Aspects of superradiant scattering off Kerr black holes

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UNIVERSIDADE DO PORTO

MASTER'S THESIS

Aspects of superradiant scattering off Kerr black holes

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Abstract

Faculdade de Ciências da Universidade do Porto

Departamento de Física e Astronomia

Master of Science

Aspects of superradiant scattering off Kerr black holes

by José SÁ

The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too…

UNIVERSIDADE DO PORTO

Resumo

Faculdade de Ciências da Universidade do Porto

Departamento de Física e Astronomia

Mestre de Ciência

Aspects of superradiant scattering off Kerr black holes

por José SÁ

Tradução em português do "Abstract" escrito em inglês mais a cima. A página é centrada vertical e horizontalmente, podendo espandir para o espaço superior da página em branco ...

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Abbreviations

BH	Black Hole
BL	Boyer-Linquist
GR	General Relativity
GW	Gravitational Wave
KR	Kerr-Newman
LIGO	Laser Interferometric Gravitational Wave Observatory
QM	Quantum Mechanics
RN	Reissner-Nordström
SWSH	Spin-Weighted Spheroidal Harmonic

Chapter 1

Superradiance

1.1 Introduction

The first direct observation of GWs by the Laser Interferometer Gravitational Wave Observatory (LIGO) was in 2015 and latter announced in 2016. The recorded event matched the signature predictions of GR for a binary system of BHs merging together in a inward spiral into a single BH [1]. These observations demonstrated not only the existence of GWs but also existence of binary stellar-mass BH systems and that these systems could merge in a time less than the known Universe age. Since then, two more similar events were detected, which assured the inauguration of a new era of GW cosmology.

Naturally, this sparked new interest in the study of binary systems and GW-related phenomena. One of these phenomena is the possibility of amplification in waves scattered off rotating and/or charged BHs, which can occur under certain conditions for scalar, electromagnetic and gravitational bosonic waves. Such effect is one of many that encompass a wide range of phenomena generally known as *superradiance*. **(How the study of superradiant scattering can be useful)**

As all bosonic waves can be reduced to the study of the same master equation (as we will see later), this work will focus primarily on electromagnetic waves in the case of a neutral rotating BH. Said choice is the most interesting from a astrophysical point of view, considering that any charged BH should be "quickly" neutralized by the surrounding interstellar plasma, due to the nature of EM interactions.

Historically, the first appearance of the concept of superradiance was in 1954, in a publication by Dicke [2]. He showed that a gas could be excited by a pulse into "superradiant states" from thermal equilibrium and then emit coherent radiation. Almost two decades later, Zel'dovich [3, 4] showed that a absorbing cylinder rotating with an angular velocity Ω could scatter an incident wave, $\psi \sim e^{-i\omega t + im\phi}$, with frequency ω if

$$\omega < m\Omega \tag{1.1}$$

would be satisfied, where m is the usual azimuthal number of the monochromatic plane wave relative to the rotation axis. In his work, he noticed that superradiance was related with dissipation of rotational energy from the absorbing object, possibly due to spontaneous pair creation at the surface. Hawking later showed that the presence of a strong electromagnetic or gravitational fields could indeed generate bosonic and fermionic pairs spontaneously. This result was possible by the efforts of Starobinsky and Deruelle [5–8], which also laid the groundwork necessary for the discovery of BH evaporation.

1.2 Klein paradox as a first example

Actually, radiation amplification can be traced to birth of Quantum Mechanics, in the beginnings of the 20th century. First studies of the Dirac equation by Klein [9] revealed the possibility of electrons propagating in a region with a sufficiently large potential barrier without the expected dampening from non-relativistic QM tunnel effect. Due to some confusion, this result was wrongly interpreted by some authors as fermionic superradiance, as if the reflected current by the barrier could be greater than the incident current. The problem was named *Klein paradox* by Sauter [10] and this misleading result was due to a incorrect calculation of the group velocities of the reflected and transmitted waves.

