A Framework for Branching Quantum States

Dr. Mark Rudolph - US Geological Survey

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1 Abstract

The concept of infinity long ago led to an understanding that proper subsets of an infinite set can be put in a one-to-one correspondence with the original set. This mechanism of 'parts' of a set 'duplicating' the original set took on deeper significance when in 1914 Hausdorff showed that the three-dimensional sphere S^2 can be decomposed into a finite number of pairwise disjoint subsets which may then be transformed by simple rotations to create two identical copies of S^2 [1].

Quantum superposition induces a branching of possibilities when another quantum system entangles with the superposition states. This phenomenon is explored in its minimal form using the geometric representation of the state of a single qubit by means of the Bloch Sphere (S^2) [2].

We present a framework for branching quantum possibilities by considering the entanglement with a single qubit in a superposition of two states (for example following the application of a Hadamard gate to a single basis state.) Using the Bloch Sphere to represent the qubit state space, we examine the mechanics used in the proof of the Hausdorff Theorem and suggest ontological implications with regard to the quantum branching of relative states.

2 Introduction

In 1914 Hausdorff published 'Bemerkung über den Inhalt von Punktmengen' in Mathematische Annalen and included it in his book Grundzüge der Mengenlehre [1]. The paper contains a theorem which states that S^2 (the surface of a 3D ball) can be decomposed into finitely many pairwise disjoint subsets which can be rotated and re-assembled into two identical instances of S^2. This 'Hausdorff Theorem' was a striking result which challenged intuitive notions of measure and surface area, and preceded the more famous 'Banach-Tarski Theorem' (1924) involving solid 3D balls and volumes. Hausdorff's insights are critical in understanding how Lebesgue non-measurable sets can form and re-assemble with unexpected consequences under rotational group actions.

The partition and duplication of S^2 presents an intriguing framework for understanding the branching of relative state possibilities for a qubit superposition entangled by another quantum system. Once defined, the partitions of S^2 never coincide, just as relative state branches formed by outside systems entangling with component states of a superposition, also never coincide.

Quantum measurement is one such relative state entanglement, but any quantum interaction can cause entanglement, so measurement is not a 'special case' and observers do not play a special role. In addition, relative state entanglement and branching eliminate any mysterious 'collapse' of the superposition and instead preserve continuous determinism in quantum mechanics.

3 Main

Let S^2 denote the unit sphere in $\mathbb{R}^3$, and let SO_3 denote the special orthogonal group in three dimensions, the group of all possible rotations about the origin in three-dimensional Euclidean space. SO_3 consists of all 3×3 matrices that preserve distances and orientations in 3D space under rotation. These matrices have two key properties: (a) Orthogonality: matrix M in SO_3 satisfies $M^{-1} = M^T$. (b) det(M)=1: rotations preserve orientation whereas reflections reverse orientation and have determinant -1.

A free group of rank 2, denoted by F_2, is the group generated by two symbols, say $\{x, y\}$, which consists of all possible finite sequences ('words') formed by concatenating the generators $\{x, y\}$ and their inverses $\{x^{-1}, y^{-1}\}$. It is assumed that all words in F_2 are 'reduced' meaning that they do not contain any concatenated pairs consisting of an element and its inverse. The group operation is left-concatenation followed by possible reduction. All free groups of the same rank are isomorphic so it is common to speak of 'the' free group of rank two F_2.

The proof of the Hausdorff Theorem given below roughly follows Weston [3, sections 3-4] and Wagon [4, chapters 1-3]. First recall that a group G acts on a set X if to each g ∈ G there corresponds a bijection X → X, also denoted by g, such that for all g,h ∈ G and x ∈ X, g(h(x)) = (gh)(x), and e(x) = x where e is the identity element of G. We begin by defining the key concept underlying all duplicative partitions of geometric spaces.

Definition 1: Let G be a group acting on a set X, and suppose E is contained in X. E is 'G-paradoxical' if for some natural numbers n and m there are pairwise-disjoint subsets of E A_1,...,A_n and B_1,...,B_m, and elements of G g_1,...,g_n and h_1,...,h_m such that E = $\bigcup_{i\leq n} g_i A_i$ and E = $\bigcup_{j\leq m} h_j B_j$

Then the Hausdorff Theorem may be stated as follows:

Theorem 1: There is a countable subset E of S^2 such that S^2-E is SO_3-paradoxical. (Note that a more elaborate but inelegant version of the proof absorbs E into one of the other partitions yielding a simpler statement that S^2 is SO_3-paradoxical. However in the present context of quantum mechanics, since the countable set E has measure zero, it can be disregarded from a probabilistic perspective.)

