Mirror Symmetry – Mathematics Proof
Personal Knowledge Base

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Abstract

This document serves as a Personal Knowledge Base in order to write my dissertation on the mathematical proof of mirror symmetry in the case of hypersurfaces of genus zero, following the Clay Mathematics Institue monograph on *Mirror Symmetry*[5]. Its purpose is to be a rigorous exposition of the material covered in [5] self-contained with respect to my mathematical knowledge before starting this project. As I learn better by writing things down, this work will help me to understand deeply and in a mathematically rigorous way the material I will have to cover for my dissertation. It will also help me when writing by serving as a centralized resource to which I can come back whenever needed.

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Chapter 1

Complex Curves

1.1 From topological surfaces to Riemann surfaces

Definition 1.1.1 (Topological surface)**.** A *topological surface* is a smooth manifold of real dimension 2 that is oriented, compact, and connected.

1.1.1 Riemannian structure

1.1.1.1 From vector bundles to tangent bundles

Definition 1.1.2 (Vector bundle)**.** Let B be a smooth manifold. A manifold E together with a smooth submersion[1] $\pi : E \to B$, onto B, is called a *vector bundle of rank k over B* (or k-vector bundle) if the following holds:

(i) there is a k-dimensional vector space V, called *typical fibre* of E, such that for any point $p \in B$ the fibre $E_p \coloneqq \pi^{-1}(p)$ of π over p is a vector space isomorphic to V;

(ii) any point $p \in B$ has a neighbourhood U, such that there is a diffeomorphism Φ_U making the following diagram commute (where pr_1 is the canonical projection on the first factor),

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\Phi_U"] \arrow[d, "\pi"'] & U \times V \arrow[d, "\mathrm{pr}_1"] \\ U \arrow[r, equal] & U \end{tikzcd}$$

which means that every fibre E_p is mapped to $\{p\} \times V$. Φ_U is called a *local trivialization* of E over U, and U a *trivializing neighbourhood* for E;

(iii) $\Phi_U|_{E_p} : E_p \to V$ is an isomorphism of vector spaces.

[1] A smooth map is called a *submersion* if its differential is surjective at each point.

Definition 1.1.3 (Section). Any smooth map $s : B \to E$ such that $\pi \circ s = \mathrm{id}_B$ is called a *section* of E. If s is only defined over a neighbourhood in B it is called a *local section.*

Definition 1.1.4 (Trivial bundle). The simplest example of an n-vector bundle is just $X \times \mathbb{R}^n$ with $\pi : X \times \mathbb{R}^n \to X$ the projection on the first factor, and the obvious vector space structure on each fibre. This is called the *trivial n-vector bundle* over X and will be denoted by $\varepsilon^n(X)$.

Definition 1.1.5 (Equivalence of vector bundles). Two vector bundles $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ are *equivalent* ($E_1 \cong E_2$) if there is a homeomorphism $h : E_1 \to E_2$ which takes each fibre $\pi_1^{-1}(p)$ isomorphically onto $\pi_2^{-1}(p)$. The map h is called an *equivalence.* A bundle equivalent to $\varepsilon^n(B)$ is called *trivial.*

Definition 1.1.6 (Bundle map). A *bundle map* from a bundle $\pi_1 : E_1 \to B_1$ to a bundle $\pi_2 : E_2 \to B_2$ is a pair of continuous maps $(\widetilde{f}, f)$, with $\widetilde{f} : E_1 \to E_2$ and $f : B_1 \to B_2$, such that

(i) the following diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{\widetilde{f}} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

(ii) for each $p \in B_1$, the induced map $\widetilde{f} : \pi_1^{-1}(p) \to \pi_2^{-1}(f(p))$ is a linear map.

Definition 1.1.7. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a differentiable map, and $p \in \mathbb{R}^n$, we define

$$f_{\star p} : \left(\begin{array}{ccc} \{p\} \times \mathbb{R}^n & \longrightarrow & \{f(p)\} \times \mathbb{R}^m \\ (p,\, v) & \longmapsto & \big(f(p),\, Df(p)(v)\big) \end{array} \right)$$

We then define the map $f_\star : \varepsilon^n(\mathbb{R}^n) \to \varepsilon^m(\mathbb{R}^m)$ which is the union of all $f_{\star p}$, and the following diagram commutes:

$$\begin{array}{ccc} \varepsilon^n(\mathbb{R}^n) & \xrightarrow{f_\star} & \varepsilon^m(\mathbb{R}^m) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^m \end{array}$$

Theorem 1.1.8 (Tangent bundle). It is possible to assign to each n-manifold M an n-vector bundle TM over M, and to each C^∞ map $f : M \to N$ a bundle map (f_*, f), such that:

(i) If $1 : M \to M$ is the identity, then $1_* : TM \to TM$ is the identity. If $g : N \to P$, then $(g \circ f)_* = g_* \circ f_*$.

(ii) There are equivalences $t^n : T\mathbb{R}^n \to \varepsilon^n(\mathbb{R}^n)$ such that for every C^∞ function $f : \mathbb{R}^n \to \mathbb{R}^m$ the following diagram commutes:

$$\begin{tikzcd}
T\mathbb{R}^n \arrow[r, "f_*"] \arrow[d, "\pi"'] & T\mathbb{R}^m \arrow[d, "\pi"] \\
\varepsilon^n(\mathbb{R}^n) \arrow[r, "f_\star"] & \varepsilon^m(\mathbb{R}^m)
\end{tikzcd}$$

(iii) If $U \subset M$ is an open submanifold, then TU is equivalent to $(TM)|_U$, and for $f : M \to N$ the map $(f|_U)_* : TU \to TN$ is just the restriction of f_*. More precisely, there is an equivalence $TU \cong (TM)|_U$ such that the natural diagrams commute, where $i : U \to M$ is the inclusion:

$$\begin{tikzcd}
TU \arrow[rr, "i_*"] \arrow[dr, "\cong"'] & & TM \\
& (TM)|_U \arrow[ur, hook] &
\end{tikzcd}$$

$$\begin{tikzcd}
TU \arrow[rr, "(f|_U)_*"] \arrow[dr, "i_*"'] & & TN \\
& TM \arrow[ur, "f_*"'] &
\end{tikzcd}$$

Proof. See Spivak, *A Comprehensive Introduction to Differential Geometry*, Vol. I [17]. □

1.1.1.2 Riemannian metric

Definition 1.1.9 (Inner product)**.** An *inner product* on a vector space V over $\mathbb{R}$ is a bilinear function from $V \times V$ to $\mathbb{R}$, denoted by $(v, w) \mapsto \langle v, w \rangle$, which is:

(i) *symmetric*, *i.e.* $\langle v, w \rangle = \langle w, v \rangle$ for all $v, w \in V$;

(ii) *non-degenerate*, *i.e.* if $v \neq 0$, then there is some $w \neq 0$ such that $\langle w, v \rangle \neq 0$.

An inner product is called *positive definite* if for all $v \neq 0$, $\langle v, v \rangle > 0$.

Definition 1.1.10 (Riemannian metric)**.** If $\pi : E \to B$ is a vector bundle, a *Riemannian metric* on E is a function $\langle\ ,\ \rangle$ assigning to each $p \in B$ a positive definite inner product $\langle\ ,\ \rangle_p$ on $\pi^{-1}(p)$, continuous in the sense that for any two continuous sections $s_1, s_2 : B \to E$, the function

$$\langle s_1, s_2 \rangle : p \longmapsto \langle s_1(p), s_2(p) \rangle_p$$

is continuous. If $E \coloneqq TM$ is the tangent bundle of a smooth manifold M, we speak of a Riemannian metric on M. In this case, we say that M is a *Riemannian manifold* (or that M has been equipped with a *Riemannian structure*).

1.1.2 Conformal structure

Definition 1.1.11 (Conformal equivalence)**.** Two Riemannian metrics R_1 and R_2 on a smooth manifold M are said to be *conformally equivalent* if there exists a map $f : M \to \mathbb{R}$ such that $R_1 = f \cdot R_2$.

This is an equivalence relation, allowing us to define conformal structures as follows:

Definition 1.1.12 (Conformal structure)**.** A *conformal structure* on a smooth manifold is an equivalence class of conformally equivalent Riemannian metrics.

Theorem 1.1.13. If M is a topological surface, then a Riemannian metric on M defines a conformal structure.

Proof. To fill. □

1.1.3 Almost complex structure

Definition 1.1.14 (Almost complex structure)**.** An *almost complex structure* on an even-dimensional smooth manifold M is an automorphism of the tangent bundle $J : TM \to TM$ such that $J^2 \coloneqq J \circ J = -\mathrm{id}_{TM}$.

This gives the tangent bundle a $\mathbb{C}$-vector space structure by defining scalar multiplication as follows:

$$\begin{array}{rcl} \mathbb{C} \times TM & \longrightarrow & TM \\ (x + iy, v) & \longmapsto & xv + yJ(v) \end{array}$$

Theorem 1.1.15. If M is a topological surface, M has an almost complex structure if, and only if, it has a conformal structure.

Proof. To fill. □

1.1.4 Complex structure

Definition 1.1.16 (Complex charts)**.** Let M be a $2n$-dimensional manifold. A *complex chart* on M is a homeomorphism

$$\varphi : U \longrightarrow V$$

of an open subset $U \subset M$ onto an open subset $V \subset \mathbb{C}^n$. Two complex charts $\varphi_i : U_i \to V_i$, $i = 1, 2$, are said to be *holomorphically compatible* if the map

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \longrightarrow \varphi_2(U_1 \cap U_2)$$

is biholomorphic[2].

Definition 1.1.17 (Complex atlas)**.** A *complex atlas* on M is a system

$$\mathcal{A} = \{\, \varphi_i : U_i \to V_i,\ i \in I \,\}$$

of charts which are holomorphically compatible and which cover M, i.e.

$$\bigcup_{i \in I} U_i = M.$$

Two complex atlases $\mathcal{A}$ and $\mathcal{A}'$ on M are said to be *analytically equivalent* if every chart of $\mathcal{A}$ is holomorphically compatible with every chart of $\mathcal{A}'$.

