Chapter Two Basic Topology

Part 2: Metric Spaces

In mathematics, space = set + structure(s).

Definition 4. Let X be a set. A function $d : X \times X \to [0, \infty)$ is called a metric (or distance) on X if

i) $\forall x, y \in X$, $d(x, y) = 0 \iff x = y$;

ii) $\forall x, y \in X$, $d(x, y) = d(y, x)$;

iii) $\forall x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$. (Triangle inequality)

In this case, (X, d) is called a metric space.

Examples. **(1)** For $x, y \in \mathbb{R}$, $d(x, y) = |x - y|$ defines a metric on $\mathbb{R}$.

More general, for $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$, $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$ is a metric on $\mathbb{R}^n$. With is metric, $\mathbb{R}^n$ is called the n-dimensional Euclidean space.

(2) Similarly, $d(x, y) = |x - y|$ defines a metric on $\mathbb{C}$.

(3) On the set $\mathbb{R}^n$ $(n > 1)$, the following are other distance functions which are all "equivalent to" the Euclidean metric d:

$$\text{for } 1 \leq p < \infty,\ d_p(x, y) = \Big(\sum_{i=1}^{n} |x_i - y_i|^p\Big)^{1/p}; \quad d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|.$$

It is clear that the Euclidean metric in (1) is just d_2.

(4) Let X be a set. Define $d : X \times X \to [0, \infty)$ by $d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$ Then d is a metric on X, and (X, d) is called a discrete metric space.

(5) Let (X, d) be a metric space and $Y \subseteq X$. Let d_Y be the restriction of d to $Y \times Y$ (i.e., $d_Y(y_1, y_2) = d(y_1, y_2)$ for all $y_1, y_2 \in Y$). Then d_Y is a metric on Y, and Y with this metric is called a subspace of X.

Some Concepts on Metric Spaces.

Let (X, d) be a metric space, $x \in X$, and $r > 0$.

(i) Define $B(x, r) = \{y \in X : d(x, y) < r\}$, called the open ball centred at x with radius r, or the r-neighborhood of x.

E.g., in $\mathbb{R}$, $B(x, r) = (x - r, x + r)$.

A general neighborhood of x is a subset U of X such that $B(x, r) \subseteq U$ for some $r > 0$.

E.g., in $\mathbb{R}$, $[x - r, x + r)$, $[x - r, x + 2r]$, etc. are all neighborhoods of x.

Exercise. For $p = 1, 2, \infty$, compare the subsets $B_p\big((0,0), 1\big)$ of $\mathbb{R}^2$, where

$$B_p\big((0,0), 1\big) \;=\; \{(x, y) \in \mathbb{R}^2 : d_p\big((x, y), (0, 0)\big) < 1\} \quad (p = 1, 2, \infty).$$

(ii) A subset G of X is called open if $\forall x \in G$, $\exists r > 0$, $B(x, r) \subseteq G$.

E.g., in $\mathbb{R}$, the interval $(0, 1)$ is open, but the interval $[0, 1)$ is not open.

$\emptyset$ and X are open in X.

(iii) Let $E \subseteq X$. x is called a limit point of E (or, cluster point, or accumulation point) if $\forall r > 0$, $B(x, r) \cap E$ contains $y \neq x$ (that is, $\big(B(x, r) - \{x\}\big) \cap E \neq \emptyset$).

We let $E' \;=\;$ the set of all limit points of E, called the derived set of E.

E.g., in $\mathbb{R}$, 0 is a limit point of $E = (0, 1)$ though 0 is not in $(0, 1)$. In this case, $E' = [0, 1]$. Also in $\mathbb{R}$, if $E = \{0, 1\}$, then $E' = \emptyset$.

(iv) A subset E of X is called closed if $E' \subseteq E$.

E.g., in $\mathbb{R}$, the set $\{0, 1\}$ is closed but $(0, 1)$ is not closed.

$\emptyset$ and X are closed in X.

Theorem 4 (Theorem 2.19). Let (X, d) be a metric space, $x \in X$ and $r > 0$. Then $B(x, r)$ is open in X.

Proof. Let $y \in B(x, r)$. We need find $\varepsilon > 0$ such that $B(y, \varepsilon) \subseteq B(x, r)$.

Let $\varepsilon = r - d(x, y)$. Then $\varepsilon > 0$ since $y \in B(x, r)$. Now $\forall z \in B(y, \varepsilon)$, we have

$$d(z, x) \leq d(z, y) + d(y, x) < \varepsilon + d(y, x) = r,$$

and thus $z \in B(x, r)$. Therefore, $B(y, \varepsilon) \subseteq B(x, r)$. Hence, $B(x, r)$ is open. $\blacksquare$

Theorem 5 (Theorem 2.23). Let E be a subset of a metric space (X, d). Then

(i) E is open $\iff$ $X - E$ is closed;

(ii) E is closed $\iff$ $X - E$ is open.

Proof. **(i)** "$\Longrightarrow$". Suppose E is open. We need prove that $(X - E)' \subseteq X - E$.

Let $x \in (X - E)'$. Then $\forall r > 0$, $\big(B(x, r) - \{x\}\big) \cap (X - E) \neq \emptyset$, which is equivalent to $B(x, r) - \{x\} \nsubseteq E$. So, $\forall r > 0$, $B(x, r) \nsubseteq E$, and thus $x \notin E$ (since E is open). That is, $x \in X - E$. Therefore, we obtain that $(X - E)' \subseteq X - E$, and hence $X - E$ is closed.

"$\Longleftarrow$". Suppose $X - E$ is closed. We need prove that $\forall x \in E, \exists r_0 > 0, B(x, r_0) \subseteq E$.

Let $x \in E$. Then $x \notin (X - E)'$, since $(X - E)' \subseteq X - E$. Thus $\exists r_0 > 0$ such that $\big(B(x, r_0) - \{x\}\big) \cap (X - E) = \emptyset$; that is, $B(x, r_0) - \{x\} \subseteq E$. Since $x \in E$, $B(x, r_0) \subseteq E$. Therefore, we prove that E is open.

