Computational Logic – Fall 2010	October 11, 2010
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Problem 1

Claim 1. *Let* $\overline{X} := \bigcap\{\, Y \subseteq X : Y$ *is* $\mathcal{X}$*-closed* $\}$*.* $\overline{X}$ *is* $\mathcal{X}$*-closed.*

Proof. It suffices to show $X_0 \subseteq \overline{X}$ and for all $H \in \mathcal{H}$ and all $x_0, \ldots, x_{k_{H-1}} \in \overline{X}$ also $H(x_0, \ldots, x_{k_{H-1}}) \in \overline{X}$. By definition, $X_0 \subseteq Y$ for all Y such that Y is $\mathcal{X}$-closed. Then, $X_0 \subseteq \bigcap\{\, Y \subseteq X : Y$ is $\mathcal{X}$-closed $\}$ and thus, $X_0 \subseteq \overline{X}$. Let $H \in \mathcal{H}$ be arbitrarily chosen and suppose $x_0, \ldots, x_{k_{H-1}} \in \overline{X}$. Then, $x_0, \ldots, x_{k_{H-1}}$ must also be in each Y for $\bigcap\{\, Y \subseteq X : Y$ is $\mathcal{X}$-closed $\}$, and since each Y is $\mathcal{X}$-closed

$$H(x_0, \ldots, x_{k_{H-1}}) \in \bigcap\{\, Y \subseteq X : Y \text{ is } \mathcal{X}\text{-closed } \}$$

Therefore,

$$H(x_0, \ldots, x_{k_{H-1}}) \in \overline{X}$$

□

Problem 2

Claim 2. *Define* $G^* : {}^{\omega}\omega \times \omega \to {}^{\omega}\omega$ *as:*

$$G^*(f, m)(0) = f(1)$$
$$G^*(f, m)(n+1) = f(G^*(f, m)(n))$$

Then G^* *is a function.*

Proof. From Theorem 1.2.12 it follows that G^* is a uniquely defined function by setting $F(0) = f(1)$ and $F(n+1) = G(F(n), n)$ where $G(z, n) = f(z)$. □

Claim 3. *There exists a unique function* $A : \omega \times \omega \to \omega$ *such that for all* $p \in \omega$ *and* $n \in \omega$*,*

$$A(0, n) = n + 1$$
$$A(p+1, 0) = A(p, 1)$$
$$A(p+1, n+1) = A(p, A(p+1, n))$$

Proof. Let $Z = {}^{\omega}\omega$ and let $z_0(n) = n + 1 \in Z$ and let $G^* : Z \times \omega \to Z$ be the function. defined in Claim 2. Then, by Theorem 1.2.12, $A^* : \omega \to Z$ is a unique function such that

$$A^*(0)(n) = z_0(n) = n + 1$$

and for all $p \in \omega$,

$$A^*(p+1)(n) = G^*(A^*(p), p)(n)$$

Then, define $A(p, n) = A^*(p)(n)$, which satisfies the claim. □

Problem 3

Claim 4. *For any sentences ϕ and ψ, $\models \phi \wedge \psi \iff \models \phi$ and $\models \psi$.*

Proof. Suppose $\models \phi \wedge \psi$. Then for all truth assignments, $V(\phi \wedge \psi) = T$, which holds if and only if $V(\phi) = T$ and $V(\psi) = T$. Therefore $\models \phi$ and $\models \psi$ follow by definition. Now suppose $\models \phi$ and $\models \psi$. Then for all truth assignments, $V(\phi) = T$ and $V(\psi) = T$. Therefore $V(\phi \wedge \psi) = T$ and thus $\models \phi \wedge \psi$ by definition. $\square$

Claim 5. *For any sentences ϕ and ψ, $\models \phi \vee \psi \iff \models \phi$ or $\models \psi$.*

Proof by counter example. Consider any sentence ϕ and let $\psi = \neg\phi$. Clearly, $\models \phi \vee \psi$ (the law of excluded middle). However, there is a truth assignment $V(\phi) = T$ which breaks the tautology $\models \psi$ and a truth assignment $W(\phi) = F$ which breaks the tautology $\models \phi$. Thus, the claim does not hold in the left to right direction. $\square$

