(1) *Proof.* Let $\varepsilon > 0$ be given. To show that f is continuous at $x = 0$, we must show that there exists $\delta > 0$ such that if $|x| < \delta$, then $|f(x)| < |\varepsilon|$.

Pick $\delta = \varepsilon$. We may consider two cases:

(a) Suppose $x \in V_\delta(0) \cap \mathbb{Q}$. Then $|f(x)| = |x| < \delta < \varepsilon$.
(b) Suppose $x \in V_\delta(0) \cap (\mathbb{R} \backslash \mathbb{Q})$. Then $|f(x)| = 0 < \varepsilon$.

In both cases, we have that $|f(x)| < \varepsilon$ so that $f(x)$ is indeed continuous at $x = 0$.

To prove that f is discontinuous everywhere else, let $c \in \mathbb{R}\backslash\{0\}$. We now consider two cases:

(a) Suppose $c \in \mathbb{R}\backslash\mathbb{Q}$. By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a sequence $(q_n) \subseteq \mathbb{Q}$ such that $q_n \to c$. Then $f(q_n) = 0$ for all $n \in \mathbb{N}$. Then $q_n \to c$, but $f(q_n) = q_n \to c \neq f(c) = 0$.
(b) Suppose $c \in \mathbb{Q}\backslash\{0\}$. For all $n \in \mathbb{N}$, let $r_n \in V_{1/n}(c)$ be an irrational number. Then

$$c - \frac{1}{n} < r_n < c + \frac{1}{n}$$

for every $n \in \mathbb{N}$. By the Squeeze Theorem, $r_n \to c$. Then $f(r_n) = 0$, but $f(r_n) \to 0 \neq f(c) = c$.

Then f is continuous at 0 and discontinuous everywhere else. □

(2-1) The sequence $\left(\frac{1}{n}\right)$ is Cauchy because it is convergent.

Observe that $f(a_n) = (n)$. Since (n) is not convergent, then it is not Cauchy.

(2-2) *Proof.* Let A be a nonempty subset of $\mathbb{R}$. Let $f : A \to \mathbb{R}$ be a uniformly continuous function, and let $(a_n) \subset A$ be Cauchy. Let $\varepsilon > 0$ be given.

Since f is uniformly continuous, there exists some $\delta > 0$ such that for every $s, t \in A$, then $|s - t| < \delta \implies |f(s) - f(t)| < \varepsilon$.

Since (a_n) is Cauchy, there exists some N such that for all $m, n > N$, then $|a_m - a_n| < \delta$.

To prove that $f(a_n)$ is Cauchy, we must show that there exists some $\hat{N}$ such that for all $m, n > \hat{N}$. Let $\hat{N} = N$. Then for all $m, n > \hat{N}$, we have $|a_m - a_n| < \delta$. Then $|f(a_m) - f(a_n)| < \varepsilon$ so that $f(a_n)$ is Cauchy. □

(3) *Proof.* For some $a, b \in \mathbb{R}$ such that $a < b$, consider some function $f : [a, b] \to \mathbb{R}$ such that is continuous and bijective. Assume for contradiction that f is not monotonic. Then for some $c \in [a, b]$ such that $a < c < b$, we may discuss two cases:

Suppose $f(a) < f(c)$ and $f(b) < f(c)$. Since $[a, c]$ and $[b, c]$ are compact intervals, and f is a continuous function, then we may use the Intermediate Value Theorem to construct numbers $s \in [a, c]$ and $t \in [b, c]$ such that $s < t$, but $f(s) = f(t)$, contradicting that f is bijective.

Suppose $f(c) < f(a)$ and $f(c) < f(b)$. Since $[a, c]$ and $[b, c]$ are compact intervals, and f is a continuous function, then we may use the Intermediate Value Theorem to construct numbers $x \in [a, c]$ and $y \in [b, c]$ such that $x < y$, but $f(x) = f(y)$, again contradicting that f is bijective.

Then f must be monotonic. □