Ecuaciones diferenciales Lineales

(26)

Definición $\frac{dy}{dx} \leftarrow$ cambio de y con respecto a x

ED lineal en y, $a(x)y' + b(x)y = c(x)$

ED lineal en x, $a(y)x' + b(y)x = c(y)$

$\frac{dx}{dy} \leftarrow$ cambio de x con respecto a y.

* Para resolver ED lineales necesitamos encontrar un factor integrante.

~~Normalizamos la ED~~

Teorema (Cálculo de factor integrante)

La ED lineal $y' + p(x)y = g(x)$ tiene al menos un factor integrante de forma $\mu = e^{\int p(x)dx}$

Demostración

factor integrante

$\mu y' + \mu p(x)y = \mu g(x)$) Queremos escribirlo de esta forma.

Si lo $(\mu y)' = \mu g(x)$

regla del producto

desarrollamos

$\mu y' + y\mu' = \mu g(x)$

* Si comparamos la (1) y (3) solo necesitamos que $\mu' = \mu p(x)$

Suponemos que $\mu = \mu(x) \neq 0$ entonces,

$$\mu' = \mu p(x)$$

$$\frac{d\mu}{dx} = \mu p(x)$$

$$\frac{d\mu}{\mu} = p(x)dx$$

$$\ln|\mu| = \int p(x)dx + C$$, solo necesitamos un factor integrante, entonces $\underline{\underline{C=0}}$.

$$\ln|\mu| = \int p(x)dx$$

$$e^{\ln|\mu|} = e^{\int p(x)dx}$$

$$\mu = \pm e^{\int p(x)dx}$$ solo necesitamos $\underline{\text{uno}}$.

$$\mu = \underbrace{e^{\int p(x)dx}}_{\text{factor integrante}}$$

Argumentos técnicos: $p(x)$ es continua, está bien definida y $\mu = \mu(x) \neq 0$.

Como lo usamos para resolver ED Lineales? (28)

ED Lineal: $a(x), b(x), c(x)$ son coeficientes que pueden ser constantes o funciones de x.

$$a(x)y + b(x)y = c(x)$$

Normalizamos,

$$y' + \frac{b(x)}{a(x)}y = \frac{c(x)}{a(x)}, \quad a(x) \neq 0$$

$$p(x) = \frac{b(x)}{a(x)}, \quad q(x) = \frac{c(x)}{a(x)}$$

$$y' + p(x)y = q(x)$$

Calculamos el factor integrante que sabemos es:

$\mu = e^{\int p(x)dx}$ y multiplicamos la ED por el factor integrante.

$$\Rightarrow \underbrace{e^{\int p(x)dx}}_{\mu(x)} y' + \underbrace{e^{\int p(x)dx}}_{\mu(x)} p(x)y = \underbrace{e^{\int p(x)dx}}_{\mu(x)} q(x)$$

$$\frac{d}{dx}\left(e^{\int p(x)dx} \cdot y\right) = e^{\int p(x)dx} q(x)$$

$$d\left(e^{\int p(x)dx} \cdot y\right) = e^{\int p(x)dx} q(x)\, dx$$

$$\int d\left(e^{\int p(x)dx} \cdot y\right) = \int e^{\int p(x)dx} q(x)\, dx$$

$$e^{\int p(x)dx} y = \int e^{\int p(x)dx} q(x)\, dx$$

$$\Rightarrow y = e^{-\int p(x)dx} \int e^{\int p(x)dx} q(x)\, dx$$ Δ forma de la solución general.

Ejemplo $y' - 2xy = x$

$a(x) = 1, \quad b(x) = -2x, \quad c(x) = x \Big\}$ coeficientes

$\Rightarrow p(x) = -2x, \quad q(x) = x$

$\mu = e^{\int p(x)dx} = e^{-2\int x dx} = e^{-x^2} \Big\}$ Factor integrante

$$e^{-x^2} y' - e^{-x^2}(2xy) = x e^{-x^2}$$

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$$\left(e^{-x^2} y\right)' = e^{-x^2}\cdot x$$

$$e^{-x^2} y' + y(-2x)e^{-x^2} = x e^{-x^2}$$

$$\int \left(e^{-x^2} y\right)' = \int x e^{-x^2}\,dx$$

$$e^{-x^2} y = \int x e^{-x^2}\,dx \qquad u = -x^2$$

$$du = -2x\,dx$$

$$-\frac{du}{2} = x\,dx$$

$$e^{-x^2} y = -\frac{1}{2}\int e^{u}\,du$$

$$e^{-x^2} y = -\frac{1}{2} e^{u} + C$$

$$y = e^{x^2}\left(-\frac{1}{2} e^{-x^2} + C\right)$$

$$\underline{y = -\frac{1}{2} + C e^{x^2}} \longleftarrow \text{Solución general}$$

Ejemplo $xy'' + 2y' = e^x, \quad x > 0$

$E(x, y', y'')$ $\quad$ $\not\exists$ y ausente

$v = y' , \quad v' = y''$

$x v' + 2v = e^x \quad \leftarrow$ ED Lineal!

