Aspects of superradiant scattering off Kerr black holes

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UNIVERSIDADE DO PORTO

MASTER'S THESIS

Aspects of superradiant scattering off Kerr black holes

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Abstract

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Departamento de Física e Astronomia

Master of Science

Aspects of superradiant scattering off Kerr black holes

by José SÁ

The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too…

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Resumo

Faculdade de Ciências da Universidade do Porto
Departamento de Física e Astronomia

Mestre de Ciência

Aspects of superradiant scattering off Kerr black holes

por José SÁ

Tradução em português do “Abstract” escrito em inglês mais a cima. A página é centrada vertical e horizontalmente, podendo espandir para o espaço superior da página em branco . . .

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BH	Black Hole
BL	Boyer-Linquist
EF	Eddington-Finkelstein
GR	General Relativity
GW	Gravitational Wave
LIGO	Laser Interferometric Gravitational Wave Observatory
NP	Newman-Penrose
SWSH	Spin-Weighted Spheroidal Harmonic

Chapter 1

Superradiance

1.1 Introduction

The first direct observation of gravitational waves (GWs) by the Laser Interferometer Gravitational Wave Observatory (LIGO) was in 2015 and latter announced in 2016. The recorded event matched the signature predictions of General Relativity (GR) for a binary system of black holes (BHs) merging together in a inward spiral into a single BH [1]. These observations demonstrated not only the existence of GWs but also existence of binary stellar-mass BH systems and that these systems could merge in a time less than the known Universe age. Since then, two more similar events were detected, which assured the inauguration of a new era of GW cosmology.

Naturally, this sparked new interest in the study of binary systems and GW-related phenomena. One of these phenomena is the possibility of amplification in waves scattered off rotating and/or charged BHs, which can occur under certain conditions for scalar, electromagnetic and gravitational bosonic waves. Such effect is one of many that encompass a wide range of phenomena generally known as *superradiance*. **(How the study of superradiant scattering can be useful)**

As all bosonic waves can be reduced to the study of the same master equation (as we will see later), this work will focus primarily on electromagnetic waves in the case of a neutral rotating BH. Said choice is the most interesting from a astrophysical point of view, considering that any charged BH should be "quickly" neutralized by the surrounding interstellar plasma, due to the nature of EM interactions.

Historically, the first appearance of the concept of superradiance was in 1954, in a publication by Dicke [2]. He showed that a gas could be excited by a pulse into "superradiant states" from thermal equilibrium and then emit coherent radiation. Almost two decades later, Zel'dovich [3, 4] showed that a absorbing cylinder rotating with an angular velocity Ω could scatter an incident wave, $\psi \sim e^{-i\omega t + im\phi}$, with frequency ω if

$$\omega < m\,\Omega \tag{1.1}$$

would be satisfied, where m is the usual azimuthal number of the monochromatic plane wave relative to the rotation axis. In his work, he noticed that superradiance was related with dissipation of rotational energy from the absorbing object, possibly due to spontaneous pair creation at the surface. Hawking later showed that the presence of a strong electromagnetic or gravitational fields could indeed generate bosonic and fermionic pairs spontaneously. This result was possible by the efforts of Starobinsky and Deruelle [5–8], which also laid the groundwork necessary for the discovery of BH evaporation.

1.2 Klein paradox as a first example

Actually, radiation amplification can be traced to birth of Quantum Mechanics, in the beginnings of the 20th century. First studies of the Dirac equation by Klein [9] revealed the possibility of electrons propagating in a region with a sufficiently large potential barrier without the expected dampening from non-relativistic tunnel effect. Due to some confusion, this result was wrongly interpreted by some authors as fermionic superradiance, as if the reflected current by the barrier could be greater than the incident current. The problem was named *Klein paradox* by Sauter [10] and this misleading result was due to a incorrect calculation of the group velocities of the reflected and transmitted waves.

Today, it is known that fermionic currents cannot be amplified for this particular problem [9, 11], result that was correctly obtained by Klein in is original paper. On the contrary, superradiant scattering can indeed occur for bosonic fields.

1.2.1 Bosons

The equation that governs bosonic wave function is the Klein-Gordon equation, which for a minimally coupled electromagnetic potential takes the form

$$(D^{\nu} D_{\nu} - \mu^2)\Phi = 0\,, \tag{1.2}$$

where the usual partial derivative becomes $D_{\nu} = \partial_{\nu} + ieA_{\nu}$ and μ is the boson mass.

The problem is greatly simplified by considering flat space-time in (1+1)-dimensions and step potential $A(x) = V\,\theta(x)\ \mathrm{d}t$, for $V > 0$ constant and wave solutions $\Phi = e^{-i\omega t}\phi$. For $x < 0$, the solution can be divided as incident and reflected, taking the form

$$\phi_{\text{inc}}(x) = \mathcal{I}\, e^{ikx}\,, \qquad \phi_{\text{refl}}(x) = \mathcal{R}\, e^{-ikx}\,, \tag{1.3}$$

in which the dispersion relation states that $k = \sqrt{\omega^2 - \mu^2}$. For $x > 0$, the transmitted wave is naturally given by

$$\psi_{\text{inc}}(x) = \mathcal{T}\, e^{iqx}\,, \tag{1.4}$$

but in this case the root sign for the momentum must be carefully chosen so that the group velocity sign of the transmitted wave matches of the incoming wave [11], *i.e.*

$$\left.\frac{\partial\omega}{\partial p}\right|_{p=q} = \frac{q}{\omega - eV} > 0\,, \tag{1.5}$$

therefore we must have that

$$q = \operatorname{sgn}(\omega - eV)\sqrt{(\omega - eV)^2 - \mu^2}\,. \tag{1.6}$$

After obtaining the continuity relations at the barrier, $x = 0$, we follow by computing the ratios of the transmitted and reflected currents relative to the incident one, which yield

$$\frac{j_{\text{refl}}}{j_{\text{inc}}} = -\left|\frac{\mathcal{R}}{\mathcal{I}}\right|^2 = -\left|\frac{1-r}{1+r}\right|^2\,, \qquad \frac{j_{\text{trans}}}{j_{\text{inc}}} = \operatorname{Re}(r)\left|\frac{\mathcal{T}}{\mathcal{I}}\right|^2 = \frac{4\operatorname{Re}(r)}{|1+r|^2}\,, \tag{1.7}$$

written as a function of the coefficient

$$r = \frac{q}{k} = \operatorname{sgn}(\omega - eV)\sqrt{\frac{(\omega - eV)^2 - \mu^2}{\omega^2 - \mu^2}}\,. \tag{1.8}$$

Hence, in the case of strong potential limit, $eV > \omega + \mu > 2\mu$, we may have $r < 0$ real and the reflected current is larger (in magnitude) than the incident wave and therefore we have amplification.

1.2.2 Fermions

Dirac noticed the that Klein-Gordon equation masked internal degrees of freedom, so he devised is own equation which describe fermions. Considering that scalar potentials do not have any impact on spin orientation [12], we need only to consider half of the spinor components in Dirac equation

$$(i\gamma^{\nu}D_{\nu} - \mu)\Psi = 0\,, \tag{1.9}$$

where μ is the fermion mass, for which a valid representation of the gamma matrices is

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.10}$$

Probing wave solutions $\Psi = e^{-i\omega t}\psi$, the incident and reflected solutions are

$$\psi_{\text{inc}}(x) = \mathcal{I}\, e^{ikx} \begin{pmatrix} 1 \\ \dfrac{k}{\omega+\mu} \end{pmatrix}, \qquad \psi_{\text{refl}}(x) = \mathcal{R}\, e^{-ikx} \begin{pmatrix} 1 \\ \dfrac{-k}{\omega+\mu} \end{pmatrix}, \tag{1.11}$$

while for $x > 0$, the transmitted wave function is written as

$$\psi_{\text{trans}}(x) = \mathcal{T}\, e^{iqx} \begin{pmatrix} 1 \\ \dfrac{q}{\omega - eV + \mu} \end{pmatrix}, \tag{1.12}$$

where was followed the same procedure as before, obtaining the same results from Eq. (1.5) through (1.7). As a result of the structure of the spinor components, the coefficient at Eq. (1.8) is modified to

$$r = \text{sgn}(\omega - eV)\frac{\omega+\mu}{\omega - eV + \mu}\sqrt{\frac{(\omega - eV)^2 - \mu^2}{\omega^2 - \mu^2}}\,, \tag{1.13}$$

and now, in the same region, $\omega > \mu$, superradiance does not occur.

Even though superradiance and spontaneous pair creation are two distinct phenomena, this result is usually interpreted using the latter, from a QFT stand point. All incident particles are completely reflected, as well as some extra due to pair creation at the barrier as a result of stimulation by the incident radiation and the presence of a strong electromagnetic field, while the resultant anti-particles are transmitted in the opposite direction, accounting for the change of sign in the transmitted current in Eq. (1.7), owing to the opposite charge they carry. This also explains the undamped transmission part.

One may think that this difference between bosons and fermions arises from the potential barrier shape, but work by other authors [10, 11, 13] shows that only the difference between the asymptotic values of the potential at infinity is essential for the process. The difference comes from intrinsic properties of these particles. The amount of fermion pairs produced in a given state, *i.e.* for a given ω, is limited by Pauli exclusion principle, while such limitation does not occur for bosons [14]. Additionally, fermionic current densities are always positive definite, while bosons can change sign because of the ambiguity of wave function describing positive and negative energy solutions.

The minimum necessary energy for this to occur, 2μ, leaves evidence that superradiance is accompanied with spontaneous pair creation and some sort of dissipation by the battery maintaining the strong electromagnetic potential, in order to maintain energy balance.

1.3 Black hole superradiance

Needs completion

Among many other cases of radiation amplification, the phenomena worked out throughout this work is an example of *rotational superradiance*. As the name suggests, it occurs in the presence of rotating objects, as is the famous example of Zel'dovich cylinder. In this case, the object in question is a Kerr black hole. This geometry is the simplest solution for a static but non-stationary BH, which breaks spherical symmetry.

Condition Eq. (1.1) was to become one of the most important results of rotational superradiance, as it presented itself in multiple examples, including in BH physics, particularly in the case of the Kerr solution.

Chapter 2

Kerr black hole

2.1 General Relativity

General Relativity is the theory of space, time and gravitation developed by Einstein in 1915. It introduced a new viewpoint on gravity and it's relation with the fabric of space-time, a *manifold* that bounded our three spatial dimensions with dimension of time. The concept challenged our deeply ingrained and intuitive notions of nature, partially because the mathematical background needed to understand the precise formulation of theory was unfamiliar to much of the Physics community at the time. This formulation corresponds to a field theory with the dynamical object of study being the metric of spacetime, $ds^2 = g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu$, connecting geometry with mass and energy through Einstein field equations. The theory inherits diffeomorphism invariance, *i.e.* remains the same theory by a active change of coordinates, which was at the core of definition of manifolds.

Immediately after, in 1916, Schwarzschild found the first solution, describing a static spherical isolated object. Then, the theory was left aside because of the numerous coupled nonlinear equations, but the astronomical discovery of compact and highly energetic objects in the 1950s breaded new interest into the somewhat dormant GR, mainly because it was thought that these quasars and compact X-ray sources had suffered some form of gravitational collapse or that strong gravitational fields were present. Soon after, the modern theory of gravitational collapse was developed in the mid-1960s, including other BHs solutions, for example Kerr's.

The theory of GR can be elegantly described in the form of the Hilbert action,

$$S_H = \frac{1}{16\pi} \int \mathrm{d}^4x \sqrt{-g}\, R\,, \tag{2.1}$$

where $g = \det(g_{\mu\nu})$ and $R = g_{\alpha\beta}R^{\alpha\beta}$ corresponds to the Ricci scalar. Naturally, the first solutions corresponded to pure gravity, usually designated as vacuum solutions, which obey

$$R_{\mu\nu} = 0\,. \tag{2.2}$$

Despite their simplicity, they enjoy some very fascinating nontrivial properties. One of which is the existence of an event horizon, a surface that separates two causally disconnected regions of spacetime.

