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ON THE IRREDUCIBLE COMPONENTS OF A SEMIALGEBRAIC SET

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In this work we define a semialgebraic set $S \subset \mathbb{R}^n$ to be irreducible if the noetherian ring $\mathcal{N}(S)$ of Nash functions on S is an integral domain. Keeping this notion we develop a satisfactory theory of irreducible components of semialgebraic sets, and we use it fruitfully to approach four classical problems in Real Geometry for the ring $\mathcal{N}(S)$: Substitution Theorem, Positivstellensätze, 17th Hilbert Problem and real Nullstellensatz, whose solution was known just in case $S = M$ is an affine Nash manifold. In fact, we give full characterizations of the families of semialgebraic sets for which these classical results are true.

Keywords: Nash function; Nash set; irreducible semialgebraic set; irreducible components of a semialgebraic set; w-Nash set; q-Nash set; substitution theorem; positivstellensätze; 17th Hilbert problem and real Nullstellensatz.

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1. Introduction

This work is motivated by the search of a satisfactory notion of *irreducibility* for semialgebraic sets, and, consequently, by the elaboration of a theory of *irreducible components* for semialgebraic sets enjoying a similar behavior to the theory of irreducible components in other settings like: algebraic sets, global analytic subsets of $\mathbb{R}^n$ and Nash subsets of open semialgebraic sets in $\mathbb{R}^n$, among others. Recall that $S \subset \mathbb{R}^n$ is a *semialgebraic set* when it has a description by a finite boolean combination of polynomial equations and inequalities. A first attempt to define what an irreducible semialgebraic set is, has been already approached in the literature (see [12]): it defines a semialgebraic set $S \subset \mathbb{R}^n$ to be irreducible if its Zariski closure $\overline{S}^{\text{zar}}$ in $\mathbb{R}^n$ (that is, the smallest algebraic subset of $\mathbb{R}^n$ containing S) is an irreducible algebraic set. With this definition the semialgebraic sets

$$S_1 = \{y^2 - x^2(x+1) = 0\}\setminus\{(-1,0)\} \quad \text{and} \quad S_2 = \{xy = 1\}$$

are irreducible, but we feel that "they should be reducible": S_1 consists of two analytic branches and S_2 is not connected.

In the algebraic, global analytic or Nash settings, a geometric object is irreducible if it is not the union of two proper geometric objects of the same nature. In our semialgebraic setting, this definition does not work because the complement of a semialgebraic set is again semialgebraic and so each semialgebraic set with at least two points would be reducible. Nevertheless, in the algebraic, global analytic or Nash settings, the irreducibility of a geometric object $X \subset \mathbb{R}^n$ is equivalent to the fact that the respective ring of polynomial, analytic or Nash functions on X is an integral domain. This equivalence suggests us to attach to each semialgebraic set S a suitable ring $\mathcal{F}(S)$ of real valued functions and to define the irreducibility of S by saying that $\mathcal{F}(S)$ is an integral domain. Thus, the first crucial decision is to choose the nature of such functions. One easily convinces himself that neither polynomials nor $\mathcal{C}^r$ semialgebraic functions for $0 \le r < +\infty$ fit our situation. In fact, it seems natural to choose the ring $\mathcal{N}(S)$ of Nash functions on S because it combines the semialgebraic flavor with the existence of an Identity Principle. Recall that $f : S \to \mathbb{R}$ is a *Nash function* if there are an open semialgebraic neighborhood U of S in $\mathbb{R}^n$ and an extension $F : U \to \mathbb{R}$ of f which is Nash on U, that is, F is a $\mathcal{C}^\infty$ function with semialgebraic graph, or equivalently, F is analytic on U and there is a nonzero polynomial $P \in \mathbb{R}[\mathtt{x}_1, \ldots, \mathtt{x}_n, \mathtt{y}]$ such that $P(x, F(x)) = 0$ for all $x \in U$.

With these preliminaries, a semialgebraic set $S \subset \mathbb{R}^n$ is *irreducible* if the ring $\mathcal{N}(S)$ is an integral domain. As one can expect our notion extends the notion of irreducibility of a Nash subset X of an open semialgebraic set $U \subset \mathbb{R}^n$. Even more, we develop a satisfactory theory of irreducible components for arbitrary semialgebraic sets and it holds that for such an X the irreducible components of X as a Nash subset of U coincide with its irreducible components as a semialgebraic set. The study of the irreducibility of semialgebraic sets is performed in Sec. 3 while the development of the theory of irreducible components of semialgebraic sets appears in Sec. 4.

We should not forget the possibility of a different approach to the irreducibility of semialgebraic sets by using analytic functions. Indeed, if $\mathcal{O}(S)$ denotes the ring of real valued functions $f : S \to \mathbb{R}$ which admit an analytic extension to an open (non-necessarily semialgebraic) neighborhood of S in $\mathbb{R}^n$, it follows from Theorem 3.2 that $\mathcal{N}(S)$ is a domain if and only if $\mathcal{A}(S)$ is a domain for every intermediate ring $\mathcal{N}(S) \subset \mathcal{A}(S) \subset \mathcal{O}(S)$ and so if we choose any of such rings to define the notion of irreducibility for semialgebraic sets we achieve the same family of irreducible semialgebraic sets. Any case, the noetherianity of $\mathcal{N}(S)$ (see Sec. 2) and the semialgebraic nature of its functions suggest to use this ring to determine the irreducibility of semialgebraic sets.

There are four fundamental classical results associated to the study of a ring of real valued functions on a real geometric object: the Substitution Theorem, the Positivstellensätze, the 17th Hilbert Problem and the real Nullstellensatz. As it is well known, if $S = M$ is an affine Nash manifold, the previous celebrated questions have been completely solved in the affirmative for its ring of Nash functions (see for instance [4, 8.5 and 8.6]). As one can expect, those results are not true for arbitrary semialgebraic sets and the second part of this work (see Secs. 5 and 6) is devoted to determine the families of semialgebraic sets for which the previous classical problems have an affirmative solution.

2. Preliminaries

In this section we introduce most of the definitions, notations and objects that appear in the subsequent sections. We also present some preliminary results that will be useful in the sequel. We begin recalling the following semialgebraic version of Tietze–Urysohn extension lemma (see [8]).

Lemma 2.1. *Let $C \subset S \subset \mathbb{R}^n$ be semialgebraic sets such that C is closed in S. Then, each continuous semialgebraic function $f: C \to \mathbb{R}$ extends to a continuous semialgebraic function $F: S \to \mathbb{R}$.*

Remark 2.2. By means of Lemma 2.1, one proves that two disjoint closed semialgebraic subsets $C_1, C_2 \subset S$ can be separated by disjoint open semialgebraic subsets U_1, U_2 of $\mathbb{R}^n$. This fact will be used freely along this work and it will be useful, for instance, to separate the connected components of a semialgebraic set.

Next, we propose a careful presentation of the ring of Nash functions of a semialgebraic set. Namely,

(2.3) Ring of Nash functions on a semialgebraic set. Let $S \subset \mathbb{R}^n$ be a semialgebraic set. We say that a function $f: S \to \mathbb{R}$ is a *Nash function on S* if there are an open semialgebraic neighborhood U of S in $\mathbb{R}^n$ and a smooth function $F: U \to \mathbb{R}$ with semialgebraic graph, that is, a Nash function in the classical sense, such that $f = F|_S$. We denote by $\mathcal{N}(S)$ the set of Nash functions on S, which endowed with the usual operations has the structure of $\mathbb{R}$-algebra. As one can expect, if $\mathtt{S}_1, \dots, \mathtt{S}_r$ are the connected components of S, we have $\mathcal{N}(S) = \bigoplus_{i=1}^r \mathcal{N}(\mathtt{S}_i)$. Of course, if $S = M$ is an affine Nash manifold, one realizes immediately, by means of a Nash tubular neighborhood of M in $\mathbb{R}^n$, that $\mathcal{N}(M)$ is the classical $\mathbb{R}$-algebra of Nash functions on M.

More generally, given two semialgebraic sets $S \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^m$, we say that a semialgebraic map $f = (f_1, \dots, f_m)$ is a *Nash map*, and we write $f \in \mathcal{N}(S, T)$, if there exist an open semialgebraic neighborhood U of S in $\mathbb{R}^n$ and a Nash map $F: U \to \mathbb{R}^m$ such that $F|_S = f$. Of course, this property is equivalent to say that $\operatorname{im} f \subset T$ and each component $f_i \in \mathcal{N}(S)$ for $i = 1, \dots, m$. Moreover, a map $f \in \mathcal{N}(S, T)$ is said to be a *Nash diffeomorphism* if there exists $g \in \mathcal{N}(T, S)$ such that $f \circ g = \mathrm{id}_T$ and $g \circ f = \mathrm{id}_S$. In such case we say that S and T are *Nash diffeomorphic*; in particular, S and T are semialgebraically homeomorphic.

Given $f \in \mathcal{N}(S)$, we denote its zeroset by $\mathcal{Z}_S(f) = \{x \in S : f(x) = 0\}$, and for each ideal $\mathfrak{a}$ of $\mathcal{N}(S)$, its zeroset is the intersection

$$\mathcal{Z}_S(\mathfrak{a}) = \bigcap_{f\in\mathfrak{a}} \mathcal{Z}_S(f) = \{x \in S : f(x) = 0,\ \forall f \in \mathfrak{a}\}.$$

As one can expect, see Lemma 2.4 below, $\mathcal{Z}_S(\mathfrak{a})$ is a semialgebraic set. Given any semialgebraic subset $T \subset S$, we denote $\mathcal{J}_S(T) = \{g \in \mathcal{N}(S) : g(x) = 0\ \forall x \in T\}$, which is an ideal of $\mathcal{N}(S)$.

Lemma 2.4. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set and let $\mathfrak{a}$ be a proper ideal of $\mathcal{N}(S)$. Then, there exists $f \in \mathfrak{a}$ such that $\mathcal{Z}_S(\mathfrak{a}) = \mathcal{Z}_S(f)$. In particular, $\mathcal{Z}_S(\mathfrak{a})$ is a nonempty semialgebraic set.*

Proof. We denote $d = \dim S$ and consider for each integer $k = 0, \ldots, d+1$ the sentence:

$\mathcal{S}_k$: *There exists a function $f_k \in \mathfrak{a}$ such that the difference $\mathcal{Z}_S(f_k)\setminus\mathcal{Z}_S(\mathfrak{a})$ is contained in a semialgebraic subset of S of dimension $< k$.*

Observe that $\mathcal{S}_{d+1}$ is obviously true by choosing $f_{d+1} = 0$, whose zeroset is S. All reduces to show that also $\mathcal{S}_0$ is true and in fact it suffices to show that $\mathcal{S}_{k+1} \Longrightarrow \mathcal{S}_k$.

