COM303: Digital Signal Processing

Lecture 17: Sampling and applications

overview

- raw sampling and aliasing
- DT processing of CT signals

Sinc Sampling

$$x[n] = \left\langle \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right), x(t) \right\rangle$$

Sinc Sampling

$$x[n] = (\mathrm{sinc}_{T_s} * x)(nT_s)$$

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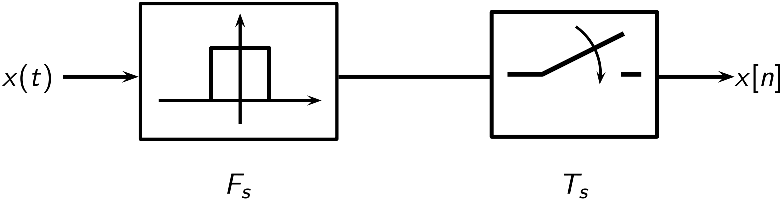
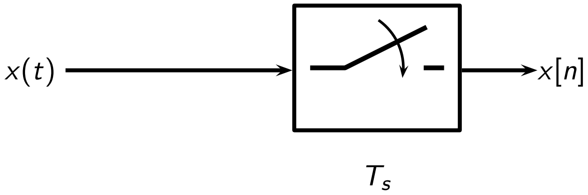


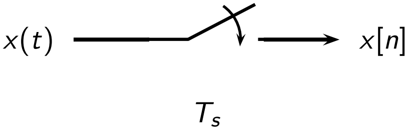
Sinc Sampling for F_s-BL signals

$$x[n] = (\mathrm{sinc}_{T_s} * x)(nT_s) = T_s\, x(nT_s)$$



"Raw" sampling - can we always do that?

$$x[n] = x(nT_s)$$



Remember the wagonwheel effect?

The continuous-time complex exponential

$$x(t) = e^{j2\pi f_0 t}$$

- always periodic, period $t_0 = 1/f_0$
- all angular speeds are allowed
- $\text{FT}\left\{e^{j2\pi f_0 t}\right\} = \delta(f - f_0)$
- highest (and only) frequency is f_0

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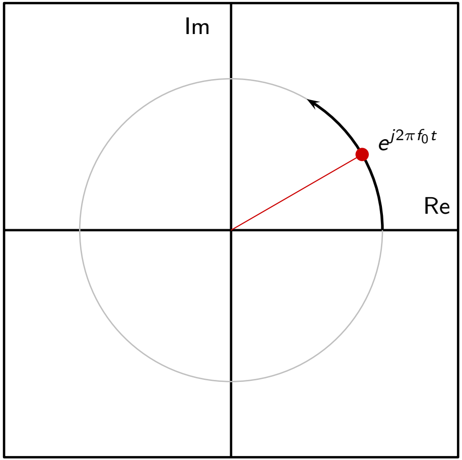

Raw samples of the continuous-time complex exponential

$$x[n] = e^{j2\pi f_0 n T_s}$$

- raw samples are snapshots at regular intervals of the rotating point
- resulting digital frequency is $\omega_0 = 2\pi f_0 T_s = 2\pi(f_0/F_s)$

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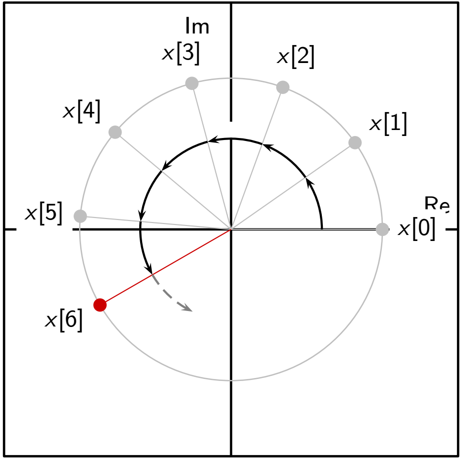
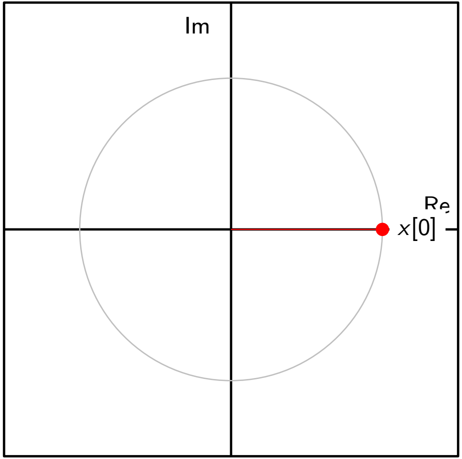
Easy: $f_0 < F_s/2 \quad \Rightarrow \quad \omega_0 < \pi$



Tricky: $F_s/2 < f_0 < F_s \quad \Rightarrow \quad \pi < \omega_0 < 2\pi$



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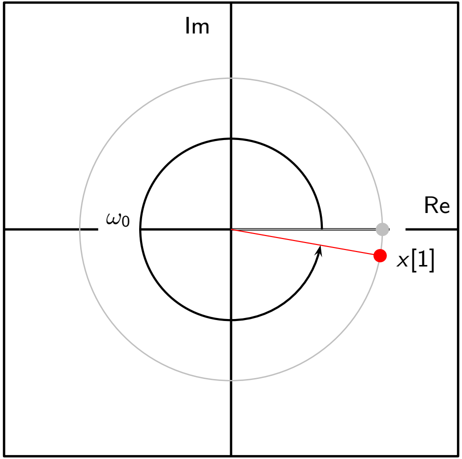

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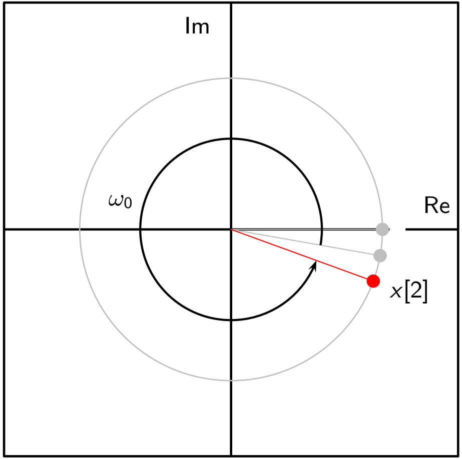

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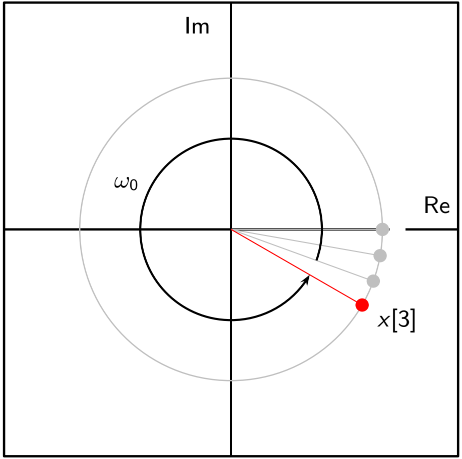

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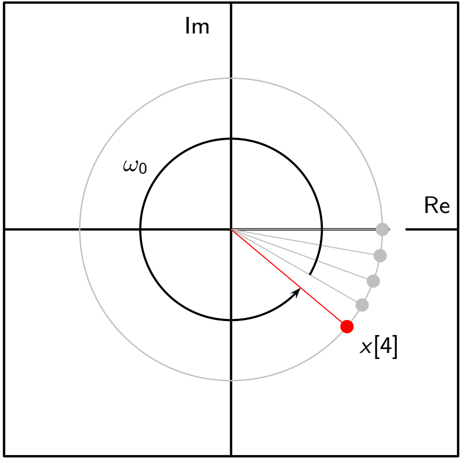

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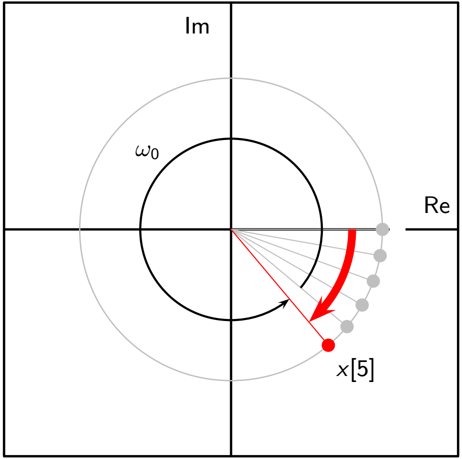
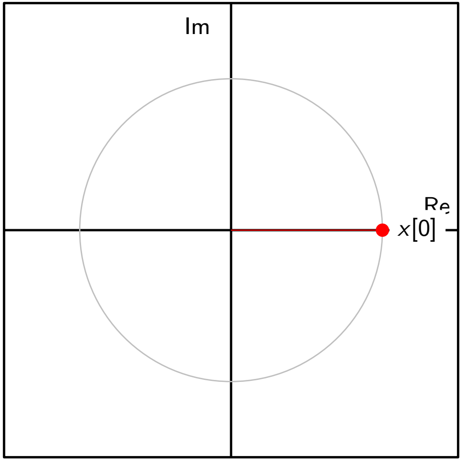


Trouble: $f_0 > F_s \quad \Rightarrow \quad \omega_0 > 2\pi$



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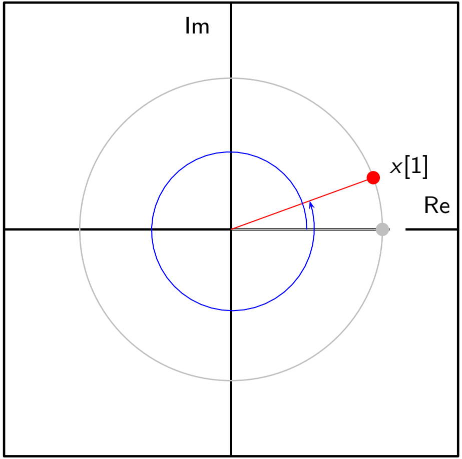

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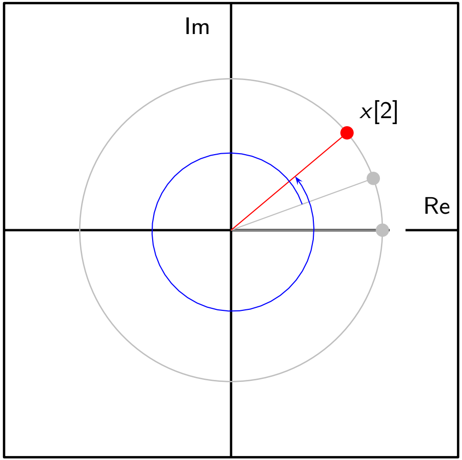

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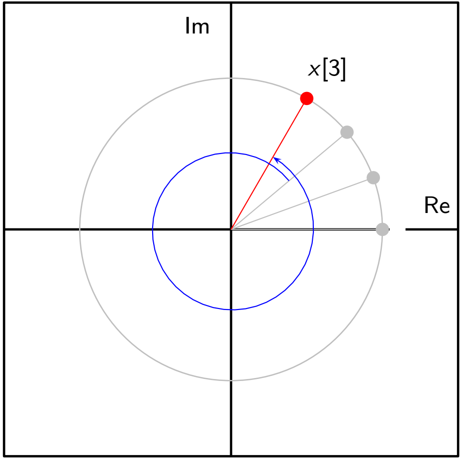

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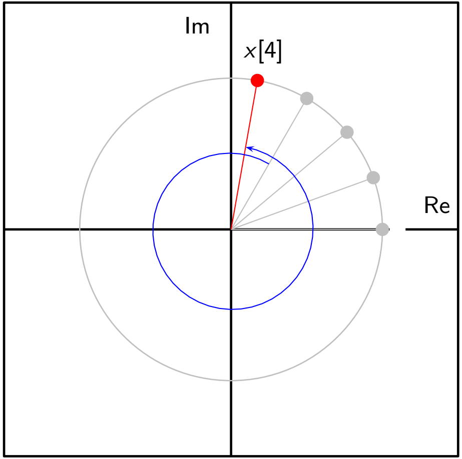

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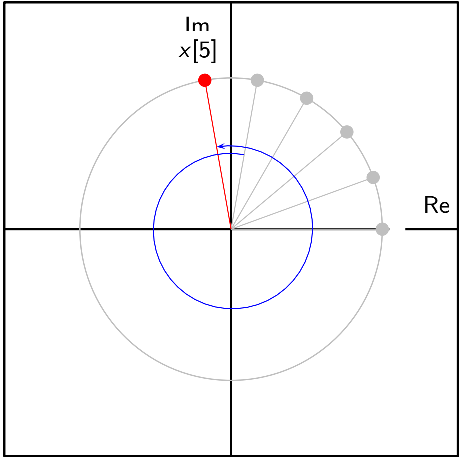
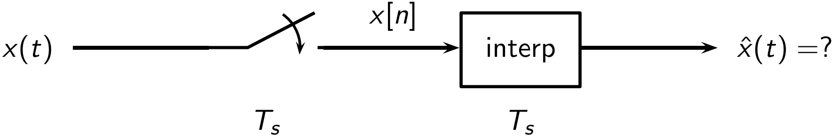


Aliasing



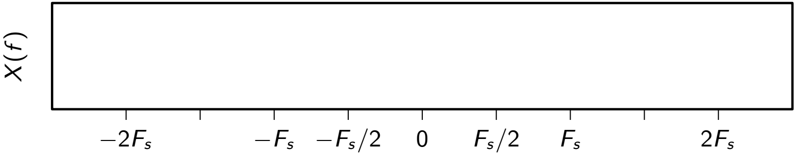
Aliasing

$$\text{pick } T_s;\ F_s = 1/T_s$$

$$\text{input: } x(t) = e^{j2\pi f_0 t}$$
$$\text{digital frequency: } \omega_0 = 2\pi f_0/F_s$$

	digital frequency	$\hat{x}(t)$	
$f_0 < F_s/2$	$0 < \omega_0 < \pi$	$e^{j2\pi f_0 t}$	
$f_0 = F_s/2$	$\omega_0 = \pi$	$e^{j2\pi f_0 t}$	
$F_s/2 < f_0 < F_s$	$\pi < \omega_0 < 2\pi$	$e^{j2\pi f_1 t}$,	$f_1 = f_0 - F_s < 0$
$f_0 > F_s$	$\omega_0 > 2\pi$	$e^{j2\pi f_2 t}$,	$f_2 = f_0 \mod F_s$

Aliasing of sinusoids: increasing the input frequency



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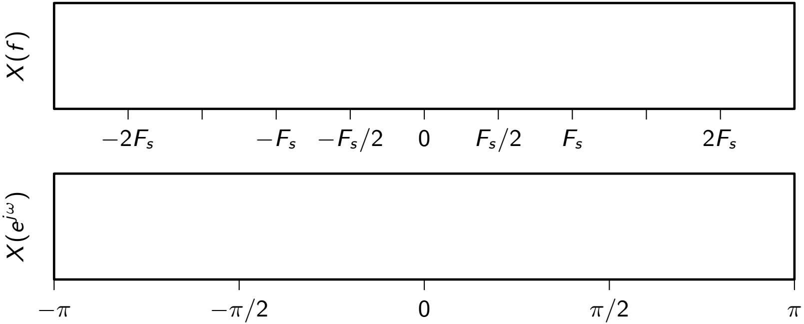

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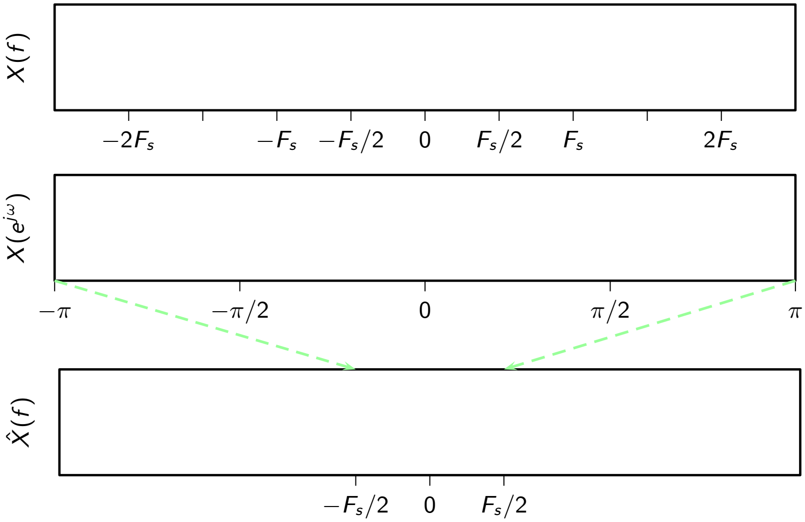

