COM-303 - Signal Processing for Communications
"Mock" Midterm Exam

- This is a no-grade, take-home midterm exam: try to work on the problems as if taking a real exam, i.e.: work uninterruptedly for 2 hours, do not use the internet and use only handwritten notes for support.
- The solution will be discussed in class after the spring break.
- There are 6 problems with different scores for a total of 100 points.

Exercise 1. (5 points)

Consider a length-N signal $\mathbf{x} = [x[0]\,x[1]\,\dots\,x[N-1]]^T$ and its DFT $\mathbf{X} = [X[0]\,X[1]\,\dots\,X[N-1]]^T$.
Consider now the length-$2N$ vector

$$\mathbf{y} = [x[0] \quad -x[0] \quad x[1] \quad -x[1] \quad x[2] \quad -x[2] \quad \dots \quad x[N-1] \quad -x[N-1]]^T$$

and express its $2N$-point DFT in terms of the N original DFT coefficients $X[k]$.

Solution:

$$\begin{aligned}
Y[k] &= \sum_{n=0}^{2N-1} y[n] e^{-j\frac{2\pi}{2N}nk} \\
&= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{2N}(2n)k} - \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{2N}(2n+1)k} \\
&= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk} - \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}\, e^{-j\frac{2\pi}{2N}k} \\
&= (1 - e^{-j\frac{\pi}{N}k}) X[k]
\end{aligned}$$

Exercise 2. (10 points)

Consider the infinite-length discrete-time signal

$$s[n] = \cos(\frac{\pi}{2}n) + \cos(\frac{5\pi}{8}n) \qquad n \in \mathbb{Z}$$

Consider now the finite-length signal

$$x[n] = s[n], \quad n = 0, 1, \dots N-1$$

Determine the minimum value for N so that the DFT $X[k]$ satisfies the following requirements:

- $X[k]$ has only four nonzero values
- the nonzero values are non-contiguous (i.e. there should be one or more zero values for $X[k]$ between the nonzero values); this corresponds to being able to resolve the frequencies of the sinusoids of the original signal.

Solution: *The original signal contains two sinusoids at frequencies*

$$\omega_0 = \frac{\pi}{2}$$
$$\omega_1 = \frac{5\pi}{8}$$

In order for the DFT of the finite-length signal to contain only four nonzero values we must choose N so that both frequencies are integer multiples of the fundamental frequency for the space of length-N signals; that is, we need to find a value for N such that

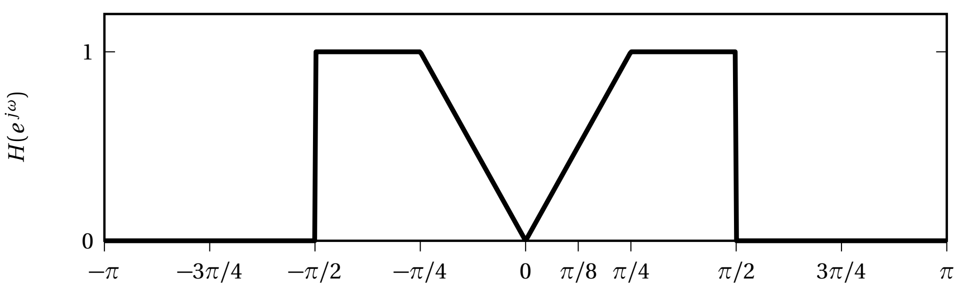
$$\omega_0 = k_0 \frac{2\pi}{N}$$
$$\omega_1 = k_1 \frac{2\pi}{N}$$

for integer values of $k_{0,1}$. By replacing the values for $\omega_{0,1}$ we have

$$k_0 = \frac{N}{4}$$
$$k_1 = \frac{5}{16}N$$

so that, in order to have $k_{0,1} \in \mathbb{N}$, N must be a multiple of 16. If we choose $N = 16$, however, we have that $k_0 = 4$ and $k_1 = 5$ which does not fulfill the requirement of having at least one zero value between the nonzero DFT values. We therefore must choose at least $N = 32$, for which $k_0 = 8$ and $k_1 = 10$.