The underlying technique behind the study of superradiance is the linearization of Einstein and/or Maxwell equations around know BHs in stationary equilibrium. These perturbations will obey a series of partial differential equations whose dynamical variables are components of the Weyl tensor, $C_{\mu\nu\rho\sigma}$, or the Maxwell field tensor, $F_{\mu\nu}$. Thanks to the NP formalism we will be able to decouple and separate the equations for both GWs and EM waves, revealing decoupled variables which contain all the information needed about the nontrivial perturbations, instead of working with all components of the field tensors.

For the gravitational case, a straightforward way of obtaining a linearized theory is to consider a background stationary BH solution, ${g_{\mu\nu}}^B$, and then expanding the field equations (2.2) using the metric

$$g_{\mu\nu} = {g_{\mu\nu}}^B + {h_{\mu\nu}}^P\,, \tag{2.3}$$

keeping only terms with are $\mathcal{O}({h_{\mu\nu}}^B)$. The indices B and P refer to the background and perturbations, respectively. As a result we are left with a wave equation in the given background.

In this work we will focus on (massless, neutral) electromagnetic waves and perturbations are performed including EM interactions through the Maxwell action

$$S_{EM} = -\frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}\,F_{\alpha\beta}F^{\alpha\beta}\,, \tag{2.4}$$

where $F_{\mu\nu}$ is the Maxwell tensor. Variation of both actions, $\delta(S_H + S_{EM}) = 0$, result in two field equations

$$\nabla_\mu F^{\mu\nu} = 0\,, \tag{2.5}$$

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = 8\pi T_{\mu\nu}\,. \tag{2.6}$$

The first equation is just the usual of Maxwell equation in curved spacetime. The latter are the Einstein field equations, reflecting the backreaction of the electromagnetic waves into the geometry through the presence of EM stress-energy tensor

$$T_{\mu\nu} = F_{\mu\alpha} F_{\nu}{}^{\alpha} - \frac{1}{4} g_{\mu\nu} F^2 \, . \tag{2.7}$$

These equation completely describe the system, but the problem is analytically untreatable, so we will resort to perturbation theory, considering the field A^μ to be small. This is a very good approximation, as near the gravitational field of stellar-mass BHs is considerably stronger compared with radiation emitted by nearby astrophysical sources. As the stress-energy tensor is quadratic in the fields, $T_{\mu\nu} \sim \mathcal{O}(A^2)$, then we can ignore the backreaction and the field equations for the metric $g_{\mu\nu}$ reduce to Eq. (2.2).

2.2 Spacetime symmetries

It was generally accepted that a perfectly spherical symmetrical star would collapse to a Schwarzschild BH. Although, at the time it was not known the effect of a slightest amount of angular momentum on a gravitational collapse. Finding a metric with intrinsic rotation could give insight to such problem. Due to the lack of spherical symmetry, the problem became much harder, and took roughly 50 years after Schwarzschild's discovery to find a metric for a rotating body. Imposing symmetries to the final metric was essential to solve the field equation.

If we represent our spacetime and corresponding fields by $(\mathcal{M}, g_{\mu\nu}, \psi)$, then the pullback f^* of the diffeomorphism $f : \mathcal{M} \to \mathcal{M}$, would give us the same physical system $(\mathcal{M}, f^* g_{\mu\nu}, f^* \psi)$. Since diffeomorphisms are just active coordinate transformations, such concept may raise some confusion, as we don't seam to obtain no new information to work with. Almost all physics theories are coordinate invariant, as is Newtonian mechanics and Special Relativity, but in such theories there is a preferable coordinate system, while the same does not hold true for GR. An analogies can be made with the path integral formalism in QFT, where special consideration is taken when summing all field configurations in order to not overcount indistinguishable configurations, as is the case of gauge field theories. A similar ambiguity can occur in GR, where two apparently different solutions which can be related by a diffeomorphism and are actually "the same", so we must be careful when deriving and analyzing any geometries.

Despite the added complexity of Einstein's field equations, it is still possible to find exact nontrivial solutions in a systematic way by considering spacetimes with symmetries with the use of Killing vector fields. A vector field ξ that obeys

$$\mathcal{L}_\xi g_{\mu\nu} = 0 \tag{2.8}$$

is called a Killing field. Locally, this expression reduces to $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$.

A *stationary* solution implies the existence of a Killing vector k that is asymptotically timelike, $k^2 > 0$, therefore allows us to normalize our vector such that $k^2 \to 1$. Unlike the case of the static spacetime, a stationary metric does not show invariance under reversal of the time coordinate, which is natural considering a system with angular momentum. Futhermore, a solution is also *axisymmetric* if is present a asymptotically spacelike Killing field m whose integral curves are closed. A solution is stationary and axisymmetric if both symmetries are present, along with commuting fields, $[k, m] = 0$, *i.e.* rotations along with the axis of symmetry commute with time translations. The commutativity of the fields implies the existence of a set of coordinates, (t, r, θ, φ), such that

$$k = \frac{\partial}{\partial t}\,, \qquad m = \frac{\partial}{\partial \varphi}\,. \tag{2.9}$$

As for direct implication of this choice of chart, components of the metric stay independent of (t, φ), in virtue of Eq. (2.8),

$$(\mathcal{L}_m g)_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial \varphi} = 0\,, \tag{2.10}$$

with the same holding true for k, hence we can write $g_{\mu\nu} = g_{\mu\nu}(r, \theta)$.

One of the major applications of Killing vectors is to find conserved charges associated with the motion along a geodesic spanned by field. These quantities are defined by taking the geodesics to regions space that are asymptotical flat, where the geometry does not affect the observer. In the case of Kerr solution, we have two Killing vectors, k and m, which are naturally associated with the total mass M and angular momentum J of the BH, respectively. This is usually done by evaluating the Komar integrals [15, 16], which can be written in a covariant way as

$$M = \frac{1}{8\pi}\int_{S^2_\infty} \star\, \mathrm{d}k^\flat = -\frac{1}{4}\lim_{r\to\infty}\int_0^\pi \mathrm{d}\theta \sqrt{-g}\, g^{t\alpha} g^{r\beta} g_{t[\alpha,\beta]}\,, \tag{2.11}$$

$$J = -\frac{1}{16\pi}\int_{S^2_\infty} \star\, \mathrm{d}m^\flat = \frac{1}{8}\lim_{r\to\infty}\int_0^\pi \mathrm{d}\theta \sqrt{-g}\, g^{t\alpha} g^{r\beta} g_{\varphi[\alpha,\beta]}\,, \tag{2.12}$$

where the usual notation $k^\flat = g_{\mu\nu} k^\mu \, \mathrm{d}x^\nu$ transforms a vector into a one-form and the operator $\star : \Omega^p(\mathcal{M}) \to \Omega^{4-p}(\mathcal{M})$ is the Hodge dual map for p-forms. In order to complete the integration in the last step is assumed (2.9) and (2.10), keeping (t, r) constant. According to the widely accepted of *no-hair conjecture* [17], these two quantities completely define a stationary (neutral) BH.

2.3 Kerr-Child coordinates

Naturally, Kerr wasn't the only one after such solution. Many presented other solutions to approximately describe a rotating star. Most of the solutions were one-parameter modifications to Schwarzschild that were not asymptotically flat except for the standard case. Simply using stationary and axisymmetric symmetries and then solving Einstein equations clearly did not suffice.

Kerr success originated in of Petrov's classification of spacetimes, which used the algebraic properties of the Weyl tensor to distinguish the solutions in 3 types, along with some subcases. He assumed that his solution would have the same classification as Schwarzschild's, associated with the geometry of isolated central objects, such as stars and BHs. From this assumption, using GR spinor techniques and only then imposing the Killing vectors in Eq. (2.9) was possible to find a new solution. Kerr's metric appear in his original paper [18] in the form

$$\begin{aligned} ds^2 = & \left(1 - \frac{2Mr}{\rho^2}\right) (\, \mathrm{d}v - a \sin^2\theta \, \mathrm{d}\chi)^2 \\ & - 2(\, \mathrm{d}v - a \sin^2\theta \, \mathrm{d}\chi)(\, \mathrm{d}r - a \sin^2\theta \, \mathrm{d}\chi) \\ & - \rho^2(\, \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\chi^2) \, , \end{aligned} \tag{2.13}$$

where a is a parameter, M is the BH mass and $\rho^2 = r^2 + a^2 \cos^2\theta$. Naturally the time Killing vector is ∂_v and ∂_χ is the axial field, entailing that $J = aM$.

Taking the limit of $a \to 0$, we reduce the metric to the Schwarzschild solution in ingoing Eddington-Finkelstein (EF) coordinates, (v, r, θ, χ), which are useful to study ingoing (to the horizon) geodesics and remove the horizon coordinate singularity. When a given metric has singularities it is not trivial to identify if is a physical singularity or if it is an artifact resultant of choice of the chart, removable by a better choice of coordinates. That being said, this raises the difficulty of finding the essential singularities. The best way to look to these singularities is to compute curvature scalar quantities, and if they diverge in one particular chart, then they diverge on all charts. Since any BH is just a vacuum solution, then the Ricci scalar vanishes, $R=0$, so we resort to the Kretschmann scalar,

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{48M(r^2-a^2\cos^2\theta)\left[(r^2-a^2\cos^2\theta)^2-16r^2M^2a^2\cos^2\theta\right]}{(r^2+a^2\cos^2\theta)^6}\,, \tag{2.14}$$

that clearly diverges for $\rho^2=0$. The Schwarzschild singularity, $r=0$, is replaced with the Kerr singularity $(r,\theta)=(0,\pi/2)$. It is not clear what is the geometry of the Kerr singularity if we interpret r and θ as being part of the ordinary spherical coordinates. Although the metric is singular, we can draw some insight considering (r,θ) constant and then the limit of $r\to 0$ through the equatorial plane,

$$ds^2|_{\text{singularity}} \sim \mathrm{d}v^2 - a^2\,\mathrm{d}\chi^2\,. \tag{2.15}$$

Hence the metric is reduced to the line element of the circle, S^1, confirming a *ring singularity* of radius a. This result implies that we may only reach the singularity $\rho^2=0$, by approaching the Kerr BH through the equatorial plane.

The Kerr-Child theory provides the "cartesian" form [19],

$$\begin{aligned} ds^2 &= \mathrm{d}\tilde{t}^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \\ &\quad - \frac{2Mr^3}{r^4+a^2z^2}\left[\mathrm{d}\tilde{t} + \frac{r(x\,\mathrm{d}x + y\,\mathrm{d}y) + a(y\,\mathrm{d}x - x\,\mathrm{d}y)}{r^2+a^2} + \frac{z}{r}\,\mathrm{d}z\right]^2\,, \end{aligned} \tag{2.16}$$

which is particularly useful to understand the singularity geometry. In this metric, r is no longer a coordinate but a function of this chart coordinates $(\tilde{t},x,y,z)$. We can relate the The Kerr-Child metric to the original Kerr solution, using

$$\tilde{t} = v - r\,, \qquad x + iy = (r-ia)e^{i\chi}\sin\theta\,, \qquad z = r\cos\theta\,, \tag{2.17}$$

which implies that $r(x,y,z)$ is implicitly given by

$$r^4 - (x^2+y^2+z^2-a^2)r^2 - a^2z^2 = 0\,. \tag{2.18}$$

This condition deserves a more in-depth analysis. For increasing r, the surfaces obeying Eq. (2.18) approximates perfect spheres as the geometry gets more and more flatter. Minkowsky flat space is immediately also guaranteed for $M=0$. On the other hand, as we approach the singularity on $z=0$ and $x^2+y^2=a^2$, rotation effects deform the surfaces into oblate spheroids ($\theta\neq\pi/2$ for the strict inequality). Such remarks are visually demonstrated in Figure 2.1.

