MATH 1830 NOTES

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UNIT 1 LIMITS

1.P Writing Equations of Lines

Introduction

Some problems can be represented by linear equations which can then be used to find solutions or make predictions. In this activity you will use slopes and ordered pairs to write expressions that model these problems.

1. You need to find a formula for the number of cars produced by a new plant that was opened 25 days ago. You are told that three days ago the plant had 450 cars on hand, and that yesterday it had 480 cars on hand. You are also told that some of these cars came from previous inventory. Assume that the plant produces cars at this same rate and has produced them at this same rate since it opened. Define “x” to be the time from today and “y” to be the number of cars on hand. Therefore, an ordered pair (x, y) will represent (time from today, number of cars on hand).

   a. Write an ordered pair using the information about the cars on hand three days ago.

   b. Write a second ordered pair using the information about the cars on hand yesterday.

   c. You need to know the rate at which the plant is producing cars. Another name for this rate of change is “slope.” Use the two ordered pairs to find the rate of change.

   d. Use the slope and one of the ordered pairs (doesn't matter which one!) to write a linear equation describing this problem. Use point-slope form: $y - y_1 = m(x - x_1)$.

   e. Write the equation in slope-intercept form: $y = mx + b$.

   f. Now write the equation in standard form: $ax + by = c$.

   g. Use the equation to find the number of cars the plant had on hand when it opened.

h. Use the equation to find the number of cars the plant will have on hand in five days. Assume in 5 days means 5 days from today.

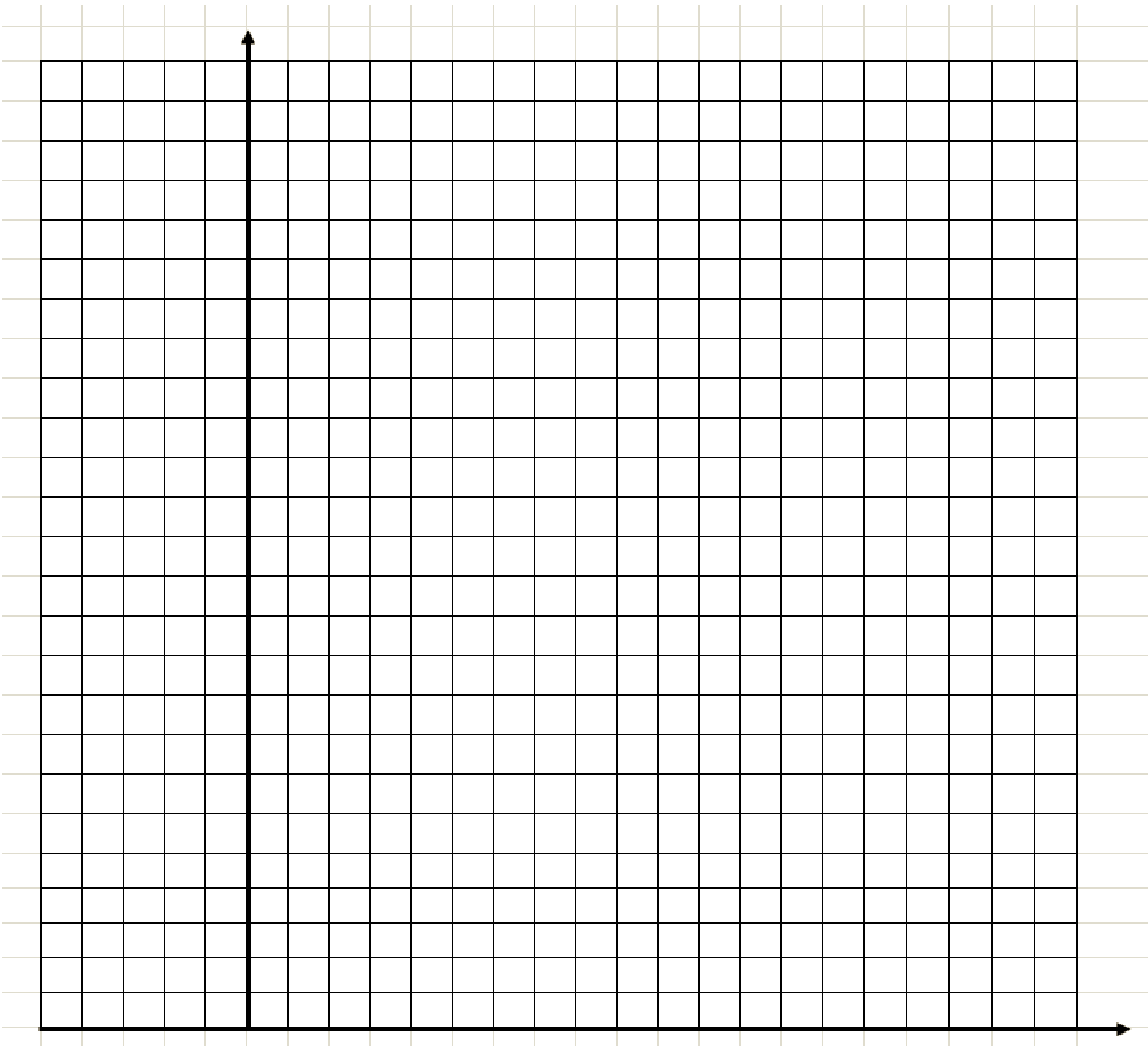
i. Use the equation to find when the plant will have 675 cars on hand.

j. Graph the equation on the interval [-3,13].

$y = 15x + 495$

x axis: time from today, in days

y axis: number of cars on hand



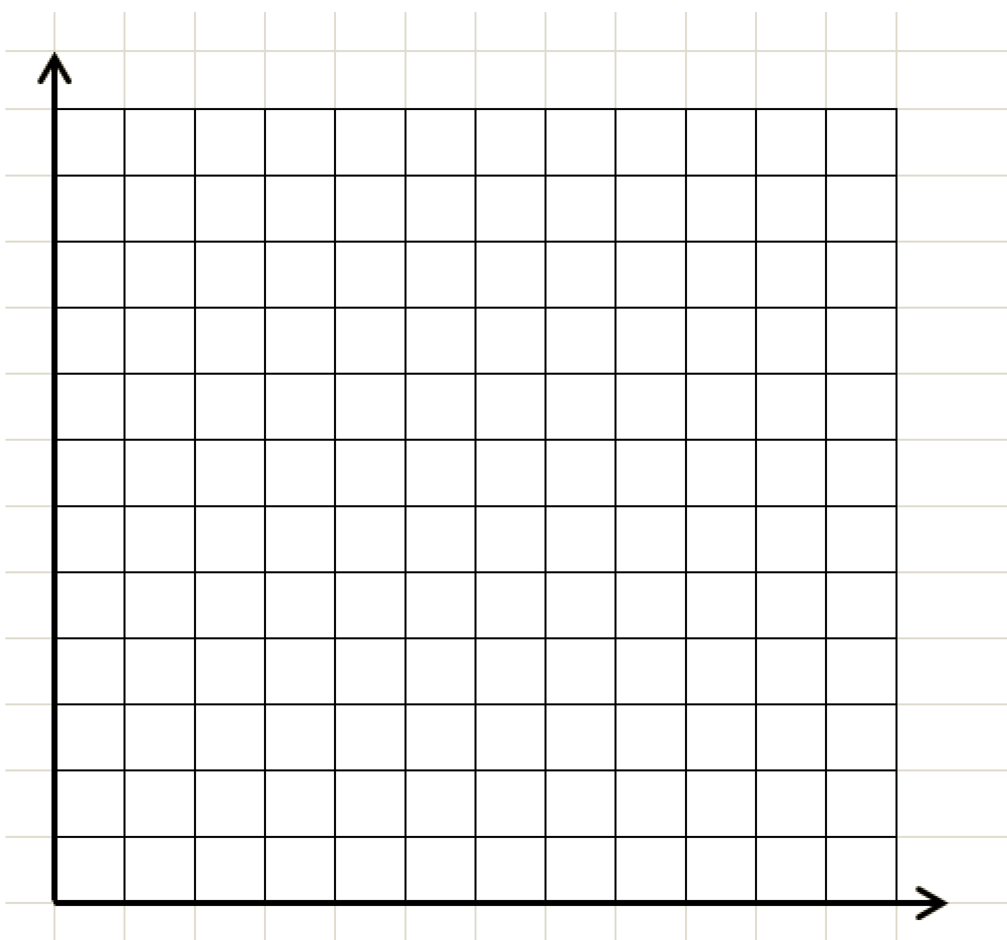
2. Ally Bank is offering its customers a 2-year Certificate of Deposit (CD) paying 1.29% (APR) with no minimum deposit. Define x to be the amount deposited and y to be the APR.

