# COM303: Digital Signal Processing

Lecture 3: Signal Processing and Vector Spaces

# Module Overview:

- signal processing as geometry
- vectors and vector spaces
- Hilbert space and basis

# Signal Models (in Physics)

Description of the evolution of a physical phenomenon

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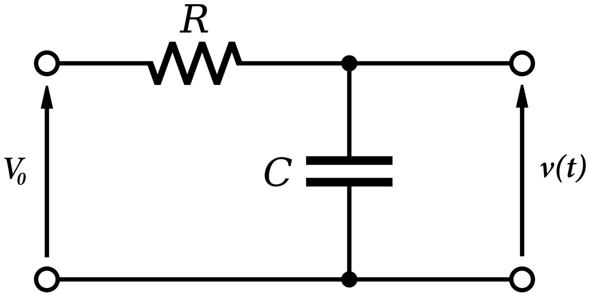



$$v(t) = V_0(1 - e^{-\frac{t}{RC}})$$

# Signal Models (in Physics)

$$f : \mathbb{R} \to \mathbb{R}$$

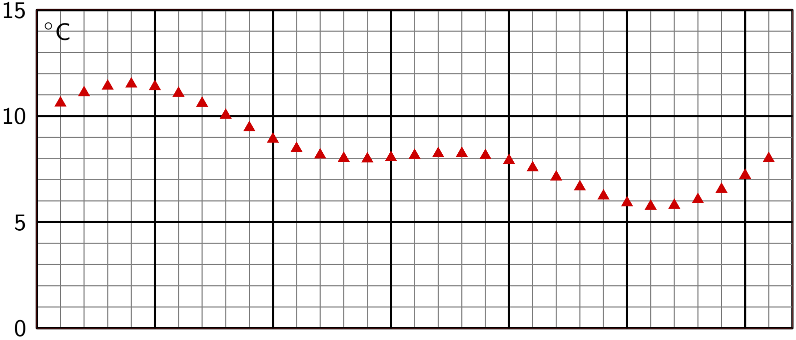
# Signal Models (in DSP)

$$\cancel{f : \mathbb{R} \to \mathbb{R}}$$

# Signal Models (in DSP)

$$x[n] = \ldots, 1.2390, -0.7372, 0.8987, 0.1798, -1.1501, -0.2642 \ldots$$

# Signal Models (in DSP)



# Discrete-Time Signal Model

$$\mathbb{C}^N$$

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# Let's talk about Vector Spaces...

Some spaces should be very familiar:

- $\mathbb{R}^2, \mathbb{R}^3$: Euclidean space, geometry
- $\mathbb{R}^N, \mathbb{C}^N$: linear algebra

Others perhaps not so much...

- $\ell_2(\mathbb{Z})$: space of square-summable infinite sequences
- $L_2([a, b])$: space of square-integrable *functions* over an interval

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# Why using vector vpaces in DSP?

Easier math and unified framework for signal processing:

- same object for different classes of signals (finite-length, finite-support, infinite, periodic)
- easy explanation of the Fourier Transform
- easy explanation of sampling and interpolation
- useful in approximation and compression
- fundamental in communication system design

# The three take-home lessons today

- vector spaces are very general objects
- vector spaces are defined by their properties
- once you know the properties are satisfied, you can use all the tools for the space

# Analogy #1: OOP

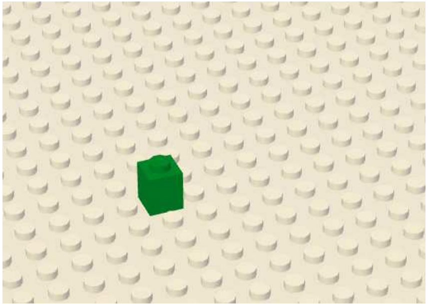
```
class Polygon(object):
    def __init__(self, num_sides, side_len=1, x=0, y=0):
        self.num_sides = num_sides
        self.side_len = side_len
        self.center = [x, y]

    def resize(self, factor):
        self.side_len *= factor

    def translate(self, x, y):
        self.center[0] += x
        self.center[1] += y

    def plot(self):
        ...
```

# Analogy #1: OOP

```
class Triangle(Polygon):
    def __init__(self):
        super(Triangle, self).__init__(3)

    ...


class Square(Polygon):
    def __init__(self):
        super(Square, self).__init__(4)

    ...
```

# Analogy #2: LEGO

basic building block:



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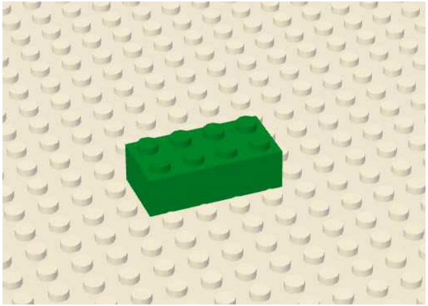
scaling (4x2):



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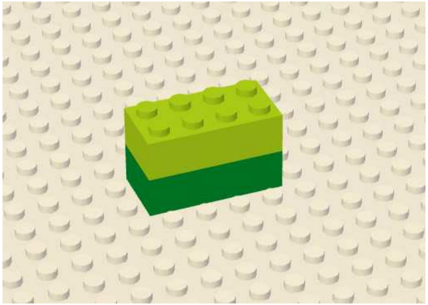
adding:



## vector spaces

# Graphical representation of a vector

# Sometimes we can

$$\mathbb{R}^2 : \qquad \mathbf{x} = \begin{bmatrix} x_0 & x_1 \end{bmatrix}^T$$

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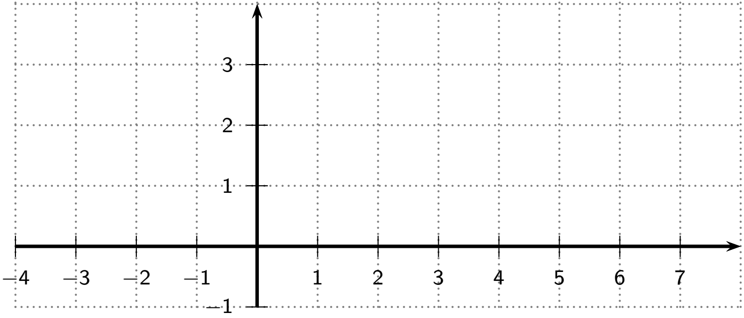

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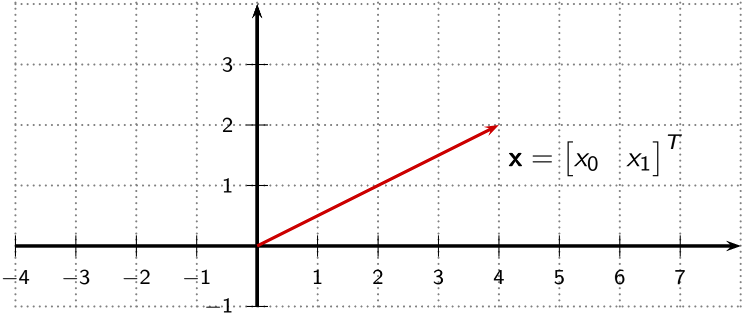

# Sometimes we can

$$\mathbb{R}^3: \qquad \mathbf{x} = \begin{bmatrix} x_0 & x_1 & x_2 \end{bmatrix}^T$$

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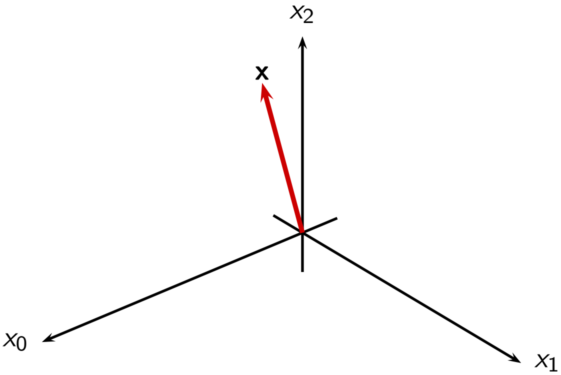

# but most of the time we can't

$$\mathbb{R}^N \text{ for } N > 3: \qquad \mathbf{x} = \begin{bmatrix} x_0 & x_1 & \dots & x_{N-1} \end{bmatrix}^T$$

# Sometimes we can

$$L_2([-1,1]): \qquad \mathbf{x} = x(t) \in \mathbb{R}, \quad t \in [-1,1]$$

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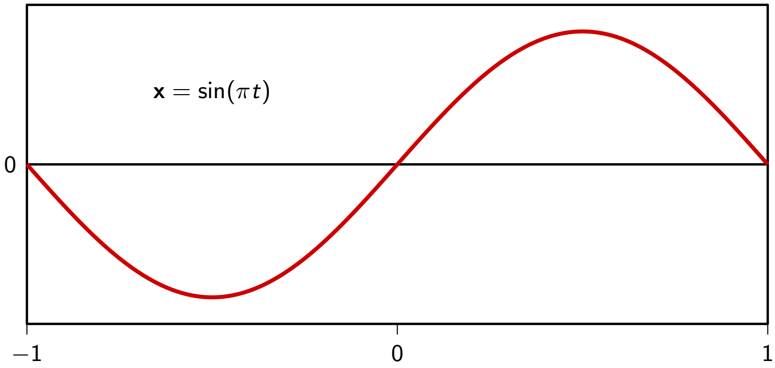

# other times we can't

$f : \mathbb{C} \to \mathbb{C}$, analytic

# Vector spaces: operational definition

Ingredients:

- the set of vectors $V$
- a set of scalars (say $\mathbb{C}$)

We need *at least* to be able to:

- resize vectors, i.e. multiply a vector by a scalar
- combine vectors together, i.e. sum them together

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# Formal properties of a vector space:

For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha, \beta \in \mathbb{C}$:

- $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
- $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
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- $\exists\, 0 \in V \quad | \quad \mathbf{x} + 0 = 0 + \mathbf{x} = \mathbf{x}$
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# Vector space example: $\mathbb{R}^N$

$$\mathbf{x} = \begin{bmatrix} x_0 & x_1 & \dots & x_{N-1} \end{bmatrix}^T$$

$$\mathbf{y} = \begin{bmatrix} y_0 & y_1 & \dots & y_{N-1} \end{bmatrix}^T$$

$$\alpha\mathbf{x} = \begin{bmatrix} \alpha x_0 & \alpha x_1 & \dots & \alpha x_{N-1} \end{bmatrix}^T$$

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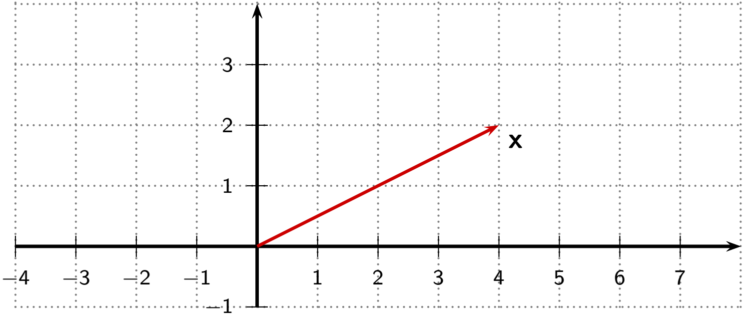

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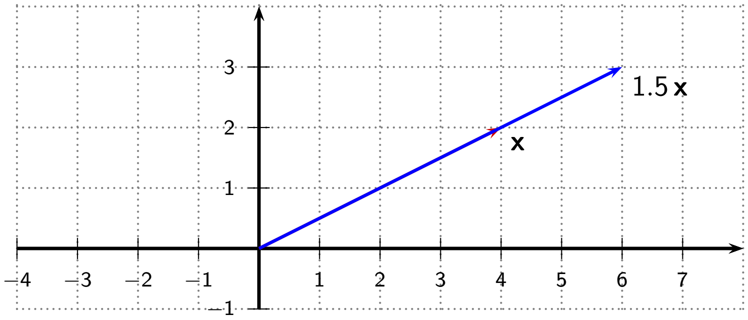

# Addition in $\mathbb{R}^2$

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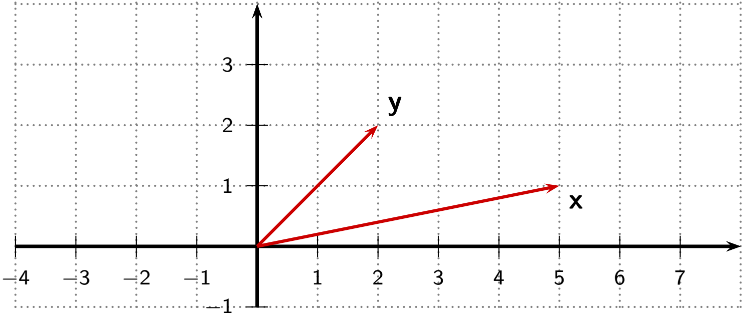

