the given specifications.

The design of the digital compensator is the activity of choosing the difference equation (3.4) or equivalently the z domain transfer function (3.53). When combined with the dynamics of the CT plant this controller must give acceptable performance of the complete closed loop system.

In the design of the digital compensator, one of two main approaches are generally taken:

The first method is to ignore any zero order holds and samplers in the control loop and do the design as if you were building a CT control system. The control structure or CT controller transfer function must then be converted to a DT compensator by some approximate technique originally developed for simulation purposes. Some of these methods called Numerical integration, Pole-zero mapping and Hold equivalence are presented in [27].

The second method is to design the compensator directly based on the discretized CT plant model as described in the previous chapter.

The first method, which is commonly applied in the industry, has certain disadvantages since it is an approximate method: By conversion of the CT-controller to a discrete equivalent, be whatever method, the z-plane poles are distorted from where they are needed. Further, we have seen in section 3.11 that a CT design method will not always ensure stable system, when the control action is discretized.

The most important drawback of these methods is that high sampling rate is necessary in order for the approximations to be valid. This is a serious limitation especially when dealing with power electronic systems controlled by standard microprocessors. In a power electronic system the time requirements of the plant impose heavy load on the microprocessor concerning execution speed.

Thus there is a need for a direct digital design procedure that is exact and accounts for the effects of data holds, computation delays, etc. The methods of the second type will fulfil these needs. In the following sections we will discuss some of these methods. These methods can also be applied when the sampling rate is slow compared to the system time constant.

4.2 COMMONLY GIVEN REQUIREMENT SPECIFICATIONS FOR POWER ELECTRONIC CONTROL SYSTEM

The most basic requirement of control system is the stability of the closed loop. In section 3.11 it was indicated that a necessary and sufficient condition for asymptotic stability of a discrete rational system was that all zeros of the characteristic equation of the system

$$1 + h_0(z) = 0$$

lie inside the unit circle in the z-plane.

Several stability criteria for testing this property exists. A stability criterion for discrete systems that is similar to the Routh criterion [26] and which can be applied to the characteristic equation directly is the

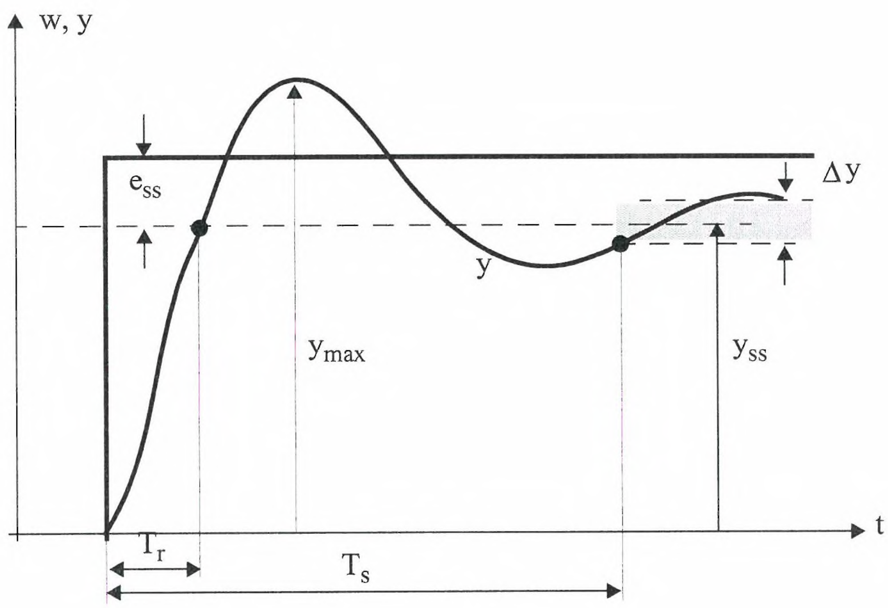
Jury test [16]. Other arithmetic stability tests are available, such as the Schur-Cohn criterion [16].

It is important to note that the Routh criterion, commonly used for CT systems, cannot be directly used to the characteristic equation expressed in the z-plane, since the stability boundary is the unit circle and not the imaginary axis. However, if the characteristic equation is expressed as a function of q, by the bilinear transformation described later, the Routh criterion can be applied.

Geometric stability investigations can also be performed. With the help of the discrete frequency response plot $h(e^{j\omega T_s})$ developed from equation (3.66), one can apply the Nyquist stability criterion and the absolute stability criterion of Tsypkin and the Circle Criterion [25].

We will return to the problem of determining stability in connection with the actual design methods described.

The quality specifications of a power electronic control system performance is usually given by its time domain behaviour. For many system the response to a reference step input is most often used. See Figur 4.1.



Figur 4.1 A typical response

The response should neither overshoot too much nor approach the final value too slowly.

The steady state accuracy e_{ss} is primarily a matter of the controller structure as showed in section 3.12.

The overshoot $y(t)_{max}$ depends primarily on the damping of the dominant poles. A second order system has a very small overshoot for $\xi = 1/\sqrt{2} \approx 0.7$. The same holds for discrete systems.

The settling time t_s, i.e. the time required for the response to settle to an error of one percent from the final value, primarily depends on the minimum negative real part of all poles.

The time scale of the step response can either be specified in the form of the rise time t_r (from 10% to 90% of the final value) or by the resonance frequency ω_0.

For a second order system, the relationship between pole locations in the s-plane and the specified values of the step response can be found.