This is an equivalence relation, allowing us to define complex structures as follows:

Definition 1.1.18 (Complex structure)**.** A *complex structure* on a $2n$-dimensional smooth manifold M is an equivalence class of analytically equivalent atlases on M, and we say that M is a *smooth n-dimensional complex manifold.*

Definition 1.1.19 (Riemann surface)**.** A topological surface with a complex structure is called a *non-singular complex curve* or *Riemann surface.*

Theorem 1.1.20. Every almost complex structure on a topological surface gives a Riemann surface.

Proof. To fill. □

[2] A map is said to be *biholomorphic* if it is a holomorphic bijection with holomorphic inverse.

Definition 1.1.21 (Holomorphic maps of complex manifolds)**.** Suppose M and N are smooth complex manifolds. A continuous mapping $f : M \to N$ is called *holomorphic* if, for every pair of charts

$$\psi_1 : U_1 \to V_1 \text{ on } M, \qquad \psi_2 : U_2 \to V_2 \text{ on } N,$$

with $f(U_1) \subset U_2$, the mapping

$$\psi_2 \circ f \circ \psi_1^{-1} : V_1 \longrightarrow V_2$$

is holomorphic in the usual sense. A mapping $f : M \to N$ is called *biholomorphic* if it is bijective and both $f : M \to N$ and $f^{-1} : N \to M$ are holomorphic. Two smooth complex manifolds M and N are said to be *isomorphic* if there exists a biholomorphic mapping $f : X \to Y$.

1.1.5 Algebraic structure

Definition 1.1.22 (Algebraic variety)**.** An *algebraic variety* $V \subset \mathbb{CP}^n$ is the locus (*ie.* the vanishing set) in $\mathbb{CP}^n$ of a collection of homogeneous polynomials $\{F_\alpha(X_0, ..., X_n)\}$.

Definition 1.1.23 (Algebraic structure)**.** A smooth manifold is said to be *algebraic* (or to carry an *algebraic structure*) if it is isomorphic to an algebraic variety.

Theorem 1.1.24. Every Riemann surface is algebraic.

The proof of this result uses the Riemann-Roch theorem, but we need to introduce a few notions to be able to state it.

1.1.6 Sheaves and cohomology groups

Definition 1.1.25 (Presheaf of Abelian groups)**.** Let X be a topological space and $\mathcal{I}$ the system of open sets in X. A *presheaf of Abelian groups* on X is a pair $(\mathscr{F}, \rho)$ consisting of:

(i) a family $\mathscr{F} = (\mathscr{F}(U))_{U \in \mathcal{I}}$ of Abelian groups, and

(ii) a family $\rho = (\rho_V^U)_{V \subset U, U, V \in \mathcal{I}}$ of group homomorphisms

$$\rho_V^U : \mathscr{F}(U) \longrightarrow \mathscr{F}(V),$$

where V is open in U, such that for all open sets $W \subset V \subset U$ in X:

$$\rho_U^U = \mathrm{id}_{\mathscr{F}(U)}, \qquad \rho_W^V \circ \rho_V^U = \rho_W^U.$$

Definition 1.1.26 (Sheaf)**.** A presheaf $\mathscr{F}$ on a topological space X is called a *sheaf* if, for every open set $U \subset X$ and every family of open subsets $(U_i)_{i\in I}$ with $U = \bigcup_{i\in I} U_i$, the following *sheaf axioms* hold:

(SA1) If $f, g \in \mathscr{F}(U)$ are such that $f|_{U_i} = g|_{U_i}$ for all $i \in I$, then $f = g$.

(SA2) If $(f_i \in \mathscr{F}(U_i))_{i\in I}$ are such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists an element $f \in \mathscr{F}(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

The elements of $\mathscr{F}(U)$ are called the *sections of* $\mathscr{F}$ *over* U for U open in X, and *global sections* if $U \coloneqq X$.

Example 1.1.27. Let X be a Riemann surface and U be an open subset of X. Then the ring $\mathcal{O}(U)$ of holomorphic functions on U together with the usual restriction mapping give the *sheaf* $\mathcal{O}$ *of holomorphic functions on* X.

Remark 1.1.28. One can also define presheaves and sheaves over other algebraic structures in a similar way; we will later extensively rely on *presheaves* and *sheaves of rings*.

Definition 1.1.29 (Cochains)**.** Let $\mathscr{F}$ be a sheaf of Abelian groups on a topological space X, and $\mathcal{U} = (U_i)_{i\in I}$ an open covering of X. For $q \in \mathbb{N}$, the *q-th cochain group* of $\mathscr{F}$ with respect to $\mathcal{U}$ is defined by

$$C^q(\mathcal{U}, \mathscr{F}) \coloneqq \prod_{(i_0,\dots,i_q)\in I^{q+1}} \mathscr{F}(U_{i_0} \cap \dots \cap U_{i_q}).$$

The elements of $C^q(\mathcal{U}, \mathscr{F})$ are called *q-cochains.* The addition of cochains is defined component-wise.

Definition 1.1.30 (Coboundary operators)**.** The *coboundary operators* δ are defined as follows:

(i) For $(f_i)_{i\in I} \in C^0(\mathcal{U}, \mathscr{F})$, define

$$\delta((f_i)_{i\in I}) = (g_{ij})_{i,j\in I}, \quad \text{where } g_{ij} = f_j|_{U_i\cap U_j} - f_i|_{U_i\cap U_j}.$$

(ii) For $(g_{ij})_{i,j\in I} \in C^1(\mathcal{U}, \mathscr{F})$, define

$$\delta((g_{ij})_{i,j\in I}) = (h_{ijk})_{i,j,k\in I}, \quad \text{where } h_{ijk} = +g_{ij}|_{U_i\cap U_j\cap U_k} - g_{ik}|_{U_i\cap U_j\cap U_k} + g_{jk}|_{U_i\cap U_j\cap U_k}.$$

Definition 1.1.31 (Cocycles and coboundaries)**.** Define

$$Z^1(\mathcal{U},\mathscr{F}) := \ker\big(\delta : C^1(\mathcal{U},\mathscr{F}) \to C^2(\mathcal{U},\mathscr{F})\big), \qquad B^1(\mathcal{U},\mathscr{F}) := \operatorname{Im}\big(\delta : C^0(\mathcal{U},\mathscr{F}) \to C^1(\mathcal{U},\mathscr{F})\big).$$

The elements of $Z^1(\mathcal{U},\mathscr{F})$ are called *1-cocycles*, and those of $B^1(\mathcal{U},\mathscr{F})$ are called *1-coboundaries.*

Proposition 1.1.32. Every coboundary is a cocycle.

Proof. Let $(g_{ij})_{i,j\in I} \in B^1(\mathcal{U},\mathscr{F})$. Then there exists $(f_i)_{i\in I} \in C^0(\mathcal{U},\mathscr{F})$ such that $g_{ij} = f_j - f_i$ for all $i,j \in I$. For $i,j,k \in I$,

$$g_{ij} = f_j - f_i, \qquad g_{ik} = f_k - f_i, \qquad g_{jk} = f_k - f_j.$$

Hence

$$g_{ij} - g_{ik} + g_{jk} = (f_j - f_i) - (f_k - f_i) + (f_k - f_j) = 0,$$

so $(g_{ij}) \in Z^1(\mathcal{U},\mathscr{F})$. $\square$

Definition 1.1.33 (First cohomology group with respect to a covering)**.** The quotient group

$$H^1(\mathcal{U},\mathscr{F}) := Z^1(\mathcal{U},\mathscr{F})/B^1(\mathcal{U},\mathscr{F})$$

is called the *first cohomology group of X with coefficients in $\mathscr{F}$ with respect to the covering $\mathcal{U}$.* Its elements are called *cohomology classes.* Two cocycles are said to be *cohomologous* if they belong to the same cohomology class.

Definition 1.1.34 (Refinements of coverings)**.** An open covering $\mathcal{V} = (V_k)_{k\in K}$ of X is called a *refinement* of an open covering $\mathcal{U} = (U_i)_{i\in I}$, denoted $\mathcal{V} < \mathcal{U}$, if for every $k \in K$ there exists $i = \tau(k) \in I$ such that $V_k \subset U_{\tau(k)}$.

Definition 1.1.35. Let $\mathcal{U},\mathcal{V}$ be open coverings with $\mathcal{V} < \mathcal{U}$ and a corresponding mapping $\tau : K \to I$ as above. We define

$$t^{\mathcal{U}}_{\mathcal{V}} : \left(\begin{array}{ccc} Z^1(\mathcal{U},\mathscr{F}) & \longrightarrow & Z^1(\mathcal{V},\mathscr{F}) \\ (f_{i,j})_{i,j\in I} & \longmapsto & (g_{k,l})_{k,l\in K} \end{array}\right)$$

such that $g_{kl} := f_{\tau k,\tau l}|_{V_k\cap V_l}$ for every $k,l \in K$. This induces a homomorphism of the cohomology groups $t^{\mathcal{U}}_{\mathcal{V}} : H^1(\mathcal{U},\mathscr{F}) \longrightarrow H^1(\mathcal{V},\mathscr{F})$, which we also denote by $t^{\mathcal{U}}_{\mathcal{V}}$.

Lemma 1.1.36. The mapping $t^{\mathcal{U}}_{\mathcal{V}} : H^1(\mathcal{U},\mathscr{F}) \to H^1(\mathcal{V},\mathscr{F})$ is independent of the choice of the refining map τ.

Proof. See O. Forster, *Lectures on Riemann Surfaces* [7]. $\square$

Lemma 1.1.37. The mapping $t^{\mathcal{U}}_{\mathcal{V}} : H^1(\mathcal{U}, \mathscr{F}) \to H^1(\mathcal{V}, \mathscr{F})$ is injective.

