(1-1) *Proof.* Let $m, n \in \mathbb{N}$. Then observe the following:

- Let $f_1(x) = x^m$ and $f_2(x) = x^m$. Since polynomial functions are differentiable on $\mathbb{R}$, then $f_1(x)$ and $f_2(x)$ are both differentiable.
- Let $f_3(x) = 1$. Since $f_2(x)$ is differentiable from above, and $f_3(x)$ is a constant function (hence, it is differentiable), then $f_4(x) = (\frac{f_3}{f_2})(x)$ is differentiable since the quotient of two functions is differentiable.
- Let $f_5(x) = \sin(x)$. Then $f_5(x)$ is differentiable on $\mathbb{R}$. If $f_6(x) = f_5(f_4(x))$, then $f_6(x)$ is differentiable since it is a composition of differentiable functions.
- Let $f_7(x) = f_1(x)f_6(x)$. Since $f_7(x)$ is a product of differentiable functions, then it is itself differentiable.

Hence $f(x)$ is differentiable on $\mathbb{R}\backslash\{0\}$. $\square$

(1-2) *Proof.* Suppose $m > 1$. Then

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0}\frac{x^m\sin(\frac{1}{x})-0}{x-0}\\ &= \lim_{x\to 0}\frac{x^m\sin(\frac{1}{x})}{x}\\ &= \lim_{x\to 0}x^{m-1}\sin\left(\frac{1}{x}\right)\end{aligned}$$

Since $m > 1$, then $m-1 > 0$. We now need to show that this limit exists. To that end, observe that for any $a \in \mathbb{R}$, then $0 \le |\sin(a)| \le 1 \implies 0 \le |x^{m-1}\sin(\frac{1}{x})| \le |x^{m-1}|$. Since

$$\lim_{x\to 0} 0 = 0, \quad \lim_{x\to 0}|x^{m-1}| = 0$$

where the second limit is justified by the continuity of $|x^{m-1}|$, then

$$\lim_{x\to 0}\left|x^{m-1}\sin\left(\frac{1}{x}\right)\right| = 0 \implies \lim_{x\to 0}x^{m-1}\sin\left(\frac{1}{x}\right) = 0$$

Then f is differentiable at 0, and $f'(0) = 0$. $\square$

(1-3) *Proof.* Suppose $m > 1 + n$. We know that

$$f'(x) = \begin{cases} mx^{m-1}\sin\left(\frac{1}{x^n}\right) + x^{m-n-1}\cos\left(\frac{1}{x^n}\right) & x \neq 0 \\ 0 & x = 0\end{cases}$$

From (1-2), we know that

$$\lim_{x\to 0} mx^{m-1}\sin(\frac{1}{x^n}) = m\cdot 0 = 0$$

By similar reasoning, we deduce that

$$\lim_{x\to 0}x^{m-n-1}\cos\left(\frac{1}{x^n}\right) = 0$$

because $m-n-1 > 0$. Then

$$\lim_{x\to 0} f'(x) = 0$$

So that $f'(x)$ is continuous at $x = 0$. $\square$

(2) *Proof.* Assume for contradiction that x_n does not converge to L. Then there exists $\varepsilon > 0$ such that for all N, there exists $n > N$ such that $|x_n - L| \geq \varepsilon$. We may construct a subsequence as follows:

- $N = 1 \implies (\exists n_1 > N)(|x_{n_1} - L| \geq \varepsilon)$
- $N = n_1 \implies (\exists n_2 > N)(|x_{n_2} - L| \geq \varepsilon)$
- $N = n_2 \implies (\exists n_3 > N)(|x_{n_3} - L| \geq \varepsilon)$
- $\dots$

It follows that no subsequence of (x_n) converges to L, contradicting the original assumption. Then (x_n) converges to L. □

(3) *Proof.* ($\Rightarrow$) Suppose that f is differentiable at some c. Then

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = L$$

for some $L \in \mathbb{R}$. Then

$$\begin{aligned}
\lim_{h\to 0} \frac{f(c+h) - f(c) - Lh}{h} &= \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} - \lim_{h\to 0} \frac{Lh}{h} \\
&= L - \lim_{h\to 0} L \\
&= 0
\end{aligned}$$

($\Leftarrow$) Suppose now that, for some $L \in \mathbb{R}$, that

$$\lim_{h\to 0} \frac{f(c+h) - f(c) - Lh}{h} = 0$$

By the Algebraic Limit Laws, we get

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} - \lim_{h\to 0} \frac{Lh}{h} = 0 \implies \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = L$$

Then f is differentiable at c, and $f'(c) = L$. □

(4) *Proof.* Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $|h - 0| < \delta$ implies $|g(h) - L| < \varepsilon$. To prove that

$$\lim_{h\to 0} g(-h) = L$$

we must show that there exists $\delta_1 > 0$ such that $|-h - 0| < \delta_1$ implies $|g(-h) - L| < \varepsilon$. Using $\hat{\delta} = \delta$, then $|-h-0| = |-h| = |h| = |h - 0| < \delta \implies |g(-h) - L| < \varepsilon$. □

(5-1) *Proof.* By (5-1), we have

$$\begin{aligned}
\lim_{-h\to 0} \frac{f(c-h) - f(c))}{-h} &= \lim_{h\to 0} \frac{f(c) - f(c-h)}{h} \\
&= f'(c)
\end{aligned}$$

□

(5-2) *Proof.* By (5-1), we have

$$\begin{aligned}
\lim_{h\to 0} \frac{f(c+h) - f(c-h)}{2h} &= \lim_{h\to 0} \frac{f(c+h) - f(c) + f(c) - f(c-h)}{2h} \\
&= \frac{1}{2} \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} + \frac{1}{2} \lim_{h\to 0} \frac{f(c) - f(c-h)}{h} \\
&= \frac{1}{2}(f'(c) + f'(c)) \\
&= f'(c)
\end{aligned}$$

□

(6-1) *Proof.* We have

$$\begin{aligned}
\lim_{h\to 0}\frac{\cos(h)-1}{h} &= \lim_{h\to 0}\frac{\cos(h)-1}{h}\cdot\frac{\cos(h)+1}{\cos(h)+1}\\
&= \lim_{h\to 0}\frac{1-\cos^2(h)}{h(\cos(h)+1)}\\
&= \lim_{h\to 0}\frac{\sin^2(h)}{h(\cos(h)+1)}\\
&= \lim_{h\to 0}\frac{\sin(h)}{h}\cdot\lim_{h\to 0}\frac{\sin(h)}{\cos(h)+1}\\
&= 1\cdot\frac{\sin(0)}{\cos(0)+1}\\
&= 0
\end{aligned}$$

□

(6-2) *Proof.* Let $c \in \mathbb{R}$ and $f(x) = \sin(x)$. Then

$$\begin{aligned}
f'(c) &= \lim_{h\to 0}\frac{\sin(c+h)-\sin(c)}{h}\\
&= \lim_{h\to 0}\frac{\sin(c)\cos(h)+\cos(c)\sin(h)-\sin(c)}{h}\\
&= \lim_{h\to 0}\frac{\sin(c)(\cos(h)-1)+\cos(c)\sin(h)}{h}\\
&= \sin(c)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(c)\lim_{h\to 0}\frac{\sin(h)}{h}\\
&= \sin(c)\cdot 0+\cos(c)\cdot 1\\
&= \cos(c)
\end{aligned}$$

Then f is differentiable at c, and $f'(c) = \cos(c)$. □