$a(x) = x, \quad b(x) = 2, \quad c(x) = e^x$

Normalizamos:

$$v' + \frac{2}{x} v = \frac{e^x}{x}, \quad p(x) = \frac{b(x)}{a(x)} = \frac{2}{x}$$

$$q(x) = \frac{c(x)}{a(x)} = \frac{e^x}{x}$$

$$\mu = e^{\int p(x)dx} = e^{2\int \frac{1}{x} dx} = e^{2\ln|x|} = \boxed{x^2}$$

$$x^2 v' + \frac{2}{x} \cdot x^2 v = \frac{e^x}{x} \cdot x^2$$

$$x^2 v' + 2xv = xe^x$$

$$(x^2 \cdot v)' = xe^x$$

$$\int (x^2 \cdot v)' = \int xe^x dx$$

$$x^2 v = \underbrace{\int xe^x dx}_{\text{por partes}}$$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$$= xe^x - \int e^x dx$$

$$= xe^x - e^x + C$$

$$= e^x(x-1) + C$$

$$\Rightarrow v = \frac{e^x}{x^2}(x-1) + Cx^{-2}$$

$$y' = \frac{e^x}{x^2}(x-1) + cx^{-2}$$

$$\int dy = \int\left(\frac{e^x}{x} - \frac{e^x}{x^2} + \frac{c}{x^2}\right)dx$$

$$y = \int \frac{e^x}{x}dx - \int \frac{e^x}{x^2}dx + c\int \frac{1}{x^2}dx$$

$$= -\frac{c}{x} + D$$

Δ

Δ por partes

$u = e^x \qquad dv = -\frac{1}{x^2}dx$

$du = e^x dx \qquad v = \frac{1}{x}$

$$-\int \frac{e^x}{x^2}dx = \frac{e^x}{x} - \int \frac{e^x}{x}dx$$

$$\Rightarrow y = \cancel{\int \frac{e^x}{x}dx} + \frac{e^x}{x} - \cancel{\int \frac{e^x}{x}dx} - \frac{c}{x} + D$$

$$y = \frac{e^x}{x} - \frac{c}{x} + D$$ $\leftarrow$ solución general

$$y(x) = \frac{e^x}{x} - \frac{c}{x} + D$$

$$y'(x) = x^{-1} \cdot e^x - c x^{-1} + D$$
$$= x^{-1} e^x + e^x(-x^{-2}) + c x^{-2}$$

$$y'(x) = \frac{e^x}{x} - \frac{e^x}{x^2} + \frac{c}{x^2}$$ ✓

Ejemplo: $(e^{2x}+1)(y'+y) = 1$

Normalizemos:

$$y' + y = \frac{1}{e^{2x}+1} \quad , \quad p(x) = 1, \; q(x) = \frac{1}{e^{2x}+1}$$

$$\mu = e^{\int dx} = e^x$$

$$e^x y' + y e^x = \frac{e^x}{e^{2x}+1}$$

$$(e^x \cdot y)' = \frac{e^x}{e^{2x}+1}$$

$$\int (e^x \cdot y)' = \int \frac{e^x}{e^{2x}+1} \, dx$$

$$e^x y = \int \frac{e^x}{(e^x)^2+1} \, dx$$

$u = e^x, \; du = e^x dx$

$$e^x y = \int \frac{du}{u^2+1}$$

trigonométrica

$$y = e^{-x} \int \frac{du}{u^2+1}$$

$$y = e^{-x} (\arctan(u) + c)$$

$$y = e^{-x} (\arctan(e^x) + c) \leftarrow$$ solución general

$$\int \frac{1}{a^2+u^2} du = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2-u^2}} du = \operatorname{arcsen}\left(\frac{u}{a}\right) + c$$

$$\int \frac{1}{u\sqrt{u^2-a^2}} du = \frac{1}{a} \sec^{-1}\left(\frac{u}{a}\right) + c$$

Recordatorio de algunos integrales con funciones trigonométricas.

