Introduction to Differential Manifolds

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ABSTRACT: Notes based on the lectures by Peter Gothen at University of Porto with personal inclusions here and there.

Contents

1 Differential Manifolds and Differentiable Maps

1.1 Review of General Topology

Let S be a set.

Definition 1.1. A topology is a collection $\mathcal{T}$ of subsets of S, called the open sets, such that:

(i) $\emptyset, S \in \mathcal{T}$, where $\emptyset$ is the empty set.

(ii) if $U_\alpha \in \mathcal{T}$, for $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$

(iii) if $U_1, \ldots, U_n \in \mathcal{T}$, $n \in \mathbb{N}$, then $\bigcap_{i=1}^n U_n \in \mathcal{T}$

Example 1.1.

(1) $S = \mathbb{R}^n$, $U \in \mathcal{T}$ iff $U \subseteq S$ is open in the usual sense.

(2) If (S, d) is a metric space, then it is a topological space.

Definition 1.2. Let $(S, \mathcal{T})$ be a topological space. A basis for the topology of S if the collection $B \subseteq \mathcal{T}$ so that any $U \in \mathcal{T}$ is the union of sets from B.

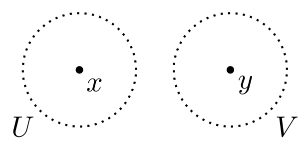
Example 1.2.

(1) $\{B(x;\epsilon) \,|\, x \in \mathbb{R}^n, \epsilon \in \mathbb{R}^+\}$ is a basis for the usual topology in $\mathbb{R}^n$.

(2) $\{B(x;\epsilon) \,|\, x \in \mathbb{Q}^n, \epsilon \in \mathbb{Q}^+\}$ is a countable basis for $\mathbb{R}^n$.

Definition 1.3. $(S, \mathcal{T})$ is second-countable if the topology $\mathcal{T}$ has a countable basis.

Definition 1.4. $(S, \mathcal{T})$ is Hausdorff if for all $x, y \in S$, with $x \neq y$, there are open sets $U, V \subseteq S$ so that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.



The second-countability and "Hausdorffness" seams to be strange requirements but they are natural conditions, since the prototype of a manifold is the Euclidean space, which has these properties. If we do not impose these then many important results do not hold. For instance, we need Hausdorff condition in order to have unique convergence of sequences and uniqueness of ODE solutions. Without second-countability, it is not certain that we can embed the manifold in a finite-dimensional Euclidean space.

Let's now define continuity in terms using topology language. Let X, Y be any two topological spaces.

Definition 1.5. A map $f : X \longrightarrow Y$ is continuous at $x \in X$ if for any open $V \subseteq Y$ containing y, there exists a open $U \subseteq X$ containing x so that $f(U) \subseteq V$.

Definition 1.6. A map $f : X \longrightarrow Y$ is continuous if it is continuous al all $x \in X$.

Proposition 1.1. A map $f : X \longrightarrow Y$ is continuous iff for all open $V \subseteq Y$, the preimage $f^{-1}(V) \equiv \{x \in X \mid f(x) \in V\} \subseteq X$ is open in X.

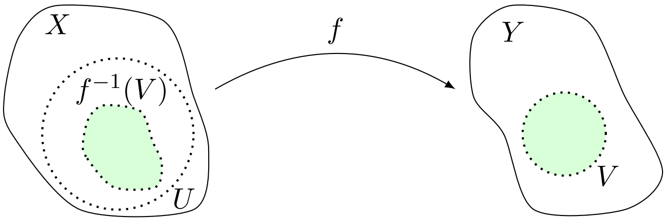


Figure 1: Add caption to this figure

Definition 1.7. A set $F \subseteq X$ is closed if $X \backslash F \subseteq X$ is open.

Proposition 1.2. A map $f : X \longrightarrow Y$ is continuous iff $f^{-1}(F) \subseteq X$ is closed in X, for every closed $F \subseteq Y$ in Y.

Definition 1.8. A continuous map $f : X \longrightarrow Y$ is called a homeomorphism iff it has a continuous inverse $g : Y \longrightarrow X$, such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. We say that X and Y are homeomorphic. We write $g = f^{-1}$ and $X \cong Y$.

