COM303: Digital Signal Processing

Lecture 6: DFS and DTFT

Overview

- periodicity in the DFT
- the DFS
- the DTFT

Overview

- periodicity in the DFT
- the DFS
- the DTFT

Overview

- periodicity in the DFT
- the DFS
- the DTFT

DFT formulas

Analysis formula:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}, \qquad k = 0, 1, \ldots, N-1$$

N-point signal in the *frequency domain*

Synthesis formula:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk}, \qquad n = 0, 1, \ldots, N-1$$

N-point signal in the *"time" domain*

DFT formulas

Analysis formula:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}, \qquad k = 0, 1, \ldots, N-1$$

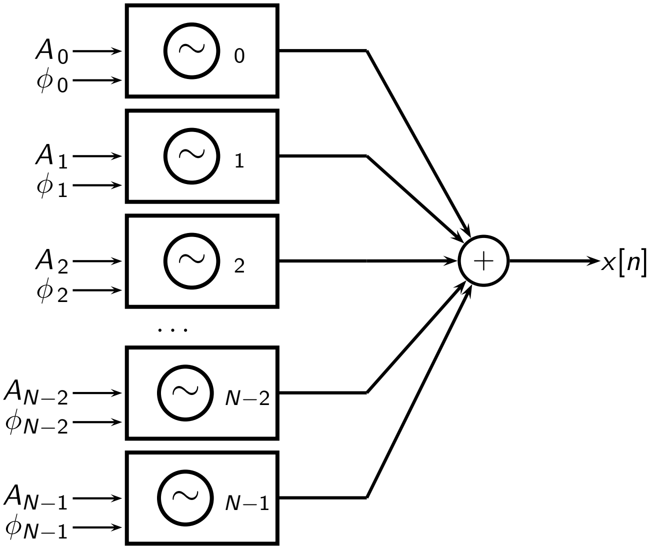
N-point signal in the *frequency domain*

Synthesis formula:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk}, \qquad n = 0, 1, \ldots, N-1$$

N-point signal in the *"time" domain*

DFT synthesis formula



Running the machine too long...

$$x[n + N] = x[n]$$

output signal is N-periodic!

Inherent periodicities in the DFT

the synthesis formula:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk}, \qquad n = 0, 1, \ldots, N-1$$

produces an N-point signal in the time domain

the analysis formula:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}, \qquad k = 0, 1, \ldots, N-1$$

produces an N-point signal in the frequency domain

Inherent periodicities in the DFT

the synthesis formula:

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk}, \qquad n \in \mathbb{Z}$$

produces an N-periodic signal in the time domain

the analysis formula:

$$\tilde{X}[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}, \qquad k = 0, 1, \ldots, N-1$$

produces an N-point signal in the frequency domain

Inherent periodicities in the DFT

the synthesis formula:

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, e^{j \frac{2\pi}{N} nk}, \qquad n \in \mathbb{Z}$$

produces an N-periodic signal in the time domain

the analysis formula:

$$\tilde{X}[k] = \sum_{n=0}^{N-1} x[n] \, e^{-j \frac{2\pi}{N} nk}, \qquad k \in \mathbb{Z}$$

produces an N-periodic signal in the frequency domain

Discrete Fourier Series (DFS)

DFS = DFT with periodicity explicit

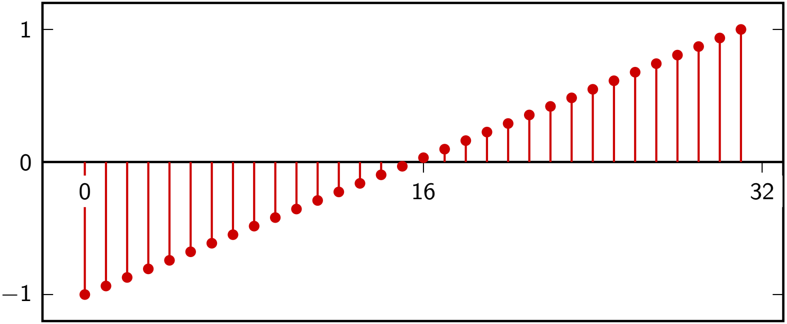
- the DFS maps an N-periodic signal onto an N-periodic sequence of Fourier coefficients
- the inverse DFS maps an N-periodic sequence of Fourier coefficients a set onto an N-periodic signal
- the DFS of an N-periodic signal is mathematically equivalent to the DFT of one period

the Fourier transform for periodic signals

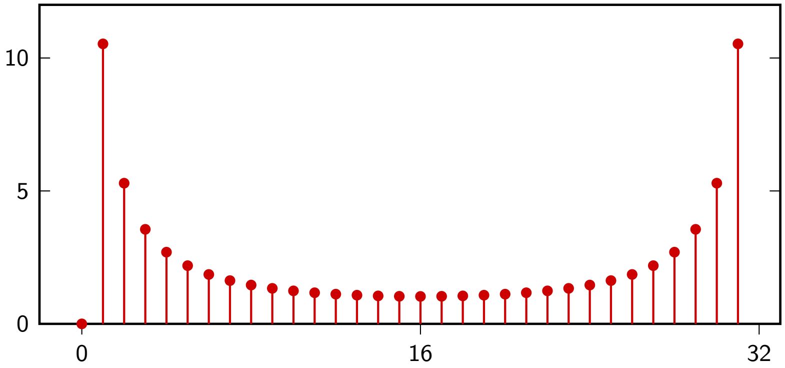
Periodic sequences: a bridge to infinite-length signals

- N-periodic sequence: N degrees of freedom
- DFS: only N Fourier coefficients capture all the information

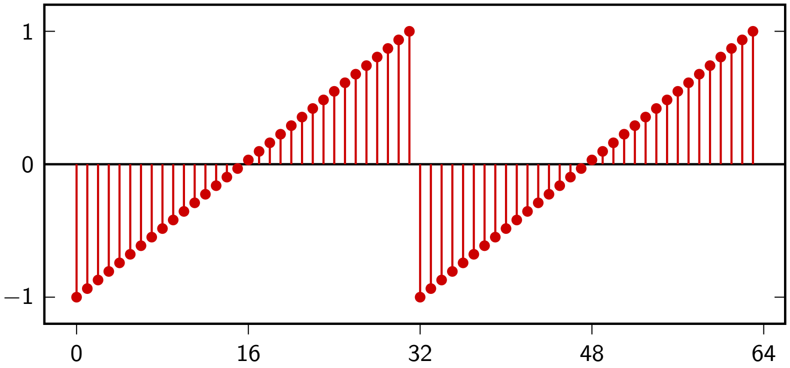
Example: 32-tap sawtooth wave



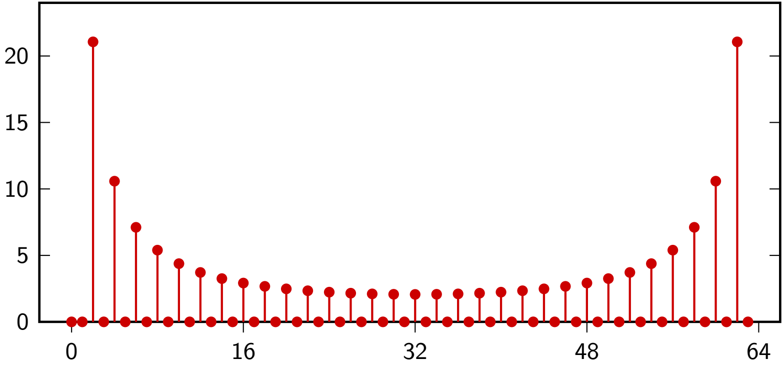
Example: DFT of 32-tap sawtooth wave



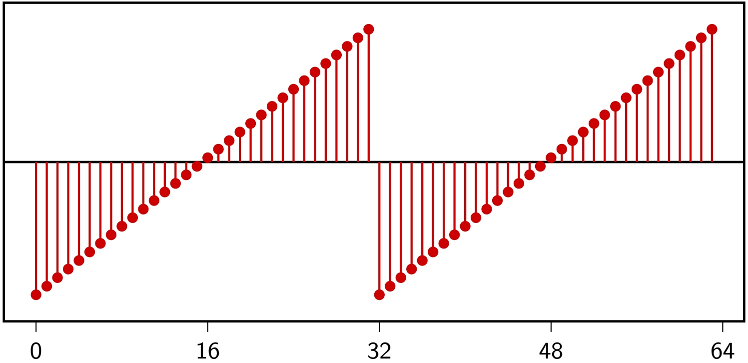
What if we take the DFT of two periods?



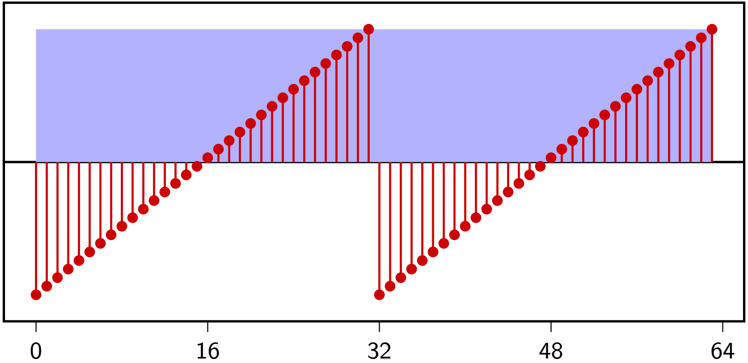
Example: 64-point DFT of two periods



DFT of two periods: intuition

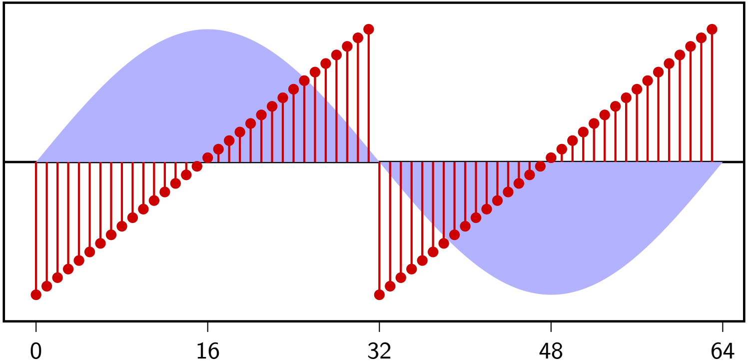


DFT of two periods: intuition



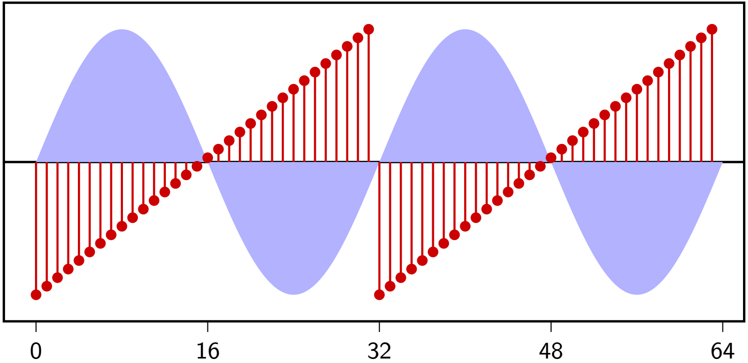
$k = 0$

DFT of two periods: intuition



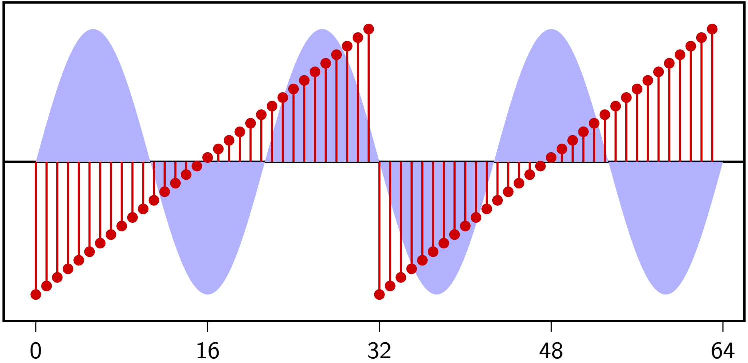
$k = 1$

DFT of two periods: intuition



$k = 2$

DFT of two periods: intuition



$k = 3$

DFT of L periods

ingredients:

- finite-length signal $x[n]$, $n = 0, 1, \ldots, N-1$
- N-periodic signal: $\tilde{x}[n] = x[n \mod N]$
- obviously $\tilde{x}[n] = \tilde{x}[n + pN]$ for all $p \in \mathbb{Z}$

DFT of L periods

$$X_L[k] = \sum_{n=0}^{LN-1} \tilde{x}[n] e^{-j\frac{2\pi}{LN}nk} \qquad k = 0, 1, 2 \ldots, LN-1$$

$$= \sum_{p=0}^{L-1} \sum_{n=0}^{N-1} \tilde{x}[n+pN] e^{-j\frac{2\pi}{LN}(n+pN)k}$$

$$= \sum_{p=0}^{L-1} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{LN}nk} e^{-j\frac{2\pi}{L}pk}$$

$$= \left(\sum_{p=0}^{L-1} e^{-j\frac{2\pi}{L}pk} \right) \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{LN}nk}$$

DFT of L periods

$$X_L[k] = \sum_{n=0}^{LN-1} \tilde{x}[n] e^{-j\frac{2\pi}{LN}nk} \qquad k = 0, 1, 2 \ldots, LN-1$$

$$= \sum_{p=0}^{L-1} \sum_{n=0}^{N-1} \tilde{x}[n+pN] e^{-j\frac{2\pi}{LN}(n+pN)k}$$

$$= \sum_{p=0}^{L-1} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{LN}nk} e^{-j\frac{2\pi}{L}pk}$$

$$= \left(\sum_{p=0}^{L-1} e^{-j\frac{2\pi}{L}pk} \right) \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{LN}nk}$$

DFT of L periods

$$X_L[k] = \sum_{n=0}^{LN-1} \tilde{x}[n] e^{-j\frac{2\pi}{LN}nk} \qquad k = 0, 1, 2 \ldots, LN-1$$

$$= \sum_{p=0}^{L-1} \sum_{n=0}^{N-1} \tilde{x}[n + pN] e^{-j\frac{2\pi}{LN}(n+pN)k}$$

$$= \sum_{p=0}^{L-1} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{LN}nk} e^{-j\frac{2\pi}{L}pk}$$

$$= \left(\sum_{p=0}^{L-1} e^{-j\frac{2\pi}{L}pk} \right) \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{LN}nk}$$

DFT of L periods

$$
\begin{aligned}
X_L[k] &= \sum_{n=0}^{LN-1} \tilde{x}[n] e^{-j\frac{2\pi}{LN}nk} \qquad k = 0, 1, 2 \ldots, LN-1 \\
&= \sum_{p=0}^{L-1} \sum_{n=0}^{N-1} \tilde{x}[n+pN] e^{-j\frac{2\pi}{LN}(n+pN)k} \\
&= \sum_{p=0}^{L-1} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{LN}nk} e^{-j\frac{2\pi}{L}pk} \\
&= \left(\sum_{p=0}^{L-1} e^{-j\frac{2\pi}{L}pk} \right) \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{LN}nk}
\end{aligned}
$$

We've seen this before

$$\sum_{p=0}^{L-1} e^{-j\frac{2\pi}{L}pk} = \begin{cases} L & \text{if } k \text{ multiple of } L \\ 0 & \text{otherwise} \end{cases}$$

(remember the orthogonality proof for the DFT basis)

DFT of L periods

if k is a multiple of L then k/L is an integer, so:

$$\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}n\frac{k}{L}} = X[k/L]$$

DFT of L periods

$$X_L[k] = \begin{cases} L\,X[k/L] & \text{if } k = 0, L, 2L, 3L, \ldots \\ 0 & \text{otherwise} \end{cases}$$

DFT and DFS

- again, all the spectral information for a periodic signal is contained in the DFT coefficients of a single period
- to stress the periodicity of the underlying signal, we use the term DFS

DFT and DFS

- again, all the spectral information for a periodic signal is contained in the DFT coefficients of a single period
- to stress the periodicity of the underlying signal, we use the term DFS

Finite-length time shifts revisited

The DFS helps us understand how to define time shifts for finite-length signals.

For an N- periodic sequence $\tilde{x}[n]$:

- $\tilde{x}[n-M]$ is well-defined for all $M \in \mathbb{N}$
- $\text{DFS}\{\tilde{x}[n-M]\} = e^{-j\frac{2\pi}{N}Mk}\tilde{X}[k]$ (easy derivation)
- $\text{IDFS}\left\{\tilde{X}[k]\right\} = \tilde{x}[n-M]$

Finite-length time shifts revisited

The DFS helps us understand how to define time shifts for finite-length signals.

For an N- periodic sequence $\tilde{x}[n]$:

- $\tilde{x}[n-M]$ is well-defined for all $M \in \mathbb{N}$
- $\text{DFS}\left\{\tilde{x}[n-M]\right\} = e^{-j\frac{2\pi}{N}Mk}\tilde{X}[k]$ (easy derivation)
- $\text{IDFS}\left\{\tilde{X}[k]\right\} = \tilde{x}[n-M]$

Finite-length time shifts revisited

The DFS helps us understand how to define time shifts for finite-length signals.

For an N- periodic sequence $\tilde{x}[n]$:

- $\tilde{x}[n-M]$ is well-defined for all $M \in \mathbb{N}$
- $\text{DFS}\left\{\tilde{x}[n-M]\right\} = e^{-j\frac{2\pi}{N}Mk}\tilde{X}[k]$ (easy derivation)
- $\text{IDFS}\left\{ \tilde{X}[k]\right\} = \tilde{x}[n-M]$

Finite-length time shifts revisited

The DFS helps us understand how to define time shifts for finite-length signals.

For an N- periodic sequence $\tilde{x}[n]$:

- $\tilde{x}[n-M]$ is well-defined for all $M \in \mathbb{N}$
- $\text{DFS}\left\{\tilde{x}[n-M]\right\} = e^{-j\frac{2\pi}{N}Mk}\tilde{X}[k]$ (easy derivation)
- $\text{IDFS}\left\{e^{-j\frac{2\pi}{N}Mk}\ \tilde{X}[k]\right\} = \tilde{x}[n-M]$

Finite-length time shifts revisited

The DFS helps us understand how to define time shifts for finite-length signals.

For an N- periodic sequence $\tilde{x}[n]$:

- $\tilde{x}[n-M]$ is well-defined for all $M \in \mathbb{N}$
- $\text{DFS}\{\tilde{x}[n-M]\} = e^{-j\frac{2\pi}{N}Mk}\tilde{X}[k]$ (easy derivation)
- $\text{IDFS}\left\{\boxed{e^{-j\frac{2\pi}{N}Mk}}\ \tilde{X}[k]\right\} = \tilde{x}[n-M]$

a delay in time becomes a *linear phase* factor in frequency

Finite-length time shifts revisited

For an N-point signal $x[n]$:

- $x[n-M]$ is *not* well-defined
- what is IDFT$\left\{ e^{-j\frac{2\pi}{N}Mk} \; X[k] \right\}$?

Finite-length time shifts revisited

For an N-point signal $x[n]$:

- $x[n-M]$ is *not* well-defined
- what is $\text{IDFT}\left\{e^{-j\frac{2\pi}{N}Mk}\,X[k]\right\}$?

Finite-length time shifts revisited

$$\text{IDFT}\left\{e^{-j\frac{2\pi}{N}Mk}\,X[k]\right\}[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{-j\frac{2\pi}{N}Mk}\,e^{j\frac{2\pi}{N}nk}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1} x[m]\,e^{-j\frac{2\pi}{N}mk}\right)\,e^{-j\frac{2\pi}{N}Mk}\,e^{j\frac{2\pi}{N}nk}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x[m]\sum_{k=0}^{N-1}\,e^{j\frac{2\pi}{N}(n-M-m)k}$$

Finite-length time shifts revisited

$$\mathrm{IDFT}\left\{e^{-j\frac{2\pi}{N}Mk}\,X[k]\right\}[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{-j\frac{2\pi}{N}Mk}\,e^{j\frac{2\pi}{N}nk}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1} x[m]\,e^{-j\frac{2\pi}{N}mk}\right)\,e^{-j\frac{2\pi}{N}Mk}\,e^{j\frac{2\pi}{N}nk}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x[m]\sum_{k=0}^{N-1}\,e^{j\frac{2\pi}{N}(n-M-m)k}$$

Finite-length time shifts revisited

$$\begin{aligned}
\mathrm{IDFT}\left\{e^{-j\frac{2\pi}{N}Mk}\,X[k]\right\}[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{-j\frac{2\pi}{N}Mk}\,e^{j\frac{2\pi}{N}nk} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1} x[m]\,e^{-j\frac{2\pi}{N}mk}\right)\,e^{-j\frac{2\pi}{N}Mk}\,e^{j\frac{2\pi}{N}nk} \\
&= \frac{1}{N}\sum_{m=0}^{N-1} x[m]\sum_{k=0}^{N-1}\,e^{j\frac{2\pi}{N}(n-M-m)k}
\end{aligned}$$

We've seen something like this before...

$$\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}nk} = \begin{cases} N & \text{if } n \text{ multiple of } N \\ 0 & \text{otherwise} \end{cases}$$

(remember the orthogonality proof for the DFT basis)

Finite-length time shifts revisited

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}((n-M)-m)k} = \begin{cases} N & \text{for } ((n-M)-m) \text{ multiple of } N \\ 0 & \text{otherwise} \end{cases}$$

- n, M are fixed
- m goes from 0 to $N-1$
- is there always a value for m that makes $((n-M)-m)$ a multiple of N?

Finite-length time shifts revisited

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}((n-M)-m)k} = \begin{cases} N & \text{for } ((n-M)-m) \text{ multiple of } N \\ 0 & \text{otherwise} \end{cases}$$

- n, M are fixed
- m goes from 0 to $N-1$
- is there always a value for m that makes $((n-M)-m)$ a multiple of N?

Modulo operator

given $C \in \mathbb{N}$, find m such that $0 \leq m < N$ and $C - m$ is a multiple of N

any integer C can be written as $C = pN + (C \mod N), \quad p \in \mathbb{N}$:

- $0 \leq (C \mod N) < N$
- $C - (C \mod N)$ is a multiple of N

Modulo operator

given $C \in \mathbb{N}$, find m such that $0 \leq m < N$ and $C - m$ is a multiple of N

any integer C can be written as $C = pN + (C \mod N), \quad p \in \mathbb{N}$:

- $0 \leq (C \mod N) < N$
- $C - (C \mod N)$ is a multiple of N

Finite-length time shifts revisited

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-M-m)k} = \begin{cases} N & \text{for } m = (n-M) \mod N \\ 0 & \text{otherwise} \end{cases}$$

$$\text{IDFT}\left\{e^{-j\frac{2\pi}{N}Mk} X[k]\right\}[n] = \frac{1}{N}\sum_{m=0}^{N-1} x[m] \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-M-m)k}$$
$$= x[(n-M) \mod N]$$

shifts for finite-length signals are “naturally” circular

Finite-length time shifts revisited

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-M-m)k} = \begin{cases} N & \text{for } m = (n-M) \mod N \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \text{IDFT}\left\{e^{-j\frac{2\pi}{N}Mk} X[k]\right\}[n] &= \frac{1}{N}\sum_{m=0}^{N-1} x[m] \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-M-m)k} \\ &= x[(n-M) \mod N] \end{aligned}$$

shifts for finite-length signals are “naturally” circular

Finite-length time shifts revisited

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-M-m)k} = \begin{cases} N & \text{for } m = (n-M) \mod N \\ 0 & \text{otherwise} \end{cases}$$

$$\text{IDFT}\left\{e^{-j\frac{2\pi}{N}Mk} X[k]\right\}[n] = \frac{1}{N} \sum_{m=0}^{N-1} x[m] \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-M-m)k}$$

$$= x[(n-M) \mod N]$$

shifts for finite-length signals are "naturally" circular

The situation so far

Fourier representation for signal classes:

- N-point finite-length: DFT
- N-point periodic: DFS
- infinite length: ?

