COM-303 - Signal Processing for Communications
Final Exam

July 8, 2016, 08:15 to 11:15, room SG1

Verify that this exam has YOUR last name on top

DO NOT OPEN THE EXAM UNTIL INSTRUCTED TO DO SO

- **Write your name** on the top left corner of **ALL the sheets you turn in**.
- There are 6 problems for a total of 100 points; the number of points is indicated for each problem.
- Please **write your derivations clearly!**
- You can have two A4 sheets of *handwritten* notes (front and back). Please **no photocopies, no books and no electronic devices**. Turn off your phone and store it in your bag.
- When you are done, simply leave your solution on your desk **with this page on top** and exit the classroom.

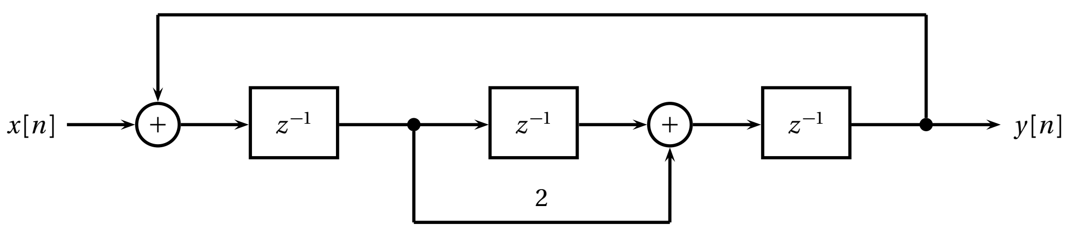
Exercise 1. (10 points)
Given a vector $\mathbf{x} = \begin{bmatrix} x_0 & x_1 & \dots & x_{N-1} \end{bmatrix} \in \mathbb{C}^N$ and its DFT $\mathbf{X} = \begin{bmatrix} X_0 & X_1 & \dots & X_{N-1} \end{bmatrix}$ show that

$$\mathbf{x}^* = \text{IDFT}\left\{\begin{bmatrix} X_0^* & X_{N-1}^* & X_{N-2}^* & \dots & X_1^* \end{bmatrix}\right\}$$

Solution:

$$\begin{aligned}
IDFT\left\{\begin{bmatrix} X_0^* & X_{N-1}^* & X_{N-2}^* & \dots & X_1^* \end{bmatrix}\right\}[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X_k^* \, e^{j\frac{2\pi}{N}n(N-k)} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X_k^* \, e^{-j\frac{2\pi}{N}nk} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} \left(X_k \, e^{j\frac{2\pi}{N}nk}\right)^* \\
&= \left(\frac{1}{N}\sum_{k=0}^{N-1} X_k \, e^{j\frac{2\pi}{N}nk}\right)^* \\
&= x_n^*
\end{aligned}$$

Exercise 2. (20 points)
Consider the causal system implemented by the following block diagram:

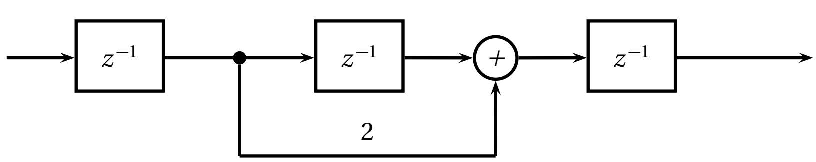


(a) compute the system's transfer function $H(z)$

(b) plot the system's poles and zeros on the complex plane

(c) determine if the system stable

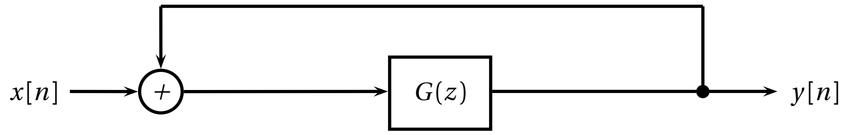
Solution: *First consider the FIR subsystem inside the feedback loop and call its transfer function $G(z)$*



From simple inspection it is easy to see that

$$G(z) = 2z^{-2} + z^{-3}$$

Now we can redraw the system as a simple feedback loop:



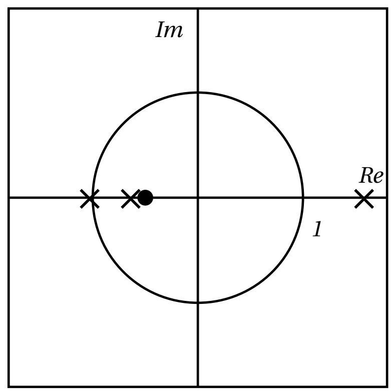
so that

$$H(z) = \frac{G(z)}{1 - G(z)} = \frac{2z^{-2} + z^{-3}}{1 - 2z^{-2} - z^{-3}}$$

Using simple factorization:

$$H(z) = 2z^{-2}\frac{1 + (1/2)z^{-1}}{1 - z^{-2} - z^{-2}(1 + z^{-1})} = 2z^{-2}\frac{1 + (1/2)z^{-1}}{(1 + z^{-1})(1 - z^{-1} - z^{-2})}$$

so that the transfer function has a zero in $z = -1/2$ *and poles in* -1 *and* $(1 \pm \sqrt{5})/2$*:*



Since one of the poles is outsite the unit circle, the system is not stable.

Exercise 3. (20 points)

The continuous-time signal

$$x_c(t) = \sum_{m=1}^{4} m \cos(2\pi f_0 m\, t),$$

with $f_0 = 300$Hz, is raw-sampled into the discrete time sequence $x[n] = x_c(nT_s)$ with $T_s = 5 \cdot 10^{-4}$s. Plot the DTFT of $x[n]$, $X(e^{j\omega})$.

Solution: *The sampling frequency is* $F_s = 1/T_s = (1/5) \cdot 10^4 Hz = 2000Hz$*. Since the continuous-time signal is just a linear combination of pure sinusoids, we can easily determine its highest frequency which is* $F_{\max} = 4{\cdot}300 = 1200Hz$*. Since* $F_{\max} > F_s/2$ *we will have aliasing. With this in mind, the easiest way to solve the problem is by working in the time domain, although of course we can operate also in the frequency domain.*

Working in the time domain: *by replacing the value for* T_s *and we have*

$$x[n] = x(nT_s) = \sum_{m=1}^{4} m \cos(\omega_m n) \qquad \omega_m = 2\pi\frac{m f_0}{F_s} = \frac{3\pi}{10}m$$

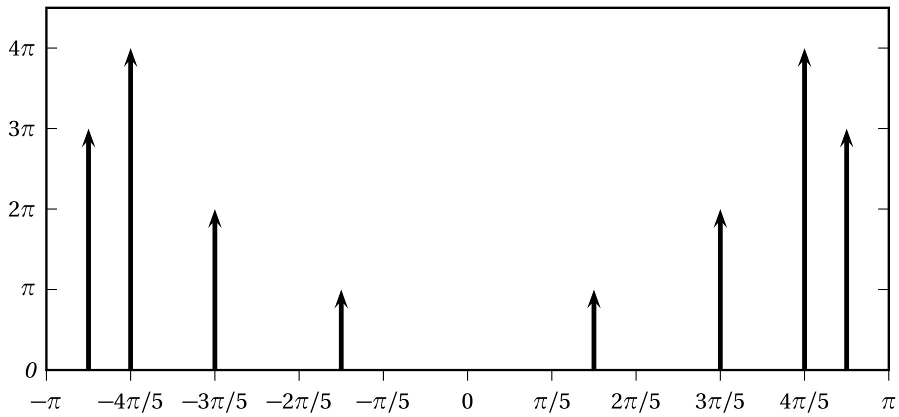
The frequencies ω_m *are less than* π *for* $m = 1,2,3$*; for* $m = 4$ *we have aliasing and we can write*

$$\cos(\frac{12}{10}\pi n) = \cos(\frac{20-8}{10}\pi n) = \cos(\frac{8}{10}\pi n)$$

so that, in the end

$$x[n] = \cos(\frac{3\pi}{10}n) + 2\cos(\frac{6\pi}{10}n) + 3\cos(\frac{9\pi}{10}n) + 4\cos(\frac{8\pi}{10}n)$$

and the spectrum is:



Working in the frequency domain: *the spectrum of the continuous-time signal is simply a collection of 8 spectral lines:*

$$X_c(j\Omega) = \pi \sum_{\substack{m=-4\\m\neq 0}}^{4} |m|\,\delta(\Omega - 2\pi f_0 m)$$