Indeed, let Y be a semialgebraic subset of S of dimension $< k+1$ containing $\mathcal{Z}_S(f_{k+1})\setminus\mathcal{Z}_S(\mathfrak{a})$ for a certain function $f_{k+1} \in \mathfrak{a}$. By [4, 2.9.10], Y is a finite union $Y = \bigcup_{i=1}^{\ell} N_i$ of connected Nash submanifolds N_i of S. Since $\mathcal{Z}_S(f_{k+1})\setminus\mathcal{Z}_S(\mathfrak{a}) \subset \bigcup_{i=1}^{\ell} N_i$ we may assume, without loss of generality, that $\mathcal{Z}_S(\mathfrak{a})$ contains no N_i. Therefore, for each index $1 \le i \le \ell$ there exists $h_i \in \mathfrak{a}$ such that $\mathcal{Z}_S(h_i)$ does not contain N_i, and so each restriction $h_i|_{N_i} \in \mathcal{N}(N_i)$ does not vanish identically on N_i. Since N_i is a connected affine Nash manifold, $\dim \mathcal{Z}_{N_i}(h_i|_{N_i}) < \dim N_i$. The Nash function $f_k = f_{k+1}^2 + \sum_{i=1}^{\ell} h_i^2 \in \mathfrak{a}$ satisfies our requirements. Indeed, let $T = \bigcup_{i=1}^{\ell} \mathcal{Z}_{N_i}(h_i|_{N_i})$ and observe that

$$\begin{aligned}\mathcal{Z}_S(f_k)\setminus\mathcal{Z}_S(\mathfrak{a}) &= \left(\mathcal{Z}_S(f_{k+1}) \cap \bigcap_{j=1}^{\ell} \mathcal{Z}_S(h_j)\right) \Big\backslash\, \mathcal{Z}_S(\mathfrak{a}) \subset \bigcup_{i=1}^{\ell} N_i \cap \bigcap_{j=1}^{\ell} \mathcal{Z}_S(h_j)\\ &\subset \bigcup_{i=1}^{\ell}(N_i \cap \mathcal{Z}_S(h_i)) = \bigcup_{i=1}^{\ell} \mathcal{Z}_{N_i}(h_i|_{N_i}) = T,\end{aligned}$$

and $\dim T < k$, because

$$\begin{aligned}\dim T &= \max\{\dim(\mathcal{Z}_{N_i}(h_i|_{N_i})) : 1 \le i \le \ell\}\\ &< \max\{\dim N_i : 1 \le i \le \ell\} = \dim Y < k+1.\end{aligned}$$

We have just proved that $\mathcal{Z}_S(\mathfrak{a}) = \mathcal{Z}_S(f)$ for a certain $f \in \mathfrak{a}$. Suppose now that $\mathcal{Z}_S(\mathfrak{a})$ is empty and let $F \in \mathcal{N}(U)$ be a Nash extension of f to an open semialgebraic neighborhood U of S in $\mathbb{R}^n$. Then, the open semialgebraic set $V = U \setminus \mathcal{Z}_U(F)$ contains S and the function $G = 1/F \in \mathcal{N}(V)$; hence, $g = G|_S \in \mathcal{N}(S)$ and $1 = gf \in \mathfrak{a}$, a contradiction $\square$

Corollary 2.5. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set. Then, each maximal ideal of $\mathcal{N}(S)$ has the form $\mathfrak{n}_a = \{f \in \mathcal{N}(S) : f(a) = 0\}$ for some point a of S.*

As a consequence of Corollary 2.5, we prove that the rings of Nash functions classify semialgebraic sets modulo Nash diffeomorphism. Namely,

Corollary 2.6. *Let $S \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^m$ be semialgebraic sets. The following assertions are equivalent*:

(i) *S and T are Nash diffeomorphic.*
(ii) *The $\mathbb{R}$-algebras $\mathcal{N}(S)$ and $\mathcal{N}(T)$ are isomorphic.*

Proof. The implication (i) $\implies$ (ii) is easy; hence, we focus on the proof of its converse. Let $\varphi : \mathcal{N}(S) \to \mathcal{N}(T)$ be an isomorphism and let $\widetilde{\varphi} : \operatorname{Spec}(\mathcal{N}(T)) \to \operatorname{Spec}(\mathcal{N}(S))$, $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$ be the homeomorphism between the Zariski spectra of $\mathcal{N}(T)$ and $\mathcal{N}(S)$ induced by φ, which preserves closed points (maximal ideals). By Corollary 2.5, the maximal ideals of $\mathcal{N}(T)$ and $\mathcal{N}(S)$ correspond to the points of T and S; hence, we get the following homeomorphism

$$g = \widetilde{\varphi}|_T : T \equiv \operatorname{Max}(\mathcal{N}(T)) \to \operatorname{Max}(\mathcal{N}(S)) \equiv S,\ q \equiv \mathfrak{m}_q \mapsto \varphi^{-1}(\mathfrak{m}_q) = \mathfrak{m}_p \equiv p,$$

which is, in fact, a Nash diffeomorphism. Indeed, write $g_i = \varphi(\mathtt{x}_i) \in \mathcal{N}(T)$ and notice that $g_i(q) = p_i$ for $i = 1, \dots, n$; this is so because the Nash function $\mathtt{x}_i - p_i \in \mathfrak{m}_p = \varphi^{-1}(\mathfrak{m}_q)$ and so $g_i - p_i = \varphi(\mathtt{x}_i - p_i) \in \mathfrak{m}_q$ for $i = 1, \dots, n$. Thus, $g(q) = p = (p_1, \dots, p_n) = (g_1, \dots, g_n)(q)$, and so $g = (g_1, \dots, g_n) \in \mathcal{N}(T, S)$. Analogously, using now $\varphi^{-1} : \mathcal{N}(T) \to \mathcal{N}(S)$, one constructs a Nash map $f \in \mathcal{N}(S, T)$ such that $f \circ g = \mathrm{id}_T$ and $g \circ f = \mathrm{id}_S$; hence, g is a Nash diffeomorphism and we are done. $\square$

To approach the noetherianity of the ring of Nash functions on a semialgebraic set (see Theorem 2.9) we introduce the ring of Nash function germs at a semialgebraic set.

(2.7) Ring of Nash function germs at a semialgebraic set. Given a semialgebraic set $S \subset \mathbb{R}^n$, germs of Nash functions at S are defined similarly as germs at a point, through open semialgebraic neighborhoods of S in $\mathbb{R}^n$; we denote by $F_S \equiv F_{U,S}$ the germ at S of a Nash function F defined on an open semialgebraic neighborhood U of S in $\mathbb{R}^n$. The collection $\mathcal{N}(\mathbb{R}^n_S)$ of all germs of Nash functions at S endowed with the natural operations has an $\mathbb{R}$-algebra structure and will be called the *ring of Nash function germs at S*. Again, as one can expect, if $\mathtt{S}_1, \dots, \mathtt{S}_r$ are the connected components of S, we have $\mathcal{N}(\mathbb{R}^n_S) = \bigoplus_{i=1}^r \mathcal{N}(\mathbb{R}^n_{\mathtt{S}_i})$ and so to study many algebraic problems concerning the ring $\mathcal{N}(\mathbb{R}^n_S)$ it is enough to consider the case in which S is connected.

Notice that $\mathcal{N}(S) \equiv \mathcal{N}(\mathbb{R}^n_S)/\mathcal{J}(S)$, where $\mathcal{J}(S) = \{F_S \in \mathcal{N}(\mathbb{R}^n_S) : F(x) = 0\ \forall x \in S\}$. Of course, it makes sense to say that an element F_S in $\mathcal{N}(\mathbb{R}^n_S)$ vanishes at a point $x \in S$ if $F(x) = 0$ and we denote $\mathcal{Z}_S(F_S) = \{x \in S : F(x) = 0\}$.

Next, fix an open semialgebraic neighborhood U of S in $\mathbb{R}^n$ and consider the restriction homomorphism $\rho_{U,S} : \mathcal{N}(U) \to \mathcal{N}(S),\ F \to F|_S$ and the natural $\mathbb{R}$-algebras homomorphism $\eta_{U,S} : \mathcal{N}(U) \to \mathcal{N}(\mathbb{R}^n_S),\ F \mapsto F_{U,S}$; of course, we have

$$\rho_{U,S} : \mathcal{N}(U) \xrightarrow{\eta_{U,S}} \mathcal{N}(\mathbb{R}^n_S) \xrightarrow{\pi} \mathcal{N}(S) \equiv \mathcal{N}(\mathbb{R}^n_S)/\mathcal{J}(S). \qquad (*)$$

Notice that $\eta_{U,S}$ is injective if and only if each connected component of U intersects S. In any case, by abuse of notation, given an ideal $\mathfrak{A}$ of $\mathcal{N}(\mathbb{R}^n_S)$ we denote by $\mathfrak{A} \cap \mathcal{N}(U)$ the preimage $\eta^{-1}_{U,S}(\mathfrak{A})$ and if $\mathfrak{a} = \mathfrak{A}/\mathcal{J}(S)$ is an ideal of $\mathcal{N}(S)$, we denote $\mathfrak{a} \cap \mathcal{N}(\mathbb{R}^n_S) = \pi^{-1}(\mathfrak{a}) = \mathfrak{A}$ and $\mathfrak{a} \cap \mathcal{N}(U) = \rho^{-1}_{U,S}(\mathfrak{a}) = \mathfrak{A} \cap \mathcal{N}(U)$. In particular, $\mathcal{J}_U(S) = \mathcal{J}(S) \cap \mathcal{N}(U)$.

Corollary 2.8. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set. Then, each maximal ideal of $\mathcal{N}(\mathbb{R}^n_S)$ has the form $\mathfrak{m}_a = \{F_S \in \mathcal{N}(\mathbb{R}^n_S) : F(a) = 0\}$ for some point $a \in S$ and the zero set of each proper ideal $\mathfrak{a}$ of $\mathcal{N}(\mathbb{R}^n_S)$ is nonempty.*

Proof. First, let $\mathfrak{m}$ be a maximal ideal of $\mathcal{N}(\mathbb{R}^n_S)$. The inclusion $\mathcal{J}(S) \subset \mathfrak{m}$ is proved straightforwardly by way of contradiction; hence, the map $\mathfrak{m} \mapsto \mathfrak{m}/\mathcal{J}(S)$ between the respective sets of maximal ideals of $\mathcal{N}(\mathbb{R}^n_S)$ and $\mathcal{N}(S)$ is well defined and, by the Correspondence Theorem for quotient rings, it is bijective. Thus, each maximal ideal of $\mathcal{N}(\mathbb{R}^n_S)$ has, by Corollary 2.5, the form in the statement. Next, if $\mathfrak{a}$ is a proper ideal of $\mathcal{N}(\mathbb{R}^n_S)$, its zero set is nonempty because $\mathfrak{a}$ is contained in a maximal ideal of $\mathcal{N}(\mathbb{R}^n_S)$. $\square$

Theorem 2.9. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set. Then, the rings $\mathcal{N}(S)$ and $\mathcal{N}(\mathbb{R}^n_S)$ are noetherian.*

Proof. First, since $\mathcal{N}(S) \equiv \mathcal{N}(\mathbb{R}^n_S)/\mathcal{J}(S)$, it is enough to prove that $\mathcal{N}(\mathbb{R}^n_S)$ is noetherian. Moreover, since $\mathcal{N}(\mathbb{R}^n_S) = \bigoplus_{i=1}^r \mathcal{N}(\mathbb{R}^n_{\mathtt{S}_i})$ where $\mathtt{S}_i$ are the connected components of S, we may assume for the rest of the proof that S is connected. Now, the proof of the noetherianity of $\mathcal{N}(\mathbb{R}^n_S)$ runs similarly to the one provided in [4, Sec. 8.7], for the noetherianity of the ring $\mathcal{N}(M)$ of Nash functions on a connected affine Nash manifold $M \subset \mathbb{R}^n$. We point out next (without complete proofs) the main steps one can follow to prove Theorem 2.9 indicating the corresponding result of [4, Sec. 8.7], whose proof is similar. Let $\mathbb{R}[\mathtt{x}] = \mathbb{R}[\mathtt{x}_1, \ldots, \mathtt{x}_n]$ denote the polynomial ring in n variables with coefficients in $\mathbb{R}$ and let $(\mathtt{x} - a) = (\mathtt{x}_1 - a_1, \ldots, \mathtt{x}_n - a_n)$ denote the maximal ideal of $\mathbb{R}[\mathtt{x}]$ of the polynomials in $\mathbb{R}[\mathtt{x}]$ which vanish at the point $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. We write $\mathbb{R}[\mathtt{x}]_{(\mathtt{x}-a)}$ to refer to the localization of $\mathbb{R}[\mathtt{x}]$ at the maximal ideal $(\mathtt{x} - a)$ and $\mathcal{N}_a$ to denote the ring of Nash function germs on $\mathbb{R}^n$ at a. Now, we fix a point $a \in S$.

