Complex Geometry. HW 7

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(11.3) Let $\text{Pic}(\mathbb{CP}^1)$ be the equivalence classes of isomorphic line bundles of $\mathbb{CP}^1$. We already know that every line bundle is isomorphic to $\mathcal{O}_{\mathbb{CP}^1}(d)$ for some $d \in \mathbb{Z}$, so these can be the representatives of the elements of the Picard Group. We also know that the tensor product is a binary operator on $\text{Pic}(\mathbb{CP}^1)$. Consider the following surjective map $\Phi : \mathbb{Z} \to \text{Pic}(\mathbb{CP}^1)$ by $d \mapsto [\mathcal{O}_{\mathbb{CP}^1}(d)]$ the class of line bundles isomorphic to $\mathcal{O}_{\mathbb{CP}^1}(d)$ We show that this is a group isomorphism, showing that the picard group is a group and is isomorphic to $\mathbb{Z}$.

Notice that $\varphi(d_1+d_2) = [\mathcal{O}_{\mathbb{CP}^1}(d_1+d_2)] = [\mathcal{O}_{\mathbb{CP}^1}(d_1)\otimes\mathcal{O}_{\mathbb{CP}^1}(d_2)] = [\mathcal{O}_{\mathbb{CP}^1}(d_1)]\otimes[\mathcal{O}_{\mathbb{CP}^1}(d_2)] = \varphi(d_1) + \varphi(d_2)$ Also notice that $\varphi(0) = [\mathcal{O}_{\mathbb{CP}^1}(0)]$ which acts as an identity on $\text{Pic}(\mathbb{CP}^1)$ with tensoring. Finally notice that $\varphi(-d) = [\mathcal{O}_{\mathbb{CP}^1}(-d)]$ which acts as the inverse to $\varphi(d) = [\mathcal{O}_{\mathbb{CP}^1}(d)]$ under tensoring. Finally notice that φ is injective by previous problems, showing if $\mathcal{O}_{\mathbb{CP}^1}(d_1) \cong \mathcal{O}_{\mathbb{CP}^1}(d_2)$ then $d_1 = d_2$.

(12.4) $H^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(d))$ was defined to be the kernel of the map δ. Notice that if $(s_1, s_2) \in \ker\delta$ then $s_1|_{U_0\cap U_1} \equiv s_2|_{U_0\cap U_1}$. And because s_1 is holomorphic at 0, and s_2 is holomorphic at ∞, it must be the case that s_1 has no worse then a removable singularity at ∞ and likewise s_2 has a removable singularity at 0. This means with these two points $s_1 \equiv s_2$ and so are equal and holomorphic everywhere.