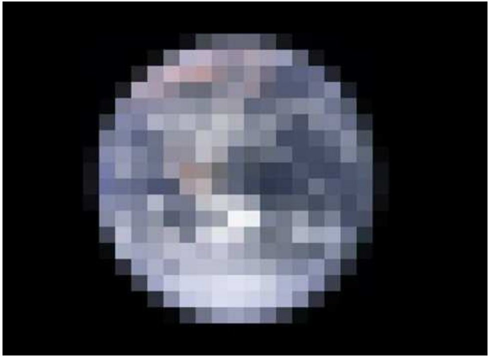
COM303: Digital Signal Processing

Lecture 16: Interpolation

overview

- the analog worldview
- interpolation of discrete-time signals
- bandlimited functions
- the sinc basis and sinc sampling

Two views of the world



Analog/continuous versus discrete/digital

Two views of the world

analog worldview:

- calculus
- distributions
- system theory
- electronics

Two views of the world

analog worldview:

- calculus
- distributions
- system theory
- electronics

digital worldview:

- arithmetic
- combinatorics
- computer science
- DSP

Two views of the world

digital worldview:

- countable integer index n
- sequences $x[n] \in \ell_2(\mathbb{Z})$
- frequency $\omega \in [-\pi, \pi]$
- DTFT: $\ell_2(\mathbb{Z}) \mapsto L_2([-\pi, \pi])$

Two views of the world

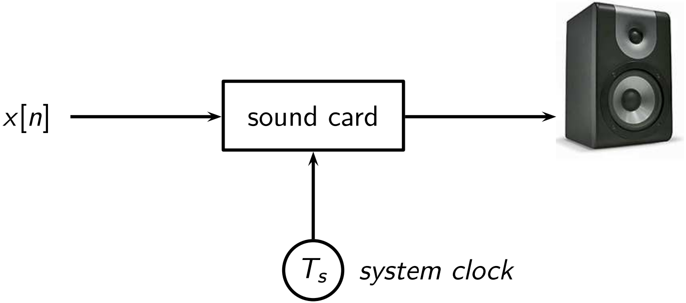
digital worldview:

- countable integer index n
- sequences $x[n] \in \ell_2(\mathbb{Z})$
- frequency $\omega \in [-\pi, \pi]$
- DTFT: $\ell_2(\mathbb{Z}) \mapsto L_2([-\pi, \pi])$

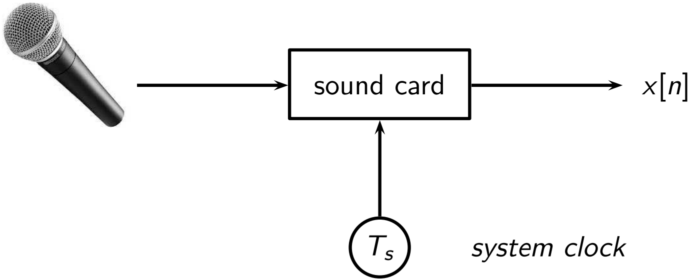
analog worldview:

- real-valued time t (sec)
- functions $x(t) \in L_2(\mathbb{R})$
- frequency $f \in \mathbb{R}$ (Hz)
- FT: $L_2(\mathbb{R}) \mapsto L_2(\mathbb{R})$

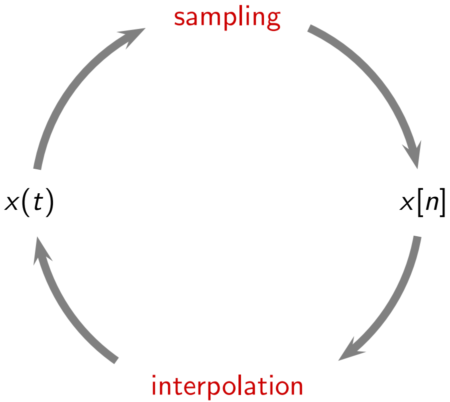
Bridging the gap: interpolation



Bridging the gap: sampling



Bridging the gap

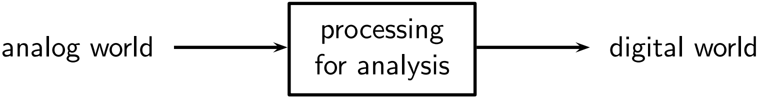


Today, processing is as digital as possible

- analog to digital
- digital to analog
- analog to digital to analog

Digital processing of signals from the analog world

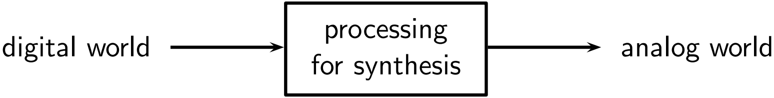
- input is continuous-time: $x(t)$
- output is discrete-time: $y[n]$
- processing is on sequences: $x[n], y[n]$



examples: storage and compression (MP3, JPG), control systems, monitoring

Digital processing of signals to the analog world

- input is discrete-time: $x[n]$
- output is continuous-time: $y(t)$
- processing is on sequences: $x[n], y[n]$



examples: music synthesizers, computer graphics, video games

Digital processing of signals from/to the analog world

- input is continuous-time: $x(t)$
- output is continuous-time: $y(t)$
- processing is on sequences: $x[n], y[n]$

$$\text{analog world} \longrightarrow \boxed{\text{processing}} \longrightarrow \text{analog world}$$

examples: telephony, VOIP, sound effects, digital photography

continuous-time signal processing

About continuous time

- time: real variable t
- signal $x(t)$: complex functions of a real variable
- finite energy: $x(t) \in L_2(\mathbb{R})$ (square integrable functions)
- inner product in $L_2(\mathbb{R})$

$$\langle x(t), y(t)\rangle = \int_{-\infty}^{\infty} x^*(t)y(t)dt$$

- energy: $||x(t)||^2 = \langle x(t), x(t)\rangle$

About continuous time

- time: real variable t
- signal $x(t)$: complex functions of a real variable
- finite energy: $x(t) \in L_2(\mathbb{R})$ (square integrable functions)
- inner product in $L_2(\mathbb{R})$

$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x^*(t)y(t)dt$$

- energy: $||x(t)||^2 = \langle x(t), x(t) \rangle$

About continuous time

- time: real variable t
- signal $x(t)$: complex functions of a real variable
- finite energy: $x(t) \in L_2(\mathbb{R})$ (square integrable functions)
- inner product in $L_2(\mathbb{R})$

$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x^*(t) y(t) dt$$

- energy: $||x(t)||^2 = \langle x(t), x(t) \rangle$

About continuous time

- time: real variable t
- signal $x(t)$: complex functions of a real variable
- finite energy: $x(t) \in L_2(\mathbb{R})$ (square integrable functions)
- inner product in $L_2(\mathbb{R})$
$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x^*(t) y(t) dt$$
- energy: $||x(t)||^2 = \langle x(t), x(t) \rangle$

About continuous time

- time: real variable t
- signal $x(t)$: complex functions of a real variable
- finite energy: $x(t) \in L_2(\mathbb{R})$ (square integrable functions)
- inner product in $L_2(\mathbb{R})$

$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x^*(t) y(t) dt$$

- energy: $||x(t)||^2 = \langle x(t), x(t) \rangle$

Analog LTI filters

$$x(t) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(t)$$

$$\begin{aligned} y(t) &= (x * h)(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \langle h^*(t-\tau), x(\tau)\rangle \end{aligned}$$

Analog LTI filters

$$x(t) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(t)$$

$$\begin{aligned} y(t) &= (x * h)(t) \\ &= \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \\ &= \langle h^*(t - \tau), x(\tau) \rangle \end{aligned}$$

Analog LTI filters

$$x(t) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(t)$$

$$\begin{aligned}
y(t) &= (x * h)(t) \\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \langle h^*(t-\tau), x(\tau) \rangle
\end{aligned}$$

Real-world frequency

frequency: number of repetitions per *second*

- f expressed in Hz (1/sec)
- alternatively, angular frequency in rad/s: $\Omega = 2\pi f$
- period for periodic signals is $T = \dfrac{1}{f} = \dfrac{2\pi}{\Omega}$

Fourier analysis

- in discrete time max angular frequency is $\pm\pi$
- in continuous time no max frequency: $f \in \mathbb{R}$
- concept is the same: similarity to sinusoidal components

$$\begin{aligned} X(f) &= \langle e^{j2\pi ft}, x(t) \rangle \\ &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \qquad \leftarrow \textit{not periodic!} \end{aligned}$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

Fourier analysis

- in discrete time max angular frequency is $\pm\pi$
- in continuous time no max frequency: $f \in \mathbb{R}$
- concept is the same: similarity to sinusoidal components

$$\begin{aligned} X(f) &= \langle e^{j2\pi ft}, x(t)\rangle \\ &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \qquad \leftarrow \textit{not periodic!} \end{aligned}$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

Fourier analysis

- in discrete time max angular frequency is $\pm\pi$
- in continuous time no max frequency: $f \in \mathbb{R}$
- concept is the same: similarity to sinusoidal components

$$\begin{aligned} X(f) &= \langle e^{j2\pi ft}, x(t) \rangle \\ &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \qquad \leftarrow \textit{not periodic!} \end{aligned}$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

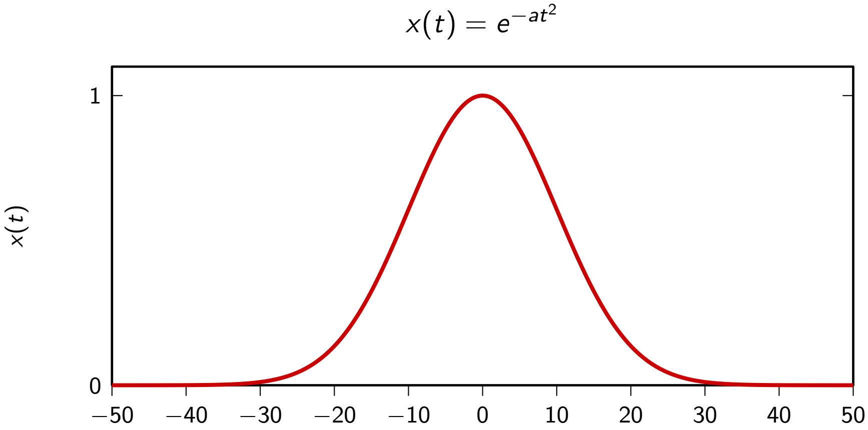
Fourier analysis (in rad/s)

$$
\begin{aligned}
X(j\Omega) &= \langle e^{j\Omega t}, x(t) \rangle \\
&= \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \qquad \leftarrow \textit{not periodic!}
\end{aligned}
$$

$$
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega
$$

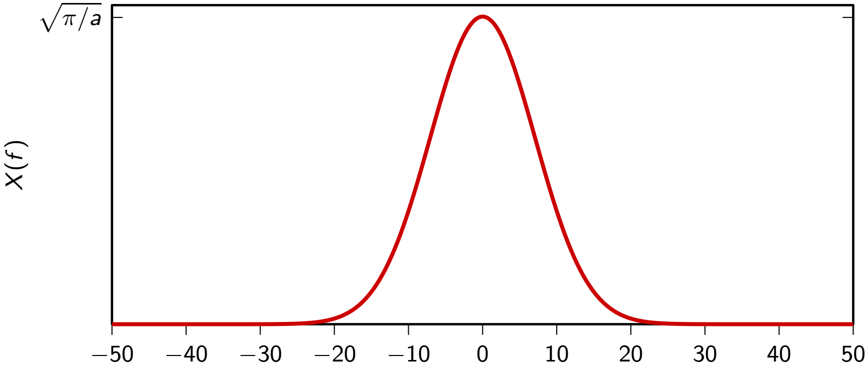
- Laplace transform computed on the imaginary axis

Example

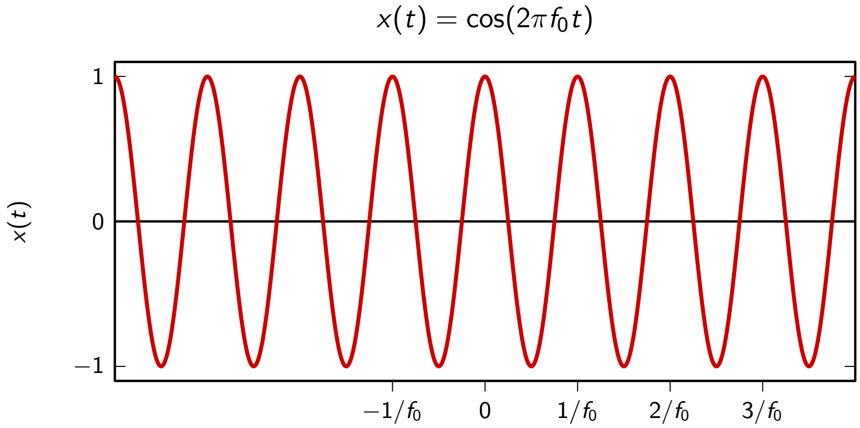


Example

$$X(f) = \sqrt{\pi/a}\, e^{-\frac{\pi^2}{a} f^2}$$

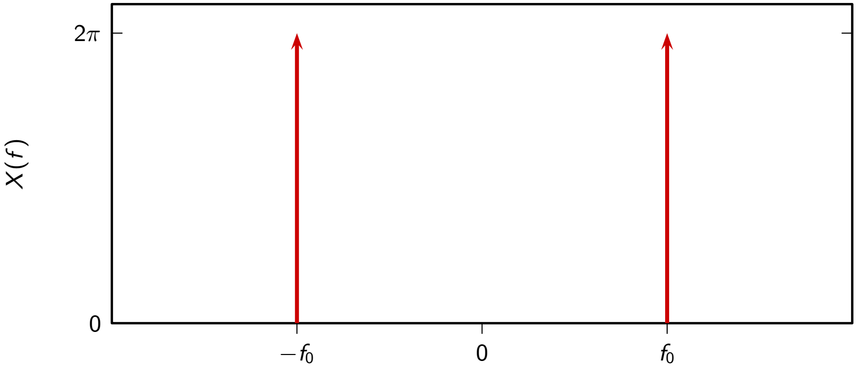


Example



Example

$$X(f) = (1/2)\delta(f \pm f_0)$$



Convolution theorem

$$x(t) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(t)$$

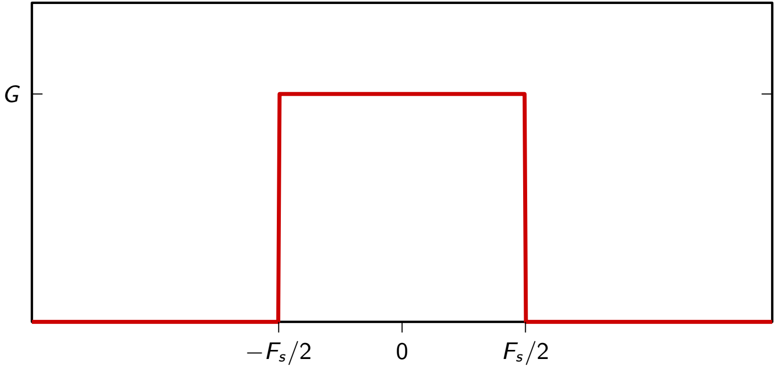
$$Y(f) = X(f)\,H(f)$$

A new concept: bandlimited functions

a continuous-time signal is bandlimited if there exists a frequency F_s such that:

$$X(f) = 0 \quad \text{for } |f| > F_s/2$$

Prototypical bandlimited function



The prototypical bandlimited function

$$\Phi(f) = G \ \mathrm{rect}\left(\frac{f}{F_s}\right)$$

$$\begin{aligned}\varphi(t) &= \int_{-\infty}^{\infty} \Phi(f) e^{j2\pi ft} df \\ &= \dots \\ &= GF_s \,\mathrm{sinc}\,(tF_s)\end{aligned}$$

