COM303: Digital Signal Processing

Lecture 13: Optimal FIR Filter design

Overview

- linear phase FIR
- the Parks-McClellan algorithm

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- linear phase FIR
- the Parks-McClellan algorithm

FIR vs IIR

IIRs: pros and cons

Pros:

- computationally efficient
- strong attenuation easy
- "natural sounding" in audio applications

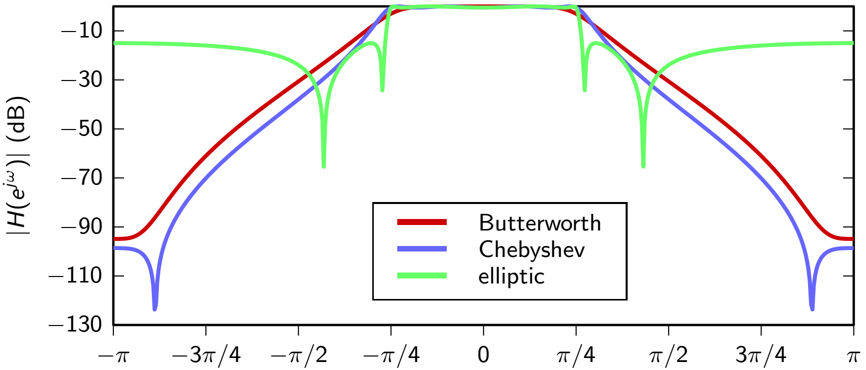
Cons:

- stability and numerical precision issues
- difficult to design for arbitrary response
- nonlinear phase

IIR design method

- based on analog filter design
- ready-made numerical tools (e.g. `b,a = sp.cheby1(4, .12, 0.25)`)
- play with order to meet the specs
- standard families: Butterworth, Chebyshev, Elliptic

4-th order IIR lowpass comparison



all filters require 9 multiplications per output sample

FIRs: pros and cons

Pros:

- always stable
- numerically precise implementations
- can be designed with linear phase
- optimal design techniques exist

Cons:

- computationally much more expensive
- because of length, significant delay (hard to use in live audio)

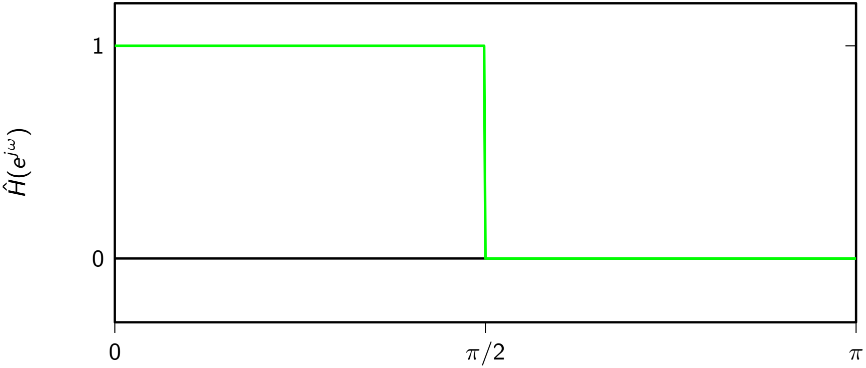
FIR design

Previous FIR designs

- impulse truncation, window method
- frequency sampling

design could not control the maximum error

The Gibbs phenomenon



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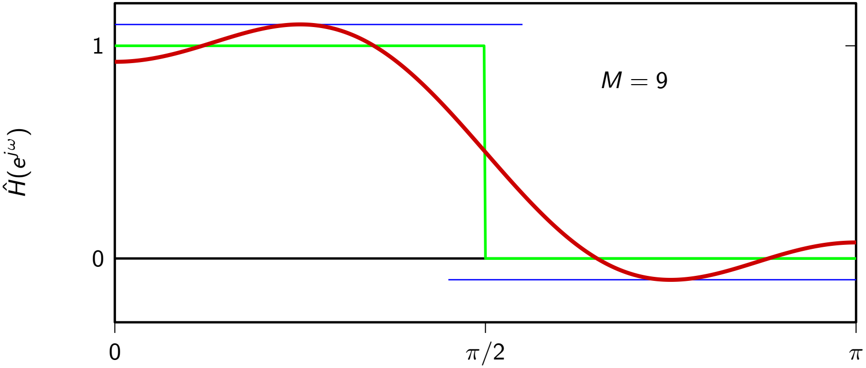

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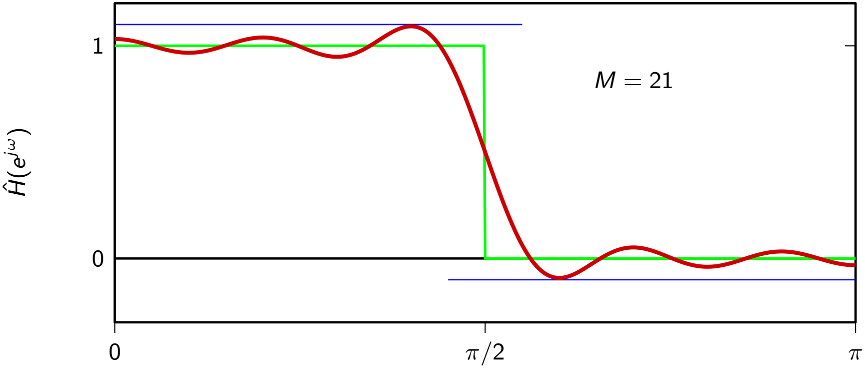

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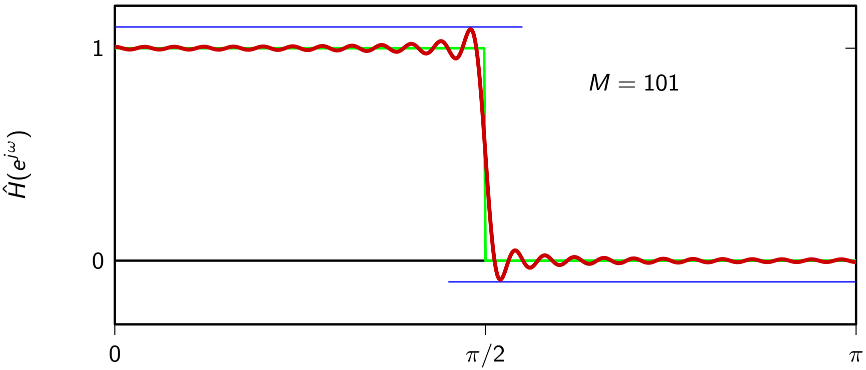

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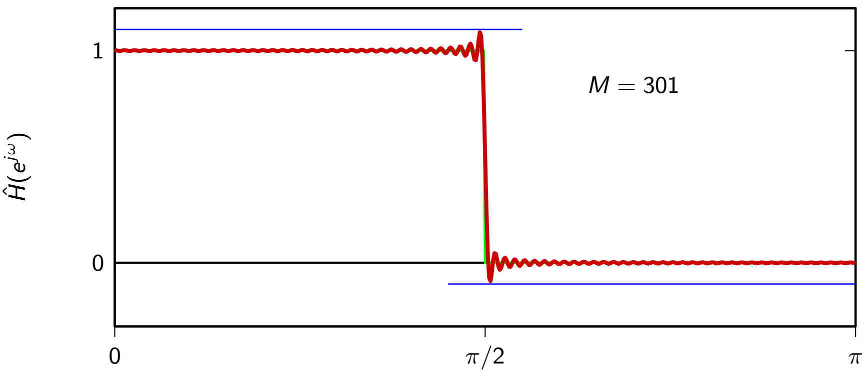