(ii) Replacing E by $X - E$ in (i), we get $X - E$ is open $\iff X - (X - E) = E$ closed. ■

In a metric space (X, d), the "closed" ball centred at x with radius r is defined by

$$B[x, r] \;=\; \{y \in X : d(x, y) \leq r\}.$$

Corollary 5. $B[x, r]$ is closed.

Proof. By Theorem 5, we only need prove that $X - B[x, r]$ is open.

Let $z \in X - B[x, r]$. Then $d(z, x) > r$. Let $\varepsilon = d(z, x) - r$. Then $\varepsilon > 0$. We prove below that $B(z, \varepsilon) \subseteq X - B[x, r]$.

Let $y \in B(z, \varepsilon)$. Then $d(y, z) < \varepsilon = d(z, x) - r$, or $d(z, x) - d(y, z) > r$. Since

$$d(z, x) \leq d(z, y) + d(y, x),$$

we obtain that

$$d(y, x) \geq d(z, x) - d(z, y) > r.$$

Hence, $y \in X - B[x, r]$. Therefore, we have $B(z, \varepsilon) \subseteq X - B[x, r]$.

Therefore, we prove that $X - B[x, r]$ is open. ■

Theorem 6 (Theorem 2.24). Let (X, d) be a metric space.

(i) If $\{G_\alpha\}$ is a family of open sets in X, then $\bigcup_\alpha G_\alpha$ is open in X.

(ii) If $G_1, \cdots, G_n$ are open sets in X, then $\bigcap_{i=1}^n G_i$ is open in X.

Remark 1. The above (i) and (ii) together with "$\emptyset$, X are open" are used as the definition of a topology on X.

Proof. **(i)** Let $x \in \bigcup_\alpha G_\alpha$. Then $x \in G_{\alpha_0}$ for some α_0. Since G_{α_0} is open, $\exists r > 0$ such that $B(x, r) \subseteq G_{\alpha_0} \subseteq \bigcup_\alpha G_\alpha$. Therefore, we obtain that $\bigcup_\alpha G_\alpha$ is open.

(ii) Let $x \in \bigcap_{i=1}^n G_i$. Then for each $1 \leq i \leq n$, $\exists r_i > 0$ such that $B(x, r_i) \subseteq G_i$. Let $r = \min\{r_1, \cdots, r_n\}$. Then $r > 0$ and $B(x, r) \subseteq G_i$ for all $1 \leq i \leq n$. Hence, $B(x, r) \subseteq \bigcap_{i=1}^n G_i$. Therefore, we prove that $\bigcap_{i=1}^n G_i$ is open. ■

By Theorems 5 and 6 together with DeMorgan's Laws, we have the following corollary on closed sets.

Corollary 6. Let (X, d) be a metric space.

(i) If $\{F_\alpha\}$ is a family of closed sets in X, then $\bigcap_\alpha F_\alpha$ is closed in X.

(ii) If $F_1, \cdots, F_n$ are closed sets in X, then $\bigcup_{i=1}^n F_i$ is closed in X.

Note that $\emptyset$ and X are both open and closed (called clopen). Therefore, $\emptyset$ is the smallest open set and X is the largest open set, and $\emptyset$ is the smallest closed set and X is the largest closed set.

Question. For any $\emptyset \subseteq E \subseteq X$, does E have the largest open/closed subset, and does E have the smallest open/closed superset?

E.g., $(0,1)$ does not have largest closed subset; $[0,1]$ does not have smallest open superset.

Definition 5. Let $E \subseteq X$. The closure of E is the set $\overline{E} = E \cup E'$.

An element x of X is called an interior point of E if $\exists r > 0$, $B(x,r) \subseteq E$. The interior of E is the set E° of all interior points of E.

- By definition, we have $\underline{E^\circ \subseteq E \subseteq \overline{E}}$.

Characterizations of Closure and Closed Sets.

Comparing with $x \in E' \iff \forall r > 0,\ \big(B(x,r) - \{x\}\big) \cap E \neq \emptyset$ (definition), we have

(I) $x \in \overline{E} \iff \forall r > 0,\ B(x,r) \cap E \neq \emptyset$. Therefore, $E_1 \subseteq E_2 \implies \overline{E_1} \subseteq \overline{E_2}$.

Proof. Since $\overline{E} = E \cup E'$, we have $x \in \overline{E} \implies \forall r > 0,\ B(x,r) \cap E \neq \emptyset$.

Conversely, suppose that $\forall r > 0$, $B(x,r) \cap E \neq \emptyset$. If $x \in E$, then $x \in \overline{E}$; if $x \notin E$, then $\forall r > 0$, $\big(B(x,r) - \{x\}\big) \cap E = B(x,r) \cap E \neq \emptyset$, that is, $x \in E' \subseteq \overline{E}$. So, in both cases, we have $x \in \overline{E}$. ■

(II) $\overline{E}$ is always closed.

Proof. To prove that $\overline{E}$ is closed, we need show that $\overline{E}' \subseteq \overline{E}$.

Let $x \in \overline{E}'$. Then $\forall r > 0$, $\big(B(x,r) - \{x\}\big) \cap \overline{E} \neq \emptyset$.

Assume that $x \notin \overline{E}$. Then, by (I), $\exists r_0 > 0$ such that $B(x,r_0) \cap E = \emptyset$. Since $B(x,r_0)$ is open, we have $B(x,r_0) \cap E' = \emptyset$. It follows that

$$B(x,r_0) \cap \overline{E} = B(x,r_0) \cap (E \cup E') = \big(B(x,r_0) \cap E\big) \cup \big(B(x,r_0) \cap E'\big) = \emptyset,$$

contradicting to $\big(B(x,r_0) - \{x\}\big) \cap \overline{E} \neq \emptyset$. Therefore, $x \in \overline{E}$. Hence, we have $\overline{E}' \subseteq \overline{E}$.