Ejemplo $\quad y - xy' = \underbrace{y^2 e^y y'}$

este término hace que la ED no sea lineal en $y, \frac{dy}{dx}$ pero es posible que sea lineal en $x, \frac{dx}{dy}$

$$y = y^2 e^y y' + xy'$$

$$y = (y^2 e^y + x)\frac{dy}{dx}$$

$$y\frac{dx}{dy} = y^2 e^y + x$$

$$yx' - x = y^2 e^y, \qquad a(y) = y,\ b(y) = -1,\ c(y) = y^2 e^y$$

$$p(y) = \frac{b(y)}{a(y)} = \frac{-1}{y}$$

$$\underbrace{x' - \frac{1}{y}x = ye^y}_{\text{ED Lineal en } x,\ \frac{dx}{dy}} \qquad q(y) = \frac{c(y)}{a(y)} = \frac{y^2 e^y}{y}$$

$$\mu(y) = e^{\int p(y)dy} = e^{-\int \frac{1}{y}dy} = e^{-\ln|y|} = \boxed{y^{-1}}$$

$$y^{-1}\cdot x' - \frac{1}{y}x\cdot y^{-1} = ye^y\cdot y^{-1}$$

$$(y^{-1}\cdot x)' = e^y$$

$$\int (y^{-1} \cdot x)' = \int e^y dy$$

$$y^{-1}x = \int e^y dy$$

$$y^{-1}x = e^y + c$$

$$x = ye^y + cy$$

$$x(y) = y(e^y + c) \leftarrow \text{solución general}$$

Classwork

Ejemplo $x^2 dy + 2xy dx = \operatorname{sen}(2x) dx$

$$\frac{a \tan\theta}{a^2 + (a^2 \tan^2\theta)}$$

$$a^2 + a^2 \tan^2\theta$$

$$a^2(1 + \tan^2\theta)$$

$$x^2\frac{dy}{dx}+2xy=\text{sen}(2x)$$

$$x^2y'+2xy=\text{sen}(2x)$$

$$y'+\frac{2}{x}y=\frac{\text{sen}(2x)}{x^2}$$

$$\mu=e^{2\int\frac{1}{x}dx}=e^{2\ln x}=\boxed{x^2}$$

$$x^2y'+\frac{2}{x}yx^2=x^2\cdot\frac{\text{sen}(2x)}{x^2}$$

$$\int(x^2\cdot y)'=\int\text{sen}(2x)$$

$$x^2y=\int\text{sen}(2x)\,dx$$

$u=2x$

$du=2dx$

$\frac{du}{2}=dx$

$$x^2y=\frac{1}{2}\int\text{sen}(u)\,du$$

$$x^2y=\frac{1}{2}(-\cos(2x))+C$$

$$y=\frac{-\cos(2x)}{2x^2}+Cx^{-2}$$

ED reducibles a lineales (Bernoulli)

Una ED de Bernoulli se puede escribir como

$$y' + p(x)y = q(x)\,y^n \quad , \underbrace{n \neq 1}_{\text{esto hace que la ED sea no lineal.}}$$

~~ED se~~

Ejemplo:

① $\dfrac{dy}{dx} + \dfrac{1}{3}y = e^x y^4$

$y' + \frac{1}{3}y = e^x y^4$ } Mover y^4 al lado izq de la ecuación.

$y^{-4}y' + \frac{1}{3}y^{-3} = e^x$ } Al lado derecho solo queda una función en términos de x.

* Para resolver vamos a hacer la siguiente sustitución: $v = y^{-3} \Rightarrow v' = -3y^{-4}y'$

Ahora sustituyo:

$\cancel{-3}\ -\frac{1}{3}v' + \frac{1}{3}v = e^x$ } Multiplique por -3.

