COM-303 - Signal Processing for Communications
Final Exam

Tuesday 20.06.2017, from 08h15 to 11h15, room CO1

Verify that this exam has YOUR last name on top

DO NOT OPEN THE EXAM UNTIL INSTRUCTED TO DO SO

- **Write your name** on the top left corner of **ALL the sheets you turn in**.
- There are 5 problems for a total of 100 points; the number of points is indicated for each problem.
- Please **write your derivations clearly!**
- You can have two A4 sheets of *handwritten* notes (front and back). Please **no photocopies, no books and no electronic devices**. Turn off your phone and store it in your bag.
- **When you are done, simply leave your solution at your place with this page on top and exit the classroom**. Don't bring the exam to the main desk.

Exercise 1. (10 points)

Assyme that your numerical package of choice (Python, Matlab, C++, etc.) implements a function `dfct(x, M, k)`, with `k`, `M` integers and `x` an array of complex values; the function computes the `k`-th DFT coefficient of the data vector `x` using an `M`-point DFT (with zero-padding if `M` > `len(x)` or with truncation otherwise):

$$\texttt{dfct(x, M, k)} = \sum_{n=0}^{\min\{M-1, \texttt{len(x)}\}} \texttt{x[n]}\, e^{-j\frac{2\pi}{M}kn}$$

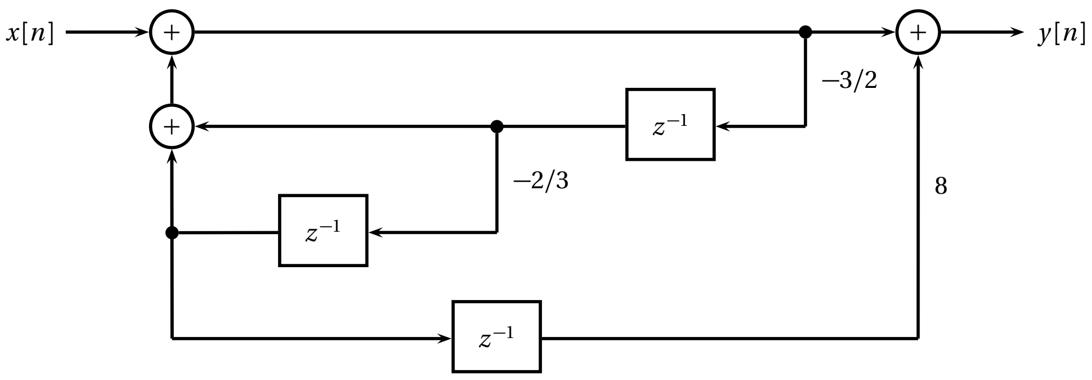
Consider now a finite-support discrete-time signal $x[n]$ that is nonzero only for $0 \le n < N$. Given a frequency value $\omega_0 = 2\pi(A/B)$, with $A, B \in \mathbb{N}$, show how you can compute the DTFT value $X(e^{j\omega_0})$ using the `dfct()` function.

Solution: *Choose an integer p so that $pB = M \ge N$; using $\omega_0 = 2\pi(pA/pB)$:*

$$\begin{aligned}
X(e^{j\omega_0}) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega_0 n} \\
&= \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{M}pAn} \\
&= \sum_{n=0}^{M-1} x[n]\, e^{-j\frac{2\pi}{M}pAn} \qquad \textit{since } x[n] = 0 \textit{ for } n \ge N \\
&= \texttt{dfct(}[x[0], x[1], \ldots, x[N-1]]\texttt{,}\ pB\texttt{,}\ pA\texttt{)}
\end{aligned}$$

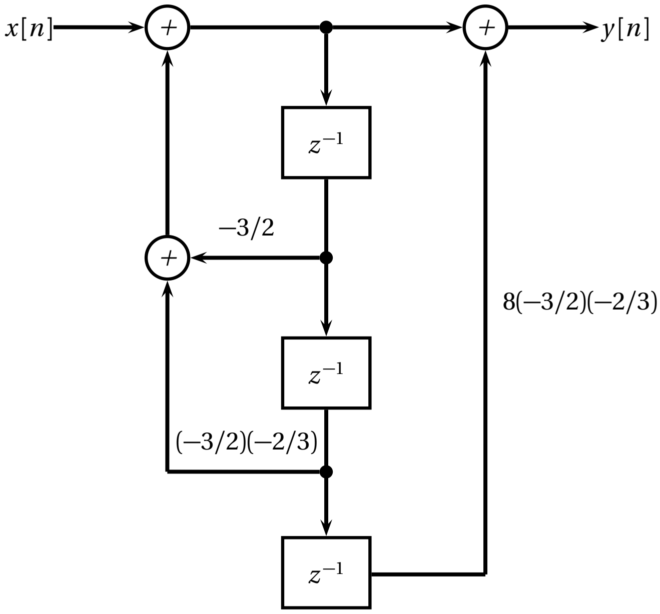
Exercise 2. (25 points)

Consider the causal system described by the following block diagram:



Compute its transfer function $H(z)$ and determine if the system is stable.

