COM303: Digital Signal Processing

Lecture 22: Digital Communication Systems (I)

overview

- the communication channel
- bandwidth constraints
- power constraints

a comparison of data rates

- Transatlantic cable:
  - 1866: 8 words per minute ($\approx$5 bps)
  - 1956: AT&T, coax, 48 voice channels ($\approx$3Mbps)
  - 2005: Alcatel Tera10, fiber, 8.4 Tbps (8.4×10^{12} bps)
  - 2012: fiber, 60 Tbps
- Voiceband modems
  - 1950s: Bell 202, 1200 bps
  - 1990s: V90, 56Kbps
  - 2008: ADSL2+, 24Mbps

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Success factors for digital communications

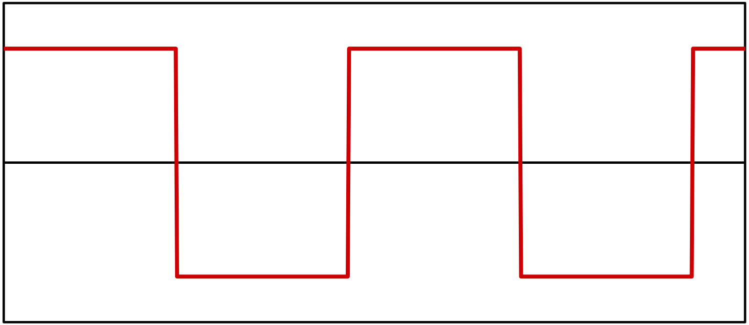
1. power of the digital paradigm
2. natural integration with information theory
3. hardware advancement

Success factors for digital communications

1) power of the DSP paradigm:

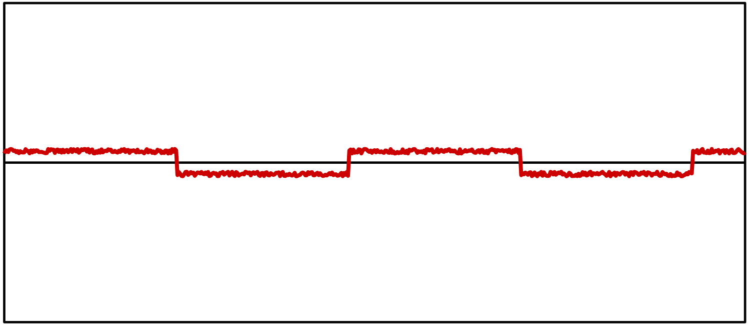
- integers are “easy” to regenerate
- good phase control
- adaptive algorithms

Regenerating signals



$x(t)$

Regenerating signals



$$x(t)/G + \sigma(t)$$

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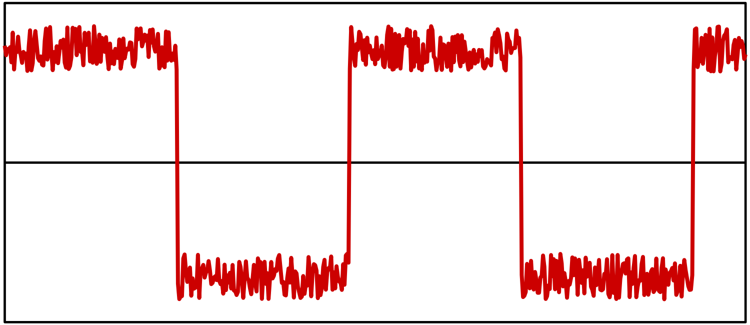
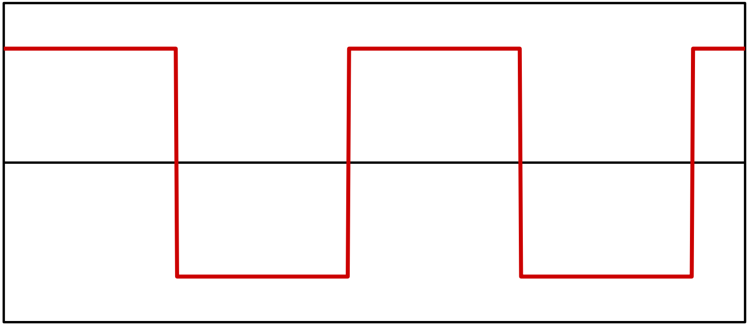


$$G[x(t)/G + \sigma(t)] = x(t) + G\sigma(t)$$

Regenerating signals



$$\hat{x}_1(t) = G\,\mathrm{sgn}[x(t) + \sigma(t)]$$

Success factors for digital communications

2) algorithmic nature of DSP is a perfect match with information theory:

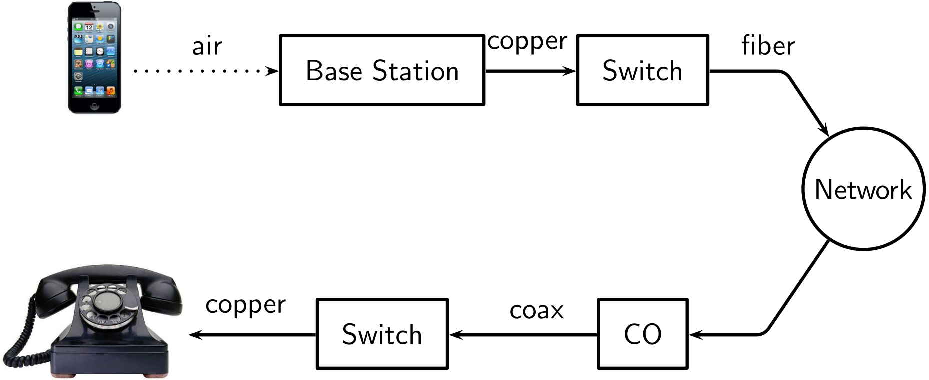
- error correction (CD's and DVD's)
- entropy coding (JPEG)

Success factors for digital communications

3) hardware advancement

- general-purpose platforms
- miniaturization
- power efficiency

The many incarnations of a conversation



The analog channel

unescapable “limits” of physical channels:

- bandwidth constraint
- power constraint

both constraints will affect the final *capacity* of the channel

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The analog channel's capacity

maximum amount of information that can be *reliably* delivered over a channel
(bits per second)

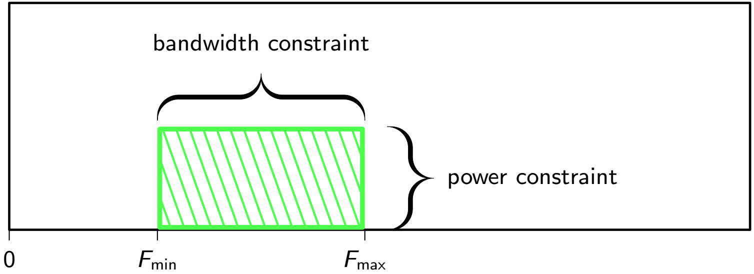
The harsh truth about reliability

we cannot design a perfect (error-free) communication system because of noise

but

we can design a system with arbitrary small error rate (e.g. 10^{-6})

The channel constraints



The design problem

- transmitted data is a sequence of digital symbols $a[n] \in \mathcal{A}$
- we will model $a[n]$ as a zero-mean white process
- we need to transform $a[n]$ into an analog signal $s(t)$ that fulfills both bandwidth and power constraints

Bandwidth vs capacity: intuition

- we want to transmit a data sequence $a[n]$ over an analog channel
- we sinc-interpolate $a[n]$ with a period T_s
- if we make T_s small we can send more symbols per unit of time...
- ... but the bandwidth of the signal will grow as $1/T_s$

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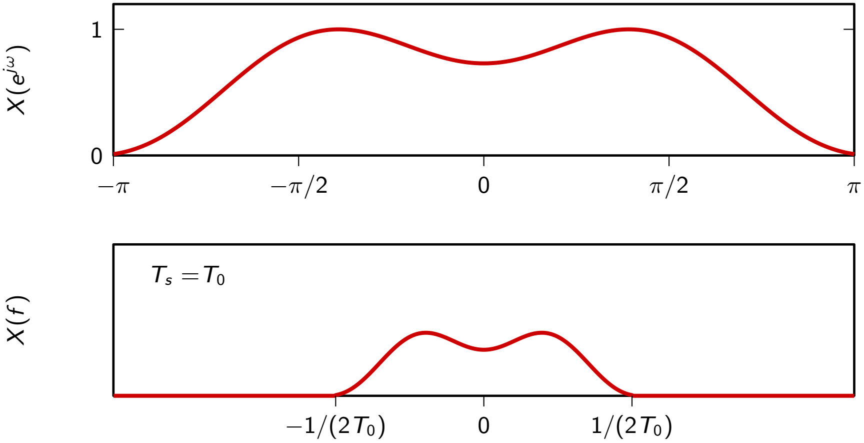
Sinc interpolation

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\,\mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

$$X(f) = \begin{cases} (1/F_s)X(e^{j2\pi f/F_s}) & \text{for } |f| \leq \frac{F_s}{2} = \frac{1}{2T_s} \\ 0 & \text{otherwise} \end{cases}$$

Spectrum of interpolated signals



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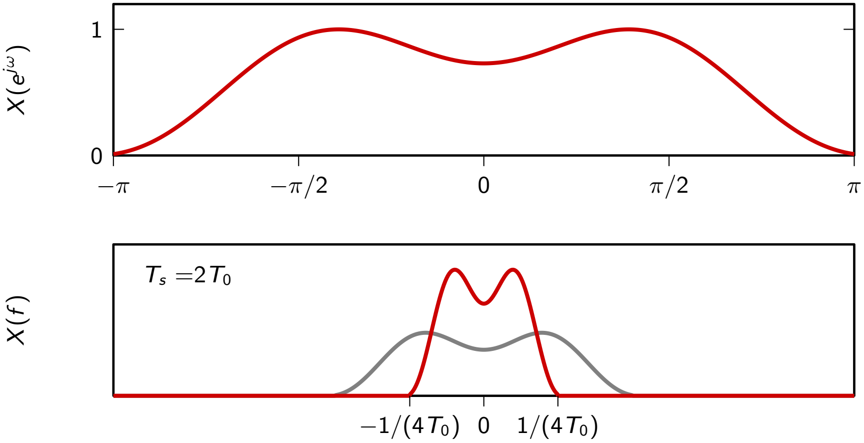