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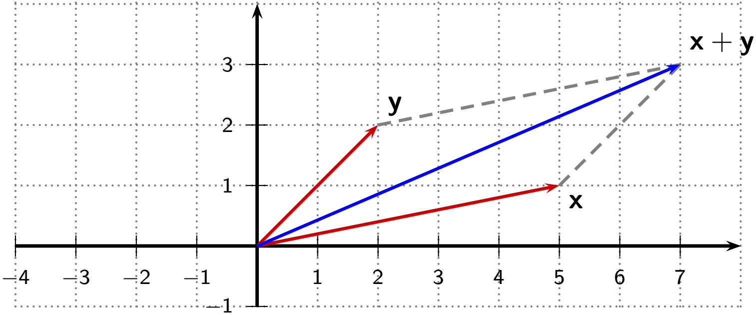
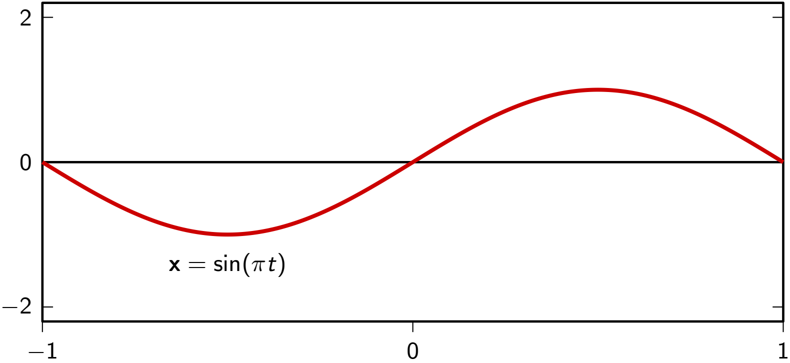


# Scalar multiplication in $L_2[-1, 1]$

$$\alpha \mathbf{x} = \alpha\, x(t)$$



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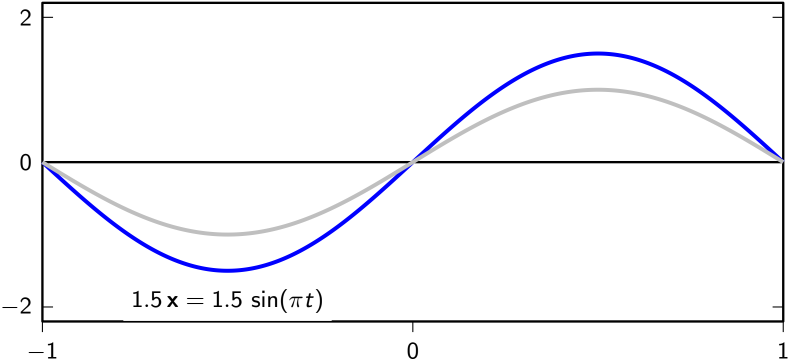

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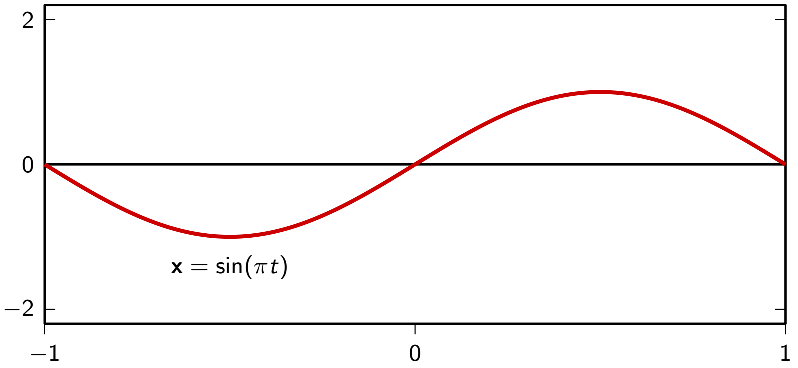

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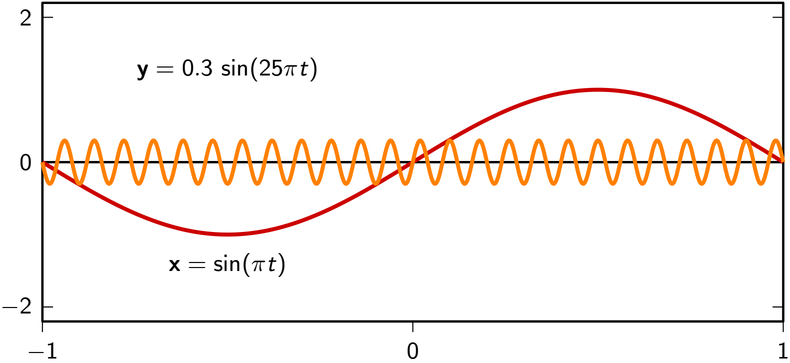

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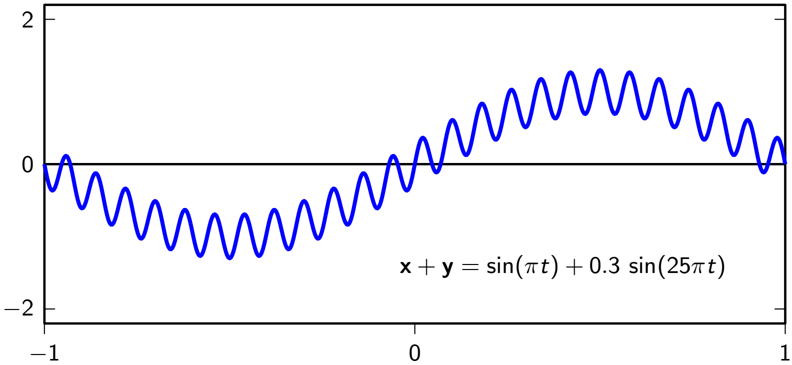

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# inner product spaces

# Inner product

$$\langle \cdot, \cdot \rangle \ : \ V \times V \to \mathbb{C}$$

- measure of similarity between vectors
- inner product is zero? vectors are *orthogonal* (maximally different)

# Formal properties of the inner product

For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha \in \mathbb{C}$:

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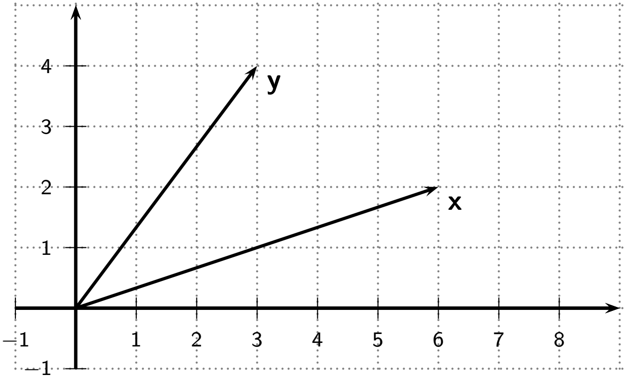
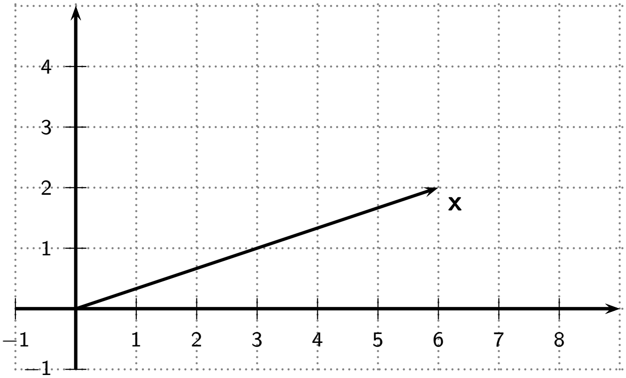
# Inner product in $\mathbb{R}^2$

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_0 y_0 + x_1 y_1$$



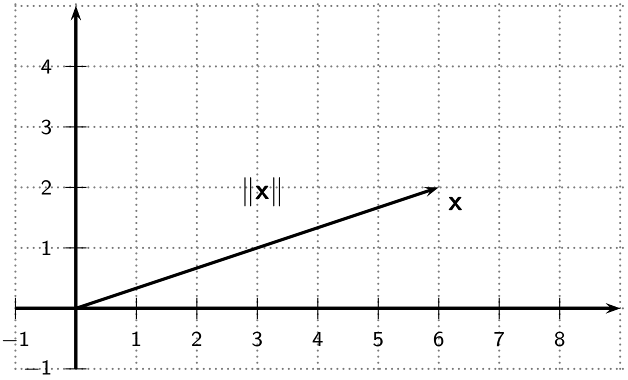
# Inner product in $\mathbb{R}^2$: the norm

$$\langle \mathbf{x}, \mathbf{x} \rangle = x_0^2 + x_1^2$$



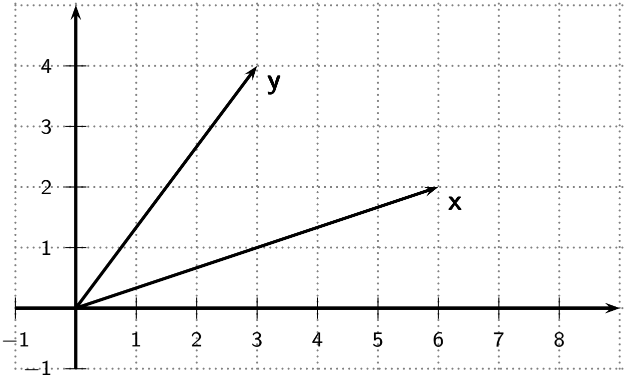
# Inner product in $\mathbb{R}^2$: the norm

$$\langle \mathbf{x}, \mathbf{x} \rangle = x_0^2 + x_1^2 = \|\mathbf{x}\|^2$$



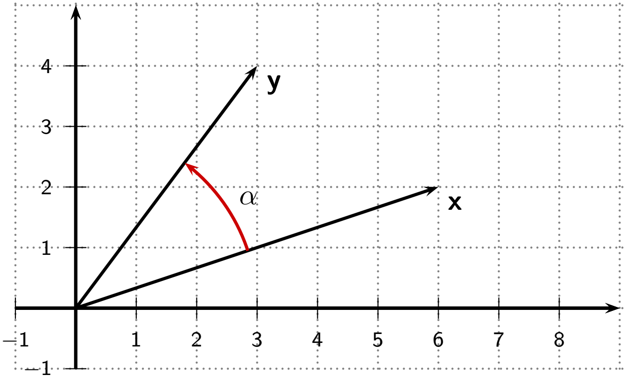
# Inner product in $\mathbb{R}^2$

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_0 y_0 + x_1 y_1$$



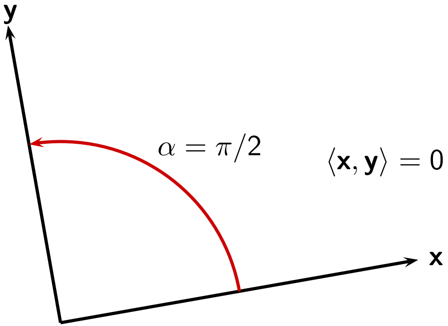
# Inner product in $\mathbb{R}^2$

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_0 y_0 + x_1 y_1 = \|\mathbf{x}\| \, \|\mathbf{y}\| \, \cos\alpha$$



# Inner product in $\mathbb{R}^2$: orthogonality

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_0 y_0 + x_1 y_1 = \|\mathbf{x}\| \, \|\mathbf{y}\| \, \cos\alpha$$



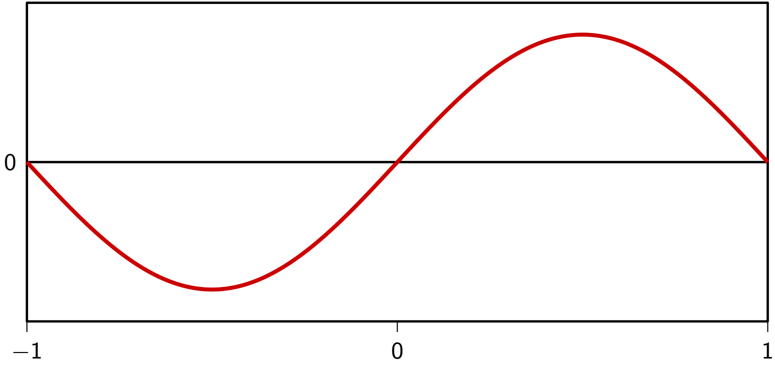

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0$$

# Inner product in $L_2[-1, 1]$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{-1}^{1} x(t)y(t)\, dt$$

# Inner product in $L_2[-1,1]$: the norm

$$\langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|^2 = \int_{-1}^{1} \sin^2(\pi t) dt = 1$$



