SUMS OF SQUARES OF ANALYTIC FUNCTION GERMS

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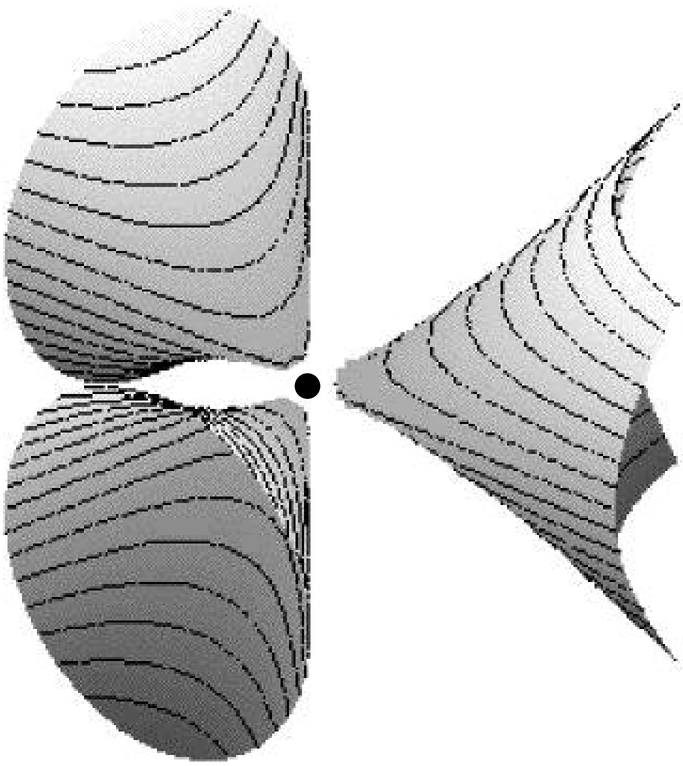


Main results

I. The Pythagoras number of an analytic surface germ is finite.

Bounds $= f$(multiplicities,codimension)

II. **List** of all surface germs in $\mathbb{R}^3$ such that psd = sos.

All of them have Pythagoras number 2.

PRELIMINARIES

1. General setting for the real spectrum

- $\mathcal{P}(A) = \left\{ \begin{array}{c} \text{psd's} \\ \text{on } A \end{array} \right\} = \{f \in A : \ f(\alpha) \geq 0 \ \forall \alpha \in \operatorname{Spec}_r(A)\}$

- $\Sigma(A) = \left\{ \begin{array}{c} \text{sums of} \\ \text{squares} \\ \text{on } A \end{array} \right\}; \qquad \Sigma_q(A) = \left\{ \begin{array}{c} \text{sums of } q \\ \text{squares} \\ \text{on } A \end{array} \right\}$

- *Pythagoras number:* $p(A) = \inf\{q \in \mathbb{N} : \Sigma(A) = \Sigma_q(A)\}$

> *Qualitative problem:* Is $\mathcal{P}(A) = \Sigma(A)$?
>
> *Quantitative problem:* To estimate $p(A)$.

Consecuencia: If $p(A) \leq p < +\infty$ then

> *Qualitative problem:*
> Determine if the equations
> $f = Y_1^2 + \cdots + Y_p^2$ for f psd
> have always a solution.

2. A little bit of history.

- **Origin:** 17th Hilbert's Problem for $\mathbb{R}(x_1, \dots, x_n)$, [E.Ar,27]

- **Generalization**

$$\boxed{\left\{\begin{array}{c}\text{Geometric formulation}\\ \text{for functions in}\\ \text{real varieties}\end{array}\right\} \overset{\text{Tarski}}{\Longleftrightarrow} \left\{\begin{array}{c}\text{Abstract formulation}\\ \text{for the real spectrum}\end{array}\right\}}$$

- **Relevant results**

$$\begin{array}{c}\text{Irreducible}\\ \text{algebraic}\\ \text{sets}\end{array} \left\{\begin{array}{ll}
\mathcal{P}\begin{pmatrix}\text{rational func.}\\ \text{real irred. var}\end{pmatrix} = \Sigma\begin{pmatrix}\text{rational func.}\\ \text{real irred. var}\end{pmatrix} & \text{[E.Ar,27]}\\[2ex]
\dim + 2 \leq p\begin{pmatrix}\text{rational func.}\\ \text{real variety}\\ \dim \geq 2\end{pmatrix} \leq 2^{\dim} & \text{[CaElPf,71]}\\[2ex]
p\begin{pmatrix}\text{polynomial func.}\\ \text{real irred. curve}\end{pmatrix} < +\infty,\ p[\mathbb{R}] = 2 & \text{[ChDLR,80]}\\[2ex]
p\begin{pmatrix}\text{polynomial func.}\\ \text{real irred. surf.}\end{pmatrix} \overset{?}{=} +\infty,\ p[\mathbb{R}^2] = +\infty & \text{[ChDLR,80]}\\[2ex]
p\begin{pmatrix}\text{polynomial func.}\\ \text{real variety}\\ \dim \geq 3\end{pmatrix} = +\infty & \text{[ChDLR,80]}
\end{array}\right.$$

$$
\text{Connected Nash varieties}
\begin{cases}
\mathcal{P}\begin{pmatrix}\text{rational Nash}\\ \text{functions}\end{pmatrix} = \mathit{\Sigma}\begin{pmatrix}\text{rational Nash}\\ \text{functions}\end{pmatrix} & \text{[Mw,76]}\\[2ex]
p\begin{pmatrix}\text{rational Nash}\\ \text{functions}\end{pmatrix} \le 2^{\dim} & \text{[BCR,87]}
\end{cases}
$$

$$
\text{Connected analytic varieties}
\begin{cases}
\mathcal{P}\begin{pmatrix}\text{mer. func.}\\ \text{compact var.}\end{pmatrix} = \mathit{\Sigma}\begin{pmatrix}\text{mer. func.}\\ \text{compact var.}\end{pmatrix} & \text{[Jw,Rz,85]}\\[2ex]
p\begin{pmatrix}\text{analytic func}\\ \text{smooth compact}\\ \text{curve}\end{pmatrix} = 2 & \text{[Jw,82]}\\[2ex]
p\begin{pmatrix}\text{analytic func.}\\ \text{non-compact}\\ \text{smooth curve}\end{pmatrix} = 1 & \text{[Jw,82]}\\[2ex]
p\begin{pmatrix}\text{analytic func}\\ \text{smooth compact}\\ \text{surface}\end{pmatrix} = 3 & \text{[BKS,81]}\\[2ex]
p\begin{pmatrix}\text{analytic func.}\\ \text{non-compact}\\ \text{smooth surface}\end{pmatrix} = 2 & \text{[BKS,81]}
\end{cases}
$$