1.2 Differential Manifolds

Definition 1.9. A topological manifold of dimension m is a second countable, Hausdorff topological space M, so that any $p \in M$ has a open neighborhood $x \ni U \subseteq M$, which is homeomorphic with $\mathbb{R}^m$. For convenience, we will sometimes write $M^m \equiv M$ such that $\dim(M) = m$.

Remark. We might as well have said homeomorphic with an open subset of $\mathbb{R}^m$, because $B(0, \epsilon) = \{x \in \mathbb{R}^m \mid \|x\| < \epsilon\}$. To see that we can use the function

$$\begin{aligned} f : B(0,\epsilon) &\longrightarrow \mathbb{R}^m \\ x &\longmapsto \frac{x}{\epsilon - \|x\|} \end{aligned}$$

Remark. The *Theorem of Invariance of Domain* states that if exists $\phi : \mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^n$, then $m = n$. Hence, the definition is consistent.

Definition 1.10. A chart on a topological manifold M^m is a tuple (U, ϕ), where $U \subseteq M$ is a open subset in M and ϕ is a homeomorphism from U to $V = \phi(U) \subseteq \mathbb{R}^m$ open in $\mathbb{R}^m$. If $0 \in V$, we say that the chart is centered on p if $\phi(0) = p$.

- (U, ϕ) is sometimes referred to as a coordinate patch.
- $(x_1(p), \dots, x_m(p)) \equiv \phi(p)$ are called local coordinates of the point p for (U, ϕ).
- ϕ^{-1} is also a homeomorphism and it is called a parametrisation.

We now are going to define the differential structure of the manifold.

Definition 1.11. A $\mathcal{C}^0$-atlas for M is a collection $\mathcal{A} = \{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ of charts so that $\bigcup_{\alpha \in A} U_\alpha = M$.

Definition 1.12. A $\mathcal{C}^k$-atlas for M is a $\mathcal{C}^0$-atlas such that

$$\theta_{\beta\alpha} \equiv \phi_\beta \circ (\phi_\alpha|_{U_{\alpha\beta}})^{-1} \, : \, \phi_\alpha(U_{\alpha\beta}) \longrightarrow \phi_\beta(U_{\alpha\beta}) \ ,$$

are differentiable of class $\mathcal{C}^k$ for all $\alpha, \beta \in A$, where $U_{\alpha\beta} \equiv U_\alpha \cap U_\beta$.

Remark.

- The mappings $\theta_{\alpha\beta}$ are called transition maps (mathematics) or coordinate transformations (physics).
- Note that $U_{\alpha\beta}$ is open in U_α and U_β, so both $\phi_\alpha(U_{\alpha\beta})$ and $\phi_\beta(U_{\alpha\beta})$ are open in $\phi_\alpha(U_\alpha)$ and $\phi_\alpha(U_\alpha)$, respectively. Therefore $\phi_\alpha(U_{\alpha\beta})$ and $\phi_\beta(U_{\alpha\beta})$ are homeomorphic open sets in $\mathbb{R}^m$.

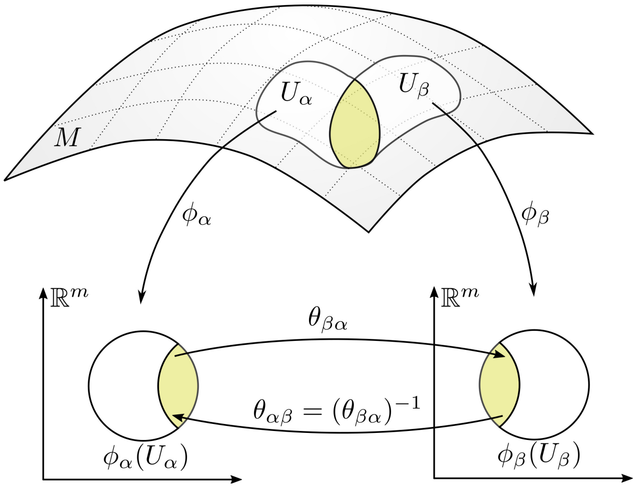


Figure 2: Figure describing homeomorphism between open sets in M and in $\mathbb{R}^m$, with transition maps $\theta_{\alpha\beta}$.

