Module 5: Inventory Theory

5.1 Components of inventory models

The inventory theory concerns placing and receiving orders of given sizes periodically. From this standpoint, an **inventory policy** answers two questions:

1. **How much** to order?
2. **When** to order?

The basis for answering these questions is the minimization of the following inventory cost function:

$$\begin{bmatrix} \text{Total} \\ \text{inventory} \\ \text{cost} \end{bmatrix} = \begin{bmatrix} \text{Ordering} \\ \text{cost} \end{bmatrix} + \begin{bmatrix} \text{Setup} \\ \text{cost} \end{bmatrix} + \begin{bmatrix} \text{Holding} \\ \text{cost} \end{bmatrix} + \begin{bmatrix} \text{Shortage} \\ \text{cost} \end{bmatrix}$$

5.2 Deterministic continuous-review models

The most common inventory situation faced by manufacturers, retailers, etc. is when stock levels are depleted over time, and then are replenished by the arrival of a batch of new units. This situation is described by the so-called **economic order quantity model**, or **EOQ model**

5.2.1 The basic economic order quantity (EOQ) model

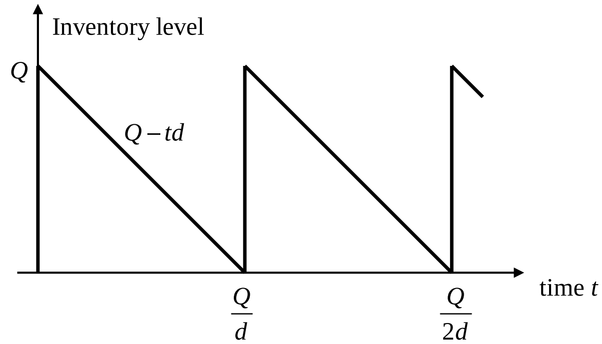
Assumptions:

- units of product are withdrawn from inventory continuously at a known constant demand rate d, such that during time t the inventory level decreases by td
- inventory is replenished when needed by ordering (purchasing or producing) a batch of Q units, where all Q units arrive simultaneously at the desired time
- the following costs are taken into consideration:

  $K =$ setup cost for ordering one batch

  $c =$ unit cost of procuring/producing one unit

  $h =$ holding cost per unit of product per unit of time held in inventory
- **continuous review** means that inventory can be replenished whenever the inventory level drops sufficiently low
- **planned** shortages are not allowed



The time between consecutive replenishments of inventory is referred to as a **cycle**. Observe that the cycle length is Q/d.

The ordering (i.e., production or purchasing) cost per cycle $= K + cQ$. The holding cost is proportional to the current inventory level $Q - td$, thus the holding cost per cycle is calculated as

$$\int_0^{Q/d} h(Q - td)\,\mathrm{d}t = hd \int_0^{Q/d} \left(\frac{Q}{d} - t\right) \mathrm{d}t = hd \left\{ \frac{Q^2}{d^2} - \frac{1}{2}\frac{Q^2}{d^2} \right\} = \frac{hQ^2}{2d}$$

Thus, the total cost per cycle is

$$\text{Total cost per cycle} = K + cQ + \frac{hQ^2}{2d}$$

so that the total cost **per unit of time** is

$$T = \frac{1}{Q/d}\left(K + cQ + \frac{hQ^2}{2d}\right) = \frac{Kd}{Q} + dc + \frac{hQ}{2}$$

The optimal value Q^* that minimizes the total cost per unit of time is the one that sets the first derivative of the above expression to 0, namely

$$-\frac{Kd}{Q^2} + \frac{h}{2} = 0 \quad \Rightarrow \quad \boxed{Q^* = \sqrt{\frac{2dK}{h}}}$$

Note:

- optimal reorder quantity Q^* increases with d and K, and decreases with h
- optimal reorder quantity does not depend on unit cost c
- since each order is for Q^* units, a total of $\frac{d}{Q^*}$ orders must be placed per unit of time
- The optimal EOQ policy is to balance the setup cost per unit of time and the holding cost per unit of time:

  $$\frac{\text{Holding cost}}{\text{unit of time}} = \frac{\text{Setup cost}}{\text{unit of time}}$$

  Indeed,

  $$\frac{\text{Holding cost}}{\text{unit of time}} = \frac{h(Q^*)^2}{2d}\Bigg/\frac{Q^*}{d} = \frac{hQ^*}{2} = \sqrt{\frac{hKd}{2}}$$

  and

  $$\frac{\text{Setup cost}}{\text{unit of time}} = \frac{K}{Q^*/d} = \sqrt{\frac{hKd}{2}}$$

Example: Camera shop

A camera store sells 1,000 cameras of particular model cameras per year. The store orders cameras from a regional warehouse. Each time an order is placed, an ordering cost of \$5 is incurred. The store pays \$5,000 for each camera, and the cost of holding \$1 worth of inventory for a year is estimated to be \$0.20.