The relation between ξ and $y(t)_{max}$ is given in [26]. In [27] the following approximate relationship is given:

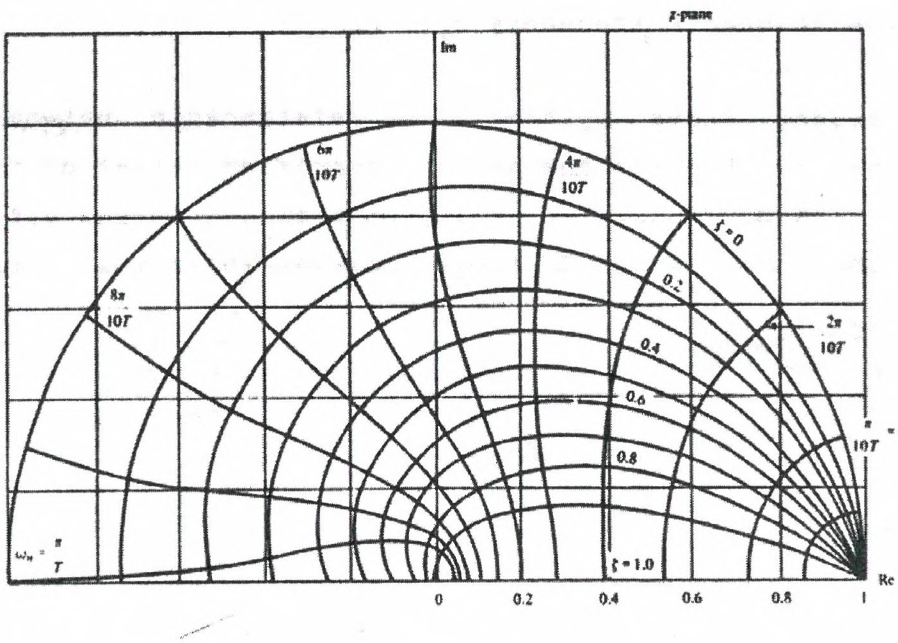
$$\begin{aligned} &\xi \approx 0{,}6\left[1 - \frac{\%\ overshoot}{100}\right] \qquad for\ y_{max} > 1{,}1 \\ &\omega_0 \approx 2{,}5/t_r \\ &\xi\omega_0 \equiv 4{,}6/t_s \end{aligned} \tag{4.4}$$

other possible formulations of the design requirements are stability margins in the frequency domain. The DT version of the Bode plot, which will be presented later, allows determination of gain and phase margins.

4.3 DESIGN METHODS BASED ON DISCRETIZED CT PLANT METHODS

The specifications given in the previous section can be carried over to design methods in the z-plane. Guidelines on the placement of poles in the z-plane can be found by the pole mapping via $z = e^{sT_s}$ as given in section 3.11. As we have seen in Figure 3.30 curves of pole locations for constant ξ are logarithmic spirals in the z-plane. Curves of constant ω_0 in the z-plane are lines draw at right angles to the constant ξ spirals. The real part of the roots of the characteristic equation, $-\xi\omega_0$, maps into a circle with radius $r = e^{-\xi\omega_0 T_s}$

The z-plane loci of poles of constant ξ and ω_0 are shown in Figur 4.2



Figur 4.2 *z-plane loci of poles of constant ξ and ω_0. Loci of constant $\xi\omega_0$ are circles about origin with radius $r = e^{-\xi\omega_0 T_s}$.*

Thus, using equations (4.4), the specified restrictions on percent overshoot, rise time and settling time for a given sampling time, T_s, can be translated into an area in the z-plane in which the poles must be located.

For higher order systems, ξ and ω_0 are considered as minimum values for all poles. Equations (4.4) only provides an initial guess for the admissible pole locations. After the first design step, the values of y_{max}, t_r and t_s are determined by simulations with the design model. If the response is not satisfactory the tendencies described in section 4.4 are then used to improve the closed loop pole locations in the following design steps.

4.3.1 DESIGN ON THE Z-PLANE BY THE ROOT LOCUS METHOD

In the previous section the favourable positions of the dominant poles of the system were established. A practical design method must also give guidelines on how to select the compensator parameters in order to shift the roots of the characteristic equation to the desired locations.

The root locus method is a well established design procedure for CT-systems. Extensive description of the method used for CT design can be found in [28]. It traces graphically the roots of the characteristic equation as some real parameter varies from zero to a large value. The characteristic equation

$$1 + h_R(z)\, h_{hp}(z) = 0$$

has exactly the same form as that found for the s-plane root locus. This implicates that the procedure of drawing the root loci in the z-plane is the same as in s-plane.

The difference between CT and DT design by root locus is, as we have stated, by the interpretation of the system stability and dynamic response. A simple example on use of the root locus method has already been given in section 3.11.

4.3.2 FREQUENCY RESPONSE TECHNIQUES - THE BILINEAR TRANSFORMATION

In the analysis of discrete linear systems we have seen that discrete and discretized transfer functions $h_0(z)$ are rational polynomial of the variable z. We have also seen that the left half of the s-plane maps into the interior of a unit circle in the z-plane by the sampling process. At high sampling rates, the z-plane poles and zeros tends to cluster on the unit circle creating numerical problems during analysis and design in the z-plane. Since the stability domain is the unit circle rather than the imaginary axis, direct application of Routh's stability criteria is not possible.

By the substitution $z = e^{j\omega T_s}$ we get the discrete frequency response $h_0(e^{j\omega T_s})$. Unfortunately these discrete frequency transfer functions are not rational functions. As the result, the simplicity of Bode design techniques is lost in the discrete domain.