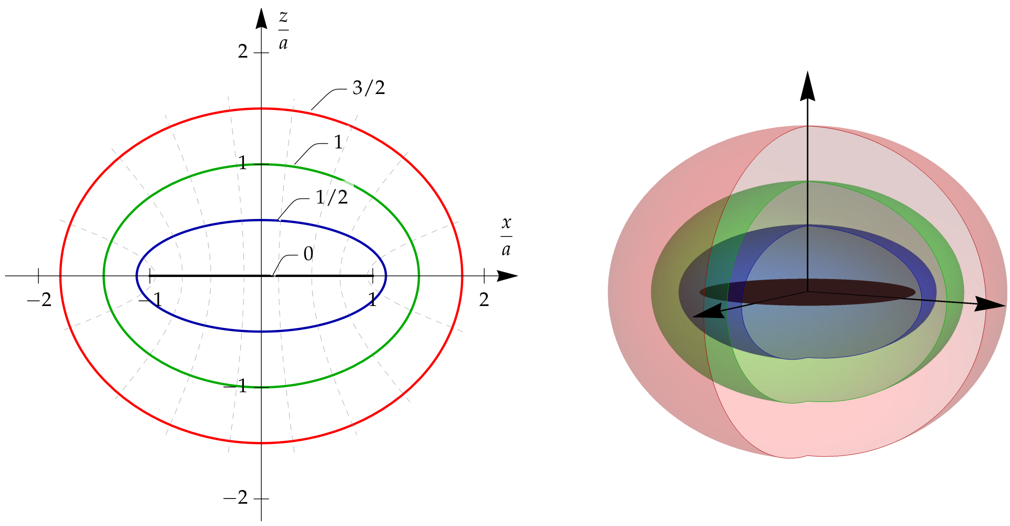

FIGURE 2.1: Contour plots of the surface $r(x, y, z)/a$ for constant values of 0, 1/2, 1, 3/2, in the Kerr-Child "cartesian" coordinates. The left plot is the intersection with $z = 0$ plane with the 3D representation (right) that spotlights the ring singularity. Dashed curves representing orthogonal constant $\theta(x, y, z)$ hypersufaces, become asymptotically affine.

Even though Kerr-Child metric takes $r > 0$ values, there is no mathematical reason to restrict r strictly to positive values. Thus, hypersurfaces of constant r can also be represented by $-r$. This means that this chart can be analytically extended to regions where $r < 0$. It is possible to obtain a *maximally extended* solution by analytic continuation and a proper collage of charts. This gives mathematical access to new spacetime regions, even tough most of them show unphysical properties.

2.4 Boyer-Linquist coordinates

Considering the problem in hand, the most suitable coordinates for work with the NP formalism, are the Boyer-Linquist coordinates [20],

$$\begin{aligned} ds^2 = \left(1 - \frac{2Mr}{\rho^2}\right) \mathrm{d}t^2 + 2a \sin^2\theta \frac{(r^2 + a^2 - \Delta)}{\rho^2} \mathrm{d}t \, \mathrm{d}\varphi \\ - \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\rho^2} \sin^2\theta \, \mathrm{d}\varphi^2 - \frac{\rho^2}{\Delta} \mathrm{d}r^2 - \rho^2 \, \mathrm{d}\theta^2 \,, \end{aligned} \tag{2.19}$$

where we define $\Delta = r^2 - 2Mr + a^2$. In order to show that these corresponds to the same solution, the change of coordinates

$$v = t + r_* \,, \qquad \chi = \varphi + \int \frac{a}{\Delta} \mathrm{d}r \,, \tag{2.20}$$

takes us back to the original Kerr form (2.13). The coordinate v is given by the known ingoing EF transformation, defined by the Regge-Wheeler coordinate, also named *tortoise* coordinate, which is very useful to construct null directions. In the case of the Kerr BL metric, it holds that

$$\frac{\mathrm{d}r_*}{\mathrm{d}r} = \frac{r^2 + a^2}{\Delta} \,. \tag{2.21}$$

These coordinates are usually referred as "Schwarzschild like", as it takes the spherical static case in standard curvature coordinates when setting $a = 0$. Time inversion symmetry is characteristic of static Schwarzschild spacetime, but the same does not hold for Kerr's. Nevertheless, this specific form is invariant under the inversion $(t, \varphi) \to (-t, -\varphi)$, also known as the *circular condition*, an intuitive notion from physical systems with angular momentum. This discrete symmetry eliminates most of the off-diagonal components of the BL metric, $g_{tr} = g_{\varphi r} = g_{t\theta} = g_{\varphi\theta} = 0$, making it the simplest to perform calculations.

Now, to study the possible horizons of Kerr BH, we will consider the one-form $n = \mathrm{d}r$ that defines normal vectors to constant radial hypersurfaces. It is easy to show that $n^2 = g^{rr}$, which implies that n is null when $\Delta = 0$, defining horizons at

$$r_\pm = M \pm \sqrt{M^2 - a^2} \,, \tag{2.22}$$

singularities at g_{rr} which we know to be removable. As a consequence, from a static observer a massless particle on an ingoing null geodesic would spiral around the BH for a infinite time, as the coordinate $t \to \infty$, never reaching $r = r_+$. This surface is the event horizon of the Kerr BH, as it separates two causally disconnected regions of spacetime, *i.e* any information from the inside this surface, will never reach any asymptotic observer. The expression for the event horizon surface also raises limitations for the amount of angular momentum a physical BH can have. We must have

$$|a| < M \,, \tag{2.23}$$

otherwise Δ would lack any real roots and would lead to a essential *naked singularity*, reachable in a finite observable time, which is forbidden by the *Weak Cosmic Censorship*.

The surface at $r = r_-$, on the other hand, is called a Cauchy horizon. In GR, a spacelike surface containing all initial conditions of spacetime (Cauchy surface) would suffice to predict all past and future events, but a Cauchy horizon separates the domain of validity of such initial conditions. Despite no information ever escaping the event horizon, it is still possible to predict events inside $r_- < r < r_+$, but such thing it is not guaranteed after crossing the Cauchy horizon. Due to this and some other unphysical features (for example, closed timelike curves and instabilities under perturbations), we need only to focus on the region outside the event horizon $r > r_+$, since only information on that region is physically reachable from a asymptotic observer point of view.

Event tough most of the Kerr BH basic properties were demonstrated, there is still no result so far showing some kind of rotation. First, consider the quantity $\xi \cdot u = \xi_\alpha u^\alpha$, where u^α is the four-velocity and ξ^α is any Killing field. Being aware of the geodesic equation, $u^\beta \nabla_\beta u^\alpha = 0$, it is easy to show that this quantity is conserved along geodesics,

$$u^\beta \nabla_\beta (\xi_\alpha u^\alpha) = u^\alpha u^\beta \nabla_\beta \xi_\alpha = \frac{u^\alpha u^\beta}{2} \left(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha \right) = 0 \,, \tag{2.24}$$

due to Killing Eq. (2.8). As a result, geodesics of a free particle in Kerr geometry will be characterized by two constants

$$E = k^\beta g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \,, \qquad -L = m^\beta g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \,, \tag{2.25}$$

where τ is the affine parameter fo the geodesic. These quantities can be interpreted the energy per mass and angular momentum per mass of the particle, respectively. Due to the circular form of the BL metric, the metric components of the coordinates (t, φ) define a product decomposition, providing the separation of previous equations,

$$\dot{t} = \frac{1}{\Delta} \left[(r^2 + a^2 + \frac{2Ma^2}{r}) E - \frac{2Ma}{r} L \right] \,, \tag{2.26a}$$

$$\dot{\varphi} = \frac{1}{\Delta} \left[\frac{2Ma}{r} E + \left(1 - \frac{2M}{r} \right) L \right] \,, \tag{2.26b}$$

specified for the equatorial plane $\theta = \pi/2$. The final equation for the geodesic is provided by the line element (2.19), which becomes also a first order ODE, after the substitution of $\dot{t}$ and $\dot{\varphi}$.

Consider now a zero angular momentum observer (ZAMO) infalling radially, with $L = 0$, then we can get the angular velocity Ω, as measured at infinity

$$\Omega = \frac{\dot{\varphi}}{\dot{t}} = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = \frac{2aM}{r^3 + a^2(2M + r)} \,. \tag{2.27}$$

Asymptotically we obtain $\Omega \to 0$, but for a finite distance, observers are forced to co-rotate with the BH. Particularly, at the event horizon, $r = r_+$, one finds that

$$\Omega_H = \frac{a}{2Mr_+} = \frac{J}{2M\left(M^2 + \sqrt{M^4 - J^2}\right)} . \tag{2.28}$$

A special linear combination of Killing vector fields,

$$\xi = k + \Omega_H m , \tag{2.29}$$

is also a null vector normal to the event horizon, $\xi^\flat = \xi_\alpha \, \mathrm{d}x^\alpha \propto \mathrm{d}r$ and $\xi^2 = 0$ at $r = r_+$, which can be written in BL coordinates as $\xi = \partial_t + \Omega_H \partial_\varphi$. Hence, on the outer horizon the integral curves of this vector obey $\xi^\alpha \partial_\alpha (\varphi - \Omega_H t) = 0$, resulting in $\varphi = \Omega_H t + const$. Since null geodesics on the horizon follow curves generated by the Killing vector ξ, then we say that the BH is "rotating" with angular velocity Ω_H.

2.5 Ergoregion and the Penrose process

One of the main characteristic that distinguishes Kerr BHs from other spherical solutions is the existence of a *ergoregion*. The surface is characterized when the Killing vector k^μ becomes spacelike, $k^2 = g_{tt} < 0$, resulting in the hypersurface boundary

$$r_{\mathrm{ergo}}(\theta) = M + \sqrt{M^2 - a^2 \cos^2\theta} . \tag{2.30}$$

This region lies outside the event horizon if $a \neq 0$, then being defined as $r_+ < r < r_{\mathrm{ergo}}(\theta)$. Notice that a static observer moves in a timelike curve with (r, θ, φ) constant, *i.e.* with tangent vector proportional to k^μ, therefore such observer cannot exist inside because the time Killing vector becomes spacelike, otherwise it would violate causality. We can see that $u^2 = g_{\alpha\beta} u^\alpha u^\beta = g_{tt}(u^t)^2 + 2g_{t\varphi} u^t u^\varphi + g_{\varphi\varphi}(u^\varphi)^2 > 0$ only occurs when $g_{t\varphi} u^\varphi > 0$, as all other therms are positive. Inside the ergoregion, $g_{t\varphi} > 0$, therefore all observers are forced to rotate in the same direction as the BH.

Despite BHs being always thought as "perfect absorvers" due to the horizon casual separation, the ergoregion allows energy extraction from the BH, through the Penrose process, an intrinsic feature of rotating BHs. Much like spontaneous pair creation and amplification at discontinuities are related but distinct effects, the Penrose process allows for a better understating of the phenomena of superradiance in GR.

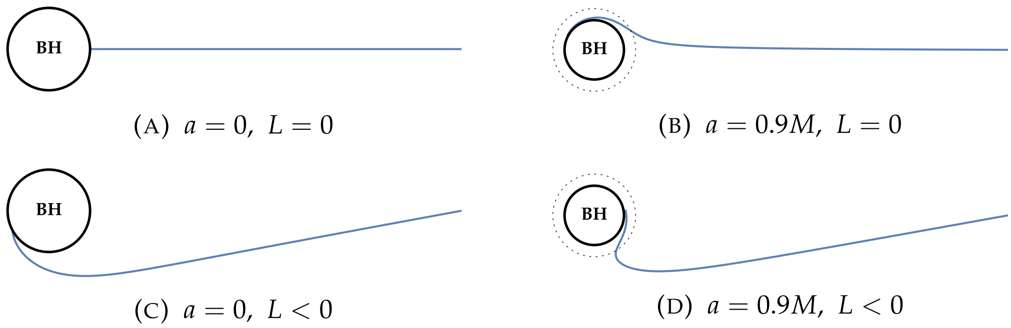

FIGURE 2.2: Illustration of the Schwarzschild (A,C) and Kerr (B,D) null equatorial infalling geodesics given by Eqs. (2.26a) and (2.26b), for $r(0) = 20M$, with emphasis on $L \neq 0$. Even starting with opposite angular momentum, the Kerr geodesic (D) is forced to co-rotate with the BH once crossed the ergoregion (dotted).

