Complex Geometry. HW 2
Ian Jorquera

(5.4) Notice that if this statement is true it is enough to show that an automorphism Φ is uniquely determined by three points P_1, P_2, P_3 being sent to $[1:0], [0:1], [1,1]$ as any general automorphism mapping P_1, P_2, P_3 to Q_1, Q_2, Q_3, can be written first as a map sending P_1, P_2, P_3 to $[1:0], [0:1], [1,1]$ then composed with a map sending $[1:0], [0:1], [1,1]$ to Q_1, Q_2, Q_3 and this second automorphism is the inverse of the map that sends Q_1, Q_2, Q_3 to $[1:0], [0:1], [1,1]$.

From the previous problems we know that every automorphism is a Mobius transformation $\Phi(X:Y) = [aX + bY : cX + dY]$. To construct a map that takes P_1, P_2, P_3 to $[1:0], [0:1], [1,1]$. Looking at the requirement that $P_1 \mapsto [0:1]$ we get the equation $0 = cX(P_1) + dY(P_1)$ and looking at $P_2 \mapsto [1:0]$ we get $0 = aX(P_2) + bY(P_2)$. And finally from $P_3 \mapsto [1:1]$ we get that $aX(P_3) + bY(P_3) = cX(P_3) + dY(P_3)$. Notice that each of these equations is invariant under scaling of the homogeneous coordinates.

This gives use the matrix

$$\begin{bmatrix} 0 & 0 & X(P_1) & Y(P_1) \\ X(P_2) & Y(P_2) & 0 & 0 \\ X(P_3) & Y(P_3) & -X(P_3) & Y(P_3) \end{bmatrix}$$

Whose kernel are vectors of the form $(a, b, c, d)^\dagger$ that define(for non-zero vectors) automorphism as desired. Notice that the rank of this matrix is 3 by the assumption that the points are distinct. and so the kernel is 1-dimension. The solutions are therefore all scaling of any non-zero vector, and because Mobius transformation are equivalent under scaling this means we have 1 unique solution.

Notice that fixing three points would have a unique solution and one possible automorphism that fixes 3 points is the identity so this must be the unique solution.

(6.3) Let $F : \mathbb{CP}^1 \to \mathbb{CP}^r$ be a regular map, meaning $F = (P_0(X:Y) : P_1(X:Y) : \cdots : P_r(X:Y))$ where each P_j is a polynomial of the same degree for all j. Now assume that the degree of each P_j is d, then with the intersection of a general hyperplane $\alpha_0 P_0(X:Y) + \alpha_1 P_1(X:Y) + \cdots + \alpha_r P_r(X:Y) = 0$ we know that this is a degree d homogeneous polynomial of degree d in two variables, and so by the fundamental theorem of algebra we know there are d solution up to multiplicity. And so the number of intersection of $F(\mathbb{CP}^1)$ with a general hyper plane is $\leq d$. Notice that the case of strict inequality happens when each polynomial $P_j(X:Y)$ shares a common root, and so the common root can be factored out reducing the degree of the polynomials by 1.