Aspects of superradiant scattering off Kerr black holes

José Sá

Supervisor: João G. Rosa

Co-supervisor: Orfeu Bertolami

November 2017

Faculdade de Ciências da Universidade do Porto

Outline

- What is superradiance ?
- Kerr black hole and the Penrose process
- Newman-Penrose formalism and the Teukolsky equation
- Analytical approximated solutions
- Numerical methods
- Scattering of planes waves
- Conclusions and future work

Introduction

Default

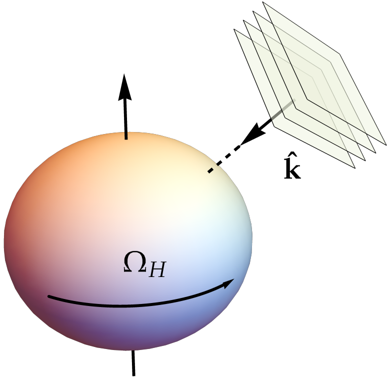
Block content.

Alert

Block content.

Example

Block content.



Kerr black hole

Boyer-Lindquist coordinates

$$ds^2 = \left(1 - \frac{2Mr}{\rho^2}\right) \mathrm{d}t^2 + 2a\sin^2\theta \frac{(r^2+a^2-\Delta)}{\rho^2} \mathrm{d}t\mathrm{d}\varphi$$

$$- \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\rho^2} \sin^2\theta \mathrm{d}\varphi^2 - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2 \mathrm{d}\theta^2$$

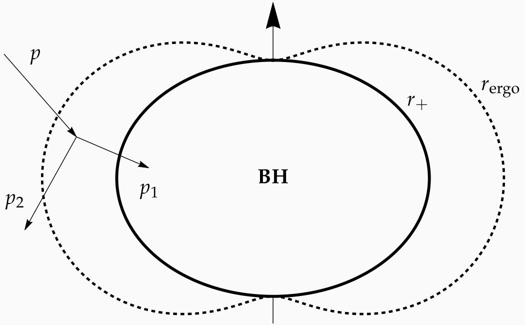
$$[\,\Delta = r^2 - 2Mr + a^2 \;,\;\; \rho^2 = r^2 + a^2\cos^2\theta\,]$$

- BH angular momentum $J = aM \quad [a = 0 \;\Rightarrow\; \text{Schwarzschild}]$
- Killing vectors ∂_t (stationary) and ∂_φ (axisymmetric)
- Horizons at $g^{rr} = 0 \;\Rightarrow\; \Delta = 0 \;\Rightarrow\; r = r_\pm \equiv M \pm \sqrt{M^2 - a^2}$
  Cosmic censorship conjecture $\;\Rightarrow\; |a| \leq M$
- Infinite redshift boundary $g_{tt} = 0 \;\Rightarrow$ Ergoregion $(g_{tt} < 0)$

$$r_+ < r < r_{\text{ergo}}(\theta) \equiv M + \sqrt{M^2 - a^2\cos^2\theta}$$

Penrose process

- Particle decays into two inside the ergoregion: $\boldsymbol{p} = \boldsymbol{p_1} + \boldsymbol{p_2}$



Killing horizon

Hypersurface $r = r_+$ with normal null vector

$$\boldsymbol{\xi} = \partial_t + \Omega_H \partial_\varphi$$

Event horizon angular momentum:

$$\Omega_H = \frac{a}{2Mr_+}$$

- We may have $E_1 = \boldsymbol{k} \cdot \boldsymbol{p_1} < 0 \;\Rightarrow E_2 = E + |E_1| > E$
- Local energy condition $\boldsymbol{\xi} \cdot \boldsymbol{p_1} > 0$ at $r = r_+ \;\Rightarrow E_1 - \Omega_H L_1 > 0$

$$\psi \sim e^{-i\omega t + im\varphi} \;\Rightarrow\; \frac{\delta J}{\delta M} = \frac{\hbar m}{\hbar \omega} \;\Rightarrow\; \boxed{\delta M < 0 \;\wedge\; \omega(\omega - \Omega_H m) < 0}$$

Energy extraction !