Solution: *The easiest way to solve the problem is to re-arrange the block diagram in standard form:*



From this, the transfer function is derived immediately as

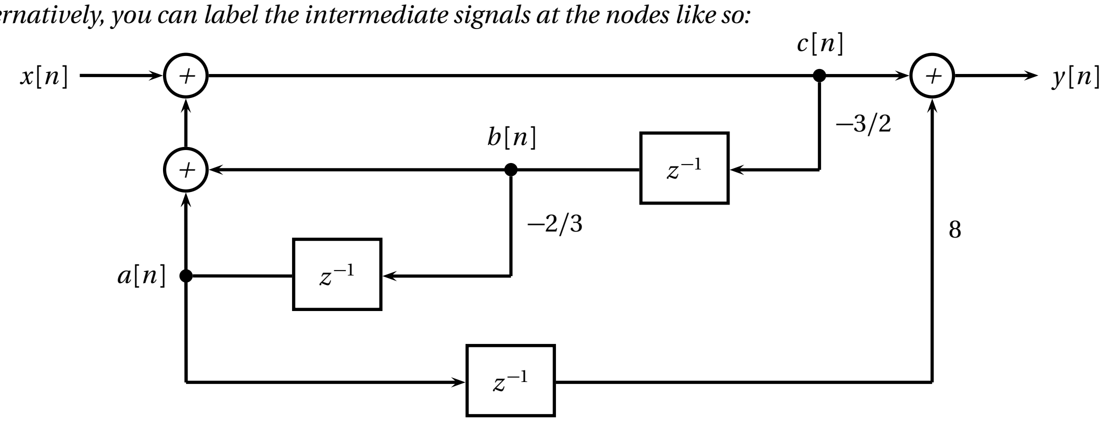
$$H(z) = \frac{1+8z^{-3}}{1+(3/2)z^{-1}-z^{-2}}$$

By finding the roots of numerator and denominator we can write

$$H(z) = \frac{(1+2z^{-1})(1-2z^{-1}+4z^{-2})}{(1+2z^{-1})(1-(1/2)z^{-1})} = \frac{1-2z^{-1}+4z^{-2}}{1-(1/2)z^{-1}}$$

so that the system is stable since it only has one pole and the pole is inside the unit circle.

Alternatively, you can label the intermediate signals at the nodes like so:

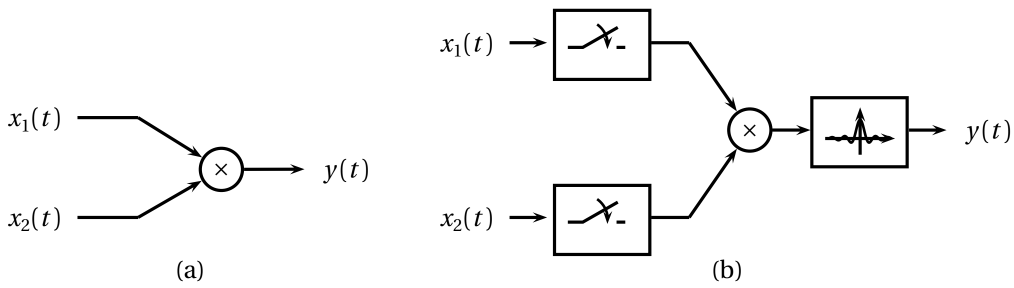


From this, using the z-transform, we can write

$$\begin{aligned}
A(z) &= -(2/3)z^{-1}B(z)\\
B(z) &= -(3/2)z^{-1}C(z)\\
C(z) &= X(z)+A(z)+B(z)\\
Y(z) &= C(z)+8z^{-1}A(z)
\end{aligned}$$

and, by substituting back, we obtain $H(z) = Y(z)/X(z)$.

Exercise 3. (15 points)
Consider a continuous-time system that computes the product of two input signals, as in figure (a) below. You decide to build a discrete-time version of the system as in figure (b), where ideal samplers and interpolator all work at the same frequency F_s.