The situation so far

Fourier representation for signal classes:

- N-point finite-length: DFT
- N-point periodic: DFS
- infinite length: ?

The situation so far

Fourier representation for signal classes:

- N-point finite-length: DFT
- N-point periodic: DFS
- infinite length: ?

the Fourier transform for infinite-length signals

DFT of increasingly long signals

- Start with the DFT. What happens when $N \to \infty$?
- $\frac{2\pi}{N}k$ becomes denser in $[0, 2\pi]$...
- In the limit $\frac{2\pi}{N}k \to \omega$:

$$\sum_n x[n]\, e^{-j\omega n} \qquad \omega \in \mathbb{R}$$

DFT of increasingly long signals

- Start with the DFT. What happens when $N \to \infty$?
- $\dfrac{2\pi}{N}k$ becomes denser in $[0, 2\pi]$...
- In the limit $\dfrac{2\pi}{N}k \to \omega$:

$$\sum_n x[n]\, e^{-j\omega n} \qquad \omega \in \mathbb{R}$$

DFT of increasingly long signals

- Start with the DFT. What happens when $N \to \infty$?
- $\dfrac{2\pi}{N}k$ becomes denser in $[0, 2\pi]$...
- In the limit $\dfrac{2\pi}{N}k \to \omega$:

$$\sum_n x[n]\, e^{-j\omega n} \quad \omega \in \mathbb{R}$$

Discrete-Time Fourier Transform (DTFT)

Formal definition:

- $x[n] \in \ell_2(\mathbb{Z})$
- define the *function* of $\omega \in \mathbb{R}$

$$F(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- inversion (when $F(\omega)$ exists):

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) e^{j\omega n} d\omega, \qquad n \in \mathbb{Z}$$

Discrete-Time Fourier Transform (DTFT)

Formal definition:

- $x[n] \in \ell_2(\mathbb{Z})$
- define the *function* of $\omega \in \mathbb{R}$

$$F(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- inversion (when $F(\omega)$ exists):

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) e^{j\omega n} d\omega, \qquad n \in \mathbb{Z}$$

Discrete-Time Fourier Transform (DTFT)

Formal definition:

- $x[n] \in \ell_2(\mathbb{Z})$
- define the *function* of $\omega \in \mathbb{R}$

$$F(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- inversion (when $F(\omega)$ exists):

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) e^{j\omega n} d\omega, \qquad n \in \mathbb{Z}$$

DTFT periodicity and notation

- $e^{j\omega n} = e^{j(\omega + 2k\pi)n} \quad \forall k \in \mathbb{N}$
- $F(\omega)$ is 2π-periodic
- to stress periodicity (and for other reasons) we will write

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- by convention, $X(e^{j\omega})$ is represented over $[-\pi, \pi]$

DTFT periodicity and notation

- $e^{j\omega n} = e^{j(\omega + 2k\pi)n} \quad \forall k \in \mathbb{N}$
- $F(\omega)$ is 2π-periodic
- to stress periodicity (and for other reasons) we will write

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- by convention, $X(e^{j\omega})$ is represented over $[-\pi, \pi]$

DTFT periodicity and notation

- $e^{j\omega n} = e^{j(\omega + 2k\pi)n} \quad \forall k \in \mathbb{N}$
- $F(\omega)$ is 2π-periodic
- to stress periodicity (and for other reasons) we will write

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- by convention, $X(e^{j\omega})$ is represented over $[-\pi, \pi]$

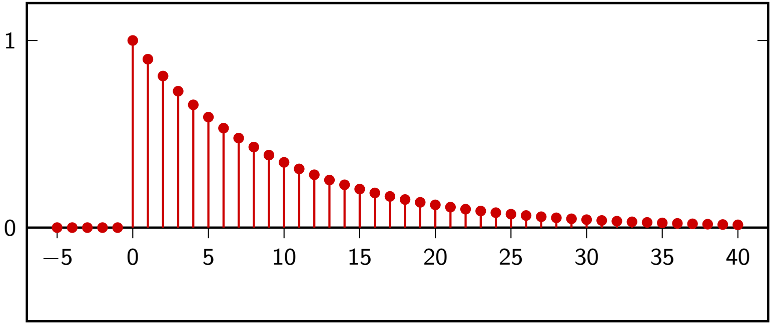
DTFT periodicity and notation

- $e^{j\omega n} = e^{j(\omega + 2k\pi)n} \quad \forall k \in \mathbb{N}$
- $F(\omega)$ is 2π-periodic
- to stress periodicity (and for other reasons) we will write

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- by convention, $X(e^{j\omega})$ is represented over $[-\pi, \pi]$

$x[n] = \alpha^n u[n], \quad |\alpha| < 1$



DTFT of $x[n] = \alpha^n u[n], \quad |\alpha| < 1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \alpha^n\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha\, e^{-j\omega})^n$$

$$= \frac{1}{1 - \alpha e^{-j\omega}}$$

DTFT of $x[n] = \alpha^n u[n], \quad |\alpha| < 1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \alpha^n\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha\, e^{-j\omega})^n$$

$$= \frac{1}{1 - \alpha e^{-j\omega}}$$

DTFT of $x[n] = \alpha^n u[n], \quad |\alpha| < 1$

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} \alpha^n\, e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} (\alpha\, e^{-j\omega})^n \\
&= \frac{1}{1 - \alpha e^{-j\omega}}
\end{aligned}
$$

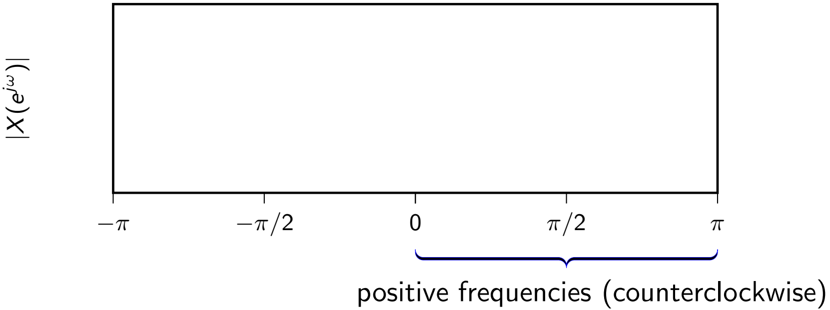
DTFT of $x[n] = \alpha^n u[n], \quad |\alpha| < 1$

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} \alpha^n\, e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} (\alpha\, e^{-j\omega})^n \\
&= \frac{1}{1 - \alpha e^{-j\omega}}
\end{aligned}
$$

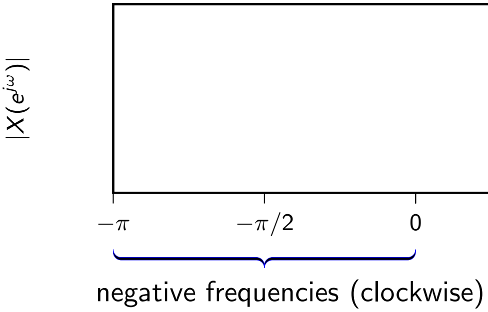
DTFT of $x[n] = \alpha^n u[n], \quad |\alpha| < 1$

$$|X(e^{j\omega})|^2 = \frac{1}{1 + \alpha^2 - 2\alpha \cos \omega}$$

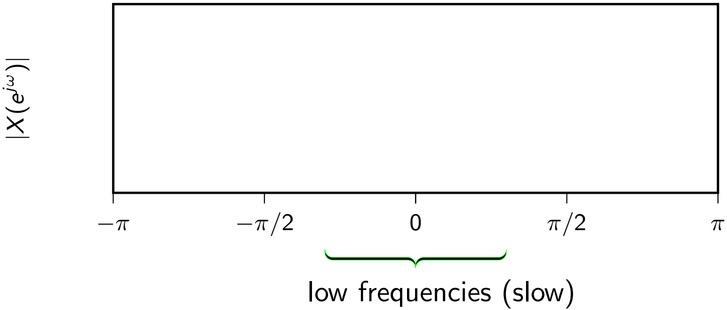
Plotting the DTFT



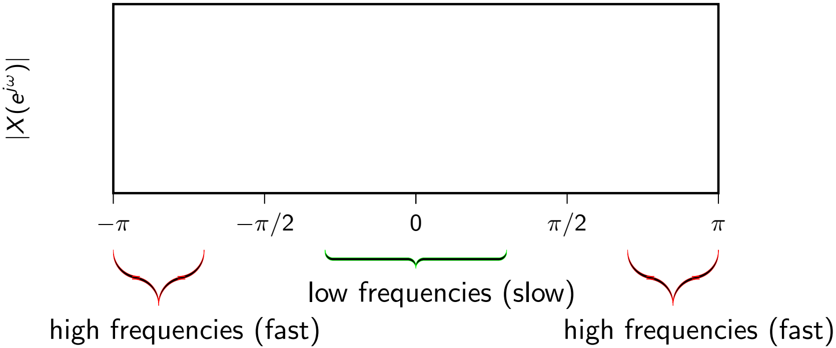
Plotting the DTFT



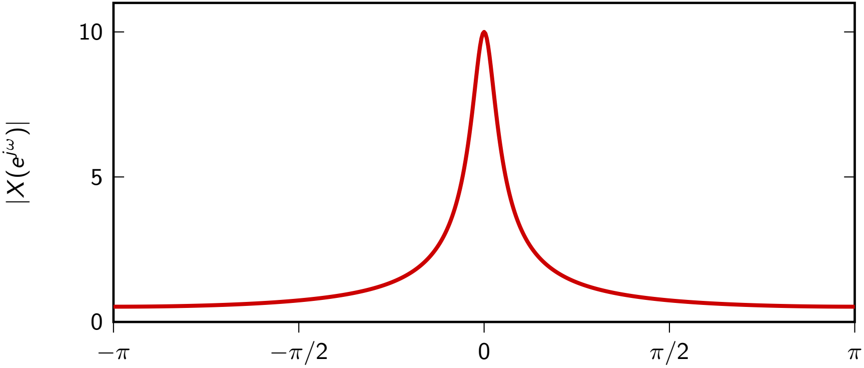
Plotting the DTFT



Plotting the DTFT

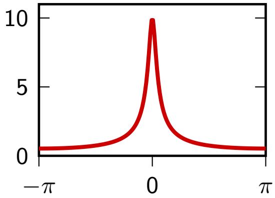


DTFT of $x[n] = \alpha^n u[n], \quad |\alpha| < 1$

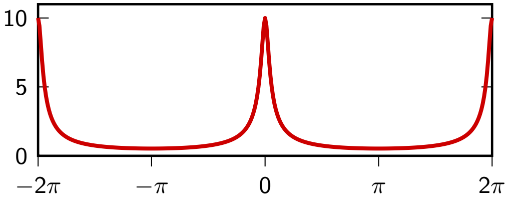


$\alpha = 0.9$

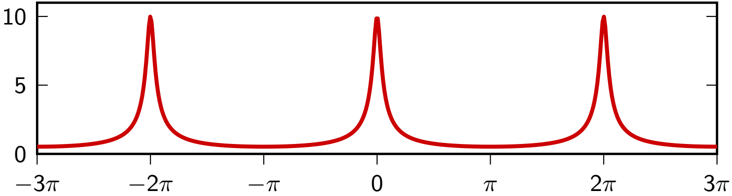
Remember the periodicity!