The second proof. We just need to prove that $X - \overline{E}$ is open.

Let $x \in X - \overline{E}$. By (I), $\exists r > 0$ such that $B(x,r) \cap E = \emptyset$. Now $\forall y \in B(x,r)$, $\exists \varepsilon_y > 0$ such that $B(y,\varepsilon_y) \subseteq B(x,r)$ and hence $B(y,\varepsilon_y) \cap E = \emptyset$. That is, $\forall y \in B(x,r)$, $y \notin \overline{E}$. Thus $B(x,r) \subseteq X - \overline{E}$. Therefore, $X - \overline{E}$ is open. ■

(III) E is closed $\iff E = \overline{E}$.

Proof. "$\Longrightarrow$". It follows from the definition of $\overline{E}$.

"$\Longleftarrow$". It holds by (II). ■

(IV) $\overline{E} = \bigcap\{F : E \subseteq F \subseteq X \text{ and } F \text{ is closed}\}$, and hence $\overline{E}$ is the smallest closed set in X containing E.

Proof. Since $\overline{E}$ is closed, we have that $\bigcap\{F : E \subseteq F \subseteq X \text{ and } F \text{ is closed}\} \subseteq \overline{E}$.

Conversely, if $E \subseteq F \subseteq X$ and F is closed, then by (I) and (III), $\overline{E} \subseteq \overline{F} = F$. Hence,

$$\overline{E} \subseteq \bigcap\{F : E \subseteq F \subseteq X \text{ and } F \text{ is closed}\}.$$

Therefore, $\overline{E} = \bigcap\{F : E \subseteq F \subseteq X \text{ and } F \text{ is closed}\}$, and $\overline{E}$ is the smallest closed set in X containing E. ■

Theorem 7 (Theorem 2.28). Let $E \subseteq \mathbb{R}$ be non-empty and bounded above. Then we have $\sup(E) \in \overline{E}$. Similarly, if $E \subseteq \mathbb{R}$ is non-empty and bounded below, then $\inf(E) \in \overline{E}$.

Proof. Let $y = \sup(E)$. To get $y \in \overline{E}$, we need show that

$$\forall \varepsilon > 0,\ B(y, \varepsilon) \cap E = (y - \varepsilon, y + \varepsilon) \cap E \neq \emptyset.$$

Let $\varepsilon > 0$. Then $\exists x \in E$, $y - \varepsilon < x$. On the other hand, we have $x \leq y < y + \varepsilon$. Therefore, $x \in (y - \varepsilon, y + \varepsilon) \cap E$, and hence $(y - \varepsilon, y + \varepsilon) \cap E \neq \emptyset$. ■

Characterizations of Interior and Open Sets.

Parallel to the results on the closure $\overline{E}$ and closed sets, we have the followings on the interior E° and open sets.

(V) $E^\circ = \bigcup\{G : G \subseteq E \text{ and } G \text{ is open}\}$, and hence E° is the largest open subset of E.

(VI) E is open $\iff E = E^\circ$.

Definition 6. Let X be a metric space, $E \subseteq X$ and $x \in X$. x is called a boundary point of E if $\forall r > 0$, $B(x,r) \cap E \neq \emptyset$ and $B(x,r) \cap (X - E) \neq \emptyset$. We use ∂E to denote the set of all boundary points of E, called the boundary of E.

By (I), we have

(VII) $\partial E = \overline{E} \cap \overline{(X - E)}$. Therefore, ∂E is closed.

(VIII) $\overline{E} = E \cup \partial E$.

Note. Though $\overline{E} = E \cup E' = E \cup \partial E$, in general, $\partial E \not\subseteq E'$ and $E' \not\subseteq \partial E$. (Examples?)

By the definition of E° and ∂E, we have

(IX) $E^\circ = E - \partial E = \overline{E} - \partial E$.

Example. Let (X, d) be a discrete metric space. Then every subset E of X is open, since

$$\forall x \in E,\ B\big(x, \tfrac{1}{2}\big) = \{x\} \subseteq E\ .$$

Hence, every subset E of X is also closed. Now we have

$$E^\circ = E = \overline{E},\ E' = \emptyset,\ \text{and}\ \partial E = \overline{E} \cap \overline{(X - E)} = E \cap (X - E) = \emptyset.$$

Chapter Two Basic Topology

Part 3: Compact Sets in Metric Spaces

Definition 7. Let X be a metric space and let $K \subseteq X$. A family $\{G_\alpha\}$ of open sets in X is called an open cover of K if $K \subseteq \bigcup_\alpha G_\alpha$. The set K is called compact if every open cover of K has a finite subcover of K; that is. if $K \subseteq \bigcup_\alpha G_\alpha$, then $\exists\, \alpha_1, \cdots, \alpha_n$ such that $K \subseteq \bigcup_{i=1}^n G_{\alpha_i}$.

Examples. **(i)** Every finite subset of X is compact.

(ii) Every infinite discrete metric space X is not compact. In fact, in this case, $\forall x \in X$, $\{x\} = B(x,1)$ is open but the open cover $\big\{\{x\} : x \in X\big\}$ of X has no finite subcover.

(iii) In $\mathbb{R}$, $[0,\infty)$ is not compact. E.g., $\{(-1,n) : n \in \mathbb{N}\}$ is an open cover of $[0,\infty)$, but it has no finite subcover of $[0,\infty)$.

(iv) In $\mathbb{R}$, $[0,1)$ is not compact. E.g., $\{(-1, 1-\frac{1}{n}) : n \in \mathbb{N}\}$ is an open cover of $[0,1)$, but it has no finite subcover of $[0,1)$.

(v) In $\mathbb{R}$, every $[a,b]$ is compact. In fact, we will see that a subset of $\mathbb{R}$ is compact if and only if it is bounded and closed.