Today, it is known that fermionic currents cannot be amplified for this particular problem [9, 11], result that was correctly obtained by Klein in is original paper. On the contrary, superradiant scattering can indeed occur for bosonic fields.

1.2.1 Bosons

The equation that governs bosonic wave function is the Klein-Gordon equation, which for a minimally coupled electromagnetic potential takes the form

$$(D^\nu D_\nu - \mu^2)\Phi = 0\,, \tag{1.2}$$

where the usual partial derivative becomes $D_\nu = \partial_\nu + ieA_\nu$ and μ is the boson mass.

The problem is greatly simplified by considering flat space-time in (1+1)-dimensions and step potential $A(x) = V\,\theta(x)\ \mathrm{d}t$, for $V > 0$ constant and wave solutions $\Phi = e^{-i\omega t}\phi$. For $x < 0$, the solution can be divided as incident and reflected, taking the form

$$\phi_{\text{inc}}(x) = \mathcal{I}\, e^{ikx}\,, \qquad \phi_{\text{refl}}(x) = \mathcal{R}\, e^{-ikx}\,, \tag{1.3}$$

in which the dispersion relation states that $k = \sqrt{\omega^2 - \mu^2}$. For $x > 0$, the transmitted wave is naturally given by

$$\psi_{\text{inc}}(x) = \mathcal{T}\, e^{iqx}\,, \tag{1.4}$$

but in this case the root sign for the momentum must be carefully chosen so that the group velocity sign of the transmitted wave matches of the incoming wave [11], *i.e.*

$$\left.\frac{\partial\omega}{\partial p}\right|_{p=q} = \frac{q}{\omega - eV} > 0\,, \tag{1.5}$$

therefore we must have that

$$q = \operatorname{sgn}(\omega - eV)\sqrt{(\omega - eV)^2 - \mu^2}\,. \tag{1.6}$$

After obtaining the continuity relations at the barrier, $x = 0$, we follow by computing the ratios of the transmitted and reflected currents relative to the incident one, which yield

$$\frac{j_{\text{refl}}}{j_{\text{inc}}} = -\left|\frac{\mathcal{R}}{\mathcal{I}}\right|^2 = -\left|\frac{1-r}{1+r}\right|^2\,, \qquad \frac{j_{\text{trans}}}{j_{\text{inc}}} = \operatorname{Re}(r)\left|\frac{\mathcal{T}}{\mathcal{I}}\right|^2 = \frac{4\operatorname{Re}(r)}{|1+r|^2}\,, \tag{1.7}$$

written as a function of the coefficient

$$r = \frac{q}{k} = \operatorname{sgn}(\omega - eV)\sqrt{\frac{(\omega - eV)^2 - \mu^2}{\omega^2 - \mu^2}}\,. \tag{1.8}$$

Hence, in the case of strong potential limit, $eV > \omega + \mu > 2\mu$, we may have $r < 0$ real and the reflected current is larger (in magnitude) than the incident wave and therefore we have amplification.

1.2.2 Fermions

Dirac noticed the that Klein-Gordon equation masked internal degrees of freedom, so he devised is own equation which describe fermions. Considering that scalar potentials do not have any impact on spin orientation [12], we need only to consider half of the spinor components in Dirac equation

$$(i\gamma^\nu D_\nu - \mu)\Psi = 0\,, \tag{1.9}$$

where μ is the fermion mass, for which a valid representation of the gamma matrices is

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.10}$$

Probing wave solutions $\Psi = e^{-i\omega t}\psi$, the incident and reflected solutions are

$$\psi_{\rm inc}(x) = \mathcal{I}\,e^{ikx}\begin{pmatrix} 1 \\ \dfrac{k}{\omega+\mu} \end{pmatrix}, \qquad \psi_{\rm refl}(x) = \mathcal{R}\,e^{-ikx}\begin{pmatrix} 1 \\ \dfrac{-\,k}{\omega+\mu} \end{pmatrix}, \tag{1.11}$$

