5.1. Calcular los polinomios de Taylor de grado 2 de las siguientes funciones centradas en los puntos que se indican:

a) $f(x) = \operatorname{sen} x$ centrado en $a = \pi$

$f'(x) = \cos x$

$f''(x) = -\operatorname{sen} x$

$f(\pi) = 0$ $\qquad T_{2,\pi}(f)(x) = -(x-\pi) = \pi - x$

$f'(\pi) = -1$

$f''(\pi) = 0$

b) $f(x) = \sqrt{1+x}$, $a = 0$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$

$f(0) = 1$ $\qquad T_{2,0}(f)(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$

$f'(0) = \frac{1}{2}$

$f''(0) = -\frac{1}{4}$

c) $f(x) = (\ln x)^2$ $a = 1$

$f'(x) = \frac{2\ln x}{x}$

$f''(x) = \frac{2 \cdot \frac{1}{x} x - 2\ln x}{x^2} = \frac{2(1-\ln x)}{x^2}$ $\qquad T_{2,1}(f)(x) = (x-1)^2$

$f(1) = 0$, $f'(1) = 0$, $f''(1) = 2$

d) $f(x) = e^x$, $a = 1$

$f'(x) = e^x$ $\qquad T_{2,1}(f)(x) = e + e(x-1) + \frac{e}{2}(x-1)^2$

$f''(x) = e^x$

$f(1) = e$, $f'(1) = e$, $f''(1) = e$

e) $f(x) = \frac{x}{1+x^2}$, $a=0$

$$f'(x) = \frac{(1+x^2) - 2x \cdot x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

$$f''(x) = \frac{-2x(1+x^2)^2 - (1-x^2)2(1+x^2) \cdot 2x}{(1+x^2)^2} = \frac{-2x(1+x^2) - 4x(1-x^2)}{(1+x^2)^3} =$$

$$= \frac{-2x - 2x^3 - 4x + 4x^3}{(1+x^2)^3} = \frac{-6x + 2x^3}{(1+x^2)^3}$$

$f(0) = 0$, $f'(0) = 1$ $f''(0) = 0$ $T_{2,0}(f) = x$

f) $f(x) = \frac{\cos x}{x+1}$, $a=0$

$$f'(x) = \frac{-\operatorname{sen} x\,(x+1) - \cos x}{(x+1)^2}$$

$$f''(x) = \frac{(-\cos x\,(x+1) - \operatorname{sen} x + \operatorname{sen} x)(x+1)^2 - 2(x+1)(-\operatorname{sen} x\,(x+1) - \cos x)}{(x+1)^4}$$

$$= \frac{-\cos(x)\,(x+1)^2 + 2\operatorname{sen} x\,(x+1) + 2\cos(x)}{(x+1)^3}$$

$f(0) = 1$ $f'(0) = -1$, $f''(0) = 1$ $T_{2,0} f(x) = 1 - x + \frac{x^2}{2}$

5.2. Determina el origen de las siguientes expresiones

En todos los casos aplicamos el teorema de Taylor y obtenemos que

$$f(x) = T_{2,a}(f)(x) + R_{2,a}(f)(x)$$

$$T_{2,a}(f)(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$R_{2,a}(f)(x) = \frac{f'''(\alpha)}{3!}(x-a)^3 \ : \alpha \in (x,a) \text{ ó } \alpha \in (a,x)$$

Si $x \simeq a \Rightarrow f'''(\alpha) \simeq f'''(a)$ y

$$\left| f(x) - T_{2,a}(f)(x) \right| = \left| R_{2,a}(f)(x) \right| = \frac{|f'''(\alpha)|}{3!}(x-a)^3$$

con lo que $f(x) \simeq T_{2,a}(f)(x)$

5.3. Encuentra una estimación del error máximo que se puede cometer al tomar:

4) $e\left(1+(x-1)+\frac{(x-1)^2}{2}\right)$ en lugar de e^x si $x \in [0.8, 1.2]$

$T_{2,1}(f)(x)$ $\qquad$ $[0.8,1.2] = [1-0.2, 1+0.2]$

$$f(x) = T_{2,1}(f)(x) + R_{2,1}(f)(x) \qquad f'''(x) = e^x$$

$\alpha \in (x,1)$ ó $(1,x)$

$$\underbrace{|f(x) - T_{2,1}(f)(x)|}_{\text{Error}} = |R_{2,1}(f)(x)| = \left|\frac{f'''(\alpha)}{3!}\right| |x-1|^3 \leq$$

$$\leq \frac{e^{1.2}}{3!} \cdot 0.2^3 \leq \frac{3.33}{3!} 0.2^3$$

5) $e\left(1+(x-1)+\frac{(x-1)^2}{2}\right)$ en lugar de e^x si $x \in [0.4, 1.6]$

