

Algebraic Topology Notes

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0.1 Notation

1. $\mathbf{1}$, The identity map.
2. I , The interval $[0, 1] \in \mathbf{R}$.
3. \amalg , Disjoint union of sets or spaces.
4. D^n , unit disk or ball in \mathbf{R}^n .
5. S^n , unit sphere in \mathbf{R}^{n+1} .
6. $\partial D^n = S^{n-1}$, the boundary of the n -disc.

Chapter 1

Geometric notions

1.1 Homotopy and Homotopy Type

Definition 1.1.1 Deformation Retraction

A **Deformation Retraction** from a subspace X onto a subspace A is a family of maps $f_t : X \rightarrow X$, $t \in I$ such that

$$f_0 = \mathbb{1}, \quad f_1(X) = A \quad \text{and} \quad f_t|_A = \mathbb{1} \quad \forall t \in I$$

The family f_t should be continuous in the sense that the associated map $X \times I \rightarrow X, (x, t) \mapsto f_t$ is continuous.

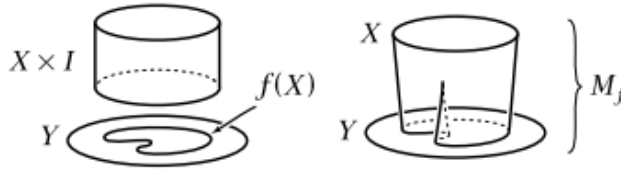
A deformation retraction allows us to deform a space onto a subspace of itself continuously. The catch is, this subspace is never affected by the retraction, only $X \setminus A$ is.

Definition 1.1.2 Mapping Cylinder

For a map $f : X \rightarrow Y$ its **Mapping Cylinder** M_f , is the quotient space of the disjoint union $X \times I \sqcup Y$, where,

$$(x, 1) \in X \times I \quad \text{identified with} \quad f(x) \in Y$$

This way, the part of M_f that is $X \times I$ has one end deformed to the shape of Y . In this manner X can deform into Y as it ‘moves’ through the tube of length 1, as seen in the image. This is properly described as sliding each point (x, t) along the segment $\{x\} \times I \subset M_f$ to the endpoint $f(x) \in Y$.



Not all deformation retractions arise from mapping cylinders though.

Definition 1.1.3 Homotopy

A **Homotopy** is a family of maps $f_t : X \rightarrow Y$, $t \in I$ such that the associated map $F : X \times I \rightarrow Y$, $F(x, t) = f_t(x)$ is continuous. Two maps $f_0, f_1 : X \rightarrow Y$ are said to be **Homotopic** if there is a homotopy f_t connecting them. it is noted $f_0 \simeq f_1$

It becomes clear that a deformation retraction is a special case of a homotopy. It is in fact, a homotopy that connects the identity map on X , $\mathbb{1} = f_0$ to its retraction on A , a map $r : X \rightarrow X$ such that $r(X) = A$ and

$r|_A = \mathbb{1}$.

Retractions are the topological analogs of projection operators in other parts of mathematics.

Definition 1.1.4 Homotopy relative to A

A homotopy is said to be **relative to** $A \subset X$ if its restriction to A is independent of t .

For example, a deformation retraction of X onto A is a homotopy **rel** A from the identity map of X to a retraction of X onto A , since $f_t|_A = \mathbb{1}$, independent of t .

Definition 1.1.5 Homotopy Equivalent Spaces

A map $f : X \rightarrow Y$ is a **Homotopy Equivalence** if there exists a map

$$g : Y \rightarrow X \quad \text{such that} \quad fg \simeq \mathbb{1} \text{ and } gf \simeq \mathbb{1}$$

In this case, the spaces X and Y are said to be **Homotopy Equivalent** or to have the same **Homotopy Type**. Notation is $X \simeq Y$.

This way, for example, several deformation retractions of the same space are homotopy equivalent; despite not being deformation retractions of each other necessarily.

Homotopy equivalence is an **equivalence relation**.

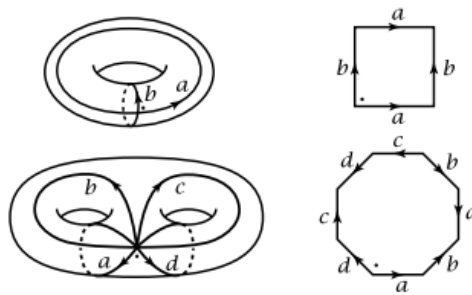
A space having the homotopy type of a single point is **Contractible**. This implies the identity map of this space is **Nullhomotopic**, meaning it's homotopic to a constant map. The converse is not true.

1.2 Cell Complexes

Definition 1.2.1 Genus

The genus of a connected, orientable surface M_g is the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected. It is equal to the number of handles (or 'holes') on it.

An orientable surface M_g of genus g can be constructed using a polygon with $4g$ sides and identifying their edges pairwise.



Definition 1.2.2 n -cells

An n -cell, at least in \mathbb{R}^n are discs D^n . These are n -dimensional spaces that can be glued together along their boundaries.

The torus can be built by first gluing two 1-cells together, a and b , resulting in a kind of 'wire' skeleton. Then, gluing to it a 2-cell (the interior of the polygon).

Definition 1.2.3 Cell Complex

A **Cell Complex** is a space X constructed by the following procedure:

1. Start with the discrete set X^0 , whose points are regarded as 0-cells.
2. Inductively, form the n -**skeleton** X^n from X^{n-1} by attaching n -cells e_α^n via maps $\varphi_\alpha : S_{n-1} \rightarrow X^{n-1}$. This means X_n is the quotient space of the disjoint union $X^{n-1} \amalg_\alpha D_\alpha^n$ of X^{n-1} with a collection of n -disks D_α^n with the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D_\alpha^n$. Thus as a set, $X^n = X^{n-1} \amalg_\alpha e_\alpha^n$ where each e_α^n is an open n -disk.
3. The process can be stopped at a finite number of steps, given $X = X^n$ for an $n < \infty$. If not, $X = \bigcup_n X^n$. Here X is given the weak topology, meaning a set $A \subset X$ is open (or closed), iff $A \cap X^n$ is open (or closed) in X^n for each n .

If $X = X^n$ for some n , X is said to be finite-dimensional, and the smallest of such n is the **Dimension** of X , the maximum dimension of cells in X .

Example 1.2.1 (1 cell complex)

A 1 cell complex is a **Graph**, consisting of 0-cells, (vertices) and 1-cells (edges).

Definition 1.2.4 Euler Characteristic

The **Euler Characteristic** of a cell complex is the number of even-dimensional cells minus the number of odd-dimensional cells.

The Euler characteristic of a cell complex depends only of its homotopy type.

Example 1.2.2 (Sphere S_n)

The sphere S^n has a structure with only two cells, e^0 and e^n , e^0 being its center point and e^n , the n -cell being attached by the constant map $S^{n-1} \rightarrow e^0$.

This is equivalent to regarding S^n as the quotient space $D/\partial D^n$, as in folding all the edge of a disk onto a single point.

Definition 1.2.5 Characteristic Map

Each cell e_α^n in a cell complex X has a **Characteristic Map** $\Phi_\alpha : D_\alpha^n \rightarrow X$, which extends the attaching map φ_α and is a homeomorphism from the interior of D_α^n onto e_α^n .

Example 1.2.3 (Characteristic map of S^n)

With S^n built as a CW-complex as presented above, the **characteristic map** for the n -cell is the quotient map $D^n \rightarrow S^n$, collapsing ∂D^n to a point.