PROOF PORTFOLIO PROBLEM SET 1
MTH337 F22 COTE

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Problem 1 *Use the axioms of Hilbert to prove the following:*
Given a line l on a plane P, the points of P not on l can be split into two disjoint sets A and B such that :

(1) for every pair of points p_1, p_2 in A, the segment p_1p_2 does not intersect l,
(2) for every pair of points p_3, p_4 in B, the segment p_3p_4 does not intersect l,
(3) given any point p_5 in A and any point p_6 in B, the segment p_5p_6 does intersect l

Proof. Let P be a plane and l be a line on P. We'll define our regions

$$A = \{\ p \in P \quad | \quad p \notin l, \text{ segment } ap \text{ does not intersect } l\ \}$$
$$B = \{\ p \in P \quad | \quad p \notin l, \text{ segment } ap \text{ does not intersect } l\ \}$$

Since $p1$, $p2 \in A$, by the definition of the region a segment between then would not intersect l. Likewise, since $p3$, $p4 \in A$ by definition of the region, a segment between them would also not intersect l.

Now lets assume the contrary to c, that is, given any point $p5$ in A and any point $p6$ in B, the segment $p5p6$ doesn't intersect l. We'll pick $p1$ from A and $p3$ from B. We'll attempt to join these points together to form the segment $p1p3$, since $p1$ can't intersect the line l and neither can $p3$ we can see it's impossible to have $p1p3$ without intersecting the line.

$\square$

Problem 2 *Use the axioms of Hilbert to prove the following:*
If in any two triangles one side and the two adjacent angles (angle-side-angle) are respectively congruent, then the triangles are congruent.

Proof. Assume we have a $\triangle ABC$ and $\triangle A'B'C'$ such that $AB \cong A'B'$, $\angle BAC \cong \angle B'A'C'$, and $\angle ABC \cong \angle A'B'C'$ We want to show AC

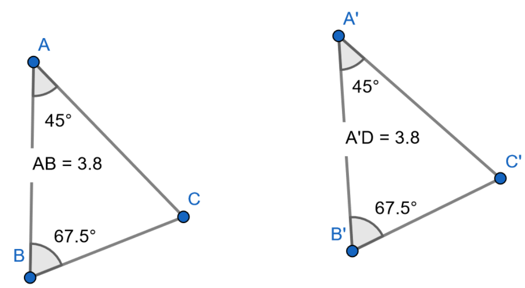

Figure 1. Two triangles with 1 equal side and two equal angles

congruent $A'C'$ and BC congruent $B'C'$. Let's assume to the contrary that this is not true. Without loss of generality we'd say AC is not congruent to $A'C'$. This would mean there is a different line from A' to a point along $B'C'$ and we'll call this C'' that is congruent to AC. So by SAS triangle congruence $\triangle ABC \cong \triangle A'B'C''$. However, by Hilbert IV,4 that cannot be the case since every ray with an angle is unique.

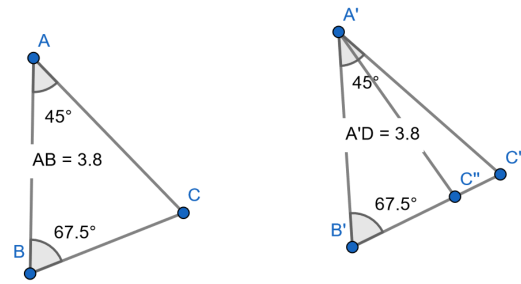

FIGURE 2. Triangle $A'B'C''$

Therefore $A'C'$ congruent AC and triangle ABC congruent triangle $A'B'C'$ by SAS. $\square$

Problem 3 *Use the axioms of Hilbert to prove the following:*
The sum of the angles of a triangle is two right angles.

Proof. We have a triangle ABC, we know that a right triangle is 90 degrees which means two right angles would equal 180 degrees.
If we extend the line BC and make a parallel line segment to BC through the point A called $l1l2$. By the converse to the alternate

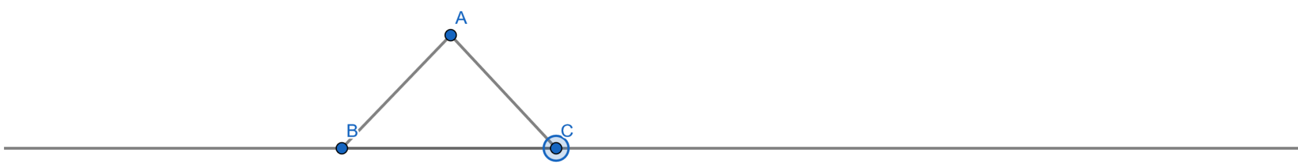

FIGURE 3. Triangle ABC with BC extended

interior angle theorem we have
angle $L1AB$ congruent angle ABC
angle $L2AC$ congruent angle ACB
Now since angle $L1AB+$ angle $L2AC+$ angle BAC = 180 degrees by the segment $l1l2$ we have
angle $ABC+$ angle $ACB+$ angle BAC = 180 or two right angles.

