MATH 1830 NOTES

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UNIT 2 DERIVATIVES

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2.P Review of Exponential and Logarithmic Functions

Notes

Three Rules for Logarithms

- Product Rule: $\ln(x * y) = \ln(x) + \ln(y)$
  - Example: $\ln(3 * 5) = \ln(3) + \ln(5)$
- Quotient Rule: $\ln(x/y) = \ln(x) - \ln(y)$
  - Example: $\ln(3/7) = \ln(3) - \ln(7)$
- Power Rule: $\ln x^y = y \ln x$
  - Example: $\ln 2^8 = 8 \ln 2$

Three Specific Logarithms to Remember

- When x<0
  - $ln(x)$ is undefined
  - $\log_b(x)$ is undefined
  - WHY?
- When x=0
  - $ln(x)$ is undefined
  - $\log_b(x)$ is undefined
  - WHY?
- When x=1
  - $ln(x) = 0$
  - $\log_b(x) = 0$
  - WHY?

Logarithm and Exponential Forms

1. Rewrite $5^3 = 125$ in logarithm notation

2. Rewrite $\log_2 32 = 5$ in exponential notation

3. Rewrite $\log 10000 = 4$ in exponential notation

4. Rewrite $\ln 148.413159 \approx 5$ in exponential notation

5. Rewrite $e^2 \approx 7.389$ in logarithm notation

<u>Review:</u>

Solve the variable to 2 decimal places

6. $A = 4000e^{0.06(8)}$

7. $34000 = Pe^{0.076(5)}$

8. $9500 = 1200e^{0.041t}$

9. $4930 = 2250e^{2.65r}$

10. $3 = e^{0.07t}$

Review:

Solve for the variable without using a calculator.

11. $y = \ln e^5$

12. $\log_5 x = -3$

13. $\log_b 5 = \frac{1}{3}$

14. $s = \ln(e)$

15. $y = \ln(\ln e)$

Use the Properties of Logarithms to Completely Expand the Term

16. $f(x) = \ln\left(xy^2\right)$

17. $g(x) = \log_5\left(\frac{25x^3}{4y^7}\right)$

18. $h(x) = 5 + 7\ln\left(\frac{2}{x}\right)$

19. $f(x) = x - \ln(ex)$

Solve for x. Check for Extraneous Solutions.

20. $\log_2(-8+4x)=4$

21. $\log\left(x^2+75\right)=2$

22. $\ln\left(x^2-35\right)=\ln(2x)$

2.1 Exponential and Logarithmic Function Applications

Introduction

Exponential functions occur frequently in science and business but are commonly used in compound interest applications.

- The value of a $1000 investment returning 8% interest compounded monthly after 12 years would be calculated using the formula