$v' - v = -3e^x$ } ED Lineal donde $p(x) = -1$

$\Rightarrow \mu = e^{\int p(x)dx} = e^{-\int dx} = \boxed{e^{-x}}$

$$(e^{-x}\cdot v)' = e^{-x}(-3e^{x})$$

$$\int (e^{-x}\cdot v)' = -3\int dx$$

$$e^{-x} v = -3x + c$$

$$v = -3xe^{x} + ce^{x}$$

$v = \frac{1}{y^3} \Rightarrow \frac{1}{y^3} = -3xe^{x} + ce^{x}$ } Solución General

(2) $y' + xy = xe^{-x^2}y^{-3}$ (*clase*)

$$y^3y' + xy^4 = xe^{-x^2}$$

$$v = y^4 \quad v' = 4y^3y'$$

$$\frac{1}{4}v' + xv = xe^{-x^2}$$

$v' + 4xv = 4xe^{-x^2}$ } ED Lineal

$$u = x^2 \quad du = 2x\,dx$$

$$\mu = e^{4\int x\,dx} = \boxed{e^{2x^2}}$$

$$= \frac{4}{2}\int e^{u}\,du$$

$$\int (e^{2x^2}\cdot v)' = \int 4x e^{2x^2} e^{-x^2}\,dx$$

$$= \frac{4}{2}e^{u} + 4C$$

$$e^{2x^2} v = 4\int x e^{x^2}\,dx$$

$$e^{2x^2} v = 2e^{x^2} + 4C$$

$\underset{y^4}{v} = 2e^{-x^2} + \underbrace{4C}_{A}\, e^{-2x^2}$ } Solución General

(3) $y' + 1 = 4e^{-y} \operatorname{sen} x$

$e^{y} y' + e^{y} = 4 \operatorname{sen} x$

$v = e^{y} \quad v' = e^{y} \cdot y'$

$v' + v = 4 \operatorname{sen} x$ } ED Lineal

$\mu = e^{\int dx} = \boxed{e^{x}}$

$\int (e^{x} \cdot v)' = 4 \int e^{x} \operatorname{sen} x \, dx$

$e^{x} v = \frac{4}{2} (\operatorname{sen} x \, e^{x} - \cos x \, e^{x}) + C$

$v = 2 \operatorname{sen} x - 2 \cos x + c$

$e^{y} = 2 \operatorname{sen} x - 2 \cos x + c e^{-y}$ } Solución General

ED Lineales (Riccatti)

Def.

Las ED's tiene forma:

$$y' + p(x)y = g(x) + r(x)y^2$$

Teorema:

Es conocida $y_1 = y_1(x)$ una solución particular no nula de la ED de Riccatti $y' + p(x)y = g(x) + r(x)y^2$

Entonces:

- El cambio de variable $y = y_1 + u$ transforma la ED de Riccatti a una ED de Bernoulli
- El cambio de variable $y = y_1 + \frac{1}{u}$ transforma la ED de Riccatti en una ED lineal de primer orden

Ejemplo

(1) $y' = \dfrac{2\cos^2 x - \operatorname{sen}^2 x + y^2}{2\cos x}$, $y_1 = \operatorname{sen} x$

~~$y' = \cos x - \frac{1}{2}$~~

~~$2\cos x\, y' = 2\cos^2 x - \operatorname{sen}^2 x + y^2$~~

$y' = \cos x - \frac{1}{2}\operatorname{sen} x \tan x + y^2$

$y' + \frac{1}{2}\operatorname{sen} x \tan x = \cos x + y^2$

$y = \underbrace{\operatorname{sen} x}_{y_1} + u$

~~$u \operatorname{sen} x + u\operatorname{se}$~~

$y' = \underbrace{\cos x}_{y_1'} + u'$

~~$\cos x + u' = \cos x - \frac{1}{2}\operatorname{sen} x \tan x + (\operatorname{sen} x + u)^2$~~

~~$u' = -\frac{1}{2}\operatorname{sen} x \tan x + (\operatorname{sen}^2 x + 2u\operatorname{sen} x + u^2)$~~

~~$u' - 2u\operatorname{sen} x = -\frac{1}{2}\frac{\operatorname{sen} x \operatorname{sen} x}{\cos x} + \operatorname{sen}^2 x + u^2$~~

~~$= -\frac{1}{2}\frac{\operatorname{sen}^2 x}{\cos x} + \operatorname{sen}^2 x + u^2$~~

~~$= \operatorname{sen}^2 x\left(-\frac{1}{2}\cos x + 1\right) + u^2$~~

$$\cos x + u' = \frac{2\cos^2 x - \cancel{\operatorname{sen}^2 x} + \cancel{\operatorname{sen}^2 x} + 2u\operatorname{sen}x + u^2}{2\cos x}$$

$$\cancel{\cos x} + u' = \cancel{\cos x} + \cancel{2}u\tan x + \frac{u^2}{2\cos x}$$

$$u' - \cancel{2}u\tan x = \frac{1}{2}\sec x \cdot u^2 \quad \Big\}\ \text{ED Lineal Bernoulli}$$