Solution:
We are given $K = \$5$, $d = 1000$ cameras per year, $h = (0.20)(\$5{,}000) = \$1{,}000$ per year, and $c = \$5{,}000$. Then

$$Q^* = \sqrt{\frac{2(1000)(5)}{(5000)(0.20)}} = \sqrt{10} = 3.16 \text{ cameras}$$

Example: Subway service

Each hour, d passengers want to take a subway from Downtown Central station toward uptown. With a cost of subway ticket c, the municipal administration places a value of h dollars on each hour that a passenger has to wait for a train. It costs K dollars send a subway train from the Downtown Central in the uptown direction. Assuming that passengers arrive at Downtown Central at a constant rate, and there are no restrictions on how many passengers a single train can carry, how many trains must be sent each hour from the Downtown Central to uptown?

Solution:
Note that

$$\begin{bmatrix} \text{Total cost} \\ \text{per hour} \end{bmatrix} = \begin{bmatrix} \text{cost of sending trains} \\ \text{per hour} \end{bmatrix} + \begin{bmatrix} \text{passenger waiting cost} \\ \text{per hour} \end{bmatrix}$$

Let Q = number of passengers present when a train arrives. The trains arrive then each Q/d hours, or d/Q trains per hour. The sending cost is one train is

$$K - cQ,$$

and the waiting cost per train is

$$\frac{hQ^2}{2d}$$

Thus, the total cost per hour is

$$\frac{1}{Q/d}\left(K - cQ + \frac{hQ^2}{2d}\right) = \frac{Kd}{Q} - dc + \frac{hQ}{2}$$

From EOQ model, the optimal value of Q^* is

$$Q^* = \sqrt{\frac{2Kd}{h}}$$

Since each train picks up Q^* passengers, d/Q^* trains must be sent each hour. For example, if $h = \$5$ per passenger per hour, $d = 1000$ passengers per hour, and $K = \$100$ per train, we find that

$$Q^* = \sqrt{\frac{2(100)(1000)}{5}} = 200$$

Then $\frac{1000}{200} = 5$ trains per hour must be sent

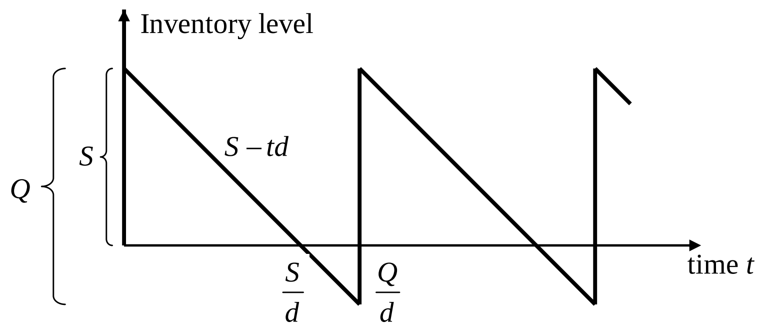
5.2.2 The EOQ model with planned shortages

In some situations, shortages (**backorders**) may be allowed. It is assumed that when a shortage occurs, the affected customers will wait for the product to become available again. Denote

$p =$ shortage cost per unit short per unit of time short

$S =$ inventory level just after a batch of Q units is added to the inventory

$Q - S =$ shortage in inventory just before a batch of Q units arrives



As before, production or ordering cost per cycle $= K + cQ$

During each cycle, the inventory is positive for a time S/d. The holding cost per cycle is then

$$\text{Holding cost per cycle} = h\frac{S}{2}\frac{S}{d} = \frac{hS^2}{2d}$$

Similarly, shortages occur during time $(Q - S)/d$. The shortage cost is

$$p\frac{Q-S}{2}\left(\frac{Q}{d} - \frac{S}{d}\right) = p\frac{(Q-S)^2}{2d}$$

The total cost per cycle is

$$\text{Total cost per cycle} = (K + cQ) + \frac{hS^2}{2d} + \frac{p(Q-S)^2}{2d}$$

and the total cost per unit of time

$$\begin{aligned}\text{Total cost per unit of time} &= \frac{1}{Q/d}\left[(K + cQ) + \frac{hS^2}{2d} + \frac{p(Q-S)^2}{2d}\right] \\ &= cd + \frac{Kd}{Q} + \frac{hS^2}{2Q} + p\frac{(Q-S)^2}{2Q}\end{aligned}$$

The optimal values of Q and S which determine the optimal reorder quantity and optimal planned shortage level, is found from

$$\begin{aligned}\frac{\partial \mathrm{TC}}{\partial S} &= \frac{hS}{Q} - \frac{p(Q-S)}{Q} = 0 \\ \frac{\partial \mathrm{TC}}{\partial Q} &= -\frac{Kd}{Q^2} - \frac{hS^2}{2Q^2} + \frac{p(Q-S)}{Q} - \frac{p(Q-S)^2}{2Q^2} = 0\end{aligned}$$

whereby we obtain

$$\boxed{Q^* = \sqrt{\frac{2Kd}{h}}\sqrt{\frac{p+h}{p}}, \quad S^* = \sqrt{\frac{2Kd}{h}}\sqrt{\frac{p}{p+h}}}$$

The maximum planned shortage is

$$Q^* - S^* = \sqrt{\frac{2Kd}{p}}\sqrt{\frac{h}{p+h}}$$

Example: Optometry clinic

Each month, the Smalltown optometry clinic sells 100 frames for eyeglasses. The clinic orders frames from a regional supplier, which charges \$15 per frame. Each order incurs an ordering cost of \$50. Smalltown optometry believes that the demand for frames can be backlogged and that the cost of being short one frame for one month is \$15. The annual holding cost for inventory is \$4.5 per frame. What is the optimal order quantity? What is the maximum shortage that will occur? What is the maximum inventory level?

