

# Review over some Math

Notes on the book 'All the Math You Missed But Need to Know for Graduate School' by Thomas A. Garrity,  
along with some personal thoughts and exercise solutions.

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# Chapter 1

## Summary over topics

### 1.1 Structure of mathematics

#### 1.1.1 Equivalence problems

Several problems or fields of study in math come down to seeing if two or more objects are the ‘same’, or detecting their differences.

**Note:-**

The concept of two objects being the same depends on the field of study.

**Equivalence** between two objects can be explained by the allowed Maps or functions between them.

**Example 1.1.1**

A square and a circle are equivalent in topology; they can be stretched out from one to the other. They are not the same in differential geometry; one is smooth (differentiable) and the other is not.

**Invariants** are properties that two or more object must satisfy if they are equivalent, however they themselves do not determine equivalence between them.

**Example 1.1.2**

In topology, two circles are not equivalent to a single circle. The two circles have two connected components, whereas the one only has one connected component. Hence, both objects are not equivalent. However, a sphere and a circle have the same number of connected components, but they are not equivalent.

#### 1.1.2 Functions

Functions describe all kinds of situations, both in the real world and in an abstract matter.

Each field of study or area in mathematics focuses in a specific kind of function. Calculus studies differentiable functions from  $\mathbb{R} \rightarrow \mathbb{R}$ , for example.

## 1.2 Topics

### 1.2.1 Linear Algebra

Linear algebra studies **Linear Transformations** and **Vector Spaces**. A linear transformation has a matrix representation, given a basis for the vector space being considered.

Two or more different matrices can represent the same transformation, but over different choices of bases of the same vector space.

The key theorem in linear algebra explains several equivalent descriptions for when a matrix, and hence the associated transformation, is invertible.

Other important aspect is understanding the occurrence and usefulness of eigenvectors and eigenvalues.

### 1.2.2 Real Analysis

The definitions of **Limit**, **Continuity**, **Differentiation** and **Integration** are given in terms of  $\epsilon$  and  $\delta$ . **Uniform Convergence** of functions is a key part of this topic.

### 1.2.3 Differentiating Vector-Valued Functions

The **Inverse Function Theorem** explains that a differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  is invertible iff the determinant of its derivative (the Jacobian) is never zero.

It is important to understand what it means for a vector-valued function to be differentiable, and why its derivative must always be a linear map, represented by the Jacobian as a matrix.

Understanding the **Implicit function theorem** and how it relates to the Inverse Function Theorem is also relevant.

### 1.2.4 Point Set Topology

The concept of **Open Sets** and how a **Topology** can be built in terms of open sets must be understood. Also how a function is said to be **Continuous** in terms of open sets is key.

The standard topology in  $\mathbb{R}^n$  and the **Heine-Borel Theorem** is covered, as well as what a metric space is and how a topology is defined using its metric.

### 1.2.5 Classical Stokes' Theorem

The calculus of vector fields covers the **Curl**, and **Divergence** of vector fields, the **Gradient** of a function and the **Path Integral** along a curve.

The **Divergence Theorem** and **Stokes' Theorem** are presented as classical extensions of the Fundamental Theorem of Calculus, it is relevant to understand why they are indeed extensions of this classic theorem.

### 1.2.6 Differential Forms and Stokes' Theorem

**Manifolds** are geometric objects. **Differential  $k$ -forms** allows calculus to be done on manifolds. There are several ways to define a manifold. How to think about and define differential  $k$ -forms, and how to take their **external derivative** is important.

Differential  $k$ -forms and external derivatives can be represented using vector fields, gradients, curls and divergences.

The statement of **Stokes' Theorem** in this context is covered, and special cases over the Divergence Theorem and Stokes' Theorem seen before.

### 1.2.7 Curvature for Curves and Surfaces

**Curvature** is a measure of the rate of change of the direction of tangent spaces of several geometric objects.

It is relevant to compute the curvature of curves on a plane, the curvature and torsion of a curve in space, and the two principal curvatures of a surface in space, in terms of the Hessian.

### 1.2.8 Geometry

Different axioms build several geometries. For example, euclidean geometry assumes that given a line  $l$  and a point  $p$  not on  $l$ , there is exactly only one other line containing  $p$  parallel to  $l$ . Elliptic and hyperbolic geometries assume different conditions.

Models for hyperbolic geometry, single elliptic geometry and double elliptic geometry are covered.