   a. Write ordered pairs that would represent depositing \$100, \$1000, and \$5000 in the CD.

   b. What is the rate of change of the APR?

   c. Find the linear equation that represents this problem.

   d. Graph the equation.

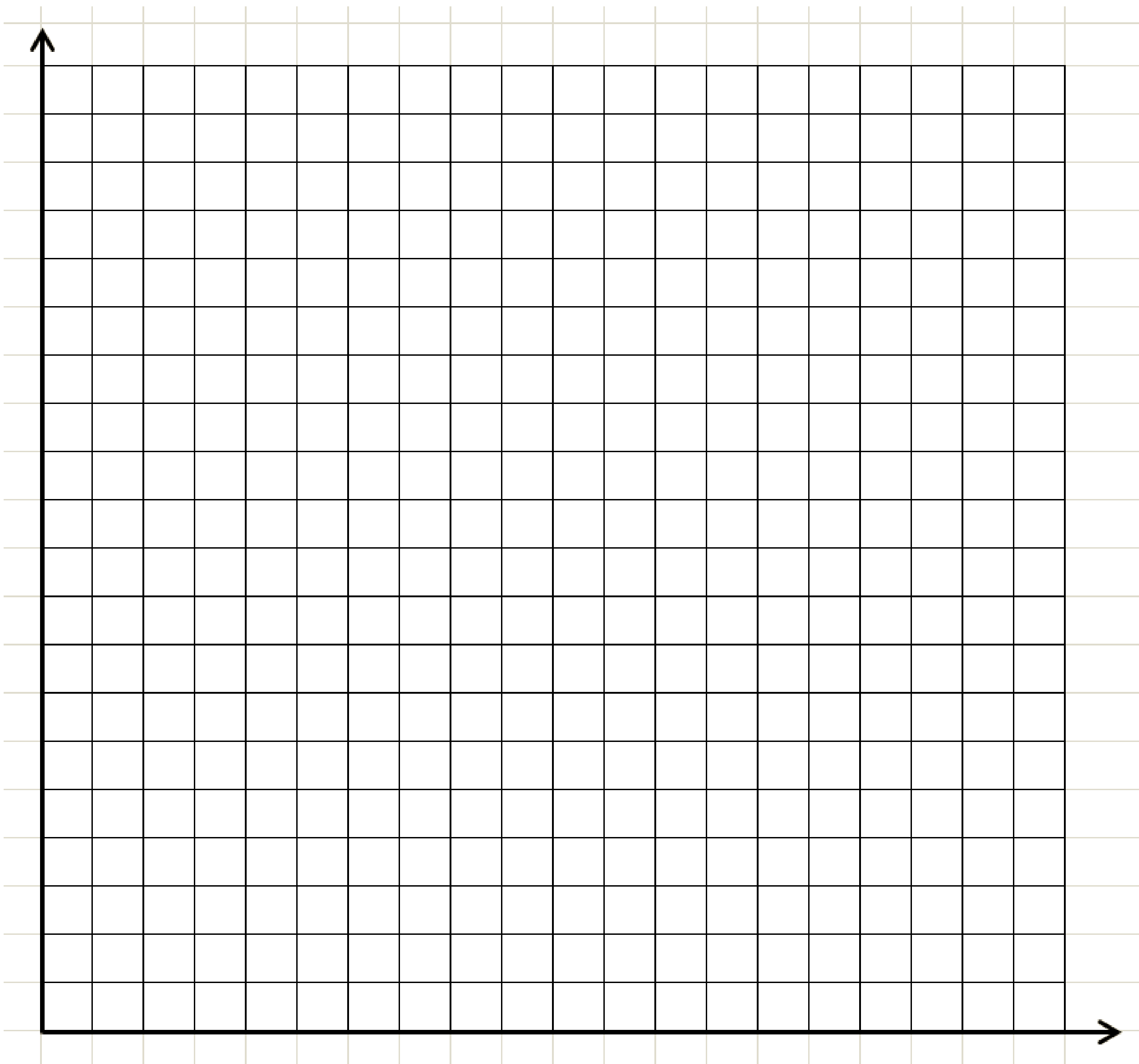


3. When Apple stock was first listed on the Nasdaq Stock Exchange on December 2, 1980, the price for one share of stock was $28.75. On June 12, 1995, the day Sam was born, his aunt gave him one share of Apple stock. On that day it was worth $44. On January 14, 2015, one share of Apple stock was selling for $109.01.

Define x to be the number of shares of Apple stock and y to be the value of the stock.

a. Write three ordered pairs to represent the value of the stock on these 3 days.

b. Graph these three points.



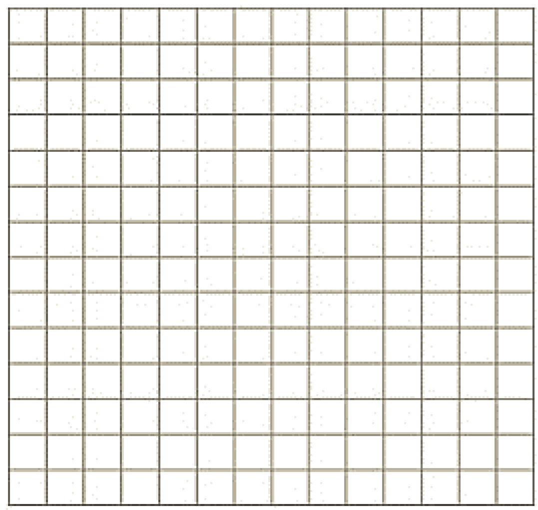
c. Write the equation of the line.

d. What is the slope of this line?

e. Is this a function? Explain.

Notes

1. Write the equation of the vertical and horizontal lines through the point (2, 4) . Graph both lines. What is the slope of each line?



2. For each of the sets of points below, find:

i. the slope of the line

ii. the slope intercept form of the line

iii. the standard form of the line

iv. the y intercept and the x intercept of the line

a. (3, 5) and (-1, 7)

i.

ii.

iii.

iv.

b. (2, 7) and (2, -3)

i.

ii.

iii.

iv.

c. (1, 3) and (0, 3)

i.

ii.

iii.

iv.

d. (-4, -8) and (-3, -4)

i.

ii.

iii.

iv.

3. You want to rent a 4"x8" U-Haul Trailer. The rental fee is $15 for the first day and $13.50 for each additional day. What is the equation relating the cost, y, to the number of days you rent the trailer?