while for $x > 0$, the transmitted wave function is written as

$$\psi_{\rm trans}(x) = \mathcal{T}\,e^{iqx}\begin{pmatrix} 1 \\ \dfrac{q}{\omega - eV + \mu} \end{pmatrix}, \tag{1.12}$$

where was followed the same procedure as before, obtaining the same results from Eq. (1.5) through (1.7). As a result of the structure of the spinor components, the coefficient at Eq. (1.8) is modified to

$$r = \mathrm{sgn}(\omega - eV)\frac{\omega+\mu}{\omega - eV + \mu}\sqrt{\frac{(\omega - eV)^2 - \mu^2}{\omega^2 - \mu^2}}\,, \tag{1.13}$$

and now, in the same region, $\omega > \mu$, superradiance does not occur.

Even though superradiance and spontaneous pair creation are two distinct phenomena, this result is usually interpreted using the latter, from a QFT stand point. All incident particles are completely reflected, as well as some extra due to pair creation at the barrier as a result of stimulation by the incident radiation and the presence of a strong electromagnetic field, while the resultant anti-particles are transmitted in the opposite direction, accounting for the change of sign in the transmitted current in Eq. (1.7), owing to the opposite charge they carry. This also explains the undamped transmission part.

One may think that this difference between bosons and fermions arises from the potential barrier shape, but work by other authors [10, 11, 13] shows that only the difference between the asymptotic values of the potential at infinity is essential for the process. The difference comes from intrinsic properties of these particles. The amount of fermion pairs produced in a given state, *i.e.* for a given ω, is limited by Pauli exclusion principle, while such limitation does not occur for bosons [14]. Additionally, fermionic current densities are always positive definite, while bosons can change sign because of the ambiguity of wave function describing positive and negative energy solutions.

The minimum necessary energy for this to occur, 2μ, leaves evidence that superradiance is accompanied with spontaneous pair creation and some sort of dissipation by the battery maintaining the strong electromagnetic potential, in order to maintain energy balance.

1.3 Black hole superradiance

Differentiating Kerr mathematics vs. physics

Among many other cases of radiation amplification, the phenomena worked out throughout this work is an example of *rotational superradiance*. As the name suggests, it occurs in the presence of rotating objects, as is the famous example of Zel'dovich cylinder. In this case, the object in question is a Kerr black hole. This geometry is the simplest solution for a static but non-stationary BH, which breaks spherical symmetry.

Condition Eq. (1.1) was to become one of the most important results of rotational superradiance, as it presented itself in multiple examples, including in BH physics, particularly in the case of the Kerr solution.

Chapter 2

Mathematical preliminaries

2.1 General Relativity

General Relativity is the theory of space, time and gravitation developed by Einstein in 1915. It introduced a new viewpoint on gravity and it's relation with the fabric of space-time, a *manifold* that bounded our three spatial dimensions with dimension of time, which was a concept that challenged our deeply ingrained and intuitive notions of nature partially because the mathematical background need to understand the precise formulation of theory was unfamiliar to much of the Physics community at the time.

This formulation corresponds to a field theory which the main object of study is the metric of the manifold, $g = g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu$ and inherited diffeomorphism invariance, which was at the core of definition of differential manifolds. Firstly, the theory was left aside because of the numerous complicated coupled nonlinear equations, but the astronomical discovery of compact and highly energetic objects in the 1950s breaded new interest into the somewhat dormant GR, mainly because it was thought that these quasars and compact X-ray sources had suffered some form of gravitational collapse or that strong gravitational fields were present. Soon after, the modern theory of gravitational collapse was developed and the first solutions of BHs were discovered in the mid-1960s, including the Schwarzschild and Kerr BHs.

The theory of GR can be elegantly described in the form of the Hilbert action

$$S_H = \frac{1}{16\pi} \int \mathrm{d}^4x \sqrt{-g}\, R\,, \tag{2.1}$$

where $g = \det(g_{\mu\nu})$ and R corresponds to the Ricci scalar. Naturally, the first solutions corresponded to pure gravity, usually designated as vacuum solutions, which obey

$$R_{\mu\nu} = 0 \,. \tag{2.2}$$

Despite their simplicity, they enjoy some very fascinating nontrivial properties. One of which is the existence of an event horizon, a surface that separates two causally disconnected regions of spacetime.