Solution:
We have that $d = 100$ frames per month, $K = 50$, $p = 15$ per frame per month, and $h = 4.5/12 = 0.375$ per month

$$Q^* = \sqrt{\frac{(2)(100)(50)}{0.375}\frac{(15+0.375)}{15}} = 165.3$$

$$Q^* - S^* = \sqrt{\frac{(2)(100)(50)}{15}\frac{0.375}{(15+0.375)}} = 4.0$$

The maximum stock level S^* = 161.3 frames

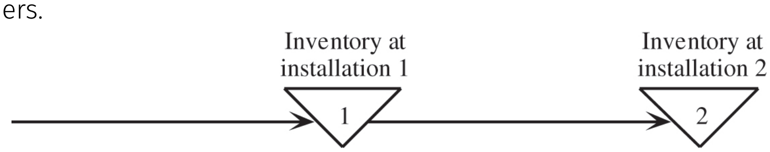
5.3 Multi-echelon models for supply chain management

Definition: Supply chain

A **supply chain** is a network of facilities that procure raw materials, transform them into intermediate goods and then final products, and finally deliver the products to customers.

5.3.1 A model for a serial two-echelon system

Consider a serial two-echelon system, where each echelon contains one installation (warehouse, factory, etc), and the product is supplied from installation 1 to installation 2, and installation 2 distributes the product to customers.



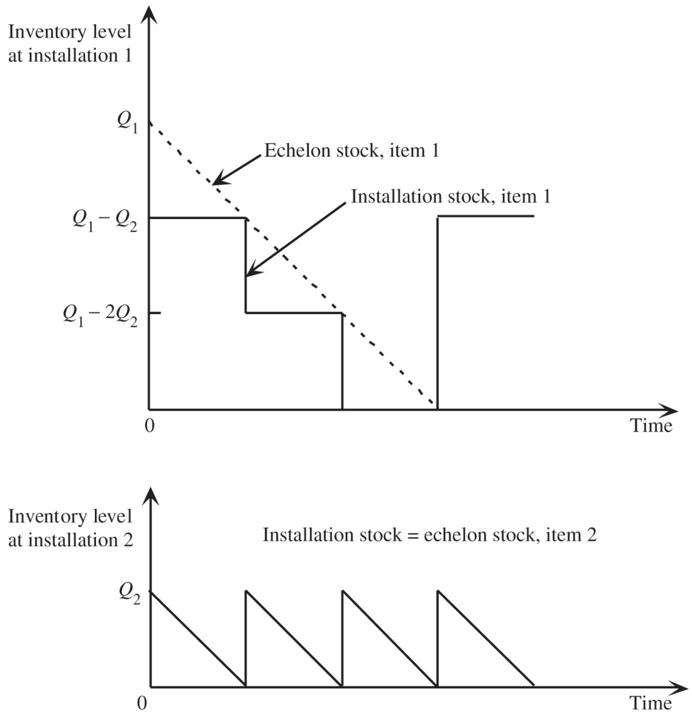
We assume that

1. The assumptions of basic EOQ model (no shortages) hold at installation 2. There is a known constant demand rate of d units per unit of time, and order quantity of Q_2 is placed periodically to replenish inventory. The relevant costs are setup cost K_2 and holding cost h_2.

2. Installation 1 uses its inventory to provide a batch of Q_2 units to installation 2 upon request. To replenish its own inventory, installation 1 places orders of size Q_1. The relevant costs at installation 1 are setup cost K_1 per order, and holding cost h_1 per unit of product per unit of time.

3. The units of product increase in value when they are received and processed at installation 2, so that $h_1 < h_2$.

The key insight: the optimal inventory policy for installation 1 is $Q_1 = nQ_2$, where n is an integer, and the order is placed as soon as inventory becomes zero at installation 1.



5.3.2 Locally optimal inventory policy

A simple (but inefficient) approach would be to optimize inventory policies at both facilities separately:

1. **Determine the optimal inventory policy (ordering quantity Q_2^*) at Installation 2:**

   From the basic EOQ we have $Q_2^* = \sqrt{\dfrac{2dK_2}{h_2}}$.