To cure this problem, Balchen [26] has suggested a bilinear transformation to a different plane called q-plane where the simplicity of the Bode design technique is regained. In the q-plane, the region of stability is once again the left half plane, so the Routh's stability criteria can be directly applied.

The q-transform technique is extensively documented in [26] so only the basic relationship and properties will be given here for later references.

The q-transform is defined by the equations:

$$q = u + jv = \frac{2}{T_s}\frac{z-1}{z+1} \qquad ; \qquad z = \frac{1+\frac{T_s}{2}q}{1-\frac{T_s}{2}q} \tag{4.5}$$

The transformation between q-plane and s-plane is given by:

$$q = \frac{2}{T_s}\frac{e^{sT_s}-1}{e^{sT_s}+1} = \frac{2}{T_s}\frac{sT_s + \frac{1}{2}(sT_s)^2 + \dots}{2 + sT_s + \frac{1}{2}(sT_s{}^2) + \dots} \tag{4.6}$$

From (4.6) we find

$$\lim_{T_s \to 0} q = s \tag{4.7}$$

In order to use the Bode-Nyquist methods we must find the relationship between the angular frequency, ω and the imaginary part of q:

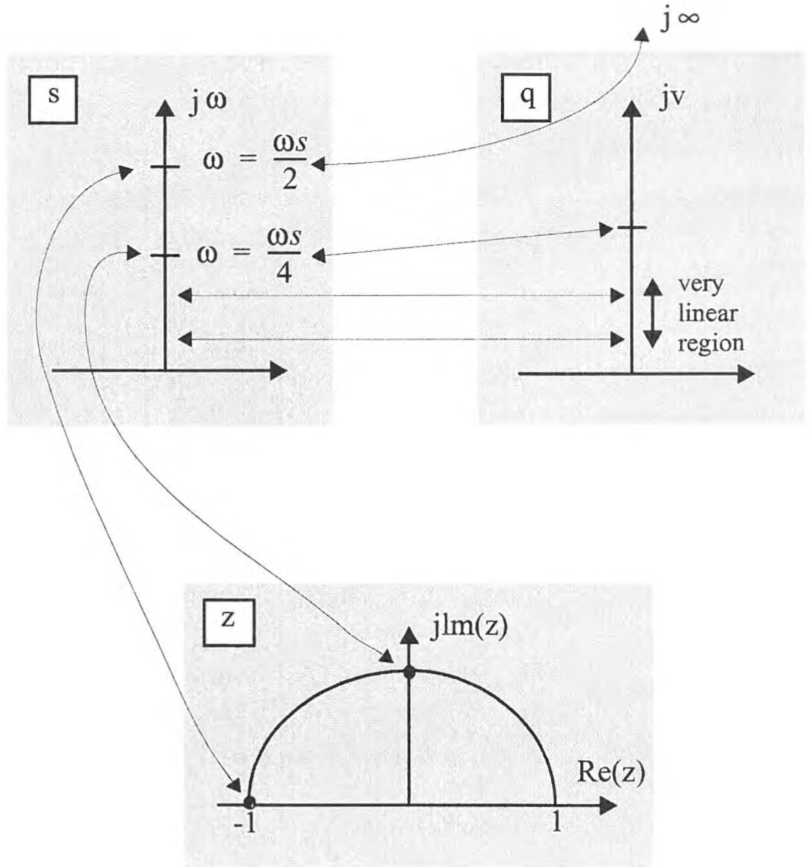
$$\begin{aligned} jv &= \frac{2}{T_s}\frac{z-1}{z+1}\bigg|_{z = e^{j\omega T_s}} = \frac{2}{T_s}\frac{e^{j\omega T_s}-1}{e^{j\omega T_s}+1} \\ &= \frac{2}{T_s}\frac{e^{j(1/2)\omega T_s} - e^{-j(1/2)\omega T_s}}{e^{j(1/2)\omega T_s} + e^{-j(1/2)\omega T_s}} \\ &= j\frac{2}{T_s}\tan\left(\frac{\omega T_s}{2}\right) = j\frac{\omega_s}{\pi}\tan\left(\pi\frac{\omega}{\omega_s}\right) \end{aligned} \tag{4.8}$$

where $\omega_s = \frac{2\pi}{T_s}$ is the angular sampling frequency.

For $|\omega| < \frac{\omega_s}{4} = \frac{\pi}{2T_s}$ we may set $v = \omega$.

For $\omega \to \frac{\omega_s}{2}$ we have $v \to \infty$.

The mapping between s, z, and q-plane are illustrated in Figur 4.3.



Figur 4.3 s, z, and q-planes showing correspondence in frequency

The frequency response technique based on Bode-Nyquist methods indicates the relative stability by gain margin and phase margin. In addition these methods suggest the designs necessary to active acceptable transient and steady state response.

The frequency resonance peak value of the closed loop transfer function in the q-plane is related, in an approximate sense, to the peak overshoot $y(t)_{max}$ in the step response. Thus, the stability margins are also an approximate, since they are related to the peak overshoot. For a second order system, the exact relationship can be found.

