(1) Consider the function $f(x) = x\sin\left(\frac{1}{x}\right)$.

(2) *Proof.* By continuity of f at c, there then exists $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \frac{f(c)}{2}$. Then

$$|f(x) - f(c)| < \frac{f(c)}{2} \implies -\frac{f(c)}{2} < f(x) - f(c) < \frac{f(c)}{2} \implies \frac{f(c)}{2} < f(x) < \frac{3f(c)}{2}$$

Since $f(c) > 0$, the conclusion follows. □

(3) *Proof.* It follows that

$$\lim_{x\to c} h(x) = \lim_{x\to c} \frac{g(x) - g(c)}{x - c} = g'(c) = h(c)$$

Then h is continuous at c. □

(4) *Proof.* Recall that a function with domain A has a local minimum at a point $x = c$ if there exists some $\delta > 0$ such that for all $x \in V_\delta(c) \cap A \implies f(x) \geq f(c)$.

Let $\varepsilon > 0$ be given. Consider the function as follows:

$$g(x) = \begin{cases} \frac{f'(x) - f'(c)}{x - c} & x \neq c \\ f''(c) & x = c \end{cases}$$

By (3), g is continuous at c. By definition, $g(c) = f''(c) > 0$. By (2), there exists $\delta > 0$ such that for all $x \in V_\delta(c)$, then $g(x) > 0$. We now consider two cases:

(a) Suppose $x \in (c - \delta, c)$. Then $x - c < 0$. If $g(x) > 0$, then $f'(x) < 0$. Then f is decreasing on this interval, and $f(x) \geq f(c)$.

(b) Suppose $x \in (c, c + \delta)$. Then $x - c > 0$. If $g(x) > 0$,, then $f'(x) > 0$. Then f is increasing on this interval, and $f(x) \geq f(c)$.

By definition, there is then a local min at c. □

(5) *Proof.* To prove that $f_n \xrightarrow{p} f$, consider some $c \in [0, 1]$. We now discuss two cases:

(a) Suppose $c \in \mathbb{R}\backslash\mathbb{Q}$. Then for any $n \in \mathbb{N}$, we have that $f_n(c) = 0$. Then $(f_n(c)) = (0)$, which obviously converges to 0. Then $f(c) = 0$.

(b) Suppose $c \in \mathbb{Q}$. Then $c = p/q$ for some $p, q \in \mathbb{N}$ such that $p \leq q$. Observe that for $n!c$ to be an integer, then $n!p/q = k$ for some $k \in \mathbb{N}$. Then $n!p = kq$. Choosing $n > q$, then $q \mid n! \implies n!c \in \mathbb{N}$. Then $(f_n(c)) = (1)$, which obviously converges to 1. Then $f(c) = 1$.

Hence $f_n \xrightarrow{p} f$. □