Proof. See O. Forster, *Lectures on Riemann Surfaces* [7]. $\square$

Proposition 1.1.38. If $\mathcal{U}, \mathcal{V}, \mathcal{W}$ are three open coverings such that $\mathcal{W} < \mathcal{V} < \mathcal{U}$, then

$$t^{\mathcal{U}}_{\mathcal{W}} = t^{\mathcal{V}}_{\mathcal{W}} \circ t^{\mathcal{U}}_{\mathcal{V}}.$$

Proof. To fill. $\square$

Definition 1.1.39 (First cohomology group of a topological space)**.** We define an equivalence relation on the disjoint union

$$\coprod_{\mathcal{U}} H^1(\mathcal{U}, \mathscr{F}),$$

where $\mathcal{U}$ ranges over all open coverings of X, by declaring $\xi \in H^1(\mathcal{U}, \mathscr{F})$ and $\eta \in H^1(\mathcal{U}', \mathscr{F})$ to be *equivalent*, written $\xi \sim \eta$, if there exists an open covering $\mathcal{V}$ of X such that $\mathcal{V} < \mathcal{U}$ and $\mathcal{V} < \mathcal{U}'$, and

$$t^{\mathcal{U}}_{\mathcal{V}}(\xi) = t^{\mathcal{U}'}_{\mathcal{V}}(\eta).$$

The set of equivalence classes is the *inductive limit* of the cohomology groups $H^1(\mathcal{U}, \mathscr{F})$, and is called the *first cohomology group of X with coefficients in $\mathscr{F}$*:

$$H^1(X, \mathscr{F}) := \varinjlim_{\mathcal{U}} H^1(\mathcal{U}, \mathscr{F}) = \left(\coprod_{\mathcal{U}} H^1(\mathcal{U}, \mathscr{F}) \right) / \sim .$$

Proposition 1.1.40. For any open covering $\mathcal{U}$ of X, the canonical mapping

$$i^{\mathcal{U}} : H^1(\mathcal{U}, \mathscr{F}) \longrightarrow H^1(X, \mathscr{F})$$

is injective. In particular, this implies that $H^1(X, \mathscr{F}) = 0$ if and only if $H^1(\mathcal{U}, \mathscr{F}) = 0$ for all open coverings $\mathcal{U}$ of X.

Proof. This follows from Lemma 1.1.37. $\square$

Definition 1.1.41 (Topological genus of a compact Riemann surface)**.** Let X be a compact Riemann surface. The *topological genus* of X is defined by

$$g := \dim H^1(X, \mathcal{O}).$$

Theorem 1.1.42. Topological surfaces are classified by their genus up to homeomorphism.

Proof. Find reference. $\square$

We can also define the

Definition 1.1.43 (The zeroth cohomology group)**.** Let $\mathscr{F}$ be a sheaf of Abelian groups on a topological space X, and let $\mathcal{U} = (U_i)_{i\in I}$ be an open covering of X. Define

$$Z^0(\mathcal{U}, \mathscr{F}) \coloneqq \ker\big(\delta : C^0(\mathcal{U}, \mathscr{F}) \to C^1(\mathcal{U}, \mathscr{F})\big), \qquad B^0(\mathcal{U}, \mathscr{F}) \coloneqq 0.$$

The quotient group

$$H^0(\mathcal{U}, \mathscr{F}) := Z^0(\mathcal{U}, \mathscr{F})/B^0(\mathcal{U}, \mathscr{F}) \cong Z^0(\mathcal{U}, \mathscr{F}).$$

is the *zeroth cohomology group of X with coefficients in $\mathscr{F}$ with respect to the covering $\mathcal{U}$.*

Proposition 1.1.44. Let $\mathscr{F}$ be a sheaf of Abelian groups on a topological space X. Then

$$H^0(X, \mathscr{F}) \cong \mathscr{F}(X).$$

Proof. See O. Forster, *Lectures on Riemann Surfaces* [7]. $\square$

1.1.7 Divisors

Definition 1.1.45 (Meromorphic function)**.** Let X be a Riemann surface and $Y \subset X$ an open subset. A *meromorphic function* on Y is a holomorphic function $f : Y' \to \mathbb{C}$, where $Y' \subset Y$ is an open subset, such that:

(i) $Y \setminus Y'$ consists only of isolated points;

(ii) for every point $p \in Y \setminus Y'$, one has

$$\lim_{x\to p} |f(x)| = \infty.$$

The points of $Y \setminus Y'$ are called the *poles* of f. The set of all meromorphic functions on Y is denoted by $\mathscr{M}(Y)$.

Definition 1.1.46 (Divisor on a Riemann surface)**.** Let X be a Riemann surface. A *divisor* on X is a mapping

$$D : X \longrightarrow \mathbb{Z}$$

such that for any compact subset $K \subset X$ there are only finitely many points $x \in K$ with $D(x) \neq 0$. With respect to pointwise addition, the set of all divisors on X forms an abelian group, denoted by $\mathrm{Div}(X)$.

Definition 1.1.47 (Order of meromorphic functions)**.** Let X be a Riemann surface and $Y \subset X$ an open subset. For a meromorphic function $f \in \mathscr{M}(Y)$ and a point $a \in Y$, we define the *order* of f as follows:

$$\mathrm{ord}_a(f) := \begin{cases} 0, & \text{if } f \text{ is holomorphic and nonzero at } a, \\ k, & \text{if } f \text{ has a zero of order}^1\ k \text{ at } a, \\ -k, & \text{if } f \text{ has a pole of order}^2\ k \text{ at } a, \\ \infty, & \text{if } f \text{ is identically zero in a neighborhood of } a. \end{cases}$$

Proposition 1.1.48 (Divisor of non-zero meromorphic functions)**.** For any non-zero meromorphic function $f \in \mathscr{M}(X)$, the mapping

$$x \longmapsto \mathrm{ord}_x(f)$$

defines a divisor on X, called the *divisor of* f and denoted by (f). The function f is said to be a *multiple* of a divisor D if $(f) \geq D$; in particular, f is holomorphic precisely when $(f) \geq 0$.

Proof. To fill. □

Definition 1.1.49 (Degree of a divisor)**.** Let X be a compact Riemann surface and define a mapping

$$\deg : \mathrm{Div}(X) \longrightarrow \mathbb{Z}, \qquad \deg D := \sum_{x \in X} D(x).$$

The *degree* of a divisor D on X is $\deg(D)$.

Definition 1.1.50 (Sheaves associated to divisors)**.** Let D be a divisor on the Riemann surface X. For any open set $U \subset X$, define

$$\mathcal{O}_D(U) := \left\{ f \in \mathscr{M}(U) : \forall x \in U \ \mathrm{ord}_x(f) \geq -D(x) \right\}.$$

Together with the natural restriction mappings, $\mathcal{O}_D$ is a sheaf. In the special case of the zero divisor $D = 0$, one has $\mathcal{O}_0 = \mathcal{O}$, the *structure sheaf.*

[1] We say that f has a *zero of order k at a* if $f(t) \underset{a}{\sim} (t-a)^k$.

[2] We say that f has a *pole of order k at a* if $f(t) \underset{a}{\sim} (t-a)^{-k}$.

1.1.8 The Riemann-Roch theorem

We can now state the

Theorem 1.1.51 (Riemann–Roch theorem)**.** Suppose D is a divisor on a compact Riemann surface X of genus g. Then $H^0(X, \mathcal{O}_D)$ and $H^1(X, \mathcal{O}_D)$ are finite-dimensional vector spaces, and

$$\dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) = 1 - g + \deg D.$$

Proof. See O. Forster, *Lectures on Riemann Surfaces* [7]. □

We can now prove Theorem 1.1.24.

Proof of Theorem 1.1.24. To fill. □

In this section we proved the following relationships between structures on topological surfaces.

$$\begin{array}{c}
\textbf{Riemannian structure} \\
\Downarrow \\
\textbf{conformal structure} \\
\Updownarrow \\
\textbf{almost complex structure} \\
\Updownarrow \\
\textbf{complex structure} \\
\Updownarrow \\
\textbf{algebraic structure}
\end{array}$$

1.2 Nodal curves

We will need to consider some *singular curves* in addition to the nice smooth curves we presented in the previous section. The nicest kind of such singular curves are so-called *nodal curves*, *ie.* curves with *nodes.* But to define what a node is we need to introduce some concepts from algebraic geometry.

1.2.1 Some algebra

Definition 1.2.1 (Algebra over a ring)**.** Let A be a ring. An *algebra over* A (or A-*algebra*, or simply *algebra*) is a set E together with:

(i) an A-module structure on E;

(ii) an A-bilinear map $E \times E \to E$, denoted $(x, y) \mapsto xy$.

The bilinear map is called the *multiplication* of the algebra.

Definition 1.2.2 (Commutative algebra)**.** An algebra E over a ring A is said to be *commutative* if $xy = yx$ for all $x, y \in E$.

Definition 1.2.3 (Algebra of finite type)**.** A commutative algebra E over a ring A is said to be of *finite type* if it is finitely generated (i.e. finitely generated as an A-module).

Definition 1.2.4 (Prime spectrum)**.** Let A be a ring. The *prime spectrum* of A, denoted $\mathrm{Spec}(A)$, is the set X of prime ideals of A, together with the topology for which the closed sets are of the form

$$V(M) := \{\, \mathfrak{p} \supseteq M : \mathfrak{p} \text{ prime ideal of } A \,\},$$

where M runs through the subsets of A. This topology is called the *Zariski topology* on X. We will denote $V(f) := V(\{f\})$ and $D(f) = X - V(f)$ for all $f \in A$.

Definition 1.2.5 (Localization of a ring)**.** Let A be a ring and $f \in A$. Let $S = \{f^n : n \geq 0\}$ be the multiplicative subset generated by f (*ie.* the set of f^n for $n \in \mathbb{N}$). We define $A_f := S^{-1}A$.