FIR: optimal minimax design

FIR filters are a digital signal processing "exclusivity".
In the 1970s Parks and McClellan developed an algorithm to design optimal FIR filters:

- linear phase
- equiripple error in passband and stopband

algorithm proceeds by **mini**mizing the **max**imum error in passband and stopband

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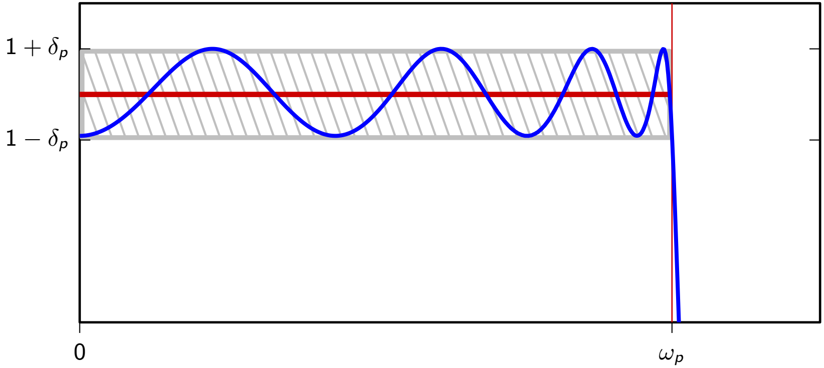
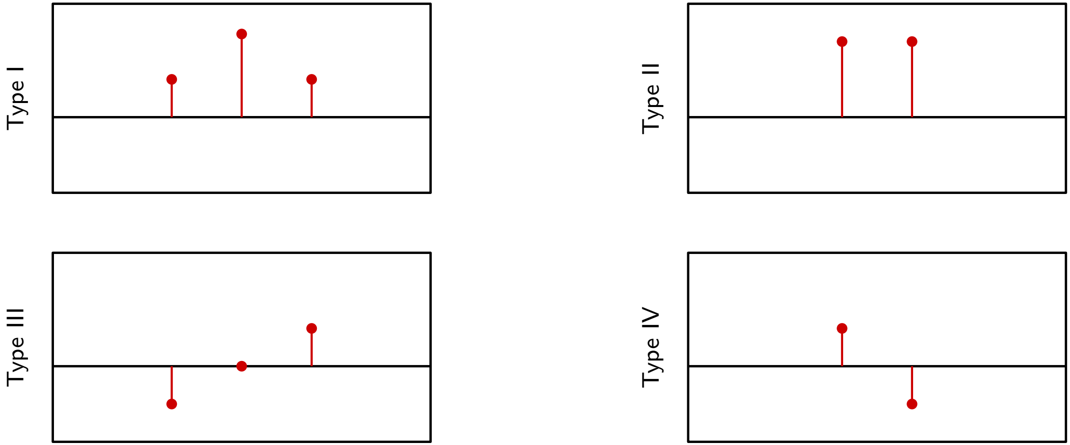
Optimal FIR will have equiripple error



Linear phase in FIRs

Symmetric or antisymmetric impulse responses have linear phase



Linear phase (Type I)

filter length is **odd**: $M = 2L + 1$

$$h[L + n] = h[L - n]$$

zero-centered filter:

$$h_d[n] = h[n + L]$$

$$h_d[n] = h_d[-n]$$

Linear phase (Type I)

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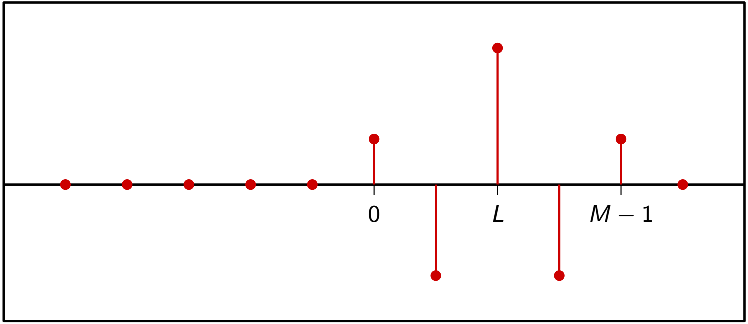
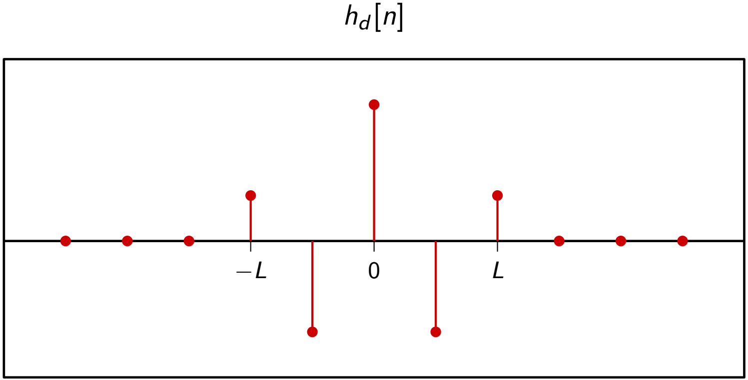
Linear phase (Type I)

$h[n]$



Linear phase (Type I)



Linear phase (Type I)

$$H_d(z) = \sum_{n=-L}^{L} h_d[n] z^{-n}$$

$$= h_d[0] + \sum_{n=1}^{L} h_d[n](z^n + z^{-n})$$

$$H_d(e^{j\omega}) = h_d[0] + \sum_{n=1}^{L} h_d[n](e^{j\omega n} + e^{-j\omega n})$$

$$= h_d[0] + 2\sum_{n=1}^{L} h_d[n] \cos \omega n \qquad \in \mathbb{R}$$

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\end{aligned}
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Linear phase (Type I)

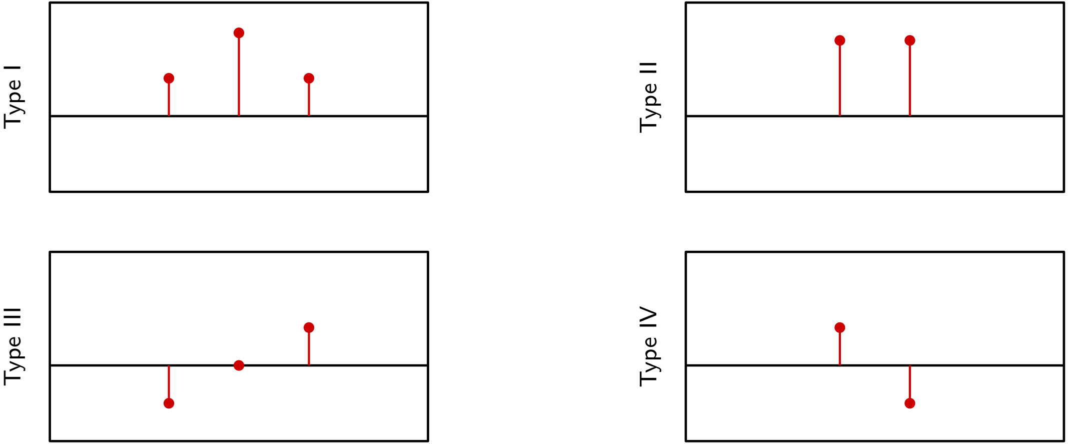
$$H(z) = z^{-L} H_d(z)$$

$$H(e^{j\omega}) = \left[h[L] + 2 \sum_{n=1}^{L} h[n+L] \cos n\omega \right] e^{-j\omega L}$$

Linear Phase FIR Filters

- L: number of points with a "companion"
- even-length FIRs: $M = 2L$ taps
- odd-length FIRs: $M = 2L + 1$ taps
- delay equal to half-length: $C = (M-1)/2$
- delay is non-integer for even-length filters!

