Name:

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COM-303 - Signal Processing for Communications
Final Exam

Saturday, June 21 2014, 09:15 to 12:15

- **Write your name** on the top left corner of **ALL sheets you turn in**, including this one. When you are done, **staple** all your sheets together **with this sheet on top**!
- You can have two A4 sheet of *handwritten* notes (front and back). Please **no photocopies, no books and no electronic devices**. Turn off your phone if you have it with you.
- There are 5 problems for a total of 100 points; the number of points for each problem is indicated next to it.
- Please write your derivations clearly, as there is partial credit.

Exercise 1. (10 points)

Consider the discrete-time signal $x[n] = \mathrm{sinc}(an)$ with $0 < a < 1$; compute the following sums:

(a) $\displaystyle\sum_{n=-\infty}^{\infty} x[n]$

(b) $\displaystyle\sum_{n=-\infty}^{\infty} x^2[n]$

the impulse response of an ideal lowpass filter with cutoff frequency ω_c is

$$h[n] = (\omega_c/\pi)\mathrm{sinc}((\omega_c/\pi)n)$$

therefore $x[n]$ is the impulse response of an ideal lowpass filter with cutoff frequency $a\pi$, scaled by $1/a$ so that $X(e^{j\omega}) = (1/a)\mathrm{rect}(\omega/(2a\pi))$. From this:

(a) $\displaystyle\sum_{n=-\infty}^{\infty} x[n] = X(e^{j\omega})|_{\omega=0} = 1/a$

(b) $\displaystyle\sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 = \frac{1}{2\pi}\int_{-a\pi}^{a\pi} a^{-2} = 1/a$ *(by using Parseval's theorem)*

Exercise 2. (10 points)

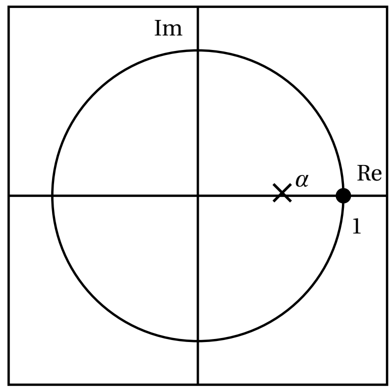
Compute the DFT of the $\mathbb{C}^4$ vector $\mathbf{x} = [1\ \ 1\ -1\ -1]^T$

the DFT matrix for $\mathbb{C}^4$ is

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

By computing the matrix-vector product $\mathbf{X} = \mathbf{W}\mathbf{x}$ *it is easy to obtain* $\mathbf{X} = [0\ \ (2-2j)\ \ 0\ \ (2+2j)]^T$

Exercise 3. (20 points)
Compute the impulse response of the causal filter with the following pole-zero plot:



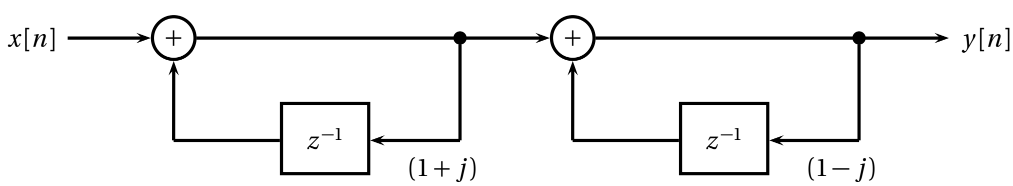
The system has a pole in $z = \alpha$ *and a zero in* $z = 1$. *We can write the transfer function of the system as*

$$H(z) = \frac{1 - z^{-1}}{1 - \alpha z^{-1}} = \frac{1}{1 - \alpha z^{-1}} - z^{-1}\frac{1}{1 - \alpha z^{-1}}$$

A first order section with a pole in $z = \alpha$ *has a transfer function* $G(z) = 1/(1 - \alpha z^{-1})$ *and impulse response* $g[n] = \alpha^n u[n]$. *Therefore the impulse response of the above system is*

$$h[n] = g[n] - g[n-1] = \alpha^n u[n] - \alpha^{n-1} u[n-1] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ \alpha^{n-1}(\alpha - 1) & n > 0 \end{cases}$$

