Equiangular Tight Frames and Mutually Unbiased Bases over Finite Fields

Ian Jorquera

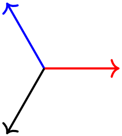
Advisor: Emily J. King
Committee: James B. Wilson
Clayton Shonkwiler
Ewan Davies

Colorado State University
College of Natural Sciences,
Mathematics Department

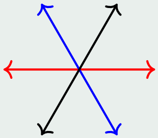
January 28, 2026

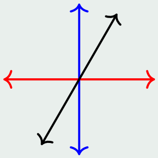
Table of Contents

Optimal Line Packings Over $\mathbb{R}$ and $\mathbb{C}$



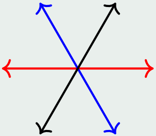
Line Packings: Pack n lines in $\mathbb{R}^d$ or $\mathbb{C}^d$, where every line is maximally spread apart.

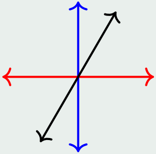




Goal: Maximize pairwise interior angles, or minimize $\cos^2\theta$

Line Packings: Pack n lines in $\mathbb{R}^d$ or $\mathbb{C}^d$, where every line is maximally spread apart.





Goal: Maximize pairwise interior angles, or minimize $\cos^2\theta$

Given n lines, represent each by a unit vector

$$\Phi = \begin{bmatrix} | & | & & | \\ \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ | & | & & | \end{bmatrix} \in \mathbb{F}^{d\times n}$$

New Goal: Minimize $\max_{i\neq j} |\langle \varphi_i, \varphi_j \rangle|^2$

Given n lines in $\mathbb{R}^d$ or $\mathbb{C}^d$, represented each by a unit vector

$$\Phi = \begin{bmatrix} | & | & & | \\ \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ | & | & & | \end{bmatrix}$$

New Goal: Minimize $\max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|^2$.

Given n lines in $\mathbb{R}^d$ or $\mathbb{C}^d$, represented each by a unit vector

$$\Phi = \begin{bmatrix} | & | & & | \\ \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ | & | & & | \end{bmatrix}$$

New Goal: Minimize $\max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|^2$.

Coherence: $\mu^2(\Phi) = \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|^2 \geq 0$
Φ is a **Grassmannian Frame** if Φ is a global minimizer for the coherence

Given n lines in $\mathbb{R}^d$ or $\mathbb{C}^d$, represented each by a unit vector

$$\Phi = \begin{bmatrix} | & | & & | \\ \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ | & | & & | \end{bmatrix}$$

New Goal: Minimize $\max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|^2$.

Coherence: $\mu^2(\Phi) = \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|^2 \geq 0$

Φ is a **Grassmannian Frame** if Φ is a global minimizer for the coherence

How To Find Grassmannian Frames:

Step 1: Find a lower bound on coherence.
Step 2: Find examples which meet the bound.

Given n lines in $\mathbb{R}^d$ or $\mathbb{C}^d$, represented each by a unit vector

$$\Phi = \begin{bmatrix} | & | & & | \\ \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ | & | & & | \end{bmatrix}$$

New Goal: Minimize $\max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|^2$.

Coherence: $\mu^2(\Phi) = \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|^2 \geq 0$

Φ is a **Grassmannian Frame** if Φ is a global minimizer for the coherence

How To Find Grassmannian Frames:

Step 1: Find a lower bound on coherence.

Step 2: Find examples which meet the bound.

$\mu^2(\Phi) \geq 0 \Rightarrow \Phi = (\varphi_j)_{j=1}^n$ orthonormal is a Grassmannian frame.

The Welch-Rankin Bound and Equinagular Tight Frames.

For $\Phi = \left[\varphi_1, \ldots, \varphi_n\right]$ in $\mathbb{F}^d$.

Welch-Rankin Bound (Welch; 1974) (Rankin; 1955)

$$\mu^2(\Phi) = \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|^2 \geq \frac{n-d}{d(n-1)}$$

The Welch-Rankin Bound and Equinagular Tight Frames.

For $\Phi = \left[\varphi_1, \ldots, \varphi_n\right]$ in $\mathbb{F}^d$.

Welch-Rankin Bound (Welch; 1974) (Rankin; 1955)

$$\mu^2(\Phi) = \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|^2 \geq \frac{n-d}{d(n-1)}$$

With equality if and only if

$$\left.\begin{array}{l} \bullet\ \textbf{Equiangular}\text{: } |\langle \varphi_i, \varphi_j \rangle|^2 = b \text{ for all } i \neq j \\ \bullet\ \textbf{Tightness}\text{: } \Phi\Phi^* = cI \end{array}\right\} \Phi \text{ is an ETF}$$

Tightness generalize the Pythagorean theorem or Parceval's identity

$$\Phi\Phi^* = cI \Leftrightarrow \sum_{i=1}^{d} \|\langle x, \varphi_i \rangle \varphi_i\|^2 = c\,\|x\|^2$$

Understanding These Optimal Line Packings

Welch-Rankin Equality: ETFs are understood in two ways

Geometrically as ETFs

- **Equiangular**: $i \neq j$
  $|\langle \varphi_i, \varphi_j \rangle|^2 = b$
- **Tightness**: $\Phi\Phi^* = cI$

Combinatorially with
$$b = \frac{n-d}{d(n-1)}$$

- $n = \#$ lines
- $d =$ dimension

Understanding These Optimal Line Packings

Welch-Rankin Equality: ETFs are understood in two ways

Geometrically as ETFs

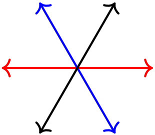
- **Equiangular**: $i \neq j$
  $|\langle \varphi_i, \varphi_j \rangle|^2 = b$
- **Tightness**: $\Phi\Phi^* = cI$

Combinatorially with

$$b = \frac{n-d}{d(n-1)}$$

- $n = \#$ lines
- $d =$ dimension

Example: Optimal line packing in $\mathbb{R}^2$ (an ETF)



$$\Phi = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\theta = \frac{2\pi}{3} \quad \text{and} \quad b = |-1/2|^2 = 1/4 = \frac{3-2}{2(3-1)}$$

When do ETFs exists?

