(I1) $\varnothing \in \mathcal{I}$

(I2) (Hereditary Property) If $I \in \mathcal{I}$ and $J \subseteq I$ then $J \in \mathcal{I}$

(I3) (Exchange): If $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$ then there exists some $x \in I_2 - I_1$ with $I_1 \cup x \in \mathcal{I}$.

A matroid generalizes the notion of linear independence in vector spaces and independence in the form of acyclic subgraphs in graphs and often the power of matroids come from applying ideas from graph theory to frames. First, we will note that every frame (more generally any collection of vectors) is a matroid with linear independence.

Example 3.3. Let $(\varphi_j)_{j=1}^n$ be a collection of vectors in $\mathbb{F}^d$. Then $M(\Phi) = ((\varphi_j)_{j=1}^n, \mathcal{I})$ with $\mathcal{I}$ the subsets of linearly dependent vectors is a matroid. We will often simplify this construction by using $M(\Phi) = ([n], \mathcal{I})$ where $\mathcal{I}$ is instead just the indices of the vectors of each subset of linearly independent vectors.

Example 3.4. Let $G = (V, E)$ be a graph. Then $M(G) = (E, \mathcal{I})$ is a matroid where E is the set of edges and $\mathcal{I}$ the subsets of the edges that form acyclic subgraphs.

Because matroids generalize the notion of independence in vector spaces and graphs it is often helpful to know when a matroid can be expressed as a collection of vectors or as a graph.

Definition 3.5. *Let $M = (E, \mathcal{I})$ be a matroid. If there exists a collection of vectors $(\varphi_j)_{j\in E}$ in $\mathbb{F}^d$ for some field $\mathbb{F}$, indexed by E such that $(\varphi_j)_{j\in I}$ is linearly independent if and only if $I \in \mathcal{I}$ the M is called representable over $\mathbb{F}$. If a matroid is representable over any field it is called regular. Likewise, if there exists a graph, not necessarily simple, $\Gamma = (V, E)$ such that $I \subseteq E$ corresponds to an acyclic subgraph if and only if $I \in \mathcal{I}$ then M is called a graphical matroid.*

A well-studied question in matroid theory is the classification of graphical and representable matroids. Theorem 3.6 is a rather surprising theorem.

Theorem 3.6. *Every graphical matroid is regular*

Proof. Let M be a graphical matroid, that is, there exists some graph $\Gamma = (V, E)$ such that $M(\Gamma) = (E, \mathcal{I}_\Gamma)$, where $\mathcal{I}_\Gamma$ are the subsets of E that form acyclic subgraphs, is the same matroid as M. Fix an ordering for the vertices and edges, that is $V = (v_j)_{j=1}^n$ and $V = (e_j)_{j=1}^m$. And we will construct an oriented incidence matrix for Γ which is the matrix $\tilde{B} = (\tilde{b}_{jk})$ where

$$\tilde{b}_{jk} = \begin{cases} 1 & \text{if } e_k \text{ connects } v_j \text{ and } v_\ell \text{ where } j < \ell \\ -1 & \text{if } e_k \text{ connects } v_j \text{ and } v_\ell \text{ where } j > \ell \\ 0 & \text{otherwise} \end{cases}$$

In a sense, this construction gives an orientation to our graph. Each column $\tilde{b}_k$ encodes an edge e_k as a vector: if e_k is an edge connecting v_i and v_j where $i < j$, we have the vector with a 1 in the ith column and a -1 in the jth column, in essence, it gives each edge an orientation based on the fixed ordering of the vertices, effectively labeling v_i as the source and v_j as the sink. Loops are encoded as zero vectors and duplicate edges are duplicate vectors.

Notice that this allows us to relate a path in a graph with a linear combination of the corresponding columns. Notice that for a path in G: a sequence $v_{j_1}e_{k_1}v_{j_2}e_{k_2}v_{j_3}$, we can add the corresponding columns associated with the edges in the path taking into account the orientations. That is if $j_1 < j_2$ the path walks along the edge according to its orientation, and if $j_1 > j_2$ the path walks against the orientation of the edge so we must multiply it by -1 when adding the edge. And so the sum of the vectors corresponding to each edge, taking into account the orientation of the edges, results in the vector with a 1 in the j_1 row and a -1 in the j_3 column, the starting the ending vertices of the path. Notice that for a cycle $v_{j_1}e_{k_1}v_{j_2}e_{k_2}\dots v_{j_r}e_{k_r}v_{j_1}$, the path $v_{j_1}e_{k_1}v_{j_2}e_{k_2}\dots v_{j_r}$ would result in a vector with a 1 in the j_1 row and a -1 in the j_r row, and so when the vector corresponding to e_{k_r} is added, taking into account the orientation, we would add a vector with 1 in the j_r row and a -1 in the j_1 row resulting in the zero vector.

Now we want to show that $M(\Gamma)$ and the matroid formed from the columns of the incidence matrix $M(\tilde{B}) = (\tilde{B}, \mathcal{I}_{\tilde{B}})$ are the same. Consider first a subset of the columns that are linearly independent:

$(b_j)_{j\in I} \in \mathcal{I}_{\tilde{B}}$. This means that no non-trivial linear combination gives the zero vector, and so no sum of edges forms a cycle. And so the set of edges $(e_j)_{j\in I} \in \mathcal{I}_\Gamma$. Conversely, consider a subset of edges $(e_j)_{j\in I} \in \mathcal{I}_\Gamma$ such that the induced subgraph is acyclic. And consider a linear combination of the corresponding columns $\sum_{j\in I} \alpha_j \tilde{b}_j = 0$. Notice that because the subgraph is acyclic it is a forest and so contains a leaf vertex v_k with an edge connecting it e_ℓ. Notice that this means that $\tilde{b}_\ell$ is the only vector with a non-zero entry in the kth row meaning $\alpha_\ell = 0$. Notice that this effectively removes e_ℓ from the graph which results in an acyclic subgraph itself. We may repeat this inductively for all edges to conclude that $\alpha_\ell = 0$ for all ℓ, meaning $(\tilde{b}_j)_{j\in I} \in \mathcal{I}_{\tilde{B}}$. □

The contrapositive of this theorem provides a way to show certain matroids are not graphical, by showing there exists a field such that a matroid is not representable.

Matroids have many *cryptomorphic* definitions, which correspond to properties that determine the matroid.

Definition 3.7. *Let $M = (E, \mathcal{I})$ be a matroid, a circuit is a subset $C \subseteq E$ such that every proper subset $I \subsetneq C$ is an independent set, that is $I \in \mathcal{I}$. And a basis is an independence set $B \in \mathcal{I}$ of maximal size, meaning for the addition of any other element $x \in E$ it is the case that $B \cup x \notin \mathcal{I}$.*

Circuits can be thought of as minimally dependent sets, and for graphical matroid correspond to cycles in graphs. Every subset of E is either an independent set or is called a *dependent set* and so contains a circuit, a minimal dependent set. The size of the smallest circuit is called the *girth* of the matroid and for a representable matroid is equivalent to the *spark* of its vectors. This allows us to interpret frames as matroids. As with bases of vector spaces and spanning trees of graphs, the bases of a matroid all have equal sizes.

