Math 601. HW A
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(1) Consider the square embedded in R^2 or C^2 with coordinates $(1,1)$, $(1,-1)$, $(-1,-1)$, and $(-1,1)$. Notice that this defines the representation $\rho : D_4 \to GL(\mathbb{C}^2)$

$$r \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad s \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Where r is the clockwise rotation and s the rotation around the y axis. Notice that the eigenvectors of $\rho(s)$ are the elementary vector e_1 and e_2 with eigenvalues -1 and 1 respectively. Notice that regardless of the underlying vector space $\mathbb{C}^2$ or $\mathbb{R}^2$ the matrix $\rho(r)$ does not share either of these eigenvalues. This means that the matrices $\rho(r)$ and $\rho(s)$ are no simultaneously diagonalizable. And so this is an irreducible representation.

(2) Let G be a finite Abelian Group and consider a representation $\rho : G \to GL(V)$ where $\dim(V) = n$.

Let $g \in G$ where $g^{|G|} = 1$, meaning as a matrix $\rho(g)^{|G|} - I = 0$, meaning the minimal polynomial of $\rho(g)$, $m_{\rho(g)}(x)$ divides $x^{|G|} - 1$, which has n distinct roots in $\mathbb{C}$, meaning $m_{\rho(g)}(x)$ has all distinct roots, and so $\rho(g)$ is diagonalizable. Because G is Abelian we know that for $g, h \in G$ that $\rho(g)\rho(h) = \rho(h)\rho(G)$, meaning every matrix in our representation commutes, and so every matrix is simultaneously diagnolizable. Which means they all share eigenvectors which form a basis for V. This gives a change of basis for each matrix into diagonal matrices, and so the representation ρ is decomposable into dimension 1 representations that are the spans of each of the distinct eigenvectors. Therefore the only irreducible representations are dimension 1 as otherwise there is a decomposition into dimension one irreducible.

(3) Let $\rho : G \to GL(\mathbb{C}^m)$ and $\sigma : G \to GL(\mathbb{C}^n)$ be representations. We can define the tensor product of these two representation by how G acts on $\mathbb{C}^m \otimes \mathbb{C}^n$ which has as a basis $\{v_j \otimes w_k | 1 \leq j \leq m \text{ and } 1 \leq k \leq n\}$ where v_j represents the jth elementary vector of $\mathbb{C}^m$ and w_k represents the kth elementary vector of $\mathbb{C}^n$. This means that $\mathbb{C}^m \otimes \mathbb{C}^n \cong \mathbb{C}^{nm}$. Furthermore we will define an ordering on the basis elements as follows $(j,k) \leq (j',k')$ if $j < j'$ or $j = j'$ and $k \leq k'$. We then define the action of G as $g \cdot (v \otimes w) = gv \otimes gw$. And the way g acts on v is as a matrix $\rho(g)v$, so $gv \otimes gw = \rho(g)v \otimes \sigma(g)w$. Now consider a group element $g \in G$ and let $A = \rho(g)$ and $B = \rho(B)$. Now consider the particular basis element $v_j \otimes w_k$ and notice that Av_j is just the jth column of A, and likewise Bw_k is the kth column of B. This means that $g(v_j \otimes w_k)$ is a linear combination of the basis element $\mathbb{C}^m \otimes \mathbb{C}^n$ where the coefficient of $v_{j'} \otimes w_{k'}$ is $a_{j'j}b_{kk'}$. And so repeating this for all basis element gives us that g maps to the matrix $A \otimes B$.

(4) First notice that the dimension of the tensor product space $(\mathbb{C}^n)^{\otimes k}$ is n^k. Notice also that for any partition λ of size k having at most n parts, the space V_λ is spanned by a basis indexed by the SYT of shape λ. Likewise the space V^λ is spanned by basis elements indexed by SSYT of shape λ. This means that basis elements of $V_\lambda \otimes V^\lambda$ are indexed by pairs (P, Q) where P is a SSYT and Q is a SYT, both of the same shape λ, with fillings of P being the numbers $1, 2, \ldots, n$ By the RSK bijection this gives we know that pairs of such tablaux are in bijection with words of length k with letters $1, \ldots, n$ with repeats. We can count the number of words as n^k as there are k digits each with n options.