Definition 1.2.6 (Local ring)**.** A *local ring* is a ring with exactly one maximal ideal.

Definition 1.2.7 (Local homomorphism)**.** If A and B are local rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ respectively, a ring homomorphism $\varphi : A \to B$ is said to be *local* if $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$.

Definition 1.2.8 (Completion of a ring)**.** Let A be a ring and I and ideal of A. The *completion of the ring* A *with respect to the ideal* I is the inverse limit

$$\widehat{A} := \varprojlim_{n \in \mathbb{N}} A/I^n$$

where the elements of $\widehat{A}$ are given by a sequence f_n such that $f_n \equiv f_{n+1}$ for all $n \in \mathbb{N}$. If A is local then we talk about *the completion of the ring* A when completing by its maximal ideal.

Definition 1.2.9 (Integrally closed domain)**.** A is an *integrally closed domain* if it is an integral domain such that every $x \in A_{\mathfrak{p}}/\mathfrak{p}$ which is the root of a monic polynomial with coefficients in A is itself in A.

Definition 1.2.10 (Normal ring)**.** A ring A is *normal* if for every prime $\mathfrak{p} \subset A$ the localisation $A_{\mathfrak{p}}$ is integrally closed in its field of fractions.

1.2.2 Sheaves of rings and ringed spaces

Definition 1.2.11 (Structure sheaf of $\operatorname{Spec}(A)$)**.** Let A be a ring, and set $X := \operatorname{Spec}(A)$. The *structure sheaf* of X, denoted $\widetilde{A}$ or $\mathcal{O}_X$, is the sheaf of rings associated[3] to the presheaf

$$\mathscr{F} : D(f) \longmapsto A_f,$$

defined on the basis $\mathcal{B} = \{D(f) : f \in A\}$ of open sets of X.

Definition 1.2.12 (The stalk of a presheaf)**.** Suppose $\mathscr{F}$ is a presheaf of rings on a topological space X, and let $a \in X$ be a point. On the disjoint union

$$\coprod_{U \ni a} \mathscr{F}(U),$$

where the union is taken over all open neighborhoods U of a, we introduce an equivalence relation $\sim_a$ by stating that two elements $f \in \mathscr{F}(U)$ and $g \in \mathscr{F}(V)$ are *equivalent around* a, written $f \sim_a g$, if there exists an open set W with

$$a \in W \subset U \cap V$$

such that $f|_W = g|_W$ (this is an equivalence relation). The set $\mathscr{F}_a$ of all equivalence classes, called the *inductive limit* of the $\mathscr{F}(U)$ as U ranges over all open neighborhoods of a, is given by

$$\mathscr{F}_a := \varinjlim_{U \ni a} \mathscr{F}(U) := \left(\coprod_{U \ni a} \mathscr{F}(U) \right) \Big/ \sim_a .$$

The set $\mathscr{F}_a$ is called the *stalk of* $\mathscr{F}$ *at the point* a.

Proposition 1.2.13. The stalk of a presheaf of rings at a point is a ring by means of operations defined on representatives.

Proof. To fill. □

[3] It is always possible to associate a sheaf canonically to a presheaf by a process called *sheafification*; see for example [18].

Definition 1.2.14 (Ringed space)**.** A *ringed space* is a topological space X together with a sheaf of rings (not necessarily commutative) $\mathcal{A}$ on X. We denote it by $(X, \mathcal{A})$, and call $\mathcal{A}$ the *structural sheaf* which we denote $\mathcal{O}_X$. We say that X is the *underlying topological space* of the ringed space $(X, \mathcal{A})$. The stalk of $\mathcal{O}_X$ at a point $x \in X$ is denoted $\mathcal{O}_{X,x}$ or simply $\mathcal{O}_x$. If the stalk $\mathcal{O}_x$ is a local ring at each point $x \in X$ we talk of a *locally ringed space.*

Definition 1.2.15 (Direct image)**.** Let X, Y be topological spaces and $\psi : X \to Y$ a continuous map. Let $\mathscr{F}$ be a presheaf of rings on X. For any open set $U \subseteq Y$, define

$$\mathscr{G}(U) := \mathscr{F}(\psi^{-1}(U)).$$

If U, V are open subsets of Y with $V \subseteq U$, let

$$\rho_V^U : \mathscr{G}(U) \longrightarrow \mathscr{G}(V)$$

be the restriction map $\mathscr{F}(\psi^{-1}(U)) \to \mathscr{F}(\psi^{-1}(V))$. The system $\mathscr{G}$ with these maps ρ_V^U forms a presheaf of rings on Y, called the *direct image* of $\mathscr{F}$ by ψ, and denoted $\psi_*(\mathscr{F})$. If $\mathscr{F}$ is a sheaf, then $\psi_*(\mathscr{F})$ is also a sheaf.

Definition 1.2.16 (Induced morphism of sheaves)**.** Under the same assumptions, if $\mathscr{F}$ and $\mathscr{G}$ are presheaves of rings on X and Y respectively, a morphism

$$u : \mathscr{G} \longrightarrow \psi_*(\mathscr{F})$$

is called a *ψ-morphism* from $\mathscr{G}$ to $\mathscr{F}$, and is also denoted $\mathscr{G} \to \mathscr{F}$.

Definition 1.2.17 (Inverse image of a presheaf of rings)**.** Let X, Y be topological spaces and $\psi : X \to Y$ a continuous map. Let $\mathscr{G}$ be a presheaf of rings on Y. We define the *inverse image* of $\mathscr{G}$ by ψ to be the pair $(\mathscr{G}', \rho)$, where $\mathscr{G}'$ is a sheaf on X and

$$\rho : \mathscr{G} \longrightarrow \mathscr{G}'$$

is a ψ-morphism satisfying the following universal property: for every sheaf $\mathscr{F}$ on X, the map

$$\mathrm{Hom}_X(\mathscr{G}', \mathscr{F}) \longrightarrow \mathrm{Hom}_Y(\mathscr{G}, \psi_*(\mathscr{F})), \qquad v \longmapsto \psi_*(v) \circ \rho,$$

is a bijection. We denote $\mathscr{G}' = \psi^*(\mathscr{G})$ and $\rho = \rho_{\mathscr{G}}$. For any homomorphism $v : \psi^*(\mathscr{G}) \to \mathscr{F}$, where $\mathscr{F}$ is a sheaf of rings on X, we will denote $v^\flat := \psi_*(v) \circ \rho_{\mathscr{G}} : \mathscr{G} \to \psi_*(\mathscr{F})$. By definition, any morphism of presheaves $u : \mathscr{G} \to \psi_*(\mathscr{F})$ is of the form $v^\flat$ for exactly one v which we will denote $u^\sharp$.

Definition 1.2.18 (Morphism of ringed spaces)**.** A *morphism of ringed spaces* between $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ is a pair

$$\Psi = (\psi, \theta)$$

where $\psi : X \to Y$ is a continuous map and $\theta : \mathcal{B} \to \mathcal{A}$ is a ψ-morphism of sheaves of rings.

Definition 1.2.19 (Derivation)**.** Let $\Psi : (X, \mathscr{A}) \to (Y, \mathscr{B})$ be a morphism of ringed spaces, and $\mathscr{F}$ be a $\mathscr{B}$-module. An *$\mathscr{A}$-derivation* (or *Y-derivation*, or more precisely a *Ψ-derivation into $\mathscr{F}$*) is a map $D : \mathscr{B} \to \mathscr{F}$ which is additive, annihilates $\Psi(\mathscr{A})$ and satisfies the Leibniz rule

$$D(ab) = aD(b) + D(a)b$$

for all a, b local sections of $\mathscr{B}$ (wherever they are both defined). We denote $\mathrm{Der}_{\mathscr{A}}(\mathscr{B}, \mathscr{F})$ the set of Ψ-derivations into $\mathscr{F}$.

Theorem 1.2.20 (Module of differentials of a morphism of ringed spaces)**.** Let $\Psi : (X, \mathscr{A}) \to (Y, \mathscr{B})$ be a morphism of ringed spaces. Then there exists a $\mathscr{B}$-module $\Omega_{\mathscr{B}/\mathscr{A}}$ called the *module of differentials of* Ψ, together with an $\mathscr{A}$-derivation d satisfying the following universal property:

$$\begin{tikzcd} \mathscr{B} \arrow[r, "d"] \arrow[dr, "\forall d'"'] & \Omega_{\mathscr{B}/\mathscr{A}} \arrow[d, dotted, "\exists! f"] \\ & \forall \mathscr{F} \end{tikzcd}$$

where the $\mathscr{F}$ are $\mathscr{B}$-modules, the d' are $\mathscr{A}$-derivations and the f are $\mathscr{B}$-module homomorphims.

Proof. See *Lemma 28.2* of the Stacks Project[18]. □

Definition 1.2.21 (Sheaf of relative Kähler differentials)**.** Let $\Psi : (X, \mathscr{A}) \to (Y, \mathscr{B})$ be a morphism of ringed spaces. The *sheaf of relative Kähler differentials* $\Omega_{X/Y}$ *of X over Y* is the module of differentials $\Omega_{\mathscr{B}/\Psi^{-1}(\mathscr{B})}$ endowed with its universal Y-derivation $d_{X/Y} : \mathscr{B} \to \Omega_{X/Y}$.

Definition 1.2.22 (Sheaf of modules generated by global sections)**.** Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of $\mathcal{O}_X$-modules. We say that $\mathscr{F}$ is *generated by global sections* if there exists a family of global sections $s_i \in \Gamma(X, \mathscr{F}) := \mathscr{F}(X)$, $i \in I$ such that its canonical projection on the stalk at any point $x \in X$ generate that stalk as an $\mathcal{O}_x$-module.