The Theorem defines a partition of S^2 into Lebesgue non-measurable sets via use of the Axiom of Choice (AC) (the role and significance of AC is discussed in section 4.) First it is shown that the free group F_2 is F_2-paradoxical. Then a pair of independent rotations is defined in SO_3 which generate a free group of rank two in SO_3. It is then shown that the free group in SO_3 can be used to partition S^2 into Lebesgue non-measurable sets A,B,C and D, and countable set E such that there exist rotations g and h in the free group contained in SO_3 with S^2-E = A $\sqcup$ gB, and S^2-E = C $\sqcup$ hD, and the unions are disjoint.

Note that S^2-E (or S^2) being SO_3-'paradoxical' is a denotative term and does not imply that the duplicative partition and re-assembly of subsets of S^2 is a mathematical paradox in which a statement contradicts itself or leads to an inconsistency. Furthermore, since the duplicative mechanics of the Hausdorff Theorem may be at the heart of the 'relative state' entanglement of quantum systems with superpositions, the methodology of the theorem may offer a new ontological intuition. (more discussion follows in section 4)

Let us begin the proof of Theorem 1 (Hausdorff Theorem). We begin by examining the free group F_2 of rank 2. This free group will play a key role in the methodology of the proof. Define W(x) to be all reduced words in F_2 which begin with x. Define W(y), W(x^{-1}) and W(y^{-1}) similarly.

(1) F_2 = e $\sqcup$ W(x) $\sqcup$ W(y) $\sqcup$ W(x^{-1}) $\sqcup$ W(y^{-1}) where e is the 'empty word' identity of F_2. The unions are pairwise disjoint.

Now consider the set x^{-1}W(x), all words of W(x) left-concatenated by x^{-1}.

Lemma 1: x^{-1}W(x) = e $\sqcup$ W(x) $\sqcup$ W(y) $\sqcup$ W(y^{-1})
Let w $\in$ W(x). Then w is a reduced word beginning with x so w cannot begin xx^{-1}, so x^{-1}w cannot begin with x^{-1}. Therefore x^{-1}W(x) $\cap$ W(x^{-1}) = $\emptyset$.

(2) Then by **(1)**, x^{-1}W(x) $\subseteq$ e $\sqcup$ W(x) $\sqcup$ W(y) $\sqcup$ W(y^{-1})

Continuing, let v = e $\sqcup$ W(x) $\sqcup$ W(y) $\sqcup$ W(y^{-1})
(a) If v=e, v $\in$ x^{-1}W(x) so x^{-1}W(x) contains e.
(b) Suppose v$\in$W(y). Then v begins with y, so xv $\in$ W(x). Then since v=x^{-1}xv $\in$ x^{-1}W(x), x^{-1}W(x) contains W(y).
(c) Next suppose v $\in$ W(x) then xv $\in$ W(x) also. Then since v=x^{-1}xv $\in$ x^{-1}W(x), x^{-1}W(x) contains W(x).
(d) finally suppose v $\in$ W(y^{-1}). Then v begins with y^{-1} so xv $\in$ W(x). Then since v=x^{-1}xv $\in$ x^{-1}W(x), x^{-1}W(x) contains W(y^{-1}).

(3) By **(a)-(d)** x^{-1}W(x) contains e $\sqcup$ W(x) $\sqcup$ W(y) $\sqcup$ W(y^{-1}).

Then by **(2)** and **(3)** the lemma is proved.

In a completely parallel demonstration **lemma 1** can also be established for W(y), W(x^{-1}) and W(y^{-1}):
y^{-1}W(y) = e $\sqcup$ W(y) $\sqcup$ W(x) $\sqcup$ W(x^{-1})
xW(x^{-1}) = e $\sqcup$ W(x^{-1}) $\sqcup$ W(y) $\sqcup$ W(y^{-1})
yW(y^{-1}) = e $\sqcup$ W(y^{-1}) $\sqcup$ W(x) $\sqcup$ W(x^{-1}) and

Theorem 2: F_2 is F_2-paradoxical. Thus two proper subsets of F_2 can be operated on and re-assembled to produce two copies of F_2:
F_2 = x^{-1}W(x) $\sqcup$ W(x^{-1}), and
F_2 = y^{-1}W(y) $\sqcup$ W(y^{-1}).