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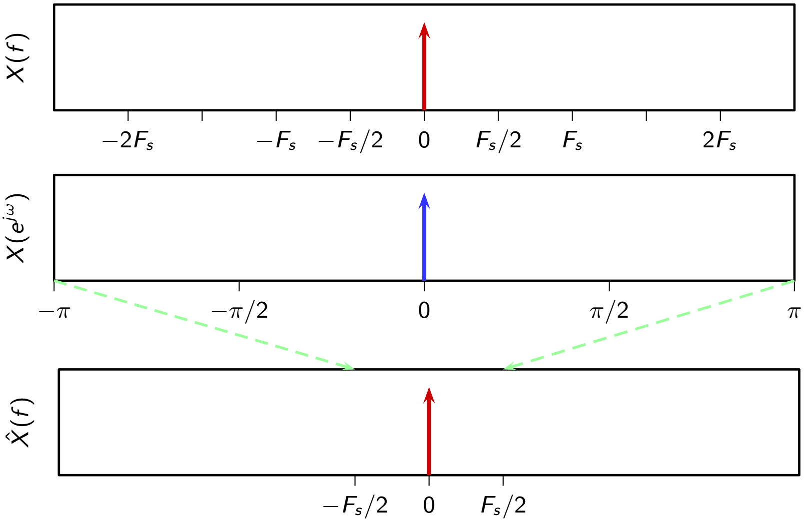

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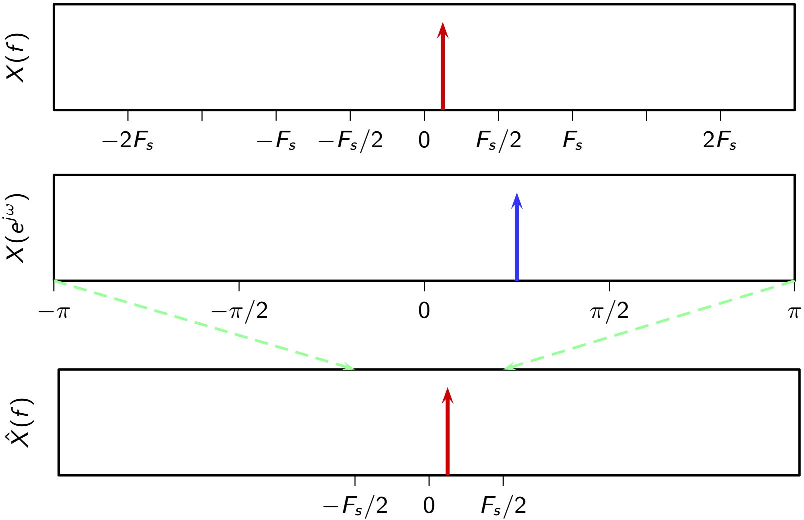

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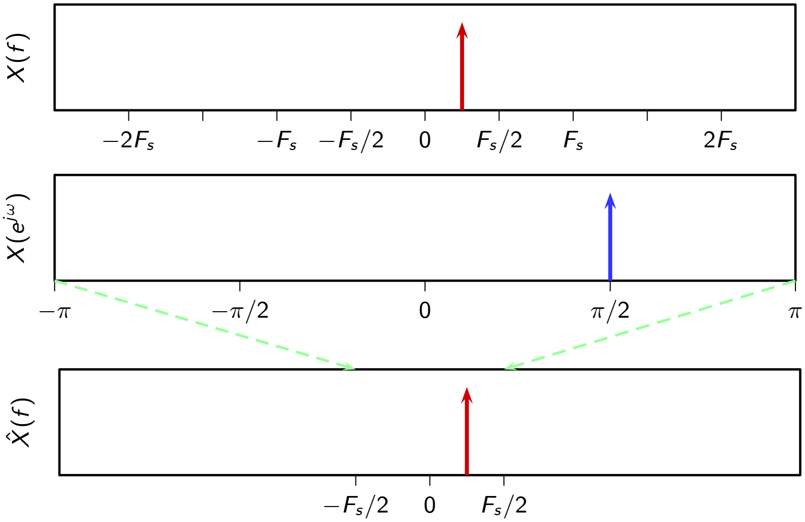

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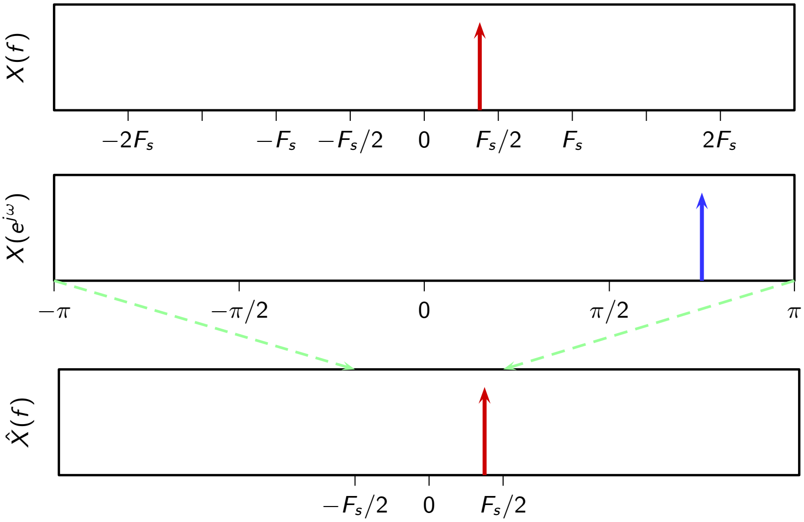

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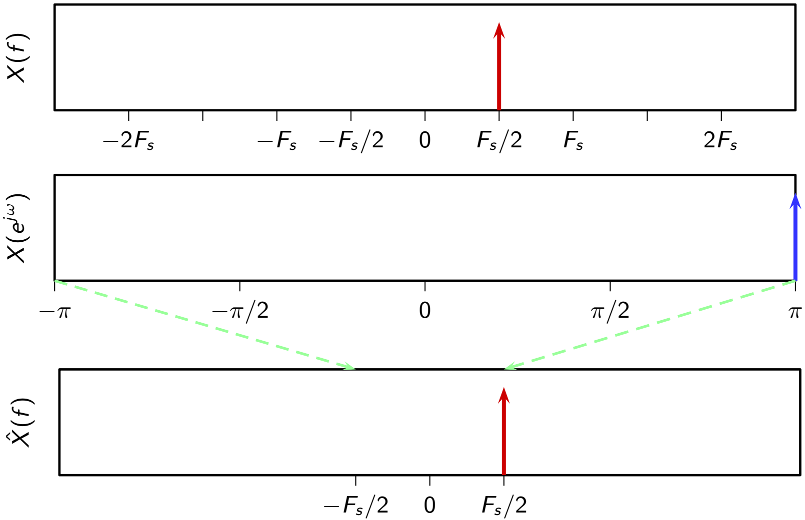

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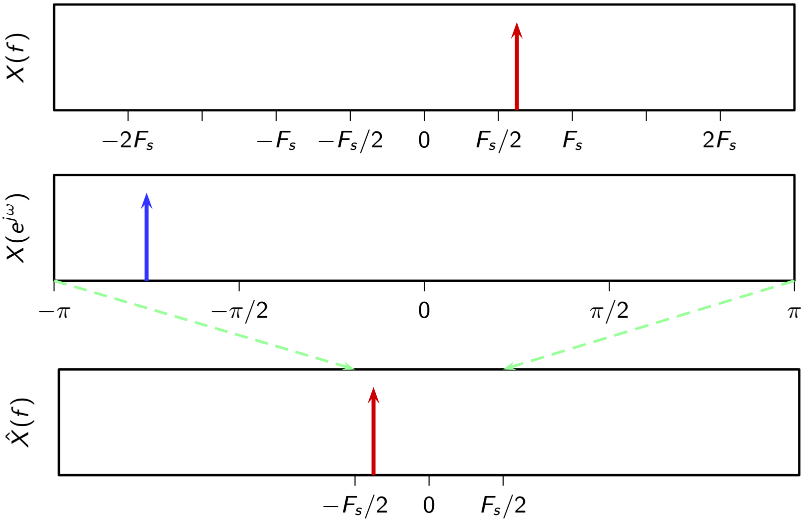

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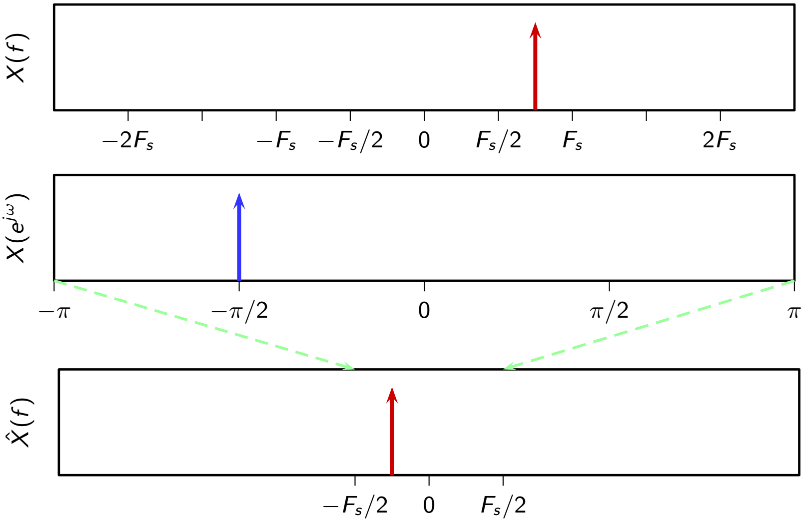

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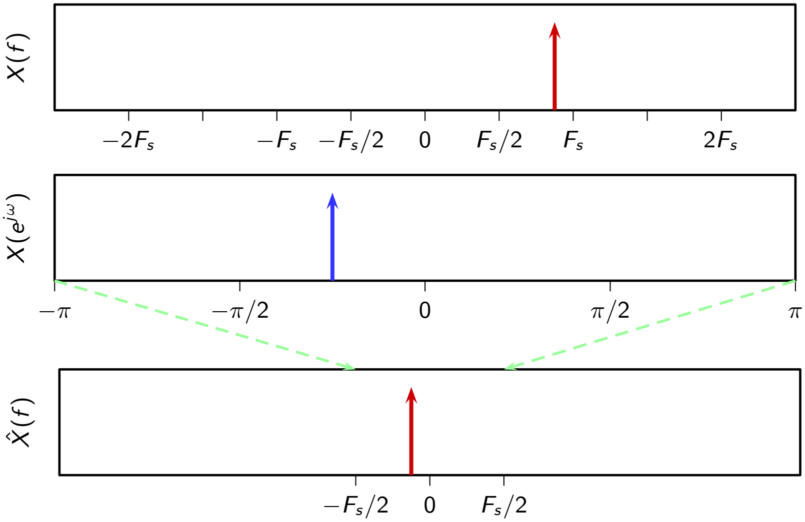

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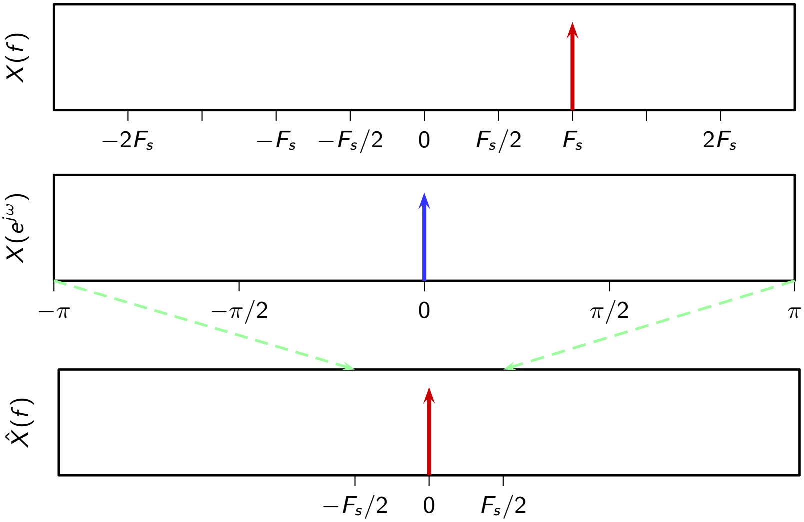

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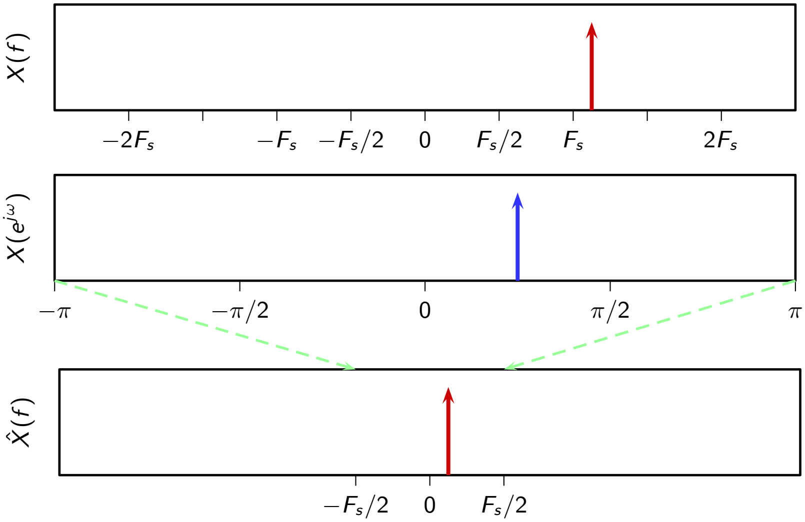

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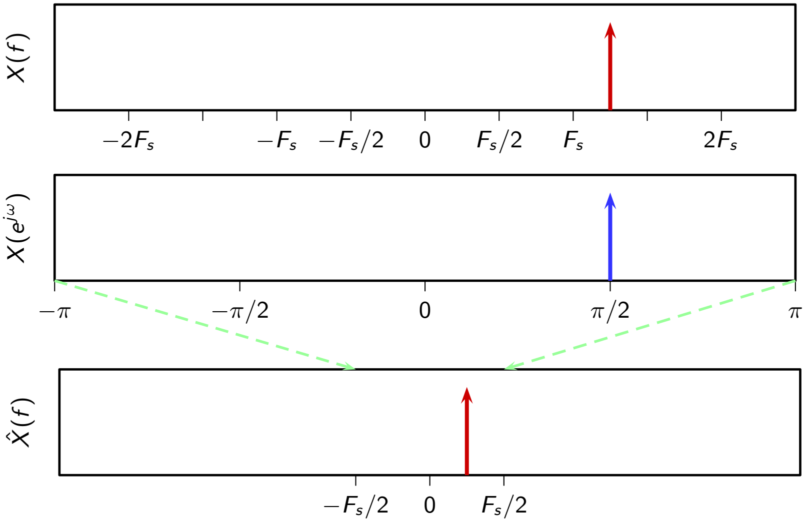

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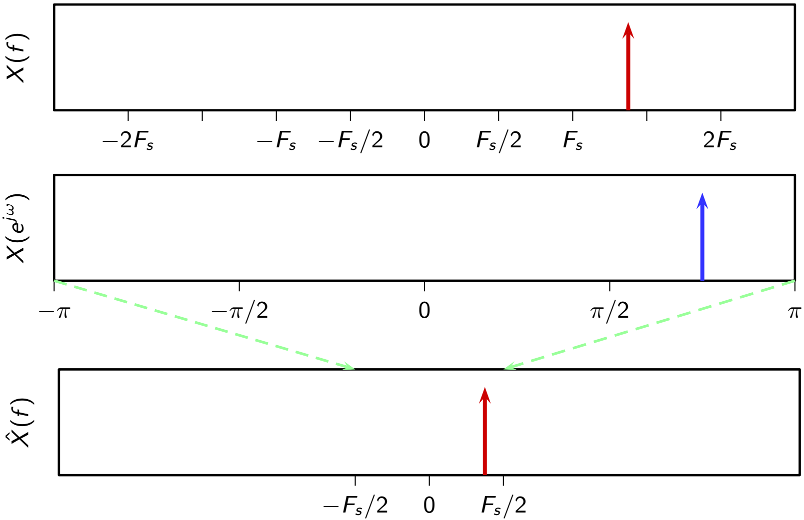

