Research Internship Report

Bridging Topology and Geometry: The Gauss–Bonnet Theorem

A Survey From Curves to Surfaces

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Abstract

The Gauss-Bonnet theorem represents one of the most profound results in differential geometry, establishing a fundamental connection between the local geometric properties of a surface and its global topological structure. This report presents a survey of the theorem, progressing from its one-dimensional analogue for plane curves to its full formulation for smooth surfaces.

We begin with the classical result that the total curvature of a simple closed curve equals 2π, developing the necessary framework of curve theory and signed curvature. The main focus then shifts to surfaces, where we establish two critical foundations: first, we prove the topological invariance of the Euler characteristic through homological methods, providing a rigorous basis for the classification of surfaces; second, we develop the differential geometry of surfaces, culminating in Gauss's Theorema Egregium, which demonstrates that Gaussian curvature depends only on the intrinsic metric structure.

The report synthesizes these developments to prove both the local and global versions of the Gauss-Bonnet theorem. The local version relates the geodesic curvature of a boundary curve and the Gaussian curvature of the enclosed region, while the global version states that for any smooth, compact, orientable surface X, the integral of the Gaussian curvature equals $2\pi\chi(X)$. This remarkable formula bridges differential geometry and topology, showing that a purely local geometric quantity, when integrated, yields a topological invariant. Throughout, we provide detailed proofs and sketches with particular attention to pedagogical clarity, filling gaps often left implicit in standard treatments.

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Chapter 1

Introduction

The Gauss-Bonnet theorem stands as a cornerstone of differential geometry, revealing an unexpected and profound relationship between the local curvature of a surface and its global topological properties. At its heart lies a simple yet striking formula: for any smooth, compact, orientable surface X, the integral of the Gaussian curvature K over the entire surface equals 2π times the Euler characteristic $\chi(X)$. This equality bridges two seemingly disparate mathematical realms: the infinitesimal geometry encoded in curvature and the global topology captured by the Euler characteristic.

The historical development of this theorem spans nearly two centuries and involves some of mathematics' most distinguished figures. Gauss first discovered a local version around 1827 in his *Disquisitiones generales circa superficies curvas*, where he proved that the Gaussian curvature is an intrinsic property of a surface meaning that it only only depends on measurements within the surface itself, not on how it sits in ambient space. This result, which Gauss himself called the *Theorema Egregium* (remarkable theorem), was revolutionary in suggesting that surfaces possess an inherent geometry independent of their embedding. Pierre Ossian Bonnet later extended Gauss's local result to a global theorem in 1848, though the topological significance through the Euler characteristic was not fully understood until later developments in algebraic topology.

The power of the Gauss-Bonnet theorem extends far beyond its original formulation. It serves as a prototype for index theorems, which relate analytical and topological invariants, culminating in the celebrated Atiyah-Singer index theorem. In physics, the theorem appears naturally in general relativity, string theory, and condensed matter physics, where topological constraints impose fundamental limitations on physical systems. The theorem's generalization to higher dimensions through the Chern-Gauss-Bonnet theorem provides crucial tools in modern differential geometry and theoretical physics, particularly in understanding gravitational instantons and topological field theories.

This report aims at presenting a self-contained presentation of the Gauss-Bonnet theorem, with particular emphasis on pedagogical clarity and mathematical rigor. We adopt a progressive approach, beginning with the simplest non-trivial case of curves in the plane before advancing to the full theorem for surfaces. This structure allows us to introduce the ideas behind key concepts such as curvature and topological invariants in their most accessible form before tackling the technical complexities of surface geometry.

The report is organized into two main parts. Part I treats the one-dimensional Gauss-Bonnet theorem for plane curves, establishing that the total curvature of a simple closed curve equals

$\pm 2\pi$ depending on the orientation of the curve. This serves both as a warm-up and as a model for understanding how local geometric information (curvature) integrates to yield global topological data (extension to the winding number). We develop the necessary machinery of curve theory, including arc-length parametrization and signed curvature, and extend the result to piecewise smooth curves with corners.

Part II, the main body of the report, addresses the Gauss-Bonnet theorem for surfaces. Chapter 4 establishes the topological foundations, where we take a somewhat unconventional approach by providing a detailed proof of the topological invariance of the Euler characteristic using homological methods. While this level of detail exceeds what is typically necessary for stating the Gauss-Bonnet theorem, it provides valuable insight into why the Euler characteristic appears in the formula and reinforces its fundamental nature as a topological invariant. This chapter also includes the classification of surfaces, showing how the Euler characteristic, together with orientability, completely determines a closed surface up to homeomorphism.

Chapter 5 develops the differential geometry of surfaces embedded in $\mathbb{R}^3$. We introduce the first and second fundamental forms, which encode the intrinsic and extrinsic geometry respectively. The highlight is the proof of Gauss's Theorema Egregium, demonstrating that Gaussian curvature can be computed entirely from the first fundamental form. This intrinsic nature of Gaussian curvature is essential for understanding why the integral in the Gauss-Bonnet theorem depends only on the topology of the surface.

Chapter 6 presents the culmination of our study: the proof of the Gauss-Bonnet theorem itself. We first establish the local version for regions bounded by simple closed curves, using Green's theorem to relate boundary integrals to area integrals. The global version then follows by triangulating the surface and carefully accounting for the contributions from each triangle. The cancellation of edge contributions and the relationship between vertex angles and the Euler characteristic complete the proof.

Throughout this report, we have endeavored to provide proofs with the maximum of details explicitly stated, addressing gaps that are often glossed over in standard treatments. Where technical computations become lengthy, we indicate their further references maintaining the logical flow of the main arguments. This approach reflects our goal of creating a resource that bridges the gap between introductory differential geometry texts and more advanced treatments, suitable for students transitioning from undergraduate to graduate-level mathematics.