$$u^{-2}u' - \cancel{2}u^{-1}\tan x = \frac{\sec x}{2}$$

$$z = u^{-1} \qquad z' = -u^{-2}u'$$

$$-z' - \cancel{2}\tan x\, z = \frac{\sec x}{2}$$

$$z' + \cancel{2}\tan x\, z = -\frac{\sec x}{2} \longrightarrow \text{sustitución simple}$$

$$\mu(x) = e^{\cancel{2}\int \tan x\,dx} = e^{-\cancel{2}\ln(\cos x)} = \boxed{\frac{1}{\cos^{\cancel{2}} x}}$$

$$\int\left(\frac{1}{\cos^{\cancel{2}} x}\cdot z\right)' = -\frac{1}{2}\int \frac{\sec^2 x\,dx}{\cancel{2\sec x}}$$

$$\sec^{\cancel{2}} x\, z = -\frac{1}{2}\int \frac{\sec^2 x\,dx}{\cancel{\cos x\,dx}}$$

$$\sec^{\cancel{2}} x\, z = -\frac{1}{2}\overset{\tan x}{\cancel{\operatorname{sen} x}} + \frac{C}{2}$$

$$\rightarrow z = -\frac{1}{2}\left(\frac{\operatorname{sen} x}{\cancel{\sec x}} + \frac{C\cos x}{\sec^{\cancel{2}} x}\right)$$

$$\frac{1}{u} = \ldots$$

$$\frac{1}{y - \operatorname{sen} x} = -\frac{1}{2}\left(\operatorname{sen} x + C\cos x\right)$$

c) $y' - y^2 + 2xy = x^2$, $y_1 = ax + b$

$y_1' = a$

$a - (ax+b)^2 + 2x(ax+b) = x^2$

$a - ((ax)^2 + 2axb + b^2) + 2ax^2 + 2xb = x^2$

$a - a^2x^2 - 2axb - b^2 + 2ax^2 + 2xb = x^2$

$(-a^2+2a)x^2 + (-2ab+2b)x + (a-b^2) = x^2$

Agrupar semejantes en ambos lados

$-a^2 + 2a = 1 \Rightarrow a^2 - 2a + 1 = 0$

$-2ab + 2b = 0 \Rightarrow \cancel{2b = 2} \cdot 2b - 2ab = 0 \Rightarrow b(1-a) = 0$

$a - b^2 = 0 \Rightarrow a = b^2$

$(a-1)^2 = 0 \Rightarrow a = 1$

$b(1-1) = 0 \Rightarrow 0 = 0 \quad \forall b \in \mathbb{R}$

$1 = b^2 \Rightarrow b = \pm 1$

$\Rightarrow y_1 = x + 1$ o $y_1 = x - 1$ $\Big\}$ Se pueden usar cualquiera de las dos.

Si $y_1 = x+1$

$\Rightarrow y = x+1+u \qquad y' = 1+u'$

$u'+1-(x+1+u)^2+2x(x+1+u)=x^2$

$u'+1-(x^2+x+xu+x+1+u+ux+u+u^2)+2x^2+2x+2xu=x^2$

$u'+\cancel{1}-\cancel{x^2}-\cancel{x}-\cancel{xu}-x-\cancel{1}-u-\cancel{ux}-u-u^2+2x^2+2x+2\cancel{x}u=x^2$

$u'-x^2-\cancel{2x}-2u-u^2+2\cancel{x^2}+\cancel{2x}=\cancel{x^2}$

$u'-2u=u^2$ } ED Lineal Bernoulli

$u^{-2}u'-2u^{-1}=1$

$z=u^{-1} \qquad z'=-u^{-2}u'$

$-z'-2z=1$

$z'+2z=-1$ } ED Lineal

$\mu = e^{2\int dx} = \boxed{e^{2x}}$

$\int (z\cdot e^{2x})' = -\int e^{2x}$

$z e^{2x} = -\frac{1}{2}\int e^{u}\,du$

$z e^{2x} = -\frac{1}{2}e^{u} + C$

$u=2x \qquad du=2dx$

$\frac{1}{u} = -\frac{1}{2} + ce^{-2x}$

$\frac{1}{y-x-1} = -\frac{1}{2} + ce^{-2x}$

ED Exactas

Teorema

$M(x,y)dx + N(x,y)dy$ es exacta si y solamente si $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Ejemplo

① $(\underbrace{3y^2+10xy^2}_{M})dx + (\underbrace{6xy-2+10x^2y}_{N})dy = 0$

Primero verificamos que es exacta.

$$\frac{\partial M}{\partial y} = 6y + 20xy \qquad \frac{\partial N}{\partial x} = 6y + 20xy$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \checkmark \quad \text{es exacta!}$$

Para resolver vamos a calcular una función potencial:

$$\frac{\partial F}{\partial x} = M(x,y) = 3y^2 + 10xy^2$$

$$\frac{\partial F}{\partial y} = N(x,y) = 6xy - 2 + 10xy^2$$

Para resolver necesitamos encontrar una función Potencial.