2. **Use Q_2^* as demand for Installation 1 to obtain the optimal inventory policy Q_1^***

   Holding cost at Installation 1 is

$$h_1 \left(\frac{1}{2}(nQ_2)\frac{nQ_2}{d} - n\frac{1}{2}Q_2\frac{Q_2}{d} \right) = h_1 \frac{n(n-1)}{2d} Q_2^2$$

   The length of period at Installation 1 is n times period at Installation 2, nQ_2/d. Installation 1 needs to replenish its stock with $Q_1 = nQ_2$ units every nQ_2/d units of time, the **variable** cost at Installation 1 is

$$C_{\text{var}\,1} = \frac{1}{nQ_2/d}\left(K_1 + h_1 \frac{n(n-1)}{2d} Q_2^2 \right) = \frac{K_1 d}{nQ_2} + \frac{h_1(n-1)Q_2}{2}$$

Minimizing $C_{\text{var}\,1}$ with respect to n, we obtain the optimal value of n^*:

$$n^* = \frac{1}{Q_2}\sqrt{\frac{2K_1 d}{h_1}}$$

Recalling that Q_2 above should be **optimal** for installation 2, we get

$$n^* = \frac{1}{Q_2^*}\sqrt{\frac{2K_1 d}{h_1}} = \sqrt{\frac{K_1 h_2}{K_2 h_1}}$$

and the optimal ordering quantity Q_1^* for installation 1 is

$$Q_1^* = [n^*]Q_2^*,$$

where $[n^*]$ is a (specially) rounded value of n^*.

Rounding procedure for n^*

$$[n^*] = \begin{cases} 1, & \text{if} \quad n^* < 1 \\ \lfloor n^* \rfloor, & \text{if} \quad n^* > 1 \ \text{ and } \ \dfrac{n*}{\lfloor n^* \rfloor} - \dfrac{\lfloor n^* \rfloor + 1}{n^*} \le 0 \\ \lfloor n^* \rfloor + 1, & \text{if} \quad n^* > 1 \ \text{ and } \ \dfrac{n*}{\lfloor n^* \rfloor} - \dfrac{\lfloor n^* \rfloor + 1}{n^*} > 0 \end{cases}$$

where $\lfloor n^* \rfloor$ is the largest integer smaller than n^*, i.e. $\lfloor n^* \rfloor \le n^* < \lfloor n^* \rfloor + 1$.

5.3.3 Globally optimal inventory policy

The globally optimal policy minimizes the total **variable cost** per unit of time at both installations simultaneously.

According to the EOQ model, the **variable** inventory cost per unit of time at Installation 2 is

$$C_{\text{var}\,2} = \frac{dK_2}{Q_2} + \frac{h_2 Q_2}{2}$$

(the **fixed** cost cd that does not depend on the choice of Q_2 is disregarded).

Similarly to the previous case, the variable cost per unit of time at Installation 1 is equal to

$$C_{\text{var}\,1} = \frac{K_1 d}{nQ_2} + h_1\frac{(n-1)Q_2}{2}$$

Total variable cost per unit of time is

$$C_{\text{var}} = C_{\text{var}\,1} + C_{\text{var}\,2} = \left(\frac{K_1}{n} + K_2\right)\frac{d}{Q_2} + [(n-1)h_1 + h_2]\frac{Q_2}{2}$$

Denote

$e_1 = h_1 =$ echelon holding cost per unit of time for installation 1

$e_2 = h_2 - h_1 =$ echelon holding cost per unit of time for installation 2

Then

$$C_{\text{var}} = \left(\frac{K_1}{n} + K_2\right)\frac{d}{Q_2} + (ne_1 + e_2)\frac{Q_2}{2}$$

This is an EOQ model with setup cost $K_1/n + K_2$ and the total unit holding cost $ne_1 + e_2$, therefore

$$Q_2^* = \sqrt{\frac{2d\left(\frac{K_1}{n} + K_2\right)}{ne_1 + e_2}}$$

Plugging expression for Q_2^* in the expression for C, we have

$$C_{\text{var}} = \sqrt{2d\left(\frac{K_1}{n} + K_2\right)(ne_1 + e_2)}$$

The optimal value of n^* that minimizes the above expression, is

$$n^* = \sqrt{\frac{K_1 e_2}{K_2 e_1}}$$

The same rounding procedure is employed to determine $[n^*]$.

Example: Two-echelon supply chain

Consider a two-echelon system consisting of a local store and a regional warehouse. The demand per unit of time is 600 units of product, the store's holding costs is $3 per unit of product per unit of time, and it costs $100 to place an order to the warehouse. The warehouse can store the product at the cost of $2 per unit of product per unit of time, and the warehouse's ordering setup cost is $1000 per order. What is the optimal inventory policy?

Solution:
Clearly, we have $d = 600$, $K_1 = 1000$, $K_2 = 100$, $h_1 = 2$, $h_2 = 3$. The locally optimal inventory policy implies

$$\begin{aligned}
Q_2^* &= \sqrt{\frac{2(600)(100)}{3}} = 200.0 \\
n^* &= \sqrt{\frac{(1{,}000)(3)}{(100)(2)}} = 3.87 \quad \Rightarrow \quad \frac{n^*}{\lfloor n^* \rfloor} - \frac{\lfloor n^* \rfloor + 1}{n^*} = \frac{3.87}{3} - \frac{4}{3.87} = 0.26 > 0 \quad \Rightarrow \quad [n^*] = 4 \\
Q_1^* &= 4 \cdot 200.0 = 800.0 \\
C_{\text{var}}^* &= 1{,}950.0
\end{aligned}$$

By employing the globally optimal inventory policy, we obtain

$$\begin{aligned}
n^* &= \sqrt{\frac{(1{,}000)(1)}{(100)(2)}} = 2.24 \quad \Rightarrow \quad \frac{n^*}{\lfloor n^* \rfloor} - \frac{\lfloor n^* \rfloor + 1}{n^*} = \frac{2.24}{2} - \frac{3}{2.24} = -0.22 < 0 \quad \Rightarrow \quad [n^*] = 2 \\
Q_2^* &= 379.0 \\
Q_1^* &= 2 \cdot 379.0 = 758.0 \\
C_{\text{var}}^* &= 1{,}897.0
\end{aligned}$$

Thus, globally optimal inventory policy results in about 2.7% savings of total variable cost C_{var}^*.