  $$A = P\left(1 + \frac{r}{n}\right)^{nt},$$

  where A is the final amount in the account

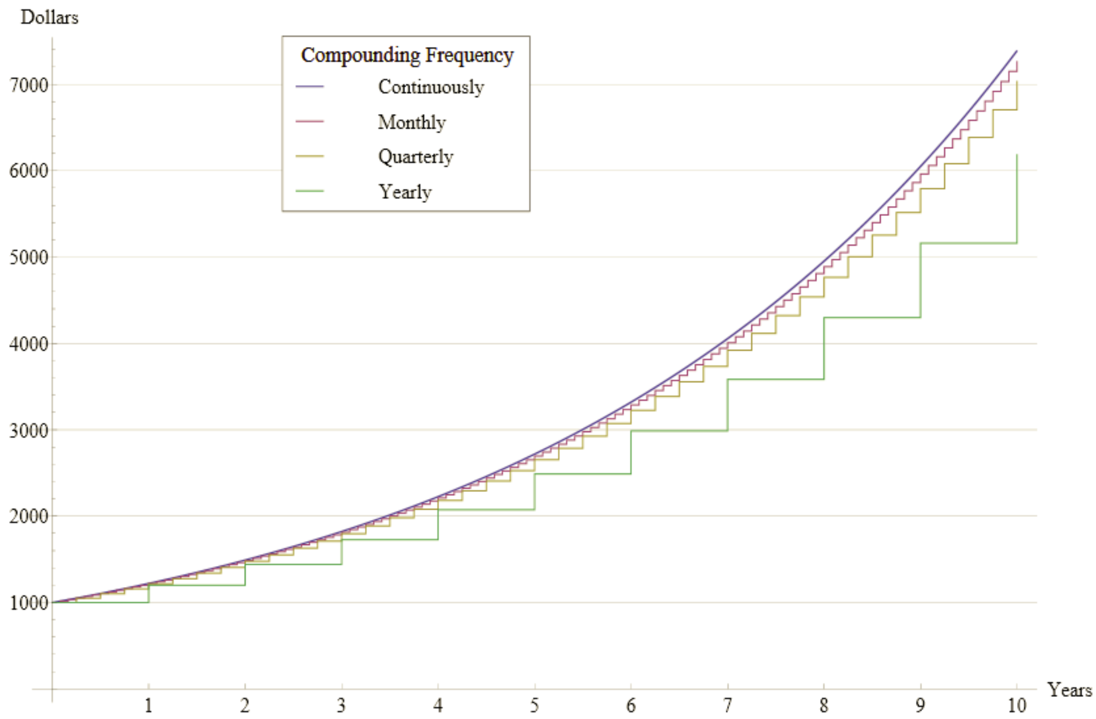
  P is the principal

  r is the interest rate

  n is the number of compounding periods per year

  and t is the number of years.

- The compounding frequency has a significant impact on the final amount of money (either saved or owed).



Compound Interest At Varying Frequencies

Starting with a principal of $1000, interest rises exponentially. Notice also that as time passes, a gap forms between the lines as less frequently-compounding methods increase at a lesser rate than more frequently-compounding methods.

Our focus will be on continuous compounding:

- What is e?
- Irrational number (similar to π)
- 2.718281828459…..
- Like π, e occurs frequently in natural phenomena
  - Growth of bacterial cultures
  - Decay of a radioactive substance
- Formal definition of e:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$\approx \; 2.718281829$$

Notes

Continuous Compounding Formula (appreciation and depreciation):

$$A = Pe^{rt}$$

CONTINUOUS COMPOUND INTEREST: Round all answers to two decimal places.

1. Hometown Bank offers a CD that earns 1.58% compounded continuously. If $10,000 is invested in this CD, how much will it be worth in 3 years?

2. Hometown Bank offers a CD that earns 1.58% compounded continuously. If $10,000 is invested in this CD, how long will it take the account to be worth $11,000?

3. Doubling Time: How long will it take money to double, if it is invested at 5% compounded continuously?

4. Doubling Rate: At what nominal rate compounded continuously must money be invested to double in 8 years?

5. How long will it take money to triple, if it is invested at 10.5% compounded continuously?

6. Radioactive Decay: A mathematical model for the decay of radioactive substances is given by

$$Q = Q_0 \ e^{rt}.$$

The continuous compound rate of decay of carbon-14 per year is r = -0.0001238. How long will it take a certain amount of carbon-14 to decay to half the original amount?

