COM303: Digital Signal Processing

Lecture 18: Multirate signal processing

overview

- ideal and practical sampling and interpolation
- bandpass sampling
- multirate signal processing

advanced topics in sampling

From Continuous to Discrete Time

$$x_c(t) \longrightarrow x[n]$$

From Continuous to Discrete Time

$$x_c(t) \longrightarrow x[n]$$

ideally

in practice

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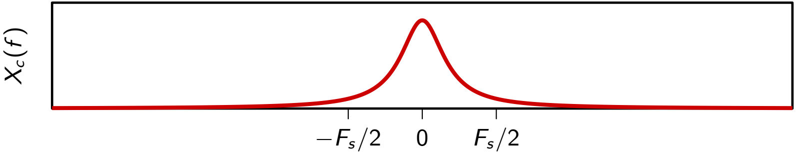
$$X(e^{j\omega}) = F_s\, X_c\left(\frac{\omega}{2\pi} F_s\right)$$

in practice

Ideal case



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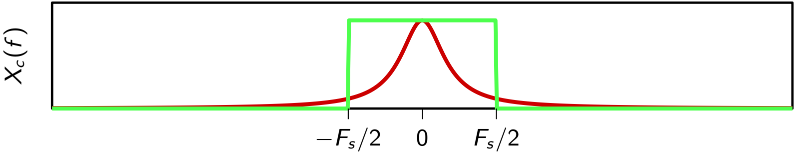

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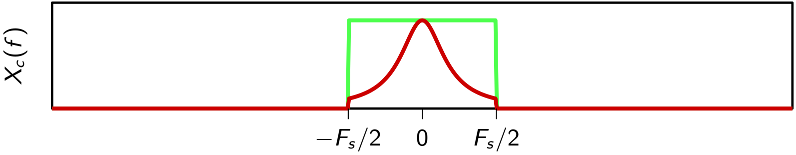

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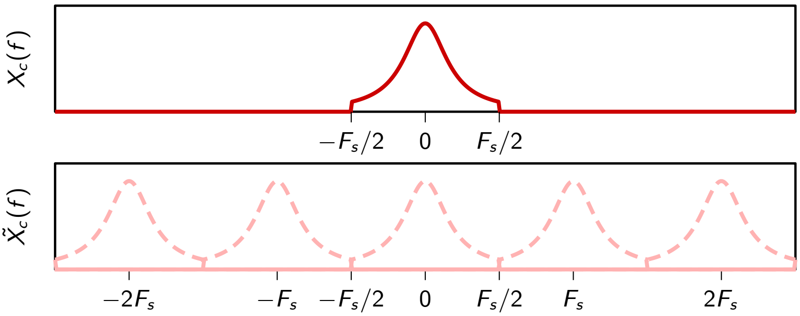

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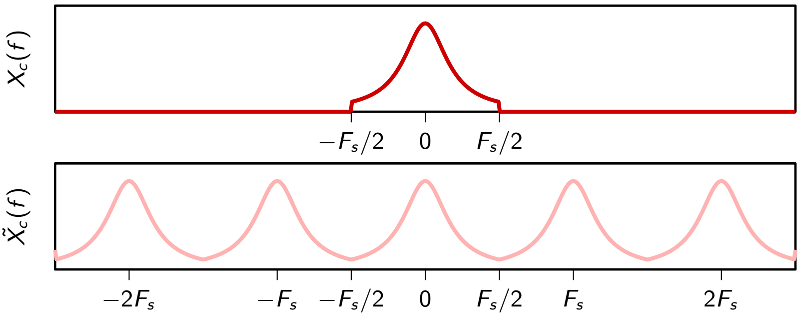

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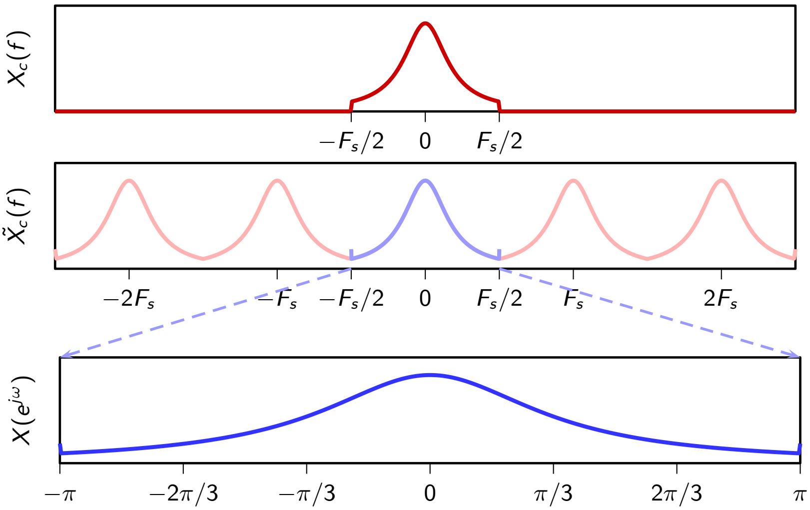

$X_c(f)$
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From Continuous to Discrete Time

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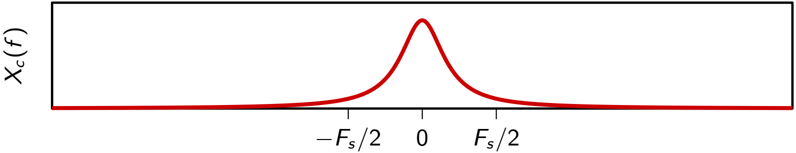
In practice



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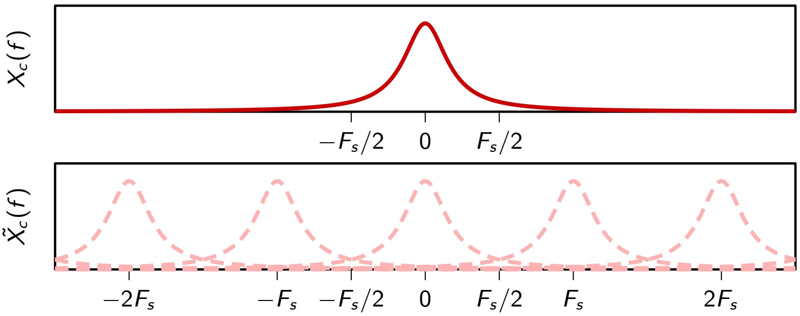

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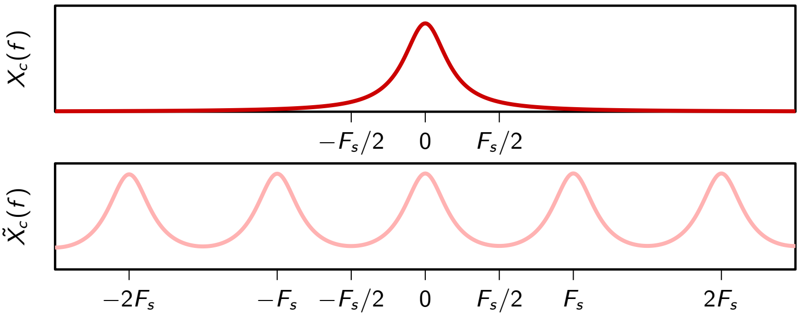

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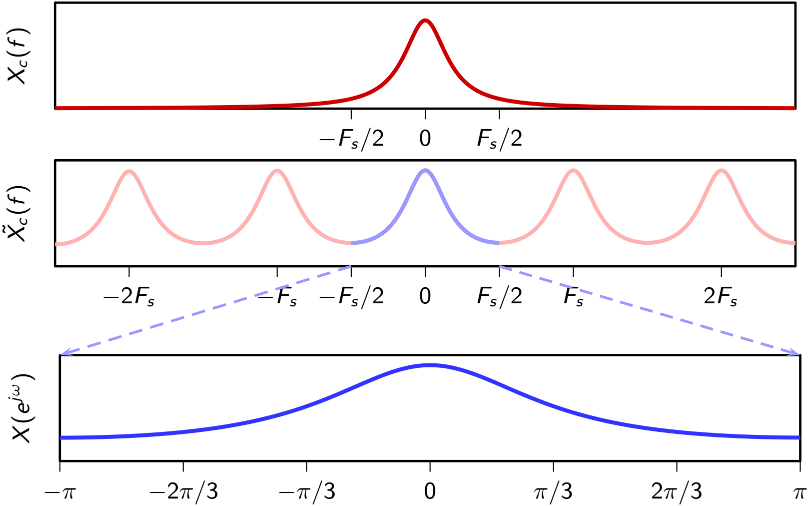

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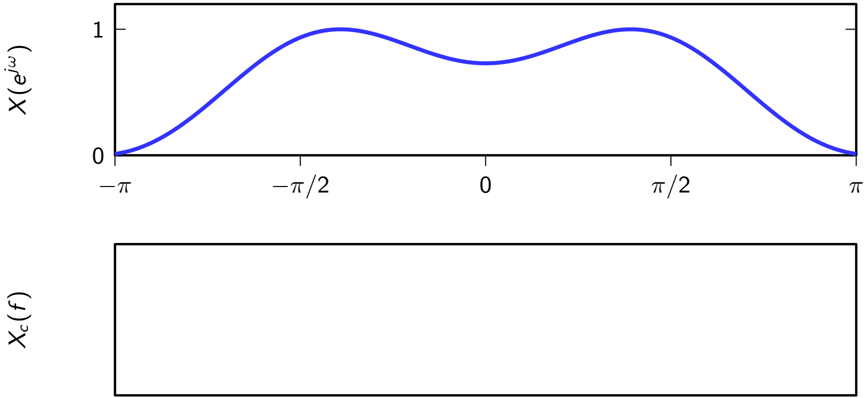
sinc interpolation



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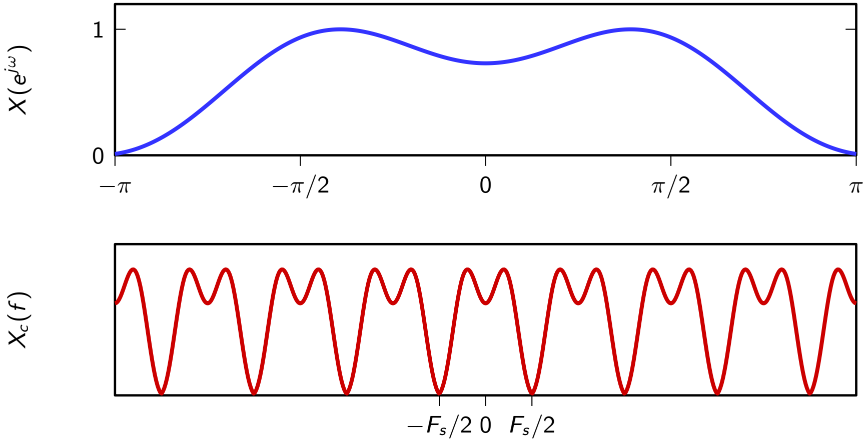

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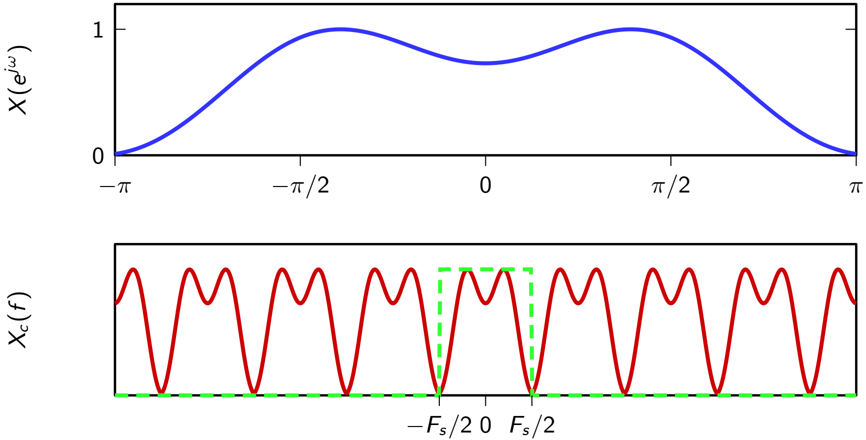

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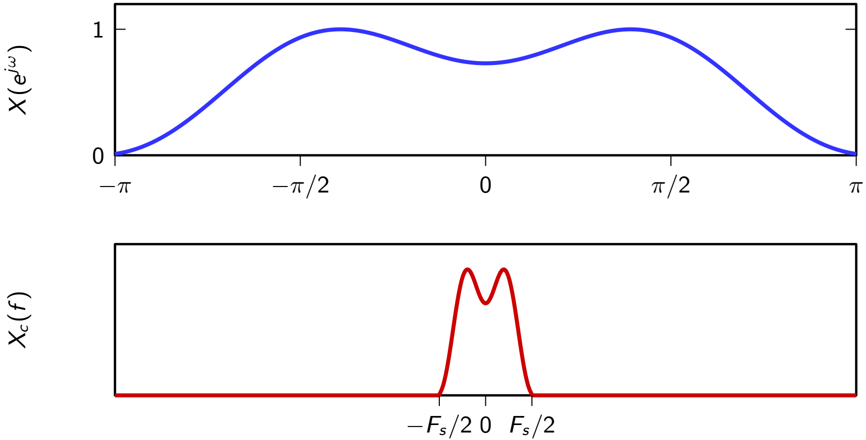

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in practice

$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

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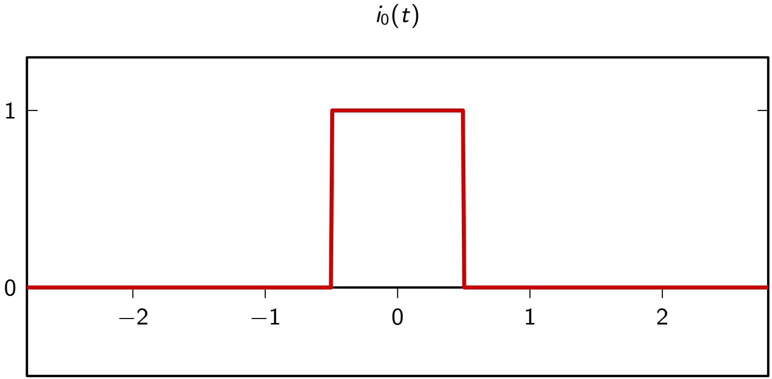
$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n]\, i\left(\frac{t - nT_s}{T_s}\right)$$

$$X_c(f) = ?$$

Practical interpolation

$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n]\, i\left(\frac{t - nT_s}{T_s}\right)$$

Local interpolators



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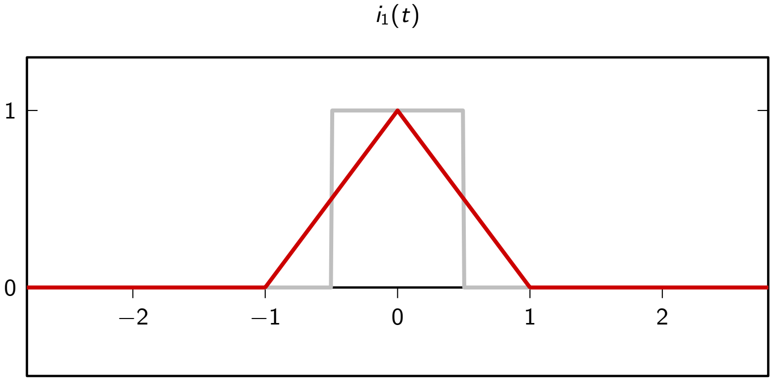

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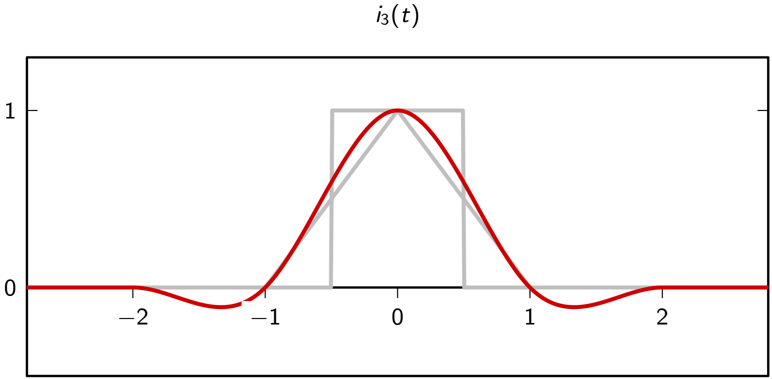

Spectral representation (I)

$$X_c(f) = \int_{-\infty}^{\infty} x_c(t)\, e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\, i\left(\frac{t - nT_s}{T_s}\right) e^{-j2\pi ft} dt$$

$$= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} i\left(\frac{t - nT_s}{T_s}\right) e^{-j2\pi ft} dt$$

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\end{aligned}
$$

Fourier Transform properties

$$\mathrm{FT}\left\{x(t/a)\right\} = aX(af)$$

$$\mathrm{FT}\left\{x(t-b)\right\} = X(f)\, e^{-j2\pi bf}$$

$$\int_{-\infty}^{\infty} i\left(\frac{t-nT_s}{T_s}\right) e^{-j2\pi ft} dt = T_s\, I\left(T_s f\right) e^{-j2\pi nT_s f}$$

Spectral representation (II)

$$X(f) = T_s \sum_{n=-\infty}^{\infty} x[n]\, I\,(T_s f)\; e^{-j2\pi n T_s f}$$

$$= \frac{1}{F_s} I \left(\frac{f}{F_s}\right) \sum_{n=-\infty}^{\infty} x[n]\, e^{-j2\pi n f/F_s}$$

$$= \frac{1}{F_s} I \left(\frac{f}{F_s}\right) X(e^{j2\pi f/F_s})$$

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\end{aligned}
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sinc interpolation

$$i(t) = \mathrm{sinc}(t)$$
$$I(f) = \mathrm{rect}(f)$$

$$X(f) = \frac{1}{F_s}\ \mathrm{rect}\left(\frac{f}{F_s}\right) X(e^{j2\pi f/F_s})$$

