Aspects of superradiant scattering off Kerr black holes

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UNIVERSIDADE DO PORTO

MASTER'S THESIS

Aspects of superradiant scattering off Kerr black holes

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Abstract

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Departamento de Física e Astronomia

Master of Science

Aspects of superradiant scattering off Kerr black holes

by José SÁ

The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too…

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Resumo

Faculdade de Ciências da Universidade do Porto

Departamento de Física e Astronomia

Mestre de Ciência

Aspects of superradiant scattering off Kerr black holes

por José SÁ

Tradução em português do "Abstract" escrito em inglês mais a cima. A página é centrada vertical e horizontalmente, podendo espandir para o espaço superior da página em branco . . .

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Abbreviations

BH	Black Hole
BL	Boyer-Linquist
EF	Eddington-Finkelstein
GR	General Relativity
GW	Gravitational Wave
KR	Kerr-Newman
LIGO	Laser Interferometric Gravitational Wave Observatory
NP	Newman-Penrose
QM	Quantum Mechanics
RN	Reissner-Nordström
SWSH	Spin-Weighted Spheroidal Harmonic

Chapter 1

Superradiance

1.1 Introduction

The first direct observation of GWs by the Laser Interferometer Gravitational Wave Observatory (LIGO) was in 2015 and latter announced in 2016. The recorded event matched the signature predictions of GR for a binary system of BHs merging together in a inward spiral into a single BH [1]. These observations demonstrated not only the existence of GWs but also existence of binary stellar-mass BH systems and that these systems could merge in a time less than the known Universe age. Since then, two more similar events were detected, which assured the inauguration of a new era of GW cosmology.

Naturally, this sparked new interest in the study of binary systems and GW-related phenomena. One of these phenomena is the possibility of amplification in waves scattered off rotating and/or charged BHs, which can occur under certain conditions for scalar, electromagnetic and gravitational bosonic waves. Such effect is one of many that encompass a wide range of phenomena generally known as *superradiance*. **(How the study of superradiant scattering can be useful)**

As all bosonic waves can be reduced to the study of the same master equation (as we will see later), this work will focus primarily on electromagnetic waves in the case of a neutral rotating BH. Said choice is the most interesting from a astrophysical point of view, considering that any charged BH should be "quickly" neutralized by the surrounding interstellar plasma, due to the nature of EM interactions.

Historically, the first appearance of the concept of superradiance was in 1954, in a publication by Dicke [2]. He showed that a gas could be excited by a pulse into "superradiant states" from thermal equilibrium and then emit coherent radiation. Almost two decades later, Zel'dovich [3, 4] showed that a absorbing cylinder rotating with an angular velocity Ω could scatter an incident wave, $\psi \sim e^{-i\omega t + im\phi}$, with frequency ω if

$$\omega < m\,\Omega \tag{1.1}$$

would be satisfied, where m is the usual azimuthal number of the monochromatic plane wave relative to the rotation axis. In his work, he noticed that superradiance was related with dissipation of rotational energy from the absorbing object, possibly due to spontaneous pair creation at the surface. Hawking later showed that the presence of a strong electromagnetic or gravitational fields could indeed generate bosonic and fermionic pairs spontaneously. This result was possible by the efforts of Starobinsky and Deruelle [5–8], which also laid the groundwork necessary for the discovery of BH evaporation.

1.2 Klein paradox as a first example

Actually, radiation amplification can be traced to birth of Quantum Mechanics, in the beginnings of the 20th century. First studies of the Dirac equation by Klein [9] revealed the possibility of electrons propagating in a region with a sufficiently large potential barrier without the expected dampening from non-relativistic QM tunnel effect. Due to some confusion, this result was wrongly interpreted by some authors as fermionic superradiance, as if the reflected current by the barrier could be greater than the incident current. The problem was named *Klein paradox* by Sauter [10] and this misleading result was due to a incorrect calculation of the group velocities of the reflected and transmitted waves.

Today, it is known that fermionic currents cannot be amplified for this particular problem [9, 11], result that was correctly obtained by Klein in is original paper. On the contrary, superradiant scattering can indeed occur for bosonic fields.

1.2.1 Bosons

The equation that governs bosonic wave function is the Klein-Gordon equation, which for a minimally coupled electromagnetic potential takes the form

$$(D^{\nu}D_{\nu} - \mu^2)\Phi = 0\,, \tag{1.2}$$

where the usual partial derivative becomes $D_{\nu} = \partial_{\nu} + ieA_{\nu}$ and μ is the boson mass.