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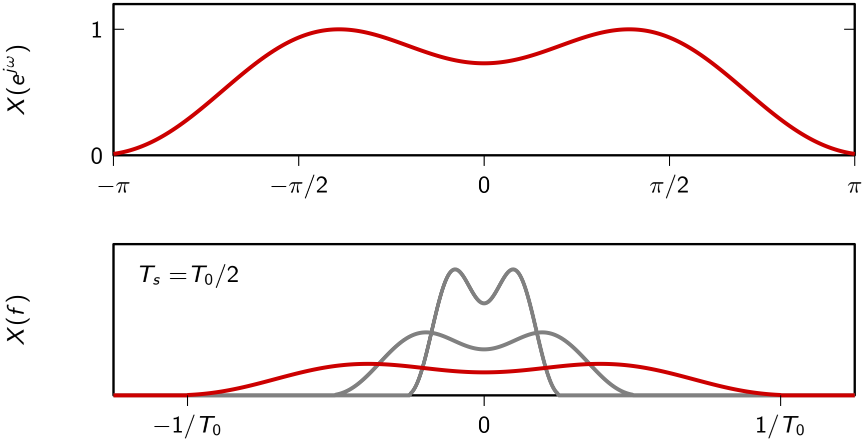

Power vs capacity: intuition

- number of symbols per second is determined by bandwidth constraint
- to increase information rate we must increase the number of possible symbols
- power is proportional to the square of the max symbol
- to keep power limited we need to pack symbols closer together
- all channels introduce noise; at the receiver we have to “guess” what was transmitted
- closer symbols have less *noise margin*, i.e., we lose reliability

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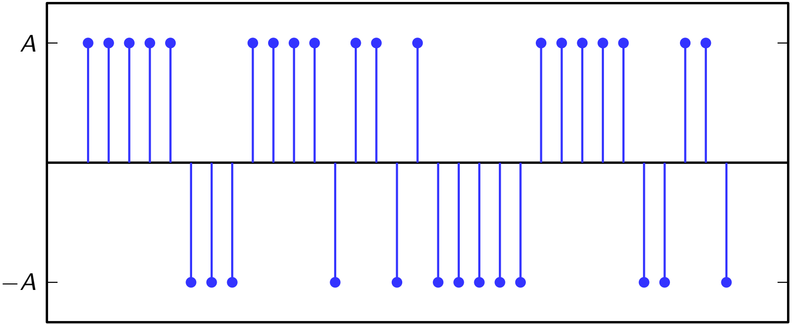

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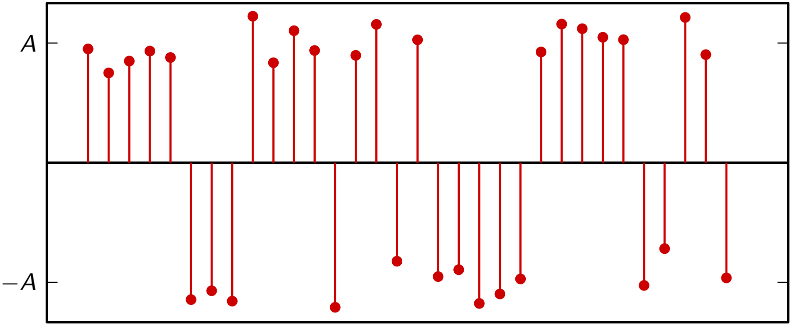

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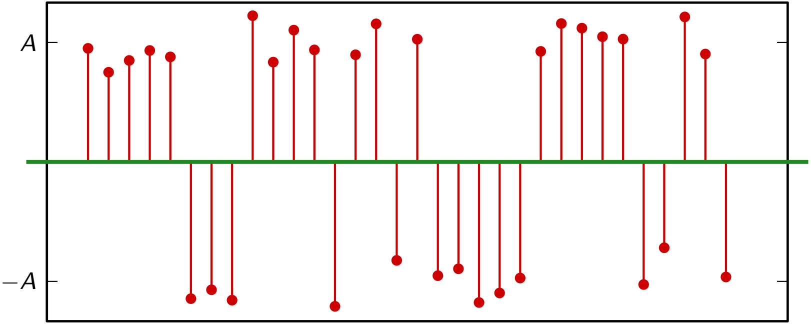

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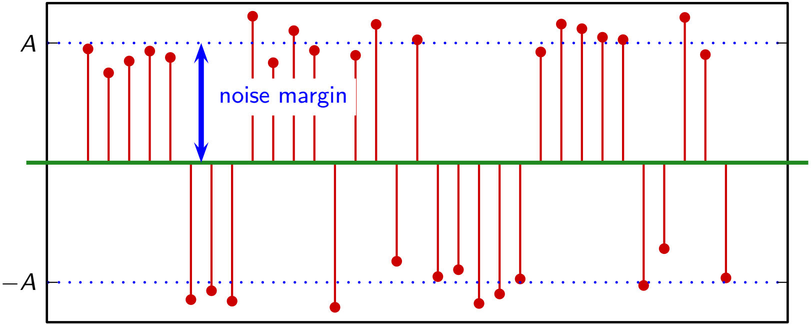

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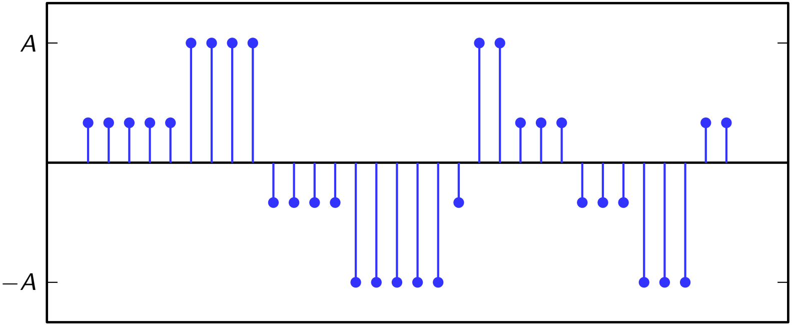

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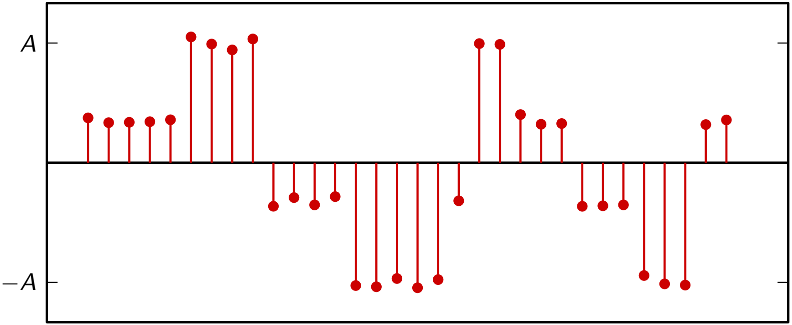

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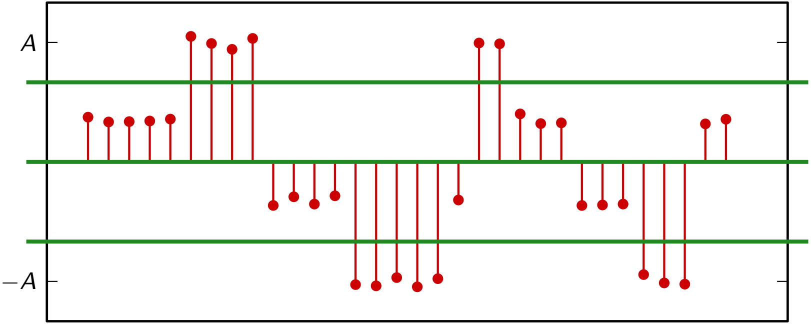

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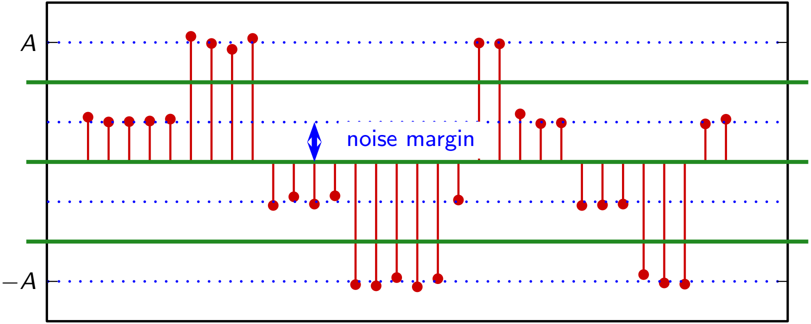
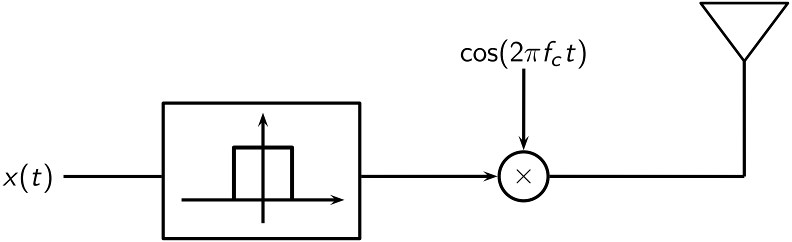


Example: the AM radio channel



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- from 530kHz to 1.7MHz
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- power limited by law:
  - daytime/nighttime
  - interference
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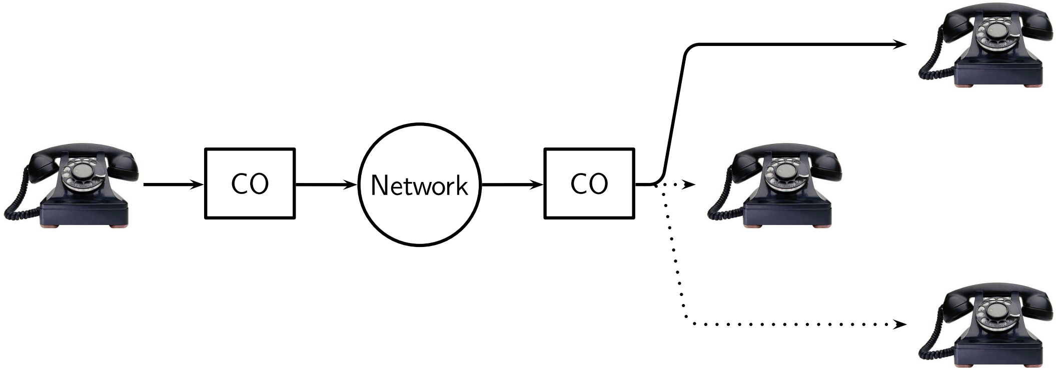

