Algebraic Topology Notes

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Contents

-1.1 Notation

1. $\mathbb{1}$, The identity map.
2. I, The interval $[0, 1] \in \mathbb{R}$.
3. $\amalg$, Disjoint union of sets or spaces.
4. D^n, unit disk or ball in $\mathbb{R}n$.
5. S^n, unit sphere in $\mathbb{R}n + 1$.
6. $\partial D^n = S^{n-1}$, the boundary of the n-disc.

Chapter 0

Geometric notions

0.1 Homotopy and Homotopy Type

Definition 0.1.1 Deformation Retraction

A **Deformation Retraction** from a subspace X onto a subspace A is a family of maps $f_t : X \to X,\ t \in I$ such that

$$f_0 = \mathbb{1}, \quad f_1(X) = A \quad \text{and} \quad f_t\big|_A = \mathbb{1}\ \forall t \in I$$

The family f_t should be continuous in the sense that the associated map $X \times I \to X, (x,t) \mapsto f_t$ is continuous.

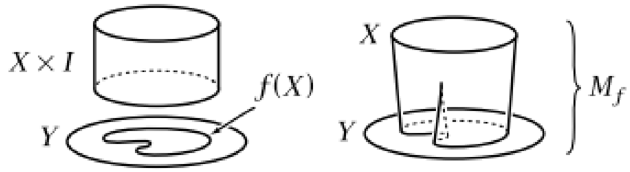
A deformation retraction allows us to deform a space onto a subspace of itself continuously. The catch is, this subspace is never affected by the retraction, only $X \setminus A$ is.

Definition 0.1.2 Mapping Cylinder

For a map $f : X \to Y$ its **Mapping Cylinder** M_f, is the quotient space of the disjoint union $X \times I \amalg Y$, where,

$$(x,1) \in X \times I \quad \text{identified with} \quad f(x) \in Y$$

This way, the part of M_f that is $X \times I$ Has one end deformed to the shape of Y. In this manner X can deform into Y as it 'moves' through the tube of length 1, as seen in the image. This is properly described as sliding each point (x,t) along the segment $\{x\} \times I \subset M_f$ to the endpoint $f(x) \in Y$.



Not all deformation retractions arise from mapping cylinders though.

Definition 0.1.3 Homotopy

A **Homotopy** is a family of maps $f_t : X \to Y,\ t \in I$ such that the associated map $F : X \times I \to Y,\ F(x,t) = f_t(x)$ is continuous. Two maps $f_0, f_1 : X \to Y$ are said to be **Homotopic** if there is a homotopy f_t connecting them. it is noted $f_0 \simeq f_1$

It becomes clear that a deformation retraction is a special case of a homotopy. It is in fact, a homotopy that connects the identity map on $X, \mathbb{1} = f_0$ to its retraction on A, a map $r : X \to X$ such that $r(X) = A$ and

$r|_A = \mathbb{1}$.

Retractions are the topological analogs of projection operators in other parts of mathematics.

Definition 0.1.4 Homotopy relative to A

A homotopy is said to be **relative to** $A \subset X$ if its restriction to A is independent of t.

For example, a deformation retraction of X onto A is a homotopy **rel** A from the identity map of X to a retraction of X onto A, since $f_t\big|_A = \mathbb{1}$, independent of t.

Definition 0.1.5 Homotopy Equivalent Spaces

A map $f : X \longrightarrow Y$ is a **Homotopy Equivalence** if there exists a map

$$g : Y \longrightarrow X \quad \text{such that} \quad fg \simeq \mathbb{1} \text{ and } gf \simeq \mathbb{1}$$

In this case, the spaces X and Y are said to be **Homotopy Equivalent** or to have the same **Homotopy Type**. Notation is $X \simeq Y$.

This way, for example, several deformation retractions of the same space are homotopy equivalent; despite not being deformation retractions of each other necessarily.

Homotopy equivalence is an **equivalence relation**.

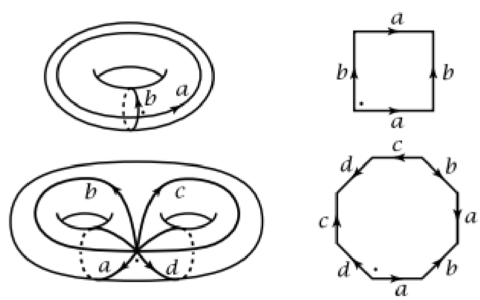
A space having the homotopy type os a single point is **Contractible**. This implies the identity map of this space is **Nullhomotopic**, meaning it's homotopic to a constant map. The converse is not true.

0.2 Cell Complexes

Definition 0.2.1 Genus

The genus of a connected, orientable surface M_g is the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected.It is equal to the number of handles (or 'holes') on it.

An orientable surface M_g of genus g can be constructed using a polygon with $4g$ sides and identifying their edges pairwise.