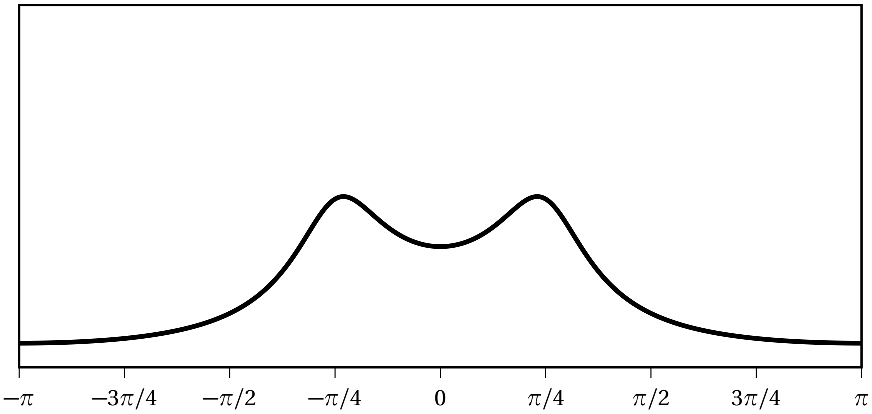
Exercise 4. (15 points)
Sketch the magnitude response of the following causal system:



The transfer function of the system is $H(z) = H_1(z)H_2(z)$ *where*

$$H_{1,2}(z) = \frac{1}{1 - (1 \pm j)z^{-1}}.$$

The system has therefore no zeros and two poles at $z = (1 \pm j)$ *or, in polar coordinates, at* $z = e^{\pm j\frac{\pi}{4}}$ *(note that the filter is not stable); its frequency response in magnitude follows the classic resonator pattern:*



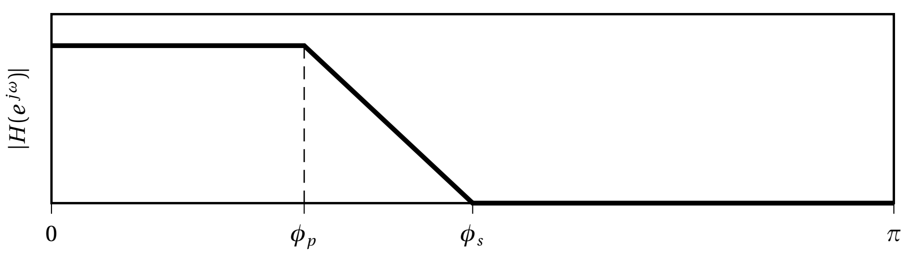
Exercise 5. (45 points)

Bellanger's Approximation is an empirical formula used to estimate the order of an equiripple lowpass filter based on its design specifications. For a lowpass filter with transition band $[\omega_p, \omega_s]$ and error tolerances of δ_p and δ_s in passband and stopband respectively, the filter order is going to be approximately

$$N \approx \frac{-2\log_{10}(10\delta_p\delta_s)}{3(\omega_s - \omega_p)/2\pi} - 1$$

Since the order is inversely proportional to the width of the transition band, "sharp" filters (i.e., filters with a narrow transition band) will require a lot of multiplications per output sample. The following questions will ask you to analyze an alternative design strategy called IFIR (Interpolated FIR), used to obtain sharp filters at a lower computational cost.

To begin with, assume you have designed an N-tap FIR lowpass $H(z)$ with the following magnitude response (we're showing just the positive frequencies and neglecting the ripples):



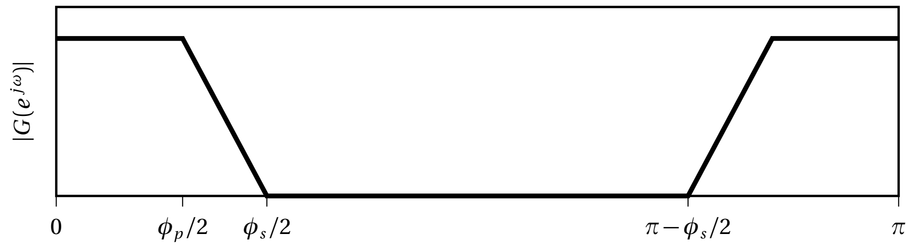
The transition band of $H(z)$ has width $\Delta_H = \phi_s - \phi_p$. Build now a derived FIR filter $G(z)$ with impulse response

$$g[n] = \begin{cases} h[n/2] & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