The problem is greatly simplified by considering flat space-time in (1+1)-dimensions and step potential $A(x) = V\,\theta(x)\ \mathrm{d}t$, for $V > 0$ constant and wave solutions $\Phi = e^{-i\omega t}\phi$. For $x < 0$, the solution can be divided as incident and reflected, taking the form

$$\phi_{\text{inc}}(x) = \mathcal{I}\,e^{ikx}\,, \qquad \phi_{\text{refl}}(x) = \mathcal{R}\,e^{-ikx}\,, \tag{1.3}$$

in which the dispersion relation states that $k = \sqrt{\omega^2 - \mu^2}$. For $x > 0$, the transmitted wave is naturally given by

$$\psi_{\text{inc}}(x) = \mathcal{T}\,e^{iqx}\,, \tag{1.4}$$

but in this case the root sign for the momentum must be carefully chosen so that the group velocity sign of the transmitted wave matches of the incoming wave [11], *i.e.*

$$\left.\frac{\partial\omega}{\partial p}\right|_{p=q} = \frac{q}{\omega - eV} > 0\,, \tag{1.5}$$

therefore we must have that

$$q = \operatorname{sgn}(\omega - eV)\sqrt{(\omega - eV)^2 - \mu^2}\,. \tag{1.6}$$

After obtaining the continuity relations at the barrier, $x = 0$, we follow by computing the ratios of the transmitted and reflected currents relative to the incident one, which yield

$$\frac{j_{\text{refl}}}{j_{\text{inc}}} = -\left|\frac{\mathcal{R}}{\mathcal{I}}\right|^2 = -\left|\frac{1-r}{1+r}\right|^2\,, \qquad \frac{j_{\text{trans}}}{j_{\text{inc}}} = \operatorname{Re}(r)\left|\frac{\mathcal{T}}{\mathcal{I}}\right|^2 = \frac{4\operatorname{Re}(r)}{|1+r|^2}\,, \tag{1.7}$$

written as a function of the coefficient

$$r = \frac{q}{k} = \operatorname{sgn}(\omega - eV)\sqrt{\frac{(\omega - eV)^2 - \mu^2}{\omega^2 - \mu^2}}\,. \tag{1.8}$$

Hence, in the case of strong potential limit, $eV > \omega + \mu > 2\mu$, we may have $r < 0$ real and the reflected current is larger (in magnitude) than the incident wave and therefore we have amplification.

1.2.2 Fermions

Dirac noticed the that Klein-Gordon equation masked internal degrees of freedom, so he devised is own equation which describe fermions. Considering that scalar potentials do not have any impact on spin orientation [12], we need only to consider half of the spinor components in Dirac equation

$$(i\gamma^{\nu} D_{\nu} - \mu)\Psi = 0\,, \tag{1.9}$$

where μ is the fermion mass, for which a valid representation of the gamma matrices is

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.10}$$

Probing wave solutions $\Psi = e^{-i\omega t}\psi$, the incident and reflected solutions are

$$\psi_{\text{inc}}(x) = \mathcal{I}\, e^{ikx} \begin{pmatrix} 1 \\ \dfrac{k}{\omega+\mu} \end{pmatrix}, \qquad \psi_{\text{refl}}(x) = \mathcal{R}\, e^{-ikx} \begin{pmatrix} 1 \\ \dfrac{-k}{\omega+\mu} \end{pmatrix}, \tag{1.11}$$

while for $x > 0$, the transmitted wave function is written as

$$\psi_{\text{trans}}(x) = \mathcal{T}\, e^{iqx} \begin{pmatrix} 1 \\ \dfrac{q}{\omega - eV + \mu} \end{pmatrix}, \tag{1.12}$$

where was followed the same procedure as before, obtaining the same results from Eq. (1.5) through (1.7). As a result of the structure of the spinor components, the coefficient at Eq. (1.8) is modified to

$$r = \text{sgn}(\omega - eV)\frac{\omega+\mu}{\omega - eV + \mu}\sqrt{\frac{(\omega - eV)^2 - \mu^2}{\omega^2 - \mu^2}}\,, \tag{1.13}$$

and now, in the same region, $\omega > \mu$, superradiance does not occur.

Even though superradiance and spontaneous pair creation are two distinct phenomena, this result is usually interpreted using the latter, from a QFT stand point. All incident particles are completely reflected, as well as some extra due to pair creation at the barrier as a result of stimulation by the incident radiation and the presence of a strong electromagnetic field, while the resultant anti-particles are transmitted in the opposite direction, accounting for the change of sign in the transmitted current in Eq. (1.7), owing to the opposite charge they carry. This also explains the undamped transmission part.

One may think that this difference between bosons and fermions arises from the potential barrier shape, but work by other authors [10, 11, 13] shows that only the difference between the asymptotic values of the potential at infinity is essential for the process. The difference comes from intrinsic properties of these particles. The amount of fermion pairs produced in a given state, *i.e.* for a given ω, is limited by Pauli exclusion principle, while such limitation does not occur for bosons [14]. Additionally, fermionic current densities are always positive definite, while bosons can change sign because of the ambiguity of wave function describing positive and negative energy solutions.

The minimum necessary energy for this to occur, 2μ, leaves evidence that superradiance is accompanied with spontaneous pair creation and some sort of dissipation by the battery maintaining the strong electromagnetic potential, in order to maintain energy balance.

1.3 Black hole superradiance

Needs completion

Among many other cases of radiation amplification, the phenomena worked out throughout this work is an example of *rotational superradiance*. As the name suggests, it occurs in the presence of rotating objects, as is the famous example of Zel'dovich cylinder. In this case, the object in question is a Kerr black hole. This geometry is the simplest solution for a static but non-stationary BH, which breaks spherical symmetry.