The rise time of the step response is related to the bandwidth of the closed loop transfer function. For a system the product $\omega_b * t_r$ is approximately constant. Thus, in order to decrease the rise time, it is necessary to increase the systems bandwidth. Once the bandwidth v_c or crossover frequency is found from the q-plane analysis, the true bandwidth in the real frequency domain is determined using equation (4.8)

$$\omega_c = \frac{2}{T_s} \arctan\left(\frac{v_c T_s}{2}\right) \tag{4.9}$$

We will use the Bode diagram techniques in the q-plane to design the controllers in the case studies. The following steps are performed:

1. The CT-transfer function of the plant $h_p(s)$ is simplified according to the rules (4.1) through (4.3).
2. The open loop pulse transfer function $h_{hp}(z)$ is derived (equation (3.76) or equation (3.87)).
3. The original open-loop pulse transfer function $h_{hp}(z)$ is transformed to $h_{hp}(q)$ through the bilinear transformation defined in (4.5).
4. By the Bode plot for $h_{hp}(q)$ with $q=jv$ the required transfer function for the controller $h_R(q)$ is determined. The open loop transfer function of the compensated system is then given by $h_0(q) = h_R(q)$ $h_{hp}(q)$. The stability margins and the system bandwidth are determined.
5. The transfer function $h_R(q)$ is transformed to $h_R(z)$.
6. The time domain difference equation corresponding to $h_R(z)$ is realized in the controller program algorithm.

Step 4 is supplemented with computer simulations taken into account possible non-linearities such as actuator saturation etc. If the requirement specifications are not satisfied, the procedure is repeated starting with step 4. The modification of $h_R(q)$ is guided by the previous given relationship between the step response and stability margins and the bandwidth.

In section 3.2 the difference equation representing the trapezoid integration operation was derived. By taking the z-transform of equation (3.9) we get:

$$h_{trapez}(z) = \frac{i(z)}{e(z)} = \frac{T_s}{2}\frac{z+1}{z-1}$$

by the bilinear transformation (4.5) we find that the q-plane representation of the integral is:

$$h_{trapez}(q) = \frac{1}{q} \tag{4.10}$$

Thus, q^{-1} exactly represents the trapezoid integral operation.

The time domain of the difference equation representing a PI control law can be written as:

$$u(k) = K_p e(k) + [u(k-1) - K_p e(k-1)] + \frac{K_p}{T_i}\frac{T_s}{2}[e(k) + e(k-1)] \tag{4.11}$$

Taking the ZT we get:

$$u(z)(1 - z^{-1}) = K_p\left[(1 - z^{-1}) + \frac{T_s}{T_i 2}(1 + z^{-1})\right]e(z)$$

$$h_R(z) = \frac{u(z)}{e(z)} = K_p + \frac{K_p}{T_i}\frac{T_s}{2}\frac{(1 + z^{-1})}{(1 - z^{-1})} \tag{4.12}$$

$$h_R(z) = \frac{K_p\left(1 + \frac{T_s}{2T_i}\right) - K_p\left(1 - \frac{T_s}{2T_i}\right)z^{-1}}{(1 - z^{-1})} = \frac{g_0 + g_1 z^{-1}}{1 + f_1 z^{-1}} \tag{4.13}$$

Taking the q-transform of (4.12) we get

$$h_R(q) = K_p + K_p\frac{1}{T_i}q^{-1} = K_p\left(\frac{1 + T_i q}{T_i q}\right) \tag{4.14}$$

which shows that the PI-control law in the q-plane have a form similar to that in the s-plane (equation (3.6)) when the integral is taken by the trapezoid operation.

4.4 CASCADED CONTROL LOOPS WITH MULTIRATE SAMPLING

By the design procedures presented in the previous sections we have assumed a single control loop. In a power electronic system we most often have more control loops usually connected in a cascaded structure. If all loops are designed with the same sampling frequency, application of the described techniques makes no problem.

As we have discussed in chapter 2, the outer loop, usually controlling a mechanical quantity, can tolerate lower sampling frequency due to the higher time constant. In order to reduce the load factor of the microprocessor we will therefore prefer to have slower sampling at the outer loop. The resulting system will then be a multirate sampled system.

No practical and complete methodology has been found in the literature which can be used for design of such systems.

Multirate systems are in reality time variant systems and as such they will be quite complex to analyse. To analyse some specific multirate systems the modified ZT can be employed [16]. The method is based on replacing samplers operating at various rates with equivalent samplers operating with a rate equal to the highest sampling rate found in the system. Although these studies are analytically detailed, a combined theory that can be used for design purposes has not been developed.

Currently the main technique employed for evaluating multirate digital control system is simulation.

However, by taking the following assumptions, the q-plane method described in the previous section can be used.

- The inner loops have higher bandwidths and sampling frequency than the outer loop.
- The inner loop as seen from the outer one is considered as a CT-system.

Simulation studies and practical experiments have shown that this approach give acceptable results from an engineering point of view.

By taking the described approach we have outlined a simple and uniform method for design of digital power electronic control systems, allowing most of the experience from CT-system design to be carried over to design in the DT domain.

Taking a further step in this direction, in the next section we will derive analytical equations for selection of optimal parameters for the discrete controller, assuming that the plant model have a given transfer function.

The method for obtaining these equations and the equations themselves have their counterpart in the CT-system design.

4.5 SELECTION OF OPTIMAL CONTROLLER PARAMETERS

A method, commonly used by practising control engineers, to select the controller parameters for a CT-controller of PID type and its derivatives is based on the modulus optimum or the symmetrical optimum criteria. The method and the results for CT-systems are presented in [29]. A more thoroughly discussion can be found in [20] and [21]. Both methods are based on the requirement that the modulus of the closed loop transfer function $m(s) = \frac{y(s)}{w(s)}$ must be very close to 1 over a wide frequency range from zero upwards.

In this section, by taking the same basic requirement as a starting point, it will be shown that similar equations can be derived for a digital controller. The formulas thus obtained will, in the limiting case, when the sampling period, T_s, approach zero be equal to the formulas for the CT-controller.

The derivation will be carried out according to the steps given in section 4.3.2. Comparison with the result from the case of CT-controller design will be done with reference to [29].