3. Analytic germs

Notations

- *Analytic ring:* $A = \mathbb{R}\{x\}/I, x = (x_1, \ldots, x_n)$

- *Zero germ:* $X = \mathcal{Z}(I) \subset \mathbb{R}^n$

- $\mathcal{J}(X) = \begin{array}{c}\text{real-radical}\\ \text{of } I\end{array} = \textit{ideal of } X = \begin{array}{c}\text{ideal of zero}\\ \text{germs of } X\end{array}$ [Ri,76]

- $\mathcal{O}(X) = \mathbb{R}\{x\}/\mathcal{J}(X) = \begin{array}{c}\text{real-reduction}\\ \text{of } A\end{array} = \begin{array}{c}\text{ring of analytic}\\ \text{germs of } X\end{array}$

- $\mathcal{M}(X) = \begin{array}{c}\text{total ring of}\\ \text{fractions of } \mathcal{O}(X)\end{array} = \begin{array}{c}\text{ring of meromorphic}\\ \text{germs of } X\end{array}$

- $\mathcal{P}(X) = \left\{\begin{array}{c}\text{psd's}\\ \text{on } X\end{array}\right\} = \mathcal{P}(\mathcal{O}(X))$ [Ri,76] [Rz,83]

- $\Sigma(X) = \Sigma(\mathcal{O}(X));\qquad \Sigma_q(X) = \Sigma_q(\mathcal{O}(X))$

- $p[X] = p(\mathcal{O}(X));\qquad p(X) = p(\mathcal{M}(X))$

Some results:

$$\text{Meromorphic germs}\quad\left\{\begin{array}{l} \mathcal{P}\begin{pmatrix}\text{meromorphic}\\ \text{func. germs}\end{pmatrix} = \Sigma\begin{pmatrix}\text{meromorphic}\\ \text{func. germs}\end{pmatrix} \quad \begin{array}{l}\text{[Ri,76]}\\ \text{[Rz,83]}\end{array} \\ \\ p(X) = 1 \qquad \text{if } d = 1 \\ \\ p(X) \leq p(\mathbb{R}^d)\cdot m \leq \begin{cases} 2m & \text{if } d = 2 \quad \text{[BoRi,75]} \\ 8m & \text{if } d = 3 \quad \text{[Jw,92]} \\ ? & \text{if } d \geq 4 \end{cases} \end{array}\right.$$

Cualitative problem for analytic rings

Curves:

$$\mathcal{P}(X) = \Sigma(X) \iff \left\{ \begin{array}{c} \text{X = union of} \\ \text{indep. lines} \end{array} \right\}$$ [Or,88] [Sch,01]

dim ≥ 3**:**

A **regular** local ring $\Rightarrow \mathcal{P}(A) \neq \Sigma(A)$ [Sch,99]

X analytic germ, $\dim X \geq 4 \Rightarrow \mathcal{P}(X) \neq \Sigma(X)$ [Rz,99]

Conjecture: $A = \mathbb{R}\{x\}/I$ analytic ring, $\dim \mathcal{Z}(I) \geq 3$
$\Rightarrow \mathcal{P}(A) \neq \Sigma(A)$

Cuantitative problem for analytic rings

Curves:

$p[X] \leq \mathrm{mult}(X)$ if X is irreducible [Qz,01]

dim ≥ 3**:**

A **regular** local ring $\Rightarrow p(A) = +\infty$ [ChDLR,80]

X analytic germ, $\dim X \geq 4 \Rightarrow p[X] = +\infty$ [Rz,83]

Conjecture: $A = \mathbb{R}\{x\}/I$ analytic ring, $\dim \mathcal{Z}(I) \geq 3$
$\Rightarrow p(A) = +\infty$

I. MAIN RESULTS ON PYTHAGORAS NUMBER

- *Strong Question* de [ChDLR,80] for modules over $R = \mathbb{R}(\{x\})[y]$, $\mathbb{R}\{x\}[y]$ y $\mathbb{R}\{x, y\}$:

Let A be a ring which is a finite generated R-module, say by m generators. Then $p(A) \leq p(R)m$.

- $A = \mathbb{R}\{x_1, \dots, x_n\}/I$, $\dim(A) = 2 \Rightarrow p(A) < +\infty$

- Let $X \subset \mathbb{R}^n$ be an analytic surface germ. Then

$$\boxed{E\big[\log_2\big(\omega(I(X)) + 1\big)\big] \leq p[X] \leq 2\,\mathrm{mult}_{\mathrm{T}}(X)^{\mathrm{codim}(X)}}$$

Moreover, $p[X] \geq 2$.

II. MAIN RESULTS ON SUMS OF TWO SQUARES

- The singular surface germs $X \subset \mathbb{R}^3$ such that $\mathcal{P}(X) = \Sigma(X)$ are exactly the following:

$$\begin{cases} z^2 - x^3 - y^5 = 0 & \text{(Brieskorn's singularity, [Rz,99])} \\ z^2 - x^3 - xy^3 = 0, & z^2 - x^3 - y^4 = 0 \end{cases}$$

$$\begin{cases} z^2 - x^2 = 0 & \text{(Two transversal planes, [Rz,99])} \\ z^2 - x^2 - y^2 = 0, & z^2 - x^2 - y^k = 0,\ k \geq 3 \end{cases}$$

$$\begin{cases} z^2 - x^2y = 0 & \text{(Whitney's umbrella, [Rz,99])} \\ z^2 - x^2y + y^3 = 0, & z^2 - x^2y - (-1)^k y^k = 0,\ k \geq 4 \end{cases}$$

- Furthermore, in **all** these cases $p[X] = p(X) = 2$.