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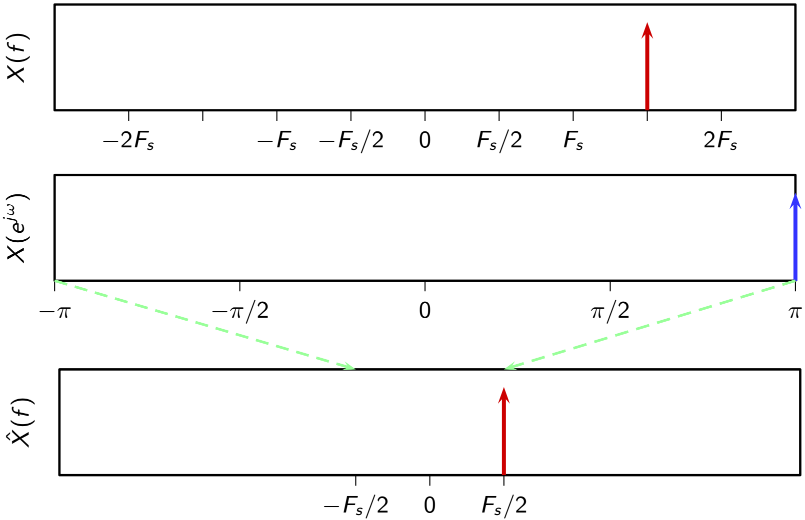

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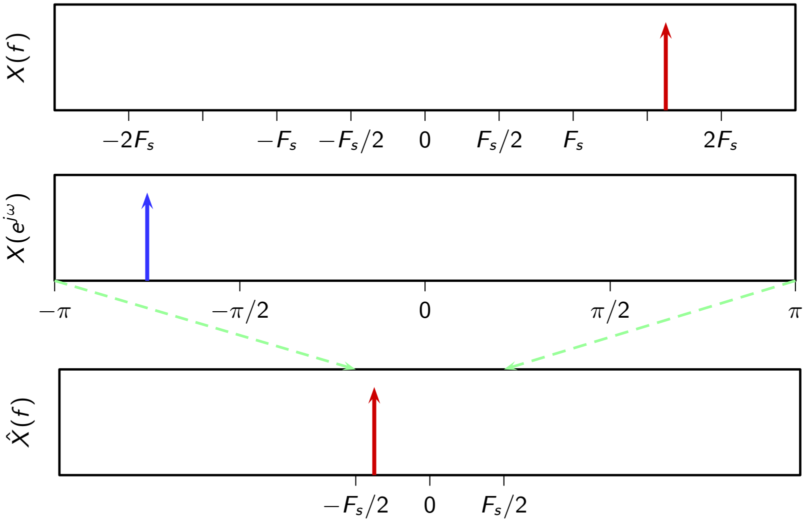

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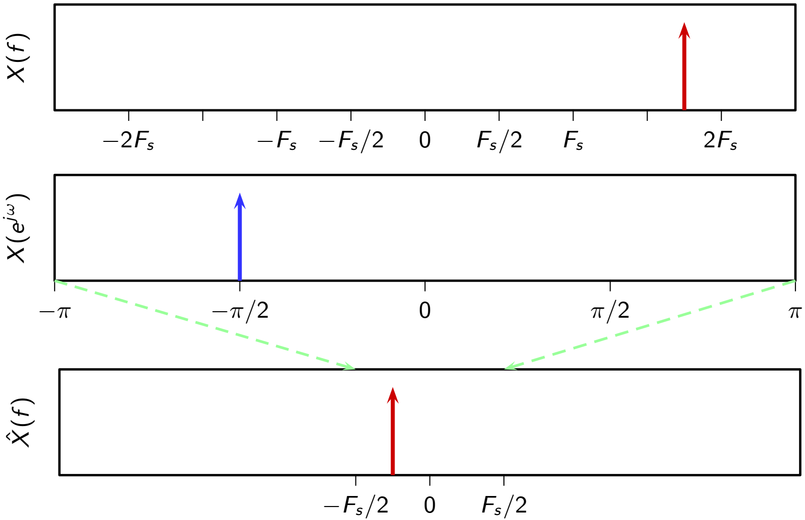

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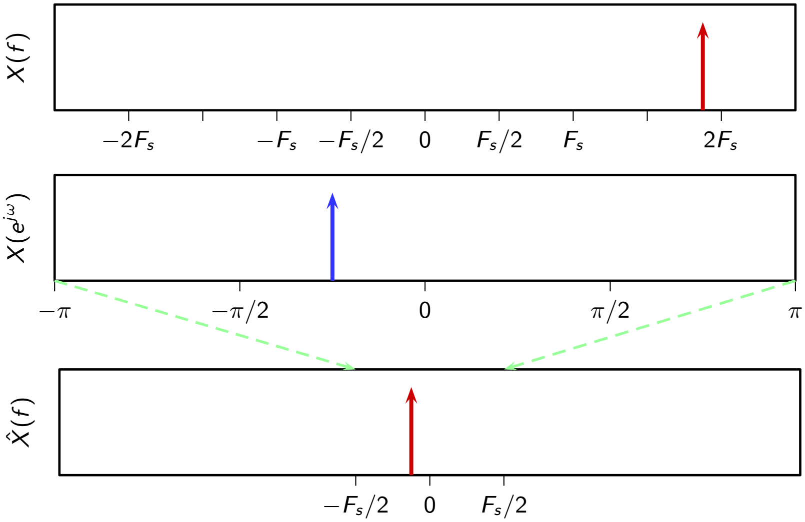

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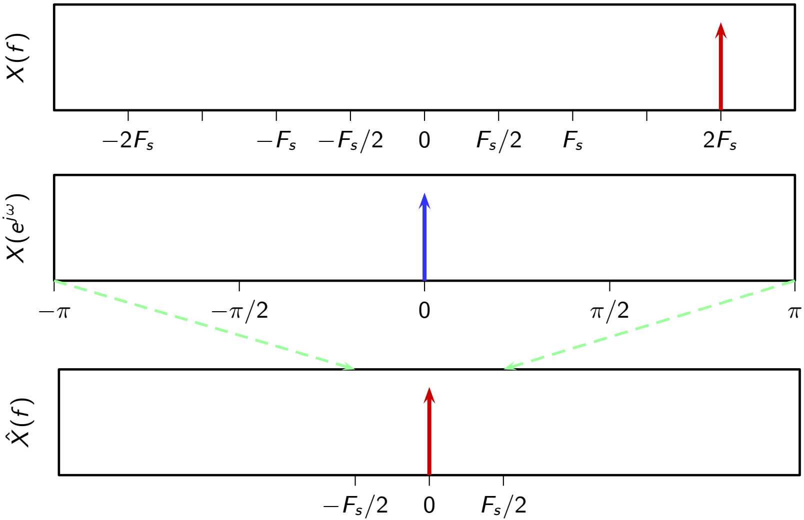

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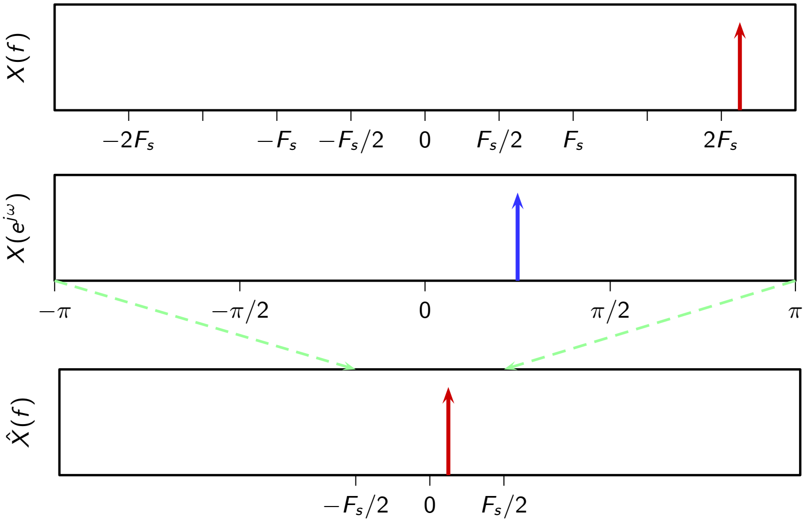
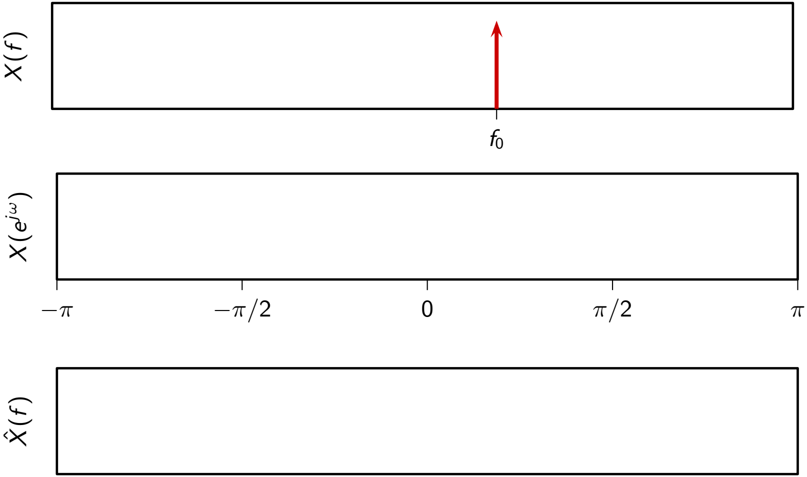


Aliasing of sinusoids: decreasing the sampling frequency



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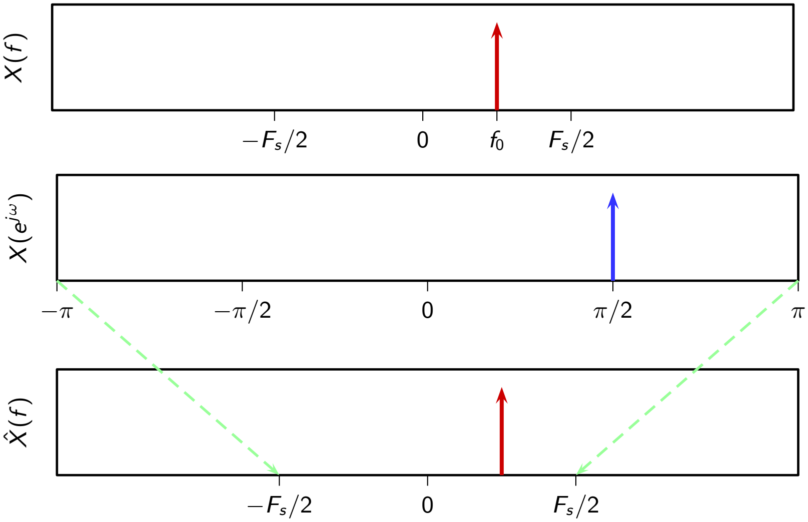

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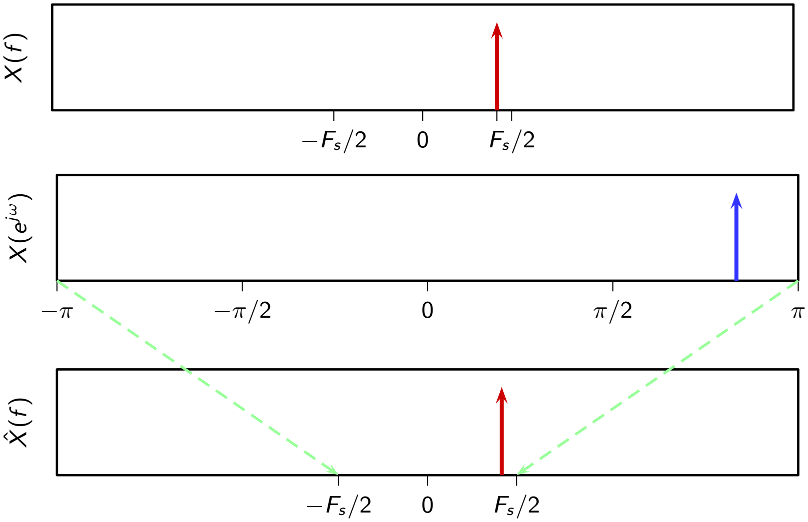

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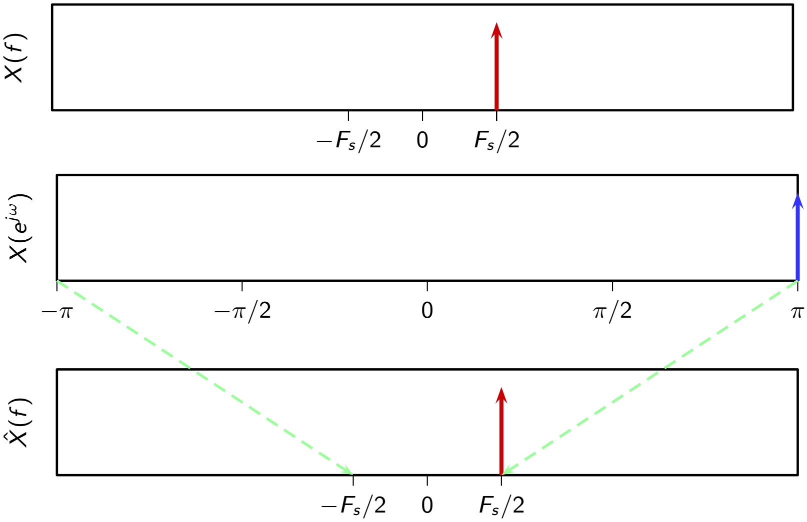

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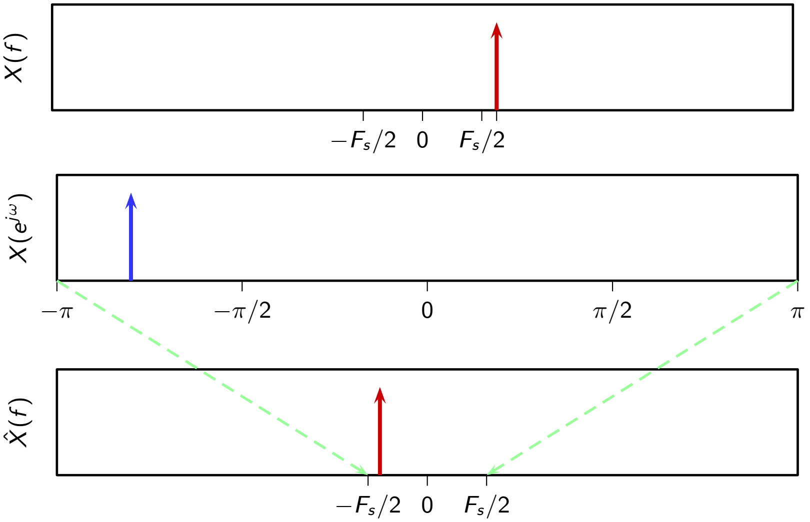

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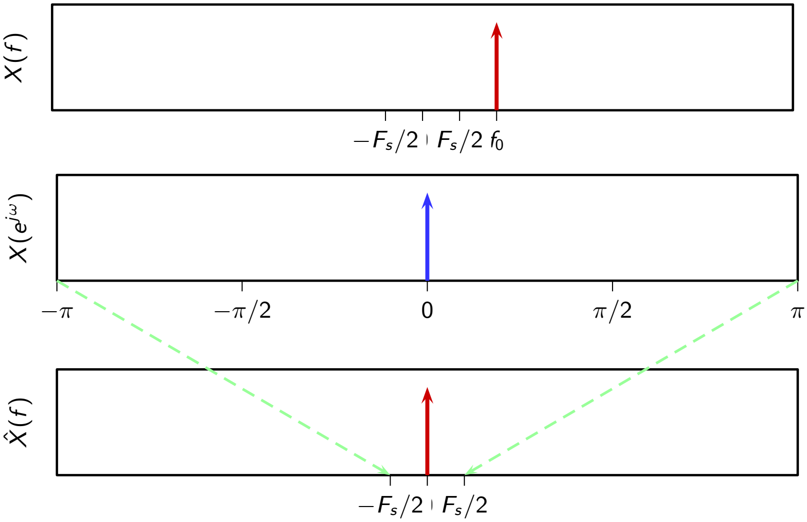

