Superradiant scattering at Kerr black holes

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MSc

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MASTER'S THESIS

Superradiant scattering at Kerr black holes

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Abstract

Faculdade de Ciências da Universidade do Porto

Departamento de Física e Astronomia

Master of Science

Superradiant scattering at Kerr black holes

by José SÁ

The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too...

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Resumo

Faculdade de Ciências da Universidade do Porto
Departamento de Física e Astronomia

Mestre de Ciência

Superradiant scattering at Kerr black holes

por José SÁ

Tradução em português do "Abstract" escrito em inglês mais a cima. A página é centrada vertical e horizontalmente, podendo espandir para o espaço superior da página em branco ...

Contents

List of Figures

List of Tables

Notation and Conventions

Units

Unit convetions

Tensors and GR related

Metric definitions and stuff

Abbreviations

QM	Quantum Mechanics
BH	Black Hole
GR	General Relativity
GW	Gravitational Wave
LIGO	Laser Interferometric Gravitational Wave Observatory
SWSH	Spin-Weighted Spheroidal Harmonic

Chapter 1

Superradiance

1.1 Introduction

The first appearance of the concept of *superradiance* was in 1954, in a publication by Dicke [1], and it is defined as the assemble of processes which result in amplified radiation. In particular, he showed that a gas could be excited by a pulse into "superradiant states" from thermal equilibrium and then emit coherent radiation.

Almost two decades later, Zel'dovich [2, 3] showed that a absorbing cylinder rotating with an angular velocity Ω could scatter incident wave with frequency ω if

$$\omega < m\Omega \tag{1.1}$$

would be satisfied, where m is the usual azimuthal number of the monochromatic plane wave relative to the rotation axis. In his study, he observed that superradiance was associated with dissipation of rotational energy from the absorbing object, possibly due to spontaneous pair creation at the surface.

Condition (1.1) was to become one of the most important results of rotational superradiance, as it presented itself in multiple examples, including in black hole (BH) physics, particularly in the case of the Kerr [**?**] solution. Furthermore, attempts of quantising (scalar/fermionic/...) fields in the Kerr geometry by Starovinsky and others, as well as thermodynamic analysis of the problem, laid seminal grounds to the discovery of BH evaporation by Hawking.

1.2 Klein paradox as a first example

Actually, radiation amplification can be traced to birth of Quantum Mechanics, in the beginnings of the 20th century. First studies of the Dirac equation by Klein [4] revealed the possibility of electrons propagating in a region with a sufficiently large potential barrier without the expected dampening from non-relativistic QM tunnel effect. Due to some confusion, this result was wrongly interpreted by some authors as fermionic superradiance, as if the reflected current by the barrier could be greater than the incident current. The problem was named *Klein paradox* by Sauter [5] and this misleading result was due to a incorrect calculation of the group velocities of the reflected and transmitted waves. Today, it is known that fermionic currents cannot be amplified for this particular problem [6], result that was correctly obtained by Klein in is original paper. On the contrary, superradiant scattering can indeed occur for bosonic fields.

1.2.1 Bosons

The equation that governs bosonic wave function is the Klein-Gordon equation, which for a minimally coupled electromagnetic potential takes the form

$$(D^\mu D_\mu - m^2)\Phi = 0\,, \tag{1.2}$$

the usual partial derivative in the becomes $D_\mu = \partial_\mu + ieA_\mu$.

The problem is greatly simplified by considering flat space-time in (1+1)-dimensions and step potential $A(x) = V\,\theta(x)\,\mathrm{d}t$, for $V > 0$ constant and wave solutions $\Phi = e^{-i\omega t}\phi$. For $x < 0$, the solution can be divided as incident and reflected, taking the form

$$\phi_{\rm inc}(x) = \mathcal{I}\,e^{ikx}\,, \qquad \phi_{\rm refl}(x) = \mathcal{R}\,e^{-ikx}\,, \tag{1.3}$$

in which the dispersion relation states that $k = \sqrt{\omega^2 - m^2}$. For $x > 0$, the transmitted wave is naturally given by

$$\psi_{\rm inc}(x) = \mathcal{T}\,e^{iqx}\,, \tag{1.4}$$

but in this case the root sign for the momentum must be carefully chosen so that the group velocity of the transmitted wave matches of the incoming wave, *i.e.*

$$\left.\frac{\partial\omega}{\partial p}\right|_{p=q} = \frac{q}{\omega - eV} > 0\,, \tag{1.5}$$

therefore we must have that

$$q = \operatorname{sgn}(\omega - eV)\sqrt{(\omega - eV)^2 - m^2}\,. \tag{1.6}$$

After obtaining the continuity relations at the barrier, $x = 0$, we follow by computing the ratios of the transmitted and reflected currents relative to the incident one, which yields

$$\frac{j_{\text{refl}}}{j_{\text{inc}}} = -\left|\frac{\mathcal{R}}{\mathcal{I}}\right|^2 = -\left|\frac{1-r}{1+r}\right|^2, \qquad \frac{j_{\text{trans}}}{j_{\text{inc}}} = \operatorname{Re}(r)\left|\frac{\mathcal{T}}{\mathcal{I}}\right|^2 = \frac{4\operatorname{Re}(r)}{|1+r|^2}, \tag{1.7}$$

written as a function of the coefficient

$$r = \frac{q}{k} = \operatorname{sgn}(\omega - eV)\sqrt{\frac{(\omega - eV)^2 - m^2}{\omega^2 - m^2}}\,. \tag{1.8}$$

Hence, for $r < 0$, in the case of strong potential limit, $eV > \omega + m > 2m$, the reflected current is larger (in magnitude) than the incident wave and therefore we have superradiance. Also, the transmitted current becomes negative. Even though superradiance and spontaneous pair creation are two distinct phenomena, this result is usually interpreted using the latter as follows: all incident particles are fully reflected as well as some extra due to pair creation at the boundary, while the resultant anti-particles are transmitted in the opposite direction, accounting for the change of sign in the current, due to the opposite charge they carry.