Part I

The One-Dimensional Gauss–Bonnet Theorem

Chapter 2

Foundations of Curve Theory

We begin with the fundamental concepts of curve theory, which will serve as our simplest model for understanding curvature and its integral properties. This part is mainly inspired by [1].

Definition 2.1 (Parametrized Curve)**.** A *parametrized curve* in $\mathbb{R}^n$ is a smooth map $\gamma : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval. The curve is *regular* if $\gamma'(t) \neq 0$ for all $t \in I$.

Even if a curve is given with some parametrization, it is possible to change it for another one as follows:

Definition 2.2 (Reparametrization)**.** A parametrized curve $\gamma : I \to \mathbb{R}^n$ is said to be reparametrized when composed on the right with a smmoth function $\delta : J \to I$.

The resulting parametrized curve $\gamma \circ \delta : J \to \mathbb{R}^n$ has the same trace as γ but which would have *travelled* at a different pace along the path. Indeed, the image is the same but the field of parameters has changed. For a regular curve, we can always reparametrize by arc length, obtaining a unit-speed curve.

Proposition 2.3 (Arc Length Parametrization)**.** *Given a regular curve $\gamma : [a, b] \to \mathbb{R}^n$, there exists a reparametrization $\tilde{\gamma} : [0, L] \to \mathbb{R}^n$ such that $\|\tilde{\gamma}'(s)\| = 1$ for all s, where $L = \int_a^b \|\gamma'(t)\|\, dt$ is the total length.*

Proof. Define $s(t) = \int_a^t \|\gamma'(\tau)\|\, d\tau$. Since γ is regular, $s'(t) = \|\gamma'(t)\| > 0$, so s is strictly increasing and has an inverse $t(s)$. Set $\tilde{\gamma}(s) = \gamma(t(s))$. Then:

$$\tilde{\gamma}'(s) = \gamma'(t(s)) \cdot t'(s) = \gamma'(t(s)) \cdot \frac{1}{\|\gamma'(t(s))\|}$$

Hence $\|\tilde{\gamma}'(s)\| = 1$. □

For plane curves, we can define a notion of signed curvature that captures both the magnitude and direction of bending. We use the second derivative of the curve to define this notion of curvature. Indeed, γ' is of unitary length so its only variations are angular: this means that γ'' will be orthogonal to γ' so in the same direction as the unit normal vectors at each point. Thus the length of γ'' indicates how much the curve pulls away from the tangent and its sign indicates the direction of the bending; this motivates the following definition:

Definition 2.4 (Signed Curvature)**.** Let $\gamma : I \to \mathbb{R}^2$ be a unit-speed curve. The *signed curvature* $\kappa(s)$ is defined by

$$\gamma''(s) = \kappa(s) N(s)$$

where $N(s)$ is the unit normal vector obtained by rotating $\gamma'(s)$ counterclockwise by $\pi/2$.

The geometric meaning of curvature becomes clear when we consider the rate of change of the tangent angle; this point of view will be presented soon by 3.2.

Finally, we give a last definition of a special case of curves which are of great use:

Definition 2.5 (Simple Closed Curve)**.** A *simple closed curve* in the plane $\mathbb{R}^2$ is the image of a continuous and injective map $\gamma : S^1 \to \mathbb{R}^2$. By the Jordan Curve Theorem, such a curve divides the plane into a bounded interior region and an unbounded exterior region. We say the curve *encloses* its interior region.

Chapter 3

The Gauss–Bonnet Theorem for Curves

We now state and prove the Gauss–Bonnet theorem for closed plane curves.

Theorem 3.1 (Gauss–Bonnet for Curves)**.** *Let $\gamma : [0, L] \to \mathbb{R}^2$ be a simple closed curve parametrized by arc length. Then*

$$\int_0^L \kappa(s)\, ds = 2\pi n$$

where $n = \pm 1$.

Before proving it, we need to state and prove the following lemma:

Lemma 3.2 (Curvature as Angular Velocity)**.** *Let $\gamma : I \to \mathbb{R}^2$ be a unit-speed curve and $\theta(s)$ the angle that $\gamma'(s)$ makes with e_1 where the canonical basis of $\mathbb{R}^2$ is $(e_1 := (1, 0), e_2 := (0, 1))$. Then $\kappa(s) = \theta'(s)$.*

Proof. As γ' is smooth and of unit length, we can write $\gamma'(s) = (\cos\theta(s), \sin\theta(s))$ for some smooth function $\theta : I \to [-\pi, \pi)$ in the canonical basis of $\mathbb{R}^2$.

Differentiating:

$$\gamma''(s) = \theta'(s)(-\sin\theta(s), \cos\theta(s)) = \theta'(s)N(s)$$

Comparing with the definition of curvature, we get $\kappa(s) = \theta'(s)$. □

Proof of Theorem 3.1. By Lemma 3.2, we have

$$\int_0^L \kappa(s)\, ds = \int_0^L \theta'(s)\, ds = \theta(L) - \theta(0)$$

Since γ is closed, $\gamma'(L) = \gamma'(0)$, which means the tangent vector has returned to its initial direction. However, it may have completed n full rotations. Thus $\theta(L) = \theta(0) + 2\pi n$, giving

$$\int_0^L \kappa(s)\, ds = 2\pi n$$

For a simple closed curve (no self-intersections), the winding number $n = \pm 1$ depending on orientation. By convention, we choose the parametrization so that $n = 1$ (counterclockwise orientation). □

It is possible to show a stronger theorem for smooth closed curves in which n is actually in $\mathbb{Z}$ and is the winding number of γ. This is a topological invariant of the curve counting the number of *oriented complete rotations* the curve makes.