(5) Recall that the lie group $B_n(\mathbb{C})$ are the invertible upper triangular matrices. Specifically we want to consider the ones whose determinant is 1 meaning the product of the diagonal is 1 Using the ϵ method we have that $\mathfrak{b}_n(\mathbb{C}) = \{X :, I+\epsilon X$ is invertable upper triangular matrix with $\det(I+\epsilon X) = 1\}$. The condition of the determinant puts the requirements that X has $\text{tr}(X) = 0$. the condition $I + \epsilon X$ being upper triangular requires that X is upper triangular. So

$$\mathfrak{b}_n(\mathbb{C}) = \{X : X \text{ is upper triangular and } \text{tr}(X) = 0\}$$

(6) Here we can use the Clebsch-Gordan which gives us that $V^3 \oplus V^5 = V^8 \oplus V^6 \oplus V^4 \oplus V^2$.

(7) (a) Recall that the represtnation $(V^1)^{\otimes n}$ can be written as the sum of irreducibles, and that the number of irreducibles is counted by the ballot words of 1s and 2s of length n. We now need to come up with a way of counting the number of irreducibles. Consider the formal character $\chi_{V^1}(q) = q + q^{-1}$. And so $\chi_{V^1}(q) = (q + q^{-1})^n$. Notice that when n is even every term of the formal character will have an even power, and when n is odd every term in the formal character will have an odd power. We can see this with induction and that multiplying the formal character by $(q+q^{-1})$ will result in the degree of every term being ± 1 of the degree of the original terms, and so changed the oddness or evenness. if n is even this would mean that every irreducible factor would have a weight 0 vector, meaning the number of irreducibles is counting by the dimension of the 0 weight space. And likewise if n is even then every irreducible factor would have a weight -1 factor meaning the number of irreducibles is counted by the dimension of the -1 weight spaces.
Now let n be a positive integer and consider the even numner $2n$ and notice that the number of irreducible factors of $(V^1)^{\otimes 2n}$ is counted by the weight 0 weight space, or the coefficient of the q^0 factor of the formal character $(q + q^{-1})^{2n}$ which is the binomial coefficient $\binom{2n}{n}$. Likewise consider the odd numner $2n + 1$ and notice that the number of irreducible factors of $(V^1)^{\otimes 2n+1}$ is counted by the weight -1 weight space, or the coefficient of the q^{-1} factor of the formal character $(q + q^{-1})^{2n+1}$ which is the binomial coefficient $\binom{2n+1}{n+1}$

(b) We know that in $V_1^{\otimes k}$ the highest weight vectors correspond to ballot words of 1s and 2s of length k. And each highest weight vector gives an irreducible representation in the decomposition of $V_1^{\otimes k}$ into irreducibles. So the number of irreducibles in the decomposition of $V_1^{\otimes 2n}$ is counted by the number of ballot words of length $2n$, of which there are $\binom{2n}{n}$ ballot words. And so there are $\binom{2n}{n}$ irreducibles in the decomposition. Likewise this means there are $\binom{2n+1}{n+1}$ irreducibles in the decomposition of $V_1^{\otimes 2n+1}$

(8) We get the following

$$\begin{tikzcd}
1\ 2\ 2\ 1\ 1\ 1\ 2\ 2\ 1\ 2\ 1\ 1\ 1\ 1 \arrow[d, shift left=1ex, "F"] \\
1\ 2\ 2\ 1\ 1\ 1\ 2\ 2\ 1\ 2\ 1\ 1\ 1\ 2 \arrow[u, shift left=1ex, "E"] \arrow[d, shift left=1ex, "F"] \\
1\ 2\ 2\ 1\ 1\ 1\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 2 \arrow[u, shift left=1ex, "E"] \arrow[d, shift left=1ex, "F"] \\
1\ 2\ 2\ 1\ 1\ 1\ 2\ 2\ 1\ 2\ 1\ 2\ 2\ 2 \arrow[u, shift left=1ex, "E"] \arrow[d, shift left=1ex, "F"] \\
1\ 2\ 2\ 1\ 1\ 2\ 2\ 2\ 1\ 2\ 1\ 2\ 2\ 2 \arrow[u, shift left=1ex, "E"] \arrow[d, shift left=1ex, "F"] \\
2\ 2\ 2\ 1\ 1\ 2\ 2\ 2\ 1\ 2\ 1\ 2\ 2\ 2 \arrow[u, shift left=1ex, "E"]
\end{tikzcd}$$