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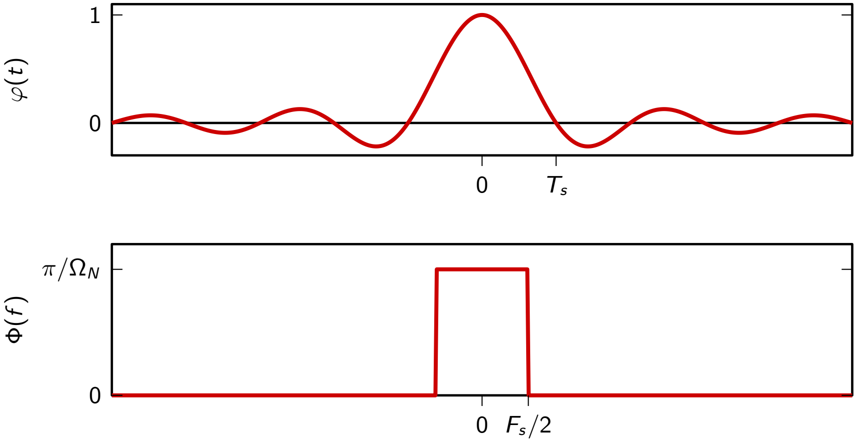

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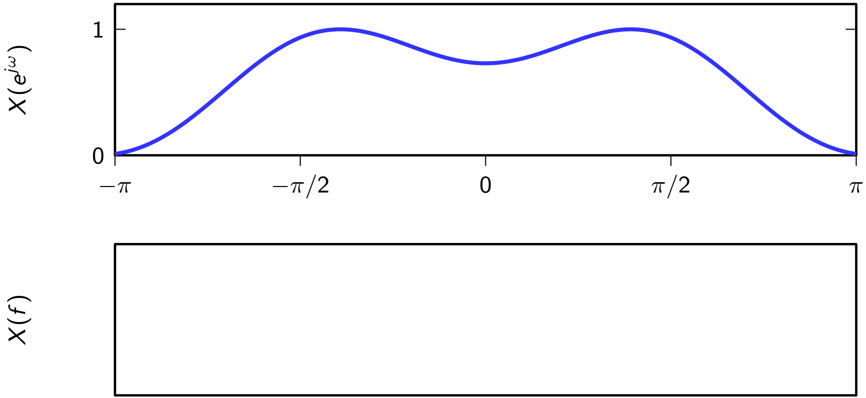

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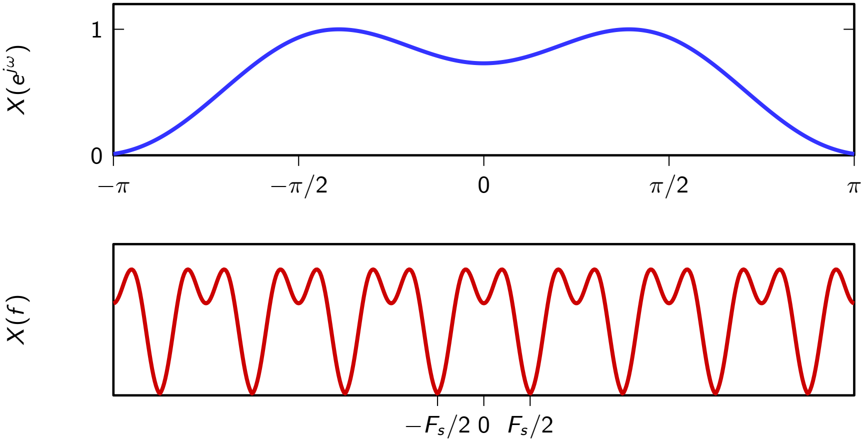

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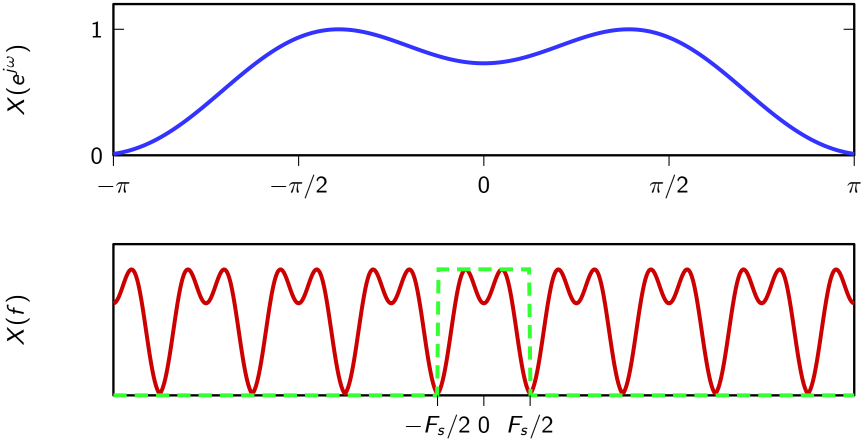

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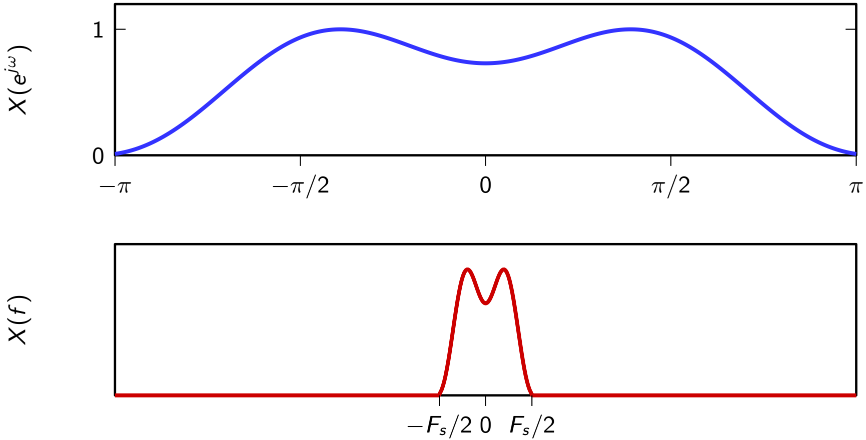

zero-order hold

$$i(t) = \mathrm{rect}(t)$$
$$I(f) = \mathrm{sinc}(f)$$

$$X(f) = \frac{1}{F_s}\, I\,(f/F_s)\; X(e^{j2\pi f/F_s})$$
$$= \frac{1}{F_s}\, \mathrm{sinc}\left(\frac{f}{F_s}\right) X(e^{j2\pi f/F_s})$$

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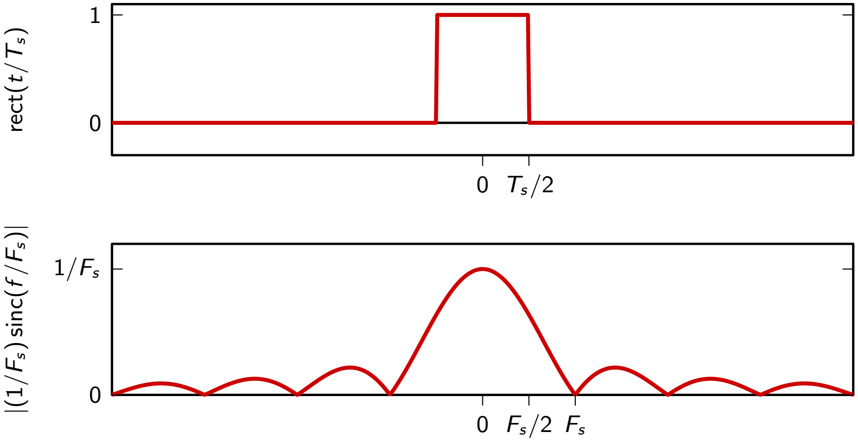

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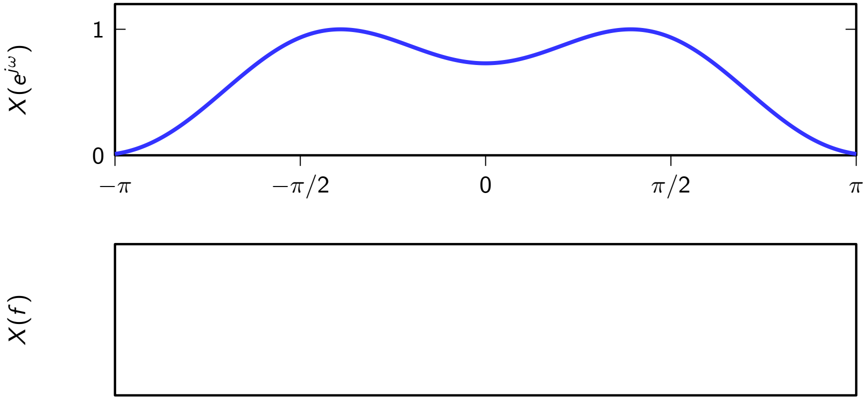

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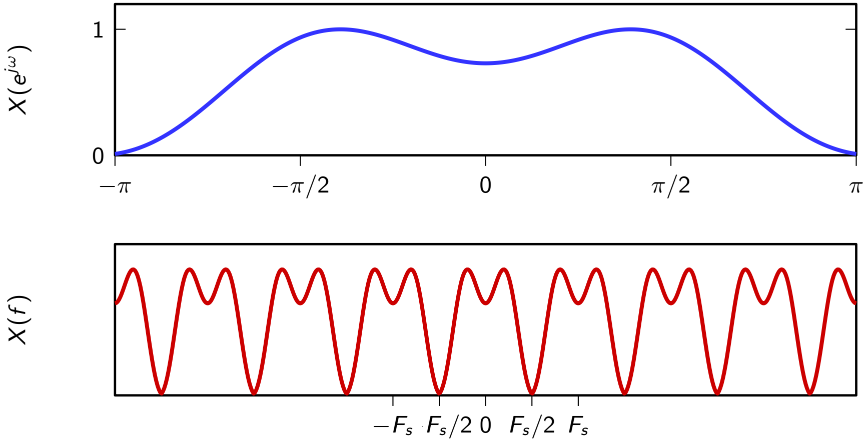

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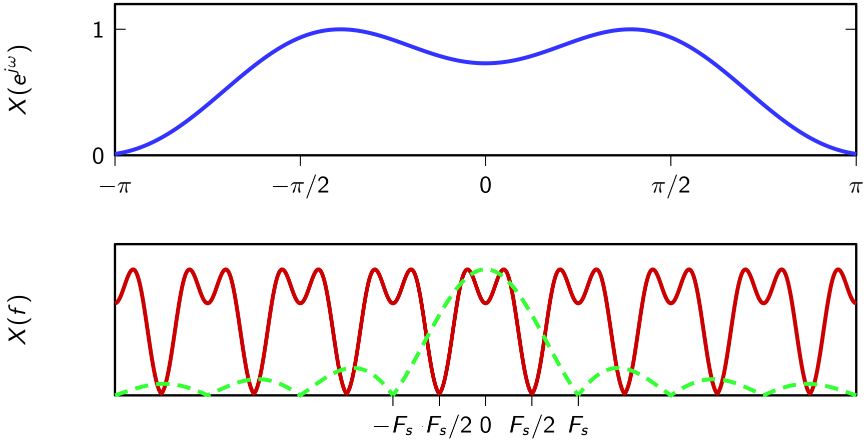

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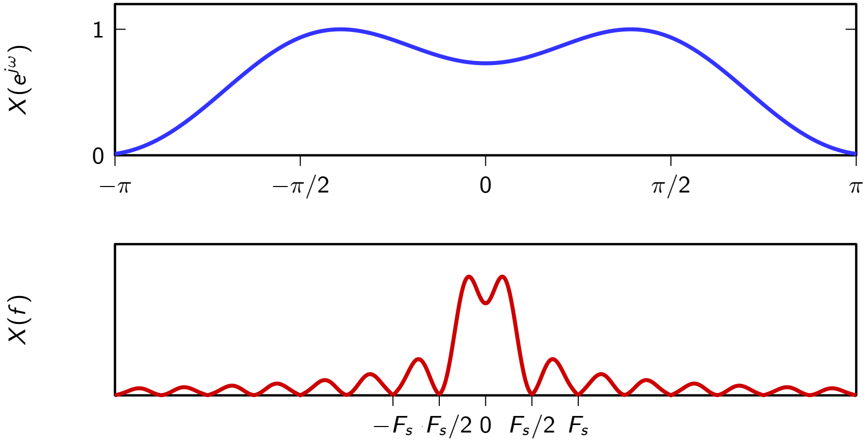
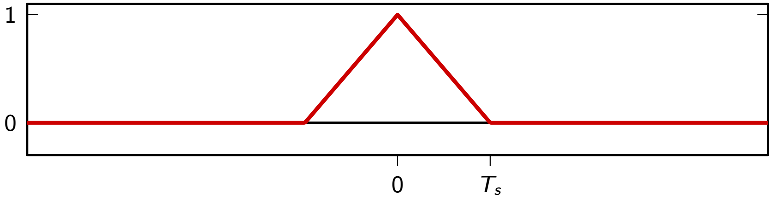
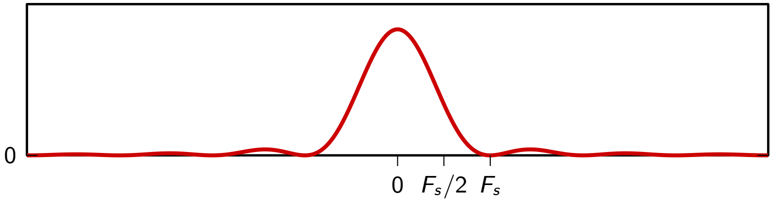


first-order interpolator





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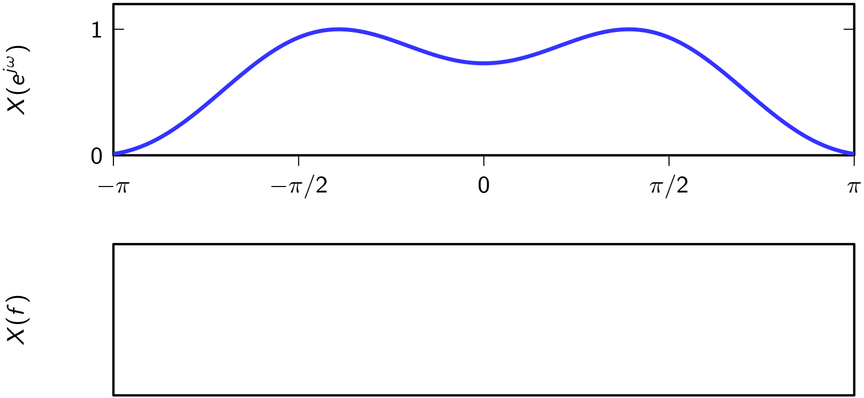

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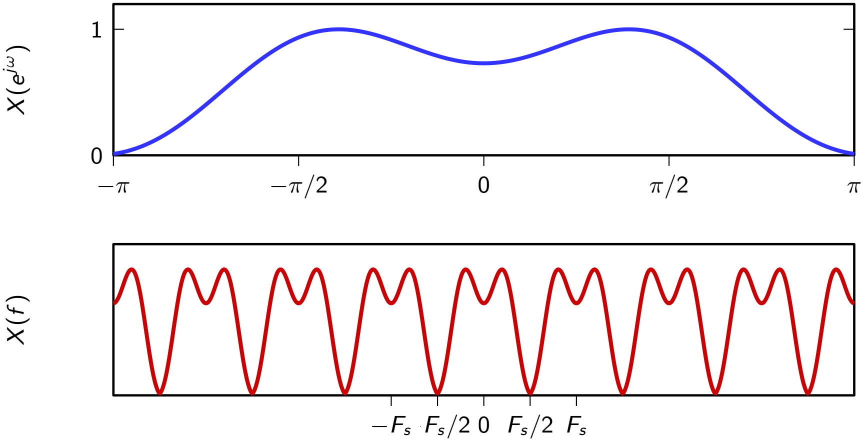

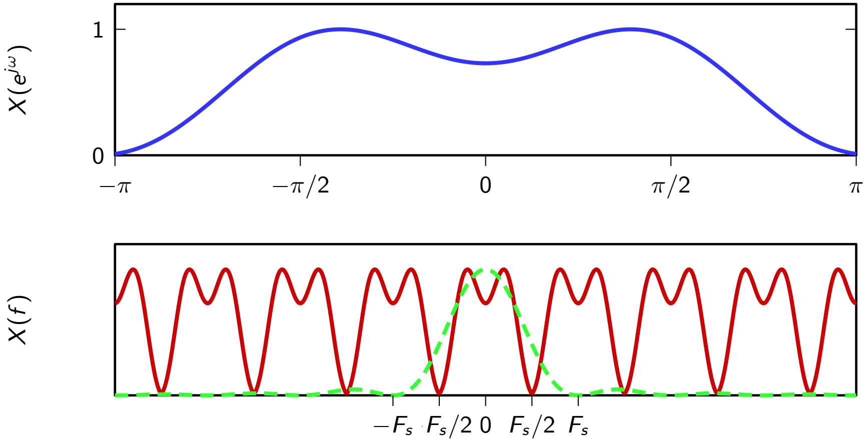
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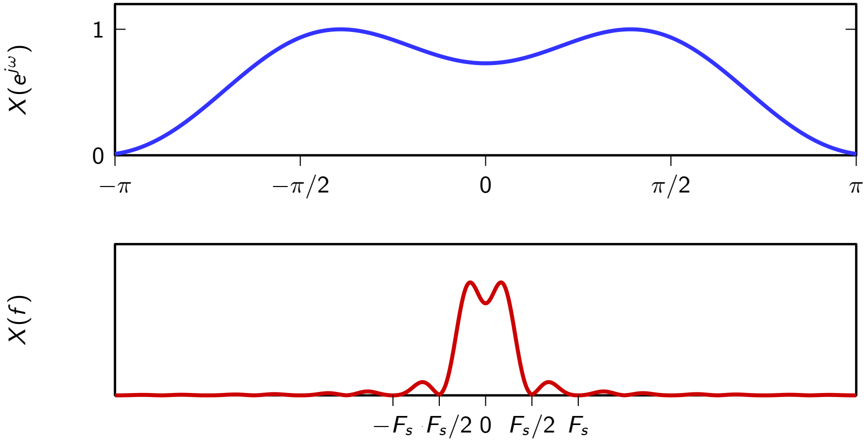

Bandpass Sampling

sampling theorem gives a *sufficient* condition

- in theory, $F_s > 2f_{\max}$
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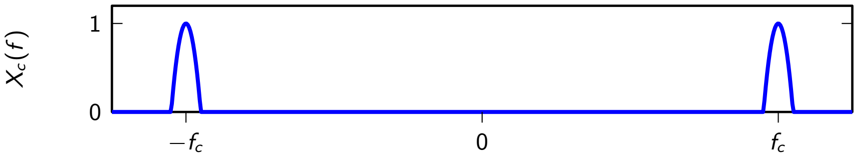