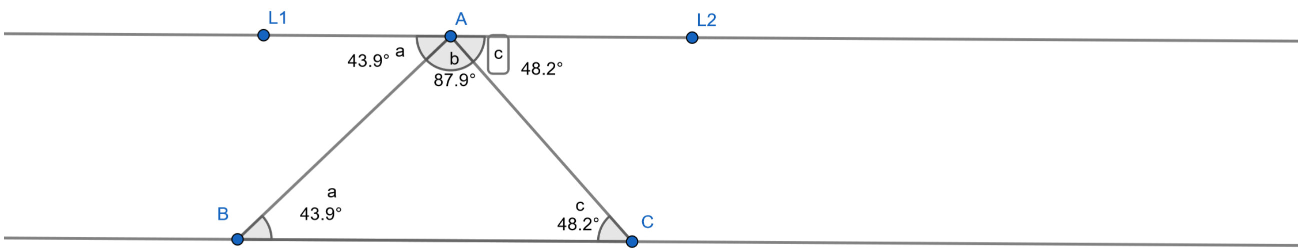

Figure 4. Triangle ABC with angle congruencies

$\square$

Problem 4 *Use the axioms of Birkhoff to prove the following:*

(1) All right angles are equal in measure
(2) Similarity for triangles is an equivalence relation

Proof. (A) Assume we have $\angle BOA$ which is equal to $\pm 90°$. If we make another angle of $\pm 1800°$ from points D, O, A, the difference of the two angles will be $\pm 90°$ according to Birkhoff's angle measure axiom.

(B) Assume we have $\triangle ABC$. Now we also have $\triangle CBA$ with a scalar of 1. Then, $AB \cong BA$, $AC \cong CA$ and $\angle BAC \cong \angle CAB$. Which shows $ABC \sim CBA$ by SAS congruency and has reflexivity.

Now lets assume we have $\triangle A'B'C'$ where $A'B' = KAB$, $A'C' = KAC$ and $\angle B'A'C' = +/- \angle BAC$ for some $k > 0$. Then by Birkhoff's similarity axiom the following is also true $B'C' = KBC$, $\angle A'B'C' = +/- \angle ABC$ and $\angle A'C'B' = +/- \angle ACB$. Now working with $\triangle ABC$ we have a similarity with $\triangle A'B'C'$ such that

$AB = (1/k)A'B'$, $AC = (1/k)A'C'$ and $\angle BAC = +/- \angle B'A'C'$ for the same k. And now by the similarity axiom we have $BC = KB'C'$, $\angle ABC = +/- angle A'B'C'$ and $\angle ACB = +/- \angle A'C'B'$ so $\triangle ABC$ is symmetric.

We have $\triangle ABC \sim \triangle A'B'C'$, now lets show $\triangle A'B'C' \sim \triangle A''B''C''$ where $A''B'' = KA'B'$, $A''C'' = KAC$ and $\angle B''A''C'' = +/- \angle B'A'C$ for some $k > 0$.

Then by Birkhoff's similarity axiom the following is also true.
$B''C'' = KB'C'$, $\angle A''B''C'' = +/- \angle A'B'C'$ and $\angle A''C''B'' = +/- \angle ACB$ which makes $A'B'C' \sim A''B''C''$.

Now since $\triangle ABC is \sim to A'B'C'$ and $A'B'C' \sim A''B''C''$

$ABC \sim A''B''C''$ by transitivity.

Since the similarity of triangles has properties of reflexivity, symmetry, and transitivity, it is in fact an equivalence relation.

$\square$

Problem 5 *Use the axioms of Birkhoff to prove the following:*

Proof. Assume we $\triangle ABC$ we'll make another $\triangle CBA$.
We'll name the vertices C,B, and A, C', B', and A' respectivly, to

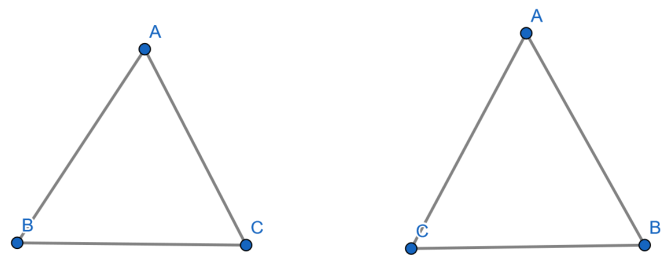

FIGURE 5. Two triangles of same vertices flipped

help in differentiating the sides.
Since $AB \cong B'A'$, $AC \cong C'A$, and $\angle BAC \cong \angle C'A'B'$
Now by SAS congruency these are equivalent triangles and it follows that
$\angle ACB \cong \angle A'B'C'$
$\angle ABC \cong \angle A'C'B'$
and so the angles opposite to the equal sides are congruent.