7. The estimated resale value R (in dollars) of a company car after t years is given by:

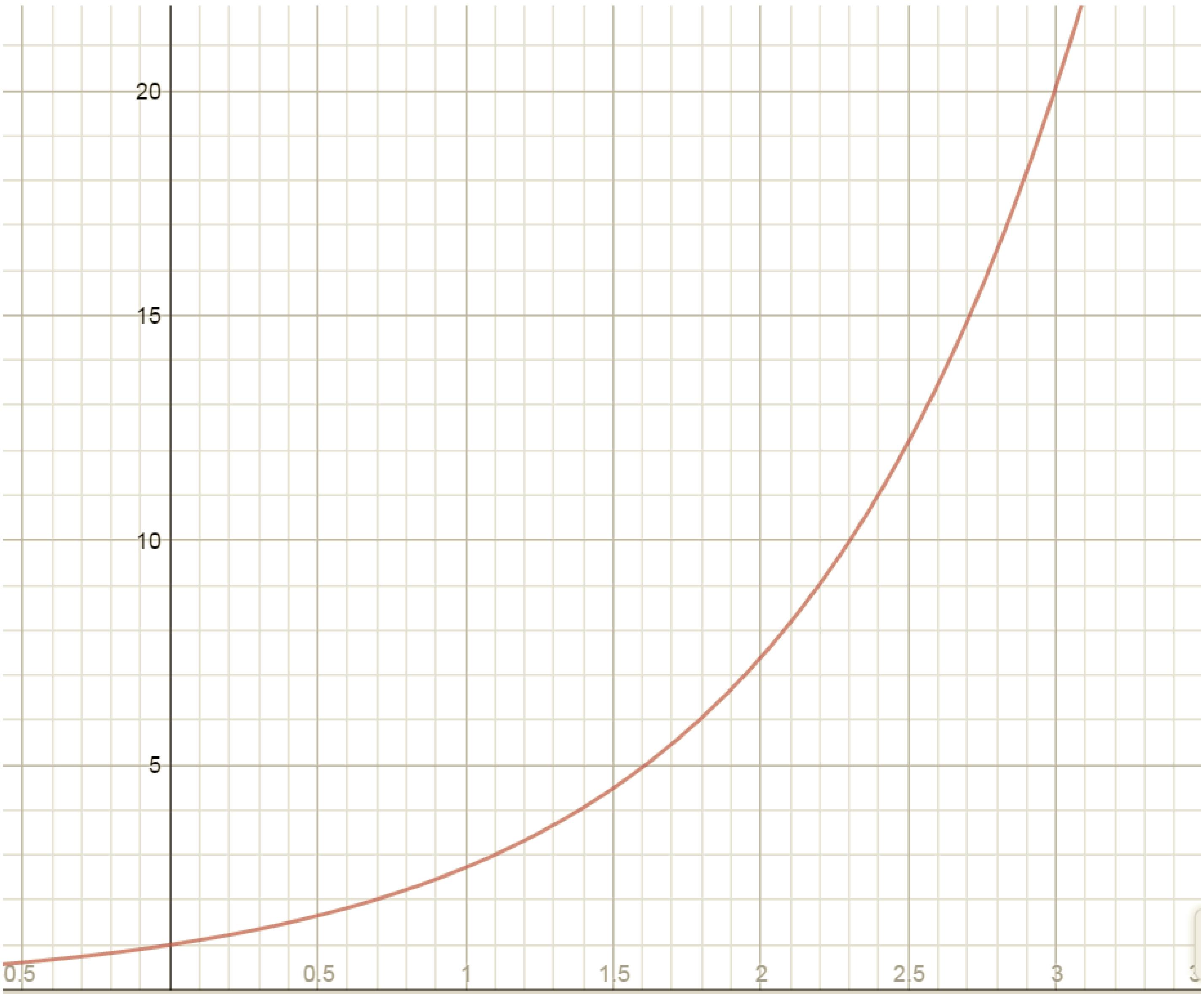
$$R(t) = 20000(0.86)^t.$$

What will be the resale value of the car after 2 years? How long will it take the car to depreciate to half the original value?

2.2 Derivatives of Exponential and Logarithmic Functions

Introduction

Finding the derivative of $f(x) = e^x$



x	f(x)=e^x
0	1
1	2.7183
2	7.3891
3	20.086
4	54.598
5	148.41
6	403.43

1. Calculate the slope of the secant line for each of the following intervals for the function $f(x) = e^x$.

   a. [1, 3]

   b. [1, 2]

   c. [1, 1.5]

2. What does the slope of the secant line represent?

3. Draw a tangent line at the point on the graph corresponding to x = 1 and calculate the slope.

4. What does the slope of the tangent line represent?

5. Compare the values of $f(1)$ and $f'(1)$. What do you notice?

Finding the derivative of $f(x) = lnx$

6. Try to find the derivative of $f(x) = lnx$ using the limit definition of the derivative, $\lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$.