It is also possible to show that the theorem extends to piecewise smooth curves by including corner contributions [1].

Theorem 3.3 (Gauss–Bonnet with Corners). *Let γ be a piecewise smooth simple closed curve with exterior angles $\alpha_1, \ldots, \alpha_k$ at the corners. Then*

$$\sum_{i=1}^{k} \alpha_i + \sum_{j=1}^{k} \int_{\gamma_j} \kappa \, ds = 2\pi$$

where γ_j are the smooth segments.

Part II

The Gauss–Bonnet Theorem for Surfaces

Chapter 4

Topological Foundations

We begin our journey in the two-dimensional world by considering topological surfaces, which are special topological spaces. This part is mainly inspired by [3].

First of all, we shall define topological spaces which are sets equipped with a structure given by a special class of subsets; more precisely:

Definition 4.1 (Topological Space)**.** A *topological space* is a set X together with a collection $\mathcal{T}$ of subsets of X (called the *open subsets* of X) such that:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$;
2. if $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$;
3. if $U_i \in \mathcal{T}$ for all $i \in I$, then $\bigcup_{i \in I} U_i \in \mathcal{T}$.

A property that we wish to have for surfaces but which is not intrinsic to the structure of topological spaces is a notion of *separation* defines below.

Definition 4.2 (Hausdorff Space)**.** A topological space X is called *Hausdorff* if whenever $x, y \in X$ with $x \neq y$, there exist open subsets $U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Hausdorff spaces have the nice property that any two distinct points can be separated in disjoint neighbourhoods. We shall now introduce continuous maps which are the mappings used to induce a topological structure on a set from a topological space.

Definition 4.3 (Continuous Map)**.** A map $f : X \to Y$ between topological spaces X and Y is called *continuous* if $f^{-1}(V)$ is an open subset of X whenever V is an open subset of Y.

A stronger class of mappings are homeomorphisms which preserve the topological structure between sets:

Definition 4.4 (Homeomorphism)**.** A map $f : X \to Y$ between topological spaces is called a *homeomorphism* if it is a bijection and both $f : X \to Y$ and its inverse $f^{-1} : Y \to X$ are continuous. In this case, we say that X is *homeomorphic* to Y.

We shall now defined special classes of topological spaces.

Definition 4.5 (Compact Space)**.** A topological space X is called *compact* if every open cover of X has a finite subcover.

A compact space is the idea that the space will never need infinitely many pieces to cover it no matter how one choose them.

Definition 4.6 ((Path-) Connectedness)**.** A topological space X is *connected* if it can not be written as the disjoint union of two non-empty sets. It is said to be *path-connected* when any two points $x_0, x_1 \in X$ can be joined by a continuous path $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

So a connected space is the idea that the space is one piece with respect to the topology and a path-connected space is such as any two points can joined by a continuous path; a stronger property would be the well-known convexity property. Indeed, a convex space is path-connected and a path-connected space is connected.

Definition 4.7 (Quotient Topology)**.** Let $\sim$ be an equivalence relation on a topological space X. If $x \in X$, let $[x]_\sim = \{y \in X : y \sim x\}$ be the equivalence class of x and let

$$X/\sim = \{[x]_\sim : x \in X\}$$

be the set of equivalence classes. Let $\pi : X \to X/\sim$ be the *quotient map* which sends an element of X to its equivalence class. Then the *quotient topology* on $X/\sim$ is given by

$$\{V \subseteq X/\sim : \pi^{-1}(V) \text{ is an open subset of } X\}.$$

A quotient topology is the idea that some points from the original space have been identified in a way that respects the initial topology; indeed, the topology on the quotient space is defined as the minimal way to make the natural projection from the space to its quotient continuous. We shall now define our notion of a surface:

Definition 4.8 (Topological Surface)**.** A *topological surface* is a Hausdorff path-connected topological space X such that each point $x \in X$ has a neighbourhood U homeomorphic to an open subset $V \subset \mathbb{R}^2$. A *closed surface* is a compact surface without boundary.

So a topological surface is just an object which *looks like* a plane locally. It is possible to define a surface in a more general setting, omitting the path-connectedness condition, but it will simplify the present exposition as the objects of interest do have this property. There are two main types of such surfaces: orientable and non-orientable surfaces. Roughly speaking it differentiates between surfaces which have *well-defined* sides from the others. To define it more rigorously we need to introduce the Möbius band:

Definition 4.9 (Möbius Band)**.** A *Möbius band* (or *Möbius strip*) is a surface which is homeomorphic to

$$(0,1) \times [0,1]/\sim$$

with the quotient topology, where $\sim$ is the equivalence relation given by

$$(x,y) \sim (s,t) \text{ iff } (x = s \text{ and } y = t) \text{ or } (x = 1 - s \text{ and } \{y,t\} = \{0,1\}).$$

The equivalence relationship used to quotient the *semi-open* square identifies the top and bottom sides in the opposite direction as one would do when building a Möbius band bending a sheet of paper. This surface has no well-defined side. It allows us to define orientability as follows.

Definition 4.10 (Orientability)**.** A surface X is *orientable* if it contains no open subset homeomorphic to a Möbius band.

Now we move on to the special case of compact surfaces and we define a subdivision of such surfaces, which in some sense identify a way to construct the surface topologically with simpler elements.