$[0.4,1.6] = [1-0.6, 1+0.6]$

Igual todo que antes, pero ahora

$\alpha \in (x,1)$ ó $(1,x)$

$$|f(x) - T_{2,1}(f)(x)| = |R_{2,1}(f)(x)| = \frac{|f'''(\alpha)|}{3!} |x-1|^3 \leq$$

$$\leq \frac{e^{1.6}}{3!}(0.6)^3 = \frac{4.96}{3!}(0.6)^3$$

5.6. Un hilo pesado, bajo la acción de la gravedad, se comba formando la catenaria $y = a\cosh\left(\frac{x}{a}\right)$. Demuestra que para valores pequeños de $|x|$ la forma que toma el hilo puede ser representada por la parábola $y = a + \frac{x^2}{2a}$

Como lo hacemos para valores pequeños de $|x|$ vamos a aproximar $y = a\cosh\left(\frac{x}{a}\right) = f(x)$ utilizando el polinomio de Taylor de grado 2 en $x=0$ de f que es $T_{2,0}(f)(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$

$f(0) = a$, $f'(x) = \operatorname{senh}\left(\frac{x}{a}\right)$, $f'(0) = 0$, $f''(x) = \frac{1}{a}\cosh\left(\frac{x}{a}\right)$ $\quad f''(0) = \frac{1}{a}$

$T_{2,0}(f)(x) = a + \frac{x^2}{2a}$

$$|f(x) - T_{2,0}(f)(x)| = |R_{2,0}(f)(x)| = \frac{|f'''(\alpha)|}{3!} x^3$$

Error

$$f''' = \frac{1}{a^2} \operatorname{senh}\left(\frac{x}{a}\right)$$

$$f'''(\alpha) \xrightarrow[x\to 0]{} f'''(0) = 0$$

Por tanto para $|x|$ próximo a 0, tenemos que $y = a + \frac{x^2}{2a}$ aproxima a $y = a\cosh\left(\frac{x}{a}\right)$.

5.6. Calcula las series de Taylor de las funciones siguientes centradas en los puntos que se indican

d) $f(x) = \cos x$, $a = \frac{\pi}{4}$

$f'(x) = -\operatorname{sen} x$

$f''(x) = -\cos x$

$f'''(x) = \operatorname{sen} x$

$f^{iv}(x) = \cos x$

$$T_{\frac{\pi}{4}}(f)(x) = \sum_{k=0}^{\infty} \frac{f^{k)}(\frac{\pi}{4})}{k!}\left(x - \frac{\pi}{4}\right)^k = \sum_{k=0}^{\infty} \frac{\frac{\sqrt{2}}{2}(-1)^{[\frac{k+1}{2}]}}{k!}\left(x - \frac{\pi}{4}\right)^k$$

$$= \frac{\sqrt{2}}{2} \sum_{k=0}^{\infty} \frac{(-1)^{[\frac{k+1}{2}]}}{k!}\left(x - \frac{\pi}{4}\right)^k$$

$f(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, $f'(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$, $f''(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$, $f'''(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, $f^{iv)}(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$$f^{k)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(-1)^{[\frac{k+1}{2}]}$$

g) $f(x) = e^{-x^2}$, $a = 0$

$$e^{-x^2} = e^{(-x^2)} = g(-x^2) \ : \ g = e^x \quad T_0(g) = \sum_{k=0}^{+\infty} \frac{1}{k!} x^k$$

Por tanto $T_0(f) = \sum_{k=0}^{+\infty} \frac{1}{k!}(-x^2)^k = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} x^{2k}$

h) $f(x) = \ln(1+x^2)$, $a=0$

$f(x) = g(x^2)$: $g(x) = \ln(1+x)$

$g'(x) = \frac{1}{1+x} = (1+x)^{-1}$

$g''(x) = -(1+x)^{-2}$

$g'''(x) = 2(1+x)^{-3}$

$g^{iv}(x) = -3 \cdot 2 (1+x)^{-4}$

$g^{v}(x) = 4 \cdot 3 \cdot 2 (1+x)^{-5}$ $\qquad g(0)=0$

$\vdots$

$g^{k)}(x) = (-1)^{k+1}(k-1)!\,(1+x)^{-k} \rightarrow g^{k)}(0) = (-1)^{k+1}(k-1)!$ si $k \geq 1$

$$T_0(g)(x) = \sum_{k=0}^{+\infty} \frac{f^{k)}(0)}{k!} x^k = \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}(k-1)!}{k!} x^k = \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k} x^k$$

Per tant, com $f(x) = g(x^2)$, tenemos $T_0(f)(x) = \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{k} x^{2k}$.