Gerzon's Bound (Lemmens, Seidel; 1973) (Gerzon)

An equiangular system of lines $\Phi = (\varphi_j)_{j=1}^{n}$ for $\mathbb{F}^d$ exists only if

$$n \leq \begin{cases} \frac{d(d+1)}{2} & \text{if } \mathbb{F}^d = \mathbb{R}^d \\ d^2 & \text{if } \mathbb{F}^d = \mathbb{C}^d \end{cases} =: Z(\mathbb{F}, d)$$

Proof sketch:

lines $\leftrightarrow$ vectors $\leftrightarrow$ rank-1 projections.

Equiangular lines $\rightarrow$ linearly independent rank-1 projections.

When do ETFs exists?

Gerzon's Bound (Lemmens, Seidel; 1973) (Gerzon)

An equiangular system of lines $\Phi = (\varphi_j)_{j=1}^n$ for $\mathbb{F}^d$ exists only if

$$n \leq \begin{cases} \frac{d(d+1)}{2} & \text{if } \mathbb{F}^d = \mathbb{R}^d \\ d^2 & \text{if } \mathbb{F}^d = \mathbb{C}^d \end{cases} =: Z(\mathbb{F}, d)$$

Conjecture (Godsil, Royle; 2001), (N. Gillespie; 2018)

In $\mathbb{R}^d$ there exists an ETF of $d(d+1)/2$ vectors if and only if $d = 2, 3, 7, 23$.

Conjecture (Zauner; 1999)

In $\mathbb{C}^d$ there exists an ETF of d^2 vectors for all d.

The Orthoplex Bound and Mutually Unbiased Bases

Orthoplex Bound

From sphere packing bounds by (Rankin; 1955), when $n > Z(\mathbb{F}, d)$

$$\mu^2(\Phi) = \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|^2 \geq \frac{1}{d}$$

The Orthoplex Bound and Mutually Unbiased Bases

Orthoplex Bound

From sphere packing bounds by (Rankin; 1955), when $n > Z(\mathbb{F}, d)$

$$\mu^2(\Phi) = \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|^2 \geq \frac{1}{d}$$

Examples which meet this bound: Mutually Unbiased Bases (MUBs)

$$B = \frac{1}{2}\left[\begin{array}{cccc|cccc|cccc|cccc|cccc} 2 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -i & -i & i & i & -i & -i & i & i \\ 0 & 0 & 2 & 0 & 1 & -1 & -1 & 1 & -i & i & i & -i & -i & i & i & -i & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 & 1 & -1 & 1 & -1 & -i & i & -i & i & -1 & 1 & -1 & 1 & -i & i & i & -i \end{array}\right]$$

$N = 5$ MUBs for $\mathbb{C}^4$ totaling $n = 20$ vectors ($Z(\mathbb{C}, 4) = 14$).

N MUBs for $\mathbb{C}^d$ form a Grassmannian Frames when $N \geq d + 1$

When do MUBs exists?

Let $\mathcal{M}_d\mathbb{F}$ be the maximum number of MUBs in $\mathbb{F}^d$.

Theorem (Ivonovic; 1981), (Wootters, Fields; 1989)

For all d

$$\mathcal{M}_d\mathbb{C} \leq d + 1$$

If $d = p^k$ a prime power then

$$\mathcal{M}_d\mathbb{C} = d + 1.$$

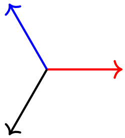
For $d = p_1^{k_1} \cdots p_r^{k_r}$ then

$$\min(p_1^{k_1} + 1, \dots, p_r^{k_r} + 1) \leq \mathcal{M}_d\mathbb{C}$$

Conjectures (Zauner; 1999)

$$\mathcal{M}_6\mathbb{C} = 3$$

“Optimal” Line Packings over Finite Fields



Discretizing Reality: Finite Field Analog to $\mathbb{R}^d$

Real IP Spaces	$\rightsquigarrow$	**Orthogonal Geometries**
$\mathbb{R}^d$	$\rightsquigarrow$	$\mathbb{F}_q^d$, where $q = p^\ell$ is odd.
Inner Products	$\rightsquigarrow$	**Non-Degenerate Symmetric Scalar Products**

Discretizing Reality: Finite Field Analog to $\mathbb{R}^d$

Real IP Spaces	$\rightsquigarrow$	**Orthogonal Geometries**
$\mathbb{R}^d$	$\rightsquigarrow$	$\mathbb{F}_q^d$, where $q = p^\ell$ is odd.

