3.5 Multinomial Opinions

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Summary

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3.5.1 The Multinomial Opinion Representation I

Definition 3.4 (Multinomial Opinion)

Let $\mathbb{X}$ be a domain larger than binary, i.e. so that $k = |\mathbb{X}| > 2$. Let X be a random variable in $\mathbb{X}$. A multinomial opinion over the random variable X is the ordered triplet $\omega_X = (\mathbf{b}_X, u_X, \mathbf{a}_X)$ where

- $\mathbf{b}_X$ is a belief mass distribution over X,
- u_X is the uncertainty mass which represents the vacuity of evidence,
- $\mathbf{a}_X$ is a base rate distribution over $\mathbb{X}$,

and the multinomial additivity requirement of Eq.(2.6) is satisfied.

A multinomial opinion has $(2k - 1)$ degrees of freedom.

3.5.1 The Multinomial Opinion Representation

The projected probability distribution of multinomial opinions is defined by:

$$\mathbf{P}_X(x) = \mathbf{b}_X(x) + \mathbf{a}_X(x)u_X, \ \forall x \in \mathbb{X}. \tag{3.12}$$

3.5.1 The Multinomial Opinion Representation

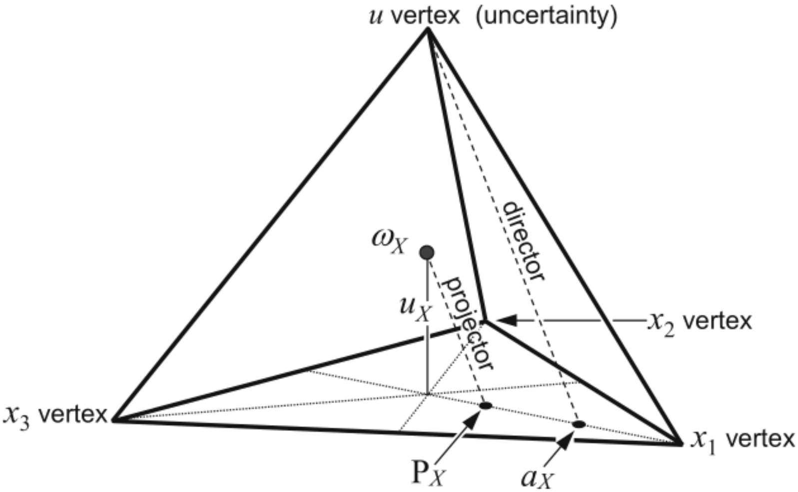


Fig. 3.3 Barycentric tetrahedron visualisation of trinomial opinion

3.5.2 The Dirichlet Multinomial Model

Definition 3.4 (Dirichlet Probability Density Function)

Let $\mathbb{X}$ be a domain consisting of k mutually disjoint values. Let α_X represent the strength vector over the values of $\mathbb{X}$, and let $\mathbf{p}_X$ denote the probability distribution over $\mathbb{X}$. With $\mathbf{p}_X$ as a k-dimensional variable, the Dirichlet PDF denoted $\mathrm{Dir}(\mathbf{p}_X, \alpha_X)$ is expressed as:

$$\mathrm{Dir}(\mathbf{p}_X, \alpha_X) = \frac{\Gamma\left(\sum_{x \in \mathbb{X}} \alpha_X(x)\right)}{\prod_{x \in \mathbb{X}} \Gamma(\alpha_X(x))} \prod_{x \in \mathbb{X}} \mathbf{p}_X(x)^{(\alpha_X(x)-1)}, \text{ where } \alpha_X(x) \geq 0, \tag{3.14}$$

with the restrictions that $\mathbf{p}_X(x) \neq 0$ if $\alpha_X(x) < 1$.

3.5.2 The Dirichlet Multinomial Model

The evidence representation of the Dirichlet PDF

The evidence representation of the Dirichlet PDF is denoted by $\mathrm{Dir}^{\mathrm{e}}_{X}(\mathbf{p}_X, \mathbf{r}_X, \mathbf{a}_X)$, where the total strength $\alpha_X(x)$ for each value $x \in \mathbb{X}$ can be expressed as

$$\alpha_X(x) = \mathbf{r}_X(x) + \mathbf{a}_X(x)W, \text{ where } \mathbf{r}_X(x) \geq 0 \ \forall x \in \mathbb{X}. \tag{3.15}$$

- $\mathbf{r}_X(x)$ is the evidence for outcome $x \in \mathbb{X}$.
- $\mathbf{a}_X$ is the base rate distribution.
- W is the non-informative prior weight.

3.5.2 The Dirichlet Multinomial Model

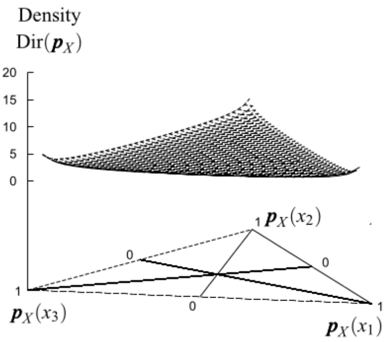
The evidence representation of the Dirichlet PDF

The expected distribution over $\mathbb{X}$ can be written as

$$\mathbf{E}_X(x) = \frac{\alpha_X(x)}{\sum\limits_{x_j \in \mathbb{X}} \alpha_X(x_i)} = \frac{\mathbf{r}_X(x) + \mathbf{a}_X(x)W}{W + \sum\limits_{x_j \in \mathbb{X}} \mathbf{r}_X(x_j)} \;\forall x \in \mathbb{X}. \tag{3.17}$$

3.5.3 Visualizing Dirichlet Probability Density Functions

Dirichlet PDFs over ternary domains are the largest that can be practically visualized.
Example: Urn with balls with three different markins: x_1, x_2 and x_3.