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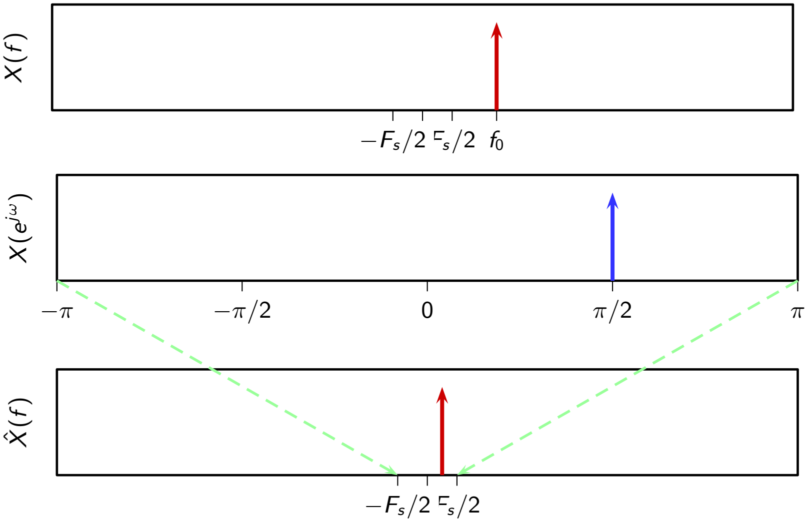
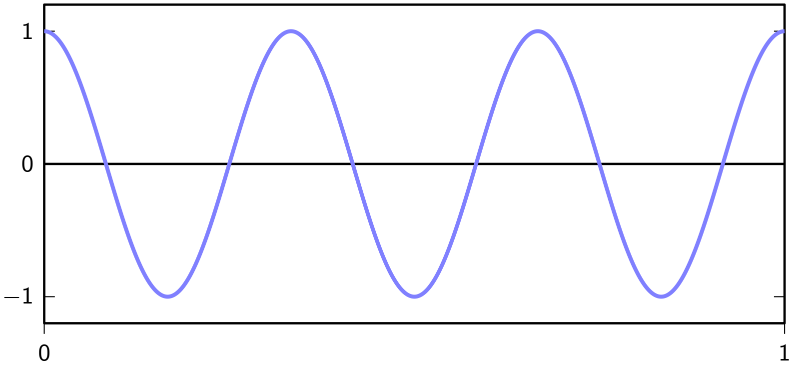


Sampling a Sinusoid

sampling frequency	digital frequency	interpolation
$F_s > 2f_0$	$0 < \omega_0 < \pi$	OK: $\hat{f}_0 = f_0$
$F_s = 2f_0$	$\omega_0 = \pi$	OK (max frequency $\hat{f}_0 = F_s$)
$f_0 < F_s < 2f_0$	$\pi < \omega_0 < 2\pi$	negative frequency: $\hat{f}_0 = f_0 - F_s$
$F_s < f_0$	$\omega_0 > 2\pi$	full aliasing: $\hat{f}_0 = f_0 \mod F_s$

Aliasing: Sampling a Sinusoid

$$x(t) = \cos(6\pi t) \qquad (f_0 = 3\text{Hz})$$



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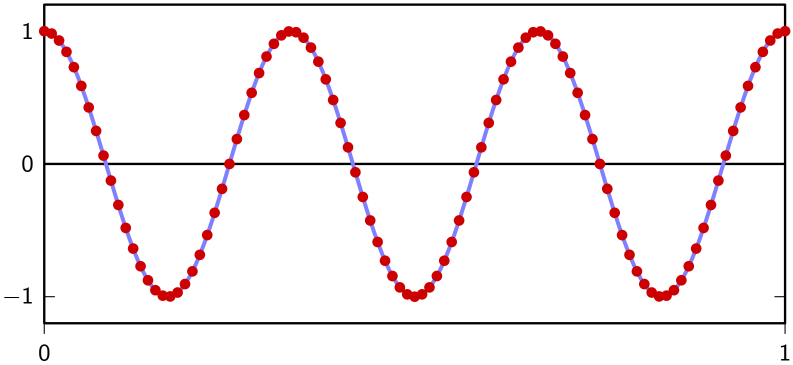
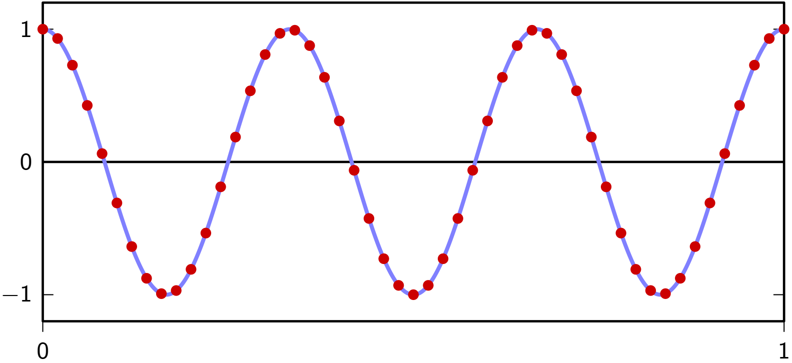


$F_s = 100\text{Hz}$

Aliasing: Sampling a Sinusoid

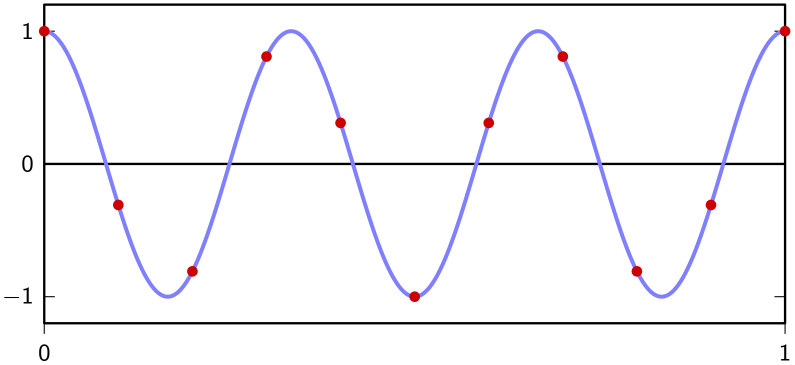
$$x(t) = \cos(6\pi t) \qquad (f_0 = 3\text{Hz})$$



$F_s = 50\text{Hz}$

Aliasing: Sampling a Sinusoid

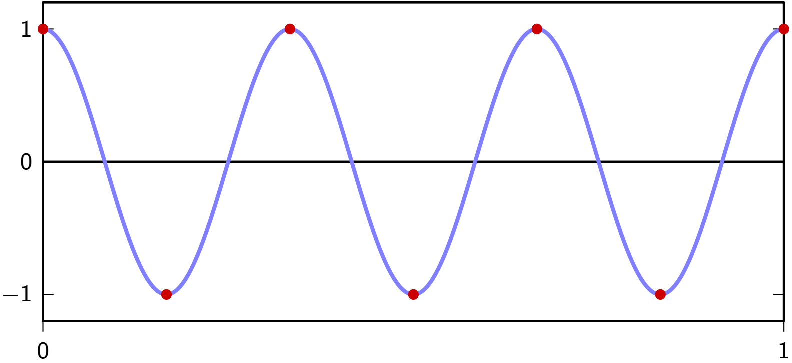
$$x(t) = \cos(6\pi t) \qquad (f_0 = 3\text{Hz})$$



$F_s = 10\text{Hz}$

Aliasing: Sampling a Sinusoid

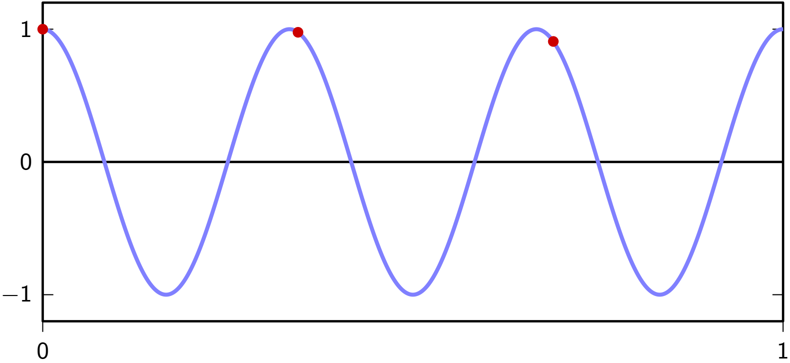
$$x(t) = \cos(6\pi t) \qquad (f_0 = 3\text{Hz})$$



$F_s = 6\text{Hz}$

Aliasing: Sampling a Sinusoid

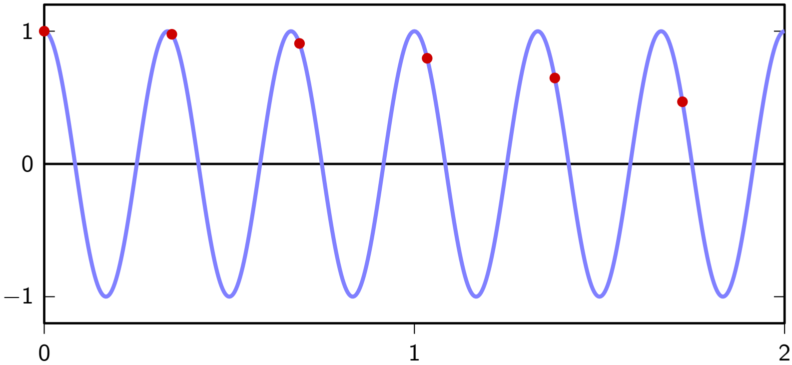
$$x(t) = \cos(6\pi t) \qquad (f_0 = 3\text{Hz})$$



$F_s = 2.9\text{Hz}$

Aliasing: Sampling a Sinusoid

$$x(t) = \cos(6\pi t) \qquad (f_0 = 3\text{Hz})$$



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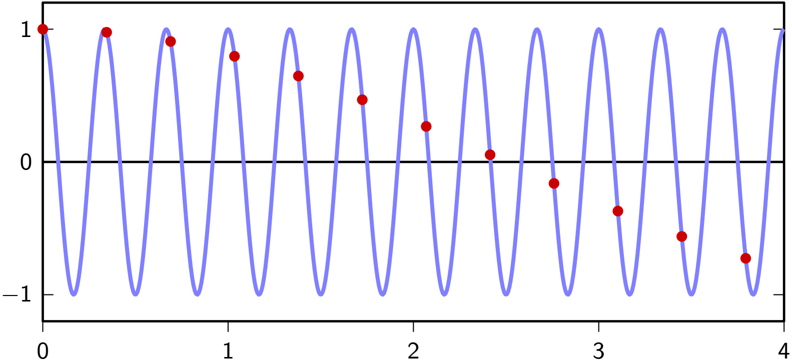
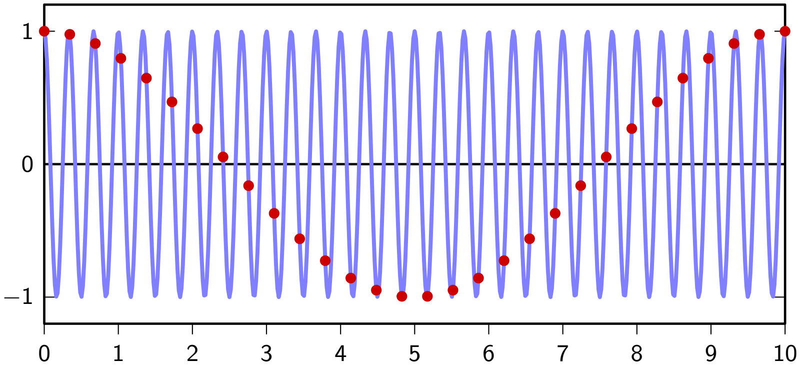


$F_s = 2.9\text{Hz}$

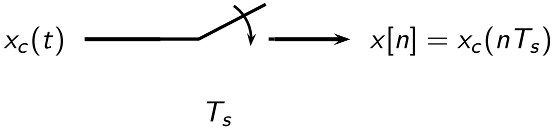
Aliasing: Sampling a Sinusoid

$$x(t) = \cos(6\pi t) \qquad (f_0 = 3\text{Hz})$$



$F_s = 2.9\text{Hz}$

Raw-sampling an arbitrary signal



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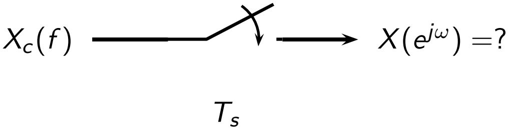
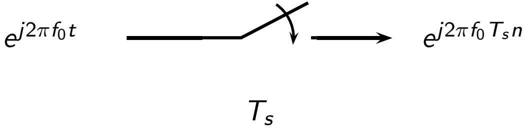


Key idea

- pick T_s (and set $F_s = 1/T_s$)
- pick $f_0 < F_s/2$



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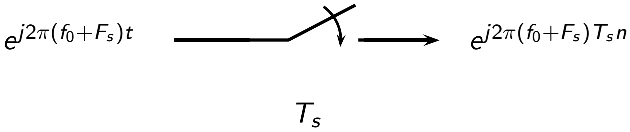

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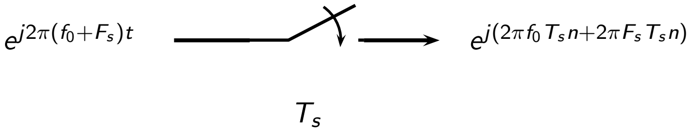

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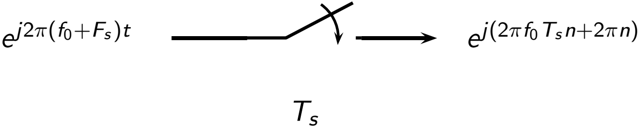

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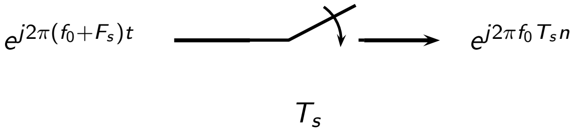

Key idea

- pick T_s (and set $F_s = 1/T_s$)
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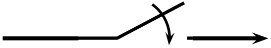
$$Ae^{j2\pi f_0 t} + Be^{j2\pi (f_0 + F_s)t}$$



$$(A+B)e^{j2\pi f_0 T_s n}$$

Spectrum of raw-sampled signals

outline: start with the inverse Fourier Transform

$$x[n] = x_c(nT_s) = \int_{-\infty}^{\infty} X_c(f) e^{j2\pi f T_s n} df$$

and manipulate the integral until it looks like

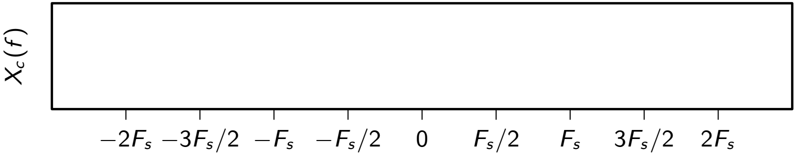
$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\omega) e^{j\omega n} d\omega$$

Spectrum of raw-sampled signals

frequencies F_s Hz apart will be aliased, so split the integration interval

$$
\begin{aligned}
x[n] &= \int_{-\infty}^{\infty} X_c(f) e^{j2\pi f\, T_s n} df \\
&= \sum_{k=-\infty}^{\infty} \int_{kF_s - \frac{F_s}{2}}^{kF_s + \frac{F_s}{2}} X_c(f) e^{j2\pi f\, T_s n} df
\end{aligned}
$$

Spectrum of raw-sampled signals



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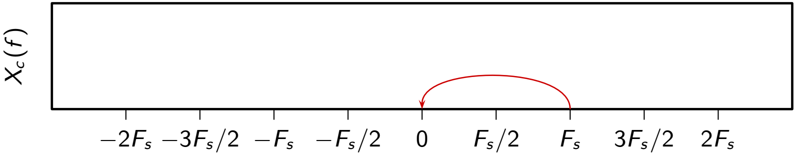

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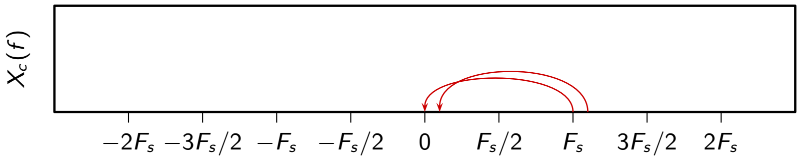

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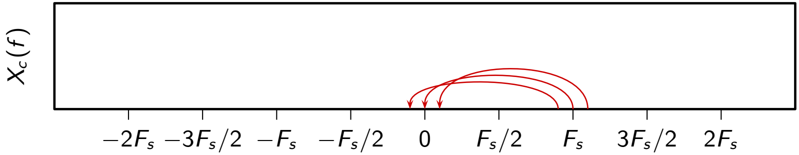