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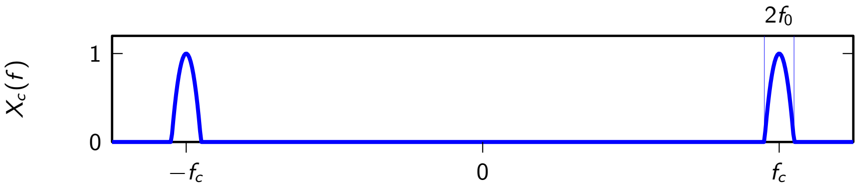

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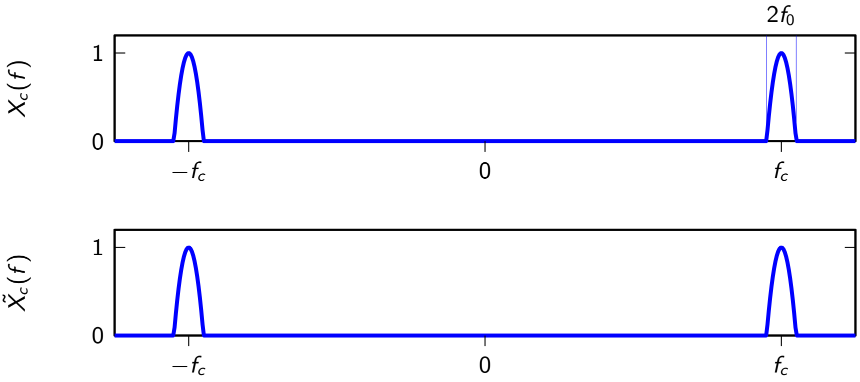

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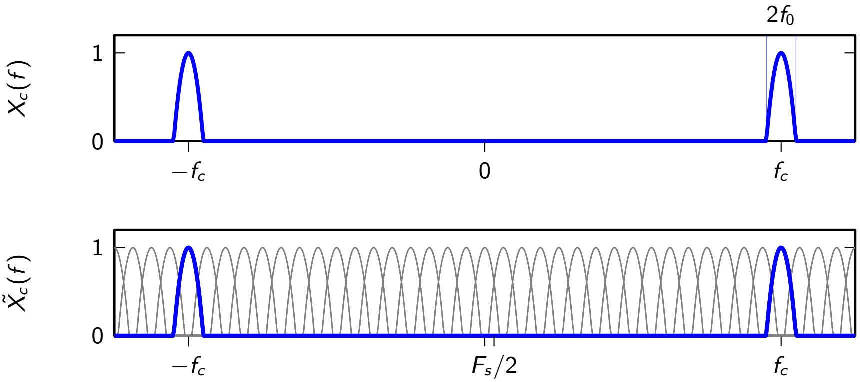

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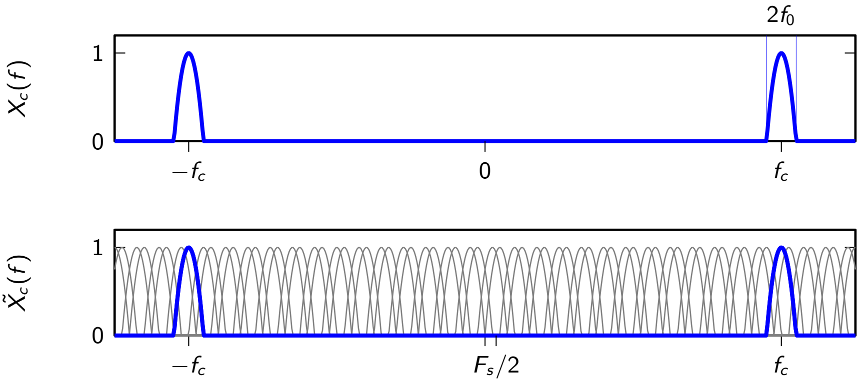

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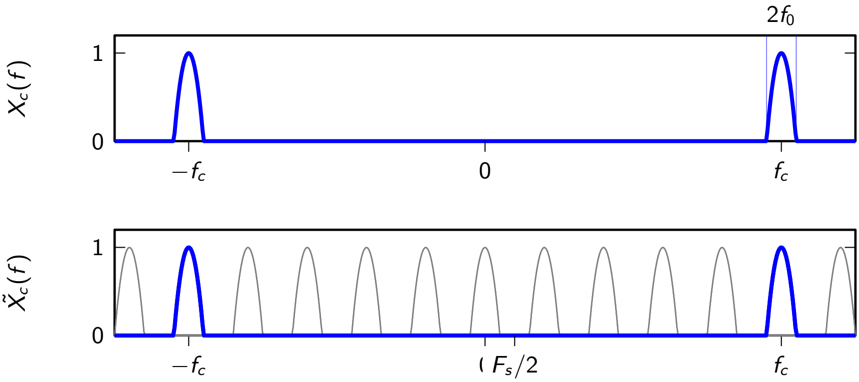

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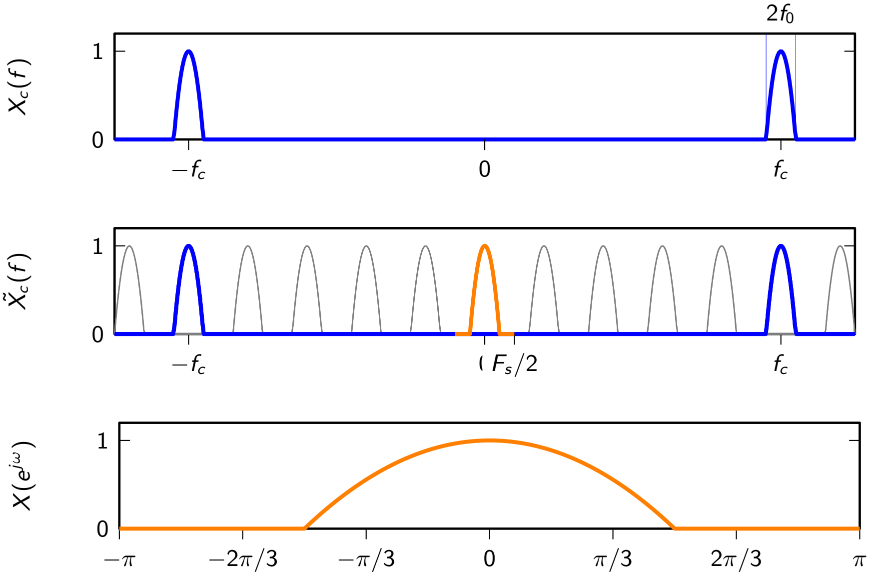

Bandpass sampling conditions

- bandpass signal: $X(f) = 0$ for $|f - f_c| > f_0$
- no alias requires at least: $F_s \geq 2f_0$
- baseband condition: $F_s = 2f_c/k$ for some $k \in \mathbb{N}$

- for $T_s = 1/F_s$, $x[n] = x(nT_s)$ is an alias-free, baseband DT version of $x(t)$

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Example: AM Channel

- AM radio band: 500kHz to 1.6MHz
- channel width is 9kHZ, i.e. $f_0 = 4.5$KHz
- take a channel at $f_c = 1.5$MHz
- in theory: $F_s \geq 2 * 1,504,500$Hz, $T_s < 10^{-6}$ seconds!

- antialias: $F_s \geq 2f_0 \Rightarrow F_s \geq 9$KHz
- baseband: $kF_s/2 = 1500$kHz

- pick $k = 300$
- $F_s = 10$KHz, $T_s = 10^{-4}$

Example: AM Channel

- AM radio band: 500kHz to 1.6MHz
- channel width is 9kHZ, i.e. $f_0 = 4.5\text{KHz}$
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multirate signal processing

Overview

- why multirate?
- upsampling
- downsampling
- applications

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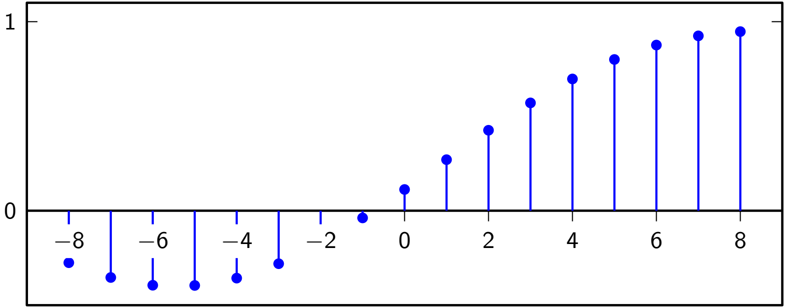
Upsampling

$$x_{NU}[n] = \begin{cases} x[k] & \text{for } n = kN, \quad k \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

$$x[n] \longrightarrow N\uparrow \longrightarrow x_{NU}[n]$$

Example: upsampling by 4



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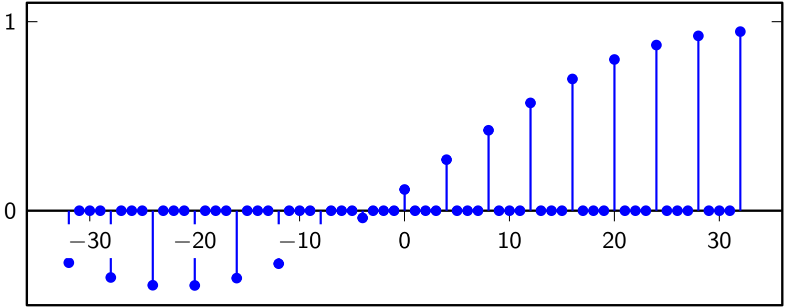

Spectral representation

$$X_{NU}(z) = \sum_{k=-\infty}^{\infty} x_{NU}[k]z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} x[k]z^{-Nk}$$

$$= X(z^N)$$

$$X_{NU}(e^{j\omega}) = X(e^{j\omega N})$$

Spectral representation

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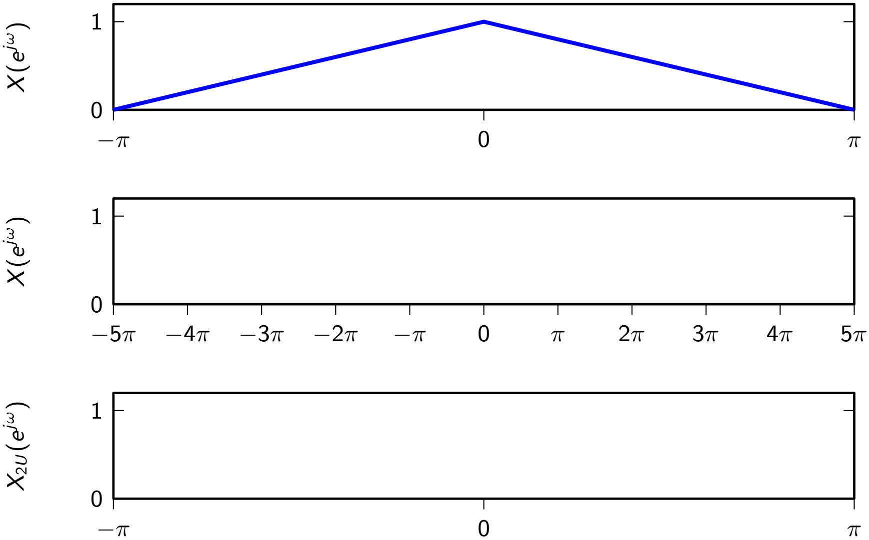
Upsampling by 2



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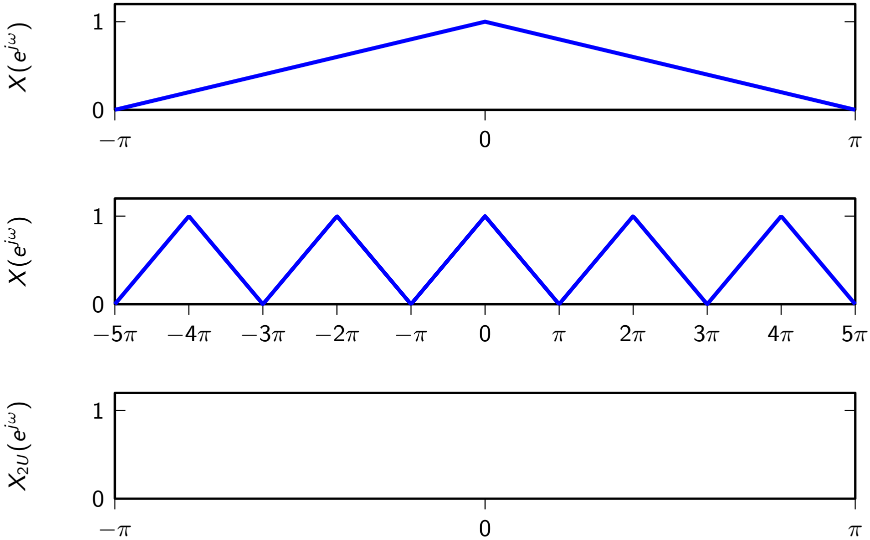
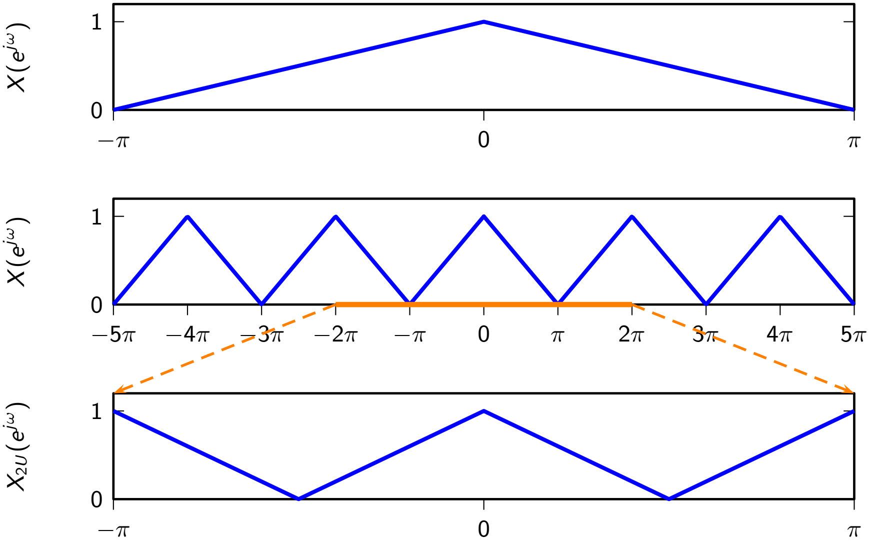
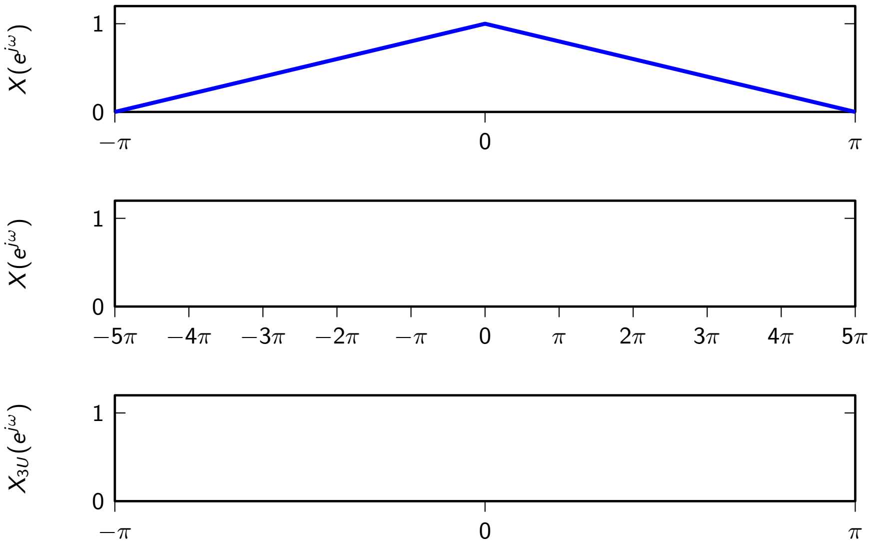


Upsampling by 2



Upsampling by 3



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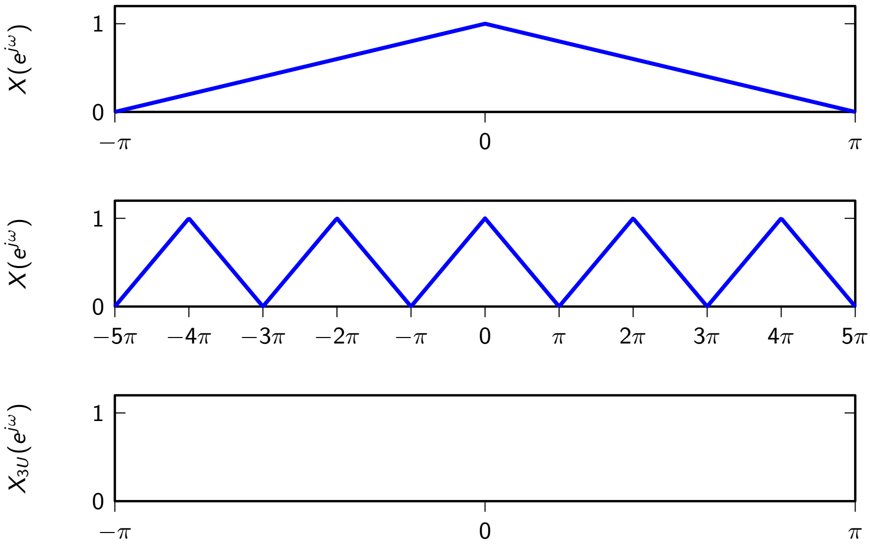