Si $dF = M(x,y)\,dx + N(x,y)\,dy$

$\Rightarrow$ Es exacta si es el diferencial total de alguna función. Estas funciones se llaman potenciales.

Tenemos:

$$M \cdot dx + N\,dy = 0;$$

Si es exacta entonces,

$$dF = M\,dx + N\,dy = 0$$

$$\Rightarrow \boxed{dF = 0}$$

$$\int dF = \int 0\,dx$$

$$\boxed{F = C} \leftarrow \text{Solución General}$$

$$dF = M\,dx + N\,dy$$

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \qquad \frac{\partial F}{\partial x} = M(x,y)$$

$$\frac{\partial F}{\partial y} = N(x,y)$$

Si tenemos

$$\frac{\partial F}{\partial x} = M(x,y)$$

integra con respecto a x: función desconocida y que depende de y.

$$F(x,y) = \int M(x,y)\,\partial x + A(y)$$

integra con respecto a y:

$$\frac{\partial F}{\partial y} = N(x,y)$$

función desconocida y que depende de x.

$$F(x,y) = \int N(x,y)\,\partial y + B(x)$$

$\int M(x,y)\,\partial x$ ← esto significa integra parcialmente con respecto a x y supone y constante.

$\int N(x,y)\,\partial y$ ← integra parcialmente con respecto a y, supone x constante

Ejemplo

① $\underbrace{(3y^2+10xy^2)}_{M(x,y)}dx+\underbrace{(6xy-2+10x^2y)}_{N(x,y)}dy=0$

$$\begin{cases} M(x,y)dx + N(x,y)dy = 0 \\ \dfrac{\partial F}{\partial x}dx + \dfrac{\partial F}{\partial y}dy = 0 \end{cases}$$

Prueba de exactitud:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\dfrac{\partial M}{\partial y} = 6y + 20xy$ $\quad$ $\left.\dfrac{\partial N}{\partial x} = 6y + 20xy\right\}$ Son exactas

Resolver: Buscar una función Potencial (F).

$$\frac{\partial F}{\partial x} = 3y^2 + 10xy^2$$

↳ Integramos

$$\int \frac{\partial F}{\partial x}dx = \int (3y^2+10xy^2)\,\partial x + A(y)$$

$$F = 3xy^2 + 5x^2y^2 + A(y)$$

↳ Derivamos con respecto a y para luego igualarla a

$\dfrac{\partial F}{\partial y} = \cancel{6y+20xy}$ $6xy-2+10x^2y$ y encontrar $A(y)$.

$$\frac{\partial F}{\partial y} = 6xy + 10x^2y + A'(y)$$

$$6xy + 10x^2y + A'(y) = \cancel{6y + 20xy}\ 6xy - 2 + 10x^2y$$

$$\Rightarrow A'(y) = -2$$

Para buscar $A'(y)$ integramos

$$\int A'(y) = -\int 2\,dy$$

$$A(y) = -2y + C$$ ← buscamos UNA función potencial por lo tanto podemos usar $c = 0$.

$$\Rightarrow \boxed{A(y) = -2y}$$

$$F = 3xy^2 + 5x^2y^2 - 2y$$

↳ Tenemos que:

$$dF = 0$$

$$\Rightarrow dF = 0$$

$$\int d(3xy^2 + 5x^2y^2 - 2y) = \int 0$$

$$3xy^2 + 5x^2y^2 - 2y = A$$ } Solución General

ED ~~E~~ Reducibles a Exactas mediante un factor integrante.

$$\widetilde{M(x,y)}\,dx + \widetilde{N(x,y)}\,dy = 0$$

$$\left.\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right\} \text{ Para que sean Exactas}$$

Pero:

$$\left.\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right\} \text{ No Exactas}$$

Teorema

Si η es un factor integrante de

$M(x,y)dx + N(x,y)dy = 0$, entonces:

$$\left.\eta(x,y)\,M(x,y)\,dx + \eta(x,y)\,N(x,y)\,dy = 0\right\} \text{ Exacta}$$

$$\frac{\partial(\eta(x,y)\,M(x,y))}{\partial y} = \frac{\partial(\eta(x,y)\,N(x,y))}{\partial x}$$