Definition 0.2.2 n-cells

An n-cell, at least in $\mathbb{R}^n$ are discs D^n. These are n-dimensional spaces that can be glues together along their boundaries.

The torus can be built by first gluing two 1-cells together, a and b, resulting in a kind of 'wire' skeleton. Then, gluing to it a 2-cell (the interior of the polygon).

Definition 0.2.3 Cell Complex

A **Cell Complex** is a space X constructed by the following procedure:

1. Start with the discrete set X^0, whose points are regarded as 0-cells.
2. Inductively, form the **n-skeleton** X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha : S_{n-1} \to X^{n-1}$. This means X_n is the quotient space of the disjoint union $X^{n-1} \amalg_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n_α with the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus as a set, $X^n = X^{n-1} \amalg_\alpha e^n_\alpha$ where each e^n_α is an open n-disk.
3. The process can be stopped at a finite number of steps, given $X = X^n$ for an $n < \infty$. If not, $X = \bigcup_n X^n$. Here X is given the weak topology, meaning a set $A \subset X$ is open (or closed), iff $A \cap X^n$ is open (or closed) in X^n for each n.

If $X = X^n$ for some n, X is said to be finite-dimensional, and the smallest of such n is the **Dimension** of X, the maximum dimension of cells in X.

Example 0.2.1 (1 cell complex)

A 1 cell complex is a **Graph**, consisting of 0-cells, (vertices) and 1-cells (edges).

Definition 0.2.4 Euler Characteristic

The **Euler Characteristic** of a cell complex is the number of even-dimensional cells minus the number of odd-dimensional cells.
The Euler characteristic of a cell complex depends only of its homotopy type.

Example 0.2.2 (Sphere S_n)

The sphere S^n has a structure with only two cells, e^0 and e^n, e^0 being its center point and e^n, the n-cell being attached by the constant map $S^{n-1} \to e^0$.
This is equivalent to regarding S^n as the quotient space $D/\partial D^n$, as in folding all the edge of a disk onto a single point.

Definition 0.2.5 Characteristic Map

Each cell e^n_α in a cell complex X has a **Characteristic Map** $\Phi_\alpha : D^n_\alpha \to X$, which extends the attaching map φ_α and is a homeomorphism from the interior of D^n_α onto e^n_α

The characteristic map homeomorphically maps the interior of the disk onto an open n-cell e^n and maps the boundary $\partial D^n = S^{n-1}$ into the $(n-1)$-skeleton X^{n-1}.

The attaching map defines how the boundary of a new n-cell is glued onto the existing skeleton (X^{n-1}), while the characteristic map defines the entire cell's position and structure within the space.

Example 0.2.3 (Characteristic map of S^n)

With S^n built as a CW-complex as presented above, the **characteristic map** for the n-cell is the quotient map $D^n \to S^n$, collapsing ∂D^n to a point.

Definition 0.2.6 Subcomplex

A **Subcomplex** of a cell complex X is a closed subspace $A \subset X$ that is a union of cells in X.

A subcomplex is a cell complex in itself, since A being closed implies that each characteristic map of it cells has image contained in A.

A pair X, A, consisting of a cell complex and a subcomplex is called a **CW pair**.

Each skeleton X^n is a subcomplex of the cell complex X.

Note:-

Cell complex constructions for an object are not unique. There are infinitely many ways to arrange cells in order to build a space. In this sense, sometimes a cell complex can or cannot be a subcomplex of another. It depends on the structure given initially for both.

Chapter 1

The Fundamental Group

1.1 Basic Constructions

Definition 1.1.1 Path

A **Path** in a space X is a continuous map $f : I \longrightarrow X$

Definition 1.1.2 Homotopy of Paths

A **Homotopy** of paths is a family of functions $f_t : I \longrightarrow X,\ 0 \leqslant t \leqslant 1$ such that

$$f_t(0) = x_0, \quad f_t(1) = x_1, \quad \text{for every } t$$

and

the associated map $F : I \times I \longrightarrow X$ defined by $F(s,t) = f_t(s)$ is continuous.

This notion is useful for having continuous deformations of a path make sense. Two paths $f_0,\ f_1$ are **Homotopic** if there is a homotopy between them. In this sense, f_1 is a continuous deformation of f_0, keeping the start and endpoint the same.

Example 1.1.1 (Linear Homotopies)

Any two paths $f_0,\ f_1$ in $\mathbb{R}^n$ with the same start and endpoints are homotopic via the homotopy

$$f_t(s) = (1-t)f_0(s) + t f_1(s)$$

This further shows that for any convex space $X \subset \mathbb{R}^n$ any $f_0,\ f_1$ in X with the same start and endpoints are homotopic.

Proposition 1.1.1

The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation.

The equivalence class of a path f, $[f]$ is called the **Homotopy Class** of f.