5.4 A stochastic continuous-review model

Stochastic continuous review model applies to situations when future demand is uncertain, but its distribution is known or can be estimated. Then, the inventory level is monitored continuously, and a new order is placed as soon as inventory level drops to the reorder point.

Before the computer age, a continuous-review system was based on a **two-bin** principle. All the units of product would be held in two bins, with capacities selected in such a way that as soon as the first bin was emptied, a new order was placed, and units would be drawn out of the second bin until the order arrived.

A continuous review model is normally based on two critical quantities:

$R =$ reorder point

$Q =$ order quantity

Then, the corresponding inventory policy is known as **(R, Q) policy** and is formulated as follows

(R, Q) policy

Whenever inventory level drops to R units, place an order for Q more units to replenish the inventory.

Assumptions In addition to the assumptions of the deterministic EOQ model, in the stochastic case it is assumed that:

1. The demand is uncertain, but its distribution is known or can be estimated.
2. There is a **lead time** between the time an order is placed and the time the ordered quantity is received.
3. If a stockout occurs before the order is received, the demand is **backlogged**, so that backorders are filled once the order arrives.

Choosing the order quantity Q Since the stochastic continuous review model is rather similar to deterministic EOQ model with allowed shortages, then the most straightforward approach to determining the value of Q in the stochastic case would be to approximate it using the deterministic model formula:

$$Q = \sqrt{\frac{2dK}{h}}\sqrt{\frac{p+h}{p}},$$

where **d = the average demand per unit of time**. It turns out that this approximation is fairly good.

Choosing the reorder point R The reorder point R depends on the desired “service level”, or measure of system performance, e.g.:

1. The probability that a stockout will not occur between the time an order is placed and the time it is received

2. The average number of stockouts per year
3. The average percentage of annual demand that can be satisfied immediately (without stockouts)
4. The average delay in filling backorders when a stockout occurs
5. The overall average delay in filling orders (where the delay without stockout is 0)

Some of the above measures are closely related:

- Measures 1 and 2 are related. If, for example, the order quantity Q has been set at 10% of average annual demand, then an average of 10 orders will be placed per year. If reorder point is chosen such that the probability of stockout is 20%, then the average number of stockouts per year will be 2.
- Measures 2 and 3 are related. Assume that, for example, that an average of 2 stockouts occur per year and the average length of a stockout is 9 days. Since $2(9) = 18$ days of stockout per year constitute about 5% of the year, then the average percentage of annual demand that can be satisfied without stockout is 95%
- Measures 3, 4, and 5 are related. Suppose that the average percentage of annual demand that can be satisfied immediately is 95 percent and the average delay in filling backorders when a stockout occurs is 5 days. Since only 5% of the customers incur this delay, the overall average delay in filling orders then would be $(0.05)(5) = 0.25$ day per order.

Let us focus on Measure 1, and denote the service level under this measure as

> $L =$ probability that a stockout will not occur between the time an order is placed and the time it arrives

Let $D =$ demand during the lead time in filling an order. Then, the desired value of R is obtained as the solution of equation

$$\boxed{F_D(R) = L, \quad \text{or} \quad P(D \leq R) = L}$$

where $F_D(\cdot)$ is the c.d.f. of the demand D. The **safety stock**, or the expected inventory level just before the order is received is computed as

$$\boxed{\text{Safety stock} = R - \mathsf{E}[D]}$$

For example, if the demand D during the lead time is distributed uniformly between a and b, then $F_D(t) = \dfrac{t-a}{b-a}$ and one has

$$R = a + L(b-a)$$

Example: TV sets

Assume that setup cost to produce speakers is $K = \$12{,}000$, the unit holding cost is $h = \$0.30$ per speaker per month, and unit shortage cost is $p = \$1.10$ per speaker per month. The lead time between ordering a production run of speakers and having the speakers ready for assembly into television sets is 1 month. The demand for speakers during the lead time is normally distributed with mean 8,000 and standard deviation of 2,000. The requirement is to avoid a stockout during the lead time in 95% of cases.