Definition 1.13 (Provisional)**.** A differential manifold of class $\mathcal{C}^k$ is a topological manifold M together with a $\mathcal{C}^k$ atlas.

Remark (Smoothness). A smooth atlas is one where the transitions functions are of the class $\mathcal{C}^\infty$, i.e. infinitely differentiable.

Example 1.3.

(1) $M = \mathbb{R}^m$ is a differential manifold, with the trivial smooth $\mathcal{C}^\infty$-atlas $\mathcal{A} = \{(\mathbb{R}^m, \mathrm{id})\}$, where $\mathrm{id}(x) = x$. Note that $\mathbb{R}^m$ is open in $\mathbb{R}^m$.

(2) Let $M = S^m \equiv \{x \in \mathbb{R}^{m+1} \,|\, \|x\| = 1\}$.

We write $x = (x_0, x_1, \ldots, x_m)$. The stereographic projections from $N = (1, 0, \ldots, 0)$ and $S = (-1, 0, \ldots, 0)$ are given from the following charts

$$\begin{aligned} \phi_N : U_N \equiv S^m \backslash \{N\} &\longrightarrow R^m \\ (x_0, \ldots, x_m) &\longmapsto \frac{1}{1 - x_0}(x_1, \ldots, x_m) \end{aligned} \qquad \begin{aligned} \phi_S : U_S \equiv S^m \backslash \{S\} &\longrightarrow R^m \\ (x_0, \ldots, x_m) &\longmapsto \frac{1}{1 + x_0}(x_1, \ldots, x_m) \end{aligned}$$

Figure 3: Insert figure about describing the stereographic projection

Fact. $\|\phi_N(p)\|\,\|\phi_S(p)\| = 1$, for all $p \in U_N \cap U_S$.

$$\Longrightarrow \ \theta_{NS}(y) = \phi_N \circ (\phi_S|_{U_{NS}})^{-1}(y) = \frac{(y_1, \ldots, y_m)}{(y_1)^2 + \cdots + (y_m)^2} = \frac{1}{\|y\|^2}\, y$$

Since this transition map is a map[1] from $\mathbb{R}^m \backslash \{0\}$ to $\mathbb{R}^m \backslash \{0\}$ of class C^∞, then S^m is a smooth m-dimensional manifold.

Note. The topology on S^m is the subspace topology, i.e. $U \subset S^m$ is defined to be open if it is of the form $U = S^m \cap V$ for some open $V \subseteq \mathbb{R}^m$.

Definition 1.14. Two charts (ϕ_α, U_α) and (ϕ_β, U_β) are said to be $\mathcal{C}^k$-compatible if $\theta_{\beta\alpha} \equiv \phi_\beta \circ (\phi_\alpha|_{U_{\alpha\beta}})^{-1}$ and $\theta_{\alpha\beta} = (\theta_{\beta\alpha})^{-1}$ are differentiable of class $\mathcal{C}^k$. Note if $U \cap V = \emptyset$, we still say the transition map is smooth.

Definition 1.15. A $\mathcal{C}^k$-atlas is maximal if it contains all $\mathcal{C}^k$-compatible charts.

Lemma 1.1. Every $\mathcal{C}^k$-atlas on a topological manifold is contained in a unique maximal $\mathcal{C}^k$-atlas.

Definition 1.16. A differentiable manifold of dimension m and of class $\mathcal{C}^k$ is topological manifold M^m with a maximal $\mathcal{C}^k$-atlas. This atlas is sometimes refer to as a differential structure.

Example 1.4.

(1) If M^m is a $\mathcal{C}^k$-manifold of dimension m, then any open $U \subseteq M^m$ is also a $\mathcal{C}^k$-manifold of dimension m.

This is clear since for any open U we can define charts

$$\phi_\alpha|_{U_\alpha \cap U} : U_\alpha \cap U \longrightarrow \phi_\alpha(U_\alpha \cap U) \subseteq \mathbb{R}^m$$

for any chart $\phi_\alpha : U_\alpha \longrightarrow \phi_\alpha(U_\alpha) \subseteq \mathbb{R}^m$ on M^m.