- Existence of **other** surface germs in embedding dimension > 3 with $\mathcal{P} = \Sigma_2$.

I. PYTHAGORAS NUMBER

1. Sums of two squares in two variables

Purpose: To show that every positive semidefinite element (=psd) of $\mathbb{R}\{x\}[y]$ can be written as a sum of two squares of elements of $\mathbb{R}\{x\}[y]$.

Notations: Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and

- $\Omega_{\mathbb{K}}$ =the ring of analytic Puiseux series over $\mathbb{K}$
- $\Phi_{\mathbb{K}} = qf(\Omega_{\mathbb{K}})$
- $\mathbb{K}(\{x\}) = qf(\mathbb{K}\{x\})$

Orderings of the field $\mathbb{R}(\{x\})$. They are exactly the following ones with real closure $\Phi_{\mathbb{R}}$:

- $(\mathbb{R}(\{x\}), <)$ ordering such that x is positive

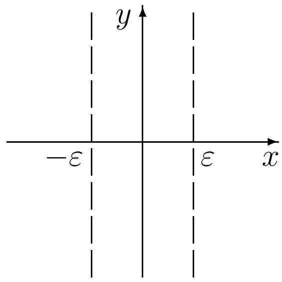
$$\begin{aligned} f = a_r x^r + a_{r+1}x^{r+1} + \cdots > 0 &\iff a_r > 0 \\ f/g > 0 &\iff fg > 0. \end{aligned}$$

- $(\mathbb{R}(\{x\}), \prec)$ ordering such that x is negative

$$\begin{aligned} f = a_r x^r + a_{r+1}x^{r+1} + \cdots \succ 0 &\iff (-1)^r a_r > 0 \\ f/g \succ 0 &\iff fg \succ 0. \end{aligned}$$

Characterization of the psd's on $\mathbb{R}(\{x\})[y]$:

Let $f \in \mathbb{R}(\{x\})[y], f \neq 0$. The following statements are equivalent:



a) f is psd on the ring $\mathbb{R}(\{x\})[y]$.

b) f is psd on the field $qf(\mathbb{R}\{x\}[y])$.

c) There exist $r \in \mathbb{N}$, $\varepsilon > 0$ such that $x^{2r}f \in \mathbb{R}\{x\}[y]$ is defined and is ≥ 0 on the vertical band $(-\varepsilon, \varepsilon) \times \mathbb{R}$.

d) For each $\xi \in \Phi_{\mathbb{R}}$ the series $f(x,\xi)$, $f(-x,\xi)$ are psd on $\Phi_{\mathbb{R}}$.

Result:

$$\boxed{\mathcal{P}(\mathbb{R}\{x\}[y]) = \Sigma_2(\mathbb{R}\{x\}[y])}$$

Sketch of proof. Let $f \in \mathcal{P}(\mathbb{R}\{x\}[y])$

- Check that $\mathrm{grad}_y(f) = n$ is odd and that the leading coefficient a_n is an square.
- Factorize f/a_n in irreducible factors in $\mathbb{R}(\{x\})[y]$.
- Check that the factors of odd multiplicity are reducible in $\mathbb{C}(\{x\})[y]$ and therefore sum of **two** squares.
- Deduce that f is sum of **two** squares in $\mathbb{R}\{x\}[y]$.

∎

2. Diagonalization in two variables

Purposes: To show that:

The positive semidefinite quadratic forms over $\mathbb{R}(\{x\})[y]$ are diagonalizable over $\mathbb{C}(\{x\})[y]$.

Consequence: A ring A which is a module with m generators over $R = \mathbb{R}(\{x\})[y]$, $\mathbb{R}\{x\}[y]$ or $\mathbb{R}\{x, y\}$ satisfies

$$p(A) \leq p(R)m = 2m$$

Essential Hint: Ideas of Djoković used to prove that:

The positive semidefinite quadratic forms over $\mathbb{R}[x]$ are diagonalizable over $\mathbb{C}[x]$.

Notations:

- $a = \langle a_1, \dots, a_n \rangle$ is the matrix with main diagonal $a_1, \dots, a_n$.

- Given $a = (a_{ij})_{1 \leq i,j \leq n}$ with coefficients in $\mathbb{C}(\{x\})[y]$, its *conjugated transposed* is:

$$\boxed{a^* = \overline{a^t} = (\overline{a_{ji}})_{1 \leq i,j \leq n}}$$

Main tool: *Diagonalization in PID's:*

Let R be a principal ideal domain and $a \in \mathfrak{M}_n(R)$ a matrix of rank r. Then, there exist two invertible matrices u, v and another diagonal one

$$e = \langle e_1, \ldots, e_r, 0, \ldots, 0 \rangle,$$

such that $e_1 | e_2 | \ldots | e_r$ and $a = uev$; moreover, the ideals

$$(e_1), (e_2), \ldots, (e_r)$$

are unique, and the elements $e_1, e_2, \ldots, e_r$ of the main diagonal of e are called invariant factors of a. *The diagonal matrix $e = \langle e_1, \ldots, e_r, 0, \ldots, 0 \rangle$ is a* matrix of invariant factors of a.

Definition: Let $a \in \mathfrak{M}_n(\mathbb{K}(\{x\})[y])$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ such that $a = a^*$:

- $\Rightarrow z^* a z \in \mathbb{R}(\{x\})[y] \ \forall z \in \mathbb{K}(\{x\})^n$
- Then, $a \geq 0 \iff z^* a z$ es psd en $\mathbb{R}(\{x\})[y] \ \forall z \in \mathbb{K}(\{x\})^n$.

Diagonalization: Given $a \in \mathfrak{M}_n(\mathbb{C}(\{x\})[y])$ such that $a \geq 0$, there exist $b \in \mathfrak{M}_n(\mathbb{C}(\{x\})[y])$ such that $a = b^*b$.

Sketch of proof. Several reduction steps:

(a) We can suppose $a \geq 0$ an dof maximum rank.