By **(1)** F_2 = e $\sqcup$ W(x) $\sqcup$ W(x^{-1}) $\sqcup$ W(y) $\sqcup$ W(y^{-1})
By **Lemma 1** x^{-1}W(x) = e $\sqcup$ W(x) $\sqcup$ W(y) $\sqcup$ W(y^{-1})
Then F_2 = x^{-1}W(x) $\sqcup$ W(x^{-1})
Similary, by a **Lemma 1** variant, y^{-1}W(y) = e $\sqcup$ W(x) $\sqcup$ W(x^{-1}) $\sqcup$ W(y)
Then again by **(1)** F_2 = y^{-1}W(y) $\sqcup$ W(y^{-1}).

Note that AC is not used in the proof of **Theorem 2**.
Note also that F_2 is countable since it is the union of a countable collection of finite sets of words (the words of length n for all n.)

Next we define two independent rotations which generate a free group of rank 2 in SO_3.

Let ϕ be defined by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & \frac{-2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & 1/3 \end{bmatrix}$$

Let ψ be defined by

$$\begin{bmatrix} 1/3 & \frac{-2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $\phi^{-1} = \phi^T$ and $\psi^{-1} = \psi^T$

Note that ϕ is a counter-clockwise rotation by acos(1/3) about the x-axis (pitch), and ψ is a counter-clockwise rotation by acos(1/3) about the z-axis (roll). We claim that $F_2(\phi, \psi)$, defined analogously to $F_2 = F_2(x, y)$, is a free group of rank 2 in SO_3. We omit the lengthy and algebraically tedious proof details, but they can be found in Wagon [5, Theorm 2.1].

We will use the F_2-paradoxical nature of $F_2(\phi, \psi)$ in SO_3 acting on S^2 to define a paradoxical partition and re-assembly of subsets of S^2.

Definition 2: For p $\in S^2$ the orbit of p with respect to a group G acting on S^2 is orbit(p) = {g(p) | g $\in$ G}.

The action of $F_2(\phi, \psi)$ on S^2 induces an equivalence relation on S^2 where p,q $\in S^2$ are equivalent, i.e. p$\sim$q, if $\exists$ $\alpha \in F_2(\phi, \psi)$ such that α(p) = q, i.e. p and q have the same orbit.

Since $F_2(\phi, \psi)$ is countable, and since orbit(p) = {α(p) | $\alpha \in F_2(\phi, \psi)$}, orbit(p) is also countable. Then because S^2 contains a continuum of points there must be uncountably many orbits.

Proposition 1: Consider the $F_2(\phi, \psi)$-orbits of S^2. Let p and q be points of S^2. If orbit(p) $\neq$ orbit(q) then orbit(p) $\cap$ orbit(q) = $\emptyset$.

Assume orbit(p) $\neq$ orbit(q) but $\exists$ s $\in$ orbit(p) $\cap$ orbit(q). Then $\exists$ α and β in $F_2(\phi, \psi)$ with α(s) = p and β(s) = q. Then $\beta\alpha^{-1}$p = q, so orbit(p) = orbit(q) a contradiction.

It is at this point we use the Axiom of Choice (AC) to select a single point from each of the uncountable $F_2(\phi, \psi)$-orbits. Call this uncountable choice set M.

From **(1)** we have $F_2(\phi, \psi)$ = e $\sqcup$ W(ϕ) $\sqcup$ W(ϕ^{-1}) $\sqcup$ W(ψ) $\sqcup$ W(ψ^{-1}).

For any $\alpha \in F_2(\phi, \psi)$ we define the set αM = {α(p) | p $\in$ M}. Then $F_2(\phi, \psi)$M = {α(p) | $\alpha \in F_2(\phi,\psi)$ and p $\in$ M} is S^2 since M contains a point in every $F_2(\phi, \psi)$-orbit. Then we have the following decomposition of S^2:

(5) $F_2(\phi, \psi)$M = eM $\sqcup$ W(ϕ)M $\sqcup$ W(ϕ^{-1}) $\sqcup$ W(ψ) $\sqcup$ W(ψ^{-1}).

However the unions are no longer pairwise disjoint. For example, if p is one of the poles of $S^2 \cap$ X-axis then p $\in$ W(ϕ)M and p $\in$ W(ϕ^{-1})M.