If the signals that are going to be processed by this system are bandlimited to $F_{\max} = 8000$Hz, determine the sampling frequency F_s so that the discrete-time system produces the same output as the continuous-time version. Explain in detail your choice. If in doubt, try the discrete-time system on the signals $x_1(t) = x_2(t) = \mathrm{sinc}(2F_{\max}t)$.

Solution: *Since the input signals are bandlimited to* 8000*Hz, it is tempting to think that the samplers and interpolator can work at* $F_s = 2F_{\max}$. *However, if we try with* $x_1(t) = x_2(t) = \mathrm{sinc}(2F_{\max}t)$ *and a sampling frequency* $F_s = 2F_{\max}$ *we have*

$$
\begin{aligned}
x_1[n] &= \mathrm{sinc}(n) = \delta[n] \\
x_2[n] &= \mathrm{sinc}(n) = \delta[n] \\
x_1[n]x_2[n] &= \delta^2[n] = \delta[n] \\
y(t) &= \sum_n x_1[n]x_2[n]\mathrm{sinc}\left(\frac{t-nT_s}{T_s}\right) = \mathrm{sinc}(2F_{\max}t) \neq \mathrm{sinc}^2(2F_{\max}t)
\end{aligned}
$$

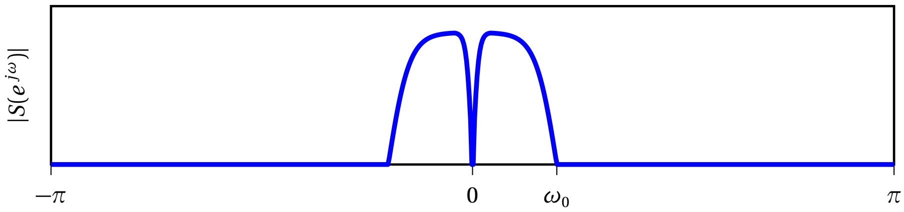
The reason is that the product of two signals usually has a wider bandwidth that the signals themselves. In fact, the dual of the convolution theorem (in continuous time) states that the Fourier transform of a product is the convolution of the Fourier transforms:

$$
FT\{x_1(t)x_2(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\Theta)X_2(j(\Omega-\Theta))d\Omega
$$

In general, therefore, the bandwidth of the product will be the sum of the bandwidths of the factors and so, for input signals limited to 8000*Hz, we will need to use a sampling frequency of at least* $F_s = 32000$*Hz.*

Exercise 4. (30 points)
In this exercise we will study two competing radio transmission schemes for audio signals; let's assume that modulation and demodulation take place in the discrete-time domain and let's ignore the D/A and A/D conversion before and after the antennas.

Consider a real-valued audio signal $s[n]$ whose DTFT, in magnitude, is shown in the following figure:



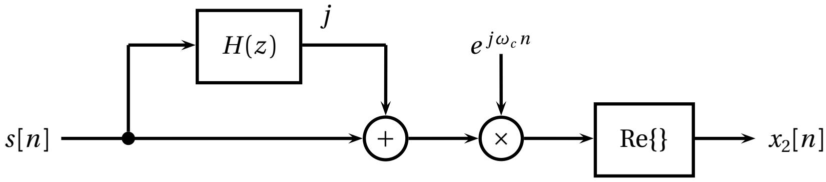
The first transmission scheme is a simple sinusoidal modulator at frequency ω_c that creates the transmitted signal $x_1[n] = s[n]\cos(\omega_c n)$.

(a) determine the minimum and the maximum values for ω_c so that the receiver can recover $s[n]$ exactly

(b) plot the magnitude DTFT of $x_1[n]$ for a value ω_c of your choice

(c) what is the size of the *total* support of $X_1(e^{j\omega})$, i.e., the total size of the frequency intervals over which $X_1(e^{j\omega})$ is nonzero? [For example, the support of $S(e^{j\omega})$ is $2\omega_0$]

Consider now the second transmission scheme shown in the following block diagram, where $H(z)$ is the ideal Hilbert filter:



(d) sketch as accurately as you can the magnitude DTFT of $x_2[n]$ for the same value ω_c you chose in (b)
Hints:
- *derive the frequency response of the initial block,* $1 + jH(e^{j\omega})$
- *to sketch the shape of the final spectrum, remember that* $\text{Re}\{x\} = (x + x^*)/2$

(e) what is the size of the total support of $X_2(e^{j\omega})$?

(f) what is the advantage of this transmission scheme with respect to the previous one?