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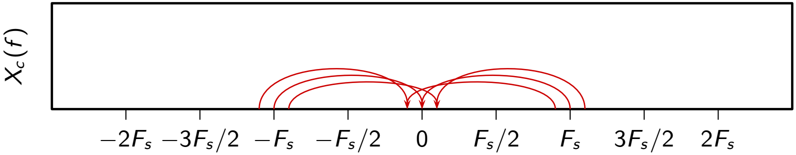

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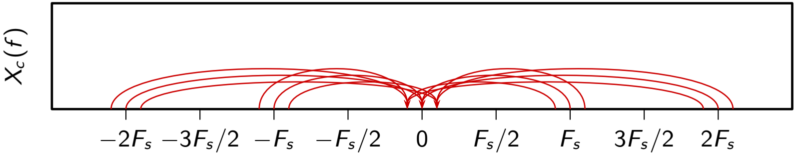


Spectrum of raw-sampled signals

$$x[n] = \sum_{k=-\infty}^{\infty} \int_{kF_s - \frac{F_s}{2}}^{kF_s + \frac{F_s}{2}} X_c(f) e^{j2\pi f\, T_s n} df$$

operate the change of variable $f \to f + kF_s$:

- integration limits become $\pm F_s/2$
- $e^{j2\pi(f - kF_s)T_s n} = e^{j2\pi f\, T_s n}$

Spectrum of raw-sampled signals

$$x[n] = \sum_{k=-\infty}^{\infty} \int_{kF_s-\frac{F_s}{2}}^{kF_s+\frac{F_s}{2}} X_c(f) e^{j2\pi f T_s n} df$$

$$= \sum_{k=-\infty}^{\infty} \int_{-F_s/2}^{F_s/2} X_c(f - kF_s) e^{j2\pi f T_s n} df$$

$$= \int_{-F_s/2}^{F_s/2} \left[\sum_{k=-\infty}^{\infty} X_c(f - kF_s) \right] e^{j2\pi f T_s n} df$$

Spectrum of raw-sampled signals

F_s-periodization of the spectrum; define:

$$\tilde{X}_c(f) = \sum_{k=-\infty}^{\infty} X_c(f - kF_s)$$

then:

$$x[n] = \int_{-F_s/2}^{F_s/2} \tilde{X}_c(f) e^{j2\pi f\, T_s n} df$$

Spectrum of raw-sampled signals

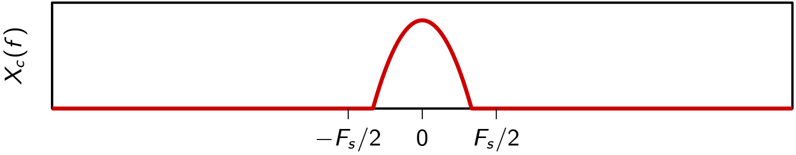
set $\omega = 2\pi f\, T_s$, so that $f = \frac{\omega}{2\pi} F_s$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_s\, \tilde{X}_c\left(\frac{\omega}{2\pi} F_s\right) e^{j\omega n} d\omega$$

$$= \text{IDTFT}\left\{F_s \tilde{X}_c\left(\frac{\omega}{2\pi} F_s\right)\right\}$$

$$X(e^{j\omega}) = F_s \tilde{X}_c\left(\frac{\omega}{2\pi} F_s\right) = F_s \sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{2\pi} F_s - kF_s\right)$$

Example: signal bandlimited to f_0 and $F_s > 2f_0$



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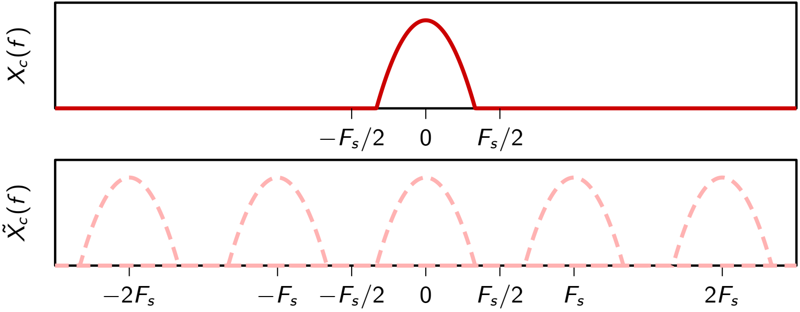

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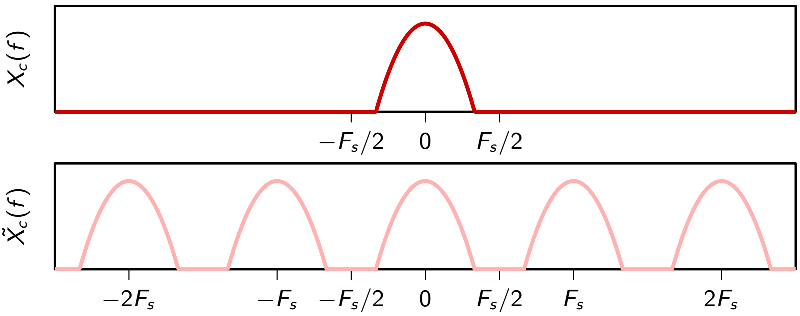

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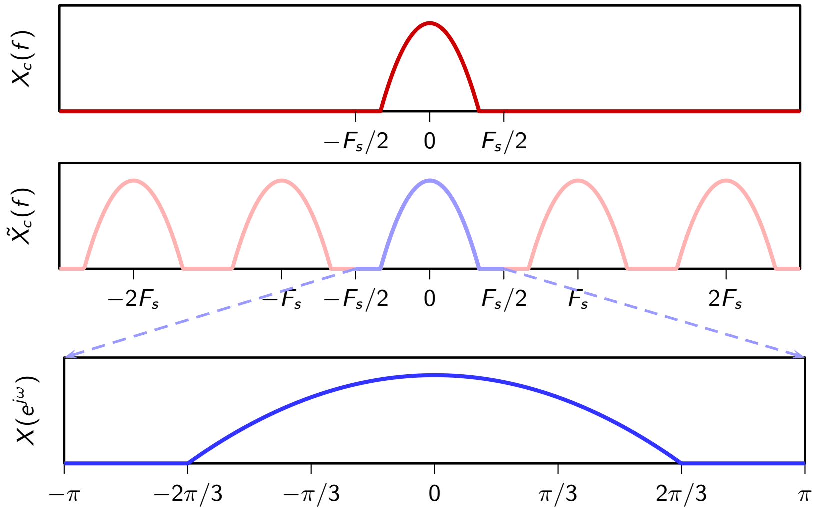

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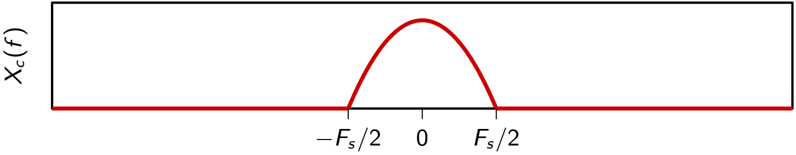

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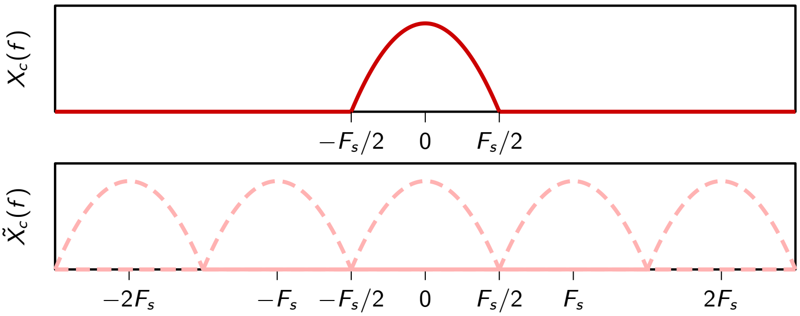

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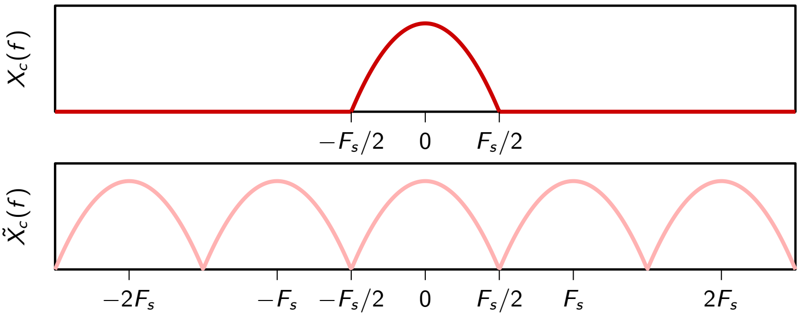

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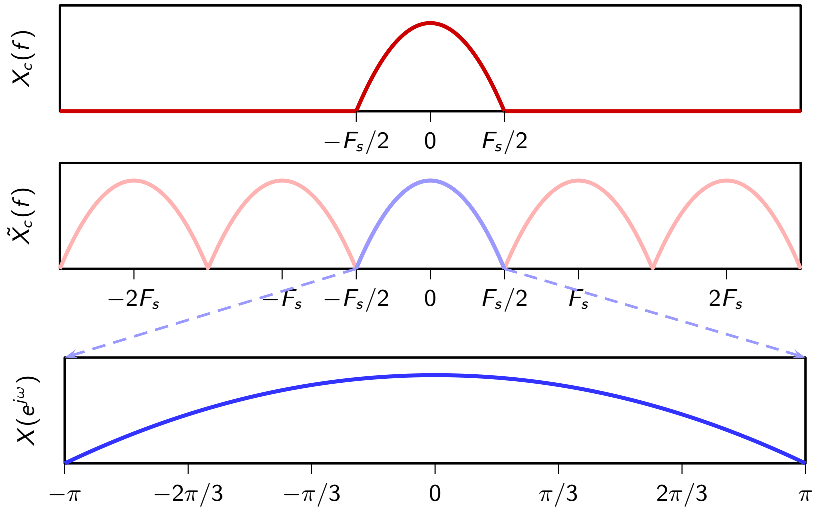

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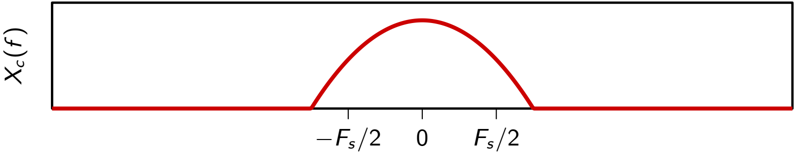

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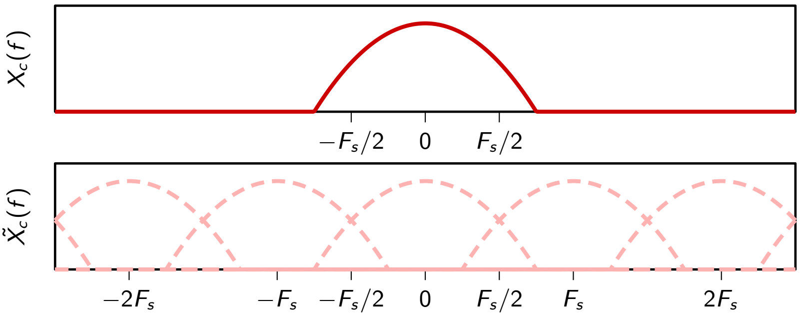

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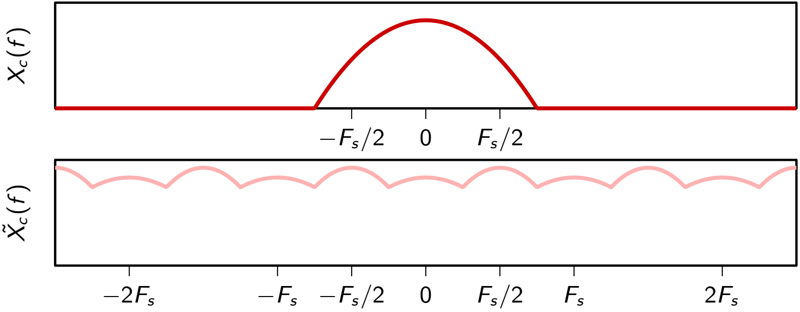

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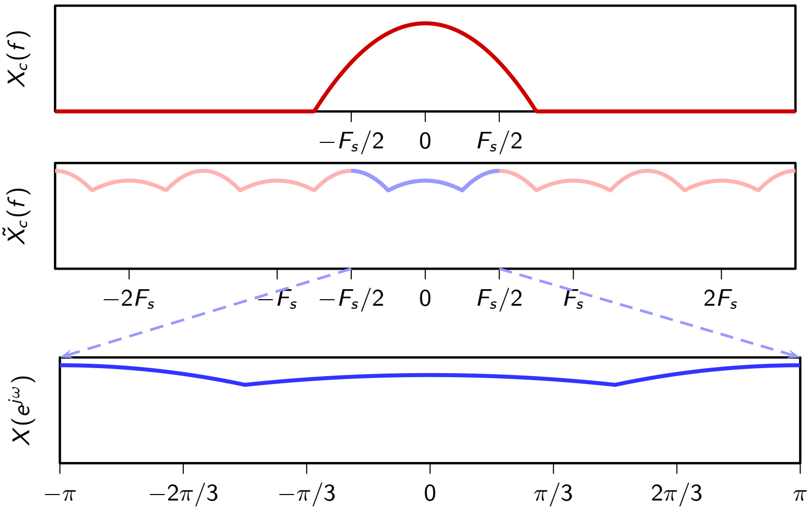
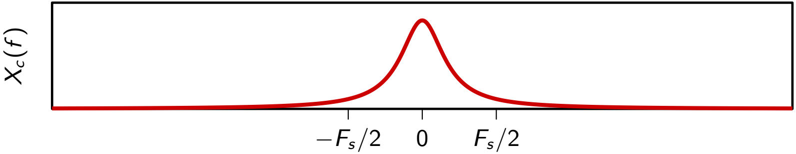


Example: non-bandlimited signal



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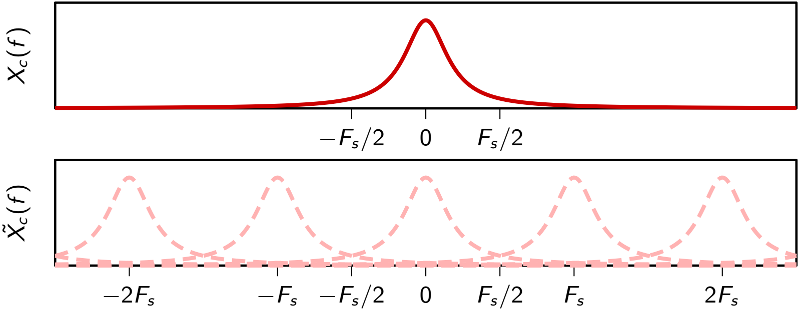

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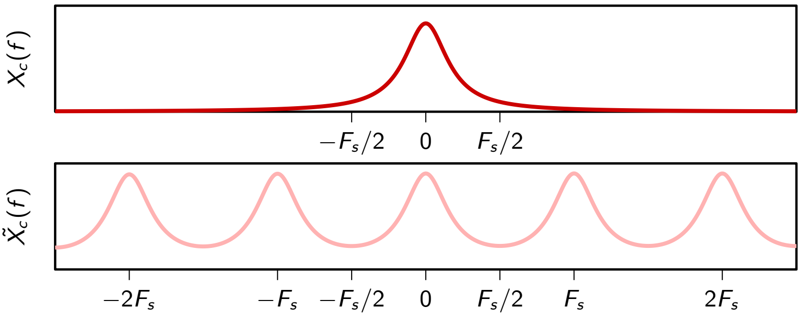

