Math 601. HW 6

Ian Jorquera

(1) The dimension of the adjoint representation of $\mathfrak{gl}_n$ is the dimension of $\mathfrak{gl}_n$ as a vector space which is n^2 as it is spanned by the matrices E_{ij} for all i, j.

(2) Recall that $\mathfrak{k}$ are all $n \times n$ matrices. First notice that the Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{gl}_n$ are the diagonal matrices. To see this let H be any matrix and let E_{ij} be the elementary matrix with a 1 is row i column j and zero everywhere else. Notice that for $[H, E_{ij}]$ to be a multiple of E_{ij}, H must be diagonal. Let the diagonal entries be $x_1, \ldots, x_n$, in which case $[H, E_{ij}] = (x_i - x_j)E_{ij}$. Notice also that diagonal matrices always commute, meaning the subalgebra of diagonal matrices is abelian, showing it is the Cartan subalgebra. We can also fix a basis for the dial space. Let the function $L_i : \mathfrak{h} \to \mathbb{C}$ be the map that takes a diagonal matrix with diagonal entries be $x_1, \ldots, x_n$ and maps it to x_i. Notice that these span the dual of the Cartan Subalgebra

Now we will determine the roots of $\mathfrak{gl}_n$, the weights of the adjoint representation. Because $\mathfrak{gl}_n$ as a vector space is spanned by the elementary matrices E_{ij}. We can look at how the Cartan subsalgebra acts on this basis, to determine the roots. From before we observed $[H, E_{ij}] = (x_i - x_j)E_{ij} = \alpha(H)E_{ij}$, where $\alpha = L_i - L_j \in \mathfrak{h}^*$. So the roots of $\mathfrak{gl}n$ are non-zero weights $\{L_i - L_j | i \neq j\}$.

Notice that these are exactly the same roots as $\mathfrak{sl}_n$. We can conclude that $\mathfrak{gl}_n$ is not semisimple because the roots only span the space of trace zero diagonal matrices which is one dimension smaller then the space of all diagonal matrices which is the cartan subalgebra of $\mathfrak{gl}_n$.

(3) We will use the definition $\mathfrak{so}_{2n+1} = \{X \in \mathbb{C}^{2n+1 \times 2n+1} : X^t S + SX = 0\}$ where

$$S = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & I_n \\ 0 & I_n & 0 \end{array}\right]$$

Notice that because S is a permutation matrix $S^{-1} = S^t$ and because S is symmetric, $S^{-1} = S^t = S$. So we can rewrite the defining equation as $X^t = -SXS$. Now let $X_1, X_2, X_3, X_4 \in \mathbb{C}^{n \times n}$, and $y_1, y_2, y_3, y_4 \in \mathbb{C}^n$ then

$$X = \left[\begin{array}{c|c|c} 0 & y_1^t & y_2^t \\ \hline y_3 & X_1 & X_2 \\ \hline y_3 & X_1 & X_2 \end{array}\right] \quad X^t = \left[\begin{array}{c|c|c} 0 & y_3^t & y_4^t \\ \hline y_1 & X_1^t & X_3^t \\ \hline y_2 & X_2^t & X_4^t \end{array}\right] \quad SXS = \left[\begin{array}{c|c|c} 0 & -y_2^t & -y_1^t \\ \hline -y_4 & -X_4 & -X_3 \\ \hline -y_3 & -X_2 & -X_1 \end{array}\right]$$

Giving us $y_1 = -y_4$, $y_2 = -y_3$, $X_1^t = -X_4$, $X_2^t = -X_2$, $X_3^t = -X_3$ which tells us about the redundant basis elements $\mathfrak{so}_{2n+1}$. The first two equations give us $2n$ basis elements, the third gives n^2 and the last two give $n^2 - n$. All combined we have $\dim(\mathfrak{so}_{2n+1}) = 2n^2 + n$.