Property A. The local rings in the diagram $\mathbb{R}[\mathtt{x}]_{(\mathtt{x}-a)} \hookrightarrow \mathcal{N}(\mathbb{R}^n_S)_{\mathfrak{m}_a} \hookrightarrow \mathcal{N}_a \hookrightarrow \widehat{\mathcal{N}_a}$ are all regular of dimension n and all the inclusions are faithfully flat.

To prove this fact proceed similarly to the proofs of [4, 8.7.12 and 8.7.16].

Property B. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{R}[\mathtt{x}]_{(\mathtt{x}-a)}$. Then, the set of prime ideals $\mathfrak{q}$ of $\mathcal{N}(\mathbb{R}^n_S)_{\mathfrak{m}_a}$ such that $\mathfrak{q}\cap\mathbb{R}[\mathtt{x}]_{(\mathtt{x}-a)}=\mathfrak{p}$ is finite. Moreover, if $\{\mathfrak{q}_1,\ldots,\mathfrak{q}_m\}$ is the set of prime ideals of $\mathcal{N}(\mathbb{R}^n_S)_{\mathfrak{m}_a}$ lying over $\mathfrak{p}$, then $\mathfrak{p}\mathcal{N}(\mathbb{R}^n_S)_{\mathfrak{m}_a}=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_m$ and $\operatorname{ht}(\mathfrak{q}_i)=\operatorname{ht}(\mathfrak{p})$ for each $1\le i\le m$.

The proof of Property B runs similarly to the one of [4, 8.7.15].

Property C. Let $\mathfrak{p}$ be a prime ideal of the polynomial ring $\mathbb{R}[\mathtt{x}]$. Then, the set of prime ideals $\mathfrak{q}$ of $\mathcal{N}(\mathbb{R}^n_S)$ such that $\mathfrak{q}\cap\mathbb{R}[\mathtt{x}]=\mathfrak{p}$ is finite. Moreover, if $\{\mathfrak{q}_1,\ldots,\mathfrak{q}_m\}$ is the set of prime ideals of $\mathcal{N}(\mathbb{R}^n_S)$ lying over $\mathfrak{p}$, then $\mathfrak{p}\mathcal{N}(\mathbb{R}^n_S)=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_m$ and $\operatorname{ht}(\mathfrak{q}_i)=\operatorname{ht}(\mathfrak{p})$ for each $1\le i\le m$.

To that end it suffices to follow the proof of [4, 8.7.17], with a psychological advantage: while in [4] the authors deal with the ring of polynomial functions on the complexification of an irreducible nonsingular real affine algebraic set V, in our case $V=\mathbb{R}^n$. Moreover, where the authors deal with a connected open semialgebraic subset M of V, we use a connected open semialgebraic neighborhood M of S in $\mathbb{R}^n$. Of course, in our case, we use Property A instead of [4, 8.7.16] and Property B instead of [4, 8.7.15].

Property D. $\mathcal{N}(\mathbb{R}^n_S)$ is a noetherian ring.

By [14, Sec. 2, Theorem II.2], it is enough to prove that all prime ideals of $\mathcal{N}(\mathbb{R}^n_S)$ are finitely generated. Let $\mathfrak{q}$ be a prime ideal of $\mathcal{N}(\mathbb{R}^n_S)$ with $\operatorname{ht}(\mathfrak{q})\ge n$. Then, $\mathfrak{q}$ is, by Property A, a maximal ideal and $\operatorname{ht}(\mathfrak{q})=n$; hence, there is, by Corollary 2.8, a point $a\in S$ such that $\mathfrak{q}=\mathfrak{m}_a$ and so $\mathfrak{m}_a\cap\mathbb{R}[\mathtt{x}]=(\mathtt{x}-a)$. Since the inclusion $\mathbb{R}[\mathtt{x}]_{(\mathtt{x}-a)}\hookrightarrow\mathcal{N}(\mathbb{R}^n_S)_{\mathfrak{m}_a}$ is, by Property A, faithfully flat, the equality $\mathfrak{q}=\mathfrak{m}_a=(\mathtt{x}-a)\mathcal{N}(\mathbb{R}^n_S)$ follows, proving that $\mathfrak{q}$ is finitely generated. Once this is proved the rest of the proof runs in the same way as [4, 8.7.18], substituting the ring $\mathcal{N}(M)$ by the ring $\mathcal{N}(\mathbb{R}^n_S)$ and using Property B instead of [4, 8.7.17]. □

(2.10) Analytic, Nash and Zariski closures of a semialgebraic set. Let $U\subset\mathbb{R}^n$ be an open semialgebraic set and let $S\subset U$ be a semialgebraic subset. Since $\mathcal{N}(U)$ is a noetherian ring, the zeroset $\mathcal{Z}_U(\mathfrak{A}\cap\mathcal{N}(U))$ is a Nash subset of U for any ideal $\mathfrak{A}$ of $\mathcal{N}(\mathbb{R}^n_S)$. Notice that $\mathcal{Z}_U(\mathcal{J}(S)\cap\mathcal{N}(U))=\mathcal{Z}_U(\mathcal{J}_U(S))$ is the *Nash closure of S in U*, that is, the smallest Nash subset of U containing S. Moreover, $\mathcal{J}_U(\mathcal{Z}_U(\mathcal{J}_U(S)))=\mathcal{J}(S)\cap\mathcal{N}(U)$.

Next, consider also the ideal $\mathcal{J}^{\mathrm{an}}_U(S)=\{F\in\mathcal{O}(U):F(x)=0,\ \forall\,x\in S\}$. We will prove that the Nash closure $\mathcal{Z}_U(\mathcal{J}_U(S))$ of S in U coincides with the zeroset $\overline{S}^{\mathrm{an}}_U$ of $\mathcal{J}^{\mathrm{an}}_U(S)$ in U, which is the smallest global analytic subset of U containing S. Consequently, for notational and conceptual simplicity we will denote by $\overline{S}^{\mathrm{an}}_U$ the Nash closure of S in U.

We denote by $\overline{S}^{\mathrm{zar}}$ the Zariski closure of S in $\mathbb{R}^n$, that is, the smallest algebraic subset of $\mathbb{R}^n$ containing S. By [4, 2.8.2], S and $\overline{S}^{\mathrm{zar}}$ have the same dimension.

Thus, since $S \subset \overline{S}_U^{\text{an}} \subset \mathcal{Z}_U(\mathcal{J}_U(S)) \subset \overline{S}^{\text{zar}}$, the dimensions of all the previous sets coincide.

(2.11) Analytic and Nash closures of a semialgebraic set germ. The situation is rather similar in the local case. Namely, let $a \in S$ and denote by $\mathcal{O}_a$, (respectively, $\mathcal{N}_a$) the ring of analytic, (respectively, Nash) functions germs on $\mathbb{R}^n$ at a. Consider the ideal

$$\mathcal{J}^{\text{an}}(S_a) = \{F_a \in \mathcal{O}_a : S_a \subset \mathcal{Z}_a(F_a)\},$$

whose zeroset $\overline{S_a}^{\text{an}}$ is the smallest analytic germ at a which contains the germ S_a. In fact, as we will check soon, $\overline{S_a}^{\text{an}}$ is a Nash set germ and so it coincides with the Nash closure of S_a, which is the zeroset germ $\overline{S_a}^{\text{Nash}}$ at a of the ideal $\mathcal{I}_a(S_a) = \{F_a \in \mathcal{N}_a : S_a \subset \mathcal{Z}_a(F_a)\}$. Thus, for simplicity, we denote by $\overline{S}_a^{\text{an}}$ the Nash closure of the germ S_a.

Let W be an open semialgebraic neighborhood of a in $\mathbb{R}^n$ such that $\dim S_a = \dim(S \cap W)$. It is clear that $S_a = (S \cap W)_a$. Now, since

$$S_a \subset \overline{S_a}^{\text{an}} \subset \overline{S_a}^{\text{Nash}} \subset (\overline{S \cap W}^{\text{zar}})_a$$

and $\dim S_a = \dim(S \cap W) = \dim \overline{S \cap W}^{\text{zar}}$, we deduce that all the involved germs have the same dimension.

Proof of (2.11). We have to prove that $\overline{S_a}^{\text{an}}$ is a Nash set germ. Let $Y_{1,a}, \ldots, Y_{r,a}$ be the irreducible components of the analytic germ $\overline{S_a}^{\text{an}}$ and let $X_{1,a}, \ldots, X_{s,a}$ be the irreducible components of the Nash germ $\overline{S_a}^{\text{Nash}}$, which are, by [4, 8.6.9], irreducible analytic germs. We may assume that $\dim Y_{1,a} = \dim S_a$. Since

$$Y_{1,a} \subset \overline{S_a}^{\text{an}} \subset \overline{S_a}^{\text{Nash}} = \bigcup_{j=1}^{s} X_{j,a}$$

and $Y_{1,a}$ is an irreducible analytic germ, we may assume that $Y_{1,a} \subset X_{1,a}$. In fact, $Y_{1,a} = X_{1,a}$ because $\dim Y_{1,a} = \dim X_{1,a}$ and $X_{1,a}$ is an irreducible analytic germ. This argument works for all the irreducible components $Y_{j,a}$ of $\overline{S_a}^{\text{an}}$ which have maximal dimension, say $Y_{1,a}, \ldots, Y_{\ell,a}$. Thus, we may assume that $Y_{j,a} = X_{j,a}$ for $j = 1, \ldots, \ell$.

Consider the semialgebraic set germ $T_a = S_a \setminus \bigcup_{j=1}^{\ell} X_{j,a} = S_a \setminus \bigcup_{j=1}^{\ell} Y_{j,a}$ which has dimension $< \dim S_a$ because it is contained in $\bigcup_{j=\ell+1}^{r} Y_{j,a}$. Arguing by induction on the dimension, we deduce that $\overline{T_a}^{\text{an}} = \overline{T_a}^{\text{Nash}}$. Finally, since $\overline{S_a}^{\text{an}} = \overline{T_a}^{\text{an}} \cup \bigcup_{j=1}^{\ell} Y_{j,a}$, we are done. $\square$

Sketch of proof of (2.10). We have to prove that $\overline{S}_U^{\text{an}}$ is a Nash subset of U. Recall that $\dim S = \dim \overline{S}_U^{\text{an}} = \dim(\mathcal{Z}_U(\mathcal{J}_U(S)))$. Now, the strategy of the proof runs in the same way as the one for germs, but using the following Property:

Property E. An irreducible Nash subset X of U is also an irreducible global analytic subset of U.