The prototypical bandlimited function

$$\Phi(f) = G \ \mathrm{rect}\left(\frac{f}{F_s}\right)$$

$$\begin{aligned}
\varphi(t) &= \int_{-\infty}^{\infty} \Phi(f) e^{j2\pi ft} df \\
&= \dots \\
&= G F_s \,\mathrm{sinc}\,(t F_s)
\end{aligned}$$

The prototypical bandlimited function

- total bandwidth: F_s
- define $T_s = 1/F_s$
- normalization: $G = 1/F_s = T_s$

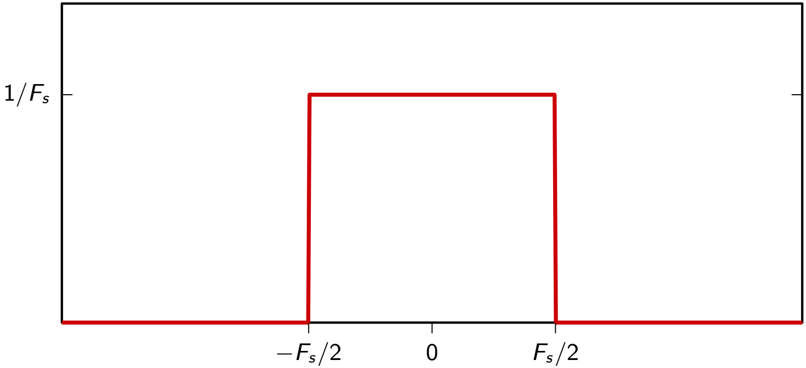
The prototypical bandlimited function

$$\Phi(f) = \frac{1}{F_s}\text{rect}\left(\frac{f}{F_s}\right)$$

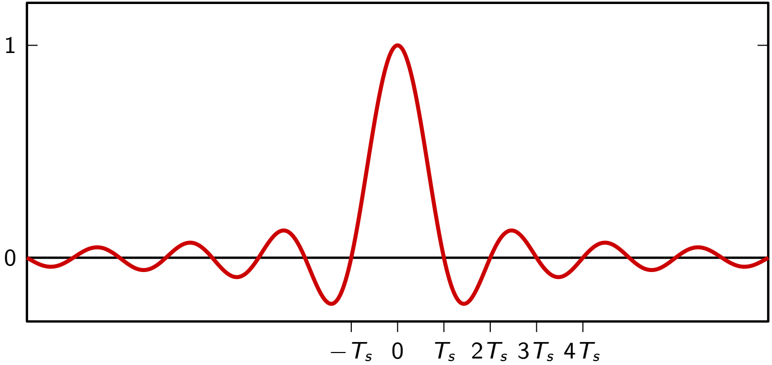
$$\varphi(t) = \text{sinc}\left(\frac{t}{T_s}\right)$$

The prototypical bandlimited function

$$F_s = 1/T_s$$



The prototypical bandlimited function

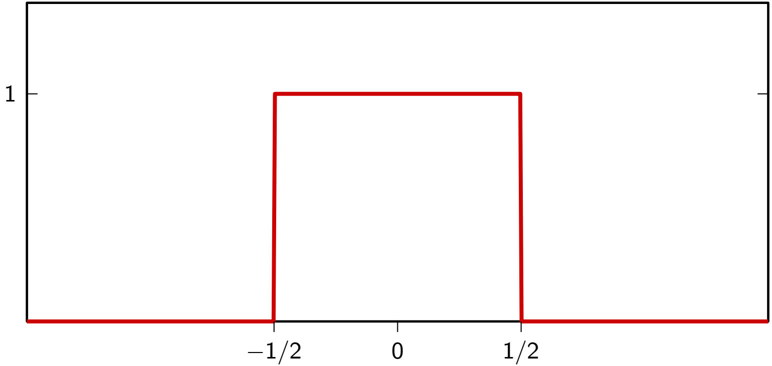


When $T_s = 1$

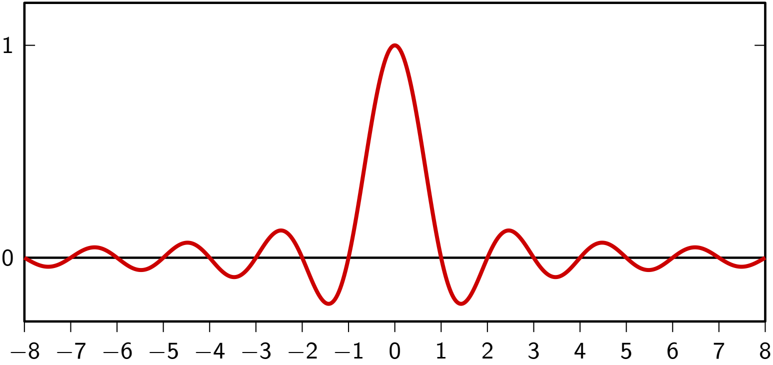
$$\Phi(f) = \mathrm{rect}\left(f\right)$$

$$\varphi(t) = \mathrm{sinc}(t)$$

The prototypical bandlimited function ($T_s = 1$)



The prototypical bandlimited function ($T_s = 1$)



interpolation

Overview:

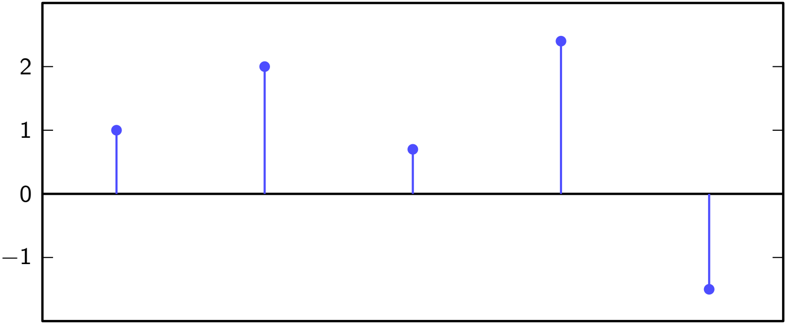
- Polynomial interpolation
- Local interpolation
- Sinc interpolation

Interpolation

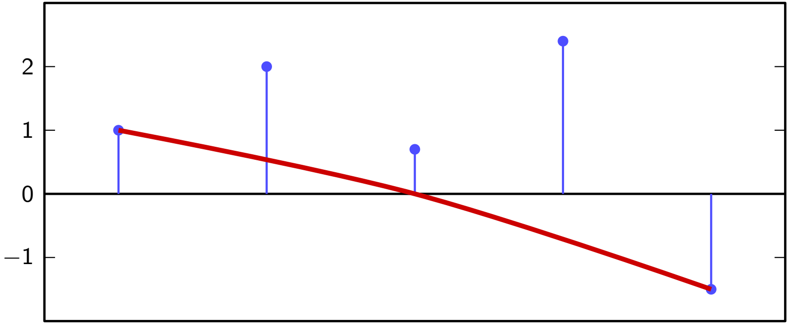
$$x[n] \longrightarrow x(t)$$

"fill the gaps" between samples

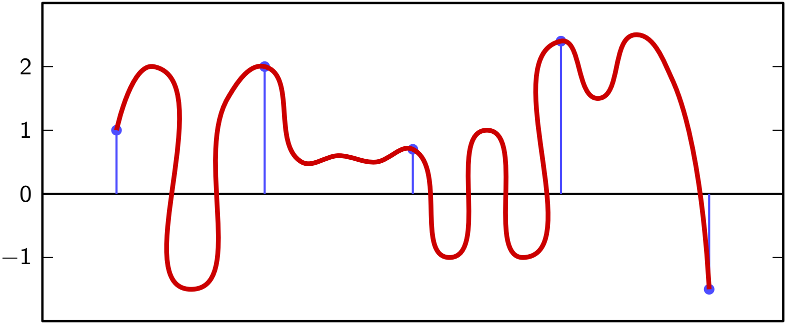
Example



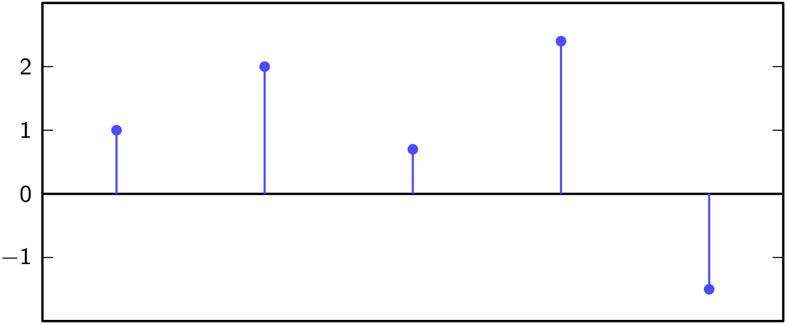
Example



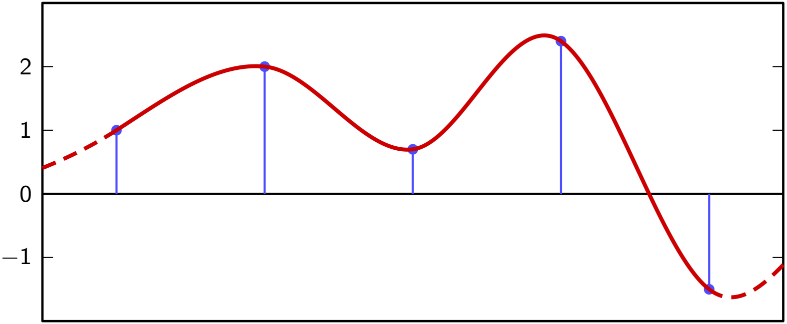
Example



Example



Example



Interpolation requirements

- decide on T_s
- make sure $x(nT_s) = x[n]$
- make sure $x(t)$ is *smooth*

Interpolation requirements

- decide on T_s
- make sure $x(nT_s) = x[n]$
- make sure $x(t)$ is *smooth*

Interpolation requirements

- decide on T_s
- make sure $x(nT_s) = x[n]$
- make sure $x(t)$ is *smooth*

Why smoothness?

- jumps (1st order discontinuities) would require the signal to move "faster than light"...
- 2nd order discontinuities would require infinite acceleration
- ...
- the interpolation should be infinitely differentiable
- "natural" solution: polynomial interpolation

Why smoothness?

- jumps (1st order discontinuities) would require the signal to move "faster than light"...
- 2nd order discontinuities would require infinite acceleration
- ...
- the interpolation should be infinitely differentiable
- "natural" solution: polynomial interpolation

Why smoothness?

- jumps (1st order discontinuities) would require the signal to move "faster than light"...
- 2nd order discontinuities would require infinite acceleration
- ...
- the interpolation should be infinitely differentiable
- "natural" solution: polynomial interpolation

Why smoothness?

- jumps (1st order discontinuities) would require the signal to move "faster than light"...
- 2nd order discontinuities would require infinite acceleration
- ...
- the interpolation should be infinitely differentiable
- "natural" solution: polynomial interpolation

Why smoothness?

- jumps (1st order discontinuities) would require the signal to move "faster than light"...
- 2nd order discontinuities would require infinite acceleration
- ...
- the interpolation should be infinitely differentiable
- "natural" solution: polynomial interpolation

Polynomial interpolation

- N points $\rightarrow$ polynomial of degree $(N-1)$
- $p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_{N-1} t^{(N-1)}$
- straightforward approach:

$$\begin{cases} p(0) = x[0] \\ p(T_s) = x[1] \\ p(2T_s) = x[2] \\ \ldots \\ p((N-1)T_s) = x[N-1] \end{cases}$$

Polynomial interpolation

- N points $\rightarrow$ polynomial of degree $(N-1)$
- $p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_{N-1} t^{(N-1)}$
- straightforward approach:

$$\begin{cases} p(0) = x[0] \\ p(T_s) = x[1] \\ p(2T_s) = x[2] \\ \ldots \\ p((N-1)T_s) = x[N-1] \end{cases}$$

Polynomial interpolation

- N points $\rightarrow$ polynomial of degree $(N-1)$
- $p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_{N-1} t^{(N-1)}$
- straightforward approach:

$$\begin{cases} p(0) = x[0] \\ p(T_s) = x[1] \\ p(2T_s) = x[2] \\ \ldots \\ p((N-1)T_s) = x[N-1] \end{cases}$$

Polynomial interpolation

Without loss of generality:

- consider a symmetric interval $I_N = [-N, \ldots, N]$
- set $T_s = 1$

$$\begin{cases} p(-N) = x[-N] \\ p(-N+1) = x[-N+1] \\ \ldots \\ p(0) = x[0] \\ \ldots \\ p(N-1) = x[N-1] \\ p(N) = x[N] \end{cases}$$

Polynomial interpolation

Without loss of generality:

- consider a symmetric interval $I_N = [-N, \dots, N]$
- set $T_s = 1$

$$\begin{cases} p(-N) = x[-N] \\ p(-N+1) = x[-N+1] \\ \dots \\ p(0) = x[0] \\ \dots \\ p(N-1) = x[N-1] \\ p(N) = x[N] \end{cases}$$

Lagrange interpolation

Let's use the power of vector spaces:

- P_N: space of degree-$2N$ polynomials over I_N
- interpolation will be a linear combination of basis vectors for P_N
- what is a good basis for *interpolation?*

Aside: N-degree polynomial bases on the interval

- naive basis: $1, t, t^2, \ldots, t^N$
- Legendre basis: orthonormal, increasing degree, good for MSE approximation
- Chebyshev basis: orthonormal, increasing degree, good for minimax approximation
- Lagrange polynomials: equal degree, interpolation property

Lagrange interpolation

- P_N: space of degree-$2N$ polynomials over I_N
- a basis for P_N is the family of $2N+1$ Lagrange polynomials

$$L_n^{(N)}(t) = \prod_{\substack{k=-N \\ k \neq n}}^{N} \frac{t-k}{n-k} \qquad\qquad n = -N, \ldots, N$$