1.1 Limits Graphically and Algebraically

Introduction

1. Box Office Receipts

   The total worldwide box-office receipts for a long running indie film are approximated by the function

   $$T(x) = \frac{120x^2}{x^2+4}$$

   where T(x) is measured in millions of dollars and x is the number of months since the movie's release. What are the total box-office receipts after:

   a. The first month?

   b. The second?

   c. The third?

   d. The hundredth?

   e. What will the movie gross in the long run? (When x is very large.)

2. Driving Costs

A study of driving costs of 1992 model subcompact cars found that the average cost (car payments, gas, insurance, upkeep, and depreciation), measured in cents/mile, is approximated by

$$C(x) = \frac{2010}{x^{2.2}} + 17.80$$

where x denotes the number of miles (in thousands of miles) the car is driven in a year. What is the average cost of driving a subcompact car:

a. 5,000 miles per year?

b. 10,000 miles per year?

c. 25,000 miles per year?

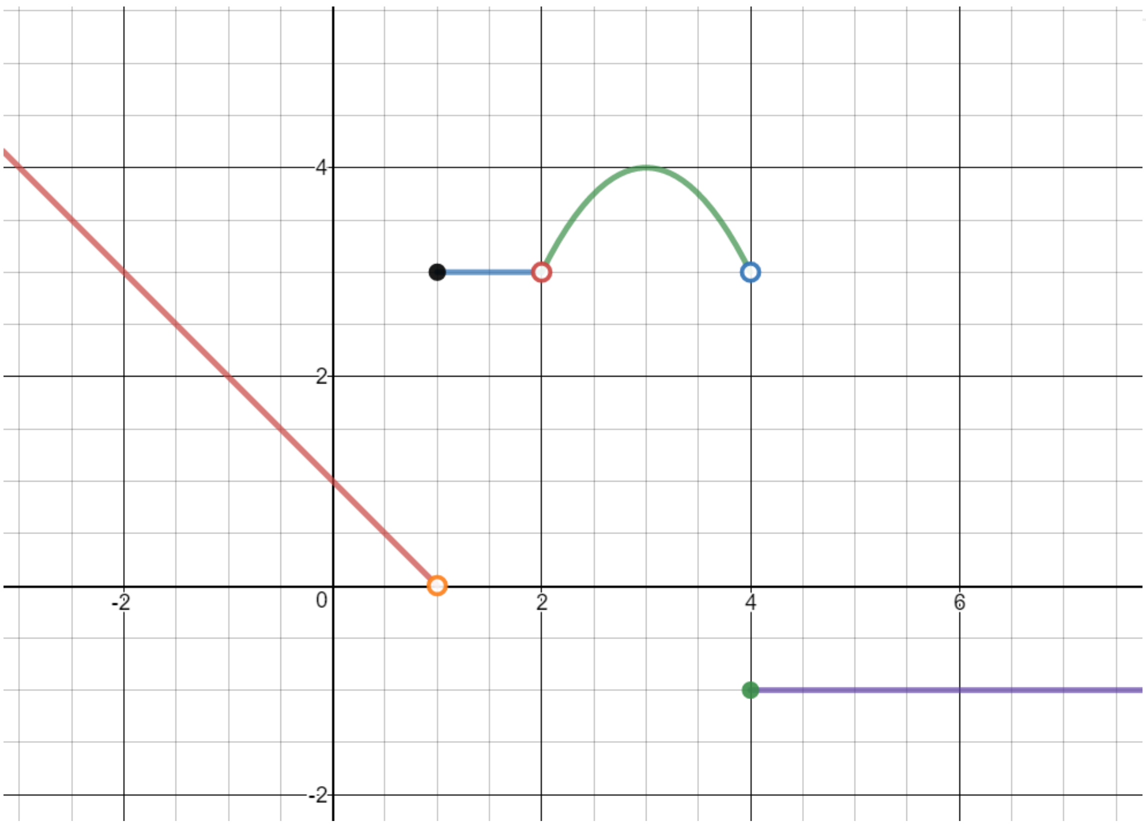
d. 50,000 miles per year?

e. What happens to the average cost as the number of miles driven increases without bound?

f. Verify by evaluating the cost when the number of miles is 1,000,000 (or any large number)

Notes

Limits: A Graphical Approach



Use the piecewise function to answer the following questions.

$$f(x) = \begin{cases} -x+1 & x < 1 \\ 3 & 1 \le x < 2 \\ -(x-3)^2+4 & 2 < x < 4 \\ -1 & x \ge 4 \end{cases}$$

Evaluate the limits graphically. If the limit does not exist, explain why.

1. $\lim_{x \to 0^-} f(x) =$
2. $\lim_{x \to 0^+} f(x) =$
3. $\lim_{x \to 0} f(x) =$
4. $f(0) =$

5. $\lim_{x \to 1^-} f(x) =$
6. $\lim_{x \to 1^+} f(x) =$
7. $\lim_{x \to 1} f(x) =$
8. $f(1) =$

9. $\lim_{x \to 2^-} f(x) =$
10. $\lim_{x \to 2^+} f(x) =$
11. $\lim_{x \to 2} f(x) =$
12. $f(2) =$
13. Is it possible to define $f(1)$ so that $\lim_{x \to 1} f(x) = f(1)$? Explain.

14. Is it possible to redefine $f(2)$ so that $\lim_{x \to 2} f(x) = f(2)$? Explain.

15. $\lim_{x \to -1} f(x) =$

16. $\lim_{x \to 4} f(x) =$

17. $\lim_{x \to 2} f(x) =$

18. $\lim_{x \to 3} f(x) =$

19. $\lim_{x \to -2} f(x) =$

Limits: An Algebraic Approach

Find each indicated quantity, if it exists.

20. $\lim_{x \to 4} x^2 - 5x + 1 =$

21. $\lim_{x \to -5} 2x^2 + 10x + 7 =$

22. $f(x) = \begin{cases} x+5 & x < -4 \\ \sqrt{x+4} & x > -4 \end{cases}$

a. $\lim\limits_{x \to -4^+} f(x) =$

b. $\lim\limits_{x \to -4^-} f(x)$

c. $\lim\limits_{x \to -4} f(x) =$

d. $f(-4) =$

23. $g(x) = \frac{x-2}{|x-2|}$

a. $\lim\limits_{x \to 2^+} g(x) =$

b. $\lim\limits_{x \to 2^-} g(x) =$

c. $\lim\limits_{x \to 2} g(x) =$

d. $g(2) =$

24. $f(x) = \frac{3x^2 + 2x - 1}{x^2 + 3x + 2}$

a. $\lim_{x \to -3} f(x) =$

b. $\lim_{x \to -1} f(x) =$

c. $\lim_{x \to 2} f(x)$

d. $\lim_{x \to -2} f(x)$

25. $\lim_{x \to 10} \frac{x^2 - 15x + 50}{(x-10)^2}$

26. Compute the following limit for the function: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$f\left(x\right) = x^2 \ + 5x - 1$$

a. Define $f\left(x + h\right)$ and $f\left(x\right)$

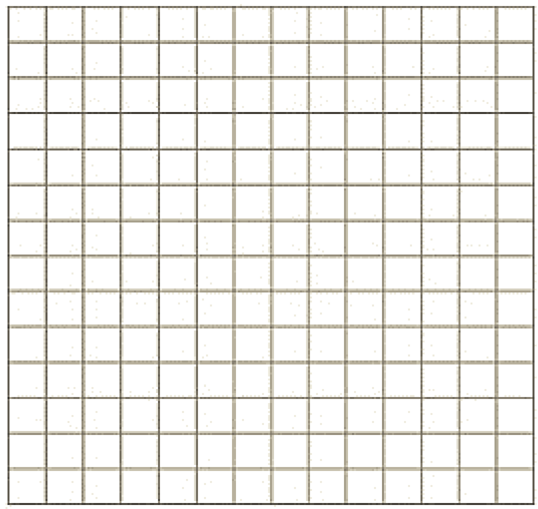
b. Calculate $f\left(x + h\right) - f\left(x\right)$

c. Divide by h

d. Evaluate the limit

27. A taxi service charges \$3.00 per mile for the first 10 miles. If the trip is over 10 miles, they charge \$5.00 per mile for every mile. Write a piecewise definition of the charge G(x) for taxi fares of x miles.