We will remove a countable set E from S^2 in order to define a pairwise-disjoint decomposition of S^2-E. Let E = {p $\in S^2$ | $\exists\ \alpha \in F_2(\phi, \psi)$ - {e} with α(p) = p}. E consists of all points of S^2 lying on axes of rotation of elements of $F_2(\phi, \psi)$. Since $F_2(\phi, \psi)$ is countable and there are exactly two poles on each axis of rotation, E is countable.

We now consider the action of $F_2(\phi, \psi)$ on S^2 - E. First we show that the action maps to S^2-E so the action is well-defined. Assume $\exists$ p $\in S^2$ - E and a rotation α in $F_2(\phi, \psi)$ with α(p) $\in$ E. By the definition of E $\exists\ \sigma$ in $F_2(\phi, \psi)$ - {e} with $\sigma\alpha$(p) = α(p). Then $\alpha^{-1}\sigma\alpha$(p) = $\alpha^{-1}\alpha$(p) = p, so p $\in$ E, a contradiction, so the action of $F_2(\phi, \psi)$ on S^2 - E produces points in S^2 - E, so is well-defined.

Continuing, we again use AC to choose a single representative point in each orbit of the action of $F_2(\phi, \psi)$ on S^2 - E. Let M' be the set of chosen representatives. To complete the proof of Theorem 1, i.e. the Hausdorff Theorem for S^2-E, the partition of $F_2(\phi, \psi)$ given in **(1)** is shown to induce a similar partition in S^2-E. For this we need the following:

Proposition 2: Let α_1 and α_2 be distinct elements in $F_2(\phi, \psi)$. Then $\alpha_1 M' \cap \alpha_2 M' = \emptyset$

Assume α_1 and $\alpha_2 \in F_2(\phi, \psi)$ and $\alpha_1 M' \cap \alpha_2 M' \neq \emptyset$. Then there is a point p in S^2-E with p in $\alpha_1 M'$ and p in $\alpha_2 M'$. Then there are points p_1 and p_2 in M' such that p=$\alpha_1(p_1)$ and p=$\alpha_2(p_2)$. From p=$\alpha_2(p_2)$ we obtain α_1^{-1}p = $\alpha_1^{-1}\alpha_2(p_2)$. Then $\alpha_1^{-1}(\alpha_1(p_1)) = \alpha_1^{-1}\alpha_2(p_2)$, thus $p_1 = \alpha^{-1}\alpha_2(p_2)$.
Therefore p_1 and p_2 belong to the same orbit.
Since M' contains a unique representative from each orbit, $p_1 = p_2$.
But then p_1 is a fixed point of $\alpha^{-1}\alpha_2$.
However, since $p_1 \in M'$, $p_1 \in S^2$-E, but then to be a fixed point, $\alpha_1^{-1}\alpha_2$ = e, so $\alpha_1 = \alpha_2$.
Thus if $\alpha_1 M' \cap \alpha_2 M' \neq \emptyset$, $\alpha_1 = \alpha_2$, so $(\alpha_1 \neq \alpha_2) \rightarrow (\alpha_1 M' \cap \alpha_2 M') = \emptyset$.

By **Propositon 1**, the partition given in **(1)** denoted in terms of $F_2(\phi, \psi)$ is:
$F_2(\phi, \psi) = \mathrm{e} \sqcup \mathrm{W}(\phi) \sqcup \mathrm{W}(\phi^{-1}) \sqcup \mathrm{W}(\psi) \sqcup \mathrm{W}(\psi^{-1})$.

Therefore S^2-E $= F_2(\phi, \psi)\ M' = eM' \sqcup \mathrm{W}(\phi)M' \sqcup \mathrm{W}(\phi^{-1})M' \sqcup \mathrm{W}(\psi)M' \sqcup \mathrm{W}(\psi^{-1})M'$.

Now we can apply **Theorem 2** to induce a $F(\phi, \psi)$-paradoxical partition of S^2-E:
$S^2 - E = \phi^{-1}(\mathrm{W}(\phi)M') \sqcup \mathrm{W}(\phi^{-1})M'$
$S^2 - E = \psi^{-1}(\mathrm{W}(\psi)M') \sqcup \mathrm{W}(\psi^{-1})M'$

which is the Hausdorff Theorem for S^2-E.