Definition 4.11 (Subdivision of a Compact Surface)**.** A subdivision of a compact surface X is a partition of X into

(i) *vertices* (finitely many points of X),

(ii) *edges* (finitely many disjoint subsets of X each homeomorphic to $(0, 1)$),

(iii) *faces* (finitely many disjoint open subsets of X each homeomorphic to the open disc).

A powerful result on compact surfaces is that we can classify them in three categories up to homeomorphism.

Theorem 4.12 (Classification of surfaces)**.** *A closed surface is either homeomorphic to the sphere, or to a connected sum of tori, or to a connected sum of projective planes.*

A proof of this theorem can be found in [6].

Not only we can classify surfaces but we can characterize them by a topological invariant known as the Euler characteristic which is defined on a subdivision. This relies on the fact that every compact surface admits a subdivision, but we will not prove this result.

Definition 4.13 (Euler Characteristic)**.** The Euler characteristic of a compact surface X with a subdivision is

$$\chi(X) = V - E + F,$$

where V, E, F are the numbers of vertices, edges, and faces respectively.

At first glance, this definition seems to depend on the specific choice of subdivision. The following fundamental theorem shows this is not the case.

Theorem 4.14 (Topological Invariance of the Euler Characteristic)**.** *The Euler characteristic $\chi(X)$ of a compact surface X is a topological invariant. That is, its value is independent of the chosen subdivision, and if two surfaces are homeomorphic, they have the same Euler characteristic.*

Proof Sketch. The idea is to give a different definition of $\chi(X)$ which makes it clear that it is a topological invariant, and then prove that the Euler characteristic of any subdivision of X is equal to $\chi(X)$ defined in this new way.

For each continuous path $f : [0,1] \to X$ define its boundary ∂f to be the formal linear combination of points $f(0) + f(1)$. If g is another map and $g(0) = f(1)$ then, with coefficients in $\mathbb{Z}/2\mathbb{Z}$, we have

$$\partial f + \partial g = f(0) + 2f(1) + g(1) = f(0) + g(1)$$

which is the boundary of the path obtained by sticking these two together. Let C_0 be the vector space of finite linear combinations of points with coefficients in $\mathbb{Z}/2\mathbb{Z}$ and C_1 the linear combinations of paths, then $\partial : C_1 \to C_0$ is a linear map. If X is connected then any two points can be joined by a path, so that $x \in C_0$ is in the image of ∂ if and only if it has an even number of terms.

Now look at continuous maps of a triangle $ABC = \Delta$ to X and the space C_2 of all linear combinations of these. The boundary of $F : \Delta \to X$ is the sum of the three paths which are the restrictions of F to the sides of the triangle. Then

$$\begin{aligned}\partial\partial F &= (F(A) + F(B)) + (F(B) + F(C)) + (F(C) + F(A)) \\ &= F(A) + (F(B) + F(B)) + (F(C) + F(C)) + F(A) \\ &= 0\end{aligned}$$

so that the image of $\partial : C_2 \to C_1$ is contained in the kernel of $\partial : C_1 \to C_0$. We define $H_1(X)$ to be the quotient space of $\ker(\partial : C_1 \to C_0)$ by $\operatorname{im}(\partial : C_2 \to C_1)$. This is clearly a topological invariant because we only used the notion of continuous functions on X to define it.

If we take X to be a surface with a subdivision, one can show that because each face is homeomorphic to a disc, any element in the kernel of $\partial : C_1 \to C_0$ can be replaced by a linear combination of edges of the subdivision upon adding something in ∂C_2.

Now we let $\mathcal{V}, \mathcal{E}$ and $\mathcal{F}$ be vector spaces over $\mathbb{Z}/2\mathbb{Z}$ with bases given by the sets of vertices, edges and faces of the subdivision, then define boundary maps in the same way

$$\partial : \mathcal{E} \to \mathcal{V} \text{ and } \partial : \mathcal{F} \to \mathcal{E}.$$

Then

$$H_1(X) \cong \frac{\ker(\partial : \mathcal{E} \to \mathcal{V})}{\operatorname{im}(\partial : \mathcal{F} \to \mathcal{E})}.$$

By the rank-nullity formula we get

$$\dim H_1(X) = \dim \mathcal{E} - \operatorname{rk}(\partial : \mathcal{E} \to \mathcal{V}) - \dim \mathcal{F} + \dim \ker(\partial : \mathcal{F} \to \mathcal{E}).$$

Because X is connected the image of $\partial : \mathcal{E} \to \mathcal{V}$ consists of sums of an even number of vertices so that

$$\dim \mathcal{V} = 1 + \operatorname{rk}(\partial : \mathcal{E} \to \mathcal{V}).$$

Also $\ker(\partial : \mathcal{F} \to \mathcal{E})$ is spanned by the sum of the faces, hence

$$\dim \ker(\partial : \mathcal{F} \to \mathcal{E}) = 1$$

so

$$\dim H_1(X) = 2 - V + E - F.$$

This shows that $V - E + F$ is a topological invariant. $\square$

Now that we have established that the Euler characteristic is a true invariant of a closed surface, we can use it to classify them.

Theorem 4.15 (Classification of Surfaces *via* Euler characteristic)**.** *A closed, connected surface is determined up to homeomorphism by its orientability and its Euler characteristic.*

Proof Sketch. The general classification theorem states that any closed, connected surface is homeomorphic to one of the following three types:

- The sphere S^2.
- A connected sum of g tori, T_g (for $g \geq 1$).
- A connected sum of g projective planes, P_g (for $g \geq 1$).

The first two types are orientable, while the third is non-orientable. To complete the proof, we show that the Euler characteristic distinguishes between surfaces within each class. We compute χ for each class using a standard planar polygon model, which provides a convenient subdivision.