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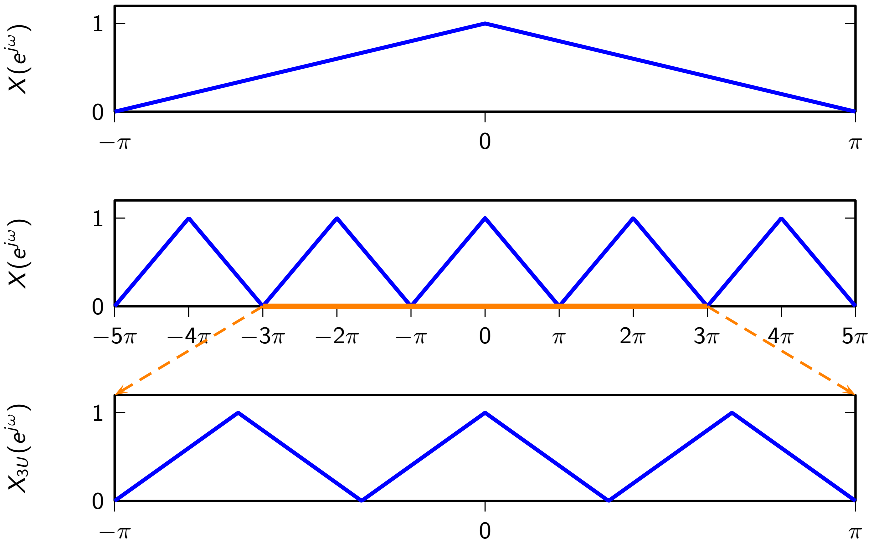
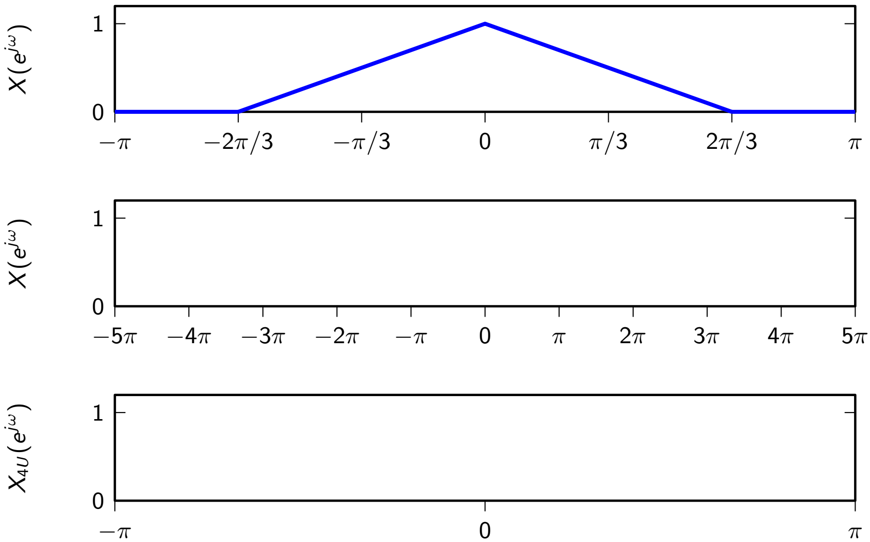
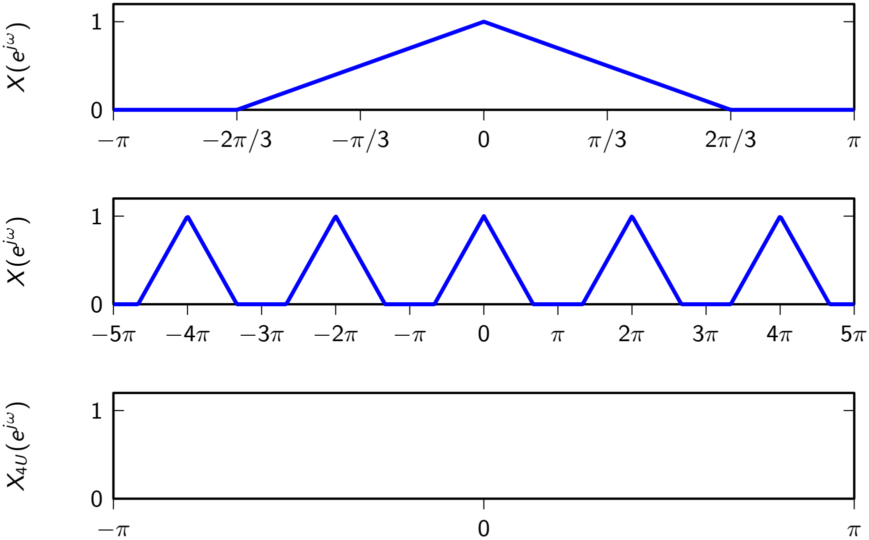


Upsampling by 4



Upsampling by 4



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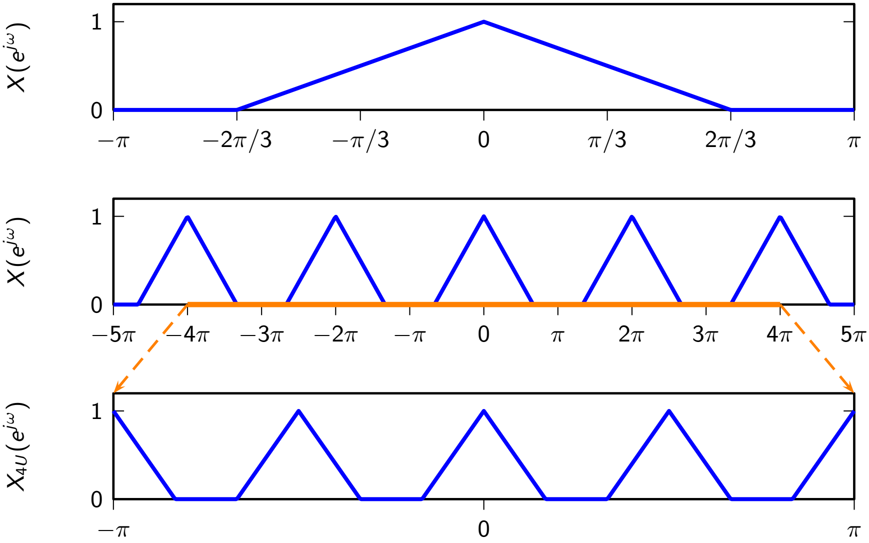

Upsampling: what we don't like

- in the time domain: zeros between nonzero samples are not "natural"
- in the frequency domain: extra replicas of the spectrum; can we get rid of them?

the two problems are the same!

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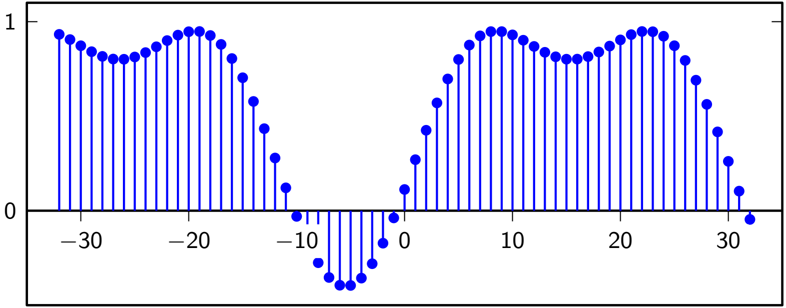
Upsampling in the time domain, revisited



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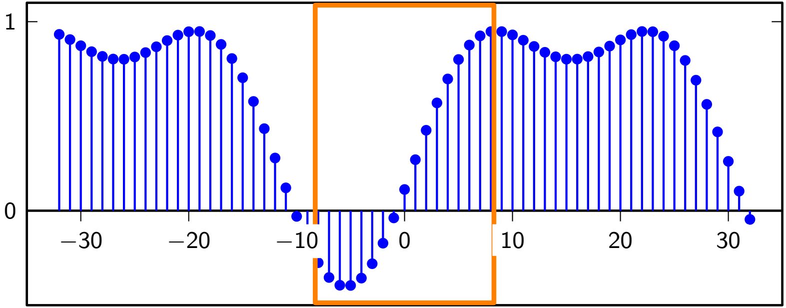

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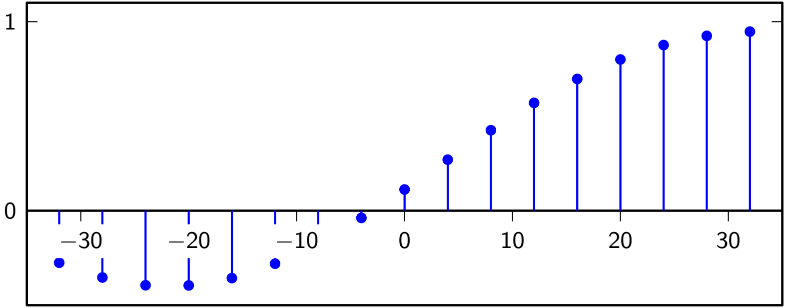

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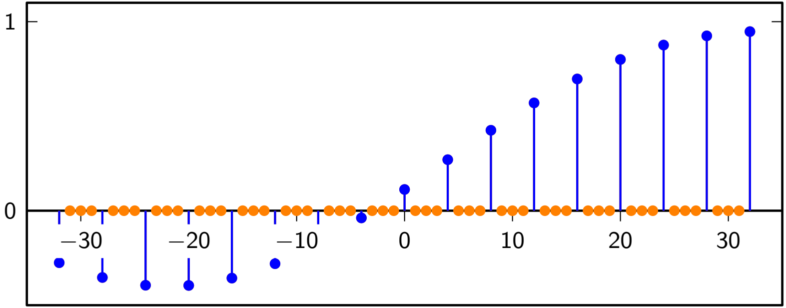

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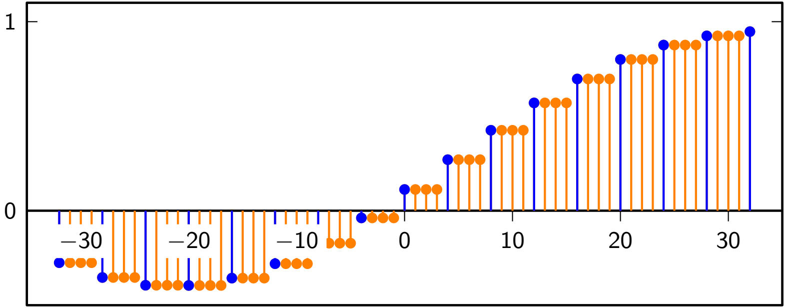
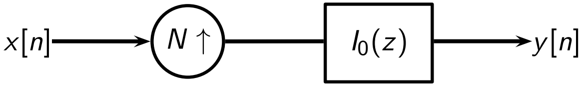
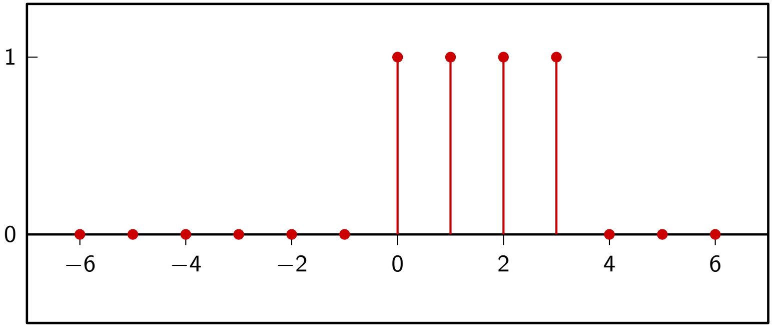


Zero-order interpolator



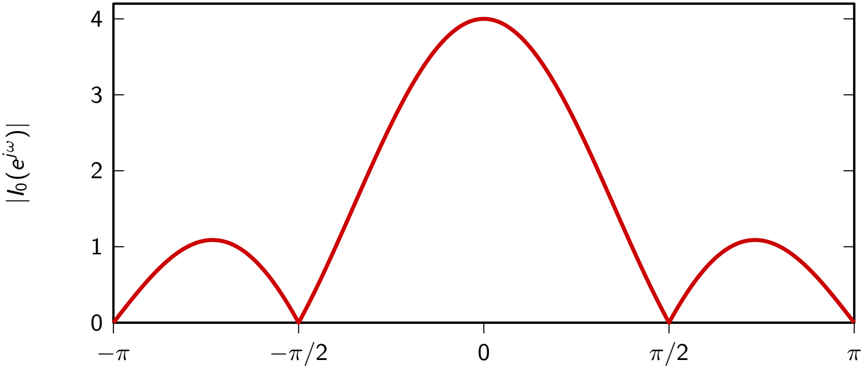
Zero-order interpolator for 4-upsampling

$$i_0[n] = u[n] - u[n-4]$$



Zero-order interpolator for 4-upsampling

$$|I_0(e^{j\omega})| = \left|\frac{\sin\left(\frac{\omega}{2}N\right)}{\sin\left(\frac{\omega}{2}\right)}\right| \qquad N = 4$$



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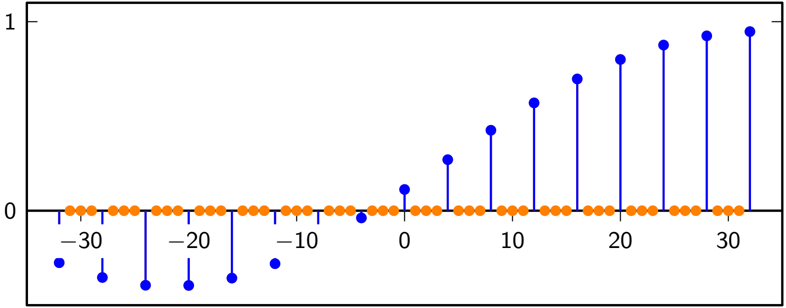

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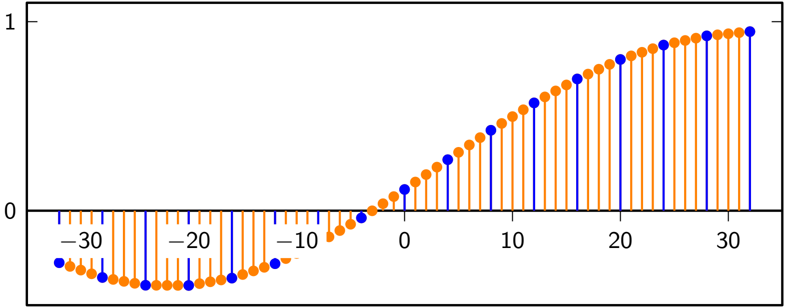
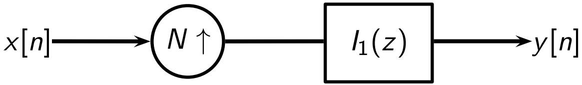
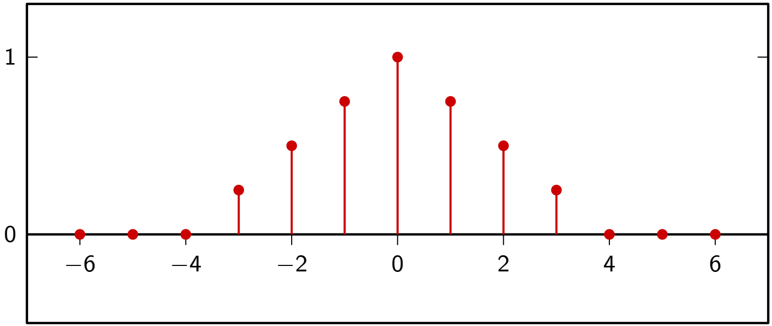


first-order interpolator



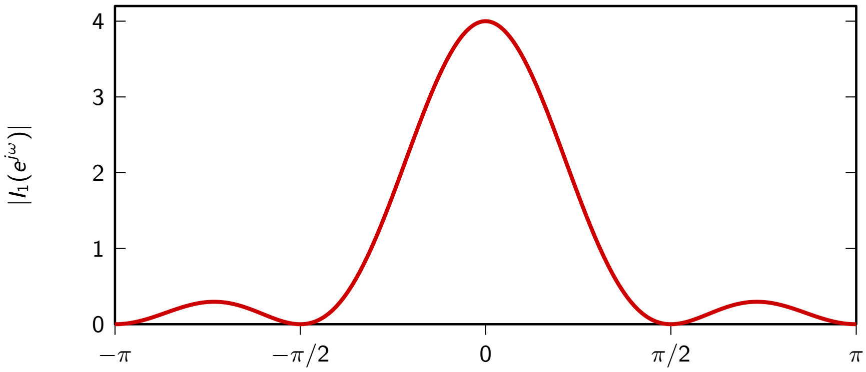
first-order interpolator for 4-upsampling

$$i_1[n] = (i_0[n] * i_0[n])/N$$

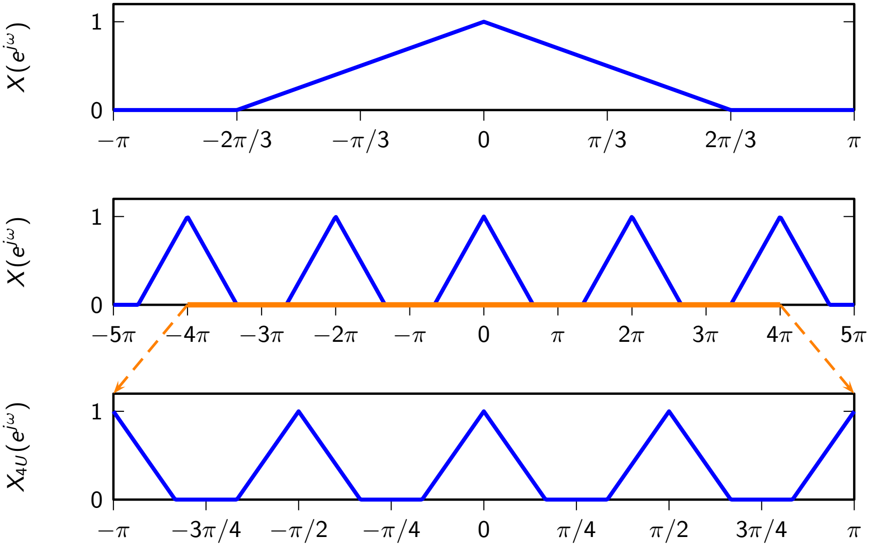


first-order interpolator for 4-upsampling

$$|I_1(e^{j\omega})| = |I_0(e^{j\omega})|^2/N \qquad N = 4$$



in the frequency domain...



