# Algebraic Topology Notes

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# Contents

## -1.1 Notation

1. $\mathbb{1}$, The identity map.
2. $I$, The interval $[0, 1] \in \mathbb{R}$.
3. $\amalg$, Disjoint union of sets or spaces.
4. $D^n$, unit disk or ball in $\mathbb{R}n$.
5. $S^n$, unit sphere in $\mathbb{R}n + 1$.
6. $\partial D^n = S^{n-1}$, the boundary of the $n$-disc.

# Chapter 0

# Geometric notions

## 0.1 Homotopy and Homotopy Type

**Definition 0.1.1** Deformation Retraction

A **Deformation Retraction** from a subspace $X$ onto a subspace $A$ is a family of maps $f_t : X \to X,\ t \in I$ such that

$$f_0 = \mathbb{1}, \quad f_1(X) = A \quad \text{and} \quad f_t\big|_A = \mathbb{1}\ \forall t \in I$$

The family $f_t$ should be continuous in the sense that the associated map $X \times I \to X, (x,t) \mapsto f_t$ is continuous.

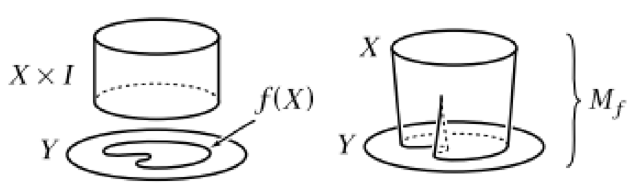
A deformation retraction allows us to deform a space onto a subspace of itself continuously. The catch is, this subspace is never affected by the retraction, only $X \setminus A$ is.

**Definition 0.1.2** Mapping Cylinder

For a map $f : X \to Y$ its **Mapping Cylinder** $M_f$, is the quotient space of the disjoint union $X \times I \amalg Y$, where,

$$(x, 1) \in X \times I \quad \text{identified with} \quad f(x) \in Y$$

This way, the part of $M_f$ that is $X \times I$ Has one end deformed to the shape of $Y$. In this manner $X$ can deform into $Y$ as it 'moves' through the tube of length 1, as seen in the image. This is properly described as sliding each point $(x,t)$ along the segment $\{x\} \times I \subset M_f$ to the endpoint $f(x) \in Y$.




Not all deformation retractions arise from mapping cylinders though.

**Definition 0.1.3** Homotopy

A **Homotopy** is a family of maps $f_t : X \to Y,\ t \in I$ such that the associated map $F : X \times I \to Y,\ F(x,t) = f_t(x)$ is continuous. Two maps $f_0, f_1 : X \to Y$ are said to be **Homotopic** if there is a homotopy $f_t$ connecting them. it is noted $f_0 \simeq f_1$

It becomes clear that a deformation retraction is a special case of a homotopy. It is in fact, a homotopy that connects the identity map on $X, \mathbb{1} = f_0$ to its retraction on $A$, a map $r : X \to X$ such that $r(X) = A$ and

$r|_A = \mathbb{1}$.

Retractions are the topological analogs of projection operators in other parts of mathematics.

**Definition 0.1.4** Homotopy relative to $A$

A homotopy is said to be **relative to** $A \subset X$ if its restriction to $A$ is independent of $t$.

For example, a deformation retraction of $X$ onto $A$ is a homotopy **rel** $A$ from the identity map of $X$ to a retraction of $X$ onto $A$, since $f_t\big|_A = \mathbb{1}$, independent of $t$.

**Definition 0.1.5** Homotopy Equivalent Spaces

A map $f : X \longrightarrow Y$ is a **Homotopy Equivalence** if there exists a map

$$g : Y \longrightarrow X \quad \text{such that} \quad fg \simeq \mathbb{1} \text{ and } gf \simeq \mathbb{1}$$

In this case, the spaces $X$ and $Y$ are said to be **Homotopy Equivalent** or to have the same **Homotopy Type**. Notation is $X \simeq Y$.

This way, for example, several deformation retractions of the same space are homotopy equivalent; despite not being deformation retractions of each other necessarily.

Homotopy equivalence is an **equivalence relation**.

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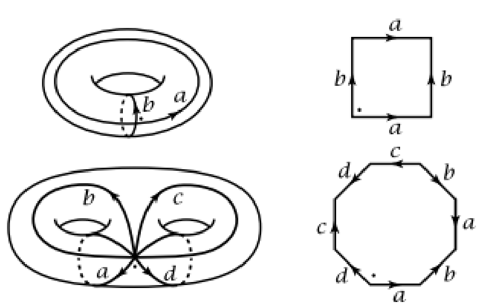
A space having the homotopy type os a single point is **Contractible**. This implies the identity map of this space is **Nullhomotopic**, meaning it's homotopic to a constant map. The converse is not true.

## 0.2 Cell Complexes

**Definition 0.2.1** Genus

The genus of a connected, orientable surface $M_g$ is the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected.It is equal to the number of handles (or 'holes') on it.

An orientable surface $M_g$ of genus $g$ can be constructed using a polygon with $4g$ sides and identifying their edges pairwise.




**Definition 0.2.2** $n$-cells

An $n$-cell, at least in $\mathbb{R}^n$ are discs $D^n$. These are $n$-dimensional spaces that can be glues together along their boundaries.