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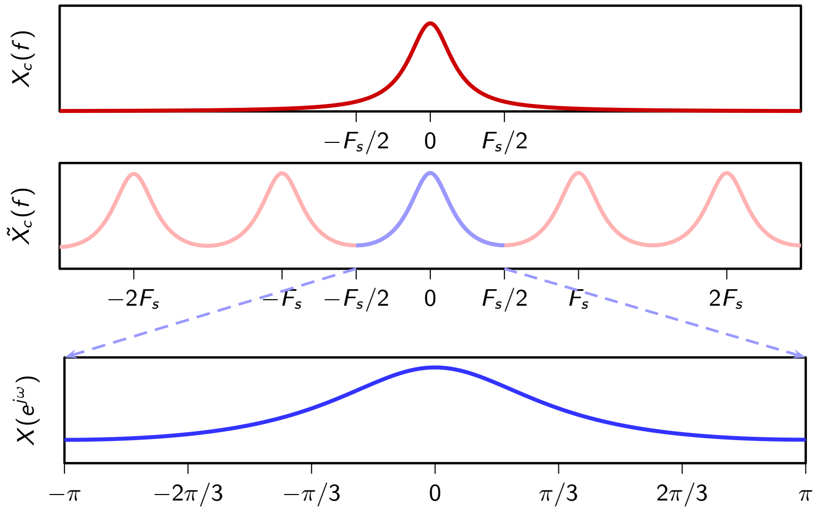

Sampling strategies

given a sampling frequency F_s

- if the signal is bandlimited to $F_s/2$ or less, raw sampling is fine (i.e. equivalent to sinc sampling up to a scaling factor T_s)
- if the signal is not bandlimited, two choices:
  - bandlimit via a lowpass filter *in the continuous-time domain* before sampling (i.e. sinc sampling)
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- aliasing introduces errors we cannot control, so the sensible choice is to bandlimit in continuous time
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Sinc Sampling and Interpolation

$$\hat{x}[n] = \left\langle \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right), x_c(t) \right\rangle \;=\; (\operatorname{sinc}_{T_s} * x_c)(nT_s)$$

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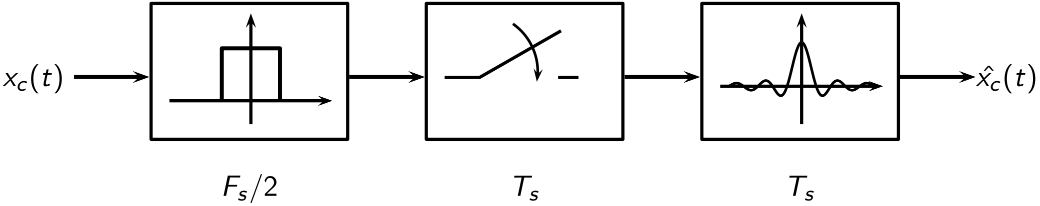
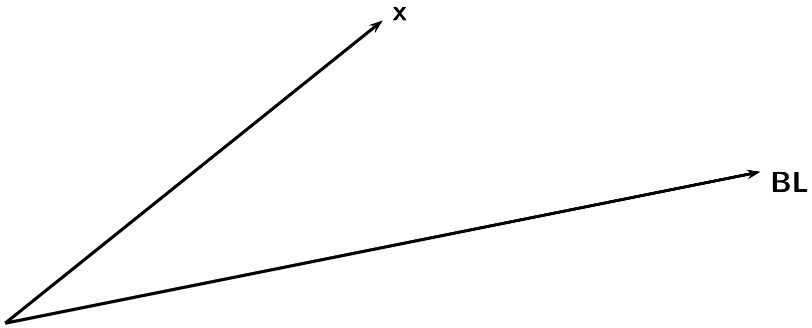


Least squares approximation with sinc sampling and interpolation



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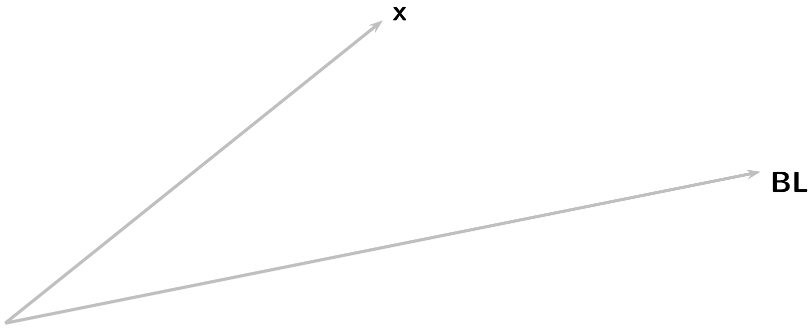

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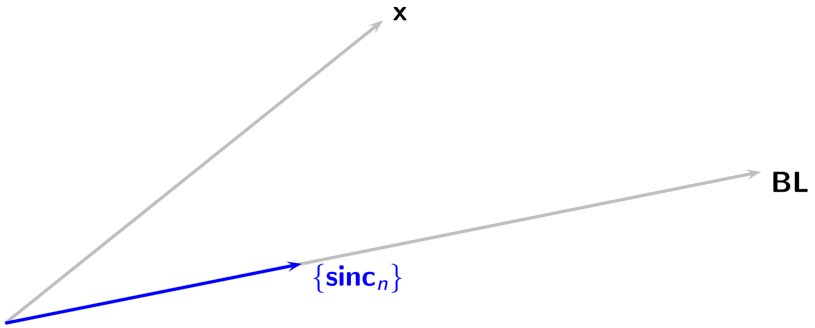

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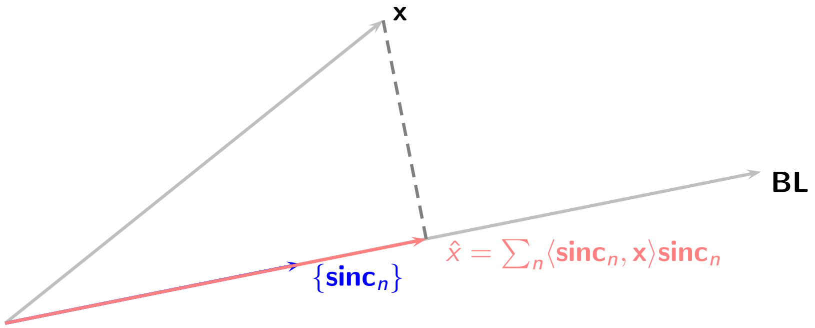

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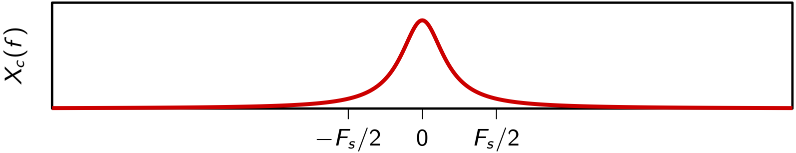

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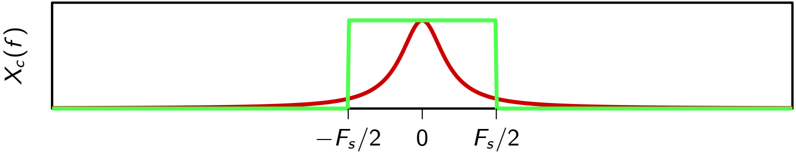

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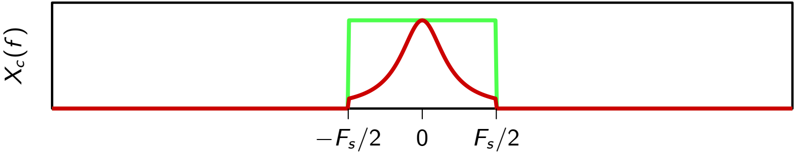

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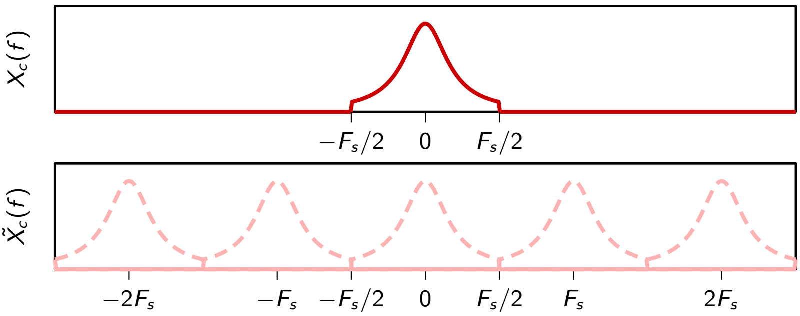

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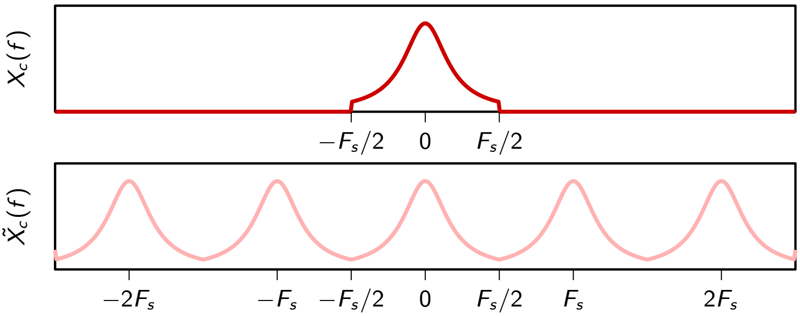

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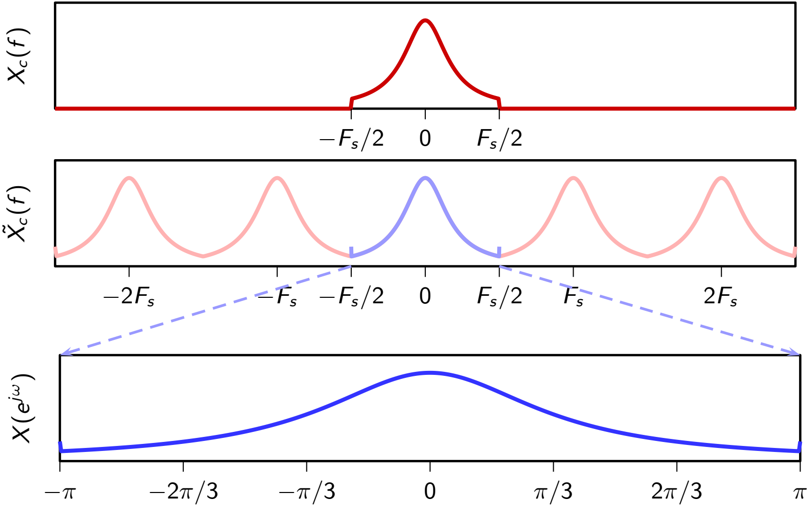

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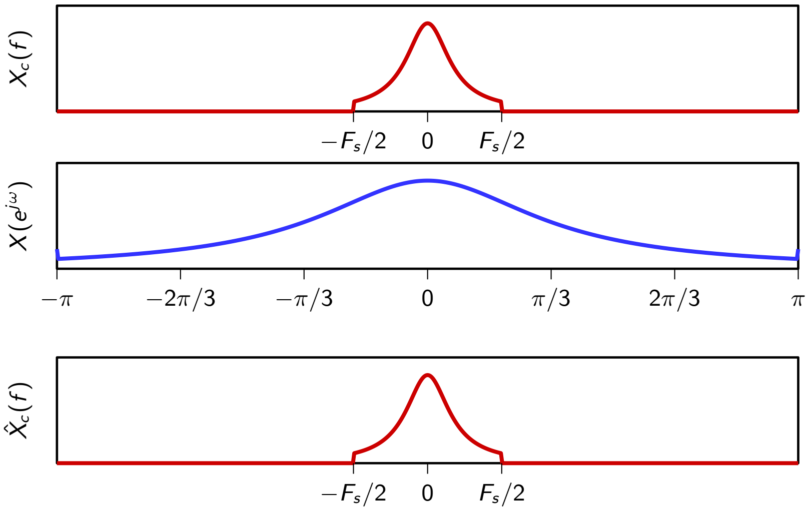

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$$X(e^{j\omega}) = X_c\left(\frac{\omega}{2\pi} F_s\right) \quad \text{prolonged by periodicity outside of } [-\pi, \pi]$$

Alternatively:

$$X(e^{j\omega}) = X_c\left(\left(\frac{\omega}{\pi} - 2\left\lfloor \frac{\omega - \pi}{2\pi} \right\rfloor - 2\right) F_s\right)$$

processing of analog signals

Overview:

- Impulse invariance
- Duality
- Examples

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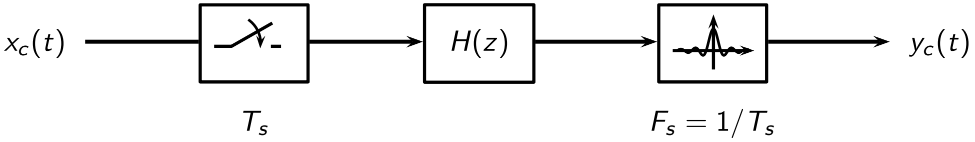
Basic setup



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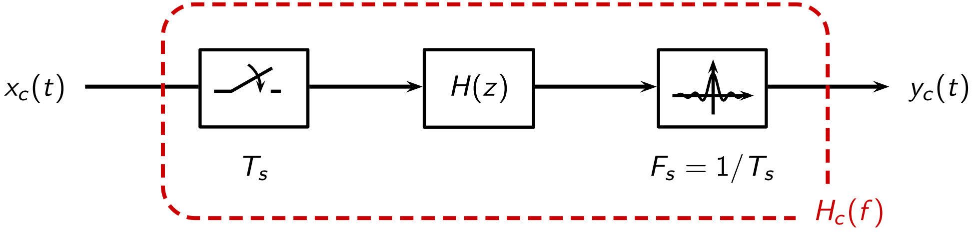

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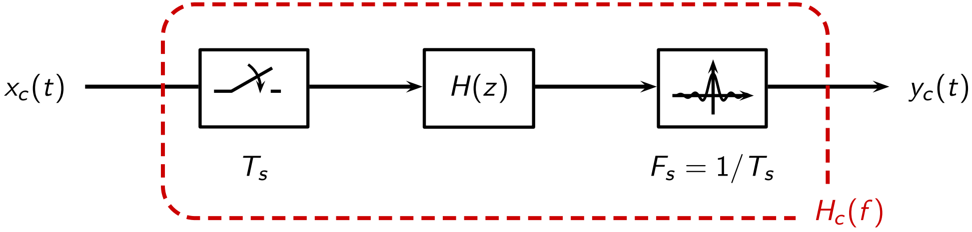
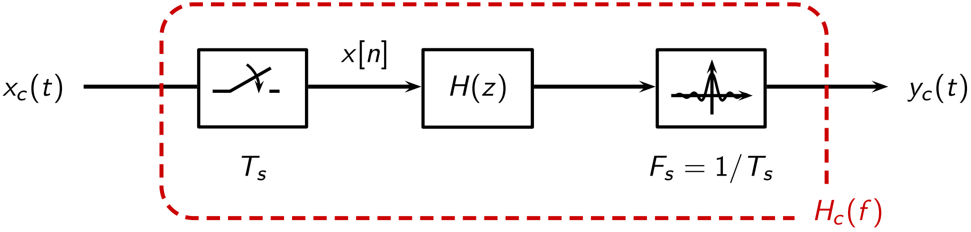


assume $x_c(t)$ is F_s-bandlimited and $T_s = 1/F_s$:

Basic setup



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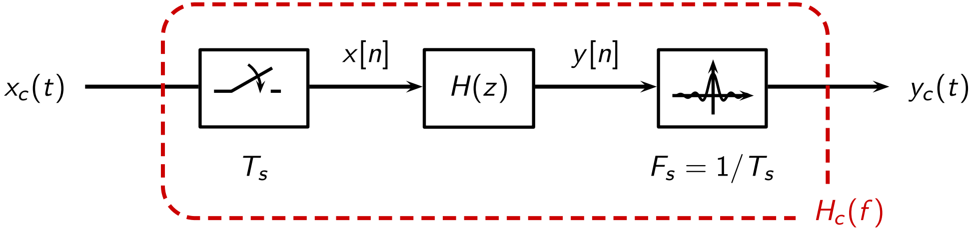

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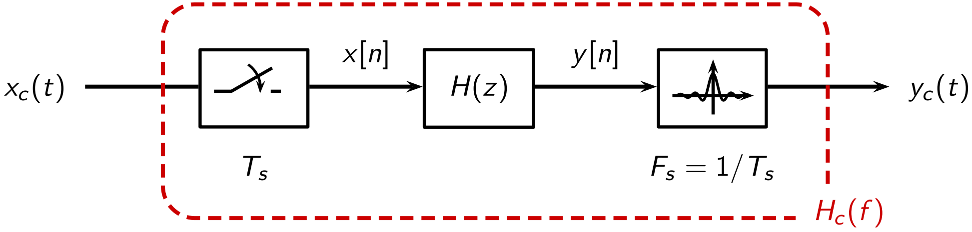

