(1) *Proof.* Let $\varepsilon > 0$ be given.

Since $f_n \xrightarrow{u} f$, there exists some N such that for all $n > N$ and for all $x \in A$, then $|f_n(x) - f(x)| < \varepsilon/3$. Particularly, for $n = N + 1$, then $|f_{N+1}(x) - f(x)| < \varepsilon/3$.

Since f_{N+1} is uniformly continuous on A, then there exists $\delta > 0$ such that for all $x, y \in A$, if $|x - y| < \delta$, then $|f_{N+1}(x) - f_{N+1}(y)| < \varepsilon/3$.

For all $x, y \in A$ such that $|x - y| < \delta$, then

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_{N+1}(x) + f_{N+1}(x) - f_{N+1}(y) + f_{N+1}(y) - f(y)| \\ &\le |f_{N+1}(x) - f(x)| + |f_{N+1}(x) - f_{N+1}(y)| + |f_{N+1}(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

Then f is uniformly continuous on A. □

(2) *Proof.* (i) $\implies$ (ii) Suppose $f_n \xrightarrow{u} f$. Let $\varepsilon > 0$ be given.

Since $f_n \xrightarrow{u} f$, there exists N such that for all $n > N$ and for all $x \in A$, then $|f_n(x) - f(x)| < \varepsilon/2$. Since this holds for all $x \in A$, it follows that $\sup |f_n(x) - f(x)| \le \varepsilon/2 < \varepsilon$. Then (ii) is true.

(ii) $\implies$ (iii) Let $\varepsilon > 0$ be given. Then there exists N such that for all $n > N$, then $\sup_{x\in A}|f_n(x) - f(x)| < \varepsilon$. Then $|\sup_{x\in A}|f_n(x) - f(x)| - 0| < \varepsilon$. This is the definition of (iii), thus (iii) holds.

(iii) $\implies$ (i) Suppose $\lim_{n\to\infty}(\sup_{x\in A}|f_n(x) - f(x)|) = 0$. Let $\varepsilon > 0$ be given.

From the assumed limit, then there exists some N such that for all $n > N$, we have that $\sup_{x\in A}|f_n(x) - f(x)| < \varepsilon$. By definition, for all $x \in A$, we have that $|f_n(x) - f(x)| \le \sup_{x\in A}|f_n(x) - f(x)|$. Then $|f_n(x) - f(x)| < \varepsilon$ when $n > N$. Then $f_n \xrightarrow{u} f$. □

(3) Since $\lim a_n \ge \lim b_n$ by the Order Limit Theorem, and $\lim b_n > 0$, then $\lim a_n \ne 0$.

(4-1) *Proof.* Suppose the convergence is uniform. Then $\lim_{n\to\infty} \sup_{x\in\mathbb{R}}|f_n(x) - 0| = 0$. Consider the sequence $(f_n(n^2))$. Clearly $\sup_{x\in\mathbb{R}}|f_n(x) - 0| \ge |f_n(n^2)| = \frac{1}{2}$. But $f_n(n^2) \to \frac{1}{2}$, which means that $\sup_{x\in\mathbb{R}}|f_n(x) - 0| \not\to 0$. Then the convergence is not uniform. □

(4-2) *Proof.* Suppose the convergence is uniform. Then $\lim_{n\to\infty} \sup_{x\in[0,1)} |f_n(x) - 0| = 0$. Consider the sequence $(f_n(1 - n^{-1}))$. Clearly $\sup_{x\in[0,1)} |f_n(x) - 0| \ge |f_n(1 - n^{-1})|$. But $f_n(1 - n^{-1}) \to e^{-1}$, which means that $\sup_{x\in[0,1)} |f_n(x) - 0| \not\to 0$. Then the convergence is not uniform. □

(5) *Proof.* Let $\varepsilon > 0$ be given.

Since $f_n \xrightarrow{u} f$ on G, there exists N_1 such that for all $n > N$ and for all $x \in G$, then $|f_n(x) - f(x)| < \varepsilon$.

Since $f_n \xrightarrow{u} f$ on H, there exists N_2 such that for all $n > N_2$ and for all $x \in H$, then $|f_n(x) - f(x)| < \varepsilon$.

Let $N = \max(N_1, N_2)$. Then for all $n > N$ and for all $x \in A = G \cup H$, we have that $|f_n(x) - f(x)| < \varepsilon$. Then $f_n \xrightarrow{u} f$ on A. □

(6) *Proof.* Let $\varepsilon > 0$ be given.

Since $a_n \to a$, there exists N such that for all $n > N$, then $|a_n - a| < \varepsilon$. Then for $n > N$ and for all $x \in A$, we have

$$\begin{aligned} |f_n(x) - (f + a)(x)| &= |(f + a_n)(x) - (f + a)(x)| \\ &= |f(x) + a_n - f(x) - a| \\ &= |a_n - a| \\ &< \varepsilon \end{aligned}$$

Then $f_n \xrightarrow{u} (f + a)$. □

(7) *Proof.* Let $\varepsilon > 0$ be given.

Since $g_k \xrightarrow{u} g$ on A, there exists N such that for all $k > N$ and for all $x \in A$, then $|g_k(x) - g(x)| < \varepsilon/2$. Then for all $k > N$ and for all $x \in A$:

$$\begin{aligned} |h_k(x) - 0| &= |g_{k+1}(x) - g_k(x)| \\ &= |g_{k+1}(x) - g(x) + g(x) - g_k(x)| \\ &\leq |g_{k+1}(x) - g(x)| + |g_k(x) - g(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Then $h_k \xrightarrow{u} 0$. □