Both the set of circuits and the set of bases as defined in definition 3.7 determine a matroid which we will see in theorems 3.10 and 3.11 and so motivate cryptomorphic definitions

Definition 3.8. *A matroid M is a pair $(E, \mathcal{C})$ where E is a finite set, called the ground set and $\mathcal{C} \subseteq \mathcal{P}(E)$ the set of circuits, such that*

(C1) $\varnothing \notin \mathcal{C}$

(C2) If $C_1, C_2 \in \mathcal{C}$ with $C_1 \subseteq C_2$ then $C_1 = C_2$

(C3) (Circuit Elemenation) If $C_1, C_2 \in \mathcal{C}$ with $e \in C_1 \cap C_2$ there exists $C_3 \in C_1 \cap C_2 - e$ such that $C_3 \in \mathcal{C}$

Definition 3.9. *A matroid M is a pair $(E, \mathcal{B})$ where E is a finite set, called the ground set and $\mathcal{B} \subseteq \mathcal{P}(E)$ the set of bases, such that*

(B1) $\mathcal{B}$ is not empty

(B2) (exchange): if $B_1, B_2 \in \mathcal{B}$ with $B_1 \neq B_2$ then for any $x \in B_1 - B_2$ there exists some $y \in B_2 - B_1$ such that $B_1 - x + y \in \mathcal{B}$

The following theorems outline how to construct the independence sets given a matroid using the circuit definition and basis definition, and showing that each definition is equivalent.

Theorem 3.10. *The Independence set definition and the circuit definitions are equivalent. That is if $M = (E, \mathcal{I})$ is a matroid, then $M = (E, \mathcal{C})$ is a matroid where $\mathcal{C} = \{C \subseteq E | C \notin \mathcal{I}, \text{ and for all } I \subsetneq C, I \in \mathcal{I}\}$ is the set of circuits.*
Likewise if $M = (E, \mathcal{C})$ is a matroid, then $M = (E, \mathcal{I})$ is a matroid where $\mathcal{I} = \{I \in E | \text{ for all } C \in \mathcal{C}, C \nsubseteq I\}$.

Proof. We will prove the first direction of this theorem. Let $M = (E, \mathcal{I})$ be a matroid satisfying (I1), (I2), and (I3). Let $\mathcal{C} = \{C \subseteq E | C \notin \mathcal{I}, \text{ and for all } I \subsetneq C, I \in \mathcal{I}\}$. First notice that $\varnothing \in \mathcal{I}$ and so by construction we know that $\varnothing \notin \mathcal{C}$ which satisfies (C1). Now let $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$. Notice that if $C_1 \subsetneq C_2$ then by construction we would have that $C_1 \in \mathcal{I}$ but by assumption we have that $C_1 \in \mathcal{C}$, meaning $C_1 = C_2$ which show (C2). Notice that this shows that if $C_1 \neq C_2$ then there must exist some $x \in C_2 - C_1$ and likewise there must exist some $y \in C_1 - C_2$ as otherwise one would be a subset of the other.

Finally let $C_1, C_2 \in \mathcal{C}$ such that $C_1 \neq C_2$ and assume there exists some $e \in C_1 \cap C_2$. We will show that there must exist some circuit containing in $(C_1 \cup C_2) - e$. Assume however that there is no such circuit, meaning it must be the case that $(C_1 \cup C_2) - e \in \mathcal{I}$, as a subset of E is either in $\mathcal{I}$ or dependent and so contains a circuit.

If $(C_1 \cup C_2) - e \in \mathcal{I}$, then consider $C_1 - y \in \mathcal{I}$ where $y \in C_1 - C_2$. Notice that $((C_1 \cup C_2) - e) - (C_1 - y) = (C_2 - C_1) + y$. This means by (I3) we may add $|C_1 - C_2|$ elements of $(C_2 - C_1) + y$ to $C_1 - y$ such that the resulting set will be in $\mathcal{I}$ and will contain either all of C_1, meaning y was added, or all of C_2, meaning all of $C_2 - C_1$ was added. In either case, the result must contain a circuit and so would not be in $\mathcal{I}$. So $(C_1 \cup C_2) - e \notin \mathcal{I}$, which means there exists a circuit contained in $(C_1 \cup C_2) - e \in \mathcal{I}$ which shows (C3). ☐

Theorem 3.11. *The Independence set definition and the basis set definitions are equivalent. That is if $M = (E, \mathcal{I})$ is a matroid, then $M = (E, \mathcal{B})$ is a matroid where $\mathcal{B} = \{I \in \mathcal{I} | I$ is maximal$\}$ the set of bases. Likewise if $M = (E, \mathcal{B})$ is a matroid, then $M = (E, \mathcal{I})$ is a matroid where $\mathcal{I} = \{I \subseteq B | B \in \mathcal{B}\}$, is the downward closure of $\mathcal{B}$.*

Example 3.12. A useful example of a matroid using the basis definition is the *uniform matroid* which is defined as $U_{d,n} = ([n], \binom{[n]}{d})$ where $\binom{[n]}{d}$ denotes all subsets of $[n]$ of size d.

The uniform matroid describes the structure of frames or more generally collections of vectors, with full spark.

Proposition 3.13. *A collection of vectors $\Phi = (\varphi_j)_{j=1}^n$ in $\mathbb{F}^d$ has full spark if and only if $M(\Phi)$ is the uniform matrix, $U_{d,n} = ([n], \binom{[n]}{d})$ by the basis definition 3.9.*

Proof. Assume $M(\Phi)$ is the uniform matroid meaning any collection of less than or equal to d vectors is linearly independent as all subsets of size d are linearly dependent and so not circuits. This also means any subset of $d+1$ vectors is a circuit, and so the girth of the matroid is $d+1$. Now assume that the collection of vectors $(\varphi_j)_{j=1}^n$ has spark $d+1$. This means the girth of the matroid $M(\Phi) = ([n], \mathcal{I})$ is $d+1$ and so every subset of $d+1$ vectors is dependent and so must contain a circuit, and because circuits are of size at least $d+1$, then every such subset is a circuit, meaning every subset of size d is in $\mathcal{I}$, which form the basis. ☐

In definition 2.17, we presented the Naimark complement, which was a complementary construction for a tight frame. As noted in proposition 2.18 Naimark complements shared many nice properties such as having equiangular vectors if and only if the original frame did. In proposition 3.17 and corollary 3.18, we will continue this duality of properties and relate their corresponding matroids and spark, which at first seems unrelated as the dimension of the underlying space of a tight frame and its Naimark complement generally differ. However, in the remainder of this section, we will show that the Naimark complement is very closely related to the dual matroid.