The torus can be built by first gluing two 1-cells together, $a$ and $b$, resulting in a kind of 'wire' skeleton. Then, gluing to it a 2-cell (the interior of the polygon).

**Definition 0.2.3** Cell Complex

A **Cell Complex** is a space $X$ constructed by the following procedure:

1. Start with the discrete set $X^0$, whose points are regarded as 0-cells.
2. Inductively, form the ***n*-skeleton** $X^n$ from $X^{n-1}$ by attaching $n$-cells $e^n_\alpha$ via maps $\varphi_\alpha : S_{n-1} \to X^{n-1}$. This means $X_n$ is the quotient space of the disjoint union $X^{n-1} \amalg_\alpha D^n_\alpha$ of $X^{n-1}$ with a collection of $n$-disks $D^n_\alpha$ with the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus as a set, $X^n = X^{n-1} \amalg_\alpha e^n_\alpha$ where each $e^n_\alpha$ is an open $n$-disk.
3. The process can be stopped at a finite number of steps, given $X = X^n$ for an $n < \infty$. If not, $X = \bigcup_n X^n$. Here $X$ is given the weak topology, meaning a set $A \subset X$ is open (or closed), iff $A \cap X^n$ is open (or closed) in $X^n$ for each $n$.

If $X = X^n$ for some $n$, $X$ is said to be finite-dimensional, and the smallest of such $n$ is the **Dimension** of $X$, the maximum dimension of cells in $X$.

**Example 0.2.1** (1 cell complex)

A 1 cell complex is a **Graph**, consisting of 0-cells, (vertices) and 1-cells (edges).

**Definition 0.2.4** Euler Characteristic

The **Euler Characteristic** of a cell complex is the number of even-dimensional cells minus the number of odd-dimensional cells.
The Euler characteristic of a cell complex depends only of its homotopy type.

**Example 0.2.2** (Sphere $S_n$)

The sphere $S^n$ has a structure with only two cells, $e^0$ and $e^n$, $e^0$ being its center point and $e^n$, the $n$-cell being attached by the constant map $S^{n-1} \to e^0$.
This is equivalent to regarding $S^n$ as the quotient space $D/\partial D^n$, as in folding all the edge of a disk onto a single point.

**Definition 0.2.5** Characteristic Map

Each cell $e^n_\alpha$ in a cell complex $X$ has a **Characteristic Map** $\Phi_\alpha : D^n_\alpha \to X$, which extends the attaching map $\varphi_\alpha$ and is a homeomorphism from the interior of $D^n_\alpha$ onto $e^n_\alpha$

The characteristic map homeomorphically maps the interior of the disk onto an open $n$-cell $e^n$ and maps the boundary $\partial D^n = S^{n-1}$ into the $(n-1)$-skeleton $X^{n-1}$.

The attaching map defines how the boundary of a new $n$-cell is glued onto the existing skeleton ($X^{n-1}$), while the characteristic map defines the entire cell's position and structure within the space.

**Example 0.2.3** (Characteristic map of $S^n$)

With $S^n$ built as a CW-complex as presented above, the **characteristic map** for the $n$-cell is the quotient map $D^n \to S^n$, collapsing $\partial D^n$ to a point.

**Definition 0.2.6** Subcomplex

A **Subcomplex** of a cell complex $X$ is a closed subspace $A \subset X$ that is a union of cells in $X$.

A subcomplex is a cell complex in itself, since $A$ being closed implies that each characteristic map of it cells has image contained in $A$.

A pair $X, A$, consisting of a cell complex and a subcomplex is called a **CW pair**.

Each skeleton $X^n$ is a subcomplex of the cell complex $X$.

**Note:-**

Cell complex constructions for an object are not unique. There are infinitely many ways to arrange cells in order to build a space. In this sense, sometimes a cell complex can or cannot be a subcomplex of another. It depends on the structure given initially for both.

# Chapter 1

# The Fundamental Group

## 1.1 Basic Constructions

**Definition 1.1.1** Path

A **Path** in a space $X$ is a continuous map $f : I \to X$

**Definition 1.1.2** Homotopy of Maps

A **Homotopy** of maps is a family of functions $f_t : I \to X,\ 0 \leqslant t \leqslant 1$ such that

$$f_t(0) = x_0, \quad f_t(1) = x_1, \quad \text{for every } t$$

and

$$\text{the associated map } F : I \times I \to X \text{ defined by } F(s, t) = f_t(s) \text{ is continuous.}$$

This notion is useful for having continuous deformations of a path make sense. Two paths $f_0,\ f_1$ are **Homotopic** if there is a homotopy between them. In this sense, $f_1$ is a continuous deformation of $f_0$, keeping the start and endpoint the same.

**Example 1.1.1** (Linear Homotopies)

Any two paths $f_0,\ f_1$ in $\mathbb{R}^n$ with the same start and endpoints are homotopic via the homotopy

$$f_t(s) = (1 - t) f_0(s) + t f_1(s)$$

This further shows that for any convex space $X \subset \mathbb{R}^n$ any $f_0,\ f_1$ in $X$ with the same start and endpoints are homotopic.

**Proposition 1.1.1**

The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation.

The equivalence class of a path $f$, $[f]$ is called the **Homotopy Class** of $f$.