Problem 4

Claim 6. *The Sheffer stroke, $\{|\}$, is adequate.*

Proof. $Sent_{\{|\}}$ is the inductive closure of the set of atomic sentences under the successor function $H_|$. $Sent_{\{|\}}$ includes all sentences up to tautological equivalence such that for every sentence $\phi \in Sent_L$ there is a sentence $\phi* \in Sent_{\{|\}}$ such that $\phi \models\mid \phi*$. Define by $Sent_{\{\neg,\vee\}}$-recursion a function $^* : Sent_{\{\neg,\vee\}} \to Sent_{\{|\}}$. For atomic ϕ let $\phi^* = \phi$ and let

$$(\neg\phi)^* = (\phi^*|\phi^*)$$
$$(\phi \vee \psi) = (\phi^*|\phi^*)|(\psi^*|\psi^*)$$

We prove tautological equivalence by $Sent_{\{\neg,\vee\}}$-induction. For atomic sentences the claim is trivially satisfied. Now suppose the induction hypothesis holds for arbitrary sentences $\phi, \psi \in Sent_{\{\neg,\vee\}}$, i.e., $\phi \models\mid \phi^*$ and $\psi \models\mid \psi^*$ for some $\phi^*, \psi^* \in Sent_{\{|\}}$. We argue that $\neg\phi \models\mid (\phi^*|\phi^*)$, and that $\phi \vee \psi \models\mid (\phi^*|\phi^*)|(\psi^*|\psi^*)$.

Assume $V(\neg\phi) = T$. Then $V(\phi) = F$ and from the induction hypothesis we know $V(\phi^*) = F$. By definition of the Sheffer stroke, $V(\phi^*|\phi^*) = T$. Alternatively suppose $V(\neg\phi) = F$. Then $V(\phi) = T$ and from the induction hypothesis we know $V(\phi^*) = T$. Therefore, from the definition of the Sheffer stroke, $V(\phi^*|\phi^*) = F$, and the claim holds. A similiar argument can be made for the opposite direction.

Assume $V(\phi \vee \psi) = T$. Then $V(\phi) = T$ or $V(\psi) = T$ and from the induction hypothesis we know $V(\phi^*) = T$ or $V(\psi^*) = T$. By definition of the Sheffer stroke, $V(\phi^*|\phi^*) = F$ or $V(\psi^*|\psi^*) = F$. Since one must be false, it follows from the defintion of the Sheffer stroke, that $V((\phi^*|\phi^*)|(\psi^*|\psi^*)) = T$. Alternatively suppose $V(\phi \vee \psi) = F$. Then, $V(\phi) = F$ and $V(\psi) = F$, and from the induction hypothesis we know $V(\phi^*) = F$ and $V(\psi^*) = F$. Therefore, $V(\phi^*|\phi^*) = T$ and $V(\psi^*|\psi^*) = T$, so $V((\phi^*|\phi^*)|(\psi^*|\psi^*)) = F$. Proof of the opposite direction is similar. $\square$

Problem 5

Claim 7. *For any truth assignment V and $\phi \in Sent_{\{\wedge,\vee,\to,\leftrightarrow\}}$, if $V(p_i) = T$ for all $p_i \in \phi$ then $V(\phi) = T$.*

Proof by $Sent_{\{\wedge,\vee,\to,\leftrightarrow\}}$ induction on ϕ. Suppose $\phi \in Sent_{\{\wedge,\vee,\to,\leftrightarrow\}}$ is atomic. Then the claim is trivially satisifed. Now suppose the claim holds for $\theta, \psi \in Sent_{\{\wedge,\vee,\to,\leftrightarrow\}}$ and that ϕ is of the form $\theta \bullet \psi$. If $\bullet = \wedge$, then for all V, $V(\theta \wedge \psi) = T$ since $V(\theta) = T$ and $V(\psi) = T$. If $\bullet = \vee$, then for all V, $V(\theta \vee \psi) = T$ since $V(\theta) = T$. If $\bullet = \to$, then $V(\theta \to \psi) = T$ since $V(\theta) = T$ and $V(\psi) = T$. Finally, if $\bullet = \leftrightarrow$, then $V(\theta \leftrightarrow \psi) = T$ since $V(\theta) = T$ and $V(\psi) = T$. □

Claim 8. *The set $\{\wedge, \vee, \to, \leftrightarrow\}$ is not adequate.*

Proof. If $Sent_{\{\wedge,\vee,\to,\leftrightarrow\}}$ is adequate then there exists a mapping $* : Sent_L \to Sent_{\{\wedge,\vee,\to,\leftrightarrow\}}$ from every sentence $\phi \in Sent_L$ to a sentence $\phi^* \in Sent_{\{\wedge,\vee,\to,\leftrightarrow\}}$ such that $\phi \mathrel{\text{⫤⊨}} \phi^*$. Then from Claim 7, for all $\phi \in Sent_L$ and all truth assignments V, if $V(p_i) = T$ for all $p_i \in \phi$, $V(\phi) = T$. However, let $\phi = \neg p_0 \wedge p_1$ and define $W(p_0) = T$ and $W(p_1) = T$. But $W(\neg p_0 \wedge p_1) = F$. It follows that $Sent_{\{\wedge,\vee,\to,\leftrightarrow\}}$ is not adequate. □