Solution:

$$Q = \sqrt{\frac{2dK}{h}}\sqrt{\frac{p+h}{p}} = \sqrt{\frac{2(8{,}000)(12{,}000)}{0.3}}\sqrt{\frac{1.1+0.3}{1.1}} = 28{,}540$$

The reorder point R is obtained from

$$0.95 = P\left(D \le R\right) = P\left(\frac{D-8000}{2000} \le \frac{R-8000}{2000}\right) = P(Z \le z_R)$$

where $Z \sim N(0,1)$ and $z_R = (R-8000)/2000$. From tables, $z_R = 1.645$, thus $R = 2000(1.645) + 8000 = 11{,}290$

The resulting amount of safety stock is $R - 8000 = 3{,}290$

5.5 A stochastic single-period model for perishable products

In real life, the demand is usually an unknown (random) value. One of the models that provides an optimal inventory policy under uncertain demand is the **newsboy problem**: every day, a newsvendor must decide how many newspapers to order from a publisher. If too few newspapers are ordered, the newsvendor may incur losses due to unsatisfied demand; if, on the other hand, the order is too large, some of the newspapers may not be sold, leading to losses due to overstocking (the unsold newspapers are worthless the next day). Note that newspapers here represent any **perishable** product, i.e. any goods that loses value after certain time period.

Assumptions

1. A single perishable product is under consideration
2. Only one time period is considered, since the (perishable) product cannot be sold later
3. There may be some initial inventory I at the beginning
4. The decision must be made with respect to the number Q of units of product to order so that they can be placed into inventory at the beginning of the period. If there are I units in the inventory already, the total number S of units in the beginning of the period is $S = I + Q$. If the initial inventory is empty ($I = 0$), we have that $S = Q$. Thus, we can regard S as the decision variable
5. The demand D is a random variable whose distribution is known
6. The following costs are taken into account:

   $K =$ setup cost for purchasing or producing the entire batch of units

   $c =$ unit cost for purchasing or producing each unit

   $h =$ holding cost per unit remaining at the end of period (includes storage costs minus salvage value)

   $p =$ shortage cost per unit of unsatisfied demand (includes lost revenue and cost of loss of customer goodwill)

First, consider a situation when there is no initial inventory ($I = 0$) and no setup costs ($K = 0$).

5.5.1 Model without initial inventory and setup cost: $I = 0,\ K = 0$

The total inventory cost is

$$cS + \tilde{p}\max\{0, D - S\} + h\max\{0, S - D\} - P\min\{S, D\},$$

where P is the price for which one unit of product is sold, and $\tilde{p}$ is the cost of loss of customer goodwill. Note that the last term above can be represented in the form

$$P\min\{S, D\} = P\big(D - \max\{0, D - S\}\big) = PD - P\max\{0, D - S\}$$

The term $P\max\{0, D - S\}$ represents the **lost revenue due to unsatisfied demand**. Hence, the total inventory cost can be rewritten as

$$cS + (\tilde{p} + P)\max\{0, D - S\} + h\max\{0, S - D\} - PD$$

Note that $(\tilde{p}+P)\max\{0, D-S\}$ represents shortage cost per unit of unsatisfied demand, including the cost of lost revenue, thus we denote $(\tilde{p} + P)$ as just p. Also, the term PD is beyond our control (does not depend on the inventory level S), thus can be disregarded when solving the problem. Finally, the total cost to be minimized by appropriately choosing level S of inventory, is

$$cS + p\max\{0, D - S\} + h\max\{0, S - D\}$$

Since demand D is random, the stock level S may be chosen based on the expected value criterion, by minimizing the expected cost of inventory $C(S)$:

$$\begin{aligned}
C_{0,0}(S) &= \mathsf{E}\big[cS + p\max\{0, D - S\} + h\max\{0, S - D\}\big] \\
&= \int_0^\infty \big(cS + p\max\{0, x - S\} + h\max\{0, S - x\}\big) f(x)\,\mathrm{d}x \\
&= cS + p\int_S^\infty (x - S) f(x)\,\mathrm{d}x + h\int_0^S (S - x) f(x)\,\mathrm{d}x,
\end{aligned}$$

where $f(x)$ is the pdf of the (continuous) random variable D:

$$P\{D \le d\} = F(d) = \int_0^d f(x)\,\mathrm{d}x,$$

and $F(\cdot)$ is the cdf of D. The optimal value S^* that minimizes the expected total inventory cost $C_{0,0}(S)$ satisfies the equation

$$\boxed{F(S^*) = \frac{p - c}{p + h}}$$

If the random demand D has a discrete distribution,

$$P\{D \le d\} = F(d) = \sum_{n=0}^{d} P\{D = n\}$$

then the optimal inventory level is the smallest integer S^* such that

$$\boxed{F(S^*) \ge \frac{p - c}{p + h}}$$

The **service level** ratio $\dfrac{p - c}{p + h}$ can be represented as

$$\frac{p - c}{p + h} = \frac{p - c}{(p - c) + (c + h)} = \frac{C_{\text{under}}}{C_{\text{under}} + C_{\text{over}}}$$

where $C_{\text{under}} = p - c$ is the cost of underordering, and $C_{\text{over}} = c + h$ is the cost of overordering.