FIR types ($L = 1$)



$M = 2L + 1 = 3$

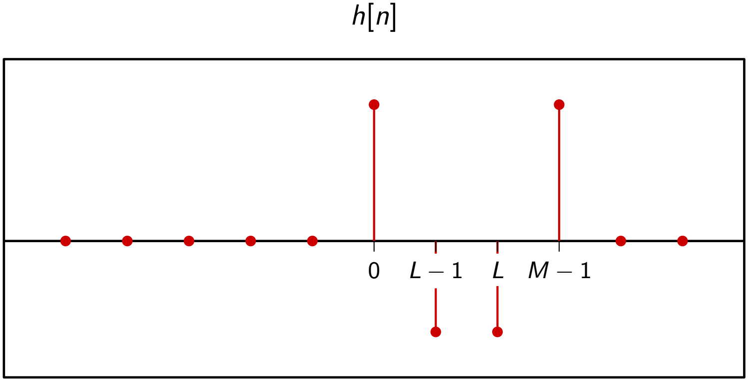
$M = 2L = 2$

Linear phase (Type II)

filter length is **even**: $M = 2L$

$$h[n] = h[2L - 1 - n]$$

Linear phase (Type II)



Linear phase (Type II)

$$
\begin{aligned}
H(z) = & h[0] & + h[1]z^{-1} & + \dots & + h[L-1]z^{-L+1} + \\
& h[2L-1]z^{-2L+1} & + h[2L-2]z^{-2L+2} & + \dots & + h[L]z^{-L} \\
= & h[0] & + h[1]z^{-1} & + \dots & + h[L-1]z^{-L+1} + \\
& h[0]z^{-2L+1} & + h[1]z^{-2L+2} & + \dots & + h[L-1]z^{-L} \\
= & \sum_{n=0}^{L-1} h[n](z^{-n} + z^{-2L+1+n}) & & &
\end{aligned}
$$

Linear phase (Type II)

$$
\begin{aligned}
H(z) =& h[0] && + h[1]z^{-1} && + \dots && + h[L-1]z^{-L+1} + \\
& h[2L-1]z^{-2L+1} && + h[2L-2]z^{-2L+2} && + \dots && + h[L]z^{-L} \\
=& h[0] && + h[1]z^{-1} && + \dots && + h[L-1]z^{-L+1} + \\
& h[0]z^{-2L+1} && + h[1]z^{-2L+2} && + \dots && + h[L-1]z^{-L} \\
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&h[0]z^{-2L+1} && + h[1]z^{-2L+2} && + \dots && + h[L-1]z^{-L} \\
= &\sum_{n=0}^{L-1} h[n](z^{-n} + z^{-2L+1+n})
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Linear phase (Type II)

$$C = (M-1)/2 = (2L-1)/2 = L - 1/2 \quad \text{(non-integer!)}$$

$$H(z) = \sum_{n=0}^{L-1} h[n](z^{-n} + z^{-2C+n})$$

$$= z^{-C} \sum_{n=0}^{L-1} h[n](z^{(C-n)} + z^{-(C-n)})$$

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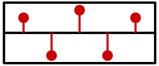
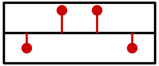
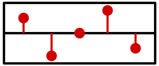
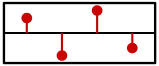
$$\begin{aligned} H(z) &= \sum_{n=0}^{L-1} h[n](z^{-n} + z^{-2C+n}) \\ &= z^{-C} \sum_{n=0}^{L-1} h[n](z^{(C-n)} + z^{-(C-n)}) \end{aligned}$$

Linear phase (Type II)

$$H(e^{j\omega}) = \left[2 \sum_{n=0}^{L-1} h[n] \cos(\omega(C - n)) \right] e^{-j\omega C}$$

$$C = L - \frac{1}{2}$$

Linear Phase FIR Filters

type	length	sym.	delay	zeros	
I	odd	S	integer		
II	even	S	non-int.		
III	odd	A	integer		
IV	even	A	non-int.		

Zero locations (all types)

- FIRs have only zeros
- $h[n] \in \mathbb{R} \Rightarrow$ if z_0 is a zero, so is z_0^*

Zero locations (Type I)

$$H(z) = z^{-L}\left[h[0] + \sum_{n=1}^{L} h[n](z^n + z^{-n})\right]$$

$$H(z^{-1}) = z^{L}\left[h[0] + \sum_{n=1}^{L} h[n](z^n + z^{-n})\right]$$

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if z_0 is a zero, so is $1/z_0$

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this is valid for all FIR types (easy to prove)

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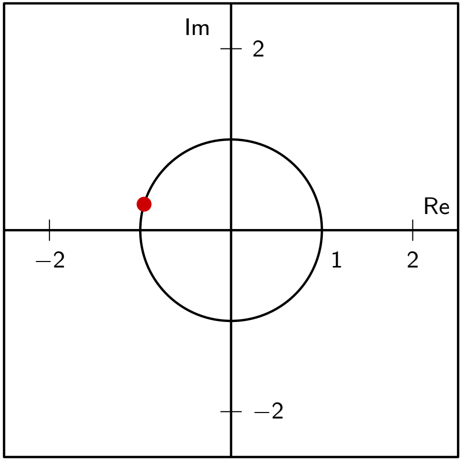
Zero locations (all types)

- if z_0 is a zero, so is z_0^*
- if z_0 is a zero, so is $1/z_0$
- if $z_0 = \rho e^{j\theta}$ is a zero so are:
  - $\rho e^{j\theta}$
  - $(1/\rho)e^{j\theta}$
  - $\rho e^{-j\theta}$
  - $(1/\rho)e^{-j\theta}$

Typical zero plot for linear-phase FIR



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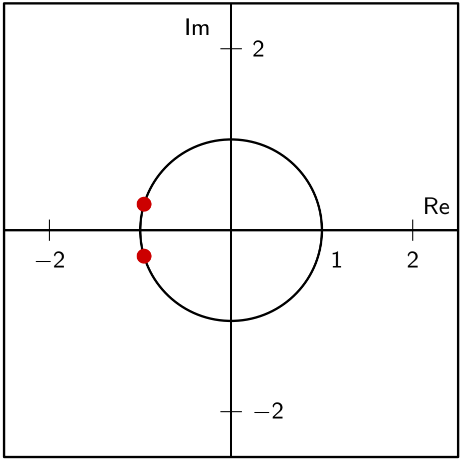

