PROOF PORTFOLIO PROBLEM SET 2
MTH337 F22 COTE

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Problem 1

Let ABC be a triangle. Let D be the midpoint of AB and E be the midpoint of CA. Show that DE is parallel to BC and that the length of DE is half the length of BC.

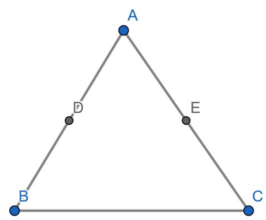


FIGURE 1. Initial triangle ABC with midpoints D and E.

Proof. First we want to show DE is half the length of BC through similarity. We'll begin by naming F the midpoint of BC, which connected to the other two midpoints of $\triangle ABC$, D and E, form $\triangle DEF$.

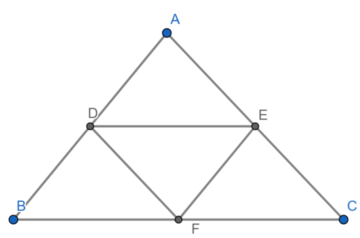


FIGURE 2. Triangle DEF.

From our midpoints, we have $BA = 2DA$, $AC = 2AE$, and as previously stated $\angle BAC \cong \angle DAE$. Now by Birkhoff's Similarity Axiom we have $ABC \sim ADE$. Therefore, $DF = \frac{1}{2}BC$, as we wanted.

Now we want to show DE is parallel to BC. We'll begin by extending the line BC on the side of B to the point B'. We'll call $\angle ABB'$, the exterior angle of $\angle ABC$, θ. By a similar argument we used to show $\triangle ABC \sim \triangle ADE$ we can say $\triangle ABC \sim \triangle EFC$ and $\triangle ABC \sim \triangle DBF$.

Since $\triangle ABC$ is similar to these three triangles by a ratio of $\frac{1}{2}$ we have

$$\angle DAE = \angle BDF = \angle FEC$$

which we'll call α,

$$\angle ADE = \angle DBF = \angle EFC$$

which we'll call β, and

$$\angle AED = \angle DFB = \angle ECF$$

which we'll call ρ.

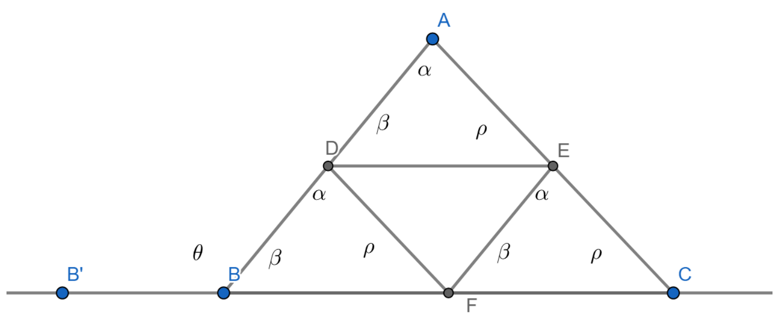


FIGURE 3. The line BC extended and angles labeled

Therefore, $\alpha + \beta + \rho = 180°$ since they are all the interior angles of a triangle.

We have

$$\angle EDF = 180° - \beta - \alpha$$

since these are supplementary angles on the line AB. So $\angle EDF = \rho$, and $\angle EDB = \alpha + \rho$.

Now since $\angle ABC = \beta$, the supplementary angle is $\alpha + \rho$ on the line BC. Therefore,

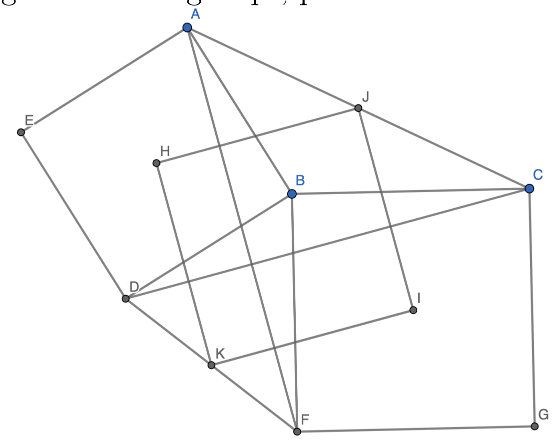
$$\theta = \alpha + \rho \ = \angle EDB$$

and by the alternate interior angle theorem, DE is parallel to BC.

□

Problem 2

Use Geogebra to construct a triangle ABC, then construct squares outside the triangle along sides AB and BC. Connect the vertices of the squares as shown in the diagram (AF, DF, DC). Construct the midpoints of AC and DF, along with the midpoints of the squares. Using the following steps, prove that $HJIK$ is a square.



(1) Prove $AF \cong DC$

Proof. We are given $\triangle ABC$ and squares along sides AB and BC. We'll construct this with lines AE and BD that will be congruent to AB and perpendicular to it. We'll then create ED congruent to AB to construct the square $ABDE$.