Inner Products	$\rightsquigarrow$	**Non-Degenerate Symmetric Scalar Products**
$\langle -, - \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ $\langle x, y \rangle = \langle y, x \rangle$ $\langle x, - \rangle : \mathbb{R}^d \to \mathbb{R}$ linear		

Discretizing Reality: Finite Field Analog to $\mathbb{R}^d$

Real IP Spaces	$\rightsquigarrow$	**Orthogonal Geometries**
$\mathbb{R}^d$	$\rightsquigarrow$	$\mathbb{F}_q^d$, where $q = p^\ell$ is odd.

Inner Products	$\rightsquigarrow$	**Non-Degenerate Symmetric Scalar Products**
$\langle -, - \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$		
$\langle x, y \rangle = \langle y, x \rangle$		
$\langle x, - \rangle : \mathbb{R}^d \to \mathbb{R}$ linear		
$\langle x, x \rangle > 0$ iff $x \neq 0$		

Discretizing Reality: Finite Field Analog to $\mathbb{R}^d$

Real IP Spaces	$\rightsquigarrow$	**Orthogonal Geometries**
$\mathbb{R}^d$	$\rightsquigarrow$	$\mathbb{F}_q^d$, where $q = p^\ell$ is odd.

Inner Products	$\rightsquigarrow$	**Non-Degenerate Symmetric Scalar Products**
$\langle -, - \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$		$\langle -, - \rangle : \mathbb{F}_q^d \times \mathbb{F}_q^d \to \mathbb{F}_q$
$\langle x, y \rangle = \langle y, x \rangle$	$\rightsquigarrow$	$\langle x, y \rangle = \langle y, x \rangle$
$\langle x, - \rangle : \mathbb{R}^d \to \mathbb{R}$ linear		$\langle x, - \rangle : \mathbb{F}_q^d \to \mathbb{F}_q$ linear
$\langle x, x \rangle > 0$ iff $x \neq 0$		

Discretizing Reality: Finite Field Analog to $\mathbb{R}^d$

Real IP Spaces	$\rightsquigarrow$	**Orthogonal Geometries**
$\mathbb{R}^d$	$\rightsquigarrow$	$\mathbb{F}_q^d$, where $q = p^\ell$ is odd.

Inner Products	$\rightsquigarrow$	**Non-Degenerate Symmetric Scalar Products**
$\langle -, - \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$		$\langle -, - \rangle : \mathbb{F}_q^d \times \mathbb{F}_q^d \to \mathbb{F}_q$
$\langle x, y \rangle = \langle y, x \rangle$	$\rightsquigarrow$	$\langle x, y \rangle = \langle y, x \rangle$
$\langle x, - \rangle : \mathbb{R}^d \to \mathbb{R}$ linear		$\langle x, - \rangle : \mathbb{F}_q^d \to \mathbb{F}_q$ linear
$\langle x, x \rangle > 0$ iff $x \neq 0$	$\rightsquigarrow$	$\langle x, y \rangle \neq 0$ for some $y \in \mathbb{F}_q^d$ iff $x \neq 0$

Discretizing Reality: Finite Field Analog to $\mathbb{R}^d$

Real IP Spaces	$\rightsquigarrow$	**Orthogonal Geometries**
$\mathbb{R}^d$	$\rightsquigarrow$	$\mathbb{F}_q^d$, where $q = p^\ell$ is odd.

Inner Products	$\rightsquigarrow$	**Non-Degenerate Symmetric Scalar Products**
$\langle -, - \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$		$\langle -, - \rangle : \mathbb{F}_q^d \times \mathbb{F}_q^d \to \mathbb{F}_q$
$\langle x, y \rangle = \langle y, x \rangle$	$\rightsquigarrow$	$\langle x, y \rangle = \langle y, x \rangle$
$\langle x, - \rangle : \mathbb{R}^d \to \mathbb{R}$ linear		$\langle x, - \rangle : \mathbb{F}_q^d \to \mathbb{F}_q$ linear
$\langle x, x \rangle > 0$ iff $x \neq 0$	$\rightsquigarrow$	$\langle x, y \rangle \neq 0$ for some $y \in \mathbb{F}_q^d$ iff $x \neq 0$

Example: Non-degeneracy as a proof of being non-zero

$V = \mathbb{F}_3^3$ with $\langle x, y \rangle = x^\intercal y$ the dot product.

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle = 0 \quad \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle = 1$$

Discretizing the Imaginary: Finite Field Analog to $\mathbb{C}^d$

Complex IP Spaces	$\rightsquigarrow$	**Unitary Geometries**
$\mathbb{C}^d$	$\rightsquigarrow$	$\mathbb{F}_{q^2}^d$, where $q = p^\ell$.
$x \mapsto \overline{x}$	$\rightsquigarrow$	$x \mapsto x^q$
Inner Products	$\rightsquigarrow$	**Non-Degenerate Hermitian Scalar Products**

Discretizing the Imaginary: Finite Field Analog to $\mathbb{C}^d$

Complex IP Spaces	$\rightsquigarrow$	**Unitary Geometries**
$\mathbb{C}^d$	$\rightsquigarrow$	$\mathbb{F}_{q^2}^d$, where $q = p^\ell$.
$x \mapsto \overline{x}$	$\rightsquigarrow$	$x \mapsto x^q$