Graph G(x) for $0 < x \ \leq 25.$



Find:

$\lim\limits_{x \to 10^-} G(x) =$

$\lim\limits_{x \to 10^+} G(x) =$

$\lim\limits_{x \to 10} G(x) =$

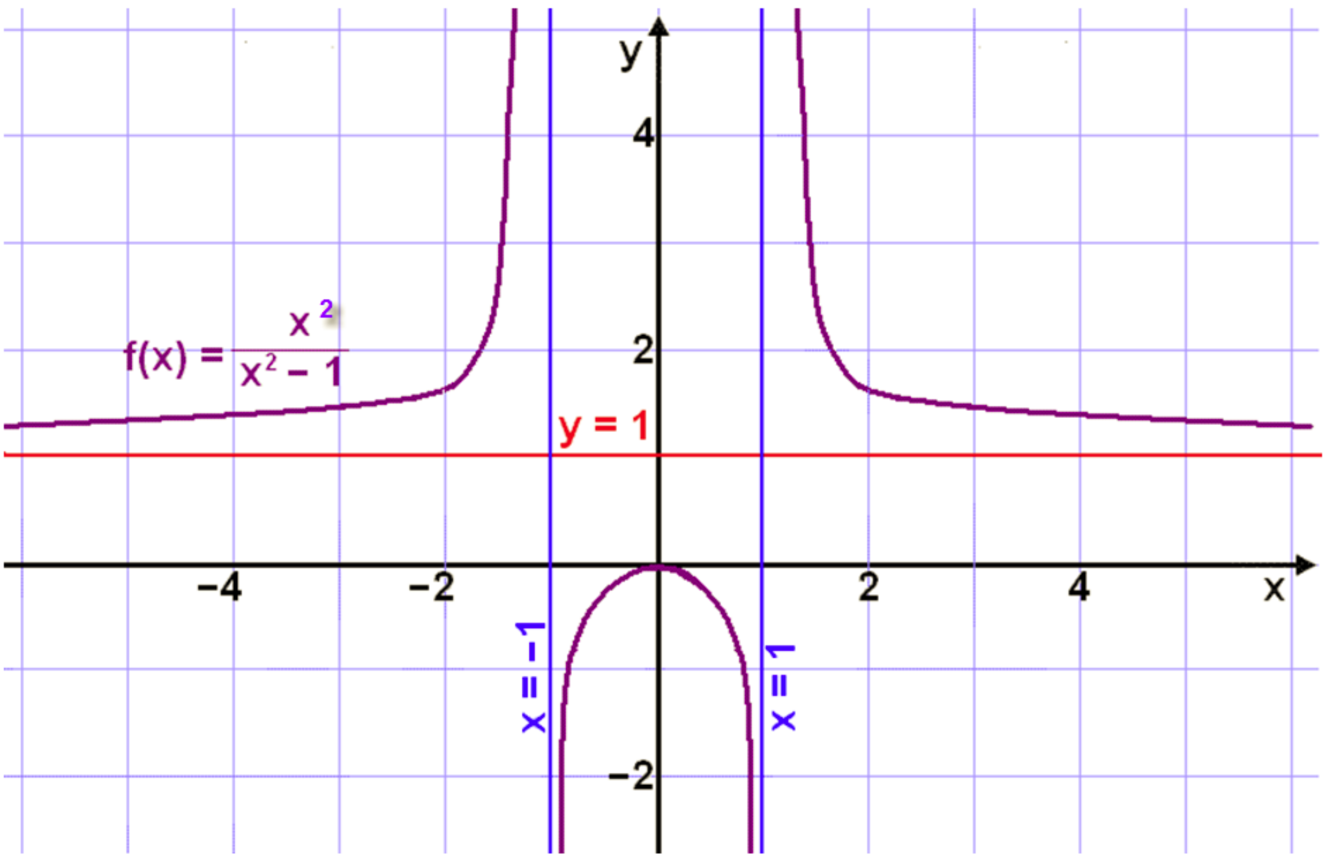
1.2 Infinite Limits and Asymptotes

Introduction

Discuss this graph with your group.

Write down everything you observe.

Be prepared to share with the class.

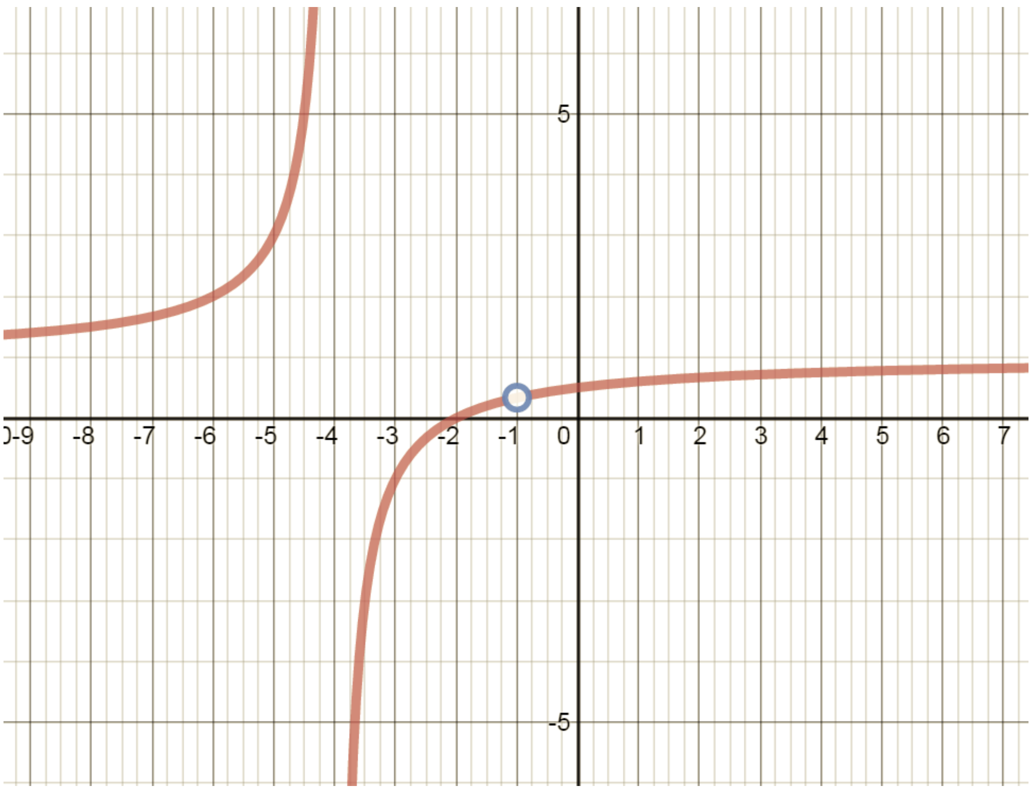


Notes

Infinite Limits, Vertical asymptotes, and Holes:

1. Describe the behavior of $f(x)$ at each zero of the denominator

$$f(x) = \frac{x^2 + 3x + 2}{x^2 + 5x + 4}$$



a. $x = -4$

b. $x = -1$

2. How would you determine whether the graph has holes and/or vertical asymptotes if you don't have a graph?

   a. With Limits:

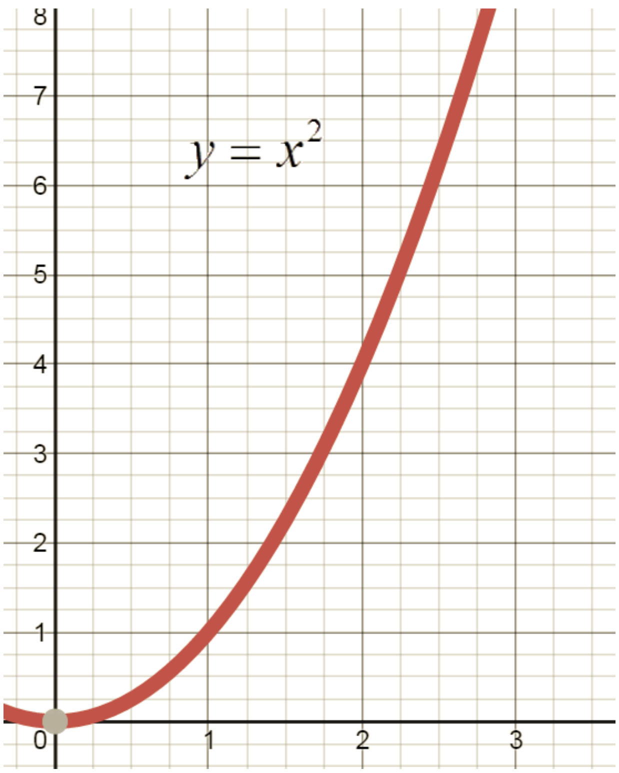
   b. Algebraically:

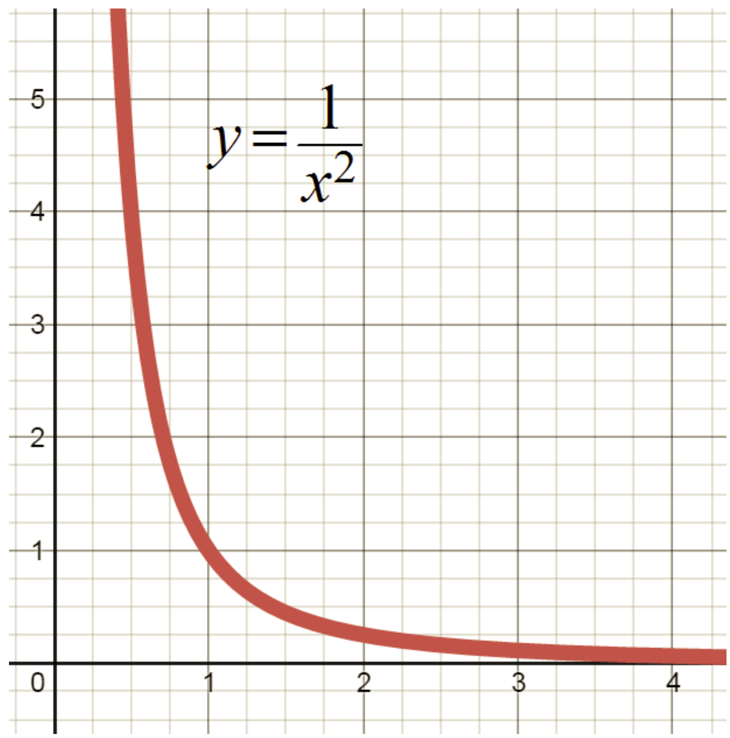
3. Describe the behavior of $f(x)$ at each zero of the denominator $f(x) = \frac{x^2 + 10}{4(x-3)^2}$

Limits at Infinity: Polynomials and Rational Expressions

Limits at Infinity: Polynomials and Rational Expressions

4. Briefly describe the behavior of the two individual functions as x approaches positive infinity?





End Behavior of Polynomials

5. Let $p\left(x\right) = 3x^3 \ - 4x^2 \ - 2x + 5,$

Find the limit of $p\left(x\right)$ as $x \rightarrow \ \infty$ and as $x \rightarrow \ -\infty$

6. Let $p\left(x\right) = -6x^4 + 3x^2 + 5$,

Find the limit of $p\left(x\right)$ as $x \rightarrow \ \infty$ and as $x \rightarrow \ -\infty$

Horizontal Asymptotes: End Behavior and Rational Expressions

7. Let $f(x) = \frac{3x^2 - 5x + 9}{2x^2 + 7}$, find the limit of $f(x)$ as $x \to \infty$ and as $x \to -\infty$

Identify the horizontal asymptotes of the following rational expression (if the horizontal asymptote exists)

8. $\lim\limits_{x \to \infty} \frac{7x^3 - x^2 + 1}{5x^3 + 6x - 7} =$

9. $\lim\limits_{x \to \infty} \frac{6x^4 - x^2 + 1}{2x^6 - 8x} =$

10. $\lim\limits_{x \to \infty} \frac{4x^5 - 9x^3 - 1}{5x^3 + 3x^2 - 7} =$

Vertical and Horizontal Asymptotes: A summary

Find all vertical asymptotes, horizontal asymptotes, and holes of the function, showing all your work:

11. $f(x) = \frac{2x^2 - 32}{x^2 + 5x + 4}$

12. $f(x) = \frac{x^2 - 9}{x^2 - 4}$

Find all vertical asymptotes, horizontal asymptotes, and holes of the function by a quick analysis:

13. $f(x) = \frac{x + 2}{x^2 + 3}$

Find all vertical asymptotes, horizontal asymptotes, and holes of the function by a quick analysis:

13. $f(x) = \frac{x + 2}{x^2 + 3}$

14. $f(x) = \frac{x^2 - 3x - 10}{x^2 - 4x - 5}$

15. $f(x) = \frac{x^2 + 5x - 14}{x - 2}$

1.3 Continuity

Introduction

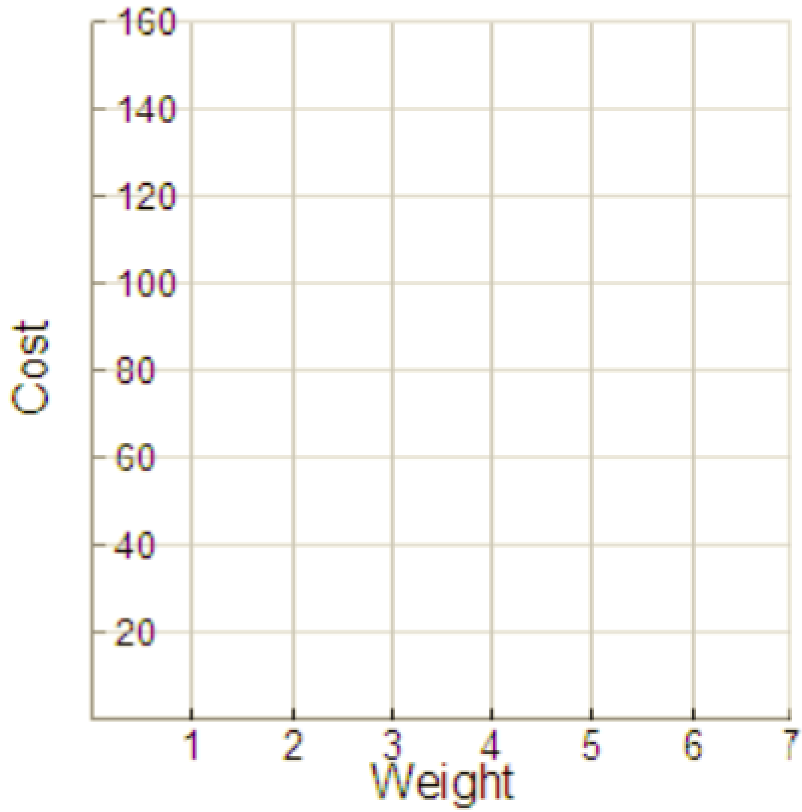
1. The table below shows the cost of mailing a letter that weighs x ounces.