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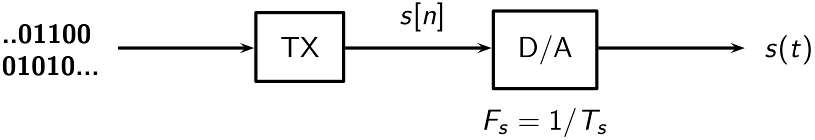
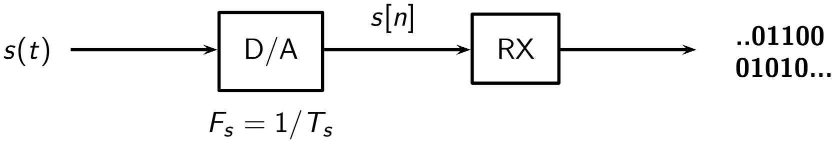
The all-digital paradigm

keep everything digital until we hit the physical channel

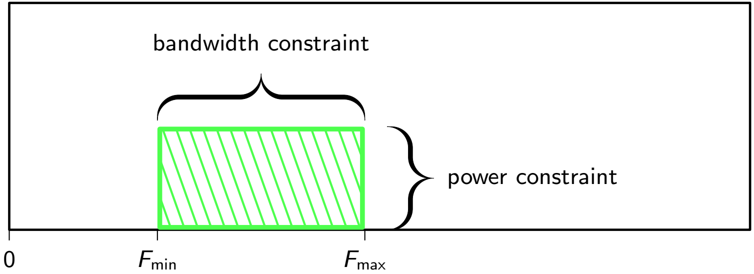


The all-digital paradigm

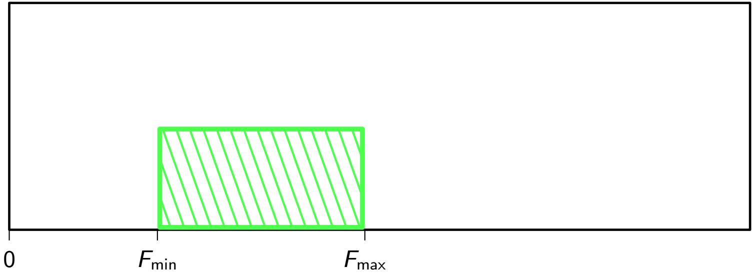
… and of course do the same for the receiver



Let's look again at the channel constraints



Converting the specs to a digital design



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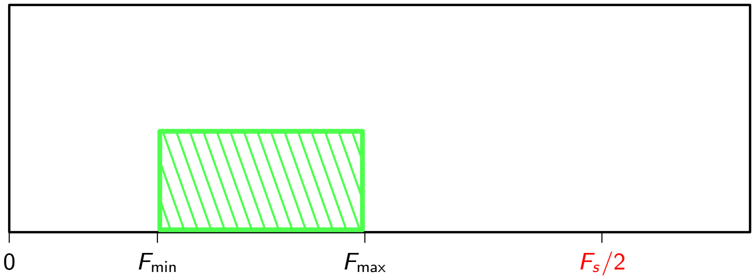

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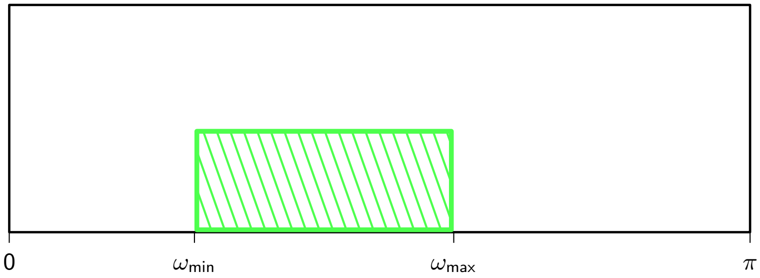

Transmitter design

some working hypotheses:

- convert the bitstream into a sequence of symbols $a[n]$ via a mapper
- model $a[n]$ as a white random sequence (add a scrambler on the bitstream to make sure)
- now we need to convert $a[n]$ into a continuous-time signal within the constraints

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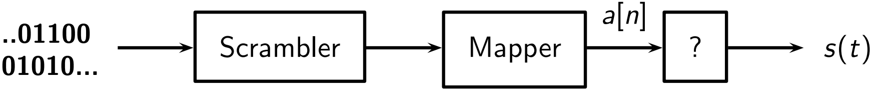
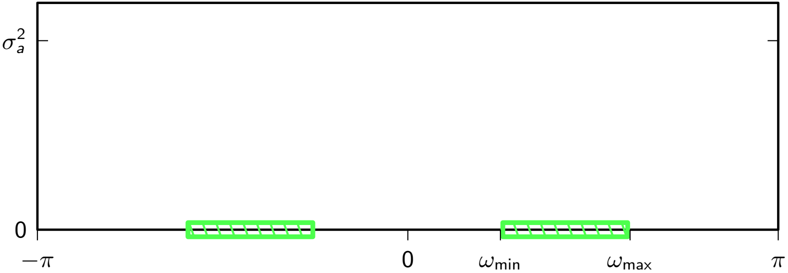


the bandwidth constraint

First problem: the bandwidth constraint

$a[n]$ white: $P_a(e^{j\omega}) = \sigma_a^2$



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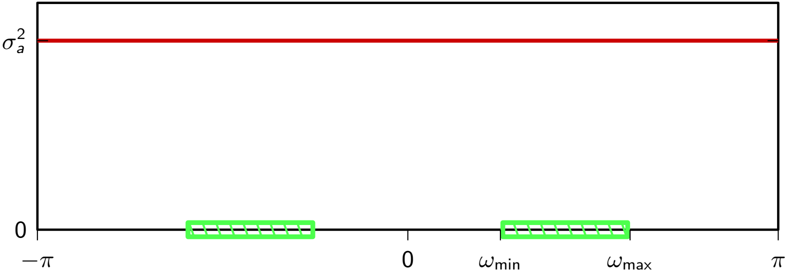

Shaping the bandwidth

Our problem:

- bandwidth constraint requires us to control the spectral support of a signal
- we need to be able to “shrink” the support of a full-band signal
- the answer is *multirate* techniques

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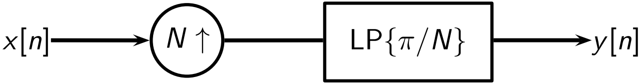
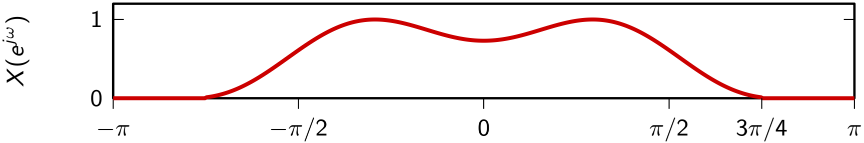
ideal digital interpolator



Upsampling by $K = 3$



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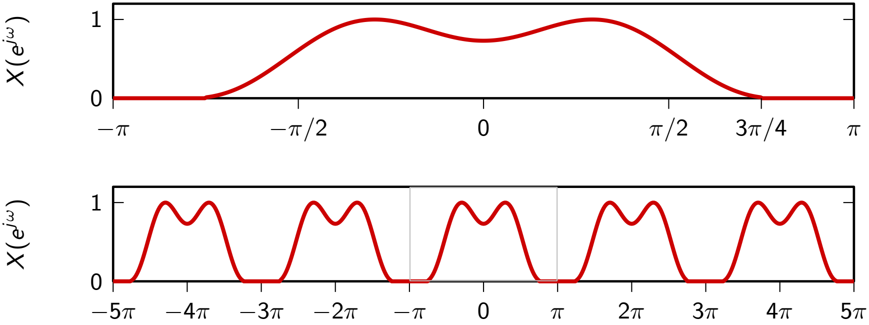

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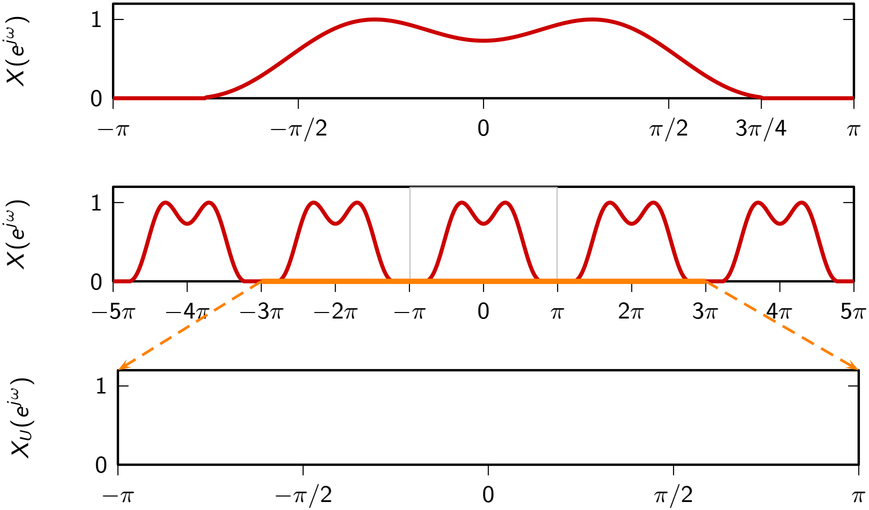