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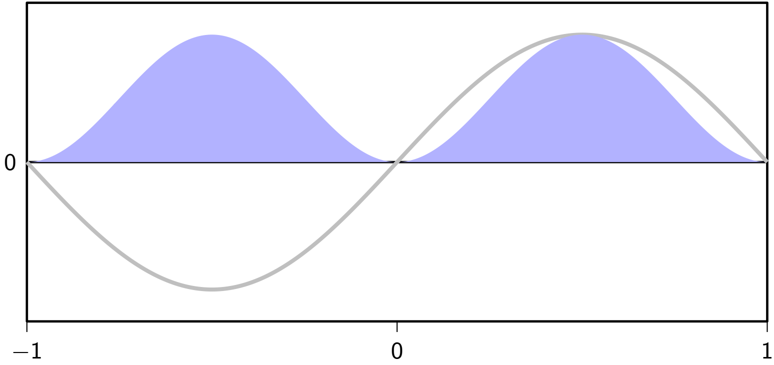
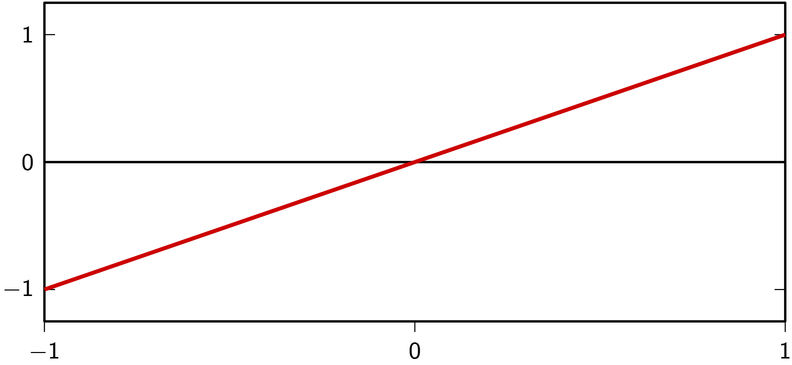


# Inner product in $L_2[-1, 1]$: the norm

$$\|\mathbf{y}\|^2 = \int_{-1}^{1} t^2 \, dt = 2/3$$



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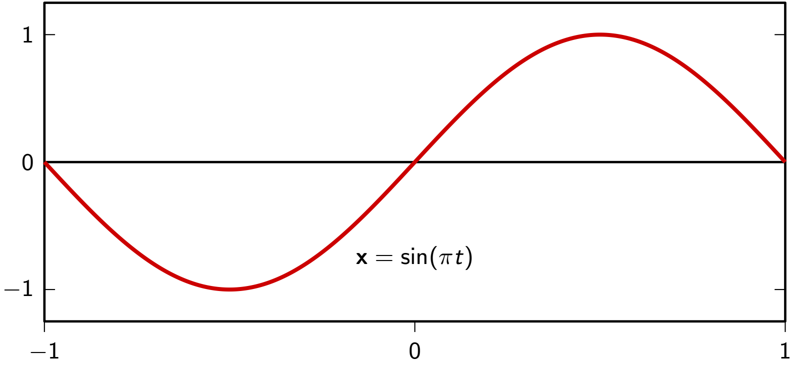

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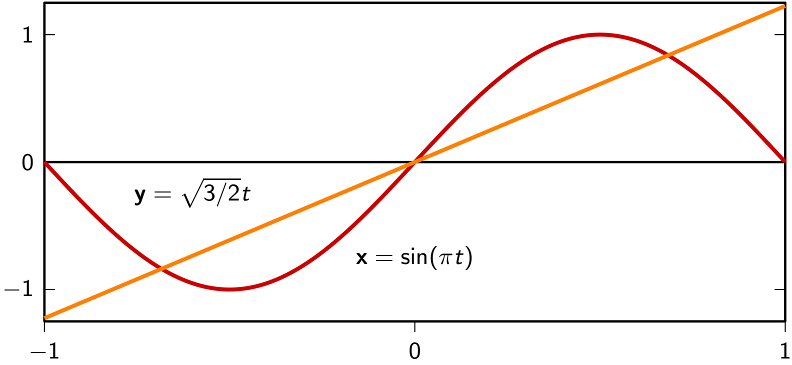
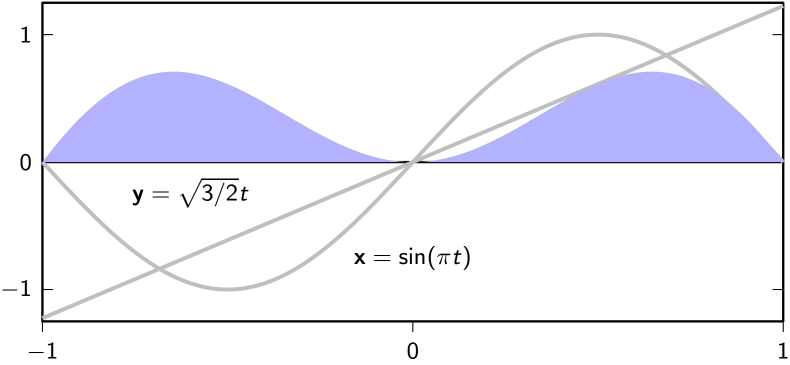


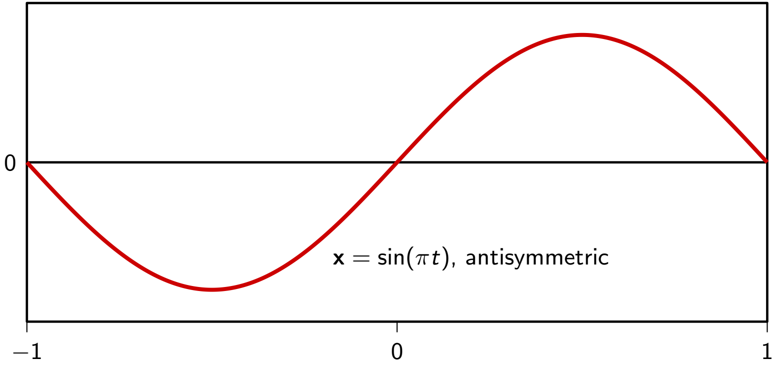
# Inner product in $L_2[-1,1]$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{-1}^{1} \sqrt{3/2}\,t \sin(\pi t) dt = (2/\pi)\sqrt{3/2} \approx 0.78 \approx \cos(38.7^\circ)$$



# Inner product in $L_2[-1, 1]$

$\mathbf{x}, \mathbf{y}$ from orthogonal subspaces:



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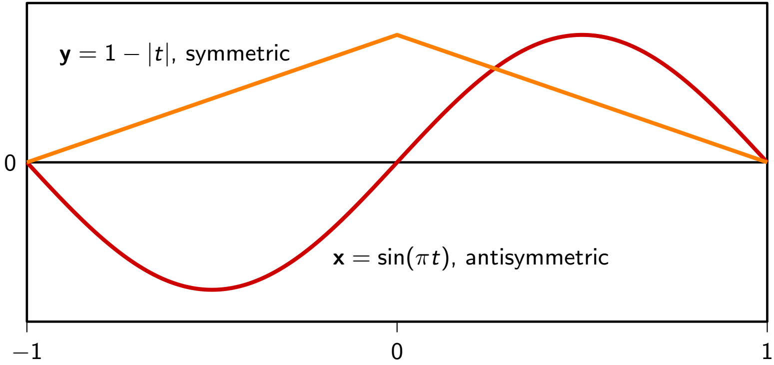

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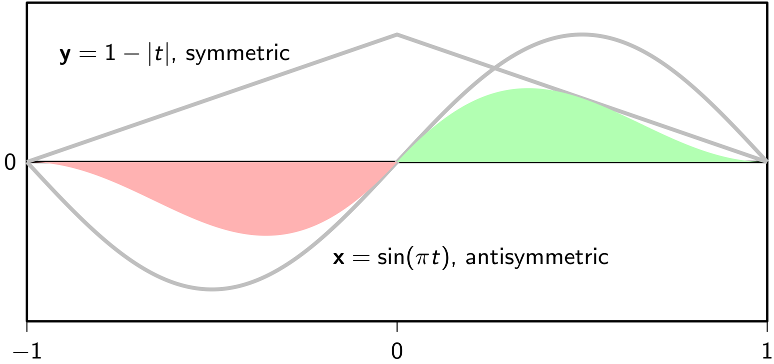
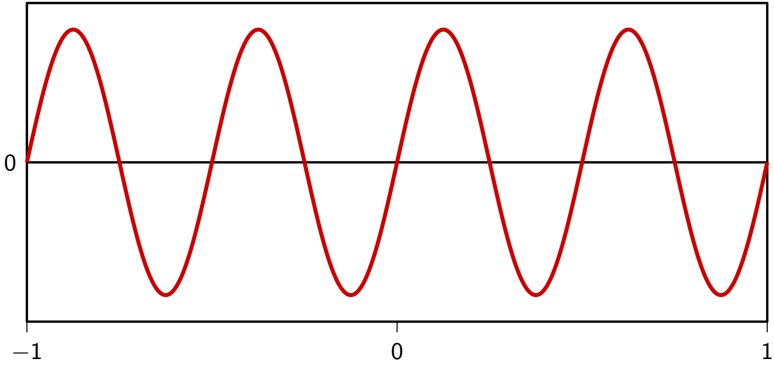


# Inner product in $L_2[-1, 1]$

sinusoids with frequencies integer multiples of a fundamental




$\mathbf{x} = \sin(4\pi t)$

# Inner product in $L_2[-1, 1]$

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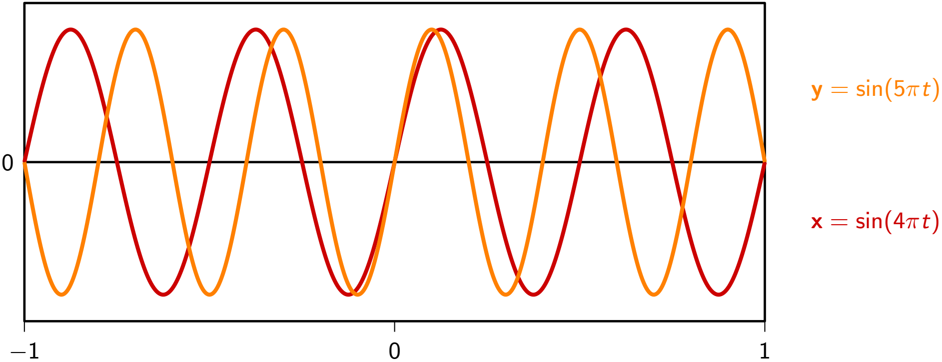

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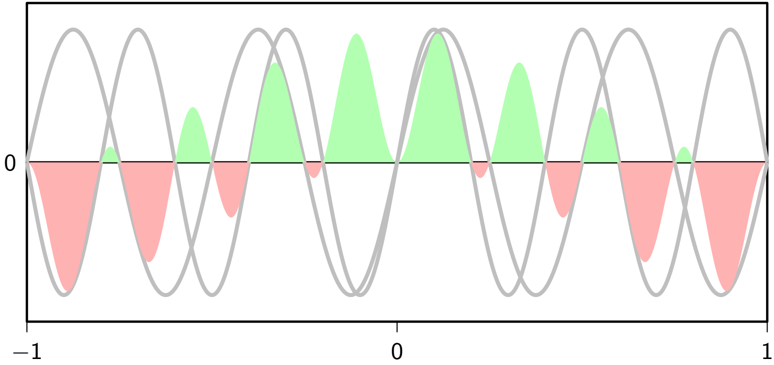
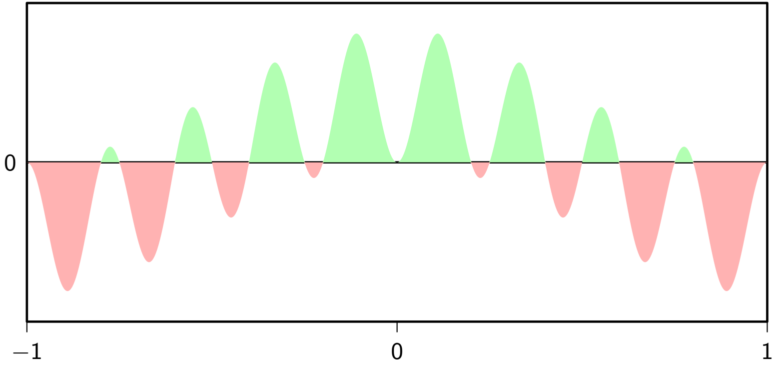