- interpolation property:

$$L_n^{(N)}(m \in \mathbb{N}) = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \qquad\qquad -N \leq n, m \leq N$$

Lagrange polynomials for I_2

$$L_{-2}^{(2)}(t) = \left(\frac{t+1}{-2+1}\right)\left(\frac{t}{-2}\right)\left(\frac{t-1}{-2-1}\right)\left(\frac{t-2}{-2-2}\right)$$

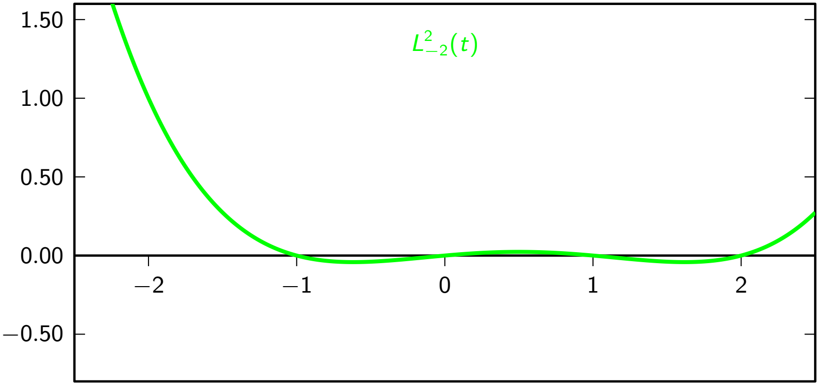
$$L_{-1}^{(2)}(t) = \left(\frac{t+2}{-1+2}\right)\left(\frac{t}{-1}\right)\left(\frac{t-1}{-1-1}\right)\left(\frac{t-2}{-1-2}\right)$$

$$L_{0}^{(2)}(t) = \left(\frac{t+2}{2}\right)\left(\frac{t+1}{1}\right)\left(\frac{t-1}{-1}\right)\left(\frac{t-2}{-2}\right)$$

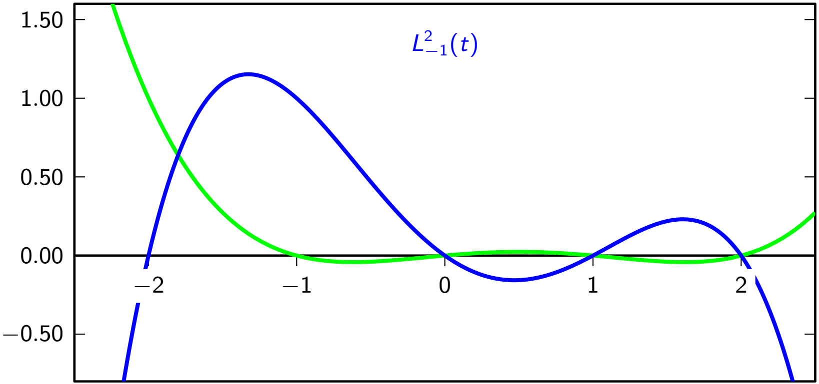
$$L_{1}^{(2)}(t) = L_{-1}^{(2)}(-t)$$

$$L_{2}^{(2)}(t) = L_{-2}^{(2)}(-t)$$

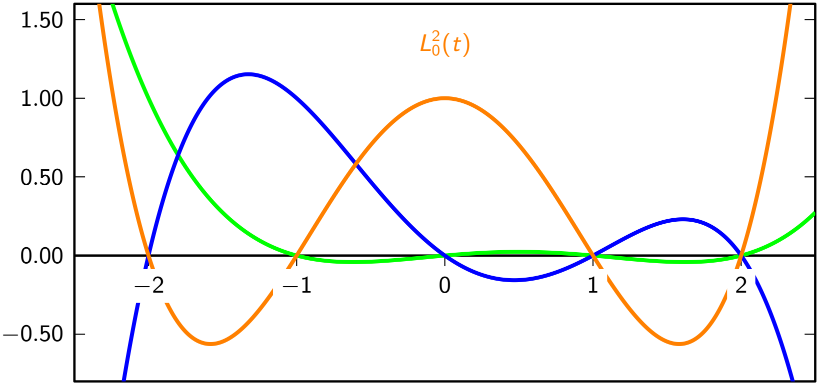
Lagrange interpolation polynomials



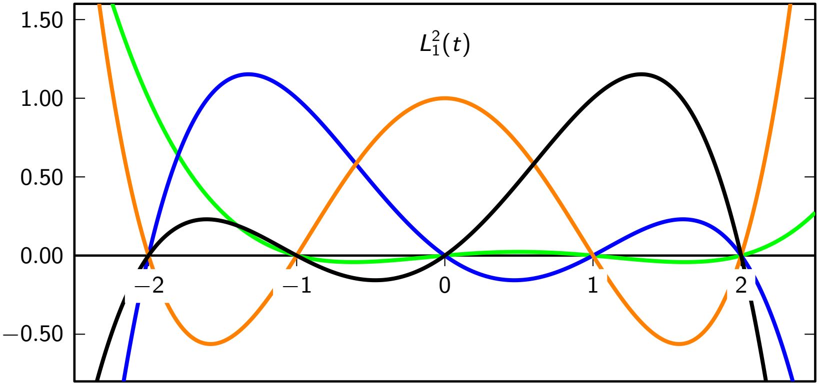
Lagrange interpolation polynomials



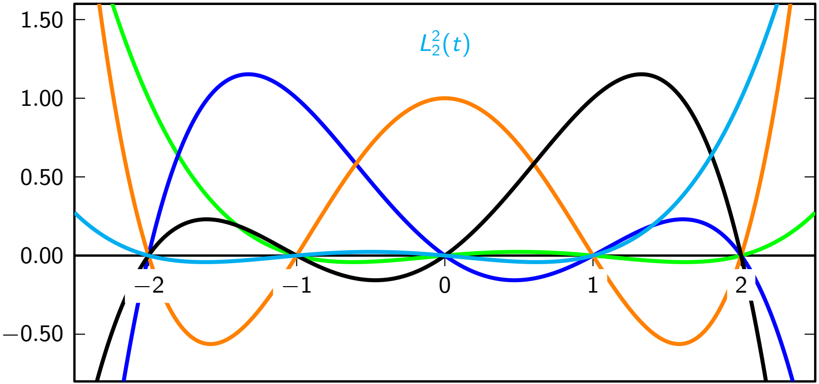
Lagrange interpolation polynomials



Lagrange interpolation polynomials



Lagrange interpolation polynomials



Lagrange interpolation

$$p(t) = \sum_{n=-N}^{N} x[n] L_n^{(N)}(t)$$

Lagrange interpolation

The Lagrange interpolation *is* the unique polynomial interpolation:

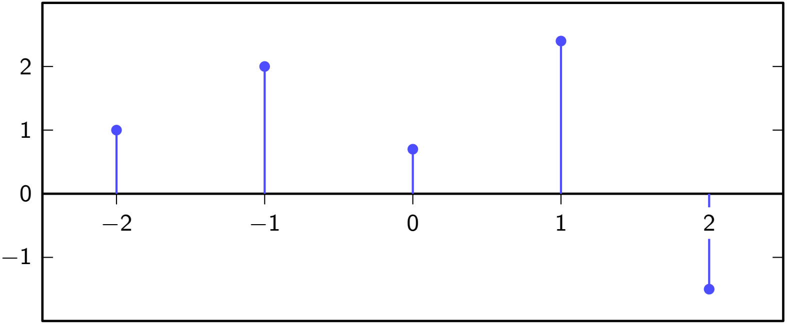
- polynomial of degree $2N$ through $2N+1$ points is unique
- the Lagrangian interpolator satisfies

$$p(n) = x[n] \qquad \text{for } -N \leq n \leq N$$

since

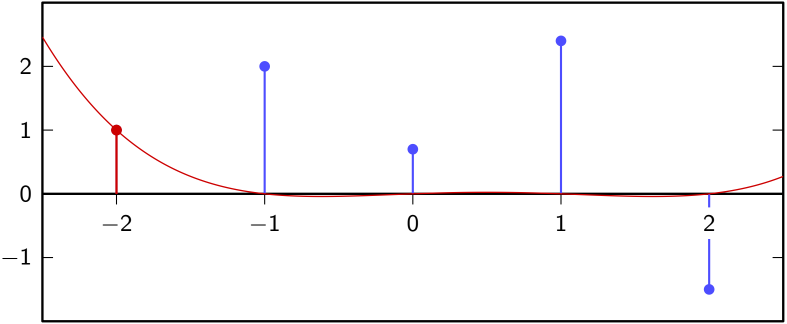
$$L_n^{(N)}(m) = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \qquad\qquad -N \leq n, m \leq N$$

Lagrange interpolation



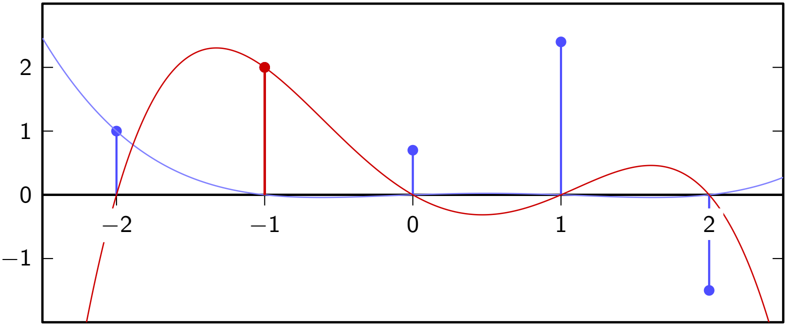
Lagrange interpolation

$$x[-2]L_{-2}^{(2)}(t)$$

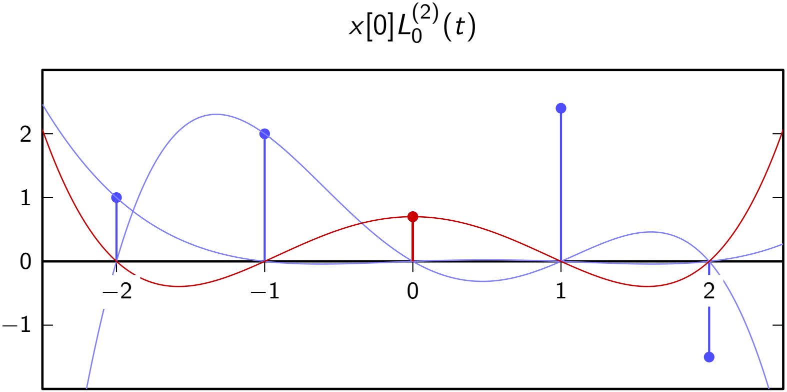


Lagrange interpolation

$$x[-1]L_{-1}^{(2)}(t)$$

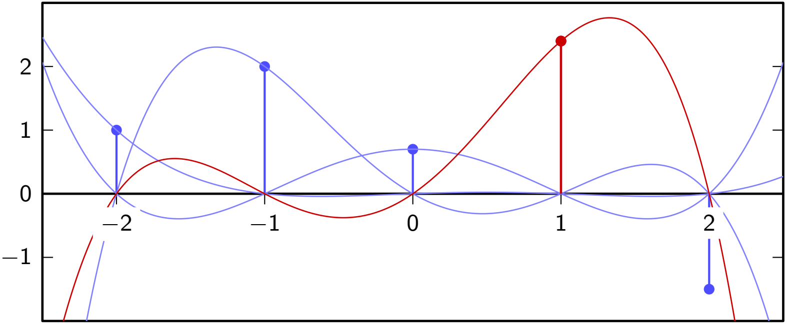


Lagrange interpolation



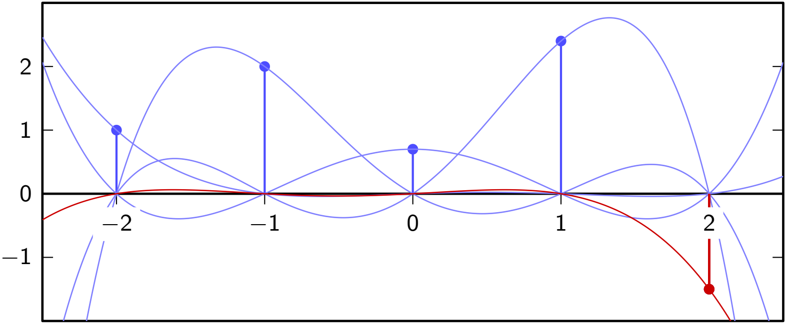
Lagrange interpolation

$x[1]L_1^{(2)}(t)$

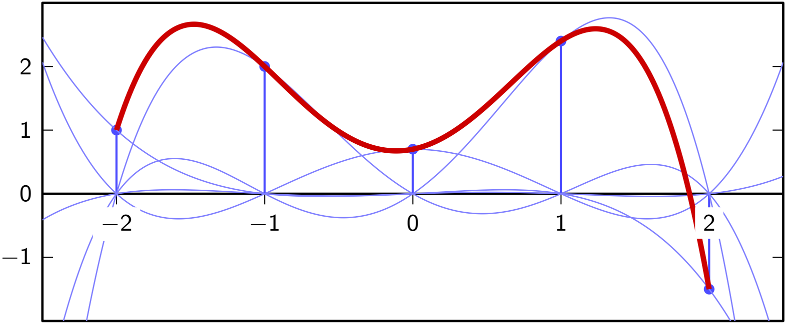


Lagrange interpolation

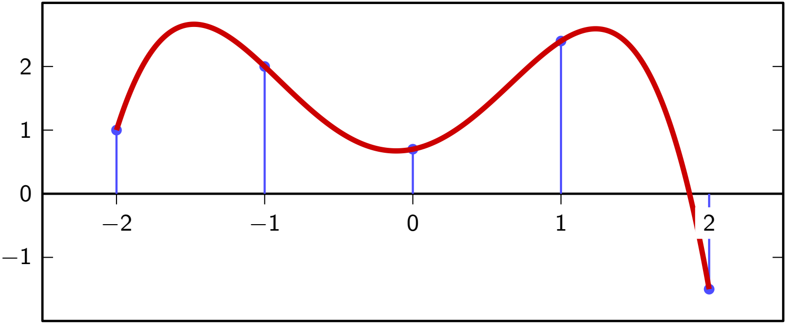
$x[2]L_2^{(2)}(t)$



Lagrange interpolation



Lagrange interpolation



Polynomial interpolation

key property:

- maximally smooth (infinitely many continuous derivatives)

drawback:

- interpolation “machine” depend on N: we need to use a different set of polynomials if the length of the dataset changes