Example: *USA Now* newsstand

The owner of newsstand wants to determine the number of *USA Now* newspapers that must be stocked at the start of each day. The owner pays 30 cents for a copy and sells it for 75 cents. Newspapers left at the end of the day are recycled for an income of 5 cents a copy. How many copies should the owner stock every morning, assuming that the demand for the day can be described as

(a) A normal distribution with mean 300 copies and standard deviation 20 copies

(b) A discrete distribution defined as

d	200	220	300	320	340
$P(D=d)$	0.1	0.2	0.4	0.2	0.1

Solution:
We have $c = 30$, selling price $P = 75$, but the holding and shortage costs h and p are not defined directly. Since no mention is made of cost of loss of customer goodwill, the shortage cost consists of only the cost of lost revenue:

$$p = P = 75$$

Similarly, holding costs are not mentioned, therefore holding cost = storage cost – salvage value:

$$h = 0 - 5 = -5$$

Then, we have

$$\frac{p-c}{p+h} = \frac{75-30}{75-5} = 0.643$$

Alternatively, one can figure that cost of underordering is $C_{\text{under}} = 75 - 30 = 45$ (net revenue per copy lost due to unmet demand), and the cost of overordering $C_{\text{over}} = 30 - 5 = 25$ (cost per copy minus salvage value), whence the optimal service level is

$$\frac{C_{\text{under}}}{C_{\text{under}} + C_{\text{over}}} = \frac{45}{45+25} = 0.643$$

Case (a) To determine the solution of equation

$$F_{N(300,20)}(S^*) = 0.643$$

where $F_{N(300,20)}$ is the cdf of normal $N(300, 20)$ distribution, we note that

$$F_{N(300,20)}(S^*) = P\{D \leq S^*\}$$

where D is a normal random variable with mean $\mu = 300$ and standard deviation $\sigma = 20$. The above expression can be computed using tables for **standard normal** distribution, i.e., normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$. To this end, we use the fact that if D is a normal $N(\mu, \sigma)$ random variable then the random variable Z defined as

$$Z = \frac{D-\mu}{\sigma}$$

is a standard normal $N(0,1)$ random variable. Thus, we transform the above equation as follows:

$$0.643 = F_{N(300,20)}(S^*) = P\{D \leq S^*\} = P\left\{\frac{D-300}{20} \leq \frac{S^*-300}{20}\right\} = P\{Z \leq z^*\} = F_{N(0,1)}(z^*)$$

where

$$z^* = \frac{S^* - 300}{20}$$

and $F_{N(0,1)}$ is the cdf of standard normal distribution. In other words, we have to find such z^* that

$$F_{N(0,1)}(z^*) = F_{N(300,20)}(S^*) = 0.643$$

Using the tables, we find that $z^* \approx 0.366$, whence $S^* = 20(0.366) + 300 \simeq 307.3$. Thus, the optimal order is about 307 copies.

Case (b) We need to determine the smallest value S^* such that

$$F(S^*) = P\{D \leq S^*\} = P\{D = 200\} + P\{D = 220\} + \ldots + P\{D = S^*\} \geq 0.643$$

We have

$$\begin{aligned}
F(200) &= P\{D \leq 200\} = P\{D = 200\} = 0.1 \\
F(220) &= P\{D \leq 220\} = P\{D = 200\} + P\{D = 220\} = 0.1 + 0.2 = 0.3 \\
F(300) &= P\{D \leq 300\} = P\{D = 200\} + P\{D = 220\} + P\{D = 300\} = 0.7 > 0.643
\end{aligned}$$

Thus, the optimal order quantity is 300 copies (based on the specified demand distribution).

5.5.2 Model with initial inventory but no setup cost: $I > 0,\ K = 0$

Denote $C_{I,0}(S)$ to be the expected inventory cost with initial inventory:

$$\begin{aligned}
C_{I,0}(S) &= c(S - I) + p\int_S^{\infty} (x - S)f(x)\,\mathrm{d}x + h\int_0^S (S - x)f(x)\,\mathrm{d}x \\
&= C_{0,0}(S) - cI
\end{aligned}$$

where $C_{0,0}(S)$ is the inventory cost with $I = K = 0$. Now, observe that in the problem of finding the optimal inventory level S^* with non-zero initial inventory $I > 0$

- the objective function $C_{I,0}(S)$ achieves its minimum in the very same point that the function $C_{0,0}(S)$ does, namely in the point S^* that satisfies $F(S^*) = \frac{p-c}{p+h}$
- but, the problem is **constrained**: the inventory level S cannot be less than already existing inventory I:

$$S \geq I$$

Thus, the optimal ordering policy is:

Optimal inventory policy with $I > 0$ and $K = 0$

1. If $I < S^*$, order $S^* - I$ to bring the inventory level up to S^*
2. If $I \geq S^*$, do not order

5.5.3 Model with initial inventory and setup costs: $I > 0,\ K > 0$

With $I > 0$ and $K > 0$, the expected inventory cost $C_{I,K}(S)$ can be written as

$$C_{I,K}(S) = K + c(S-I) + p\int_S^\infty (x-S)f(x)\,\mathrm{d}x + h\int_0^S (S-x)f(x)\,\mathrm{d}x \quad \text{if order is placed } (S > I)$$

$$C_{I,K}(I) = p\int_I^\infty (x-I)f(x)\,\mathrm{d}x + h\int_0^I (I-x)f(x)\,\mathrm{d}x \quad \text{if no order is placed } (S = I)$$

Using the introduced above function

$$C_{0,0}(S) = cS + p\int_S^\infty (x-S)f(x)\,\mathrm{d}x + h\int_0^S (S-x)f(x)\,\mathrm{d}x,$$

