

Collapsed Segmented LDA (CSLDA):

$$p(S, Z, W | \alpha, \beta, \eta, \sigma) = \left[\prod_k \frac{\Gamma(W\beta)}{\Gamma(\beta)^W \cdot \Gamma(N_k + W\beta)} \prod_w \Gamma(N_{kw} + \beta) \right] \\ \times \left[\prod_d \mathcal{N}\left(s_d \mid \eta^T \cdot \frac{N_{dk}}{N_d}, \sigma\right) \frac{\Gamma(K\alpha)}{\Gamma(\alpha)^K \cdot \Gamma(N_d + K\alpha)} \prod_k \Gamma(N_{dk} + \alpha) \right]$$

Where we used $\frac{N_{dk}}{N_d} \equiv \bar{Z}_d$

$$\log p(S, Z, W | \alpha, \beta, \eta, \sigma) = \\ \left[\sum_k \log \Gamma(W\beta) - W \log \Gamma(\beta) - \log \Gamma(N_k + W\beta) + \sum_w \log \Gamma(N_{kw} + \beta) \right] \\ + \left[\sum_d \underbrace{-\log \sigma - \frac{1}{2} \log(2\pi) - \frac{\left(s_d - \eta^T \cdot \frac{N_{dk}}{N_d}\right)^2}{2\sigma^2}}_{\text{Normal distribution}} + \right. \\ \left. \log \Gamma(K\alpha) - K \log \Gamma(\alpha) - \log \Gamma(N_d + K\alpha) + \sum_k \log \Gamma(N_{dk} + \alpha) \right]$$

Maximum a posteriori (MAP) estimate for the η hyperparameter:

$$\nabla_{\eta_k} \log p(S, Z, W | \alpha, \beta, \eta, \sigma) = \sum_d \frac{\frac{N_{dk}}{N_d} \left(s_d - \eta^T \frac{N_{d\cdot}}{N_d}\right)}{\sigma^2} = \\ = \sum_d \frac{s_d \frac{N_{dk}}{N_d}}{\sigma^2} - \sum_d \frac{\frac{N_{dk}}{N_d} \left(\eta^T \frac{N_{d\cdot}}{N_d}\right)}{\sigma^2} = 0 \\ \Rightarrow \sum_d s_d \frac{N_{dk}}{N_d} = \sum_d \frac{N_{dk}}{N_d} \left(\sum_{k'} \eta_{k'} \frac{N_{dk'}}{N_d} \right) = \sum_d \frac{N_{dk}}{N_d} \left(\eta_k \frac{N_{dk}}{N_d} + \sum_{k' \neq k} \eta_{k'} \frac{N_{dk'}}{N_d} \right) \\ \Rightarrow \sum_d s_d \frac{N_{dk}}{N_d} = \eta_k \sum_d \left(\frac{N_{dk}}{N_d} \right)^2 + \sum_d \left(\frac{N_{dk}}{N_d} \sum_{k' \neq k} \eta_{k'} \frac{N_{dk'}}{N_d} \right) \\ \Rightarrow \sum_d \left(s_d \frac{N_{dk}}{N_d} - \frac{N_{dk}}{N_d} \sum_{k' \neq k} \eta_{k'} \frac{N_{dk'}}{N_d} \right) = \eta_k \sum_d \left(\frac{N_{dk}}{N_d} \right)^2 \\ \Rightarrow \sum_d \frac{N_{dk}}{N_d} \left(s_d - \sum_{k' \neq k} \eta_{k'} \frac{N_{dk'}}{N_d} \right) = \eta_k \sum_d \left(\frac{N_{dk}}{N_d} \right)^2 \\ \Rightarrow \eta_k = \frac{\sum_d \frac{N_{dk}}{N_d} \left(s_d - \sum_{k' \neq k} \eta_{k'} \frac{N_{dk'}}{N_d} \right)}{\sum_d \left(\frac{N_{dk}}{N_d} \right)^2}$$

Trying to apply the previous formula as an update rule for η does not converge. Instead, the following update can be used:

$$\eta_k^{new} \leftarrow (1 - \gamma)\eta_k^{old} + \gamma \frac{\sum_d \frac{N_{dk}}{N_d} \left(s_d - \sum_{k' \neq k} \eta_{k'} \frac{N_{dk'}}{N_d} \right)}{\sum_d \left(\frac{N_{dk}}{N_d} \right)^2 + \epsilon}$$

With $1 \gg \gamma > 0$ in order for the previous series to converge and $1 \gg \epsilon > 0$ is a smoothing constant.

Gibbs sampler:

$$\begin{aligned} p(z_{di} = k | Z^{\setminus i}, S, W, \alpha, \beta, \eta, \sigma) &\propto p(z_{di} = k, Z_{-i}, S, W, \alpha, \beta, \eta, \sigma) \\ &\propto \left[\prod_{k'} \frac{\prod_w \Gamma(N_{k'w}^{\setminus i} + 1(k' = k \wedge w = w_{di}) + \beta)}{\Gamma(N_{k'}^{\setminus i} + 1(k' = k) + W\beta)} \right] \times \\ &\mathcal{N} \left(s_d \left| \eta^T \cdot \frac{N_{dk'}^{\setminus i} + 1(k' = k)}{N_d}, \sigma \right. \right) \prod_{k'} \Gamma(N_{dk'}^{\setminus i} + 1(k' = k) + \alpha) \end{aligned}$$