STUDENT-NAME

SCIPER

COM-303 - Signal Processing for Communications
Final Exam

Tuesday 21.06.2018, from 16h15 to 19h15

rooms **CM1120** for last names beginning with the letter A to F inclusive, **C01** otherwise

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Verify that this exam has YOUR last name on top

DO NOT OPEN THE EXAM UNTIL INSTRUCTED TO DO SO

- **Write your name** on the top left corner of **ALL the sheets you turn in**.
- There are 5 problems for a total of 100 points; the number of points is indicated for each problem.
- Please **write your derivations clearly!**
- You can have two A4 sheets of *handwritten* notes (front and back). Please **no photocopies, no books and no electronic devices**. Turn off your phone and store it in your bag.
- **When you are done, simply leave your solution at your place with this page on top and exit the classroom**. Do NOT bring the exam to the main desk.

Exercise 1. (15 points)

Consider the finite-support sequence

$$x[n] = \begin{cases} 1/6 & \text{for } 0 \le n < 6 \\ 0 & \text{otherwise} \end{cases}$$

Next, consider the family of complex-valued finite-support sequences

$$x_k[n] = x[n]\, e^{-j\omega_k n}$$

where $\omega_k = (2\pi/6)k$.

(a) sketch $|X(e^{j\omega})|$, the magnitude of the DTFT of $x[n]$; be as precise as possible

(b) sketch $|X_k(e^{j\omega})|$, the magnitude of the DTFT of $x_k[n]$, for $k = 1$ and $k = 4$

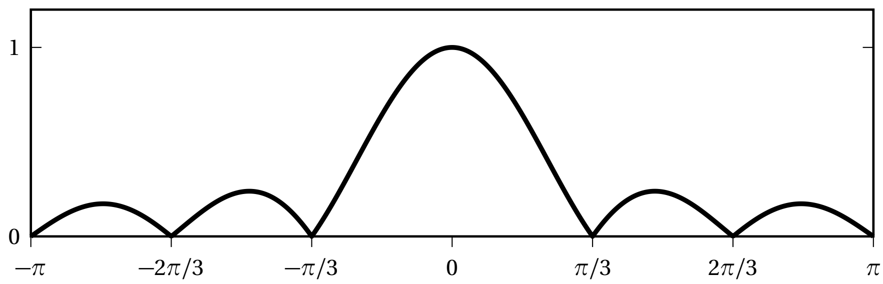
(c) prove that $\displaystyle\sum_{k=0}^{5} X_k(e^{j\omega}) = 1$

Solution:

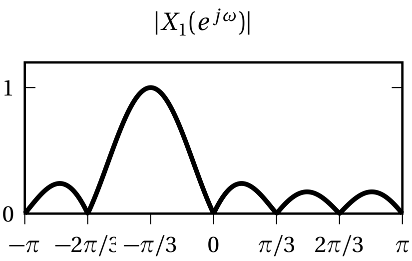
(a) the sequence corresponds to the impulse response of a moving average filter of length six; the magnitude response is

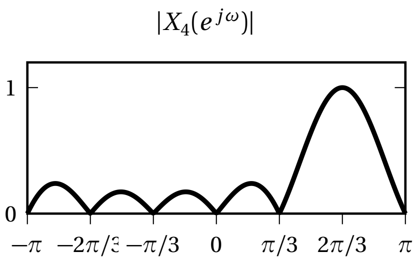
$$|X(e^{j\omega})| = \left|\frac{1}{6}\,\frac{\sin(3\omega)}{\sin(\omega/2)}\right|$$

so it will be equal to zero for $\omega = \pm\pi/3, \pm 2\pi/3, \pm\pi$ and equal to 1 (by continuity) for $\omega = 0$:



(b) multiplication by $e^{-j\omega_k n}$ in time corresponds to a left shift by $\omega_k = k(\pi/3)$ in frequency. Because of the 2π-periodicity of the spectrum, the shift appears as a circular shift over the $[-\pi, \pi]$ range:

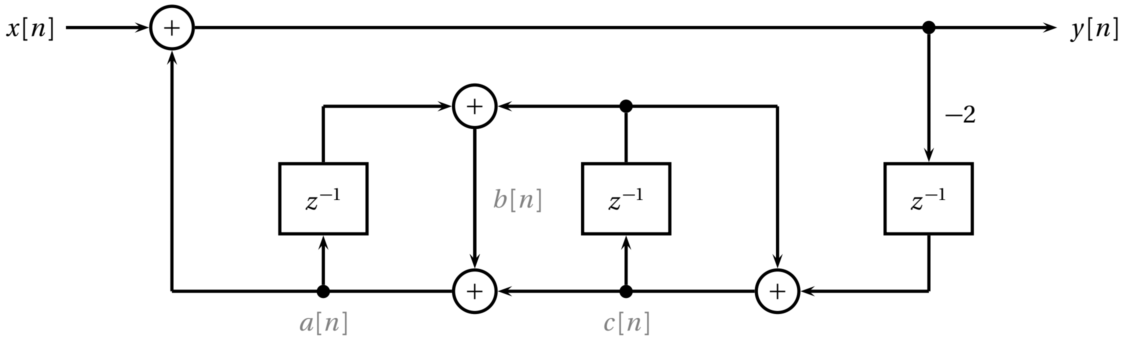




(c)

$$\begin{aligned}
\sum_{k=0}^{5} X_k(e^{j\omega}) &= \sum_{k=0}^{5} \mathrm{DTFT}\{x_k[n]\} \\
&= \mathrm{DTFT}\left\{\sum_{k=0}^{5} x_k[n]\right\} \qquad \textit{(by linearity)} \\
&= \mathrm{DTFT}\left\{\frac{1}{6}\sum_{k=0}^{5} e^{-j\frac{2\pi}{6}nk}\right\} \\
&= \mathrm{DTFT}\left\{\frac{1}{6}\mathrm{DFT}\{1\}\right\} \qquad \textit{(DFT in } \mathbb{C}^6\textit{)} \\
&= \mathrm{DTFT}\{\delta[n]\} = 1
\end{aligned}$$

Exercise 2. (25 points)
Consider the causal system described by the following block diagram:



Compute its transfer function $H(z) = Y(z)/X(z)$.

Solution: *Consider the intermediate signals $a[n]$, $b[n]$, $c[n]$ as in the above figure. In the z-domain we have*

$$\begin{aligned}
Y(z) &= X(z) + A(z) \\
A(z) &= B(z) + C(z) \\
B(z) &= z^{-1}A(z) + z^{-1}C(z) \\
C(z) &= z^{-1}C(z) - 2z^{-1}Y(z)
\end{aligned}$$

Using the third equation with the second

$$A(z) = z^{-1}A(z) + z^{-1}C(z) + C(z) \quad \Rightarrow \quad A(z) = \frac{1+z^{-1}}{1-z^{-1}}C(z)$$

while the fourth equation gives

$$C(z) = \frac{-2z^{-1}}{1-z^{-1}}Y(z)$$

Replacing these results in the first equation:

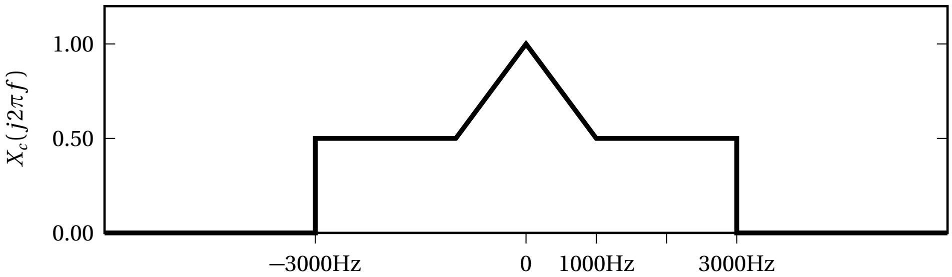
$$Y(z) = X(z) - 2z^{-1}\frac{1+z^{-1}}{(1-z^{-1})^2}Y(z)$$

$$\left[1+2z^{-1}\frac{1+z^{-1}}{(1-z^{-1})^2}\right]Y(z)=\left[\frac{1-2z^{-1}+z^{-2}+2z^{-1}+2z^{-2}}{(1-z^{-1})^2}\right]Y(z)=X(z)$$