7. Complete the table below to try to find the derivative of $f(x) = lnx$.

   (Use your calculator and let h=0.00001 to represent h→0)

x	$\dfrac{\ln(x+h) - lnx}{h}$	$\lim\limits_{h \to 0} \dfrac{\ln(x+h) - lnx}{h}$
1	$\dfrac{\ln(1+0.00001) - ln1}{0.00001}$	1
2		
3		
4		
5		

8. Based on your results what do you think the rule for the derivative of $f(x) = lnx$ is?

Notes

Derivatives of Exponentials and Logarithms

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}b^x = b^x \ln b \qquad (b > 0,\ b \neq 1)$$

$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad (x > 0)$$

$$\frac{d}{dx}\log_b x = \left(\frac{1}{\ln b}\right)\left(\frac{1}{x}\right) \qquad (x > 0,\ b > 0,\ b \neq 1)$$

1. Find $f'(x)$ when $f(x) = 3x^3 + 4x^2 - 5x + 8$

2. Find $f'(x)$ when $f(x) = 4\ln x - x^3 + 2x$

3. Find $f'(x)$ when $f(x) = \ln x + 5e^x - 7x^2$

4. Find $f'(x)$ when $f(x) = \ln x^8 - 3\ln x$

Properties of Logarithms:

Use appropriate properties of logarithms to expand $f(x)$ and then find $f'(x)$.

5. $f(x) = 9 + 5\ \ln \frac{1}{x}$

6. $f(x) = x - 2\ \ln 5x$

Tangent Lines:

Find the equation of the line tangent to the graph of f at the indicated value of x.

7. $f(x) = e^{x} + 2$ at $x = 0$

8. $f(x) = 1 \ + \ln x^6$ at $x = e$

Applications:

9. The estimated resale value R (in dollars) of a company car after t years is given by

$$R(t) = 24000(0.84)^t$$

What is the instantaneous rate of depreciation (in dollars per year) after: 1 year? 2 years? 3 years?

2.3 Derivatives of Products and Quotients

Introduction

1. The manager of a miniature golf course is planning to raise the ticket price per game. At the current price of $6.50, an average of 81 rounds is played each day. The manager's research suggests that for every $0.50 increase in price, an average of four fewer games will be played each day. Based on this information, find the function that represents revenue from rounds of mini golf, where n represents the number of $0.50 increases in price.

    a. What must you do with this revenue function in order to find the rate of change?

    b. Find the rate of change for this revenue function when the manager increases the price of a round of mini golf by $1.50.

2. Find the rate of change for the function $y = (x^2 + 1)(x^2 - 2x + 1)$

3. The cost of manufacturing x MP3 players per day is represented by the function

$$C(x) = 0.01x^2 + 42x + 300 \quad 0 \leq x \leq 300.$$

   a. Determine the average cost function.

   b. Determine the marginal average cost function. What did you have to do to the average cost function in order to find the marginal average cost function?

4. Suppose the function $V(t) = \frac{50{,}000+6t}{1+0.4t}$ represents the value, in dollars, of a new car t years after it is purchased. Determine the rate of change in the value of the car.

Notes

Derivatives of Products and Quotients

Rewriting a Function as a Product or Quotient

1. Rewrite as a product: $f(x) = 5e^x \ + \ 10x^2 \ e^x \ + \ 25x^4 \ e^x$

2. Rewrite as one quotient: $f\left(x\right) = 3x^{-4}\ln x$

The Product Rule

If $y = f(x) \cdot g(x)$,

then $y' = f'(x) \cdot g(x) \; + \; f(x) \cdot g'(x)$

The Quotient Rule

If $y = \frac{f(x)}{g(x)}$,

then $y' = \frac{f'(x)\; g(x) \; - \; f(x)\; g'(x)}{[g(x)]^2}$

Two Methods for Finding the Derivative:

Find the derivative two different ways.

a. Simplify first and use the power rule.
b. Use the product or quotient rule.

3. $m(x) = 2x^3 \left(x^5 - 2\right)$

a.

b.