Definition 8. A subset E of a metric space is called bounded if $\exists r > 0$ and $x_0 \in X$ such that $E \subseteq B(x_0, r)$.

Note. $\forall x_0 \in X$, $X = \bigcup_{r>0} B(x_0, r) = \bigcup_{n=1}^{\infty} B(x_0, n)$.

Theorem 8. Every compact set in a metric space is bounded and closed.

Therefore, if E is either unbounded or non-closed, then E is not compact.

Proof. Let K be a compact set in a metric space (X,d). Pick $x_0 \in X$.

Claim 1: $\exists r_0 > 0$, $K \subseteq B(x_0, r_0)$.

Since K is compact and $K \subseteq X = \bigcup_{r>0} B(x_0, r)$, $\exists\, r_1, \cdots, r_n > 0$ such that $K \subseteq \bigcup_{i=1}^n B(x_0, r_i)$.

Let $r_0 = \max\{r_1, \cdots, r_n\}$. Then $r_0 > 0$ and $K \subseteq \bigcup_{i=1}^n B(x_0, r_i) = B(x_0, r_0)$.

Claim 2: K is closed (i.e., $\overline{K} \subseteq K$).

Assume that $\overline{K} \nsubseteq K$. Then $\exists\, x \in \overline{K}$ such that $x \notin K$. In this case, $\forall y \in K$, $d(y,x) > 0$. Let $r_y = d(y,x)/2$. Then $\forall y \in K$, $B(x, r_y) \cap B(y, r_y) = \emptyset$. Now $\{B(y, r_y) : y \in K\}$ is an open cover of K. Since K is compact , $\exists\, y_1, \cdots, y_n \in K$ such that $K \subseteq \bigcup_{i=1}^{n} B(y_i, r_{y_i})$.

Let $r = \min\{r_{y_1}, \cdots, r_{y_n}\}$. Then $r > 0$ and for $1 \le i \le n$, we have

$$B(x,r) \cap B(y_i, r_{y_i}) \;\subseteq\; B(x, r_{y_i}) \cap B(y_i, r_{y_i}) \;=\; \emptyset,$$

i.e., $B(x,r) \cap B(y_i, r_{y_i}) = \emptyset$ for $i = 1, \cdots, n$. Thus we get

$$B(x,r) \cap K \;\subseteq \bigcup_{i=1}^{n} \big(B(x,r) \cap B(y_i, r_{y_i})\big) \;=\; \emptyset,$$

that is, $B(x,r) \cap K = \emptyset$, contradicting that $x \in \overline{K}$. Therefore, $\overline{K} \subseteq K$ ■

Remark 2. The converse of Theorem 8 is not true). E.g., any infinite discrete space X is closed and bounded (since $X = B(x_0, 2)$ for any $x_0 \in X$), but X is not compact.

Theorem 9. Let K be a compact set in metric space and let F be a closed subset of K. Then F is compact.

Proof. Let $\{G_\alpha\}$ be an open cover of F. Then $\{X - F\} \cup \{G_\alpha\}$ is an open cover of K. Since K is compact, $\exists\, \alpha_1, \cdots, \alpha_n$ such that

$$K \;\subseteq\; (X - F) \cup \big(G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}\big).$$

Now $F \;=\; F \cap K \;\subseteq\; \big(F \cap (X - F)\big) \cup \big(G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}\big) \;=\; G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$. Hence, $\{G_\alpha\}$ has a finite subcover $\{G_{\alpha_1}, \cdots, G_{\alpha_n}\}$ of F. Therefore, F is compact. ■

Corollary 7. Let E be a subset of a compact metric space. Then

$$E \text{ is compact} \iff E \text{ is closed.}$$

<u>Theorem 10</u>. Let $a, b \in \mathbb{R}$ be such that $a < b$. Then $[a, b]$ is compact in $\mathbb{R}$.

<u>Proof</u>. Assume that $\{G_\alpha\}$ is an open cover of $[a, b]$ which has no finite subcover of $[a, b]$.

Let $c = (a + b)/2$. Then either $[a, c]$ or $[c, b]$ cannot be covered by finitely many G_α. We denote this subinterval of $[a, b]$ by $[a_1, b_1]$. Replacing $[a, b]$ by $[a_1, b_1]$ and continuing this process, we get a sequence $\{[a_n, b_n]\}$ of closed intervals such that

1) $[a, b] \supseteq [a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots$;

2) each $[a_n, b_n]$ cannot be covered by finitely many G_α;

3) $b_n - a_n = (b - a)/2^n$.

Now we have

$$a \le a_1 \le a_2 \le a_3 \le \cdots \le b_3 \le b_2 \le b_1 \le b.$$

Let E be the set of all a_n, and let $x = \sup(E)$. Then $a_n \le x \le b_n$ for all n (note that each b_n is an upper bound of E). That is, $x \in [a_n, b_n]$ for all n.

On the other hand, $x \in G_{\alpha_0}$ for some α_0 because $x \in [a, b]$. Since G_{α_0} is open, $\exists r > 0$ such that $(x - r, x + r) \subseteq G_{\alpha_0}$. Choose n_0 such that $(b - a)/2^{n_0} < r$. Then we have

$$x - a_{n_0} \le b_{n_0} - a_{n_0} = (b - a)/2^{n_0} < r, \text{ or } \underline{a_{n_0} > x - r},$$

$$\text{and} \quad b_{n_0} - x \le b_{n_0} - a_{n_0} = (b - a)/2^{n_0} < r, \text{ or } \underline{b_{n_0} < x + r}.$$

So, we that $[a_{n_0}, b_{n_0}] \subseteq (x - r, x + r) \subseteq G_{\alpha_0}$, contradicting that $[a_{n_0}, b_{n_0}]$ cannot be covered by finitely many G_α. Therefore, $\{G_\alpha\}$ has a finite subcover of $[a, b]$.

Therefore, $[a, b]$ is compact. ■

<u>Corollary 8</u>. Let $E \subseteq \mathbb{R}$. Then E is compact $\iff$ E is bounded and closed.