Teorema

Si tenemos $M(x,y)dx + N(x,y)dy = 0$, entonces:

$$\left.\frac{M_y - N_x}{N}\right\} \text{ Si el resultado es función de } x \text{ solamente} \Rightarrow \cancel{\eta(x)}\quad \underbrace{\eta(x)}_{\text{es un Factor Integrante}} = e^{\int \frac{M_y - N_x}{N}dx}$$

$\dfrac{M_y - N_x}{-M}$ } si es función de y solamente $\Rightarrow$ $\underbrace{\eta(y)}_{\text{es un Factor Integrante}} = e^{\int \frac{M_y - N_x}{-M} dy}$

es un Factor Integrante

Ejemplo

① $\underbrace{(3x + 2y^2)}_{M} dx + \underbrace{2xy}_{N} dy = 0$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ } Pruebe de exactitud

$\dfrac{\partial M}{\partial y} = 4y \quad \dfrac{\partial N}{\partial x} = 2y$ } no son exactas

$\dfrac{M_y - N_x}{N} = \dfrac{4y - 2y}{2xy} = \dfrac{2y}{2xy} = \dfrac{1}{x}$ } función de $\underline{\underline{x}}$ solamente

Entonces:

$\eta(x) = e^{\int \frac{1}{x} dx} = e^{\ln|x|} = \boxed{x} \leftarrow$ Factor Integrante

$x(3x + 2y^2)dx + (2xy)x\, dy = 0$ } Esto debería hacer la ED Exacta

$\underbrace{(3x^2 + 2xy^2)}_{M} dx + \underbrace{2x^2y}_{N} dy = 0$

$\dfrac{\partial M}{\partial y} = 4xy \quad \dfrac{\partial N}{\partial x} = 4xy$ } Exacta!

$$\frac{\partial F}{\partial x} = 3x^2 + 2xy^2$$

$$\int \partial F = \int (3x^2 + 2xy^2)\,\partial x + A(y)$$

$$F = x^3 + x^2y^2 + A(y)$$

$$\cancel{x^3 + x^2y^2 + A'(y) = \underbrace{2x^2y}_{\frac{\partial F}{\partial y}}}$$

$$\cancel{\Rightarrow A'(y) =}$$

$$\frac{\partial F}{\partial y} = 2x^2y + A'(y)$$

$$2x^2y + A'(y) = \underbrace{2x^2y}_{\frac{\partial F}{\partial y}}$$

$$\Rightarrow A'(y) = 0$$

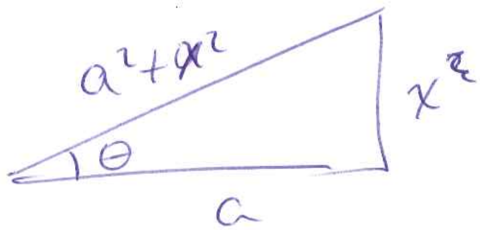
$$\int A'(y) = \int 0$$

$$A(y) = C, \quad C = 0$$

$$\boxed{A(y) = 0} \qquad \int dF = \int (x^3 + x^2y^2) = \int 0$$

$$x^3 + x^2y^2 = C \Big\} \text{ Solución General}$$

$$\int \frac{1}{a^2+x^2}\,du$$



$\tan\theta = \frac{x}{a}$ $\qquad$ $x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

$\theta = \arctan\frac{x}{a}$

$$a\int \frac{\sec^2\theta\, d\theta}{a^2+a^2\tan^2\theta} = a\int \frac{\sec^2\theta}{a^2(1+\tan^2\theta)}\,d\theta$$

$$= \frac{1}{a}\int \frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta$$

$$= \frac{1}{a}\int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta = \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + c$$

$$= \frac{1}{a}\arctan\frac{x}{a} + c$$

Ejemplo

Halle una función $N(y)$ tal que $h(x,y)=3xy^2$ sea un factor integrante de la ED.