For this proof we would need to introduce planar models of surfaces. A planar model has one face ($F = 1$). If its $2n$ sides are identified in pairs, it has n edges ($E = n$). The number of vertices (V) depends on how the corners are identified. For polygons in normal form, all vertices are identified to a single point, so $V = 1$. In this case, $\chi = V - E + F = 1 - n + 1 = 2 - n$.

Case 1: Connected sum of g tori (T_g, orientable)
The normal planar model is a $4g$-gon with sides identified as $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$. Here, $2n = 4g$, so $n = 2g$. All vertices are identified to one.

$$\chi(T_g) = 2 - n = 2 - 2g.$$

(For $g = 1$, the torus, $\chi = 0$. For $g = 2$, $\chi = -2$, etc.)

Case 2: Connected sum of g projective planes (P_g, non-orientable)
The normal planar model is a $2g$-gon with sides identified as $a_1 a_1 \dots a_g a_g$. Here, $2n = 2g$, so $n = g$. Again, all vertices are identified to one.

$$\chi(P_g) = 2 - n = 2 - g.$$

(For $g = 1$, the projective plane, $\chi = 1$. For $g = 2$, the Klein bottle, $\chi = 0$, etc.)

Case 3: The sphere (S^2, orientable)
A simple planar model is a 2-gon with identification aa^{-1}. Here, $n = 1$. However, the two vertices are not identified, forming two distinct vertex classes ($V = 2$). The general formula for normal forms does not apply. We compute directly:

$$\chi(S^2) = V - E + F = 2 - 1 + 1 = 2.$$

Since orientability splits the surfaces into two families, and the Euler characteristic takes a unique value for each g within each family, these two invariants completely determine the surface up to homeomorphism. $\square$

Chapter 5

Differential Geometry of Surfaces

We now move on to smooth surfaces; they are a special case of topological spaces on which one can do calculus. This part is mainly inspired by [3].

5.1 Smooth Structures and Parametrizations

So we begin by defining the notion of a smooth surface

Definition 5.1 (Smooth Surface in $\mathbb{R}^3$)**.** A *smooth surface in* $\mathbb{R}^3$ is a subset $X \subset \mathbb{R}^3$ such that each point has a neighbourhood $U \subset X$ and a map $r : V \to \mathbb{R}^3$ from an open set $V \subseteq \mathbb{R}^2$ such that

- $r : V \to U$ is a homeomorphism
- $r(u, v) = (x(u, v), y(u, v), z(u, v))$ has derivatives of all orders
- at each point $\mathbf{r}_u = \partial r/\partial u$ and $r_v = \partial r/\partial v$ are linearly independent.

We can see that the fact that X is a topological surface resides in the fact that r defines a homeomorphism. The last two conditions tell us that the parametrization is smooth and that we can define a tangent place at each point. So by the *implicit function theorem*, this regularity condition has a powerful consequence: around any point p on the surface, we can find new smooth coordinates (x_1, x_2, x_3) for the ambient $\mathbb{R}^3$ in which the surface is simply the plane $\{x_3 = 0\}$.

Now consider two overlapping parameterizations of the surface. Each gives rise to its own coordinate system (x_1, x_2, x_3) where the surface is $\{x_3 = 0\}$, and (x'_1, x'_2, x'_3) where the surface is $\{x'_3 = 0\}$. The coordinate change $(x_1, x_2, x_3) \mapsto (x'_1, x'_2, x'_3)$ is a smooth map between open sets of $\mathbb{R}^3$ that necessarily maps the plane $\{x_3 = 0\}$ to the plane $\{x'_3 = 0\}$.

Restricting to the surface itself, we can forget the third coordinate (it's always zero) and work with just (x_1, x_2) and (x'_1, x'_2). The transition map between these 2D coordinate systems is then just a smooth map. This is precisely the map $\varphi_{U'} \circ \varphi_U^{-1}$, where φ_U assigns (x_1, x_2) coordinates to points in U and $\varphi_{U'}$ assigns (x'_1, x'_2) coordinates to points in U'.

This observation reveals that the essential property of a smooth surface is not its embedding in $\mathbb{R}^3$, but rather the smooth compatibility of its coordinate charts. This motivates the following definition of a smooth surface:

Definition 5.2. A *smooth surface* is a surface with a class of homeomorphisms φ_U such that each map $\varphi_{U'}\varphi_U^{-1}$ is a smoothly invertible homeomorphism.

Definition 5.3. A *smooth map* between smooth surfaces X and Y is a continuous map $f : X \to Y$ such that for each smooth coordinate system φ_U on U containing $x \in X$ and ψ_W defined in a neighborhood of $f(x)$ on Y, the composition

$$\psi_W \circ f \circ \varphi_U^{-1}$$

is smooth.

5.2 The First Fundamental Form

Now let's get back to $\mathbb{R}^3$ and rigorously define the notion of tangent plane we mentionned:

Definition 5.4 (Tangent Plane)**.** The tangent plane of a surface at $a \in X$ is the vector space spanned by $r_u(a), r_v(a)$ for a parametrization $r(u, v)$.

An equivalent definition can be found in [1] where the tangent space at a point is defined as the space containing every tangent vector to a smooth path through the point; one can show that it is a plane. It is easy to see then that it is the same plane as defined above. This definition has the merit of highlighting that the tangent plane does not define on the choice of parametrization, whereas r_u and r_v do.

As well as the tangent plane, we define at each point the unit normals to describe the geometry of the surface:

Definition 5.5 (Unit Normals)**.** The two unit normals to a surface at (u, v) are

$$\pm\frac{r_u \wedge r_v}{|r_u \wedge r_v|}.$$

From now on we will only consider the outward normal $n := +\dfrac{r_u \wedge r_v}{|r_u \wedge r_v|}$.