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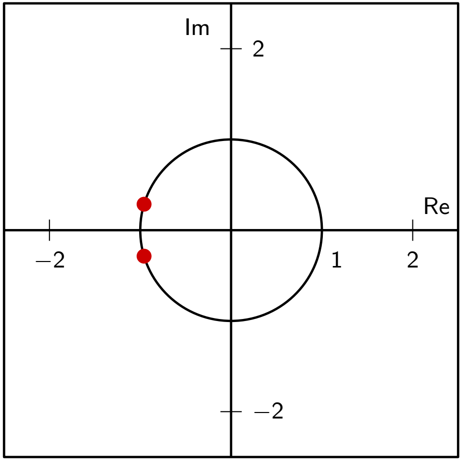

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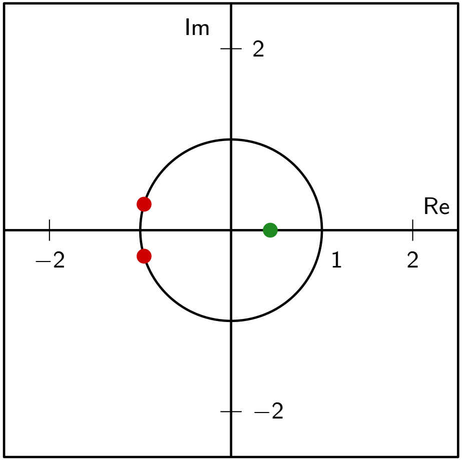

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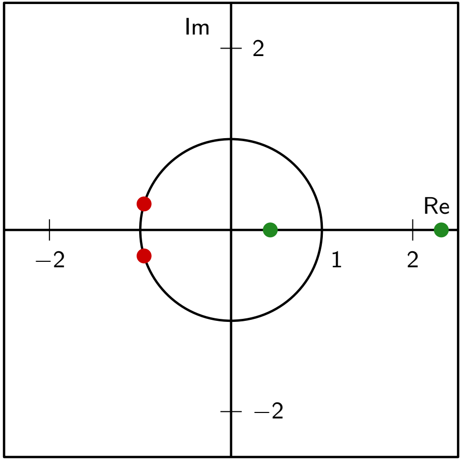

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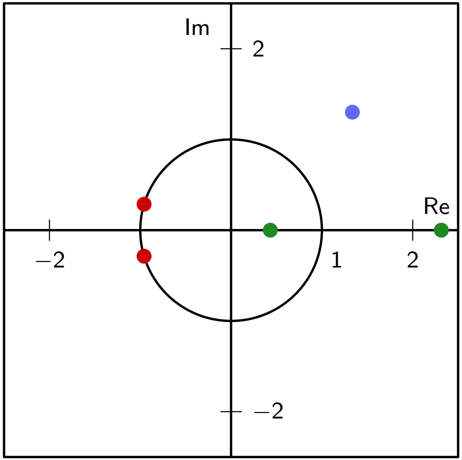

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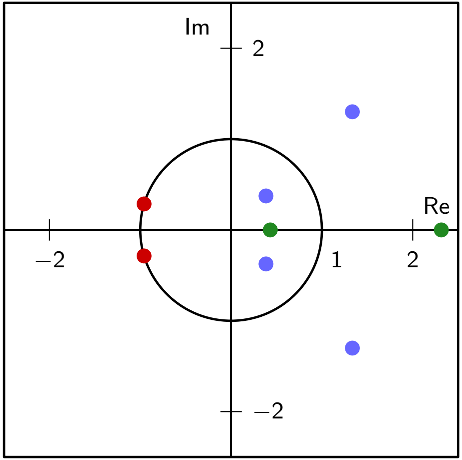

Zero locations (Type II)

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$$C = L - 1/2$$

$$\begin{aligned} H(z^{-1}) &= z^{2C} H(z) \\ &= z^{2L-1} H(z) \end{aligned}$$

$$H(-1) = (-1)^{2L-1} H(-1) = -H(-1)$$

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Zero locations (Type II)

type-II FIRs always have a zero at $\omega = \pi$

Zero locations (Type III)

$$H(z) = z^{-L} \left[\sum_{n=1}^{L} h[n](z^{n} - z^{-n}) \right]$$

$$H(z^{-1}) = z^{L} \left[\sum_{n=1}^{L} h[n](-z^{n} + z^{-n}) \right]$$

$$= -z^{2L} H(z)$$

$$H(1) = -H(1) \implies H(1) = 0$$

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Zero locations (Type III)

type-III FIRs always have a zero at $\omega = 0$ and $\omega = \pi$

Zero locations (Type IV)

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$$H(1) = -H(1) \implies H(1) = 0$$

Zero locations (Type IV)

$$C = L - 1/2$$

$$H(z) = z^{-C} \sum_{n=0}^{L-1} h[n](z^{(C-n)} - z^{-(C-n)})$$

$$H(z^{-1}) = z^{C} \left[\sum_{n=0}^{L-1} h[n](-z^{(C-n)} + z^{-(C-n)} \right]$$

$$= -z^{2C} H(z)$$

$$H(1) = -H(1) \implies H(1) = 0$$

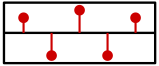
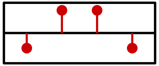
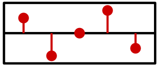
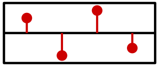
Zero locations (Type III)

type-IV FIRs always have a zero at $\omega = 0$

Zero locations

Filter Type	**Relation**	**Constraint on Zeros**
Type I	$H(z^{-1}) = z^{M-1}H(z)$	No constraints
Type II	$H(z^{-1}) = z^{M-1}H(z)$	Zero at $z = -1$ (i.e. $\omega = \pi$)
Type III	$H(z^{-1}) = -z^{M-1}H(z)$	Zeros at $z = \pm 1$ (i.e. at $\omega = \pi$, $\omega = 0$)
Type IV	$H(z^{-1}) = -z^{M-1}H(z)$	Zero at $z = 1$ (i.e. $\omega = 0$)

Linear Phase FIR Filters

type	length	sym.	delay	zeros	
I	odd	S	integer		
II	even	S	non-int.	$\pm\pi$	
III	odd	A	integer	$0, \pm\pi$	
IV	even	A	non-int.	0	

optimal FIR filter design

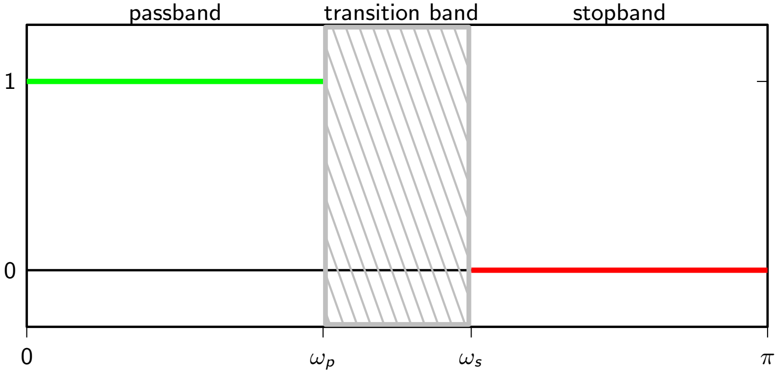
How do we design linear-phase FIRs?

