PROOF PORTFOLIO PROBLEM SET 1
MTH337 F22 COTE

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Problem 1 *Use the axioms of Hilbert to prove the following:*

Given a line l on a plane P, the points of P not on l can be split into two disjoint sets A and B such that :

(1) for every pair of points p_1, p_2 in A, the segment p_1p_2 does not intersect l,
(2) for every pair of points p_3, p_4 in B, the segment p_3p_4 does not intersect l,
(3) given any point p_5 in A and any point p_6 in B, the segment p_5p_6 does intersect l

Proof. Let P be a plane and l be a line on P. Theorem 5 of Hilbert let's us use the point a as another point of the same region. We'll define our sets

$$A = \{\ p \in P \quad | \quad p \notin l, \text{ segment } ap \text{ does not intersect } l\ \}$$

$$B = \{\ p \in P \quad | \quad p \notin l, \text{ segment } ap \text{ does intersect } l\ \}$$

Also by Theorem 5, l divides the points of P into two regions and allows the disjoint sets to be true. Since every point of region A to any point of region B has one point on line l allows set B to hold. Also, any two points a, a' of the same region determine a segment aa' containing no point of l which allows set A to be true.

Therefore, the disjoint sets A and B are true.

□

Problem 2 *Use the axioms of Hilbert to prove the following:*
If in any two triangles one side and the two adjacent angles (angle-side-angle) are respectively congruent, then the triangles are congruent.

Proof. Assume we have a $\triangle ABC$ and $\triangle A'B'C'$ such that $AB \cong A'B'$, $\angle BAC \cong \angle B'A'C'$, and $\angle ABC \cong \angle A'B'C'$.

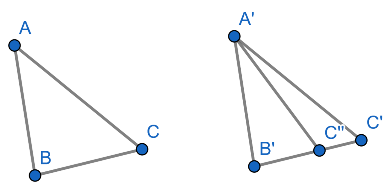

Figure 1. Triangles ABC and A'B'C'

We want to show $\triangle ABC \cong \triangle A'B'C'$ by showing $AC \cong A'C'$. Let's assume to the contrary that this is not true. Without loss of generality we'd say $AC \ncong A'C'$. This would mean there is a different line from A' to a point along $B'C'$, we'll call this C'', that is congruent to AC.

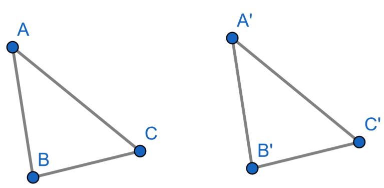

Figure 2. Triangle $A'B'C''$

However, by Hilbert IV,4 that cannot be the case since every ray with an angle is unique. So there can't be two sides congruent to AC with an angle of measure $\angle BAC$. Therefore $A'C' \cong AC$. We also have $AB \cong A'B'$ and $\angle BAC \cong \angle B'A'C$ which means $\triangle ABC \cong \triangle A'B'C'$ by SAS triangle congruence. $\square$

Problem 3 *Use the axioms of Hilbert to prove the following:*
The sum of the angles of a triangle is two right angles.

Proof. We have $\triangle ABC$.We know that a right triangle is 90° which means two right angles would equal 180°.

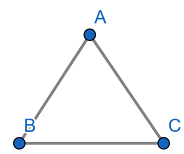

FIGURE 3. Triangle ABC

If we extend the line BC and make a parallel line segment to BC through the point A called l with points l_1 and l_2.

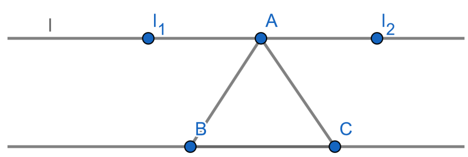

FIGURE 4. Triangle ABC with parellel line l

By the converse of the Alternate Interior Angle Theorem we have

$$\angle L_1AB \cong \angle ABC \text{ and } \angle L_2AC \cong \angle ACB.$$

Now since

$$\angle L_1AB + \angle L_2AC + \angle BAC = 180°.$$

We can say,

$$\angle ABC + \angle ACB + \angle BAC = 180°$$

or two right angles. □

Problem 4 *Use the axioms of Birkhoff to prove the following:*

(1) All right angles are equal in measure
(2) Similarity for triangles is an equivalence relation

Proof. (1) Assume we have $\angle BOA$ which is equal to $\pm 90°$. If we make another angle of $\pm 180°$ from points $\angle DOA$, the difference of the two angles will be $\pm 90°$ according to Birkhoff's Angle Measure Axiom. Therefore, all right angles are equal in measure.