so that finally

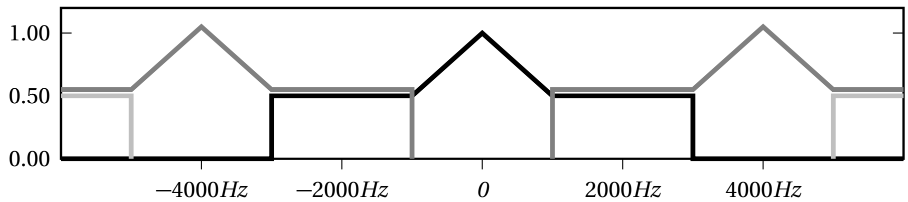
$$H(z)=\frac{(1-z^{-1})^2}{1+3z^{-2}}$$

Exercise 3. (25 points)
The following figure shows the real-valued, symmetric spectrum $X_c(j\Omega)$ of the continuous-time signal $x_c(t)$; the signal is bandlimited to 3000Hz and the spectrum is plotted as a function of the frequency in Hertz $f = \Omega/2\pi$:

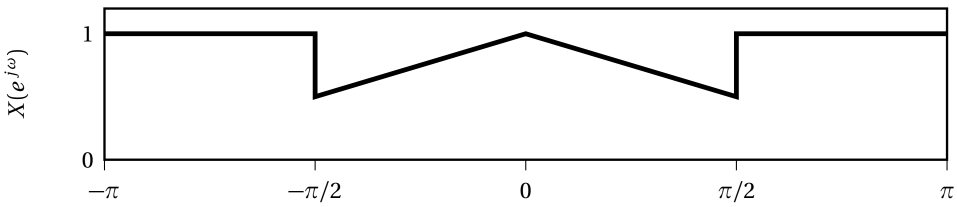


The signal $x_c(t)$ is now sampled at a frequency $F_s = 1/T_s = 4000$Hz. Write the expression for the discrete-time sequence $x[n] = x_c(nT_s)$.

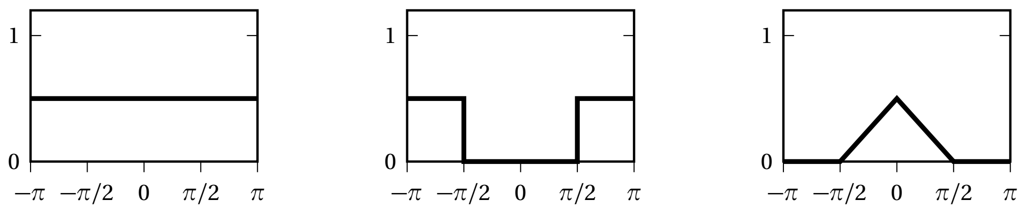
Solution: *The signal is bandlimited to* 3000*Hz but* $F_s = 4000$*Hz, which is less than twice the maximum positive frequency, so we will have aliasing. The aliased spectrum will consist of the sum of copies of the original spectrum placed at all multiples of* F_s *as in the following figure (spectral replicas are shown a bit shifted in amplitude for clarity):*



The spectrum of the discrete-time sampled sequence will be the sum of the replicas between -2000 and 2000Hz, rescaled to $[-\pi, \pi]$*:*



We can now decompose the spectrum as $X(e^{j\omega}) = A(e^{j\omega}) + B(e^{j\omega}) + C(e^{j\omega})$ *where the three components are as in the following pictures:*



$A(e^{j\omega})$ $\quad$ $B(e^{j\omega})$ $\quad$ $C(e^{j\omega})$

By simple inspection we can observe that

$$A(e^{j\omega}) = 1/2$$
$$B(e^{j\omega}) = (1/2)(1 - \text{rect}(\omega/\pi))$$

$C(e^{j\omega})$ *is a triangular shape with support* $[-\pi/2, \pi/2]$*; this can be obtained by convolving in frequency two rectangular shapes with half the support, i.e.*

$$C(e^{j\omega}) = c\,\text{rect}(\omega/(\pi/2)) * \text{rect}(\omega/(\pi/2))$$

where c *is a normalizing constant so that* $C(e^{j\omega}) = 1/2$ *for* $\omega = 0$*; the value of the convolution in zero is*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \text{rect}^2(\sigma/(\pi/2))d\sigma = \frac{1}{4}$$