Definition 1.2.23 (Sheaf of modules locally generated by sections)**.** Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of $\mathcal{O}_X$-modules. We say that $\mathscr{F}$ is *locally generated by sections* if for every $x \in X$ there exists an open neighbourhood $U \ni x$ such that $\mathscr{F}|_U$ is globally generated as a sheaf of $\mathcal{O}_X|_U$-modules.

1.2.3 Affine schemes, schemes and morphisms

Definition 1.2.24 (Affine scheme)**.** A ringed space $(X, \mathcal{O}_X)$ is called an *affine scheme* if it is isomorphic to a ringed space of the form

$$(\mathrm{Spec}(A), \widetilde{A}),$$

where A is a ring.

Proposition 1.2.25. If an affine scheme $(X, \mathcal{O}_X)$ is isomorphic to $(\mathrm{Spec}(A), \widetilde{A})$ for a ring A, then it is *canonically associated* to the ring A. We call it the *ring of the affine scheme* and write it $\mathcal{O}_X(X) = A$.

Proof. See A. Grothendieck & J. A. Dieudoné *EGA I* Th. (I.I.3.7) [11]. □

Definition 1.2.26 (Affine open subsets)**.** Given a ringed space $(X, \mathcal{O}_X)$, an open subset $V \subset X$ is called *affine* if the ringed space $(V, \mathcal{O}_X|_V)$ is an affine scheme.

Definition 1.2.27 (Scheme)**.** A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ such that every point $x \in X$ has an affine open neighborhood.

Definition 1.2.28 (Irreducibile scheme)**.** A nonempty scheme is said to be *irreducible* if its underlying topological space is, *ie.* if it cannot be expressed as the union of two proper closed subsets. The empty scheme is not considered to be irreducible. An *irreducible component of a scheme* is an irreducible component of its underlying topological space, *ie.* an irreducible subspace maximal for inclusion.

Definition 1.2.29 (Dimension of a scheme)**.** The *dimension* of a scheme is the dimension of its underlying topological space X, *ie.* the supremum of the numbers $n \in \mathbb{N}$ such that there exists a a chain $Z_0 \subset ... \subset Z_n$ of distinct irreducible closed subsets in X.

Definition 1.2.30 (Reduced scheme)**.** A scheme X is *reduced* if every local ring $\mathcal{O}_x$ is for $x \in X$, *ie.* if fro all $x \in X$ and $y \in \mathcal{O}_x$, $y = 0$ only if $y \cdot y = 0$.

Definition 1.2.31 (Scheme over a base)**.** Let S be a scheme. We say that X is a *scheme over* S if it comes equipped with a morphism of schemes $X \to S$, called the *structure morphism*. If R is a ring, X is a *scheme over* R if it is a scheme over $\mathrm{Spec}(R)$.

Definition 1.2.32 (Morphism of schemes)**.** Given two schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, a *morphism of schemes* is a morphism of ringed spaces (ψ, θ) such that for every $x \in X$, the homomorphism of rings $\theta_x^\sharp : \mathcal{O}_{\psi(x)} \longrightarrow \mathcal{O}_x$ is local.

Definition 1.2.33 (Morphism of scheme over a base)**.** Let S be a scheme. We say that $f : X \to Y$ is a *morphism of schemes over* S if the following diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, "h"] \\ & S \end{tikzcd}$$

commutes with g, h the structure morphisms of X, Y respectively.

Definition 1.2.34 (Fibre product of schemes)**.** Given morphisms of schemes $f : X \to S$ a,d $g : Y \to S$, the *fibre product of* X *and* Y is a scheme $X \times_S Y$ together with projection morphisms $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ satisfying the following universal property.

$$\begin{tikzcd} \forall T \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_S Y \arrow[r, "q"'] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & S \end{tikzcd}$$

In other words, the structural morphisms of any scheme T over X and Y factor uniquely through $X \times_S Y$.

Definition 1.2.35 (Diagonal morphism)**.** Given a scheme X over S, the *diagonal morphism* $\Delta_{X/S} : X \to X \times_S X$ is the unique morphism of schemes satisfying the following universal property.

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!"] \arrow[drr, "\mathrm{id}_X"] \arrow[ddr, "\mathrm{id}_X"'] & & \\ & X \times_S X \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"'] & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & S \end{tikzcd}$$

Definition 1.2.36 (Sheaf of relative Kähler differentials)**.** Let $f : X \to S$ be a morphism of schemes. The *sheaf of relative Kähler differentials* $\Omega_{X/S}$ *of* X *over* S is the sheaf of differentials of X over S viewing f as a morphism of ringed spaces, equipped with the universal S-derivation $d_{X/S} : \mathcal{O}_S \to \Omega_{X/S}$.

Definition 1.2.37 (Closed immersion)**.** Let $\psi : X \to Y$ be a morphism of schemes. It is called a *closed immersion* if:

(i) The map ψ is a homeomorphism of the underlying topological spaces.

(ii) The map $\psi_* : \mathcal{O}_Y \to \psi_*(\mathcal{O}_X)$ is surjective.

(iii) The sheaf of $\mathcal{O}_Y$-modules $\ker(\psi_*)$ is locally generated by sections.

Definition 1.2.38 (Separated morphism)**.** Let $f : X \to S$ be a morphism of schemes. We say that f is *separated* if the diagonal morphism $\Delta_{X/S}$ is a closed immersion.

Definition 1.2.39 (Separated scheme)**.** A scheme X is *separated* if the morphism $X \to \mathrm{Spec}(\mathbb{Z})$ is.

Definition 1.2.40 (Morphism of finite type)**.** A morphism of schemes $f : X \to Y$ is said to be of *finite type* if Y can be covered by open affine subsets $\{V_\alpha\}$ such that for each α, the preimage $f^{-1}(V_\alpha)$ is a union of affine open subsets $\{U_{\alpha i}\}$, and for each i, the ring $A(U_{\alpha i})$ is an algebra of finite type over $A(V_\alpha)$. We then say that X is a *scheme of finite type over* Y.

Definition 1.2.41 (Algebraic k-schemes)**.** Let k be a field. An *algebraic* k*-scheme* is a scheme X of finite type over k. The field k is called the *base field* of X.

Definition 1.2.42 (Normal scheme)**.** A scheme X is *normal* if every local ring $\mathcal{O}_x$ for $x \in X$ is normal.

1.2.4 Nodal Curves

Definition 1.2.43 (Complex curve)**.** A *complex curve* is a reduced, separated scheme of finite type over $\mathrm{Spec}(\mathbb{C})$ such that all irreducible components have dimension 1.

Definition 1.2.44 (Node)**.** Let k be an algebraically closed field, X be an algebraic 1-dimensional k-scheme, and $x \in X$ be a closed point[4]. We say that x defines a *multicross singularity* if the completion $\widehat{\mathcal{O}}_{X,x}$ is isomorphic to

$$A \coloneqq \{(f_1, \dots, f_n) \in k[[t]] \times \cdots \times k[[t]] \mid f_1(0) = \cdots = f_n(0)\}$$

for some $n \geq 2$. We say that x is a *node*, or an *ordinary double point*, or that x *defines a nodal singularity* if $n = 2$.

Proposition 1.2.45. If C is a complex curve, $q \in C$ is a node if, and only if, there exists a neighbourhood of $q \in C(\mathbb{C})$ which is biholomorphic to a neighbourhood of the origin in the locus $\{(x, y) : x \cdot y = 0\} \subset \mathbb{C}^2$.

Proof. To fill/find reference. □

Definition 1.2.46 (Smooth point)**.** Too hard for now.

Proposition 1.2.47. If C is a complex curve, $q \in C$ is a smooth point if, and only if, there exists a neighbourhood of $q \in C(\mathbb{C})$ which is biholomorphic to $\mathbb{C}$.

Proof. See later. □

Definition 1.2.48 (Nodal curve)**.** A complex curve is *nodal* if all its points are either smooth or nodal.

Definition 1.2.49 (Normalization of a nodal curve)**.** Le C be a nodal curve. A morphism $\nu : \tilde{C} \to C$ is called a *normalization morphism* and $\tilde{C}$ the *normalization of* C if $\tilde{C}$ is normal and if every dominant[5]morphism $f : D \to C$ with D normal factors uniquely through ν:

$$\begin{tikzcd} D \arrow[r, "\forall f"] \arrow[d, dotted, "\exists!"'] & C \\ \tilde{C} \arrow[ur, "\pi"'] & \end{tikzcd}$$

The preimages in $\tilde{C}$ of the nodes of C are called the *node-branches.*

Proposition 1.2.50. Under the same assumptions and notations as in the previous definition, if $\tilde{C} \coloneqq \cup C_i$ is the decomposition of $\tilde{C}$ into (connected) Riemann surfaces, the $\nu(C_i)$ are the irreducible components of C.

Proof. To fill. □

Definition 1.2.51 (Dual graph)**.** The *dual graph* of a nodal curve is given by its irreducible components as vertices (labelled with their genera) and the nodes as vertices.

[4] We say that a point x of a topological space X is *closed* if the singleton $\{x\}$ is closed in X. It can be showed that in the Zariski topology on $\mathrm{Spec}(A)$ for some ring A, the closed points are exactly the maximal ideals of A; see for example Proposition 3.5 in the nLab article on the Zariski topology.

[5] A morphism of schemes $f : X \to Y$ is called *dominant* if $f(X)$ is dense into the underlying topological space Y.

Definition 1.2.52 (Geometric genus)**.** Let C be a smooth complex projective curve. We define its *geometric genus* as

$$p_g(C) \coloneqq h^{1,0}(C) \coloneqq \dim H^0(C, \Omega^1_C),$$

where $\Omega^1_C \coloneqq \Omega_{C,\mathbb{C}} \coloneqq \Omega_{C/\mathrm{Spec}(\mathbb{C})}$ is the *cotangent bundle of* C (the sheaf of relative Kähler differentials of C over $\mathbb{C}$, *ie.* over $\mathrm{Spec}(\mathbb{C})$). For a singular curve, the geometric genus is defined to be the one of its normalization.

