MATH 601. HW 3

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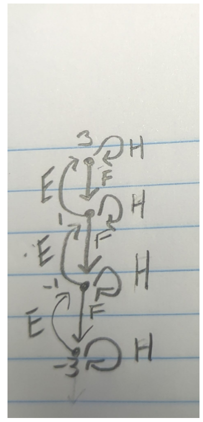
(1) (a) The following is the diagram representing V^3



(b) Consider the representation $\rho : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}_2(\mathbb{C}) = M_{4\times 4}(\mathbb{C})$ where we will fix the basis $\{v_3, v_1, v_{-1}, v_{-3}\}$ In which case we know that we must map to matrices of the form

$$F \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}$$

$$E \mapsto \begin{bmatrix} 0 & d & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$H \mapsto \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Furthermore we can use the first requirement of part c that $[\rho(E), \rho(F)] = \rho(H)$ to find that $a = f = 3$, $c = d = 1$ and $b = e = 2$ is a solution, giving

$$F \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$E \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$H \mapsto \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

(c) We can then check this in fact works as a representation with matlab

```
>> F=[0 0 0 0; 3 0 0 0; 0 2 0 0; 0 0 1 0];
>> E= [0 1 0 0; 0 0 2 0; 0 0 0 3; 0 0 0 0 ];
>> H = [3 0 0 0; 0 1 0 0; 0 0 -1 0; 0 0 0 -3];
>> E*F-F*E

ans =

     3     0     0     0
     0     1     0     0
     0     0    -1     0
     0     0     0    -3

>> H*E-E*H

ans =

     0     2     0     0
     0     0     4     0
     0     0     0     6
     0     0     0     0

>> H*F-F*H

ans =

     0     0     0     0
    -6     0     0     0
     0    -4     0     0
     0     0    -2     0
```

(3) First notice that the formal character of V^2 is $\chi_{V^2}(q) = q^2 + 1 + q^{-2}$ And

$$
\begin{aligned}
\chi_{(V^2)^{\otimes 3}}(q) &= (\chi_{V^2}(q))^3 \\
&= (q^2 + 1 + q^{-2})^3 \\
&= q^6 + 3q^4 + 6q^2 + 7 + 6q^{-2} + 3q^{-4} + q^{-6} \\
&= (q^6 + q^4 + q^2 + 1 + q^{-2} + q^{-4} + q^{-6}) + 2(q^4 + q^2 + 1 + q^{-2} + q^{-4}) + 3(q^2 + 1 + q^{-2}) + 1 \\
&= \chi_{V^6}(q) + \chi_{V^4}(q) + \chi_{V^4}(q) + \chi_{V^2}(q) + \chi_{V^2}(q) + \chi_{V^2}(q) + \chi_{V^0}(q)
\end{aligned}
$$

Meaning $(V^2)^{\otimes 3} = V^6 \oplus V^4 \oplus V^4 \oplus V^2 \oplus V^2 \oplus V^2 \oplus V^0$