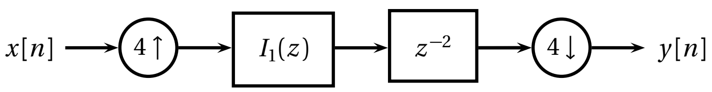
so that $c = 2$.
From the relationship $\text{IDTFT}\{\text{rect}(\omega/\omega_b)\} = (\omega_b/(2\pi))\text{sinc}((\omega_b/(2\pi))n)$ *we have*

$$b[n] = (1/2)[\delta[n] - (1/2)\text{sinc}(n/2)]$$
$$c[n] = 2[(1/4)\text{sinc}(n/4)]^2$$

so that finally:

$$x[n] = \delta[n] - (1/4)\text{sinc}(n/2) + (1/8)\text{sinc}^2(n/4)$$

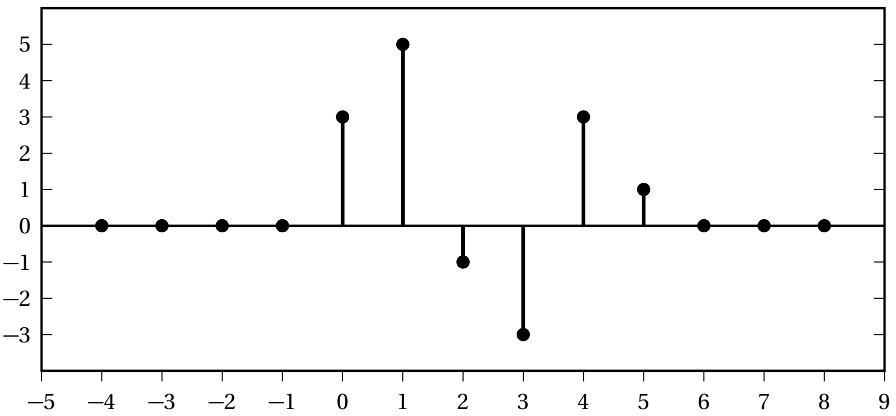
Exercise 4. (20 points)
Consider the following multirate processing system:



where $I_1(z)$ is the first-order discrete-time interpolator with impulse response

$$i_1[n] = \begin{cases} 1 - |n|/4 & \text{for } |n| < 4 \\ 0 & \text{otherwise.} \end{cases}$$

Assume $x[n]$ is the finite-support signal shown here:



Compute the values of $y[n]$ for $0 \le n \le 6$, showing your calculation method.

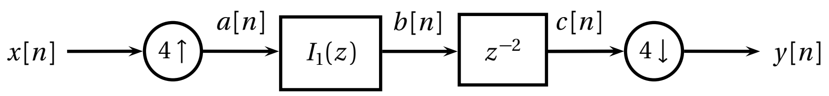
Solution: *Intuitively, the upsampling by four followed by $I_1(z)$ creates a linear interpolation over three "extra" samples between the original values the ("connect-the-dots" strategy). The delay by two followed by the down-sampler selects the midpoint of each interpolation interval. As a whole, the chain implements a fractional delay of half a sample using a linear interpolator so that*

$$y[n] = \frac{x[n] + x[n-1]}{2}$$

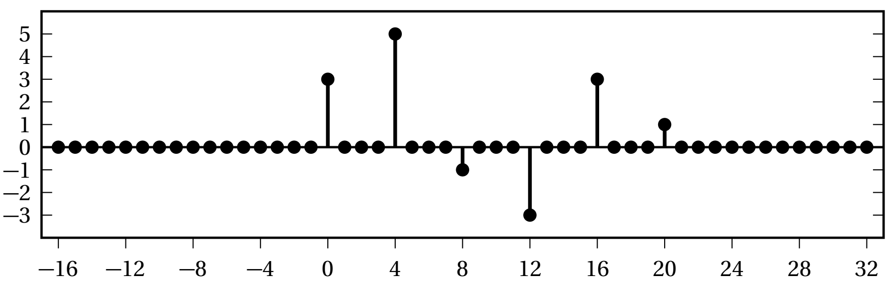
The required values are therefore:

$$y[0] = 1.5, \quad y[1] = 4, \quad y[2] = 2, \quad y[3] = -2, \quad y[4] = 0, \quad y[5] = 2, \quad y[6] = 0.5$$