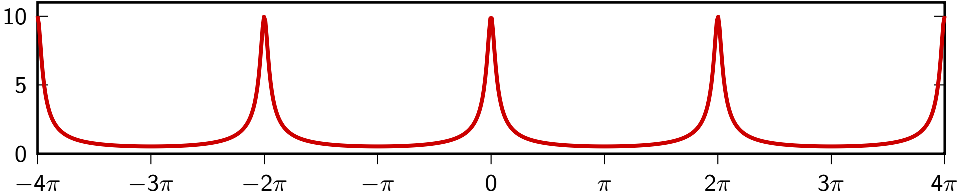
Remember the periodicity!



Remember the periodicity!



Remember the periodicity!



DTFT intuition and properties

Overview:

- DTFT Existence
- Properties
- DTFT as basis expansion

Discrete-Time Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- when does it exist?
- is it a change of basis?

Discrete-Time Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- when does it exist?
- is it a change of basis?

Existence easy for absolutely summable sequences

$$
\begin{aligned}
|X(e^{j\omega})| &= \Big|\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}\Big| \\
&\leq \sum_{n=-\infty}^{\infty} |x[n]\, e^{-j\omega n}| \\
&= \sum_{n=-\infty}^{\infty} |x[n]| \\
&< \infty
\end{aligned}
$$

Existence easy for absolutely summable sequences

$$
\begin{aligned}
|X(e^{j\omega})| &= \left|\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}\right| \\
&\leq \sum_{n=-\infty}^{\infty} |x[n]\, e^{-j\omega n}| \\
&= \sum_{n=-\infty}^{\infty} |x[n]| \\
&< \infty
\end{aligned}
$$

Existence easy for absolutely summable sequences

$$
\begin{aligned}
|X(e^{j\omega})| &= \Big| \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \Big| \\
&\le \sum_{n=-\infty}^{\infty} |x[n]\, e^{-j\omega n}| \\
&= \sum_{n=-\infty}^{\infty} |x[n]| \\
&< \infty
\end{aligned}
$$

Existence easy for absolutely summable sequences

$$
\begin{aligned}
|X(e^{j\omega})| &= \left|\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}\right| \\
&\leq \sum_{n=-\infty}^{\infty} |x[n]\, e^{-j\omega n}| \\
&= \sum_{n=-\infty}^{\infty} |x[n]| \\
&< \infty
\end{aligned}
$$

Inversion easy for absolutely summable sequences

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n} d\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega k}\right) e^{j\omega n} d\omega \\
&= \sum_{k=-\infty}^{\infty} x[k] \int_{-\pi}^{\pi} \frac{e^{j\omega(n-k)}}{2\pi} d\omega \\
&= x[n]
\end{aligned}
$$

Inversion easy for absolutely summable sequences

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega k}\right) e^{j\omega n} d\omega$$

$$= \sum_{k=-\infty}^{\infty} x[k] \int_{-\pi}^{\pi} \frac{e^{j\omega(n-k)}}{2\pi} d\omega$$

$$= x[n]$$

Inversion easy for absolutely summable sequences

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega k}\right) e^{j\omega n} d\omega$$

$$= \sum_{k=-\infty}^{\infty} x[k] \int_{-\pi}^{\pi} \frac{e^{j\omega(n-k)}}{2\pi} d\omega$$

$$= x[n]$$

Inversion easy for absolutely summable sequences

$$\int_{-\pi}^{\pi} \frac{e^{j\omega m}}{2\pi} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega = 1 \qquad \text{for } m = 0$$

$$= \frac{1}{2\pi} \frac{1}{jm} \left. e^{j\omega m} \right|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \frac{1}{jm} \left(e^{j\pi m} - e^{-j\pi m} \right) = 0 \qquad \text{otherwise}$$

the DTFT as the limit of the DFS

Synopsis

- $x[n]$ absolutely summable $\Rightarrow X(e^{j\omega})$ exists formally
- $x[n]$ absolutely summable $\Rightarrow$ we can *periodize* it into $\tilde{x}_N[n]$
- natural Fourier representation for $\tilde{x}_N[n]$ is DFS
- DFS of $\tilde{x}_N[n]$ turns out to be $X(e^{j\omega})$ at $\omega = (2\pi/N)k$
- as N grows to infinity $\tilde{x}_N[n]$ becomes $x[n]$
- as N grows to infinity natural Fourier representation becomes $X(e^{j\omega})$

Synopsis

- $x[n]$ absolutely summable $\Rightarrow X(e^{j\omega})$ exists formally
- $x[n]$ absolutely summable $\Rightarrow$ we can *periodize* it into $\tilde{x}_N[n]$
- natural Fourier representation for $\tilde{x}_N[n]$ is DFS
- DFS of $\tilde{x}_N[n]$ turns out to be $X(e^{j\omega})$ at $\omega = (2\pi/N)k$
- as N grows to infinity $\tilde{x}_N[n]$ becomes $x[n]$
- as N grows to infinity natural Fourier representation becomes $X(e^{j\omega})$

Synopsis

- $x[n]$ absolutely summable $\Rightarrow X(e^{j\omega})$ exists formally
- $x[n]$ absolutely summable $\Rightarrow$ we can *periodize* it into $\tilde{x}_N[n]$
- natural Fourier representation for $\tilde{x}_N[n]$ is DFS
- DFS of $\tilde{x}_N[n]$ turns out to be $X(e^{j\omega})$ at $\omega = (2\pi/N)k$
- as N grows to infinity $\tilde{x}_N[n]$ becomes $x[n]$
- as N grows to infinity natural Fourier representation becomes $X(e^{j\omega})$

Synopsis

- $x[n]$ absolutely summable $\Rightarrow X(e^{j\omega})$ exists formally
- $x[n]$ absolutely summable $\Rightarrow$ we can *periodize* it into $\tilde{x}_N[n]$
- natural Fourier representation for $\tilde{x}_N[n]$ is DFS
- DFS of $\tilde{x}_N[n]$ turns out to be $X(e^{j\omega})$ at $\omega = (2\pi/N)k$
- as N grows to infinity $\tilde{x}_N[n]$ becomes $x[n]$
- as N grows to infinity natural Fourier representation becomes $X(e^{j\omega})$

Synopsis

- $x[n]$ absolutely summable $\Rightarrow X(e^{j\omega})$ exists formally
- $x[n]$ absolutely summable $\Rightarrow$ we can *periodize* it into $\tilde{x}_N[n]$
- natural Fourier representation for $\tilde{x}_N[n]$ is DFS
- DFS of $\tilde{x}_N[n]$ turns out to be $X(e^{j\omega})$ at $\omega = (2\pi/N)k$
- as N grows to infinity $\tilde{x}_N[n]$ becomes $x[n]$
- as N grows to infinity natural Fourier representation becomes $X(e^{j\omega})$

Synopsis

- $x[n]$ absolutely summable $\Rightarrow X(e^{j\omega})$ exists formally
- $x[n]$ absolutely summable $\Rightarrow$ we can *periodize* it into $\tilde{x}_N[n]$
- natural Fourier representation for $\tilde{x}_N[n]$ is DFS
- DFS of $\tilde{x}_N[n]$ turns out to be $X(e^{j\omega})$ at $\omega = (2\pi/N)k$
- as N grows to infinity $\tilde{x}_N[n]$ becomes $x[n]$
- as N grows to infinity natural Fourier representation becomes $X(e^{j\omega})$

Some intuition

With $x[n]$ absolutely summable we can build arbitrarily "periodized" sequences:

$$\tilde{x}_N[n] = \sum_{p=-\infty}^{\infty} x[n + pN]$$

clearly $\tilde{x}_N[n] = \tilde{x}_N[n + N]$

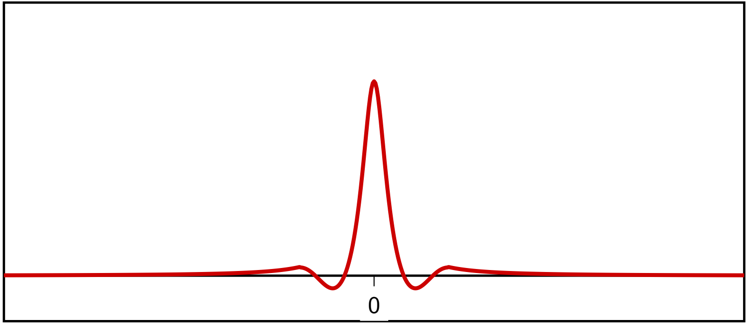
Some intuition

With $x[n]$ absolutely summable we can build arbitrarily "periodized" sequences:

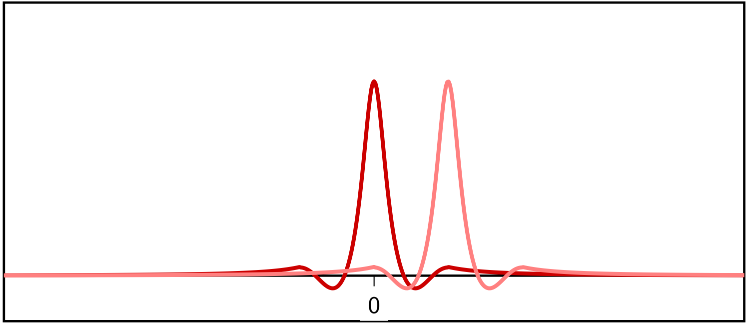
$$\tilde{x}_N[n] = \sum_{p=-\infty}^{\infty} x[n + pN]$$