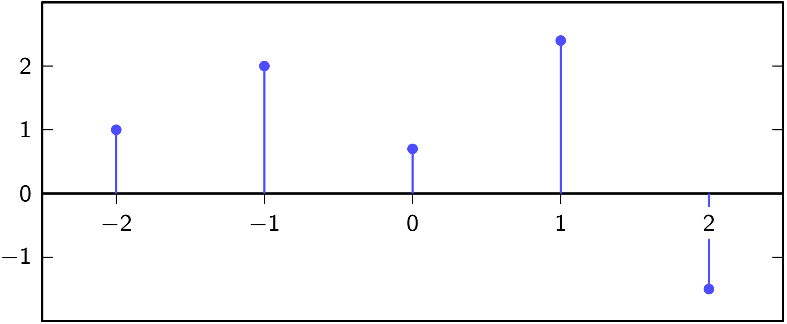
Relaxing the interpolation requirements

- decide on T_s
- make sure $x(nT_s) = x[n]$
- make sure $x(t)$ is *smooth*

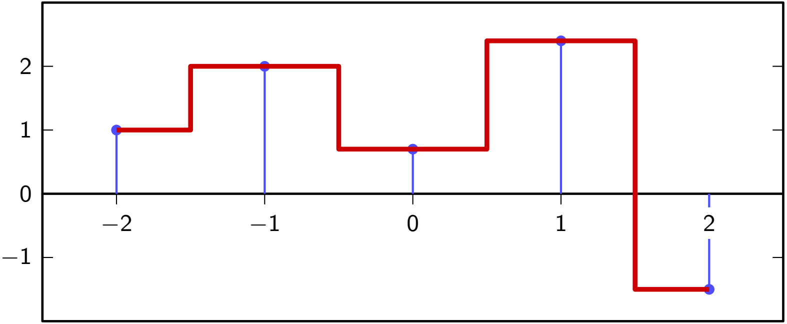
Relaxing the interpolation requirements

- decide on T_s
- make sure $x(nT_s) = x[n]$
- make sure $x(t)$ is *smooth*

Zero-order interpolation



Zero-order interpolation



Zero-order interpolation

- $x(t) = x[\lfloor t + 0.5 \rfloor], \qquad -N \leq t \leq N$
- $x(t) = \sum_{n=-N}^{N} x[n] \, \mathrm{rect}(t - n)$
- interpolation kernel: $i_0(t) = \mathrm{rect}(t)$
- $i_0(t)$: "zero-order hold"
- interpolator's support is 1
- interpolation is not even continuous

Zero-order interpolation

- $x(t) = x[\,\lfloor t + 0.5 \rfloor\,], \qquad -N \le t \le N$
- $x(t) = \sum_{n=-N}^{N} x[n]\,\mathrm{rect}(t - n)$
- interpolation kernel: $i_0(t) = \mathrm{rect}(t)$
- $i_0(t)$: "zero-order hold"
- interpolator's support is 1
- interpolation is not even continuous

Zero-order interpolation

- $x(t) = x[\lfloor t + 0.5 \rfloor], \qquad -N \le t \le N$
- $x(t) = \sum_{n=-N}^{N} x[n]\,\mathrm{rect}(t-n)$
- interpolation kernel: $i_0(t) = \mathrm{rect}(t)$
- $i_0(t)$: "zero-order hold"
- interpolator's support is 1
- interpolation is not even continuous

Zero-order interpolation

- $x(t) = x[\lfloor t + 0.5 \rfloor], \qquad -N \leq t \leq N$
- $x(t) = \sum_{n=-N}^{N} x[n] \operatorname{rect}(t - n)$
- interpolation kernel: $i_0(t) = \operatorname{rect}(t)$
- $i_0(t)$: "zero-order hold"
- interpolator's support is 1
- interpolation is not even continuous

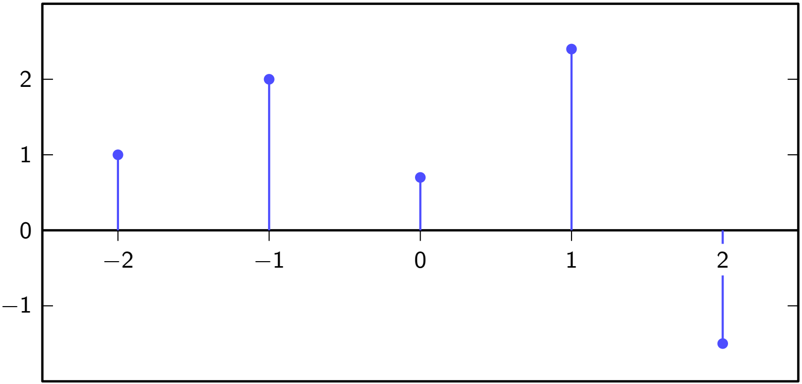
Zero-order interpolation

- $x(t) = x[\,\lfloor t + 0.5 \rfloor\,], \qquad -N \leq t \leq N$
- $x(t) = \sum_{n=-N}^{N} x[n]\,\mathrm{rect}(t - n)$
- interpolation kernel: $i_0(t) = \mathrm{rect}(t)$
- $i_0(t)$: "zero-order hold"
- interpolator's support is 1
- interpolation is not even continuous

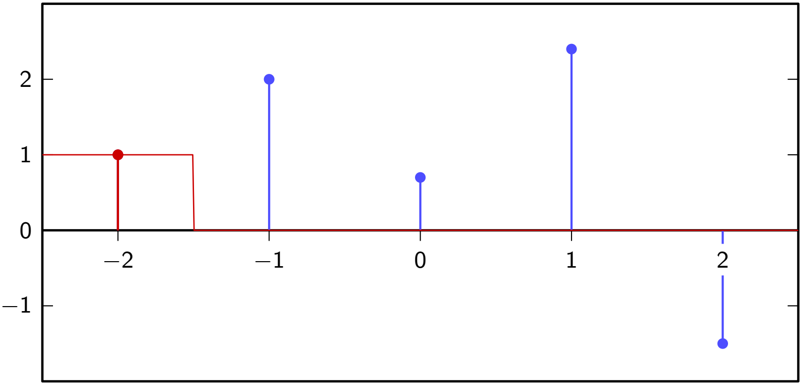
Zero-order interpolation

- $x(t) = x[\,\lfloor t + 0.5 \rfloor\,], \qquad -N \leq t \leq N$
- $x(t) = \sum_{n=-N}^{N} x[n]\,\mathrm{rect}(t - n)$
- interpolation kernel: $i_0(t) = \mathrm{rect}(t)$
- $i_0(t)$: “zero-order hold”
- interpolator’s support is 1
- interpolation is not even continuous

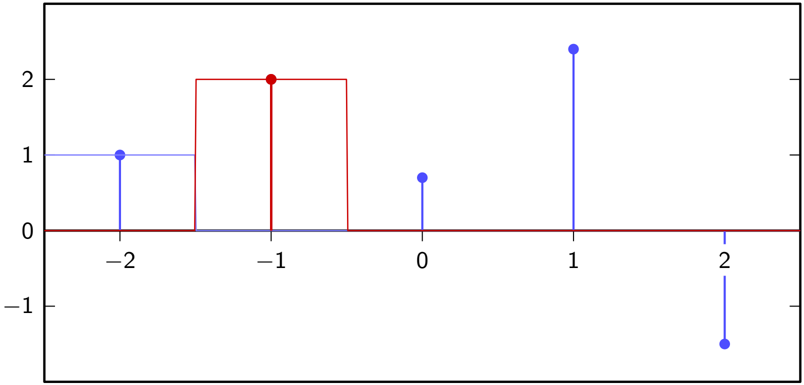
Zero-order interpolation



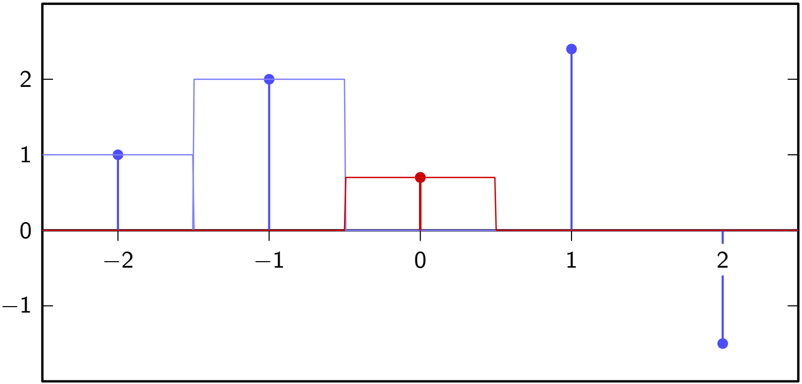
Zero-order interpolation



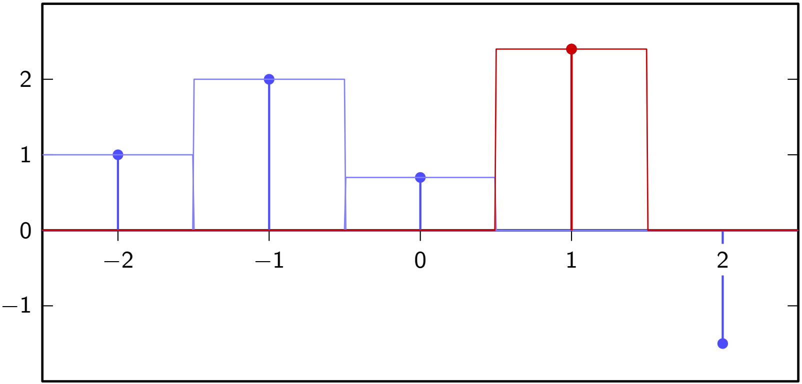
Zero-order interpolation



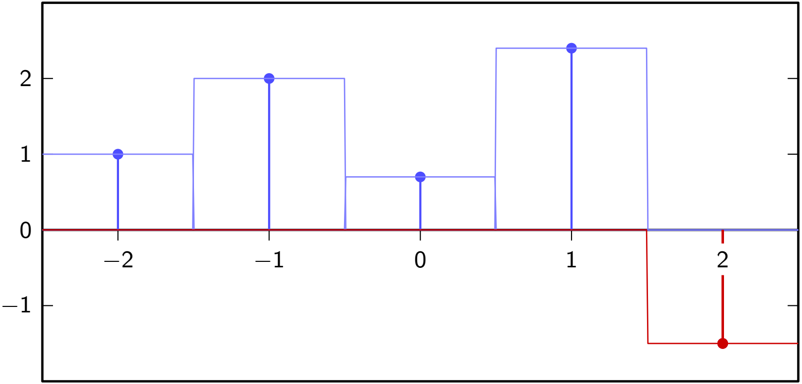
Zero-order interpolation



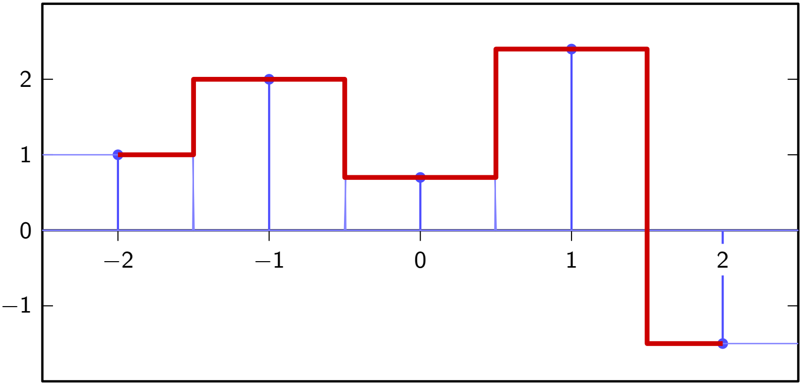
Zero-order interpolation



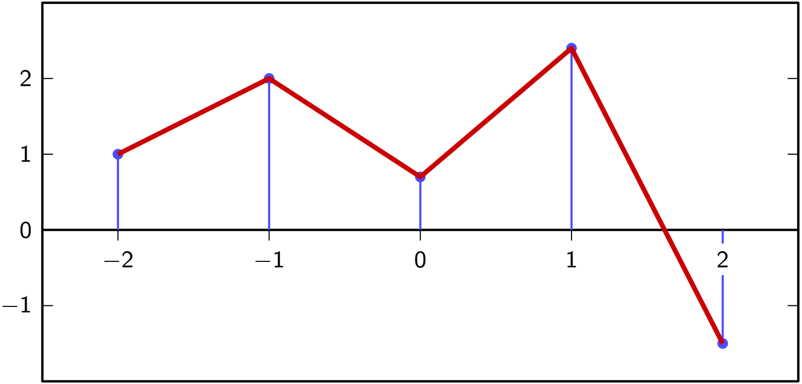
Zero-order interpolation



Zero-order interpolation



First-order interpolation



First-order interpolation

- "connect the dots" strategy
- $x(t) = \sum_{n=-N}^{N} x[n]\, i_1(t-n)$
- interpolation kernel:

$$i_1(t) = \begin{cases} 1 - |t| & |t| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- interpolator's support is 2
- interpolation is continuous but derivative is not

First-order interpolation

- "connect the dots" strategy
- $x(t) = \sum_{n=-N}^{N} x[n]\, i_1(t-n)$
- interpolation kernel:

$$i_1(t) = \begin{cases} 1 - |t| & |t| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- interpolator's support is 2
- interpolation is continuous but derivative is not

First-order interpolation

- “connect the dots” strategy
- $x(t) = \sum_{n=-N}^{N} x[n]\, i_1(t-n)$
- interpolation kernel:

$$i_1(t) = \begin{cases} 1 - |t| & |t| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- interpolator’s support is 2
- interpolation is continuous but derivative is not

First-order interpolation

- "connect the dots" strategy
- $x(t) = \sum_{n=-N}^{N} x[n]\, i_1(t-n)$
- interpolation kernel:

$$i_1(t) = \begin{cases} 1 - |t| & |t| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- interpolator's support is 2
- interpolation is continuous but derivative is not

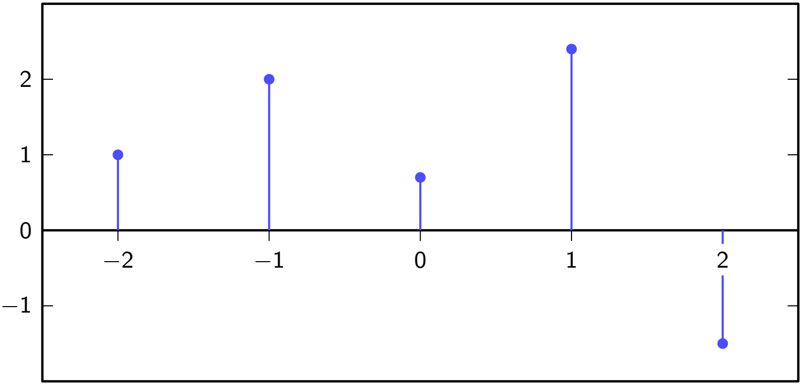
First-order interpolation

- "connect the dots" strategy
- $x(t) = \sum_{n=-N}^{N} x[n]\, i_1(t-n)$
- interpolation kernel:

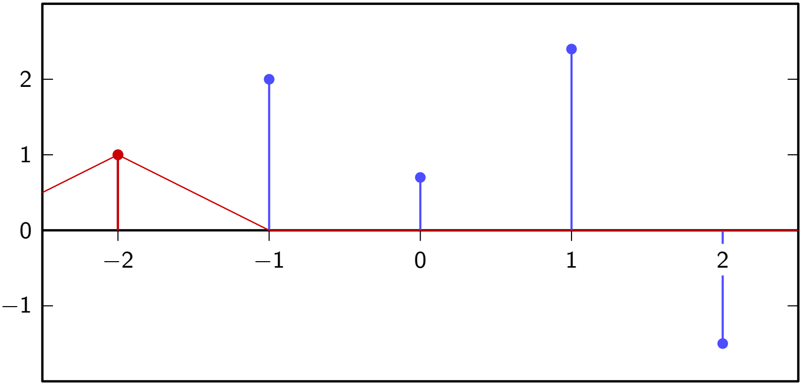
$$i_1(t) = \begin{cases} 1 - |t| & |t| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- interpolator's support is 2
- interpolation is continuous but derivative is not

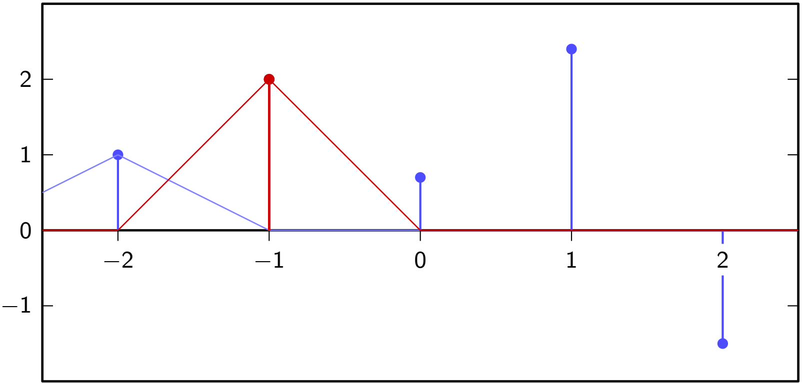
First-order interpolation



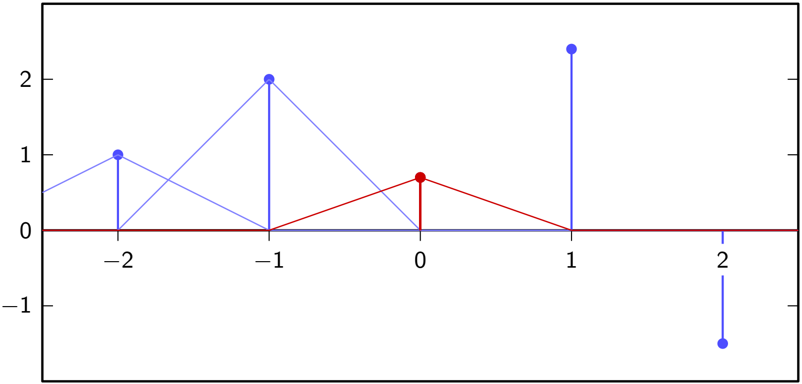
First-order interpolation



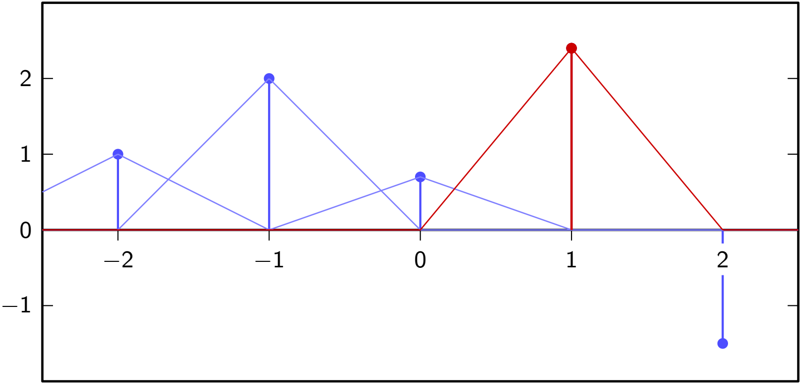
First-order interpolation



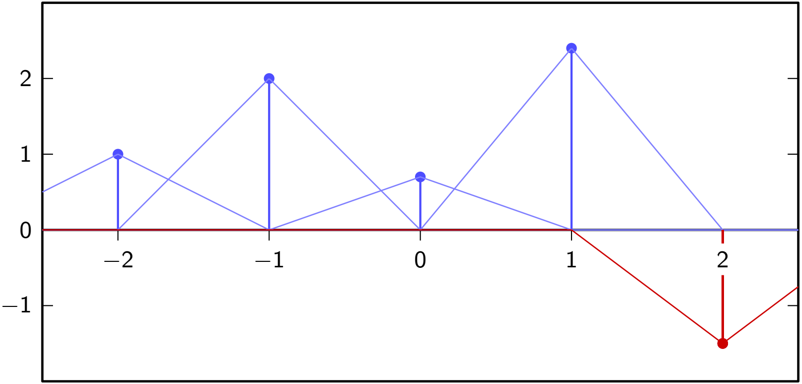
First-order interpolation



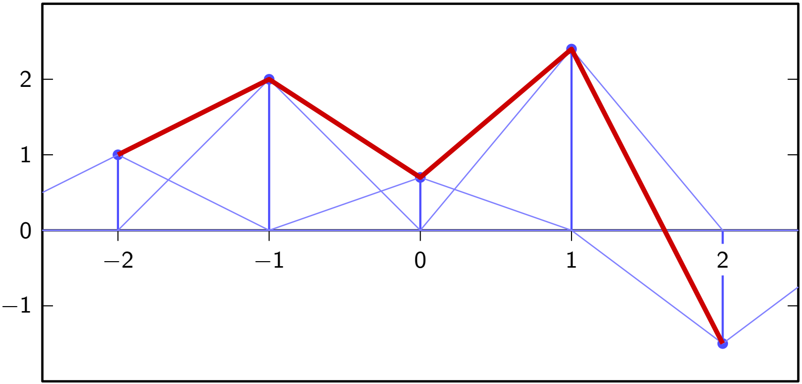
First-order interpolation



First-order interpolation



First-order interpolation



Third-order interpolation

- $x(t) = \sum_{n=-N}^{N} x[n]\, i_3(t-n)$
- interpolation kernel obtained by splicing two cubic polynomials
- interpolator's support is 4
- interpolation is continuous up to second derivative

Third-order interpolation

- $x(t) = \sum_{n=-N}^{N} x[n]\, i_3(t-n)$
- interpolation kernel obtained by splicing two cubic polynomials
- interpolator's support is 4
- interpolation is continuous up to second derivative

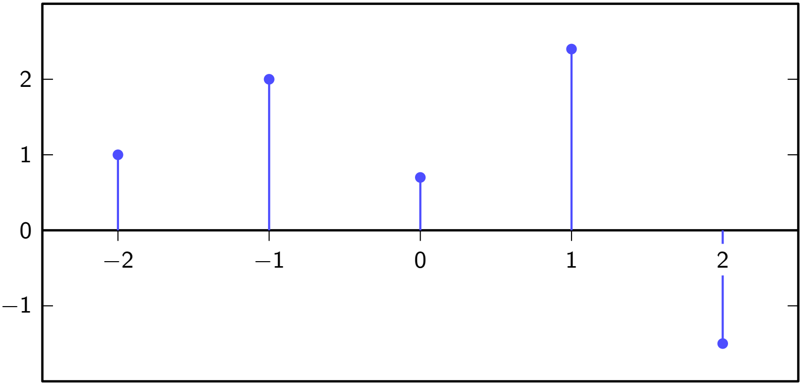
Third-order interpolation

- $x(t) = \sum_{n=-N}^{N} x[n]\, i_3(t-n)$
- interpolation kernel obtained by splicing two cubic polynomials
- interpolator's support is 4
- interpolation is continuous up to second derivative

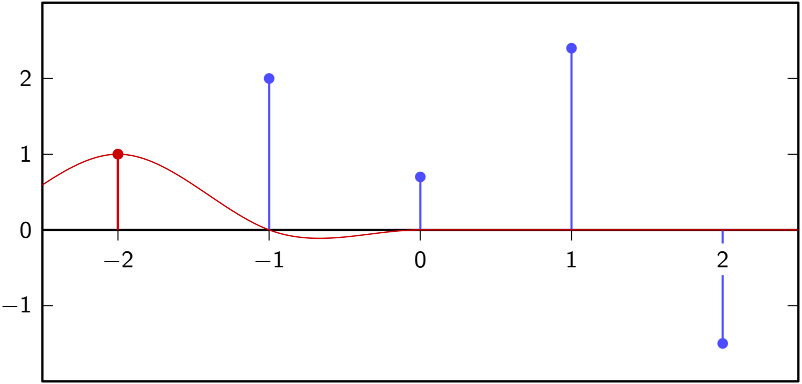
Third-order interpolation

- $x(t) = \sum_{n=-N}^{N} x[n]\, i_3(t-n)$
- interpolation kernel obtained by splicing two cubic polynomials
- interpolator's support is 4
- interpolation is continuous up to second derivative

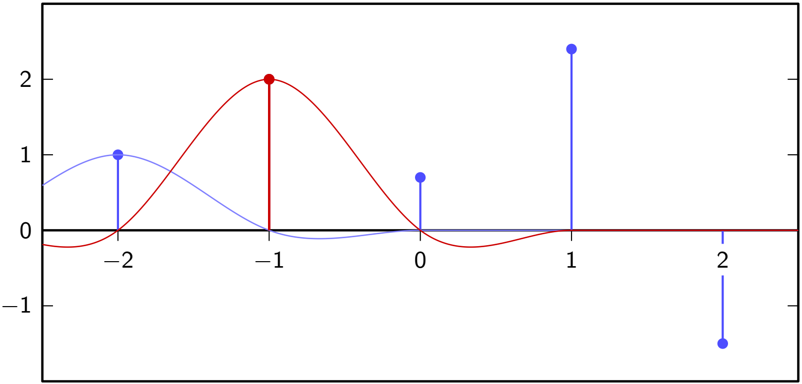
Third-order interpolation



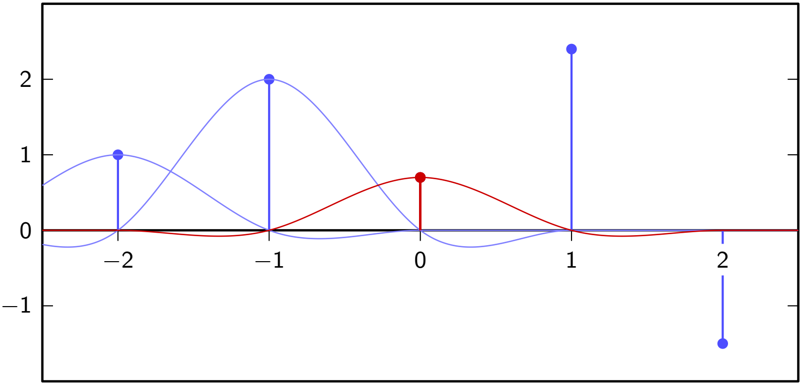
Third-order interpolation



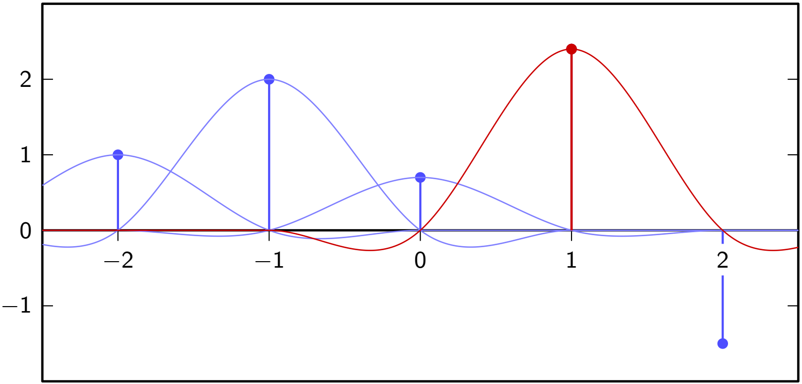
Third-order interpolation



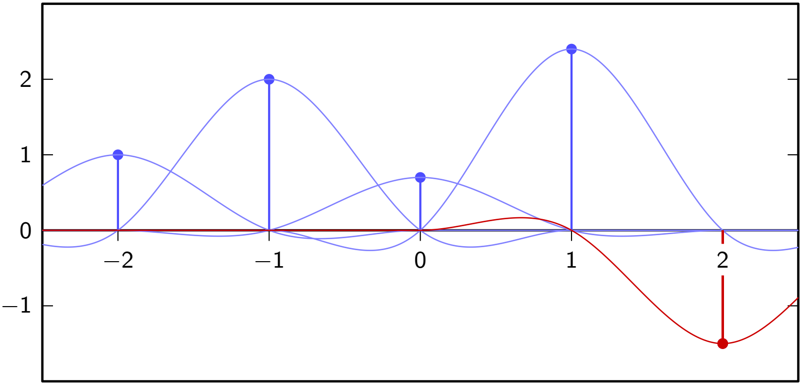
Third-order interpolation



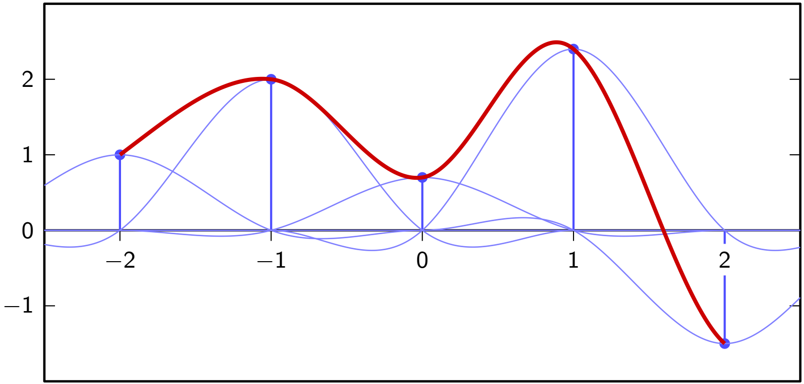
Third-order interpolation



Third-order interpolation



Third-order interpolation



Local interpolation schemes

$$x(t) = \sum_{n=-N}^{N} x[n]\, i_c(t-n)$$

Kernel must satisfy the interpolation property:

- $i_c(0) = 1$
- $i_c(m) = 0$ for m a nonzero integer.