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- $Y_c(f) = (1/F_s)\, Y(e^{j2\pi f/F_s})$

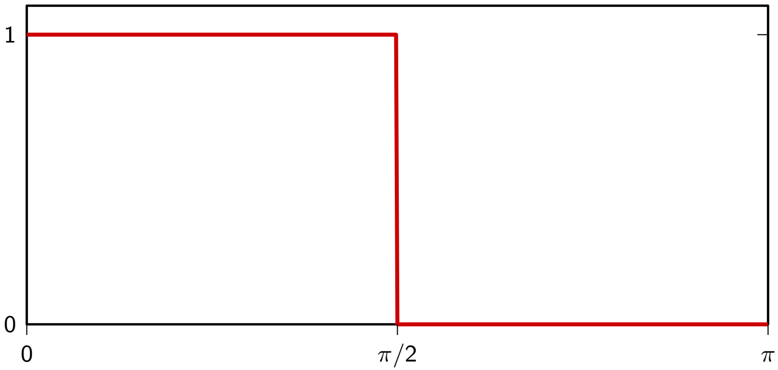
Impulse invariance

$$Y_c(f) = X_c(f)\, H(e^{j2\pi f/F_s})$$

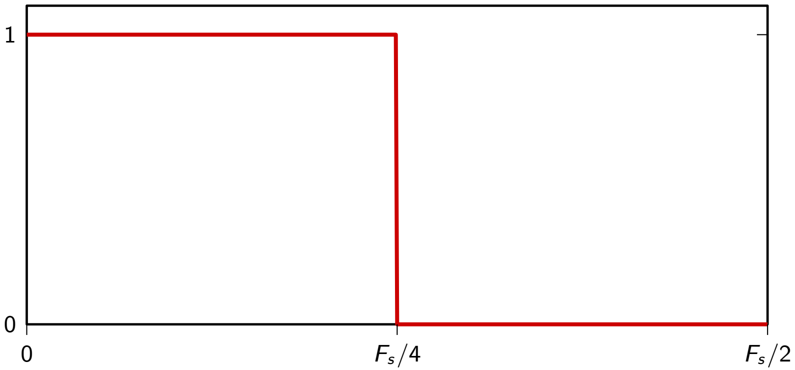
Impulse invariance

$$H_c(f) = H(e^{j2\pi f/F_s})$$

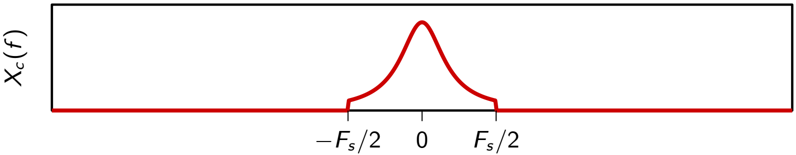
Impulse invariance



Impulse invariance



DT processing of CT signals



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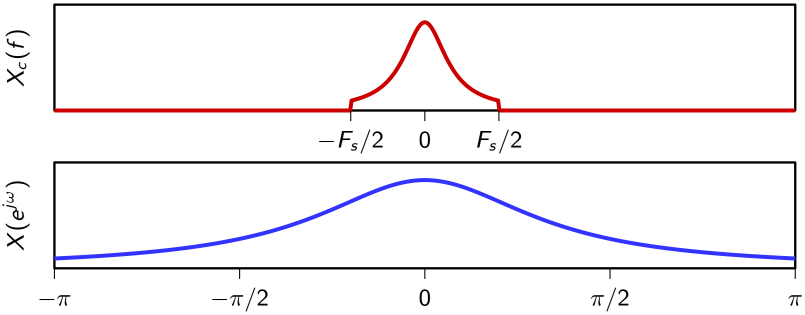

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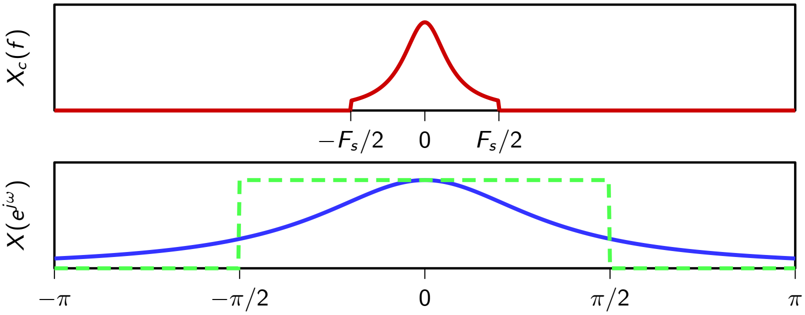

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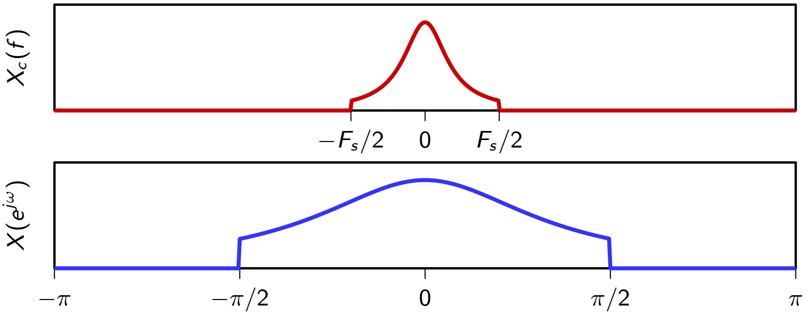

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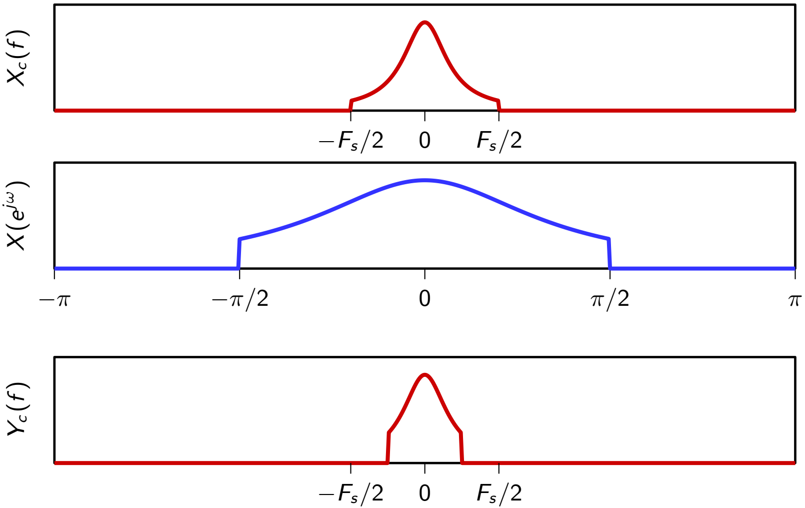

Example

quickly design a discrete-time filter to isolate a band of frequencies between 4000 and 5000Hz; input signals are bandlimited to 7KHz.

Example

- 7KHz band limit $\Rightarrow$ we can use any sampling frequency above 14KHz
- pick $F_s = 16KHz$
- we need a bandpass with a 1000Hz bandwidth
- start with some lowpass with cutoff $f_c = 500$Hz
- modulate it to center it on $f_0 = 4500$Hz

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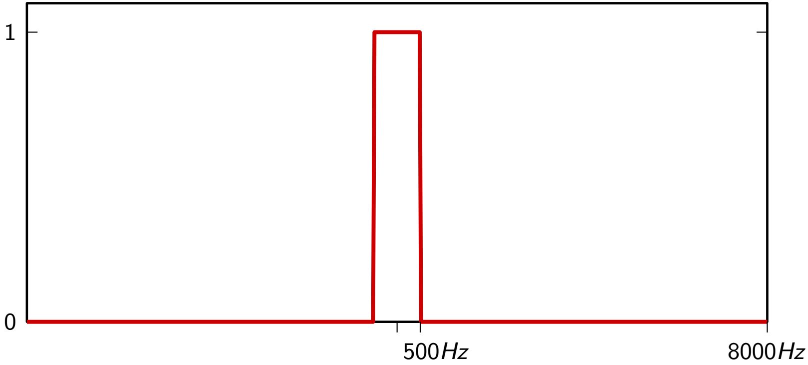

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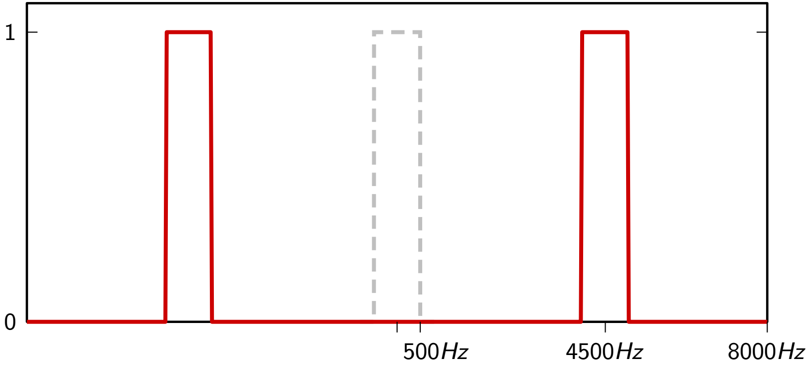

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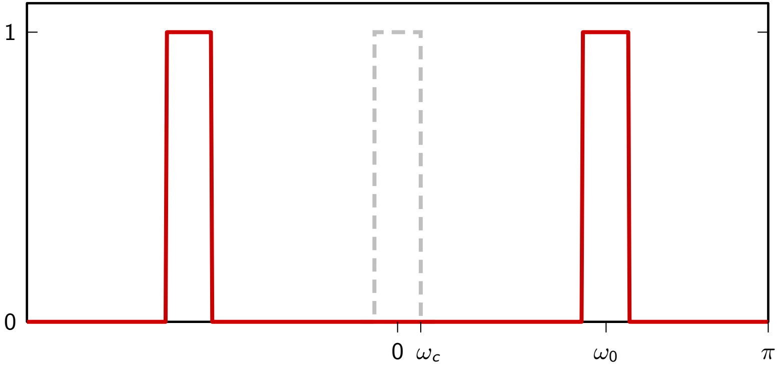

Example

- $\omega_c = 2\pi \frac{f_c}{F_s} = 2\pi \frac{500}{16000} = 0.0625\pi$
- $\omega_0 = 2\pi \frac{4500}{16000} = 0.5625\pi$
- modulate the impulse response of an ideal lowpass with cutoss ω_c by $2\cos\omega_0 n$
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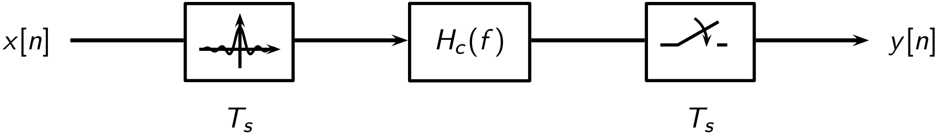
Duality



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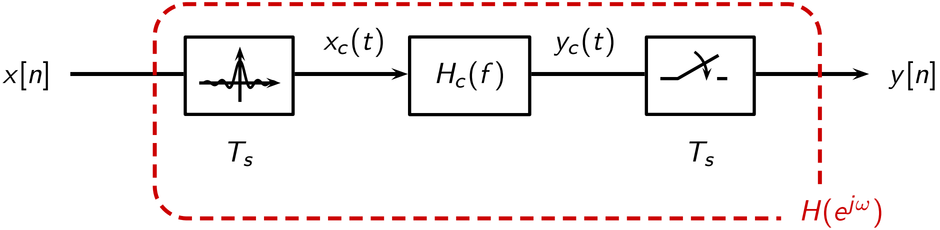

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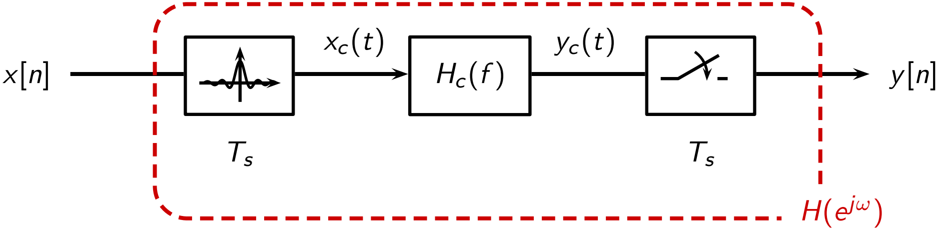
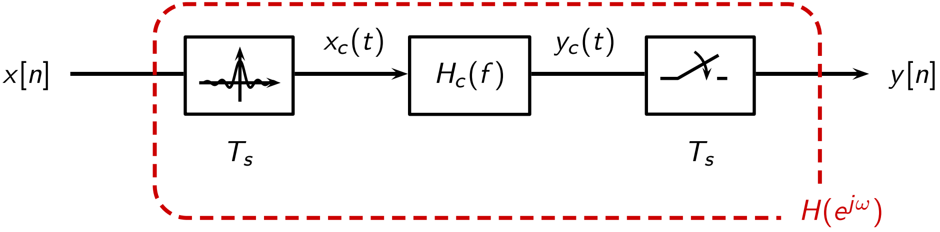


we can pick any T_s so pick $T_s = 1$:

Duality



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- $X_c(f) = X(e^{j2\pi f})$

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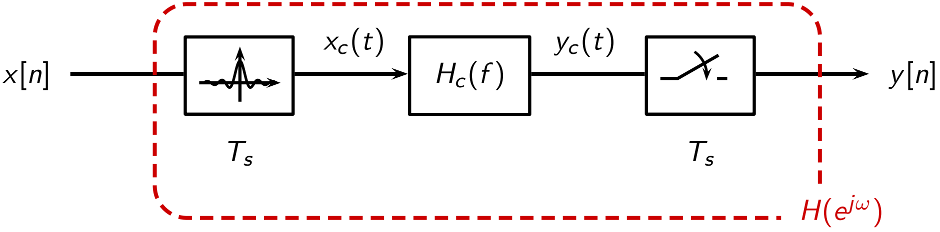

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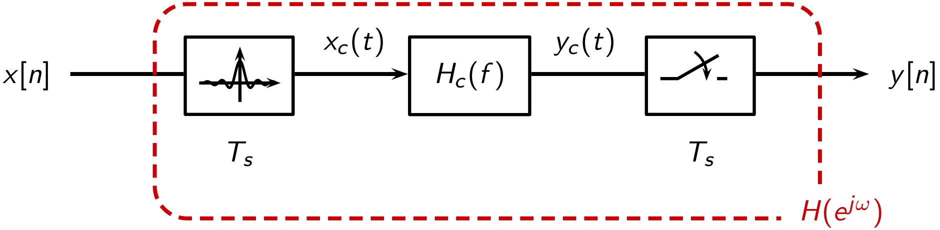


we can pick any T_s so pick $T_s = 1$:

- $X_c(f) = X(e^{j2\pi f})$
- $Y_c(f) = X_c(f)H_c(f)$
- LTI systems cannot change the bandwidth $\Rightarrow Y(e^{j\omega}) = Y_c(\frac{\omega}{2\pi})$

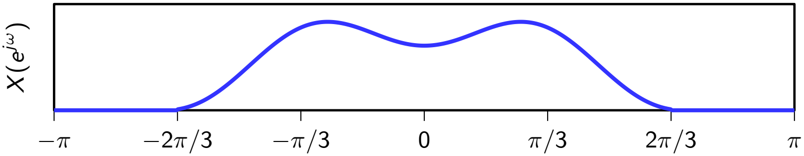
Duality

$$Y(e^{j\omega}) = X(e^{j\omega})\, H_c\left(\frac{\omega}{2\pi}\right)$$

Duality

$$H(e^{j\omega}) = H_c\left(\frac{\omega}{2\pi}\right)$$

CT processing of DT signals



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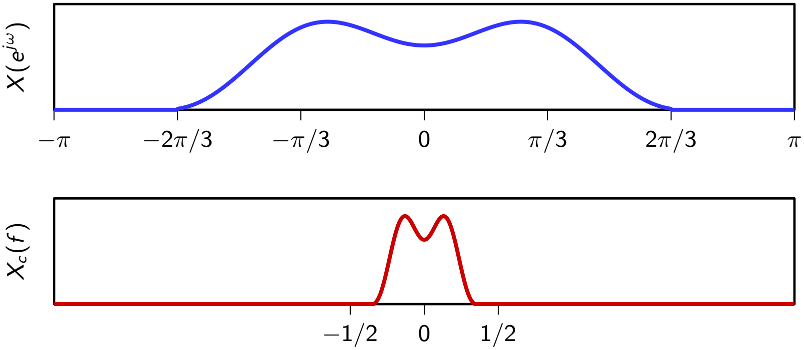