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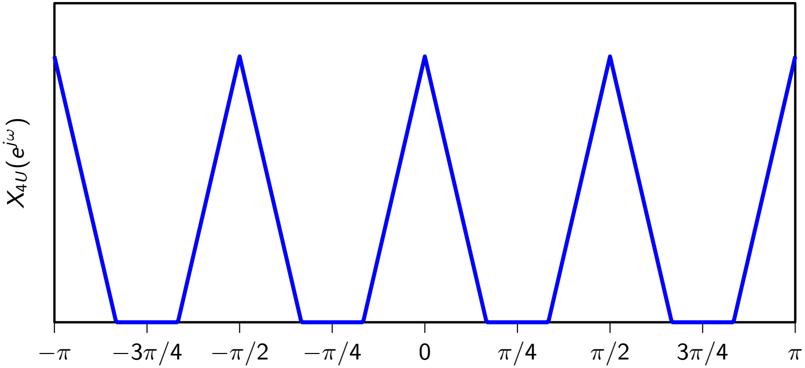

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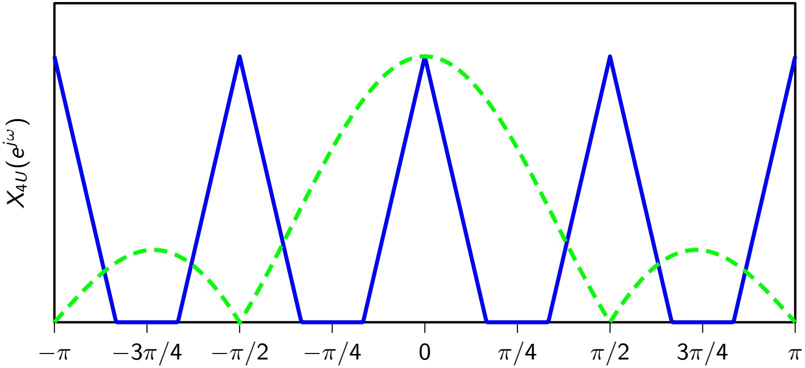

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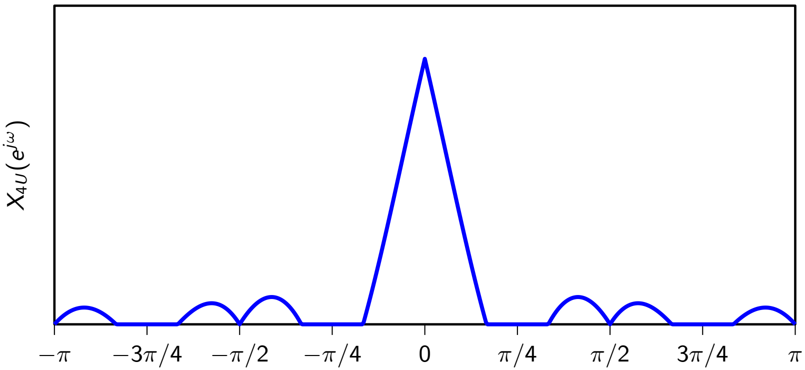

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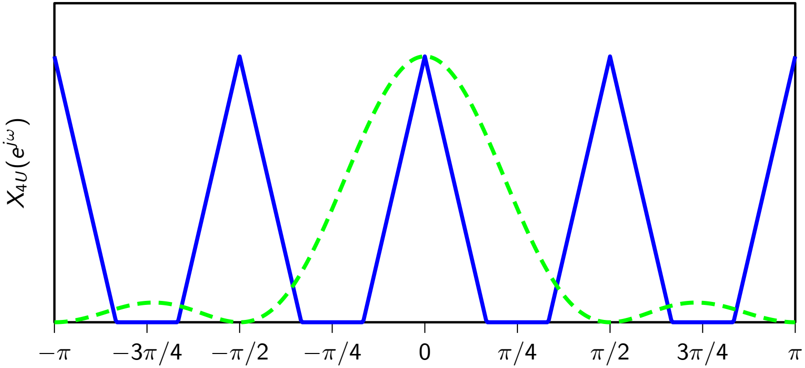

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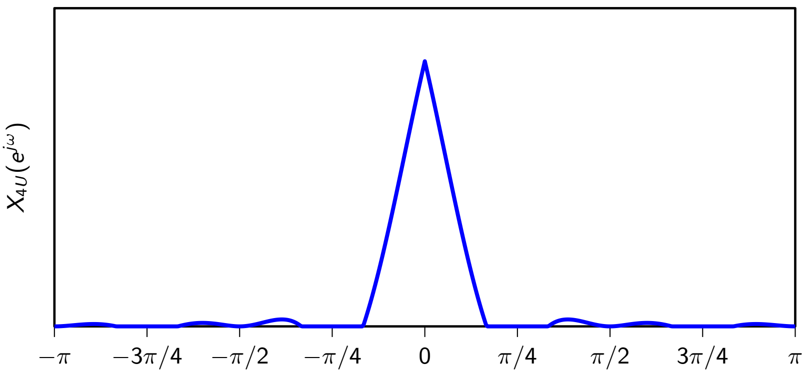

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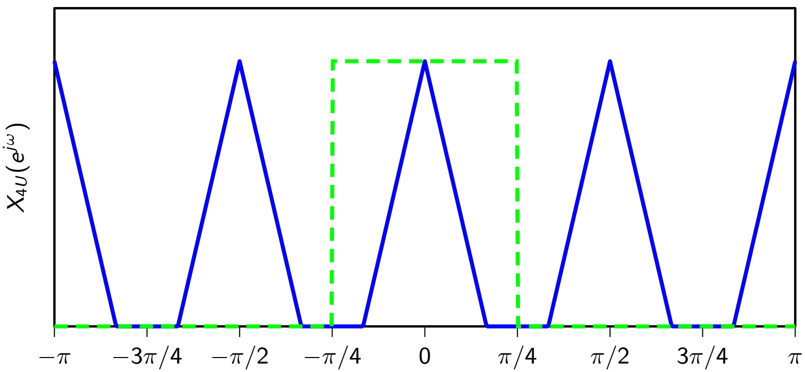

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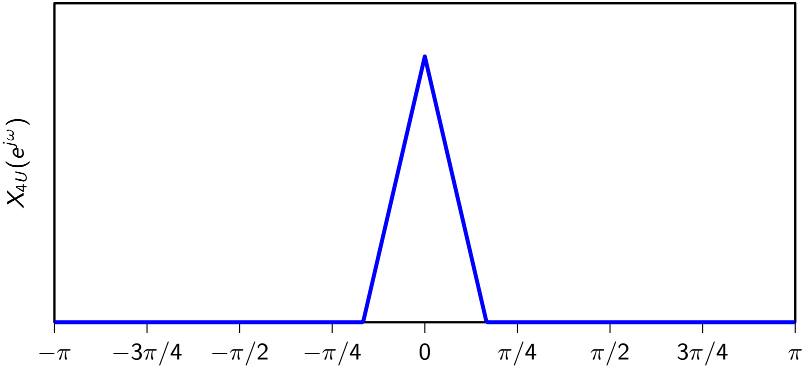
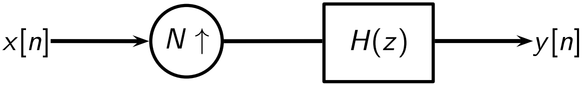


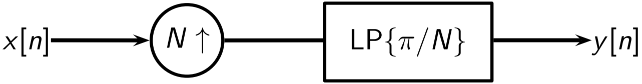
ideal digital interpolator



$$H(e^{j\omega}) = \mathrm{rect}(\omega N/2\pi)$$

$$h[n] = (1/N)\,\mathrm{sinc}(n/N)$$

ideal digital interpolator

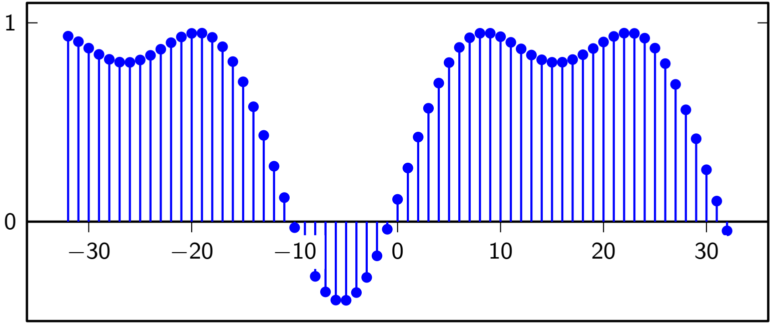


Downsampling

$$x_{ND}[n] = x[nN]$$

$$x[n] \longrightarrow N\downarrow \longrightarrow x_{ND}[n]$$

Example: downsampling by 4



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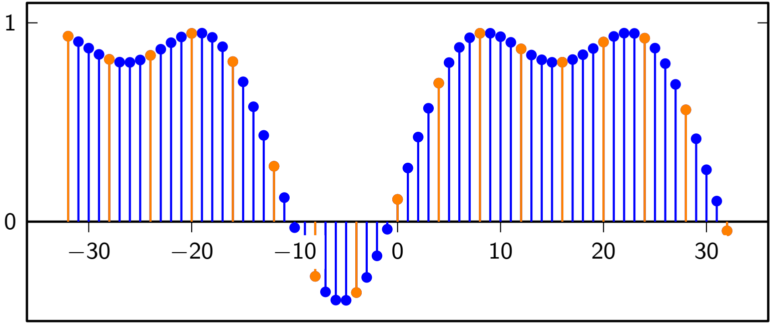

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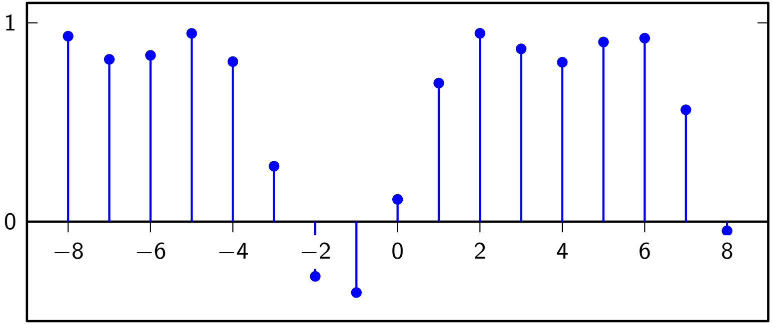

Spectral representation

$$X_{ND}(z) = \sum_{k=-\infty}^{\infty} x[kN]z^{-k} = \quad ?$$

if we can compute

$$A(z) = \sum_{k=-\infty}^{\infty} x[kN]z^{-kN}$$

then

$$X_{ND}(z) = A(z^{1/N})$$

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$$= \sum_{k=-\infty}^{\infty} \xi[k]x[k]z^{-k}$$

$$\xi[n] = \begin{cases} 1 & \text{for } n = kN \\ 0 & \text{otherwise} \end{cases}$$

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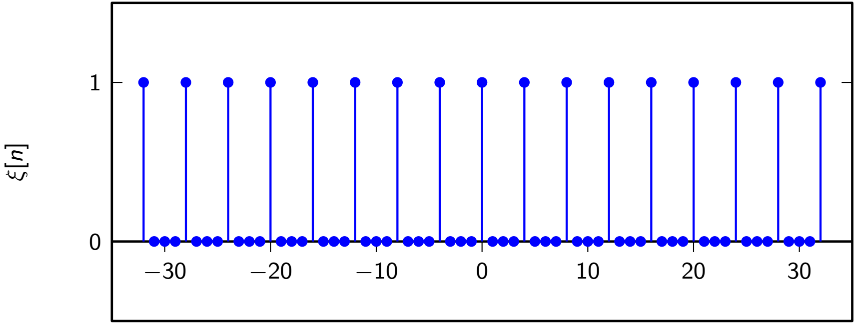
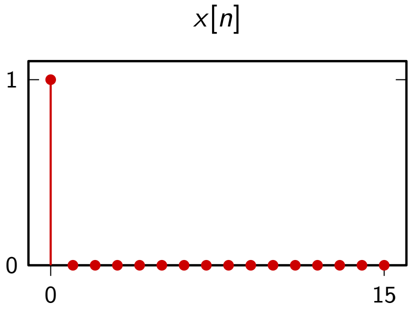
$$

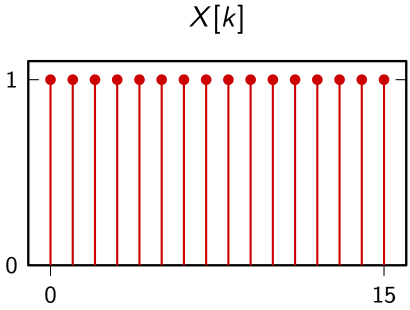
$\xi[n]$ for $N = 4$



Blast from the past: DFT of $x[n] = \delta[n], \quad x[n] \in \mathbb{C}^N$

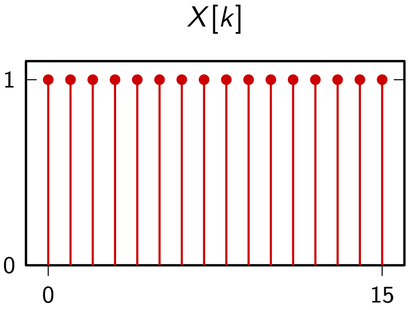
$$X[k] = \sum_{n=0}^{N-1} \delta[n]\, e^{-j\frac{2\pi}{N}nk} = 1$$

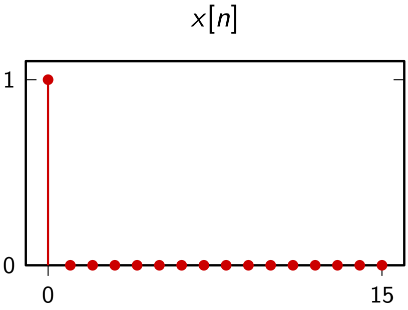




From the other side:

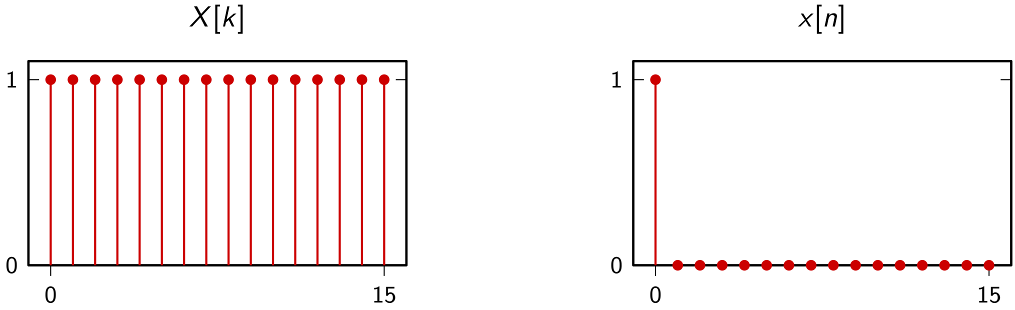
$$\mathrm{IDFT}\left\{1\right\} = \frac{1}{N}\sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}mn} = \delta[n]$$





From the other side:

$$\text{IDFT}\left\{1\right\} = \frac{1}{N}\sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}mn} = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases} \qquad n = 0, \ldots, N-1$$



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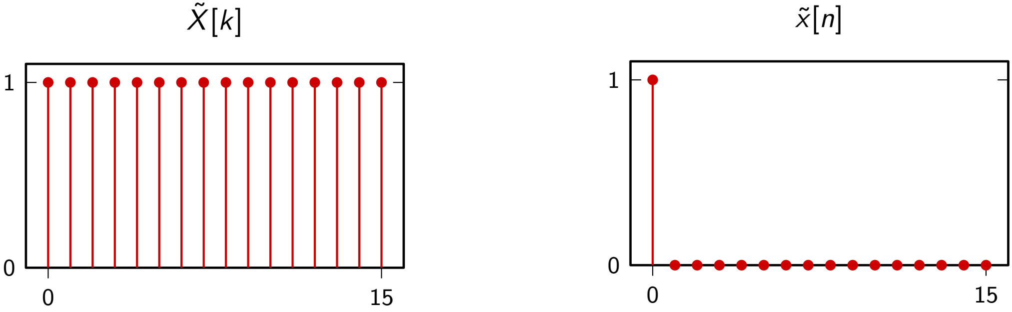

Spectral representation

$$\xi[n] = \frac{1}{N} \sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}mn}$$

$$\begin{aligned} A(z) &= \sum_{k=-\infty}^{\infty} \xi[k]x[k]z^{-k} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{k=-\infty}^{\infty} x[k]e^{j\frac{2\pi}{N}mk} z^{-k} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} X(e^{-j\frac{2\pi}{N}m} z) \end{aligned}$$

Spectral representation

$$\xi[n] = \frac{1}{N}\sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}mn}$$

$$A(z) = \sum_{k=-\infty}^{\infty} \xi[k]x[k]z^{-k}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=-\infty}^{\infty} x[k]e^{j\frac{2\pi}{N}mk}z^{-k}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} X(e^{-j\frac{2\pi}{N}m}z)$$

Spectral representation

$$\xi[n] = \frac{1}{N}\sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}mn}$$

$$\begin{aligned}
A(z) &= \sum_{k=-\infty}^{\infty} \xi[k]x[k]z^{-k} \\
&= \frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=-\infty}^{\infty} x[k]e^{j\frac{2\pi}{N}mk}z^{-k} \\
&= \frac{1}{N}\sum_{m=0}^{N-1} X(e^{-j\frac{2\pi}{N}m}z)
\end{aligned}$$

Spectral representation

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&= \frac{1}{N}\sum_{m=0}^{N-1} X(e^{-j\frac{2\pi}{N}m}z)
\end{aligned}$$

Spectral representation

$$A(e^{j\omega}) = \frac{1}{N}\sum_{m=0}^{N-1} X\left(e^{j(\omega - \frac{2\pi}{N}m)}\right)$$

Spectral representation

$$X_{ND}(z) = A(z^{1/N}) = \frac{1}{N}\sum_{m=0}^{N-1} X(e^{-j\frac{2\pi}{N}m}\, z^{\frac{1}{N}})$$

$$X_{ND}(e^{j\omega}) = \frac{1}{N}\sum_{m=0}^{N-1} X(e^{j(\frac{\omega-2\pi m}{N})})$$