### 1.2.9 Countability and the Axiom of Choice

The concept of a **Countably Infinite** set is presented. For example, the Integers and Rationals are, while the Real numbers are not.

The statement of **The Axiom of Choice** and its (sometimes weird) equivalences should be covered.

### 1.2.10 Elementary Number Theory

An introduction to **Modular Arithmetic** is given. Important explanations as why there are infinitely many primes, what is a **Diophantine Equation**, what is the **Euclidean Algorithm** and how it is linked to **Continuous Fractions** are covered.

### 1.2.11 Algebra

**Groups** and their properties should be studied, as well as understanding how these represent geometric symmetries. Basics over Groups, **Rings** and **Fields** are covered.

**Galois Theory** explains how finding the roots of polynomials and finite group theory are linked. **Representation Theory** is also covered, establishing a relation between abstract groups and groups of matrices.

### 1.2.12 Algebraic Number Theory

Definitions and examples of **Algebraic Number Fields** are presented. Each of these have themselves an analog of the integers, but not always do these integers have a unique factorization.

### 1.2.13 Complex Analysis

Complex Analysis presents what it means for a function  $f : U \subseteq \mathbb{C} \rightarrow \mathbb{C}$  to be **Analytic**, and equivalent conditions that imply analyticity; featuring the **Cauchy-Riemann Equations** and the **Cauchy Integral Formula**.

An analytic function  $f : U \rightarrow \mathbb{C}$  such that  $f'(z_0) \neq 0$ , then at  $z_0$  the function  $f$  is **Conformal**, (angle preserving), when viewed as a function from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ .

### 1.2.14 Analytic Number Theory

The definition of the **Riemann Zeta Function**, its relation to prime numbers, and the statement of the **Riemann Hypothesis** are presented.

### 1.2.15 Lebesgue Integration

The concepts of **Lebesgue Measure** and **Lebesgue Integration** are presented, as well as the **Lebesgue Dominating Convergence Theorem**. It is important to cover what a set of measure zero is, and what the

‘almost everywhere’ notion means.

### 1.2.16 Fourier Analysis

It is important to understand how to find the **Fourier Series** of a periodic function, the **Fourier Integral** of a function, the **Fourier Transform**, and how they all relate to **Hilbert Spaces**. Their use in helping solve differential equations is also relevant.

### 1.2.17 Differential Equations

Several real world problems can be modeled by systems of differential equations. Almost all **Ordinary Differential Equations** have solutions (and simple formulas for them). In contrast, **Partial Differential Equations** do not have general solutions, only in very specific cases; furthermore, sometimes it is not known if there even exists a solution for them.

Important classes of partial differential equations are the heat equation, the wave equation and the Laplacian.

### 1.2.18 Combinatorics and Probability Theory

Elementary combinatorics and probability theory are covered. The **Binomial Theorem** for polynomials is very useful for computations, and occasionally appears as a step for proofs in several branches of mathematics.

**Sample Spaces** and **Random Variables**, **Expected Value** and **Variance** are key concepts. The **Central Limit Theorem** links integral calculus to the subject.

### 1.2.19 Algorithms

Understanding what the **Complexity** of an algorithm is, and the question **P=NP** is relevant. Some graph theory and algorithms over graphs are covered, as well as why graphs are relevant in the representation of problems.

**Numerical Analysis** algorithms and methods are important for computing approximations for problems in mathematics.

### 1.2.20 Category Theory

Category Theory is a method for thinking about mathematics. Each area of mathematics could be put in the language of categories, being described by its relevant mathematical objects and the morphisms between them, with the results being described by diagrams of arrows.

## Chapter 2

# Linear Algebra

### Note:-

The basic mathematical object is a **Vector Space**, the basic map is a **Linear Transformation**, and the objective is to find equivalent conditions for **Invertibility of Matrices**

Linear Algebra is one of the most relevant (if not the most relevant from my experience) area of mathematics. It allows us to solve systems of linear equations. However, as interesting as solving linear equations might be, the power of linear algebra comes when the abstraction of several different mathematical objects can be represented as vector spaces, hence they can be modified and mapped to and from by linear transformations.

Here, solving the systems (and knowing when a solution for them exists) becomes way more interesting.

## 2.1 Basic concepts

$\mathbb{R}^n$  is the most basic example of a vector space. It is the set of all  $n$ -tuples of real numbers

$$\{(x_1, \dots, x_n) : x_i \in \mathbb{R}\}$$

Each  $n$ -tuple is usually called a **vector**, and two can be added to get another one

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

or they could be multiplied by a number  $\lambda$ , usually called a **scalar**.

$$\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$

The natural map from some  $\mathbb{R}^n$  to a  $\mathbb{R}^m$  is matrix multiplication by a  $n \times m$  matrix. Matrix multiplication is **linear**.