clearly $\tilde{x}_N[n] = \tilde{x}_N[n + N]$

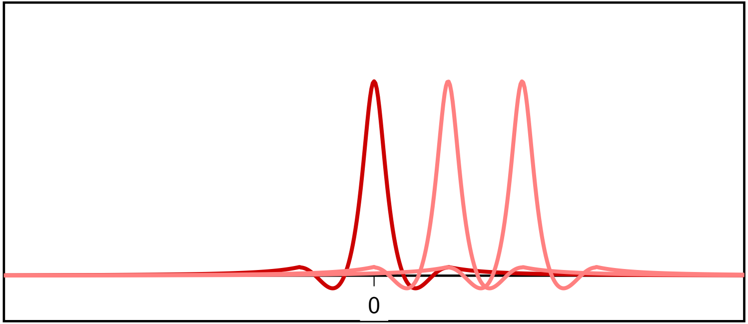
Periodization



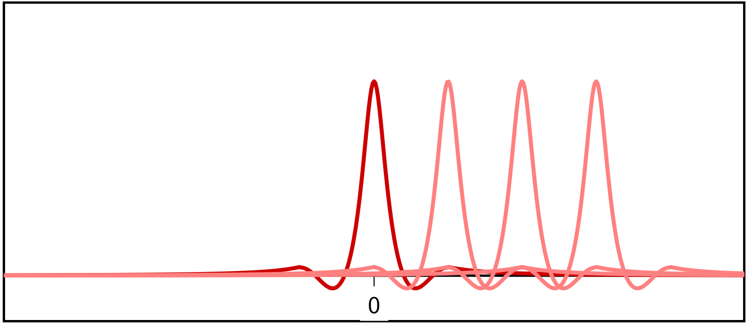
Periodization



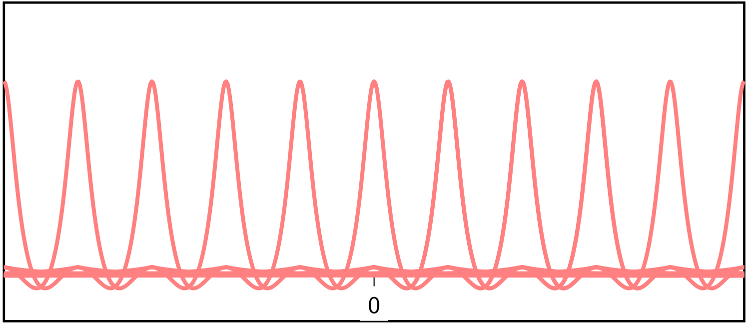
Periodization



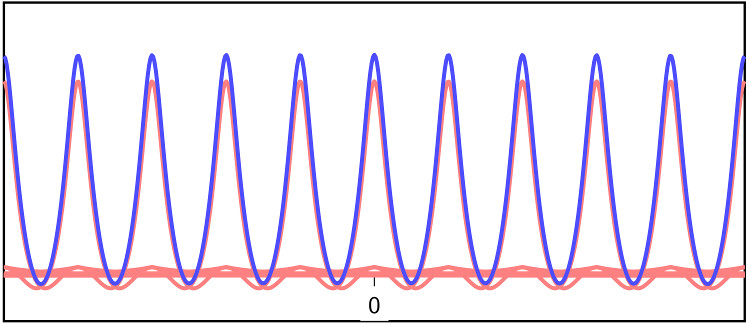
Periodization



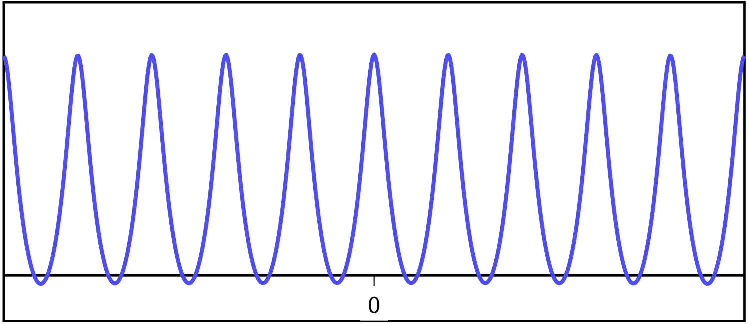
Periodization



Periodization

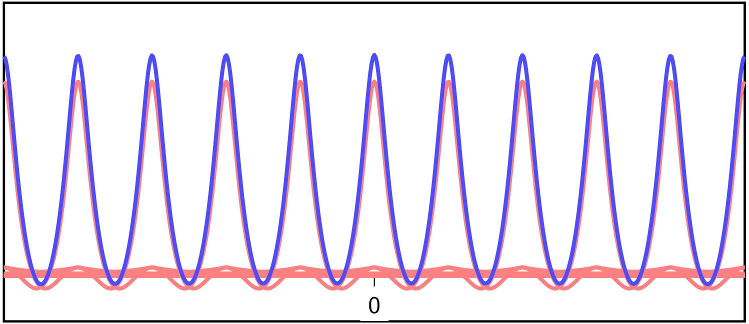


Periodization



Periodization

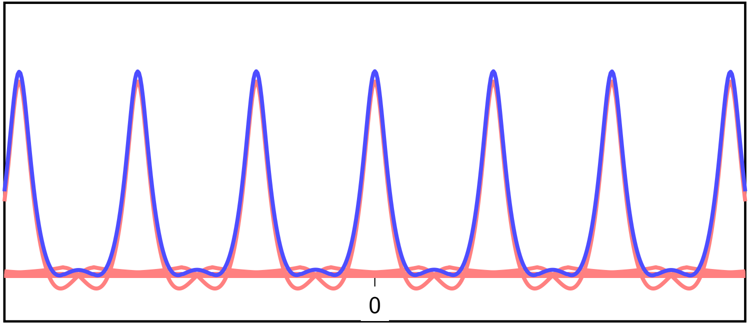
Let N grow large...



$N = 10$

Periodization

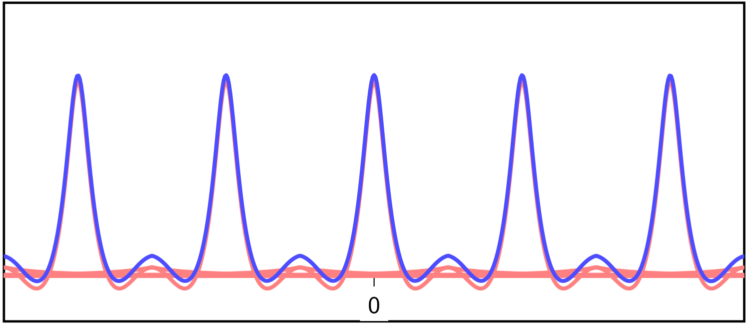
Let N grow large...



$N = 16$

Periodization

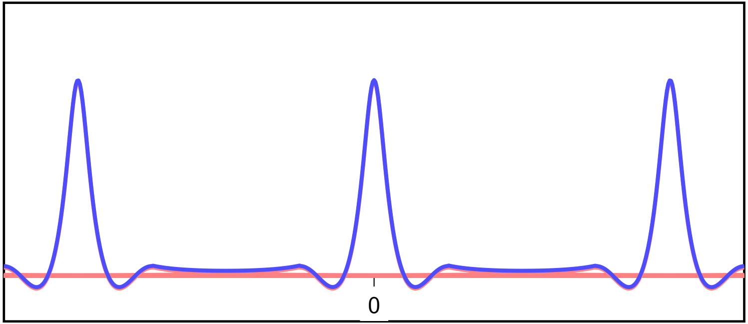
Let N grow large...



$N = 20$

Periodization

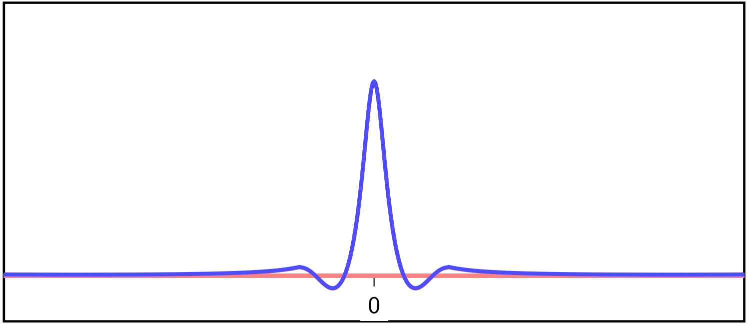
Let N grow large...



$N = 40$

Periodization

... as N grows, $\tilde{x}_N[n] \to x[n]$



$N = 80$

From DFS to DTFT

Natural spectral representation for $\tilde{x}_N[n]$ is the DFS:

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}_N[n] e^{-j\frac{2\pi}{N}nk}$$

$$= \sum_{n=0}^{N-1} \sum_{p=-\infty}^{\infty} x[n+pN] e^{-j\frac{2\pi}{N}nk}$$

$$= \sum_{p=-\infty}^{\infty} \sum_{n=0}^{N-1} x[n+pN] e^{-j\frac{2\pi}{N}(n+pN)k}$$

(remember $e^{j\alpha} = e^{j(\alpha+2K\pi)} \quad \forall K$)

From DFS to DTFT

Natural spectral representation for $\tilde{x}_N[n]$ is the DFS:

$$
\begin{aligned}
\tilde{X}[k] &= \sum_{n=0}^{N-1} \tilde{x}_N[n] e^{-j\frac{2\pi}{N}nk} \\
&= \sum_{n=0}^{N-1} \sum_{p=-\infty}^{\infty} x[n+pN] e^{-j\frac{2\pi}{N}nk} \\
&= \sum_{p=-\infty}^{\infty} \sum_{n=0}^{N-1} x[n+pN] e^{-j\frac{2\pi}{N}(n+pN)k}
\end{aligned}
$$

(remember $e^{j\alpha} = e^{j(\alpha+2K\pi)} \quad \forall K$)

From DFS to DTFT

Natural spectral representation for $\tilde{x}_N[n]$ is the DFS:

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}_N[n] e^{-j\frac{2\pi}{N}nk}$$

$$= \sum_{n=0}^{N-1} \sum_{p=-\infty}^{\infty} x[n+pN] e^{-j\frac{2\pi}{N}nk}$$

$$= \sum_{p=-\infty}^{\infty} \sum_{n=0}^{N-1} x[n+pN] e^{-j\frac{2\pi}{N}(n+pN)k}$$

(remember $e^{j\alpha} = e^{j(\alpha+2K\pi)} \quad \forall K$)

From double sum to single sum

we can always write for all $N \in \mathbb{N}^+$

$$\sum_{m=-\infty}^{\infty} y[m] = \sum_{p=-\infty}^{\infty} \sum_{n=0}^{N-1} y[n + pN]$$

Example (N=4)

p \ n	0	1	2	3
...				
-1				
0				
1				
2				
...				

$$m = n + 4p$$

Example (N=4)

	n				
		0	1	2	3
	...				
	-1				
p	0	0			
	1				
	2				
	...				

$$m = n + 4p$$

Example (N=4)

n

p	0	1	2	3
...				
-1				
0	0	1		
1				
2				
...				

$$m = n + 4p$$

Example (N=4)

p \ n	0	1	2	3
...				
-1				
0	0	1	2	
1				
2				
...				

$$m = n + 4p$$

Example (N=4)

n

		0	1	2	3
	. . .				
	-1				
p	0	0	1	2	3
	1				
	2				
	. . .				

$$m = n + 4p$$

Example (N=4)

n

p

	0	1	2	3
...				
-1				
0	0	1	2	3
1	4			
2				
...				

$$m = n + 4p$$

Example (N=4)

n

p

	0	1	2	3
...				
-1				
0	0	1	2	3
1	4	5		
2				
...				

$$m = n + 4p$$

Example (N=4)

n

p	0	1	2	3
...				
-1				
0	0	1	2	3
1	4	5	6	
2				
...				

$$m = n + 4p$$

Example (N=4)

	n = 0	1	2	3
...				
-1				
p = 0	0	1	2	3
1	4	5	6	7
2				
...				

$$m = n + 4p$$

Example (N=4)

p \ m	0	1	2	3
...				
-1	-4	-3	-2	-1
0	0	1	2	3
1	4	5	6	7
2	8	9	10	11
...				

$$n = 4p + m$$

From DFS to DTFT

$$\tilde{X}[k] = \sum_{p=-\infty}^{\infty} \sum_{n=0}^{N-1} x[n + pN] e^{-j\frac{2\pi}{N}(n+pN)k}$$

$$= \sum_{m=-\infty}^{\infty} x[m] e^{-j\frac{2\pi}{N}km}$$

$$= X(e^{j\omega})\big|_{\omega=\frac{2\pi}{N}k}$$

From DFS to DTFT

$$\tilde{X}[k] = \sum_{p=-\infty}^{\infty} \sum_{n=0}^{N-1} x[n + pN] e^{-j\frac{2\pi}{N}(n+pN)k}$$