(2) If $U \subseteq \mathbb{R}^m$ is open and $g : U \longrightarrow \mathbb{R}$ is of class $\mathcal{C}^k$, then the graph of g

$$G = \{(x, g(x)) \,|\, x \in U\} \subseteq \mathbb{R}^{m+1}$$

is also a m-dimensional manifold.

Use the single chart $\phi : G \longrightarrow U$, $(x, y) \longmapsto x$, where $(x, y) \in U \times \mathbb{R}$. Note that $\phi^{-1}(x) = (x, g(x))$.

(3) *Level hypersufaces* $f(p) = 0$

$$f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R} \qquad p = (x_1, \cdots, x_n, z) \longmapsto f(p)$$

If $\frac{\partial z}{\partial f}(p_0) \neq 0$, for $p_0 \in f^{-1}(0) = \{p \in \mathbb{R}^{n+1} \,|\, f(p) = 0\}$, then exists an open neighbourhood of p_0, $V \subseteq \mathbb{R}^{n+1}$, so that

$$f^{-1}(0) \cap V = \{(x, g(x)) \,|\, x \in U\} \equiv U \times g(U)$$

[1] This is called the inversion map and is actually one-to-one in the m-dimensional punctured Euclidean space $\mathbb{R}^m \backslash \{0\}$. It maps the region $0 < \|y\| < 1$ to the region $\|y\| > 1$ and vice-versa, while leaving the unit sphere unchanged.

for a unique $\mathcal{C}^k$-function $g : U \longrightarrow \mathbb{R}$, for some open $U \subseteq \mathbb{R}^n$. Hence if $\nabla f(p) \neq 0 \ \ \forall p \in f^{-1}(0)$, then $f^{-1}(0)$ will be a is a $\mathcal{C}^k$-manifold of dimension n.

Figure 4: Insert figure about graph of a function

1.3 Differential Maps

Definition 1.17. Let U be open in $\mathbb{R}^n$. A function $f : U \longrightarrow \mathbb{R}^m$ is differentiable at $x \in U$ if there is a linear map $D_x f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ so that

$$f(x+h) - f(x) = D_x f(h) + o(h) \quad ,$$

where $\lim_{h \to 0} \frac{o(h)}{\|h\|} = 0$. We may write $D_x f(h) \equiv D_x f \cdot h$ to emphasize the linearity. A function $f : U \longrightarrow \mathbb{R}^m$ is a differentiable iff it is differentiable in all $x \in U$.

Definition 1.18. Let M^m be a smooth manifold. A function $f : M \longrightarrow \mathbb{R}$ is differentiable at $p \in M$ if for some chart (U, ϕ), with $p \in U$, the function

$$f \circ \phi^{-1} : \phi(U) \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}$$

is differentiable at $x = \phi(p)$. A function $f : M \longrightarrow \mathbb{R}$ is a differentiable iff it is differentiable in all $p \in M$.

Definition 1.19. A continuous map $f : M^m \longrightarrow N^n$ is differentiable at $p \in M$ if for some charts (U, ϕ), with $p \in U \subseteq M$, and (V, ψ), with $f(p) \in V \subseteq N$, the function in local coordinates

$$\psi \circ f \circ \phi^{-1} : \phi(U) \subseteq \mathbb{R}^m \longrightarrow \psi(V) \subseteq \mathbb{R}^n$$

is differentiable at $x = \phi(p)$.

Note. f continuous means that we can take (U, ϕ) so that $f(U) \subseteq V$.

Figure 5: Insert figure about describing differentiability using charts

Definition 1.20. A diffeomorphism is a differentiable map $f : M^m \longrightarrow N^n$ which has a differentiable inverse $g = f^{-1} : N^n \longrightarrow M^m$. We say that M and N are diffeomorphic.

Example 1.5.

(1) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto x$ is a trivial diffeomorphism

(2) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto x^3$ is a homeomorphism but not a diffeomorsphim because $f^{-1}(y) = \sqrt[3]{y}$ is no differentiable at $y = 0$.

Notation.