We will do the calculations for a first order plant and a second order plant with an integrating part. In both cases a dead time is also associated to the plant transfer function. The reason is, as we have seen in section 3.7, that the power electronic converter have an associated dead time part. In addition the time delay due to the calculations in the microprocessor can be taken into account by an equivalent dead time to the plant.

If the plant transfer function have a higher degree than two, the method can also be used, assuming that the smaller time constant of the plant can be compensated by zeros of the higher order controller transfer function. In this case we end up with the basic open loop transfer functions discussed in this section.

Thus, it is assumed that these basic transfer functions in an approximate sense, will be representative for most plants encountered in a power electronic system.

4.5.1 DESIGN RULES FOR A DIGITAL CONTROLLER BASED ON A DISCRETE EQUIVALENCE TO MODULUS OPTIMUM

Firstly we will assume that there is no integrating effect in the plant transfer function. The following CT-transfer function will be considered.

$$h_p(s) = \frac{K}{(1+sT)}e^{-sT_d} \tag{4.15}$$

Including the power converter driving this plant we get:

$$h_{hp}(s) = \frac{1-e^{-sT_d}}{s}K\frac{e^{-sT_d}}{1+sT} \tag{4.16}$$

The pulse transfer function of this has already been calculated, see equation (3.92).

$$h_{hp}(z,\varepsilon) = z^{-m}K\frac{b_0+b_1z^{-1}}{1-az^{-1}} \tag{4.17}$$

where we have:

$$a = e^{-\frac{T_s}{T}} = 1-\frac{T_s}{T}+\frac{1}{2}\left(\frac{T_s}{T}\right)^2+\ldots \quad ; \qquad 0<a<1 \tag{4.18}$$

$$b_0 = \left(1-e^{-\varepsilon\frac{T_s}{T}}\right) = (1-a^{\varepsilon}) \qquad ;0<b_0<1 \tag{4.19}$$

$$b_1 = \left(e^{-\varepsilon\frac{T_s}{T}}-e^{-\frac{T_s}{T}}\right) = (a^{\varepsilon}-a) \qquad ;0<b_1<1 \tag{4.20}$$

$$T_d = (m-\varepsilon)T_s \qquad , m = 1,2,3\ldots \; and \; 0<\varepsilon\le 1 \tag{4.21}$$

The plant transfer function in the q-plane is:

$$h_{hp}(q) = h_{hp}(z, \varepsilon)\Bigg|_{z = \frac{1 + \frac{T_s}{2}q}{1 - \frac{T_s}{2}q}}$$

$$= K\frac{b_0 + b_1\left(\dfrac{1 - \frac{T_s}{2}q}{1 + \frac{T_s}{2}q}\right)}{1 - a\left(\dfrac{1 - \frac{T_s}{2}q}{1 + \frac{T_s}{2}q}\right)}\left[\frac{1 + \frac{T_s}{2}q}{1 - \frac{T_s}{2}q}\right]^{-m} \tag{4.22}$$

$$= K\frac{(b_0 + b_1)}{(1 - a)}\frac{\left[1 + \left(\dfrac{b_0 - b_1}{b_0 + b_1}\right)\dfrac{T_s}{2}q\right]}{\left[1 + \left(\dfrac{1 + a}{1 - a}\right)\dfrac{T_s}{2}q\right]}\left[\frac{1 + \frac{T_s}{2}q}{1 - \frac{T_s}{2}q}\right]^{-m}$$

From (4.18) through (4.21) we find

$$b_0 + b_1 = 1 - a \tag{4.23}$$

$$b_0 - b_1 = 1 - 2a^{\varepsilon} + a \tag{4.24}$$

$$\frac{b_0 - b_1}{b_0 + b_1} = \frac{1 - 2a^{\varepsilon} + a}{1 - a} = \frac{1 - 2e^{-\frac{T_s}{T}\varepsilon} + e^{-\frac{T_s}{T}}}{1 - e^{-\frac{T_s}{T}}} = \beta \tag{4.25}$$

For practical sampling values, $T_s < T$, we find using (4.18) and (4.21) that

$$(\beta \approx 2\varepsilon - 1) \qquad ; \; (-1 < \beta \le 1) \tag{4.26}$$

Using a PI controller with the q-transform, according to equation (4.14) we get for the open loop transfer function:

$$h_0(q) = h_R(q)h_{hp}(q)$$
$$= K_pK\frac{[1+qT_i]\left[1+\beta\frac{T_s}{2}q\right]}{qT_i\left[1+\left(\frac{1+a}{1-a}\right)\frac{T_s}{2}q\right]}\left[\frac{1+\frac{T_s}{2}q}{1-\frac{T_s}{2}q}\right]^{-m} \quad (4.27)$$

The integrating time of the controller will be chosen to compensate the plant pole.Thus we have:

$$T_i = \frac{T_s}{2}\left(\frac{1+a}{1-a}\right) = \frac{T_s}{2}\left(\frac{1+e^{-\frac{T_s}{T}}}{1-e^{-\frac{T_s}{T}}}\right) \quad (4.28)$$

By substitution of equation (4.18) we may write.

$$T_i = \frac{T_s}{2}\left(\frac{1+1-\left(\frac{T_s}{T}\right)}{1-1+\left(\frac{T_s}{T}\right)}\right) = T-\frac{T_s}{2} \quad (4.29)$$