Definition 1.2.53 (Arithmetic genus)**.** Let C be a complex projective curve. The *arithmetic genus of* C is defined as

$$p_a(C) \coloneqq 1 - \dim H^0(C, \mathcal{O}_C) + \dim H^1(C, \mathcal{O}_C).$$

Proposition 1.2.54. If C is a smooth curve, the geometric and arithmetic genera are the same. If moreover C is a Riemann surface, they equal its topological genus.

Proof. To fill. □

Proposition 1.2.55. Let C be a curve with δ nodes, $\nu : \tilde{C} \to C$ its normalization and $b_1, ..., b_{2\delta}$ its node-branches. We have the following short exact sequence of sheaves

$$0 \to \nu^* \mathcal{O}_C \to \mathcal{O}_{\tilde{C}} \to \oplus \mathcal{O}_{b_i} \to 0$$

called the *normalization exact sequence.*

Proof. To fill. □

Exercise 1.2.56. Suppose C is a curve with δ nodes such that $\tilde{C}$ has n components of genera $g_1, ..., g_n$. Show that $p_a(C) = \sum (g_i - 1) + \delta + 1$.

Solution. To fill. □

Chapter 2

The moduli space of curves

2.1 Some (2-)category theory

Definition 2.1.1 (Grothendieck topology)**.** A *Grothendieck topology* $\mathcal{T}$ consists of a category T and a set $\mathrm{Cov}(\mathcal{T})$ of families $\{U_i \xrightarrow{\phi_i} U\}_{i \in I}$ of morphisms of T called *coverings* (where in each covering the codomain is fixed) satisfying:

(T1) If ϕ is an ismorphism, then $\{\phi\} \in \mathrm{Cov}(\mathcal{T})$.

(T2) If $\{U_i \to U\} \in \mathrm{Cov}(\mathcal{T})$ and $\{V_{ij} \to U_i\} \in \mathrm{Cov}(\mathcal{T})$ for each $i \in I$, then the family $\{V_{ij} \to U\}$ obtained by composition is in $\mathrm{Cov}(\mathcal{T})$.

(T3) If $\{U_i \to U\} \in \mathrm{Cov}(\mathcal{T})$ and $V \to U$ is arbitrary then $U_i \times_U V$ exists and $\{U_i \times_U V \to V\} \in \mathrm{Cov}(\mathcal{T})$.

Definition 2.1.2 (Site)**.** A *Grothendieck site* (or just *site*) is a category together with a Grothendieck topology on it.

Definition 2.1.3 (Exactness of a diagram)**.** A diagram $A \xrightarrow{f} B \overset{g_1}{\underset{g_2}{\rightrightarrows}} C$ in a category C is *exact* if $g_1 \circ f = g_2 \circ f$ and the following universal property is verified for all h such that $g_1 \circ h = g_2 \circ h$.

$$\begin{tikzcd} & \forall X \arrow[dl, dotted, "\exists!"'] \arrow[d, "\forall h"] & \\ A \arrow[r, "f"'] & B \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & C \end{tikzcd}$$

Definition 2.1.4 (Sheaf on a site)**.** Let $\mathcal{T}$ be a site and $\mathcal{C}$ a category with products. A *presheaf* on $\mathcal{T}$ with values in $\mathcal{C}$ is a functor $\mathscr{F} : \mathcal{T}^{\mathrm{op}} \to \mathcal{C}$. A *sheaf* $\mathscr{F}$ is a presheaf satisfying the following axiom:

(S) If $\{U_i \to U\} \in \mathrm{Cov}(\mathcal{T})$ then the diagram

$$\mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \times_U U_j)$$

is exact.

Definition 2.1.5 (2-catgeory)**.** A *2-category* $\mathcal{C}$ consists of a class of objects $\mathrm{ob}(\mathcal{C})$ and a category $\mathrm{Hom}(X, Y)$ for each pair of objects $X, Y \in \mathcal{C}$ which objects are called *1-morphisms* $X \xrightarrow{f} Y$ and morphisms are called *2-morphisms*

$$\begin{tikzcd} X \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & Y \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$

such that

(i) Given a diagram $X \xrightarrow{f} Y \xrightarrow{g} Z$ there exists $X \xrightarrow{g \circ f} Z$ called the *composition of 1-morphisms*, and this composition is associative.

(ii) For each object X there is a 1-morphism id_X called the *identity 1-morphism* such that $f \circ \mathrm{id}_X = \mathrm{id}_X \circ f = f$.

(iii) Given a diagram

$$\begin{tikzcd} X \arrow[r, bend left=60, "f"{name=U}] \arrow[r, "g"{name=M, description}] \arrow[r, bend right=60, "h"'{name=D}] & Y \arrow[Rightarrow, from=U, to=M, "\alpha"] \arrow[Rightarrow, from=M, to=D, "\beta"] \end{tikzcd}$$

there exists a *vertical composition of 2-morphisms*

$$\begin{tikzcd} X \arrow[r, bend left=60, "f"{name=U}] \arrow[r, bend right=60, "h"'{name=D}] & Y \arrow[Rightarrow, from=U, to=D, "\beta \circ \alpha"] \end{tikzcd}$$

and this composition is associative.

(iv) Given a diagram

$$
\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & Z
\end{tikzcd}
$$

there exists a *horizontal composition of 2-morphisms*

$$
\begin{tikzcd}
X \arrow[r, bend left, "f' \circ f"] \arrow[r, bend right, "g' \circ g"'] \arrow[r, phantom, "\Downarrow\beta \bullet \alpha"] & Z
\end{tikzcd}
$$

and this composition is associative.

(v) For every 1-morphism f there is a 2-morphism id_f called the *identity 2-morphism* such that $\alpha \circ \mathrm{id}_f = \mathrm{id}_f \circ \alpha = \alpha$ for any morphism α, and we have $\mathrm{id}_g \bullet \mathrm{id}_f = \mathrm{id}_{g \circ f}$.

(vi) Given a diagram

$$
\begin{tikzcd}
X \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & Y \arrow[r, bend left=60, "f'"] \arrow[r, "g'" description] \arrow[r, bend right=60, "h'"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha'"] \arrow[r, phantom, shift right=3, "\Downarrow\beta'"] & Z
\end{tikzcd}
$$

then the horizontal and vertical compositions of 2-morphisms are *compatible*, *ie.* $(\beta' \circ \beta) \bullet (\alpha' \circ \alpha) = (\beta' \bullet \alpha') \circ (\beta \bullet \alpha)$.

Definition 2.1.6 (Equivalence of objects)**.** Two objects X and Y of a 2-category are called *equivalent* if there exists two 1-morphisms $f : X \to Y$ and $g : Y \to X$ and two 2-isomorphisms (*ie.* invertible morphisms) $\alpha : g \circ f \Rightarrow \mathrm{id}_X$ and $\beta : f \circ g \Rightarrow \mathrm{id}_Y$.

Definition 2.1.7 (Commutative diagram of 1-morphisms)**.** A *commutative diagram of 1-morphisms* in a 2-category is a diagram

$$
\begin{tikzcd}
& Y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\alpha"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"'] & {} & Z
\end{tikzcd}
$$

such that $\alpha : g \circ f \Rightarrow h$ is a 2-isomorphism.

Definition 2.1.8 (2-functor)**.** A *2-functor* F between two 2-categories $\mathcal{C}$ and $\mathcal{C}'$ is a law that for each object $X \in \mathcal{C}$ gives an object $F(X) \in \mathcal{C}'$, for each 1-morphism $f : X \to Y$ in $\mathcal{C}$, gives a 1-morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{C}'$, and for each 2-morphism $\alpha : f \Rightarrow g$ in $\mathcal{C}$ gives a 2-morphism $\alpha' : F(f) \Rightarrow F(g)$ in $\mathcal{C}'$ and such that:

(2F1) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, $\forall X \in \mathcal{C}$.

(2F2) $F(\mathrm{id}_f) = \mathrm{id}_{F(f)}$ for any 1-morphism f in $\mathcal{C}$.

(2F3) For every diagram $X \xrightarrow{f} Y \xrightarrow{g} Z$ there exists a 2-isomorphism $\epsilon_{fg} : F(g) \circ F(f) \Rightarrow F(g \circ f)$ making the following diagram commute

$$\begin{tikzcd} & F(Y) \arrow[dr, "F(g)"] \arrow[d, phantom, "\Downarrow \epsilon_{gf}"] & \\ F(X) \arrow[ur, "F(f)"] \arrow[rr, "F(g \circ f)"'] & & F(Z) \end{tikzcd}$$

and such that

(i) $\epsilon_{f,\mathrm{id}_X} = \epsilon_{\mathrm{id}_Y,g} = \mathrm{id}_{F(f)}$.

(ii) ϵ is *associative*, *ie.* the following diagram commutes.

$$\begin{tikzcd} F(h) \circ F(g) \circ F(f) \arrow[r, Rightarrow, "\epsilon_{hg} \times \mathrm{id}"] \arrow[d, Rightarrow, "\mathrm{id} \times \epsilon_{gf}"'] & F(h \circ g) \circ F(f) \arrow[d, Rightarrow, "\epsilon_{h\circ g,f}"] \\ F(h) \circ F(g \circ f) \arrow[r, Rightarrow, "\epsilon_{h,g\circ f}"'] & F(h \circ g \circ f) \end{tikzcd}$$

(2F4) For any pair of 2-morphisms $\alpha : f \Rightarrow f'$, $\beta : g \Rightarrow g'$, we have $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$.