(b) We can suppose $a \geq 0$ is invertible.

(c) We can suppose $a \in \mathfrak{M}_n(\mathbb{R}(\{x\}))$, $a \geq 0$ and diagonal.

(d) In the hypotheses of *(c)* there exist $g \in \mathfrak{M}_n(\mathbb{R}(\{x\}))$ diagonal such that $a = gg = g^*g$. ■

Main result:

Let $L_1, \dots, L_r$ linear forms in n variables over $R = \mathbb{R}(\{x\})[y]$, $\mathbb{R}\{x\}[y]$ ó $\mathbb{R}\{x, y\}$, and $\varphi = L_1^2 + L_2^2 + \cdots + L_r^2$. There exist linear forms $Q_1, Q_2, \dots, Q_{2n}$ over R such that

$$\varphi = Q_1^2 + Q_2^2 + \cdots + Q_{2n}^2.$$

Sketch of proof.

(a) $R = \mathbb{R}(\{x\})[y]$. Let $a \in \mathfrak{M}_n(\mathbb{R}(\{x\})[y])$ be the matrix associated to φ which is ≥ 0:

- There exists $b \in \mathfrak{M}_n(\mathbb{C}(\{x\})[y])$ such that $a = b^* b$.
- $b = b_1 + ib_2$ with $b_1, b_2 \in \mathfrak{M}_n(\mathbb{R}(\{x\})[y])$ and then
$$a = (b_1^t - ib_2^t)(b_1 + ib_2) = b_1^t b_1 + b_2^t b_2 + i(b_1^t b_2 - b_2^t b_1)$$

and thus:

$$\begin{aligned} a &= b_1^t b_1 + b_2^t b_2 \\ b_1^t b_2 &= b_2^t b_1 \end{aligned}$$

Therefore

$$\varphi = Q_1^2 + Q_2^2 + \ldots + Q_{2n}^2.$$

(b) $R = \mathbb{R}\{x\}[y]$. Proceed as above and check that the denominator is a unit of $\mathbb{R}\{x\}$.

(c) $R = \mathbb{R}\{x, y\}$. Take suitable jets and use *M. Artin's Approximation Theorem.* ■

3. Multiplicities

Purpose: Recall the general properties of the *multiplicity* of a local ring, and deduce a particular description for analytic rings.

(3.1) Multiplicity of a local ring. Let A be a local ring with residue field κ and maximal ideal $\mathfrak{m}$ and M a module f.g. over A. The *characteristic function of M* is

$$L_M : k \mapsto \dim_\kappa(M/\mathfrak{m}^k M).$$

There exists $Q_M \in \mathbb{Q}[T]$ such that

$$L_M(k) = Q_M(k) \;\text{ for }\; k >> 0$$

$Q_M =$ *characteristic polynomial of M*:

- $\mathrm{grad}(Q_M) = d = \dim_A(M) =$ Krull's dimension of $A/(\mathrm{ann}\, M)$.
- Leading coefficient $= \mathfrak{e}(M)$.
- *Multiplicity of M*:

$$\boxed{\mathrm{mult}(M) = d!\, \mathfrak{e}(M) \geq 1}$$

(3.2) Total multiplicity. Suppose $M = A/I$ where I is a radical ideal of height r:

$$I = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_s \cap \mathfrak{p}_{s+1} \cap \ldots \cap \mathfrak{p}_l$$

$\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ associated prime ideals of height r,

$\mathfrak{p}_{s+1}, \ldots, \mathfrak{p}_l$ associated prime ideals of height $> r$.

We have:

$$\boxed{\mathrm{mult}(M) = \textstyle\sum_{i=1}^{s} \mathrm{mult}(A/\mathfrak{p}_i)}$$

Consequence: The usual notion of *multiplicity* forgets the associated prime ideals which do not have minimal height, and thus, we must consider the *total multiplicity*:

$$\boxed{\mathrm{mult}_{\mathrm{T}}(M) = \textstyle\sum_{i=1}^{l} \mathrm{mult}(A/\mathfrak{p}_i)}$$

(3.3) Multiplicity for analytic rings. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_n$ of height $r = n - d$. After a suitable linear change:

$$\boxed{\mathrm{mult}(\mathcal{O}_n/\mathfrak{p}) = [qf(\mathcal{O}_n/\mathfrak{p}) : qf(\mathcal{O}_d)]}$$

4. Bounds

Results:

1. *Lower bound*: Let $X \subset \mathbb{R}^n$ be an analytic germ, $n \geq 3$. Then

$$\boxed{p[X] \geq E\big[\log_2\big(\omega(I(X)) + 1\big)\big]}$$

Sketch of proof. Proceed by *way of contradiction* using homogeneous polynomials of $\mathbb{R}[x, y, z]$ of degree $q = 2^p - 2$ which are sum of squares of polynomials, but not less than p [ChDLR,80].

■

2. *Upper bound*: Let $X \subset \mathbb{R}^n$ be a surface germ. Then

$$\boxed{p[X] \leq 2\,\mathrm{mult}_{\mathrm{T}}(X)^{\mathrm{codim}(X)}}$$

Sketch of proof. It suffices to bound γ_X = minimal number of generators of $\mathcal{O}(X)$ as $\mathbb{R}\{x, y\}$-module, using

$$\boxed{\gamma_X \leq \mathrm{mult}_{\mathrm{T}}(X)^{\mathrm{codim}(X)}}$$

and the results of section I.2. ■

5. Examples

Purpose: To show that there is not an upper bound of the Pythagoras number of a surface germ depending only on the multiplicity. Therefore **the total multiplicity is necessary**.

Result: For each $q \in \mathbb{N}$ there exist a analytic surface germ $X \subset \mathbb{R}^3$of multiplicity 1 and Pythagoras number $\geq q$.