Particularly, in this work we will also include electromagnetic (massless, neutral) wave interactions, which are described by the Maxwell action

$$S_{EM} = -\frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}\, F_{\mu\nu}F^{\mu\nu} \,, \tag{2.3}$$

where $F_{\mu\nu}$ is the Maxwell tensor. Variation of both actions, *i.e.* $\delta(S_H + S_{EM}) = 0$, give rise to two field equations

$$\nabla_\mu F^{\mu\nu} = 0 \,, \tag{2.4}$$

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = 8\pi T_{\mu\nu} \,. \tag{2.5}$$

The first equation is just the usual of Maxwell equation in curved spacetime. The second equation reflects the backreaction of the electromagnetic waves into the geometry through the presence of a energy momentum tensor

$$T_{\mu\nu} = F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}F^2 \,. \tag{2.6}$$

These equation completely describe the system, but they are coupled and nonlinear so will be specializing to perturbation theory, considering the field A^μ to be small. Because the stress-energy tensor is quadratic in the fields, $T_{\mu\nu} \sim \mathcal{O}(A^2)$, then we can ignore the backreaction and the field equations for the metric $g_{\mu\nu}$ reduce to Eq. (2.2). This is a very good approximation, since near black holes of stellar-mass proportions the gravitational field is considerably strong compared with radiation emitted by nearby sources. Therefor we need only to focus on the Maxwell equation in a static background. In order to to able to solve this equations we will resort to the Newman-Penrose formalism, which is suited to study any kind of radiation in curved spacetime.

2.2 Kerr black hole

(More on stationary and Kerr). Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

2.2.1 Spacetime symmetries

If we represent our spacetime by $(\mathcal{M}, g_{\mu\nu}, \psi)$, then the pullbash f^* of the diffeomorphism $f : \mathcal{M} \to \mathcal{M}$, would give us the same physical system $(\mathcal{M}, f^* g_{\mu\nu}, f^*\psi)$. Since diffeomorphisms are just active coordinate transformations, such concept may raise some confusion, as we don't seam to obtain no new information to work with. Almost all physics theories are coordinate invariant, as is Newtonian mechanics and Special Relativity, but in such theories there is a preferable coordinate system, while the same does not hold true for GR. An analogies can be made with the path integral formalism in QFT, where special consideration is taken when summing all field configurations in order to not overcount indistinguishable configurations, as is the case of gauge field theories. A similar ambiguity can occur in GR, where two apparently different solutions which can be related by a diffeomorphism and are actually "the same", so we must be careful when deriving and analyzing any geometries.

Despite the added complexity of Einstein's field equations, it is still possible to find exact nontrivial solutions in a systematic way by considering spacetimes with symmetries with the use of Killing vector fields. A vector field ξ that obeys

$$\mathcal{L}_\xi g = 0 \tag{2.7}$$

is called a Killing field. Locally, this expression reduces to $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$.

A *stationary* solution implies the existence of a Killing vector k that is asymptotically timelike, $k^2 < 0$, therefore allows us to normalize our vector such that $k^2 \to -1$. Unlike the case of the static spacetime, a stationary metric does not show invariance under reversal of the time coordinate, which is natural considering a system with angular momentum. Futhermore, a solution is also *axisymmetric*, due to the presence of a asymptotically spacelike Killing field m whose integral curves are closed. A solution is stationary and axisymmetric if both symmetries are present, along with commuting fields, $[k, m] = 0$, *i.e.* rotations along with the axis of symmetry commute with time translations. The commutativity of the fields implies the existence of a set of coordinates, (t, r, θ, ϕ), such that

$$k = \frac{\partial}{\partial t}\,, \qquad m = \frac{\partial}{\partial \phi}\,. \tag{2.8}$$

As for direct implication of this choice of chart, components of the metric stay independent of (t, ϕ), in virtue of Eq. (2.7),

$$(\mathcal{L}_m g)_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial \phi} = 0\,, \tag{2.9}$$

with the same holding true for k, hence we can write $g_{\mu\nu} = g_{\mu\nu}(r, \theta)$.