Considering a particle with rest mass μ and four-momentum $p^\alpha = \mu u^\alpha$, we may identify the constant of motion

$$E = k \cdot p = \mu(g_{tt}p^t + g_{t\varphi}p^\varphi) \, . \tag{2.31}$$

as it's energy measured by a stationary observer at infinity, due to relations (2.25). As shown above, the Killing vector is asymptotically timelike but is spacelike inside the ergoregion, thus $g_{tt} < 0$. For a future-directed geodesic, $p^0 = \mu u^0 > 0$, the energy beyond the ergosurface needs not to be positive. Suppose, by any means, that such particle manages to decay inside the ergoregion into two other, with momenta p_1 and p_2. Contracting with k, implies that $E = E_1 + E_2$. Supposing that the first of the particles has negative energy, $E_1 < 0$, then

$$E_2 = E + |E_1| > E \, . \tag{2.32}$$

I can be shown that the particle with negative energy (bounded) must fall into the BH while the other may escape the ergoregion, with greater energy than the particle sent in. Energy is conserved by making the BH absorb the particle with negative energy, therefore resulting in a net energy extraction.

To understand the limits of the Penrose process, we use the fact that a stationary observer near the horizon, must follow orbits of ξ, given by Eq. (2.29). Although a particle may have negative energy as measured from an asymptotic observer, the stationary one at the horizon must measure a positive energy, as both he and the particle must follow the same orbits, which implies that $\xi \cdot p_1 \leq 0$. The BH will have a variation of mass $\delta M = E_1$ and angular momentum $\delta J = L_1$, where $L_1 = -m \cdot p_1$ is the particle angular momentum.

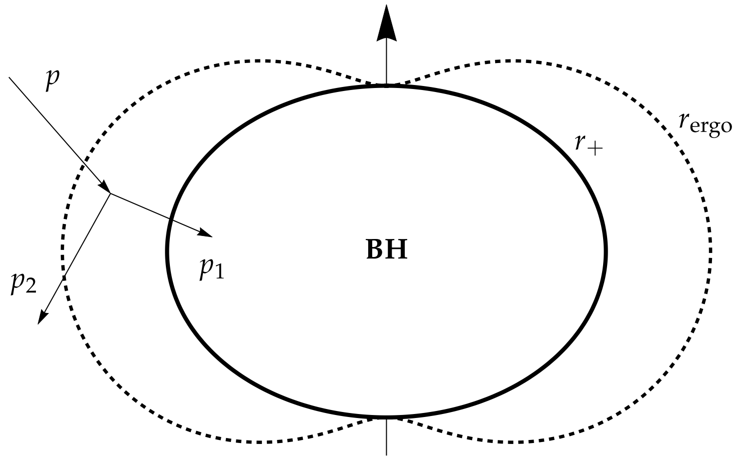

FIGURE 2.3: Illustration of the Penrose process, with ergoregion (dotted) and event horizon surfaces parameterized in Kerr-Child cartesian coordinates.

As a result,

$$\delta J \leq \frac{2M\left(M^2 + \sqrt{M^4 - J^2}\right)}{J}\delta M\,, \tag{2.33}$$

which is equivalent to $\delta\left(M^2 + \sqrt{M^4 - J^2}\right) \geq 0$. This quantity is usually refereed as the "area" of the event horizon $A = 4\pi(r_+^2 + a^2) = 8\pi\left(M^2 + \sqrt{M^4 - J^2}\right)$. Energy extraction from the Penrose process is limited by the requirement that the horizon area must always increase, which is a special case of the second law of BH mechanics. For this process to occur (and therefore superradiance), we must have a rotating BH, which is guaranteed to have a ergoregion.

Chapter 3

Teukolsky master equation

3.1 Newman-Penrose formalism

Study of gravitational and electromagnetic perturbations in a BH background were performed long before Kerr found his solution, for other spacetimes such as Schwarzschild's. Despite it's simplicity, the procedure involved was already algebraically tedious. In the Kerr case, the metric was far more complicated, making the problem almost untreatable.

Fortunately, the NP formalism [21] provides an alternative method of studying perturbations. Results as a natural introduction of spinor techniques into GR, after the choice of a null complex tetrad basis,

$$e_a = (e_a)^\mu \frac{\partial}{\partial x^\mu} \qquad (a = 1, 2, 3, 4) \,, \tag{3.1}$$

where all quantities will be projected, *i.e.* for the Weyl tensor we define

$$C_{abcd} = (e_a)^\alpha (e_b)^\beta (e_c)^\gamma (e_d)^\delta C_{\alpha\beta\gamma\delta} \,. \tag{3.2}$$

Penrose believed that the light-cone was the essential element of the spacetime, thus it was of importance to find null directions. The basis consisted in two real vectors, l and n, and two complex conjugate vectors m and $\bar{m}$. Besides satisfying

$$l^2 = n^2 = m^2 = \bar{m}^2 = 0 \,, \tag{3.3}$$

orthogonality conditions of NP formalism require

$$l \cdot m = l \cdot \bar{m} = n \cdot m = n \cdot \bar{m} = 0 \,. \tag{3.4}$$

Still we are left with the ambiguity raised by multiplication of scalar functions to each vector, therefore its customary to impose normalization conditions to the basis,

$$l \cdot n = 1\,, \qquad m \cdot \bar{m} = -1\,. \tag{3.5}$$

This formalism is a special case of tetrad calculus, where we can identify the new basis as $(l, n, m, \bar{m})$. The "metric" for manipulating tetrad indices, η_{ab}, is defined by all restrictions provided above,

$$g^{\mu\nu} = \eta^{ab}(e_a)^{\mu}(e_b)^{\nu} = l^{\mu}n^{\nu} + n^{\mu}l^{\nu} - m^{\mu}\bar{m}^{\nu} - \bar{m}^{\mu}m^{\nu}\,. \tag{3.6}$$

Additionally this vectors define new directional derivatives. We will depart shortly from standard notation [22–24], by redefining these derivatives as

$$\mathbb{D} = \nabla_l\,, \qquad \mathbb{\Delta} = \nabla_n\,, \qquad \mathbb{\delta} = \nabla_m\,, \qquad \bar{\mathbb{\delta}} = \nabla_{\bar{m}}\,. \tag{3.7}$$

More details and definitions on the tetrad formalism can be found in Appendix A.

3.1.1 Kinnersley tetrad

The Riemann tensor may have up to twenty non-vanishing components. We know that ten of these are present in the symmetric Ricci tensor, that is intrinsically connected to matter and energy. The other components are pure gravitational degrees of freedom and are encoded in the Weyl tensor. It becomes the most useful object when the Ricci tensor vanishes, such as vacuum solutions and source-free gravitational waves. In order to remove the Ricci tensor degrees of freedom, the tensor must be constructed trace-free,

$$\eta^{ad}C_{abcd} = C_{1bc2} + C_{1bc2} - C_{3bc4} - C_{4bc3} = 0\,. \tag{3.8}$$

Together with the other symmetries inherited from the Riemann tensor, for instance the first Bianchi identity, $C_{a[bcd]} = 0$, it is possible to vanish some components and rewrite others such that only ten degrees of freedom remain. As a result, in NP formalism the Weyl tensor can be represented by five complex scalars, usually chosen as

$$\begin{aligned}
\psi_0 &= -C_{1313} = -C_{\alpha\beta\gamma\delta}\, l^{\alpha}m^{\beta}l^{\gamma}m^{\delta}\,, \qquad & \psi_1 &= -C_{1213} = -C_{\alpha\beta\gamma\delta}\, l^{\alpha}n^{\beta}l^{\gamma}m^{\delta}\,,\\
\psi_2 &= -C_{1342} = -C_{\alpha\beta\gamma\delta}\, l^{\alpha}m^{\beta}\bar{m}^{\gamma}n^{\delta}\,, \qquad & \psi_3 &= -C_{1242} = -C_{\alpha\beta\gamma\delta}\, l^{\alpha}n^{\beta}\bar{m}^{\gamma}n^{\delta}\,,\\
\psi_4 &= -C_{2424} = -C_{\alpha\beta\gamma\delta}\, n^{\alpha}\bar{m}^{\beta}n^{\gamma}\bar{m}^{\delta}\,.
\end{aligned} \tag{3.9}$$

The complex conjugates can be obtained by doing the replacement $3 \rightleftarrows 4$, by exchanging m with $\bar{m}$ and vice-versa. Weyl tensor has a unique decomposition in therms of a linear combination of NP scalars and tensorial product of two-forms such as $l_{[\mu}n_{\nu]}$ and $m_{[\rho}\bar{m}_{\sigma]}$. It is clear that the values that these five complex scalars take is completely dependent on the choice of tetrad frame.

BH solutions are "type D" spacetimes according to Petrov's classification, which was a major restriction necessary to the discovery of Kerr's metric. For these spacetimes it is possible to find two different doubly-degenerate principal directions of the Weyl tensor, which we choose to be the real vectors of the tetrad, l and n [25]. These yield

$$C_{\mu\alpha\beta[\nu}l_{\rho]}l^{\alpha}l^{\beta} = 0\,, \qquad C_{\mu\alpha\beta[\nu}n_{\rho]}n^{\alpha}n^{\beta} = 0\,. \tag{3.10}$$

In NP formalism terms, this implies, respectively,

$$\psi_0 = \psi_1 = 0\,, \qquad \psi_3 = \psi_4 = 0\,. \tag{3.11}$$

Finding the principal directions may not be trivial, but we can apply successive local transformations of the six-parameter Lorentz group in order to rotate the tetrad vectors. This procedure allows for the simplification of the Weyl tensor by vanishing NP scalars, "locking" the orientation of the tetrad frame. Weyl scalar ψ_2 becomes invariant under boosts in the principal directions. These keep the light-cone structure intact by maintaining the direction of l and n unchanged (up to multiplication of scalar functions), being useful to change between ingoing and outgoing frames [24]. Kinnersly solved type D vaccum field equations [26], finding a suitable tetrad

$$\begin{aligned} l &= \left(\frac{r^2+a^2}{\Delta},\, 1,\, 0,\, \frac{a}{\Delta}\right)\,, \\ n &= \frac{1}{2\rho^2}\left(r^2+a^2,\, -\Delta,\, 0,\, a\right)\,, \\ m &= \frac{1}{\sqrt{2}\bar{\rho}^2}\left(ia\sin\theta,\, 0,\, 1,\, i\csc\theta\right)\,, \end{aligned} \tag{3.12}$$

where $\bar{\rho} = r + ia\cos\theta$ and $\rho^2 \equiv |\bar{\rho}|^2 = \bar{\rho}\bar{\rho}^*$.

The NP formalism provides a full set of first-order coupled diferential equations, relating the NP scalar (Weyl and Maxwell tensors) and the spin-coeficients resultant of the Kinnersley tetrad. These equations result from second Bianchi identity, $C_{\mu\nu[\rho\sigma;\lambda]} = 0$, and Eq. (2.5). To study GWs, instead of perturbing the background metric, the NP formalism provides a natural way of performing perturbations by modification of the tetrad,

$l = l^B + l^P$, $n = n^B + n^P$, etc., and also the NP scalars, $\psi_a = \psi_a{}^B + \psi_a{}^P$, maintaining only first-order terms. The formalism reveals decoupled equations for $\psi_0{}^P$ and $\psi_4{}^P$, which implies that these dynamic variables are the only independent degrees of freedom of the GWs.

3.1.2 Maxwell equations

We focus with more detail on EM perturbations with a fixed background because is a simpler procedure and then we will tie with the same master equation that also describes GW perturbations.