Condition Eq. (1.1) was to become one of the most important results of rotational superradiance, as it presented itself in multiple examples, including in BH physics, particularly in the case of the Kerr solution.

Chapter 2

Mathematical preliminaries

2.1 General Relativity

General Relativity is the theory of space, time and gravitation developed by Einstein in 1915. It introduced a new viewpoint on gravity and it's relation with the fabric of space-time, a *manifold* that bounded our three spatial dimensions with dimension of time, which was a concept that challenged our deeply ingrained and intuitive notions of nature, partially because the mathematical background needed to understand the precise formulation of theory was unfamiliar to much of the Physics community at the time.

This formulation corresponds to a field theory with the dynamical object of study being the metric of the manifold, $ds^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$, and inherits diffeomorphism invariance, *i.e.* remains the same theory by a active change of coordinates, which was at the core of definition of differential manifolds.

Immediately after, in 1916, Schwarzschild found the first solution. Then, the theory was left aside because of the numerous coupled nonlinear equations, but the astronomical discovery of compact and highly energetic objects in the 1950s breaded new interest into the somewhat dormant GR, mainly because it was thought that these quasars and compact X-ray sources had suffered some form of gravitational collapse or that strong gravitational fields were present. Soon after, the modern theory of gravitational collapse was developed in the mid-1960s, including other BHs solutions, including Kerr's.

The theory of GR can be elegantly described in the form of the Hilbert action

$$S_H = \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\,R\,, \tag{2.1}$$

where $g = \det(g_{\mu\nu})$ and R corresponds to the Ricci scalar. Naturally, the first solutions corresponded to pure gravity, usually designated as vacuum solutions, which obey

$$R_{\mu\nu} = 0 \, . \tag{2.2}$$

Despite their simplicity, they enjoy some very fascinating nontrivial properties. One of which is the existence of an event horizon, a surface that separates two causally disconnected regions of spacetime.

The underlying techniques behind the study of superradiance is the linearization of Einstein and/or Maxwell equations around know BHs in stationary equilibrium. These perturbation will obey a series of partial equations whose dynamical variables are components of the Weyl tensor, $C_{\mu\nu\rho\sigma}$, or the Maxwell field tensor. Thanks to the Newman-Penrose (NP) formalism we will be able to decouple and separate the equations for both GWs and EM waves, revealing decoupled variables which contain all the information needed about the nontrivial perturbations, instead of working with all components of the field tensors.

For the gravitational case, a straightforward way of obtaining a linearized theory is to consider a background stationary BH solution, ${g_{\mu\nu}}^B$, and then expanding the field equations (2.2) using the metric $\tilde{g}_{\mu\nu} = {g_{\mu\nu}}^B + h_{\mu\nu}$, keeping only terms with are $\mathcal{O}(h_{\mu\nu})$, which leads to a wave equation in the particular background chosen.

Particularly, in this work we will focus on (massless, neutral) electromagnetic waves and perturbations are performed including EM interactions through the Maxwell action

$$S_{EM} = -\frac{1}{4} \int \mathrm{d}^4x \sqrt{-g}\, F_{\alpha\beta} F^{\alpha\beta} \, , \tag{2.3}$$

where $F_{\mu\nu}$ is the Maxwell tensor. Variation of both actions, $\delta(S_H + S_{EM}) = 0$, result in two field equations

$$\nabla_\mu F^{\mu\nu} = 0 \, , \tag{2.4}$$

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = 8\pi T_{\mu\nu} \, . \tag{2.5}$$

The first equation is just the usual of Maxwell equation in curved spacetime. The latter is the Einstein field equation, which reflects the backreaction of the electromagnetic waves into the geometry through the presence of EM stress-energy tensor

$$T_{\mu\nu} = F_{\mu\alpha} {F_\nu}^\alpha - \frac{1}{4} g_{\mu\nu} F^2 \, . \tag{2.6}$$

These equation completely describe the system, but the problem is analytically untreatable, so will be resorting to perturbation theory, considering the field A^μ to be small. This is a very good approximation, since near the gravitational field of stellar-mass BHs is considerably strong compared with radiation emitted by nearby astrophysical sources. As the stress-energy tensor is quadratic in the fields, $T_{\mu\nu} \sim \mathcal{O}(A^2)$, then we can ignore the backreaction and the field equations for the metric $g_{\mu\nu}$ reduce to Eq. (2.2).

2.2 Kerr black hole

It was generally accepted that a perfectly spherical symmetrical star would collapse to a Schwarzschild BH. Although, at the time it was not known the effect of a slightest amount o angular momentum on a gravitational collapse of a star. Finding a metric with intrinsic rotation could give insight to such problem. Due to the lack of spherical symmetry, the problem became much harder, and took roughly 50 years after Schwarzschild's discovery to find a metric for a rotating body. Imposing symmetries to the final metric were essential to solve the field equation.