$\mathbf{y} = \sin(5\pi t)$

# Inner product in $L_2[-1, 1]$

sinusoids with frequencies integer multiples of a fundamental



# Norm vs Distance

- inner product defines a norm: $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$
- norm defines a distance: $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$

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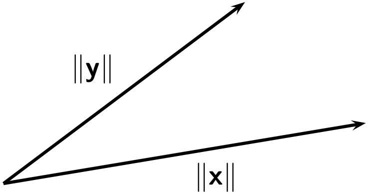
# Norm and distance in $\mathbb{R}^2$

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x_0^2 + x_1^2}$$

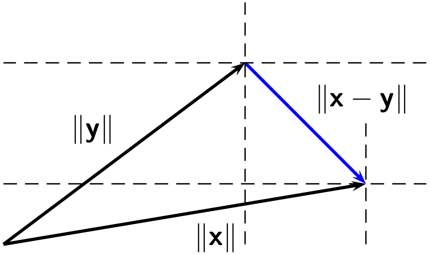
# Norm and distance in $\mathbb{R}^2$

$$\|\mathbf{y}\| = \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle} = \sqrt{y_0^2 + y_1^2}$$



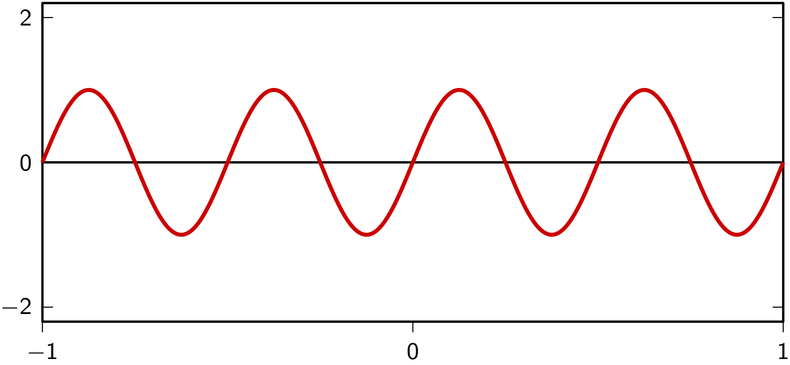
# Norm and distance in $\mathbb{R}^2$

$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_0 - y_0)^2 + (x_1 - y_1)^2}$$



# Distance in $L_2[-1, 1]$: the Mean Square Error

$$\|\mathbf{x} - \mathbf{y}\|^2 = \int_{-1}^{1} |x(t) - y(t)|^2 \, dt$$

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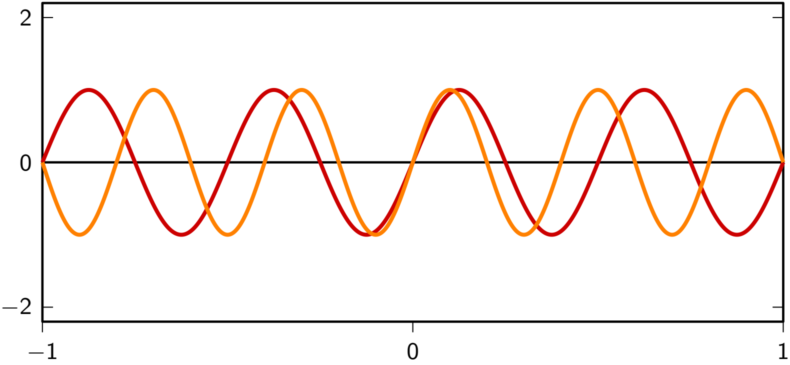
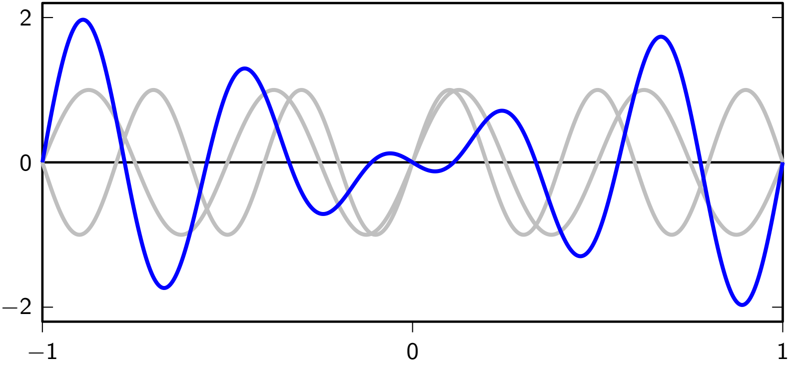



$\mathbf{y} = \sin(5\pi t)$

$\mathbf{x} = \sin(4\pi t)$

# Distance in $L_2[-1, 1]$: the Mean Square Error

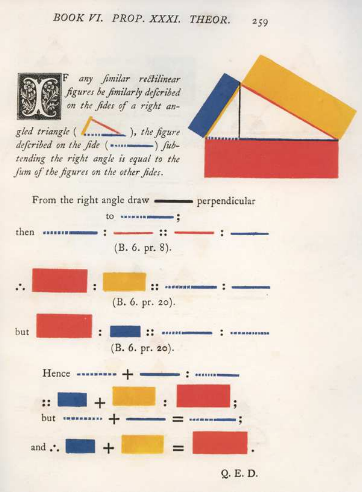
$$\|\mathbf{x} - \mathbf{y}\|^2 = \int_{-1}^{1} |x(t) - y(t)|^2 \, dt = 2$$



# A familiar result

Pythagorean theorem:

$$
\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 &= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle \\
&= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \text{ for } \mathbf{x} \perp \mathbf{y}
\end{aligned}
$$




From Euclid's elements by Oliver Byrne (1810 - 1880)

# Inner product for signals

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=0}^{N-1} x^*[n]y[n]$$

well defined for all finite-length vectors (i.e. vectors in $\mathbb{C}^N$)

# Inner product for signals

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=-\infty}^{\infty} x^*[n]y[n]$$

careful: sum may explode!

# Inner product for signals

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=-\infty}^{\infty} x^*[n]y[n]$$

We require sequences to be *square-summable*: $\sum |x[n]|^2 < \infty$

Space of square-summable sequences: $\ell_2(\mathbb{Z})$

# bases

# Bases

linear combination is the basic operation in vector spaces:

$$\mathbf{g} = \alpha\,\mathbf{x} + \beta\,\mathbf{y}$$

can we find a set of vectors $\{\mathbf{w}^{(k)}\}$ so that we can write *any* vector as a linear combination of the $\{\mathbf{w}^{(k)}\}$?

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can we find a set of vectors $\{\mathbf{w}^{(k)}\}$ so that we can write *any* vector as a linear combination of the $\{\mathbf{w}^{(k)}\}$?

# The canonical $\mathbb{R}^2$ basis

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{e}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{e}^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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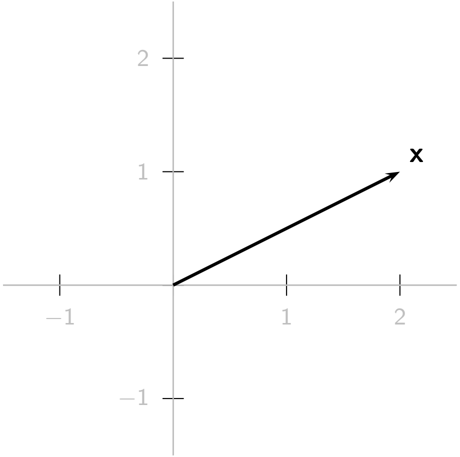
# The canonical $\mathbb{R}^2$ basis

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\mathbf{x} = 2\mathbf{e}^{(0)} + \mathbf{e}^{(1)}$$



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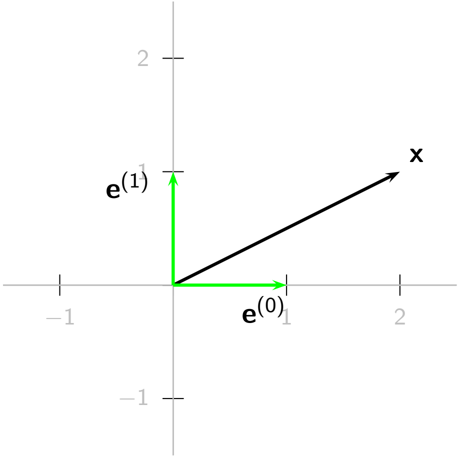

# Another $\mathbb{R}^2$ basis

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\alpha_1 = x_1 - x_2, \quad \alpha_2 = x_2$$

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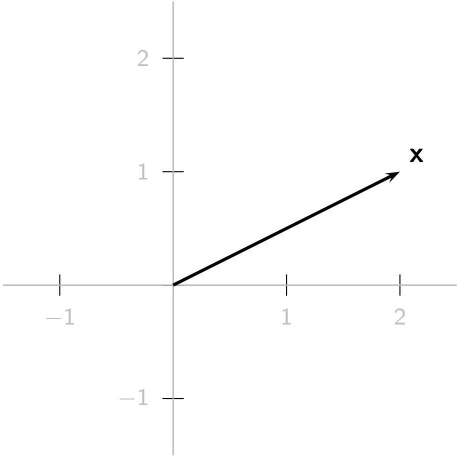

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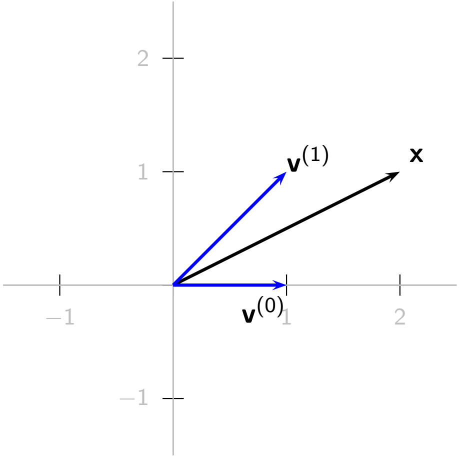

# But this is not a basis for $\mathbb{R}^2$...

$$\mathbf{g}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{g}^{(1)} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} \text{ whenever } x_2 \neq 0$$

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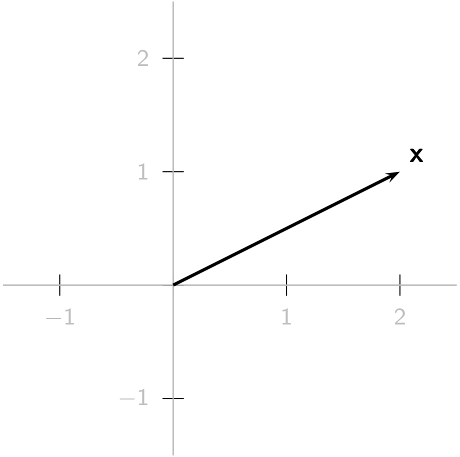
$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\mathbf{x} \neq \alpha_1 \mathbf{g}^{(0)} + \alpha_2 \mathbf{g}^{(1)}$$



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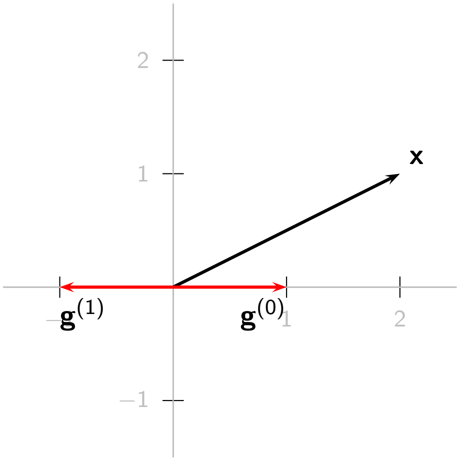