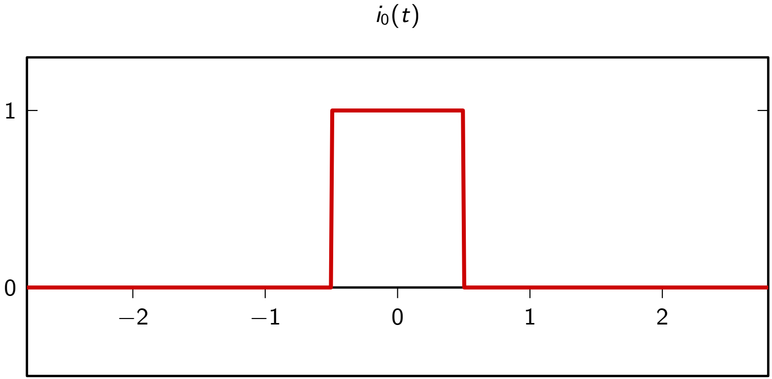
Local interpolation schemes

$$x(t) = \sum_{n=-N}^{N} x[n]\, i_c(t-n)$$

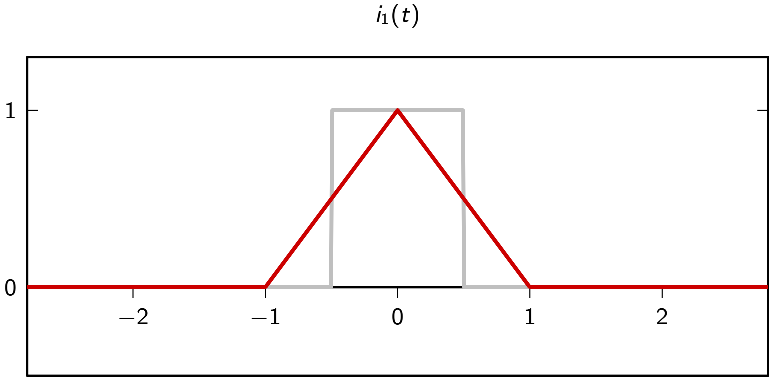
Kernel must satisfy the interpolation property:

- $i_c(0) = 1$
- $i_c(m) = 0$ for m a nonzero integer.

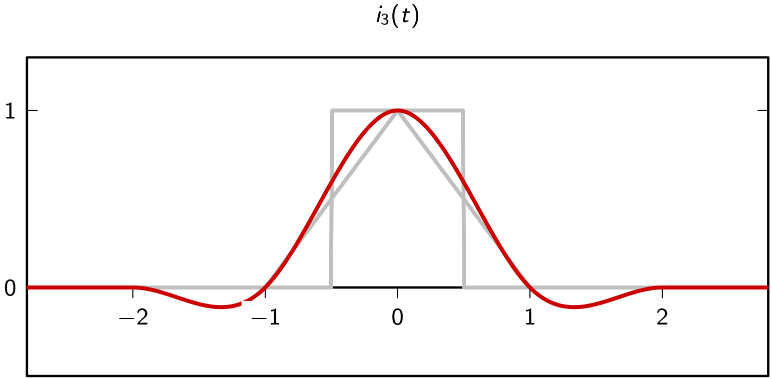
Local interpolators



Local interpolators



Local interpolators



Local interpolation

key property:

- same interpolating function independently of N

drawback:

- lack of smoothness

Polynomial interpolation

key property:

- maximally smooth (infinitely many continuous derivatives)

drawback:

- interpolation kernels depend on N

A remarkable result:

$$\lim_{N\to\infty} L_n^{(N)}(t) = f(t-n)$$

in the limit, local and global interpolation are the same!

A remarkable result:

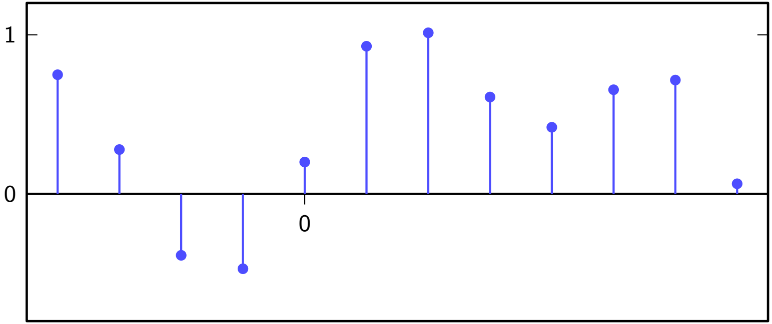
$$\lim_{N\to\infty} L_n^{(N)}(t) = \operatorname{sinc}(t-n)$$

in the limit, local and global interpolation are the same!

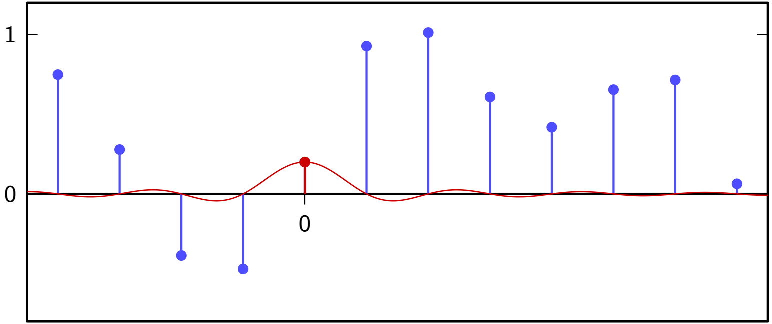
Sinc interpolation formula

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

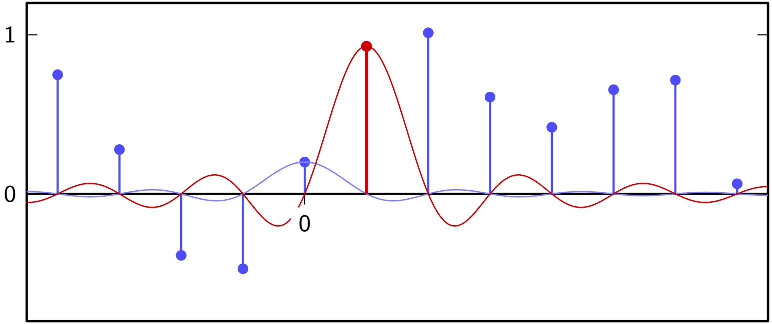
Sinc interpolation



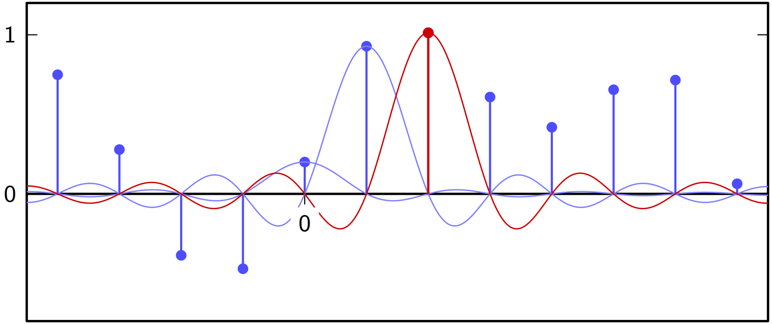
Sinc interpolation



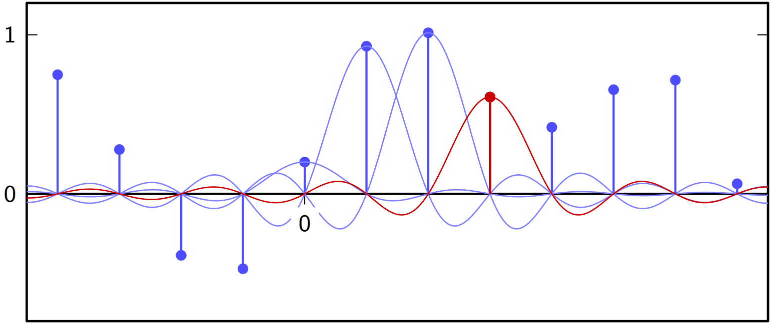
Sinc interpolation



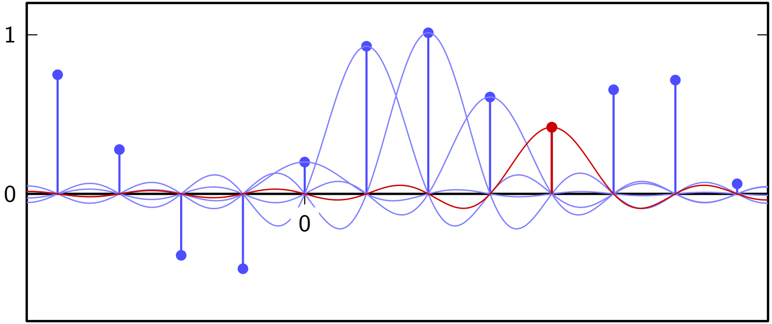
Sinc interpolation



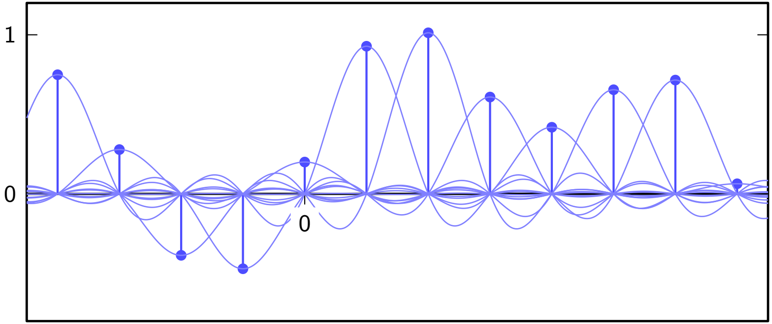
Sinc interpolation



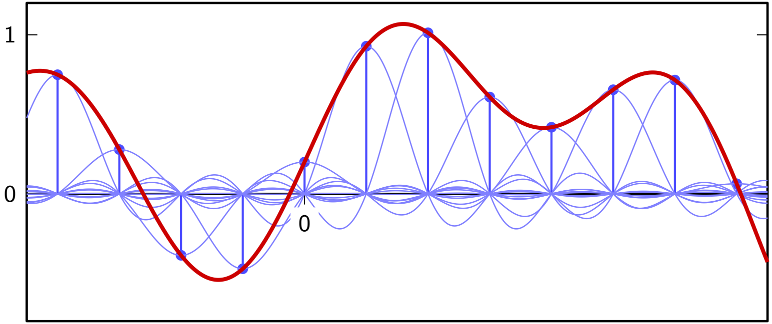
Sinc interpolation



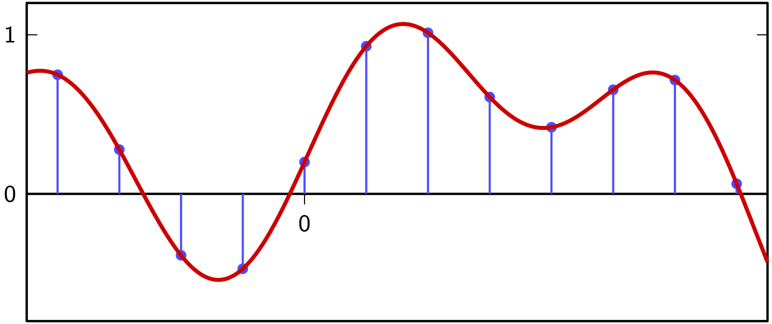
Sinc interpolation



Sinc interpolation



Sinc interpolation



Convergence: graphical "proof"

$$L_n^{(N)}(t) = \prod_{\substack{k=-N \\ k \neq n}}^{N} \frac{t-k}{n-k}$$

$$L_0^N(t) = \prod_{\substack{k=-N \\ k \neq 0}}^{N} \frac{t-k}{-k} = \prod_{k=-N}^{-1} \frac{t-k}{-k} \prod_{k=1}^{N} \frac{t-k}{-k}$$

$$= \prod_{k=1}^{N} \frac{t+k}{k} \prod_{k=1}^{N} \frac{t-k}{-k}$$

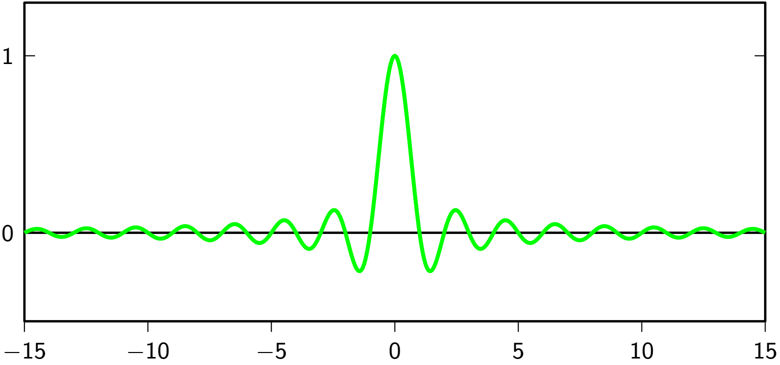
$$= \prod_{k=1}^{N} \left(1 - \frac{t^2}{k^2}\right)$$

Convergence: graphical "proof"

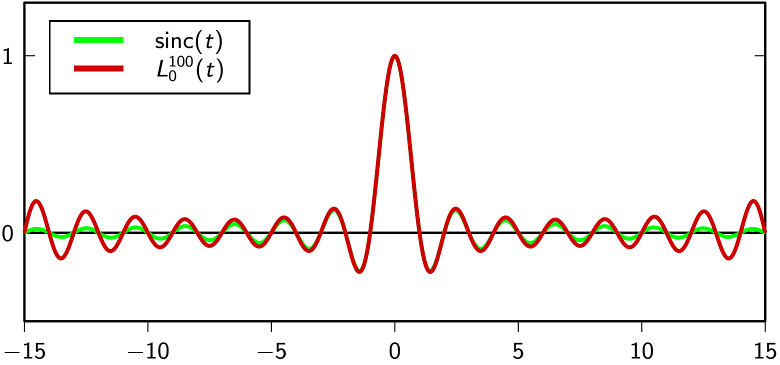
$$L_n^{(N)}(t) = \prod_{\substack{k=-N \\ k\neq n}}^{N} \frac{t-k}{n-k}$$

$$\begin{aligned}
L_0^N(t) &= \prod_{\substack{k=-N \\ k\neq 0}}^{N} \frac{t-k}{-k} = \prod_{k=-N}^{-1} \frac{t-k}{-k} \prod_{k=1}^{N} \frac{t-k}{-k} \\
&= \prod_{k=1}^{N} \frac{t+k}{k} \prod_{k=1}^{N} \frac{t-k}{-k} \\
&= \prod_{k=1}^{N} \left(1 - \frac{t^2}{k^2}\right)
\end{aligned}$$