2.2.1 Spacetime symmetries

If we represent our spacetime by $(\mathcal{M}, g_{\mu\nu}, \psi)$, then the pullbash f^* of the diffeomorphism $f : \mathcal{M} \to \mathcal{M}$, would give us the same physical system $(\mathcal{M}, f^* g_{\mu\nu}, f^* \psi)$. Since diffeomorphisms are just active coordinate transformations, such concept may raise some confusion, as we don't seam to obtain no new information to work with. Almost all physics theories are coordinate invariant, as is Newtonian mechanics and Special Relativity, but in such theories there is a preferable coordinate system, while the same does not hold true for GR. An analogies can be made with the path integral formalism in QFT, where special consideration is taken when summing all field configurations in order to not overcount indistinguishable configurations, as is the case of gauge field theories. A similar ambiguity can occur in GR, where two apparently different solutions which can be related by a diffeomorphism and are actually "the same", so we must be careful when deriving and analyzing any geometries.

Despite the added complexity of Einstein's field equations, it is still possible to find exact nontrivial solutions in a systematic way by considering spacetimes with symmetries with the use of Killing vector fields. A vector field ξ that obeys

$$\mathcal{L}_\xi g = 0 \tag{2.7}$$

is called a Killing field. Locally, this expression reduces to $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$.

A *stationary* solution implies the existence of a Killing vector k that is asymptotically timelike, $k^2 < 0$, therefore allows us to normalize our vector such that $k^2 \to -1$. Unlike the case of the static spacetime, a stationary metric does not show invariance under reversal of the time coordinate, which is natural considering a system with angular momentum. Futhermore, a solution is also *axisymmetric*, due to the presence of a asymptotically spacelike Killing field m whose integral curves are closed. A solution is stationary and axisymmetric if both symmetries are present, along with commuting fields, $[k, m] = 0$, *i.e.* rotations along with the axis of symmetry commute with time translations. The commutativity of the fields implies the existence of a set of coordinates, (t, r, θ, ϕ), such that

$$k = \frac{\partial}{\partial t}\,, \qquad m = \frac{\partial}{\partial \phi}\,. \tag{2.8}$$

As for direct implication of this choice of chart, components of the metric stay independent of (t, ϕ), in virtue of Eq. (2.7),

$$(\mathcal{L}_m g)_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial \phi} = 0\,, \tag{2.9}$$

with the same holding true for k, hence we can write $g_{\alpha\beta} = g_{\alpha\beta}(r, \theta)$.

One of the major applications of Killing vectors is to find conserved charges associated with the motion along a geodesic spanned by field. These quantities are defined by taking the geodesics to regions space that are asymptotical flat, where the geometry does not affect the observer. In the case of Kerr solution, we have two Killing vectors, k and m, which are naturally associated with the total mass M and angular momentum J of the BH, respectively. This is usually done by evaluating the Komar integrals [15, 16], which can be written a covariant way as

$$M = -\frac{1}{8\pi}\int_{S^2_\infty} \star\, \mathrm{d}k^\flat \;=\; \frac{1}{4}\lim_{r\to\infty}\int_0^\pi \mathrm{d}\theta \sqrt{-g}\, g^{t\alpha} g^{r\beta} g_{t[\alpha,\beta]}\,, \tag{2.10}$$

$$J = \frac{1}{16\pi}\int_{S^2_\infty} \star\, \mathrm{d}m^\flat = -\frac{1}{8}\lim_{r\to\infty}\int_0^\pi \mathrm{d}\theta \sqrt{-g}\, g^{t\alpha} g^{r\beta} g_{\phi[\alpha,\beta]}\,, \tag{2.11}$$

where the usual notation $k^\flat = g(k, \cdot\,) = g_{\mu\nu} k^\mu \, \mathrm{d}x^\nu$ transforms a vector into a one-form, and the operator $\star : \Omega^p(\mathcal{M}) \to \Omega^{4-p}(\mathcal{M})$ is the Hodge dual map for p-forms. In order to complete the integration in the last step is assumed (2.8) and (2.9), keeping (t, r) constant. According to the widely accepted of *no-hair conjecture* [17], these two quantities completely define a stationary (neutral) BH.

2.2.2 Kerr-Child coordinates

Naturally, Kerr wasn't the only after such solution. Many presented other metrics to approximately describe a rotating star. Most of the solutions were modified one-parameter modification to Schwarzschild that were not flat in case of the standard case. Simply using stationary and axisymmetric symmetries and then solving the Einstein's equations clearly wouldn't suffice.