□

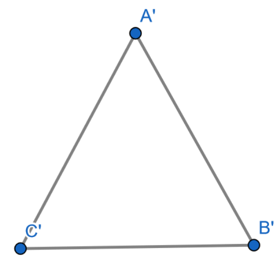

FIGURE 6. Triangles $C'B'A'$

Problem 1 *(10 points)*
1. Let ABC be a triangle. Let D be the midpoint of AB and E be the midpoint of CA. Show that DE is parallel to BC and that the length of DE is half the length of BC.

Proof. We have the triangle and midpoints. We'll also call F the midpoint of BC, which forms $\triangle DEF$ with the other midpoints of $\triangle ABC$

Since D is the midpoint of AB, $BD \cong AD$. Likewise, $AE \cong EC$ and $BF \cong FC$

An angle is congruent to itself, so $\angle ABC \cong \angle DBF$ which we'll call β

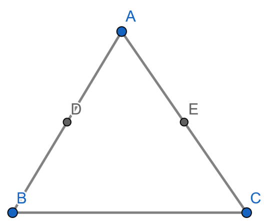

FIGURE 7. Initial triangle ABC with midpoints D and E

Now since the length of $BA = 2DB$, $BC = 2BF$, and $\angle ABC \cong \angle DBF$
by birkhoffs similarity axiom we have $ABC \sim DBF$
Therefore $DF = 1/2AC$, $\angle DFB = \angle ACB$ which we'll call ρ, and $\angle BDF = \angle BAC$ which we'll call α

Likewise we have $\triangle EFC \sim \triangle ABC$ and $\triangle ADE \sim \triangle ABC$

Therefore, $\alpha + \beta + \rho = 180°$ since they are all the angles of the triangles DBF, EFC, and ADE.

Since this is true, $\angle FDE = 180° - \alpha - \beta$ which also equals ρ. We can now say $\angle BDE = \alpha + \rho$.
Now if we extend lines DE and BC, we also have the exterior angle of $\angle ABC$ on the line BC which we'll call ω is $\rho + \alpha$ by the exterior angle theorem.
Thus, we have the angles $\angle BDE = \omega$ and by the converse of the alternate interior angle theorem would make the lines DE and BC parallel.

$\square$

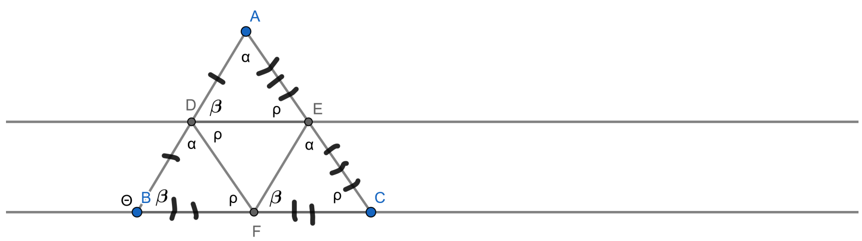

Figure 8. Extending the lines DE, BC

Problem 2 *(10 points)*
2. Use Geogebra to construct a triangle ABC, then construct squares outside the triangle along sides AB and BC. Connect the vertices of the squares as shown in the diagram (AF , DF , DC). Construct the midpoints of AC and DF , along with the midpoints of the squares. Using the following steps, prove that $HJIK$ is a square.

Proof. We are given $\triangle ABC$ and squares along sides AB and BC. We'll construct this with lines AE, BD, and connect these by ED to construct the square $ABDE$.

Likewise for the other square we'll construct lines BF, CG, and FG to construct $BCGF$.

We'll now add lines AF, DC, and DF.

Now we want to show $AF \cong DC$. To help with this we'll label $\angle DBF$ as α.

We have $AB \cong BD$, $BF \cong BC$, and we have $\angle DBC = \alpha + 90°$ since $\angle FBC$ is a right angle. Similarly, $\angle ABF = \alpha + 90°$ since $\angle AB$ since it is also a right angle. We then have $\angle DBC \cong \angle ABF$.

By these three congruencies we have $\triangle ABF \cong DBC$ by SAS congruency. Which also shows $AF \cong DC$. □