Sketch of proof. Two steps:

(a) There exist a curve germ $Y \subset \mathbb{R}^3$ such that $p[Y] \geq q$.

$$\forall k \geq 1 \; \exists \; Y_k \; \begin{array}{c}\text{irreducible}\\ \text{curve}\\ \text{germ}\end{array} : \omega(\mathcal{J}(Y_k)) > k.$$ Take, for instance,

$Y_k : (t^a, t^b, t^c), a = p$ prime$\geq k^2 + 2,\; b = p(p-1) + k,\; c = p^2 + 1$

(b) Let $X = Y \cup \{z = 0\}$, which a is an analytic germ of dimension 2 and satisfies

$\operatorname{mult}(X) = \operatorname{mult}(\{z = 0\}) = 1,$

$\operatorname{mult}_{\mathrm{T}}(X) = \operatorname{mult}(Y) + 1,$

$p[X] \geq p[Y] \geq q$ (since $\mathcal{O}(Y)$ is a quotient of $\mathcal{O}(X)$). ∎

II. SUMS OF TWO SQUARES

STRATEGY

Generation of the list.

Two steps:

- If $X \subset \mathbb{R}^n$ is an analytic singular germ such that $\mathcal{P}(X) = \Sigma(X)$, then

$$\boxed{\omega(\mathcal{J}(X)) = 2}$$

- If $X \subset \mathbb{R}^3$ has $\dim X = 2$, $\mathcal{P}(X) = \Sigma(X)$ and $\omega(\mathcal{J}(X)) = 2$, then X is one of the germs of the list [Rz,99].

General method to attack $\mathcal{P} = \Sigma_2$.

1. ***Polynomial reduction*:** Consider S_X the *algebraic surface associated* to X, and prove that the set of *positive definite* polynomials in S_X, considered as germs, is *dense* in the set of psd function germs on X.

2. ***Blowing-up*:** using a suitable blowing-up, we obtain a biregular equivalence between a dense open set of S_X and a dense open set of the plane or of Brieskorn's singularity.

3. ***Solution for the polynomial case*:** using the previous biregular equivalence, the fact that the plane and Brieskorn's singularity have the property $\mathcal{P} = \Sigma_2$ and certain standard equations of sum of squares, we show that every *positive definite* polynomial over S_X is a sum of two squares of analytic function germs on X.

4. ***Solution of the general case*:** we extend the previous property for polynomials to analytic function germs by means of M. Artin's Approximation Theorem .

For the two planes an Whitney's umbrella the new proof is a kind of *limit argument* using the corresponding result for its deformations.

1. The cone

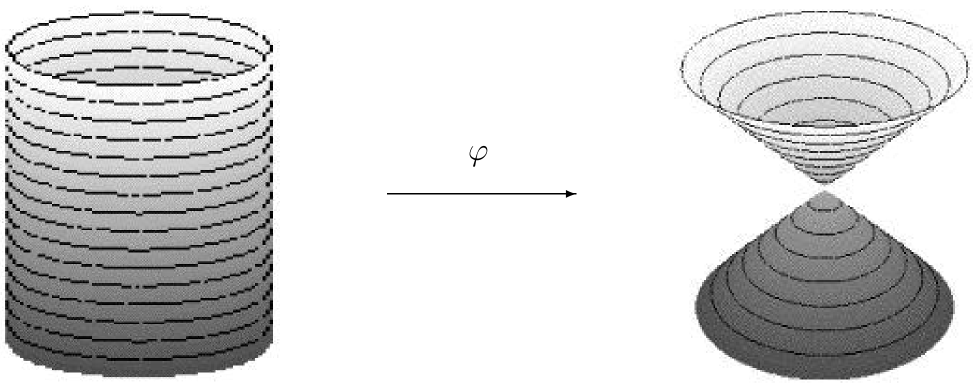
Result: If $X : z^2 = x^2 + y^2$ is the cone singularity, then any non negative function germ on X can be written as a sum of two squares of analytic function germs.

Sketch of proof. Several steps:

(a) If $f, g \in \mathbb{R}[x, y]$ and $f + zg$ (*cannonical expression* of an element of $\mathcal{O}(X)$) is psd on $X \Rightarrow$ there exists $m \geq 0$, such that $z^m(f + zg)$ is a sum of two squares in $\mathcal{O}(X)$.

Consider the blowing-up

$C : u^2 + v^2 = 1$ $\qquad\qquad$ $X : x^2 + y^2 = z^2$



$$(u, v, t) \xrightarrow{\quad\quad\varphi\quad\quad} (ut, vt, t)$$

$$\left(\frac{x}{z}, \frac{y}{z}, z\right) \overset{\quad\quad\psi\quad\quad}{\rule{8em}{0.4pt}} (x, y, z)$$

The used argument is a kind of parametrized revission of the classical proof of Polya of the fact that every psd polynomial on the circunference is a sum of two squares of polynomials.

(b) Get rid of the denominator z^m.

(c) Check that the set of psd polynomials on X is *dense* in $\mathcal{P}(X)$, using for that *Newton-Puiseux's algorithm.*

(d) Apply M. Artin's Approximation Theorem to solve the analytic case. $\blacksquare$

2. Polynomial reduction

Notations: Let $X \subset \mathbb{R}^3$ be an analytic surface germ at the origin with equation $z^2 = F(x,y)$, $F \in \mathbb{R}[x,y]$

$S_X = \{(x,y,z) \in \mathbb{R}^3 |\ z^2 - F(x,y) = 0\}$ is the *algebraic surface associated to* X (it satisfies $(S_X)_0 = X$),

$\mathcal{P}(S_X)$={polynomials $P(x,y) + zQ(x,y) \geq 0$ in S_X}.