Kerr success originated in of Petrov's classification of spacetimes, which used the algebraic properties of the Weyl tensor to distinguish the solutions in 3 types, along with some subcases. He assumed that his solution would have the same classification as Schwarzschild's, which associated with the gravitational fields of isolated massive objects, such as stars and BHs. From this assumption, using GR spinor techniques in Newman-Penrose formalism, a only then imposing the Killing vectors in Eq. (2.8), was possible to find a new solution. Kerr's metric appear in his original paper in the form

$$\begin{aligned} ds^2 = &- \left(1 - \frac{2Mr}{\rho^2}\right)(\,\mathrm{d}v - a\sin^2\theta\,\mathrm{d}\chi)^2 \\ &+ 2(\,\mathrm{d}v - a\sin^2\theta\,\mathrm{d}\chi)(\,\mathrm{d}r - a\sin^2\theta\,\mathrm{d}\chi) \\ &+ \rho^2(\,\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\chi^2)\,, \end{aligned} \tag{2.12}$$

where a is a parameter, M is the Komar mass and $\rho^2 = r^2 + a^2\cos^2\theta$. Naturally the time Killing vector is ∂_v and ∂_χ is the axial field, which implies that $J = aM$.

Taking the limit of $a \to 0$, we reduce the metric to the Schwarzschild solution in ingoing Eddington-Finkelstein coordinates, (v, r, θ, χ), which are useful to study ingoing (to the horizon) geodesics and remove the horizon coordinate singularity. If a given metric has singularities, then it is not trivial to identify if is a physical singularity or just an artifact resultant of choice of the chart, which can simply be removed by a better choice of coordinates. That being said, this raises the difficulty of finding the essential singularities. The best way to look to these singularities is to compute curvature scalar quantities, and if they diverge in one chart then they diverge on all charts. Since any BH is just a vacuum solution, then the Ricci scalar vanishes, $R = 0$, so we resort to the Kretschmann scalar,

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{48M(r^2 - a^2\cos^2\theta)\left[(r^2 - a^2\cos^2\theta)^2 - 16r^2M^2a^2\cos^2\theta\right]}{(r^2 + a^2\cos^2\theta)^6}\,, \tag{2.13}$$

that diverges for $\rho^2 = 0$. The Schwarzschild singularity, $r = 0$, is replaced with the Kerr singularity $(r, \theta) = (0, \pi/2)$. It is not clear what is the geometry of the Kerr singularity if we interpret r and θ as being part of the ordinary spherical coordinates. Although the metric is singular, considering (t, r, θ) constant and then the limit of $r \to 0$ through the equatorial plane, the metric

$$ds^2|_{\text{singularity}} \sim a^2\,\mathrm{d}\chi^2 \tag{2.14}$$

is reduced to the line element of the circle, S^1, confirming a *ring singularity* of radius a. This result implies that only approaching the a Kerr BH through the equatorial plane we may reach the singularity $\rho^2 = 0$.

The Kerr-Child "cartesian" form,

$$\begin{aligned} ds^2 = &- \mathrm{d}\tilde{t}^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ &+ \frac{2Mr^3}{r^4 + a^2z^2}\left[\mathrm{d}\tilde{t} + \frac{r(x\,\mathrm{d}x + y\,\mathrm{d}y) + a(y\,\mathrm{d}x - x\,\mathrm{d}y)}{r^2 + a^2} + \frac{z}{r}\,\mathrm{d}z\right]^2\,, \end{aligned} \tag{2.15}$$

is useful to really observe the ring singularity. In this metric, r is no longer a coordinate but a function of this chart coordinates $(\tilde{t}, x, y, z)$. We can relate the The Kerr-Child metric to the original Kerr solution, using

$$\tilde{t} = v - r\,, \qquad x + iy = (r - ia)e^{i\chi}\sin\theta\,, \qquad z = r\cos\theta\,, \tag{2.16}$$

which implies that $r(x, y, z)$ is implicitly given by

$$r^4 - (x^2 + y^2 + z^2 - a^2)r^2 - a^2z^2 = 0\,. \tag{2.17}$$

This condition deserves a more in-depth analysis. For increasing r, the surfaces obeying Eq. (2.17) approximates perfect spheres as the geometry get more and more flat, as is observed in (2.15). Minkowsky flat space is also guaranteed for $M = 0$. On the other hand, as we approach the singularity on $z = 0$ and $x^2 + y^2 = a^2$, rotation effects deform the surfaces into oblate spheroids (for the strict inequality, the singularity is removable). Such remarks are visually demonstrated in Figure 2.1.

Even thought both metrics $r > 0$, there is no mathematical reason to restrict r strictly