Newman-Penrose formalism

- EM perturbations $A^\mu \ll 1 \;\Rightarrow\; R_{\mu\nu} = \mathcal{O}(A^2) \;\Rightarrow$ fixed background
- Projection of tensors onto a tetrad frame of complex vectors $(\mathfrak{l}, \mathfrak{n}, \mathfrak{m}, \bar{\mathfrak{m}})$

$$\underbrace{\mathbf{E}\,,\,\mathbf{B}}_{\text{6 real components}} \longrightarrow \boxed{\begin{aligned} \phi_0 &= F_{\mu\nu}\mathfrak{l}^\mu \mathfrak{m}^\nu \\ \phi_1 &= \tfrac{1}{2}F_{\mu\nu}(\mathfrak{l}^\mu \mathfrak{n}^\nu - \mathfrak{m}^\mu \bar{\mathfrak{m}}^\nu) \\ \phi_2 &= F_{\mu\nu}\bar{\mathfrak{m}}^\mu \mathfrak{n}^\nu \end{aligned}}$$

- **Kinnersley tetrad**

$$\mathfrak{l} = \frac{1}{\Delta}\Big(r^2 + a^2,\, \Delta,\, 0,\, a\Big) \qquad \mathfrak{m} = \frac{1}{\sqrt{2}\bar{\rho}}\Big(i\,a\sin\theta,\, 0,\, 1,\, i\csc\theta\Big)$$

$$\mathfrak{n} = \frac{1}{2\rho^2}\Big(r^2 + a^2,\, -\Delta,\, 0,\, a\Big) \qquad \bar{\mathfrak{m}} = \mathfrak{m}^* \qquad [\,\bar{\rho} = r + i\,a\cos\theta\,]$$

Tetrad particularly suitable for the study of incoming and outgoing radiation $\Rightarrow$ equations decouple

$$\mathfrak{l} \sim \partial_t + \partial_r \qquad 2\,\mathfrak{n} \sim \partial_t - \partial_r$$

Newman-Penrose formalism

$\nabla_\mu F^{\mu\nu} = 0\,,\ \nabla_{[\mu} F_{\nu\rho]} = 0 \ \longrightarrow\ $ 4 complex first-order coupled equations

$$\Big\Downarrow \qquad \mathcal{D}_n\,,\mathcal{D}_n^\dagger = \partial_r \mp iK/\Delta + 2n(r-M)/\Delta \qquad \mathcal{L}_n\,,\mathcal{L}_n^\dagger = \partial_\theta \mp Q + n\cot\theta$$

Eliminate ϕ_1 and rewrite $\Phi_0 = \phi_0\,,\ \Phi_2 = 2(\bar{\rho}^*)^2\,\phi_2$:

- 2 equations with eigenvalue $\lambda\!\!\bar{}$

$$\left[\Delta\mathcal{D}_1\mathcal{D}_1^\dagger + \mathcal{L}_0^\dagger\mathcal{L}_1 + 2i\omega(r + ia\cos\theta)\right]\Phi_0 = 0$$
$$\left[\Delta\mathcal{D}_0^\dagger\mathcal{D}_0 + \mathcal{L}_0\mathcal{L}_1^\dagger - 2i\omega(r + ia\cos\theta)\right]\Phi_2 = 0$$

- 2 equations with relative normalization $\mathcal{B} = \sqrt{\lambda\!\!\bar{}^{\,2} - 4a^2\omega^2 + 4a\omega m}$

$$\mathcal{L}_0\mathcal{L}_1\Phi_0 = \mathcal{D}_0\mathcal{D}_0\Phi_2$$
$$\mathcal{L}_0^\dagger\mathcal{L}_1^\dagger\Phi_2 = \Delta\mathcal{D}_0^\dagger\mathcal{D}_0^\dagger\Delta\Phi_0$$

Whats the form of the eigenvalue $\lambda\!\!\bar{}$?

Teukolsky equation

General perturbation solution $\Rightarrow \Upsilon_s = \int \mathrm{d}\omega \sum_{\ell,m} e^{-i\omega t + im\varphi}\, {}_sS_{\ell m}(\theta)\, {}_sR_{\ell m}(r)$

- Scalar ($s = 0$)
- Electromagnetic ($s = \pm 1$)
- Gravitational ($s = \pm 2$)

$$\boxed{\Upsilon_{+1} = \phi_0 \qquad \Upsilon_{-1} = 2(\bar{\rho}^*)^2 \phi_2}$$

Angular equation $[\,z = \cos\theta\ ,\ c = a\omega\,]$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[(1-z^2)\frac{\mathrm{d}\,{}_sS_{\ell m}}{\mathrm{d}z}\right] + \left[(cz)^2 - 2csz - \frac{(m+sz)^2}{1-z^2} + s + {}_s\mathcal{A}_{\ell m}\right] {}_sS_{\ell m} = 0$$