Proposition 3.14. *Let $M = (E, \mathcal{B})$ be a matroid. Then $M^* = (E, \mathcal{B}^*)$, called the dual matroid, where $\mathcal{B}^* = \{E - B | B \in \mathcal{B}\}$ is a matroid.*

Proof. Let $M = (E, \mathcal{B})$ and let $M^* = E, \mathcal{B}^*$ where $\mathcal{B}^* = \{E - B | B \in \mathcal{B}\}$. Notice that $\mathcal{B}^*$ because $\mathcal{B}$ is non-empty. Consider bases $B_1, B_2 \in \mathcal{B}$ such that $B_1 \neq B_2$ meaning $E - B_1 \neq E - B_2$ Notice that this means for any $x \in (E - B_1) - (E - B_2)$ that $x \in B_2 - B_1$ meaning there must exists some $y \in B_1 - B_2 = (E - B_2) - (E - B_1)$ such that $B_2 - x + y$ is a basis in $\mathcal{B}$ meaning $E - B_2 - y + x \in \mathcal{B}^*$. So M^* is a matroid. ☐

Example 3.15. As an example we will look at the uniform matroid of $U_{d,n} = ([n], \binom{[n]}{d})$ where $\mathcal{B}^* = \{[n] - B | B \in \binom{[n]}{d}\} = \binom{[n]}{n-d}$ meaning $U_{d,n}^* = U_{n-d,n}$

In the construction of the dual matroid, we use basis elements of M to create the basis element of M^*, that is we may say that the complement of a basis (over E) is a basis of the dual matroid. It is also useful to think about dual matroid under different definitions, that is we want to understand the complement of a circuit and a complement of an independent set.

Proposition 3.16. *Let M be a matroid with $\mathcal{I}$ its independent sets and $\mathcal{C}$ its circuits. Then for $I \in \mathcal{I}$, the set $E - I$ is a spanning set of M^* and for $C \in \mathcal{C}$, the set $E - C$ is a hyperplane of M^*, a maximally non-spanning independent set of M^*.*

As we will see in proposition 2.18 and in later propositions in section 3, Naimark complements share many of the same properties as the original frame.

Proposition 2.18. *Let $\Phi = (\varphi_j)_{j=1}^n$ be a tight frame for $\mathbb{F}^d$ with frame bound A and Naimark complement $\Psi = (\psi_j)_{j=1}^n$ in $\mathbb{F}^{n-d}$. Then Ψ is a tight frame with frame bound A and likewise, Φ is the Naimark complement of Ψ. Furthermore, if Φ is equiangular or equal-norm then so is Ψ.*

Proof. If Ψ is a tight frame we know that $\Phi\Phi^* = AI_d$, meaning there are d eigenvalues that are A. And because the singular values of Φ and Φ^* agree, the non-zero eigenvalues of $\Phi\Phi^*$ and $\Phi^*\Phi$ agree and so there are d eigenvalues of A and $n-d$ of 0. Since Ψ is the Naimark complement of Φ we know that $\Phi^*\Phi + \Psi^*\Psi = AI$ and so $\Psi^*\Psi = AI - \Phi^*\Phi$ has $n-d$ non-zero eigenvalues that are A and d eigenvalues that are 0. To see this notice that $\operatorname{tr}(\Psi^*\Psi) = \operatorname{tr}(AI) - \operatorname{tr}(\Phi^*\Phi) = (n-d)A$. Notice also that the only eigenvalues of $\Psi^*\Psi$ are 0 and A: that is for x an eigenvalue for some vector v, $AIv - \Phi^*\Phi v = xIv$ which means $(A-x)Iv = \Phi^*\Phi v$ and so $A - x$ is an eigenvalue of $\Phi^*\Phi$ meaning $A - x$ is either A or 0 so x is either 0 or A. This therefore means Ψ is a frame of $\mathbb{F}^{n-d}$ with singular values $\sigma_1 = \sigma_{n-d} = \sqrt{A}$ and so Ψ is a tight frame with frame bound A. Because Ψ is a tight frame and Φ satisfies $\Psi^*\Psi + \Phi^*\Phi = AI$ we have the Φ is the Naimark complement of Ψ.
From the definition of Naimark complement, we know that

$$\left\langle \begin{pmatrix} \varphi_j \\ \psi_j \end{pmatrix}, \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix} \right\rangle = \langle \varphi_j, \varphi_k \rangle + \langle \psi_j, \psi_k \rangle = A\delta_{j,k}$$

Now assume that Φ is equiangular with angle $\alpha < 1$. This means when $j \neq k$ we have that $|\langle \psi_j, \psi_k \rangle| = |\langle \varphi_j, \varphi_k \rangle| = \alpha < 1$ and so Ψ is equiangular. Likewise if Φ is equal norm, with norm β we have for any j that $\langle \psi_j, \psi_j \rangle = A - \langle \varphi_j, \varphi_j \rangle = A - \beta$ and so Ψ is equal norm. ☐

Naimark complements are not necessarily unique, as in the construction of the orthogonal projection in the proof of Naimark's theorem, 2.16, it is entirely possible to construct multiple projections resulting in different frames.

3 Matroids

A key advantage to interpreting data with a frame, as opposed to an orthonormal basis, is to make the data more robust to noise and loss of information. Many frames achieve this robustness through redundancies. However, in proposition 2.12, we saw that a frame was equivalent to a collection of vectors that spans a vector space. And it is often important to distinguish good frames from bad frames with notions of geometric and algebraic spread. In definition 2.6 we defined an equiangular frame that encapsulations an understanding of good geometric spread. However, this does not guarantee a good algebraic spread. Algebraic spread encapsulates how mutually linearly dependent a collection of data is, and is formalized with the definition of spark.

Definition 3.1. *Let $\Phi = (\varphi_j)_{j=1}^n$ be a collection of vectors in $\mathbb{F}^d$, then the spark of Φ is defined as*

$$\operatorname{spark} \Phi = \min\{m \mid (\varphi_{j_k})_{k=1}^m \subseteq (\varphi_j)_{j=1}^n \text{ linearly dependent, } j_1 < j_2 < \cdots < j_m\}$$

If $\operatorname{spark}(\Phi) = d + 1$ *then we say Φ is full spark.*

This means the spark is the size of the smallest subset of linearly dependent vectors. A frame being full spark means any subset of d vectors forms a basis. In general $1 \leq \operatorname{spark}(\Phi) \leq \operatorname{rank}(\Phi) + 1$, which suggests that while the rank encapsulates the maximal (linear) independence of a collection of vectors the spark in a sense encapsulates the worst-case, or minimal dependence, of a collection of vectors. The spark captures a sense of how mutually redundant the vectors are.

As we will see throughout this section, matroids act as important and useful tools in encapsulating the dependence of a frame and its algebraic spread.

Definition 3.2. *A matroid M is a pair $(E, \mathcal{I})$ where E is a finite set, called the ground set and $\mathcal{I} \subseteq \mathcal{P}(E)$ the independent sets, such that*

Proof. Recall that the sum of the eigenvalues of a matrix is equal to the trace of the matrix and that the trace of a product of matrices is invariant under cyclic permutations of the matrices, meaning

$$\sum_{k=1}^{d} \lambda_k = \operatorname{tr}(S) = \operatorname{tr}(\Phi\Phi^*) = \operatorname{tr}(\Phi^*\Phi) = \operatorname{tr}(G) = \sum_{j=1}^{n} \|\varphi_j\|^2$$

Furthermore notice that if $(\varphi_j)_{j=1}^n$ is an FUNTF with frame bound A we know from proposition 2.11 that $S = AI$ and so $\operatorname{tr}(S) = \operatorname{tr}(AI) = dA$. And like wise we know that $\operatorname{tr}(S) = \sum_{j=1}^n \|\varphi_j\|^2 = n$, so $n = dA$ meaning $A = n/d$. $\square$

To conclude this section we will highlight a connection between Parseval frames and orthonormal bases through orthonormal projection maps.