answer: with the Parks-McClellan algorithm

let's work with an example:

- type I
- zero phase (work with $H_d(z)$)
- lowpass characteristc

Remember the realistic specs



Setting up the problem

Intuition #1: z-transform a finite-degree polynomial in z

$$H_d(z) = h_d[0] + \sum_{n=1}^{L} h_d[n](z^n + z^{-n}) = Q_M(z)$$

Setting up the problem

Intuition #2: Fourier transform also a finite-degree polynomial

$$H_d(e^{j\omega}) = h_d[0] + 2\sum_{n=1}^{L} h_d[n]\cos\omega n$$

$$\cos 2\omega = 2\cos^2\omega - 1$$

$$\cos 3\omega = 4\cos^3\omega - 3\cos\omega$$

$$\cos 4\omega = \dots$$

$$H_d(e^{j\omega}) = P_L(x)\big|_{x=\cos\omega}$$

Setting up the problem

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\cos 3\omega &= 4\cos^3\omega - 3\cos\omega \\
\cos 4\omega &= \ldots
\end{aligned}$$

$$H_d(e^{j\omega}) = P_L(x)\big|_{x=\cos\omega}$$

Setting up the problem

Intuition #3: we want

$$P_L(x) \approx D(x)$$

filter design becomes polynomial fitting!

Finding the polynomial

$$H_d(e^{j\omega}) = h_d[0] + 2\sum_{n=1}^{L} h_d[n] \cos \omega n$$

Step 1: Chebyshev polynomials

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$\dots$$

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

Step 1: Chebyshev polynomials

fundamental property:

$$T_n(\cos\omega) = \cos n\omega$$

Step 1: Chebyshev polynomials

$$H_d(e^{j\omega}) = h_d[0] + \sum_{n=1}^{L} 2h_d[n] \cos n\omega$$

$$P(x) = h_d[0] + \sum_{n=1}^{L} 2h_d[n] T_n(x)\Big|_{x=\cos\omega}$$

$$H_d(e^{j\omega}) = P(x)\Big|_{x=\cos\omega}$$

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$$H_d(e^{j\omega}) = P(x)\Big|_{x=\cos\omega}$$

Example for a 7-tap filter

$$H_d(e^{j\omega}) = a + 2b\cos\omega + 2c\cos 2\omega + 2d\cos 3\omega$$

$$= a\,T_0(\cos\omega) + 2b\,T_1(\cos\omega) + 2c\,T_2(\cos\omega) + 2d\,T_3(\cos\omega)$$

$$= a + 2b\cos\omega + 2c(2\cos^2\omega - 1) + 2d(4\cos^3\omega - 3\cos\omega)$$

$$= (a - 2c) + (2b - 6d)\cos\omega + 4c\cos^2\omega + 8d\cos^3\omega$$

$$= \left[(a - 2c) + (2b - 6d)x + 4c\,x^2 + 8d\,x^3\right]_{x=\cos\omega}$$

Example for a 7-tap filter

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\begin{aligned}
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&= a + 2b\cos\omega + 2c(2\cos^2\omega - 1) + 2d(4\cos^3\omega - 3\cos\omega) \\
&= (a - 2c) + (2b - 6d)\cos\omega + 4c\cos^2\omega + 8d\cos^3\omega \\
&= [(a - 2c) + (2b - 6d)x + 4c\,x^2 + 8d\,x^3]_{x=\cos\omega}
\end{aligned}
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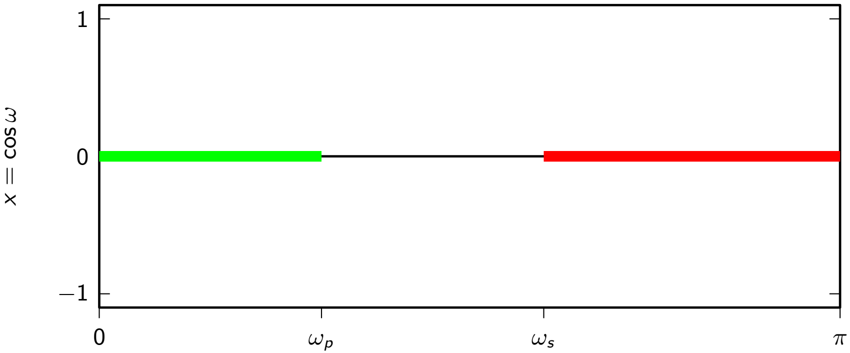
\end{aligned}
$$

Step 2: Convert the specs



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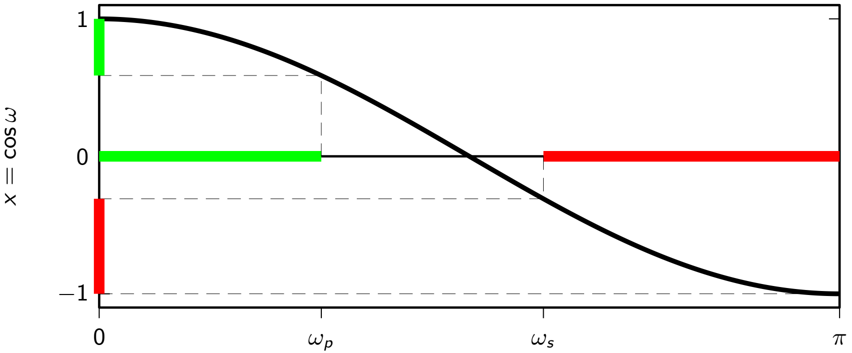

Step 2: Convert the specs

If $x = \cos\omega$

$$I_p = [0,\ \omega_p] \to I_p' = [\cos\omega_p,\ 1]$$

$$I_s = [\omega_p,\ \pi] \to I_s' = [-1,\ \cos\omega_s]$$

Step 2: Convert the specs

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$$I_p = [0,\, \omega_p] \rightarrow I'_p = [\cos\omega_p,\, 1]$$

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Step 2: Convert the specs

We want

$$P(x) \approx 1 \quad \text{for } x \in I'_p$$

$$P(x) \approx 0 \quad \text{for } x \in I'_s$$

Global error function

$$E(x) = P(x) - D(x)$$

with

$$D(x) = \begin{cases} 1 & \text{for } x \in I'_p \\ 0 & \text{for } x \in I'_s \end{cases}$$

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We could try this...

standard fitting of a degree-L polynomial:

- pick $L+1$ points over the two intervals
- build the Vandermode matrix

$$\mathbf{A} = \begin{bmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^L \\ 1 & x_1 & x_1^2 & \ldots & x_1^L \\ \vdots & & & & \\ 1 & x_L & x_L^2 & \ldots & x_L^L \end{bmatrix}$$