Problem 6

Claim 9. $At(\phi) \cap At(\psi) \neq \emptyset$

Proof. From the assumption that $\not\models \neg\phi$, it follows that there exists some truth assignment V such that $V(\phi) = T$ and from the assumption that $\not\models \psi$, it follows that there exists some truth assignment W such that $W(\psi) = F$. Suppose by way of contradiction that $At(\phi) \cap At(\psi) = \emptyset$. Construct a truth assignment X for atomic sentences as follows: $X(p_i) = V(p_i)$ for all atomic sentences $p_i \in \phi$ and $X(p_j) = W(p_j)$ for all atomic sentences $p_j \in \psi$. Then $X(\phi) = V(\phi) = T$ and $X(\psi) = V(\psi) = F$. But this is a contradiction since we assumed $\phi \models \psi$. □

Claim 10. *Suppose that $\phi \models \psi$ but neither $\models \neg\phi$ nor $\models \psi$. Then there exists a sentence θ such that (i) $\phi \models \theta$, (ii) $\theta \models \psi$ and (iii) $At(\theta) \subseteq At(\phi) \cap At(\psi)$.*

Proof by induction on the number of atoms that appear in ϕ but not in ψ. Suppose $n = 0$, then $\theta = \phi$ and therefore,

$$\phi \models \theta$$
$$\theta \models \psi$$
$$At(\theta) \subseteq At(\phi) \cap At(\psi)$$

Now suppose the induction hypothesis holds for $k \geq 0$ and let $n = k$. Existentially quantify over one of the atoms, p_i that appear in ϕ but not in ψ

$$\phi' = \phi[T/p_i] \vee \phi[F/p_i]$$

Note that $\phi^{'} \mathrel{\models\!\!\!|} \phi$, but $\phi^{'}$ has one atom less than ϕ, and therefore the induction hypothesis applies to $\phi^{'}$, i.e., there exists a θ such that,

$$\phi^{'} \models \theta$$
$$\theta \models \psi$$
$$At(\theta) \subseteq At(\phi^{'}) \cap At(\psi)$$

Then,

$$\phi \models \theta$$
$$\theta \models \psi$$
$$At(\theta) \subseteq At(\phi) \cap At(\psi)$$

since $\phi \mathrel{\models\!\!\!|} \phi^{'}$. □

Problem 7

Claim 11. *Every finite set of sentences has an independent equivalent subset.*

Proof by induction on the cardinality of Γ. If $|\Gamma| = 0$, the claim holds trivially. Now suppose the induction hypothesis holds for $|\Gamma| = n$ and consider some Γ such that $|\Gamma| = n + 1$. If Γ is independent we are done since it is also equivalent to itself. Otherwise, there exists some sentence ϕ such that $\Gamma - \{\phi\} \models \phi$. But $|\Gamma - \{\phi\}| = n$ and therefore the induction hypothesis holds, i.e., there exists an independent equivalent subset $\Gamma^{'}$ of $\Gamma - \{\phi\}$. Note that $\Gamma^{'}$ is also an independent equivalent subset of Γ because $\Gamma - \{\phi\}$ and Γ are equivalent (since by assumption $\Gamma - \{\phi\} \models \phi$). □

Claim 12. *There exists an infinite set of sentences which has no independent equivalent subset.*

Proof. Let $\Gamma = \bigcup\{p_i\}$, for all atomic sentences p_i (and $i \in \omega$). Γ does not have an independent equivalent subset since for all $i \in \omega$, $\Gamma - \{p_i\} \not\models p_i$. □

Claim 13. *For every (finite or infinite) set of sentences, there exists an independent equivalent set (not necessarily a subset).*

Proof. Let Γ be a set of sentences. Define $\Gamma^I = \bigcap \{\Delta \mid \Delta \mathrel{\models\!\!\!|} \Gamma\}$. By construction Γ^I is the smallest equivalent set. Now suppose Γ^I is not independent. Then there exists some $\phi \in \Gamma^I$ such that $\Gamma^I - \{\phi\} \models \phi$. But $\Gamma^I - \{\phi\}$ is equivalent to Γ and smaller than Γ^I. This is a contradiction. It follows that Γ^I is an independent equivalent set of Γ. □