Definition 2.15. *A linear operator $P : V \to V$ is called a projection if $P^2 = P$. Furthermore, a projection P is called an orthogonal projection if P is Hermitian, that is $P = P^*$.*

Theorem 2.16. *(Naimark's Theorem). The collection $(\varphi_j)_{j=1}^n$ is a Parseval frame for $V = \mathbb{F}^d$ if and only if there exists an inner product space $W \supseteq V$ with orthonormal basis $(e_j)_{j=1}^n$ and orthogonal projection $P : W \to W$ onto V such that $\varphi_j = Pe_j$ for all j.*

Proof. We will first show the $\Leftarrow$ direction: Assume $V \subseteq W$, meaning W is a super set of dimension n with basis $(e_j)_{j=1}^n$ and $P : W \to W$ an orthogonal projection onto V, which means P fixes V. Notice that for any $x \in V \subseteq W$ we have that

$$\sum_{j=1}^{n} |\langle x, Pe_j\rangle|^2 = \sum_{j=1}^{n} |\langle P^*x, e_j\rangle|^2 = \sum_{j=1}^{n} |\langle Px, e_j\rangle|^2 = \sum_{j=1}^{n} |\langle x, e_j\rangle|^2 = \|x\|^2$$

which follows from proposition 2.5, Parseval's equality, as $(e_j)_{j=1}^n$ is an orthonormal basis. This means the collection of vectors $(Pe_j)_{j=1}^n$ is a Parseval frame for $V = \mathbb{F}^d$.

Now for the $\Rightarrow$ direction: assume that $(\varphi_j)_{j=1}^n$ is a Parseval frame for $\mathbb{F}^d$ with synthesis operator Φ and by proposition 2.11 we know that $S = \Phi\Phi^* = I_d$, meaning the d rows of Φ are n-dimensional orthonormal vectors. Case 1: if $d = n$ then notice that because $S = I_d$ we know that the rows of Φ form an orthonormal basis for $\mathbb{F}^d$ and so Φ is an orthonormal matrix meaning $(\varphi_j)_{j=1}^n$ is an orthonormal basis and so with the identity map from V to V as the orthogonal projection we satisfy the statement. Case 2: Assume that $n > d$ then it is the case that the d n-dimensional orthonormal row vectors can be extended to an orthonormal basis over $\mathbb{F}^n$. That is there exists $n-d$ n-dimensional vectors $(v_\ell)_{\ell=1}^{n-d}$ such that the rows of Φ and $(v_\ell)_{\ell=1}^{n-d}$ form an orthonormal basis. Let Ψ be the matrix whose rows are $(v_\ell)_{\ell=1}^{n-d}$ and we will denote the columns as $(\psi_j)_{j=1}^n$. Notice the matrix

$$X = \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}$$

is invertible as $XX^* = I$ and so $X^*X = I$ and so the columns of X form an orthonormal basis for $\mathbb{F}^n$ which are the vectors

$$\left(\begin{pmatrix} \varphi_j \\ \psi_j \end{pmatrix}\right)_{j=1}^{n}$$

Finally, consider the projection of $\mathbb{F}^n$ onto the first d coordinates which would map this orthonormal basis onto $(\varphi_j)_{j=1}^n$ as desired. $\square$

This theorem can be extended to tight frames with frame bound A by first rescaling to get a Parseval frame. This statement is sometimes referred to as Witt's extension theorem. This theorem provides a very powerful duality of tight frames, which are the vectors (ψ_j), which share many of the properties of the original frame.

Definition 2.17. *Let $(\varphi_j)_{j=1}^n$ be a tight frame for $\mathbb{F}^d$ with corresponding synthesis operator Φ with frame bound A and consider vectors $(\psi_j)_{j=1}^n$ in $\mathbb{F}^{n-d}$ with associated matrix Ψ. If $\Phi^*\Phi + \Psi^*\Psi = AI$ then we call Ψ a Naimark complement of Φ.*

Proof. Let $I \in \mathcal{I}$, meaning there exists some basis B such that $I \subseteq B$. This means that $E - B \subseteq E - I$, and so the complement of an independence set contains a basis in M^* and so is a spanning set of M^* and the converse is also true.

Let C be a circuit of M, and notice that $E - C$ cannot be a spanning set as if it were $E - (E - C) = C$ would be an independence set. However, notice that the removal of any element $e \in C$ would give $C - e \in \mathcal{I}$ meaning $E - C + e$ would be a spanning set for all $c \notin E - C$. Meaning $E - C$ is a maximal non-spanning set, a hyperplane. $\square$

In frame theory, matroid duals connect tight frames with their Naimark complements.

Proposition 3.17. *Let $\Phi = (\varphi_j)_{j=1}^n$ be a tight frame with frame bound A for $\mathbb{F}^d$ with $n > d$ with Naimark complement $(\psi_j)_{j=1}^n$ in $\mathbb{F}^{n-d}$. Then $M(\Phi)^* = M(\Psi)$.*

Proof. Let $X = \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}$ as in the proof of theorem 2.16. And let $(u_j)_{j=1}^n = \left(\begin{pmatrix} \varphi_j \\ \psi_j \end{pmatrix}\right)_{j=1}^n$ denote the columns of X. Notice that $X^*X = \Phi^*\Phi + \Psi^*\Psi = AI$ which means the vectors (u_j) are pairwise orthogonal. Now because both $M(\Phi)$ and $M(\Psi)$ are matroids over the same number of vectors, n, we will assume that both have the same ground set $E = [n]$, which indexes the vectors. Because $n > d$ there must exist a subset of vectors in $(\varphi_j)_{j=1}^n$ that is linearly dependent, meaning it contains a circuit. Let $D \in [n]$ be the indices of a dependent set, meaning $(\varphi_j)_{j\in D}$ is linearly dependent. And so there exists $\alpha_j \in \mathbb{F}$ for all $j \in D$ such that not all α_j are zero and

$$\sum_{j\in D} \alpha_j \varphi_j = 0$$

However because the vectors (u_j) form an orthonormal basis we have that

$$\sum_{j\in D} \alpha_j u_j \neq 0 \text{ and so } \sum_{j\in D} \alpha_j \psi_j \neq 0$$