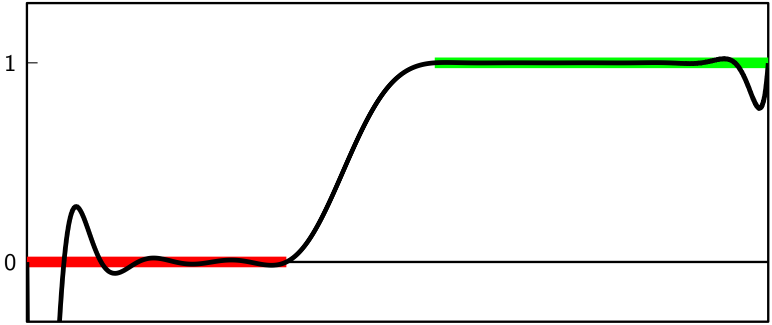
- solve the interpolation problem

$$\mathbf{Ap} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

… but it wouldn't work

- (direct methods numerically unstable)
- interpolation minimizes the MSE but not the maximum error

max error vs MSE



Brilliant idea: minimize max error

$$E = \min_{P(x)} \max_{x \in I'_p \cup I'_s} \{|P(x) - D(x)|\}$$

Alternation Theorem

$P(x)$ is the minimax approximation to $D(x)$ if and only if $P(x) - D(x)$ alternates $L + 2$ times between $+E$ and $-E$ in $I'_p \cup I'_s$

Why Alternation Theorem is key

- check candidates: if $P(x)$ satisfies the AT, we're done
- leads to a numerical algorithm to find $P(x)$: the Remez Exchange

Step 3: the Remez Algorithm

suppose we *knew* the positions of the alternations; then we could solve

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^L & \epsilon \\ 1 & x_1 & x_1^2 & \dots & x_1^L & -\epsilon \\ & & & \vdots & & \\ 1 & x_L & x_L^2 & \dots & x_L^L & (-1)^L\epsilon \end{bmatrix} \mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

and find both the polynomial coefficients and E

Step 3: the Remez Algorithm

obviously we don't know the positions of the alternations; but we can start with a guess

- solve the system of equation for the guessed x_i
- check if the solution satisfies the alternation theorem; if so, we're done
- otherwise, find the extrema of the error and use the locations as new guess; repeat

Example

- $M = 9$ $(L = 4)$
- $\omega_p = 0.4\pi$
- $\omega_s = 0.6\pi$
- we need at least $L + 2 = 6$ alternations
- 2 alternations always at band edges (otherwise specs not fulfilled)
- guess the other 4 and apply remez

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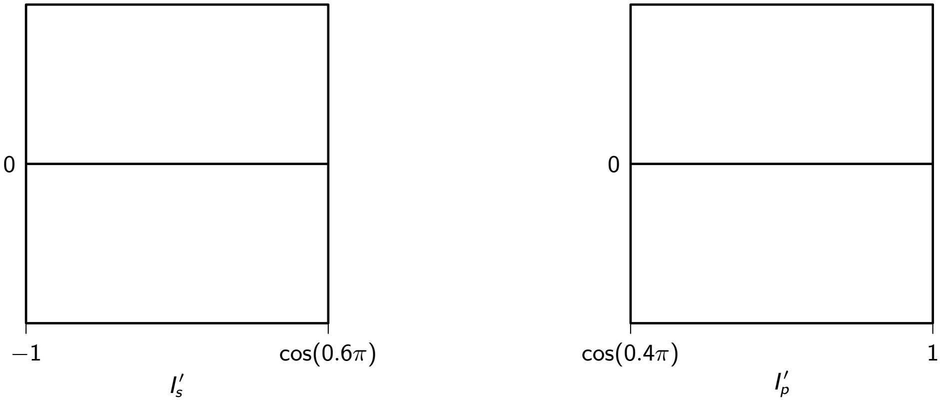
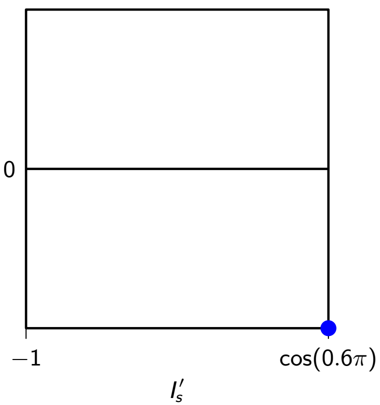
Remez exchange algorithm



Remez exchange algorithm



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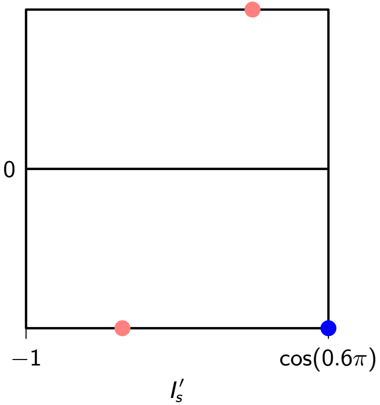

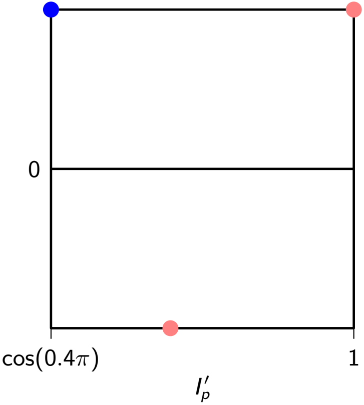

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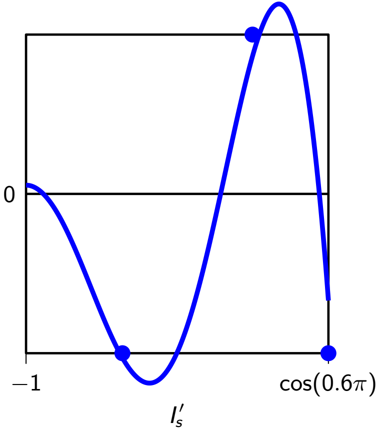

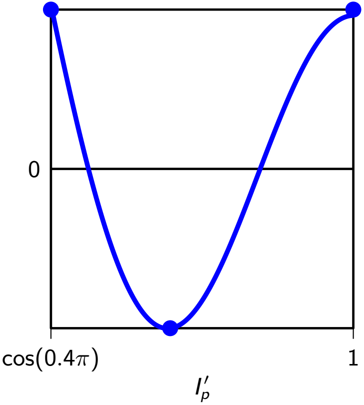

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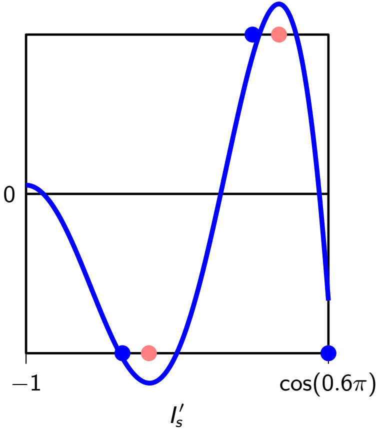