- $c = 0$ $\Rightarrow$ Spherical symmetry (closed form)

$$e^{im\varphi}\, {}_sS_{\ell m}(\theta) = {}_sY_{\ell m}(\theta,\varphi) \qquad {}_s\mathcal{A}_{\ell m} = \ell(\ell+1) - s(s+1)$$

- $c \neq 0$ $\Rightarrow$ Series approximation or numerical methods (Leaver/Spectral)

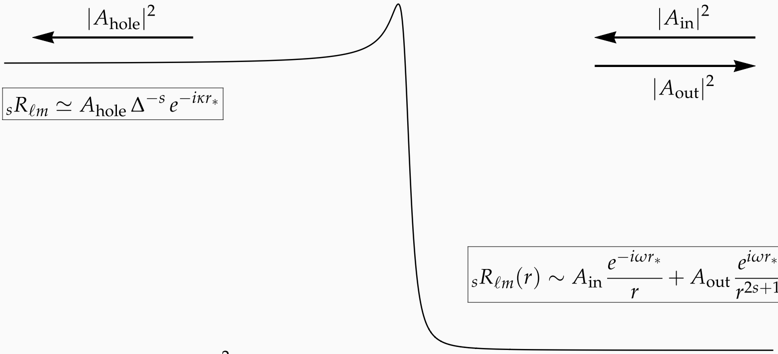
$${}_s\mathcal{A}_{\ell m} = \ell(\ell+1) - s(s+1) - \frac{2ms^2}{\ell(\ell+1)}c + \mathcal{O}(c^2)$$

Teukolsky equation

Radial equation $\frac{1}{\Delta^s}\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta^{s+1}\frac{\mathrm{d}\,{}_sR_{\ell m}}{\mathrm{d}r}\right)+\left[\frac{K^2-2is(r-M)K}{\Delta}+4is\omega r-\lambda\right]{}_sR_{\ell m}=0$

$$\Rightarrow \left(\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}+V_{\mathrm{eff}}\right){}_sU_{\ell m}=0 \qquad \left[\frac{\mathrm{d}r_*}{\mathrm{d}r}=\frac{r^2+a^2}{\Delta}\right]$$

where $\lambda={}_s\mathcal{A}_{\ell m}-2ma\omega+a^2\omega^2$ and ${}_sU_{\ell m}=\sqrt{\Delta^s(r^2+a^2)}\,{}_sR_{\ell m}$



$$\boxed{{}_sR_{\ell m}\simeq A_{\mathrm{hole}}\,\Delta^{-s}\,e^{-i\kappa r_*}}$$

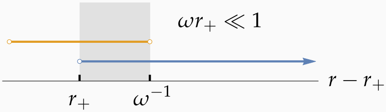
$$\boxed{{}_sR_{\ell m}(r)\sim A_{\mathrm{in}}\frac{e^{-i\omega r_*}}{r}+A_{\mathrm{out}}\frac{e^{i\omega r_*}}{r^{2s+1}}}$$

$$V_{\mathrm{eff}}\simeq\left(\kappa-is\frac{r_+-M}{2Mr_+}\right)^2$$

$$V_{\mathrm{eff}}\sim\omega^2+\frac{2is\omega}{r}$$

Matching of coefficients

- Solving radial equation for $r - r_+ \gg r_+$ (far) and $r - r_+ \ll \omega^{-1}$ (near)
- Extend validity of the solutions to opposite regions and match dominant monomials of $x^{\ell-s}$ and $x^{-\ell-1-s}$ $\Rightarrow$ valid for modes $\bar{\omega} \equiv \omega r_+ \ll 1$



- Obtain loss/gain factor $\boxed{{}_{\pm 1}Z_{\ell m} = \frac{\mathrm{d}E_{\text{out}}}{\mathrm{d}t} \Big/ \frac{\mathrm{d}E_{\text{in}}}{\mathrm{d}t} - 1}$

$$ {}_{\pm 1}Z_{\ell m} \simeq -4\underbrace{\bar{\omega}(\bar{\omega} - m\bar{\Omega}_H)}_{\substack{\text{superradiant}\\ \text{condition}}} \underbrace{(2-\tau)(2\bar{\omega}\tau)^{2\ell}\left[\frac{(\ell-1)!(\ell+1)!}{(2\ell)!(2\ell+1)!}\right]^2 \prod_{n=1}^{\ell}\left(n^2 + \frac{4\varpi^2}{\tau^2}\right)}_{\text{always positive}} $$