<u>Proof</u>. "$\Longrightarrow$". It holds by Theorem 8.

"$\Longleftarrow$". Suppose E is bounded and closed. Then $E \subseteq [a, b]$ for some a, $b \in \mathbb{R}$ with $a < b$. Since $[a, b]$ is compact (Theorem 10) and E is closed, by Theorem 9, E is compact. ■

Theorem 11. Let K be a compact set in a metric space. Then for any infinite subset A of K, $A' \cap K \neq \emptyset$.

Proof. Assume that A is an infinite subset of K but $A' \cap K = \emptyset$.

Then $\forall x \in K$, since $x \notin A'$, $\exists r_x > 0$, $\big(B(x, r_x) - \{x\}\big) \cap A = \emptyset$. Now $\{B(x, r_x) : x \in K\}$ is an open cover of K, and hence $\exists x_1, \cdots, x_n \in K$ such that $K \subseteq \displaystyle\bigcup_{i=1}^{n} B(x_i, r_{x_i})$.

Since A is infinite, we can pick $y \in A - \{x_1, \cdots, x_n\}$. Now $y \in K$, but for all $1 \leq i \leq n$, since $\big(B(x_i, r_{x_i}) - \{x_i\}\big) \cap A = \emptyset$, we have $y \notin B(x_i, r_{x_i})$, contradicting that $K \subseteq \displaystyle\bigcup_{i=1}^{n} B(x_i, r_{x_i})$. Therefore, $A' \cap K \neq \emptyset$. ■

Remark 3. The converse of Theorem 11 is also true.

Weierstrass Theorem. Every bounded infinite subset of $\mathbb{R}$ has a limit point.

Proof. Suppose A is a bounded infinite subset of $\mathbb{R}$. Then $A \subseteq [a, b]$ for some $a, b \in \mathbb{R}$ with $a < b$. Since $[a, b]$ is compact, by Theorem 11, $A' \cap [a, b] \neq \emptyset$, and hence $A' \neq \emptyset$. ■

Chapter Three   Numerical Sequences and Series

Part 1: Covnvergent Sequences

Definition 1. Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is convergent if $\exists x \in X$ such that $\forall \varepsilon > 0$, $\exists N = N(\varepsilon) \in \mathbb{N}$, $\forall n \geq N$, $d(x_n, x) < \varepsilon$.

In this case, we say that $\{x_n\}$ converges to x, and write $\lim_{n\to\infty} x_n = x$ (or $x_n \to x$).

The above is the so-called ε-N description of $x_n \to x$.

We say that $\{x_n\}$ is divergent if $\{x_n\}$ is not convergent; that is, $\forall x \in X$, $\{x_n\}$ does not converge to x.

Basic Facts on Convergence and Divergence.

(i) Changing finitely many terms in a sequence does not affect its convergence or limit.

(ii) $x_n \to x$ in $X \iff d(x_n, x) \to 0$ in $\mathbb{R}$.

(iii) Geometry description:

$x_n \to x$ means that $\forall \varepsilon > 0$, there are at most finitely many terms x_n outside the ball $B(x, \varepsilon)$;

$x_n \not\to x$ means that $\exists \varepsilon_0 > 0$, there are infinitely many terms x_n outside the ball $B(x, \varepsilon_0)$.

(iv) "$x_n \not\to x$" is the negation of "$x_n \to x$", and it means that either $\{x_n\}$ is divergent or $x_n \to y$ but $y \neq x$.

(v) The ε-N description of $x_n \not\to x$: $\exists \varepsilon_0 > 0$, $\forall N$, $\exists n \geq N$, $d(x_n, x) \geq \varepsilon_0$.

The above n can be denoted by n_N and can be chosen inductively such that $n_1 < n_2 < \cdots$.

(vi) The ε-N description of divergence of $\{x_n\}$: $\forall x \in X$, $\exists \varepsilon_0 > 0$, $\forall N$, $\exists n \geq N$, $d(x_n, x) \geq \varepsilon_0$.

Here, $\varepsilon_0 = \varepsilon_0(x)$ and $n = n(N, x)$.

Examples.

1) $\lim_{n\to\infty} \dfrac{1}{n} = 0.$

Proof. Given $\varepsilon > 0$. Let $N = 1 + \left[\frac{1}{\varepsilon}\right]$ ($[\cdot]$ denotes the integer part function). Then $N > \frac{1}{\varepsilon}$, or $\frac{1}{N} < \varepsilon$. Thus $\forall n \geq N$, we have $\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$. Therefore, $\lim_{n\to\infty} \dfrac{1}{n} = 0.$ ∎

2) $\lim_{n\to\infty} \left(1 + \dfrac{(-1)^n}{n}\right) = 1.$

Proof. Given $\varepsilon > 0$. Let N be the same as in 1) above. Then $\forall n \geq N$, we have

$$\left|1 + \tfrac{(-1)^n}{n} - 1\right| = \left|\tfrac{(-1)^n}{n}\right| = \tfrac{1}{n} \leq \tfrac{1}{N} < \varepsilon.$$

Therefore, $\lim_{n\to\infty} \left(1 + \dfrac{(-1)^n}{n}\right) = 1.$ ∎

3) Let $x_n = c$ for all n. Then $\lim_{n\to\infty} x_n = c$.

Proof. Given $\varepsilon > 0$. Let $N = 1$. Then $\forall n \geq N$, we have $|x_n - c| = 0 < \varepsilon$. Therefore, $\lim_{n\to\infty} x_n = c.$ ∎

4) Let $x_n = (-1)^n$. Then $\{x_n\}$ is divergent.

Proof. We need prove that $\forall x \in \mathbb{R}$, $x_n \not\to x$. Let $x \in \mathbb{R}$.

If $x = 1$, let $\varepsilon_0 = 2$. In this case, $\forall N$, we choose $n = 2N + 1 > N$, and thus

$$|x_n - 1| = |-1-1| = 2 \geq \varepsilon_0.$$

Hence, $x_n \not\to 1$.