Exercise 3. (25 points)
Determine the impulse response $h[n]$ of an ideal filter whose real-valued frequency response $H(e^{j\omega})$ is shown in the following figure:



Solution: *The real-valued frequency response in the figure can be decomposed as the difference between an ideal lowpass with cutoff frequency $\pi/2$ and an ideal filter with triangular characteristic and support between $-\pi/4$ and $\pi/4$:*

$$H(e^{j\omega}) = \text{rect}(\omega/\pi) - T(e^{j\omega})$$

In order to find the expression for $T(e^{j\omega})$, consider that the convolution of a rect shape with itself produces a triangular shape with twice the support:

$$\text{rect}(t) = \begin{cases} 1 & \textit{for } |t| < 1/2 \\ 0 & \textit{otherwise} \end{cases}$$

$$\text{rect}(t) * \text{rect}(t) = \begin{cases} 1 - |t| & \textit{for } |t| < 1 \\ 0 & \textit{otherwise} \end{cases}.$$

In our case, using a lowpass with cutoff $\pi/8$:

$$\begin{aligned}\mathrm{rect}\left(\frac{\omega}{\pi/4}\right) * \mathrm{rect}\left(\frac{\omega}{\pi/4}\right) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{rect}\left(\frac{\sigma}{\pi/4}\right)\mathrm{rect}\left(\frac{\omega-\sigma}{\pi/4}\right) d\sigma \\ &= \begin{cases} 1/8 - |\omega/2\pi| & \text{for } |\omega| < \pi/4 \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

We can therefore write

$$H(e^{j\omega}) = \mathrm{rect}\left(\frac{\omega}{\pi}\right) - 8\,\mathrm{rect}\left(\frac{\omega}{\pi/4}\right) * \mathrm{rect}\left(\frac{\omega}{\pi/4}\right)$$

In order to find the impulse response, simply remember that convolution in frequency is multiplication in time:

$$h[n] = (1/2)\mathrm{sinc}(n/2) - (1/8)\mathrm{sinc}^2(n/8)$$

Exercise 4. (10 points)

Consider a filter with frequency response $H(e^{j\omega}) = \cos 2\omega + 3\cos 5\omega$.

(a) write out the impulse response $h[n]$

(b) what is the delay introduced by the filter?

(c) write the frequency response of a causal implementation of the filter.

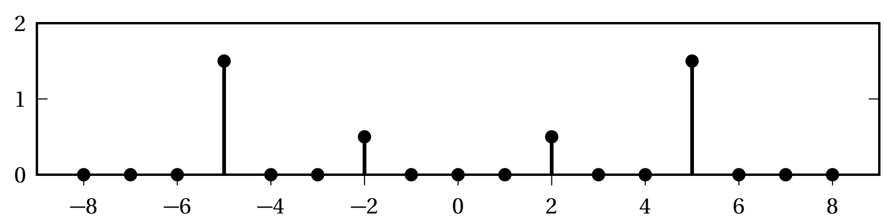
Solution: *From the frequency response*

$$H(e^{j\omega}) = \frac{1}{2}e^{j2\omega} + \frac{1}{2}e^{-j2\omega} + \frac{3}{2}e^{j5\omega} + \frac{3}{2}e^{-j5\omega}$$

it's immediate to derive the transfer function

$$H(z) = \frac{3}{2}z^5 + \frac{1}{2}z^2 + \frac{1}{2}z^{-2} + \frac{3}{2}z^{-5}$$

so that the FIR impulse response looks like so:



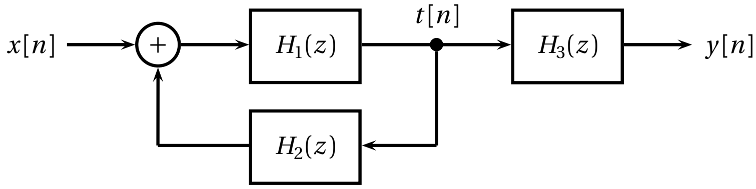
The frequency response of the filter is real-valued; therefore its phase response is zero and so the filter introduces no delay. This is also apparent by the shape of the impulse response, which is symmetrical around zero.