Now we want to show that the vectors in (ψ_j) with indices $[n] - D$ do not span $\mathbb{F}^{n-d}$ (This aligns with what we expect to see based on the interpretation in proposition 3.16 assuming the statement is true). So let $\tilde{v} = \sum_{j\in D} \alpha_j \psi_j$ and let $k \in [n] - D$ and notice that

$$\begin{aligned} \langle \tilde{v}, \psi_k \rangle = \left\langle \sum_{j\in D} \alpha_j \psi_j, \psi_k \right\rangle &= \left\langle \sum_{j\in D} \alpha_j (u_k - \varphi_k), u_k - \varphi_k \right\rangle \\ &= \left\langle \sum_{j\in D} \alpha_j u_k, u_k \right\rangle - \left\langle \sum_{j\in D} \alpha_j \varphi_k, \varphi_k \right\rangle = \sum_{j\in D} \alpha \langle u_j, u_k\rangle - \langle 0, \varphi_k\rangle = 0 \end{aligned}$$

which means that the non-zero vector $\tilde{v}$ is orthogonal to the space $(\psi_j)_{j\in[n]-D}$ and so not in its span, meaning the vectors with indices $[n] - D$ is not a spanning set. Notice that this means for any $S \subseteq [n]$ that is a spanning set in $M(\Psi)$, the set $[n] - S$ must be an independence set by the contrapositive of the above. Furthermore if S was a basis of $M(\Psi)$, meaning $|S| = n - d$, then $|[n] - S| = d$ meaning it would be a maximal independence set in $M(\Phi)$, a basis. Recall that Naimark complements are reflective and so we know that Φ is the Naimark complement of Ψ, and so it is also the case that for any basis of $M(\Phi)$ the complement is a basis in $M(\Psi)$. And so $M(\Phi)^* = M(\Psi)$. $\square$

This proposition has a very useful corollary for full spark tight frames.

Corollary 3.18. *A tight frame $\Phi = (\varphi_j)_{j=1}^n$ is full spark if and only if its Naimark complements $\Psi = (\psi_j)_{j=1}^n$ is full spark.*

Proof. Recall that if Ψ is the Naimark complement of Φ then Φ is the Naimark complement of Ψ and so it suffices to only show one direction.

Assume Φ is full spark which means that $M(\Phi) = U_{d,n}$. And so $M(\Psi) = M(\Phi)^* = U_{d,n}^* = U_{n-d,n}$ and so from proposition 3.13 Ψ has full spark. $\square$

4 Framing the Theory

Now that we have developed some of the frame theory needed we will look at an example of a real frame in $\mathbb{R}^6$, and its Naimark complement in $\mathbb{R}^{10}$.

$$\Phi = \frac{1}{\sqrt{3}} \left(\begin{array}{cccc|cccc|cccc|cccc} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{array}\right)$$

Example 4.1. Φ is an unit-norm ETF with frame bound $A = \frac{8}{3}$ and $\text{spark}(\Phi) = 4$.

Proof. First to show that Φ is tight frame we will compute the frame operator. Notice that the inner product of any two rows will mostly be the product of zeros, but it may be the case that two columns will agree and two will disagree in sign, meaning the inner product of distinct rows is 0. Notice also that the norm squared of every row is $8 \cdot \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{8}{3}$. So Φ is a tight frame by proposition 2.11.

Now we need to show that this is an equal angular tight frame. We can do this by noting that every two distinct columns will have a non-zero element in 1 or 3 entries and in the case of 1 entry the sign will disagree and in the case of 3, 1 entry will agree in sign and the other 2 with disagree so we find that $\langle \varphi_j, \varphi_k \rangle = -\frac{1}{3}$ for all $j \neq k$ and $\langle \varphi_j, \varphi_j \rangle = 1$ for all j. And so Φ is a finite unit norm equiangular tight frame.

Finally, notice if we index these vectors from left to right by [16] we may notice that $1, 2, 3$ and 4 form a circuit. Furthermore, no three vectors from the same block are linearly dependent and for any three vectors in different blocks, each vector has one dimension that only that vector reaches. And in the case that two vectors are from the same block, they would not be linearly dependent and the third would reach dimensions the others do not. And so no three vectors are linearly dependent. So the smallest circuit is of size 4, meaning the spark is 4. $\square$

Now consider a Naimark complement which we will call Ψ. We know that Ψ is a frame for $\mathbb{R}^{10}$ with 16 vectors. Using the properties of Naimark complements and matroids we can analyze the properties of the Naimark complement without directly computing the possible vectors.

Example 4.2. The frame Ψ is an equal-norm ETF with frame found $A = \frac{8}{3}$ and has $\text{spark}(\Psi) = 8$

Proof. By proposition 2.18 we know that Ψ is an equal-norm ETF with frame bound $A = \frac{8}{3}$. By the proof of 2.18 we know that the norm of the vectors of Ψ is $\frac{8}{3} - 1 = \frac{5}{3}$, which can also be seen with proposition 2.14. So we need only compute the spark. We will look at the corresponding matroid where we know that $M(\Psi) = M(\Phi)^*$. And from proposition 3.16 we know that the smallest circuit in $M(\Psi)$ corresponds to the maximal hyperplane of $M(\Phi)$. So it suffices to find the maximal number of columns of Φ that do not span $\mathbb{R}^3$. Notice that any 2 blocks of vectors span all but one of the dimensions corresponding to the non-zero entries, and any ninth vector not in the blocks spans $\mathbb{R}^{10}$. This means any 2 blocks is a maximal hyperplane of size 8. And because a maximal hyperplane H corresponds to a minimal circuit by $E - H$ which would have a size of $16 - 8 = 8$, and so the spark of Ψ is 8. $\square$

Notice that for $\mathbb{R}^{10}$ a full spark frame would have a spark of 11.

A motivation behind this example is in the computability of spark, for frames. In general computing the girth of a matroid can be very difficult: for graphical matroids, girth can be computed rather easily, but for representable matroids, it can be very computationally difficult. In general, for linear matroids finding girth is w[1]-hard with respect to the girth or rank of the matroid. And FPT with respect to a combination of the rank of the matroid and the size of the underlying field. Meaning that finding good algebraic spread in frames is very hard unless you have a small dimensional space and the frame is on a small field.

Naimark complements can reduce the dimension. Given dim $d < n$ and $n - d < d$ then studying the Naimark complement is likely easier, at least heuristically. Although in this case, we are computing cogirth, which is the problem of finding non-spanning sets, it is still the case that the dimension of the space can be dramatically reduced.

follows from the imposed order of the singular values on Σ. Likewise for the same reason if x is the last column, the dth column, of U then $\|\Sigma^* U^* x\|^2 = \|\Sigma^* \mathbf{e}_d\|$ is minimized. Case 1: assume that $n \geq d$ in which case Σ has d diagonal elements, which means $\|\Sigma^* y\|^2 = \|\Sigma^* \mathbf{e}_d\| = \|\sigma_d \mathbf{e}_d\| = \sigma_d^2$ which again follows from the imposed order of the singular values in Σ. Case 2: If $n < d$ then $\|\Sigma^* U^* x\|^2 = \|\Sigma^* \mathbf{e}_d\|^2 = 0$ this follows from Σ^* being a diagonal matrix with $n < d$ diagonal elements so the dth column of Σ^* is zero, and so $\Sigma^* \mathbf{e}_d = 0$. Notice that for $(\varphi_j)_{j=1}^n$ to be a frame we must have $n \geq d$ which means there will be a dth singular value in the SVD in which case we would have have

$$\sigma_d^2 \|x\|^2 \leq \sum_{j=1}^n |\langle x, \varphi_j \rangle|^2 \leq \sigma_1^2 \|x\|^2$$

with optimal bounds $A = \sigma_d^2$ and $B = \sigma_1^2$. This concludes the first part of the proof.