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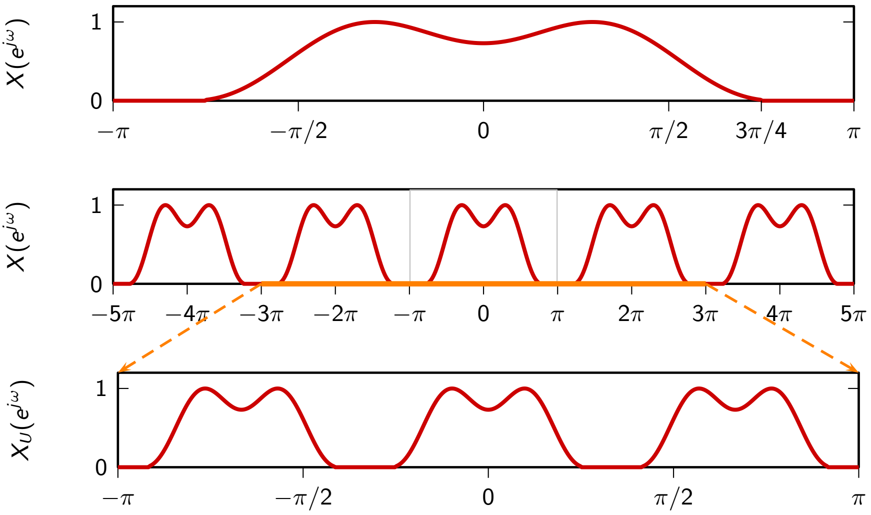

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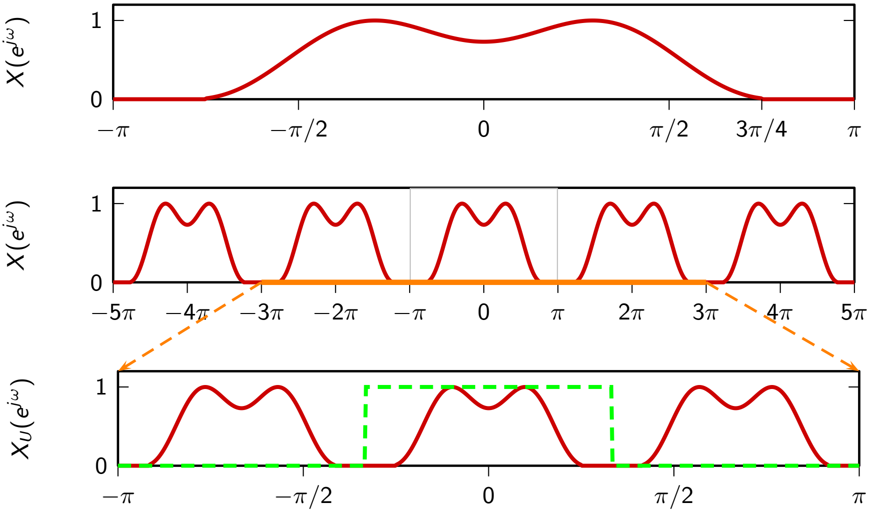

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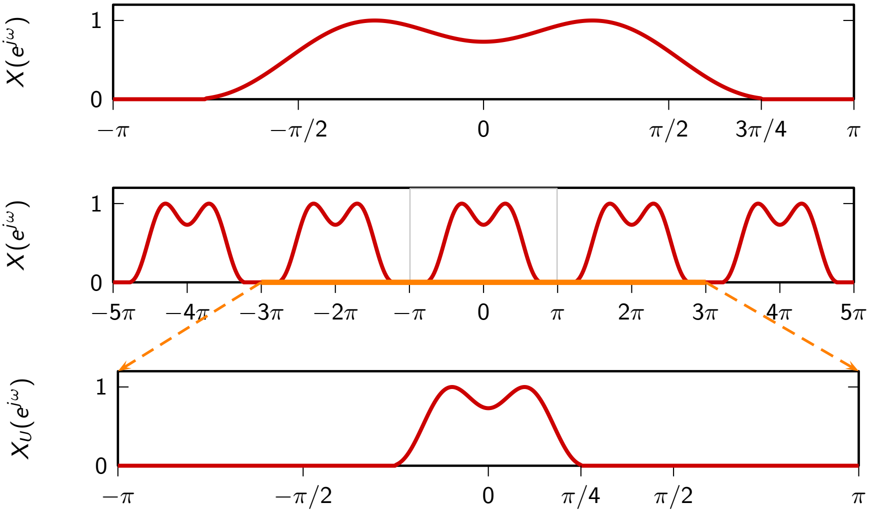
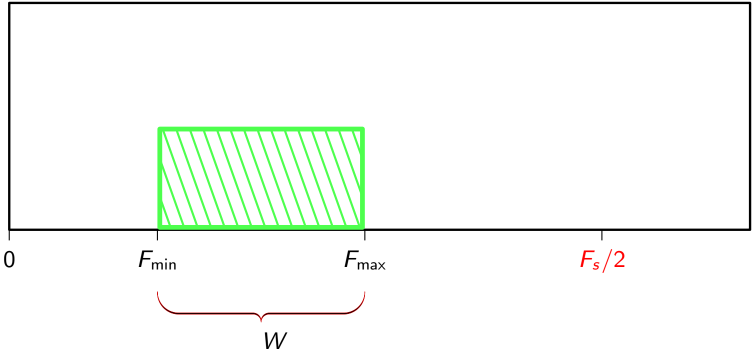


Fulfilling the bandwidth constraint



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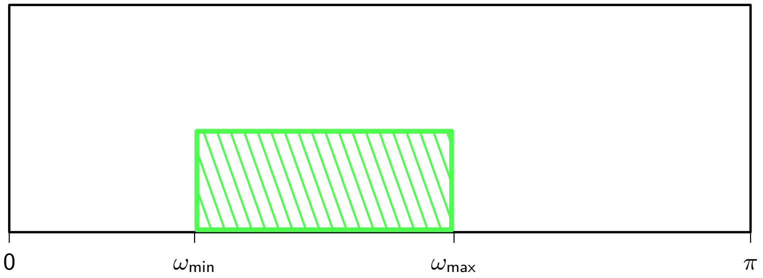

Here's a neat trick

let $W = F_{\max} - F_{\min}$; pick F_s so that:

- $F_s > 2F_{\max}$ (obviously)
- $F_s = KW$, $K \in \mathbb{N}$
- $\omega_{\max} - \omega_{\min} = 2\pi\frac{W}{F_s} = \frac{2\pi}{K}$
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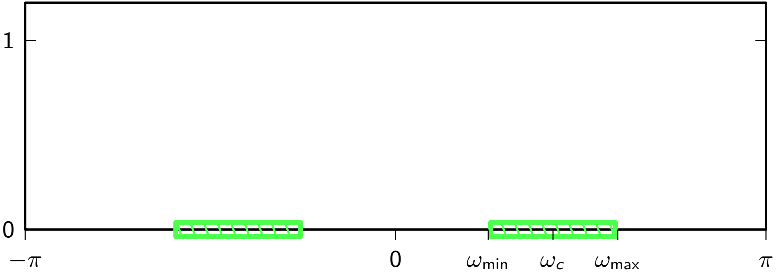

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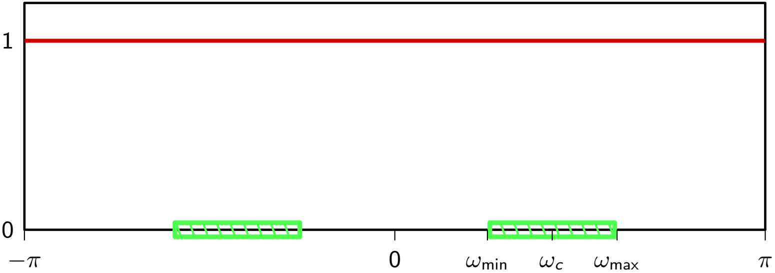

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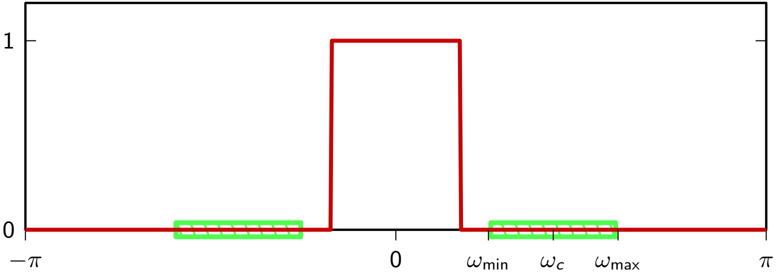
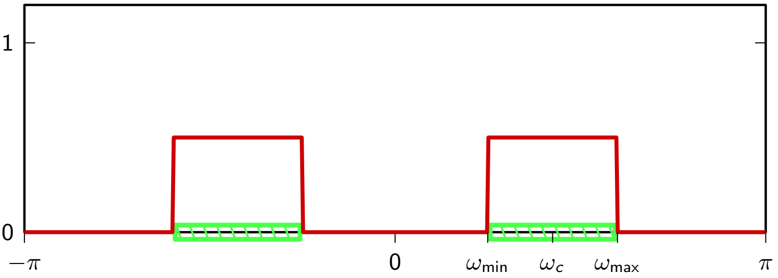
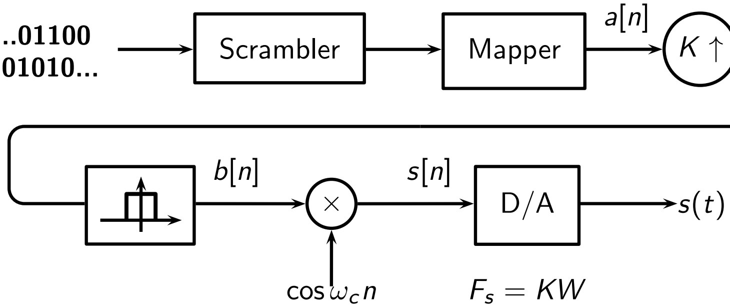