To prove this, we proceed as follows. By [7, Corollary 2], $\mathfrak{p} = \mathcal{J}_U(X)\mathcal{O}(U) \subset \mathcal{J}_U^{\mathrm{an}}(X)$ is a prime ideal of $\mathcal{O}(U)$. Now, since $\dim X = \dim \overline{X}^{\mathrm{zar}}$, we can choose a point $x \in X \setminus \operatorname{Sing}(\overline{X}^{\mathrm{zar}})$, which is a regular point of $\overline{X}^{\mathrm{zar}}$ in the sense of [4, 3.3.9]. Notice that the germs X_x and $\overline{X}_x^{\mathrm{zar}}$ coincide. Now, by [4, 3.3.10(iii)], we deduce that the analytic ring $\mathcal{O}_x/(\mathfrak{p}_x\mathcal{O}_x)$ is regular of dimension d; hence $\mathfrak{p}\mathcal{O}_x$ is a real ideal in the sense of [4, 4.1.3]. Thus, by [1, 3.1], we deduce that $\mathfrak{p} = \mathcal{J}_U^{\mathrm{an}}(X)$. Since $\mathfrak{p}$ is prime, it follows that $X = Z_U(\mathcal{J}_U(X)) = Z_U(\mathcal{J}_U(\mathcal{J}_U^{\mathrm{an}}(X))) = X_U^{\mathrm{an}}$ is an irreducible global analytic subset of U. $\square$

When we have referred to a Nash set S we have always involved certain open semialgebraic neighborhood on which S has finitely many Nash equations. To get rid of this fact, we introduce the following result.

(2.12) Nash sets. Let $S \subset \mathbb{R}^n$ be a semialgebraic set. The following assertions are equivalent:

(i) There is an open semialgebraic neighborhood V of S in $\mathbb{R}^n$ such that S is a Nash subset of V.
(ii) There is an open semialgebraic neighborhood V of S in $\mathbb{R}^n$ such that $S = \overline{S}_V^{\mathrm{an}}$.
(iii) The semialgebraic set $U_0 = \mathbb{R}^n \setminus (\mathrm{Cl}_{\mathbb{R}^n}(S) \setminus S)$ is an open neighborhood of S in $\mathbb{R}^n$ and S is a Nash subset of U_0.
(iv) S is locally compact and a Nash subset of each open semialgebraic open neighborhood of S on which S is closed.

If S satisfies one of the above conditions (and hence all), we say that S is a Nash set.

Proof. The implications (i) $\Longrightarrow$ (ii), (iii) $\Longrightarrow$ (iv) and (iv) $\Longrightarrow$ (i) are straightforward. Only the implication (ii) $\Longrightarrow$ (iii) requires some explanation. Suppose that $S = \overline{S}_V^{\mathrm{an}}$ for an open semialgebraic subset V of $\mathbb{R}^n$ containing S. Then, S is closed in V and so it is locally compact; hence, $\mathrm{Cl}_{\mathbb{R}^n}(S) \setminus S$ is closed in $\mathbb{R}^n$ and so $U_0 = \mathbb{R}^n \setminus (\mathrm{Cl}_{\mathbb{R}^n}(S) \setminus S)$ is an open semialgebraic subset of $\mathbb{R}^n$ containing S as a closed subset. In fact, U_0 is the largest open subset of $\mathbb{R}^n$ in which S is closed; hence, $V \subset U_0$ and, since S is a Nash subset of V it follows, by [21, II.5.3], that S is a Nash subset of U_0. $\square$

We end this section by recalling some main properties of the set of regular points of a semialgebraic set, that will be useful in the sequel.

(2.13) Regular points of a semialgebraic set. Let $S \subset \mathbb{R}^n$ be a d-dimensional semialgebraic set. A point $x \in S$ is a *regular point* of S if there is an open neighborhood V^x of x in S analytically diffeomorphic to $\mathbb{R}^d$. We denote by $\operatorname{Reg}(S)$ *the set of regular points of* S and by $\delta(S) = S \setminus \operatorname{Reg}(S)$ the *set of nonregular points of* S. By [22], $\operatorname{Reg}(S)$ is a nonempty open semialgebraic subset of S, and since it is an analytic manifold, we deduce from [21, 1.3.9], that $\operatorname{Reg}(S)$ is an affine Nash manifold. Moreover, $\delta(S)$ is a semialgebraic set of dimension $\leq d-1$ closed in S. Of course, $\operatorname{Reg}(S') \subset \operatorname{Reg}(S)$ and $\delta(S') \subset \delta(S)$ for each semialgebraic set $S' \subset S$ which is open in S; moreover, if $S \subset \mathbb{R}^n$ is pure dimensional, $S \subset \operatorname{Cl}_{\mathbb{R}^n}(\operatorname{Reg}(S) \setminus T)$ for each semialgebraic subset T of S with $\dim T \leq d-1$. In particular, $\operatorname{Reg}(S)$ is a dense subset of every pure dimensional semialgebraic set S.

3. Irreducible Semialgebraic Sets

In this section we introduce and explore the notion of irreducible semialgebraic set.

(3.1) Irreducibility of semialgebraic sets. Given a semialgebraic set $S \subset \mathbb{R}^n$, we say that S is *irreducible* if the ring $\mathcal{N}(S)$ is an integral domain; otherwise, S is *reducible*. One deduces straightforwardly the following facts concerning irreducibility:

(i) Irreducible semialgebraic sets are connected, because the ring of Nash functions of a disconnected semialgebraic set is the direct sum of the rings of Nash functions of its connected components (see (2.3)), and so, it contains zero divisors. In particular, an affine Nash manifold is irreducible if and only if it is connected.

(ii) The Zariski closure of an irreducible semialgebraic set is irreducible as an algebraic set. Of course, there are many irreducible algebraic sets which are reducible as semialgebraic sets; for instance, the hyperbola $S = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$.

(iii) Any semialgebraic set compressed between an irreducible semialgebraic set and its closure in $\mathbb{R}^n$ is also irreducible.

(iv) The image of an irreducible semialgebraic set under a Nash map is also irreducible. In particular, the irreducibility of semialgebraic sets is preserved by Nash diffeomorphisms.

(v) Let $T \subset S \subset \mathbb{R}^n$ be semialgebraic sets such that T is irreducible. Then, $\mathcal{J}_S(T)$ is a prime ideal of $\mathcal{N}(S)$, because $\mathcal{N}(T)$ is an integral domain and $\mathcal{J}_S(T)$ is the kernel of the restriction homomorphism $\mathcal{N}(S) \to \mathcal{N}(T),\ f \mapsto f|_T$.

(vi) A Nash set X is irreducible as a semialgebraic set if and only if it is irreducible as a Nash subset of $U_0 = \mathbb{R}^n \setminus (\operatorname{Cl}_{\mathbb{R}^n}(X) \setminus X)$. If such is the case, X is, by (v), irreducible as a Nash subset of each open semialgebraic neighborhood of X in $\mathbb{R}^n$ in which X is closed.

Proof of statement (vi). Just, the "if" implication requires some comment. Suppose that X is reducible as a semialgebraic set. Then, there is an open semialgebraic neighborhood $V \subset U_0$ of X and $F_1, F_2 \in \mathcal{N}(V)$ such that $F_1|_X, F_2|_X \not\equiv 0$, but $F_1|_X F_2|_X \equiv 0$. Consider $X_i = \{F_i = 0\} \cap X \subsetneq X$ and note that $X = X_1 \cup X_2$. Since X_1 and X_2 are closed in X, so are in U_0. Thus, by (2.12), each X_i is a Nash subset of U_0, against the irreducibility of X as a Nash subset of U_0. □

Recall that $\mathcal{O}(S)$ denotes the $\mathbb{R}$-algebra of analytic functions on S, that is, the collection of real valued functions on S which admit an analytic extension to an open neighborhood of S in $\mathbb{R}^n$, endowed with the usual operations. Our main results in this section, whose proofs are postponed, are the following:

Theorem 3.2. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set. Then S is irreducible if and only if $\mathcal{O}(S)$ is an integral domain.*

Proposition 3.3. *Let $M \subset \mathbb{R}^n$ be a d-dimensional affine Nash manifold and let $S \subset M$ be a d-dimensional semialgebraic set. Then, S is irreducible if and only if S is connected.*

In fact, we will prove also Proposition 3.4, which is a more general result than Proposition 3.3 because each affine Nash manifold admits by [21, VI.2.11], an structure of affine nonsingular real algebraic variety (and so it can be understood as a normal algebraic set).

Proposition 3.4. *Let $X \subset \mathbb{R}^n$ be a normal algebraic set and let $S \subset X$ be a semialgebraic subset of X of its same dimension. Then, S is irreducible if and only if S is connected.*

Corollary 3.5. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set and let $(Y \subset \mathbb{R}^m, \pi)$ be the normalization of $\overline{S}^{\text{zar}}$. Suppose that $\pi(\pi^{-1}(S)) = S$. Then, S is irreducible if and only if there exists a connected component T of $\pi^{-1}(S)$ such that $\pi(T) = S$.*

Before proving the previous results we need some preparation. We begin with some useful characterizations of the semialgebraic irreducibility.

Lemma 3.6. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set. The following assertions are equivalent*:

(i) *S is irreducible.*
(ii) *For each open semialgebraic neighborhood U of S in $\mathbb{R}^n$, the Nash closure $\overline{S}^{\text{an}}_U$ of S in U is irreducible.*
(iii) *Each Nash function $f \in \mathcal{N}(S)$ such that $\dim \mathcal{Z}_S(f) = \dim S$ is identically zero.*

Proof. (i) $\Longrightarrow$ (ii) By (3.1)(v), $\mathcal{J}_U(S)$ is a prime ideal of $\mathcal{N}(U)$; hence, $\overline{S}^{\text{an}}_U = \mathcal{Z}_U(\mathcal{J}_U(S))$ is, by (3.1)(vi), an irreducible Nash subset of U.

(ii) $\Longrightarrow$ (iii) Let $F : U \to \mathbb{R}$ be a Nash extension of f to some open semialgebraic neighborhood U of S in $\mathbb{R}^n$. Since $\dim \overline{S}^{\text{an}}_U = \dim S = d$, the Nash set $\operatorname{Sing}(\overline{S}^{\text{an}}_U)$ has dimension $\leq d-1$. Consider the d-dimensional Nash submanifold $M = \operatorname{Reg}(\mathcal{Z}_S(f)) \setminus \operatorname{Sing}(\overline{S}^{\text{an}}_U)$ of the d-dimensional Nash manifold $\operatorname{Reg}(\overline{S}^{\text{an}}_U)$. Since F vanishes identically on M and $\overline{S}^{\text{an}}_U$ is an irreducible Nash subset of U, we deduce that $F|_{\overline{S}^{\text{an}}_U} \equiv 0$, and so $f = F|_S \equiv 0$.