The open loop transfer function is:

$$h_0(q) = K_0\left[\frac{1+\beta\frac{T_s}{2}q}{\frac{T_s}{2}q}\right]\left[\frac{1-\frac{T_s}{2}q}{1-\frac{T_s}{2}q}\right]^{-m} \quad (4.30)$$

where

$$K_0 = \frac{K_pK}{\left(\frac{1+a}{1-a}\right)} = \frac{K_pKT_s}{2T_i} \quad (4.31)$$

In order to calculate the optimal gain, K_R, of the controller, we may consider the closed loop transfer function

$$m(jv) = \frac{Y(jv)}{w(jv)} = \frac{h_0(jv)}{1+h_0(jv)} = \frac{1}{1+\frac{1}{h_0(jv)}} \quad (4.32)$$

The denominator will be written as:

$$\left[1 + \frac{1}{h_0(jv)}\right] = \left|\frac{1}{h_0}\right| \cos\varphi + j\left|\frac{1}{h_0}\right| \sin\varphi + 1$$

where

$$= -arc\tan\left(\frac{\beta T_s}{2} v\right) + 2m\ \ arc\tan\left(\frac{vT_s}{2}\right) + \frac{\pi}{2} \approx \frac{\pi}{2} + \frac{vT_s}{2}(2m - \beta \qquad (4.33)$$

where we have assumed $v < \frac{\omega_s}{4}$ or equivalently $\frac{vT_s}{2} < \frac{\pi}{4}$

$$|m(jv)| = \frac{1}{\sqrt{\left(1 + \left|\frac{1}{h_0}\right| \cos\varphi\right)^2 + \left|\frac{1}{h_0}\right|^2 \sin\varphi^2}} = \frac{1}{\sqrt{\left|\frac{1}{h_0}\right|^2 + 2\left|\frac{1}{h_0}\right| \cos\varphi + 1}} \qquad (4.34)$$

According to the optimum modulus criteria we must have $|m(jv)|$ equal to 1. This requires:

$$\left|\frac{1}{h_0}\right|\left(\left|\frac{1}{h_0}\right| + 2\cos\varphi\right) = 0 \qquad (4.35)$$

Thus we must have

$$|h_0| = -\frac{1}{2\cos\varphi} \qquad (4.36)$$

Using the trigonometric identity

$$\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin(\alpha) \approx -\alpha$$

From (4.33) and (4.36) we get

$$|h_0| = \frac{1}{2\frac{vT_s}{2}(2m - \beta)}$$

Substituting (4.30) and (4.31) gives:

$$|h_0| = \frac{K_pKT_s}{2T_i} \frac{\sqrt{1 + \left(\beta\frac{T_s}{2}v\right)^2}}{\frac{T_s}{2}v} = \frac{1}{2\frac{vT_s}{2}(2m - \beta)} \qquad (4.37)$$

By neglecting the term $\left(\beta\frac{T_s}{2}v\right)^2$ this gives the optimal controller gain

$$K_p = \frac{T_i}{KT_s(2m-\beta)} \tag{4.38}$$

Using (4.25), (4.18) and (4.21) we may write:

$$T_s(2m-\beta) = T_s(2m-2\varepsilon+1) = 2T_d + T_s$$

Thus the controller gain may be written as

$$K_p = \frac{T_i}{K(2T_d+T_s)} = \frac{\left(T-\frac{T_s}{2}\right)}{K(2T_d+T_s)} \tag{4.39}$$

We will now calculate the crossover frequency, v_c, in the q-plane. Using (4.30) we may write

$$l_0(jv) = \frac{K_pKT_s}{2T_i}\frac{\sqrt{1+\left(\beta\frac{T_s}{2}v\right)^2}}{\frac{T_s}{2}v}e^{j\left[\frac{\pi}{2}-2m\left(\arctan\left(\frac{vT_s}{2}\right)\right)+\arctan\left(\frac{\beta T_s}{2}\right)\right]} \tag{4.40}$$

At the crossover point we have

$$\frac{K_pKT_s}{2T_i}\frac{\sqrt{1+\left(\beta\frac{T_s}{2}v_c\right)^2}}{\frac{T_s}{2}v_c} = 1 \tag{4.41}$$

Neglecting the term $\left(\frac{\beta T_s}{2}v_c\right)^2$ we find

$$\begin{aligned} v_c &\approx \frac{K_pK}{T_i} = \frac{T_i}{K(2T_d+T_s)}*\frac{K}{T_i} \\ v_c &\approx \frac{1}{(2T_d+T_s)} \end{aligned} \tag{4.42}$$

From (4.8) we find:

$$v_c = \frac{2}{T_s}\tan\left(\frac{\omega_c T_s}{2}\right) \approx \omega_c \approx \frac{1}{2T_d + T_s}$$

The result of the calculations are summarized in Table 4.1

Table 4.1 Optimal values of T_i, K_p and ω_c

$\dfrac{Ke^{-sT_d}}{(1+sT)}$	Optimal controller integrating time T_i	Optimal controller gain K_p	Crossover frequency ω_c
slow sampling down to $\dfrac{1}{T_s} \approx \dfrac{1}{T}$	$\dfrac{T_s}{2}\left[\dfrac{1+e^{-\frac{T_s}{T}}}{1-e^{-\frac{T_s}{T}}}\right]$	$\dfrac{T_i}{KT_s\left[2m - \dfrac{1-2e^{-\frac{T_s}{T}}\varepsilon + e^{-\frac{T_s}{T}}}{1-e^{-\frac{T_s}{T}}}\right]}$	$\dfrac{2}{T_s}arc\tan\left(\dfrac{T_s}{2}v_c\right)$
Practical sampling frequencies $\dfrac{1}{T_s} > \dfrac{1}{T}$	$T - \dfrac{T_s}{2}$	$\dfrac{\left(T - \dfrac{T_s}{2}\right)}{K(2T_d + T_s)}$	$\dfrac{1}{2T_d + T_s}$
Fast sampling $T_s \to 0$	T	$\dfrac{T_i}{2KT_d}$	$\dfrac{1}{2T_d}$

4.5.2 DESIGN RULES FOR A DIGITAL CONTROLLER BASED ON A DISCRETE EQUIVALENCE TO SYMMETRICAL OPTIMUM

In this section the following plant model will be considered:

$$h_p(s) = \frac{K}{sT_0(1+sT)}e^{-sT_d} \tag{4.43}$$

The plant is to be driven by a power electronic converter, constituting the hold element. The dead time due to the converter and the computer calculations are assumed to be included in the plant delay.

