Algebraic Topology Notes

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Contents

0.1 Notation

1. $\mathbb{1}$, The identity map.
2. I, The interval $[0,1] \in \mathbb{R}$
3. $\amalg$, Disjoint union of sets or spaces.

Chapter 1

Geometric notions

Definition 1.0.1 Deformation Retraction

A **Deformation Retraction** from a subspace X onto a subspace A is a family of maps $f_t : X \to X,\ t \in I$ such that

$$f_0 = \mathbb{1}, \quad f_1(X) = A \quad \text{and} \quad f_t\big|_A = \mathbb{1}\ \forall t \in I$$

The family f_t should be continuous in the sense that the associated map $X \times I \to X, (x,t) \mapsto f_t$ is continuous.

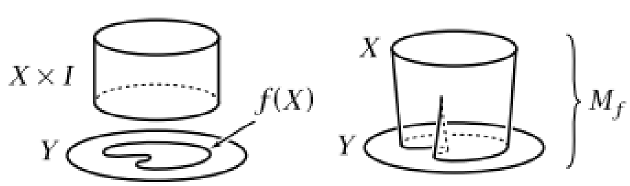
A deformation retraction allows us to deform a space onto a subspace of itself continuously. The catch is, this subspace is never affected by the retraction, only $X \setminus A$ is.

Definition 1.0.2 Mapping Cylinder

For a map $f : X \to Y$ its **Mapping Cylinder** M_f, is the quotient space of the disjoint union $X \times I \amalg Y$, where,

$$(x,1) \in X \times I \quad \text{identified with} \quad f(x) \in Y$$

This way, the part of M_f that is $X \times I$ Has one end deformed to the shape of Y. In this manner X can deform into Y as it 'moves' through the tube of length 1, as seen in the image. This is properly described as sliding each point (x,t) along the segment $\{x\} \times I \subset M_f$ to the endpoint $f(x) \in Y$.



Not all deformation retractions arise from mapping cylinders though.

Definition 1.0.3 Homotopy

A **Homotopy** is a family of maps $f_t : X \to Y,\ t \in I$ such that the associated map $F : X \times I \to Y,\ F(x,t) = f_t(x)$ is continuous. Two maps $f_0, f_1 : X \to Y$ are said to be **Homotopic** if there is a homotopy f_t connecting them. it is noted $f_0 \simeq f_1$

It becomes clear that a deformation retraction is a special case of a homotopy. It is in fact, a homotopy that connects the identity map on $X, \mathbb{1} = f_0$ to its retraction on A, a map $r : X \to X$ such that $r(X) = A$ and $r|_A = \mathbb{1}$.

Retractions are the topological analogs of projection operators in other parts of mathematics.

Definition 1.0.4 Homotopy relative to A

A homotopy is said to be **relative to** $A \subset X$ if its restriction to A is independent of t.

For example, a deformation retraction of X onto A is a homotopy **rel** A from the identity map of X to a retraction of X onto A, since $f_t\big|_A = \mathbb{1}$, independent of t.

Definition 1.0.5 Homotopy Equivalent Spaces

A map $f : X \to Y$ is a **Homotopy Equivalence** if there exists a map

$$g : Y \to X \quad \text{such that} \quad fg \simeq \mathbb{1} \text{ and } gf \simeq \mathbb{1}$$

In this case, the spaces X and Y are said to be **Homotopy Equivalent** or to have the same **Homotopy Type**. Notation is $X \simeq Y$.

This way, for example, several deformation retractions of the same space are homotopy equivalent; despite not being deformation retractions of each other necessarily.

Homotopy equivalence is an **equivalence relation**.

A space having the homotopy type os a single point is **Contractible**. This implies the identity map of this space is **Nullhomotopic**, meaning it's homotopic to a constant map. The converse is not true.