(2F5) For any pair of 2-morphisms $\alpha : f \Rightarrow f'$, $\beta : g \Rightarrow g'$, the following diagram commutes.

$$\begin{tikzcd} F(g) \circ F(f) \arrow[r, Rightarrow, "F(\beta) \bullet F(\alpha)"] \arrow[d, Rightarrow, "\epsilon_{gf}"'] & F(g') \circ F(f') \arrow[d, Rightarrow, "\epsilon_{g',f'}"] \\ F(g \circ f) \arrow[r, Rightarrow, "F(\beta \bullet \alpha)"'] & F(g' \circ f') \end{tikzcd}$$

2.2 Stacks

Definition 2.2.1 (Presheaf in the 2-catehory of groupoids)**.** Let **Gpd** be the 2-category of groupoids (categories with all arrows invertible) with 1-morphisms functors between groupoids and 2-morphims natural transformations between these functors. A *presheaf in groupoids* (also called *quasi-functor*) is a 2-functor $\mathscr{F} : \mathbf{Sch_S}^{\mathrm{op}} \to$ **Gpd** where $\mathbf{Sch_S}$ is the 2-category of S-schemes with 1-morphisms S-morphisms and the only 2-morphisms are the identities.

Definition 2.2.2 (Stack)**.** A *stack* is a sheaf of groupoids, *ie.* a 2-functor (presheaf) that satisfies the following sheaf axioms. Let $\{U_i \to U\}_{i \in I}$ be a covering of U in the site $\mathbf{Sch_S}$. Then:

(S1) **(Glueing of morphisms)** If X and Y are two objects of $\mathscr{F}(U)$, and $\varphi_i : X|_{U_i} \to Y|_{U_i}$ are morphisms such that $\varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}}$, then there exists a morphism $\eta : X \to Y$ such that $\eta|_{U_i} = \varphi_i$.

(S2) **(Monopresheaf)** If X and Y are two objects of $\mathscr{F}(U)$, and $\varphi, \psi : X \to Y$ are morphisms such that $\varphi|_{U_i} = \psi|_{U_i}$, then $\varphi = \psi$.

(S3) **(Glueing of objects)** If X_i are objects of $\mathscr{F}(U_i)$ and $\varphi_{ij} : X_j|_{U_{ij}} \to X_i|_{U_{ij}}$ are morphisms satisfying the cocycle condition $\varphi_{ij}|_{U_{ijk}} \circ \varphi_{jk}|_{U_{ijk}} = \varphi_{ik}|_{U_{ijk}}$, then there exists an object X of $\mathscr{F}(U)$ and isomorphisms $\varphi_i : X|_{U_i} \cong X_i$ such that $\varphi_{ji}|_{U_{ij}} \circ \varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}}$.

It is possible to give an equivalent definition of stacks, and it is convenient to have both in minds for different perspectives. We need to first define categories fibred on groupoids:

Definition 2.2.3. A *category over* $\mathbf{Sch_S}$ is a category $\mathscr{F}$ together with a functor

$$p_{\mathscr{F}} : \mathscr{F} \to \mathbf{Sch_S}.$$

If X is an object (resp. ϕ a morphism) of $\mathscr{F}$, and $p_{\mathscr{F}}(X) = B$ (resp. $p_{\mathscr{F}}(\phi) = f$), we say that X lies over B (resp. ϕ lies over f).

Definition 2.2.4 (Category fibred in groupoids)**.** A category $\mathscr{F}$ over $\mathbf{Sch_S}$ is called a *category fibred in groupoids* if:

(i) For every $f : B' \to B$ in $\mathbf{Sch_S}$ and every object X with $p_{\mathscr{F}}(X) = B$, there exists at least one object X' and a morphism $\phi : X' \to X$ such that $p_{\mathscr{F}}(X') = B'$ and $p_{\mathscr{F}}(\phi) = f$:

$$\begin{tikzcd} \exists X' \arrow[r, dotted, "\exists\phi"] \arrow[d] & \forall X \arrow[d] \\ B' \arrow[r, "\forall f"'] & B \end{tikzcd}$$

(ii) For every diagram

$$\begin{tikzcd} X_3 \arrow[r, "\Psi"] \arrow[d] & X_1 \arrow[d] & X_2 \arrow[l, "\phi"'] \arrow[dl] \\ B_3 \arrow[r, "f\circ f'"] \arrow[dr, "f'"'] & B_1 & \\ & B_2 \arrow[u, "f"] & \end{tikzcd}$$

(where $p_{\mathscr{F}}(X_i) = B_i$, $p_{\mathscr{F}}(\phi) = f$, $p_{\mathscr{F}}(\psi) = f \circ f'$), there exists a unique morphism $\varphi : X_3 \to X_2$ with $\psi = \phi \circ \varphi$ and $p_{\mathscr{F}}(\varphi) = f'$. This implies that X' in (i) is unique up to canonical isomorphism, so we will denote $f{*}X \coloneqq X'$.

Definition 2.2.5 (Fibres)**.** Let B be an S-scheme. We define $\mathscr{F}(B)$, the *fibre of $\mathscr{F}$ over B*, to be the subcategory of $\mathscr{F}$ whose objects lie over B and morphisms over id_B.

Proposition 2.2.6. For any $B \in \mathbf{Sch_S}$, $\mathscr{F}(B)$ is fibred in groupoids. Alson $B \to \mathscr{F}(B)$ is a presheaf of groupoids.

Definition 2.2.7 (Stack)**.** A *stack* is a category $\mathscr{F}$ fibred on groupoid that satisfies:

(S1') (**Prestacks**) For all schemes B and pairs of objects X, Y of $\mathscr{F}$ over B, the contravariant functor

$$\mathrm{Iso}_B(X,Y) : \mathbf{Sch_B} \longrightarrow \mathbf{Sets}, \quad (f : B' \to B) \longmapsto \mathrm{Hom}(f^*X, f^*Y)$$

is a sheaf on the site $\mathbf{Sch_B}$.

(S2') (**Effective descent**) Descent data is effective (*ie.*, condition S3 in Definition 2.2.2).

Definition 2.2.8 (Morphisms of stacks)**.** A *morphism of stacks* $f : \mathscr{F} \to \mathscr{G}$ is a functor between the underlying categories such that $p_{\mathscr{G}} \circ f = p_{\mathscr{F}}$. A commutative diagram of stacks is a diagram

$$\begin{tikzcd} & \mathscr{G} \arrow[dr, "g"] \arrow[d, Rightarrow, "\alpha"] & \\ \mathscr{F} \arrow[ur, "f"] \arrow[rr, "h"'] & {} & \mathscr{H} \end{tikzcd}$$

such that $\alpha : g \circ f \Rightarrow h$ is an isomorphism of functors. If f is an equivalence of categories, then we say that the stacks $\mathscr{F}$ and $\mathscr{G}$ are *isomorphic*. We denote by $\mathrm{Hom}_S(\mathscr{F}, \mathscr{G})$ the category whose objects are morphisms of stacks and whose morphisms are natural transformations.

Definition 2.2.9 (Stack associated to a scheme)**.** Given an S-scheme U, we define a stack U from $\mathbf{Sch_U}$ by defining the functor $p_U : \mathbf{Sch_U} \to \mathbf{Sch_S}$ which sends the U-scheme $f : B \to U$ to the S-scheme $B \xrightarrow{f} U \to S$.

Lemma 2.2.10. If a stack has an object with an automorphism other than the identity, it cannot be represented by a scheme.

Proof. By definition, the only automorphisms of the stack associated to a scheme are identitites. □

Definition 2.2.11 (fibre product of stacks)**.** Given two morphisms $f_1 : \mathscr{F}_1 \to \mathscr{G}$ and $f_2 : \mathscr{F}_2 \to \mathscr{G}$, we define their *fibre product* $\mathscr{F}_1 \times_{\mathscr{G}} \mathscr{F}_2$ (with projections to $\mathscr{F}_1$ and $\mathscr{F}_2$) as the stack whose objects are triples (X_1, X_2, α) where X_i are objects of $\mathscr{F}_i$ that lie over the same scheme U, and $\alpha : f_1(X_1) \to f_2(X_2)$ is an isomorphism in $\mathscr{G}$. A morphism $(X_1, X_2, \alpha) \to (Y_1, Y_2, \beta)$ is a pair (ϕ_1, ϕ_2) of morphisms $\phi_i : X_i \to Y_i$ that lie over the same morphism $f : U \to V$ and such that $\beta \circ f_1(\phi_1) = f_2(\phi_2) \circ \alpha$. The fibre product satisfies the usual universal property.

2.3 More algebra and schemes

Definition 2.3.1 (Flatness for rings)**.** A module N over a ring A is said to be *flat* if the functor $M \mapsto M \otimes_A N$ is exact. A morphism $f : A \to B$ of rings is said to be *flat* if the functor $M \mapsto M \otimes_A B$ is exact.

Definition 2.3.2 (Flatness for schemes)**.** A morphism $f : X \to Y$ of schemes is said to be *flat at $x \in X$* if the associated morphism of local rings $\mathcal{O}_{f(x)} \to \mathcal{O}_x$ is flat. The morphism f is said to be *flat* if it is flat at all points $x \in X$.

Definition 2.3.3 (Algebraic independence over a field)**.** Let K/k be a field extension. A collection $\{x_i\}_{i \in I}$ of K is called *algebraically independent* over k if the map $k[X_i]_{i \in I} \to K$ which maps $X_i \mapsto x_i$ for all $i \in I$ is injective.

Definition 2.3.4 (Transcendence basis)**.** Let K/k be a field extension. A *transcendence basis* of K/k is a collection of elements $\{x_i\}_{i \in I}$ which are algebraically independent over k and such that the extension $K/k(x_i)_{i \in I}$ is algebraic where $k(x_i)_{i \in I}$ is the field of fractions of the polynomial ring $k[x_i]_{i \in I}$.

Definition 2.3.5 (Separable extension)**.** Let K/k be a field extension. We say an irreducible polynomial P over k is *separable* if it is relatively prime to its derivatives. Given $\alpha \in K$ algebraic over k, we say that α is *separable* over k if its minimal polynomial is separable over k. We say that K is separable over k if all its elements are separable over k.