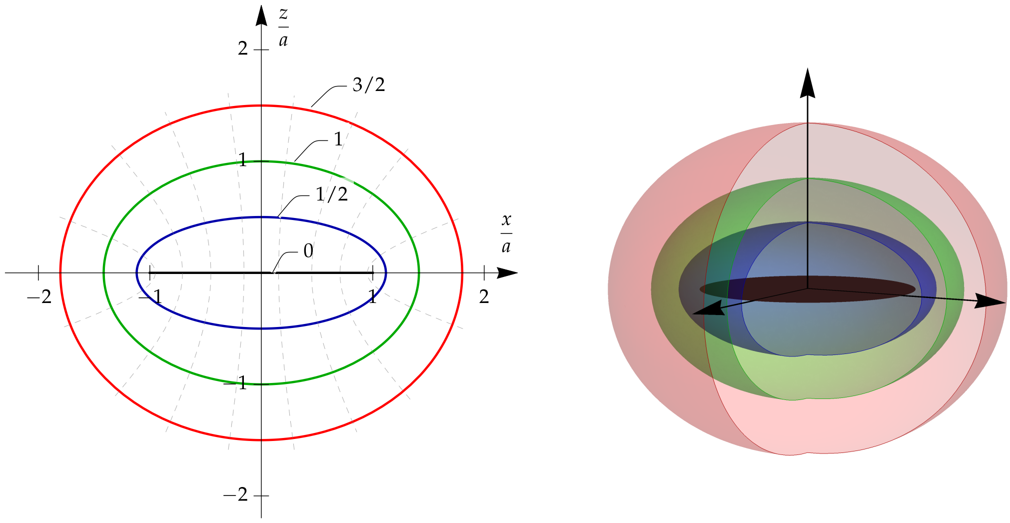


FIGURE 2.1: Contour plots of the surface $r(x, y, z)$ for constant values of 0, 1/2, 1, 3/2, in the Kerr-Child "cartesian" coordinates. The left plot is the intersection with $z = 0$ plane with the 3D representation (right) that spotlights the ring singularity. Dashed curves, representing orthogonal constant $\theta(x, y, z)$ hypersufaces, become asymptotically affine.

to positive values. Particularly for Kerr-Child, hypersurfaces of constant r can be represented also by $-r$. This means that this chart can be analytically extended to regions where $r < 0$. From this procedure and a proper collage of charts it is possible to achieve a *maximally extended* solution, with gives mathematical access to new spacetime regions, even tough most of them show unphysical properties.

2.2.3 Boyer-Linquist coordinates

Considering the problem in hand, the most suitable coordinates for work with the NP formalism, are the Boyer-Linquist coordinates

$$\begin{aligned} ds^2 = &- \left(\mathrm{d}t - \frac{2Mr}{\rho^2} \right) \mathrm{d}t^2 - 2a \sin^2\theta \frac{(r^2 + a^2 - \Delta)}{\rho^2} \mathrm{d}t\, \mathrm{d}\phi \\ &+ \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\rho^2} \sin^2\theta\, \mathrm{d}\phi^2 + \frac{\rho^2}{\Delta} \mathrm{d}r^2 + \rho^2\, \mathrm{d}\theta^2 \,, \end{aligned} \tag{2.18}$$

where we define $\Delta = r^2 - 2Mr + a^2$. In order to show that these corresponds to the same solution, the change of coordinates

$$v = t + r_* \,, \qquad \chi = \phi + \int \frac{a}{\Delta} \mathrm{d}r \,, \tag{2.19}$$

takes us back to the original Kerr form (2.12). The coordinate v is given by the known ingoing EF transformation, defined by the Regge-Wheeler coordinate, also named *tortoise* coordinate, which is very useful to construct null directions. In the case of the Kerr BL metric, it holds that

$$\frac{\mathrm{d}r_*}{\mathrm{d}r} = \frac{r^2 + a^2}{\Delta} \,. \tag{2.20}$$

These coordinates are usually referred as "Schwarzschild like", as it takes the spherical static case in standard curvature coordinates when setting $a = 0$. Time inversion symmetry is characteristic of static Schwarzschild spacetime, but the same does not hold for Kerr's. Nevertheless, this specific form is invariant under the inversion $(t, \phi) \to (-t, -\phi)$, also known as the *circular condition*, an intuitive notion from physical systems with angular momentum. This discrete symmetry eliminates most of the off-diagonal components of the BL metric, $g_{tr} = g_{\phi r} = g_{t\theta} = g_{\phi\theta} = 0$, making it the simplest to perform calculations.

Now, to study the possible horizons of Kerr BH, the one-form $n = \mathrm{d}r$, defines normal vectors to constant radial hypersurfaces. It is easy to show that $n^2 = g^{rr}$, which implies that n is null when $\Delta = 0$, defining horizons at

$$r_{\pm} = M \pm \sqrt{M^2 - a^2} \,, \tag{2.21}$$

singularities at g_{rr} which we know to be removable. As a consequence, from a static observer a massless particle on an ingoing null geodesic would spiral around the BH for a infinite time, as the coordinate $t \to \infty$, never reaching $r = r_+$. This surface is the event horizon of the Kerr BH, as it separates two causally disconnected regions of spacetime, *i.e* any information from the inside this surface, will never reach any asymptotic observer. The expression for the event horizon surface also raises limitations for the amount of angular momentum a physical BH can have. We must have

$$|a| < M \,, \tag{2.22}$$

otherwise Δ would lack any real roots and would lead to a essential *naked singularity*, reachable in a finite observable time, which is forbidden by the *Weak Cosmic Censorship*.