We now define some intrinsic tools of the surface, meaning objects defined independently of the ambient space and thus characterize the geometry of the surface.

Definition 5.6 (Smooth Curve and Length)**.** A *smooth curve lying in the surface* is a map $t \mapsto (u(t), v(t))$ with derivatives of all orders such that $\gamma(t) = r(u(t), v(t))$ is a parametrized curve in $\mathbb{R}^3$. Its arc length from $t = a$ to $t = b$ is

$$\int_a^b |\gamma'(t)|\, dt = \int_a^b \sqrt{E\, u'(t)^2 + 2F\, u'(t)v'(t) + G\, v'(t)^2}\, dt,$$

where $E = r_u \cdot r_u$, $F = r_u \cdot r_v$, $G = r_v \cdot r_v$.

Definition 5.7 (First Fundamental Form)**.** For a parametrization $r(u, v)$, the first fundamental form is

$$I = E\, du^2 + 2F\, du\, dv + G\, dv^2,$$

where $E = r_u \cdot r_u$, $F = r_u \cdot r_v$, $G = r_v \cdot r_v$.

Definition 5.8 (Area)**.** The area of $r(U) \subset \mathbb{R}^3$ is

$$\int_U |r_u \wedge r_v| \, dudv = \int_U \sqrt{EG - F^2} \, dudv.$$

The area element $dA = |r_u \wedge r_v| \, dudv$ can be expressed in terms of the first fundamental form. We use Lagrange's identity, which states that for any two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, we have $|\mathbf{a} \wedge \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$. Applying this to our basis vectors:

$$|r_u \wedge r_v|^2 = |r_u|^2 \, |r_v|^2 - (r_u \cdot r_v)^2$$

By the definitions of E, F, G, this is precisely:

$$|r_u \wedge r_v|^2 = (r_u \cdot r_u)(r_v \cdot r_v) - (r_u \cdot r_v)^2 = EG - F^2$$

Taking the square root gives the desired relation:

$$|r_u \wedge r_v| = \sqrt{EG - F^2} \tag{1}$$

Definition 5.9 (Isometric Surfaces)**.** Two surfaces X, X' are isometric if there exists a smooth homeomorphism $f : X \to X'$ mapping curves to curves of the same length.

5.3 Curvature

Similarly as with curves, an interesting aspect of the geometry of a surface is to study its curvature, being how it bends in the space. We first define the second fundamental form which testifies on how the ambient spaces *acts* on the surface before defining the Gaussian curvature.

Definition 5.10 (Second Fundamental Form)**.** The second fundamental form is

$$II = L \, du^2 + 2M \, du \, dv + N \, dv^2,$$

where $L = r_{uu} \cdot n$, $M = r_{uv} \cdot n$, $N = r_{vv} \cdot n$.

Definition 5.11 (Gaussian Curvature)**.** The Gaussian curvature of a surface in $\mathbb{R}^3$ is

$$K = \frac{LN - M^2}{EG - F^2}.$$

A powerful result on the Gaussian curvature is that while apparently depending on extrinsic data from the second fundamental form, it is an intrinsic property of the surface. This means that we can also define it for abstract surfaces with analogous tools.

Theorem 5.12 (Theorema Egregium (Gauss))**.** *The Gaussian curvature K depends only on the first fundamental form and its derivatives.*

Sketch Proof. The Gaussian curvature K is defined using the second fundamental form, which measures how the surface pulls away from its tangent plane. This is captured by the change in the unit normal vector n. Our goal is to show that this quantity can be computed purely from E, F, G and their derivatives. We will do this by analyzing the commutator of tangential derivatives, $[\nabla_v, \nabla_u] = \nabla_v \nabla_u - \nabla_u \nabla_v$, which acts like a "curvature" operator within the surface itself. Let's consider locally a smooth family of tangent vectors $a = f r_u + g r_v$. We use the tangential derivative to differentiate with respect to u or v in the tangent plane:

$$\nabla_u a = a_u - (n \cdot a_u)n = a_u + (n_u \cdot a)n$$

The crucial observation is that this tangential derivative only depends on E, F, G and their derivatives. Differentiating $\nabla_u a$ tangentially with respect to v and doing the same for $\nabla_v a$ with respect to u, we find that the commutator is:

$$\nabla_v \nabla_u a - \nabla_u \nabla_v a = (n_u \cdot a)n_v - (n_v \cdot a)n_u = (n_u \wedge n_v) \wedge a.$$

The term $n_u \wedge n_v$ is normal to the tangent plane, so we can write $n_u \wedge n_v = \lambda n$ for some scalar λ. A calculation then shows that $\lambda\sqrt{EG - F^2} = LN - M^2$. Since the operator $(\nabla_v \nabla_u - \nabla_u \nabla_v)$ is intrinsic (definable only from E, F, G), the quantity λ must be intrinsic. This implies that $(LN - M^2)/\sqrt{EG - F^2}$ is intrinsic, and therefore K itself is intrinsic. □

Finally we define a notion of curvature for curves on a surface:

Definition 5.13 (Geodesic Curvature)**.** The geodesic curvature κ_g of a smooth curve γ in X is the component of its acceleration vector that lies in the tangent plane of the surface. For a unit-speed curve, $\kappa_g = t' \cdot (n \wedge t)$, where $t = \gamma'$ is the unit tangent vector and n the surface normal.