Reducción polinomial: Let $X \subset \mathbb{R}^3$ be an analytic germ of equation $z^2 - F(x,y) = 0$, $F \in \mathbb{R}[x,y]$, $F(0,0) = 0$. If $k \geq 1$

$$\boxed{\mathcal{P}(S_X) \subset \Sigma_k(X) \Rightarrow \mathcal{P}(X) = \Sigma_k(X)}$$

Sketch of proof. Several steps:

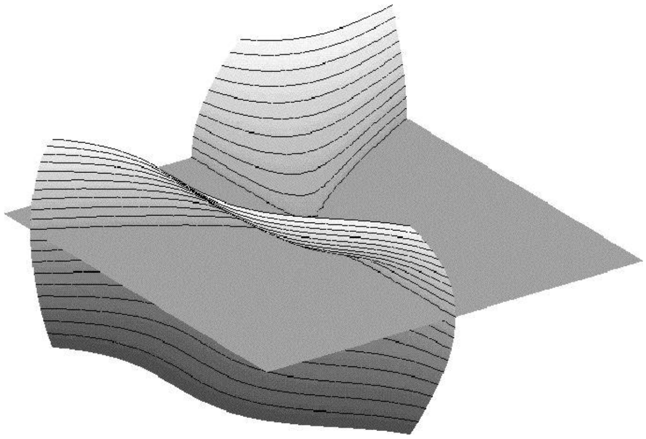
(a) Weak polynomial density: If $Z \subset \mathbb{R}^2$ is a closed semianalytic set germ and $f \in \mathcal{O}(Z)$ id positive defined or *pd* on Z (that is, $f|_{Z\setminus\{0\}} > 0$), there exist $r \in \mathbb{N}$ such that if $g \equiv f \mod (x,y)^r$ $\Rightarrow g$ is pd in Z.

(b) Study of $\mathcal{O}(X)$*:*

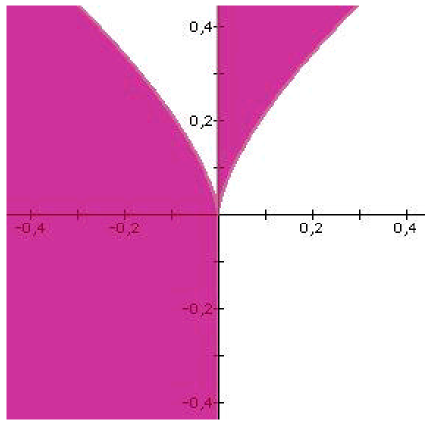
$\mathcal{O}(X) = \{f(x,y) + zg(x,y) : \ f,g \in \mathbb{R}\{x,y\}\}$

$\mathcal{P}(X) = \{f + zg : \ f \in \mathcal{P}(F \geq 0), f^2 - Fg^2 \in \mathcal{P}(\mathbb{R}^2)\}.$

$X : z^2 = F(x,y)$



$Z : F(x,y) \geq 0$



(c) Strong polynomial density: If $\varphi = f + zg \in \mathcal{P}(X)$ then for each $m \geq 1$ there exist a polynomial

$$h_m = P(x,y) + zQ(x,y) \in \mathcal{P}(S_X)$$

such that $\omega(\varphi - h_m) \geq m$ (using *(a)*, *(b)*).

(d) Use the hypothese, the step *(c)* and apply M. Artin's Approximation Theorem to solve the equations

$$\varphi = f + zg = X_1^2 + \cdots + X_k^2 + (z^2 - F)Y$$

en $\mathbb{R}\{x,y,z\}$. ∎

Example of application of our method

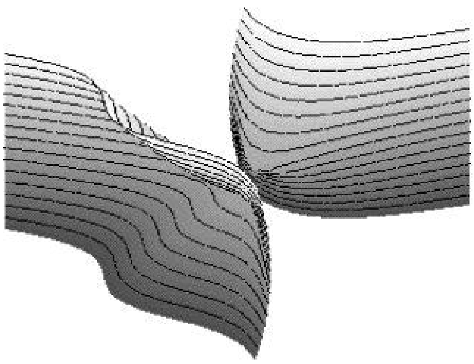
Let us see that $\mathcal{P}(X : z^2 - x^3 - xy^3 = 0) = \Sigma_2(X)$.

1. ***Polynomial reduction:*** it suffices to prove that

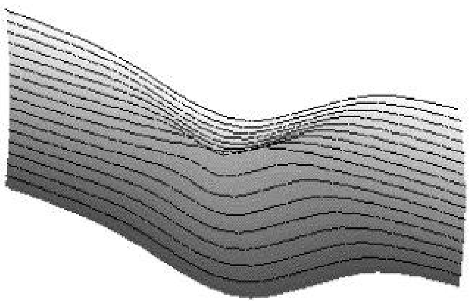
$$\mathcal{P}(S_X) \subset \Sigma_2(X).$$

2. ***Blowing-up:***

$X : z^2 = x^3 + xy^3$ $\qquad\qquad$ $E_8 : v^2 = x^5 + y^3$



$$\xrightarrow{\phi}$$



$$(x, y, z) = \left(x, \frac{u}{x}, \frac{v}{x}\right) \longrightarrow (x, xy, xz) = (x, u, v)$$

3. ***Solution of the polynomial case:*** If $P+zQ \in \mathcal{P}(S_X)$ using the previous transform and the fact that

$$\mathcal{P}((E_8)_o) = \Sigma_2((E_8)_o)$$

it can be shown that $r \geq 0$, $\alpha' s, \beta' s, q_0 \in \mathbb{R}\{x,y\}$ such that

$$x^{2r}(P+zQ) = (\alpha_0 + z\alpha_1)^2 + (\beta_0 + z\beta_1)^2 - (z^2 - x^3 - xy^3)q_0.$$

To end up we get rid of x^{2r}. Comparing coefficient with respect to z we obtain

(0) $x^{2r}P = \alpha_0^2 + \beta_0^2 + q_0(x^3 + xy^3) = \alpha_0^2 + \beta_0^2 + q_0 x(x^2 + y^3)$

(1) $x^{2r}Q = 2(\alpha_0\alpha_1 + \beta_0\beta_1)$

(2) $q_0 = \alpha_1^2 + \beta_1^2$

Now:

(0) $\Rightarrow x|\alpha_0^2 + \beta_0^2 \Rightarrow x|\alpha_0, \beta_0 \Rightarrow x|q_0$.

(2) $\Rightarrow x|\alpha_1^2 + \beta_1^2 \Rightarrow x|\alpha_1, \beta_1 \Rightarrow x^2|q_0$.