In NP formalism, all Maxwell equations, $F_{[\mu\nu;\rho]} = 0$ and Eq. (2.5), reduce to

$$F_{[ab|c]} = 0 \,, \qquad \eta^{bc} F_{ab|c} = 0 \,. \tag{3.13}$$

The Mawell tensor $F_{\mu\nu}$ has a total of six components which encodes the vector quantities of the electric and the magnetic fields. We may reduce the equation using three complex NP scalars,

$$\begin{aligned} &\phi_0 = F_{13} = F_{\alpha\beta} l^\alpha m^\beta \,, \qquad \phi_1 = \tfrac{1}{2}(F_{12} + F_{43}) = \tfrac{1}{2} F_{\alpha\beta}(l^\alpha n^\beta + \bar{m}^\alpha m^\beta) \,, \\ &\phi_2 = F_{42} = F_{\alpha\beta} \bar{m}^\alpha n^\beta \,. \end{aligned} \tag{3.14}$$

Considering all posible combinations of NP indices in (3.13), we gather eight equations, double the amount of necessary relations. This occurs because the conjugates ϕ_0^*, ϕ_1^*, ϕ_2^* are coupled in these equations. Eliminating every term of the form $F_{23|a}$ or $F_{14|b}$,

$$\phi_{2|1} = \phi_{1|4} \,, \tag{3.15a}$$

$$\phi_{1|2} = \phi_{2|3} \,, \tag{3.15b}$$

$$\phi_{1|1} = \phi_{0|4} \,, \tag{3.15c}$$

$$\phi_{0|2} = \phi_{1|3} \,. \tag{3.15d}$$

We may expand explicitly the left-hand side of Eq. (3.15a),

$$\begin{aligned} \phi_{2|1} &= \phi_{2,1} - \eta^{ab}(\gamma_{a41} F_{b2} + \gamma_{a21} F_{4b}) \\ &= \phi_{2,1} - (\gamma_{241} F_{12} + \gamma_{121} F_{42}) + (\gamma_{341} F_{42} + \gamma_{421} F_{43}) \\ &= \phi_{2,1} + 2F_{42}\left(\frac{\gamma_{341} + \gamma_{211}}{2}\right) + 2\gamma_{421}\left(\frac{F_{12} + F_{43}}{2}\right) \\ &= \mathbb{D}\phi_2 + 2\varepsilon\phi_2 - 2\pi\phi_1 \,, \end{aligned} \tag{3.16}$$

where we used the antisymmetry of the spin connection, $\gamma_{abc} = -\gamma_{bac}$. The right-hand side yields

$$\begin{aligned}
\phi_{1|4} &= \phi_{1,4} - \tfrac{1}{2}\eta^{ab}(\gamma_{a14}F_{b2} + \gamma_{a24}F_{1b} + \gamma_{a44}F_{b3} + \gamma_{a34}F_{4b}) \\
&= \phi_{1,4} - \tfrac{1}{2}(\gamma_{144}F_{23} + \gamma_{134}F_{42} + \gamma_{214}F_{12} + \gamma_{234}F_{41}) \\
&\qquad + \tfrac{1}{2}(\gamma_{314}F_{42} + \gamma_{414}F_{42} + \gamma_{324}F_{14} + \gamma_{424}F_{13}) \\
&= \phi_{2,1} - \gamma_{244}F_{13} + \gamma_{314}F_{42} \\
&= \bar{\delta}\phi_1 - \lambda\phi_0 + \tau\phi_2 \, .
\end{aligned} \tag{3.17}$$

The spin coefficients ε, π, λ, τ, along with other NP definitions are found in Appendix B.1. If we repeat the same expansion for the other Maxwell equations, we gather the set

$$\mathbb{D}\phi_2 - \bar{\delta}\phi_1 = -\lambda\phi_0 + 2\pi\phi_1 + (\varrho - 2\varepsilon)\phi_2 \, , \tag{3.18a}$$

$$\triangle\phi_1 - \delta\phi_2 = \nu\phi_0 - 2\mu\phi_1 + (2\beta - \tau)\phi_2 \, , \tag{3.18b}$$

$$\mathbb{D}\phi_1 - \bar{\delta}\phi_0 = (\pi - 2\alpha)\phi_0 + 2\varrho\phi_1 - \kappa\phi_2 \, , \tag{3.18c}$$

$$\triangle\phi_0 - \delta\phi_1 = (2\gamma - \mu)\phi_0 - 2\tau\phi_1 + \sigma\phi_2 \, . \tag{3.18d}$$

The Kinnersley tetrad guarantees that $\kappa = \sigma = \lambda = \nu = 0$, decoupling all equations above. After substitution of all spin coefficients,

$$\left(\mathbb{D} + \frac{1}{\bar{\rho}^*}\right)\phi_2 = \left(\bar{\delta} + \frac{2ia\sin\theta}{\sqrt{2}(\bar{\rho}^*)^2}\right)\phi_2 \, , \tag{3.19a}$$

$$\left(\triangle - \frac{\Delta}{\rho^2\bar{\rho}^*}\right)\phi_1 = \left[\delta + \frac{1}{\sqrt{2}\bar{\rho}}\left(\cot\theta - \frac{ia\sin\theta}{\bar{\rho}^*}\right)\right]\phi_2 \, , \tag{3.19b}$$

$$\left(\mathbb{D} + \frac{2}{\bar{\rho}^*}\right)\phi_1 = \left[\bar{\delta} + \frac{1}{\sqrt{2}\bar{\rho}^*}\left(\cot\theta - \frac{ia\sin\theta}{\bar{\rho}^*}\right)\right]\phi_0 \, , \tag{3.19c}$$

$$\left[\triangle + \frac{\Delta}{2\rho^2}\left(\frac{1}{\bar{\rho}^*} - \frac{2(r-M)}{\Delta}\right)\right]\phi_0 = \left(\delta + \frac{2ia\sin\theta}{\sqrt{2}\bar{\rho}\bar{\rho}^*}\right)\phi_1 \, . \tag{3.19d}$$

An important consequence of the symmetries of Kerr spacetime allows for a wave decomposition of the form $\phi_0, \phi_1, \phi_2 \sim e^{-i\omega t + im\varphi}$. Therefore, the four differential operators group into radial $(\mathbb{D}, \triangle)$ and angular $(\delta, \bar{\delta})$. The procedure for separation of the Maxwell equations can be further simplified by introducing new operators

$$\begin{aligned}
\mathcal{D}_n &= \partial_r - \frac{i\mathcal{K}}{\Delta} + 2n\frac{r-M}{\Delta} \, , \qquad & \mathcal{D}_n^\dagger &= \partial_r + \frac{i\mathcal{K}}{\Delta} + 2n\frac{r-M}{\Delta} \, , \\
\mathcal{L}_n &= \partial_\theta - \mathcal{Q} + n\cot\theta \, , \qquad & \mathcal{L}_n^\dagger &= \partial_\theta + \mathcal{Q} + n\cot\theta \, ,
\end{aligned} \tag{3.20}$$

where we define the functions $\mathcal{K} = (r^2 + a^2)\omega - ma$, $\mathcal{Q} = a\omega \sin\theta - m\csc\theta$. In this definition, n is a non-negative integer. These operators are related to the tetrad by

$$\mathbb{D} = \mathcal{D}_0\,, \qquad \mathbb{\Delta} = -\frac{\Delta}{2\rho^2}\mathcal{D}_0^\dagger\,, \qquad \mathbb{\delta} = \frac{1}{\sqrt{2}\bar{\rho}}\mathcal{L}_0^\dagger\,, \qquad \bar{\mathbb{\delta}} = \frac{1}{\sqrt{2}\bar{\rho}^*}\mathcal{L}_0\,, \tag{3.21}$$

as a result of the substitutions $\partial_t \to -i\omega$, $\partial_\varphi \to im$. We may use the fact that $\mathcal{D}_n$ and $\mathcal{L}_n$ act mostly as radial and angular derivatives, respectively, to deduce the properties

$$\mathcal{D}_n\Delta = \Delta\mathcal{D}_{n+1}\,, \tag{3.22a}$$

$$\mathcal{L}_n \sin\theta = \sin\theta\,\mathcal{L}_{n+1}\,, \tag{3.22b}$$

$$\left(\mathcal{D}_n + \frac{q}{\bar{\rho}^*}\right)\frac{1}{(\bar{\rho}^*)^p} = \frac{1}{(\bar{\rho}^*)^p}\left(\mathcal{D}_n + \frac{q-p}{\bar{\rho}^*}\right)\,, \tag{3.22c}$$

$$\left(\mathcal{L}_n + \frac{iqa\sin\theta}{\bar{\rho}^*}\right)\frac{1}{(\bar{\rho}^*)^p} = \frac{1}{(\bar{\rho}^*)^p}\left(\mathcal{L}_n + \frac{i(q-p)a\sin\theta}{\bar{\rho}^*}\right)\,, \tag{3.22d}$$

$$\left(\mathcal{D}_n + \frac{q}{\bar{\rho}^*}\right)\left(\mathcal{L}_n + \frac{iqa\sin\theta}{\bar{\rho}^*}\right) = \left(\mathcal{L}_n + \frac{iqa\sin\theta}{\bar{\rho}^*}\right)\left(\mathcal{D}_n + \frac{q}{\bar{\rho}^*}\right)\,, \tag{3.22e}$$

for any integers p, q, n, holding also for either $\mathcal{D}_n^\dagger$ or $\mathcal{L}_n^\dagger$.

In order to achieve the separable form, we still need to perform a replacement of the Maxwell NP scalars by new dynamical variables

$$\Phi_0 = \phi_0\,, \qquad \Phi_1 = \sqrt{2}\bar{\rho}^*\phi_1\,, \qquad \Phi_2 = 2(\bar{\rho}^*)^2\phi_2\,, \tag{3.23}$$

and using properties (3.22c) and (3.22d), we go from Eqs. (3.19) to

$$\left(\mathcal{D}_0 - \frac{1}{\bar{\rho}^*}\right)\Phi_2 = \left(\mathcal{L}_0 + \frac{ia\sin\theta}{\bar{\rho}^*}\right)\Phi_1\,, \tag{3.24a}$$

$$\Delta\left(\mathcal{D}_0^\dagger + \frac{1}{\bar{\rho}^*}\right)\Phi_1 = -\left(\mathcal{L}_1^\dagger - \frac{ia\sin\theta}{\bar{\rho}^*}\right)\Phi_2\,, \tag{3.24b}$$

$$\left(\mathcal{D}_0 + \frac{1}{\bar{\rho}^*}\right)\Phi_1 = \left(\mathcal{L}_1 - \frac{ia\sin\theta}{\bar{\rho}^*}\right)\Phi_0\,, \tag{3.24c}$$

$$\Delta\left(\mathcal{D}_1^\dagger - \frac{1}{\bar{\rho}^*}\right)\Phi_0 = -\left(\mathcal{L}_0^\dagger + \frac{ia\sin\theta}{\bar{\rho}^*}\right)\Phi_1\,. \tag{3.24d}$$

Now we may use commutatively property (3.22e) together with (3.22a) to separate the equations for Φ_0 and Φ_2. In order to obtain the first equation, we must first apply the operator $(\mathcal{L}_0^\dagger + ia\sin\theta/\bar{\rho}^*)$ to Eq. (3.24c) and then use the commutativity relation to substitute Eq. (3.24d). Similarly, applying $(\mathcal{L}_0 + ia\sin\theta/\bar{\rho}^*)$ to Eq. (3.24b) we obtain the final equation. Together yield

$$\left[\Delta\mathcal{D}_1\mathcal{D}_1^\dagger + \mathcal{L}_0^\dagger\mathcal{L}_1 + 2i\omega(r + ia\cos\theta)\right]\Phi_0 = 0\,, \tag{3.25}$$

$$\left[\Delta\mathcal{D}_0^\dagger\mathcal{D}_0 + \mathcal{L}_0\mathcal{L}_1^\dagger - 2i\omega(r + ia\cos\theta)\right]\Phi_2 = 0\,. \tag{3.26}$$

Still, there is another way of combining equations, *i.e* Eq. (3.24b) with (3.24d) and the remaining two form the set

$$\mathcal{L}_0\mathcal{L}_1\Phi_0 = \mathcal{D}_0\mathcal{D}_0\Phi_2\,, \tag{3.27}$$

$$\mathcal{L}_0^\dagger\mathcal{L}_1^\dagger\Phi_2 = \Delta\mathcal{D}_0^\dagger\mathcal{D}_0^\dagger\Delta\Phi_0\,. \tag{3.28}$$

Thus, we went from four first-order differential equations relating three NP scalars to four second-order differential equations, two of each decoupled, eliminating the need for the scalar Φ_1. The last two equations imply that each one of the complex NP scalars contains all the information necessary to describe a EM wave (two polarizations). One may think that we only need one of each group of equations to solve all perturbations, as of today no closed form solution has been found. Thus the problem has to be tackled using approximations or numerical methods, recurring to all last four equations.