To make the filter causal, we need to add a delay so that all the nonzero values of the impulse response are for positive values of the index. This can be achieved with a delay by five, so that the resulting frequency response becomes

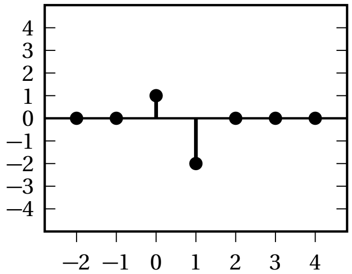
$$H_c(e^{j\omega}) = e^{-j5\omega}H(e^{j\omega})$$

Exercise 5. (30 points)

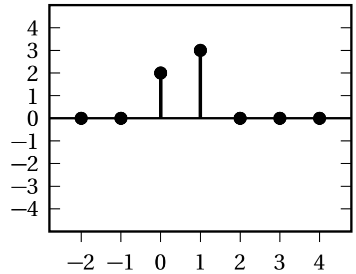
Consider the following system



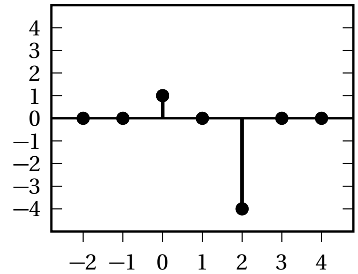
where the finite-length impulse responses of the three filters $H_1(z)$, $H_2(z)$, $H_3(z)$ are as in the following figures:



$H_1(z)$



$H_2(z)$



$H_3(z)$

(a) compute the global transfer function of the system

(b) sketch its pole-zero plot

(c) determine if the system stable

Solution: *To find the overall transfer function of the system let's first consider the auxiliary signal $t[n]$ as in the figure. In the z-domain we have*

$$Y(z) = H_3(z)T(z)$$

and

$$T(z) = [X(z) + H_2(z)T(z)]H_1(z)$$

from which

$$H(z) = \frac{H_1(z)H_3(z)}{1 - H_1(z)H_2(z)}$$

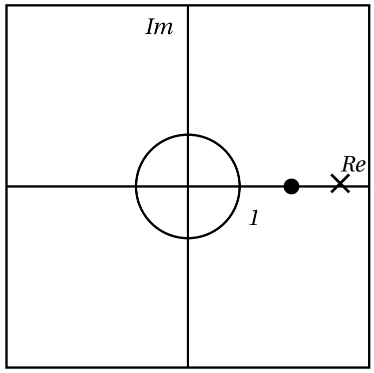
Let's consider now the individual transfer functions; all filters are FIR filter and from the plots we have

$$H_1(z) = 1 - 2z^{-1}$$
$$H_2(z) = 2 + 3z^{-1}$$
$$H_3(z) = 1 - 4z^{-2}$$

By plugging these values in the expression for the transfer function we have

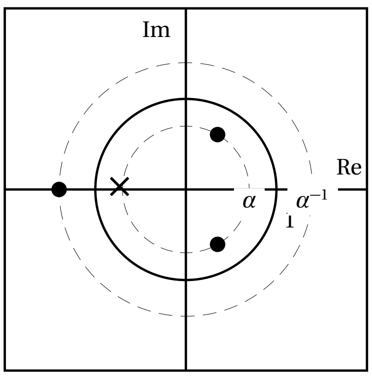
$$\begin{aligned} H(z) &= \frac{(1-2z^{-1})(1-4z^{-2})}{-1+z^{-1}+6z^{-2}} \\ &= \frac{(1-2z^{-1})(1-2z^{-1})(1+2z^{-1})}{(1+2z^{-1})(1-3z^{-1})} \\ &= \frac{(1-2z^{-1})^2}{(1-3z^{-1})} \end{aligned}$$

The pole-zero plot is the following

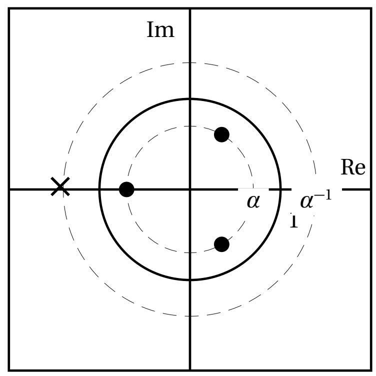


and, since the pole is outside the unit circle, the system is unstable.