Weight	Cost
$0 < x \leq 1$	49¢
$1 < x \leq 2$	70¢
$2 < x \leq 3$	91¢
$3 < x \leq 4$	112¢
$4 < x \leq 5$	133¢

a. Complete the table of letters with the following weights.

Weight	Cost
.98	
1.26	
2.55	
3.01	
4.29	

b. Graph the function on the following page.

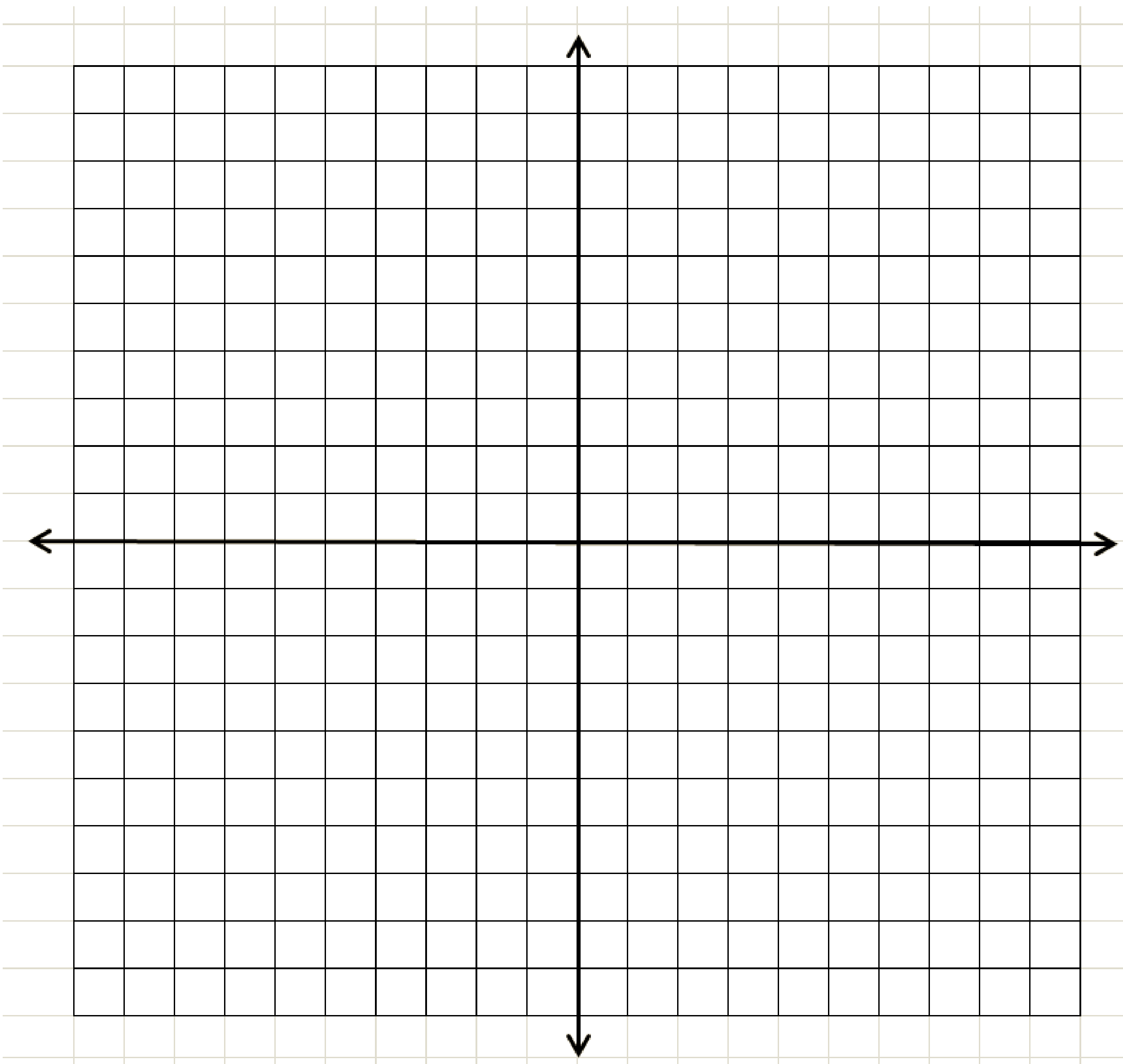


Source http://www.stamps.com/usps/postage-rate-increase/

2. Given: $f(x) = \frac{3x^2-12x-15}{x^2-3x-10}$

a. From looking at the given function (and NOT graphing), where would you expect to see vertical asymptote(s)?

b. Graph the function. Where are the vertical asymptote(s)?



c. Do you get the same answer for a & b? Why or why not?

d. What is the horizontal asymptote(s) for this function?

Notes

Informal Definition: Continuity

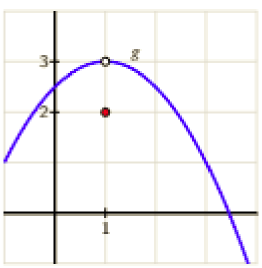
A function is continuous over an interval if the graph over the interval can be drawn without removing the pencil from the paper.

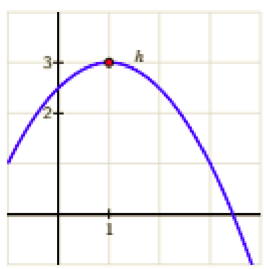
Formal Definition: Continuity

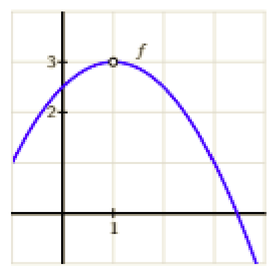
A function, f(x), is continuous at the point $x = c$ if all three of the following requirements are met:

1. $\lim\limits_{x \to c} f(x)$ exists
2. $f(c)$ exists
3. $\lim\limits_{x \to c} f(x) = f(c)$

Examples of Continuous and Discontinuous Functions:

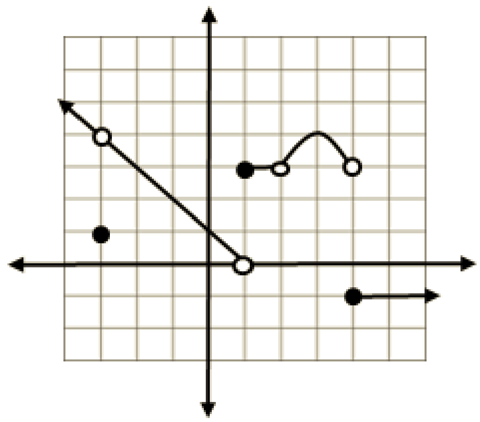






Active Calculus https://open.umn.edu/opentextbooks/

In Groups: Use the formal definition of continuity to discuss the continuity of the function whose graph is shown below.