# What about infinte-dimensional spaces?

$$\mathbf{x} = \sum_{k=-\infty}^{\infty} \alpha_k \, \mathbf{w}^{(k)}$$

# A basis for $\ell_2(\mathbb{Z})$

$$\mathbf{e}^{(k)} = \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

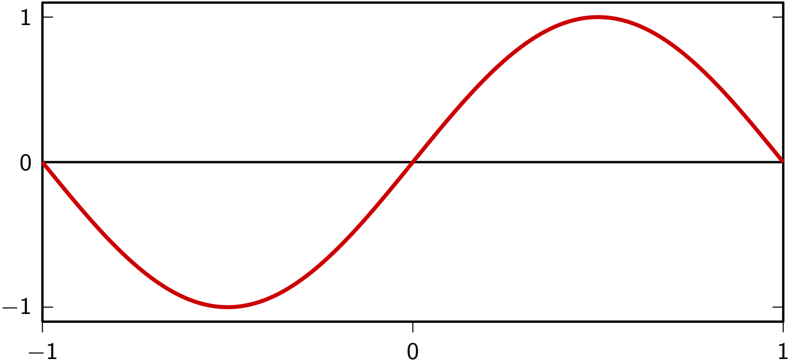
$k$-th position, $k \in \mathbb{Z}$

# What about functional vector spaces?

$$f(t) = \sum_k \alpha_k \, h^{(k)}(t)$$

# A basis for the functions over an interval?

$$\sum_{k=0}^{N} (1/k)\mathbf{v}^{(k)}, \qquad \mathbf{v}^{(k)} = \sin(\pi(2k+1)t), \quad t \in [-1, 1]$$




$N = 0$

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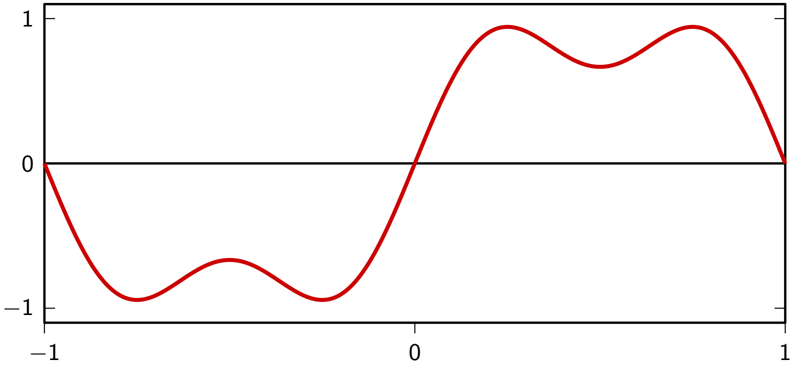
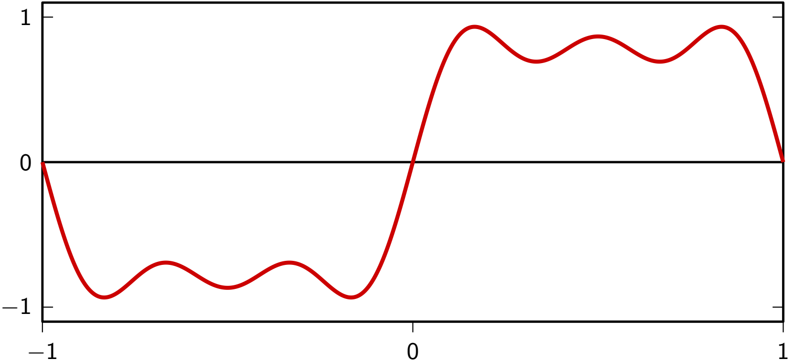



$N = 1$

# A basis for the functions over an interval?

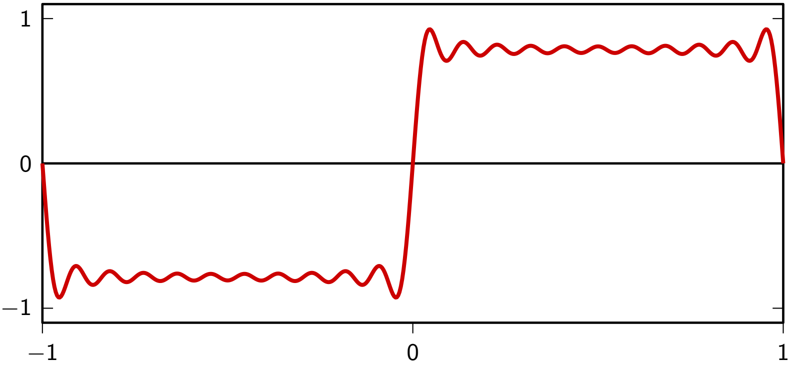
$$\sum_{k=0}^{N} (1/k)\mathbf{v}^{(k)}, \qquad \mathbf{v}^{(k)} = \sin(\pi(2k+1)t), \quad t \in [-1,1]$$




$N = 2$

# A basis for the functions over an interval?

$$\sum_{k=0}^{N}(1/k)\mathbf{v}^{(k)}, \qquad \mathbf{v}^{(k)} = \sin(\pi(2k+1)t), \quad t \in [-1,1]$$




$N = 10$

# A basis for the functions over an interval?

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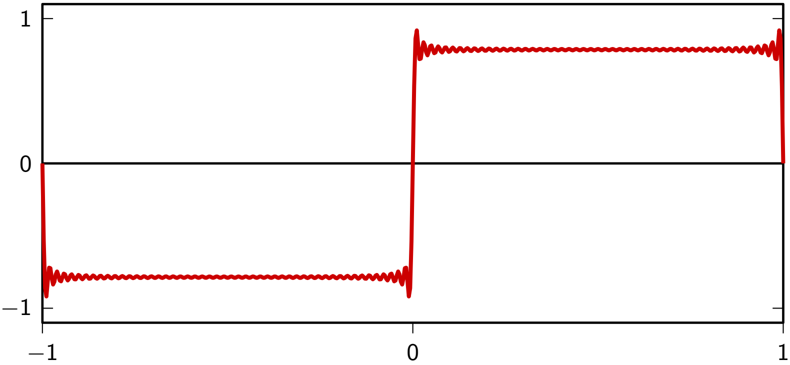
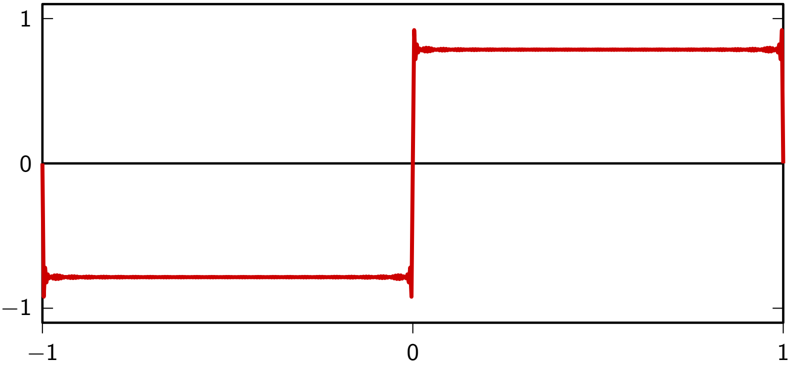



$N = 50$

# A basis for the functions over an interval?

$$\sum_{k=0}^{N}(1/k)\mathbf{v}^{(k)}, \qquad \mathbf{v}^{(k)} = \sin(\pi(2k+1)t), \quad t \in [-1,1]$$




$N = 150$

# Bases: formal definition

Given:

- a vector space $H$
- a set of $K$ vectors from $H$: $W = \{\mathbf{w}^{(k)}\}_{k=0,1,\ldots,K-1}$

$W$ is a basis for $H$ if:

1. we can write for *all* $\mathbf{x} \in H$:

$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)}, \quad \alpha_k \in \mathbb{C}$$

2. the coefficients $\alpha_k$ are unique

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# Bases: formal definition

Unique representation implies linear independence:

$$\sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)} = 0 \qquad \Longleftrightarrow \qquad \alpha_k = 0, \;\; k = 0, 1, \ldots, K-1$$

# Special bases

Orthogonal basis:

$$\langle \mathbf{w}^{(k)}, \mathbf{w}^{(n)} \rangle = 0 \text{ for } k \neq n$$

Orthonormal basis:

$$\langle \mathbf{w}^{(k)}, \mathbf{w}^{(n)} \rangle = \delta[n-k]$$

(we can always orthonormalize a basis via the Gram-Schmidt algorithm)

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# Basis expansion

$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)}$$

how do we find the $\alpha$'s ?

Orthonormal bases are the best:

$$\alpha_k = \langle \mathbf{w}^{(k)}, \mathbf{x} \rangle$$

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# Change of basis

$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)} = \sum_{k=0}^{K-1} \beta_k \mathbf{v}^{(k)}$$

if $\{\mathbf{v}^{(k)}\}$ is orthonormal:

$$\begin{aligned}
\beta_h &= \langle \mathbf{v}^{(h)}, \mathbf{x} \rangle \\
&= \langle \mathbf{v}^{(h)}, \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)} \rangle \\
&= \sum_{k=0}^{K-1} \alpha_k \langle \mathbf{v}^{(h)}, \mathbf{w}^{(k)} \rangle
\end{aligned}$$

# Change of basis

$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)} = \sum_{k=0}^{K-1} \beta_k \mathbf{v}^{(k)}$$

if $\{\mathbf{v}^{(k)}\}$ is orthonormal:

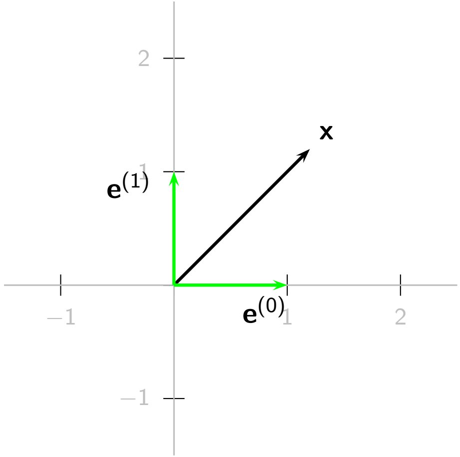
$$\begin{aligned}
\beta_h &= \langle \mathbf{v}^{(h)}, \mathbf{x} \rangle \\
&= \langle \mathbf{v}^{(h)}, \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)} \rangle \\
&= \sum_{k=0}^{K-1} \alpha_k \langle \mathbf{v}^{(h)}, \mathbf{w}^{(k)} \rangle
\end{aligned}$$

# Change of basis

$$
\begin{aligned}
\beta_h &= \sum_{k=0}^{K-1} \alpha_k \langle \mathbf{v}^{(h)}, \mathbf{w}^{(k)} \rangle \\
&= \sum_{k=0}^{K-1} \alpha_k \, c_{hk} \\
&= \begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0(K-1)} \\ & & \vdots & \\ c_{(K-1)0} & c_{(K-1)1} & \cdots & c_{(K-1)(K-1)} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{K-1} \end{bmatrix}
\end{aligned}
$$

# Change of basis: example

- canonical basis $E = \{\mathbf{e}^{(0)}, \mathbf{e}^{(1)}\}$
- $\mathbf{x} = \alpha_0 \mathbf{e}^{(0)} + \alpha_1 \mathbf{e}^{(1)}$



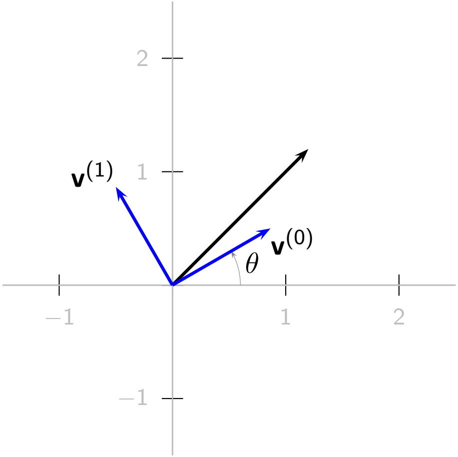
# Change of basis: example

- canonical basis $E = \{\mathbf{e}^{(0)}, \mathbf{e}^{(1)}\}$
- $\mathbf{x} = \alpha_0 \mathbf{e}^{(0)} + \alpha_1 \mathbf{e}^{(1)}$
- new basis $V = \{\mathbf{v}^{(0)}, \mathbf{v}^{(1)}\}$ with

$$\mathbf{v}^{(0)} = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix}^T$$

$$\mathbf{v}^{(1)} = \begin{bmatrix} -\sin\theta & \cos\theta \end{bmatrix}^T$$

- $\mathbf{x} = \beta_0 \mathbf{v}^{(0)} + \beta_1 \mathbf{v}^{(1)}$



# Change of basis: example

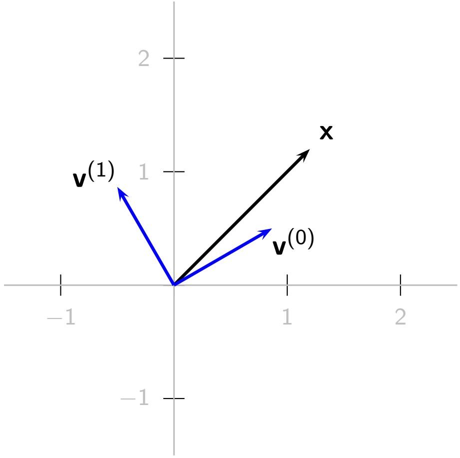
- new basis is orthonormal:

$$c_{hk} = \langle \mathbf{v}^{(h)}, \mathbf{e}^{(k)} \rangle$$

- in compact form:

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \mathbf{R}\alpha$$

- $\mathbf{R}$: rotation matrix
- key fact: $\mathbf{R}^T\mathbf{R} = \mathbf{I}$



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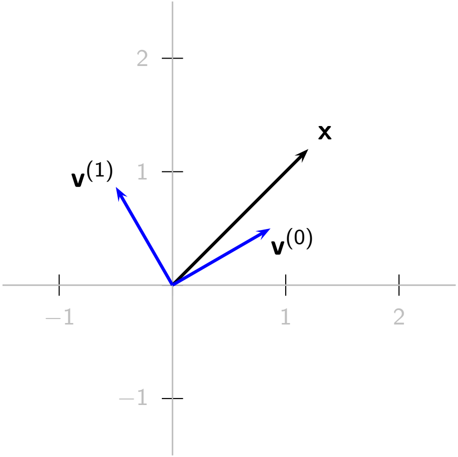

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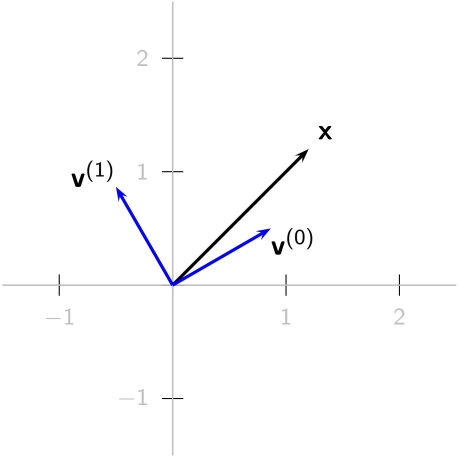

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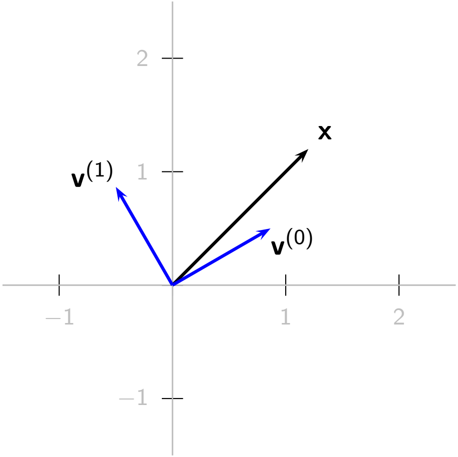

# Norm and energy

In $\mathbb{C}^N$

$$\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle = \sum_{k=0}^{K-1} |x_k|^2$$

*(remember the definition of energy for discrete-time signals)*

# Parseval's Theorem (conservation of energy)

If $\{\mathbf{w}^{(k)}\}$ is orthonormal and $\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)}$

$$\begin{aligned} \|\mathbf{x}\|^2 &= \langle \mathbf{x}, \mathbf{x} \rangle \\ &= \sum_{k=0}^{K-1} |\alpha_k|^2 \end{aligned}$$

energy is conserved across a change of basis

# Conservation of energy: example

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \mathbf{R}\boldsymbol{\alpha}$$

- square norm in canonical basis: $\|\mathbf{x}\|^2 = \alpha_0^2 + \alpha_1^2$
- square norm in rotated basis: $\|\mathbf{x}\|^2 = \beta_0^2 + \beta_1^2$
- let's verify Parseval:

$$\begin{aligned} \beta_0^2 + \beta_1^2 &= \boldsymbol{\beta}^T \boldsymbol{\beta} \\ &= (\mathbf{R}\boldsymbol{\alpha})^T (\mathbf{R}\boldsymbol{\alpha}) \\ &= \boldsymbol{\alpha}^T (\mathbf{R}^T \mathbf{R}) \boldsymbol{\alpha} \\ &= \boldsymbol{\alpha}^T \boldsymbol{\alpha} = \alpha_0^2 + \alpha_1^2 \end{aligned}$$

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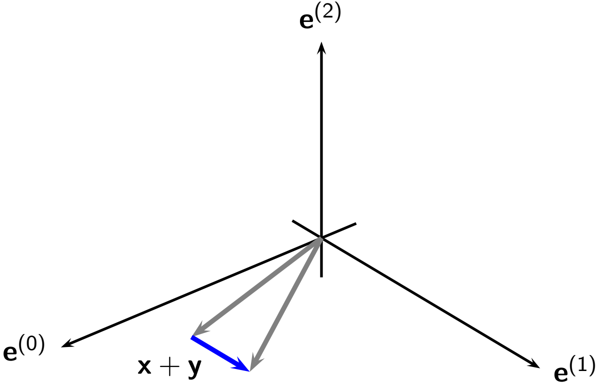
# subspaces and approximations

# Vector subspace

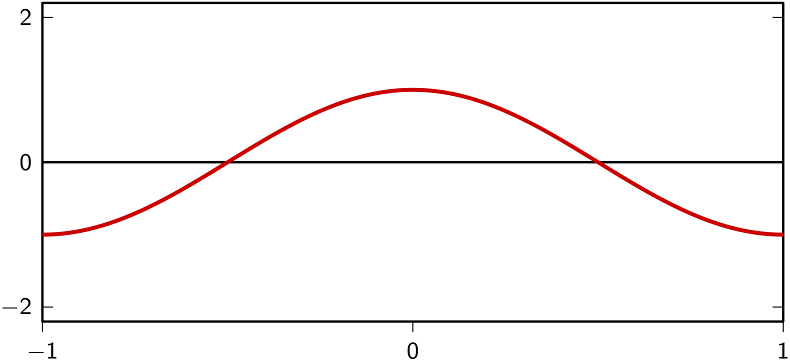
a subset of vectors *closed* under addition and scalar multiplication

# Example in Euclidean Space

intuition: $\mathbb{R}^2 \subset \mathbb{R}^3$



# Subspace of symmetric functions over $L_2[-1,1]$




$\mathbf{x} = \cos(\pi t)$

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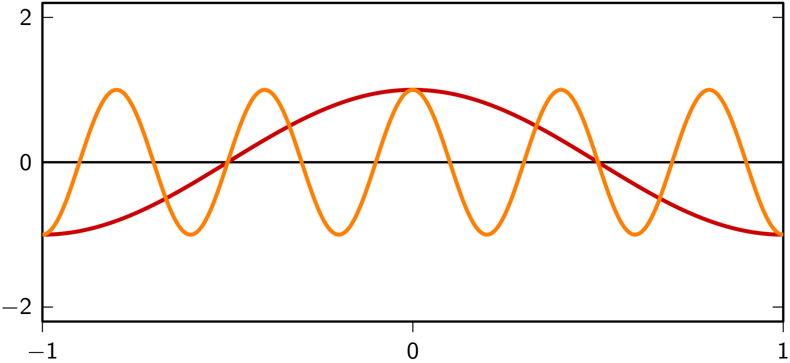


$$\mathbf{y} = \cos(5\pi t)$$

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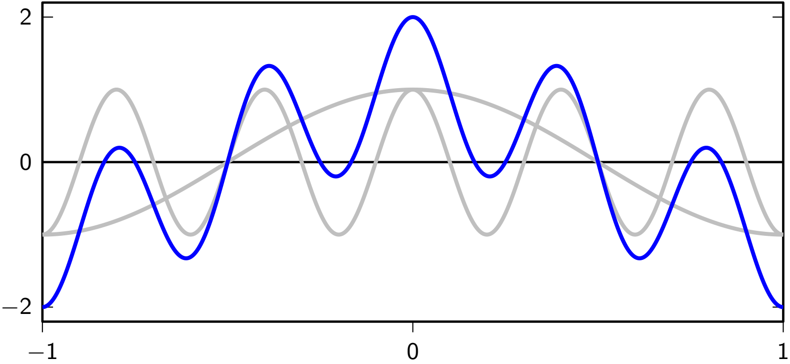



$\mathbf{x} + \mathbf{y}$, symmetric

# Subspaces have their own basis

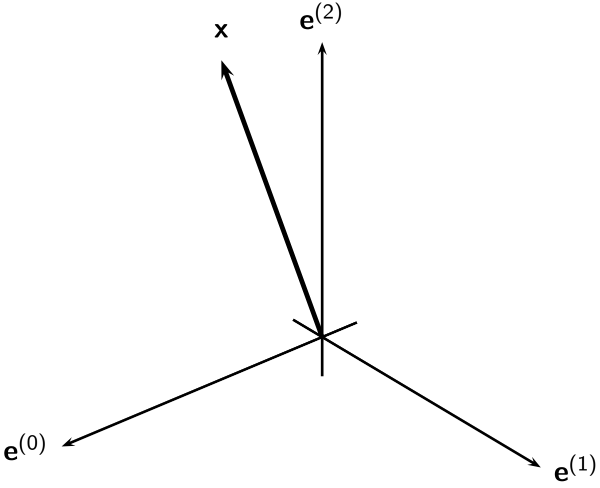
$$\mathbf{e}^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{e}^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

basis vector for the plane in $\mathbb{R}^3$

# Approximation

Problem:

- vector $\mathbf{x} \in V$
- subspace $S \subseteq V$
- approximate $\mathbf{x}$ with $\hat{\mathbf{x}} \in S$



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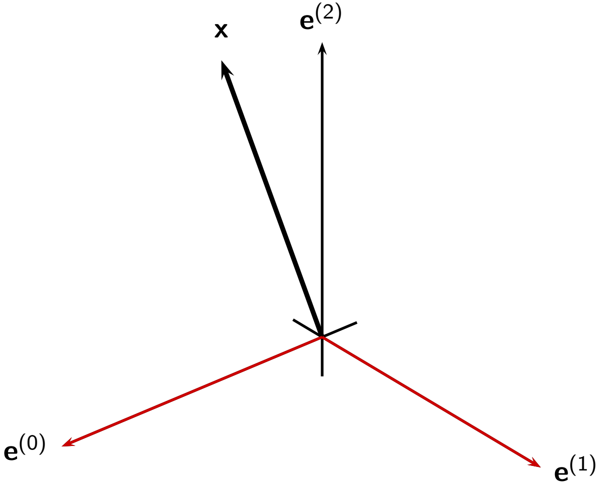

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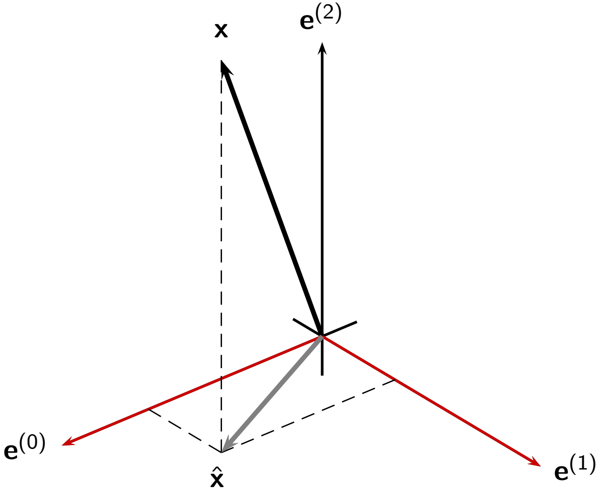

# Least-Squares Approximation

- $\{\mathbf{s}^{(k)}\}_{k=0,1,\ldots,K-1}$ orthonormal basis for $S$
- orthogonal projection:

$$\hat{\mathbf{x}} = \sum_{k=0}^{K-1} \langle \mathbf{s}^{(k)}, \mathbf{x} \rangle \, \mathbf{s}^{(k)}$$

orthogonal projection is the "best" approximation over $S$

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- orthogonal projection has minimum-norm error:

$$\arg\min_{\mathbf{y}\in S} \|\mathbf{x} - \mathbf{y}\| = \hat{\mathbf{x}}$$

- error is orthogonal to approximation:

$$\langle \mathbf{x} - \hat{\mathbf{x}}, \hat{\mathbf{x}} \rangle = 0$$

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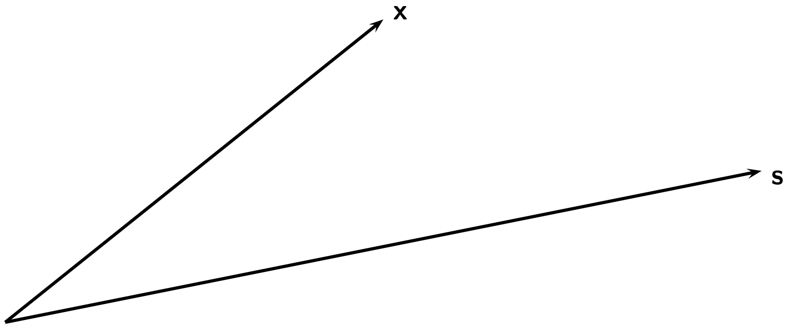