For the second part notice that the frame operator $S = (\Phi^*)^* \Phi^*$, meaning S is the gram matrix of the analysis operator Φ^*, meaning it represents inner products between conjugates of the rows. That is $S = (\langle {\Phi_k}^*, {\Phi_j}^* \rangle)_{jk}$, where Φ_k represents the kth row of Φ and so from equation 2 we have that

$$(\langle \Phi_k^*, {\Phi_j}^* \rangle)_{jk} = S = U \Sigma \Sigma^* U^* = U A I_d U^* = A U U^* = A I_d$$

Where the last three steps follow from the fact that the frame being tight means the singular values must all be $\sqrt{A}$. Furthermore, under conjugate symmetry, we know that $(\langle \Phi_j, \Phi_k \rangle)_{jk} = (\langle \Phi_k^*, {\Phi_j}^* \rangle)_{jk}$ This is equivalent to the rows of Φ being linearly independent and $\|\Phi_j\| = \sqrt{\langle \Phi_j, \Phi_j \rangle} = \sqrt{A}$ for all j $\square$

Notice that the second part of the proof of this proposition shows that a frame is tight if and only if the frame operator is a positive multiple of the identity, more specifically when $S = AI$ where A is the frame bound. We have also shown that frames must have at least as many vectors as the dimension, we can strengthen this by observing that the first d singular values must be non-zero.

Corollary 2.12. *Let $(\varphi_j)_{j=1}^n$ be a collection of vectors in $\mathbb{F}^d$. Then $(\varphi_j)_{j=1}^n$ is a frame if and only if $(\varphi_j)_{j=1}^n$ spans $\mathbb{F}^d$.*

Proof. Notice that $(\varphi_j)_{j=1}^n$ spans $\mathbb{F}^d$ if and only if the rank of Φ is d And this is the case if and only if there are d non-zero singular values, and by proposition 2.11 this is the case if and only if $(\varphi_j)_{j=1}^n$ is a frame. $\square$

As another corollary we can combine frames to get a new frame, oftentimes retaining the same properties.

Corollary 2.13. *Let $(\varphi_j)_{j=1}^n$ and $(\psi_j)_{j=1}^m$ be tight frames with frame bounds A and B respectively. Their concatenation $(\varphi_1, \ldots, \varphi_n, \psi_1, \ldots \psi_m)$ is a tight frame with frame bound $A + B$.*

Proof. From proposition 2.11, we know that the frame operators for both frames are multiples of the identity matrix: $\Phi\Phi^* = AI_d$ and $\Psi\Psi^* = BI_d$. Notice that the union of both frames has as its synthesis matrix the augmented matrix $\begin{pmatrix} \Phi & \Psi \end{pmatrix}$. And so the frame operator

$$\begin{pmatrix} \Phi & \Psi \end{pmatrix} \begin{pmatrix} \Phi & \Psi \end{pmatrix}^* = \begin{pmatrix} \Phi & \Psi \end{pmatrix} \begin{pmatrix} \Phi^* \\ \Psi^* \end{pmatrix} = \Phi\Phi^* + \Psi\Psi^* = AI_d + BI_d = (A+B)I_d$$

Therefore the union of both tight frames is a tight frame with frame bound $A + B$. $\square$

Along similar lines as in the previous propositions and corollaries, we can use the matrices created in definition 2.8 to learn other useful properties about frames and in particular FUNTFs.

Proposition 2.14. *Let $(\varphi_j)_{j=1}^n$ be a frame for $\mathbb{F}^d$ with synthesis operator Φ. If $(\lambda_k)_{k=1}^d$ are the eigenvalues of the frame operator S then*

$$\sum_{k=1}^d \lambda_k = \sum_{j=1}^n \|\varphi_j\|^2$$

And if $(\varphi_j)_{j=1}^n$ is a FUNTF then the frame bound is n/d

This suggests that *nice* frames which are not orthonormal basis are those that are FUNTF but are not Parseval or those that are Parseval but not unit-norm, and in general both cases are equivalent up to rescaling.

In frame theory it is often useful to describe frames with matrices, four are of particular importance for this paper

Definition 2.8. *Let $(\varphi_j)_{j=1}^n$ be a collection of vectors in an inner product space V over a field $\mathbb{F}$*

- $\Phi : \mathbb{F}^n \to V$ *is the matrix whose columns are the vectors $(\varphi_j)_{j=1}^n$, in which case by right multiplication $c \mapsto \sum_{j=1}^n c_j \varphi_j$ (Synthesis operator)*
- $\Phi^* : V \to \mathbb{F}^n$ *and by right multiplication takes $x \mapsto (\sum_{j=1}^d \overline{\Phi_{jk}} x_j)_{k=1}^n = (\langle x, \varphi_k \rangle)_{k=1}^n$ (Analysis operator)*
- $S = \Phi\Phi^* : V \to V$*, by right multiplication $x \mapsto \sum_{j=1}^n \langle x, \varphi_j \rangle \varphi_j$ (Frame operator)*
- $G = \Phi^*\Phi = (\langle \varphi_k, \varphi_j \rangle)_{jk}$*, and so right multiplication maps $x \to (\sum_{k=1}^n c_k \langle \varphi_k, \varphi_j \rangle)_{j=1}^n$ (Gram Matrix)*

In this paper we will often denote a frame, or a collection of vectors, by its synthesis operator, that is we may write $\Phi = (\varphi_j)_{j=1}^n$ and use Φ to denote both the associated matrix and the frame. As we will see in proposition 2.11, singular values, and singular value decompositions play an important role in frame theory.

Definition 2.9. *Let B be a $m \times n$ matrix with elements in $\mathbb{F}$. Let $(\lambda_j)_j^n$ be the n eigenvalues of B^*B. In which case we case we call $(\sigma_j)_j^{\min(n,m)}$ the singular values of B where $\sigma_j = \sqrt{\lambda_j}$.*

Definition 2.10. *Let B be a $m \times n$ matrix with elements in $\mathbb{F}$. The singular value decomposition, often referred to as the SVD, is a factorization*

$$B = U\Sigma V^*$$

Where U is an $m \times m$ unitary matrix, V is an $n \times n$ unitary matrix, and Σ is a $m \times n$ diagonal matrix with non-increasing non-negative real diagonal entries. The diagonal values of Σ, $(\sigma_j)_{j=1}^{\min(m,n)}$ are the first $\min(m,n)$ singular values of B. Furthermore, the number of non-zero singular values is equal to the rank of B.

Every matrix has a singular value decomposition but in general, this decomposition is not unique. However with a fixed ordering on the singular values, the matrix Σ is uniquely determined by the matrix B, and we can gain a lot of information from the singular values and the existence of an SVD of Φ.