1. Continuity at $x = -3$

2. Continuity at

3. Continuity at $x=1$

4. Continuity at $x = 2$

5. Continuity at $x = 3$

6. Continuity at $x = 4$

Rules for Continuity

- Constant functions $f(x) = k$ are continuous for all x
  Example: $f(x) = -2$
- Power functions $f(x) = x^n$, where n is a positive integer, are continuous for all x

  Example: $f(x) = x^5$
- Polynomial Functions are continuous for all x
  Example: $f(x) = 2x^3 - 5x + 1$
- Rational Function s are continuous for all x except where the denominator = 0
  Example: $f(x) = \frac{x^2+5}{x-3}$ Where numerator and denominator are polynomials
- $\sqrt[n]{f(x)}$ functions are continuous for all x where n is an odd positive integer > 1
  Example: $\sqrt[3]{x}$
- $\sqrt[n]{f(x)}$ functions are continuous for all x where n is an even positive integer and f(x) is positive
  Example: $\sqrt[4]{x}$

Assessing Continuity

Are the functions continuous? Use the Rules for Continuity to explain your answers..

7. $h\left(x\right)=5-3x$

8. $n\left(x\right)=\ \frac{x-3}{x^2\ +\ 2x-15}$

9. $f\left(x\right)=\ \sqrt{25-x^2}$

10. $g\left(x\right)=\ \sqrt[3]{x^2-4}$

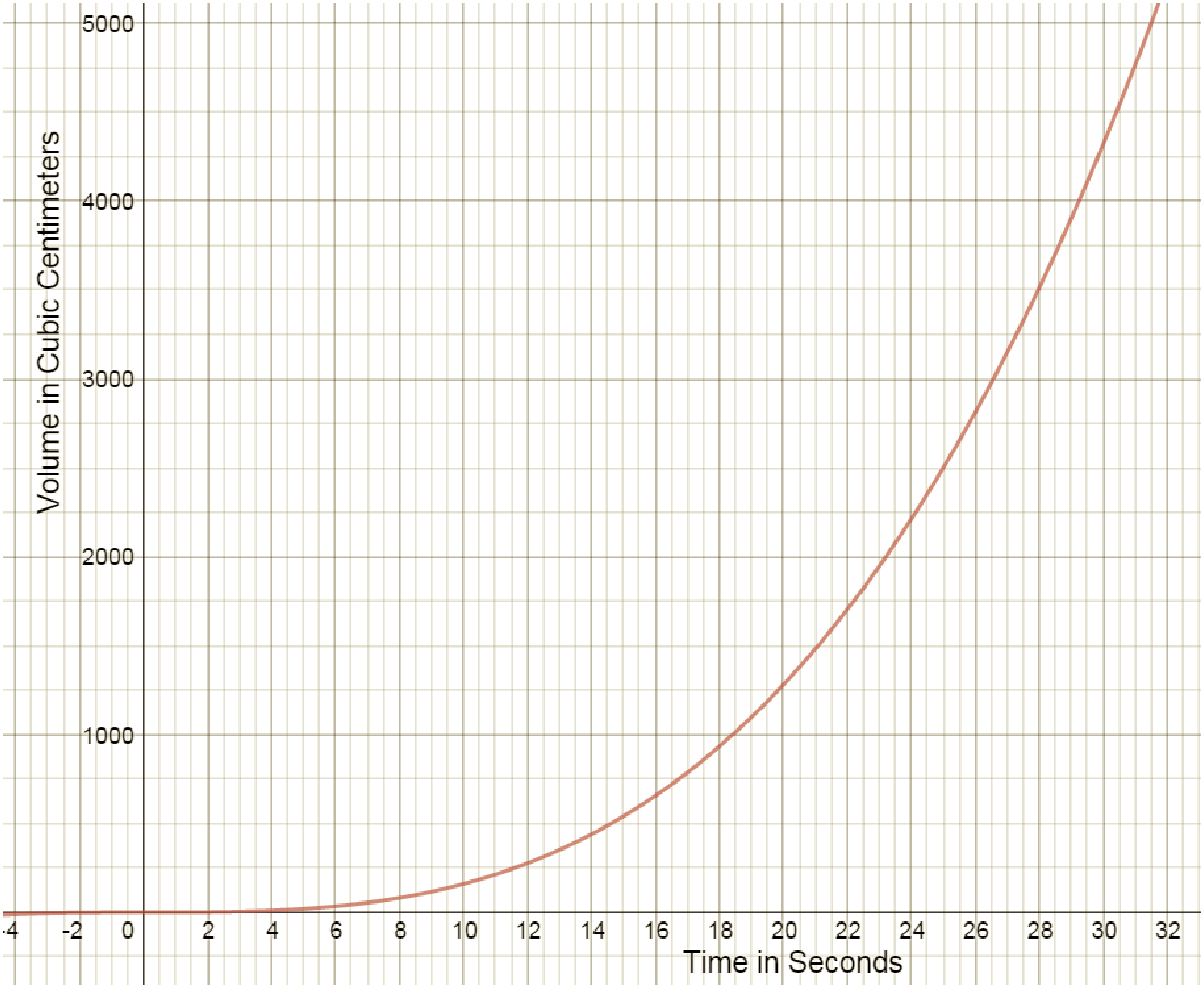
1.4 Definition of Derivatives

Introduction

A decorative birthday balloon is being filled with helium. The table shows the volume of the helium in the balloon at 3 second intervals for 30 seconds.

t(seconds)	V (cubic meters)
0	0
3	4.2
6	33.5
9	113.0
12	267.9
15	523.3
18	904.3
21	1436.0
24	2143.6
27	3052.1
30	4186.7

This function can be approximated by the equation
$y = 0.16x^3 + 0.0003x^2 - 0.007x + 0.0161$ (graphed below)



1. What are the dependent and independent variables for this problem? In what units is the rate of changed expressed?

2. A secant line is a line that intersects two points on a curve. Draw a secant line on the graph for each of the following. Calculate the slope of the secant line for each of the following intervals.

    a. 21 s to 30 s

    b. 21 s to 27 s

    c. 21 s to 24 s

3. What does the slope of the secant line represent?

4. A tangent line is a line that intersects a curve at only one point. Draw a tangent line at the point on the graph corresponding to 21 s and estimate the slope of this line.

5. What does the slope of the tangent line represent?

6. Compare the secant slopes to the slope of the tangent line. What do you notice?

Source http://mysite.science.uottawa.ca/iabde083/ch01.pdf

Notes

Limit Definition of the Derivative of a Function: 4 Step Process

Step 1 Find $f\left(x+h\right)$

Step 2 Find $f\left(x+h\right)-f\left(x\right)$

Step 3 Find $\frac{f(x+h)-f(x)}{h}$

Step 4 Find $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$

1. Use the limit definition of the derivative to find $f'\left(x\right)$, given $f\left(x\right)=\ x^2\ -3x-2$

2. Use the limit definition of the derivative to find $f'(x)$, given $f(x) = 5x^2 + 2x - 8$

Write the equation of the tangent line at x=2.

3. Use the limit definition of the derivative to find $f'(x)$, given $f(x) = -4 \quad + x + 2$

<u>Write the equation of the tangent line at x = 1</u>

4. Use the limit definition of the derivative to find $f'(x)$, given $f(x) = \frac{6}{x} - 2$

Find the equation of the tangent line at $x = 3$

5. Friedman Academy is a charter school in South Texas. The school is not at capacity, and enrollment increases monthly. The charter school's enrollment is modeled by

$$y = f(x) = x^2 + x + 100$$

Where y is in students and x is in years since 2008.

   a. Estimate the average change in enrollment (the average rate of change of y with respect to x) from the year 2010 to the year 2012.

   b. Find the function for the instantaneous rate of change in enrollment using the limit definition of the derivative.

   c. Use the equation to calculate the instantaneous change in enrollment in 2010, 2011, and 2012.

6. Suppose an object moves along the y axis so that its location is $y = f(x) = 2x^2 + 3x$ at time x. y is in meters and x is in seconds

   a. Find the average velocity (the average rate of change of y with respect to x) for x changing from 2 to 4 sec.

   b. use the limit definition of the derivative to find the instantaneous velocity.

   c. The instantaneous velocity at x = 2 seconds, 3 seconds, and 4 seconds.