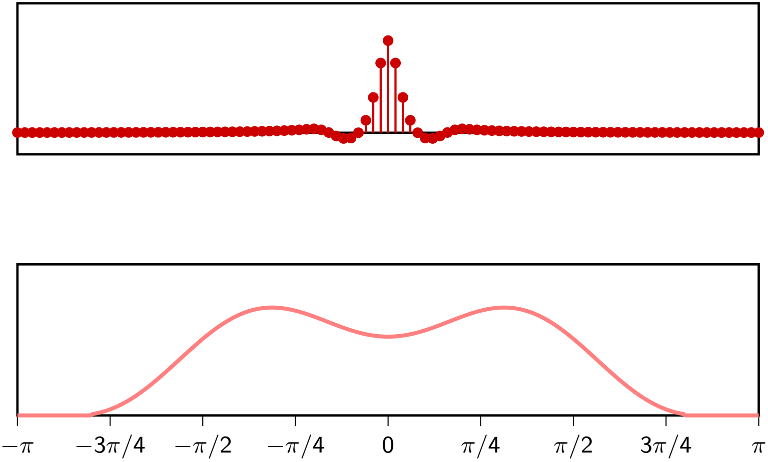
$$= \sum_{m=-\infty}^{\infty} x[m] e^{-j\frac{2\pi}{N}km}$$

$$= X(e^{j\omega})\big|_{\omega=\frac{2\pi}{N}k}$$

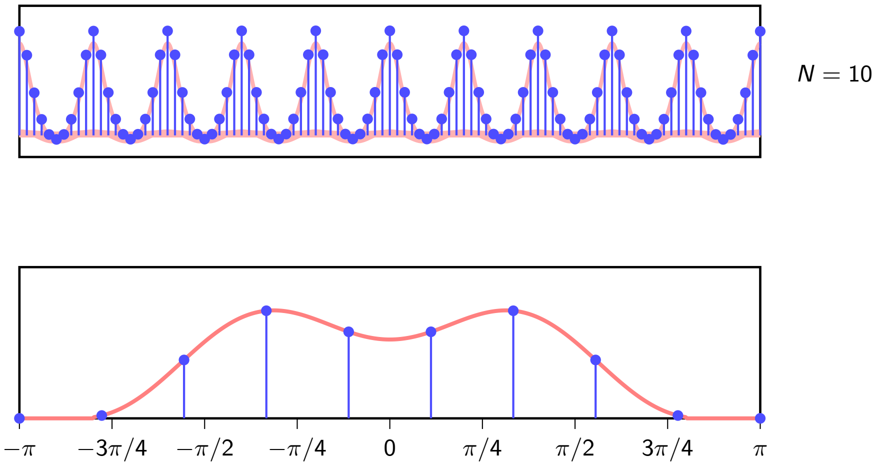
From DFS to DTFT

$$\begin{aligned}
\tilde{X}[k] &= \sum_{p=-\infty}^{\infty} \sum_{n=0}^{N-1} x[n+pN] e^{-j\frac{2\pi}{N}(n+pN)k} \\
&= \sum_{m=-\infty}^{\infty} x[m] e^{-j\frac{2\pi}{N}km} \\
&= X(e^{j\omega})\big|_{\omega=\frac{2\pi}{N}k}
\end{aligned}$$

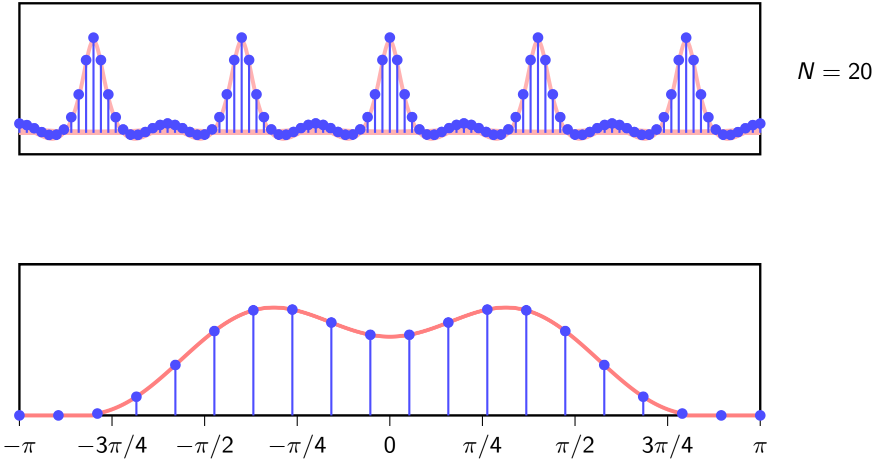
From DFS to DTFT



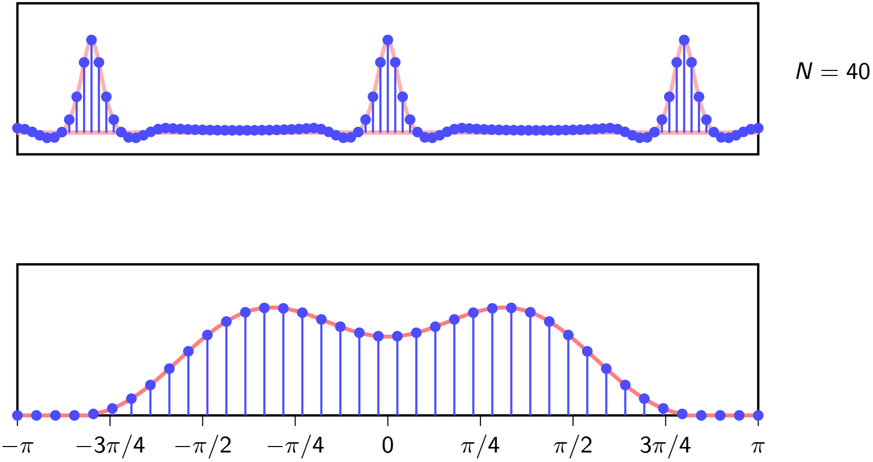
From DFS to DTFT



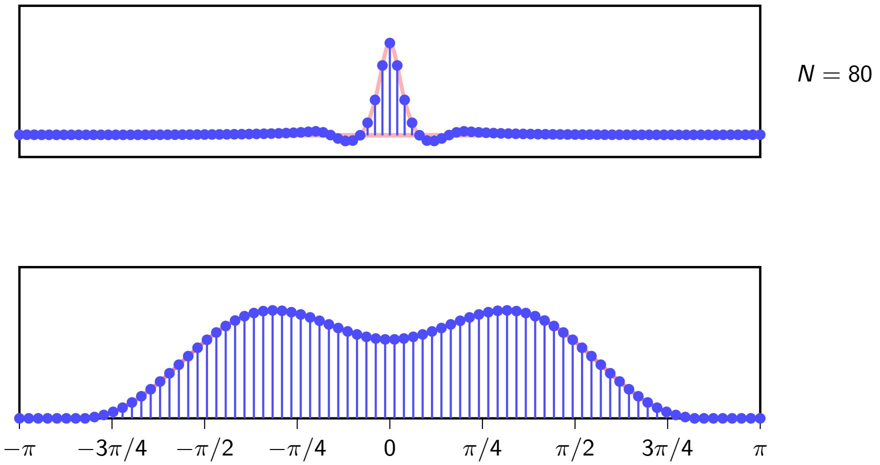
From DFS to DTFT



From DFS to DTFT



From DFS to DTFT



From DFS to DTFT

- we're comfortable with DFS: change of basis, energy conservation, etc.
- as N grows, $\tilde{x}_N[n] \rightarrow x[n]$ and the spectral representation "becomes" the DTFT
- we can retain the "change of basis" paradigm for the DTFT

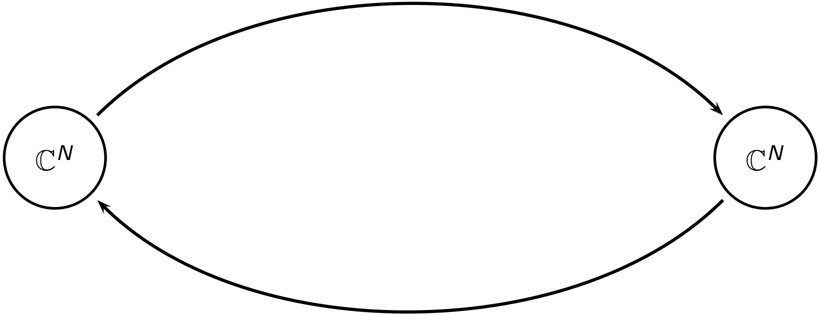
From DFS to DTFT

- we're comfortable with DFS: change of basis, energy conservation, etc.
- as N grows, $\tilde{x}_N[n] \rightarrow x[n]$ and the spectral representation "becomes" the DTFT
- we can retain the "change of basis" paradigm for the DTFT

From DFS to DTFT

- we're comfortable with DFS: change of basis, energy conservation, etc.
- as N grows, $\tilde{x}_N[n] \rightarrow x[n]$ and the spectral representation "becomes" the DTFT
- we can retain the "change of basis" paradigm for the DTFT

Review: DFT



Review: DFT

$$
\begin{tikzcd}
\mathbb{C}^N \arrow[rr, bend left, "X[k] = \langle e^{j\frac{2\pi}{N}nk}, x[n] \rangle"] & & \mathbb{C}^N \arrow[ll, bend left]
\end{tikzcd}
$$

Review: DFT

$$\begin{tikzcd}[column sep=huge] \mathbb{C}^N \arrow[r, bend left, "X[k] = \langle e^{j\frac{2\pi}{N}nk}, x[n]\rangle"] & \mathbb{C}^N \arrow[l, bend left, "x[n] = (1/N)\sum X[k]\, e^{j\frac{2\pi}{N}nk}"] \end{tikzcd}$$

Review: DFT

$$\begin{tikzcd}[column sep=huge]
\mathbb{C}^N \arrow[r, bend left, "X[k] = \langle e^{j\frac{2\pi}{N}nk}, x[n]\rangle"] & \mathbb{C}^N \arrow[l, bend left, "x[n] = (1/N)\sum X[k]\, e^{j\frac{2\pi}{N}nk}"]
\end{tikzcd}$$

basis: $\{e^{j\frac{2\pi}{N}nk}\}_k$

Review: DFS

$$\begin{tikzcd}[column sep=huge]
\tilde{\mathbb{C}}^N \arrow[r, bend left, "\tilde{X}[k] = \langle e^{j\frac{2\pi}{N}nk}, \tilde{x}[n] \rangle"] & \mathbb{C}^N \arrow[l, bend left, "\tilde{x}[n] = (1/N)\sum \tilde{X}[k]\, e^{j\frac{2\pi}{N}nk}"]
\end{tikzcd}$$

basis: $\{e^{j\frac{2\pi}{N}nk}\}_k$

What about the DTFT?

- formally DTFT is an inner product in $\mathbb{C}^\infty$:

$$\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = \langle e^{j\omega n}, x[n] \rangle$$

- "basis" is an infinite, uncountable basis: $\{e^{j\omega n}\}_{\omega\in\mathbb{R}}$
- something "breaks down": we start with sequences but the transform is a function
- we used absolutely summable sequences but DTFT exists for all square-summable sequences (proof is rather technical)

What about the DTFT?

- formally DTFT is an inner product in $\mathbb{C}^\infty$:

$$\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = \langle e^{j\omega n}, x[n] \rangle$$

- "basis" is an infinite, uncountable basis: $\{e^{j\omega n}\}_{\omega \in \mathbb{R}}$
- something "breaks down": we start with sequences but the transform is a function
- we used absolutely summable sequences but DTFT exists for all square-summable sequences (proof is rather technical)

What about the DTFT?