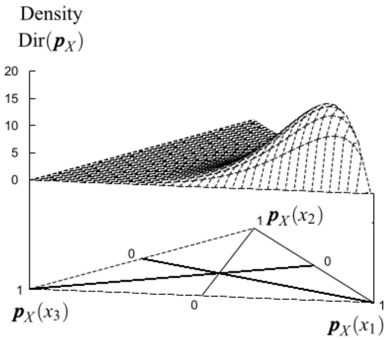


(a) Non-informative prior Dirichlet PDF

$$\mathbf{a}_X(x) = 1/k = 1/3$$
$$\mathbf{r}_X(x_1) = \mathbf{r}_X(x_1) = \mathbf{r}_X(x_1) = 0$$
$$\mathbf{E}_X(x_1) = 1/3$$

3.5.3 Visualising Dirichlet Probability Density Functions

Dirichlet PDFs over ternary domains are the largest that can be practically visualized.
Example: Urn with balls with three different markins: x_1, x_2 and x_3.

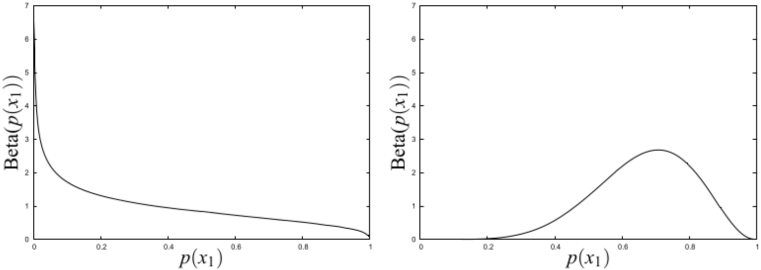


(b) Posterior Dirichlet PDF

$$\mathbf{a}_X(x) = 1/k = 1/3$$
$$\mathbf{r}_X(x_1) = 6$$
$$\mathbf{r}_X(x_2) = 1$$
$$\mathbf{r}_X(x_3) = 1$$
$$\mathbf{E}_X(x_1) = 2/3$$

3.5.4 Coarsening Example: From Ternary to Binary

Same case as before, but $x_1 = x_1$ and $\overline{x}_1 = \{x_2, x_3\}$



(a) Non-informative prior Beta PDF

(b) After six balls marked x_1, and two balls marked x_2 or x_3

Fig. 3.5 Prior and posterior Beta PDF

$\mathbf{E}_X(x_1) = 2/3$, same as before.

3.5.5 Mapping Between Multinomial Opinion and Dirichlet PDF

Requirements

$$\mathbf{P}_X = \mathbf{E}_X \tag{3.19}$$

$$\sum_{x \in \mathbb{X}} \mathbf{r}_X(x) \longrightarrow \infty \Rightarrow \sum_{x \in \mathbb{X}} \mathbf{b}_X(x) \longrightarrow 1 \tag{3.21}$$

$$\sum_{x \in \mathbb{X}} \mathbf{r}_X(x) \longrightarrow \infty \Rightarrow u_X = 0 \tag{3.22}$$

3.5.5 Mapping Between Multinomial Opinion and Dirichlet PDF

Definition 3.6 ((Mapping: Multinomial Opinion $\leftrightarrow$ Dirichlet PDF)

Let $\omega_X = (\mathbf{b}_X, u_X, \mathbf{a}_X)$ be a multinomial opinion and let $\mathrm{Dir}^{\mathrm{e}}_X(\mathbf{p}_X, \mathbf{r}_X, \mathbf{a}_X)$ be a Dirichlet PDF, both over the same variable $X \in \mathbb{X}$. These are equivalent through the following mapping,

$$\forall x \in \mathbb{X}$$

$$\begin{cases} \mathbf{b}_X(x) & = \dfrac{\mathbf{r}_X(x)}{W + \sum\limits_{x_i \in \mathbb{X}} \mathbf{r}_X(x_i)} \\ u_X & = \dfrac{W}{W + \sum\limits_{x_i \in \mathbb{X}} \mathbf{r}_X(x_i)} \end{cases} \Leftrightarrow \begin{cases} \begin{cases} \mathbf{r}_X(x) = \dfrac{W\mathbf{b}_X(x)}{u_X} \\ 1 = u_X = \sum\limits_{x_i \in \mathbb{X}} \mathbf{b}_X(x_i) \end{cases} & \text{if } u_X \neq 0 \\ \begin{cases} \mathbf{r}_X(x) = \mathbf{b}_X(x) \cdot \infty \\ 1 = \sum\limits_{x_i \in \mathbb{X}} \mathbf{b}_X(x_i) \end{cases} & \text{if } u_X \neq 0 \end{cases} \tag{3.23}$$

3.5.6 Uncertainty-Maximisation