Inner Products	$\rightsquigarrow$	**Non-Degenerate Hermitian Scalar Products**
$\langle -, - \rangle : \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}$		$\langle -, - \rangle : \mathbb{F}_{q^2}^d \times \mathbb{F}_{q^2}^d \to \mathbb{F}_{q^2}$
$\boldsymbol{\langle x, y \rangle = \overline{\langle y, x \rangle}}$	$\rightsquigarrow$	$\boldsymbol{\langle x, y \rangle = \langle y, x \rangle^q}$
$\langle x, - \rangle : \mathbb{C}^d \to \mathbb{C}$ linear		$\langle x, - \rangle : \mathbb{F}_{q^2}^d \to \mathbb{F}_{q^2}$ linear
$\langle x, x \rangle > 0$ iff $x \neq 0$	$\rightsquigarrow$	$\langle x, y \rangle \neq 0$ for some $y \in \mathbb{F}_{q^2}^d$ iff $x \neq 0$

Discretizing the Imaginary: Finite Field Analog to $\mathbb{C}^d$

Complex IP Spaces	$\rightsquigarrow$	**Unitary Geometries**
$\mathbb{C}^d$	$\rightsquigarrow$	$\mathbb{F}_{q^2}^d$, where $q = p^\ell$.
$x \mapsto \overline{x}$	$\rightsquigarrow$	$x \mapsto x^q$

Inner Products	$\rightsquigarrow$	**Non-Degenerate Hermitian Scalar Products**
$\langle -, - \rangle : \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}$		$\langle -, - \rangle : \mathbb{F}_{q^2}^d \times \mathbb{F}_{q^2}^d \to \mathbb{F}_{q^2}$
$\langle x, y \rangle = \overline{\langle y, x \rangle}$	$\rightsquigarrow$	$\langle x, y \rangle = \langle y, x \rangle^q$
$\langle x, - \rangle : \mathbb{C}^d \to \mathbb{C}$ linear		$\langle x, - \rangle : \mathbb{F}_{q^2}^d \to \mathbb{F}_{q^2}$ linear
$\langle x, x \rangle > 0$ iff $x \neq 0$	$\rightsquigarrow$	$\langle x, y \rangle \neq 0$ for some $y \in \mathbb{F}_{q^2}^d$ iff $x \neq 0$

Helpful Notation

$z \in \mathbb{C}$ we have $\lvert z\rvert^2 = z\overline{z}$	$\rightsquigarrow$	$x \in \mathbb{F}_{q^2}$ we have $N(x) = xx^q$
$x \in \mathbb{R}$ we have $\lvert x\rvert^2 = x^2$	$\rightsquigarrow$	$x \in \mathbb{F}_q$ we have $N(x) = x^2$

Types of Geometry

Definition

A $\mathbb{F}$-vector space V is called **non-degenerate** if it has a non-degenerate symmetric/Hermitian scalar product.

We can always pick a basis $\{e_1, \ldots, e_d\}$ for V, identifying $V = \mathbb{F}^d$.

Types of Geometry

Definition

A $\mathbb{F}$-vector space V is called **non-degenerate** if it has a non-degenerate symmetric/Hermitian scalar product.

We can always pick a basis $\{e_1, \ldots, e_d\}$ for V, identifying $V = \mathbb{F}^d$.

Case O

$V = \mathbb{F}_q^d$
$\langle x, y \rangle = x^\intercal M y$, where $M^\intercal = M$ and is invertible.

There is a basis for V such that $M = \mathrm{Diag}(1, \ldots, 1, \delta)$

- $\delta = 1$ is a square
- δ is not a square

Types of Geometry

Definition

A $\mathbb{F}$-vector space V is called **non-degenerate** if it has a non-degenerate symmetric/Hermitian scalar product.

We can always pick a basis $\{e_1, \ldots, e_d\}$ for V, identifying $V = \mathbb{F}^d$.

Case O

$V = \mathbb{F}_q^d$
$\langle x, y \rangle = x^\intercal M y$, where $M^\intercal = M$ and is invertible.

There is a basis for V such that $M = \mathrm{Diag}(1, \ldots, 1, \delta)$

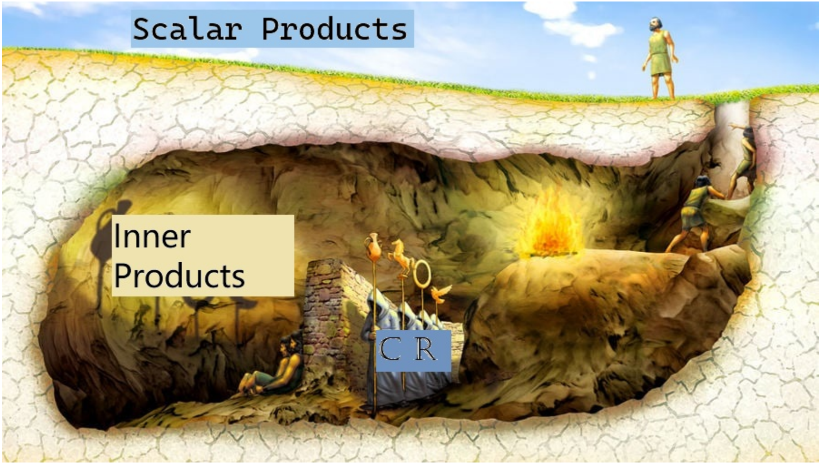
- $\delta = 1$ is a square
- δ is not a square

Case U

$V = \mathbb{F}_{q^2}^d$
$\langle x, y \rangle = x^* M y$, where $M^* = M$ and is invertible.