- formally DTFT is an inner product in $\mathbb{C}^\infty$:

$$\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = \langle e^{j\omega n}, x[n] \rangle$$

- "basis" is an infinite, uncountable basis: $\{e^{j\omega n}\}_{\omega \in \mathbb{R}}$
- something "breaks down": we start with sequences but the transform is a function
- we used absolutely summable sequences but DTFT exists for all square-summable sequences (proof is rather technical)

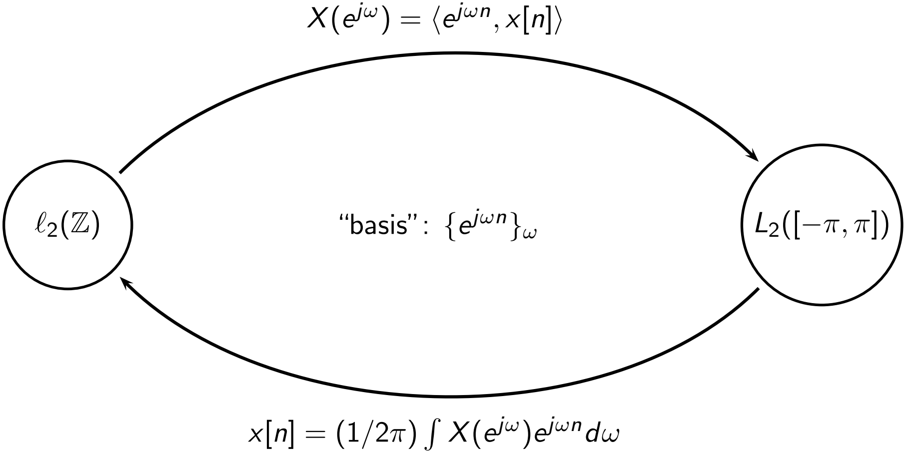
What about the DTFT?

- formally DTFT is an inner product in $\mathbb{C}^\infty$:

$$\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = \langle e^{j\omega n}, x[n] \rangle$$

- "basis" is an infinite, uncountable basis: $\{e^{j\omega n}\}_{\omega \in \mathbb{R}}$
- something "breaks down": we start with sequences but the transform is a function
- we used absolutely summable sequences but DTFT exists for all square-summable sequences (proof is rather technical)

DTFT


$X(e^{j\omega}) = \langle e^{j\omega n}, x[n] \rangle$
$\ell_2(\mathbb{Z})$
"basis": $\{e^{j\omega n}\}_\omega$
$L_2([-\pi, \pi])$
$x[n] = (1/2\pi) \int X(e^{j\omega}) e^{j\omega n} d\omega$

DTFT properties

- linearity

$$\text{DTFT}\{\alpha x[n] + \beta y[n]\} = \alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$$

- time shift

$$\text{DTFT}\{x[n-M]\} = e^{-j\omega M} X(e^{j\omega})$$

- modulation (dual)

$$\text{DTFT}\{e^{j\omega_0 n} x[n]\} = X(e^{j(\omega-\omega_0)})$$

DTFT properties

- linearity

$$\text{DTFT}\{\alpha x[n] + \beta y[n]\} = \alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$$

- time shift

$$\text{DTFT}\{x[n-M]\} = e^{-j\omega M} X(e^{j\omega})$$

- modulation (dual)

$$\text{DTFT}\{e^{j\omega_0 n} x[n]\} = X(e^{j(\omega-\omega_0)})$$

DTFT properties

- linearity

$$\text{DTFT}\{\alpha x[n] + \beta y[n]\} = \alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$$

- time shift

$$\text{DTFT}\{x[n-M]\} = e^{-j\omega M} X(e^{j\omega})$$

- modulation (dual)

$$\text{DTFT}\{e^{j\omega_0 n} x[n]\} = X(e^{j(\omega-\omega_0)})$$

DTFT properties

- time reversal

$$\text{DTFT}\{x[-n]\} = X(e^{-j\omega})$$

- conjugation

$$\text{DTFT}\{x^*[n]\} = X^*(e^{-j\omega})$$

DTFT properties

- time reversal

$$\mathrm{DTFT}\{x[-n]\} = X(e^{-j\omega})$$

- conjugation

$$\mathrm{DTFT}\{x^*[n]\} = X^*(e^{-j\omega})$$

Some particular cases:

- if $x[n]$ is symmetric, the DTFT is symmetric:

$$x[n] = x[-n] \iff X(e^{j\omega}) = X(e^{-j\omega})$$

- if $x[n]$ is real, the DTFT is Hermitian-symmetric:

$$x[n] = x^*[n] \iff X(e^{j\omega}) = X^*(e^{-j\omega})$$

- in other words: if $x[n]$ is real, the magnitude of the DTFT is symmetric:

$$x[n] \in \mathbb{R} \iff |X(e^{j\omega})| = |X(e^{-j\omega})|$$

- finally, if $x[n]$ is real and symmetric, the DTFT is also real and symmetric!

Some particular cases:

- if $x[n]$ is symmetric, the DTFT is symmetric:

$$x[n] = x[-n] \Longleftrightarrow X(e^{j\omega}) = X(e^{-j\omega})$$

- if $x[n]$ is real, the DTFT is Hermitian-symmetric:

$$x[n] = x^*[n] \Longleftrightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$$

- in other words: if $x[n]$ is real, the magnitude of the DTFT is symmetric:

$$x[n] \in \mathbb{R} \Longleftrightarrow |X(e^{j\omega})| = |X(e^{-j\omega})|$$

- finally, if $x[n]$ is real and symmetric, the DTFT is also real and symmetric!

Some particular cases:

- if $x[n]$ is symmetric, the DTFT is symmetric:

$$x[n] = x[-n] \iff X(e^{j\omega}) = X(e^{-j\omega})$$

- if $x[n]$ is real, the DTFT is Hermitian-symmetric:

$$x[n] = x^*[n] \iff X(e^{j\omega}) = X^*(e^{-j\omega})$$

- in other words: if $x[n]$ is real, the magnitude of the DTFT is symmetric:

$$x[n] \in \mathbb{R} \iff |X(e^{j\omega})| = |X(e^{-j\omega})|$$

- finally, if $x[n]$ is real and symmetric, the DTFT is also real and symmetric!

Some particular cases:

- if $x[n]$ is symmetric, the DTFT is symmetric:

$$x[n] = x[-n] \Longleftrightarrow X(e^{j\omega}) = X(e^{-j\omega})$$

- if $x[n]$ is real, the DTFT is Hermitian-symmetric:

$$x[n] = x^*[n] \Longleftrightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$$

- in other words: if $x[n]$ is real, the magnitude of the DTFT is symmetric:

$$x[n] \in \mathbb{R} \Longleftrightarrow |X(e^{j\omega})| = |X(e^{-j\omega})|$$

- finally, if $x[n]$ is real and symmetric, the DTFT is also real and symmetric!

DTFT as basis expansion

Some things are OK:

- $\text{DFT}\{\delta[n]\} = 1$
- $\text{DTFT}\{\delta[n]\} = \langle e^{j\omega n}, \delta[n] \rangle = 1$

DTFT as basis expansion

Some things are OK:

- $\text{DFT}\{\delta[n]\} = 1$
- $\text{DTFT}\{\delta[n]\} = \langle e^{j\omega n}, \delta[n]\rangle = 1$

DTFT as basis expansion

Some things aren't:

- $\text{DFT}\{1\} = \delta[n]$
- $\text{DTFT}\{1\} = \sum_{n=-\infty}^{\infty} e^{-j\omega n} = ?$
- problem: too many interesting sequences are *not* square summable!

DTFT as basis expansion

Some things aren't:

- DFT $\{1\} = \delta[n]$
- DTFT $\{1\} = \sum_{n=-\infty}^{\infty} e^{-j\omega n} = ?$

- problem: too many interesting sequences are *not* square summable!

DTFT as basis expansion

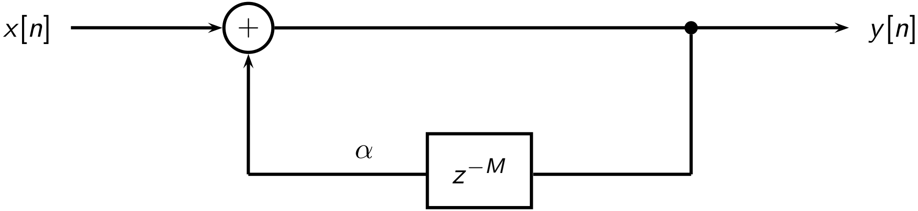
Some things aren't:

- $\text{DFT}\{1\} = \delta[n]$
- $\text{DTFT}\{1\} = \sum_{n=-\infty}^{\infty} e^{-j\omega n} = ?$

- problem: too many interesting sequences are *not* square summable!

The DTFT of the Karplus-Strong output

Karplus-Strong revisited



$$y[n] = \alpha\, y[n-M] + x[n]$$

Karplus-Strong revisited

- choose a signal $\bar{x}[n]$ that is nonzero only for $0 \leq n < M$
- generated signal is infinite-length but not periodic:

$$y[n] = \bar{x}[0], \bar{x}[1], \ldots, \bar{x}[M-1], \alpha\bar{x}[0], \alpha\bar{x}[1], \ldots, \alpha\bar{x}[M-1], \alpha^2\bar{x}[0], \alpha^2\bar{x}[1], \ldots$$

- what is the DTFT of this signal?

Karplus-Strong revisited

- choose a signal $\bar{x}[n]$ that is nonzero only for $0 \leq n < M$
- generated signal is infinite-length but not periodic:

$$y[n] = \underbrace{\bar{x}[0], \bar{x}[1], \ldots, \bar{x}[M}_{1^{\text{st}}\text{ period}} - 1], \underbrace{\alpha\bar{x}[0], \alpha\bar{x}[1], \ldots, \alpha\bar{x}[M}_{2^{\text{nd}}\text{ period}} - 1], \underbrace{\alpha^2\bar{x}[0], \alpha^2\bar{x}[1], \ldots}_{\ldots}$$

- what is the DTFT of this signal?

Karplus-Strong revisited

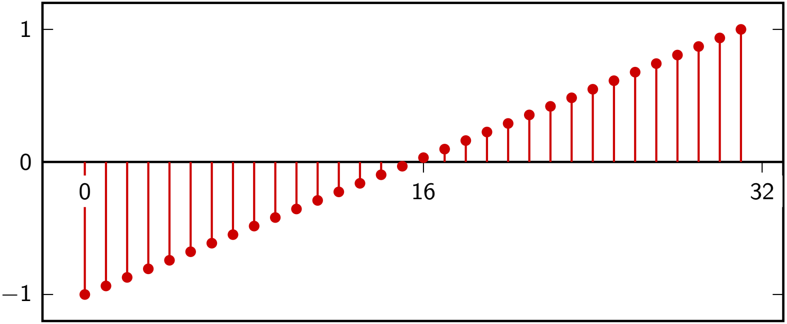
- choose a signal $\bar{x}[n]$ that is nonzero only for $0 \le n < M$
- generated signal is infinite-length but not periodic:

$$y[n] = \underbrace{\bar{x}[0], \bar{x}[1], \ldots, \bar{x}[M-1]}_{1^{\text{st}}\ \text{period}}, \underbrace{\alpha\bar{x}[0], \alpha\bar{x}[1], \ldots, \alpha\bar{x}[M-1]}_{2^{\text{nd}}\ \text{period}}, \underbrace{\alpha^2\bar{x}[0], \alpha^2\bar{x}[1], \ldots}_{\ldots}$$

- what is the DTFT of this signal?

