MATH 502. HW 10
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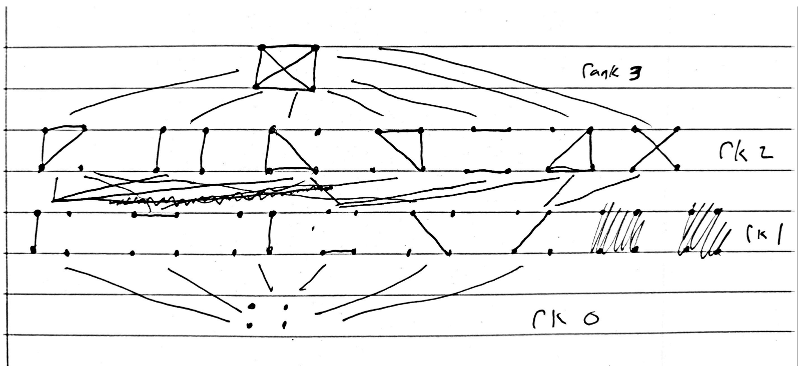
(1) Let $M = (E, \mathcal{B})$ and let $M^* = E, \mathcal{B}^*$ where $\mathcal{B}^* = \{E - B | B \in \mathcal{B}\}$. Notice that $\mathcal{B}^*$ because $\mathcal{B}$ is non-empty. Consider bases $B_1, B_2 \in \mathcal{B}$ such that $B_1 \neq B_2$ meaning $E - B_1 \neq E - B_2$ Notice that this means for any $x \in (E - B_1) - (E - B_2)$ that $x \in B_2 - B_1$ meaning there must exists some $y \in B_1 - B_2 = (E - B_2) - (E - B_1)$ such that $B_2 - x + y$ is a basis in $\mathcal{B}$ meaning $E - B_2 - y + x \in \mathcal{B}^*$. So M^* is a matroid .

(2) 

(3) Let X, and Y be flats of a matroid $M = (E, \text{cl})$, that is $\text{cl}(X) = X$ and $\text{cl}(Y) = Y$. We will show that on the lattice of flats $X \wedge Y = X \cap Y$. First, we will show that $X \cap Y$ is a common lower bound. Notice first that $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, and from the closure axioms we know that this means that $\text{cl}(X \cap Y) \subseteq \text{cl}(X) = X$ and $\text{cl}(X \cap Y) \subseteq \text{cl}(Y) = Y$. Meaning $\text{cl}(X \cap Y) \subseteq X \cap Y$ and by the closure axioms we know that $X \cap Y \subseteq \text{cl}(X \cap Y)$, and so $X \cap Y$ is a flat. Assume also that there existed a flat W such that $W \subseteq X$ and $W \subseteq Y$. This would necessarily means that $W \subseteq X \cap Y$, and so $X \wedge Y = X \cap Y$.

Now we will show that on the lattice of flats $X \vee Y = cl(X \cup Y)$. Notice first the by the closure axioms that because $X \subseteq X \cap Y$ and $Y \subseteq X \cap Y$ we have that $\text{cl}(X) = X \subseteq \text{cl}(X \cap Y)$ and $\text{cl}(Y) = Y \subseteq \text{cl}(X \cap Y)$. And so $\text{cl}(X \cap Y)$ is a common upper bound. Assume also that there existed a flat W such that $X \subseteq W$ and $Y \subseteq W$ which means that $X \cup Y \in W$. and by the closure axioms we know that $\text{cl}(X \cup Y) \in \text{cl}(W) = W$ and so $X \vee Y = \text{cl}(X \cup Y)$.

(5) Suppose $\{x, y\}$ and $\{y, z\}$ are circuits such that no x, y, z alone is a circuit. By C3 axioms we know that $\{x, y\} \cup \{x, y\} - y = \{x, z\}$ contains a circuit. But because no singleton or the empty set is a circuit we know that $\{x, z\}$ must be a circuit.

(7) First, notice that because the greedy algorithm will always pick elements of E that form an independence set it will always form a basis by the axioms I3. So let $B \in \mathcal{B}$ be a basis obtained with a greedy algorithm, and let B^* be a basis of minimal weight. Assume that $\text{wt}(B) = \text{wt}(B^*)$ then we are done. Otherwise assume that $\text{wt}(B) > \text{wt}(B^*)$ with means that $B \neq B^*$ Notice that for any element $e \in B - B^*$ by the basis exchange axiom, we know that there exists some $y \in B^* - B$ such that $B - e + y$ is a basis. Notice furthermore that at the step e was chosen by the greedy algorithm we may have also chosen y, and because e was chosen minimally we know that $\text{wt}(e) \leq \text{wt}(y)$. meaning $\text{wt}(B) \leq \text{wt}(B - e + y)$. We may repeat this process for all $e \in B - B^*$ which replaces B to be B^* in which case we find that $\text{wt}(B) \leq \text{wt}(B^*)$. And so B the basis generated by the greedy algorithm finds minimal bases.