Fulfilling the bandwidth constraint



Transmitter design, continued

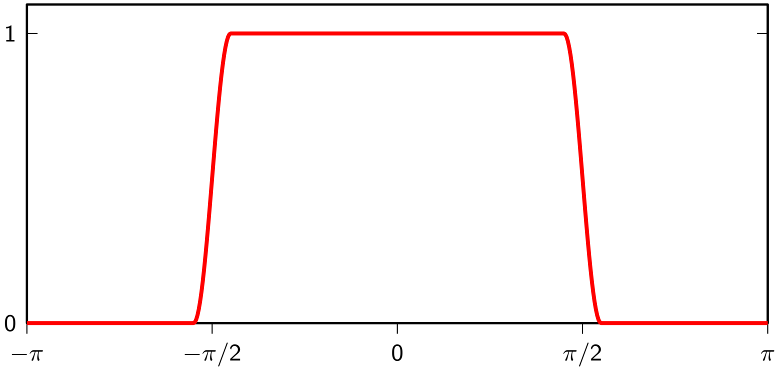


Raised Cosine

$$H_{K,\beta}(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \dfrac{\pi(1-\beta)}{K} \\ \frac{1}{2}\left(1 + \cos\left(\dfrac{K|\omega| - (1-\beta)\pi}{\beta}\right)\right) & \dfrac{\pi(1-\beta)}{K} \leq |\omega| \leq \dfrac{\pi(1+\beta)}{K} \\ 0 & |\omega| > \dfrac{\pi(1+\beta)}{K} \end{cases}$$

Raised Cosine

$H_{2,\beta}(e^{j\omega})$



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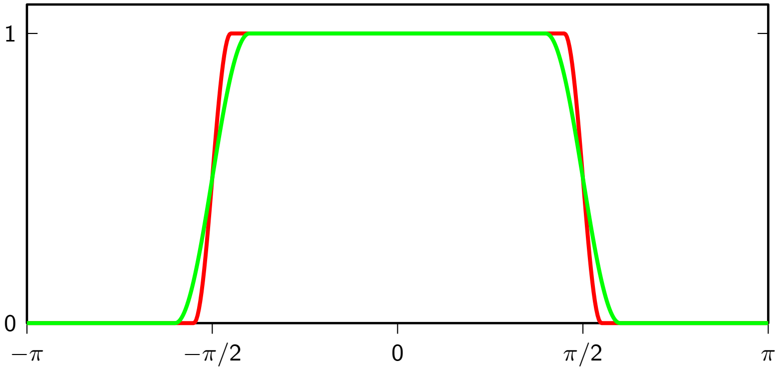

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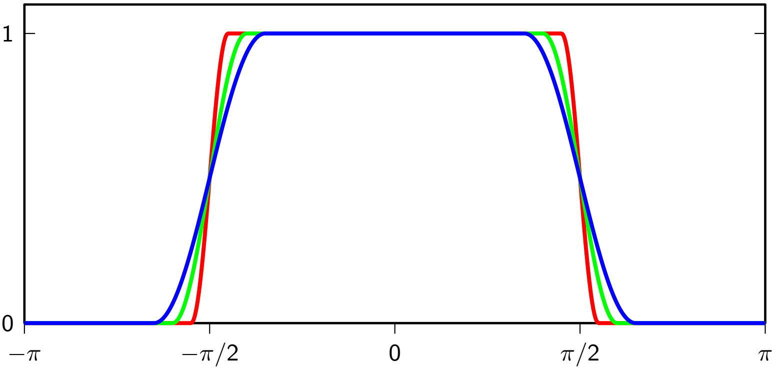

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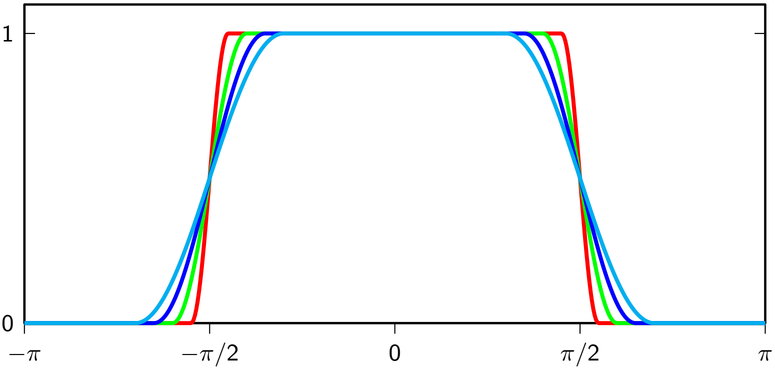


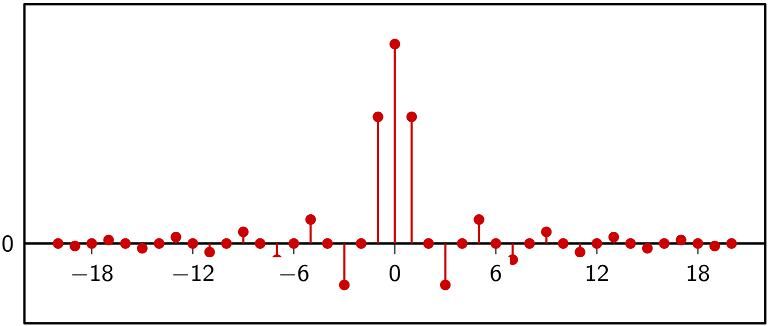
Raised Cosine

$$h_{K,\beta}[nK] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

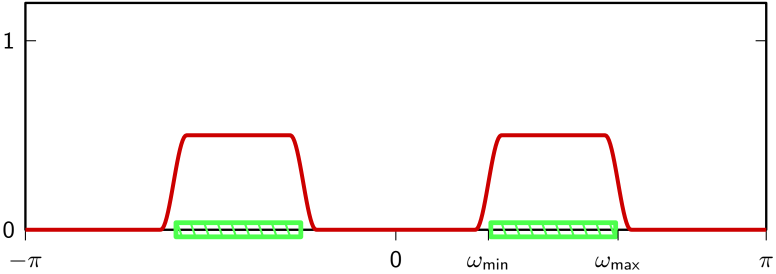
$$h_{K,\beta}[nK] \propto \frac{1}{(\beta n)^2}$$

Raised Cosine: $1/n^2$ decay

$h_{2,\beta}[n]$



Spectral shaping with raised cosine



the power constraint

Overview:

- Noise and probability of error
- Signaling alphabet and power
- QAM signaling

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Transmission reliability

- transmitter sends a sequence of symbols $a[n]$
- receiver obtaines a sequence $\hat{a}[n]$
- even if no distortion we can't avoid noise: $\hat{a}[n] = a[n] + \eta[n]$
- when noise is large, we make an error

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depends on:

- power of the noise wrt power of the signal
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Mappers and slicers

mapper:

- split incoming bitstream into chunks
- assign a symbol $a[n]$ from a finite alphabet $\mathcal{A}$ to each chunk

slicer:

- receive a value $\hat{a}[n]$
- decide which symbol from $\mathcal{A}$ is “closest” to $\hat{a}[n]$
- piece back together the corresponding bitstream

Example: two-level signaling

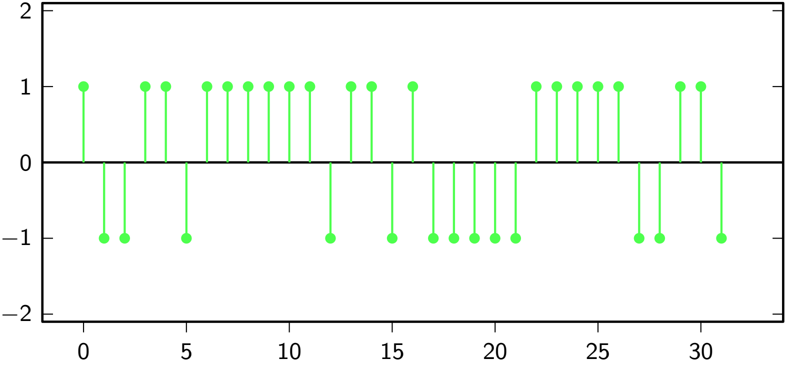
mapper:

- split incoming bitstream into single bits
- $a[n] = G$ if the bit is 1, $a[n] = -G$ if the bit is 0

slicer:

- n-th bit $= \begin{cases} 1 & \text{if } \hat{a}[n] > 0 \\ 0 & \text{otherwise} \end{cases}$

Example: two-level signaling



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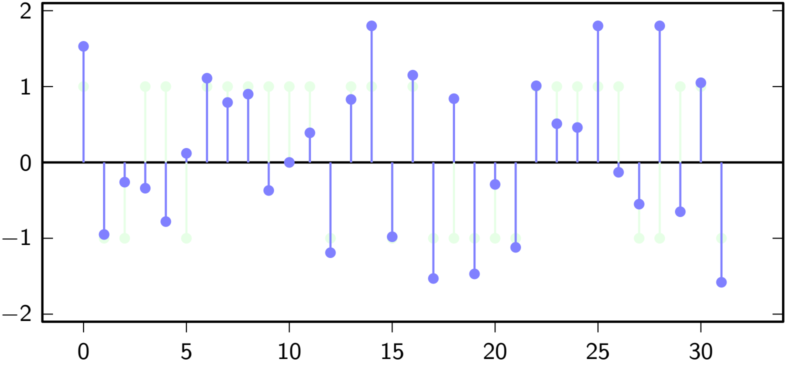

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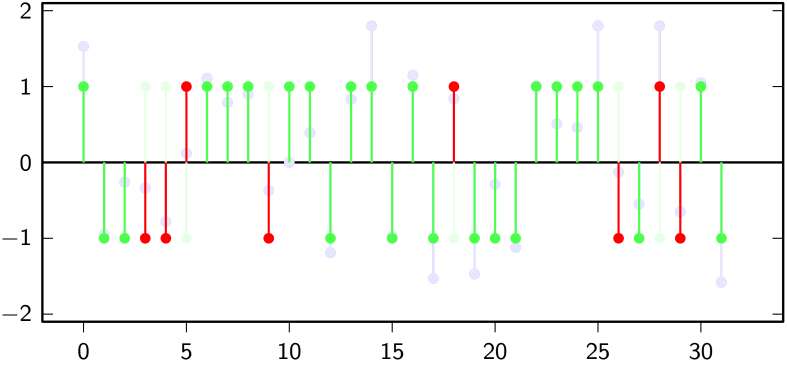

Example: two-level signaling

let's look at the probability of error after making some hypotheses:

- $\hat{a}[n] = a[n] + \eta[n]$
- bits in bitstream are equiprobable
- noise and signal are independent
- noise is additive white Gaussian noise with zero mean and variance σ_0