One of the major applications of Killing vectors is to find conserved charges associated with the motion along a geodesic spanned by field. These quantities are defined by taking the geodesics to regions space that are asymptotical flat, where the geometry does not affect the observer. In the case of Kerr solution, we have two Killing vectors, k and m, which are naturally associated with the total mass M and angular momentum J of the BH, respectively. This is usually done by evaluating the Komar integrals [15, 16], which can be written a covariant way as

$$M = -\frac{1}{8\pi}\int_{S^2_\infty} \star\, \mathrm{d}k^\flat \;=\; \frac{1}{4}\lim_{r\to\infty}\int_0^\pi \mathrm{d}\theta\,\sqrt{-g}\, g^{t\alpha} g^{r\beta} g_{t[\alpha,\beta]}\,, \tag{2.10}$$

$$J = \frac{1}{16\pi}\int_{S^2_\infty} \star\, \mathrm{d}m^\flat = -\frac{1}{8}\lim_{r\to\infty}\int_0^\pi \mathrm{d}\theta\,\sqrt{-g}\, g^{t\alpha} g^{r\beta} g_{\phi[\alpha,\beta]}\,, \tag{2.11}$$

where the usual notation $k^\flat = g(k, \cdot\,) = g_{\mu\nu}k^\mu\, \mathrm{d}x^\nu$ transforms a vector into a 1-form and $\star : \Omega^p(\mathcal{M}) \to \Omega^{n-k}(\mathcal{M})$ is the Hodge dual map. In order to complete the integration in the last step is assumed (2.8) and (2.9). According to the widely accepted of *no-hair conjecture* [17], these two quantities completely define a stationary (neutral) BH.

2.2.2 Metric and ring singularity

In little assumption, we were able to fix some conditions for the family stationary rotating BH solutions.

2.2.3 Ergoregion and the Penrose process

2.3 Newman-Penrose formalism

2.3.1 Kinnersly tetrad

2.3.2 Spin coefficients

2.3.3 Maxwell equations

Chapter 3

Teukolsky master equation

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Chapter 4

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4.1.1 Leaver method

4.1.2 Spectral

4.2 Radial ansatz

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Chapter 5

Scattering problem

5.1 Plane wave decomposition

Appendix A

Spin-weighted spherical harmonics

SWSHs play an important role BH physics and was first introduced by Teukolsky when considering non-scalar wave perturbations on a Kerr background, obtaining a separable master equation in four dimensions. After the usual change of coordinates, the polar differential equation goes as

$$\frac{1}{S}\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}S}{\mathrm{d}x}\right) + (cx)^2 - 2csx - \frac{(m+sx)^2}{1-x^2} + s = -\lambda \tag{A.1}$$

with $x = \cos\theta$, where λ is the eigenvalue for a given SWSH solution. Periodic boundary conditions on the azimuthal wave function constrains m to the integers.

A.1 Connection with spheroidal harmonics

By setting $s = 0$ (scalar) and $c = 0$ (spherical), then it's clear that Eq. (A.1) appears as a generalization of the spherical harmonics equation. In this last case, the solution are given by the associated Legendre polynomials, $P_\ell^m(x)$, for which the eigenvalue is $\ell(\ell+1)$, restricted to the condition of $|m| \leq \ell$. The closed form for spherical harmonics, after normalization, is

$$ {}_0Y_\ell^m(x) = (-1)^m \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(x) \tag{A.2}$$

where P_ℓ^m are the associated Legendre polynomials which can be obtained using the famous Rodrigues' formula.

A.2 Spin raising/lowering differential operators

A.3 Generalized addition of angular momentum formula

A.4 Some useful harmonics

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