(1) (a) *Proof.* Let β be an upper bound of B. Since $A \subseteq B$, then it follows that for all $a \in A$, then $a \leq \beta$. Then every upper bound of B is an upper bound of A. In particular, $\sup(B)$ is an upper bound of A. By definition of a supremum, then $\sup(A) \leq \sup(B)$. $\square$

(b) *Proof.* Let β be a lower bound of B. Since $A \subseteq B$, then it follows that for all $a \in A$, then $\beta \leq a$. Then every lower bound of B is a lower bound of A. In particular, $\inf(B)$ is a lower bound of A. By definition of an infimum, then $\inf(B) \leq \inf(A)$. $\square$

(c) *Proof.* Since $a \leq b$ for all $a \in A$ and $b \in B$, then each b is an upper bound of A so that $\sup(A) \leq b$. Furthermore each a is a lower bound of B so that $a \leq \inf(B)$. Then $\sup(A) \leq \inf(B)$. $\square$

(d) *Proof.* Since there is some b such that for all $a \in A$ we have that $a \leq b$, then $\sup A \leq b$. By definition, $b \leq \sup(B)$. Then $\sup(A) \leq \sup(B)$. $\square$

(e) *Proof.* Since there is some b such that for all $a \in A$ we have that $a \geq b$. Then $b \leq \inf(A)$. Since $b \geq \inf(B)$ for $b \in B$, then $\inf(B) \leq \inf(A)$. $\square$

(2) (a) Observe that

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} = \inf\{0, 1\} = 1$$
$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} = \sup\{1\} = 1$$

Then

$$L(f, P) = \sum_{k=1}^{n} m_k \Delta x_k = \sum_{k=1}^{n} 0 \Delta x_k = 0$$
$$U(f, P) = \sum_{k=1}^{n} M_k \Delta x_k = \sum_{k=1}^{n} 1 \Delta x_k = 1$$

Then

$$L(f) = \sup\{L(f, P) : P \in \Pi\} = \sup\{0\} = 0$$
$$U(f) = \inf\{L(f, P) : P \in \Pi\} = \inf\{1\} = 1$$

Since $L(f) \neq U(f)$, then f is not Riemann integrable.

(3) (a) Consider the equipartition (P_n) of $[0, 1]$. Then

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} = \frac{(k-1)^2}{n^2}$$
$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} = \frac{k^2}{n^2}$$

Then

$$\begin{aligned} L(f, P_n) &= \sum_{k=1}^{n} m_k \Delta x_k = \sum_{k=1}^{n} \frac{(k-1)^2}{n^2} \cdot \frac{1}{n} \\ &= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - n(n+1) + n\right) \\ &= \frac{2n^3 + 3n^2 + n}{6n^3} \\ U(f, P_n) &= \sum_{k=1}^{n} M_k \Delta x_k = \sum)k = 1^n \frac{k^2}{n^2} \cdot \frac{1}{n} \\ &= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

(b) It's clear that $L(f,P_n) \to 1/3$ and $U(f,P_n) \to 1/3$. Then

$$\int_0^1 x^2 \mathrm{d}x = \frac{1}{3}$$

(4) (a) *Proof.* Let $q \in [0,1] \cap \mathbb{Q}$. Let (a_n) be a sequence of irrational numbers such that $a_n \to q$. Then $f(a_n) = f(0) \not\to 1 = f(q)$. Then g is discontinuous on $[0,1] \cap \mathbb{Q}$. □

(b) Since g is continuous on $[0,1] \cap \{\mathbb{R} \backslash \mathbb{Q}\}$ and discontinuous on $[0,1] \cap \mathbb{Q}$, and $\mathbb{Q}$ is a countable set of points, then g is Riemann integrable on $[0,1]$.

(c) $(f \circ g)(c) = f(g(c)) = f(1/n) = 1$, where $c \in [0,1] \cap \mathbb{Q}$. If $c \in [0,1] \cap \{\mathbb{R} \backslash \mathbb{Q}\}$, then $(f \circ g)(c) = f(g(c)) = f(0) = 0$. $f \circ g$ is the Dirichlet function.

(5) *Proof.* By definition, $L(f) = \sup\{L(f,P) : P \in \Pi\}$ and $U(f) = \inf\{L(f,P) : P \in \Pi\}$. Similarly, $L(g) = \sup\{L(g,P) : P \in \Pi\}$ and $U(g) = \inf\{U(g,P) : P \in \Pi\}$. Furthermore, define the quantities

$$\begin{aligned} m_{k,f} &= \inf\{f(x) : x \in [x_{k-1}, x_k]\} \\ M_{k,f} &= \sup\{f(x) : x \in [x_{k-1}, x_k]\} \\ m_{k,g} &= \inf\{g(x) : x \in [x_{k-1}, x_k]\} \\ M_{k,g} &= \sup\{g(x) : x \in [x_{k-1}, x_k]\} \end{aligned}$$

Then $m_{k,f} \leq m_{k,g}$ and $M_{k,f} \leq M_{k,g}$ by $f \leq g$ on $[a,b]$. Then

$$L(f) = \sup\left\{\sum_{k=1}^{n} m_{k,f} \Delta x_k : P \in \Pi\right\} \leq \sup\left\{\sum_{k=1}^{n} M_{k,g} \Delta x_k : P \in \Pi\right\} = L(g)$$

A similar line of reasoning shows that $U(f) \leq U(g)$. □

(6) Since $0 \to 0$ trivially, and $2r_n(b-a) \to 0$ since $r_n \to 0$. then Squeeze Theorem asserts that $U(f) - L(f) \to 0 \implies U(f) = L(f)$.