(4) We will use the definition $\mathfrak{so}_5 = \{X \in \mathbb{C}^{5 \times 5} : X^t S + SX = 0\}$ where

$$S = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & I_2 \\ 0 & I_2 & 0 \end{array}\right]$$

This along with the work from the previous problems gives us a basis

$$\begin{aligned}\{E_{12}-E_{41}, E_{13}-E_{51}, E_{14}-E_{21}, E_{15}-E_{31},\\ E_{22}-E_{44}, E_{23}-E_{54}, E_{32}-E_{45}, E_{33}-E_{55},\\ E_{25}-E_{34}, E_{43}-E_{52}\}\end{aligned}$$

Where the first row are the basis elements determining the blocks for the ys, the second for X_1 and X_4 and third line for X_2 and X_3. So this is the basis for the adjoint representation. To see how this corresponds the the adjoint representation, we need to determine the roots, which means we need determine what the Cartan Subalgebra. Because of the definition

$$\mathfrak{h} = \{ \left[\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & a & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ \hline 0 & 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 & -b \end{array}\right] \}$$

This also means the dual of the Cartan is spanned by L_1, L_2 which maps diagonal matrices to a and b respectively So we can look at how this multiplies with a general matrix in $\mathfrak{so}_{2n+1}$

```
H =

[0, 0, 0,  0,  0]
[0, a, 0,  0,  0]
[0, 0, b,  0,  0]
[0, 0, 0, -a,  0]
[0, 0, 0,  0, -b]
```

```
A =

[ 0,  g, h,  i,  j]
[-i,  0, e,  0,  d]
[-j,  f, 0, -d,  0]
[-g,  0, c,  0, -f]
[-h, -c, 0, -e,  0]
```

```
[H,A] =

[   0,       -a*g,          -b*h,           a*i,       b*j]
[-a*i,          0,   a*e - b*e,              0, a*d + b*d]
[-b*j, b*f - a*f,             0, - a*d - b*d,           0]
[ a*g,          0, - a*c - b*c,              0, a*f - b*f]
[ b*h, a*c + b*c,             0,     b*e - a*e,          0]
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This shows us that the Roots are $\{\pm L_1, \pm L_2, \pm L_1 \pm L_2\}$ which is exactly the root system for type B. To see this a bit more we can now that L_1 and L_2 are orthogonal and span a real 2 dimensional space, so the roots are exactly the drawing for type B we already know and love. We also have a choice of positive simple roots being $L_1, L_2 - L_1$.

(7) The corresponding Dynkin Diagram for $\mathfrak{sl}_4$ is 3 dots, so there are 3 unit norm simple roots, 2 are pairwise orthogonal and all angles are $2/3\pi$. We can denote them as

$$s_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, s_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, s_3 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1/\sqrt{2} \end{bmatrix}$$

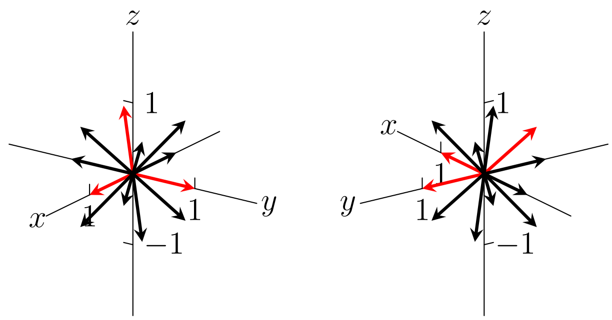
And are shown in red in th diagram below. Next we can add in the negatives of these roots. And finally consider the reflections of the simple roots where we get the following

$$s_1 + s_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/\sqrt{2} \end{bmatrix}, s_1 + s_3 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix}$$

and there negatives. And then again we can look at the reflections of these with the simple roots and get

$$s_1 + s_2 + s_3 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix}$$

This gives us a total of 12 roots. We know that the roots of $\mathfrak{sl}_4$ are $\{L_i - L_j | i \neq j\}$, meaning there are 12 total roots, and so we know the root system would be the following



(9) Notice the the Weyl group would have the following presentation

$$\langle s_0, s_1 | s_0, s_1, s_0s_1s_0s_1s_0s_1 = s_1s_0s_1s_0s_1s_0\rangle$$

And so we can enumerate all group element using the simplification given by the relations, the elements would be

$$\{1,\ s_0,\ s_1,\ s_0s_1,\ s_0s_1s_0,\ s_0s_1s_0s_1,\ s_0s_1s_0s_1s_0,\ s_0s_1s_0s_1s_0s_1\ s_1s_0,\ s_1s_0s_1,\ s_1s_0s_1s_0,\ s_1s_0s_1s_0s_1\}$$

And so $|W_{G_2}| = 12$.