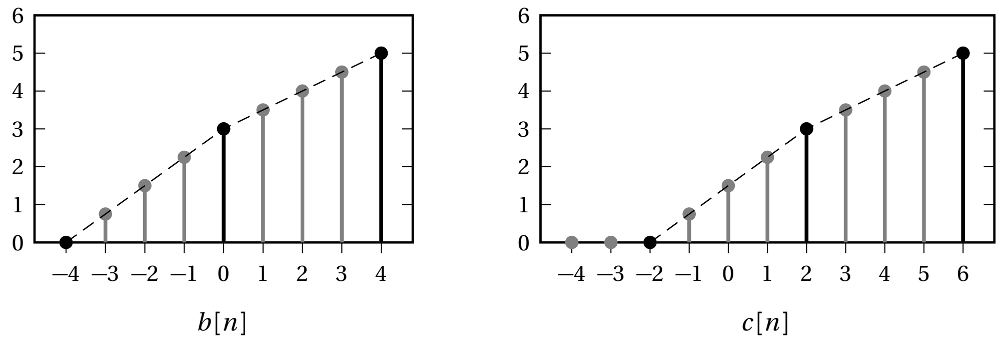
For a proof, label the intermediate signals in the processing chain like so:



We can either proceed graphically or analytically. Graphically, which is the easiest way, we can start by plotting $a[n]$:



Since the interpolator $I(z)$ has finite support of length 7, we can concentrate on the interval $[-4, 4]$ and extend the result to the other points. Linear interpolation fills in the gaps while the delay shifts the interpolated signal by two towards the right:



The downsampler selects the points in $c[n]$ where n is a multiple of four, which are the midpoints between original data values:

$$y[0] = c[0] = b[-2] = (x[0] + x[-1])/2 = 1.5$$
$$y[1] = c[4] = b[2] = (x[1] + x[0])/2 = 4$$
$$\ldots$$

Alternatively, we can proceed analytically as follows. The z-transform of $c[n]$ is

$$C(z) = X(z^4)z^{-2}I(z)$$

and, after the downsampler, we have

$$\begin{aligned}
Y(z) &= \frac{1}{4}\sum_{m=0}^{3} C\left(e^{-j\frac{2\pi}{4}m}\, z^{\frac{1}{4}}\right) \\
&= \frac{1}{4}\sum_{m=0}^{3} X(z)\, e^{-j\frac{2\pi}{4}2m} z^{-\frac{1}{2}}\, I\left(e^{-j\frac{\pi}{2}m}\, z^{\frac{1}{4}}\right) \\
&= X(z)\frac{1}{4}z^{-\frac{1}{2}}\left[I\left(z^{\frac{1}{4}}\right) - I\left(-jz^{\frac{1}{4}}\right) + I\left(-z^{\frac{1}{4}}\right) - I\left(jz^{\frac{1}{4}}\right)\right]
\end{aligned}$$

The transfer function of the interpolator is

$$I(z) = 1 + (1/4)(z + z^{-1}) + (1/2)(z^2 + z^{-2}) + (3/4)(z^3 + z^{-3})$$

and therefore

$$\begin{aligned}
I\left(z^{\frac{1}{4}}\right) &= 1 + (1/4)(z^{1/4} + z^{-1/4}) + (1/2)(z^{1/2} + z^{-1/2}) + (3/4)(z^{3/4} + z^{-3/4}) \\
I\left(-z^{\frac{1}{4}}\right) &= 1 - (1/4)(z^{1/4} + z^{-1/4}) + (1/2)(z^{1/2} + z^{-1/2}) - (3/4)(z^{3/4} + z^{-3/4}) \\
I\left(-jz^{\frac{1}{4}}\right) &= 1 + (j/4)(z^{1/4} + z^{-1/4}) - (1/2)(z^{1/2} + z^{-1/2}) - (3j/4)(z^{3/4} + z^{-3/4}) \\
I\left(jz^{\frac{1}{4}}\right) &= 1 - (j/4)(z^{1/4} + z^{-1/4}) - (1/2)(z^{1/2} + z^{-1/2}) + (3j/4)(z^{3/4} + z^{-3/4})
\end{aligned}$$

Finally,

$$I\left(z^{\frac{1}{4}}\right) + I\left(-z^{\frac{1}{4}}\right) - I\left(-jz^{\frac{1}{4}}\right) - I\left(jz^{\frac{1}{4}}\right) = 2(z^{1/2} + z^{-1/2})$$

so that

$$Y(z) = X(z)\frac{1}{4}z^{-\frac{1}{2}}\left[2(z^{1/2} + z^{-1/2})\right] = \frac{1 + z^{-1}}{2}X(z).$$

Exercise 5. (15 points)

In this exercise we will study a data transmission scheme known as *phase modulation* (PM). Consider a discrete-time signal $x[n]$, with the following properties:

- $|x[n]| < 1$ for all n
- $X(e^{j\omega}) = 0$ for $|\omega| < \alpha$, with α small.

