Math 601. HW 1
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(2) Let B_2 be the upper triangular matrices in $GL_2(\mathbb{C})$. First to show that there is not decomposition notice that the matrices $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$ do not share both of their eigenvectors and therefore V can not be decomposed as a G module.

However notice we can act on the 1-dim subspace spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the action is closed on this subspace as this vector is a common eigenvector for all upper triangular matrices meaning this repn on B_2 is reducible.

(3) Notice that B_n acts on $\mathbb{C}^n$. However notice that for $1 \leq \ell < n$ the group B_n also acts on the ℓ-dimensional space spanned by the first ℓ elementary vectors. This follows from B_n being the upper triangular matrices, and so action by B_n only add the rows of the vector upward Therefore every space spanned by the first ℓ elementary vectors is a subrepresentation.

(5) Consider the permutation representation of $S_3 \to GL_3(\mathbb{C})$.

Because the image of this representation are the permutation matrices, we know that the all 1s vector is an eigenvector. This means with a change of basis matrix $\begin{pmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ which will isolate the eigenvector in the third component with the other two columns being orthogonal, we can block diagonalize each matrix in our representation to get the following decomposed representation. Computations done using a computer

$$() \mapsto \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \quad (1,2) \mapsto \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(1,3) \mapsto \left(\begin{array}{cc|c} 1 & 0 & 0 \\ -1 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \quad (2,3) \mapsto \left(\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(1,2,3) \mapsto \left(\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \quad (1,3,2) \mapsto \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

(7) Let G be a Lie Group. Let N be an open neighborhood(or an open set really.) containing e. Consider any element $g \in G$ which defines a diffeomorphism $g \cdot - : G \to G$, therefore a homeomorphism. Notice that this means that for any $n \in \langle N \rangle$ we have that $n \cdot N$ is an open set as N was open. Notice also that $n \cdot N$ is contained in $\langle N \rangle$. This means that for an $n \in \langle N \rangle$ there is an open set $n \cdot N \subseteq \langle N \rangle$ containing n and so $\langle N \rangle$ is open. Equivalently this means $\langle N \rangle = \bigcup_{n \in \langle N \rangle} nN$ and is therefore open.

Now we will show that $\langle N \rangle$ is closed or $\langle N \rangle^c$ is open. Consider a $p \in \langle N \rangle^c$, then $p \cdot N$ is open. Notice also that it is distinct from $\langle N \rangle$ as if there existed an element $x \in p \cdot N \cap \langle N \rangle$ then $x = p \cdot n_0 = \prod_{i=1}^k n_i$ in which case $p = \prod_{i=1}^k n_i n_0^{-1} \in \langle N \rangle$ which is not the case. So every point p is contained in an open set $p \cdot N$ contained in $\langle N \rangle^c$ so $\langle N \rangle^c$ is open.

Finally because $\langle N \rangle$ is both open and closed, contains e and G is connected we know $\langle N \rangle = G$