Definition 2.3.6 (Separably generated extension)**.** Let K/k be a field extension. We say that K is *separably generated* over k if there exists a transcendence basis $\{x_i\}_{i \in I}$ of K/k such that the extension $K/k(x_i)_{i \in I}$ is a separable algebraic extension.

If A is a local ring, let's denote its maximal ideal by $r(A)$ and its residue class field $A/r(A)$ by $k(A)$.

Definition 2.3.7 (Unramified morphism of local rings)**.** A morphism $f : A \to B$ of local rings is said to be *unramified* if $f(r(A))B = r(B)$ and $k(B)$ is a finite, separable extension of $k(A)$.

Definition 2.3.8 (Unramified morphism of schemes)**.** A morphism $f : X \to Y$ of schemes is said to be *unramified at* $x \in X$ if it is of finite type at x and the associated morphism of local rings at x $\mathcal{O}_{f(x)} \to \mathcal{O}_x$ is unramified. The morphism f is said to be *unramified* if it is unramified at all $x \in X$.

Definition 2.3.9 (Etale morphism)**.** A morphism $f : X \to Y$ of schemes is said to be *étale at* $x \in X$ if it is flat and unramified at x. The morphism f is said to be *étale* if it is étale at all its points.

Definition 2.3.10 (Quasi-compactedness)**.** We say that a topological space X is *quasi-compact* if every open covering of X has a finite subcover. We say that a continuous map $f : X \to Y$ is *quasi-compact* if the inverse image $f^{-1}(V)$ of any quasi-compact open $V \subset Y$ is quasi-compact. We say that a morphism of schemes is *quasi-compact* if the underlying map of topological spaces is.

2.4 Algebraic spaces and representability

Definition 2.4.1 (S-space)**.** An *S-space* is a sheaf (as in Definition 2.1.4) from the site $\mathbf{Sch_S}$ to $\mathbf{Sets}$, and *morphisms of S-spaces* or morphisms of sheaves (*ie.* natural transformations of functors respecting the sheaf axiom).

Definition 2.4.2 (Quptient S-space)**.** An *equivalence relation* in the category of S-spaces consists of two S-spaces R and U and a morphism of S-spaces $\delta : R \to U \times_S U$ such that for any S-scheme B the map $\delta(B) : R(B) \to U(B) \times_S U(B)$ is the graph of an equivalence relation between sets. A *quotient S-space* for such an equivalence relation is the sheaf cokernel of the following diagram.

$$R \underset{p_2 \circ \delta}{\overset{p_1 \circ \delta}{\rightrightarrows}} U$$

Definition 2.4.3 (Algebraic space)**.** With the same notations, an S-space is called an *algebraic space* if it is the quotient S-space form an equivalence relation such that R and U are S-schemes, $p_1 \circ \delta$ and $p_2 \circ \delta$ are étale, and δ is a quasi-compact morphism.

Below is an equivalent, more *geometric* defintion of algebraic spaces.

Definition 2.4.4 (Algebraic space)**.** An S-space $\mathscr{F}$ is called an *algebraic space* if there exists a scheme U (*atlas*) and a morphism of S-spaces $u : U \to \mathscr{F}$ such that:

1. For any S-scheme V and morphism $V \to \mathscr{F}$, the (sheaf) fibre product $U \times_{\mathscr{F}} V$ is representable by a scheme, and the map $U \times_{\mathscr{F}} V \to V$ is an étale morphism of schemes.

2. The morphism $U \times_{\mathscr{F}} U \to U \times_S U$ is quasi-compact.

Remark 2.4.5. We recover the first definition by taking $R \coloneqq U \times_{\mathscr{F}} U$.

Remark 2.4.6. We can associate a stack to any algebraic space by viewing $\mathbf{Sch_S}$ as a 2-category and adding identity arrows to every element of a set in $\mathbf{Sets}$ making them groupoids.

Definition 2.4.7 (Representability)**.** A stack $\mathcal{X}$ is said to be *representable by* an algebraic space (resp. scheme) if there exists an algebraic space (resp. scheme) X such that the stack associated to X is isomorphic to $\mathcal{X}$. If P is a property of algebraic spaces (resp. schemes) and $\mathcal{X}$ is a representable stack, we say that $\mathcal{X}$ has property P if X has P. A morphism of stacks $f : \mathscr{F} \to \mathscr{G}$ is *representable* if for all objects U in $\mathbf{Sch_S}$ and morphisms $U \to \mathscr{G}$, the fibre product stack $U \times_{\mathscr{G}} \mathscr{F}$ is representable by an algebraic space.

Definition 2.4.8 (Diagonal)**.** Let $\Delta_{\mathscr{F}} : \mathscr{F} \to \mathscr{F} \times_S \mathscr{F}$ be the diagonal morphism. A morphism from a scheme U to $\mathscr{F} \times_S \mathscr{F}$ is equivalent to two objects X_1, X_2 of $\mathscr{F}(U)$. Taking the fibre product of these we have

$$\begin{tikzcd} \mathrm{Iso}_U(X_1, X_2) \arrow[r] \arrow[d] & \mathscr{F} \arrow[d, "\Delta_{\mathscr{F}}"] \\ U \arrow[r] & \mathscr{F} \times_S \mathscr{F} \end{tikzcd}$$

2.5 Deligne-Mumford stacks

Definition 2.5.1 (Etale topology)**.** We name *étale topology* on $\mathbf{Sch_S}$ the Grothendieck topology such that for every covering $\{U_i \xrightarrow{\phi_i} U\}$, ϕ_i is étale and the family is *jointly surjective, ie.* $\bigcup_{i \in I} \phi_i(U_i) = U$.

Definition 2.5.2 (Deligne-Mumford stack)**.** Let $\mathbf{Sch_S}$ be the category of S-schemes with the étale topology, and let $\mathscr{F}$ be a stack. Assume:

(DM1) The diagonal $\Delta_{\mathscr{F}}$ is representable, quasi-compact, and separated.

(DM2) There exists a scheme U (called an *atlas*) and an étale surjective morphism $u : U \to \mathscr{F}$.

Then $\mathscr{F}$ is called a *Deligne–Mumford stack*.

2.6 The mooduli stack $\mathcal{M}_g$ of non-singular Riemann surfaces

Theorem 2.6.1. The space of isomorphism classes of genus $g > 1$ non-singular Riemann surfaces is a smooth Deligne-Mumford stack.

Proof. See DM. □

Definition 2.6.2 (Moduli-space of non-singular Riemann surfaces)**.** The space of isomorphism classes of genus $g > 1$ non-singular Riemann surfaces with its structure of a Deligne-Mumford stack, denoted $\mathcal{M}_g$ is called the *moduli-space of genus g non-singular Riemann surfaces.*

Definition 2.6.3 (Isotopy group)**.** The isotropy group $\mathrm{Iso}([C])$ of the equivalence class $[C] \in \mathcal{M}_g$ of a curve C is the automorphism group $\mathrm{Aut}(C) := \{\phi : C \to C \text{ biholomorphic}\}$ of C.

Definition 2.6.4 (Orbifold)**.** To fill.

Proposition 2.6.5. For $g > 2$, $\mathcal{M}_g$ is an orbifold.

Proof. To fill. □

Proposition 2.6.6. $\mathcal{M}_g$ is connected and of dimension $3g - 3$.

Proof. To fill. □

Proposition 2.6.7. The space $\mathcal{M}_g$ is not compact.

Proof. To fill. □

2.7 The deligne-Mumford compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$

Definition 2.7.1 (Stable curve)**.** A *stable curve* is a connected nodal curve such that

(i) every irredicuble component of geometric genus 0 has at least three node-branches.

(ii) every irreducible component of geometric genus 1 has at least one node-branch.

Theorem 2.7.2. Stability is equivalent to the automorphism group being finite.

Proof. To fill. □

Proposition 2.7.3. A stable curve must have genus at lest 2.

Proof. To fill. □

Theorem 2.7.4. The moduli space of stable curves $\overline{\mathcal{M}}_g$ is a connected, irreducible, compact, non-singular Deligne-Mumfor stack of dimension $3g - 3$.

Proof. To fill. □

2.8 The moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable pointed curves

Definition 2.8.1 (Stable pointed curves)**.** An *n-pointed curve* is a nodal curve with n distinct labelled non-singular points called *marked points*. A *special point* of a component of a pointed curve is a point on the normalization of the component that is either a node-branch or (the preimage of) a marked point. A pointed curve is *stable* if every genus 0 irreducible component has at least three special points, and every genus 1 irreducible component has at least one special point.

Theorem 2.8.2. Stability is equivalent to the automorphism group being finite.

Proof. To fill. □

In the dual graph of pointed curves, the marked points are represented by labelled half-edges from the vertex representing their component.

Proposition 2.8.3. There are no stable n-pointed genus g curves if $2g-2+n \leq 0$.

Proof. To fill. □

Theorem 2.8.4. The moduli space $\overline{\mathcal{M}}_{g,n}$ of stable n-pointed genus g curves is an irreducible, compact, non-singular Deligne-Mumford stack of dimension $3g-3+n$.

Proof. To fill. □

Definition 2.8.5 (Forgetfull morphism)**.** If $n_1 \geq n_2$, and $2g-2+n_2 > 0$, we define the *forgetful morphism*

$$\nu_{n_1,n_2} : \overline{\mathcal{M}}_{g,n_1} \to \overline{\mathcal{M}}_{g,n_2}$$

by mapping a point $[(C, p_1, ..., p_{n_1})] \in \overline{\mathcal{M}}_{g,n_1}$ by removing the genus 0 vertices of the dual graph of $(C, p_1, ..., p_{n_1})$.

Definition 2.8.6 (Universal curve)**.** The forgetful morphism $\nu : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the *universal curve* over $\overline{\mathcal{M}}_{g,n}$.

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