Chapter 6

The Gauss–Bonnet Theorem

We now how all the tools to prove the main theorem of this work: the Gauss-Bonnet theorem. This part is mainly inspired by [3].

6.1 Local Version

We begin with its local version which relates the curvature of a curve and the region it encloses on the surface. This will be used as the proof for the global Gauss-Bonnet theorem relies on subdividing the surface in such regions. More precisely:

Theorem 6.1 (Local Gauss-Bonnet). *Let $X \subset \mathbb{R}^3$ be a smooth surface. Let γ be a smooth simple closed curve on a coordinate neighbourhood of X enclosing a region R. Then*

$$\int_\gamma \kappa_g \, ds = 2\pi - \int_R K \, dA$$

where ds is the element of arc-length of γ and dA the element of area of R.

Proof. Let's choose a parametrization r and a unit length tangent vector field, for example $e = r_u/\sqrt{E}$. Then $e, n \wedge e$ is an orthonormal basis for each tangent space. Since e has unit length, $\nabla_u e$ is tangential and orthogonal to e so there are functions P, Q such that

$$\nabla_u e = P \, n \wedge e, \quad \nabla_v e = Q \, n \wedge e.$$

The idea is to apply Green's formula to P and Q with the area element $dA := \sqrt{EG - F^2} dudv$:

$$\int_\gamma Pdu + Qdv = \int_R (Q_u - P_v) dA$$

The right-hand side becomes

$$\int_\gamma (Pu' + Qv') ds = \int_\gamma (u' \nabla_u e + v' \nabla_v e) \cdot (n \wedge e) ds = \int_\gamma e' \cdot (n \wedge e) ds \tag{4}$$

by definition of $e' := \nabla_{\gamma(s)} e = u' \nabla_u e + v' \nabla_v e$ by the chain rule.

Let t be the unit tangent to γ (parallel to γ'), and write it relative to the orthonormal basis

$$t = \cos\theta \, e + \sin\theta \, n \wedge e.$$

So

$$t' = -\theta' \sin(\theta)\, e + \cos(\theta)\, e' + \theta' \cos(\theta)\,(n \wedge e) + \sin(\theta)\,\frac{d}{ds}(n \wedge e).$$

and

$$t' \cdot (n \wedge e) = \cos\theta\, e' \cdot (n \wedge e) + \cos\theta\, \theta'.$$

But t' is orthogonal to t so we can rewrite

$$t' = \alpha n + \kappa_g n \wedge t = \alpha n + \kappa_g(\cos\theta\, n \wedge e - \sin\theta\, e)$$

with the geodesic curvature of γ defined by $\kappa_g = t' \cdot (n \wedge t)$ and so

$$t' \cdot (n \wedge e) = \kappa_g \cos(\theta)$$

giving

$$\kappa_g \cos(\theta) = \cos\theta\, \mathbf{e}' \cdot (n \wedge e) + \theta' \cos\theta$$

and finally

$$\kappa_g = e' \cdot (n \wedge e) + \theta'.$$

We can therefore write (4) as

$$\int_\gamma (\kappa_g - \theta')ds$$

and by a similar reasoning from the proof of 3.1 as θ changes by 2π on going round the curve, this is

$$\int_\gamma \kappa_g ds - 2\pi.$$

To compute the right hand side of Green's formula, note that

$$\nabla_v \nabla_u \mathbf{e} = \nabla_v (P\, n \wedge e = P_v\, n \wedge e + P\, n \wedge \nabla_v e = P_v\, n \wedge e + PQ\, n \wedge (n \wedge e)$$

since $n_v \wedge e$ is normal. Interchanging the roles of u and v and subtracting we obtain

$$(\nabla_v \nabla_u - \nabla_u \nabla_v)e = (P_v - Q_u) n \wedge e$$

and from (1) this is equal to $K\sqrt{EG - F^2}$.

Applying Green's theorem and using $dA = \sqrt{EG - F^2} dudv$ gives the result. □

We now apply this result to curvilinear triangles as we will use a triangulation, a special kind of subdivision of a surface made of triangles, to prove the main result.

Definition 6.2 (Curvilinear Triangle)**.** A *curvilinear triangle* on a smooth surface X is a region $R \subset X$ whose boundary ∂R is a simple closed curve composed of three smooth arcs $\gamma_1, \gamma_2, \gamma_3$. These arcs meet at three distinct points $p_1, p_2, p_3 \in X$, called the *vertices*. The *internal angle* α_i at vertex p_i is the angle between the two tangent vectors of the boundary arcs meeting at that point.

Corollary 6.3 (Curvilinear Triangle)**.** *The sum of the internal angles α_i of a curvilinear triangle is*

$$\sum_i \alpha_i = \pi + \int_R K\, dA + \int_{\partial R} \kappa_g\, ds.$$

Proof Sketch. Let the curvilinear triangle have a boundary γ which is formed by three smooth curves, and let the interior angles at the vertices be $\alpha_1, \alpha_2, \alpha_3$. Since γ is not smooth at the vertices, we cannot directly apply the previous theorem.

Instead, we construct a new smooth curve γ_ϵ by rounding off each of the three corners with a small circular arc of radius ϵ. This new curve encloses a region R_ϵ. By Theorem 4.4, we have:

$$\int_{\gamma_\epsilon} \kappa_g ds + \int_{R_\epsilon} K dA = 2\pi$$

We now examine what happens in the limit as $\epsilon \to 0$. The region R_ϵ approaches the original triangular region R, so

$$\lim_{\epsilon \to 0} \int_{R_\epsilon} K dA = \int_R K dA$$

The integral along the curve γ_ϵ can be split into the parts along the original sides (now slightly shortened) and the three new corner arcs, c_1, c_2, c_3.

$$\int_{\gamma_\epsilon} \kappa_g ds = \int_{\text{sides}} \kappa_g ds + \sum_{i=1}^{3} \int_{c_i} \kappa_g ds$$

As $\epsilon \to 0$, the integral over the sides approaches the integral over the original boundary of the triangle: $\int_{\text{sides}} \kappa_g ds \to \int_\gamma \kappa_g ds$.

