Math 601. HW 3

Ian Jorquera

(1) For $\mathfrak{sl}_2$ representation, with weight spaces corresponding to words of 1's and 2s, a word to be lowest weight means that the lowering operator F sends the word to 0 this is precisely the case where the word is anti-ballot, where for all prefixes of the words the number of 2s is greater than or equal to the number of 1s. To see that this is in fact lowest weight let $w = w_1 w_2 \dots w_n$ be a word of 1s and 2s that is anti-ballot. Meaning every prefix has at least as many 2s as 1s. Notice that this means that for every prefix, every 1 will be paired with a 2. This means that there are no unpaired 1s Applying F would then result in 0. Now assume that w is a word that is not anti-ballot. Meaning there is a prefix $w_1 \dots w_i$ that contains more 1s then 2s. We may assume that $w_i = 1$ and this suffix is of minimal length in which case w_i would be the left most unpaired 1, as there are less 2s then 1s in the prefix. This means that $Fw = w_1 w_2 \dots w_{i-1} 2 w_{i+1} \dots w_n$ and so w did not represent a lowest weight vector.

For $\mathfrak{sl}_3$ representation, with weight spaces corresponding to words of 1's 2s and 3s, a word to be lowest weight means that the lowering operators F_1, and F_2 both send the word to 0 this is precisely the case where the word is anti-ballot in both 1s and 2s ignoring 3s and anti-ballot in 2s and 3s ignoring the 1s, where for all prefixes of the words the number of 2s is greater than or equal to the number of 1s and the number of 3s is greater than or equal to the number of 2s. To see that this is in fact lowest weight let $w = w_1 w_2 \dots w_n$ be a word of 1s, 2s, and 3s that is anti-ballot. Meaning every prefix has at least as many 2s as 1s and 3s as 2. Notice that this means that for every prefix, every 1 will be paired with a 2 and every 2 will be paired with a 3. This means that there are no unpaired 1s. Applying F_1 would then result in 0. This also means that there are no unpaired 2s. Applying F_2 would then result in 0. Now assume that w is a word that is not anti-ballot. Meaning there is a prefix $w_1 \dots w_i$ that contains more 1s then 2s or more 2s then 3s. Applying F_1 or F_2 would result in the first unpaired 1 or 2 being switched to a 2 or 3 respectively. In either case this would mean that w did not represent a lowest weight vector.

(2) For $\mathfrak{sl}_3$ tableau crystals, the possible tableau will have shape $\lambda = \lambda_1, \lambda_2, \lambda_3$ where the first λ_1 columns of will have 3 rows and will be filled with content $3, 2, 1$ reading down. The next $\lambda_2 - \lambda_1$ columns will have 2 rows and will be filled with content $3, 2$ reading down. Finally the remaining columns $\lambda_3 - \lambda_2$ will all have 1 row and be filled with 3s. These are the lowest weight Tableau, as there would be no unpaired 2s or unpaired 1s in the reading words.

This means that irreducible $\mathfrak{sl}_3$ representations have unique lowest weights because any irreducible representation have weight spaces corresponding to a particular shape λ which has a unique filling as described above.

(3) First we will order the basis $\mathfrak{sl}_3$ in the following way $E_{12}, E_{13}, E_{23}, E_{21}, E_{31}, E_{32}, H_{12}, H_{23}$ In which case we compute the following commutators: $[E_{12}, E_{2k}] = E_{1k}$ when $k \neq 1$. We also have that $[E_{12}, E_{31}] = -[E_{31}, E_{12}] = -E_{32}$ and $[E_{12}, E_{\ell k}] = 0$ for all other cases $\ell \neq 2$. $[E_{12}, E_{21}] = H_{12}$ And $[E_{12}, H_{12}] = -2E_{12}$ and $[E_{12}, H_{23}] = E_{12}$ This all gives us the following matrix. missing

$$\begin{array}{c|cccccccc|}
 & E_{12} & E_{13} & E_{23} & E_{21} & E_{31} & E_{32} & H_{12} & H_{23} \\
\hline
E_{12} & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \\
E_{13} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
E_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
E_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
E_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
E_{32} & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
H_{12} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
H_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

(4) Consider the embedding $\iota : \mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_3$ Notice first that this map is a linear map which is clearly an embedding. idk maybe show but that would be tedious So we must only show that the map respects the lie for the basis elements of $\mathfrak{sl}_2$ Notice that

$$[\iota(E),\iota(F)] = \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix} - \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix} = \begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix} = \iota(H) = \iota([\iota(E),\iota(F)])$$

$$[\iota(H),\iota(F)] = \begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix} - \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix} = \begin{bmatrix}0&0&0\\-2&0&0\\0&0&0\end{bmatrix} = \iota(-2F) = \iota([\iota(H),\iota(F)])$$

$$[\iota(H),\iota(E)] = \begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix} - \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix} = \begin{bmatrix}0&2&0\\0&0&0\\0&0&0\end{bmatrix} = \iota(2E) = \iota([\iota(H),\iota(E)])$$

All other products are accounted for by properties of the Lie Bracket, because the lie bracket is bilinear and anti-symmetric, with every element lie bracketed with itself being zero.

(7) (a) The homogenous symmetric function $h_d(x_1,\dots,x_n)$ corresponds to the schur function $S_{(d)}(x_1,\dots,x_n)$ because the SSYT of shape (d) allows repeats

(b) Recall that the formal character of the irreducible representation $V^{(a,b)}$ is the schur polynomial on x_1,x_2,x_3 for tableau of shape λ. Meaning $V^{(\mu_1,0)}$ has as its character $S_{(\mu_1)}(x_1,x_2,x_3) = h_{\mu_1}(x_1,x_2,x_3)$. And because the character of a tensor product is the product of the characters we have that the character of $V^{(\mu_1,0)} \otimes V^{(\mu_2,0)} \otimes \cdots \otimes V^{(\mu_k,0)}$ is $h_{\mu_1}(x_1,x_2,x_3) \cdot h_{\mu_2}(x_1,x_2,x_3) \cdots h_{\mu_k}(x_1,x_2,x_3) = h_\mu(x_1,x_2,x_3)$