This proves

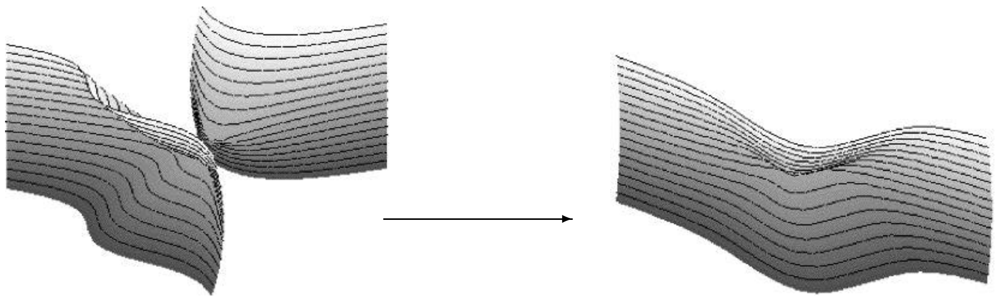
$$x^{2(r-1)}(P+zQ) = (\alpha_0' + z\alpha_1')^2 + (\beta_0' + z\beta_1')^2 - (z^2 - x^3 - xy^3)q_0'.$$

Applying this process $r-1$ times more, we deduce

$$(P+zQ) = (a_0 + za_1)^2 + (b_0 + zb_1)^2 - (z^2 - x^3 - xy^3)h_0.$$

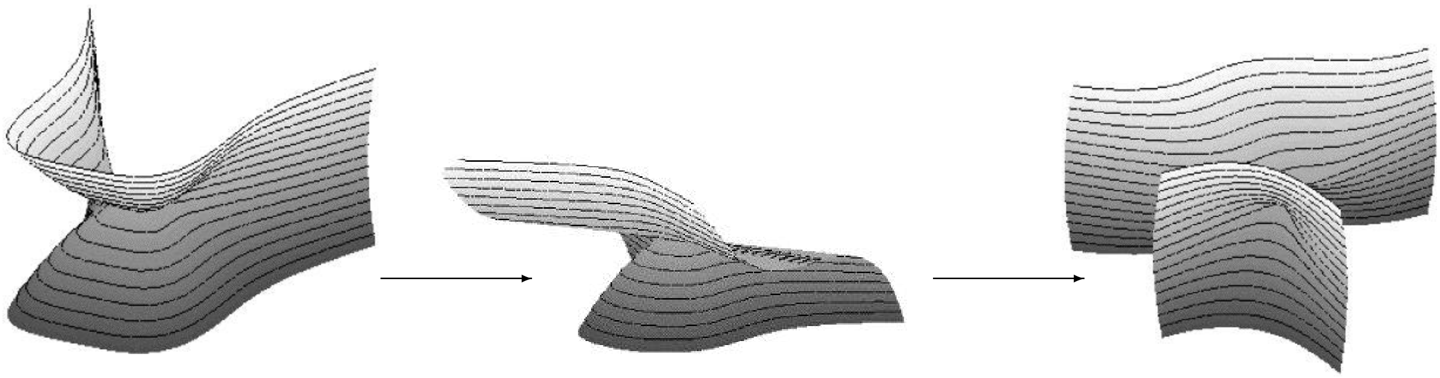
$\blacksquare$

3. Brieskorn's and it blowings-up



$X : z^2 = x^3 + xy^3$ $\qquad$ $E_8 : v^2 = x^5 + y^3$

Blowing-up

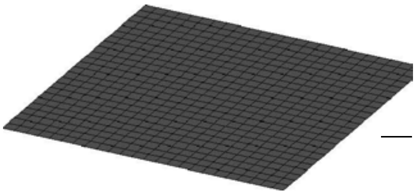


$Y : z^2 = x^3 + y^4$ $\qquad$ $Z : z^2 = x^3 - zy^2$ $\qquad$ $X : u^2 = -z^3 + zx^3$

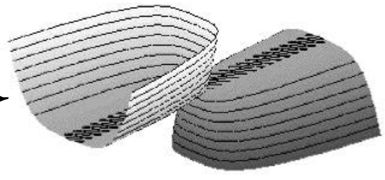
Change of coordinates	*Blowing-up*

4. Two transversal planes and its deformations

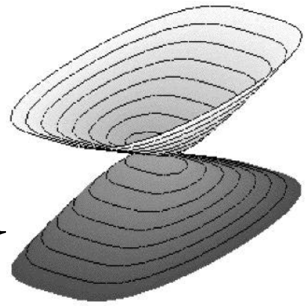
$k = 2l$



$\mathbb{R}^2$



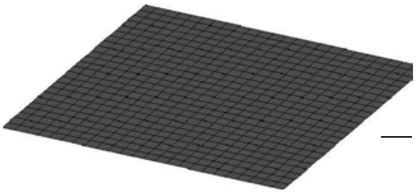
$Y_k : y^{2l} = uv$



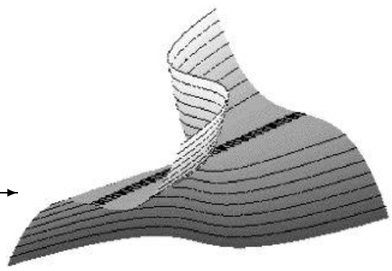
$X_k : z^2 = x^2 + y^{2l}$

Blowing-up	*Change of coordinates*

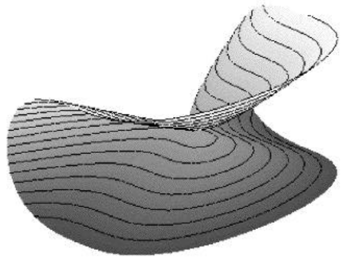
$k = 2l + 1$



$\mathbb{R}^2$

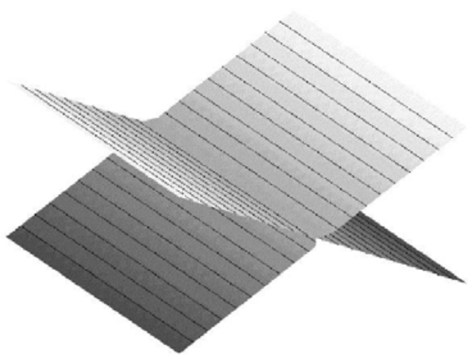


$Y_k : y^{2l+1} = uv$



$X_k : z^2 = x^2 + y^{2l+1}$

Blowing-up	*Change of coordinates*



$X : z^2 = x^2$

Solved by *limit argument*

Limit argument

For the two transversal planes $\mathcal{P}(X : z^2 - x^2 = 0) = \Sigma_2(X)$ we proceed as follows:

Let $f + zg \in \mathcal{P}(z^2 - x^2 = 0)$. There exists $m_0 \geq 1$ such that $\forall m \geq m_0$ there exists $r \geq 2m$ such that the function germ $f + (x^2 + y^2)^m + zg \in \mathcal{O}(X_{2r})$ is pd in $X_{2r} : z^2 - x^2 - y^{2r} = 0$.