- $\mathcal{C}^k(M, N) = \{f : M \longrightarrow N \mid f \text{ is differentiable of class } \mathcal{C}^k\}$

- $\mathcal{C}^k(M) = \mathcal{C}^k(M, \mathbb{R})$

Example 1.6. The General Linear group manifold $GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$

It is clear that $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ is a smooth manifold since the multiplication map is smooth. Now to check $GL(n, \mathbb{R})$ not that $\det : M_n(\mathbb{R}) \longrightarrow R$ is a continuous map and we can write

$$GL(n, \mathbb{R}) = \det{}^{-1}\big((-\infty, 0)\big) \bigsqcup \det{}^{-1}\big((0, +\infty)\big) \ .$$

The fact that preimage open set will also be open, then $GL(n, \mathbb{R})$ is also open in $M_n(\mathbb{R})$. Therefore $GL(n, \mathbb{R})$ is a smooth manifold with two disconnected components.

1.4 Submanifolds and the Submersion Theorem

Theorem 1.1 (Inverse Function)**.** Let $f : U \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be smooth and $x \in U$ so that $D_x f : \mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^n$ is an isomorphism. Then $m = n$ and f is a *local diffeomorphism*, i.e., there exists an open neighbourhood $V \subseteq U$ so that

$$f|_V : V \longrightarrow f(V)$$

is a diffeomorphism onto the open set $f(V) \subseteq \mathbb{R}^n$.

$$\begin{tikzcd}
\mathbb{R}^n \supseteq V \arrow[r, "f"] \arrow[d, shift left, "\phi = f"] & f(V) \subseteq \mathbb{R}^n \\
\mathbb{R}^n \supseteq \phi(V) \arrow[u, shift left, "\phi^{-1}"] \arrow[ur, bend right, "f \circ \phi^{-1} = \mathrm{id}"']
\end{tikzcd}$$

Therefore, for small enough open set it is possible to pick the chart $\phi = f$ in which the expansion of f is the identity map.

Proof. Add an outline of the proof □

Definition 1.21. The rank of a smooth map $f : M^m \longrightarrow N^n$ at $p \in M$ is the rank of the linear map

$$D_x(\psi \circ f \circ \phi^{-1}) : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

for some chart (U, ϕ) of M such that $p \in U$ and some chart (V, ψ) of N such that $f(p) \in V$, with $x = \phi(p)$.

Remark. The rank of a linear map $L : V \longrightarrow W$ at $x \in V$ is given by $\dim \operatorname{im}(D_x L)$

Figure 6: Insert figure about describing ranks using charts

Definition 1.22. Map $f : M^m \longrightarrow N^n$ is a *submersion* if it is subjective and the rank of f for all $p \in M$ is n.

Notation. For the next result we introduce the standard symbols for inclusions of factors into products and projections from products to factors:

$$\iota_2 : \mathbb{R}^p \longrightarrow \mathbb{R}^m \times \mathbb{R}^p \ ; \quad y \longmapsto (0, y)$$
$$\pi_2 : \mathbb{R}^m \times \mathbb{R}^p \longrightarrow \mathbb{R}^p \ ; \quad (x, y) \longmapsto y$$

Corollary 1.1 (Submersion Theorem)**.** If $0 \in U \subseteq \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^p$ and if $f : U \longrightarrow \mathbb{R}^q$ is smooth with $Df(0) \circ \iota_2 \equiv D_2 f(0) : \mathbb{R}^p \longrightarrow \mathbb{R}^q$ a linear isomorphism, then $p = q$ and there is a local diffeomorphism ϕ around $0 \in \mathbb{R}^n$ so that $f \circ \phi = \pi_2 \Leftrightarrow f \circ \phi(x, y) = y$.

Proof. Let $\psi(x, y) = (x, f(x, y))$, where $(x, y) \in \mathbb{R}^m \times \mathbb{R}^p$. Then the matrix of $D\psi(0, 0)$ has the form

$$J\psi(0,0) = \begin{pmatrix} \mathrm{Id}_m & 0 \\ J_1 f(0,0) & J_2 f(0,0) \end{pmatrix} ,$$

where $J_i f(0, 0)$ is the Jacobian matrix of $D_i f(0, 0)$. Because $\det J_2 f(0, 0) \neq 0$, then $D\psi(0, 0)$ is an isomorphism and so, by theorem 1.1, ϕ is a local diffeomorphism at $(0, 0)$.

□

1.5 Compactness

1.6 Product and Quocient Manifolds

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References

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