Example: two-level signaling

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\begin{aligned}
P_{\text{err}} &= P[\ \eta[n] < -G \ \mid\ n\text{-th bit is } 1\ ]\, P[\ n\text{-th bit is } 1\ ] + \\
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transmitted power

$$\begin{aligned}\sigma_s^2 &= G^2\, P[n\text{-th bit is 1}] + G^2\, P[n\text{-th bit is 0}] \\ &= G^2\end{aligned}$$

$$P_{\text{err}} = Q(\sigma_s/\sigma_0) = Q(\sqrt{\text{SNR}})$$

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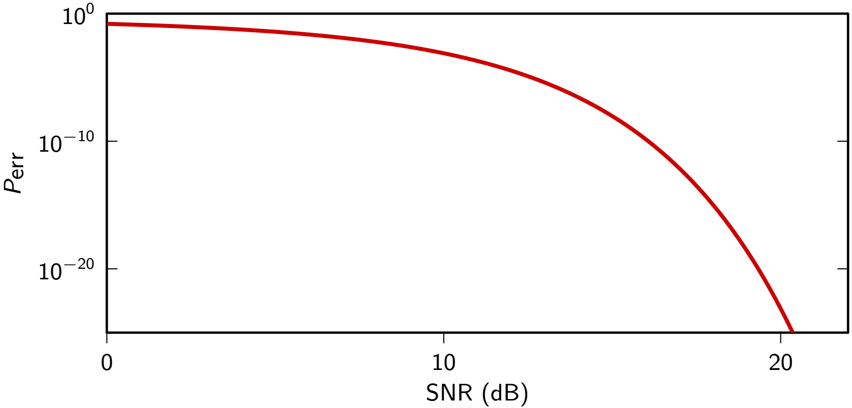
Probability of error



Lesson learned:

- to reduce the probability of error increase SNR
- can't control noise so we must increase G
- increasing G increases the power
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- binary signaling is not very efficient (one bit at a time)
- to increase the throughput we can use multilevel signaling
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PAM

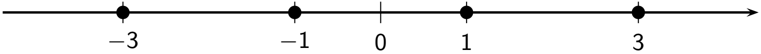
mapper:

- split incoming bitstream into chunks of M bits
- chunks define a sequence of integers $k[n] \in \{0, 1, \ldots, 2^M - 1\}$
- $a[n] = G((-2^M + 1) + 2k[n])$ (odd integers around zero)

slicer:

- $a'[n] = \arg\min_{a \in \mathcal{A}} [|\hat{a}[n] - a|]$

PAM, $M = 2, G = 1$



- distance between points is $2G$
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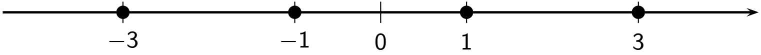

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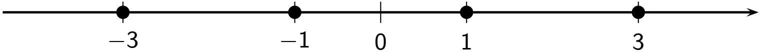

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From PAM to QAM

- error analysis for PAM along the lines of binary signaling
- can we increase the throughput even further?
- here's a wild idea, let's use complex numbers

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QAM

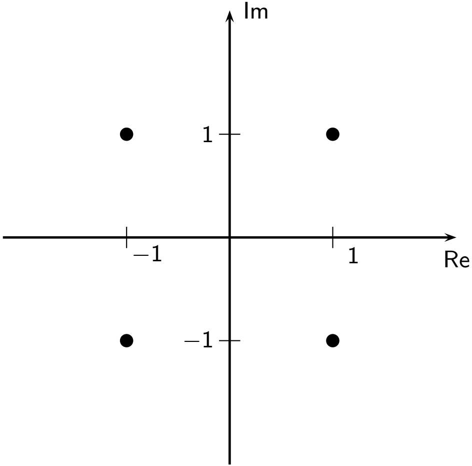
mapper:

- split incoming bitstream into chunks of M bits, M even
- use $M/2$ bits to define a PAM sequence $a_r[n]$
- use the remaining $M/2$ bits to define an independent PAM sequence $a_i[n]$
- $a[n] = G(a_r[n] + ja_i[n])$

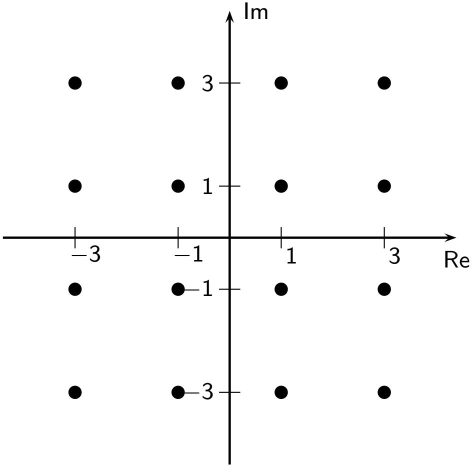
slicer:

- $a'[n] = \arg\min_{a \in \mathcal{A}} [|\hat{a}[n] - a|]$

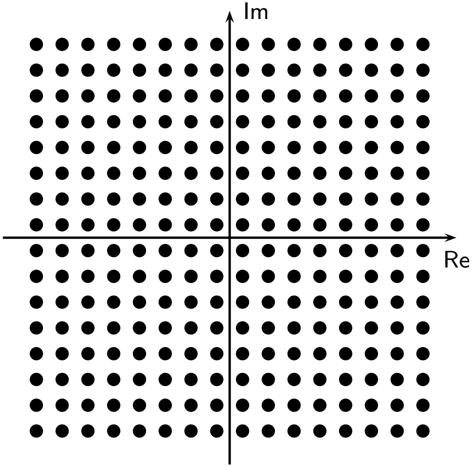
QAM, $M = 2$, $G = 1$



QAM, $M = 4, G = 1$



QAM, $M = 8, G = 1$



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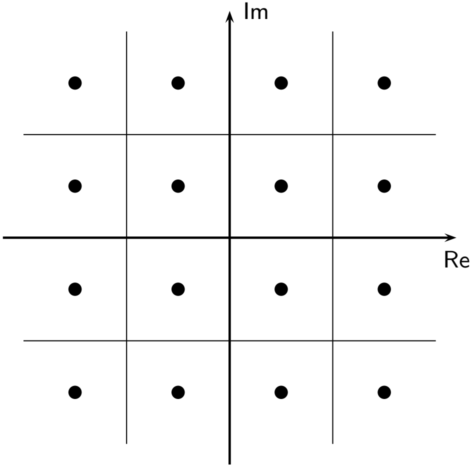

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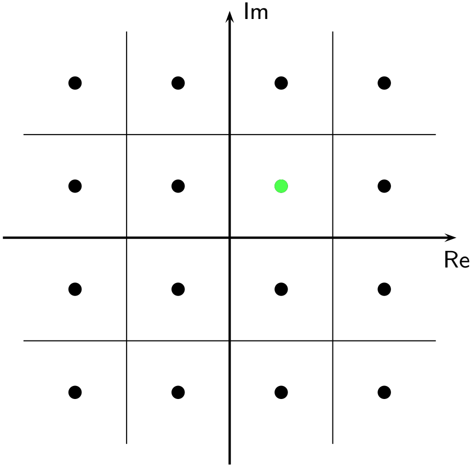

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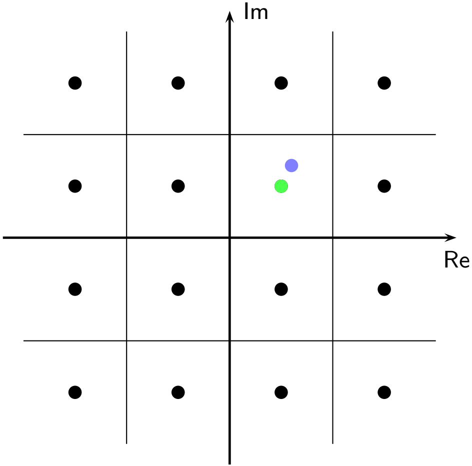

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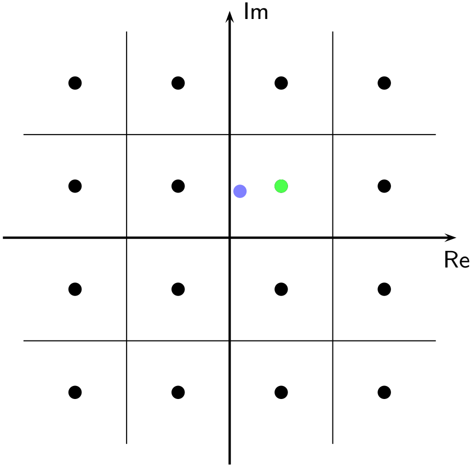

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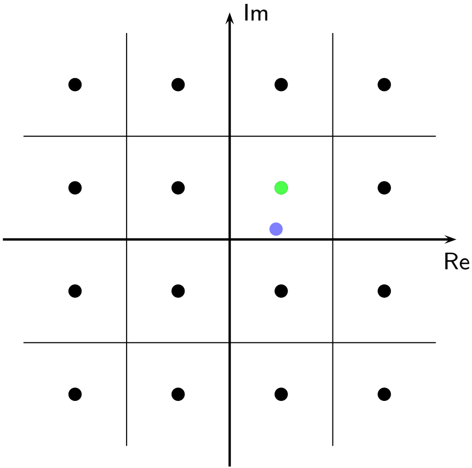

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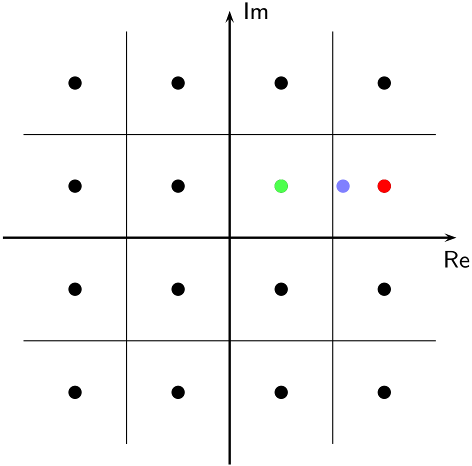
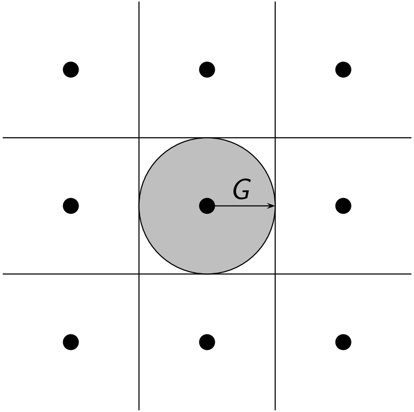