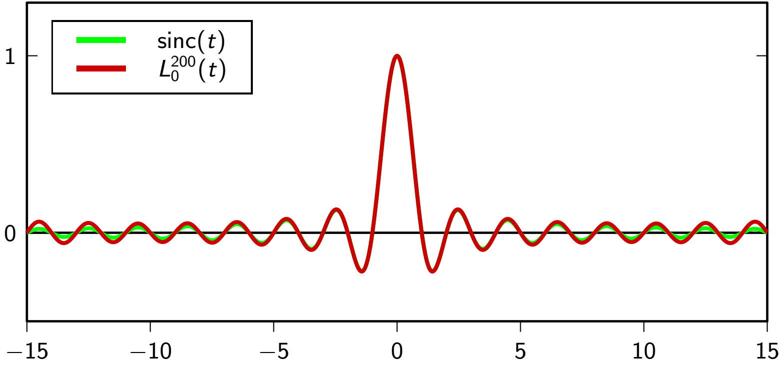
Convergence: graphical “proof”



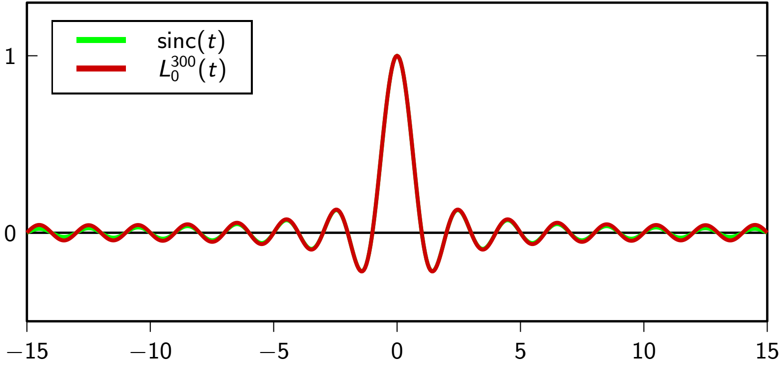
Convergence: graphical "proof"



Convergence: graphical "proof"



Convergence: graphical "proof"



Convergence: mathematical intuition

- $\text{sinc}(t-n)$ and $L_n^{(\infty)}(t)$ share an infinite number of zeros:

$$\text{sinc}(m-n) = \delta[m-n] \qquad m, n \in \mathbb{Z}$$

$$L_n^{(N)}(m) = \delta[m-n] \qquad m, n \in \mathbb{Z}, \quad -N \leq n, m \leq N$$

Convergence: Euler's "proof" (1748)

very cute (if non-rigorous) proof – see handout or book for details

Convergence: rigorous proof

uses the properties of Fourier series expansions – see handout or book for details

bandlimited functions and sampling

Overview:

- Spectrum of interpolated signals
- Space of bandlimited functions
- Sinc sampling
- The sampling theorem

Sinc interpolation

the ingredients:

- discrete-time signal $x[n]$, $n \in \mathbb{Z}$ (with DTFT $X(e^{j\omega})$)
- interpolation interval T_s
- the sinc function

the result:

- a smooth, continuous-time signal $x(t)$, $t \in \mathbb{R}$

what does the spectrum of $x(t)$ look like?

Sinc interpolation

the ingredients:

- discrete-time signal $x[n]$, $n \in \mathbb{Z}$ (with DTFT $X(e^{j\omega})$)
- interpolation interval T_s
- the sinc function

the result:

- a smooth, continuous-time signal $x(t)$, $t \in \mathbb{R}$

what does the spectrum of $x(t)$ look like?

Sinc interpolation

the ingredients:

- discrete-time signal $x[n]$, $n \in \mathbb{Z}$ (with DTFT $X(e^{j\omega})$)
- interpolation interval T_s
- the sinc function

the result:

- a smooth, continuous-time signal $x(t)$, $t \in \mathbb{R}$

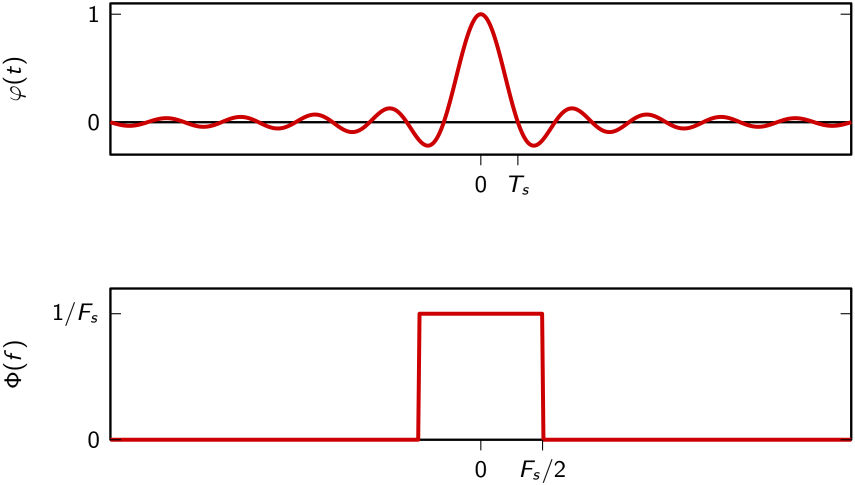
what does the spectrum of $x(t)$ look like?

Key facts about the sinc

$$\varphi(t) = \operatorname{sinc}\left(\frac{t}{T_s}\right) \qquad \longleftrightarrow \qquad \Phi(f) = \frac{1}{F_s}\operatorname{rect}\left(\frac{f}{F_s}\right)$$

$$T_s = \frac{1}{F_s}$$

Key facts about the sinc



Sinc interpolation

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

Spectral representation (I)

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc}\left(\frac{t-nT_s}{T_s}\right) e^{-j2\pi ft} dt \\
&= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \operatorname{sinc}\left(\frac{t-nT_s}{T_s}\right) e^{-j2\pi ft} dt \\
&= \sum_{n=-\infty}^{\infty} x[n] \left(\frac{1}{F_s}\right) \operatorname{rect}\left(\frac{f}{F_s}\right) e^{-j2\pi f\, nT_s}
\end{aligned}
$$

Spectral representation (I)

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\, \mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right) e^{-j2\pi ft} dt$$

$$= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right) e^{-j2\pi ft} dt$$

$$= \sum_{n=-\infty}^{\infty} x[n] \left(\frac{1}{F_s}\right) \mathrm{rect}\left(\frac{f}{F_s}\right) e^{-j2\pi f\, nT_s}$$

Spectral representation (I)

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\, \mathrm{sinc}\left(\frac{t-nT_s}{T_s}\right) e^{-j2\pi ft}\, dt \\
&= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \mathrm{sinc}\left(\frac{t-nT_s}{T_s}\right) e^{-j2\pi ft}\, dt \\
&= \sum_{n=-\infty}^{\infty} x[n] \left(\frac{1}{F_s}\right) \mathrm{rect}\left(\frac{f}{F_s}\right) e^{-j2\pi f\, nT_s}
\end{aligned}
$$

Spectral representation (I)

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right) e^{-j2\pi ft} dt \\
&= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right) e^{-j2\pi ft} dt \\
&= \sum_{n=-\infty}^{\infty} x[n] \left(\frac{1}{F_s}\right) \operatorname{rect}\left(\frac{f}{F_s}\right) e^{-j2\pi f\, nT_s}
\end{aligned}
$$

Spectral representation (II)

$$X(f) = \sum_{n=-\infty}^{\infty} x[n] \left(\frac{1}{F_s}\right) \mathrm{rect}\left(\frac{f}{F_s}\right) e^{-j2\pi f\, nT_s}$$

$$= T_s\, \mathrm{rect}\left(\frac{f}{F_s}\right) \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi(f/F_s)n}$$

$$= \begin{cases} T_s\, X(e^{j2\pi f/F_s}) & \text{for } |f| \leq F_s/2 \\ 0 & \text{otherwise} \end{cases}$$

Spectral representation (II)

$$X(f) = \sum_{n=-\infty}^{\infty} x[n] \left(\frac{1}{F_s}\right) \text{rect}\left(\frac{f}{F_s}\right) e^{-j2\pi f\, nT_s}$$

$$= T_s \,\text{rect}\left(\frac{f}{F_s}\right) \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi(f/F_s)n}$$

$$= \begin{cases} T_s X(e^{j2\pi f/F_s}) & \text{for } |f| \leq F_s/2 \\ 0 & \text{otherwise} \end{cases}$$

Spectral representation (II)

$$X(f) = \sum_{n=-\infty}^{\infty} x[n] \left(\frac{1}{F_s}\right) \mathrm{rect}\left(\frac{f}{F_s}\right) e^{-j2\pi f\, nT_s}$$

$$= T_s \,\mathrm{rect}\left(\frac{f}{F_s}\right) \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi(f/F_s)n}$$

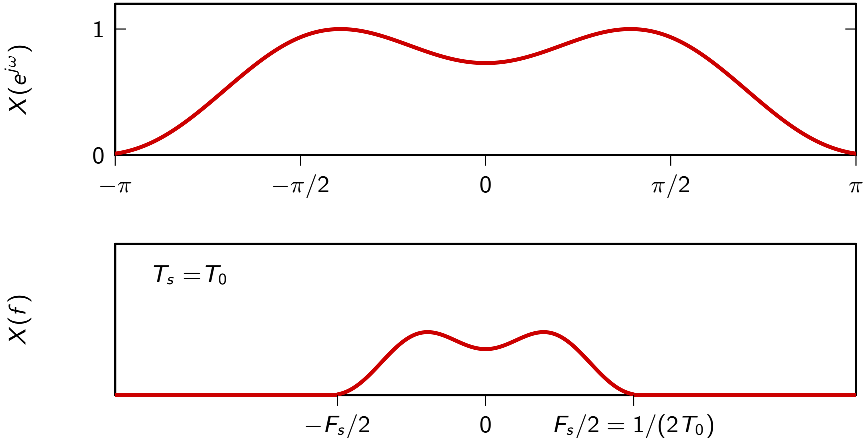
$$= \begin{cases} T_s\, X(e^{j2\pi f/F_s}) & \text{for } |f| \leq F_s/2 \\ 0 & \text{otherwise} \end{cases}$$

Spectral representation (III)

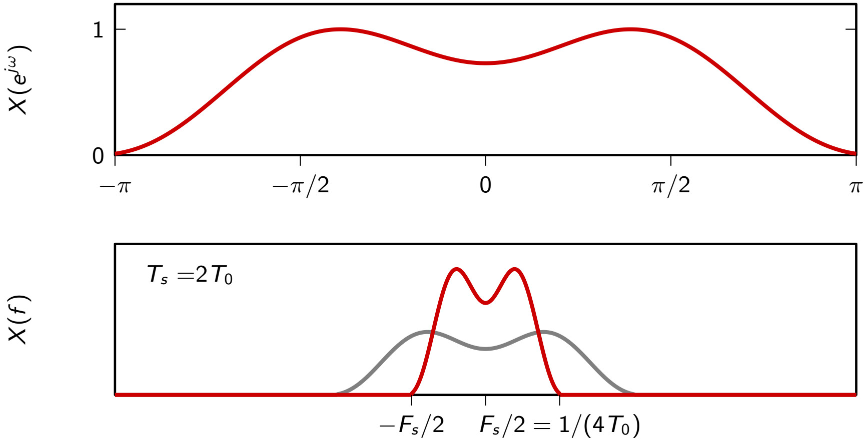
$$X(f) = \begin{cases} T_s\, X(e^{j2\pi f/F_s}) & \text{for } |f| \leq F_s/2 \\ 0 & \text{otherwise} \end{cases}$$

- map $\omega = \pi$ to $f = F_s/2$
- scale spectrum by T_s (total energy constant)
- rect keeps only the baseband copy of the periodic digital spectrum

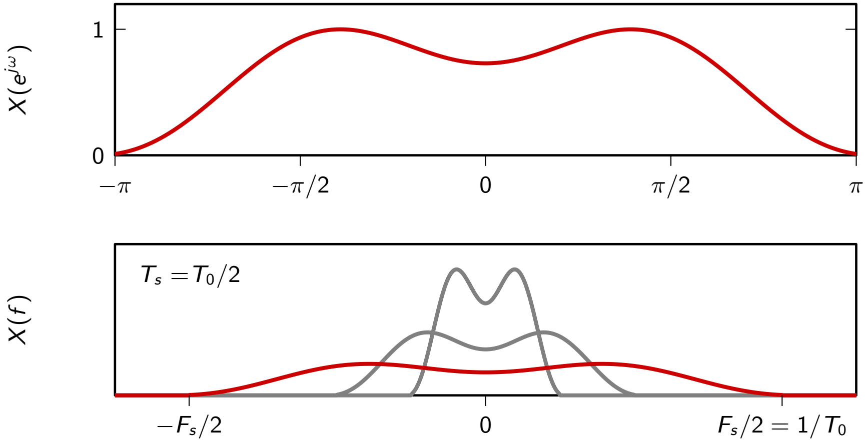
Spectrum of interpolated signals



Spectrum of interpolated signals



Spectrum of interpolated signals



Spectrum of interpolated signals

pick interpolation period T_s:

- $X(f)$ is F_s-bandlimited, with $F_s = 1/T_s$
- fast interpolation (T_s small) $\rightarrow$ wider spectrum
- slow interpolation (T_s large) $\rightarrow$ narrower spectrum
- (for those who remember...) it's like changing the speed of a record player

Spectrum of interpolated signals

pick interpolation period T_s:

- $X(f)$ is F_s-bandlimited, with $F_s = 1/T_s$
- fast interpolation (T_s small) $\rightarrow$ wider spectrum
- slow interpolation (T_s large) $\rightarrow$ narrower spectrum
- (for those who remember...) it's like changing the speed of a record player

Spectrum of interpolated signals

pick interpolation period T_s:

- $X(f)$ is F_s-bandlimited, with $F_s = 1/T_s$
- fast interpolation (T_s small) $\rightarrow$ wider spectrum
- slow interpolation (T_s large) $\rightarrow$ narrower spectrum
- (for those who remember...) it's like changing the speed of a record player

Spectrum of interpolated signals

pick interpolation period T_s:

- $X(f)$ is F_s-bandlimited, with $F_s = 1/T_s$
- fast interpolation (T_s small) $\rightarrow$ wider spectrum
- slow interpolation (T_s large) $\rightarrow$ narrower spectrum
- (for those who remember...) it's like changing the speed of a record player

Space of bandlimited functions

$$x[n] \in \ell_2(\mathbb{Z}) \xrightarrow{\quad T_s \quad} \begin{array}{c} x(t) \in L_2(\mathbb{R}) \\ F_s\text{-BL} \end{array}$$