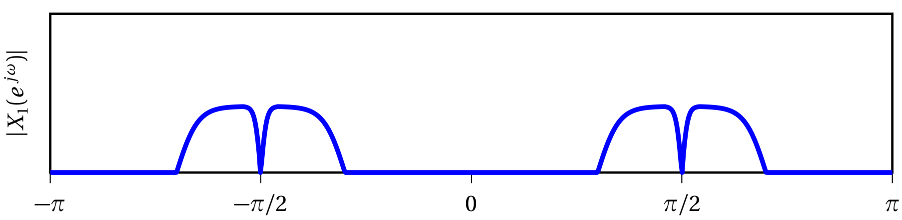
The modulation scheme we just analyzed is called Single Sideband modulation (SSB)

(g) design a block diagram of a demodulator that exactly recovers $s[n]$ from the SSB signal $x_2[n]$

Solution:

(a) In order not to have overlap between the two copies of the baseband spectrum create by the sinusoidal modulation it must be $\omega_0 < \omega_c < \pi - \omega_0$

(b) choosing $\omega_c = \pi/2$,

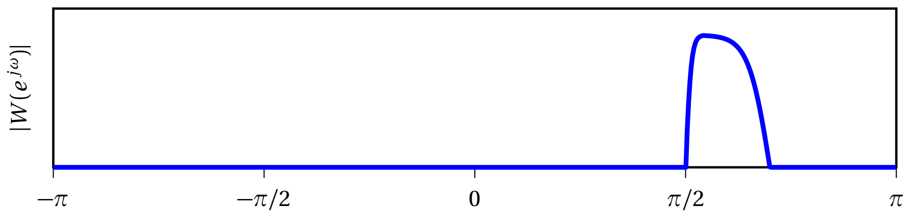


(c) *the total support has size* $4\omega_0$

(d) *remember that the Hilbert filter's frequency response is* $H(e^{j\omega}) = -j\,\mathrm{sign}(\omega)$ *so that*

$$1 + jH(e^{j\omega}) = \begin{cases} 2 & \omega \geq 0 \\ 0 & \omega < 0 \end{cases}$$

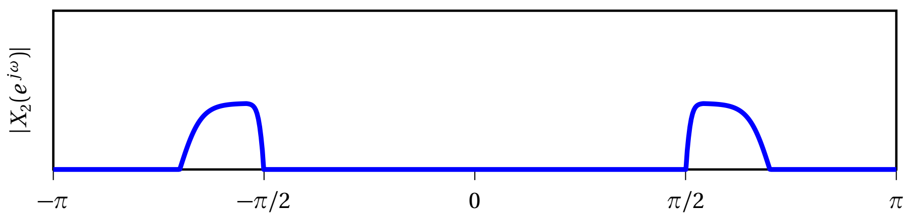
In other words, the first block in the processing chain kills the negative frequencies of the spectrum. Call $w[n]$ *the signal after multiplication by the complex exponential; its spectrum looks like so:*



The transmitted signal is $x_2[n] = \mathrm{Re}\{w[n]\} = (w[n] + w^*[n])/2$ *so that its DTFT is*

$$X_2(e^{j\omega}) = [W(e^{j\omega}) + W^*(e^{-j\omega})]/2$$

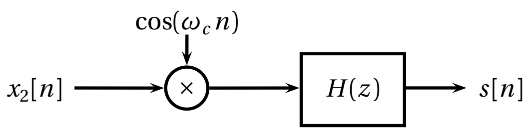
which, in magnitude, looks like so:



(e) *the total support in this case has size* $2\omega_0$*, like the original baseband*

(f) *the advantage of the second modulation scheme is to use half the bandwidth of the first, so that it utilizes the channel more efficiently*

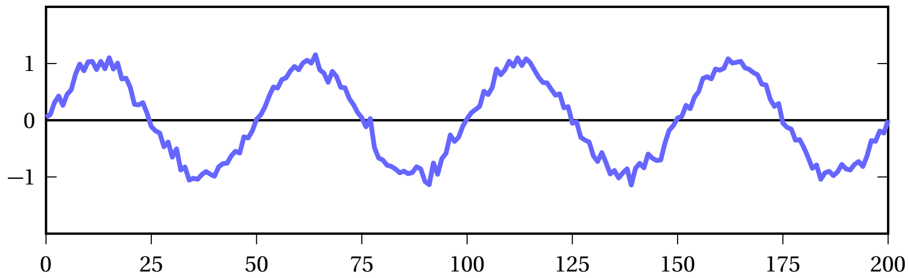
(g) *an SSB signal like the one we just generated can be demodulated using a standard sinusoidal demodulation circuit:*



where $H(z)$ *is a lowpass filter with cutoff frequency* ω_0.