Spectral representation

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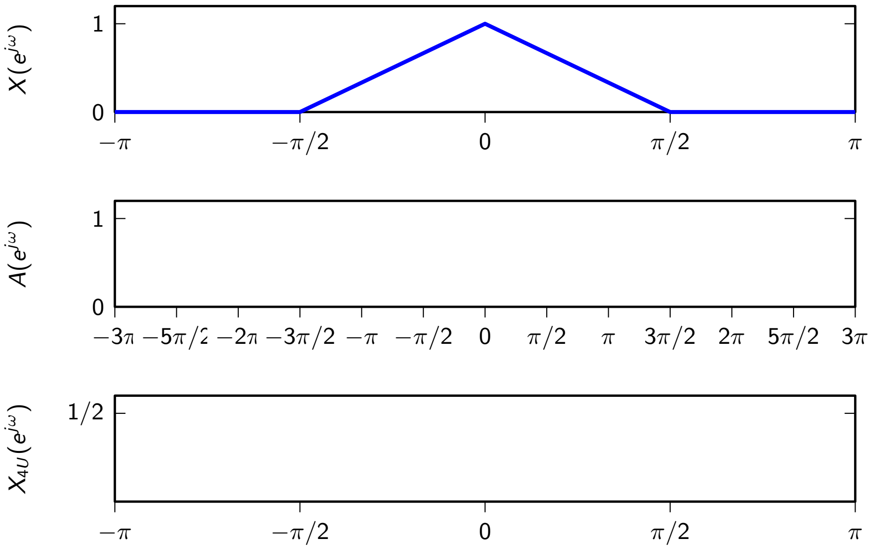
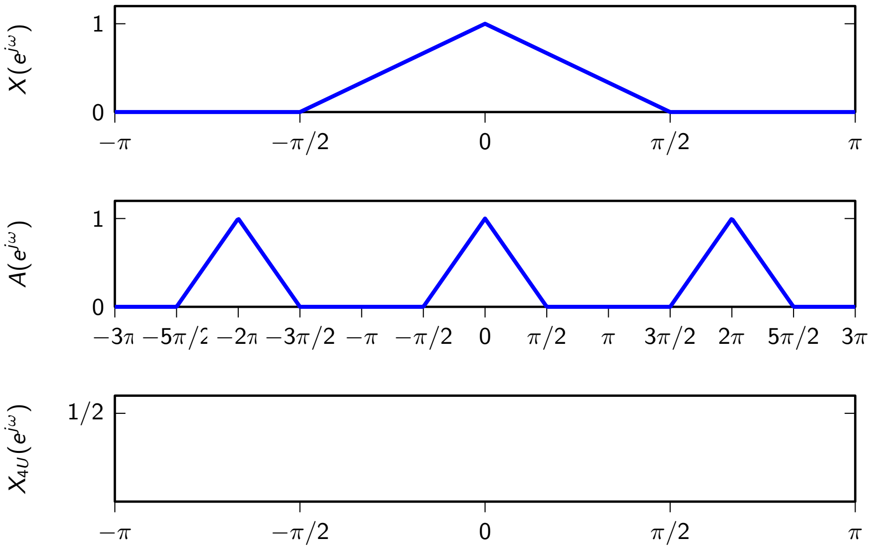
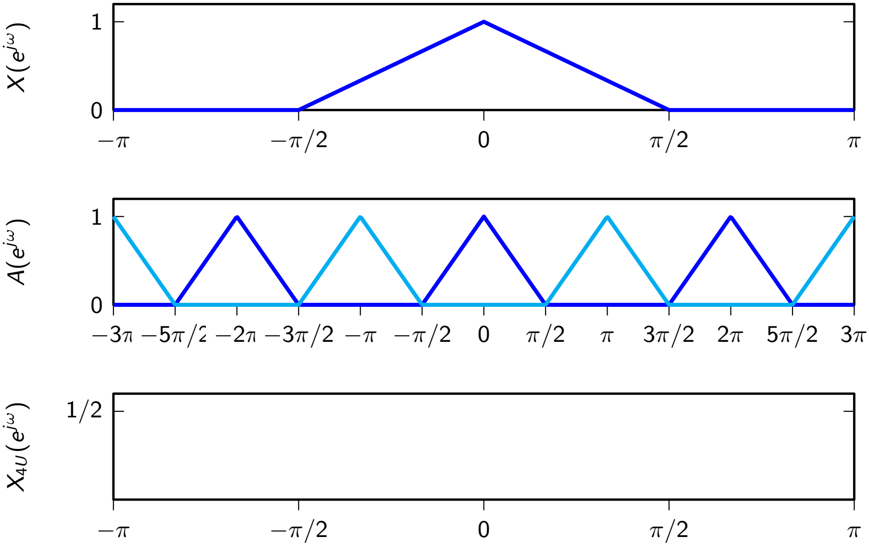
Downsampling by 2



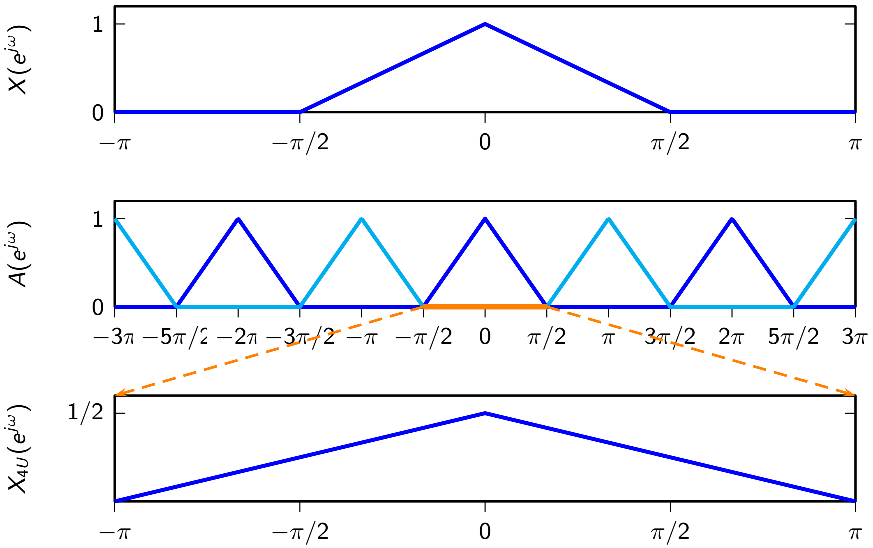
Downsampling by 2



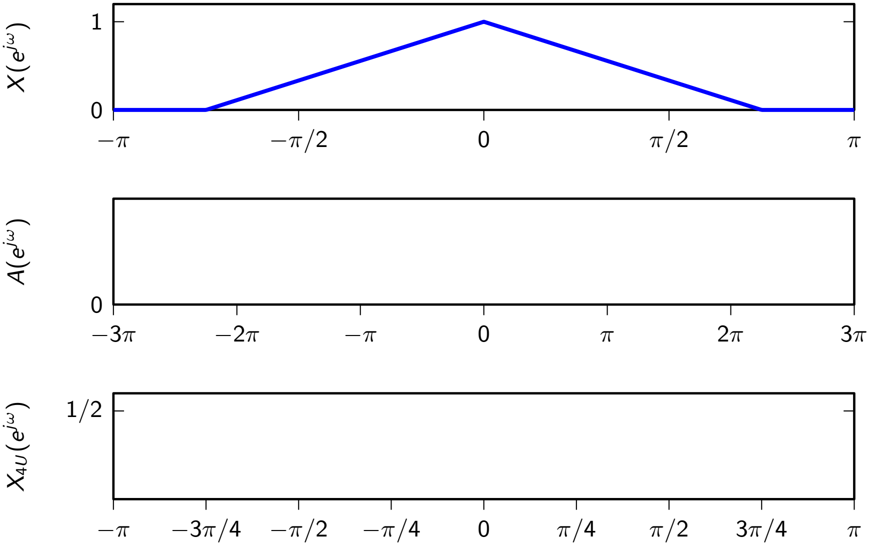
Downsampling by 2



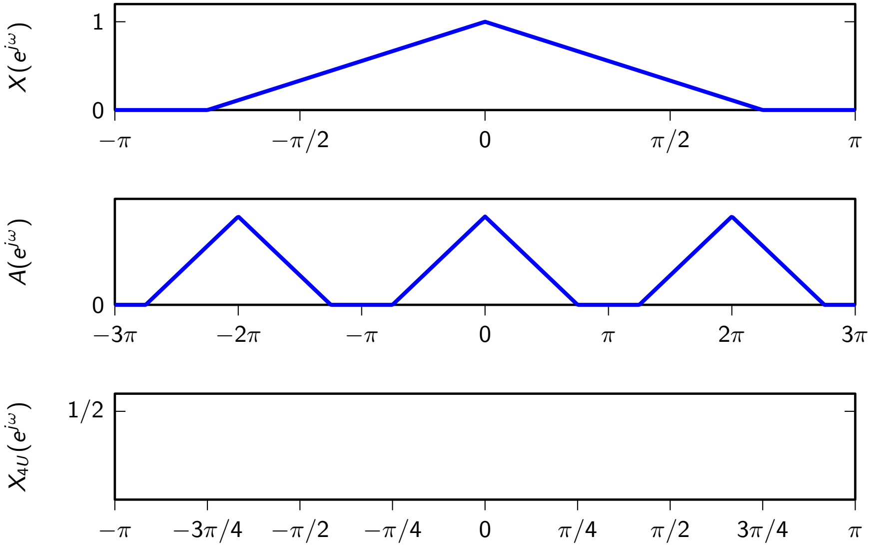
Downsampling by 2



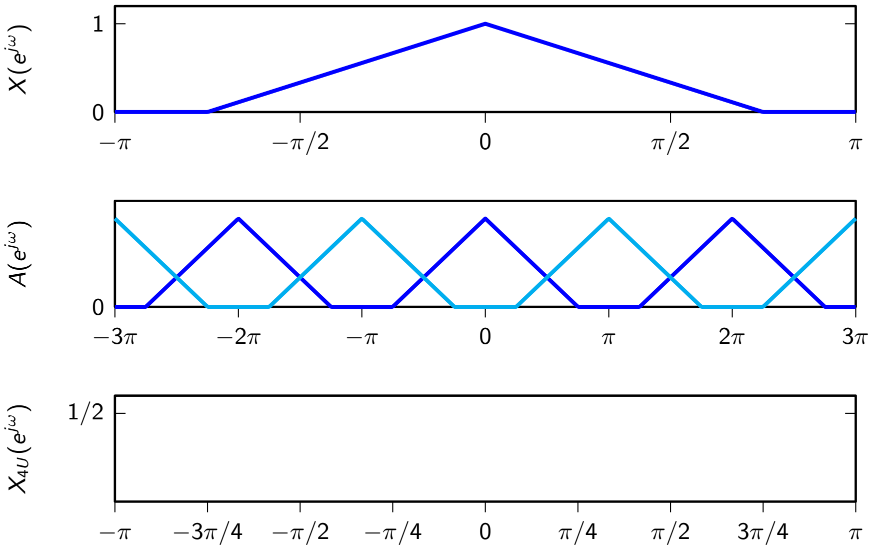
Downsampling by 2 with aliasing



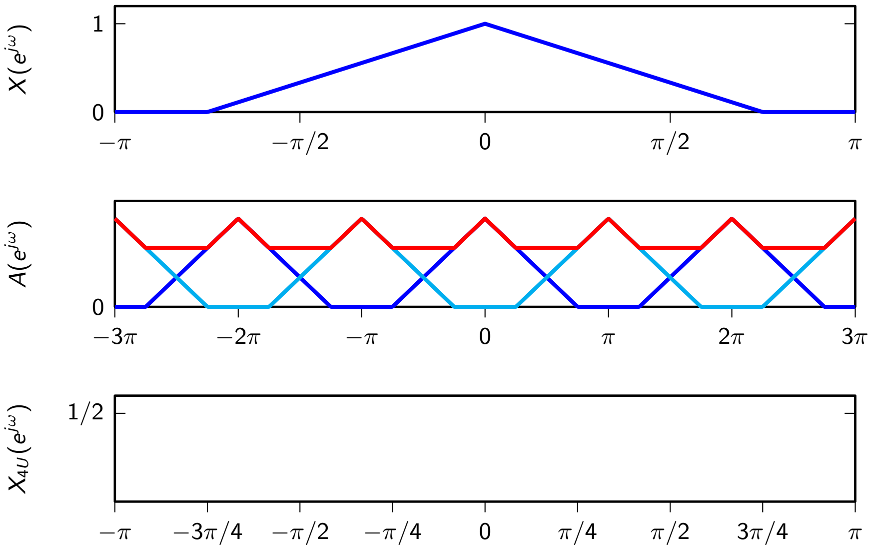
Downsampling by 2 with aliasing



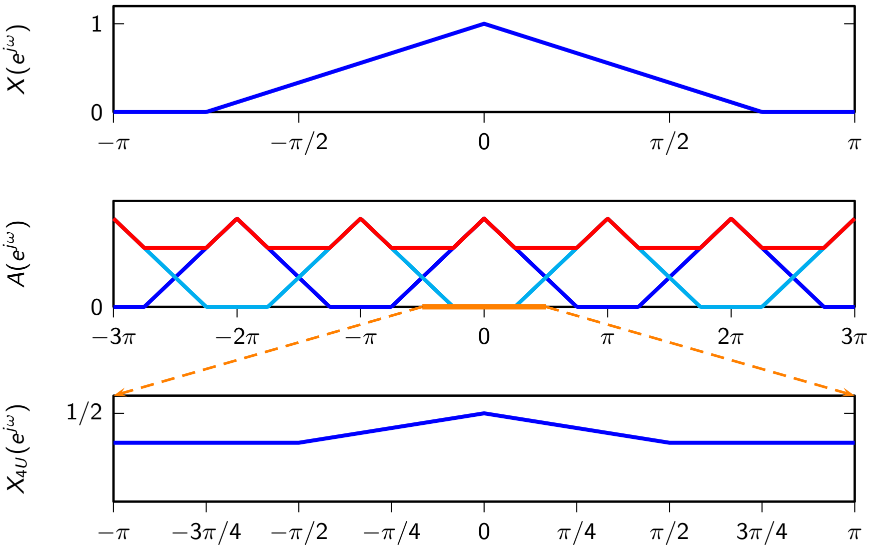
Downsampling by 2 with aliasing



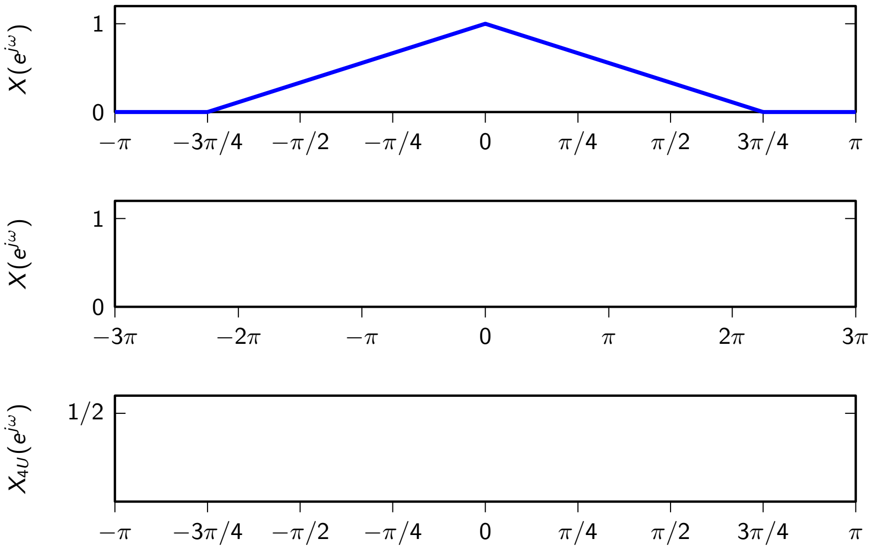
Downsampling by 2 with aliasing



Downsampling by 2 with aliasing



Downsampling by 2 with antialiasing filter



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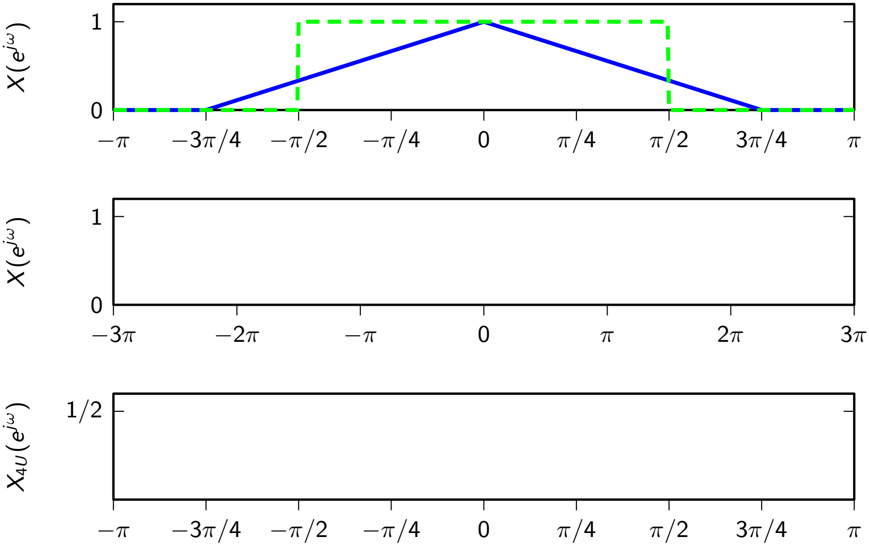

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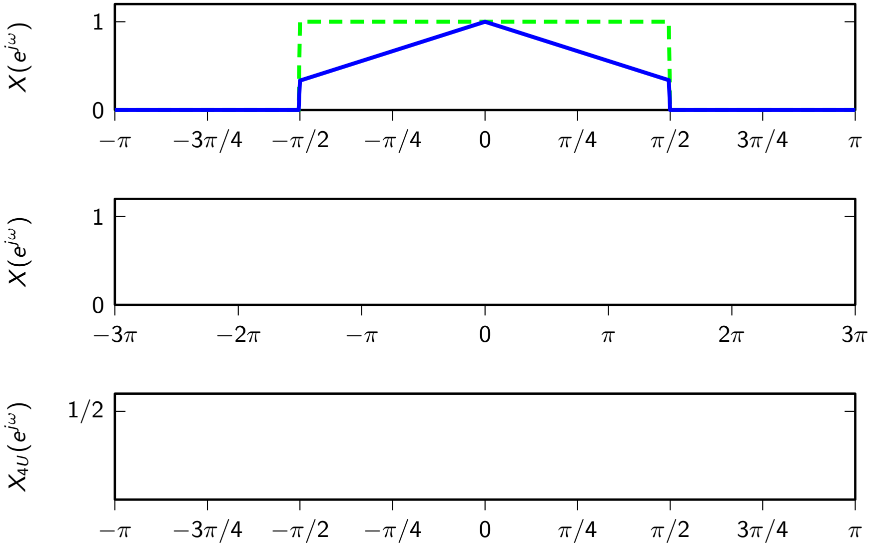