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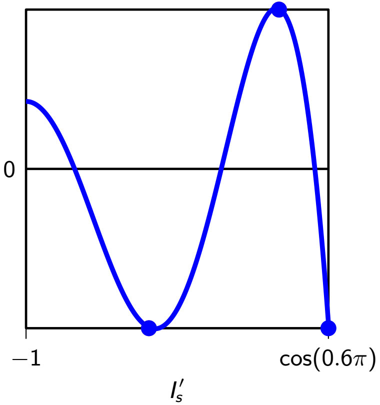
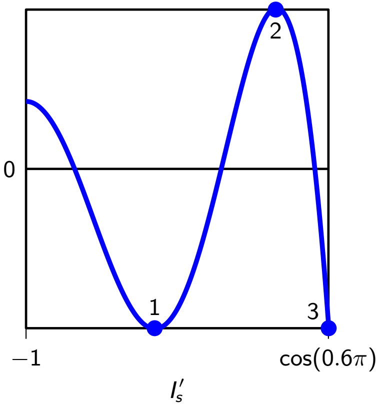
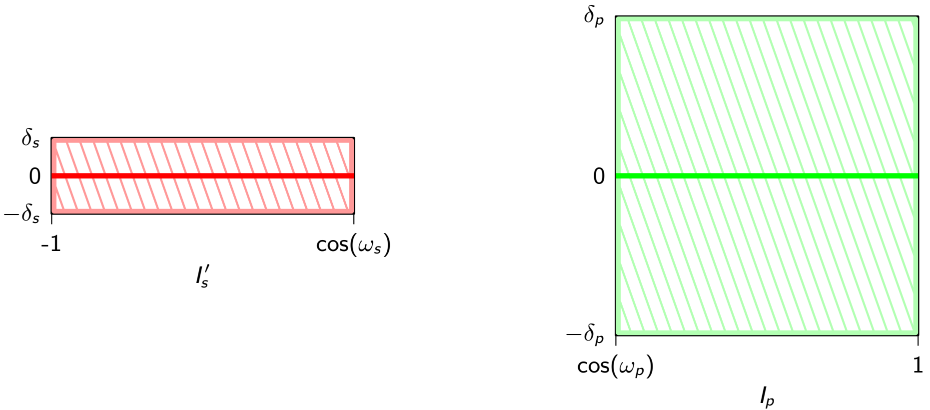


Passband and Stopband Error



Tuning the error

generally, we want to pay more attention to the error in stopband or passband



Goal: fit $E(x)$ within the boxes.

Tuning the error

The Alternation Theorem works also with a weighting function:

$$W(x) = \begin{cases} 1 & \text{for } x \in I'_p \\ \delta_p/\delta_s & \text{for } x \in I'_s \end{cases}$$

The updated minimization problem:

$$\min \max_{x \in I'_p \cup I'_s} \{|W(x)[P(x) - D(x)]|\}$$

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Parks-McClellan Algorithm; the full recipe for lowpass

User data:

Parks-McClellan Algorithm; the full recipe for lowpass

User data:

- filter length $M = 2L + 1$

Parks-McClellan Algorithm; the full recipe for lowpass

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Parks-McClellan Algorithm; the full recipe for lowpass

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Parks-McClellan Algorithm; the full recipe for lowpass

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Run Parks-McClellan algorithm; obtain:

- M filter coefficients

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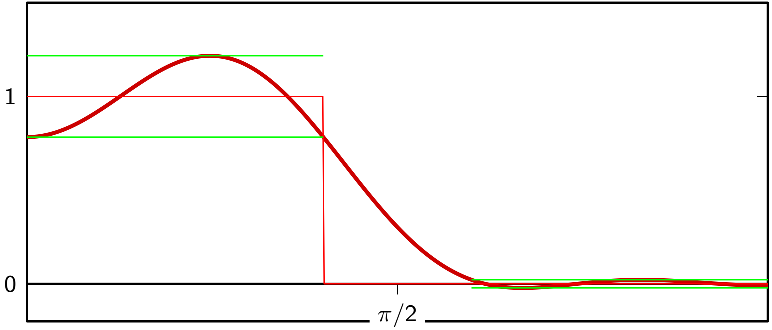
Run Parks-McClellan algorithm; obtain:

- M filter coefficients
- stopband and passband tolerances δ_s and δ_p
- If error too big, increase M and retry.

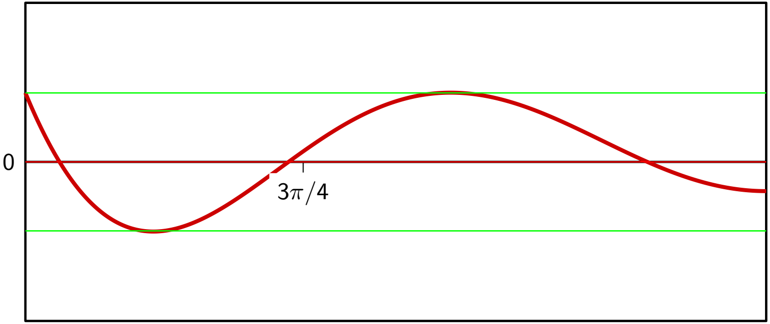
Example revisited

- $M = 9$ $(L = 4)$
- $\omega_p = 0.4\pi$
- $\omega_s = 0.6\pi$
- $\delta_s/\delta_p = 1/10$

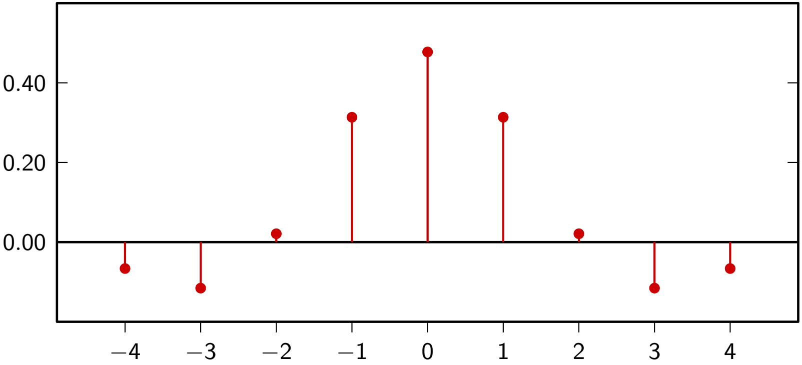
Final Result



Final Result (stopband)



Final Result (Impulse Response)



Minimax lowpass filter (recap)

Magnitude response:

- equiripple in passband and stopband

Design parameters:

- order N (number of taps)
- passband edge ω_p
- stopband edge ω_s
- ratio of passband to stopband error δ_p/δ_s

Design test criterion:

- passband max error
- stopband max error

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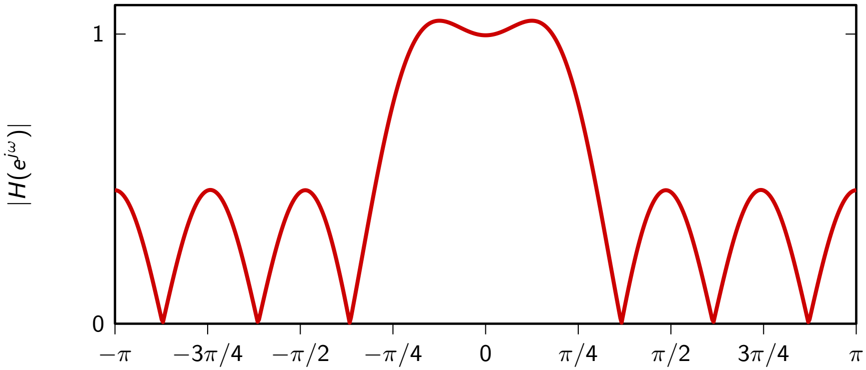
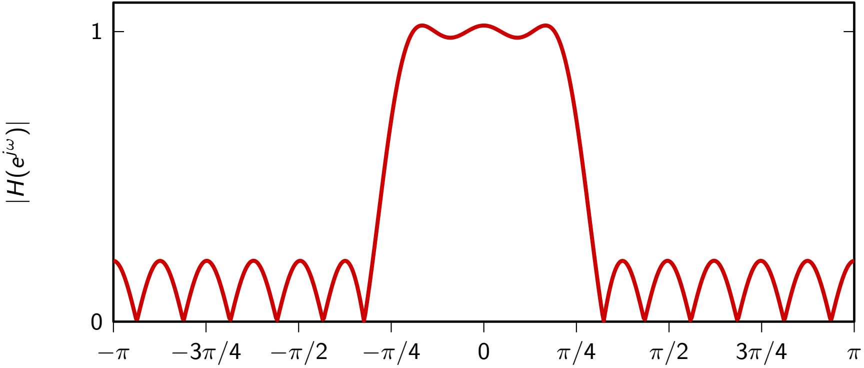
Minimax lowpass example

$$N = 9, \omega_p = 0.2\pi, \omega_s = 0.3\pi, \delta_p/\delta_s = 10$$



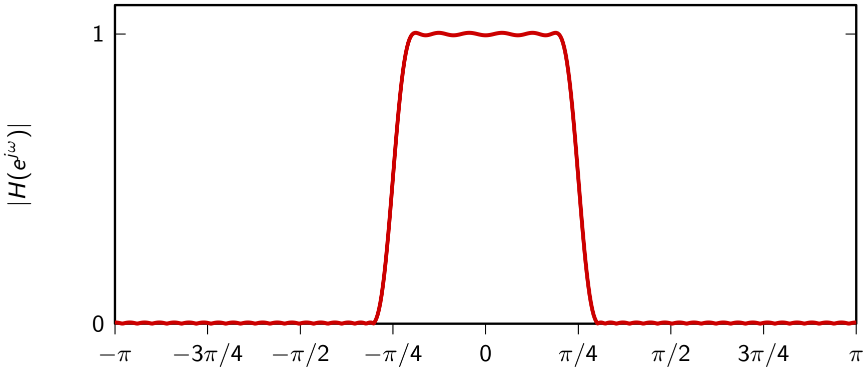
Minimax lowpass example

$$N = 19, \omega_p = 0.2\pi, \omega_s = 0.3\pi, \delta_p/\delta_s = 10$$



Minimax lowpass example

$$N = 51, \omega_p = 0.2\pi, \omega_s = 0.3\pi, \delta_p/\delta_s = 1$$



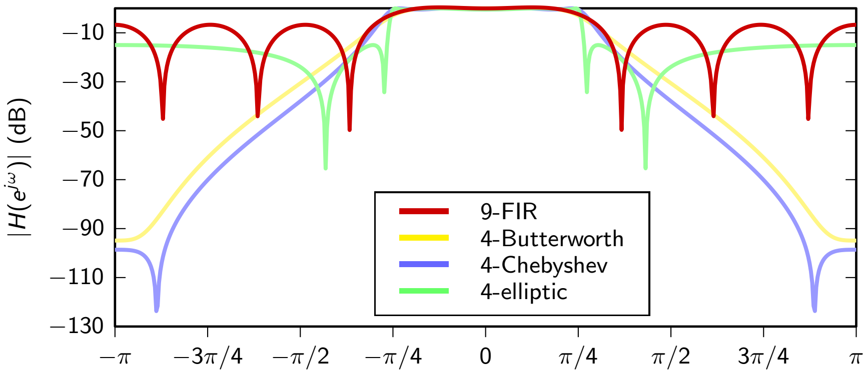
Magnitude response in decibels

- filter max passband magnitude G
- filter attenuation expressed in decibels as:

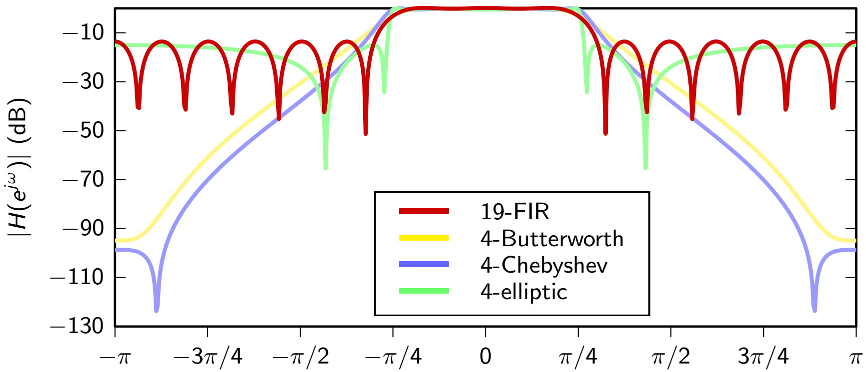
$$A_{\text{dB}} = 20 \log_{10}(|H(e^{j\omega})|/G)$$

- useful to compare attenuations between filters

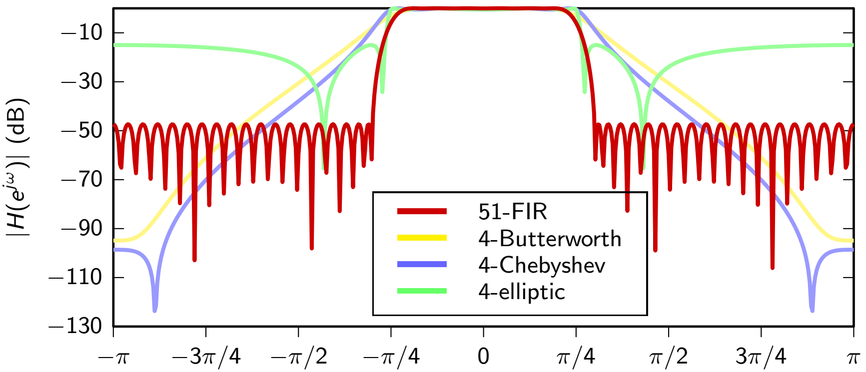
Lowpass comparison, $\omega_c = \pi/4$, log scale



Lowpass comparison, $\omega_c = \pi/4$, log scale



Lowpass comparison, $\omega_c = \pi/4$, log scale



Life beyond lowpass

The IIR and FIR methods we just described can be used to design more general filter types than lowpass, with only minor modifications

- IIR bandpass and highpass can be obtain by modulating the lowpass response
- optimal FIR bandpass and highpass can be designed by the Parks-McClellan algorithm
- optimal FIR can also be designed with piecewise linear magnitude response
- the literature on filter design is vast: this is just the tip of the iceberg!

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Life beyond lowpass

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Play with the demo!

Play with the interactive minimax filter design demo

`https://github.com/prandoni/COM303/`