Numerical methods

- Dependent variables ($\mathcal{J} = a/M$, $\bar{\omega}$, ℓ, m)
- ${}_{\pm 1}R_{\ell m} = (r_+)^{\mp 1}\, x^{\mp 1 - i\varpi/\tau} f_\pm(x) \Rightarrow$ removes singular points of the eq.
- Integrate from $\epsilon \ll 1$ (stiffness) up to $x_\infty = 2\pi/\bar{\omega} \times 200$

$$f_\pm(\epsilon) = \sum_{n=0}^{N_H} \underbrace{\left(\frac{a_n}{a_0}\right)}_{\ell, m, \omega, a} \epsilon^n \;\; \Rightarrow f_\pm(\epsilon) \simeq 1\,,\;\; f_\pm{}'(\epsilon) \simeq 0$$

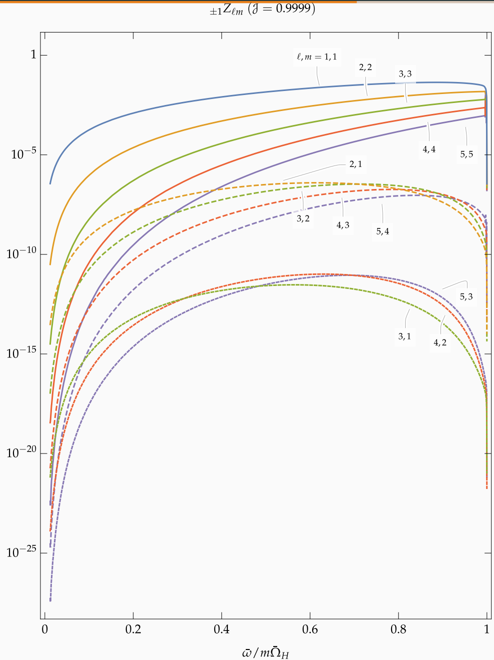
- Integrate from the horizon $x_0 = \epsilon = 10^{-12}$ onward

$${}_{\pm 1}Z_{\ell m} = \frac{\mathcal{B}^2 \tau^4}{4\varpi^2(\tau^2 + 4\varpi^2)} \left| \frac{f_-(x_\infty)}{f_+(x_\infty)} \right|^2 - 1$$

- Using conservation of energy $\frac{\mathrm{d}E_{\text{in}}}{\mathrm{d}t} - \frac{\mathrm{d}E_{\text{out}}}{\mathrm{d}t} = \frac{\mathrm{d}E_{\text{hole}}}{\mathrm{d}t}$

$${}_{\pm 1}Z_{\ell m} = -\frac{\bar{\omega}\tau^2}{\varpi} \left| \frac{1}{f_+(x_\infty)} \right|^2 = -\left(1 + \frac{\mathcal{B}^2 \tau^2}{4\bar{\omega}\varpi(\tau^2 + 4\varpi^2)} \left| f_-(x_\infty) \right|^2 \right)^{-1}$$

Suppp


$_{\pm 1}Z_{\ell m}$ ($\mathcal{J} = 0.9999$)
$\ell, m = 1,1$
2,2
3,3
4,4
5,5
2,1
3,2
4,3
5,4
5,3
3,1
4,2
1
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-25}
0
0.2
0.4
0.6
0.8
1
$\bar{\omega}/m\bar{\Omega}_H$

Newman-Penrose formalism

$$\nabla_\mu F^{\mu\nu} = 0\,,\ \nabla_{[\mu} F_{\nu\rho]} = 0 \ \longrightarrow\ \text{4 first-order coupled equations } (\phi_0, \phi_1, \phi_2)$$

$$\Downarrow$$

Teukolsky's equation

$$\frac{1}{\Delta^s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial \Upsilon_s}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Upsilon_s}{\partial\theta}\right) - \left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\Upsilon_s}{\partial t^2}$$
$$-\frac{4Mar}{\Delta}\frac{\partial^2\Upsilon_s}{\partial t\partial\varphi} - \left(\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right)\frac{\partial^2\Upsilon_s}{\partial\varphi^2} + 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\Upsilon_s}{\partial t}$$
$$+2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\Upsilon_s}{\partial\varphi} - (s^2\cot^2\theta - s)\Upsilon_s = 0$$

Spin-weight s	$\Upsilon_s = \int \mathrm{d}\omega \sum_{\ell,m} e^{-i\omega t + im\varphi}\, {}_sS_{\ell m}(\theta)\, {}_sR_{\ell m}(r)$
$+1$	$\Phi_0 \equiv \phi_0$
-1	$\Phi_2 \equiv 2(\bar\rho^*)^2\phi_2$

$\rightarrow$ Describes other perturbations: scalar ($s=0$), GWs ($s=\pm 2$)

Conclusions / Future work

Questions ?