Due to the nature of the operators $\mathcal{D}_n$ and $\mathcal{L}_n$, we may separate the equations for $\Phi_0 \sim R_{+1}(r)S_{+1}(\theta)$ and $\Phi_2 \sim R_{-1}(r)S_{-1}(\theta)$ into two pairs of equations,

$$\left(\Delta\mathcal{D}_0\mathcal{D}_0^\dagger + 2i\omega r\right)\Delta R_{+1} = \lambda\Delta R_{+1}\,, \tag{3.29a}$$

$$\left(\mathcal{L}_0^\dagger\mathcal{L}_1 - 2a\omega\cos\theta\right)S_{+1} = -\lambda S_{+1}\,, \tag{3.29b}$$

and

$$\left(\Delta\mathcal{D}_0^\dagger\mathcal{D}_0 - 2i\omega r\right)R_{-1} = \lambda R_{-1}\,, \tag{3.30a}$$

$$\left(\mathcal{L}_0\mathcal{L}_1^\dagger + 2a\omega\cos\theta\right)S_{-1} = -\lambda S_{-1}\,, \tag{3.30b}$$

where λ is a separation constant. We use the property (3.22a) in to obtain Eq. (3.29a). The constant λ must be real, as the angular differential operators $\mathcal{L}_n$ are also real. Notice that we not distinguish the separation constants of both equations. Performing the transformation $\theta \to \pi - \theta$, the angular operators transforms as $\mathcal{L}_0^\dagger\mathcal{L}_1 \to \mathcal{L}_0\mathcal{L}_1^\dagger$. Then if $S_{+1}(\theta)$ is a solution for Eq. (3.29b) for a given separation constant λ, this implies that $\tilde{S}_{-1}(\theta) = S_{+1}(\pi - \theta)$ is a solution for Eq. (3.30b) for the same constant. In other words, the separation constant must me the same for both equations. Also, solutions R_{-1} and

ΔR_{+1} obey the same complex conjugate equations due to $\mathcal{D}_n^\dagger = (\mathcal{D}_n)^*$.

The second-order equations relating Φ_0 and Φ_2 can be separated in the same fashion. Naturally, the separation constant will differ from the eigenvalue Eqs. (3.29) and (3.30). Using the same substitutions made previously, we divide each equation by the corresponding ansatz to obtain

$$\frac{\mathcal{L}_0\mathcal{L}_1 S_{+1}}{S_{-1}} = \frac{\Delta\mathcal{D}_0\mathcal{D}_0 R_{-1}}{\Delta R_{+1}} = \mathcal{B}\,, \tag{3.31}$$

$$\frac{\mathcal{L}_0^\dagger\mathcal{L}_1^\dagger S_{-1}}{S_{+1}} = \frac{\Delta\mathcal{D}_0^\dagger\mathcal{D}_0^\dagger \Delta R_{+1}}{R_{-1}} = \mathcal{B}\,. \tag{3.32}$$

The separation constant $\mathcal{B}$ is real and equal for both equations. This claim rests on the same arguments of the eigenvalue λ. We also make the angular functions S_{-1}, S_{+1} equally normalized. We may observe the latter by assuming two different separation constants B_1, B_2. Then, we have

$$\begin{aligned}(\mathcal{B}_1)^2\int_0^\pi d\theta\sin\theta\,(S_{-1})^2 &= \int_0^\pi d\theta\sin\theta\,(\mathcal{L}_0\mathcal{L}_1 S_{+1})^2\\ &= \int_0^\pi d\theta\sin\theta\,(\mathcal{L}_0^\dagger\mathcal{L}_1^\dagger\mathcal{L}_0\mathcal{L}_1 S_{+1})S_{+1}\\ &= \mathcal{B}_1\mathcal{B}_2\int_0^\pi d\theta\sin\theta\,(S_{+1})^2\,,\end{aligned} \tag{3.33}$$

where we used integration by parts twice. Thus $(\mathcal{B}_1)^2 = \mathcal{B}_1\mathcal{B}_2 = \mathcal{B}^2$. We can compute the coefficient by computing the operation

$$\begin{aligned}\mathcal{L}_0^\dagger\mathcal{L}_1^\dagger\mathcal{L}_0\mathcal{L}_1 &= \mathcal{L}_0^\dagger(\mathcal{L}_0\mathcal{L}_1^\dagger - 4a\omega\cos\theta)\mathcal{L}_1\\ &= \mathcal{L}_0\mathcal{L}_1^\dagger(-\lambda + 2a\omega\cos\theta) - 4a\omega\cos\theta\,\mathcal{L}_0^\dagger\mathcal{L}_1 + 4a\omega\sin\theta\,\mathcal{L}_1\\ &= -\lambda\,\mathcal{L}_0\mathcal{L}_1^\dagger + 2a\omega\left[\cos\theta\,\mathcal{L}_0\mathcal{L}_1^\dagger - \sin\theta(\mathcal{L}_1 + \mathcal{L}_1^\dagger)\right]\\ &\qquad\qquad - 4a\omega\cos\theta\,\mathcal{L}_0^\dagger\mathcal{L}_1 + 4a\omega\sin\theta\,\mathcal{L}_1\\ &= (-\lambda + 2a\omega\cos\theta)\mathcal{L}_0\mathcal{L}_1^\dagger + 4a\omega\mathcal{Q}\sin\theta\\ &= (-\lambda + 2a\omega\cos\theta)(-\lambda - 2a\omega\cos\theta) + 4a\omega(-a\omega\sin^2\theta + m)\\ &= \lambda^2 - 4a^2\omega^2 + 4a\omega m = \mathcal{B}^2\,,\end{aligned} \tag{3.34}$$

applied on the angular function S_{+1}. The commutation relations between the angular operators can by found directly or by noticing that $[e_a, e_b] = (\gamma^c{}_{ba} - \gamma^c{}_{ab})e_c$. Then using Eq. (3.29b) it is possible to eliminate the second-order angular operators. To obtain $\mathcal{B}^2$, the same procedure could be done for S_{-1} or the the radial functions.

3.1.3 Mode decomposition

Will be more profitable to study Eqs. (3.25) and (3.26) as a special case of the Teukolsky master equation which describes all the linearized perturbations arround the Kerr geometry. The generality of this equation is the primary reason for the focus on the EM case. The treatment for GWs differs in the perturbation formalism only in algebraic complexity, resulting in the same master equation. With the Teukolsky master equation we can proceed considering general perturbations, but there are several numerical and analytical details that make EM waves and GWs differ later on.

The general equation reads

$$\begin{aligned}
&\frac{1}{\Delta^s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial \Upsilon_s}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Upsilon_s}{\partial\theta}\right)-\left[\frac{(r^2+a^2)^2}{\Delta}-a^2\sin^2\theta\right]\frac{\partial^2\Upsilon_s}{\partial t^2}\\
&-\frac{4Mar}{\Delta}\frac{\partial^2\Upsilon_s}{\partial t\partial\varphi}-\left(\frac{a^2}{\Delta}-\frac{1}{\sin^2\theta}\right)\frac{\partial^2\Upsilon_s}{\partial\varphi^2}+2s\left[\frac{M(r^2-a^2)}{\Delta}-r-ia\cos\theta\right]\frac{\partial\Upsilon_s}{\partial t}\\
&+2s\left[\frac{a(r-M)}{\Delta}+\frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\Upsilon_s}{\partial\varphi}-(s^2\cot^2\theta-s)\Upsilon_s=0\,,
\end{aligned}\tag{3.35}$$

where s is the field *spin weight* and each field quantity Υ_s is related to the NP scalars as shown in the Table 3.1. Depending on the spin weight, the equation may describe massless scalar ($s=0$) or Dirac fields ($s=\pm\frac{1}{2}$), as well as electromagnetic ($s=\pm1$) or gravitational waves ($s=\pm2$). Substituting the spin-weight for the EM waves we obtain Eqs. (3.25) and (3.26).

s	Υ_s
$+1$	$\Phi_0=\phi_0$
-1	$\Phi_2=2(\bar{\rho}^*)^2\phi_2$
$+2$	$\Psi_0=\psi_0$
-2	$\Psi_4=(\bar{\rho}^*)^4\psi_4$

TABLE 3.1: Performance at peak F-measure

Obviously, Teukolsky equation is explicitly independent of t and φ, thus Υ_s accepts a decomposition in $e^{-i\omega t+im\varphi}$, which we already assumed in the EM case to separate the equations. Stationary and axisymmetry of the spacetime geometry guarantees this form. The azimuthal wave number m must be an integer, due to periodic boundary conditions on the BL coordinate φ. We may separate all perturbations in a completely general mode decomposition

$$\Upsilon_s = \int \mathrm{d}\omega\, e^{-i\omega t} \left(\sum_{\ell,m} e^{im\varphi}\, {}_sS_{\ell m}(\theta) {}_sR_{\ell m}(r) \right) . \tag{3.36}$$

The integer ℓ plays a role in labelling all possible solutions for the eigenvalue problem of both radial and angular equations,

$$\frac{1}{\Delta^s} \frac{\mathrm{d}}{\mathrm{d}r} \left(\Delta^{s+1} \frac{\mathrm{d}\, {}_sR_{\ell m}}{\mathrm{d}r} \right) + \left[\frac{\mathcal{K}^2 - 2is(r-M)\mathcal{K}}{\Delta} + 4is\omega r - {}_s\mathcal{F}_{\ell m} \right] {}_sR_{\ell m} = 0 \,, \tag{3.37}$$

$$\frac{1}{\sin\theta} \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\sin\theta \frac{\mathrm{d}\, {}_sS_{\ell m}}{\mathrm{d}\theta} \right) + \left[a^2\omega^2 \cos^2\theta - 2sa\omega\cos\theta - \frac{(m+s\cos\theta)^2}{\sin^2\theta} + s + {}_s\mathcal{A}_{\ell m} \right] {}_sS_{\ell m} = 0 \,. \tag{3.38}$$

The radial and angular eigenvalues are related to the separation constant on Eqs. (3.29) and (3.30) through

$${}_s\mathcal{F}_{\ell m} = {}_s\mathcal{A}_{\ell m} - 2ma\omega + a^2\omega^2 \underset{(s=\pm 1)}{=} \lambda - s(s+1) \,. \tag{3.39}$$

Due to the form of the angular equation, the eigenvalues ${}_s\mathcal{C}_{\ell m}$, ${}_s\mathcal{A}_{\ell m}$ as well as the function ${}_sS_{\ell m}(\theta)$ depends also on the coupling $a\omega$. Clearly, the same does not hold for the radial function ${}_sR_{\ell m}(r)$.

3.2 Spin-Weighted Spheroidal Harmonics

To shed some light into the explicit form of λ, we will need to dive into the eigenvalue problem for the angular equation. We may transform the Eq. (3.38) into a more familiar form using the change of coordinate $z = \cos\theta$ and renaming the BH-wave coupling $c = a\omega$, obtaining

$$\frac{\mathrm{d}}{\mathrm{d}z} \left[(1-z^2) \frac{\mathrm{d}\, {}_sS_{\ell m}}{\mathrm{d}z} \right] + \left[(cz)^2 - 2csz - \frac{(m+sz)^2}{1-z^2} + s + {}_s\mathcal{A}_{\ell m} \right] {}_sS_{\ell m} = 0 \,. \tag{3.40}$$

We may also use freely ${}_sS_{\ell m}(\cos\theta) \equiv {}_sS_{\ell m}(\theta)$. We will consider c real as we are analyzing superradiance of EM waves in the vacuum, although we could generalize the spin-weighted spheroidal harmonic (SWSH) equation to imaginary c values to describe waves in a particular medium.

