(1) *Proof.* Suppose (a_n) is a sequence where $a_n > 0$ for all $n \in \mathbb{N}$. Suppose that $\sum_{n=1}^{\infty} a_n$ converges. Then $a_n \to 0$. Particularly, there exists some $N \in \mathbb{N}$ such that for all $n > N$, then $|a_n - 0| = a_n < 1$. Then $(a_n)^2 < a_n < 1$. Since $\sum a_n$ is convergent, and the sequence $b_n = (a_n)^2$ is a sequence such that $b_n < a_n$ for all $n > N$, then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}(a_n)^2$ by the Comparison Test. □

(2) Consider the sequence $(a_n) = \frac{1}{n}$. Then $\sum_{n=1}^{\infty} a_n$ diverges, but $\sum_{n=1}^{\infty}(a_n)^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(3) *Proof.* Consider the sum $a_n = u_n - u_{n-1}$ for all $n \geq 2$, and $a_n = u_1$. Then for any $n \geq 2$, we have that

$$a_n = u_n - u_{n-1} = \ln(n) - \sum_{k=1}^{n} \frac{1}{k} - \ln(n-1) + \sum_{k=1}^{n-1} \frac{1}{k} = \ln\left(\frac{n}{n-1}\right) - \frac{1}{n} = -\frac{1}{n} - \ln\left(1 - \frac{1}{n}\right)$$

Consider the sequence $b_n = \frac{1}{n^2}$. Then

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} \frac{-\frac{1}{n} - \ln\left(1 - \frac{1}{n}\right)}{\frac{1}{n^2}} \implies \lim_{t\to 0} \frac{-t - \ln(1-t)}{t^2} = \frac{1}{2}$$

Since $\sum b_n$ is a convergent sequence, then by the Limit Comparison Test, $\sum a_n$ also converges, thereby u_n converges. □

(4) *Proof.* Let $(s_n) \subseteq \mathbb{R}$ be the sequence of partial sums of (a_n). Let $(b_n) = (n)$ be a strictly increasing sequence that is divergent in $\mathbb{R}$. Observe that $s_n - s_{n-1} = a_n$, and $b_n - b_{n-1} = n - (n-1) = 1$. Then by Stolz-Cesáro Theorem, we have

$$\lim_{n\to\infty} \frac{s_n - s_{n-1}}{b_n - b_{n-1}} = \lim_{n\to\infty} a_n = a \implies \lim_{n\to\infty} \frac{s_n}{b_n} = \lim_{n\to\infty} A_n = a$$

□

(5-1) *Proof.* By Stolz-Cesáro Theorem, the sequence of arithmetic means

$$\sigma_n = \frac{\sum_{i=1}^{n} s_i}{n}$$

converges to L, since $s_n \to L$ by $\sum a_n = L$. Then $\sum a_n$ is Cesáro convergent to L. □

(5-2) *Proof.* Consider the sequence $a_n = (-1)^n$. Consider the sequence of partial sums s_n of a_n for $n \geq 1$. Consider now the sequence (σ_n) of arithmetic means of s_n. Then

$$(\sigma_n) = \left(-1, -\frac{1}{2}, -\frac{2}{3}, -\frac{1}{2}, -\frac{3}{5}, \ldots\right) = \begin{cases} -\frac{n+1}{2n} & n = 2k-1, k \in \mathbb{N} \\ -\frac{1}{2} & n = 2k, k \in \mathbb{N} \end{cases}$$

Taking the subsequence (σ_{2n}), this clearly converges to $-\frac{1}{2}$. Taking the subsequence (σ_{2n-1}), we want to show that this converges to $-\frac{1}{2}$. Let $\varepsilon > 0$. Then for all $n > \frac{1}{2\varepsilon}$, we have that

$$\begin{aligned} \left|-\frac{n+1}{2n} + \frac{1}{2}\right| &= \left|-\frac{n+1}{2n} + \frac{n}{2n}\right| \\ &= \left|-\frac{1}{2n}\right| \\ &< \varepsilon \end{aligned}$$

so that $\sigma_{2n-1} \to -\frac{1}{2}$. Then $\sigma_n \to -\frac{1}{2}$, and $\sum a_n$ is Cesáro convergent to $-\frac{1}{2}$. □