There is a basis for V such that $M = \mathrm{Diag}(1, \ldots, 1)$

Plato's Allegory of the Inner Product



Plato's Allegory of the Inner Product

Inner Product Spaces:

- Subspaces of inner product spaces are inner product spaces
- $\Phi = \left[\varphi_1, \ldots, \varphi_n\right]$ and its Gram matrix $\Phi^*\Phi = [\langle \varphi_i, \varphi_j \rangle]$ give "equivalent information".

Plato's Allegory of the Inner Product

Inner Product Spaces:

- Subspaces of inner product spaces are inner product spaces
- $\Phi = \left[\varphi_1, \ldots, \varphi_n\right]$ and its Gram matrix $\Phi^*\Phi = [\langle \varphi_i, \varphi_j \rangle]$ give "equivalent information".

Case O and U: Not the case. Consider an orthogonal geometry $V = \mathbb{F}_3^4$ with $\langle x, y \rangle = x^\mathsf{T} y$

$$\Phi = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 2 & 2 \\ 0 & 1 \end{bmatrix} \quad \Phi^\dagger \Phi = [\langle \varphi_i, \varphi_j \rangle] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\operatorname{im} \Phi \subseteq V$ is degenerate.

Frame Theory (Greaves, Iverson, Jasper, Mixon; 2022), (**J**, King; 2025)

Let $\Phi = \left[\varphi_1, \varphi_2 \ldots, \varphi_n\right]$ from $V = \mathbb{F}^d$, $a, b, c \in \mathbb{F}$. Then Φ is a

- **frame** for $\operatorname{im}\Phi$ if $\operatorname{im}\Phi$ is non-degenerate $\Leftrightarrow \mathrm{rk}(\Phi) = \mathrm{rk}(\Phi^\dagger\Phi)$
- c-**tight frame** for $\operatorname{im}\Phi$ if $\Phi(\Phi^\dagger\Phi) = c\Phi$
- (a, b)-**equiangular** if
  - $\langle\varphi_j, \varphi_j\rangle = a$ for all j
  - $N(\langle\varphi_j, \varphi_k\rangle) = b$ for all $j \neq k$
- (a, b, c)-**equiangular tight frame(ETF)** if all the above.

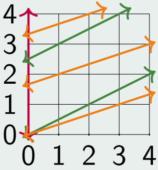
Frame Theory (Greaves, Iverson, Jasper, Mixon; 2022), (**J**, King; 2025)

Let $\Phi = \left[\varphi_1, \varphi_2 \ldots, \varphi_n\right]$ from $V = \mathbb{F}^d$, $a, b, c \in \mathbb{F}$. Then Φ is a

- **frame** for $\operatorname{im}\Phi$ if $\operatorname{im}\Phi$ is non-degenerate $\Leftrightarrow \mathsf{rk}(\Phi) = \mathsf{rk}(\Phi^\dagger\Phi)$
- c-**tight frame** for $\operatorname{im}\Phi$ if $\Phi(\Phi^\dagger\Phi) = c\Phi$
- (a, b)-**equiangular** if
  - $\langle\varphi_j, \varphi_j\rangle = a$ for all j
  - $N(\langle\varphi_j, \varphi_k\rangle) = b$ for all $j \neq k$
- (a, b, c)-**equiangular tight frame(ETF)** if all the above.

Example: $V = \mathbb{F}_5^2$ with $\langle x, y\rangle = x^\intercal My$, where $M = \mathrm{Diag}(1, 3)$

$$\Phi = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix} \qquad \Phi^\dagger\Phi = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$



Φ is an $(2, 1, 3)$-ETF for $\mathbb{F}_5^2$ of $n = 3$ vectors.

Frame Theory: ETFs in case O and U

$V = \mathbb{F}_3^4$ with $\langle x, y \rangle = x^\intercal M y$, where $M = \mathrm{Diag}(1,1,1,2)$

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Φ is an $(0,1,0)$-ETF for $\mathbb{F}_3^4$ of $n = 10$ vectors.

$V = \mathbb{F}_{3^2}^5$ with $\langle x, y \rangle = x^* y$. Ψ is a $(0,1,0)$-ETF of 16 vectors.

$$\Psi = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & a & a & a & a & a^3 & a^3 & a^3 & a^3 \\ a & a & a^5 & a^5 & a^5 & a^5 & a^5 & a^5 & 1 & 1 & 1 & 1 & a^6 & a^6 & a^6 & a^6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & a & a & a^5 & a^5 & a^3 & a^3 & a^7 & a^7 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & a & a^5 & a & a^5 & a^3 & a^7 & a^3 & a^7 \\ 0 & 0 & a^2 & a^6 & 0 & 0 & 0 & 0 & a^7 & a^3 & a^3 & a^7 & a & a^5 & a^5 & a \end{bmatrix}$$

When do ETFs exists?

Gerzon's Bound (Greaves, Iverson, Jasper, Mixon; 2022)

An (a, b)-equiangular system of lines $\Phi = (\varphi_j)_{j=1}^n$, where $a^2 \neq b$, for $V = \mathbb{F}^d$ exists only if

$$n \leq \begin{cases} \frac{d(d+1)}{2} & \text{if } \mathbb{F} = \mathbb{F}_q \text{ and } V \text{ is in case O} \\ d^2 & \text{if } \mathbb{F} = \mathbb{F}_{q^2} \text{ and } V \text{ is in case U} \end{cases} =: Z(\mathbb{F}, d)$$

Proof sketch: practically the same as before.

There is no upper bound when $a^2 = b$, there exists large examples.

“Optimality” in Case U

Over finite fields, there is no notion of coherence to be optimized. We are merely mimicking what we once knew to be “optimal.”