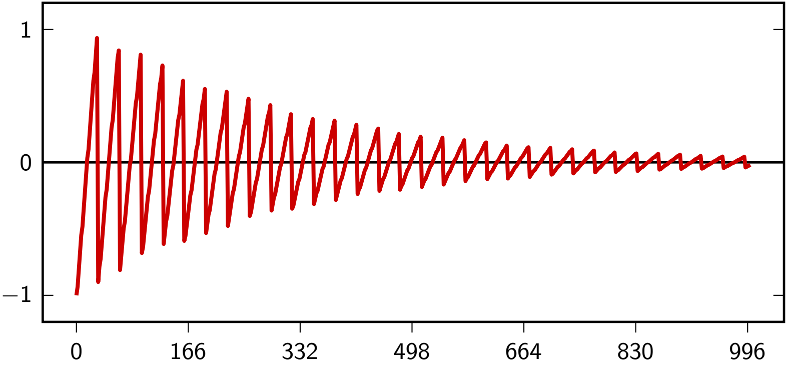
KS revisited: 32-tap sawtooth wave

$$x[n] = 2n/(M-1) - 1, \quad n = 0, 1, \ldots, M-1$$



KS revisited: decay $\alpha = 0.9$

$$y[n] = \alpha^{\lfloor n/M \rfloor} \bar{x}[n \mod M]\, u[n]$$



DTFT of KS signal

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]\, e^{-j\omega n}$$

Same trick we used before:

$$\sum_{m=-\infty}^{\infty} y[m] = \sum_{p=-\infty}^{\infty} \sum_{n=0}^{N-1} y[n+pN]$$

DTFT of KS signal

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]\, e^{-j\omega n}$$

$$= \sum_{p=0}^{\infty} \sum_{m=0}^{M-1} \alpha^p \bar{x}[m]\, e^{-j\omega(pM+m)}$$

$$= \sum_{p=0}^{\infty} \alpha^p e^{-j\omega Mp} \sum_{m=0}^{M-1} \bar{x}[m] e^{-j\omega m}$$

$$= \sum_{p=0}^{\infty} \alpha^p e^{-j\omega Mp} \sum_{m=-\infty}^{\infty} \bar{x}[m] e^{-j\omega m}$$

$$= A(e^{j\omega M})\, \bar{X}(e^{j\omega})$$

DTFT of KS signal

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]\, e^{-j\omega n}$$

$$= \sum_{p=0}^{\infty} \sum_{m=0}^{M-1} \alpha^p \bar{x}[m]\, e^{-j\omega(pM+m)}$$

$$= \sum_{p=0}^{\infty} \alpha^p e^{-j\omega Mp} \sum_{m=0}^{M-1} \bar{x}[m] e^{-j\omega m}$$

$$= \sum_{p=0}^{\infty} \alpha^p e^{-j\omega Mp} \sum_{m=-\infty}^{\infty} \bar{x}[m] e^{-j\omega m}$$

$$= A(e^{j\omega M})\, \bar{X}(e^{j\omega})$$

DTFT of KS signal

$$
\begin{aligned}
Y(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} y[n]\, e^{-j\omega n} \\
&= \sum_{p=0}^{\infty} \sum_{m=0}^{M-1} \alpha^p \bar{x}[m]\, e^{-j\omega(pM+m)} \\
&= \sum_{p=0}^{\infty} \alpha^p e^{-j\omega Mp} \sum_{m=0}^{M-1} \bar{x}[m] e^{-j\omega m} \\
&= \sum_{p=0}^{\infty} \alpha^p e^{-j\omega Mp} \sum_{m=-\infty}^{\infty} \bar{x}[m] e^{-j\omega m} \\
&= A(e^{j\omega M})\, \bar{X}(e^{j\omega})
\end{aligned}
$$

DTFT of KS signal

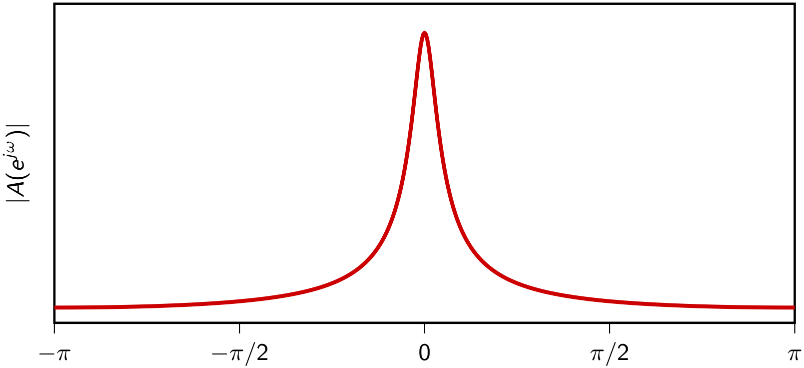
$$
\begin{aligned}
Y(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} y[n]\, e^{-j\omega n} \\
&= \sum_{p=0}^{\infty} \sum_{m=0}^{M-1} \alpha^p \bar{x}[m]\, e^{-j\omega(pM+m)} \\
&= \sum_{p=0}^{\infty} \alpha^p e^{-j\omega Mp} \sum_{m=0}^{M-1} \bar{x}[m] e^{-j\omega m} \\
&= \sum_{p=0}^{\infty} \alpha^p e^{-j\omega Mp} \sum_{m=-\infty}^{\infty} \bar{x}[m] e^{-j\omega m} \\
&= A(e^{j\omega M})\, \bar{X}(e^{j\omega})
\end{aligned}
$$

DTFT of KS signal

$$
\begin{aligned}
Y(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} y[n]\, e^{-j\omega n} \\
&= \sum_{p=0}^{\infty} \sum_{m=0}^{M-1} \alpha^p \bar{x}[m]\, e^{-j\omega(pM+m)} \\
&= \sum_{p=0}^{\infty} \alpha^p e^{-j\omega Mp} \sum_{m=0}^{M-1} \bar{x}[m] e^{-j\omega m} \\
&= \sum_{p=0}^{\infty} \alpha^p e^{-j\omega Mp} \sum_{m=-\infty}^{\infty} \bar{x}[m] e^{-j\omega m} \\
&= A(e^{j\omega M})\, \bar{X}(e^{j\omega})
\end{aligned}
$$

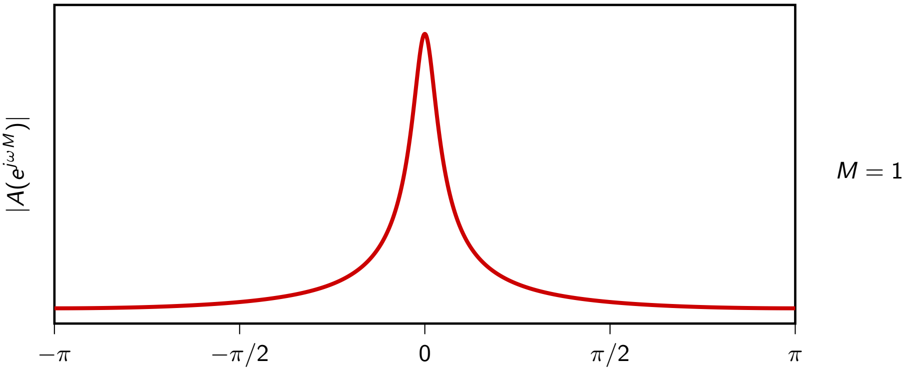
We know the first term

$$A(e^{j\omega}) = \mathrm{DTFT}\left\{\alpha^n\, u[n]\right\} = \frac{1}{1-\alpha e^{-j\omega}}$$



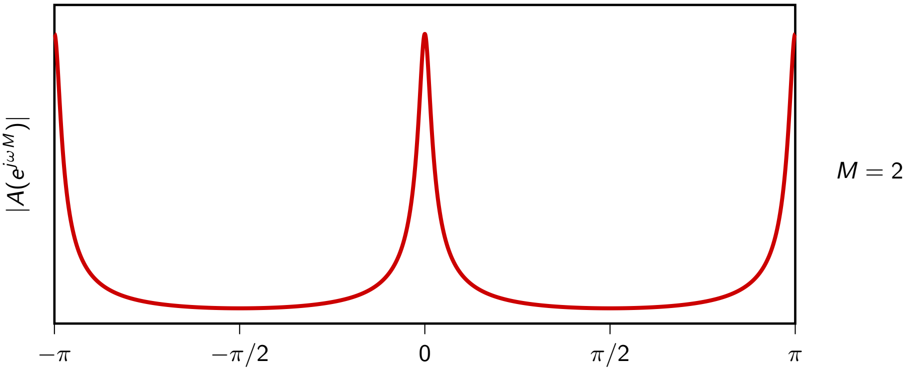
We know the first term

$A(e^{j\omega M})$ rescales the frequency axis: periodicity!



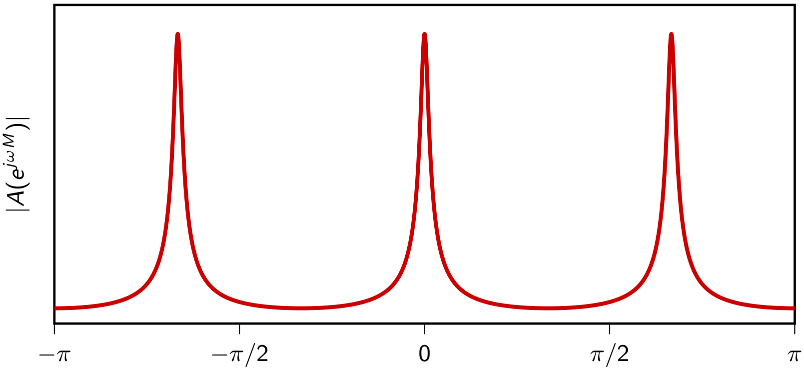
We know the first term

$A(e^{j\omega M})$ rescales the frequency axis: periodicity!



We know the first term

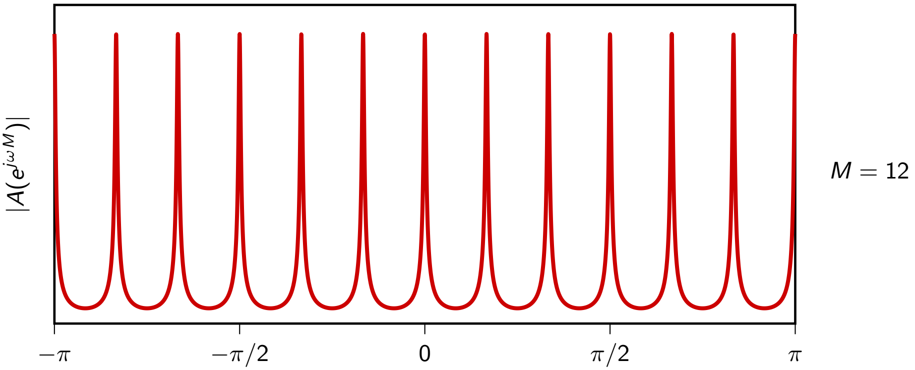
$A(e^{j\omega M})$ rescales the frequency axis: periodicity!



$M = 3$

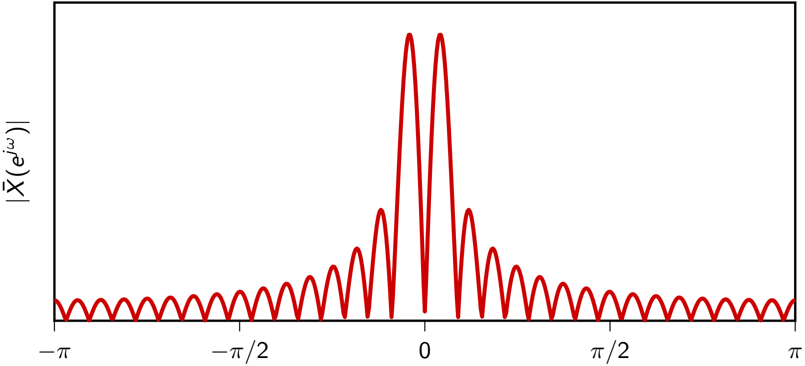
We know the first term

$A(e^{j\omega M})$ rescales the frequency axis: periodicity!



Second term is left as an exercise

$$\bar{X}(e^{j\omega}) = e^{-j\omega}\left(\frac{M+1}{M-1}\right)\frac{1-e^{-j(M-1)\omega}}{(1-e^{-j\omega})^2} - \frac{1-e^{-j(M+1)\omega}}{(1-e^{-j\omega})^2}$$



DTFT of KS with decay

$$Y(e^{j\omega}) = A(e^{j\omega M})\bar{X}(e^{j\omega})$$