$$\left(\frac{x}{y^2}-\frac{1}{xy}\right)dx+N(y)\,dy=0$$

$$\underbrace{3xy^2}_{\text{factor integrante}}\left(\frac{x}{y^2}-\frac{1}{xy}\right)dx+\underbrace{3xy^2}_{\text{factor integrante}}N(y)\,dy=0$$

$$\underbrace{(3x^2-3y)}_{M}dx+\underbrace{3xy^2N(y)}_{N}dy=0$$

$$\underbrace{\frac{\partial M}{\partial y}=-3 \qquad \frac{\partial N}{\partial x}=6xyN(y)}_{\text{Deben ser iguales}}$$

$$-3=\cancel{6x}\,3y^2N(y)$$

$$\Rightarrow \boxed{N(y)=-\frac{1}{\cancel{2}y^2}}$$

$$(3x^2-3y)dx-\frac{3xy^2}{y^2}dy=0$$

$$\frac{\partial M}{\partial y}=-3 \qquad \frac{\partial N}{\partial x}=-3$$

Resolve:

$$\frac{\partial F}{\partial y} = -3x$$

$$\int \partial F = -3x \int \partial y$$

$$F = -3xy + B(x)$$

$$\frac{\partial F}{\partial x} = -3y + B'(x)$$

$$3x^2 - 3y = -3y + B'(x)$$

$$\int B'(x) = \int 3x^2 \, dx$$

$$\boxed{B(x) = x^3}$$

$$\Rightarrow F = -3xy + x^3$$

$$dF = 0$$

$$\boxed{F = C}$$

<u>Ejemplo</u>

① Hallar un factor integrante de la forma $\eta = \eta(x^2+y^2)$ para la ED

$$\underbrace{-y}_{M}\,dx+\underbrace{x}_{N}\,dy=0 \ , \quad xy>0$$

$$\left.\frac{\partial M}{\partial y} = -1 \qquad \frac{\partial N}{\partial x} = 1 \ \right\} \text{no son exactas}$$

$$\underbrace{-y\,\eta(x^2+y^2)}_{M}\,dx + \underbrace{x\,\eta(x^2+y^2)}_{N}\,dy = 0$$

$$\frac{\partial M}{\partial y} = -y\cdot \eta'(x^2+y^2)(2y) + \eta(x^2+y^2)(-1)$$

$$\frac{\partial N}{\partial x} = x\,\eta'(x^2+y^2)(2x) + \eta(x^2+y^2)$$

$$-2y^2\,\eta'(x^2+y^2) - \eta(x^2+y^2) = 2x^2\eta'(x^2+y^2) + \eta(x^2+y^2)$$

$$-2\eta(x^2+y^2) = (2x^2+2y^2)\,\eta'(x^2+y^2)$$

$$\frac{\eta'(x^2+y^2)}{\eta(x^2+y^2)} = \frac{-2}{2x^2+2y^2} = -\frac{1}{x^2+y^2}$$

$$v = x^2+y^2 \qquad \frac{\eta'(v)}{\eta(v)} = -\frac{1}{v}$$

$$\frac{d\eta}{dr}\cdot\frac{1}{\eta}=-\frac{1}{r}$$

$$\int\frac{d\eta}{\eta}=-\int\frac{dr}{r}$$

$$\ln|\eta|=-\ln|r|+c$$

$$\Rightarrow \eta=\frac{A}{r}\ ,\quad A=\pm e^{c}$$

Escogemos $A=0$:

$$\Rightarrow \eta=\frac{1}{r}=\frac{1}{x^2+y^2}$$ } La forma que inicialmente queríamos

$$\eta(x^2+y^2)=\frac{1}{x^2+y^2}$$ } Factor Integrante

$$\underbrace{-\frac{y}{x^2+y^2}}_{M}dx+\underbrace{\frac{x}{x^2+y^2}}_{N}dy=0$$

$$\frac{\partial M}{\partial y}=\frac{-(x^2+y^2)+y(2y)}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x}=\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

} Son exactas

$$\frac{\partial F}{\partial y} = \frac{x}{x^2+y^2}$$

$$\int \partial F = \int \frac{x}{x^2+y^2}\, \partial y$$

$$F = x \int \frac{1}{x^2+y^2}\, \partial y$$

$$F = \cancel{x} \arctan \frac{y}{x} + B(x)$$

$$\frac{\partial F}{\partial x} = \cancel{x} - \frac{y}{x^2+y^2} + B'(x)$$

$$-\frac{y}{x^2+y^2} + B'(x) = -\frac{y}{x^2+y^2}$$

$$\Rightarrow B'(x) = 0$$

$$B(x) = c, \quad c = 0$$

función potencial

$$\Rightarrow F = \arctan\left(\frac{y}{x}\right)$$ Solución General

$$dF = 0$$

$$\boxed{F = C}$$