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(1) Let $\lambda = (5, 4, 1)$, then the element of $S^{\lambda}V$

$$(e_1 \wedge e_3 \wedge e_4) \otimes (e_1 \wedge e_3) \otimes (e_2 \wedge e_4) \otimes (e_2 \wedge e_4) \otimes e_4 = \begin{array}{|c|c|c|c|c|}\cline{1-1} 4 \\ \cline{1-4} 3 & 3 & 4 & 4 \\ \hline 1 & 1 & 2 & 4 & 4 \\ \hline \end{array}$$

(2) Because the SSYT with the elementary vectors as the filling form a basis, the dimension of $V^{(k,k)}$ is the number of SSYT of shape (k, k) where the entries can be no greater then 3, one for each elementary vector. So we need to count the number of SSYT. If the bottom row is all 1s then the top would be all 2s or part 2s and part 3s, meaning there are $k+1$ ways to fill the top row with 2s then 3s possible with no 2s or 3s. Notice that it is not possible for there to be any 3s on the bottom so we need only consider the case where there are 1s and 2s. If there are r 1s on the bottom and $k-r$ 2s this would for there to be 3s above each 2, and above the ones there could be a combination of 1s and 2s, of which there would be $r+1$ ways to have 1s then 2 above the 1s. Putting this together

$$\dim V^{(k,k)} = \sum_{r=0}^{k} r + 1 = \frac{(k+1)(k+2)}{2}$$

where we are iterating of there being r 1s on the bottom row.

(3) We can follow the algorithm from class, utilizing the equivalence of ant filling with the sum of all the column swaps

$$\begin{array}{|c|c|}\hline 5 & 6 \\ \hline 4 & 3 \\ \hline 1 & 2 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 3 & 4 \\ \hline 2 & 1 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 4 & 3 \\ \hline 1 & 2 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 4 & 6 \\ \hline 3 & 5 \\ \hline 2 & 1 \\ \hline \end{array}$$

$$= \left(\begin{array}{|c|c|}\hline 5 & 6 \\ \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 3 & 6 \\ \hline 2 & 5 \\ \hline 1 & 4 \\ \hline \end{array}\right) - \left(\begin{array}{|c|c|}\hline 4 & 6 \\ \hline 3 & 5 \\ \hline 1 & 2 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 4 & 6 \\ \hline 2 & 5 \\ \hline 1 & 3 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 3 & 6 \\ \hline 2 & 5 \\ \hline 1 & 4 \\ \hline \end{array}\right) + \begin{array}{|c|c|}\hline 3 & 6 \\ \hline 2 & 5 \\ \hline 1 & 4 \\ \hline \end{array}$$

$$= \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 3 & 6 \\ \hline 2 & 5 \\ \hline 1 & 4 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 4 & 6 \\ \hline 3 & 5 \\ \hline 1 & 2 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 4 & 6 \\ \hline 2 & 5 \\ \hline 1 & 3 \\ \hline \end{array}$$

(6) First notice that with the commutator bracket we have that $[Y, X] = YX - XY = -(XY - YX) = -[X, Y]$ and so is skew symmetric. Notice also that

$$\begin{aligned}
&[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] \\
&= X[Y, Z] - [Y, Z]X + Y[Z, X] - [Z, X]Y + Z[X, Y] - [X, Y]Z \\
&= X(YZ - ZY) - (YZ - ZY)X + Y(ZX - XZ) - (ZX - XZ)Y + Z(XY - YX) - (XY - YX)Z \\
&= XYZ - XZY - YZX + ZYX + YZX - YXZ - ZXY + XZY + ZXY - ZYX - XYZ + YXZ \\
&= 0
\end{aligned}$$

which shows that this commutator satisfies the Jacobi Identity.