4. $r(x) = \frac{x^5+4}{x^2}$

a.

b.

Find $f'(x)$ using the Product Rule.

5. $n(x) = 7x^2\left(2x^3+5\right)$

6. $h\left(x\right) = 4x^3\ e^x$

7. $s\left(x\right)=2x^5\ln x$

8. $v(x) = (8x+1)\left(3x^2\ -7\right)$

<u>Find $f'(x)$ using the Quotient Rule.</u>

9. $b(x) = \frac{4x}{3x+8}$

10. $c(x) = \frac{x^2 - 9}{x^2 + 1}$

11. $h(x) = \frac{1+e^x}{1-e^x}$

12. $j(x) = \frac{3x}{4+\ln x}$

13. Find $\frac{dy}{dw}$ for $y = \frac{2w^4 - w^3}{6w-1}$

14. Explain how $f'(x)$ can be found without using the quotient rule: $f(x) = \frac{3}{x^3}$

Tangent Lines

15. $r\left(x\right)=\left(5-4x\right)\left(1+3x\right)$

a. Find $r'\left(x\right)$

b. Find the equation of the line tangent to the graph of r at $x = 2$.

c. Find the values of x where $r'\left(x\right)=0$

16. $h\left(x\right) = \frac{3x-7}{2x-1}$

a. Find $h'\left(x\right)$

b. Find the equation of the line tangent to the graph of h at $x = 2$.

c. Find the values of x where h′(x) = 0

Derivatives with Radicals

17. Find y' for $y = \frac{6\sqrt[3]{x}}{2x^2 - 5x + 1}$

18. Find $\frac{dy}{dx}$ for $y = \frac{2x^2 - 2x + 3}{\sqrt[4]{x}}$

Applications

19. A cable company has installed a new television system in a city. The total number N (in thousands) of subscribers t months after the installation of the system is given by $N\left(t\right) = \ \frac{178t}{t+5}$

    a. Find $N'(t)$

    b. Find $N\left(12\right)$ and $N'\left(12\right)$. Write a brief interpretation of these results.

    c. Use the results above to estimate the total number of subscribers after 13 months.

20. According to economic theory, the supply x of a quantity in a free market increases as the price p increases. Suppose the number x of baseball gloves a retail chain is willing to sell per week at a price of $p is given by

$$x = \frac{100p}{0.1p+1} \qquad 30.00 \le p \le 190.00$$

a. Find $\frac{dx}{dp}$

b. Find the supply and the instantaneous rate of change (IRC) of supply with respect to price when the price is $40. Write a brief verbal interpretation of these results.

c. Use the results above to estimate the supply if the price is increased to $41.

2.4 The Chain Rule

Introduction

1. The gas tank of a parked pickup truck develops a leak. The amount of gas, in liters, remaining in the tank after t hours is represented by the function $V(t) = 90\left(1 - \frac{t}{18}\right)^2 \quad 0 \le t \le 18$. How fast is the gas leaking from the tank after 12 hours?

2. Andrew and David are training to run a marathon. They both go on a run on Sunday mornings at precisely 7 A.M. Andrew's house is 22 km south of David's. One Sunday morning, Andrew leaves his house and runs west at 7 km/hr. The distance between the two runners can be modeled by the function

$$s(t) = \sqrt{130t^2 - 396t + 484},$$

where s is in kilometers and t is in hours. Determine the rate at which the distance between the two runners is changing.

Notes

General Derivative Rules Using the Chain Rule

$$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} \cdot f'(x)$$

$$\frac{d}{dx}\ln[f(x)] = \frac{1}{f(x)} \cdot f'(x)$$

$$\frac{d}{dx}e^{f(x)} = e^{f(x)} \cdot f'\left(x\right)$$

Fill in the blank with an expression that will make the indicated equation valid. Then simplify.

1. $\frac{d}{dx}\ (3-7x)^6 = 6(3-7x)^5$ _____
2. $\frac{d}{dx}\ e^{5x-3} = \ e^{5x-3}$______

3. $\frac{d}{dx}\ ln\left(x^2-x^4\right)=\ \frac{1}{x^2-x^4}$ ______

Find $f'(x)$ and simplify.

