MATH 601. HW 3

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(1) (b) Consider the representation $\rho : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}_2(\mathbb{C}) = M_{4\times 4}(\mathbb{C})$ where we will fix the basis $\{v_3, v_1, v_{-1}, v_{-3}\}$ In which case we know that we must map to matrices of the form

$$F \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}$$

$$E \mapsto \begin{bmatrix} 0 & d & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$H \mapsto \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Furthermore we can use the first requirement of part c that $[\rho(E), \rho(F)] = \rho(H)$ to find that $a = f = 3$, $c = d = 1$ and $b = e = 2$ is a solution, giving

$$F \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$E \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$H \mapsto \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

(c) We can now check explicitly that the above map is a representation with matlab

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>> F=[0 0 0 0; 3 0 0 0; 0 2 0 0; 0 0 1 0];
>> E= [0 1 0 0; 0 0 2 0; 0 0 0 3; 0 0 0 0 ];
>> H = [3 0 0 0; 0 1 0 0; 0 0 -1 0; 0 0 0 -3];
>> E*F-F*E

ans =

     3     0     0     0
     0     1     0     0
     0     0    -1     0
     0     0     0    -3

>> H*E-E*H

ans =

     0     2     0     0
     0     0     4     0
     0     0     0     6
     0     0     0     0

>> H*F-F*H

ans =

     0     0     0     0
    -6     0     0     0
     0    -4     0     0
     0     0    -2     0
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(3) First notice that the formal character of V^2 is $\chi_{V^2}(q) = q^2 + 1 + q^{-2}$ And

$$\begin{aligned}
\chi_{(V^2)^{\otimes 3}}(q) &= (\chi_{V^2}(q))^3 \\
&= (q^2 + 1 + q^{-2})^3 \\
&= q^6 + 3q^4 + 6q^2 + 7 + 6q^{-2} + 3q^{-4} + q^{-6} \\
&= (q^6 + q^4 + q^2 + 1 + q^{-2} + q^{-4} + q^{-6}) + 2(q^4 + q^2 + 1 + q^{-2} + q^{-4}) + 3(q^2 + 1 + q^{-2}) + 1 \\
&= \chi_{V^6}(q) + \chi_{V^4}(q) + \chi_{V^4}(q) + \chi_{V^2}(q) + \chi_{V^2}(q) + \chi_{V^2}(q) + \chi_{V^0}(q)
\end{aligned}$$

Meaning $(V^2)^{\otimes 3} = V^6 \oplus V^4 \oplus V^4 \oplus V^2 \oplus V^2 \oplus V^2 \oplus V^0$

(4) Consider the Tensor product $V^n \otimes V^m$, we will assume WLOG that $n \geq m$. Let the vectors $v_n, v_{n-2}, \ldots, v_{-2}$ be the weight vectors of V^n and $w_m, w_{m-2}, \ldots w_m$ be the weight vectors of V^m. Notice that for $k = 0, \ldots, m$ a vector in the tensor product of the form

$$v_{n-2k} \otimes w_m - v_{n-2k+2} \otimes w_{m-2} + \cdots + (-1)^k v_n \otimes w_{m-2} = \sum_{i=0}^{k} (-1)^i v_{n-2(k-i)} \otimes w_{m-2i}$$

is a highest weight vector with weight $n + m - 2k$. To see this notice that

$$\begin{aligned}
E(\sum_{i=0}^{k} (-1)^i v_{n-2(k-i)} \otimes w_{m-2i}) &= \sum_{i=0}^{k} (-1)^i E(v_{n-2(k-i)} \otimes w_{m-2i}) \\
&= \sum_{i=0}^{k} (-1)^i (v_{n-2(k-i)+2} \otimes w_{m-2i} + v_{n-2(k-i)} \otimes w_{m-2i+2})
\end{aligned}$$

And notice that this is a telescoping series, so the only terms remaining in the sum are $v_{n-2k} \otimes w_{m+2}$ and $v_{n+2} \otimes w_{m-2k}$ but $w_{m+2} = E(w_m) = 0$ and likewise $v_{n+2} = E(v_n) = 0$ and so the entire sum is 0, meaning these vectors are in fact highest weight.

Now we will determine the weight of these vectors, so notice that

$$\begin{aligned}
H(\sum_{i=0}^{k}(-1)^i v_{n-2(k-i)} \otimes w_{m-2i}) &= \sum_{i=0}^{k}(-1)^i H(v_{n-2(k-i)} \otimes w_{m-2i}) \\
&= \sum_{i=0}^{k}(-1)^i((n-2(k-i))v_{n-2(k-i)} \otimes w_{m-2i} + (m-2i)v_{n-2(k-i)} \otimes w_{m-2i}) \\
&= \sum_{i=0}^{k}(-1)^i(n-2(k-i)+m-2i)v_{n-2(k-i)} \otimes w_{m-2i} \\
&= (n+m-2k)\sum_{i=0}^{k}(-1)^i v_{n-2(k-i)} \otimes w_{m-2i}
\end{aligned}$$

showing that for each k we have a weight vector with weight $n+m-2k$

The last thing to check is that these are all the hightest weight vectors, which we will do by counting dimensions. For each k the corresponding vector with weight $n+m-2k$ gives a chain of length $n+m-2k+1$. This means if we sum the number of weight vector in all of the chains we would have a total of $(n+1)(m+1)$ linearly independent vectors, and because the dimension of $V^n \oplus V^m$ is $(m+1)(n+1)$ there can not be any additional chains and so there cant be any additional highest weight vectors. This gives is the Clebsch-Gordan rule because the irreducible we formed from each chain are exactly $V^n \oplus V^m = V^{n+m} \oplus V^{n+m-2} \oplus \cdots \oplus V^{n-m}$

(5) Let $w = w_1 w_2 \dots w_n$ be a word of 1s and 2s that is ballot. Meaning every suffix has at least as many 1s as 2s. Notice that this means that for every suffix, every 2 will be paired with a 1. This means that there are no unpaired 2s Applying E would then result in 0. Now assume that w is a word that is not ballot. Meaning there is a suffix $w_i \dots w_n$ that contains more 2s then 1s. We may assume that $w_i = 2$ and this suffix is of minimal length in which case w_i would be the right most unpaired 2, as there are less 1s then 2s in the suffix. This means that $Ew = w_1 w_2 \dots w_{i-1} 1 w_{i+1} \dots w_n$ and so w did not represent a highest weight vector.