Math 670 HW #1

Due 11:59 PM Friday, February 21

1. A smooth manifold M is called *orientable* if there exists a collection of coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ so that, for every α, β such that $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) = W \neq \emptyset$, the differential of the change of coordinates $\phi_\beta^{-1} \circ \phi_\alpha$ has positive determinant.

   (a) Show that for any n, the sphere S^n is orientable.

   Proof. □

   (b) Prove that, if M and N are smooth manifolds and $f : M \to N$ is a local diffeomorphism at all points of M, then N being orientable implies that M is orientable. Is the converse true?

   Proof. Becasue N is orientable, there is an atlas $\{(V_\beta, \psi_\beta)\}$ for N such that any change of variables has positive determinant. Now we will consider an atlas $\{(U_\alpha, \phi_\alpha)\}$ for M. Any point $p \in M$, there exists chart (U, ϕ) and (V, ψ) where $p \in \phi(U)$ and $f(p) \in \psi(V)$ and $f : \phi(U) \to \psi(V)$ is a diffeomorphism. Now consider a second chart (U_2, ϕ_2) containing the point p. Now we want to show that $\phi_2^{-1} \circ \phi$ defined on $U \cap U_2$ has positive determinant. Let (V_2, ψ_2) be a chart containing $f(\phi_2(U_2))$. Notice that from chasing diagrams we have that

   $$\phi_2^{-1} \circ \phi = \phi_2^{-1} \circ f^{-1} \circ \psi_2 \circ \psi_2^{-1} \circ \psi \circ \psi^{-1} \circ f \circ \phi$$

   on $U \cap U_2$ in which case

   $$\det(\phi_2^{-1} \circ \phi) = \det(\phi_2^{-1} \circ f^{-1} \circ \psi_2) \det(\psi_2^{-1} \circ \psi) \det(\psi^{-1} \circ f \circ \phi)$$

   And becasue

   □

2. Supply the details for the proof that, if $F : \mathrm{Mat}_{d\times d}(\mathbb{C}) \to \mathcal{H}(d)$ is given by $F(U) = UU^*$ (where U^* is the conjugate transpose [a.k.a., Hermitian adjoint] of U), then the unitary group

   $$U(d) = F^{-1}(I_{d\times d})$$

   is a submanifold of $\mathrm{Mat}_{d\times d}(\mathbb{C})$ of dimension d^2. (Hint: it may be helpful to remember that a Hermitian matrix M can always be written as $M = \frac{1}{2}(M + M^*)$.)

   Proof. □

3. Let M be a compact manifold of dimension n and let $f : M \to \mathbb{R}^n$ be a smooth map. Prove that f must have at least one critical point.

4. Prove that, if X, Y, and Z are smooth vector fields on a smooth manifold M and $a, b \in \mathbb{R}$, $f, g \in C^\infty(M)$, then

   (a) $[X, Y] = -[Y, X]$ (anticommutivity)
   (b) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ (linearity)
   (c) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity)
   (d) $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$.