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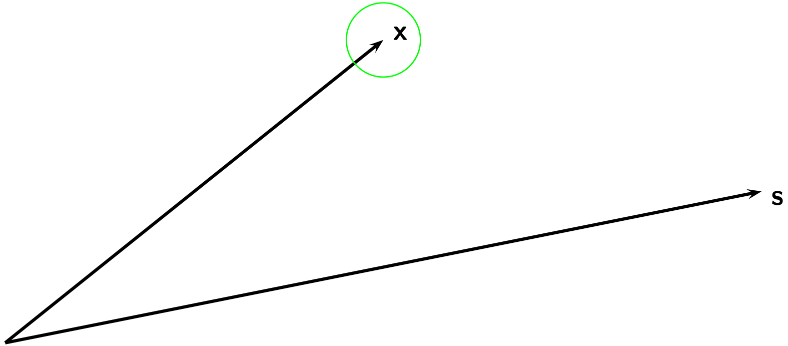

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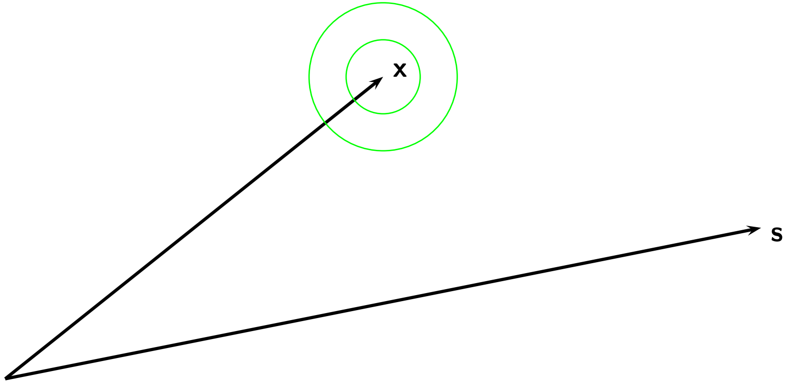

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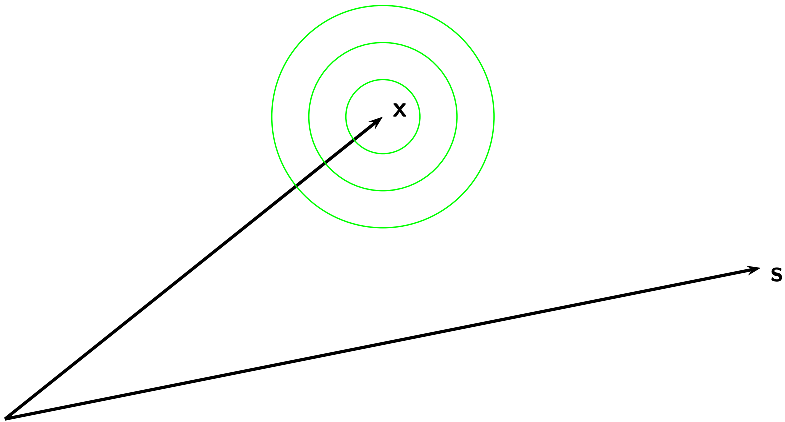

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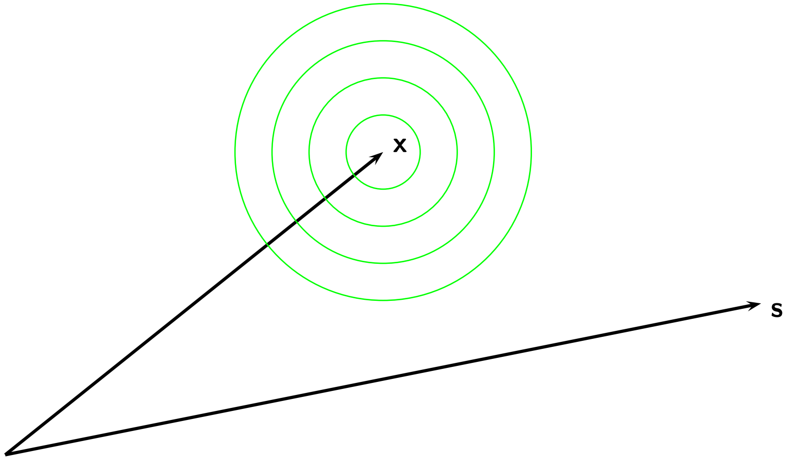

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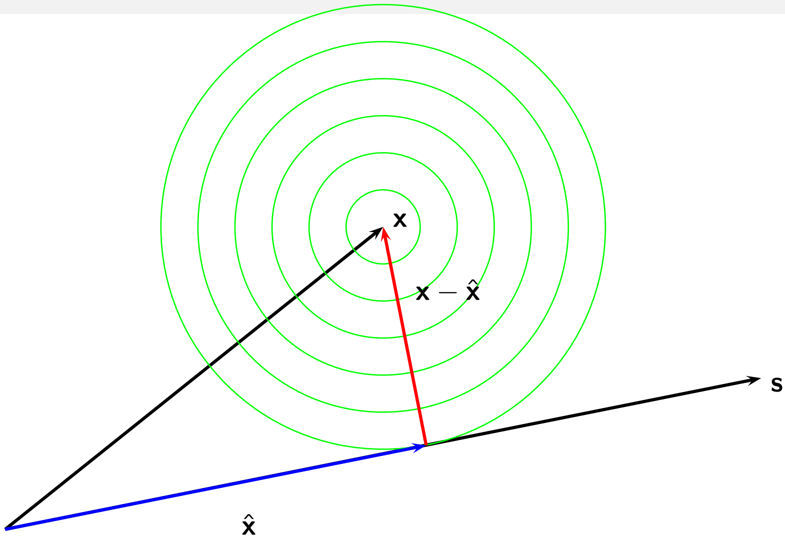

# Example: polynomial approximation

- vector space $P_N[-1,1] \subset L_2[-1,1]$
- $\mathbf{p} = a_0 + a_1 t + \ldots + a_N t^N$
- a self-evident, naive basis: $\mathbf{s}^{(k)} = t^k, \quad k = 0, 1, \ldots, N$
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# Building an orthonormal basis

Gram-Schmidt orthonormalization procedure:

$$\{\mathbf{s}^{(k)}\} \xrightarrow{\qquad\qquad} \{\mathbf{u}^{(k)}\}$$

original set orthonormal set

Algorithmic procedure: at each step $k$

1. $\mathbf{p}^{(k)} = \mathbf{s}^{(k)} - \sum_{n=0}^{k-1} \langle \mathbf{u}^{(n)}, \mathbf{s}^{(k)} \rangle \, \mathbf{u}^{(n)}$

2. $\mathbf{u}^{(k)} = \mathbf{p}^{(k)} / \|\mathbf{p}^{(k)}\|$

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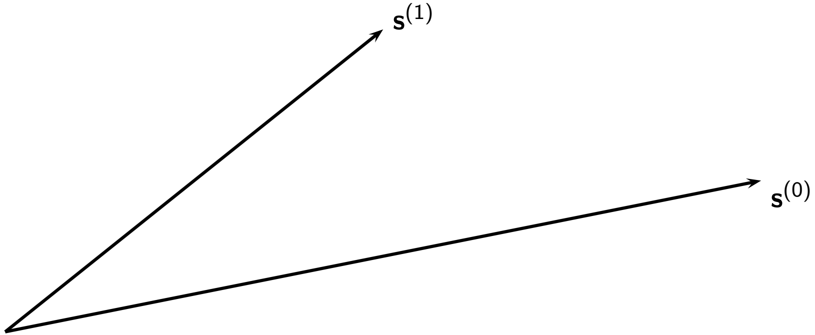

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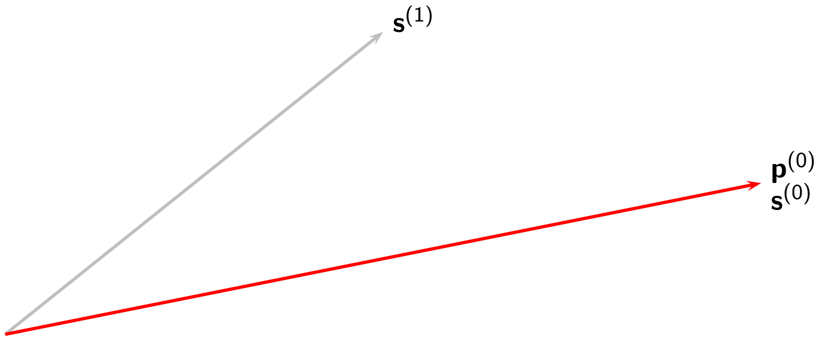

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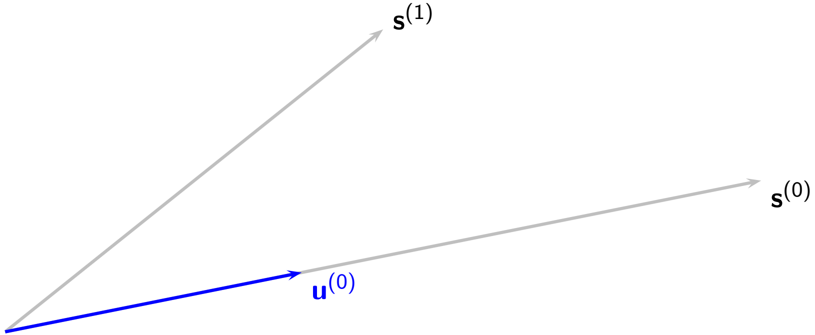

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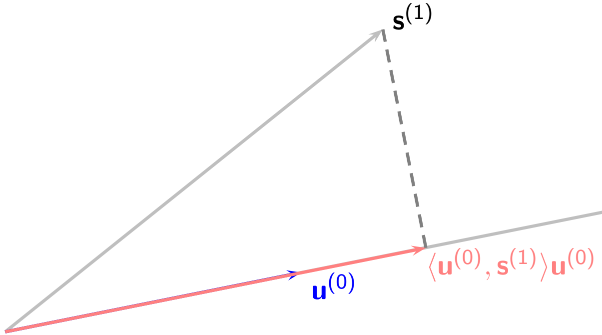

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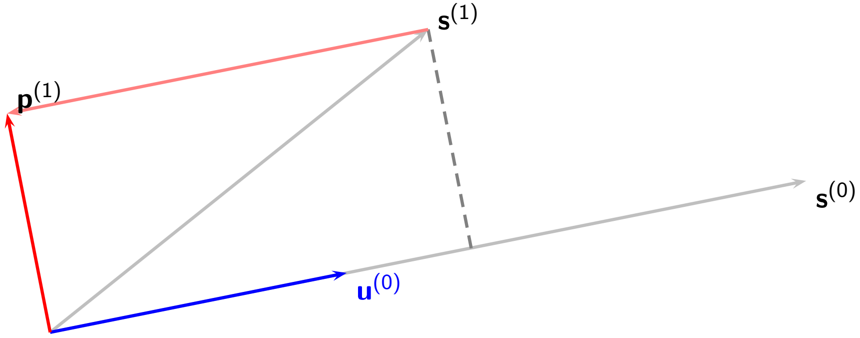

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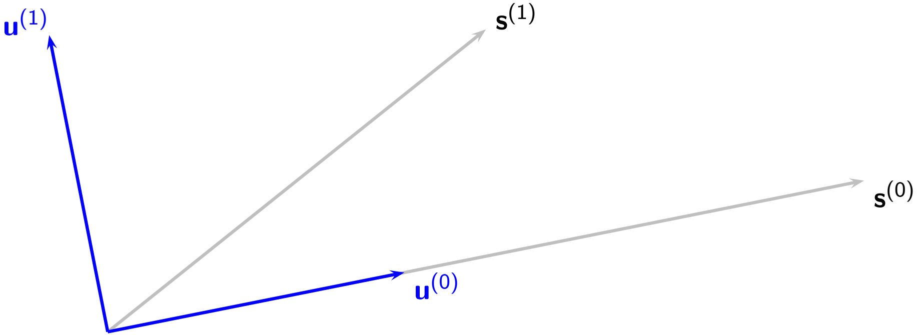