(iii) $\Longrightarrow$ (i) Let $f_1, f_2 \in \mathcal{N}(S)$ such that $f_1 f_2 \equiv 0$. Pick a point $x \in \operatorname{Reg}(S)$. Since the Nash germ S_x is irreducible, we may assume that f_1 vanishes identically on an open semialgebraic neighborhood of x in S. Thus, $\dim \mathcal{Z}_S(f_1) = \dim S$ and so $f_1 \equiv 0$; hence, S is irreducible. $\square$

Next, we prove the following result from which Proposition 3.3 follows as a particular case. Namely,

Proposition 3.7. *Let $S \subset T \subset \mathbb{R}^n$ be semialgebraic sets of the same dimension such that T is pure dimensional and the Nash set germ $\overline{T_x}^{\text{an}}$ is irreducible for each point $x \in T$. Then, S is irreducible if and only if S is connected.*

Proof. The irreducibility of S implies its connectedness. Conversely, suppose now that S is connected. To prove that S is irreducible it suffices, by Lemma 3.6, to show that each $f \in \mathcal{N}(S)$ with $\dim \mathcal{Z}_S(f) = \dim S$ is identically zero.

Let $F \in \mathcal{N}(U)$ be a Nash extension of f to some open semialgebraic neighborhood U of S in $\mathbb{R}^n$. Let T_1 be the connected component of $U \cap T$ containing S and let $W \subset U$ be an open semialgebraic neighborhood of S in $\mathbb{R}^n$ such that $W \cap T = T_1$. Since $f = (F|_{W\cap T})|_S$, it is enough to show that $F|_{W\cap T}$ is identically zero.

Indeed, by hypothesis, the Nash germ $\overline{T_x}^{\text{an}}$ is irreducible for each point $x \in W \cap T$. Hence, for each $x \in W \cap T$ there exist an open semialgebraic neighborhood $V^x \subset W$ of x and a representative $X^x \subset V^x$ of $\overline{T_x}^{\text{an}}$ which is an irreducible Nash set in V^x and $\dim X^x = \dim \overline{T_x}^{\text{an}}$. Denote $Z = \mathcal{Z}_{W\cap T}(F|_{W\cap T})$ and observe that

$$\dim(W \cap T) \geq \dim Z \geq \dim \mathcal{Z}_S(f) = \dim S = \dim T \geq \dim(W \cap T);$$

hence, all the inequalities above become equalities and we may choose $x_0 \in Z \subset T$ such that $\dim Z_{x_0} = \dim T = \dim T_{x_0}$. Since $\overline{T_{x_0}}^{\text{an}}$ is irreducible, $Z_{x_0} \subset \overline{T_{x_0}}^{\text{an}}$ and $\dim Z_{x_0} = \dim \overline{T_{x_0}}^{\text{an}}$, we deduce that $\overline{T_{x_0}}^{\text{an}} \subset \mathcal{Z}(F_{x_0})$; hence, F is identically zero on the (irreducible) chosen representative X^{x_0} of $\overline{T_{x_0}}^{\text{an}}$.

Finally, let us show that F vanishes identically on $W \cap T$. Fix a point $x \in W \cap T$ and let us check that $F(x) = 0$. Since $W \cap T$ is connected, there exist finitely many points $x_1, \dots, x_r, x_{r+1} = x$ in $W \cap T$ such that $V^{x_k} \cap V^{x_{k+1}} \cap T \neq \varnothing$ for $k = 0, \dots, r$. Let us prove inductively that $F|_{X^{x_k}} \equiv 0$ for $k = 0, \dots, r+1$. We have already proved that $F|_{X^{x_0}} \equiv 0$. Thus, assume that $F|_{X^{x_k}} \equiv 0$ and let us check that $F|_{X^{x_{k+1}}} \equiv 0$. Since T is pure dimensional,

$$\dim(V^{x_k} \cap V^{x_{k+1}} \cap T) = \dim T = \dim \overline{T_{x_{k+1}}}^{\text{an}} = \dim X^{x_{k+1}}.$$

Moreover, $V^{x_k} \cap V^{x_{k+1}} \cap T \subset X^{x_k} \cap X^{x_{k+1}}$. Thus, since $F|_{X^{x_k}} \equiv 0$ and $X^{x_{k+1}}$ is an irreducible Nash subset of $V^{x_{k+1}}$, we deduce that $F|_{X^{x_{k+1}}} \equiv 0$. Hence, $F|_{W\cap T} \equiv 0$, as wanted. $\square$

Notice that in the previous result Proposition 3.7 we need the pure dimensionality of the "ambient" T but we do not require any other dimensional restriction apart from the equality $\dim S = \dim T$. The following example shows that Proposition 3.7 is false in general if T is not pure dimensional.

Example 3.8. Consider the semialgebraic subset of $\mathbb{R}^3$:

$$S = T = \{((z-1)(x^2+y^2) - y^3)((z+1)(x^2+y^2) - y^3) = 0\}.$$

A straightforward computation shows that $\overline{T}_p^{\mathrm{an}}$ is irreducible for all $p \in T$. However, S is not irreducible because $\mathcal{N}(S)$ is not an integral domain. Indeed, the Nash functions f_k on S of respective formulae $f_k = (z+(-1)^k)(x^2+y^2) - y^3$ for $k = 1, 2$ are not identically zero on S but $f_1 f_2 \equiv 0$.

Next, we approach the proof of Proposition 3.4. The existence of normal algebraic sets which are not pure dimensional (see Example 3.9) blocks us to use Proposition 3.7 to achieve Proposition 3.4 as a consequence. Conversely, the existence of irreducible germs which are not normal hinder us to obtain Proposition 3.7 as a corollary of Proposition 3.4. The subsequent example is inspired in one already proposed in [23, Esempio, p. 211].

Example 3.9. Consider the real algebraic set $X_{\mathbb{R}} = \{w^2 - z(x^2+y^2) = 0\} \subset \mathbb{R}^4$ and its algebraic complexification $X_{\mathbb{C}} = \{w^2 - z(x^2+y^2) = 0\} \subset \mathbb{C}^4$. The set of singular points of $X_{\mathbb{C}}$ is the complex algebraic set $\mathrm{Sing}(X_{\mathbb{C}}) = \{x = 0, y = 0, w = 0\} \cup \{x^2+y^2 = 0, z = 0, w = 0\} \subset \mathbb{C}^4$ which has codimension 2 in $X_{\mathbb{C}}$. Since $X_{\mathbb{C}}$ is a complex irreducible analytic hypersurface, we deduce, by [17], that $X_{\mathbb{C}}$ is a normal complex analytic set. This implies that $X_{\mathbb{R}}$ is a real normal algebraic set. However, since the points of $X_{\mathbb{R}}$ satisfy the equation $w^2 = z(x^2+y^2)$, one checks straightforwardly that the set germs $X_{\mathbb{R},p}$ at the points $p \in X_{\mathbb{R}}$ of the form $p = (0,0,z,0)$ with $z < 0$ have dimension 1. Thus, $X_{\mathbb{R}}$ is a normal algebraic set which is not pure dimensional.

(3.10) Normalization. We recall here for the sake of the reader well-known results about the normalization of an algebraic set. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $Z_{\mathbb{K}} \subset \mathbb{K}^n$ be an algebraic set and $\mathfrak{a}$ an ideal of $\mathbb{K}[\mathtt{x}] = \mathbb{K}[\mathtt{x}_1, \dots, \mathtt{x}_n]$. We denote

$$\mathcal{I}_{\mathbb{K}}(Z_{\mathbb{K}}) = \{F \in \mathbb{K}[\mathtt{x}] : F(z) = 0\ \forall\, z \in Z_{\mathbb{K}}\} \quad \text{and}$$
$$\mathcal{Z}_{\mathbb{K}^n}(\mathfrak{a}) = \{z \in \mathbb{K}^n : F(z) = 0\ \forall\, F \in \mathfrak{a}\}.$$

Given a real algebraic set $X_{\mathbb{R}} \subset \mathbb{R}^n$, we denote $X_{\mathbb{C}} = \{z \in \mathbb{C}^n : F(z) = 0,\ \forall\, F \in \mathcal{I}_{\mathbb{R}}(X_{\mathbb{R}})\}$. Let $\sigma = \sigma_n : \mathbb{C}^n \to \mathbb{C}^n, z = (z_1, \dots, z_n) \mapsto \overline{z} = (\overline{z_1}, \dots, \overline{z_n})$ be the complex conjugation in $\mathbb{C}^n$; we say that a subset $A \subset \mathbb{C}^n$ is *σ-invariant* if $\sigma(A) = A$. Of course, if $X_{\mathbb{R}}$ is a real algebraic set, then $X_{\mathbb{C}}$ is σ-invariant.

Proposition 3.11 (Normalization). *Let $X_{\mathbb{R}} \subset \mathbb{R}^n$ be an irreducible algebraic set and consider the integral domain $A_{\mathbb{K}} = \mathbb{K}[\mathtt{x}]/\mathcal{I}_{\mathbb{K}}(X_{\mathbb{K}})$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{K}_{\mathbb{K}}$ be the field of fractions of $A_{\mathbb{K}}$ and denote by $\widetilde{A}_{\mathbb{K}}$ the integral closure of $A_{\mathbb{K}}$ in $\mathcal{K}_{\mathbb{K}}$. Then*:

(i) $\mathcal{I}_{\mathbb{C}}(X_{\mathbb{C}}) = \mathcal{I}_{\mathbb{R}}(X)\mathbb{C}[\mathtt{x}]$, $A_{\mathbb{C}} = A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = A_{\mathbb{R}}[\sqrt{-1}]$ *and* $\mathcal{K}_{\mathbb{C}} = \mathcal{K}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{K}_{\mathbb{R}}[\sqrt{-1}]$.

(ii) *$\widetilde{A}_{\mathbb{C}}$ is the integral closure of $A_{\mathbb{R}}$ in $\mathcal{K}_{\mathbb{C}}$ and $\widetilde{A}_{\mathbb{C}} = \widetilde{A}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \widetilde{A}_{\mathbb{R}}[\sqrt{-1}]$.*

(iii) *There exist an integer $m \geq n$ and irreducible algebraic sets $Y_{\mathbb{K}} \subset \mathbb{K}^m$ such that if $\mathtt{y} = (\mathtt{y}_1, \ldots, \mathtt{y}_{m-n})$, we have:*

(1) *$\widetilde{A}_{\mathbb{R}} \cong \mathbb{R}[\mathtt{x}, \mathtt{y}]/\mathcal{I}_{\mathbb{R}}(Y_{\mathbb{R}})$ and $\widetilde{A}_{\mathbb{C}} \cong \mathbb{C}[\mathtt{x}, \mathtt{y}]/\mathcal{I}_{\mathbb{C}}(Y_{\mathbb{C}})$.*

(2) *The projection $p_{\mathbb{K}} : \mathbb{K}^m \to \mathbb{K}^n$ onto the first n coordinates of $\mathbb{K}^m$ induces by restriction a proper map $\pi_{\mathbb{K}} : Y_{\mathbb{K}} \to X_{\mathbb{K}}$ with finite fibers. Moreover, the set $Y_{\mathbb{K}} \setminus \pi_{\mathbb{K}}^{-1}(\operatorname{Sing}(X_{\mathbb{K}}))$ is an analytic submanifold of $\mathbb{K}^m$ and the restriction map $\pi_{\mathbb{K}}| : Y_{\mathbb{K}} \setminus \pi_{\mathbb{K}}^{-1}(\operatorname{Sing}(X_{\mathbb{K}})) \to X_{\mathbb{K}} \setminus \operatorname{Sing}(X_{\mathbb{K}})$ is a biregular diffeomorphism.*

(3) *The map $\pi_{\mathbb{C}}$ is surjective and $\operatorname{Cl}_{\mathbb{R}^n}(\operatorname{Reg}(X_{\mathbb{R}})) \subset \operatorname{im} \pi_{\mathbb{R}}$.*

In what follows $(Y_{\mathbb{R}}, \pi_{\mathbb{R}})$ will be called the *normalization of* $X_{\mathbb{R}}$ and $(Y_{\mathbb{C}}, \pi_{\mathbb{C}})$ the *normalization of the complexification $X_{\mathbb{C}}$ of $X_{\mathbb{R}}$*.