1.2.2 Fermions

For a minimally coupled electromagnetic potential, the usual partial derivative in the Dirac equation becomes $D_\mu = \partial_\mu + ieA_\mu$ in order to preserve gauge invariance of the theory. Thus

$$(i\gamma^\mu D_\mu - m)\Psi = 0 \tag{1.9}$$

where m is the fermion mass. The problem is greatly simplified by considering flat spacetime in (1+1)-dimensions, for which a valid representation of the gamma matrices is

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.10}$$

Following chronologically, Klein [4] used Dirac equation to study electrons in a step potential $A(x) = V\,\theta(x)\,\mathrm{d}t$, for $V > 0$ constant and plane wave solutions $\Psi = e^{-iEt}\psi$. For

$x < 0$, the solution can be divided as incident and reflected, taking the form

$$\psi_{\text{inc}}(x) = \mathcal{I}\, e^{ikx} \begin{pmatrix} 1 \\ \dfrac{k}{E+m} \end{pmatrix} \qquad \psi_{\text{refl}}(x) = \mathcal{R}\, e^{-ikx} \begin{pmatrix} 1 \\ \dfrac{-\,k}{E+m} \end{pmatrix} \tag{1.11}$$

while for $x > 0$, the transmitted wave function is written as

$$\psi_{\text{trans}}(x) = \mathcal{T}\, e^{iqx} \begin{pmatrix} 1 \\ \dfrac{q}{E-eV+m} \end{pmatrix} \tag{1.12}$$

where $q = [(E-eV)^2 - m^2]^{1/2}$, by solving the eigenvalue problem. Defining

$$r = \frac{q}{k}\,\frac{E+m}{E-eV+m}\,, \tag{1.13}$$

we can write the continuity condition for the complete solution at the barrier $x = 0$

$$\mathcal{I} + \mathcal{R} = \mathcal{T}\,, \qquad \mathcal{I} - \mathcal{R} = r\,\mathcal{T}\,, \tag{1.14}$$

which determines the coefficients. The computation of the Dirac currents yields

$$\frac{j_{\text{trans}}}{j_{\text{inc}}} = \frac{4\,\mathrm{Re}(r)}{|1+r|^2}\,, \qquad \frac{j_{\text{refl}}}{j_{\text{inc}}} = -\left|\frac{1-r}{1+r}\right|^2\,, \tag{1.15}$$

with conservation of probabilities currents assured

$$j_{\text{inc}} + j_{\text{refl}} + j_{\text{trans}} = 0 \tag{1.16}$$

1.3 Black hole superradiance

Chapter 2

Mathematical preliminaries

2.1 General Relativity

2.2 Kerr black hole

2.3 Kinnersley tetrad

2.4 Newman-Penrose formalism

Chapter 3

Teukolsky master equation

3.1 Angular solutions

3.2 Asymptotic radial solution

3.3 Amplification factor Z_{slm}

Chapter 4

Numerical method

4.1 Eigenvalues

4.1.1 Leaver method

4.1.2 Spectral

4.2 Radial ansatz

4.3 Amplification factor as a first test

Chapter 5

Scattering problem

5.1 Plane wave decomposition

Appendix A

Spin-weighted spherical harmonics

SWSHs play an important role BH physics and was first introduced by Teukolsky when considering non-scalar wave perturbations on a Kerr background, obtaining a separable master equation in four dimensions. After the usual change of coordinates, the polar differential equation goes as

$$\frac{1}{S}\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}S}{\mathrm{d}x}\right)+(cx)^2-2csx-\frac{(m+sx)^2}{1-x^2}+s=-\lambda \tag{A.1}$$

with $x=\cos\theta$, where λ is the eigenvalue for a given SWSH solution. Periodic boundary conditions on the azimuthal wave function constrains m to the integers.

A.1 Connection with spheroidal harmonics

By setting $s=0$ (scalar) and $c=0$ (spherical), then it's clear that (A.1) appears as a generalization of the spherical harmonics equation. In this last case, the solution are given by the associated Legendre polynomials, $P_\ell^m(x)$, for which the eigenvalue is $\ell(\ell+1)$, restricted to the condition of $|m|\leq\ell$. The closed form for spherical harmonics, after normalization, is

$${}_0Y_\ell^m(x)=(-1)^m\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(x) \tag{A.2}$$

where P_ℓ^m are the associated Legendre polynomials which can be obtained using the famous Rodrigues' formula.

A.2 Spin raising/lowering differential operators

A.3 Generalized addition of angular momentum formula

A.4 Some useful harmonics

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