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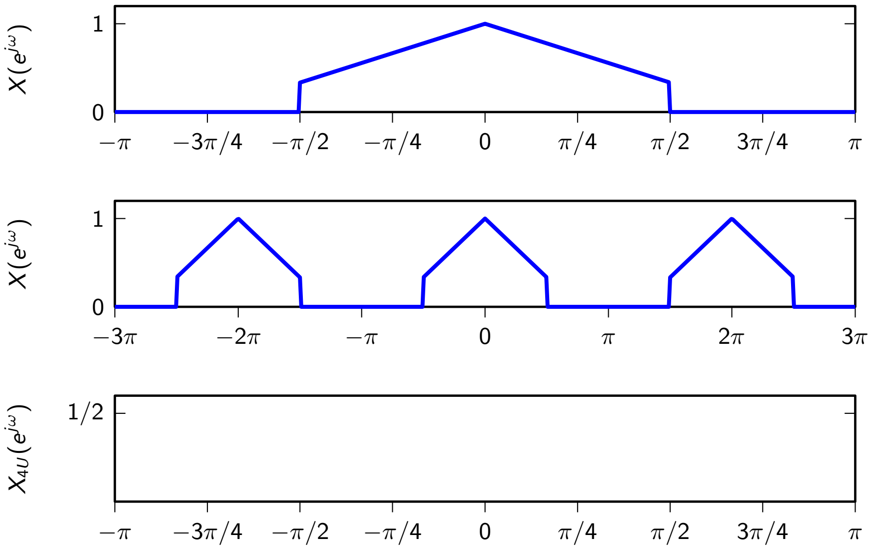
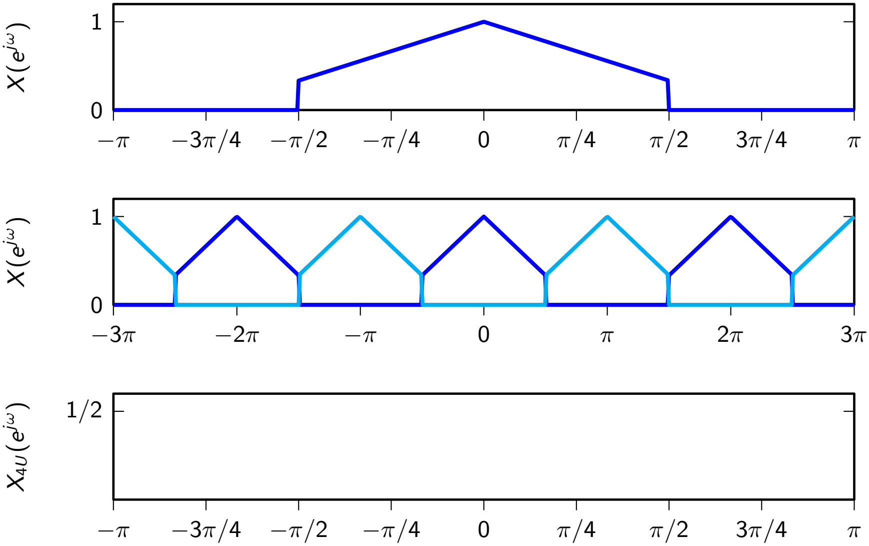
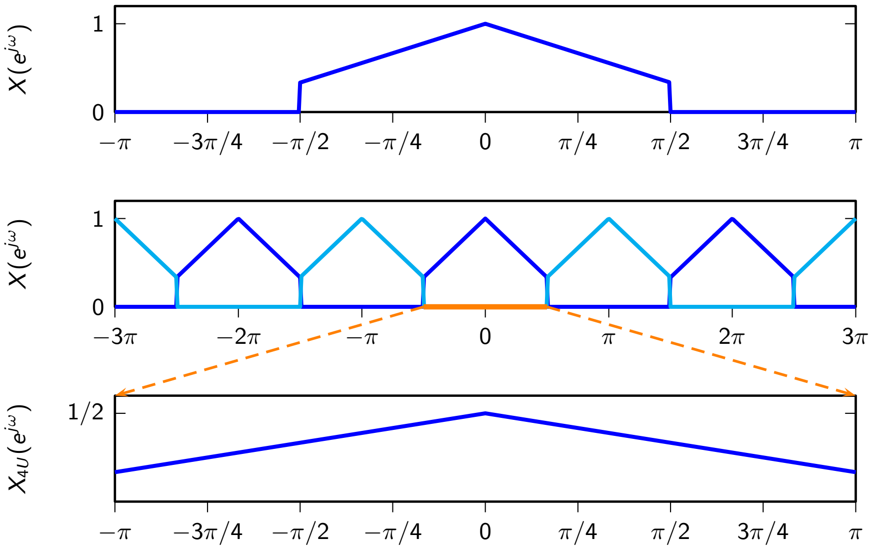


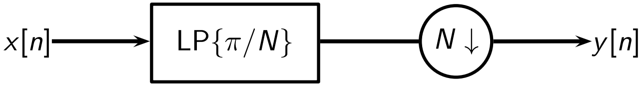
Downsampling by 2 with antialiasing filter



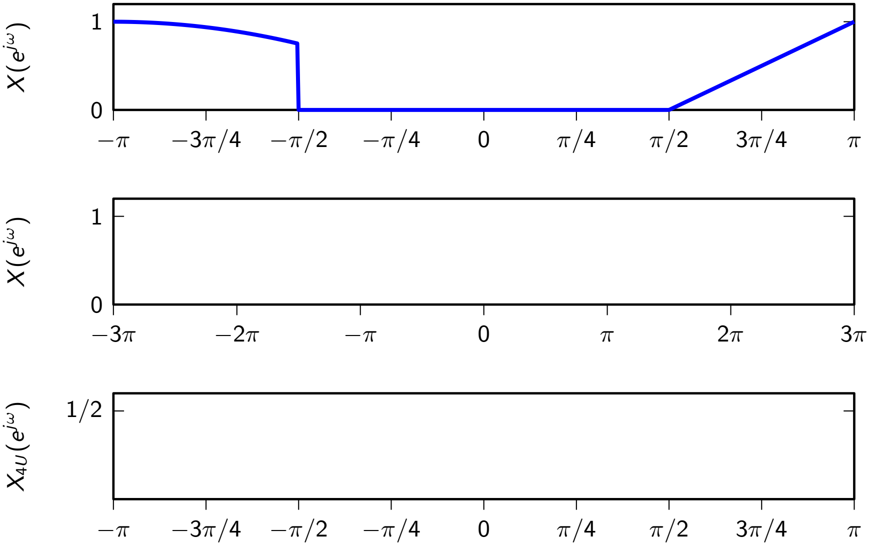
Downsampling by 2 with antialiasing filter



Downsampling



Downsampling by 2, careful with highpass signals



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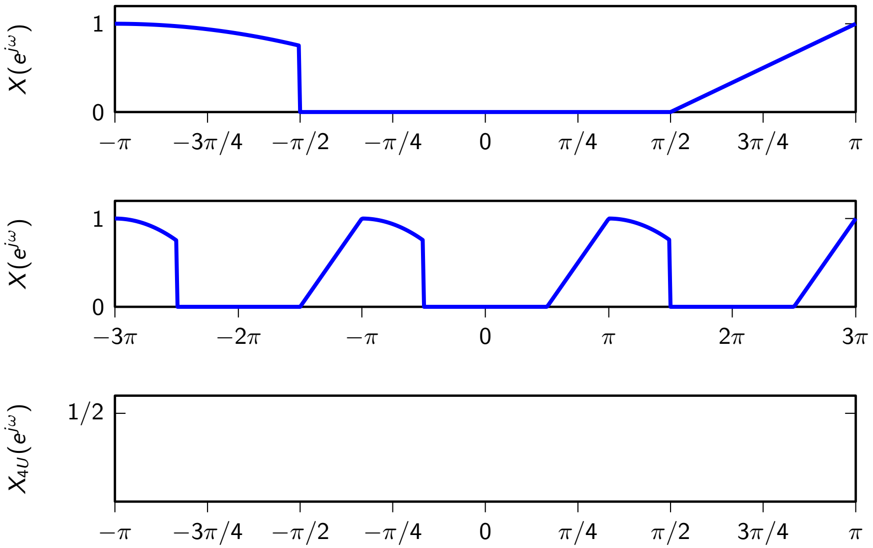

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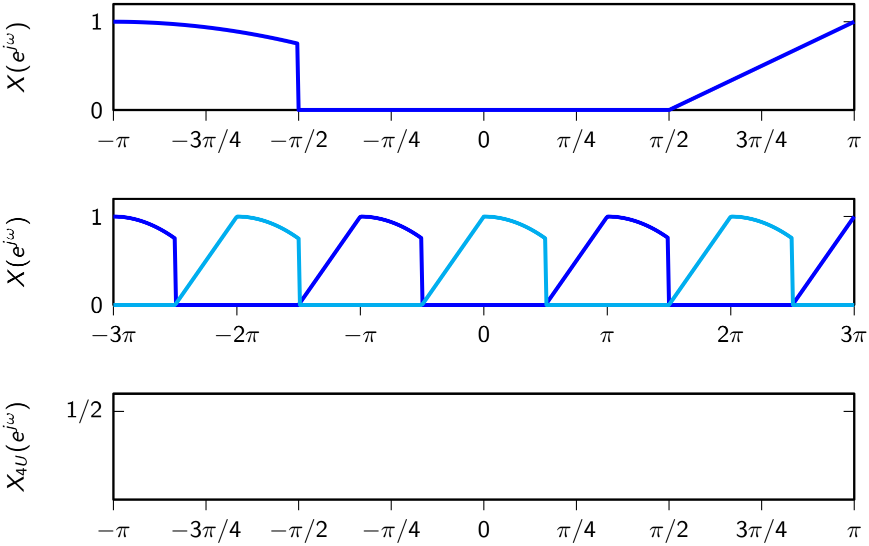

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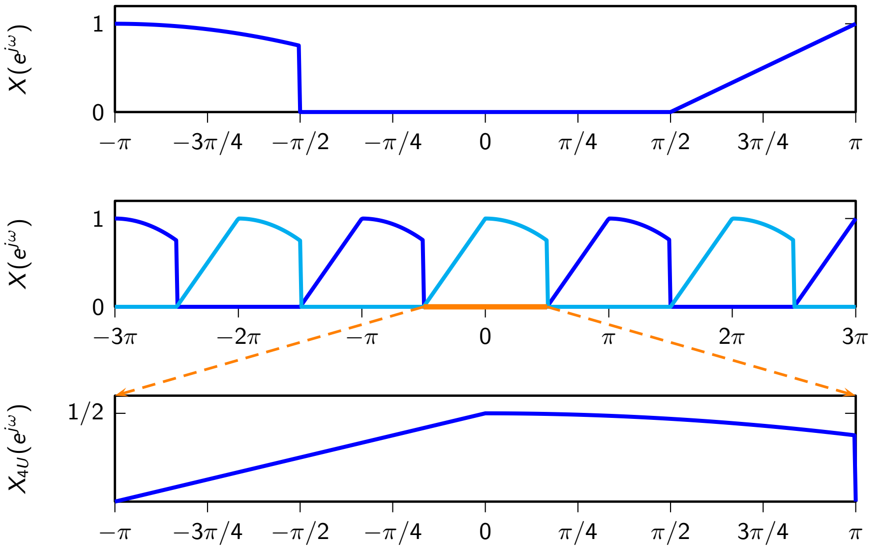
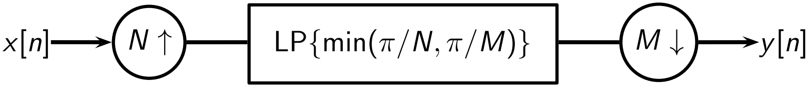


Rational Sampling Rate Change

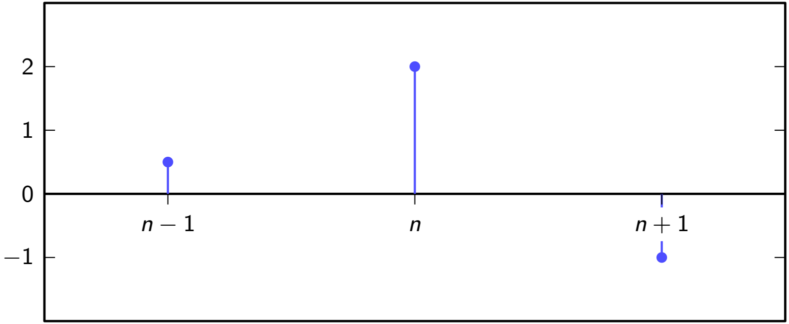


Rational Sampling Rate Change

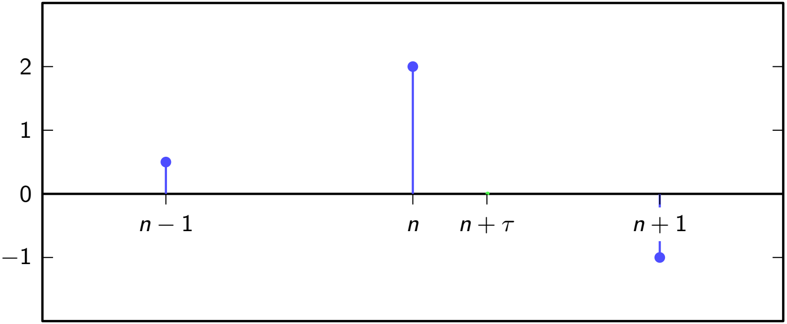
Example CD to DVD:

- CD: $F_s = 44100\text{Hz}$
- DVD: $F_S = 48000\text{Hz}$
- $\dfrac{N}{M} = \dfrac{160}{147}$
- in practice, we use time-varying local interpolation

Subsample Interpolation



Subsample Interpolation



Subsample Interpolation

- we want to compute $x(n+\tau)$, with $|\tau| < 1/2$
- local Lagrange approximation around n

$$x_L(n;t) = \sum_{k=-N}^{N} x[n+k]L_k^{(N)}(t)$$

$$L_k^{(N)}(t) = \prod_{\substack{i=-N \\ i\neq n}}^{N} \frac{t-i}{k-i} \qquad\qquad k = -N, \ldots, N$$

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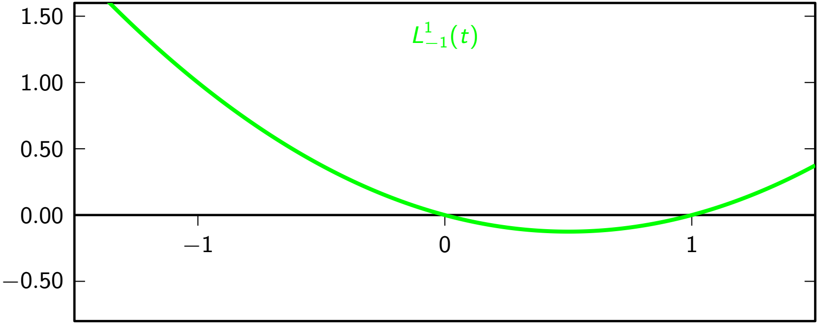
2nd-order Lagrange interpolation polynomials ($N = 1$)



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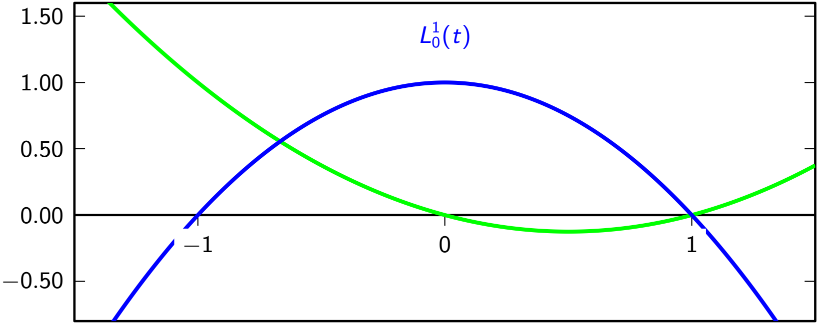

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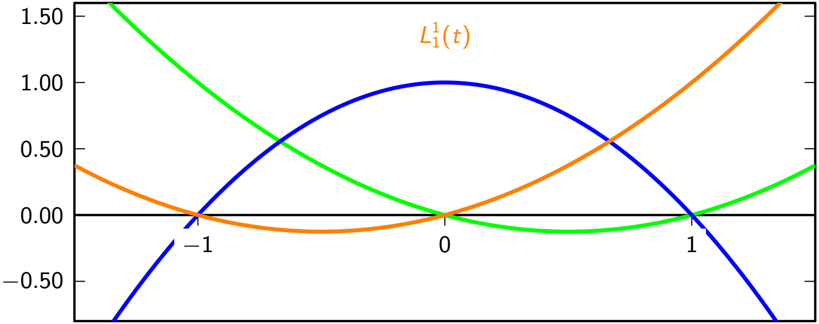
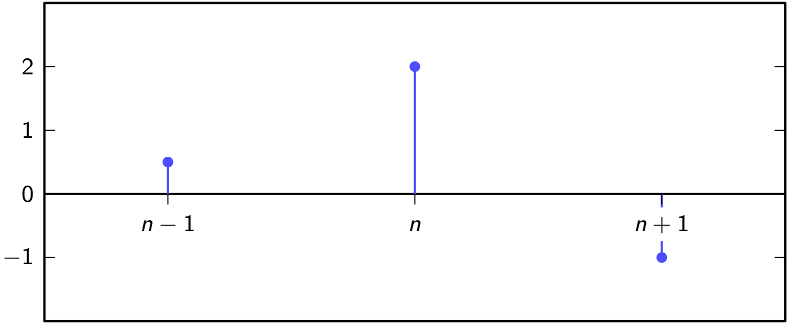


Lagrange interpolation ($N = 1$)



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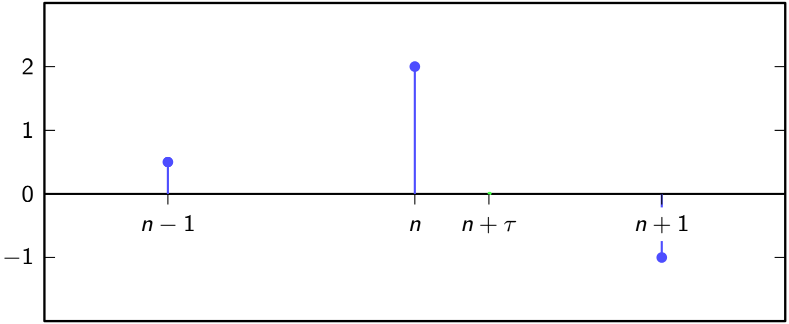

