Math 670 HW #1

Due 11:59 PM Friday, February 21

1. A smooth manifold M is called *orientable* if there exists a collection of coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ so that, for every α, β such that $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) = W \neq \emptyset$, the differential of the change of coordinates $\phi_\beta^{-1} \circ \phi_\alpha$ has positive determinant.

   (a) Show that for any n, the sphere S^n is orientable.

   Proof. Here we will consider theatlas from the notes, generated by $\{(\mathbb{R}^n, \phi_N), (\mathbb{R}^n, \phi_S)\}$ Notice that because $\phi_N(\mathbb{R}^2) \cap \phi_S(\mathbb{R}^n) = S^2 - \{N, S\}$ we can consider the change of coordinates map which is $(\phi_N \circ \phi_S)(\vec{x}) = \frac{1}{||\vec{x}||^2}\vec{x}$. Notice that becasue this is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ the differential is the Jacobian which is just $\frac{1}{||\vec{x}||^2}I$, which has positive determinant.

   □

   (b) Prove that, if M and N are smooth manifolds and $f : M \to N$ is a local diffeomorphism at all points of M, then N being orientable implies that M is orientable. Is the converse true?

   Proof. Becasue N is orientable, there is an atlas $\{(V_\beta, \psi_\beta)\}$ for N such that any change of variables has differential with positive determinant. Now we will consider an atlas $\{(U_\alpha, \phi_\alpha)\}$ for M. Any point $p \in M$, there exists chart (U, ϕ) and (V, ψ) where $p \in \phi(U)$ and $f(p) \in \psi(V)$ and $f : \phi(U) \to \psi(V)$ is a diffeomorphism. Now consider a second chart (U_2, ϕ_2) containing the point p. Now we want to show that the differential of $\phi_2^{-1} \circ \phi$ defined on $U \cap U_2$ has positive determinant. Let (V_2, ψ_2) be a chart containing $f(\phi_2(U_2))$. Notice that from chasing diagrams we have that

   $$\phi_2^{-1} \circ \phi = \phi_2^{-1} \circ f^{-1} \circ \psi_2 \circ \psi_2^{-1} \circ \psi \circ \psi^{-1} \circ f \circ \phi$$

   on $U \cap U_2$ in which case we can determine that the differential at any point $p \in U \cap U_2$ is e

   □

2. Supply the details for the proof that, if $F : \mathrm{Mat}_{d\times d}(\mathbb{C}) \to \mathcal{H}(d)$ is given by $F(U) = UU^*$ (where U^* is the conjugate transpose [a.k.a., Hermitian adjoint] of U), then the unitary group

   $$U(d) = F^{-1}(I_{d\times d})$$

   is a submanifold of $\mathrm{Mat}_{d\times d}(\mathbb{C})$ of dimension d^2. (Hint: it may be helpful to remember that a Hermitian matrix M can always be written as $M = \frac{1}{2}(M + M^*)$.)

   Proof. Notice first that $\mathrm{Mat}_{d\times d}(\mathbb{C})$ is a real manifold with dimension $2d^2$. Likewise $\mathcal{H}(d)$ is a real manifold with with dimension d, this can be computed by directly entries that satisfy $M = M^*$, where the diagonal has to be all real entries. Next recall that $T_I\mathrm{Mat}_{d\times d} \cong \mathrm{Mat}_{d\times d}$. We can also use the a defining equation $M = \frac{1}{2}(M + M^*)$ to determine that any curve

$\alpha(t)$ that satisfies this relation would have a derivative at $t = 0$ equal to $\frac{d}{dt}|_{t=0}[\alpha(t)] = \frac{d}{dt}|_{t=0}\left[\frac{1}{2}(\alpha(t) + \alpha^*(t))\right] = \frac{1}{2}(\alpha'(0) + \alpha'(t)^*)$ and so $T_I\mathcal{H}(d) \cong \mathcal{H}(d)$

Here we want to use the level set theorem. So we will show that $I \in \mathcal{H}(d)$ is a regular point. Notice that $F(I) = I$. To show I is regular we will show that the differential $dF_I : T_I\mathrm{Mat}_{d\times d} \to T_I\mathcal{H}(d)$ is surjective. Consider a smooth curve $\alpha(t)$ through I with velocity v in $\mathrm{Mat}_{d\times d}$. This would give us the curve $\beta(t) = F \circ \alpha(t) = \alpha(t)\alpha(t)^*$. Where the derivative at $t = 0$ is $\frac{d}{dt}|_{t=0}[\beta(t)] = \alpha'(0)\alpha(0)^* + \alpha(0)\alpha'(0)^* = v + v^*$. This shows that for any $v \in T_I\mathcal{H}(d) \cong \mathcal{H}(d)$ we have that the tangent vector $\frac{1}{2}v$ would map to the tangent vector v. This shows that dF_I is surjective.

So from the level set theorem we have that $F^{-1}(I) = \{UU^* = 1\} = U(d)$ is a submanifold of dimension d^2. □

3. Let M be a compact manifold of dimension n and let $f : M \to \mathbb{R}^n$ be a smooth map. Prove that f must have at least one critical point.

   Proof. First notice that $f(M) \subseteq \mathbb{R}^n$ is compact because f is continuous and M is compact. This means the image $f(M)$ is closed and bounded. Let q be a point of the boundary of $f(M)$, and p and point that maps to q. Notice that this means that there is a direction v in the tangent space $T_q\mathbb{R}^n$ that would point out of $f(M)$, meaning any curve $\beta(t)$ through q with velocity v would leave $f(M)$. So v is not in the image of the differential df_p and so the differential is not surjective. And so p is a critical point. □

4. Prove that, if X, Y, and Z are smooth vector fields on a smooth manifold M and $a, b \in \mathbb{R}$, $f, g \in C^\infty(M)$, then

   (a) $[X, Y] = -[Y, X]$ (anticommutivity)

   (b) *Proof.* $[X, Y] = XY - YX = -(YX - XY) = -[X, Y]$ □

   (c) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ (linearity)

   (d) *Proof.* Let $a, b \in \mathbb{R}$ then notice that because $\mathcal{X}(M)$ is a $\mathbb{R}$-module and XY is well defined(as a distributive product on vector fields that gives back a differential operator) we have that $[aX + bY, Z] = (aX + bY)Z - Z(aX + bY) = aXZ + bYZ - aZX - bZY = a[X, Z] + b[Y, Z]$ □

   (e) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity)

   Proof.

$$
\begin{aligned}
&[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y] = [XY-YX,Z]+[YZ-ZY,X]+[ZX-XZ,Y]\\
&= (XY-YX)Z-Z(XY-YX)+(YZ-ZY)X-X(YZ-ZY)+(ZX-XZ)Y-Y(ZX-XZ)\\
&= XYZ-YXZ-ZXY-ZYX+YZX-ZYX-XYZ-XZY+ZXY-XZY-YZX-YXZ\\
&= 0
\end{aligned}
$$

□

   (f) $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$.

Proof. This follows from the fact that X is a first order differential operator, and so we can utilize the product rule in the case $X(gY(h)) = X(g)Y(h) + gX(Y(h))$ where $h \in C^\infty(M)$ and so we can simply write $X(gY) = (Xg)Y + gXY$ which gives us

$$\begin{aligned}
[fX, gY] &= fX(gY) - gY(fX) \\
&= f(Xg)Y + fgXY - g(Yf)X - gfYX \\
&= fg[X, Y] + f(Xg)Y - g(Yf)X
\end{aligned}$$

$\square$