QAM, probability of error

$$\hat{a}[n] = a[n] + \eta[n]$$

$$\begin{aligned} P_{\text{err}} &= 1 - P[|\operatorname{Re}(\eta[n])| < G \,\wedge\, |\operatorname{Im}(\eta[n])| < G] \\ &= 1 - \int_D f_\eta(z)\, dz \end{aligned}$$

QAM, probability of error, circular approximation



QAM, probability of error

$$P_{\text{err}} \approx 1 - \int_{|z|<G} f_\eta(z)\, dz \qquad\qquad f_\eta(z) = \frac{1}{\pi\sigma_0^2} e^{-\frac{|z|^2}{\sigma_0^2}}$$

$z \rightarrow \rho e^{j\theta}$

$$= 1 - \int_0^{2\pi} d\theta \int_0^G \frac{\rho}{\pi\sigma_0^2} e^{-\frac{\rho^2}{\sigma_0^2}} d\rho$$

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transmitted power (all symbols equiprobable and independent):

$$\begin{aligned}\sigma_s^2 &= G^2 \frac{1}{2^M} \sum_{a \in \mathcal{A}} |a|^2 \\ &= G^2 \frac{2}{3}(2^M - 1)\end{aligned}$$

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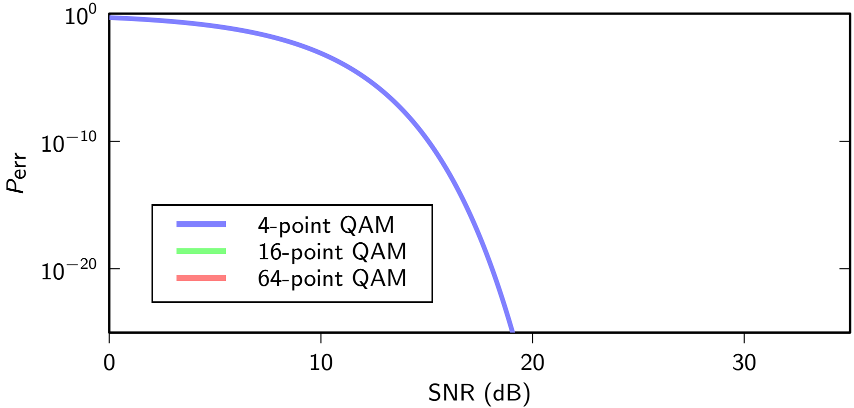

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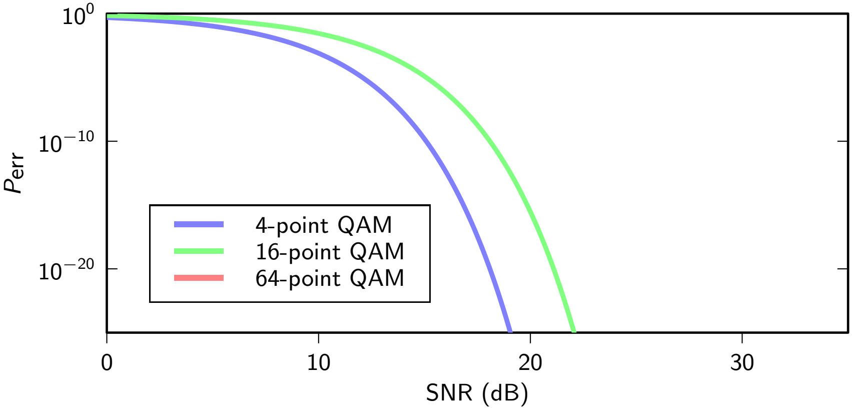

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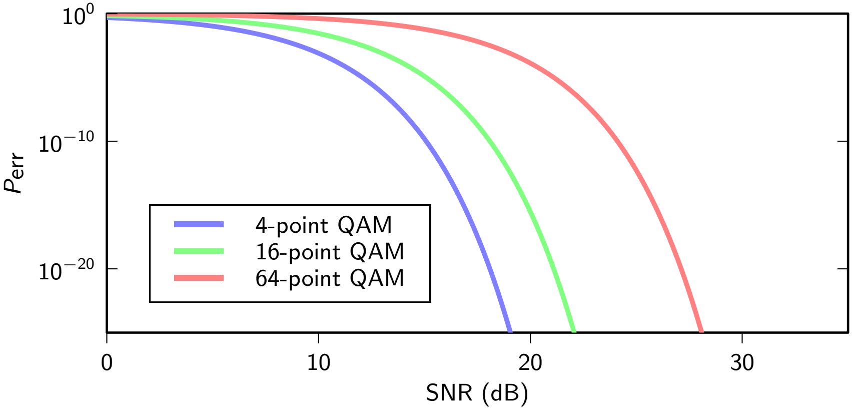

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- pick a probability of error you can live with (e.g. 10^{-6})
- find out the SNR imposed by the channel's power constraint
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QAM, the recipe

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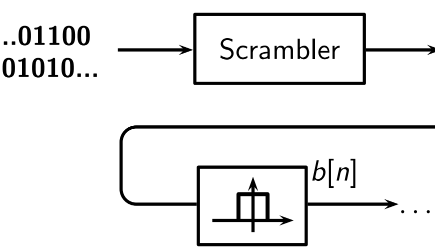
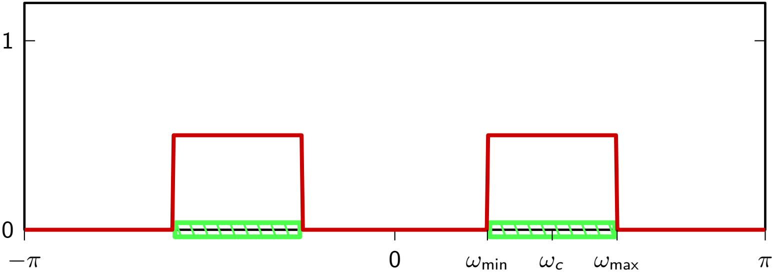
a QAM transmitter

QAM transmitter design

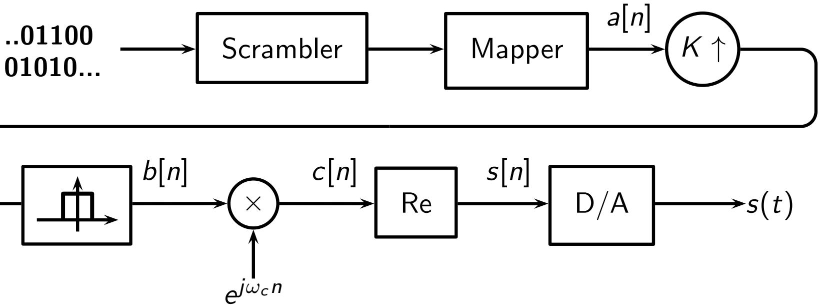


$b[n] = b_r[n] + jb_i[n]$ is a complex-valued baseband signal...

Fulfilling the bandwidth constraint



QAM transmitter, final design



The transmitted passband signal

$$c[n] = b[n]\, e^{j\omega_c n}$$

$$\begin{aligned} s[n] &= \mathrm{Re}\{c[n]\} \\ &= \mathrm{Re}\{(b_r[n] + jb_i[n])(\cos\omega_c n + j\sin\omega_c n)\} \\ &= b_r[n]\cos\omega_c n - b_i[n]\sin\omega_c n \end{aligned}$$

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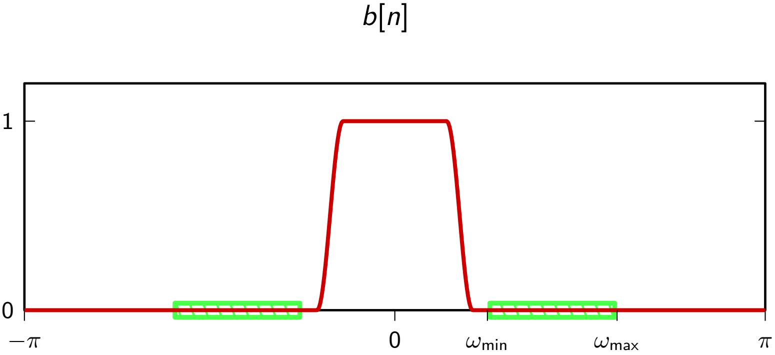
From complex baseband to real passband signal



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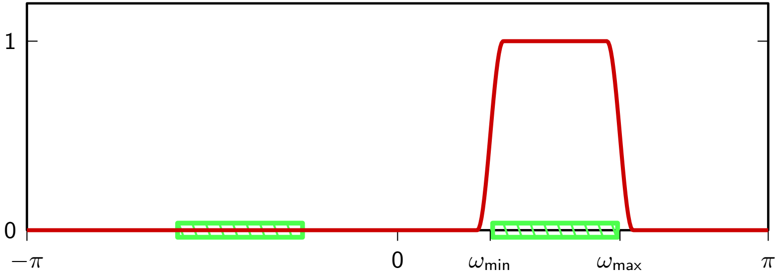

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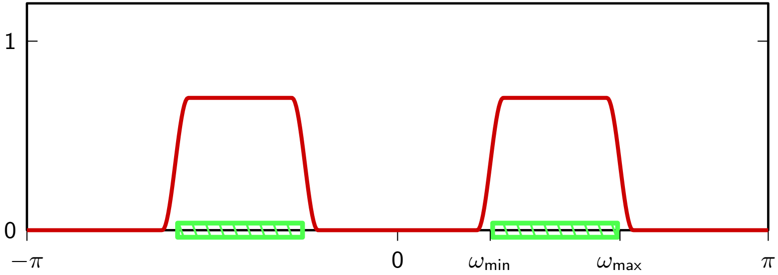
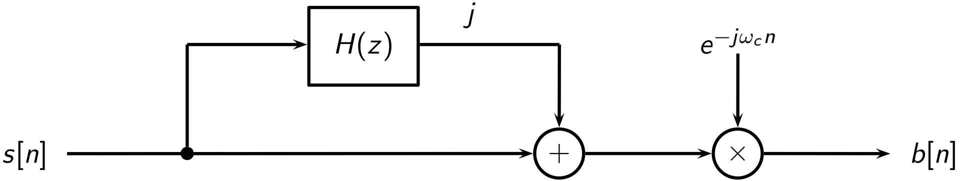


Can we go back?

