Math 601. HW 3
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(1) Let w be a word of 1s and 2s. Let b be the position of some 2 in the word. And consider the case where the 2 is not paired with some 1. This must also mean there are no unpaired 2s to the left of position b. So the removal of b does not unbracket any 1s. Now consider the case that the 2 at position b was paired with a 1 at position $c > b$. In which case we must consider a few cases.

First assume that the only 2s to the left of b are also bracketed with a 1 also to the left of b. In this case removing the 2 at b will result in the 1 at c being unbracketed but no others, and no 2 bracketing with the 1 at c.

next assume that there exists an unbracketed 2 at position a to the left of b, such that a is also the first 2 to the left of b that is not bracketed with a 1 also to the left of b. In this case the removal of b will result in the 2 at a bracketing with the 1 at c. So no new 1s are unbracketed.

Finally assume that there is a 2 at position $a < b$ which is paired with a 1 at position $d > c$ such that a is the first such 2 to the left of b. Notice that the removal of b will result in the 2 at a pairing with the 1 at c. And if there is another 2 (minimally) to the left of a not paired with a 1 to the left of a then that 2 would pair with the 1 at d. If that 2 was itself paired with a 1 to the right of d then this process would repeat in an inductive fashion, otherwise it wouldn't be paired with a 1 at all and there would be no new 1s left unpaired. If there wasn't an unpaired 2 to the left of a then d would be a new unpaired 1 but no other 1s would be unpaired.

This also shows that at most one new 1 will be unpaired with the removal of any 2.

(2) For word crystals, φ_i counts the number of $i+1$s that are not paired with an i. And likewise ϵ_i counts the number of is that are not paired with $i + 1$. This means word crystals are seminormal, because $\varphi_i(w)$ counts the number of unpaired $i+1$s which means we can apply the operator E_i that many times. Likewise F_i can be applied for each unpaired i of which there are $\epsilon_i(w)$.

(3) Recall first that the $wt : B \to \mathbb{Z}\langle L_1, L_2, \ldots, L_n \rangle / (L_1 + \cdots + L_n)$ is the function that maps any word $w \mapsto (\#1s, \#2s, \ldots, \#ns)$. Notice that if x, y are two words such that $e_i(x) = y$ and $x = f_i(y)$ Notice that x is equal to the word y but with the right most i not paired with an $i + 1$ changed to an $i + 1$. Notice that the weight functions would be as follows, if $wt(x) = (w_1, w_2, \ldots, w_n)$ then $wt(y) = (w_1, w_2, \ldots, w_i + 1, w_{i+1} - 1, w_n) = w(x) + \alpha_i$ Notice also that because word crystals satisfy axiom S0, we also know that this means $\epsilon_i(y) = \epsilon_i(x) - 1$ and $\varphi_i(y) = \varphi_i(x) + 1$. This shows that Word crystals satisfy K1.

Now to see word crystals satisfy K2, let x be a word. Recall that $wt(x) = (\#1s, \#2s, \ldots, \#ns)$. Notice that because the weights classes formed by the relation $L_1 + \cdots + L_n = 0$, let k be the number of paired $i + 1$ with an i in which case we have that $wt(x) = (\#1s - k, \#2s - k, \cdots - k, \#ns - k)$ where $wt(x)_i$ represents the number of is that are not paired with a $i + 1$, or in other words $\varphi_i(x)$ and $wt(x)_{i+1}$ represents the number of $i + 1$s that are not paired with a i, or in other words $\epsilon_i(x)$. So we have $wt(x)_i - wt(x)_{i+1} = wt(x)_i - wt(x)_{i+1} = (\#is - k) - (\#i + 1s - k)$ which is closed under the relation of adding or subtraction a constant to each of the terms in the weight. This proves that word crystal Satisfy K1 and K2.