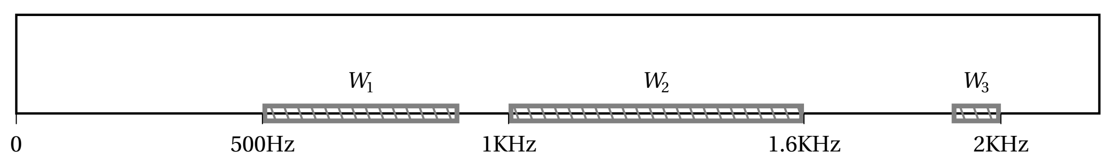
By sampling with a period T_s we have the standard periodization formula:

$$\begin{aligned}
X(e^{j\omega}) &= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega+2\pi k}{T_s}\right)\\
&= \frac{\pi}{T_s}\sum_{\substack{m=-4\\m\neq 0}}^{4} |m| \sum_{k=-\infty}^{\infty} \delta\left(\frac{\omega+2\pi k}{T_s} - 2\pi f_0 m\right)\\
&= \pi \sum_{\substack{m=-4\\m\neq 0}}^{4} |m| \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi(f_0 T_s)m + 2\pi k)\\
&= \frac{1}{2}\sum_{\substack{m=-4\\m\neq 0}}^{4} |m|\tilde{\delta}(\omega - \frac{3\pi}{10}m)
\end{aligned}$$

which yields the same plot as before. Note that we had to use the scaling property of the Dirac delta, $\delta(t/\alpha) \equiv \alpha\delta(t)$, which is perhaps not widely known. Hence the preference for the time-domain approach.

Exercise 4. (20 points)

In this exercise you will need to design a data transmission system for the analog channel sketched in this picture (positive frequencies only):



The channel has three usable bands with the following characteristics:

- band W_1, from 500Hz to 900Hz

- band W_2, from 1000Hz to 1600Hz
- band W_3, from 1900Hz to 2000Hz

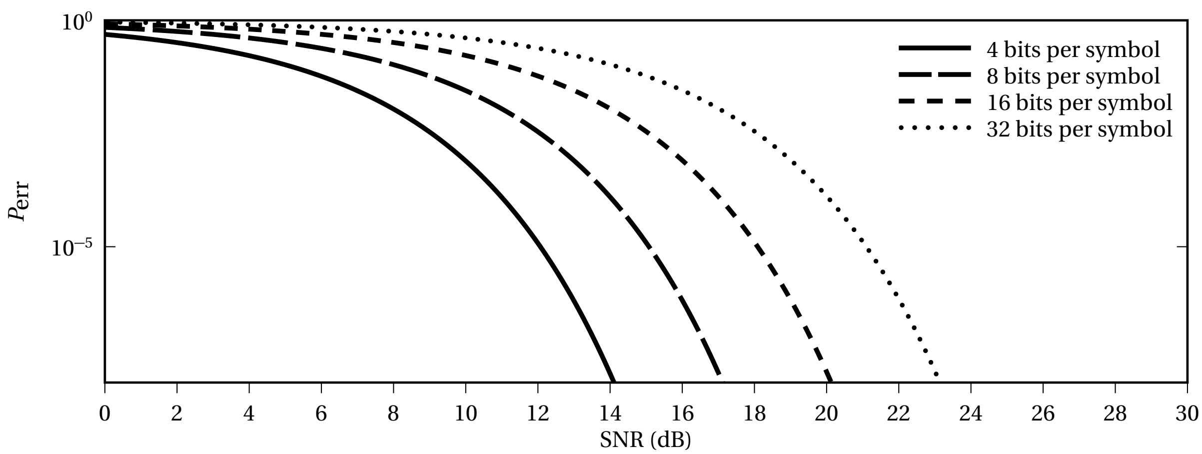
To transmit the data you can use one or more configurable passband data transmitters. For each transmitter you can set the following parameters:

1- the center frequency for the trasmission band (in Hz)
2- the number of symbols per second (i.e. the symbol rate or Baud rate)
3- the number of bits per symbol (4, 8, 16, or 32)

Each transmitter can also adjust its gain so that its transmitted signal reaches the maximum power allowed by the channel's power constraint. Because of different noise levels across the spectrum the resulting SNR levels for each subband are

- band W_1: SNR of 16dBs
- band W_2: SNR of 10dBs
- band W_3: SNR of 20dBs

The operational characteristic for the data transmitters is shown in this chart, where each curve shows the attainable probability of error as a function of the SNR; each curve corresponds to a different bit-per-symbol operation mode:

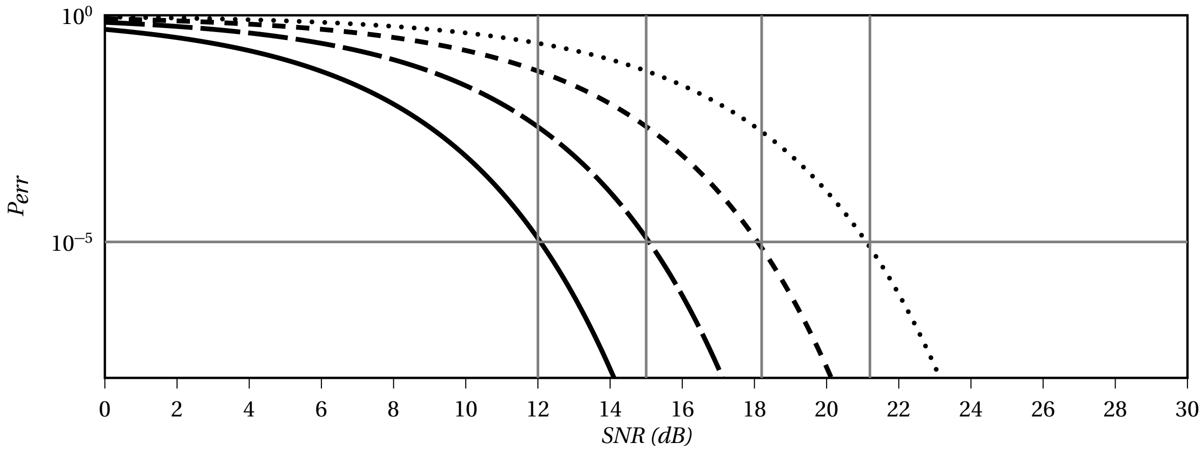


We want the transmission scheme to operate at an overall probability of error of 10^{-5} or less. Determine the maximum achievable transmission rate (in bits per second).

Solution: *Since we have three independent subbands, we will use a separate transmitter for each one. The number of symbols per second for each transmitter will be equal to the (positive) width of each subband while the trasmission center frequency will be equal to each subband's center frequency*

- *band W_1: 400 symbols per second, center frequency $f_1 = 700Hz$*
- *band W_2: 600 symbols per second, center frequency $f_1 = 1300Hz$*
- *band W_3: 100 symbols per second, center frequency $f_1 = 1950Hz$*

To compute the total throughput we need to determine how many bits per symbol can be sent over each sub-channel at the given reliability figure. Since we need to operate at most at $P_e = 10^{-5}$ we can find the required minimum SNR for each operation curve by looking at the intersection with the line $P_e = 10^{-5}$:



From the plot we can see that:

- *to transmit at 4 bits per symbol we need at least* $\approx 12dB$ *SNR*
- *to transmit at 8 bits per symbol we need at least* $\approx 15dB$ *SNR*
- *to transmit at 16 bits per symbol we need at least* $\approx 18dB$ *SNR*
- *to transmit at 32 bits per symbol we need at least* $\approx 21dB$ *SNR*

therefore:

- *band* W_1*: SNR of 16dBs* $\Rightarrow$ *we can transmit at 8 bits per symbol*
- *band* W_2*: SNR of 10dBs* $\Rightarrow$ *we cannot transmit since all possible rates will have* $P_e > 10^{-5}$ *at this SNR*
- *band* W_3*: SNR of 20dBs* $\Rightarrow$ *we can transmit at 16 bits per symbol*

The final capacity of the transmission scheme is therefore $R = W_1 \cdot 8 + W_3 \cdot 16 = 4800bps$*.*

Exercise 5. (20 points)

We have seen that one of the fundamental ingredients of JPEG compression is the *deadzone* quantizer, i.e. a quantizer which has a quantization interval centered around zero. To see the effects of the deadzone quantizer on SNR consider the following problem.