Proposition 2.11. *Let $(\varphi_j)_{j=1}^n$ be a collection of vectors in $\mathbb{F}^d$ with Φ its associated $d \times n$ (synthesis operator) matrix with SVD $\Phi = U\Sigma V^*$, where the diagonal entries of Σ are decreasing. Then Φ is the synthesis operator for the frame $(\varphi_j)_{j=1}^n$ with optimal bounds A and B if and only if $n \geq d$, $\sigma_d^2 = A$ and $\sigma_1^2 = B$. Furthermore, $(\varphi_j)_{j=1}^n$ is a tight frame with frame bound A if and only if the rows of Φ are orthogonal with norm $\sqrt{A}$.*

Proof. Notice first that with the SVD of Φ we can write

$$S = \Phi\Phi^* = U\Sigma V^*(U\Sigma V^*)^* = U\Sigma V^* V^* \Sigma^* U^* = U\Sigma\Sigma^* U^* \tag{2}$$

And to compute the frame bounds we notice that for any $x \in F^d$ the frame operator allows us to express $\langle Sx, x \rangle = \left\langle \sum_{j=1}^n \langle x, \varphi_j \rangle \varphi_j, x \right\rangle = \sum_{j=1}^n \langle x, \varphi_j \rangle \langle \varphi_j, x \rangle = \sum_{j=1}^n | \langle x, \varphi_j \rangle |^2$ meaning

$$\sum_{j=1}^n | \langle x, \varphi_j \rangle |^2 = \langle Sx, x \rangle = \langle U\Sigma\Sigma^* U^* x, x \rangle = \langle \Sigma^* U^* x, \Sigma^* U^* x \rangle = \|\Sigma^* U^* x\|^2$$

And so our frame bounds are determined by the maximum and minimum of $\|\Sigma^* U^* x\|^2$. Notice that we need only consider vectors x with unit norm, in which case $\|U^* x\| = 1$. To maximize $\|\Sigma^* y\|$ we may choose the vector $y = \mathbf{e}_1$, as the diagonal entries of Σ are ordered in decreasing order, and because $U^*U = I$ we know that if x is equal to the first column of U then $U^* x = \mathbf{e}_1$ which maximizes $\|\Sigma^* U^* x\|^2 = \|\sigma_1 \mathbf{e}_1\| = \sigma_1^2$, which

References

[1] Emily King. *Algebraic, Geometric, and Combinatorial Methods in Frame Theory.* Unpublished, 2020.

[2] Fahad Panolan, M. S. Ramanujan, and Saket Saurabh. On the Parameterized Complexity of Girth and Connectivity Problems on Linear Matroids. In Frank Dehne, Jörg-Rüdiger Sack, and Ulrike Stege, editors, *Algorithms and Data Structures*, Lecture Notes in Computer Science, pages 566–577, Cham, 2015. Springer International Publishing.

Proposition 2.5. *Let V be a finite inner product space with an orthonormal collection of vectors $(e_j)_{j=1}^n$. The following are equivalent*

(1) if $x \in V$ is such that $\langle x, e_j \rangle = 0$ for all $j \in [n]$ then $x = 0$

(2) $(e_j)_{j=1}^n$ spans V. (Notice that this means $(e_j)_{j=1}^n$ is an orthonormal basis, meaning $n = \dim V$)

(3) $x = \sum_{j=1}^n \langle x, e_j \rangle e_j$ for all $x \in V$

(4) $\|x\|^2 = \sum_{j=1}^n |\langle x, e_j \rangle|^2$ for all $x \in V$ (Parseval's equality)

In many applications of orthonormal bases (3) is of critical importance and highlights why orthonormal bases are often more desirable than general bases. (4) is relevant in defining a frame and acts as a generalization of the Pythagorean theorem so we will provide a proof that (4) is equivalent to the other 3 parts.

Proof. (3) $\Rightarrow$ (4): First for any $x \in V$ let $x = \sum_{j=1}^n \langle x, e_j \rangle e_j$ and notice that from the Pythagorean theorem $\|x\|^2 = \left\| \sum_{j=1}^n \langle x, e_j \rangle e_j \right\|^2 = \sum_{j=1}^n |\langle x, e_j \rangle|^2$.

We will show (4) $\Rightarrow$ (1) by contrapositive: Assume there exists a vector $x \in V$ such that $\langle x, e_j \rangle = 0$ for all $j \in [n]$ but $\|x\| > 0$. However notice that $\sum_{j=1}^n |\langle x, e_j \rangle|^2 = \sum_{j=1}^n 0 = 0$, And so the (4) is not true. $\square$

2.3 Frames

Now that we have seen some useful properties of orthonormal basis we can generalize them to create what is called a frame. In this case, we will define a frame with a weakened version of Parseval's equality which was seen in Proposition 2.5, where we showed that Parseval's equality was a necessary and sufficient condition to show that a finite collection of orthonormal vectors was an orthonormal basis. The results in this paper are well known results, see [1] as a reference.

Definition 2.6. *A finite collection of vectors $(\varphi_j)_j$ from $\mathbb{F}^d$ is a frame for $\mathbb{F}^d$ if there exists optimal constants $0 < A \leq B < \infty$ such that*

$$A\|x\|^2 \leq \sum_{j=1}^n |\langle x, \varphi_j \rangle|^2 \leq B\|x\|^2 \quad \forall x \in \mathbb{F}^d \tag{1}$$

where A is called the lower frame bound and B is called the upper frame bound.

- *A frame is tight if $A = B$, in which case A is used to denote the single frame bound, and Parseval if $A = B = 1$.*
- *A finite frame is equal-norm if there exists some β such that $\|\varphi_j\| = \beta$ for all j and unit-norm if $\beta = 1$.*
- *A unit-norm collection of vectors (φ_j) is called equiangular if there exists some $\alpha \geq 0$ such that $\langle \varphi_j, \varphi_k \rangle = \alpha$ when $j \neq k$.*

We may denote a finite unit-norm tight frame as FUNTF and an equiangular tight frame as ETF. Notice that from this definition every orthonormal basis is a Parseval unit-norm frame. This is an equivalence.

Proposition 2.7. *A collection of vectors $(\varphi_j)_j$ from $\mathbb{F}^d$ is a Parseval unit-norm frame if and only if it is an orthonormal basis.*

Proof. First notice that any orthonormal basis satisfies Parseval's equality and therefore is a Parseval frame. An orthonormal basis also has unit-norm vectors.

So now assume that $(\varphi_j)_j$ is a unit-norm Parseval frame. This means for some fixed k we have $1 = \|\varphi_k\| = \sum_{j=1}^n |\langle \varphi_k, \varphi_j \rangle|^2 = |\langle \varphi_k, \varphi_k \rangle| + \sum_{j \neq k} |\langle \varphi_k, \varphi_j \rangle|^2 = 1 + \sum_{j \neq k} |\langle \varphi_k, \varphi_j \rangle|^2$ Which means $\sum_{j \neq k} |\langle \varphi_k, \varphi_j \rangle|^2 = 0$ and because $|\langle \varphi_k, \varphi_j \rangle|^2 \geq 0$ for all j, we know that $\langle \varphi_k, \varphi_j \rangle = 0$ for all j and all fixed k. Meaning $(\varphi_j)_j$ is an orthonormal collection of vectors satisfying Parseval's equality and so is an orthonormal basis by Proposition 2.5. $\square$

Notice we do not require linearity in the second terms, as instead using properties (1) and (2) we get a form of conjugate linearity (often referred to as anti-linearity) where $\langle x, \alpha y\rangle = \overline{\langle \alpha y, x\rangle} = \overline{\alpha}\overline{\langle y, x\rangle} = \overline{\alpha}\,\langle x, y\rangle$ and $\langle x, y+z\rangle = \overline{\langle y+z, x\rangle} = \overline{\langle y, x\rangle} + \overline{\langle z, x\rangle} = \langle x, y\rangle + \langle x, z\rangle$.