4 Discussion

A key question for the application of a Hausdorff Theorem based mechanism for branching quantum states is whether the results for the qubit state space can be generalized to higher-dimensional Hilbert spaces, and to superpositions of cardinal greater than two.

It is beyond the scope of this paper to develop mechanics for quantum superposition branching in all physical cases and contexts. However, there are intriguing mathematical results which suggest ample paradoxical partitioning capability in higher dimensional and infinite dimensional cases. First, by de Groot and Decker [5], SO_n $n \geq 3$ contains free groups of arbitrary rank including uncountable rank. The proof uses the axiom of choice and extends Hausdorff's 1914 construction for SO_3 (see proof in section 3.) via transfinite induction. Second, by Wagon [4, Corollary 6.9], assuming AC, for any cardinal κ such that $3 \leq \kappa \leq 2^{\aleph_0}$ and any $n \geq 1$, S^n is κ-divisible with respect to SO_{n+1}.

Note: a set X is κ-divisible with respect to a group G acting on X if X splits into κ pairwise disjoint G-congruent subsets.

Note: A,B members of X are G-congruent if there exists a g$\in$G such that B = gA, i.e. A can be transformed into B by the action of an element g$\in$G.

It should also be noted also that S^1 is not SO_2-paradoxical, but is countably-SO_2-paradoxical (Wagon [4, Theorem 1.5]) where Definition 1 (see section 3.) is extended to allow countably many pairwise-disjoint pieces. Thus E is countably G-paradoxical if there exist pairwise disjoint $\{A_i\}_{i=0,...}$ and $\{B_i\}_{i=0,...}$ in E, and

$\{g_i\}_{i=0,\ldots}$ and $\{h_i\}_{i=0,\ldots}$ in G such that E=$\bigcup_{i<\omega} g_i A_i$ and E=$\bigcup_{i<\omega} h_i B_i$. From this result we conclude that the set of quantum phases is countably-paradoxical.

A second key question is the role of the Axiom of choice (AC) in the mechanics of branching relative states. It is known that the Hausdorff Theorem is not a theorem in ZF + ¬AC, but is (as shown in section 3.) a theorem of ZF + AC. By Solovay [6] there is a model of ZF in which all sets of reals are Lebesgue measurable, and hence the Hausdorff Theorem is false, but it involves assuming the existence of an 'inaccessible cardinal' (IC). However the consistency of ZF+IC cannot be proved. This is a deep subject and beyond the scope of this paper, but a preliminary discussion can be found in Wagon [4, chapter 13].

It may be tempting to reject the Axiom of Choice to remove the mechanism for the deterministic generation of quantum relative state branching, but the consequences of that rejection are stark and disturbing. A brief listing of cautionary consequences begins by noting that the existence of orthonormal bases in infinite-dimensional Hilbert spaces depends on Zorn's Lemma, which is equivalent to AC. Further, many results in functional analysis and quantum mechanics rely on the Hahn-Banach Theorem, also equivalent to AC.

More basic to quantum mechanics is that many results in probability theory such as countable additivity and the Law of Large Numbers rely on one form of AC or another. Even the existence of a basis in a vector space is in question. In ZF (i.e. without AC), it is consistent that there exist vector spaces that do not have a basis. Further, without AC, there exist linearly independent sets that cannot be extended to a basis, causing many classical proofs to fail. Also, without AC, it is possible for a vector space to have two bases having different cardinals, breaking the uniqueness of basis dimension. For a discussion of many difficulties after rejection of AC see Jech [7, chapter 10].

References

[1] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig: Veit and Comp., 1914.

[2] F. Bloch, 'Nuclear Induction', Physical Review 1946, volume 70, issue 7, pp 460-474.

[3] T. Weston, 'The Banach-Tarski Paradox,' Unpublished manuscript, Available: https://citeseerx.ist.psu.edu/document?repid=rep1&type=pdf&doi=5ab85e37c6e1da73049610cf [Accessed: Mar. 9, 2025].

[4] S. Wagon, *The Banach-Tarski Paradox*, Cambridge University Press, 1985.

[5] J. de Groot, J., and T. Dekker, 'Free subgroups of the orthogonal group.' Compositio Mathematica 12 (1954-1956): 134-136.

[6] R. M. Solovay, 'A Model of Set Theory in which every set of reals is Lebesgue measurable', Ann. Math. 92 (1970), 1-56.

[7] T. Jech, *Axiom of Choice*, originally published: Amsterdam: North-Holland Pub. Co., 1973.