(a) express $G(z)$ in terms of $H(z)$

$G(z)$ is obtained by upsampling the impulse response by a factor of 2; therefore $G(z) = H(z^2)$

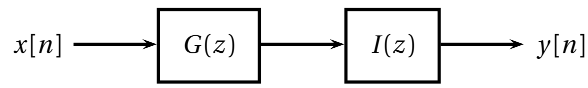
(b) sketch the magnitude response $|G(e^{j\omega})|$; you don't need to draw the ripples but clearly show the band edges and their values



(c) assuming that multiplications by zero can be ignored, what is the number of multiplications per output sample required by $G(z)$?

N multiplications per output sample

Consider now the following cascade, used to implement a complete IFIR filter:



(d) describe filter $I(z)$ so that the cascade $G(z)I(z)$ is equivalent to a simple lowpass filter

$I(z)$ should be a lowpass filter that removes the high frequency image introduced by the upsampling.

(e) specify the passband and stopband frequencies of the global filter implemented by the cascade

the global filter is a lowpass with band edges:

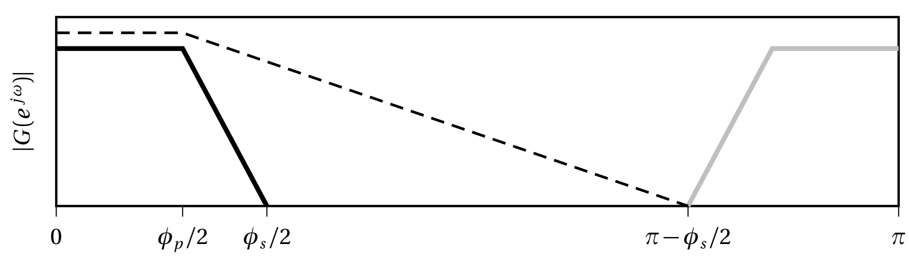
$$\omega_p = \phi_p/2$$
$$\omega_s = \phi_s/2$$

(f) find the passband and stopband frequencies of $I(z)$ so as to maximize its transition band

to minimize the computational cost of the cascade we can keep the transition band as wide as possible. We could use the following values, for instance:

$$\theta_p = \phi_p/2$$
$$\theta_s = \pi - \phi_s/2$$

for a transition band of $\Delta_I = \pi - (\phi_s + \phi_p)/2$.



Consider now the following specifications for a lowpass filter:

$$\omega_p = 0.3\pi$$
$$\omega_s = 0.31\pi$$
$$\delta_p = \delta_s = 0.01$$

and compare a direct FIR with an IFIR implementation.

(g) estimate the order of a direct equiripple implementation of the filter using Bellanger's formula

$$
\begin{aligned}
N &\approx \frac{-2\log_{10}(10\cdot 10^{-2}\cdot 10^{-2})}{3(0.31-0.3)\pi/2\pi}-1 \\
&= \frac{6}{0.015}-1=399
\end{aligned}
$$

(h) now consider an IFIR implementation: give the specifications for the initial IFIR filter $H(z)$ (i.e. the values of ϕ_p and ϕ_s to use in the design of $H(z)$)

the initial filter has double the passband and stopband frequencies, i.e.

$$
\begin{aligned}
\phi_p &= 0.6\pi \\
\phi_s &= 0.62\pi
\end{aligned}
$$

(i) estimate the order of an equiripple implementation of $H(z)$

$$N_H \approx \frac{6}{0.03}-1=199$$

(j) assume an optimal equiripple design for $I(z)$ using the maximum transition band Δ_I you found before and using $\delta_p=\delta_s=0.01$; estimate the order of $I(z)$

$\Delta_I = \pi-(\phi_p+\phi_s)/2 = 0.39\pi$

$$N_I \approx \frac{6}{3\cdot 0.39\pi/2\pi}-1=\frac{6}{3\cdot 0.39\pi/2\pi}-1\approx 9$$

(k) using the above estimations, determine the number of operations per output sample of the IFIR cascade

we will need 199 multiplications for $H(z)$ and 9 for $I(z)$ for a total of 208 multiplications