(6) Consider the sum

$$\sum_{k=1}^{\infty} \frac{\sum_{n=1}^{k} \frac{1}{n}}{(k+1)(k+2)}$$

Denote the sum in the numerator as u_k. Consider the sum $b_k = \frac{1}{k}$, and notice that

$$\frac{1}{(k+1)(k+2)} = \frac{1}{k+1} - \frac{1}{k+2} = b_{k+1} - b_{k+2}$$

Then the sum can be written as

$$\sum_{k=1}^{\infty} u_k(b_{k+1} - b_{k+2})$$

Let s_n denote the partial sums of this sequence. Analyzing the partial sums:

$$\begin{aligned}
s_1 &= u_1(b_2 - b_3) \\
s_2 &= u_1(b_2 - b_3) + u_2(b_3 - b_4) \\
&= u_1 b_2 + b_3(u_2 - u_1) - u_2 b_4 \\
&= u_1 b_2 + b_3 b_2 - u_2 b_4 \\
s_3 &= u_1(b_2 - b_3) + u_2(b_3 - b_4) + u_3(b_4 - b_5) \\
&= u_1 b_2 + b_3(u_2 - u_1) + b_4(u_3 - u_2) - u_3 b_5 \\
&= u_1 b_2 + b_3 b_2 + b_4 b_3 - u_3 b_5 \\
s_4 &= u_1(b_2 - b_3) + u_2(b_3 - b_4) + u_3(b_4 - b_5) + u_4(b_5 - b_6) \\
&= u_1 b_2 + b_3(u_2 - u_1) + b_4(u_3 - u_2) + b_5(u_4 - u_3) - u_4 b_6 \\
&= u_1 b_2 + b_3 b_2 + b_4 b_3 + b_5 b_4 - u_4 b_6
\end{aligned}$$

Notice that $b_k b_{k+1} = b_{k+1} - b_k$. Then

$$s_4 = u_1 b_2 + b_3 - b_2 + b_4 - b_3 + b_5 - b_4 - u_4 b_6 = u_1 b_2 - b_2 + b_5 - u_4 b_6$$

We then have a telescoping series scenario with some left over terms. In other words, we have

$$s_n = u_1 b_2 - b_2 + b_{n+1} - u_n b_{n+2}$$

Clearly $b_n \to 0$, so as $n \to \infty$, then $s_n = u_1 b_2 - b_2 - u_n b_{n+2}$. We then must evaluate $u_n b_{n+2}$. Observe that $0 \le u_n b_{n+2} \le u_n b_n$, since b_n is a decreasing function. Looking at $u_n b_n$, we have that

$$u_n b_n = \frac{\sum_{i=1}^{n} \frac{1}{i}}{n}$$

which converges to 0 by Stolz-Cesáro Theorem. Then $u_n b_{n+2} \to 0$, and $s_n = u_1 b_2 - b_2$. Then

$$\sum_{k=1}^{\infty} \frac{\sum_{n=1}^{k} \frac{1}{n}}{(k+1)(k+2)} = (1)\left(\frac{1}{2}\right) - \frac{1}{2} = 0$$

(7) *Proof.* Consider a sequence (a_n) such that the sequence $b_n = \sqrt[n]{|a_n|}$ converges to some $L \in \mathbb{R}$. Suppose $0 \le L < 1$. Consider some $\alpha \in \mathbb{R}$ such that $L < \alpha < 1$. Since $b_n \to L$, there exists some $N \in \mathbb{N}$ such that for all $n > N$, then $b_n < \alpha \implies |a_n| < \alpha^n$. Since $\alpha < 1$, then the series $\sum \alpha^n$ is convergent. By Comparison test, then $\sum |a_n|$ is convergent, consequently proving that $\sum a_n$ is convergent. Hence $\sum a_n$ is absolutely convergent. $\square$