The pulse transfer function is given by equation (3.87).

$$h_{hp}(z,\varepsilon) = (1-z^{-1})z^{-m}z_{\varepsilon}\left[\frac{K}{s^2T_0(1+sT)}\right] \tag{4.44}$$

where

$$T_d = (m-\varepsilon)T_s \qquad ;m=\ 1,2,3\ldots\ and \quad 0<\varepsilon\le 1 \tag{4.45}$$

From table in [24] giving the advanced ZT and after some algebraic manipulations we arrive:

$$h_{hp(z,\varepsilon)} = \frac{KT}{T_0}z^{-m}\frac{b_0+b_1z^{-1}+b_2z^{-2}}{(1-z^{-1})(1-az^{-1})} \tag{4.46}$$

where we have:

$$b_0 = \left[\varepsilon\frac{T_s}{T} - 1 + e^{-\varepsilon\frac{T_s}{T}}\right] \qquad ;0\le b_0\le 1 \tag{4.47}$$

$$b_1 = \left[\frac{T_s}{T}\left(1-\varepsilon-\varepsilon e^{-\frac{T_s}{T}}\right)+1-2e^{-\varepsilon\frac{T_s}{T}}+e^{-\frac{T_s}{T}}\right] \qquad ;\ 0\le b_1\le 1 \tag{4.48}$$

$$b_2 = \left[e^{-\varepsilon\frac{T_s}{T}} - \frac{T_s}{T}e^{-\frac{T_s}{T}}(1-\varepsilon) - e^{-\frac{T_s}{T}}\right] \qquad ;\ 0\le b_2\le 1 \tag{4.49}$$

$$a = e^{-\frac{T_s}{T}} = 1-\frac{T_s}{T}+\frac{1}{2}\left(\frac{T_s}{T}\right)^2+\ldots \qquad ;\ 0<a<1 \tag{4.50}$$

Taking the q-transform of (4.46) we get:

$$h_{hp}(q) = \frac{KT}{T_0}\frac{\left[b_0+b_1\left(\dfrac{1-\frac{T_s}{2}q}{1+\frac{T_s}{2}q}\right)+b_2\left(\dfrac{1-\frac{T_s}{2}q}{1+\frac{T_s}{2}q}\right)^2\right]}{\left[1-\left(\dfrac{1-\frac{T_s}{2}q}{1+\frac{T_s}{2}q}\right)\right]\left[1-a\left(\dfrac{1-\frac{T_s}{2}q}{1+\frac{T_s}{2}q}\right)\right]}\left[\frac{1-\frac{T_s}{2}q}{1+\frac{T_s}{2}q}\right]^m \tag{4.51}$$

$$h_{hp}(q) = \frac{KT\left[b_0\left(1+\frac{T_s}{2}q\right)^2 + b_1\left(1-\frac{T_s}{2}q\right)\left(1+\frac{T_s}{2}q\right) + b_2\left(1-\frac{T_s}{2}q\right)^2\right]}{T_0 T_s(1-a)q\left[1+\left(\frac{1+a}{1-a}\right)\frac{T_s}{2}q\right]}\left[\frac{1-\frac{T_s}{2}q}{1+\frac{T_s}{2}q}\right]^m \tag{4.52}$$

It can be shown that

$$b_0\left(1+\frac{T_s}{2}q\right)^2 + b_1\left(1+\frac{T_s}{2}q\right)\left(1-\frac{T_s}{2}q\right) + b_2\left(1-\frac{T_s}{2}q\right)^2 \tag{4.53}$$

can be written as

$$(b_0+b_1+b_2)\left(1+\beta_1\frac{T_s}{2}q\right)\left(1+\beta_2\frac{T_s}{2}q\right) \tag{4.54}$$

And that

$$\begin{aligned} \beta_1\beta_2 &= \frac{b_0-b_1+b_2}{b_0+b_1+b_2} \\ \beta_1+\beta_2 &= \frac{2(b_0-b_2)}{b_0+b_1+b_2} \\ |\beta_1|&, |\beta_2| < 1 \end{aligned} \tag{4.55}$$

From (4.47) through (4.50) we find

$$b_0+b_1+b_2 = \frac{T_s}{T}(1-a) \qquad ;\ 0 < b_0+b_1b_2 < 1 \tag{4.56}$$

$$\beta_1+\beta_2 = \frac{2}{T_s}\left[\varepsilon T_s - T + \frac{aT_s}{1-a}\right] \tag{4.57}$$

Substituting (4.54) and (4.56) in (4.52) we get:

$$h_{hp}(q) = \frac{K}{T_0 q}\frac{\left(1+\beta_1\frac{T_s}{2}q\right)\left(1+\beta_2\frac{T_s}{2}q\right)}{\left[1+\left(\frac{1+a}{1-a}\right)\frac{T_s}{2}q\right]}\left[\frac{1+-\frac{T_s}{2}q}{1+\frac{T_s}{2}q}\right]^m \tag{4.58}$$