Proof. Statement (i) is a straightforward computation while (ii) follows at once from [10, 13.13]. Finally, statement (iii) is an almost immediate consequence of [18, Sec. 1, Theorem 1], [15, Theorem 1.5] and of [10, 4.13], in what concerns the second part of (iii.2). □

Remarks 3.12. (i) Of course, the previous result extends straightforwardly to arbitrary (non-necessarily irreducible) algebraic sets in the natural way using the total ring of fractions of $A_{\mathbb{K}}$ instead of the field of fractions of $A_{\mathbb{K}}$ (used when $X_{\mathbb{R}}$ is irreducible).

(ii) More generally, one defines the normalization of a (complex) analytic space (see [16, VI.2]) in the following way. Recall that an analytic space $(X, \mathcal{O}_X)$ is *normal* if for all $x \in X$ the local analytic ring $\mathcal{O}_{X,x}$ is integrally closed. A *normalization* (Y, π) of an analytic space $(X, \mathcal{O}_X)$ is a normal analytic space $(Y, \mathcal{O}_Y)$ together with a proper surjective holomorphic map $\pi : Y \to X$ with finite fibers such that $Y \setminus \pi^{-1}(\operatorname{Sing}(X))$ is dense in Y and $\pi| : Y \setminus \pi^{-1}(\operatorname{Sing}(X)) \to X \setminus \operatorname{Sing}(X)$ is an analytic isomorphism. In [16, VI.2. Lemma 2 and VI.3. Theorem 4], it is proved the uniqueness and the existence of the analytic normalization (Y, π) of an analytic space X.

In particular, if Z is an open subset of X, then $(\pi^{-1}(Z), \pi|)$ is the normalization of Z. Moreover, if T is a connected component of $\pi^{-1}(Z)$, the map $\pi|_T : T \to Z$ is proper and by Remmert's proper mapping theorem (see [16, VII.2. Theorem 2]) we deduce that $\pi(T)$ is a complex analytic subspace of Z. Moreover, T is irreducible because it is connected and normal; hence, $T \setminus (\pi|_T)^{-1}(\operatorname{Sing}(\pi(T)))$ is, by [16, IV.1. Corollary 2], connected. Thus,

$$\pi(T \setminus (\pi|_T)^{-1}(\operatorname{Sing}(\pi(T)))) = \pi(T) \setminus \operatorname{Sing}(\pi(T)) = \operatorname{Reg}(\pi(T))$$

is connected and $\pi(T)$ is an irreducible complex analytic space (see [16, IV.1. Corollary 1]). In particular, one checks straightforwardly that $\pi(T)$ is an irreducible component of Z and that $(T, \pi|_T)$ is the normalization of $\pi(T)$.

(iii) If $X \subset \mathbb{C}^n$ is a complex algebraic set and (Y, π) is the algebraic normalization of X, then by means of [16, VI.2. Definition 2 and Lemma 2], [10, 4.13], [24] and [2, VII.2.2(d)], one deduces that (Y, π) coincides with the analytic normalization of X.

Proof of Proposition 3.4. We know that the irreducibility of S implies its connectedness. Conversely, suppose now that S is connected. To prove that S is irreducible it suffices, by Lemma 3.6, to show that each $f \in \mathcal{N}(S)$ with $\dim \mathcal{Z}_S(f) = \dim S$ is identically zero.

Let $F \in \mathcal{N}(U)$ be a Nash extension of f to some open semialgebraic neighborhood U of S in $\mathbb{R}^n$. Let X_1 be the connected component of $U \cap X$ containing S and let $W \subset U$ be a connected open semialgebraic neighborhood of S such that $W \cap X = X_1$. Let $F_{\mathbb{C}} : \Omega \to \mathbb{C}$ be a holomorphic extension of F to some σ-invariant connected open neighborhood Ω of S in $\mathbb{C}^n$ such that $\Omega \cap \mathbb{R}^n = W$; we may assume that $\Omega \cap X_{\mathbb{C}}$ is connected.

Next, since $X = X_{\mathbb{R}}$ is normal, also $X_{\mathbb{C}}$ is normal; hence, $\mathcal{O}(\mathbb{C}^n_z)/(\mathcal{I}_{\mathbb{C}}(X_{\mathbb{C}})\mathcal{O}(\mathbb{C}^n_z))$ is normal for all $z \in X_{\mathbb{C}}$ (see [3, 5.13], [2, VII.2.2(d)] and [24]). Observe that, $X_{\mathbb{C}}$ being a complex analytic set, it is coherent and so $\mathcal{O}(X_{\mathbb{C},z}) \cong \mathcal{O}(\mathbb{C}^n_z)/(\mathcal{I}_{\mathbb{C}}(X_{\mathbb{C}})\mathcal{O}(\mathbb{C}^n_z))$ for all $z \in X_{\mathbb{C}}$. Since $\Omega \cap X_{\mathbb{C}}$ is a connected and normal analytic space, it is irreducible.

Now, since $\dim \mathcal{Z}_S(f) = \dim S = \dim X$, the function F is identically zero on an open neighborhood in X of a regular point x_0 of $W \cap X$. Hence, $F_{\mathbb{C}}$ vanishes identically on an open neighborhood of x_0 in $\Omega \cap X_{\mathbb{C}}$. Since the latter set is an irreducible complex analytic space, we conclude that $F_{\mathbb{C}}|_{\Omega \cap X_{\mathbb{C}}}$ is identically zero, and so is $f = (F_{\mathbb{C}}|_{\Omega \cap X_{\mathbb{C}}})|_S$. □

Before proving Corollary 3.5 from Proposition 3.4, we need the following preliminary result.

Lemma 3.13. *Let $S \subset T \subset E \subset \mathbb{R}^n$ be semialgebraic sets such that S is irreducible and let $T_1, \ldots, T_r \subset T$ be semialgebraic sets such that $T = \bigcup_{i=1}^r T_i$ and $T_i = \mathcal{Z}_E(\mathcal{J}_E(T_i))$ for $i = 1, \ldots, r$. Then, there exists $i = 1, \ldots, r$ such that $S \subset T_i$.*

Proof. Indeed, since $S \subset T$ is irreducible, $\mathcal{J}_E(S)$ is a prime ideal of $\mathcal{N}(E)$ and the intersection $\bigcap_{i=1}^r \mathcal{J}_E(T_i) = \mathcal{J}_E(T) \subset \mathcal{J}_E(S)$; hence, we may assume that $\mathcal{J}_E(T_1) \subset \mathcal{J}_E(S)$ and so $S \subset \mathcal{Z}_E(\mathcal{J}_E(S)) \subset \mathcal{Z}_E(\mathcal{J}_E(T_1)) = T_1$, as wanted. □

Proof of Corollary 3.5. For the "if" part note that, by (3.1)(iv), it is enough to see that T is irreducible. For that it suffices, by Proposition 3.4, to check that $\dim Y = \dim T$, which follows from:

$$\dim Y = \dim \overline{S}^{\mathrm{zar}} = \dim S = \dim \pi(T) \leq \dim T \leq \dim Y.$$

Conversely, suppose S irreducible and let $U_1, \ldots, U_r \subset \mathbb{R}^m$ be finitely many nonempty open pairwise disjoint semialgebraic sets such that $\{\pi^{-1}(S) \cap U_i : 1 \leq i \leq r\}$ is the collection of the connected components of $\pi^{-1}(S)$. Consider the closed semialgebraic subset $C = Y \setminus (U_1 \cup \cdots \cup U_r)$ of Y; clearly, C does not intersect $\pi^{-1}(S)$. Since $\pi : Y \to \overline{S}^{\text{zar}}$ is a proper map and C and $\overline{S}^{\text{zar}}$ are, respectively, closed in $\mathbb{R}^m$ and $\mathbb{R}^n$, the projection $\pi(C)$ of C is a closed semialgebraic subset of $\mathbb{R}^n$ which does not intersect S. Consider the open semialgebraic set $U = \mathbb{R}^n \setminus \pi(C)$ of $\mathbb{R}^n$ and the Nash subset $X = \overline{S}^{\text{zar}} \setminus \pi(C)$ of U, which contains S. By Lemma 3.13, there exists an irreducible Nash component Z of X which contains S. By [18, Sec. 2, Theorem 3], there exists a connected component Y_0 of $\pi^{-1}(Z)$ such that $\pi(\pi^{-1}(Z)) = \pi(Y_0)$, and since $Y_0 \subset U_1 \cup \cdots \cup U_r$ is connected, we may assume that $Y_0 \subset U_1$. Then

$$S = \pi(\pi^{-1}(S)) \subset \pi(\pi^{-1}(Z)) = \pi(Y_0) \subset \pi(U_1),$$

and so $S = \pi(\pi^{-1}(S) \cap U_1)$; hence, $T = \pi^{-1}(S) \cap U_1$ is a connected component of $\pi^{-1}(S)$ such that $\pi(T) = S$, as wanted. $\square$

Examples 3.14. (i) The assumption $\pi(\pi^{-1}(S)) = S$ in Corollary 3.5 is a necessary condition to relate connectedness with irreducibility. Consider the Whitney umbrella $X : z^2 = yx^2$ in $\mathbb{R}^3$. Its normalization is $\pi : \mathbb{R}^2 \to X$, $(t, s) \mapsto (t, s^2, st)$ and $X = \operatorname{im} \pi \cup \{x = 0, z = 0\}$. The semialgebraic set $S = X \setminus \{(0, -1, 0)\}$ is reducible because it is not connected. However, $\pi^{-1}(S) = \mathbb{R}^2$ is connected.

(ii) Consider the umbrella $X : z^2 = (y-2)(x^2 - y^2 - y^3)^2$ in $\mathbb{R}^3$. Its normalization is

$$\pi : \mathbb{R}^2 \to X,\ (t, s) \mapsto (t, s^2 + 2, s(t^2 - (s^2 + 2)^2 - (s^2 + 2)^3))$$

and $X = \operatorname{im} \pi \cup \{x^2 - y^2 - y^3 = 0, z = 0\}$. One could think that the semialgebraic set $S = X \setminus \{(0, -1, 0), (\sqrt{2}, 1, 0)\}$ is reducible because, although S is connected, the set of points of S of local dimension 1 is reducible. However, as a straightforward consequence of the following result Theorem 3.15 one deduces that S is irreducible.

In the rest of this section we prove the main Theorem 3.2. The key result is the following.