Spherically symmetric problems allow for a decomposition using spherical harmonics $Y_{\ell m}(\theta, \varphi)$ of angular dependent functions with finite boundary conditions. These have innumerable applications in physics such as the hydrogen atom or the description of anisotropies in the cosmic microwave background. By setting $s = 0$ and $c = 0$ (spherical), then it is clear that solution for Eq. (3.40) are given by the associated Legendre polynomials, $P_\ell^m(z)$, and the therefore it is a generalization of the spherical harmonics, *i.e.*

$$e^{im\varphi}\,{}_0S_{\ell m}(\theta) \underset{(c=0)}{=} Y_{\ell m}(\theta,\varphi)\,, \qquad {}_0\mathcal{A}_{\ell m} \underset{(c=0)}{=} \ell(\ell+1)\,. \tag{3.41}$$

The value of ℓ are non-negative integers, with the restriction of $\ell \geq |m|$. Since this generalization must hold for the spherical case, we require that spin-weighted spheroidal harmonics (SWSHs) are normalized for any spin s and coupling c,

$$\int_0^\pi \mathrm{d}\theta \sin\theta\, |\,{}_sS_{\ell m}(\theta)|^2 = \int_{-1}^1 \mathrm{d}z\, |\,{}_sS_{\ell m}(z)|^2 = \frac{1}{2\pi}\,. \tag{3.42}$$

Still on subject of spherical harmonics ($c = 0$), perturbations of any type in Schwarzschild spacetime are described using spin-weighted *spherical* harmonics, ${}_sY_{\ell m}(\theta,\varphi)$. Due to the shared symmetries with spherical harmonics it is possible to find a closed form for $s \neq 0$ harmonics, which is a well studied problem. We can raise and lower spin-weight with the use of the operators commonly denoted as $\bar{\eth}$ and $\eth$, respectively, applied on $Y_{\ell m}$,

$$\begin{aligned} (\bar{\eth})^s Y_{\ell m} &= \sqrt{\frac{(\ell+s)!}{(\ell-s)!}}\,{}_sY_{\ell m}\,, \\ (-1)^s(\eth)^s Y_{\ell m} &= \sqrt{\frac{(\ell+s)!}{(\ell-s)!}}\,{}_{-s}Y_{\ell m}\,, \end{aligned} \tag{3.43}$$

limited by $|s| \leq \ell$. Eqs. (3.31) and (3.32) are closely related to former, as applications of operators $\mathcal{L}_s^\dagger$ and $\mathcal{L}_s$, generalize $\bar{\eth}$ and $\eth$, respectively.

Thus, for the non-spherical symmetry we could in principle obtain all SWSHs if we knew the closed form for ${}_0S_{\ell m}$ and its eigenvalues. The problem lies in the fact that no such decomposition of elementary function was found. This require us to follow numerical and approximate methods to find the values of ${}_sS_{\ell m}$.

The major advances on the study of the eigenvalues of the SWSHs was performed by Leaver in 1985. Working out the asymptotical and critical behavior of the equation, we observe that equation the equation diverges at the poles, $z = \pm 1$, where the it takes the form $(1 \mp z)\,{}_sS_{\ell m}{}'(z) \sim \mp\frac{1}{4}(m \pm s)^2\,{}_sS_{\ell m}(z)$. In order to guarantee that SWSH is everywhere analytical, Leaver proposed the series expansion at $x = -1$,

$${}_sS_{\ell m}(z) = e^{cz}(1+z)^{k_-}(1-z)^{k_+}\sum_{p=0}^{\infty} a_p(1+z)^p\,, \tag{3.44}$$

where $k_\pm = \frac{1}{2}|m \pm s|$. The exponential in the ansatz accounts for the large z behavior of the equation. Substituting in the angular equation, we obtain a three-term recurrence relation between the expansion coefficients a_p and the boundary condition at $x = -1$,

$$\alpha_p a_{p+1} + \beta_p a_p + \gamma_p a_{p-1} = 0 \,, \qquad \alpha_0 a_1 + \beta_0 a_0 = 0 \,, \tag{3.45}$$

where

$$\begin{aligned} \alpha_p &= -2(1+p)(1+2k_+ + p) \,, \\ \beta_p &= (k_- + k_+ + p - s)(1 + k_- + k_+ + p + s) \\ &\qquad\qquad - 2c(1 + 2k_- + 2p + s) - c^2 - {}_s\mathcal{A}_{\ell m} \,, \\ \gamma_p &= 2c(k_- + k_+ + p + s) \,. \end{aligned} \tag{3.46}$$

We then find an equation for the eigenvalue ${}_s\mathcal{A}_{\ell m}$ with explicit dependence on m, s and c, by combining the previous relations into a continued fraction,

$$\beta_0 = \frac{\alpha_0\gamma_1}{\beta_1 -}\frac{\alpha_1\gamma_2}{\beta_2 -}\frac{\alpha_2\gamma_3}{\beta_3 -}\cdots \left(\equiv \frac{\alpha_0\gamma_1}{\beta_1 - \frac{\alpha_1\gamma_2}{\beta_2 - \frac{\alpha_2\gamma_3}{\beta_3 - \dots}}} \right) \,. \tag{3.47}$$

We can also consider the r-th inversion of Eq. (3.47),

$$\beta_r - \frac{\alpha_{r-1}\gamma_r}{\beta_{r-1} -}\frac{\alpha_{r-2}\gamma_{r-1}}{\beta_{r-2} -}\cdots\frac{\alpha_1\gamma_2}{\beta_1 -}\frac{\alpha_0\gamma_1}{\beta_0} = \frac{\alpha_r\gamma_{r+1}}{\beta_{r+1} -}\frac{\alpha_{r+1}\gamma_{r+2}}{\beta_{r+2} -}\cdots \,. \tag{3.48}$$

These equations involve an infinite fraction that depends explicitly on c, m, s, which leads to suspicion that this leads to a infinite spectra. This is a close parallel to the spherical case, where we have a infinite number of harmonics, although no proof has been found. If we notice that $\gamma_p \propto c$, then the zero order expansion of the eigenvalue expansion on for $c \ll 1$ leads to the equation $\beta_r = 0$. In the spherical geometry, this corresponds to truncating the series at r, since $\gamma_p = 0$ for any p. Since the the eigenvalue root depends on integer r,

$${}_s\mathcal{A}_{\ell m} = \ell(\ell+1) - s(s+1) + \mathcal{O}(a\omega) \,, \tag{3.49}$$

implying the discretization of the spectra, where we identified $\ell = r + k_+ + k_-$. Since $r \geq 0$, then we must have $\ell \geq \max\{|m|, |s|\}$, *i.e.* the leading contribution for the monopole expansion is the dipole for EM waves and the quadrupole for GWs. Changing r for ℓ corresponds simply to a relabeling of the eigenfunctions in order to match the values of the spectra when $c = 0$ and also when $s = 0$.

(MORE ON PERTURBATIONS $a\omega \sim 1$)

Will be useful expanding ${}_s\mathcal{A}_{\ell m}$ in high order terms in order to obtain the series coefficients for the eigenvalue (Appendix D). Up to sixth order, the only the zero-order term depends on the sign of the spin weight as it occurs in Eq. (3.39). In the general angular equation, inversion of spin corresponds to inversion of poles, *i.e.* stays invariant under the transformation $(s, z) \to (-s, -z)$. Under this transformation, the eigenvalue must obey $s + {}_s\mathcal{A}_{\ell m} = -s + {}_{-s}\mathcal{A}_{\ell m}$, thus it is beneficial to define

$$ {}_s\mathcal{E}_{\ell m} = {}_s\mathcal{A}_{\ell m} - s(s+1) \, , \tag{3.50} $$

to also exploit the symmetry of spin inversion in numerical computations.

3.3 Analytic radial expansions

3.3.1 Near horizon approximation

3.3.2 Assymptotic expansion

3.4 Amplification factor Z_{slm}

Chapter 4

Numerical method

4.1 Eigenvalues

4.1.1 Leaver method

4.1.2 Spectral

4.2 Radial equation

4.3 Amplification factor as a first test

Chapter 5

Scattering problem

5.1 Plane wave decomposition

Appendix A

Tetrad techniques

A.1 Noncoordinate representation

The standard way of expressing quantities in GR was to use a local coordinate basis. This corresponds to use

$$\frac{\partial}{\partial x^\mu} \qquad (x^\mu = t, r, \theta, \varphi) \tag{A.1}$$

as our vector basis. One-form basis can be defined the usual way. The tetrad formalism allows for an alternative choice of a *noncoordinate* basis, by introducing a set of linear independent four-vectors,

$$e_a = (e_a)^\mu \frac{\partial}{\partial x^\mu} \qquad (a = 1, 2, 3, 4)\ . \tag{A.2}$$

We will use Greek alphabet $(\alpha, \beta, \gamma, \dots)$ for the coordinate components and the Latin alphabet $(a, b, c, \dots)$ for the tetrad components. The tetrad fields also defined directional derivatives, for example for any scalar field f

$$f_{,a} = e_a(f) = (e_a)^\mu \frac{\partial f}{\partial x^\mu} = (e_a)^\mu f_{,\mu}\ . \tag{A.3}$$

However, this formalism must not be mistaken with as change of coordinates, $y = \phi(x)$, such that $(e_a)^\mu = \partial x^\mu / \partial y^a$, since those coordinates may not exist. A tetrad frame is a pointwise rotation of the coordinate frame, *i.e.* the concept is related to passive transformations rather that active (diffeomorphisms).

Given any tensor field $F_{\mu\nu}$, we can obtain its tetrad components by projecting it onto the tetrad frame,

$$F_{ab} = (e_a)^\mu (e_b)^\nu F_{\mu\nu} \,. \tag{A.4}$$

We may invert this expression, by defining the inverse tetrad, $(e^a)_\mu$, such that

$$(e_a)^\mu (e^b)_\mu = (e_a)^\mu (e^b)^\nu g_{\mu\nu} = \delta_a{}^b \,, \tag{A.5}$$

hence invariant quantities remain unchanged,

$$A^2 = A^\mu A_\mu = A^a (e_a)^\mu \, A_b (e^b)_\mu = A^a A_a \,. \tag{A.6}$$

We can then substitute the manifold metric for the tetrad "metric"

$$\eta_{ab} = e_a \cdot e_b = g_{\mu\nu} (e_a)^\mu (e_b)^\nu \,, \tag{A.7}$$

which can be used for raising/lowering tetrad indices,

$$A^a = \eta^{ab} A_b \,, \tag{A.8}$$

and to contract tetrad components, such as $\eta_{ab} A^a A^b = A^2$. This implies that we may return to the original metric using

$$g^{\mu\nu} = \eta^{ab} (e_a)^\mu (e_b)^\nu \,. \tag{A.9}$$

By analyzing the underlying symmetries of spacetime, one may choose a basis makes the components of η_{ab} constant, which is particularly important for the NP formalism. Going forward, we will assume that this is the case.

A.2 Spin connection

The analogy with the coordinate basics breaks when applying the a directional derivative of a tetrad components,

$$A_{a,b} = (e_b)^\nu \partial_\nu A_a = (e_b)^\nu \nabla_\nu \left[(e_a)^\mu A_\mu \right] = (e_a)^\mu (e_b)^\nu A_{\mu;\nu} + (e_c)^\mu (e_a)_{\mu;\nu} (e_b)^\nu A^c \,, \tag{A.10}$$

due to extra terms resultant of the tetrad derivatives. Tetrad decompositions, A^a, are scalars and must not be mistaken as vector fields such as $(e_a)^\mu$ or A^μ. These extra terms can be written using the spin connection

$$\gamma_{cab} = (e_c)^{\mu}(e_a)_{\mu;\nu}(e_b)^{\nu} \,, \tag{A.11}$$

which is antisymmetric in the first two indices,

$$\gamma_{cab} = -\gamma_{acb} \,, \tag{A.12}$$

due to the metric compatibility of the covariant derivative, $\nabla_{\mu} g_{\nu\rho} = 0$. Nonetheless, this only holds if η_{ab} is constant, otherwise we would have $\gamma_{abc} + \gamma_{bac} = \eta_{ab,c}$

The other term is called the *intrinsic* derivative of A_a in the direction of e_b, being defined as the projection of the tensor $A_{\mu;\nu}$ in the tetrad frame,

$$A_{a|b} = (e_a)^{\mu}(e_b)^{\nu} A_{\mu;\nu} \tag{A.13}$$

If we have a higher rank tensor, $F_{\mu\nu}$, we can generalize the intrinsic derivative inverting Eq. (A.10) and generalizing for multiple indices with the use of the spin connection,

$$F_{ab|c} = F_{ab,c} - \eta^{nm}(F_{nb}\gamma_{mac} + F_{an}\gamma_{mbc}) \,. \tag{A.14}$$

Obviously, the spin connection replaces the Christoper symbols, $\Gamma^{\rho}_{\mu\nu}$, in the tetrad formalism, although they are fundamentally different. We will avoid the computation of the Christoper symbols because every equation involving a covariant derivative will become an intrinsic derivative in NP formalism due to tetrad projections. This will become useful during calculations as we generally need $\frac{1}{2}d^2(d+1)$ computations to fully define the latter, while the spin connection has $\frac{1}{2}d^2(d-1)$ independent components. For $d = 4$, the use of the spin connection implies 16 components less to work with.