"Optimality" in Case U

Over finite fields, there is no notion of coherence to be optimized. We are merely mimicking what we once knew to be "optimal."

Theorem (Greaves, Iverson, Jasper, Mixon; 2022)

If Φ is a ETF of n vectors for $\mathbb{C}^d$ then there exists ETFs of n vectors in $\mathbb{F}_{q^2}^d$, in Case U, for infinity many fields with distinct characteristics.

"Optimality" in Case U

Over finite fields, there is no notion of coherence to be optimized. We are merely mimicking what we once knew to be "optimal."

Theorem (Greaves, Iverson, Jasper, Mixon; 2022)

If Φ is a ETF of n vectors for $\mathbb{C}^d$ then there exists ETFs of n vectors in $\mathbb{F}_{q^2}^d$, in Case U, for infinity many fields with distinct characteristics.

Theorem (J)

If B is a collection of N MUBs for $\mathbb{C}^d$ then there exists N MUBs in $\mathbb{F}_{q^2}^d$, in Case U, for infinity many fields with distinct characteristics.

"Optimality" in Case U

Over finite fields, there is no notion of coherence to be optimized. We are merely mimicking what we once knew to be "optimal."

Theorem (Greaves, Iverson, Jasper, Mixon; 2022)

If Φ is a ETF of n vectors for $\mathbb{C}^d$ then there exists ETFs of n vectors in $\mathbb{F}_{q^2}^d$, in Case U, for infinity many fields with distinct characteristics.

Theorem (J)

If B is a collection of N MUBs for $\mathbb{C}^d$ then there exists N MUBs in $\mathbb{F}_{q^2}^d$, in Case U, for infinity many fields with distinct characteristics.

MUBs Conjecture Rephrased

In $\mathbb{C}^6$ there does exists more then 3 MUBs.

“Optimality” in Case U

Over finite fields, there is no notion of coherence to be optimized. We are merely mimicking what we once knew to be “optimal.”

Theorem (Greaves, Iverson, Jasper, Mixon; 2022)

If Φ is a ETF of n vectors for $\mathbb{C}^d$ then there exists ETFs of n vectors in $\mathbb{F}_{q^2}^d$, in Case U, for infinity many fields with distinct characteristics.

Theorem (J)

If B is a collection of N MUBs for $\mathbb{C}^d$ then there exists N MUBs in $\mathbb{F}_{q^2}^d$, in Case U, for infinity many fields with distinct characteristics.

MUBs Conjecture Rephrased

There are at most finitely many fields with distinct characteristics where $\mathbb{F}_{q^2}^6$ has more then 3 MUBs.

A Flash Back: Understanding ETFs

ETFs are understood in two ways

Geometrically as ETFs

- **Equiangular**: $i \neq j$
  $|\langle \varphi_i, \varphi_j \rangle|^2 = b$
- **Tightness**: $\Phi\Phi^* = cI$

Combinatorially with

$$b = \frac{n-d}{d(n-1)}$$

- $n = \#$ lines
- $d =$ dimension

Do we get this over Finite Fields?

Short answer: No.

Long answer: Yes!

On the Failure of a Welch-Rankin Equality

Theorem (Greaves, Iverson, Jasper, Mixon; 2022)

If Φ is a (a, b, c)-ETF for $V = \mathbb{F}^d$ then $d(n-1)b \equiv (n-d)a^2$
(if the field is nice: $b = \frac{n-d}{d(n-1)}a^2$)

On the Failure of a Welch-Rankin Equality

Theorem (Greaves, Iverson, Jasper, Mixon; 2022)

If Φ is a (a,b,c)-ETF for $V=\mathbb{F}^d$ then $d(n-1)b \equiv (n-d)a^2$ (if the field is nice: $b=\frac{n-d}{d(n-1)}a^2$)

Example: $V=\mathbb{F}_5^7$ with $\langle x,y\rangle = x^{\intercal}y$

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 4 & 0 & 2 \\ 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 2 & 3 & 2 & 2 & 3 \\ 1 & 1 & 0 & 2 & 3 & 4 & 4 & 1 \end{bmatrix}$$

Φ is an $(2,1)$-equiangular frame for V.
It satisfies $b \equiv 1 \equiv \frac{1}{49}2^2 \equiv \frac{n-d}{d(n-1)}a^2$.

On the Failure of a Welch-Rankin Equality

Theorem (Greaves, Iverson, Jasper, Mixon; 2022)

If Φ is a (a, b, c)-ETF for $V = \mathbb{F}^d$ then $d(n-1)b \equiv (n-d)a^2$ (if the field is nice: $b = \frac{n-d}{d(n-1)}a^2$)

Example: $V = \mathbb{F}_5^7$ with $\langle x, y \rangle = x^\intercal y$

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 4 & 0 & 2 \\ 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 2 & 3 & 2 & 2 & 3 \\ 1 & 1 & 0 & 2 & 3 & 4 & 4 & 1 \end{bmatrix}$$

Φ is an $(2, 1)$-equiangular frame for V.
It satisfies $b \equiv 1 \equiv \frac{1}{49}2^2 \equiv \frac{n-d}{d(n-1)}a^2$. But Φ is not a tight frame

A New Hope: Using Sums of Triple Products

- Triple Product: $\Delta(\varphi_j, \varphi_k, \varphi_\ell) = \langle \varphi_j, \varphi_k \rangle \langle \varphi_k, \varphi_\ell \rangle \langle \varphi_\ell, \varphi_j \rangle$
- Sums of triple products have been used to study the algebraic properties of frames by (Appleby et. al.; 2011), (Zhu; 2015), and (King; 2019).