4. $f(x) = \left(8x^2 - 7\right)^5$

5. $f(x) = e^{3x^2 + 2x + 5}$

6. $f(x) = 2\ln\left(9x^2 - 5x + 21\right)$

7. $f(x) = (4x - 5\ lnx)^7$

Horizontal Tangents

Finding the Equation of the Tangent Line

a. Find the y value by calculating $f(a)$: $(a, f(a))$
b. Find the slope of the tangent line by calculating $f'(a)$: $m_{tan} = f'(a)$
c. Equation of the tangent line: $y - f(a) = f'(a)\,(x - a)$

Finding the Value(s) where the Tangent Line is Horizontal

a. Set $f'(x) = 0$
b. Solve for x
c. Verify that each x is in the domain of $f(x)$ and $f'(x)$

Find $f'(x)$ and simplify. Then find the equation of the tangent line to the graph of $f(x)$ at the given value of x. Find the values of x where the tangent line is horizontal.

8. $f(x) = (3x+13)^{1/2}$ at $x = 4$

Horizontal Tangent

9. $f(x) = 3e^{2x^2 + 5x - 4}$ $x = 0$

Horizontal Tangent:

10. $f(x) = \ln\left(1 - x^2 + 2x^4\right)$ at $x = 1$

Horizontal Tangent

Set each factor equal to zero.

Find the indicated derivative and simplify.

11. $\frac{d}{dt}\ 3\left(2t^4\ + t^2\ \right)^{-5}$

12. $\frac{dh}{dw}$ if $h\left(w\right) = \ \sqrt[5]{8w - 1}$

13. $h'\left(x\right)$ if $h\left(x\right) = \ \frac{e^{4x}}{x^3\ + 9x}$

14. $\frac{d}{dx}\left[x^5 \ \ ln\left(3+x^5\ \right)\right]$

15. $G'\left(t\right)$ if $G\left(t\right)=\ \left(t-e^{9t}\right)^2$

16. y' if $y=\ \left[ln\left(x^2\ +7\right)\right]^{4/5}$

17. $\frac{d}{dw} \ \frac{1}{(w^2\ -5)^3}$

Horizontal Tangents

Find $f'(x)$ and simplify. Then find the equation of the tangent line to the graph of $f(x)$ at the given value of x. Find the values of x where the tangent line is horizontal.

18. $f(x) = x^2\ (3 - 2x)^4 \qquad x = 1$

Horizontal Tangent

19. $f(x) = \frac{x^4}{(2x-5)^2}$ $\qquad x = 2$

Horizontal Tangent

20. $f\left(x\right)=e^{\sqrt{x}}$ when $x=1$

Horizontal tangent

21. $f(x) = \sqrt{x^2 + 4x + 5}$ at $x = 0$

Horizontal tangent

Applications

22. COST FUNCTION: The total cost (in hundreds of dollars) of producing x pairs of sandals per week is: $C\left(x\right) = 6 + \sqrt{3x + 25}$ when $0 \ \leq x \ \leq 30$

    a. Find $C'\left(x\right)$

    b. Find $C'\ \left(17\right)$ and $C'\ \left(26\right)$. Interpret the results.

23. PRICE DEMAND EQUATION: The number of large pumpkin spice drinks (x) people are willing to buy per week from a local coffee shop at a price of p (in dollars) is given by:

$x = 1000 - 60(p+25)^{1/2}$ when $3.50 \ \leq p \ \leq 6.25$

a. Find $\frac{dx}{dp}$

b. Find the demand and the instantaneous rate of change of demand with respect to price when the price is \$4.50. Write a brief interpretation of these results.

24. BIOLOGY: A yeast culture at room temperature (68°F) is placed in a refrigerator set at a constant temperature of 38°F. After t hours, the temperature, T, of the culture is given approximately by

    $T = 25e^{-0.62t} + 40$ when $t \geq 0.$

    What is the rate of change of temperature of the culture at the end of 1 hour? At the end of 5 hours?