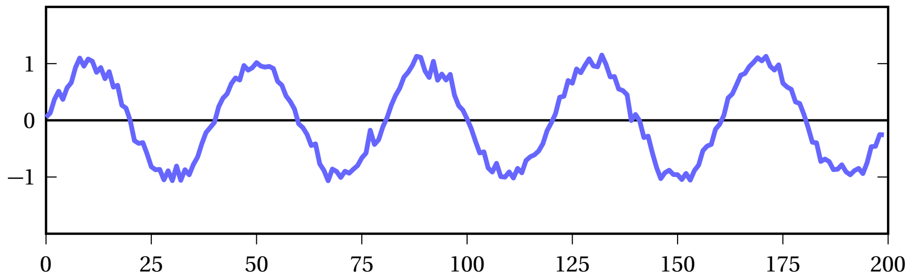
Exercise 5. (20 points)

Consider the signal $x[n] = \sin(\omega_0 n) + s[n]$ where $\omega_0 = 2\pi/50$ and $s[n]$ is a zero-mean, Gaussian white noise sequence with power spectral density $P_s(e^{j\omega}) = \sigma^2 = 10^{-2}$. A realization of the signal is plotted in the following figure:

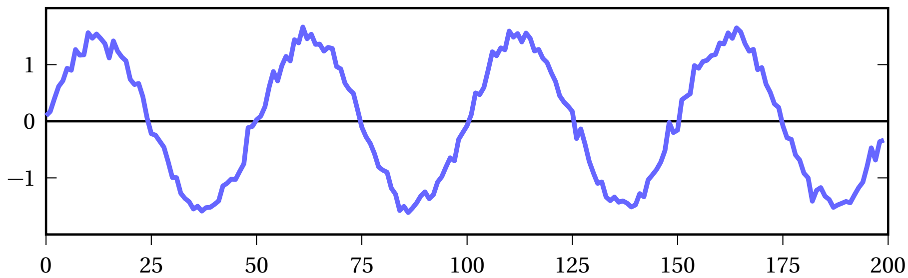


The signal is filtered with a stable, real-valued, causal LTI system whose frequency response satisfies $|H(e^{j\omega})| \leq 1$ for all frequencies. For each of the following plots, explain if the signal in the plot could be the result of filtering $x[n]$ with $H(e^{j\omega})$; explain your answers in detail.

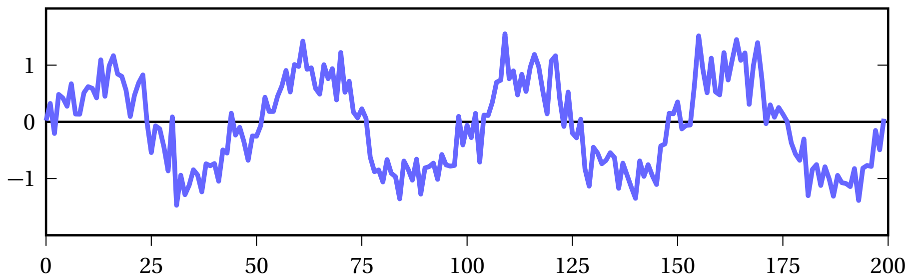
(a)



(b)



(c)



Solution: *The filter will act on the deterministic sinusoidal component and on the noise component indepen-*

dently; therefore the output can be written as

$$\begin{aligned} y[n] &= h[n] * (\sin(\omega_0 n) + s[n]) \\ &= A\sin(\omega_c + \theta) + h[n] * s[n] \\ &= A\sin(\omega_c + \theta) + v[n] \end{aligned}$$

where $A = |H(e^{j\omega_c})|$ *and* $\theta = \angle H(e^{j\omega_c})$.

(a) *NO: the frequency of the sinusoidal component in the output is clearly different than the frequency of the input; since a filter cannot change the frequency of a sinusoidal input this signal cannot be a valid output*

(b) *NO: the amplitude of the sinusoidal component has increased, but this cannot happen since by design* $|H(e^{j\omega})| \leq 1$ *for all* ω

(c) *NO: the variance of the noise seems to have increased in the output. However the variance of the output noise is*

$$\begin{aligned} \sigma_v^2 &= r_v[0] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 P_s(e^{j\omega}) d\omega \\ &= \sigma_s^2 \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 d\omega \\ &\leq \sigma_s^2 \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega \\ &\leq \sigma_s^2 \end{aligned}$$

so the variance of the output noise cannot increase after filtering with the given $H(e^{j\omega})$.