The surface at $r = r_-$, on the other hand, is called a Cauchy horizon. In GR, a spacelike surface containing all initial conditions of spacetime (Cauchy surface) would suffice to predict all past and future events, but a Cauchy horizon separates the domain of validity of such initial conditions. Despite no information ever escaping the event horizon, it is still possible to predict events inside $r_- < r < r_+$, but such thing it is not guaranteed after crossing the Cauchy horizon. Due to this and some other unphysical features (for example, closed timelike curves and instabilities under perturbations), we need only to focus on the region outside the event horizon $r > r_+$, since only information on that region is physically reachable from a stationary observer point of view.

Event tough most of the Kerr BH basic properties were demonstrated, there is still no result so far showing some kind of rotation. First, consider the quantities $\xi_\alpha u^\alpha$, where u^α is the four-velocity and ξ^α is a Killing field. Being aware of the geodesic equation, $u^\beta \nabla_\beta u^\alpha = 0$, it is easy to show that this quantity is conserved along geodesics, *i.e.*

$$u^\beta \nabla_\beta (\xi_\alpha u^\alpha) = u^\alpha u^\beta \nabla_\beta \xi_\alpha = \frac{u^\alpha u^\beta}{2} (\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha) = 0 \,, \tag{2.23}$$

due to Killing Eq. (2.7). As a result, geodesics of a free particle in Kerr geometry will be characterized by two constants

$$-\varepsilon = k^\beta g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \,, \qquad \ell = m^\beta g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \,, \tag{2.24}$$

where τ is the affine parameter fo the geodesic. These quantities can be interpreted the energy per mass and angular momentum per mass of the particle, respectively. Due to the circular form of the BL metric, the metric components of the coordinates (t, ϕ) define a product decomposition, providing the separation of previous equations,

$$\dot{t} = \frac{1}{\Delta} \left[(r^2 + a^2 + \frac{2Ma^2}{r})\varepsilon - \frac{2Ma}{r} \ell \right] \,, \tag{2.25}$$

$$\dot{\phi} = \frac{1}{\Delta} \left[\frac{2Ma}{r} \varepsilon + \left(1 - \frac{2M}{r} \right) \ell \right] \,, \tag{2.26}$$

specified for the equatorial plane $\theta = \pi/2$. The final equation for the geodesic is provided by the line element (2.18), which becomes also a first order ODE, after the substitution of $\dot{t}$ and $\dot{\phi}$. Consider now a zero angular momentum observer (ZAMO) infalling radially, with $\ell = 0$, then we can get the angular velocity Ω, as measured at infinity

$$\Omega = \frac{\dot{\phi}}{\dot{t}} = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2aM}{r^3 + a^2(2M + r)} \,. \tag{2.27}$$

Asymptotically we obtain $\Omega \to 0$, coincident with a static observer measurement, but for a finite distance, observers are forced to co-rotate with the BH. Particularly, at the event horizon, $r = r_+$, one finds that

$$\Omega_H = \frac{a}{2Mr_+} = \frac{a}{2M(M + \sqrt{M^2 - a^2})} \,. \tag{2.28}$$

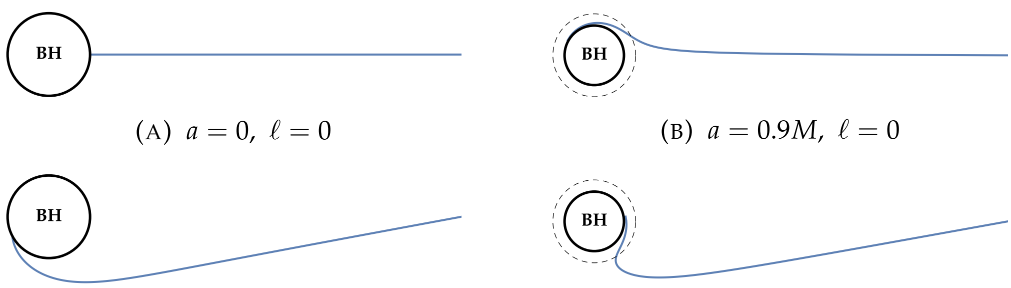
The field Killing vector $\xi = k + \Omega_H m$ is a special linear combination that is also a null vector normal to the event horizon, $\xi^\flat = \xi_\alpha \, \mathrm{d}x^\alpha \propto \mathrm{d}r$ and $\xi^2 = 0$ at $r = r_+$, which can be written in BL coordinates as $\xi = \partial_t + \Omega_H \partial_\phi$. Hence, on the outer horizon the integral curves of this vector obey $\xi^\alpha \partial_\alpha(\phi - \Omega_H t) = 0$, resulting in $\phi \propto \Omega_H t$. Since null geodesics on the horizon follow curves generated by the Killing vector ξ, then we say that the BH is "rotating" with angular velocity Ω_H.