Theorem 3.15. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set and let $(Y_{\mathbb{C}} \subset \mathbb{C}^m, \pi_{\mathbb{C}})$ be the normalization of $X_{\mathbb{C}} = \overline{S}_{\mathbb{C}}^{\text{zar}}$. Then, S is irreducible if and only if there exists a connected component $\mathtt{S}$ of $\pi_{\mathbb{C}}^{-1}(S)$ such that $\pi_{\mathbb{C}}(\mathtt{S}) = S$.*

Proof. Since we are concerned only with the (complex) normalization of the complex algebraic set $X = X_{\mathbb{C}}$, we erase in this proof all the subindices $\mathbb{C}$ for the sake of simplicity. To prove the equivalence in the statement, we may use that X is an irreducible complex algebraic set. Indeed, if S is irreducible, the irreducibility of X is clear. Conversely, if $\pi(\mathtt{S}) = S$ for some connected component $\mathtt{S}$ of $\pi^{-1}(S)$, denote by Z the connected component of Y which contains $\mathtt{S}$. Since Y is a normal complex algebraic set, Z is an irreducible component of Y (see [20, VII.2.1. Corollary]); hence, since (Y, π) is the normalization of X, $\pi(Z)$ is an irreducible component of X. Therefore, since $S \subset \pi(Z)$, we deduce that $X = \pi(Z)$ is irreducible. In particular, the complex algebraic sets X and Y are pure dimensional, say of (complex) dimension $d = \dim_{\mathbb{R}}(S)$. By Lemma 3.6, all reduces to prove:

Property F. For each open semialgebraic neighborhood U of S in $\mathbb{R}^n$ the Nash closure $\overline{S}_U^{\text{an}}$ is an irreducible Nash subset of U if and only there exists a connected component $\mathtt{S}$ of $\pi^{-1}(S)$ such that $\pi(\mathtt{S}) = S$.

Indeed, suppose first that $\pi(\mathtt{S}) = S$ for some connected component $\mathtt{S}$ of $\pi^{-1}(S)$. Let U be an open semialgebraic neighborhood of S in $\mathbb{R}^n$ and fix a σ-invariant open neighborhood Ω of S in $\mathbb{C}^n$ such that $\Omega \cap \mathbb{R}^n = U$. By Remarks 3.12(iii), $(\pi^{-1}(X \cap \Omega), \pi|)$ is the normalization of $X \cap \Omega$; moreover, $X \cap \Omega \cap \mathbb{R}^n = X \cap U$ is a Nash subset of U of dimension d. Since $\mathtt{S}$ is connected, it is contained in a connected component T of $\pi^{-1}(X \cap \Omega)$, which has dimension d and is pure dimensional, because so is Y. By Remarks 3.12(ii), $\pi(T)$ is an irreducible component of $X \cap \Omega$ of dimension $d = \dim_{\mathbb{C}}(T)$. Since $S = \pi(\mathtt{S}) \subset \pi(T) \subset X$, we have $\overline{S}_U^{\text{an}} \subset \pi(T) \cap U$. Now, since $\pi(T)$ is irreducible and has dimension d, $\pi(T)$ is the complex analytic closure of $\overline{S}_U^{\text{an}}$ in Ω. Thus, since this holds for all σ-invariant open neighborhood Ω of $\overline{S}_U^{\text{an}}$ in $\mathbb{C}^n$ we deduce that $\overline{S}_U^{\text{an}}$ is an irreducible global analytic subset of U and so an irreducible Nash subset of U.

Conversely, let us construct foremost a suitable open semialgebraic neighborhood U of S in $\mathbb{R}^n$ to prove the existence of a connected component $\mathtt{S}$ of $\pi^{-1}(S)$ such that $\pi(\mathtt{S}) = S$ applying the hypothesis that $\overline{S}_U^{\text{an}}$ is an irreducible Nash subset of U.

Indeed, note first that $S = \pi(\pi^{-1}(S))$ because π is surjective. We identify $\mathbb{C}^p$ with $\mathbb{R}^{2p}$ for $p = n, m$ and so Y and $\pi^{-1}(S)$ can be understood as semialgebraic subsets of $\mathbb{R}^{2m}$, while X can be seen as a semialgebraic subset of $\mathbb{R}^{2n}$. Let $\mathtt{S}_1, \dots, \mathtt{S}_r$ be the connected components of $\pi^{-1}(S)$. Since π commutes with complex conjugation and $S \subset \mathbb{R}^n \subset \mathbb{C}^n$ is σ-invariant, also $\pi^{-1}(S)$ is σ-invariant. Let $\Delta'_1, \dots, \Delta'_r$ be pairwise disjoint open semialgebraic subsets of $\mathbb{R}^{2m}$ such that $\mathtt{S}_i \subset \Delta'_i$ for $i = 1, \dots, r$. Denote $\Delta' = \bigcup_{i=1}^r \Delta'_i$ and $\Delta = \Delta' \cap \sigma(\Delta')$; observe that Δ is a σ-invariant semialgebraic neighborhood of $\pi^{-1}(S)$ in $\mathbb{R}^{2m}$ and $\{\Delta_i = \Delta'_i \cap \Delta\}_{i=1}^r$ is a collection of pairwise disjoint open semialgebraic neighborhoods of the $\mathtt{S}_i$'s. Next, consider the σ-invariant closed semialgebraic subset $C = Y \setminus \Delta$ of Y, which does not intersect $\pi^{-1}(S)$. Since π is proper and σ-invariant, $\pi(C)$ is a σ-invariant closed semialgebraic subset of X. In fact, $S \cap \pi(C) = \varnothing$, and so $\pi^{-1}(S) \cap \pi^{-1}(\pi(C)) = \varnothing$. Substituting C by the σ-invariant closed semialgebraic set $\pi^{-1}(\pi(C))$, we may further assume that $C = \pi^{-1}(\pi(C))$; hence, the restriction map $\pi|_{Y \setminus C} : Y \setminus C \to X \setminus \pi(C)$ is proper and surjective. Consider the open set $\Omega = \mathbb{C}^n \setminus \pi(C)$ and the open semialgebraic neighborhood $U = \Omega \cap \mathbb{R}^n$ of S in $\mathbb{R}^n$.

By assumption $\overline{S}_U^{\rm an}$ is an irreducible Nash subset of U and so, by Property E, $\overline{S}_U^{\rm an}$ is an irreducible global analytic subset of U. Hence, the complex analytic closure Z of $\overline{S}_U^{\rm an}$ in Ω is irreducible, it is contained in X and has dimension d. Moreover, $(Y' = \pi^{-1}(X\cap\Omega), \pi|)$ is the (analytic) normalization of $X\cap\Omega$ (see Remarks 3.12(iii)).

Property G. Our goal is to prove that: There is a connected component K of $\pi^{-1}(X\cap\Omega)$ such that $Z=\pi(K)$.

Once this is proved observe that $S\subset \overline{S_U}^{\rm an}\subset Z=\pi(K)$, and since $K\subset\bigcup_{i=1}^r\Delta_i$ is connected, then $K\subset\Delta_i$ for some $i=1,\dots,r$. This, together with the fact that $\mathtt{S}_j\cap K\subset\mathtt{S}_j\cap\Delta_i=\varnothing$ if $i\neq j$, implies that $S=\pi(\mathtt{S}_i)$ for some $i=1,\dots,r$, as wanted.

Thus, we are reduced to prove Property G. Indeed, since $\dim_{\mathbb C}(\operatorname{Sing}(X))<d=\dim_{\mathbb C}(Z)$, we deduce that $Z\setminus\operatorname{Sing}(X)$ is an (open) connected and dense subset of Z (see [16, IV.1. Corollary 2]). In particular, also $\pi^{-1}(Z\setminus\operatorname{Sing}(X))$ is a connected subset of Y', and so it is contained in one of its connected components, say K. In fact, we prove next that this is the connected component K we are seeking.

Observe that K is an irreducible component of the pure dimensional normal (complex) analytic space Y' and $\dim_{\mathbb C}(\pi^{-1}(\operatorname{Sing}(X)))<d=\dim_{\mathbb C}(Y')$. Thus, K is, by [16, IV.1. Corollary 2], the closure in Y' of a connected component of $Y'\setminus\pi^{-1}(\operatorname{Sing}(X))$. But Z being an irreducible component of $X\cap\Omega$ it follows from [16, IV.1. Theorem 1], that $Z\setminus\operatorname{Sing}(X)$ is a connected component of $\operatorname{Reg}(X\cap\Omega)=X\cap\Omega\setminus\operatorname{Sing}(X)$. Hence, $\pi^{-1}(Z\setminus\operatorname{Sing}(X))$ is a connected component of $Y'\setminus\pi^{-1}(\operatorname{Sing}(X))$ and so K is the closure in Y' of $\pi^{-1}(Z\setminus\operatorname{Sing}(X))$.

Since the restriction map $\pi|:Y'\to X\cap\Omega$ is proper, we have

$$\pi(K)=\operatorname{Cl}_{X\cap\Omega}(\pi(\pi^{-1}(Z\setminus\operatorname{Sing}(X))))=\operatorname{Cl}_{X\cap\Omega}(Z\setminus\operatorname{Sing}(X))=Z,$$

and we are done. □

Proof of Theorem 3.2. Since $\mathcal N(S)\subset\mathcal O(S)$, it is clear that if $\mathcal O(S)$ is an integral domain, then so is $\mathcal N(S)$; hence, S is irreducible. Conversely, assume that S is irreducible and let $(Y\subset\mathbb C^m,\pi)$ be the (algebraic) normalization of $X=\overline{S}_{\mathbb C}^{\rm zar}$, introduced in Proposition 3.11(iii). Since S is an irreducible semialgebraic set, there exists, by Theorem 3.15, a connected component $\mathtt{S}$ of $\pi^{-1}(S)$ such that $\pi(\mathtt{S})=S$. Write $d=\dim_{\mathbb R}S=\dim_{\mathbb R}\overline{S}_{\mathbb R}^{\rm zar}=\dim_{\mathbb C}X$.

Next, let $f,g\in\mathcal O(S)$ be two analytic functions on S such that $fg\equiv 0$ and let $\Omega\subset\mathbb C^n$ be an open neighborhood of S in $\mathbb C^n$ on which f,g have holomorphic extensions $F,G:\Omega\to\mathbb C$. By Remarks 3.12(iii), $(Y'=\pi^{-1}(X\cap\Omega),\pi|)$ is the (analytic) normalization of $X\cap\Omega$. Let Z be the connected component of Y' which contains (the connected set) $\mathtt{S}$; notice that Z is an irreducible component of the normal complex analytic space Y' and in particular it is an irreducible normal analytic space.