A vector space V along with an inner product often denoted $(V, \langle -, -\rangle)$, is called an *inner product space*. Throughout this paper, we will assume all vector spaces are inner product spaces and hence just refer to an inner product space as a vector space V. We will also consider norms induced by inner products.

Proposition 2.2. *Let $(V, \langle -, -\rangle)$ be an inner product space with V a vector space over a subfield of $\mathbb{C}$. Then $\|-\| : V \to \mathbb{R}$ where $\|x\| = \sqrt{\langle x, x\rangle}$ is a well-defined norm.*

An important example of an inner product is the standard inner product which is defined by $\langle x, y\rangle = y^*x$, which in Euclidean space is called the dot product and measures the cosine of the angle between vectors taking into account the magnitude of the vectors. And its induced norm measures the length of a vector. In general inner products generalize the notion of how similar two vectors are, with an inner product of zero indicating orthogonality or that vectors have no similarity. Norms generalize the notion of the magnitude or the length of a vector.

Complex inner products are an example of sesquilinear forms, meaning an inner product over a finite-dimensional vector space V is of the form $\langle x, y\rangle = y^*Ax$ for a Hermitian positive definite matrix A. However notice that we may decompose $A = U^*U$, by the spectral theorem and the fact that A is positive definite, where U is invertible, meaning $\langle x, y\rangle = y^*Ax = y^*U^*Ux = (Uy)^*(Ux)$. This shows that any inner product looks like the standard inner product with a change of basis. So we may assume that all inner products are the standard inner product on the elementary basis, up to a change of basis.

Definition 2.3. *Suppose V is an inner product space with $v, w \in V$ then v and w are called orthogonal when $\langle v, w\rangle = 0$. A collection of vectors $(e_j)_j$ is called orthogonal if $\langle e_j, e_k\rangle = 0$ for $j \neq k$. If in addition $\|e_j\| = 1$ for all j, then $(e_j)_j$ is called orthonormal. If $(e_j)_j$ is a basis for V then it is called an orthonormal basis.*

For any finite-dimensional vector space $\mathbb{F}^n$, the standard basis is orthonormal and is often denoted as $(e_j)_{j=1}^n$. However, in this paper, we may use $(e_j)_{j=1}^n$ to denote any orthonormal basis.

2.2 Properties of Orthogonality

In this part, we will look at important properties of orthonormal vectors which we will use to motivate the definition of a frame.

Lemma 2.4. *(Pythagorean theorem) Let V be an inner product space with its induced norm, with a finite collection of orthogonal vectors $(x_j)_{j=1}^n$ then*

$$\left\|\sum_{j=1}^n x_j\right\|^2 = \sum_{j=1}^n \|x_j\|^2$$

Proof. This follows from repeated application of the standard Pythagorean theorem. To see this we will use induction on n. Notice first that when $n = 1$ this follows trivially from $(x_j)_{j=1}^1$ having 1 vector. For $n = 2$, notice that for orthogonal vectors $x, y \in V$ we have that $\|v + w\|^2 = \langle x + y, x + y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \langle x, x\rangle + \langle y, y\rangle = \|x\|^2 + \|y\|^2$. Now fix $n > 2$ and notice that

$$\sum_{j=1}^{n+1} \|x_j\|^2 = \sum_{j=1}^{n} \|x_j\|^2 + \|x_{n+1}\|^2 = \left\|\sum_{j=1}^n x_j\right\|^2 + \|x_{n+1}\|^2 = \left\|\sum_{j=1}^n x_j + x_{n+1}\right\|^2 = \left\|\sum_{j=1}^{n+1} x_j\right\|^2$$

where the last step follows from the $n = 2$ case. ☐

Notice that any finite orthonormal collection of vectors $(e_j)_{j=1}^n$, is linearly independent. To see a proof sketch of this assume that $a_1e_1 + \cdots + a_ne_n = 0$, which means that $a_1e_1 + \cdots + a_{n-1}e_{n-1} = -a_ne_n$, Using orthogonality of $(e_j)_{j=1}^n$ we may show $\langle a_1e_1 + \cdots + a_{n-1}e_{n-1}, -a_ne_n\rangle = 0$, which must mean $a_n = 0$ and $a_1e_1 + \cdots + a_{n-1}e_{n-1} = 0$. This can be repeated to show $a_1 = \cdots = a_n = 0$. This shows that orthogonality implies linear independence. Other properties of orthonormal vectors, in particular orthonormal bases, are shown below without complete proof.

Who Framed Matroid Rabbit

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May 30, 2023

Abstract

In this paper, we motivate and develop definitions and fundamental results in frame theory using linear algebra. We then present definitions and results from matroid theory and use them to further understand the linear independence of frames and their Naimark complements.

1 Introduction

An orthonormal basis is a fundamental concept in linear algebra that plays a crucial role in many computational tasks, as these bases span the entire underlying space and their vectors are pairwise orthogonal. For example, orthonormal bases have many applications in information theory especially with compression algorithms that use nonstandard orthonormal basis to better align with the structure of the data. Frames aim to generalize orthonormal bases by adding redundancy which can act as a method of error and noise mitigation and often align with the structure of the data even more so than an orthonormal basis, which is a motivation behind dictionary learning algorithms.

This paper aims to explore the theory and applications of frames, building on concepts from linear algebra and combinatorics. In Section 2, we define orthonormal bases and discuss some of the properties that motivate the use of frames. We then introduce frames in Section 2.3, exploring their linear algebraic properties and providing proofs for many of the fundamental theorems in frame theory. In Section 3, we study the structure of frames through their linear independence, using matroids. Finally, in Section 4, we provide an example of an equiangular tight frame and its associated Naimark complement, using matroids to analyze their structure and properties.

2 Frames

2.1 Orthonormal Bases

A frame is a generalization of an orthonormal basis often adding redundancy. To start we will define an orthonormal basis and then prove Parseval's equality which motivates the definition of a frame.

Throughout this paper, we will be looking at vector spaces that are also inner product spaces, which allows us to define orthogonality. We will restrict our focus to vector spaces with coefficients that are subfields of $\mathbb{C}$, usually $\mathbb{R}$ or $\mathbb{C}$, however, the theorems and definitions we will develop may apply to finite inner product spaces over other fields.

Definition 2.1. *An inner product on a vector space V over $K \subseteq \mathbb{C}$ is a mapping $\langle -, - \rangle : V \times V \to K$ such that for all $x, y, z \in V$ and $\alpha \in K$ we have*

(1) conjugate symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$ *where* $\overline{(\cdot)} : \mathbb{C} \to \mathbb{C}$ *denotes the complex conjugation.*

(2) linearity in the first term: $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ *and* $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

(3) positive definiteness: $\langle x, x \rangle > 0$ *for all* $x \neq 0$