According to (4.2) the dynamic behaviour will be approximately the same if this equation is written as:

$$h_{hp}(q) \approx \frac{K}{T_0 q(1 + T_\Sigma q)} \tag{4.59}$$

where

$$T_\Sigma = \left(\frac{1+a}{1-a}\right)\frac{T_s}{2} - (\beta_1 + \beta_2)\frac{T_s}{2} + 2m\frac{T_s}{2} \tag{4.60}$$

Substituting (4.57)

$$T_\Sigma = \frac{1+a}{1-a}\frac{T_s}{2} - \varepsilon T_s + T - \frac{aT_s}{1-a} + 2m\frac{T_s}{2} \tag{4.61}$$

$$T_\Sigma = T + \frac{1}{2}T_s + (m - \varepsilon)T_s = \left(T + T_d + \frac{T_s}{2}\right) \tag{4.62}$$

Selection of the controller parameters will be based on the equivalent transfer function (4.59). Using a PI-controller, the open loop transfer function is approximated to

$$h_0(jv) \approx \frac{KK_p}{T_0 T_i q^2} \frac{(1 + T_i q)}{(1 + T_\Sigma q)} \tag{4.63}$$

In this case compensation of T_Σ is not possible because this would lead to an oscillating system.

It is shown in [29] that the optimum control parameters for the CT-case

$$T_i = 4T_\Sigma = 4\left(T + T_d + \frac{T_s}{2}\right) \tag{4.64}$$

$$Kp = \frac{T_0}{2KT_\Sigma} = \frac{T_0}{2K\left(T + T_d + \frac{T_s}{2}\right)} \tag{4.65}$$

when they are optimized according to the symmetrical optimum criteria.

Selecting these parameters for the controller, the resulting open loop transfer function will be:

$$h_0(j\nu) = \frac{KK_p}{T_0 4T_\Sigma (j\nu)^2} \frac{\left(1 + j4T_\Sigma \nu\right)\left(1 + j\beta_1 \frac{T_s}{2}\nu\right)\left(1 + j\beta_2 \frac{T_s}{2}\nu\right)}{\left(1 + j\left(\frac{1+a}{1-a}\right)\frac{T_s}{2}\nu\right)} \left[\frac{1 - j\frac{T_s}{2}\nu}{1 + j\frac{T_s}{2}\nu}\right]^m \tag{4.66}$$

We will now calculate the crossover frequency of the open loop transfer function. In order to do that, we need the modulus of $h_0(j\nu)$, which can be written as:

$$\left|h_0(j\nu)\right| = \frac{1}{8T_\Sigma^2} \frac{\sqrt{1 + \left(4T_\Sigma \nu\right)^2}\sqrt{\left[1 - \beta_1\beta_2\left(\frac{T_s}{2}\right)^2 \nu^2\right]^2 + (\beta_1 + \beta_2)^2\left(\frac{T_s}{2}\right)^2 \nu^2}}{\nu^2 \sqrt{1 + \left(\frac{1+a}{1-a}\right)^2\left(\frac{T_s}{2}\right)^2 \nu^2}} \tag{4.67}$$

By the method of symmetrical optimum we know that the crossover frequency must lie between $\frac{1}{4T_\Sigma} < \nu_c < \frac{1}{T_\Sigma}$.

In calculating the crossover frequency we will neglect all time constants smaller than T_Σ. These time constants will be of importance only at higher frequencies above the cross-over frequency. Thus we have:

$$\frac{\sqrt{1 + 4^2 (T_\Sigma \nu_c)^2}}{8(T_\Sigma \nu_c)^2} \approx 1$$

$$1 + 16(T_\Sigma \nu_c)^2 = 8^2 (T_\Sigma \nu_c)^4 \tag{4.68}$$

giving

$$T_\Sigma \nu_c = \frac{1}{2}$$

$$\nu_c \approx \frac{1}{2T_\Sigma} = \frac{1}{2\left(T + T_d + \frac{T_s}{2}\right)} \tag{4.69}$$

In the real frequency plane the cross-over frequency will be

$$\omega_c = \frac{2}{T_s} arc\tan\left(v_c \frac{T_s}{2}\right) \approx \frac{1}{2T_\Sigma} \tag{4.70}$$

The result of the above calculations are summarized in Table 4.2.

Table 4.2 Optimal values of T_i, K_p and ω_c

$\dfrac{Ke^{-sT_d}}{T_0 s(1+sT)}$	Optimal controller integrating time T_i	Optimal controller gain K_p	Crossover frequency ω_c
Slow sampling $\frac{1}{T_s} \approx \frac{1}{T}$	$4\left(T+T_d+\frac{T_s}{2}\right)$	$\dfrac{T_0}{2K\left(T+T_d+\frac{T_s}{2}\right)}$	$\frac{2}{T_s} arc\tan\left(\dfrac{T_s}{4\left(T+T_d+\frac{T_s}{2}\right)}\right)$
Practical sampling frequencies $\frac{1}{T_s} > \frac{1}{T}$	$4\left(T+T_d+\frac{T_s}{2}\right)$	$\dfrac{T_0}{2K\left(T+T_d+\frac{T_s}{2}\right)}$	$\dfrac{1}{2\left(T+T_d+\frac{T_s}{2}\right)}$
Fast Sampling $T_s \to 0$	$4(T+T_d)$	$\dfrac{T_0}{2K(T+T_d)}$	$\dfrac{1}{2(T+T_d)}$