MATH 601. HW 2
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(3.3) Let $\phi_{21} = \varphi_2 \circ \varphi_1^{-1}$ and consider an element $x_0 \in (\varphi_1(U_1 \cap U_2), x) = \mathbb{C} - \{0\}$ The map φ_1^{-1} takes a point $x_0 \neq 0$ on the x axis and maps it the line ℓ through the origin that contains the point $(x_0, 1)$, which is the line that intersection the line $Y = 1$ at an x value of x_0. The the map φ_2 takes this line ℓ and maps it the the y value at the intersection of ℓ and the line $X = 1$, that is a point of the form $(1, y_0)$ which has the value $y_0 = \frac{1}{x_0}$. So $\phi_{21}(z) = \frac{1}{z}$ for all $z \neq 0$.

(4.2) Let $f : \mathbb{CP}^1 \to \mathbb{C}$ be a meramorphic function meaning with both charts $f \circ \varphi_j^{-1} : \mathbb{C} \to \mathbb{C}$ is meramorphic for $j = 1, 2$. Consider first the zeros of $f \circ \varphi_j^{-1}$, the collection $\{x \in \mathbb{C} | f \circ \varphi_j^{-1}(x) = 0\}$. We know that either $f \equiv 0$ or the set of zeros is discrete. And because $\mathbb{CP}^1$ is compact we know that under φ_1^{-1} that $\{\varphi_1^{-1}(x) \in \mathbb{C} | f \circ \varphi_1^{-1}(x) = 0\}$ must therefore be finite, as the only discrete sets are the finite sets. This is also true for the poles of f as the poles must also be discrete and so there can only be finitely many poles.

Because $f \circ \varphi_1^{-1}$ has finitely many poles and zeros we know we factor out the zeros and poles as

$$f \circ \varphi_1^{-1}(x) = \frac{\displaystyle\prod_{i=1}^{n}(x - a_i)}{\displaystyle\prod_{j=1}^{m}(x - b_j)} h(x)$$

where $h(x)$ is a holomorphic function on $\mathbb{C}$ with no zeros. Under the transition function we know that

$$f \circ \varphi_1^{-1}\left(\frac{1}{x}\right) = \frac{\displaystyle\prod_{i=1}^{n}(1 - a_i y)}{\displaystyle\prod_{j=1}^{m}(1 - b_j y)} h\left(\frac{1}{y}\right)$$

where $h\left(\frac{1}{y}\right)$ is holomorphic everywhere except possible $y = 0$. Because $f \circ \varphi_2^{-1}(y)$ is meromorphic we know that it can not have an essential singularity at $y = 0$ and so $h\left(\frac{1}{y}\right)$ can not have an essential singularity at $y = 0$. Furthermore $y = 0$ can not be a pole as if it were we could factor out a $\frac{1}{y^k}$ which would correspond to x^k in the affine coord x. But this cant be the case as we have already factored out all the zeros. So $y = 0$ is a removable singularity, meaning we can add that point and then $h\left(\frac{1}{y}\right)$ would be holomorphic on both the affine coordinates of x and y, meaning h must be constant by problem (4.1). So

$$f \circ \varphi_1^{-1}(x) = c \cdot \frac{\displaystyle\prod_{i=1}^{n}(x - a_i)}{\displaystyle\prod_{j=1}^{m}(x - b_j)}$$

To get an equivalent interpretation we can homogenize using the fact that the affine coordinate x was defined as $x = X/Y$ which gives us

$$f(X:Y) = c \cdot \frac{\prod_{i=1}^{n}(X/Y - a_i)}{\prod_{j=1}^{m}(X/Y - b_j)} = c \cdot \frac{Y^m \prod_{i=1}^{n}(X - a_i Y)}{Y^n \prod_{j=1}^{m}(X - b_j Y)}$$

Notice also that this means that both the top and bottom are degree $n+m$ homogenous polynomials. We can further simplify, but the degree of the top and bottom will be the same.