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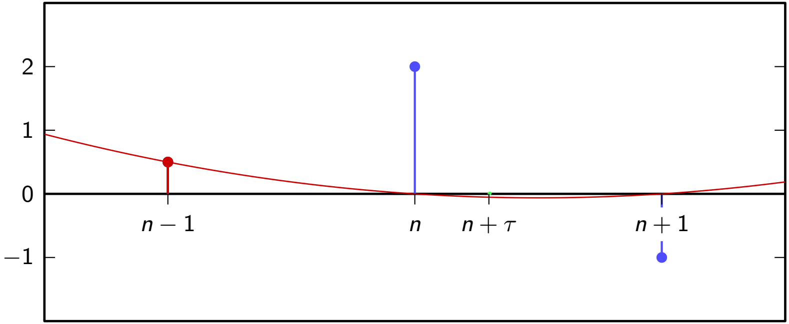

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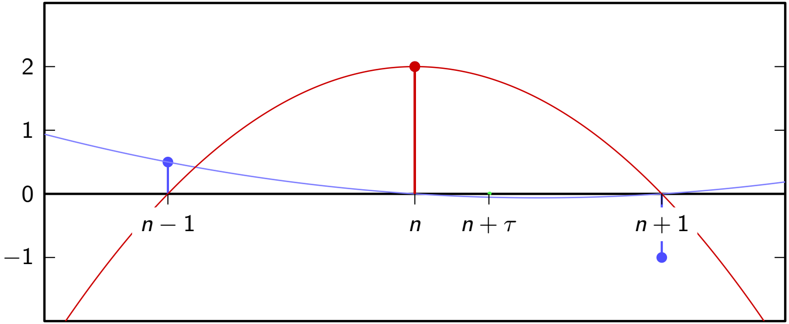

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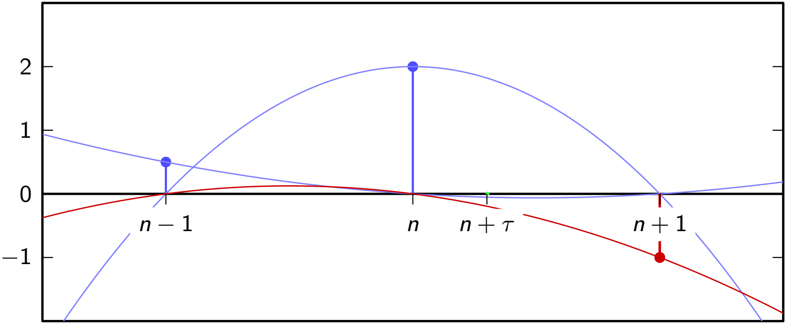

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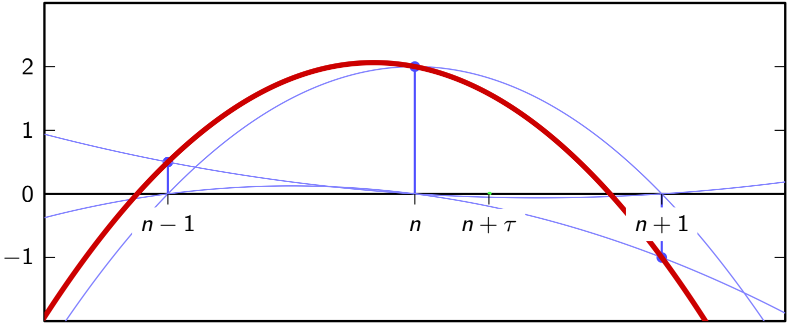

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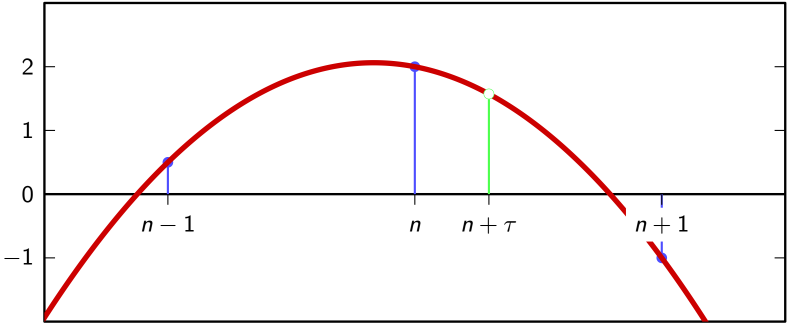

Lagrange interpolation as an FIR

- $x(n+\tau) \approx x_L(n;\tau)$
- $x_L(n;\tau) = \sum_{k=-N}^{N} x[n+k]L_k^{(N)}(\tau) = \sum_{k=-N}^{N} x[n-k]L_{-k}^{(N)}(\tau)$
- define $d_\tau[k] = L_{-k}^{(N)}(\tau)$, $k = -N, \ldots, N$
- $x_L(n;\tau) = \sum_{k=-N}^{N} x[n-k]d_\tau[k]$
- $x_L(n;\tau) = (x * d_\tau)[n]$
- $d_\tau[k]$ is a $(2N+1)$-tap FIR (dependent on τ)

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Example ($N = 1$, second order approximation)

$$L_{-1}^{(1)}(t) = t\frac{t-1}{2}$$

$$L_{0}^{(1)}(t) = (1-t)(1+t)$$

$$L_{1}^{(1)}(t) = t\frac{t+1}{2}$$

Example ($N = 1$, second order approximation)

$$d_{0.2}[n] = \begin{cases} 0.12 & n = -1 \\ 0.96 & n = 0 \\ -0.08 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

CD to DVD, revisited

for every 147 CD samples, generate 160 DVD samples

Fractional resampling: algorithm

sampling rate change A/B: for every B input point, generate A output points

Initialization (pattern will repeat every A output points)

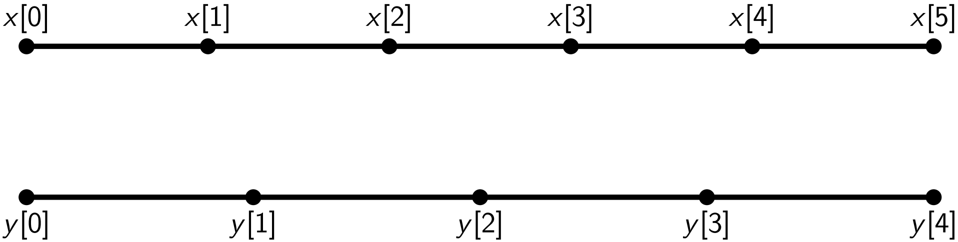
- for each output index $0 \leq m < A$ find closest (< 0.5) input point
- generate a FIR interpolation filter using difference between output and anchor

For every block of A output values $y[n]$:

- use all filters in turn
- generate output points

Fractional resampling: sample rate reduction

Downsampling: $\dfrac{A}{B} = \dfrac{4}{5}$



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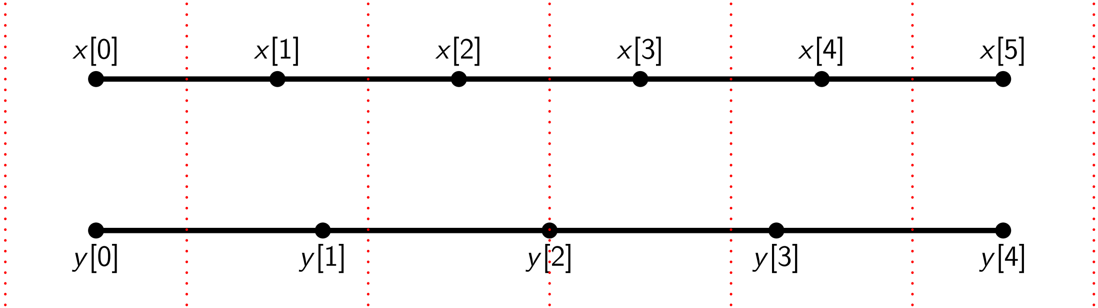

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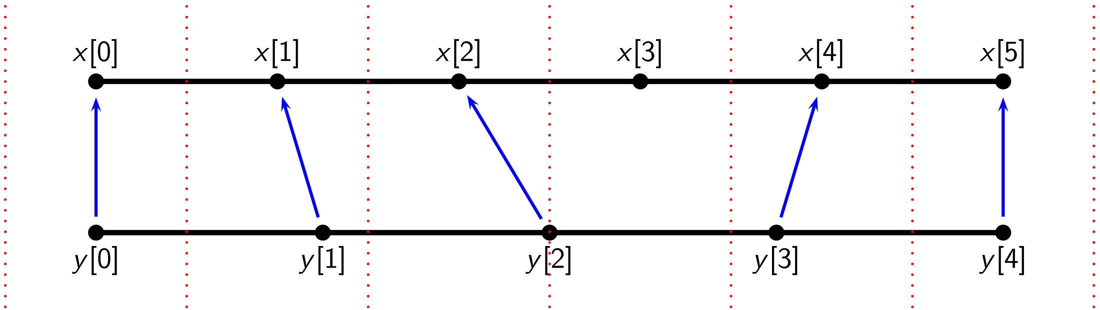
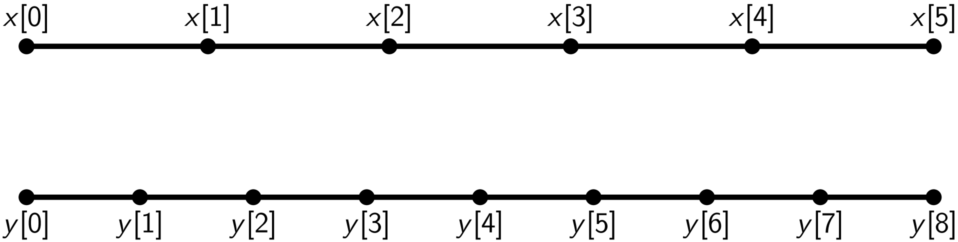


Fractional resampling: sample rate increase

$$\text{Upsampling: } \frac{A}{B} = \frac{8}{5}$$



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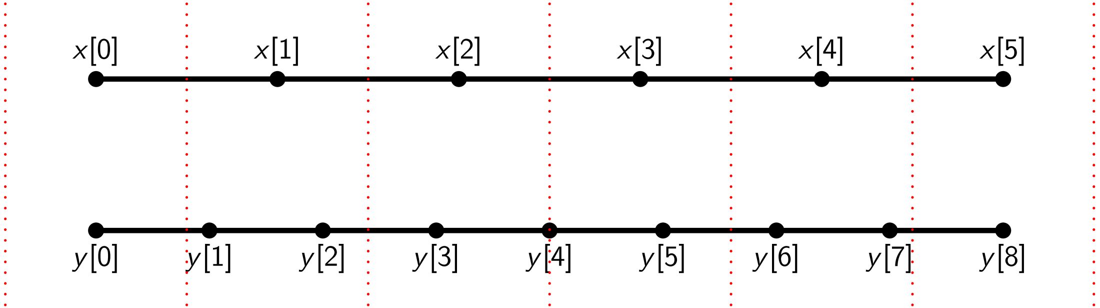

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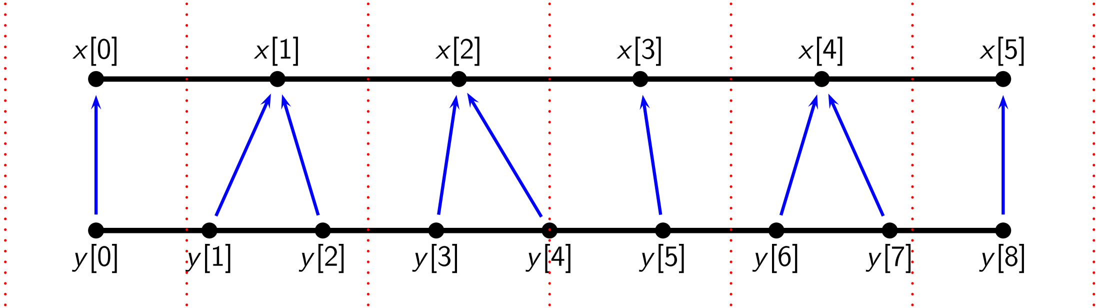
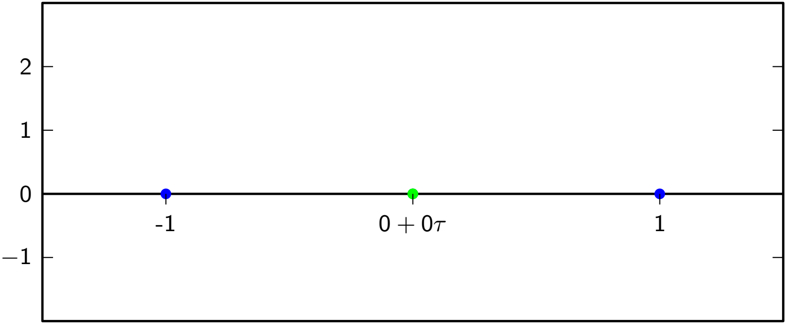
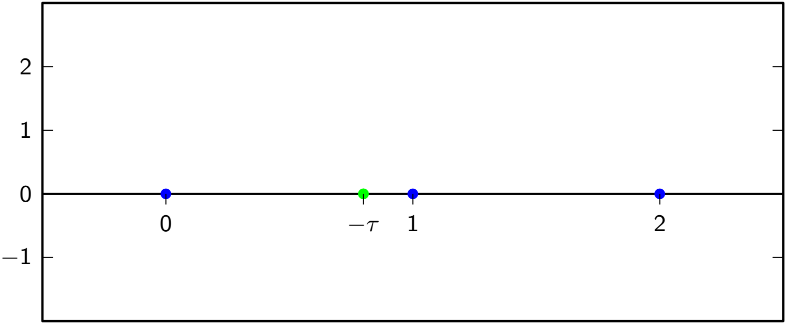


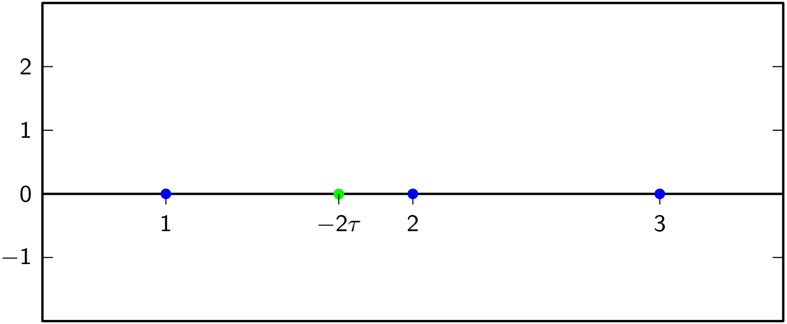
CD to DVD, revisited



CD to DVD, revisited, $\tau[1] = 0.06875$



CD to DVD, revisited



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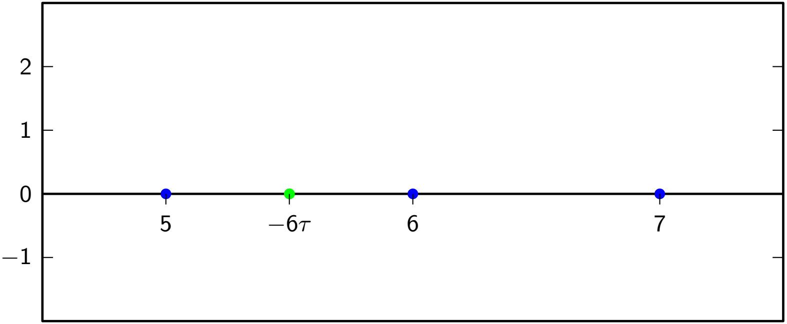
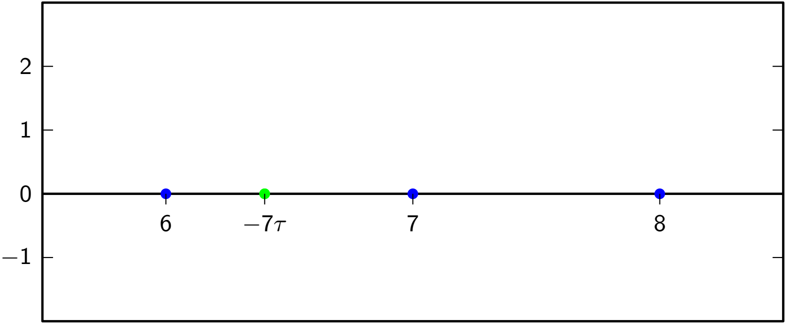
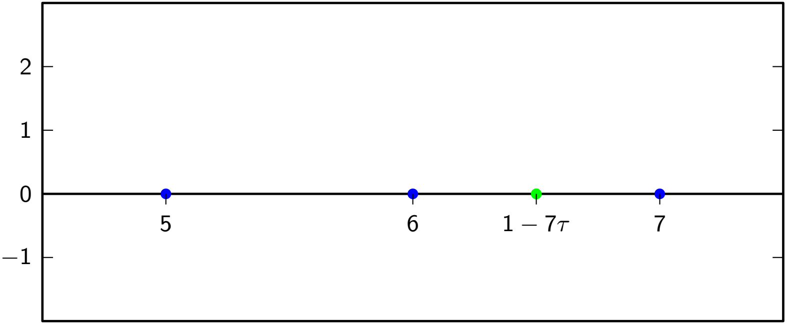


CD to DVD, revisited

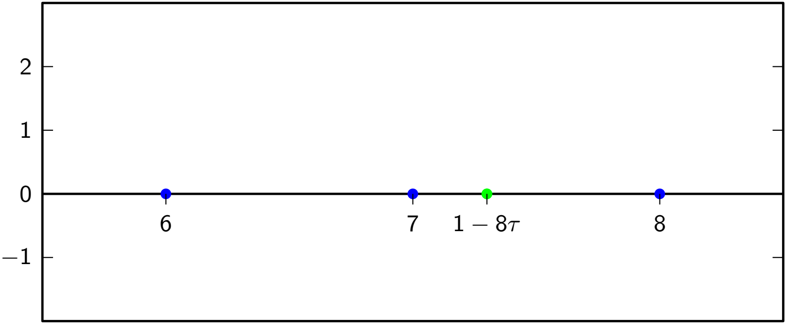


but $-7\tau < -0.5$

CD to DVD, revisited: repeat a sample



CD to DVD, revisited



CD to DVD, revisited

efficient local interpolation with 160 3-tap filters, used in sequence