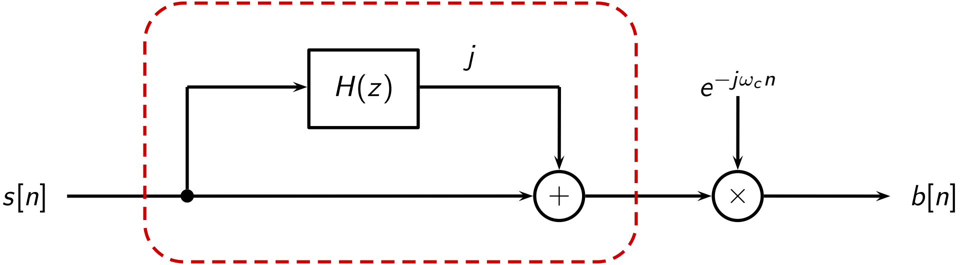
receiver obtains $s[n]$
can we recover the complex baseband $b[n]$ from $s[n]$?

easiest way: Hilbert demodulation

Hilbert demodulation



Hilbert demodulation



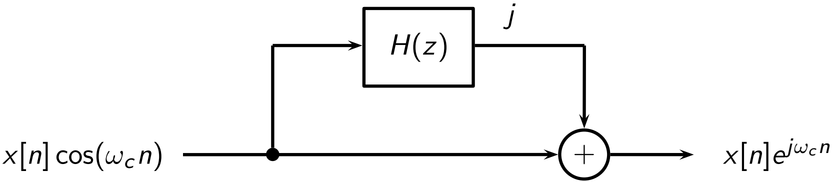
$$H_A(e^{j\omega}) = 1 + jH(e^{j\omega})$$

Hilbert demodulation

$$H(e^{j\omega}) = \begin{cases} -j & \omega \geq 0 \\ j & \omega < 0 \end{cases}$$

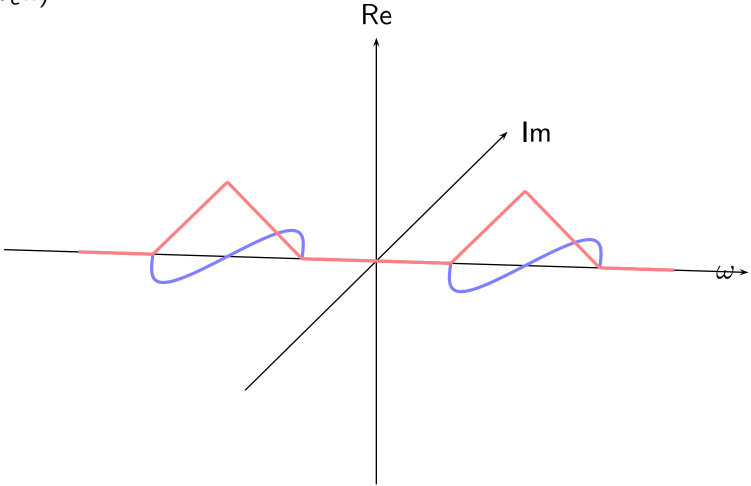
$$H_A(e^{j\omega}) = 1 + jH(e^{j\omega}) = \begin{cases} 2 & \omega \geq 0 \\ 0 & \omega < 0 \end{cases}$$

Hilbert demodulation (recap)



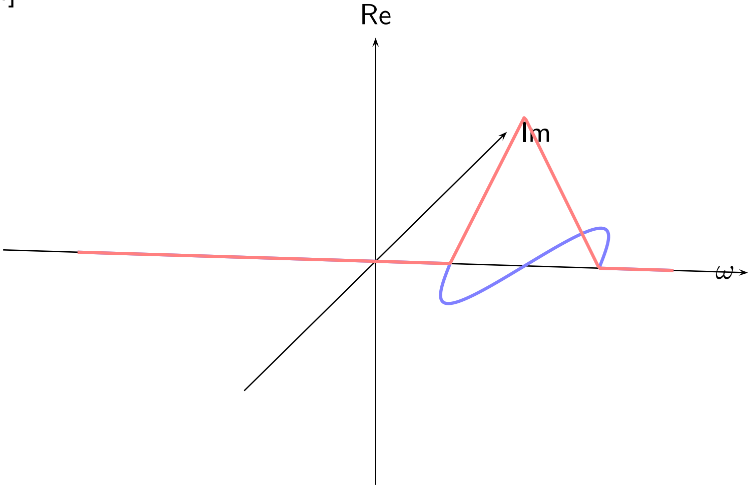
Hilbert demodulation

$y[n] = x[n]\cos(\omega_c n)$



Hilbert demodulation

$jy[n] * h[n] + y[n]$



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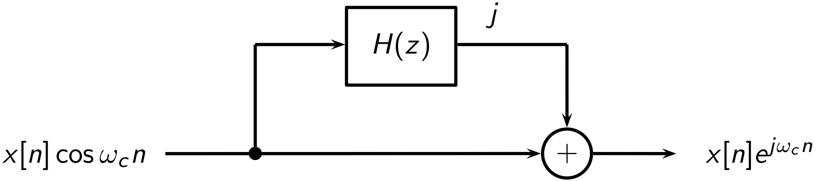

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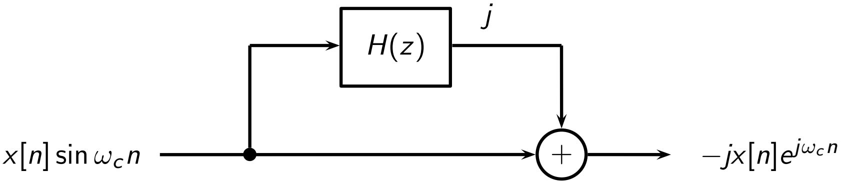

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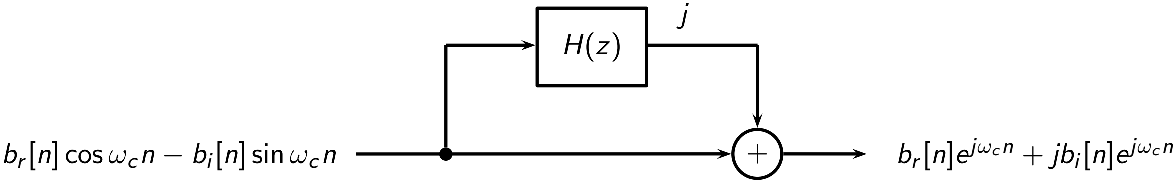


$b_r[n]\cos\omega_c n - b_i[n]\sin\omega_c n$

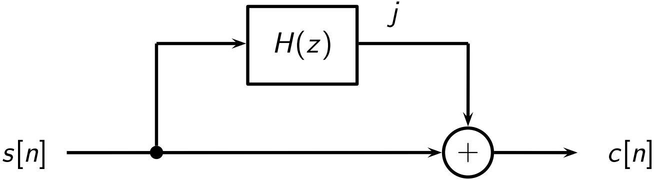
$H(z)$

j

$+$

$b_r[n]e^{j\omega_c n} + jb_i[n]e^{j\omega_c n}$

Hilbert demodulation



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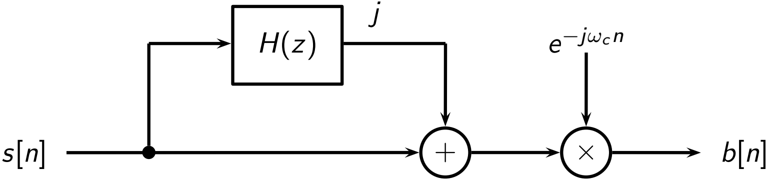
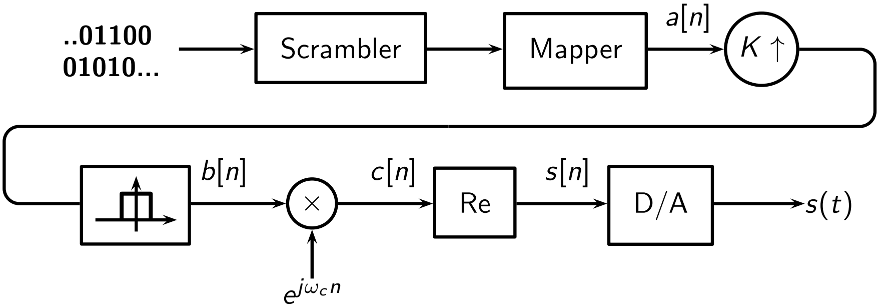
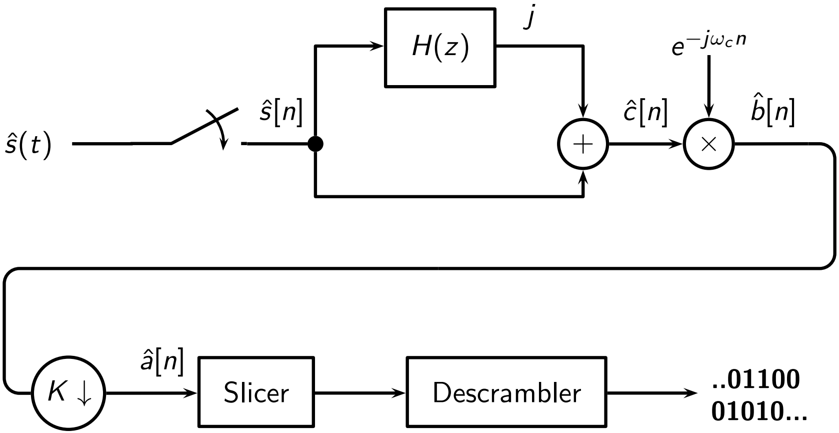


QAM transmitter, final design



QAM receiver, idealized design



Example: the V.32 voiceband modem

- analog telephone channel: $F_{\min} = 450\text{Hz}$, $F_{\max} = 2850\text{Hz}$
- usable bandwidth: $W = 2400\text{Hz}$, center frequency $F_c = 1650\text{Hz}$
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- we used very specific design choices to derive the throughput
- what is the best one can do?
- Shannon's capacity formula is the upper bound

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