(1) *Proof.* Consider the function $h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$. Then

$$h(a) = f(b)g(a) - f(a)g(a) - g(b)f(a) + g(a)f(a) = f(b)g(a) - f(a)g(b)$$

and

$$h(b) = f(b)g(b) - f(a)g(b) - g(b)f(b) + g(a)f(b) = f(b)g(a) - f(a)g(b)$$

Since $h(a) = h(b)$, and h is a differentiable function by the Algebraic Differentiability Theorem, then by Rolle's Theorem, there exists $c \in (a, b)$ such that $h'(c) = 0$. This consequently implies that $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$. $\square$

(2) *Proof.* Assume for contradiction that f is not strictly increasing. Then there exists $a, b \in I$ such that $a < b$, but $f(a) > f(b)$. By the MVT, there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$. But $f(a) > f(b) \implies f(b) - f(a) < 0$, and $a < b \implies b - a > 0$. Then $f'(c) < 0$, contradicting that $f'(x) > 0$ for every $x \in I$. Then f must be strictly increasing. $\square$

(3-1) *Proof.* Suppose f is differentiable on $\mathbb{R}$, and it is Lipschitz continuous. Then

$$|f'(x)| = \lim_{x \to h} \left| \frac{f(x+h) - f(x)}{h} \right|$$

By Lipschitz continuity of f, we have:

$$\begin{aligned} &\leq \lim_{x \to h} \left| \frac{C|x + h - x|}{h} \right| \\ &= |C \operatorname{sgn}(h)| \\ &= C \end{aligned}$$

Hence $f'(x)$ is bounded. $\square$

(3-2) *Proof.* Suppose f is differentiable on $\mathbb{R}$, and f' is bounded. Consider any $u, v \in \mathbb{R}$. By the MVT, then there exists $t \in (u, v)$ such that $f(u) - f(v) = f'(c)(u - v)$. Then

$$f'(c) = \frac{f(u) - f(v)}{u - v} \implies |f'(c)| = \frac{|f(u) - f(v)|}{|u - v|}$$

Since f' is bounded by C, then

$$\frac{|f(u) - f(v)|}{|u - v|} = f'(c) \leq C \implies |f(u) - f(v)| \leq C|u - v|$$

Hence f is Lipschitz continuous. $\square$

(4-1) *Proof.* Observe that $f(0) = 0$ and $f(-1) = -5$. Consider the closed interval $[0, 1] \subset \mathbb{R}$. Since f is a polynomial, then it is continuous on $\mathbb{R}$. Furthermore, $[0, 1]$ is a closed and bounded interval so that it is compact. By the IVT, there exists some $c \in [0, 1]$ such that $f(c) = 0$, where $f(a) < 0 < f(b)$. Then there exists a solution to $f(x) = 0$. $\square$

(4-2) *Proof.* By (4-1), there exists a point $t \in \mathbb{R}$ such that $f(t) = 0$. Assume for contradiction that there exists another $s \neq t$ such that $f(s) = 0$. Since f is a polynomial, then it is continuous on $\mathbb{R}$. Furthermore, $[s, t]$ is a closed and bounded interval so that it is compact. By Rolle's Theorem, there exists some $c \in [s, t]$ such that $f'(c) = 0$. Consider $f'(x) = 5x^4 + 3x^2 - 2x + 5$. By the given hint, we have that $3x^2 - 2x + 5 > 0$ for all $x \in \mathbb{R}$. Similarly, $5x^4 \geq 0$ for all $x \in \mathbb{R}$. Then $f'(x) > 0$ for all $x \in \mathbb{R}$, contradicting Rolle's Theorem. Hence, there is no other solution to $f(x) = 0$ except t. $\square$

(5) The second and third inequalities in the statement utilize all 3 points.

(6) *Proof.* Assume for contradiction that f has at least $n+1$ solutions, say $x_1, x_2, \dots, x_{n+1}$ such that $x_1 < x_2 < \dots < x_{n+1}$. Then $f(x_i) = 0$ for every i. By Rolle's Theorem, there then exists points $y_i \in (x_i, x_{i+1})$ for $1 \le i \le n$ such that $f'(y_i) = 0$. But then f' has n solutions, contradicting that it had $n-1$ solutions at most. Then f has at most n solutions. □

(7-1) *Proof.* Consider the following:

- Let $f_1(x) = \frac{x}{2}$. Since f_1 is a polynomial, then it is continuous and differentiable everywhere.
- Let $f_2(x) = x^2$. Since f_2 is a polynomial, then it is continuous and differentiable everywhere.
- Let $f_3(x) = \sin(x)$ and $f_4(x) = \frac{1}{x}$. Since f_3 is a sine function, it is continuous and differentiable everywhere. Since f_4 is a rational function, it is continuous and differentiable on its domain of $\mathbb{R}\backslash\{0\}$. Then $f_5(x) = f_3(f_4(x))$ is continuous and differentiable on $\mathbb{R}\backslash\{0\}$.
- Let $f_6(x) = f_2(x)f_5(x)$. Since f_6 is a product of continuous and differentiable functions, then it follows by the Algebraic Differentiability Theorem that it is also continuous and differentiable.
- Let $f(x) = f_1(x) + f_6$. Since f is a sum of continuous and differentiable functions, then it follows by the Algebraic Differentiability Theorem that f is also continuous and differentiable.

Then by the above, $\frac{x}{2} + x^2 \sin\frac{1}{x}$ is differentiable when $x \neq 0$. □

(7-2) *Proof.* Using the definition of the derivative:

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{\frac{x}{2} - x^2\sin(\frac{1}{x}) - 0}{x - 0} \\ &= \lim_{x\to 0} \left(\frac{1}{2} - x\sin\left(\frac{1}{x}\right)\right) \end{aligned}$$

Using previous results:

$$= \frac{1}{2} > 0$$

Then $f'(0) > 0$. □

(7-3) *Proof.* Observe that

$$f'(x) = \begin{cases} \frac{1}{2} + 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ \frac{1}{2} & x = 0 \end{cases}$$

Assume for contradiction that there exists an interval around 0 such that $f'(x) \ge 0$. Then there exists $\delta > 0$ such that $(\forall x \in V_\delta(0))(f'(x) \ge 0)$. Consider the sequence $(a_n) = \frac{1}{2n\pi}$. Since $a_n \to 0$, there exists N such that $n > N \implies a_n < \delta$. But then

$$f'(a_n) = \frac{1}{2} + \frac{1}{n\pi}\cdot 0 - 1 = -\frac{1}{2} < 0$$

Which means that there exists points in $V_\delta(0)$ such that $f'(x) \ngeq 0$. □