2.2.4 Ergoregion and the Penrose process

One of the main characteristic that distinguishes Kerr BH from other spherical solutions is the existence of a *ergoregion* which allows for energy extraction from th BH an therefore superradiance. The surface is characterized when the Killing vector k^μ becomes spacelike, $k^2 = g_{tt} < 0$, resulting in the hypersurface boundary

$$r_{\text{ergo}}(\theta) = M + \sqrt{M^2 - a^2 \cos^2\theta}\,. \tag{2.29}$$

This region lies outside the event horizon if $a \neq 0$, then being defined as $r_+ < r < r_{\text{ergo}}(\theta)$. Notice that a static observer moves in a timelike curve with (r, θ, ϕ) constant, *i.e.* with tangent vector proportional to k^μ, therefore such observer cannot exist inside because the time Killing vector becomes spacelike, otherwise it would violate causality. We can see that $u^2 = g_{\alpha\beta}u^\alpha u^\beta = g_{tt}(u^t)^2 + 2g_{t\phi}u^t u^\phi + g_{\phi\phi}(u^\phi)^2 < 0$ only occurs when $g_{t\phi}u^\phi < 0$, as all other components are positive. Inside the ergoregion, $g_{t\phi} < 0$, therefore all observers are forced to rotate in the same direction as the BH.



(A) $a = 0,\ \ell = 0$

(B) $a = 0.9M,\ \ell = 0$

(C) $a = 0,\ \ell < 0$

(D) $a = 0.9M,\ \ell < 0$

FIGURE 2.2: Illustration of the Schwarzschild (A,C) and Kerr (B,D) null equatorial infalling geodesics given by Eqs. (2.25) and (2.26), for $r(0) = 20M$, with emphasis on $\ell \neq 0$. Even starting with opposite angular momentum, the Kerr geodesic (D) is forced to co-rotate with the BH once crossed the ergoregion (dashed).

Despite BHs being always thought as "perfect absorvers" due to casual separation of the horizon, but the ergoregion is a region that allows energy extraction from the BH, through the Penrose process, an intrinsic feature of rotating BHs. Much like spontaneous pair creation and amplification at discontinuities are related but distinct effects, the Penrose process allows for a better understating of superradiance and why it is relevance in GR. Considering a particle with rest mass μ and four-momentum $p^\alpha = \mu u^\alpha$, we may identify the constant of motion

$$\varepsilon = -k_\alpha p^\alpha = -\mu g_{tt} p^t - g_{t\phi} p^\phi \, . \tag{2.30}$$

as it's energy, as measured by a stationary observer at infinity, due to relations (2.24). As shown above the Killing vector is asymptotically timelike but is spacelike inside the ergoregion, thus $g_{tt} > 0$. Since the geodesic is future-directed, $p^0 = \mu u^0 > 0$ and the energy beyond the ergosurface needs not to be positive. Suppose, by any means, that such particle manages to decay inside the ergoregion into two other, with momenta p_1 and p_2, such that $p = p_1 + p_2$. Contracting with k, implies that $\varepsilon = \varepsilon_1 + \varepsilon_2$. Supposing that the first of the particles has negative energy, $\epsilon_1 < 0$, then

$$\varepsilon_2 = \varepsilon + |\varepsilon_1| > \varepsilon \, . \tag{2.31}$$

I can be shown that the particle with negative energy (bounded) must fall into the BH while the other may escape the ergoregion, with greater energy than the particle sent in. Energy is conserved by making the BH absorb the particle with negative energy, therefore resulting in a net energy extraction.

From now on, all results will be provided using BL coordinates.

2.3 Newman-Penrose formalism

2.3.1 Kinnersly tetrad

2.3.2 Spin coefficients

2.3.3 Maxwell equations

Chapter 3

Teukolsky master equation

3.1 Angular solutions

3.2 Asymptotic radial solution

3.3 Amplification factor Z_{slm}

Chapter 4

Numerical method

4.1 Eigenvalues

4.1.1 Leaver method

4.1.2 Spectral

4.2 Radial ansatz

4.3 Amplification factor as a first test

Chapter 5

Scattering problem

5.1 Plane wave decomposition

Appendix A

Spin-weighted spherical harmonics

SWSHs play an important role BH physics and was first introduced by Teukolsky when considering non-scalar wave perturbations on a Kerr background, obtaining a separable master equation in four dimensions. After the usual change of coordinates, the polar differential equation goes as

$$\frac{1}{S}\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}S}{\mathrm{d}x}\right)+(cx)^2-2csx-\frac{(m+sx)^2}{1-x^2}+s=-\lambda \tag{A.1}$$

with $x=\cos\theta$, where λ is the eigenvalue for a given SWSH solution. Periodic boundary conditions on the azimuthal wave function constrains m to the integers.

A.1 Connection with spheroidal harmonics

By setting $s=0$ (scalar) and $c=0$ (spherical), then it's clear that Eq. (A.1) appears as a generalization of the spherical harmonics equation. In this last case, the solution are given by the associated Legendre polynomials, $P_\ell^m(x)$, for which the eigenvalue is $\ell(\ell+1)$, restricted to the condition of $|m|\leq\ell$. The closed form for spherical harmonics, after normalization, is

$${}_0Y_\ell^m(x)=(-1)^m\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(x) \tag{A.2}$$

where P_ℓ^m are the associated Legendre polynomials which can be obtained using the famous Rodrigues' formula.

A.2 Spin raising/lowering differential operators

A.3 Generalized addition of angular momentum formula

A.4 Some useful harmonics

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