Exercise 6. (20 points)
The figures below show the pole-zero plots of two filters, $H_1(z)$ and $H_2(z)$. The poles and the zeros lie on circles of radius α and $1/\alpha$. Sketch as accurately as you can the magnitude responses of both filters, highlighting the differences if any.



$H_1(z)$



$H_2(z)$

Solution:
The frequency response of $H_1(z)$ and $H_2(z)$ can be decomposed into a cascade of two filters: the first filter is determined by the two zeros in the right half-plane while the second filter accounts for the pole-zero pair in the left half-plane. The two complex-conjugate zeros in the right half-plane are at z_0 and z_0^* with $z_0 = \alpha e^{j\omega_0}$ for some angle ω_0; their response is described by the transfer function:

$$\begin{aligned} B(z) &= (1 - z_0 z^{-1})(1 - z_0^* z^{-1}) \\ &= 1 - 2\alpha\cos(\omega_0) z^{-1} + \alpha^2 z^{-2}. \end{aligned}$$

Take now the pole-zero pair in the left half-plane in the plot for $H_1(z)$; the resulting transfer function is

$$A(z) = \frac{1 - (1/\alpha)z^{-1}}{1 - \alpha z^{-1}}$$

In the second plot, the roles of the pole and the zero are reversed; we can thus write

$$\begin{aligned} H_1(z) &= B(z)A(z) \\ H_2(z) &= B(z)/A(z) \end{aligned}$$

The magnitude response of $A(z)$ is that of an allpass filter; from

$$A(z) = \frac{1 - (1/\alpha)z^{-1}}{1 - \alpha z^{-1}} = \frac{z^{-1}}{\alpha} \cdot \frac{z^{-1} - \alpha}{z - \alpha}$$

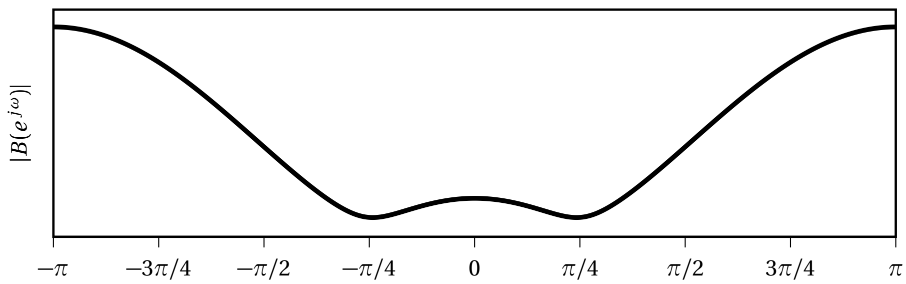
we can derive

$$|A(e^{j\omega})| = \left|\frac{e^{-j\omega}}{\alpha}\right| \cdot \left|\frac{e^{-j\omega}-\alpha}{e^{j\omega}-\alpha}\right| = |\alpha|^{-1}\left|\frac{[e^{-j\omega}-\alpha]}{[e^{-j\omega}-\alpha]^*}\right| = |\alpha|^{-1}$$

so that the only difference between the magnitude responses of $H_1(z)$ and $H_2(z)$ is a scaling factor:

$$|H_1(e^{j\omega})| = |B(e^{j\omega})|/|\alpha|$$
$$|H_2(e^{j\omega})| = |\alpha||B(e^{j\omega})|$$

The shape of the magnitude response will be the same for both filters and is determined by $B(z)$, which is a simple notch filter, sketched here for $\alpha = 0.8$ and $\omega_0 = \pi/4$:



Note however that the phase response of $H_1(z)$ and $H_2(z)$ will be different because of the phase response introduced by $A(z)$ and $1/A(z)$.