Appendix B

Additional Newman-Penrose definitions and computations

In this appendix we will present important computations of NP formalism in Kerr background (2.19), using the Kinnersley tetrad defined in (3.12). We will find useful in the one form conversion from the Kinnersley vectors, $(e_a)^\flat = (e_a)_\mu \, \mathrm{d}x^\mu$,

$$
\begin{aligned}
l^\flat =& \frac{1}{\Delta}\left(\Delta, -\rho^2, 0, -a\Delta\sin^2\theta\right) , \\
n^\flat =& \frac{1}{2\rho^2}\left(\Delta, \rho^2, 0, -a\Delta\sin^2\theta\right) , \\
m^\flat =& \frac{1}{\sqrt{2}\bar{\rho}}\left(ia\sin\theta, 0, -\rho^2, -i(r^2+a^2)\sin\theta\right) ,
\end{aligned}
\tag{B.1}
$$

where $\Delta = r^2 - 2Mr + a^2,\ \bar{\rho} = r + ia\cos\theta,\ \rho^2 = \bar{\rho}\bar{\rho}^*$.

B.1 Spin coefficients

The spin connection is defined as the covariant derivative of the tetrad field projected onto the tetrad frame. For example, we write $\gamma_{412} = \bar{m}^\mu l_{\mu;\nu} n^\nu = \bar{m}^\mu n^\nu \nabla_\nu l_\mu = \bar{m}^\mu \Delta l_\mu$. It has 24 components due to the antisymmetry of the first tetrad indices. These can be encapsulated using 12 complex variables,

$$
\begin{aligned}
\kappa =&\gamma_{311} , & \varrho =&\ \gamma_{314} , & \varepsilon =&\tfrac{1}{2}(\gamma_{211}+\gamma_{341}) , \\
\sigma =&\gamma_{313} , & \mu =&\gamma_{243} , & \gamma =&\tfrac{1}{2}(\gamma_{212}+\gamma_{342}) , \\
\lambda =&\gamma_{311} , & \tau =&\ \gamma_{312} , & \alpha =&\tfrac{1}{2}(\gamma_{214}+\gamma_{344}) , \\
\nu =&\gamma_{311} , & \pi =&\gamma_{241} , & \beta =&\tfrac{1}{2}(\gamma_{213}+\gamma_{343}) .
\end{aligned}
\tag{B.2}
$$

The computation of these coefficients can be done without computation of the Christopher symbols associated with the covariant derivative. This is cleverly avoided by observing that for any torsion-free connection, $(e_b)_{[\mu;\nu]} = (e_b)_{[\mu,\nu]}$. Therefore we define

$$\lambda_{abc} = (e_a)^{\mu}(e_c)^{\nu}\left[(e_b)_{\mu,\nu} - (e_b)_{\nu,\mu}\right] . \tag{B.3}$$

The computation of the various spin coefficients can be easily performed noticing that $\lambda_{abc} = \gamma_{abc} - \gamma_{cba}$, which can be inverted to

$$\gamma_{abc} = \frac{1}{2}\left(\lambda_{abc} + \lambda_{cab} - \lambda_{bca}\right) . \tag{B.4}$$

All relevant non-vanishing λ-symbols can be computed by simple coordinate derivatives on the one-form basis,

$$\begin{aligned}
\lambda_{122} &= -\frac{1}{\rho^4}\left[(r-M)\rho^2 - r\Delta\right] , & \lambda_{314} &= -\frac{2ia\cos\theta}{\rho^2} , \\
\lambda_{132} &= \frac{i\sqrt{2}ar\sin\theta}{\rho^2\bar{\rho}} , & \lambda_{324} &= -\frac{ia\Delta\cos\theta}{\rho^4} , \\
\lambda_{213} &= -\frac{\sqrt{2}a^2\cos\theta\sin\theta}{\rho^2\bar{\rho}} , & \lambda_{334} &= \frac{(ia + r\cos\theta)\csc\theta}{\sqrt{2}\bar{\rho}^2} , \\
\lambda_{243} &= -\frac{\Delta}{2\rho^2\bar{\rho}} , & \lambda_{341} &= -\frac{1}{\bar{\rho}} .
\end{aligned} \tag{B.5}$$

All other necessary symbols may be found by the symmetry $\lambda_{abc} = -\lambda_{cba}$ or by complex conjugation ($3 \rightleftarrows 4$). For example, for computing the spin coefficient μ, we need to use the relation $\lambda_{432} = -\lambda_{234} = -(\lambda_{243})^*$. Assembling all symbols, we obtain

$$\begin{gathered}
\kappa = \sigma = \lambda = \nu = 0 , \\
\varrho = -\frac{1}{\bar{\rho}} , \qquad \mu = -\frac{\Delta}{2\rho^2\bar{\rho}^*} , \qquad \tau = -\frac{ia\sin\theta}{\sqrt{2}\rho^2} , \qquad \pi = \frac{ia\sin\theta}{\sqrt{2}(\bar{\rho}^*)^2} , \\
\varepsilon = 0 , \qquad \gamma = \mu + \frac{r-M}{2\rho^2} , \qquad \alpha = \pi - \beta^* , \qquad \beta = \frac{\cot\theta}{2\sqrt{2}\bar{\rho}} .
\end{gathered} \tag{B.6}$$

Appendix C

Spin-weighted spherical harmonics

SWSHs play an important role BH physics and was first introduced by Teukolsky when considering non-scalar wave perturbations on a Kerr background, obtaining a separable master equation in four dimensions. After the usual change of coordinates, the polar differential equation goes as

$$\frac{1}{S}\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\,\frac{\mathrm{d}S}{\mathrm{d}x}\right)+(cx)^2-2csx-\frac{(m+sx)^2}{1-x^2}+s=-\lambda \tag{C.1}$$

with $x=\cos\theta$, where λ is the eigenvalue for a given SWSH solution. Periodic boundary conditions on the azimuthal wave function constrains m to the integers.

C.1 Connection with spheroical harmonics

By setting $s=0$ (scalar) and $c=0$ (spherical), then it's clear that Eq. (C.1) appears as a generalization of the spherical harmonics equation. In this last case, the solution are given by the associated Legendre polynomials, $P_\ell^m(x)$, for which the eigenvalue is $\ell(\ell+1)$, restricted to the condition of $|m|\leq\ell$. The closed form for spherical harmonics, after normalization, is

$$ {}_0Y_\ell^m(x)=(-1)^m\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(x) \tag{C.2}$$

where P_ℓ^m are the associated Legendre polynomials can be obtained using Rodrigues' formula.

C.2 Spin raising/lowering differential operators

C.3 Generalized addition of angular momentum formula

C.4 Some useful harmonics

Appendix D

Eigenvalue small-c expansion

Some text

$$f_0 = \ell(\ell+1) - s(s+1) \tag{D.1a}$$

$$f_1 = -\frac{2ms^2}{\ell(\ell+1)} \tag{D.1b}$$

$$f_2 = H(\ell+1) - H(\ell) - 1 \tag{D.1c}$$

$$f_3 = 2ms^2\left[\frac{H(\ell)}{(\ell-1)\ell^2(\ell+1)} - \frac{H(\ell+1)}{\ell(\ell+1)^2(\ell+2)}\right] \tag{D.1d}$$

$$\begin{aligned} f_4 = 4m^2s^4&\left[\frac{H(\ell+1)}{\ell^2(\ell+1)^4(\ell+2)^2} - \frac{H(\ell)}{(\ell-1)^2\ell^4(\ell+1)^2}\right] - \frac{H(\ell)^2}{2\ell} + \frac{H(\ell+1)^2}{2(\ell+1)} \\ &+ \frac{(\ell-1)H(\ell-1)H(\ell)}{2\ell(2\ell-1)} + \frac{H(\ell)H(\ell+1)}{2\ell(\ell+1)} - \frac{(\ell+2)H(\ell+1)H(\ell+2)}{2(\ell+1)(2\ell+3)} \end{aligned} \tag{D.1e}$$

$$\begin{aligned}
f_5 &= 8m^3s^6\left[\frac{H(\ell)}{(\ell-1)^3\ell^6(\ell+1)^3} - \frac{H(\ell+1)}{\ell^3(\ell+1)^6(\ell+2)^3}\right] \\
&\quad + 2ms^2\left[\frac{3H(\ell)^2}{2(\ell-1)\ell^3(\ell+1)} - \frac{3H(\ell+1)^2}{2\ell(\ell+1)^3(\ell+2)} + \frac{(3\ell+7)H(\ell+1)H(\ell+2)}{2\ell(\ell+1)^3(\ell+3)(2\ell+3)}\right. \\
&\qquad \left. - \frac{(3\ell-4)H(\ell-1)H(\ell)}{2(\ell-2)\ell^3(\ell+1)(2\ell-1)} - \frac{(7\ell^2+7\ell+4)H(\ell)H(\ell+1)}{2(\ell-1)\ell^3(\ell+1)^3(\ell+2)}\right]
\end{aligned} \tag{D.1f}$$

$$\begin{aligned}
f_6 &= \frac{16m^4s^8}{\ell^4(\ell+1)^4}\left[\frac{H(\ell+1)}{(\ell+1)^4(\ell+2)^4} - \frac{H(\ell)}{(\ell-1)^4\ell^4}\right] \\
&\quad + \frac{4m^2s^4}{\ell^2(\ell+1)^2}\left[\frac{3H(\ell+1)^2}{(\ell+1)^3(\ell+2)^2} - \frac{3H(\ell)^2}{(\ell-1)^2\ell^3} - \frac{(3\ell^2+14\ell+17)H(\ell+1)H(\ell+2)}{(\ell+1)^3(\ell+2)(\ell+3)^3(2\ell+3)}\right. \\
&\quad \left. + \frac{(11\ell^4+22\ell^3+31\ell^2+20\ell+6)H(\ell)H(\ell+1)}{(\ell-1)^2\ell^3(\ell+1)^3(\ell+2)^2} + \frac{(3\ell^2-8\ell+6)H(\ell-1)H(\ell)}{(\ell-2)^2(\ell-1)\ell^3(2\ell-1)}\right] \\
&\quad + \frac{H(\ell+1)^3}{2(\ell+1)^2} - \frac{H(\ell)^3}{2\ell^2} - \frac{(\ell-1)^2H(\ell-1)^2H(\ell)}{4\ell^2(2\ell-1)^2} + \frac{(\ell-1)(7\ell-3)H(\ell-1)H(\ell)^2}{4\ell^2(2\ell-1)^2} \\
&\quad + \frac{(2\ell^2+4\ell+3)H(\ell)^2H(\ell+1)}{4\ell^2(\ell+1)^2} - \frac{(2\ell^2+1)H(\ell)H(\ell+1)^2}{4\ell^2(\ell+1)^2} \\
&\quad - \frac{(\ell+2)(7\ell+10)H(\ell+1)^2H(\ell+2)}{4(\ell+1)^2(2\ell+3)^2} + \frac{(\ell+2)^2H(\ell+1)H(\ell+2)^2}{4(\ell+1)^2(2\ell+3)^2} \\
&\quad + \frac{(\ell+3)H(\ell+1)H(\ell+2)H(\ell+3)}{12(\ell+1)(2\ell+3)^2} + \frac{(\ell+2)(3\ell^2+2\ell-3)H(\ell)H(\ell+1)H(\ell+2)}{4\ell(\ell+1)^2(2\ell+3)^2} \\
&\quad - \frac{(\ell-1)(3\ell^2+4\ell-2)H(\ell-1)H(\ell)H(\ell+1)}{4\ell^2(\ell+1)(2\ell-1)^2} - \frac{(\ell-2)H(\ell-2)H(\ell-1)H(\ell)}{12\ell(2\ell-1)^2}
\end{aligned} \tag{D.1g}$$

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