A New Hope: Using Sums of Triple Products

- Triple Product: $\Delta(\varphi_j, \varphi_k, \varphi_\ell) = \langle \varphi_j, \varphi_k \rangle \langle \varphi_k, \varphi_\ell \rangle \langle \varphi_\ell, \varphi_j \rangle$
- Sums of triple products have been used to study the algebraic properties of frames by (Appleby et. al.; 2011), (Zhu; 2015), and (King; 2019).

Let $\mathbb{F}$ be a field with $\mathrm{char}\mathbb{F} \nmid dn$, and $V = \mathbb{F}^d$ in case O or U.

Theorem (J; 2025)

Let $\Phi = \left[\varphi_1, \ldots, \varphi_n\right]$ for V be an (a, b)-equiangular frame for V $(a \neq 0)$. Then Φ is an $(a, b, na/d)$-ETF if and only if

-
-

A New Hope: Using Sums of Triple Products

- Triple Product: $\Delta(\varphi_j, \varphi_k, \varphi_\ell) = \langle \varphi_j, \varphi_k \rangle \langle \varphi_k, \varphi_\ell \rangle \langle \varphi_\ell, \varphi_j \rangle$
- Sums of triple products have been used to study the algebraic properties of frames by (Appleby et. al.; 2011), (Zhu; 2015), and (King; 2019).

Let $\mathbb{F}$ be a field with $\operatorname{char}\mathbb{F} \nmid dn$, and $V = \mathbb{F}^d$ in case O or U.

Theorem (J; 2025)

Let $\Phi = \left[\varphi_1, \ldots, \varphi_n\right]$ for V be an (a, b)-equiangular frame for V $(a \neq 0)$. Then Φ is an $(a, b, na/d)$-ETF if and only if

- $d(n-1)b = (n-d)a^2$
-

A New Hope: Using Sums of Triple Products

- Triple Product: $\Delta(\varphi_j, \varphi_k, \varphi_\ell) = \langle \varphi_j, \varphi_k \rangle \langle \varphi_k, \varphi_\ell \rangle \langle \varphi_\ell, \varphi_j \rangle$
- Sums of triple products have been used to study the algebraic properties of frames by (Appleby et. al.; 2011), (Zhu; 2015), and (King; 2019).

Let $\mathbb{F}$ be a field with $\operatorname{char}\mathbb{F} \nmid dn$, and $V = \mathbb{F}^d$ in case O or U.

Theorem (J; 2025)

Let $\Phi = \left[\varphi_1, \ldots, \varphi_n\right]$ for V be an (a, b)-equiangular frame for V $(a \neq 0)$. Then Φ is an $(a, b, na/d)$-ETF if and only if

- $d(n-1)b = (n-d)a^2$
- $\displaystyle\sum_{\ell=1}^{n} \langle \varphi_j, \varphi_k \rangle \langle \varphi_k, \varphi_\ell \rangle \langle \varphi_\ell, \varphi_j \rangle = \frac{nab}{d}$ for all $j \neq k$

Applications of this Welch-Rankin Equality

Theorem (J; 2025)

Let $\Phi = \left[\varphi_1, \ldots, \varphi_n\right]$ for $\mathbb{F}^d$ be an (a, b)-equiangular frame

- $\mathrm{char}\mathbb{F} \nmid d(d+1)$, and $a \neq 0$, $a^2 \neq b$
- $\Gamma \subseteq \Phi$ of $d+1$ vectors, where $\langle \varphi_j, \varphi_k \rangle \langle \varphi_k, \varphi_\ell \rangle \langle \varphi_\ell, \varphi_j \rangle$ is constant for all distinct $\varphi_j, \varphi_k, \varphi_\ell \in \Gamma$

Then Γ is a regular d-simplex, an $(a, b, \frac{(d+1)a}{d})$-ETF

The converse of this is also true: a d-simplex has equal triple products.

Applications of this Welch-Rankin Equality

Theorem (**J**; 2025)

Let $\Phi = \left[\varphi_1, \ldots, \varphi_n\right]$ for $\mathbb{F}^d$ be an (a, b)-equiangular frame

- $\mathrm{char}\,\mathbb{F} \nmid d(d+1)$, and $a \neq 0$, $a^2 \neq b$
- $\Gamma \subseteq \Phi$ of $d+1$ vectors, where $\langle \varphi_j, \varphi_k \rangle \langle \varphi_k, \varphi_\ell \rangle \langle \varphi_\ell, \varphi_j \rangle$ is constant for all distinct $\varphi_j, \varphi_k, \varphi_\ell \in \Gamma$

Then Γ is a regular d-simplex, an $(a, b, \frac{(d+1)a}{d})$-ETF

The converse of this is also true: a d-simplex has equal triple products.

More Results (**J**; 2025)

- Determine existence of k-simplices contained in Φ (which span subspaces when $k < d$).
- Show that certain, case O, ETFs with $n = d(d+1)/2$ vectors give rise to combinatorial 4-designs.