Likewise for the other square we'll construct lines BF, CG congruent to BC and each parallel to BC. Now we'll add FG congruent to BC to construct the square $BCGF$. We'll now add lines AF, DC, and DF.

Now we want to show $AF \cong DC$. To help with this we'll label $\angle DBF$ as α.

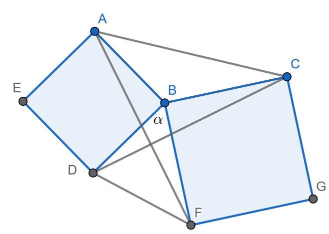


FIGURE 4. Angle α in triangle

Now we have

$$\angle DBC = \alpha + 90^\circ$$

since $\angle FBC$ is a right angle. Similarly,

$$\angle ABF = \alpha + 90^\circ$$

since $\angle AB$ since it is also a right angle. We then have $\angle DBC \cong \angle ABF$.

Now since $\angle DBC \cong \angle ABF$, $AB \cong BD$, and $BF \cong BC$ we have $\triangle ABF \cong \triangle DBC$ by SAS congruency. Since these two triangles are congruent, $DC \cong AF$, as we wanted. □

(2) Prove $AF \perp DC$ (hint: Consider line FC)

Proof. We want to show AF is perpendicular to DC by showing they intersect at a 90° angle at the point of intersection which we'll label Q. We'll begin by recalling from part 1 that $\triangle ABF \cong \triangle DBC$. From this we have the following angle congruencies. We have

$$\angle BAF \cong \angle BDC$$

and

$$\angle BCD \cong \angle BFA.$$

We'll label $\angle BCD$ and $\angle BFA$ as α.

Let's now construct the line FC which forms $\triangle BFC$. Also from part 1, we have $BC \cong BF$ which makes $\triangle BFC$ an isosceles triangle. We also have $\angle FBC = 90^\circ$ since it is also an angle of the square $BCGF$. Now since $\triangle BFC$ is isosceles and $\triangle BFC = 90^\circ$, the other two interior angles $\angle BFC$ and $\angle BCF$ must be equivalent and their angle sum must equal $180^\circ - \angle FBC$. Now the angles $\angle BFC$ and $\angle BCF$ must both equal 45°.

Now we'll form a new triangle $\triangle QCF$. Since $\angle QFC$ contains $\angle BFC$ and $\angle BFA$, which don't overlap,

$$\angle QFC = \alpha + \angle BFC.$$

Also, $\angle QCF$ contains $\angle BCF$ minus the angle measure of $\angle BCD$. In other words,

$$\angle QCF = \angle BCF - \alpha.$$

Now since

$$\angle QFC + \angle QCF = \alpha + \angle BFC + \angle BCF - \alpha$$

we have

$$\angle QFC + \angle QCF = 90^\circ.$$

Which shows $\angle FQC = 90°$ by the total sum of interior angle of a triangle having to be 180°. Therefore AF and DC do intersect on a 90°

□

(3) Using these results along with a previous result from this exam, prove $HJIK$ is a square (hint: consider triangle AFC and the first problem from the exam).

Proof. Lets consider $\triangle AFC$. We can also form the $\triangle CJI$.Since, J is the midpoint of AC and I is the midpoint of FC JI is half the length of AF, as we proved in problem 1. Also, $\angle ACF \cong \angle JCI$ and by Birkhoff's Similarity Axiom $\triangle AFC \sim \triangle CJI$.

Similarly, we can form $\triangle AFD$ and $\triangle DHK$. Since H and K are also midpoints, we have HK is half the length of AF which makes $JI \cong HK$. Also, $\triangle AFD \sim DHK$.

Again, we can form $\triangle DCF$ and $\triangle KIF$. Since K and I are midpoints we have KI is half of DC and $\triangle DCF \sim \triangle KIF$.

Lastly, we can form $\triangle ADC$ and $\triangle AHJ$. Since H and J are midpoints we have HJ is half of DC and $\triangle ADC \sim AHJ$. This also makes $HJ \cong KI$. Since $DC \cong AF$ from part 1,

$$HJ \cong KI \cong JI \cong HK.$$

Now we want to show, all four vertices of the square are 90°. We will do this by using our proved theorem from problem 1. Again, since $\triangle CJI \sim \triangle AFC$, $JI \parallel AF$.

Similarly,

$$HK \parallel AF,\ HJ \parallel DC, \text{ and } KI \parallel DC.$$

From part 2, we also know $AF \perp FC$. Using this fact, we have $JI \perp KI$ which makes their intersection, I at 90°, $HK \perp KI$, which makes their intersection K at 90°, $HJ \perp HK$, which makes their intersection, H at 90°, and $HJ \perp JI$, which makes their intersection J at 90°.

Therefore, $HJIK$ is a square.

□