(4) Using the above for φ_i and ϵ_i for word crystal of words consisting of $1, \ldots, n$ we can consider the length axiom (a). let $|i - j| > 1$ and assume $y = e_i(x)$ which means y is the word after applying E_i to the word x, which changes the left most unpaired $i + 1$ to an i. Notice that because $|i - j| > 1$ this can only have affected the number of paired and unpaired $i - 1$s and $i + 1$s, and not the number of js. So $\epsilon_j(x) = \epsilon_j(y)$ and $\varphi_j(x) = \varphi_j(y)$ as desired. For axiom (b) consider the case where $|i - j| = 1$ and specifically when $j = i + 1$ in which case $y = e_i(x)$ is the word x with the left most unpaired $i + 1 = j$ changed to an i, considering pairing of js with is. Notice there are two cases if j was paired with some $j + 1$ or not. If it was then the number of j that are unpaired with a $j + 1$ does not change, but this makes it so there are 1 addition $j + 1$ that is unpaired, so $\varphi_j(x) = \varphi_j(y)$ and $\epsilon_j(x) + 1 = \epsilon_j(y)$. In the other case if j was not paired with some $j + 1$, then the number js that are unpaired would decrease by 1 but the number of $j + 1$s that are unpaired would not change as no pairings were broken in the removal of the j. So $\varphi_j(x) - 1 = \varphi_j(y)$ and $\epsilon_j(x) = \epsilon_j(y)$

Finally notice that the case where $j = i - 1$. $y = e_i(x)$ is the word x with the left most unpaired $i + 1$ changed to an i, considering pairing of $i + 1$s with is. Notice there are two cases if the new i pairs with some j or not. If it pairs with a j then the number of $j + 1 = i$s, unpaired with a j does not change, but this makes it so there is 1 addition j that is now paired, so the number of unpaired js decreases by one, so $\varphi_j(x) - 1 = \varphi_j(y)$ and $\epsilon_j(x) = \epsilon_j(y)$. In the other case if the new i does not pairs with a j then the number of $j + 1 = i$s, unpaired with a j increases by 1, but no additional j are now paired or unpaired, so the number of unpaired js does not change, so $\varphi_j(x) = \varphi_j(y)$ and $\epsilon_j(x) + 1 = \epsilon_j(y)$.

(7) (a) Assume that w and v are knuth equivalent, and that they differ by at most one knuth move. We may also assume that both words are made up of distinct characters, as if they were not we can relabel with in the following way $iij \cdots \mapsto i(i + 1)(j + 1) \ldots$ where for any consecutive numbers we keep the first the same and add 1 to all remaining characters. Let a be the smallest character in the words w and v. We will consider the possible knuth moves, considering the removal of this smallest letter a.

First consider the case where the smallest letter a occurs in w and v in the same place, outside of any knuth move. Meaning the knuth move(or possible no move at all) making w and v equivalent is not affected by the removal of a. And so with the removal of a the two words are still equivalent.

Now assume again a is the smallest letter and for $a < b < c$ that $w = \ldots acb \ldots$ and $v = \ldots cab \ldots$ then removal of a would result in $w' = \ldots cb \ldots$ and $v' = \ldots cb \ldots$, meaning w and v with the removal of a are the same word, and so would be knuth equivalent.

Now assume for a, the smallest letter and for $a < b < c$ that $w = \ldots bac \ldots$ and $v = \ldots bca \ldots$ then removal of a would result in $w' = \ldots bc \ldots$ and $v' = \ldots bc \ldots$, meaning w and v with the removal of a are the same word, and so would be knuth equivalent.

(b) In order for the concatenated reading word $\mathrm{rw}(S)\mathrm{rw}(T)$ to be ballot every suffix has to be ballot. And because $\mathrm{rw}(T)$ is a suffix, it must be ballot. SSYT that have ballot reading words are highest weight tableau.

Now notice that if we had the pair (T, T) of tableau of of shape μ that are height weight, the RSK bijection would standardize the right tableau to be of the formed of counting from 1 to $|\mu|$ starting from bottom left reading right then moving up a row. Then we would unbump according to tht tableau, and at every step we would have a suffix that is ballot, because taking out a letter j from the unstandardized tableau would require

that we first had to unbump a letter from all rows below it. The un-standardizing would then gives the unique weakly increasing word of the content μ.

(c)(iii) This map is well defined because RSK is a bijection, meaning the two line arrow created in 7b is uniquely determined by the tableau T. And then the process of inserting the bottom row of the two line array also unique determines the resulting tableau by RSK, so then the resulting skew tableau must be well-defined as the letters $a_1, \ldots, a_{|\mu|}$ are unique determined.