Differential Manifolds

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Abstract: Abstract test...

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1 Differential Manifolds and differentiable maps

1.1 Review of General Topology

Let S be a set.

Definition 1.1. A topology is a collection $\mathcal{T}$ of subsets of S, called the open sets, such that:

(i) $\phi, S \in \mathcal{T}$, where $\phi \equiv \{\ \}$ is the empty set.

(ii) if $U_\alpha \in \mathcal{T}$, for $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$

(iii) if $U_1, \ldots, U_n \in \mathcal{T}$, $n \in \mathbb{N}$, then $\bigcap_{i=1}^{n} U_n \in \mathcal{T}$

Example 1.1.

1) $S = \mathbb{R}^n$, $U \in \mathcal{T}$ *iff* $U \subseteq S$ *is open in the usual sense.*

2) If (S, d) *is a metric space, then it is a topological space.*

Definition 1.2. Let $(S, \mathcal{T})$ be a topological space. A basis for the topology of S if the collection $B \subseteq \mathcal{T}$ so that any $U \in \mathcal{T}$ is the union of sets from B.

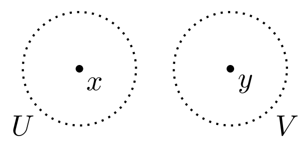
Example 1.2.

1) $\{\, B(x;\epsilon) \mid x \in \mathbb{R}^n, \epsilon \in \mathbb{R}^+ \,\}$ *is a basis for the usual topology in* $\mathbb{R}^n$.

2) $\{\, B(x;\epsilon) \mid x \in \mathbb{Q}^n, \epsilon \in \mathbb{Q}^+ \,\}$ *is a countable basis for* $\mathbb{R}^n$.

Definition 1.3. $(S, \mathcal{T})$ is second countable if the topology $\mathcal{T}$ has a countable basis.

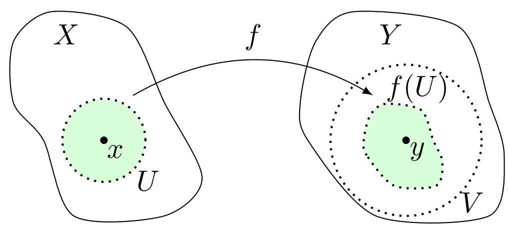
Definition 1.4. $(S, \mathcal{T})$ is Hausdorff if for all $x, y \in S$, with $x \neq y$, there are open sets $U, V \subseteq S$ so that $x \in U$, $y \in V$ and $U \cap V = \phi$.



In a Hausdorff space, we can always distinguish two points by enclosing them on open balls that do not intersect.

Let's now define continuity in terms using topology language. Let X, Y be any two topological spaces.

Definition 1.5. $f : X \longrightarrow Y$ is continuous at $x \in X$ if for any open $V \subseteq Y$ containing y, there exists a open $U \subseteq X$ containing x so that $f(U) \subseteq V$.



Definition 1.6. $f : X \longrightarrow Y$ is continuous if it is continuous al all $x \in X$.

Proposition 1.1. *$f : X \longrightarrow Y$ is continuous iff for all open $V \subseteq Y$, the preimage $f^{-1}(V) \equiv \{ x \in X \mid f(x) \in V \} \subseteq X$ is open in X.*

Definition 1.7. $F \subseteq X$ is closed if $X \backslash F \subseteq X$ is open.

Proposition 1.2. *$f : X \longrightarrow Y$ is continuous iff $f^{-1}(F) \subseteq X$ is closed in X, for every closed $F \subseteq Y$ in Y.*

Definition 1.8. A continuous map $f : X \longrightarrow Y$ is called a homeomorphism iff it has a continuous inverse $g : Y \longrightarrow X$, such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. We say that X and Y are homeomorphic. We write $g = f^{-1}$ and $X \cong Y$.

1.2 Differential Manifolds

Definition 1.9. A topological manifold of dimension m is a second countable, Hausdorff topological space M, so that any $p \in M$ has a open neighborhood $x \ni U \subseteq M$, which is homeomorphic with $\mathbb{R}^m$.

Remark. We mightas well have said homeomorphic with an open subset of $\mathbb{R}^m$, because $B(0, \epsilon) = \{ x \in \mathbb{R}^m \mid \|x\| < \epsilon \} \cong \mathbb{R}^m$. To see that we can use the function

$$\begin{aligned} f : B(0, \epsilon) &\longrightarrow \mathbb{R}^m \\ x &\longmapsto \frac{x}{\epsilon - \|x\|} \end{aligned}$$

Remark. The *Theorem of Invariance of Domain* states that if $\phi :\ \mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^n$, then $m = n$. Hence the definition is consistent.