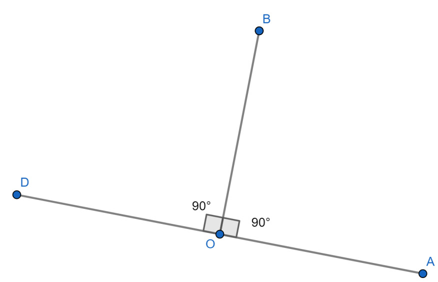

FIGURE 5. Angles DOA and BOA

(2) Assume we have $\triangle ABC$. We have, $AB \cong BA$, $AC \cong CA$ and $\angle BAC \cong \angle CAB$, which we'll call angle α. This shows $\triangle ABC \sim \triangle ABC$ with a scalar of 1 and similarity for triangles is reflexive.

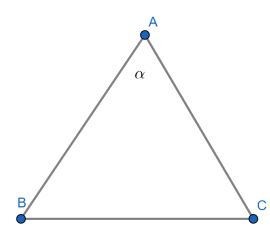

FIGURE 6. Triangle ABC

Now lets assume we have $\triangle A'B'C'$ where $A'B' = kAB$, $A'C' = kAC$ and $\angle B'A'C' = \pm\angle BAC$ for some $k > 0$. Now considering $\triangle ABC$, we have a similarity with $\triangle A'B'C'$ such that

$$AB = \frac{1}{k}A'B', AC = \frac{1}{k}A'C', \text{ and } \angle BAC = \pm\angle B'A'C'$$

for the same k. Therefore, similarity for triangles is symmetric.

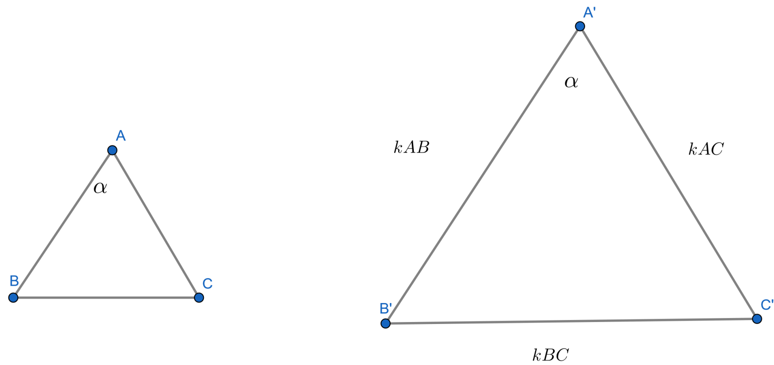

FIGURE 7. Triangle ABC and triangle A'B'C'

We have $\triangle ABC \sim \triangle A'B'C'$, and lets assume $\triangle A'B'C' \sim \triangle A''B''C''$ where $A''B'' = lA'B'$, $A''C'' = lA'C'$, and $\angle B''A''C'' = \pm\angle B'A'C'$ for some $l > 0$. Now it is also true that,

$$A''B'' = l(kAB),\ A''C'' = l(kAC), \text{ and } B''A''C'' = \pm\angle BAC.$$

Since $(lk) > 0$, similarity for triangles is also transitive.

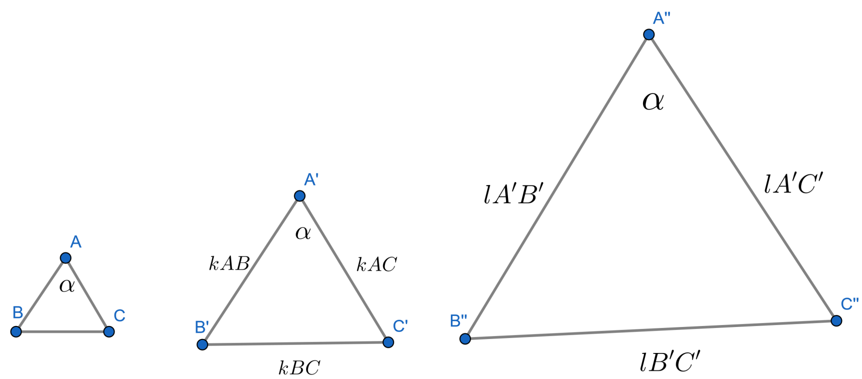

FIGURE 8. Triangles ABC, A'B'C', and A"B"C"

Since the similarity of triangles has properties of reflexivity, symmetry, and transitivity, it is in fact an equivalence relation.

$\square$

Problem 5 *Use the axioms of Birkhoff to prove the following:*
If two sides of a triangle are congruent, then the angles opposite those sides are con-gruent.

Proof. Assume we have $\triangle ABC$.

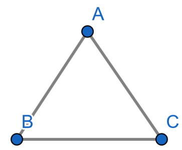

FIGURE 9. Triangle ABC

We have $AB \cong BA$, $AC \cong CA$. In this triangle its also true that $\angle BAC \cong \angle CAB$. So by SAS congruency $\triangle ABC \cong \triangle CBA$ and it follows that

$$\angle ACB \cong \angle BCA, \text{ and } \angle ABC \cong \angle CBA$$

Therefore, the angles opposite to the equal sides are congruent.

$\square$