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Gram-Schmidt orthonormalization of the naive basis: $\{\mathbf{s}^{(k)}\} \rightarrow \{\mathbf{u}^{(k)}\}$

- $\mathbf{s}^{(0)} = 1$
  - $\mathbf{p}^{(0)} = \mathbf{s}^{(0)} = 1$
  - $\|\mathbf{p}^{(0)}\|^2 = 2$
  - $\mathbf{u}^{(0)} = \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2}$
- $\mathbf{s}^{(1)} = t$
  - $\langle \mathbf{u}^{(0)}, \mathbf{s}^{(1)} \rangle = \int_{-1}^{1} t/\sqrt{2} = 0$
  - $\mathbf{p}^{(1)} = \mathbf{s}^{(1)} = t$
  - $\|\mathbf{p}^{(1)}\|^2 = 2/3$
  - $\mathbf{u}^{(1)} = \sqrt{3/2}\, t$
- $\mathbf{s}^{(2)} = t^2$
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  - $\mathbf{p}^{(2)} = \mathbf{s}^{(2)} - (2/3\sqrt{2})\mathbf{u}^{(0)} = t^2 - 1/3$
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# Legendre polynomials

The Gram-Schmidt algorithm leads to an orthonormal basis for $P_N([-1,1])$

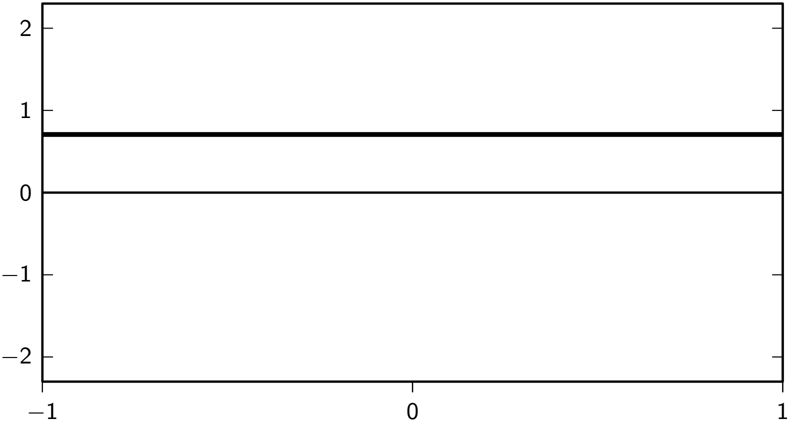
$$\mathbf{u}^{(0)} = \sqrt{1/2}$$

$$\mathbf{u}^{(1)} = \sqrt{3/2}\,t$$

$$\mathbf{u}^{(2)} = \sqrt{5/8}(3t^2 - 1)$$

$$\mathbf{u}^{(3)} = \ldots$$

# Legendre Polynomials



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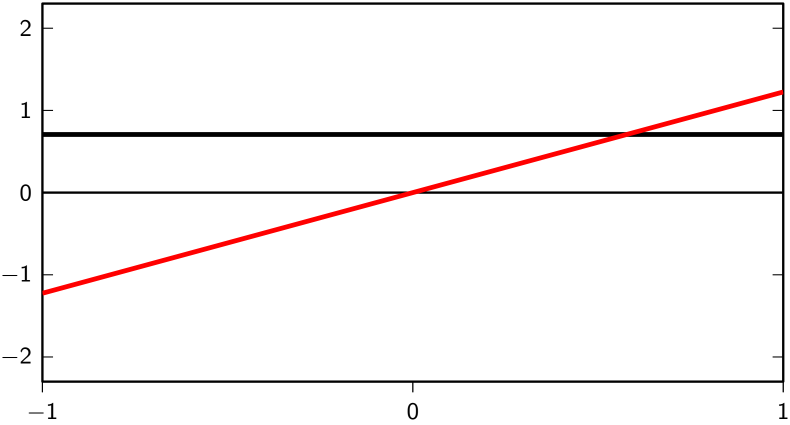

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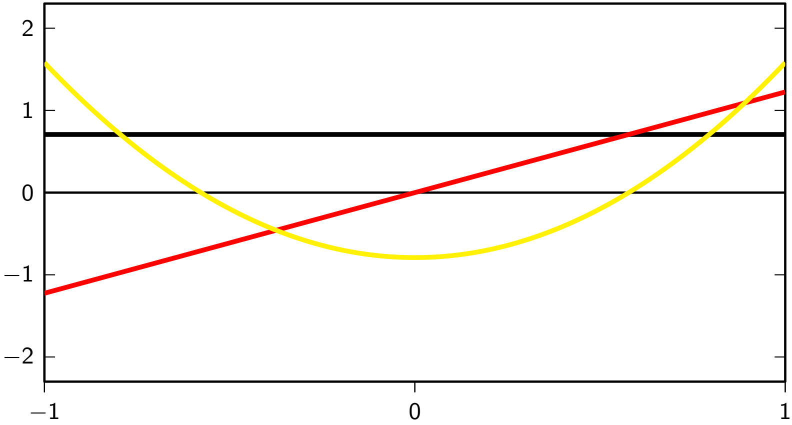

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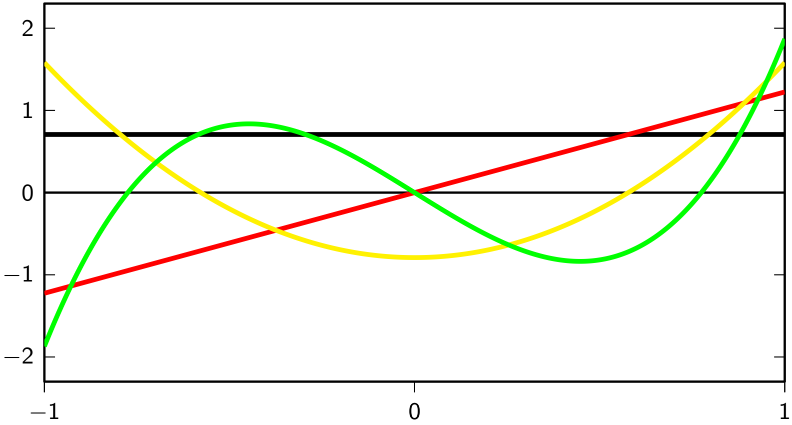

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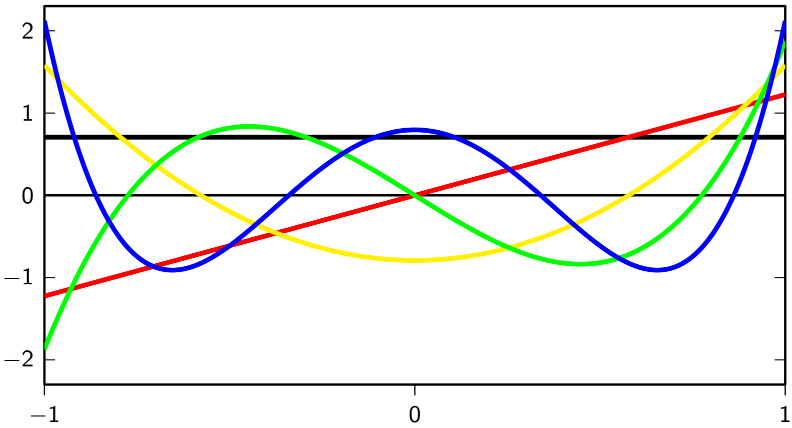

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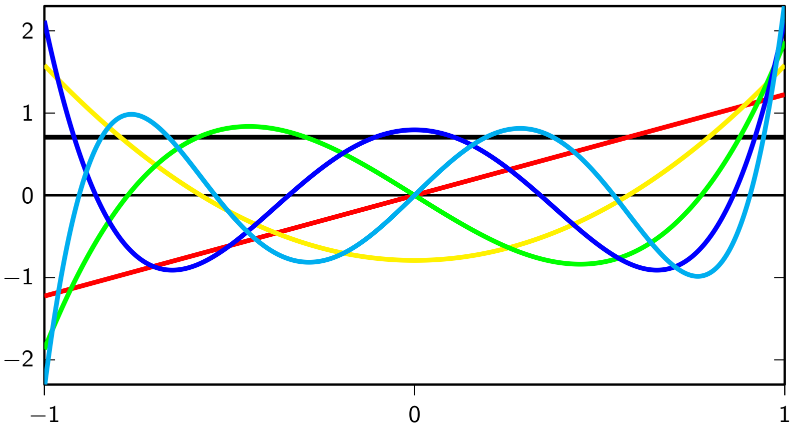

# Orthogonal projection over $P_2[-1, 1]$

$$\alpha_k = \langle \mathbf{u}^{(k)}, \mathbf{x} \rangle = \int_{-1}^{1} u_k(t) \sin t \, dt$$

- $\alpha_0 = \langle \sqrt{1/2}, \sin t \rangle = 0$
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Using the orthogonal projection over $P_2[-1,1]$:

$$\sin t \to \alpha_1 \mathbf{u}^{(1)} \approx 0.9035\, t$$

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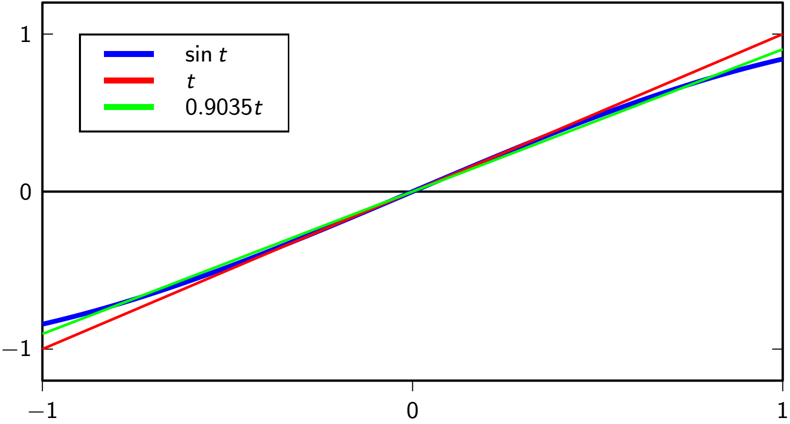
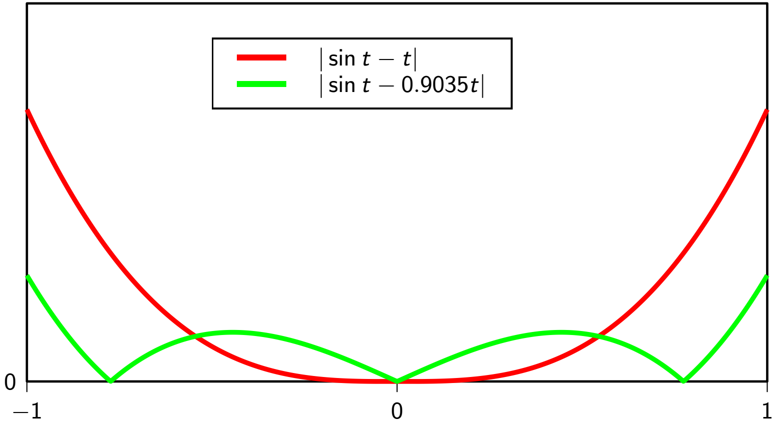


# Approximation error



# Error norm

Orthogonal projection over $P_2[-1,1]$:

$$\|\sin t - \alpha_1\, \mathbf{u}^{(1)}\| \approx 0.0337$$

Taylor series:

$$\|\sin t - t\| \approx 0.0857$$

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# Hilbert space

# Hilbert Space – the ingredients:

1. a vector space: $H(V, \mathbb{C})$
2. an inner product: $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$
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# Completeness

limiting operations must yield vector space elements

Example of an *incomplete* space: the set of rational numbers

$$x_n = \sum_{k=0}^{n} \frac{1}{k!} \in \mathbb{Q} \qquad \text{but} \qquad \lim_{n\to\infty} x_n = e \notin \mathbb{Q}$$

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# Signals in Hilbert Space

Why did we do all this?

- finite-length and periodic signals live in $\mathbb{C}^N$
- infinite-length signals live in $\ell_2(\mathbb{Z})$
- different bases are different observation tools for signals
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