(7) (a) Recall that the lie group $SO_n(\mathbb{C})$ was defined as $SO_n(\mathbb{C}) = \{M | \det(M) = 1, A^\dagger A = I\}$ where $(-)^\dagger$ is the standard transpose (Not the conjugate transpose). Using the ϵ method we have that $\mathfrak{so}_n(\mathbb{C}) = \{X : \det(I + \epsilon X) = 1, (I + \epsilon X)^\dagger (I + \epsilon X) = I\}$. The condition $\det(I + \epsilon X) = 1$ is equivalent to $\operatorname{tr}(X) = 0$ and the condition $I = (I + \epsilon X)^\dagger (I + \epsilon X) = I + \epsilon X + \epsilon X^\dagger$ and so the matrices that satisfy this condition are the skew symmetric matrices, the matrices satisfying $X = -X^\dagger$ so $\mathfrak{so}_n(\mathbb{C}) = \{X : \operatorname{tr} X = 0, X^\dagger = -X\}$. Also notice that if $X^\dagger = -X$ then the diagonal entries must all be zero, and so the $\operatorname{tr}(X) = 0$ condition is guaranteed.

$$\mathfrak{so}_n(\mathbb{C}) = \{X : X^\dagger = -X\}$$

(b) Recall that the lie group $Sp_{2n}(\mathbb{C})$ was defined as $Sp_{2n}(\mathbb{C}) = \{M \in GL_n(\mathbb{C}) | M^\dagger \Omega M = \Omega\}$. Using the ϵ method we have that $\mathfrak{sp}_{2n}(\mathbb{C}) = \{X : (I + \epsilon X)^\dagger \Omega (I + \epsilon X) = \Omega\}$. The condition $\Omega = (I + \epsilon X)^\dagger \Omega (I + \epsilon X) = \Omega + \epsilon X^\dagger \Omega + \epsilon \Omega X$ and so the matrices that satisfy this condition are the ? matrices, the matrices satisfying $X^\dagger \Omega = -\Omega X$ so

$$\mathfrak{sp}_n(\mathbb{C}) = \{X : X^\dagger \Omega = -\Omega X\}$$

(c) Recall that the lie group $T_n(\mathbb{C})$ were the invertible diagonal matrices. Using the ϵ method we have that $\mathfrak{t}_n(\mathbb{C}) = \{X :, I + \epsilon X \text{ is invertable diagonal matrix}\} \cong (\mathbb{C}^*)^n$. The condition of inevitability puts no requirements on X and the condition $I + \epsilon X$ being diagonal requires that X is diagonal. So

$$\mathfrak{t}_n(\mathbb{C}) = \{X : X \text{ is diagonal}\} \cong \mathbb{C}^n$$

(d) Recall that the lie group $B_n(\mathbb{C})$ were the invertible upper triangular matrices. Using the ϵ method we have that $\mathfrak{b}_n(\mathbb{C}) = \{X :, I + \epsilon X \text{ is invertable upper triangular matrix}\}$. The condition of inevitability puts no requirements on X and the condition $I + \epsilon X$ being upper triangular requires that X is upper triangular. So

$$\mathfrak{b}_n(\mathbb{C}) = \{X : X \text{ is upper triangular}\}$$

(8) First recall that $\mathfrak{so}_n(\mathbb{C}) = \{X : X^\dagger = -X\}$. Now recall that the lie group $O_n(\mathbb{C})$ was defined as $O_n(\mathbb{C}) = \{M | A^\dagger A = I\}$ where $(-)^\dagger$ is again the standard transpose. Using the ϵ method we have that $\mathfrak{o}_n(\mathbb{C}) = \{X : (I + \epsilon X)^\dagger (I + \epsilon X) = I\}$. The condition $I = (I + \epsilon X)^\dagger (I + \epsilon X) = I + \epsilon X + \epsilon X^\dagger$ and so the matrices that satisfy this condition are the skew symmetric matrices, the matrices satisfying $X = -X^\dagger$ so $\mathfrak{o}_n(\mathbb{C}) = \{X : X^\dagger = -X\}$. And because in both cases the Lie bracket is the commutator we know that this two lie algebras are equal, and therefore isomorphic.

This does contradict the bijection correspondence because the matrices satisfying $A^\dagger A = I$ must satisfy $(\det(A))^2 = 1$ which is only the case when $\det(A) = \pm 1$ because we are looking at the standard transpose and not the conjugate transpose. Meaning $O_n(\mathbb{C})$ is not a connected Lie group.