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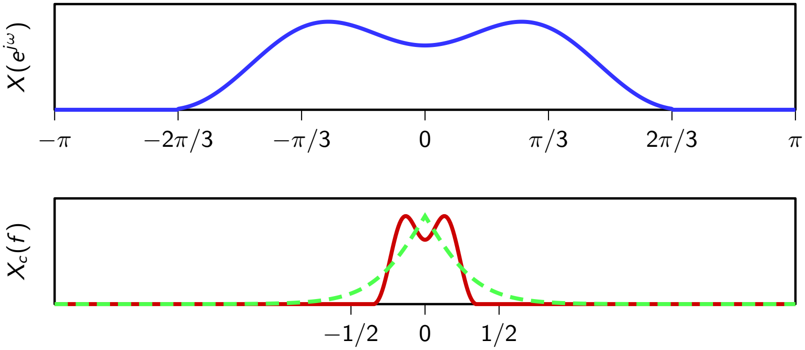

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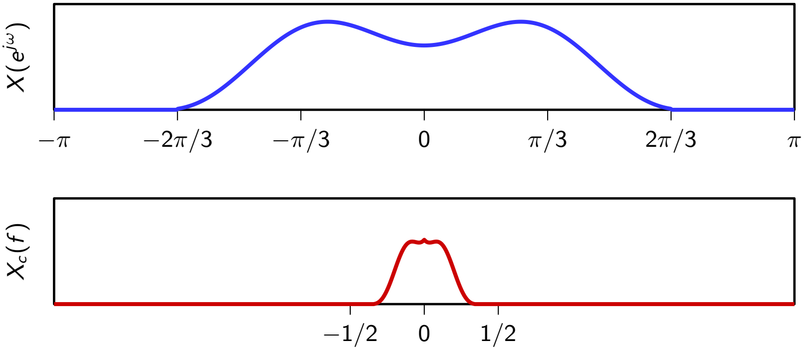

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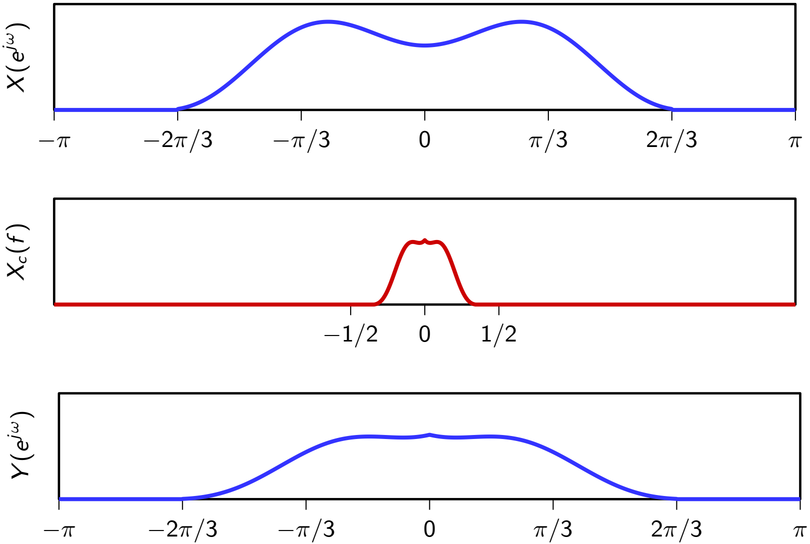
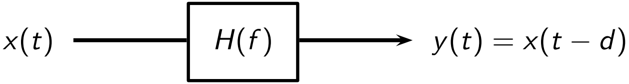


Delays in continuous time

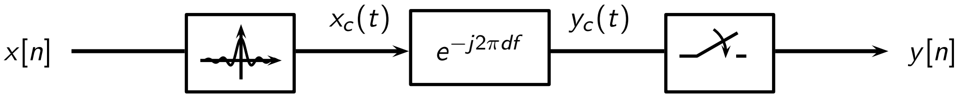


Delays in continuous time

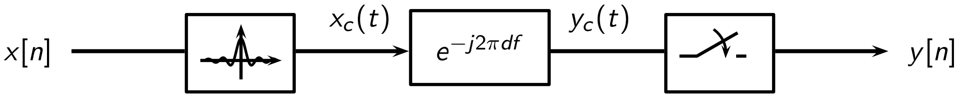
$$x(t) \longrightarrow \boxed{H(f)} \longrightarrow y(t) = x(t-d)$$

- in continuous time, delays are well defined for all $d \in \mathbb{R}$
- $H(f) = e^{-j2\pi df}$

Interpretation of fractional delay by duality

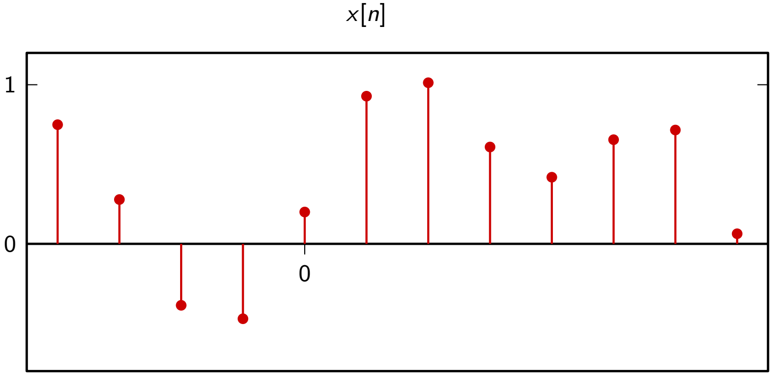


Interpretation of fractional delay by duality



- chain interpolates $x[n]$, delays the result by d and resamples
- equivalent filter $H(e^{j\omega}) = H_c(\omega/(2\pi)) = e^{-j\omega d}$
- that's how a discrete-time fractional delay works internally

Example: fractional delay



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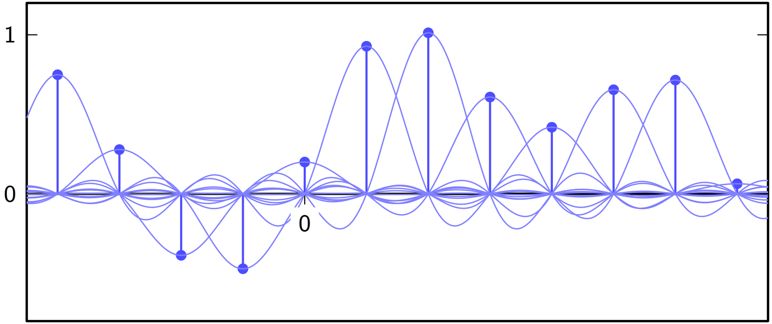

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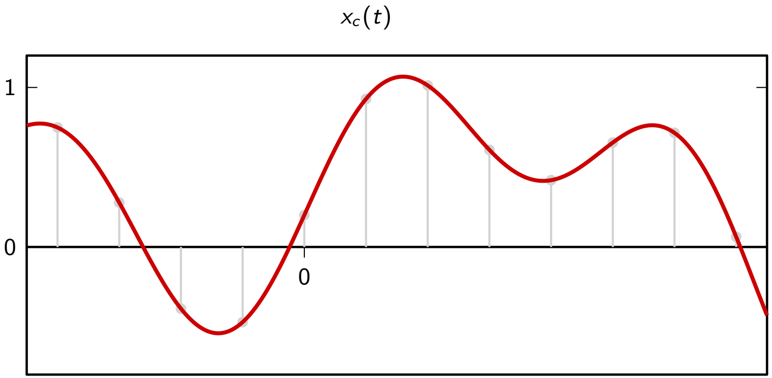

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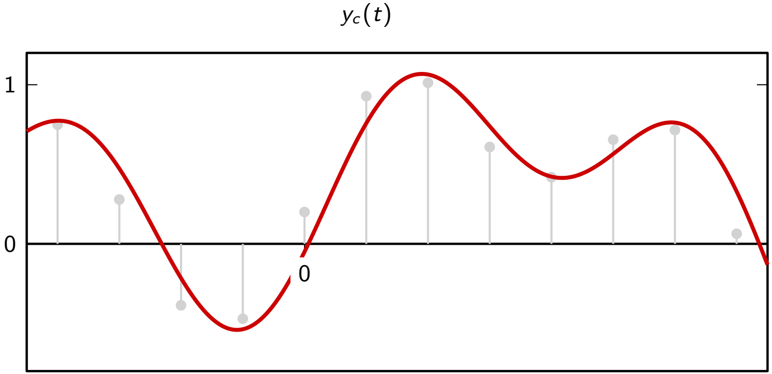

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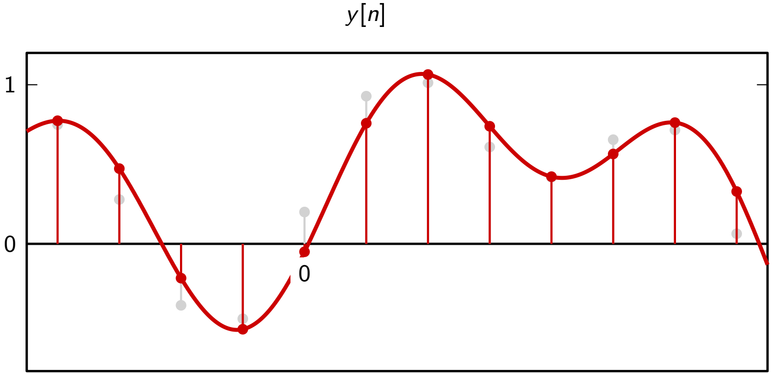

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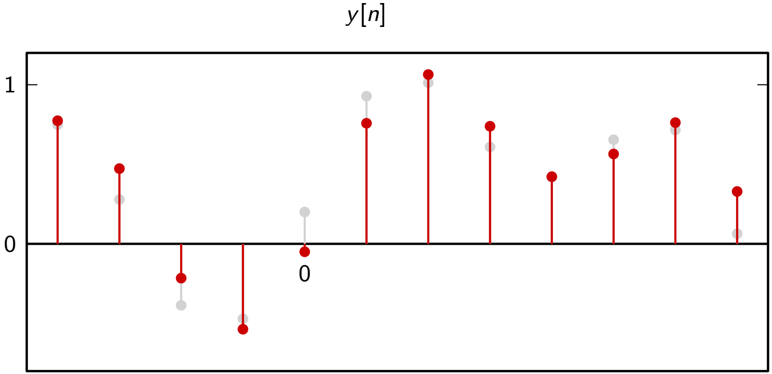
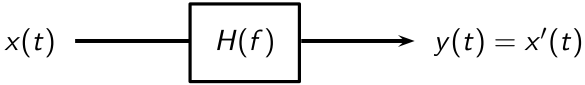


Example: fractional delay

- $h[n] = \text{sinc}(n - d)$
- to delay a discrete-time signal by a fraction of a sample we need an ideal filter!
- efficient approximations exist (e.g. cubic local interpolation)

Differentiation in continuous time

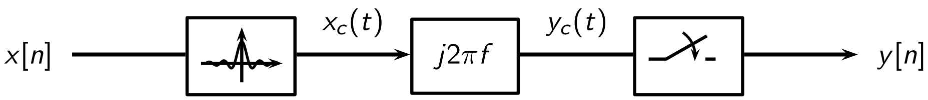


Differentiation in continuous time

$$x(t) \longrightarrow \boxed{H(f)} \longrightarrow y(t) = x'(t)$$

- easy to show that FT $\{x'(t)\} = j2\pi f\, X(f)$
- $H(f) = j2\pi f$

By duality



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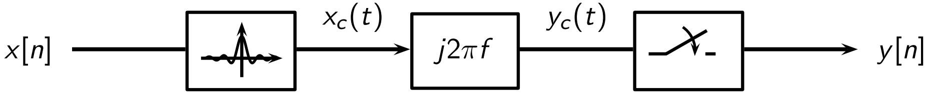
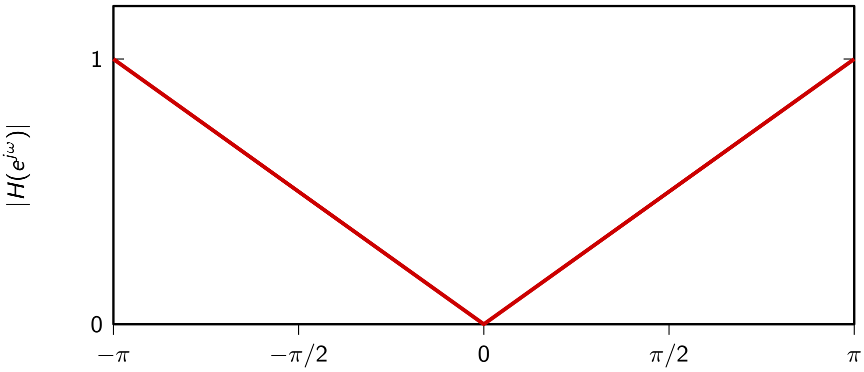


- chain interpolates $x[n]$, differentiates the result by d and resamples
- equivalent filter $H(e^{j\omega}) = H_c(\omega/(2\pi)) = j\omega$
- equivalent filter defines a digital differentiator

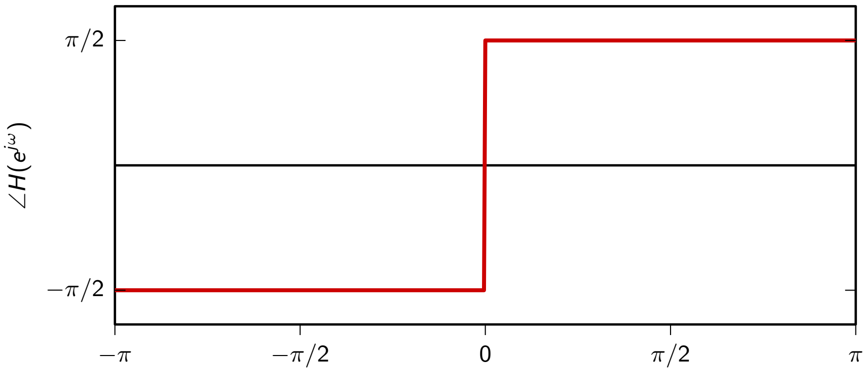
Digital differentiator, magnitude response

$$H(e^{j\omega}) = j\omega$$



Digital differentiator, phase response

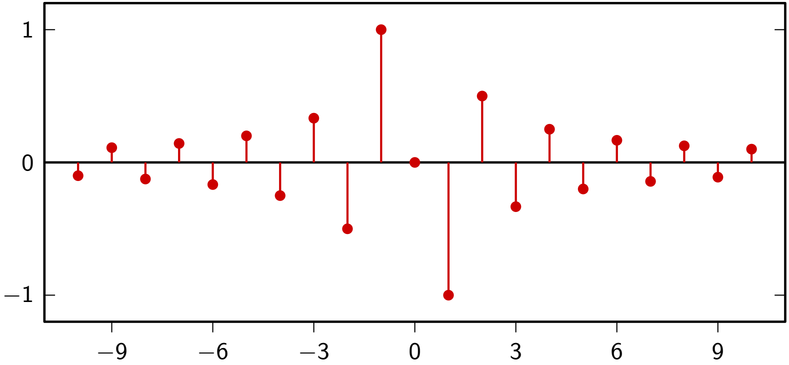
$$H(e^{j\omega}) = j\omega$$



Digital differentiator, impulse response

$$
\begin{aligned}
h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} j\omega e^{j\omega n}\, d\omega \\
&= \ldots \textit{(integration by parts)} \ldots \\
&= \begin{cases} 0 & n = 0 \\ \dfrac{(-1)^n}{n} & n \neq 0 \end{cases}
\end{aligned}
$$

Digital differentiator, impulse response



Digital differentiator

- the digital differentiator is again an ideal filter!
- many approximations exist, with different properties

Wrap up

- Continuous-time processing of discrete-time sequences
- Discrete-time processing of continuous-time signals
- Jumping back and forth using sampling and interpolation
- In practice: Many applications of processing continuous-time signals in discrete time!