MUBs over Finite Fields

Definition: $B = \left[B_0 | \cdots | B_{N-1}\right]$ is a collection of N MUB for $V = \mathbb{F}_{q^2}^d$, in case U, if

- $B_j = \begin{bmatrix} u_1 & \cdots & u_d \end{bmatrix}$ is an orthonormal basis: $u, u' \in B_j$ has $\langle u, u \rangle = 1$, and $\langle u, u' \rangle = 0$.
- $u \in B_j$ and $v \in B_k$ we have $N(\langle u, v \rangle) = d^{-1}$.

MUBs over Finite Fields

Definition: $B = \left[B_0 | \cdots | B_{N-1}\right]$ is a collection of N MUB for $V = \mathbb{F}_{q^2}^d$, in case U, if

- $B_j = \begin{bmatrix} u_1 & \cdots & u_d \end{bmatrix}$ is an orthonormal basis: $u, u' \in B_j$ has $\langle u, u \rangle = 1$, and $\langle u, u' \rangle = 0$.
- $u \in B_j$ and $v \in B_k$ we have $N(\langle u, v \rangle) = d^{-1}$.

As matrices, each B_j is unitary: $B_j^* B_j = I$.
So

$$B_0^* B = \left[B_0^* B_0 | \cdots | B_0^* B_{n-1}\right] = \left[I | \hat{B}_1 | \cdots | \hat{B}_{n-1}\right]$$

is also a MUB.

MUBs over Finite Fields

Definition: $B = \left[B_0 | \cdots | B_{N-1}\right]$ is a collection of N MUB for $V = \mathbb{F}_{q^2}^d$, in case U, if

- $B_j = \begin{bmatrix} u_1 & \cdots & u_d \end{bmatrix}$ is an orthonormal basis: $u, u' \in B_j$ has $\langle u, u \rangle = 1$, and $\langle u, u' \rangle = 0$.
- $u \in B_j$ and $v \in B_k$ we have $N(\langle u, v \rangle) = d^{-1}$.

As matrices, each B_j is unitary: $B_j^* B_j = I$.
So

$$B_0^* B = \left[B_0^* B_0 | \cdots | B_0^* B_{n-1}\right] = \left[I | \hat{B}_1 | \cdots | \hat{B}_{n-1}\right]$$

is also a MUB.
Let α be a field element such that $N(\alpha) = d$ in which case

- $\alpha \hat{B}_j$ is a Hadamard matrix
- $\alpha \hat{B}_j \hat{B}_k$ is also a Hadamard matrix.

When do Hadamards Exist?

Definintion: Hadamard Matrix

a $d \times d$ matrix $H = [h_{jk}]$ with entries in $\mathbb{F}_{q^2}$ ($\mathrm{char}\mathbb{F}_{q^2} \nmid d$) is called a Hadamard if

- $H^* H = dI$
- Each entry $N(h_{jk}) = h_{jk}^{q+1} = 1$.

We can rescale H to be in the form

$$\begin{bmatrix} 1 & h_{12} & \cdots & h_{1d} \\ 1 & h_{22} & \cdots & h_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & h_{d2} & \cdots & h_{dd} \end{bmatrix}$$

Each column is a vanishing sum of d $(q+1)$-roots of unity.

When do Vanishing Sums of Roots of Unity Exist?

Theorem (Lam, Leung; 1996), (J)

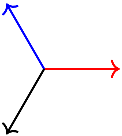
Fix a finite field $\mathbb{F}_{q^2}$ ($q = p^\ell$), and a prime r distinct from p such that $p^{r-1} \neq 1 \pmod{r^2}$.
A $d \times d$ Hadamard matrix with entries r^mth roots of unity exists only if $d \in \mathbb{N}r + \mathbb{N}p$ and $p \nmid d$.

Proof sketch: The cyclotomic polynomial $\Phi_{r^m}(x)$ is irreducible over $\mathbb{F}_p$.

This gives finitely many results (on the order of 2^{16}) in terms of MUBs such as

- $\mathcal{M}_7\mathbb{F}_{16^2} = 1$
- $\mathcal{M}_{15}\mathbb{F}_{16^2} = 1$
- $\mathcal{M}_{15}\mathbb{F}_{256^2} = 1$
- $\mathcal{M}_{6005}\mathbb{F}_{65536^2} = 1$

Future Work



Future Work: When do Vanishing Sums of Roots of Unity Exist?

- Over finite fields, $\Phi_m(x)$ is almost never irreducible, but there are more cases to explore where it is.
- There is limited known about when vanishing sums of n roots of unity exists for small n ($d = 6$ is often "small")
- For a field $\mathbb{F}_{q^2}$, q is often odd, which meaning the non-existence of vanishing sums can not rule out the existence of 6×6 Hadamards. But they can tell us more about the entries.

Future Work: ETFs over Quaternions

Many authors have studied frame theoretic objection over $\mathbb{H}$:
(Hoggar; 1976-1998),(Khatirinejad Fard; 2008), (Et-Taoui; 2020), (Iverson, King, Mixon; 2021), (Waldron; 2024)

- Welch-Rankin Bound works: ETFs are optimal again. (Waldron; 2024)
- Gerzon's Bound: no more then $2d^2 - d$ equiangular lines can exists in $\mathbb{H}^d$. (Waldron; 2024)
- Ex: An ETF of $n = 2(2)^2 - 2 = 6$ lines exists in $\mathbb{H}^2$ (Khatirinejad Fard; 2008) (Et-Taoui; 2020).

My future contributions:

- Use and study the alternating projections algorithm in this context, for finding ETFs with non-trivial symmetry groups.

Questions

$$\Phi = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

An $(0, 1, 1)$-ETF for $\mathbb{F}_3^3$