In the following, we suppose $x \neq 1$. Then $\varepsilon_0 = |1 - x| > 0$. In this case, $\forall N$, we choose $n = 2N > N$, and thus

$$|x_n - x| = |1 - x| = \varepsilon_0 \geq \varepsilon_0.$$

Hence, $x_n \not\to x$.

Therefore, $\forall x \in \mathbb{R}$, $x_n \not\to x$, and thus $\{x_n\}$ is divergent. ∎

5) Let $x_n = n$. Then $\{x_n\}$ is divergent.

Proof. We need prove that $\forall x \in \mathbb{R}$, $x_n \not\to x$. Let $x \in \mathbb{R}$.

Take $\varepsilon_0 = 1$. Then $\forall N$, choose $n = \max\{N, [\,|x|\,] + 2\}$. Thus $n \geq N$, and we have

$$|x_n - x| \;=\; |n - x| \;\geq\; n - |x| \;\geq\; n - \big([\,|x|\,] + 1\big) \;\geq 1 \;=\; \varepsilon_0.$$

Therefore, $\forall x \in \mathbb{R}$, $x_n \not\to x$, and hence $\{x_n\}$ is divergent. $\blacksquare$

Definition 2. Let $\{x_n\}$ be a sequence in $\mathbb{R}$.

We write $\lim\limits_{n\to\infty} x_n = \infty$ (or $x_n \to \infty$) if $\underline{\forall M \in \mathbb{R},\, \exists N,\, \forall n \geq N,\, x_n > M}$.

Similarly, we write $\lim\limits_{n\to\infty} x_n = -\infty$ (or $x_n \to -\infty$) if $\underline{\forall M \in \mathbb{R},\, \exists N,\, \forall n \geq N,\, x_n < M}$.

- Clearly, $x_n \to \pm\infty \implies \{x_n\}$ is divergent. E.g., in 5), $x_n \to \infty$ and hence it is divergent. However, the sequence in 4) is divergent, but $x_n \not\to \pm\infty$.

Theorem 1 (Uniqueness of Limit). Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X. If $x_n \to x$ and $x_n \to y$, then $x = y$.

Proof. Assume that $x_n \to x$ and $x_n \to y$, but $x \neq y$. Then $d(x, y) > 0$. Let $\varepsilon = d(x, y)/2$. Then $\varepsilon > 0$ and $B(x, \varepsilon) \cap B(y, \varepsilon) = \emptyset$. For this ε, we have

$$\exists N_1,\, \forall n \geq N_1,\, d(x_n, x) < \varepsilon \;\; (\text{since } x_n \to x),$$

$$\exists N_2,\, \forall n \geq N_2,\, d(x_n, y) < \varepsilon \;\; (\text{since } x_n \to y).$$

Let $n_0 = \max\{N_1, N_2\}$. Then $d(x_{n_0}, x) < \varepsilon$ and $d(x_{n_0}, y) < \varepsilon$, and thus we have

$$d(x, y) \;\leq\; d(x, x_{n_0}) + d(x_{n_0}, y) \;=\; d(x_{n_0}, x) + d(x_{n_0}, y) \;<\; 2\varepsilon \;=\; d(x, y),$$

that is, $d(x, y) < d(x, y)$, a contradiction. Therefore, $x = y$. $\blacksquare$

Theorem 2. Any convergent sequence in a metric space is bounded.

Proof. Suppose $x_n \to x$ in a metric space (X, d). We prove that $\exists M > 0$, $\forall n$, $x_n \in B(x, M)$.

For $\varepsilon = 1$, $\exists N$, $\forall n \geq N$, $d(x_n, x) < 1$. Let $M = 1 + \max\{d(x_1, x), \cdots, d(x_{N-1}, x)\}$. Then $\forall n$, we have $d(x_n, x) < M$. $\blacksquare$

Remark 1. The converse of Theorem 2 is not true. E.g., $x_n = (-1)^n$.

Theorem 3 (Squeeze Theorem in $\mathbb{R}$). Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences in $\mathbb{R}$ such that $a_n \to x$, $c_n \to x$, and $\forall n$, $a_n \leq b_n \leq c_n$. Then $b_n \to x$.

Note. Here either $x \in \mathbb{R}$ or $x = \pm\infty$. We only prove the case when $x \in \mathbb{R}$. Also, in the theorem, we can only require that $a_n \leq b_n \leq c_n$ holds for all $n \geq n_0$ for some n_0.

Proof. Given $\varepsilon > 0$. Since $a_n \to x$ and $c_n \to x$, $\exists N_1,\, N_2 \in \mathbb{N}$ such that

$$\forall n \geq N_1,\ |a_n - x| < \varepsilon \quad \text{and} \quad \forall n \geq N_2,\ |c_n - x| < \varepsilon.$$

Let $N = \max\{N_1, N_2\}$. Then $\forall n \geq N$, we have $|a_n - x| < \varepsilon$ and $|c_n - x| < \varepsilon$; that is,

$$x - \varepsilon < a_n < x + \varepsilon \ \text{ and } \ x - \varepsilon < c_n < x + \varepsilon.$$

In this case, we have $x - \varepsilon < a_n \leq b_n \leq c_n < x + \varepsilon$; that is,

$$x - \varepsilon < b_n < x + \varepsilon, \ \text{ or } \ |b_n - x| < \varepsilon.$$

So, we prove that $\forall \varepsilon > 0$, $\exists N$, $\forall n \geq N$, $|b_n - x| < \varepsilon$. Therefore, $b_n \to x$. ■

Corollary 1. If $0 \leq a_n \leq b_n$ for all n and $b_n \to 0$, then $a_n \to 0$.

Question. If $0 \leq a_n \leq b_n$ for all n and $b_n \to 1$, is $\{a_n\}$ convergent?