1.5 Derivatives: The Power Rule

Notes

Basic Differentiation Properties

Three Equivalent Terms:

If $y = f(x)$, you can use any of these to represent the derivative $y' = f'(x) = \frac{dy}{dx}$

THE POWER RULE: If $f(x) = x^n$ then $f'(x) = nx^{n-1}$

Using the Power Rule Find the indicated derivative:

1. $g(x) = x^4$

2. $y = 2x^3$

3. $\frac{d}{dx}(5)$

4. $y = \frac{1}{x^7} = x^{-7}$

5. $y = \frac{x^4}{16}$

6. $y = 8 + 3t - 5t^3$

7. $g(x) = 6x^{-5} - 5x^{-4}$

8. $\frac{d}{dx}\left(\frac{4x^3}{10} - \frac{2}{3x^4}\right)$

9. $H'(w)$ if $H(w) = \frac{5}{w^6} - 7\sqrt{w}$

10. $\frac{d}{du}\left(7u^{2/3} + 4u^{-3/5}\right)$

11. Find and approximate the value(s) of x where the graph of f has a horizontal tangent line. Use a graphing calculator to verify. $f(x) = 2x^2 - 5x$

12. A company's total sales (in millions of dollars) t months from now are given by
$S(t) = 0.032t^4 + 0.5t^3 + 2.8t^2 + 9t - 4$

a. Find $S'(t)$

b. Find $S(4)$ and $S'(4)$ Write a brief verbal interpretation of these results.

c. Find $S(8)$ and $S'(8)$ Write a brief verbal interpretation of these results.

13. A company decides to develop a cost equation based on the quantity of the product produced in a day. They collected the following data:

Quantity	20	35	50	65	80	95	110
Cost	642.35	766.48	858.82	928.83	1005.32	1078.82	1140.79

a. Enter the data in a graphing calculator and find a cubic regression equation for the data.
   Let x represent the quantity produced in a day.

   Let y represent the daily cost for production.

b. If $y = F(x)$ denotes the regression equation found in part A, find $F(70)$ and $F'(70)$

c. Write a brief verbal interpretation of these results

1.6 Marginal Analysis in Business and Economics

Introduction

The cost, in dollars, of producing x frozen fruit yogurt bars per day can be modeled by the function

$$C(x) = 3450 + 1.5x - 0.0001x^2 \quad when \quad 0 \leq \ x \ \leq 5000.$$

The revenue from selling x yogurt bars is

$$R(x) = 3.25x.$$

Cost, Revenue, & Profit

1. What does 3450 represent in the cost function?

2. What does 3.25 represent in the revenue function?

3. What is the cost of producing 0 yogurt bars? What is the revenue generated from selling this many bars? What is the profit for selling 0 bars?

4. What is the cost of producing 1000 yogurt bars? What is the revenue generated from selling this many bars? What is the profit for selling 1000 bars?

5. What is the cost of producing 5000 yogurt bars? What is the revenue generated from selling this many bars? What is the profit for selling 5000 bars?

6. Find the model for the profit function, $P(x)$.

7. Answer the following questions about the model:

   When is the profit function positive?

   When is it negative?

   What important information does this provide the business owner?

Average Cost, Revenue, & Profit

8. It is clear from 3, 4, & 5 that the cost, revenue, and profit change based on the number of yogurt bars produced/sold. How would you determine the AVERAGE cost, revenue, and profit?

   Use these equations generated earlier:

   $$C(x) = 3450 + 1.5x - 0.0001x^2 \quad when \quad 0 \leq \ x \ \leq 5000$$

   $$R(x) = 3.25x$$

   $$P(x) = 0.0001x^2 + 1.75x - 3450$$

   a.

   b.

   c.

9. Find the average cost and explain the result:

   a. $\bar{C}(0) =$

   b. $\bar{C}(1000) =$

   c. $\bar{C}(5000) =$

10. Is there a difference in the average cost when the number of bars produced changes? Why or why not?

Marginal Cost, Revenue, & Profit

Use these equations generated earlier:

$$C(x) = 3450 + 1.5x - 0.0001x^2 \quad when \quad 0 \leq \ x \ \leq 5000$$

$$R(x) = 3.25x$$

$$P(x) = 0.0001x^2 + 1.75x - 3450$$

11. Determine $C'\,(x)$. What does this represent?

12. Determine $C'\,(1000)$ Interpret this result.

13. When is $C'\,(x) = 0$? Explain your answer.

14. Determine $R'\,(x)$. What does this represent?

15. Determine $P'\,(x)$. What does this represent?

16. How do marginal cost, revenue, & profit relate to our discussions about average rate of change and instantaneous rate of change?

Marginal Average Cost, Revenue, & Profit

Use these equations generated earlier:

$$C(x) = 3450 + 1.5x - 0.0001x^2 \quad when \quad 0 \leq \ x \ \leq 5000$$

$$\cancel{R(x) = 3.25x}$$

$$P(x) = 0.0001x^2 + 1.75x - 3450$$

17. Determine mathematical model for the marginal average cost.

18. Find the average cost when 100 bars are produced and the marginal average cost when 100 bars are produced. Interpret these quantities.

19. Based on your answers from the question above, estimate the average cost per bar at a production level of 101 bars per day.

Notes

Cost and Marginal Cost

1. A small company manufactures bicycle helmets. The total weekly cost (in dollars) of producing x helmets is given by:

$$C(x) = 9870 + 85x - 0.05x^2$$

   a. **Find the Marginal Cost Function:**

   b. **Using the Marginal Cost Function: Find the marginal cost at a production level of 550 helmets per week. Interpret the results.**

   c. **Using the Cost Function: Find the exact cost of producing the 551st item.**

Marginal Cost, Revenue and Profit

2. A company's market research department recommends the manufacture and marketing of a new 3 meter lightening to USB power cord. After suitable test marketing, the research department presents the following price demand equation:

$$p = 12 - 0.001x$$

where x is demand at price p. The financial department provides the cost function that includes a fixed cost of $5600 (tooling and overhead) and variable costs of $1.80 per power cord (materials, labor, marketing, transportation, storage):

$$C\left(x\right) = 5600 + 1.80x$$

a. **Marginal Cost Function: Find and interpret the Marginal Cost Function**

b. **Revenue Function: Find the Revenue Function as a function of x.**

c. **Marginal Revenue Function: Find the Marginal Revenue function and find the marginal revenue at x = 2000, x = 5000, and x = 7000. Interpret the results.**

d. **Cost and Revenue Functions: Graph the cost function and the revenue function in the same coordinate system. Find the intersection points of these two graphs and interpret the results.**

e. **Profit Function: Find the profit function. Sketch the graph.**

f. **Marginal Profit Function: Find the marginal profit function and evaluate the marginal profit at $x = 1000$, $x = 4000$, and $x = 6000$. Interpret the results.**

Marginal Average Cost, Revenue, and Profit:

3. A shop manufactures performance bikes. The manager estimates that the total cost (in dollars) of producing b bikes is:

$$C\left(b\right) = 1200 + 25b - 0.14b^2$$

   a. **Average Cost: Find the average cost function, $\bar{C}\left(b\right)$. Calculate $\bar{C}\left(7\right)$ and interpret the results.**

   b. **Marginal Average Cost: Find the Marginal Average Cost Function,$\bar{C}'\left(b\right)$. Calculate $\bar{C}'\left(7\right)$ and interpret the results.**

   c. **Application: Use the results above to estimate the average cost per bicycle at a production level of 8 bikes.**