The integral $\int_{c_i} \kappa_g ds$ represents the total turning of the tangent vector along the corner arc c_i. In the limit as the arc shrinks to a vertex, this turning is precisely the external angle δ_i at that vertex. The external angle is related to the internal angle by $\delta_i = \pi - \alpha_i$.

Taking the limit of the entire equation for γ_ϵ as $\epsilon \to 0$, we get:

$$\left(\int_\gamma \kappa_g ds + \sum_{i=1}^{3} \delta_i \right) + \int_R K dA = 2\pi$$

Now, we substitute $\delta_i = \pi - \alpha_i$:

$$\int_\gamma \kappa_g ds + \sum_{i=1}^{3} (\pi - \alpha_i) + \int_R K dA = 2\pi$$

$$\int_\gamma \kappa_g ds + 3\pi - \sum_{i=1}^{3} \alpha_i + \int_R K dA = 2\pi$$

Finally, rearranging the terms to solve for the sum of the internal angles, we get:

$$\sum_{i=1}^{3} \alpha_i = \pi + \int_R K dA + \int_\gamma \kappa_g ds$$

This completes the proof. ☐

6.2 Global Version

It is now time to state the main result:

Theorem 6.4 (Global Gauss-Bonnet). *If $X \subset \mathbb{R}^3$ is a smooth, compact, orientable surface without boundary, then*

$$\int_X K\,dA = 2\pi\chi(X)$$

Before proving it, let's observe how powerful this result is. On the left hand-side we integrate the Gaussian curvature of the surface which is intrinsic by the Theorema Egregium. On the right-hand side we simply get the Euler chacateristic, a topological invariant, multiplied by 2π. This theorem states that we can recover all the topological information of a surface just by integrating its Gaussian curvature which is an intrinsic quantity. We shall now prove the theorem:

Proof Sketch. Take a smooth triangulation so that each triangle is inside a coordinate neighbourhood and apply Theorem 6.3 to each triangle. The integrals of κ_g on the edges cancel because the orientation on the edge from adjacent triangles is opposite (this is for Green's theorem: we use the anticlockwise orientation on γ for every triangle). The theorem gives the total sum of internal angles as

$$\pi F + \int_X K dA,$$

where F is the number of triangles (faces) of the triangulation.

But around each vertex the internal angles add to 2π so we have

$$2\pi V = \pi F + \int_X K dA,$$

where V is the number of vertices of the triangulation.

As our faces are triangles whose sides meet in pairs there are $3F/2$ edges (every edge is counted twice). Hence

$$2\pi\chi(X) = 2\pi(V - E + F) = \pi F + \int_X K dA - 3\pi F + 2\pi F = \int_X K dA.$$

☐

Chapter 7

Extensions and Future Directions

The Gauss-Bonnet theorem extends far beyond the smooth surfaces we have studied. For piecewise smooth surfaces with corners, the formula incorporates exterior angles; for abstract Riemannian manifolds, it requires no embedding in $\mathbb{R}^3$. Most remarkably, the Chern-Gauss-Bonnet theorem generalizes to even-dimensional manifolds M^{2n}, stating that $\int_M \Omega = (2\pi)^n \chi(M)$, where Ω is the Pfaffian of the curvature form. This profound extension, developed by S.S. Chern in the 1940s using differential forms and fiber bundles, reveals that our theorem is the first instance of a fundamental principle relating characteristic classes to topological invariants.

The Gauss-Bonnet theorem also serves as a special case of even grander unifying results. The Atiyah-Singer index theorem, proved in 1963, encompasses it while connecting the analytical properties of differential operators to topological invariants through characteristic classes—a connection with profound implications for quantum field theory, from computing chiral anomalies to understanding D-brane charges in string theory. In algebraic geometry, the Grothendieck-Riemann-Roch theorem provides an algebraic analogue, replacing differential operators with coherent sheaves and integration with pushforward in K-theory. When specialized to Riemann surfaces, it recovers the classical Riemann-Roch theorem, revealing deep connections between differential and algebraic geometry.

These extensions illuminate the fundamental unity of mathematics. The simple observation that a closed curve has total curvature 2π contains within it the seeds of differential geometry, algebraic topology, mathematical physics, and algebraic geometry. The principle that local geometric data integrates to yield global topological invariants, whether expressed through curvature integrals, index formulas, or characteristic classes, pervades modern mathematics. For those continuing in mathematics, particularly in algebraic geometry where these ideas are central, the geometric intuition developed through the Gauss-Bonnet theorem provides invaluable guidance. The formula $\int_X K\, dA = 2\pi\chi(X)$ thus represents not an endpoint but a gateway to the deeper structures that unite all of mathematics.

Bibliography

[1] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, 1976.

[2] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[3] N. Hitchin, *Geometry of Surfaces (B3a course notes)*, Mathematical Institute, University of Oxford, 2013.

[4] W. Kühnel, *Differential Geometry: Curves – Surfaces – Manifolds*, Second Edition, American Mathematical Society, 2006.

[5] G. Rotskoff, *The Gauss–Bonnet Theorem: A Survey*, Unpublished manuscript, 2008.

[6] J. Huang, *Classification of Surfaces*, Unpublished manuscript, 2008.