Corollary 2. Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X. If $d(x_n, x) \leq c_n$ for all n and $c_n \to 0$, then $x_n \to x$ in X.

Theorem 4. Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}$ such that $a_n \to a$, $b_n \to b$ and $\forall n$, $a_n \leq b_n$. Then $a \leq b$.

Proof. Assume that $a > b$, Take $\varepsilon_0 = (a - b)/2$. Then $\varepsilon_0 > 0$. For this ε_0, $\exists N_1,\, N_2$ such that

$$\forall n \geq N_1,\ |a_n - a| < \varepsilon_0 \quad \text{and} \quad \forall n \geq N_2,\ |b_n - b| < \varepsilon_0.$$

Let $N = \max\{N_1, N_2\}$. Then $|a_N - a| < \varepsilon_0$ and $|b_N - b| < \varepsilon_0$. In this case, we have

$$b_N < b + \varepsilon_0 = a - \varepsilon_0 < a_N; \ \text{ that is, } \ b_N < a_N,$$

a contradiction. Therefore, we have $a \leq b$. ■

Exercise. We can give a direct proof of Theorem 4 by showing that $a \leq b + \varepsilon$ for all $\varepsilon > 0$.

Theorem 5. Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}$ such that $a_n \to a$ and $b_n \to b$. Then

(i) $a_n + b_n \to a + b$ and $a_n - b_n \to a - b$;

(ii) $a_n b_n \to ab$, and $ca_n \to ca$ for all $c \in \mathbb{R}$;

(iii) $\dfrac{a_n}{b_n} \to \dfrac{a}{b}$ if $b \neq 0$ and $b_n \neq 0$ for all n.

Proof. **(i)** Given $\varepsilon > 0$. Then $\exists N_1$, N_2 such that

$$\forall n \geq N_1,\ |a_n - a| < \varepsilon/2 \quad \text{and} \quad \forall n \geq N_2,\ |b_n - b| < \varepsilon/2.$$

Let $N = \max\{N_1, N_2\}$. Then $\forall n \geq N$, we have

$$|(a_n \pm b_n) - (a \pm b)| = |(a_n - a) \pm (b_n - b)| \leq |a_n - a| + |b_n - b| < \varepsilon.$$

(ii) Given $\varepsilon > 0$. Choose η such that $0 < \eta < \min\{1, \varepsilon\,(1 + |a| + |b|)^{-1}\}$. Then $\exists N_1$, N_2 such that

$$\forall n \geq N_1,\ |a_n - a| < \eta \quad \text{and} \quad \forall n \geq N_2,\ |b_n - b| < \eta.$$

Let $N = \max\{N_1, N_2\}$. Then $\forall n \geq N$, we have

$$\begin{aligned} |a_n b_n - ab| &= |(a_n - a)(b_n - b) + (a_n - a)b + a(b_n - b)| \\ &\leq |a_n - a||b_n - b| + |a_n - a||b| + |a||b_n - b| \\ &< \eta^2 + \eta|b| + |a|\eta = \eta(\eta + |a| + |b|) \\ &< \eta(1 + |a| + |b|) < \varepsilon. \end{aligned}$$

Therefore, $a_n b_n \to ab$. In particular, if $b_n = c$ for all n, then $b_n \to c$ and hence $ca_n \to ca$.

(iii) First, we prove that $\frac{1}{b_n} \to \frac{1}{b}$. Given $\varepsilon > 0$. Since $b \neq 0$, $|b| > 0$. Let

$$\beta = \min\left\{ |b|/2,\ |b|^2\varepsilon/2 \right\}.$$

Then $\beta > 0$. So, $\exists N$, $\forall n \geq N$, $|b_n - b| < \beta$, and thus

$$|b_n| = |(b_n - b) + b| \geq |b| - |b_n - b| \geq |b|/2$$

and

$$\left|\frac{1}{b_n} - \frac{1}{b_n}\right| = \left|\frac{b - b_n}{b_n b}\right| = \frac{|b_n - b|}{|b_n|\,|b|} \leq \frac{2|b_n - b|}{|b|^2} < \frac{2\beta}{|b|^2} \leq \varepsilon.$$

Therefore, $\dfrac{1}{b_n} \to \dfrac{1}{b}$. It follows from (ii) that $\dfrac{a_n}{b_n} = a_n \dfrac{1}{b_n} \longrightarrow a\dfrac{1}{b} = \dfrac{a}{b}$. ■

To prove some basic limits, we need the following Binomial Theorem and Binomial Inequality. They can be proved by using the PMI.

Binomial Theorem. For all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$(1+x)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n.$$

Binomial Inequality. For all $x \geq -1$ and $n \in \mathbb{N}$, we have $(1+x)^n \geq 1 + nx$.

Four Basic Limits.

(i) $\forall p > 0,\ \dfrac{1}{n^p} \to 0.$

Proof. Given $\varepsilon > 0$. Choose $N = 1 + \left[\dfrac{1}{\varepsilon^{1/p}}\right]$. Then $N > \dfrac{1}{\varepsilon^{1/p}}$, or $\dfrac{1}{N^p} < \varepsilon$. Now $\forall n \geq N$, we have $\left|\dfrac{1}{n^p} - 0\right| = \dfrac{1}{n^p} \leq \dfrac{1}{N^p} < \varepsilon$. Therefore, $\dfrac{1}{n^p} \to 0$. ∎

(ii) If $a \in \mathbb{R}$ and $|a| < 1$, then $a^n \to 0$.

Proof. Suppose $|a| < 1$. Then $b = \frac{1}{|a|} - 1 > 0$. By Binomial Inequality, we have

$$(1+b)^n \geq 1 + nb; \text{ that is, } \tfrac{1}{|a|^n} \geq 1 + nb, \text{ or } |a^n| = |a|^n \leq \tfrac{1}{1+nb} < \tfrac{1}{nb}.$$

Since $\frac{1}{nb} \to 0$, by Squeeze Theorem, $|a^n - 0| = |a^n| \to 0$ and hence $a^n \to 0$. ∎

(iii) $n^{\frac{1}{n}} \to 1$.