Space of bandlimited functions

$$x[n] \in \ell_2(\mathbb{Z}) \quad \overset{T_s}{\underset{?}{\longleftrightarrow}} \quad \begin{array}{c} x(t) \in L_2(\mathbb{R}) \\ F_s\text{-BL} \end{array}$$

Let's lighten the notation

for a while we will proceed with $T_s = 1$ (so that $F_s = 1$ as well)
(derivations in the general case are in the book)

The road to the sampling theorem

claims:

- the space of 1-bandlimited functions is a Hilbert space
- the functions $\varphi^{(n)}(t) = \text{sinc}(t - n)$, with $n \in \mathbb{Z}$, form a basis for the space
- if $x(t)$ is 1-BL, the sequence $x[n] = x(n)$, with $n \in \mathbb{Z}$, is a sufficient representation (i.e. we can reconstruct $x(t)$ from $x[n]$)

The road to the sampling theorem

claims:

- the space of 1-bandlimited functions is a Hilbert space
- the functions $\varphi^{(n)}(t) = \text{sinc}(t - n)$, with $n \in \mathbb{Z}$, form a basis for the space
- if $x(t)$ is 1-BL, the sequence $x[n] = x(n)$, with $n \in \mathbb{Z}$, is a sufficient representation (i.e. we can reconstruct $x(t)$ from $x[n]$)

The road to the sampling theorem

claims:

- the space of 1-bandlimited functions is a Hilbert space
- the functions $\varphi^{(n)}(t) = \text{sinc}(t - n)$, with $n \in \mathbb{Z}$, form a basis for the space
- if $x(t)$ is 1-BL, the sequence $x[n] = x(n)$, with $n \in \mathbb{Z}$, is a sufficient representation (i.e. we can reconstruct $x(t)$ from $x[n]$)

The space 1-BL

- clearly a vector space because 1-BL $\subset L_2(\mathbb{R})$ (and linear combinations of 1-BL functions are 1-BL functions)
- inner product is standard inner product in $L_2(\mathbb{R})$
- completeness... that's more delicate

The space 1-BL

- clearly a vector space because 1-BL $\subset L_2(\mathbb{R})$ (and linear combinations of 1-BL functions are 1-BL functions)
- inner product is standard inner product in $L_2(\mathbb{R})$
- completeness... that's more delicate

The space 1-BL

- clearly a vector space because 1-BL $\subset L_2(\mathbb{R})$ (and linear combinations of 1-BL functions are 1-BL functions)
- inner product is standard inner product in $L_2(\mathbb{R})$
- completeness... that's more delicate

The space of 1-BL functions

recap:

- inner product:

$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x^*(t)y(t)dt$$

- convolution:

$$(x * y)(t) = \langle x^*(\tau), y(t - \tau) \rangle$$

A basis for the 1-BL space

$$\varphi^{(n)}(t) = \mathrm{sinc}(t-n), \qquad n \in \mathbb{Z}$$

$$\begin{aligned}
\langle \varphi^{(n)}(t), \varphi^{(m)}(t) \rangle &= \langle \varphi^{(0)}(t-n), \varphi^{(0)}(t-m) \rangle \\
&= \langle \varphi^{(0)}(t-n), \varphi^{(0)}(m-t) \rangle \\
&= \int_{-\infty}^{\infty} \mathrm{sinc}(t-n)\,\mathrm{sinc}(m-t)\, dt \\
&= \int_{-\infty}^{\infty} \mathrm{sinc}(\tau)\,\mathrm{sinc}((m-n)-\tau)\, d\tau \\
&= (\mathrm{sinc} * \mathrm{sinc})(m-n)
\end{aligned}$$

A basis for the 1-BL space

$$\varphi^{(n)}(t) = \mathrm{sinc}(t-n), \qquad n \in \mathbb{Z}$$

$$\begin{aligned}
\langle \varphi^{(n)}(t), \varphi^{(m)}(t) \rangle &= \langle \varphi^{(0)}(t-n), \varphi^{(0)}(t-m) \rangle \\
&= \langle \varphi^{(0)}(t-n), \varphi^{(0)}(m-t) \rangle \\
&= \int_{-\infty}^{\infty} \mathrm{sinc}(t-n)\,\mathrm{sinc}(m-t)\,dt \\
&= \int_{-\infty}^{\infty} \mathrm{sinc}(\tau)\,\mathrm{sinc}((m-n)-\tau)\,d\tau \\
&= (\mathrm{sinc} * \mathrm{sinc})(m-n)
\end{aligned}$$

A basis for the 1-BL space

$$\varphi^{(n)}(t) = \mathrm{sinc}(t-n), \qquad n \in \mathbb{Z}$$

$$\begin{aligned}
\langle \varphi^{(n)}(t), \varphi^{(m)}(t) \rangle &= \langle \varphi^{(0)}(t-n), \varphi^{(0)}(t-m) \rangle \\
&= \langle \varphi^{(0)}(t-n), \varphi^{(0)}(m-t) \rangle \\
&= \int_{-\infty}^{\infty} \mathrm{sinc}(t-n)\,\mathrm{sinc}(m-t)\,dt \\
&= \int_{-\infty}^{\infty} \mathrm{sinc}(\tau)\,\mathrm{sinc}((m-n)-\tau)\,d\tau \\
&= (\mathrm{sinc} * \mathrm{sinc})(m-n)
\end{aligned}$$

A basis for the 1-BL space

$$\varphi^{(n)}(t) = \mathrm{sinc}(t-n), \qquad n \in \mathbb{Z}$$

$$\begin{aligned}
\langle \varphi^{(n)}(t), \varphi^{(m)}(t) \rangle &= \langle \varphi^{(0)}(t-n), \varphi^{(0)}(t-m) \rangle \\
&= \langle \varphi^{(0)}(t-n), \varphi^{(0)}(m-t) \rangle \\
&= \int_{-\infty}^{\infty} \mathrm{sinc}(t-n)\,\mathrm{sinc}(m-t)\,dt \\
&= \int_{-\infty}^{\infty} \mathrm{sinc}(\tau)\,\mathrm{sinc}((m-n)-\tau)\,d\tau \\
&= (\mathrm{sinc} * \mathrm{sinc})(m-n)
\end{aligned}$$

A basis for the 1-BL space

$$\varphi^{(n)}(t) = \mathrm{sinc}(t-n), \qquad n \in \mathbb{Z}$$

$$\begin{aligned}
\langle \varphi^{(n)}(t), \varphi^{(m)}(t) \rangle &= \langle \varphi^{(0)}(t-n), \varphi^{(0)}(t-m) \rangle \\
&= \langle \varphi^{(0)}(t-n), \varphi^{(0)}(m-t) \rangle \\
&= \int_{-\infty}^{\infty} \mathrm{sinc}(t-n)\,\mathrm{sinc}(m-t)\,dt \\
&= \int_{-\infty}^{\infty} \mathrm{sinc}(\tau)\,\mathrm{sinc}((m-n)-\tau)\,d\tau \\
&= (\mathrm{sinc} * \mathrm{sinc})(m-n)
\end{aligned}$$

A basis for the 1-BL space

now use the convolution theorem knowing that:

$$\mathrm{FT}\left\{\mathrm{sinc}(t)\right\} = \mathrm{rect}\left(f\right)$$

$$\begin{aligned}
(\mathrm{sinc} * \mathrm{sinc})(m-n) &= \int_{-\infty}^{\infty} \mathrm{rect}^2(f)\, e^{j2\pi f(m-n)}\, df \\
&= \int_{-1/2}^{1/2} e^{j2\pi f(m-n)}\, df \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(m-n)}\, d\Omega \\
&= \begin{cases} 1 & \text{for } m = n \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

A basis for the 1-BL space

now use the convolution theorem knowing that:

$$\mathrm{FT}\left\{\mathrm{sinc}(t)\right\} = \mathrm{rect}\left(f\right)$$

$$\begin{aligned}
(\mathrm{sinc} * \mathrm{sinc})(m-n) &= \int_{-\infty}^{\infty} \mathrm{rect}^2(f)\, e^{j2\pi f(m-n)}\, df \\
&= \int_{-1/2}^{1/2} e^{j2\pi f(m-n)}\, df \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\Omega(m-n)}\, d\Omega \\
&= \begin{cases} 1 & \text{for } m = n \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

A basis for the 1-BL space

now use the convolution theorem knowing that:

$$\text{FT}\{\text{sinc}(t)\} = \text{rect}(f)$$

$$\begin{aligned}
(\text{sinc} * \text{sinc})(m-n) &= \int_{-\infty}^{\infty} \text{rect}^2(f)\, e^{j2\pi f(m-n)} df \\
&= \int_{-1/2}^{1/2} e^{j2\pi f(m-n)} df \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\Omega(m-n)} d\Omega \\
&= \begin{cases} 1 & \text{for } m = n \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

A basis for the 1-BL space

now use the convolution theorem knowing that:

$$\text{FT}\left\{\text{sinc}(t)\right\} = \text{rect}\left(f\right)$$

$$\begin{aligned}
(\text{sinc} * \text{sinc})(m-n) &= \int_{-\infty}^{\infty} \text{rect}^2(f)\, e^{j2\pi f(m-n)}\, df \\
&= \int_{-1/2}^{1/2} e^{j2\pi f(m-n)}\, df \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(m-n)}\, d\Omega \\
&= \begin{cases} 1 & \text{for } m = n \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Space of bandlimited functions

$$x[n] \in \ell_2(\mathbb{Z}) \quad \longleftrightarrow \quad \begin{array}{c} x(t) \in L_2(\mathbb{R}) \\ \text{1-BL} \end{array}$$

Sampling as a basis expansion

for any $x(t) \in$ 1-BL:

$$\langle \varphi^{(n)}(t), x(t) \rangle = \langle \mathrm{sinc}(t-n), x(t) \rangle = \langle \mathrm{sinc}(n-t), x(t) \rangle$$

$$= (\mathrm{sinc} * x)(n)$$

$$= \int_{-\infty}^{\infty} \mathrm{rect}\,(f)\, X(f) e^{j2\pi fn} df$$

$$= \int_{-\infty}^{\infty} X(f) e^{j2\pi fn} df$$

$$= x(n)$$

Sampling as a basis expansion

for any $x(t) \in$ 1-BL:

$$\begin{aligned}
\langle \varphi^{(n)}(t), x(t) \rangle &= \langle \mathrm{sinc}(t-n), x(t) \rangle = \langle \mathrm{sinc}(n-t), x(t) \rangle \\
&= (\mathrm{sinc} * x)(n) \\
&= \int_{-\infty}^{\infty} \mathrm{rect}\,(f)\, X(f) e^{j2\pi fn} df \\
&= \int_{-\infty}^{\infty} X(f) e^{j2\pi fn} df \\
&= x(n)
\end{aligned}$$

Sampling as a basis expansion

for any $x(t) \in$ 1-BL:

$$\begin{aligned}
\langle \varphi^{(n)}(t), x(t) \rangle &= \langle \text{sinc}(t-n), x(t) \rangle = \langle \text{sinc}(n-t), x(t) \rangle \\
&= (\text{sinc} * x)(n) \\
&= \int_{-\infty}^{\infty} \text{rect}\,(f)\, X(f) e^{j2\pi fn} df \\
&= \int_{-\infty}^{\infty} X(f) e^{j2\pi fn} df \\
&= x(n)
\end{aligned}$$

Sampling as a basis expansion

for any $x(t) \in$ 1-BL:

$$\begin{aligned}
\langle \varphi^{(n)}(t), x(t) \rangle &= \langle \mathrm{sinc}(t-n), x(t) \rangle = \langle \mathrm{sinc}(n-t), x(t) \rangle \\
&= (\mathrm{sinc} * x)(n) \\
&= \int_{-\infty}^{\infty} \mathrm{rect}\,(f)\, X(f) e^{j2\pi f n} df \\
&= \int_{-\infty}^{\infty} X(f) e^{j2\pi f n} df \\
&= x(n)
\end{aligned}$$

Sampling as a basis expansion

for any $x(t) \in$ 1-BL:

$$
\begin{aligned}
\langle \varphi^{(n)}(t), x(t) \rangle &= \langle \mathrm{sinc}(t-n), x(t) \rangle = \langle \mathrm{sinc}(n-t), x(t) \rangle \\
&= (\mathrm{sinc} * x)(n) \\
&= \int_{-\infty}^{\infty} \mathrm{rect}\,(f)\, X(f) e^{j2\pi f n} df \\
&= \int_{-\infty}^{\infty} X(f) e^{j2\pi f n} df \\
&= x(n)
\end{aligned}
$$

Sampling as a basis expansion, 1-BL

Analysis formula:

$$x[n] = \langle \mathrm{sinc}(t-n), x(t) \rangle$$

Synthesis formula:

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \, \mathrm{sinc}(t-n)$$

Sampling as a basis expansion, F_s-BL

Analysis formula:

$$x[n] = \left\langle \mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right), x(t) \right\rangle = T_s\, x(nT_s)$$

Synthesis formula:

$$x(t) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} x[n]\, \mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

The sampling theorem

- the space of F_s-bandlimited functions is a Hilbert space
- set $T_s = 1/F_s$
- the functions $\varphi^{(n)}(t) = \text{sinc}\left(\frac{t - nT_s}{T_s}\right)$ form a basis for the space
- for any $x(t) \in F_s$-BL the coefficients in the sinc basis are the (scaled) samples $T_s\, x(nT_s)$

for any $x(t) \in F_s$-BL, a sufficient representation is the sequence $x[n] = x(nT_s)$

The sampling theorem

- the space of F_s-bandlimited functions is a Hilbert space
- set $T_s = 1/F_s$
- the functions $\varphi^{(n)}(t) = \mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right)$ form a basis for the space
- for any $x(t) \in F_s$-BL the coefficients in the sinc basis are the (scaled) samples $T_s\, x(nT_s)$

for any $x(t) \in F_s$-BL, a sufficient representation is the sequence $x[n] = x(nT_s)$

The sampling theorem, corollary

- F_s-BL $\subseteq$ F-BL for any $F \geq F_s$

for any $x(t) \in F_s$-BL, a sufficient representation is the sequence
$x[n] = x(nT_s)$ for any $T_s \leq 1/F_s$

The sampling theorem, corollary

- F_s-BL $\subseteq F$-BL for any $F \geq F_s$

$$\text{for any } x(t) \in F_s\text{-BL, a sufficient representation is the sequence}$$
$$x[n] = x(nT_s) \text{ for any } T_s \leq 1/F_s$$

The sampling theorem, again

any signal $x(t)$ whose highest frequency component is F_N Hz
can be sampled with no loss of information
using a sampling frequency $F_s \geq 2F_N$ (i.e. a sampling period $T_s \leq 1/(2F_N)$)

F_N is called the Nyquist frequency of the signal.