A PM transmitter with carrier frequency ω_c works by producing the signal

$$y[n] = \mathscr{P}_{\omega_c}\{x[n]\} = \cos(\omega_c n + k\,x[n])$$

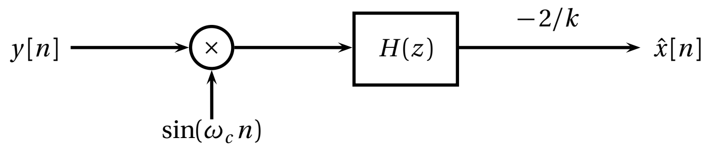
where k is a small positive constant; in other words, the data signal $x[n]$ is used to modify the instantaneous *phase* of a sinusoidal carrier. The advantage of this modulation technique is that it builds a signal with constant envelope (namely, a sinusoid with fixed amplitude) which results in a greater immunity to noise; this is the same principle behind the better quality of FM radio versus AM radio. However phase modulation is less "user friendly" than standard amplitude modulation because it is nonlinear.

(a) Show that phase modulation is *not* a linear operation.

Because of nonlinearity, the spectrum of the signal produced by a PM transmitter cannot be expressed in simple mathematical form. For the purpose of this exercise you can simply assume that the PM signal occupies the frequency band $[\omega_c - \gamma, \omega_c + \gamma]$ (and, obviously, the symmetric interval $[-\omega_c - \gamma, -\omega_c + \gamma]$) with

$$\gamma \approx 2(k+1)\alpha.$$

To demodulate a PM signal the following scheme is proposed, in which $H(z)$ is a lowpass filter with cutoff frequency equal to α:



(b) Show that $\hat{x}[n] \approx x[n]$. Assume that $\omega_c \gg \alpha$ and that k is small, say $k = 0.2$. (You may find it useful to express trigonometric functions in terms of complex exponentials if you don't recall the classic trigonometric identities. Also, remember that $\sin x \approx x$ for x sufficiently small).

Solution:

(a) Given a signal $x[n]$ fulfilling the magnitude and bandwidth requirements, if PM was a linear operation, for any scalar $\beta \in \mathbb{R}$ we should have

$$\mathscr{P}_{\omega_c}\{\beta\,x[n]\} = \beta\,\mathscr{P}_{\omega_c}\{x[n]\}.$$

However, irrespective of $x[n]$, $|\mathscr{P}_{\omega_c}\{\cdot\}| \le 1$. Since we can always pick a value for β so that the right-hand side of the equality takes values larger than one, the equality cannot hold in general.

(b) Nonlinear operators make it impossible to proceed analytically in the frequency domain. In the time domain, however, the signal after the multiplier is

$$\begin{aligned} d[n] &= y[n]\sin(\omega_c n) \\ &= \cos(\omega_c n + k\,x[n])\sin(\omega_c n) \\ &= (1/2)\sin(\omega_c n + k\,x[n] + \omega_c n) - (1/2)\sin(\omega_c n + k\,x[n] - \omega_c n) \\ &= (1/2)\sin(2\omega_c n + k\,x[n]) - (1/2)\sin(k\,x[n]) \\ &\approx (1/2)\sin(2\omega_c n + k\,x[n]) - (k/2)x[n] \end{aligned}$$

where we have used the small-angle approximation for the sine since $|k\,x[n]| < 0.2$. The signal $d[n]$ now contain a baseband component and a PM component at twice the carrier frequency, which is eliminated by the lowpass filter:

$$\hat{x}[n] = (-2/k)h[n] * d[n] \approx x[n].$$

[Note: we used the trigonometric identity $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$. This can be easily derived by developing the product $(e^{j\alpha} + e^{-j\alpha})(e^{j\beta} - e^{-j\beta})/(2j)$.]