It is satisfied that

$$f + (x^2 + y^2)^m + zg = \alpha^2 + \beta^2 - (z^2 - x^2 - y^{2r})h$$

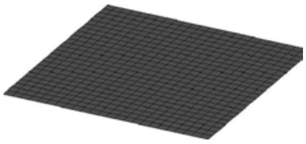
and then

$$f + zg \equiv \alpha^2 + \beta^2 - (z^2 - x^2)h \qquad \mathrm{mod}\ (x, y)^{2m}$$

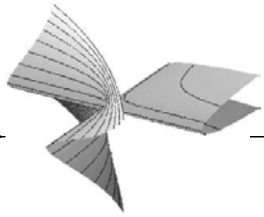
By M. Artin's Approximation Theorem

$$f + zg = a^2 + b^2 - (z^2 - x^2)q.$$

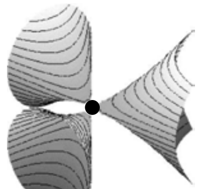
5. Whitney's umbrella and its deformations



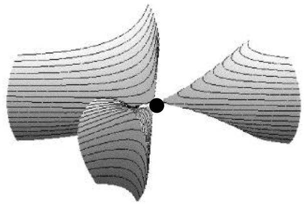
$\mathbb{R}^2$



$M_k : w^2(x - y^k) = xy$

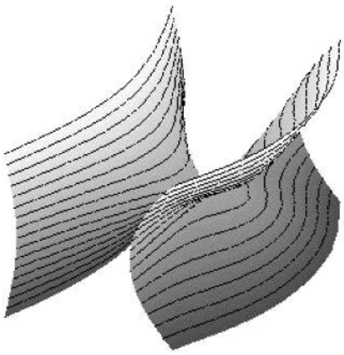


$Z_k : z^2 = xy(x - y^k)$

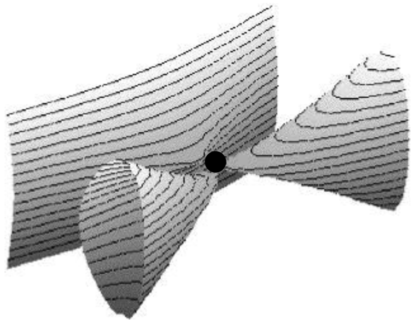


$X_k : z^2 = x^2y - y^{2k+1}$

$2^{\underline{nd}}$ *Blowing-up*	$1^{\underline{st}}$ *Blowing-up*	*Change of coordinates*

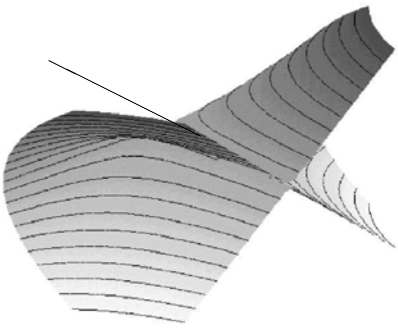


$Y_k : z^2 = x^2y + y^{2k}$



$X_k : w^2 = z^2y - y^{2k+1}$

Blowing-up



$X : z^2 = x^2y$

Solved by *limit argument*

6. Germs in higher codimension

Purpose: To prove that $\forall n \in \mathbb{N}$ there exist a surface germ with emb dim $= n+1$ such that $\mathcal{P} = \Sigma_2$.

Germs and associated surfaces: *Veronese's real cones*

Germs: $X_n = (S_n)_o \subset \mathbb{R}^{n+1}_o$,

$$S_n : \; F_{ij} = x_i x_j - x_{i-1} x_{j+1}, \qquad 1 \leq i \leq j \leq n-1$$

Associated algebraic surfaces: $S_{X_n} = S_n$.

Parametrization of the complexification of S_n*:*

$$\gamma(z,w) = (z^n, z^{n-1}w, \ldots, zw^{n-1}, w^n)$$

Parametrizations of S_n*:*

- n par:
  $\gamma^+ = \gamma_{|\mathbb{R}^2}$ parametrizes $S_n \cap \{x_0 \geq 0\}$,
  $\gamma^- = -\gamma_{|\mathbb{R}^2}$ parametrizes $S_n \cap \{x_0 \leq 0\}$.
- n impar:
  $\gamma^+ = \gamma_{|\mathbb{R}^2}$ parametrizes S_n,
  $\gamma^- = -\gamma_{|\mathbb{R}^2}$ parametrizes S_n.

Result: $\mathcal{P}(X_n) = \Sigma_2(X_n)$

a) Polynomial reduction: It is enough to prove:

$$\mathcal{P}(S_n) \subset \Sigma_2(X_n).$$

b) Blowing-up:

$$\begin{aligned} \phi_n : \mathbb{R}^2 \setminus \{x_0 = 0\} &\rightarrow S_n \setminus \{x_0 = 0\} \\ (x_0, x_1) &\mapsto \left(x_0, x_1, \frac{x_1^2}{x_0}, \dots, \frac{x_1^k}{x_0^{k-1}}, \dots, \frac{x_1^n}{x_0^{n-1}}\right). \end{aligned}$$

c) Solution of the polynomial case: If $f \in \mathcal{P}(S_n)$, using the previous transform, check that

$$x_0^{2r} f \equiv (a^2 + b^2) \mod I(X_n)$$

Using the parametrization γ^+ verify that we can divide by x_0^{2r}.