### Example 2.1.1 (Left Matrix Multiplication)

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix}$$

Where  $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \in \mathbb{R}^2$  and  $\begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix} \in \mathbb{R}^3$ .

Systems of linear equations can be expressed as a left matrix multiplication. Our goal is to find the correct value for each  $x_i$ .

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

This system becomes  $Ax = b$ , where

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

When  $m > n$  (more equations than unknown variables) there are usually no solutions to the system (except in some linearly dependent cases). Conversely, when  $m < n$ , we expect to have several solutions.

The most common situation though, is when  $m = n$ . In this case, finding if an inverse to  $A$ ,  $A^{-1}$  exists is our objective, because then finding the vector  $x$  is very straightforward.

$$Ax = b \implies A^{-1}Ax = A^{-1}b \implies x = A^{-1}b$$

Hence, finding when a square matrix has an inverse is the same as determining when an  $n \times n$  system of linear equations has a solution.

## 2.2 Vector Spaces and Linear Transformations

### Definition 2.2.1: Vector Space

A set  $V$  is a **Vector Space** over a field  $K$  (usually  $\mathbb{R}$  or  $\mathbb{C}$ ) if there are maps:

1.  $\cdot : K \times V \rightarrow V$  multiplication by scalar,  $av$  for  $a \in K$ ,  $v \in V$
2.  $+: V \times V \rightarrow V$  vector sum,  $v + u$  for  $v, u \in V$

with the properties:

- (a)  $\exists 0 \in V : 0 + v = v \ \forall v \in V$
- (b)  $\forall v \in V \ \exists (-v) \in V : (-v) + v = 0$
- (c)  $\forall v, u \in V \ v + u = u + v$
- (d)  $\forall v, u \in V, \ \forall a \in K \ a(u + v) = au + av$
- (e)  $\forall v \in V, \ \forall a, b \in K \ a(bv) = (ab)v$
- (f)  $\forall v \in V, \ \forall a, b \in K \ (a + b)v = av + bv$
- (g)  $\exists 1 \in K \ \forall v \in V \ 1v = v$

These are all usual properties easily recognizable in the vector space  $\mathbb{R}^n$  over the field  $\mathbb{R}$ .



### Definition 2.2.2: Linear Transformation

A **Linear Transformation**  $T$  is a map from any vector space  $V$  to any other vector space  $W$ , such that

$$T(a_1u + a_2v) = a_1T(u) + a_2T(v)$$

for any two vectors and scalars.

Left matrix multiplications as seen before is a linear transformation.

### Definition 2.2.3: Subspace

A subset  $U \subset V$  is a **Subspace** if it itself is a vector space.

It is way easier to check this condition however,

**Proposition 2.2.1**  $U \subset V$  is a subspace if it contains the zero vector and it's closed under addition and scalar multiplication

Given a linear transformation  $T$  from a vector space  $V$  to another  $W$ , two natural subspaces are the kernel and image.

### Definition 2.2.4: Kernel & Image of a Linear Map

Given a linear map  $T : V \rightarrow W$ , its **Kernel** is

$$\ker(T) = \{v \in V \mid T(v) = 0 \in W\}$$

and its **Image** is

$$\text{Im}(T) = \{T(v) \in W \mid v \in V\}$$

Clearly both are subsets of  $V$  and  $W$  respectively, and closed under addition and multiplication by scalars.

### Example 2.2.1 (Derivative as linear map)

In the vector space  $C^k[0, 1]$ , the set of all real-valued functions with domain over  $[0, 1]$  such that the  $k$ th derivative exists and is continuous, the derivative is a linear transformation from  $C^k[0, 1]$  to  $C^{k-1}[0, 1]$ .

$$\frac{d}{dx} : C^k[0, 1] \rightarrow C^{k-1}[0, 1]$$

Then the kernel of this linear transformation are all constant functions.

For example, in the differential equation

$$f'' + 3f' + 2f = 0$$

we can define the linear map

$$T = \frac{d^2}{dx^2} + 3\frac{d}{dx} + 2I : C^2[0, 1] \rightarrow C^0[0, 1]$$

Hence, the solution lies in  $\ker(T)$ .