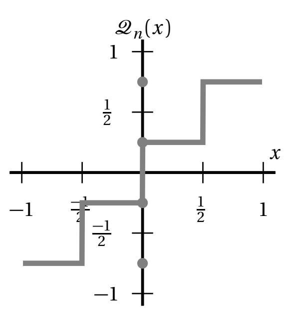
Assume $x[n]$ is an i.i.d. discrete-time signal whose values are over the $[-1, 1]$ interval. Consider the following uniform 2-bit quantizers for the interval:

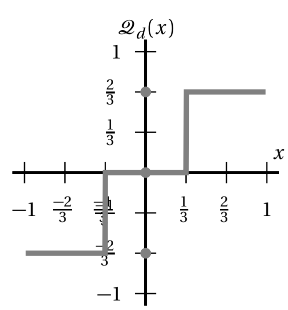
normal quantizer

$$\mathcal{Q}_n(x) = \begin{cases} 3/4 & \text{if } 1/2 \le x \le 1 \\ 1/4 & \text{if } 0 \le x < 1/2 \\ -1/4 & \text{if } -1/2 \le x < 0 \\ -3/4 & \text{if } -1 \le x < -1/2 \end{cases}$$

deadzone quantizer

$$\mathcal{Q}_n(x) = \begin{cases} 2/3 & \text{if } 1/3 \le x \le 1 \\ 0 & \text{if } |x| < 1/3 \\ -2/3 & \text{if } -1 \le x \le -1/3 \end{cases}$$





Both quantizers operate at two bits per sample but the deadzone quantizer "wastes" a fraction of a bit since it has only 3 quantization intervals instead of 4; for a uniformly distributed input, therefore, the SNR of the deadzone quantizer is smaller than the SNR of the standard quantizer. Assume now that the probability distribution for each input sample is the following:

$$P[x[n]=\alpha]=\begin{cases}0 & \text{if } |\alpha|>1\\ p & \text{if } |\alpha|=0\\ (1-p)/2 & \text{otherwise}\end{cases}$$

In other words, each sample is either zero with probability p or drawn from a uniform distribution over the $[-1,1]$ interval; we can express this distribution as a pdf like so:

$$f(x)=\frac{1-p}{2}+p\delta(x)$$

Determine the minimum value of p for which it is better to use the deadzone quantizer, i.e. the value of p for which the SNR of the deadzone quantizer is larger that the SNR of the uniform quantizer.

Hint: remember that the formula for the MSE of a scalar quantizer over the $[-1,1]$ interval (under the usual hypotheses of iid samples with pdf $f(x)$) is

$$\sigma^2=\int_{-1}^{1}(\mathcal{Q}(x)-x)^2 f(x)\,dx$$

For a uniform quantizer with M quantization levels and uniform input distribution $f(x)=1/2$, we also know that

$$\sigma^2=\int_{-1}^{1}(\mathcal{Q}(x)-x)^2\frac{1}{2}\,dx=\frac{\Delta^2}{12}=\frac{(2/M)^2}{12}=\frac{1}{3M^2}$$

Solution: *First of all, since the input signal is the same, in order to compare SNRs we just need to compare the mean square errors of the two quantizers and find out the value of p for which the deadzone quantizer's MSE is smaller than the normal quantizer's MSE.*

The number of quantization levels in the two quantizers are $M=M_n=4$ for the normal 2-bit quantizer and $M=M_d=3$ for the deadzone quantizer. Let's compute the MSE for the normal quantizer using the composite pdf for the input

$$\begin{aligned}\sigma_n^2&=\int_{-1}^{1}(\mathcal{Q}_n(x)-x)^2\left(\frac{1-p}{2}+p\delta(x)\right)dx\\&=(1-p)\int_{-1}^{1}(\mathcal{Q}_n(x)-x)^2\frac{1}{2}\,dx+p\int_{-1}^{1}(\mathcal{Q}_n(x)-x)^2\,\delta(x)\,dx\\&=(1-p)\frac{1}{3M_n^2}+p[Q_n(0)]^2\\&=(1-p)\frac{1}{48}+p\frac{1}{16}\end{aligned}$$

where we have used the fact that the normal quantizer maps zero to $1/4$; similarly, for the deadzone quantizer (which maps zero to zero):

$$\begin{aligned}\sigma_d^2&=\int_{-1}^{1}(\mathcal{Q}_d(x)-x)^2\left(\frac{1-p}{2}+p\delta(x)\right)dx\\&=(1-p)\frac{1}{3M_d^2}+p[Q_d(0)]^2\\&=(1-p)\frac{1}{27}\end{aligned}$$

from which we find

$$\sigma_d^2<\sigma_n^2\quad \textit{for}\quad p>\frac{21}{102}\approx 20\%$$