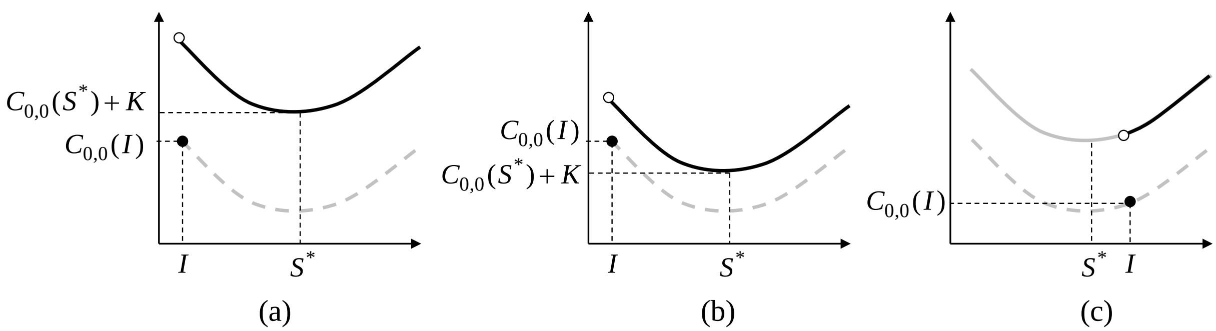
we can rewrite the expression for the expected inventory cost $\widetilde{C}(S)$ as

$$C_{I,K}(S) = \begin{cases} K + C_{0,0}(S) - cI, & \text{if } S > I \text{ (order is placed)} \\ C_{0,0}(I) - cI, & \text{if } S = I \text{ (do not order)} \end{cases}$$

Note that the constant term $(-cI)$ does not influence the location of the minimum point of the function $C_{I,K}(S)$, thus we can consider minimization of the function

$$C_{I,K}(S) = \begin{cases} K + C_{0,0}(S), & \text{if } S > I \text{ (order is placed)} \\ C_{0,0}(I), & \text{if } S = I \text{ (do not order)} \end{cases}$$

over values S such that $S \geq I$.



We have three possibilities:

(a) $C_{0,0}(I) \leq C_{0,0}(S^*) + K \quad \Rightarrow \quad$ do not order

(b) $C_{0,0}(I) > C_{0,0}(S^*) + K \quad \Rightarrow \quad$ order up to level S^*

(c) $S^* \leq I \quad \Rightarrow \quad$ do not order

The above three variants can be reduced to a two-choice strategy, just like the one for $K = 0,\ I > 0$, by introducing a value s^* that satisfies

$$\boxed{C_{0,0}(s^*) = C_{0,0}(S^*) + K \quad \text{and} \quad s^* < S^*}$$

“(s, S) policy” (optimal inventory policy with $I \geq 0$ and $K > 0$)

1. If $s^* > I$, order $S^* - I$ to bring the inventory level up to S^*
2. If $s^* \leq I$, do not order

Example: Perishable item with uniform demand

The daily demand for an item has a uniform $U(0,10)$ distribution. The unit holding cost of the item during the period is \$0.50, and the unit penalty cost of running out of stock is \$4.50. A fixed cost of \$5 is incurred each time an order is placed. Determine the optimal inventory policy if there is an initial inventory I at hand.

Solution:
The pdf of the uniform $U(0,10)$ distribution is

$$f(x) = \begin{cases} \frac{1}{10}, & 0 \leq x \leq 10 \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$F(d) = P\{D \leq d\} = \int_{-\infty}^{d} f(x)\,\mathrm{d}x = \int_0^d \frac{1}{10}\,\mathrm{d}x = \frac{d}{10},$$

We have $p = 4.5$, $h = 0.5$; the purchasing cost c is not mentioned, thus assume $c = 0$. Then

$$\frac{p-c}{p+h} = \frac{4.5-0}{4.5+0.5} = 0.9$$

therefore

$$F(S^*) = 0.9 \quad \Leftrightarrow \quad \frac{S^*}{10} = 0.9 \quad \Leftrightarrow \quad \boxed{S^* = 9}$$

The expected cost function $C_{0,0}(S)$ is

$$\begin{aligned} C_{0,0}(S) &= p\int_S^{\infty} (x-S)f(x)\,\mathrm{d}x + h\int_0^S (S-x)f(x)\,\mathrm{d}x \\ &= 4.5\int_S^{10} (x-S)\frac{1}{10}\,\mathrm{d}x + 0.50\int_0^S (S-x)\frac{1}{10}\,\mathrm{d}x \\ &= 0.25S^2 - 4.5S + 22.5 \end{aligned}$$

The value of s^* is determined by solving the equation

$$C_{0,0}(s^*) = K + C_{0,0}(S^*)$$

or, equivalently,

$$0.25(s^*)^2 - 4.5s^* + 22.5 = 5 + 0.25(S^*)^2 - 4.5S^* + 22.5 \quad \Leftrightarrow \quad \boxed{(s^*)^2 - 18s^* + 61 = 0}$$

The solutions of this quadratic equation are $s^* = 4.53$ and $s^* = 13.47$. The last value is discarded, since it is greater than S^*. The optimal policy is order $9 - I$, if $I < 4.53$, or do not order.