Proof. Let $x_n = n^{\frac{1}{n}} - 1$. Then $x_n \geq 0$ and $n^{\frac{1}{n}} = 1 + x_n$. We prove that $x_n \to 0$ to obtain $n^{\frac{1}{n}} \to 1$. By Binomial Theorem, we have

$$n = (1+x_n)^n = 1 + nx_n + \tfrac{n(n-1)}{2!}x_n^2 + \cdots + x_n^n \geq \tfrac{n(n-1)}{2}x_n^2.$$

So, $\forall n \geq 2$, we have $x_n^2 \leq \frac{2}{n-1}$, or $0 \leq x_n \leq \frac{\sqrt{2}}{\sqrt{n-1}}$. By Squeeze Theorem, $x_n \to 0$. ∎

(iv) If $p > 0$, then $p^{\frac{1}{n}} \to 1$.

Proof. Pick N such that $\frac{1}{N} \leq p \leq N$. Then $\forall n \geq N$, we have $\frac{1}{n} \leq \frac{1}{N} \leq p \leq N \leq n$ and hence $\dfrac{1}{n^{1/n}} \leq p^{1/n} \leq n^{1/n}$. By Squeeze Theorem and the limit in (iii), $p^{\frac{1}{n}} \to 1$. ∎

Definition 3. A sequence $\{x_n\}$ in $\mathbb{R}$ is called increasing if $a_n \le a_{n+1}$ for all n, and is called decreasing if $a_n \ge a_{n+1}$ for all n.

- An increasing sequence is bounded below (since $a_1 \le a_n$ for all n), and a decreasing is bounded above (since $a_n \le a_1$ for all n).

Theorem 6 (**Monotone Convergence Theorem**). Every bounded monotone sequence in $\mathbb{R}$ is convergent. More precisely, we have

(i) If $\{a_n\}$ is $\uparrow$ and bounded, then $a_n \to \sup\limits_{k\ge 1} a_k$;

(ii) If $\{a_n\}$ is $\downarrow$ and bounded, then $a_n \to \inf\limits_{k\ge 1} a_k$.

Proof. We prove (i) only. (ii) can be proved similarly.

Suppose $\{a_n\}$ is a bounded increasing sequence in $\mathbb{R}$. Let $a = \sup\limits_{k\ge 1} a_k$. Then $a \in \mathbb{R}$. We prove that $a_n \to a$. Given $\varepsilon > 0$. Then $\exists N$, $a_N > a - \varepsilon$. Thus $\forall n \ge N$, we have

$$a - \varepsilon < a_N \le a_n \le a < a + \varepsilon;$$

that is, $|a_n - a| < \varepsilon$. Therefore, $a_n \to a$. ∎

Remark 2. If $\{a_n\}$ is $\uparrow$ and unbounded, then $a_n \to \infty$;

if $\{a_n\}$ is $\downarrow$ and unbounded, then $a_n \to -\infty$.

Example. Let $a_1 = 1$ and $a_{n+1} = \sqrt{1 + a_n}$ $(n \ge 1)$. Prove that $\{a_n\}$ is convergent in $\mathbb{R}$.

Proof. Using PMI, we prove that for all n, $a_n < 2$ and $a_{n+1} \ge a_n$, and hence $\{a_n\}$ is convergent by the MCT.

For $n = 1$, we have $a_1 < 2$ and $a_2 = \sqrt{2} \ge a_1$.

Assume that $a_k < 2$ and $a_{k+1} \ge a_k$ for some $k \in \mathbb{N}$. Then we have

$$a_{k+1} = \sqrt{1 + a_k} < \sqrt{1 + 2} = \sqrt{3} < 2$$

and

$$a_{k+2} = \sqrt{1 + a_{k+1}} \ge \sqrt{1 + a_k} = a_{k+1}.$$

Therefore, by PMI, $a_n < 2$ and $a_{n+1} \ge a_n$ hold for all $n \in \mathbb{N}$. ∎

Questions. (i) $\lim\limits_{n\to\infty} a_n = \ ?$ (ii) What is the conclusion if $a_1 = 3$?

Theorem 7. Let (X, d) be a metric space, $E \subseteq X$ and $x \in X$. Then

(i) $x \in \overline{E} \iff \exists$ a sequence $\{x_n\}$ in E such that $x_n \to x$;

(ii) $x \in E' \iff \exists$ a sequence $\{x_n\}$ in $E - \{x\}$ such that $x_n \to x$.

Proof. **(i)** "$\Longrightarrow$". Suppose $x \in \overline{E}$. Then $\forall n$, $B\big(x, \frac{1}{n}\big) \cap E \neq \emptyset$ and hence we can pick $x_n \in B\big(x, \frac{1}{n}\big) \cap E$. Now $\{x_n\}$ is a sequence in E and $d(x_n, x) < \frac{1}{n}$ for all n. Therefore, $x_n \to x$.

"$\Longleftarrow$". Suppose $\{x_n\}$ is a sequence in E such that $x_n \to x$. Let $r > 0$. Then $\exists N$, $\forall n \geq N$, $d(x_n, x) < r$. In particular, we have $x_N \in B(x, r) \cap E$. So, $B(x, r) \cap E \neq \emptyset$. Therefore, $x \in \overline{E}$.

(ii) This can be proved by replacing E by $E - \{x\}$ in the above. ■

Remark 3. If $x \in E'$, then $\forall r > 0$, $\big(B(x, r) - \{x\}\big) \cap E$ is an infinite set. So, in (ii), the sequence $\{x_n\}$ can be taken as a sequence of distinct terms.

Corollary 3. let (X, d) be a metric space and $E \subseteq X$. Then

$$E \text{ is closed} \iff \big[\forall \text{ sequence } \{x_n\} \text{ in } E,\, x_n \to x \implies x \in E\big].$$