Moreover, $\operatorname{Sing}(\overline{S}_{\mathbb R}^{\rm zar})\cup(S\setminus\operatorname{Reg}(S))$ has dimension $\leq d-1$ and $M=\operatorname{Reg}(S)\setminus\operatorname{Sing}(\overline{S}_{\mathbb R}^{\rm zar})\neq\varnothing$ is a d-dimensional submanifold of $\operatorname{Reg}(\overline{S}_{\mathbb R}^{\rm zar})$; hence, M

is an open subset of $\operatorname{Reg}(\overline{S}^{\text{zar}}_{\mathbb{R}})$. Pick a point $x_0 \in M$; since the analytic germ $S_{x_0} = \overline{S}^{\text{zar}}_{\mathbb{R},x_0}$ is regular and the product $f_{x_0}g_{x_0}$ is identically zero on $\overline{S}^{\text{zar}}_{\mathbb{R},x_0}$, we may assume that f_{x_0} is identically zero on $\overline{S}^{\text{zar}}_{\mathbb{R},x_0}$; hence, F is identically zero on a neighborhood of x_0 in $X \cap \Omega$. Note also that, since $\operatorname{Sing}(X) \cap \mathbb{R}^n = \operatorname{Sing}(\overline{S}^{\text{zar}}_{\mathbb{R}})$, x_0 is a regular point of the complex analytic space $X \cap \Omega$. Thus, the map $(F \circ \pi)|_{Y'}$ is identically zero on an open neighborhood of the unique point y_0 in the fiber under π of x_0. Since $x_0 \in S = \pi(\mathtt{S})$ and $\mathtt{S} \subset Z$, we have $y_0 \in Z$. Thus, $(F \circ \pi)|_Z$ is identically zero on an open neighborhood of y_0 in the irreducible analytic space Z; hence, $(F \circ \pi)|_Z \equiv 0$ and so $f = F|_S \equiv 0$. Therefore, $\mathcal{O}(S)$ is an integral domain and we are done. □

4. Irreducible Components of a Semialgebraic Set

The next natural step is to introduce the notion of *irreducible components* of a semialgebraic set. Of course, such notion should behaves as the analogous one in the algebraic, global analytic or Nash settings. Namely,

(4.1) Irreducible components of a semialgebraic set. Given a semialgebraic set S, a finite family $\{S_1, \dots, S_\ell\}$ of semialgebraic subsets of S is said to be *a family of irreducible components of* S if the following conditions are fulfilled:

(1) Each S_i is irreducible.
(2) If $T \subset S$ is an irreducible semialgebraic set which contains S_i, then $S_i = T$.
(3) $S_i \neq S_j$ if $i \neq j$.
(4) $S = \bigcup_{i=1}^{\ell} S_i$.

Remarks 4.2. (i) Condition (2) together with (3.1)(iii) implies that each irreducible component S_i is closed in S, and conditions (2) and (3) imply that $S_i \not\subset S_j$ if $i \neq j$.

(ii) If X is a Nash set and $X_1, \dots, X_\ell$ are the irreducible components of X as a Nash subset of an open semialgebraic neighborhood U of X in $\mathbb{R}^n$ in which X is closed, then, $\{X_1, \dots, X_\ell\}$ is a family of irreducible components of X as a semialgebraic set.

All reduces to check that the family $\{X_1, \dots, X_\ell\}$ satisfies the conditions in (4.1). Just condition (2) requires some comment. Let $X_1 \subset T \subset X$ be an irreducible semialgebraic set. By Lemma 3.13, there exists $j = 1, \dots, r$ such that $X_1 \subset T \subset X_j$; hence, $j = 1$ and $T = X_1$, as wanted.

The following result proves the existence and uniqueness of the family of irreducible components of an arbitrary semialgebraic set.

Theorem 4.3. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set. Then, there exists the family of irreducible components $\{S_1, \ldots, S_\ell\}$ of S and satisfies*

(i) $S_i = \mathcal{Z}_S(\mathcal{J}_S(S_i))$ *for* $i = 1, \ldots, \ell$.
(ii) $\mathcal{J}_S(S_1),\ldots,\mathcal{J}_S(S_\ell)$ *are the minimal prime ideals of* $\mathcal{N}(S)$.

We postpone the proof of Theorem 4.3, and obtain first some consequences.

Remarks 4.4. (i) By Lemma 3.13 and the minimality of the ideals $\mathcal{J}_S(S_i)$, once deduces that $S_i \not\subset \bigcup_{j\neq i} S_j$ for each $i = 1, \ldots, \ell$.

(ii) *The family of the irreducible components of a semialgebraic set is unique.* Let $\{S_1, \ldots, S_\ell\}$ be the family of irreducible components proposed by Theorem 4.3 and let $\{T_1, \ldots, T_r\}$ be another family of irreducible components of S satisfying the conditions in (4.1). Then, by Lemma 3.13 (see conditions (1) and (4) in (4.1)), we deduce that each $T_i \subset S_j$ for some $j = 1, \ldots, \ell$ depending on i; hence, by condition (2) in (4.1), we have $T_i = S_j$ and after reordering the indices, we may assume $r \leq \ell$ and $T_i = S_i$ for $i = 1, \ldots, r$ (see also condition (3) in (4.1)). Now, since $S = \bigcup_{i=1}^r S_i$ and using (i), we deduce that $r = \ell$, and we are done.

(iii) The irreducible components of a pure dimensional semialgebraic set need not to be pure dimensional. Let $X = X_1 \cup X_2 \cup X_3 \subset \mathbb{R}^3$, where

$$X_1 = [-1, 1] \times [-2, 2] \times \{0\}, \quad X_2 = [-2, -1] \times \{-1, 1\} \times [-1, 1],$$
$$\text{and} \quad X_3 = [1, 2] \times \{-1, 1\} \times [-1, 1].$$

Using Proposition 3.3 and Theorem 4.3 one sees straightforwardly that the irreducible components of X are the intersections $X \cap \{x_3 = 0\}$, $X \cap \{x_2 = 1\}$ and $X \cap \{x_2 = -1\}$, and none of them is pure dimensional.

Corollary 4.5 (The 1-dimensional case). *Let $S \subset \mathbb{R}^n$ be a* 1*-dimensional semialgebraic set and let $(Y \subset \mathbb{R}^m, \pi)$ be the normalization of $\overline{S}^{\text{zar}} \subset \mathbb{R}^n$. Let $\mathtt{S}_1, \ldots, \mathtt{S}_r$ be the* 1*-dimensional connected components of $\pi^{-1}(S)$. Then, $S_1 = \pi(\mathtt{S}_1), \ldots, S_r = \pi(\mathtt{S}_r)$ are the* 1*-dimensional irreducible components of S and the isolated points of S are the zero dimensional ones.*

Proof. Since the irreducible components of a semialgebraic set are connected, it is clear that each isolated point of S is an irreducible component of S. Let T be the finite set of isolated points of $\overline{S}^{\text{zar}}$. Since $\overline{S}^{\text{zar}}$ has dimension 1, the map $\pi : Y \to \overline{S}^{\text{zar}} \setminus (T \setminus \pi(\pi^{-1}(T)))$ is surjective. By Corollary 3.5, the semialgebraic sets $\pi(\mathtt{S}_1), \ldots, \pi(\mathtt{S}_r)$ are irreducible. Observe that, since (Y, π) is the normalization of $\overline{S}^{\text{zar}}$, the union of any two of the semialgebraic sets $\pi(\mathtt{S}_i)$ is reducible. Moreover, since $\dim \pi(\mathtt{S}_i) = 1$, we deduce that $\{\pi(\mathtt{S}_1), \ldots, \pi(\mathtt{S}_r)\}$ is the family of one dimensional irreducible components of S. $\square$

The clue to prove Theorem 4.3 is the following result, whose proof is postponed too.

Lemma 4.6. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set and let $\mathfrak{a}$ be an ideal of $\mathcal{N}(S)$. Then, the semialgebraic set $\mathcal{Z}_S(\mathfrak{a})$ is irreducible if and only if $\mathcal{J}_S(\mathcal{Z}_S(\mathfrak{a}))$ is a prime ideal of $\mathcal{N}(S)$.*

Proof of Theorem 4.3. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell$ be the minimal prime ideals of the noetherian ring $\mathcal{N}(S)$. Then, $(0) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_\ell$ and so $S = \bigcup_{i=1}^{\ell} \mathcal{Z}_S(\mathfrak{p}_i)$. Next, let us check that: *Each semialgebraic set $S_i = \mathcal{Z}_S(\mathfrak{p}_i)$ is irreducible and $\mathcal{J}_S(\mathcal{Z}_S(\mathfrak{p}_i)) = \mathfrak{p}_i$ for $i = 1, \ldots, \ell$.*

By Lemma 4.6, it is enough to show that $\mathcal{J}_S(\mathcal{Z}_S(\mathfrak{p}_i)) \subset \mathfrak{p}_i$. Fix first $i = 1, \ldots, \ell$ and for each $j \neq i$, let $h_j \in \mathfrak{p}_j \backslash \mathfrak{p}_i$. Let now $g \in \mathcal{J}_S(\mathcal{Z}_S(\mathfrak{p}_i))$ and observe that $g \prod_{j \neq i} h_j \equiv 0$ on S. Hence, $g \prod_{j \neq i} h_j \in \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_\ell \subset \mathfrak{p}_i$ and so, since $\mathfrak{p}_i$ is a prime ideal and $h_j \notin \mathfrak{p}_i$ for $j \neq i$, we conclude that $g \in \mathfrak{p}_i$.

Next, note that the semialgebraic sets $S_1, \ldots, S_\ell$ satisfy conditions (1), (3), (4) in (4.1) and let us check that they also satisfy condition (2) in (4.1).

Indeed, let $S_i \subset T \subset S$ be an irreducible semialgebraic set. By Lemma 3.13, there exists $1 \leq j \leq r$ such that $S_i \subset T \subset S_j$ and so $\mathfrak{p}_j \subset \mathfrak{p}_i$. Since $\mathfrak{p}_i$ is a minimal prime ideal, we deduce that $\mathfrak{p}_j = \mathfrak{p}_i$ and so $S_i = T = S_j$, as wanted. □

Thus, to complete the construction of the irreducible components of a semialgebraic set, we are reduced to prove Lemma 4.6, which is mainly based in the following result.

Lemma 4.7. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set and let $\mathfrak{a}$ be an ideal of $\mathcal{N}(S)$; denote $T = \mathcal{Z}_S(\mathfrak{a})$. Then, for each $f \in \mathcal{N}(T)$ there exists $g \in \mathcal{N}(S)$ such that $\mathcal{Z}_T(f) = \mathcal{Z}_S(g)$.*

Proof of Lemma 4.6. Recall that, by (3.1), if $T = \mathcal{Z}_S(\mathfrak{a})$ is irreducible, then $\mathcal{J}_S(T)$ is prime. Conversely, assume that $\mathcal{J}_S(T)$ is prime and let $f_1, f_2 \in \mathcal{N}(T)$ such that $f_1 f_2 = 0$. By Lemma 4.7, there exist Nash functions $g_1, g_2 \in \mathcal{N}(S)$ such that $\mathcal{Z}_S(g_i) = \mathcal{Z}_T(f_i)$ for $i = 1, 2$. Thus, $\mathcal{Z}_S(g_1 g_2) = \mathcal{Z}_T(f_1 f_2) = T$ and so $g_1 g_2 \in \mathcal{J}_S(T)$. Since $\mathcal{J}_S(T)$ is a prime ideal, we may assume that $g_1 \in \mathcal{J}_S(T)$. Hence, $\mathcal{Z}_T(f_1) = \mathcal{Z}_S(g_1) = T$ and so $f_1 = 0$. This way, $\mathcal{N}(T)$ is an integral domain and T is irreducible, as wanted. □

Again, we need an auxiliary result to prove Lemma 4.7. Namely,

Lemma 4.8. *Let $C \subset S \subset U \subset \mathbb{R}^n$ be semialgebraic sets such that C is closed in S and U is open in $\mathbb{R}^n$. Let $W \subset U$ be an open semialgebraic neighborhood of C in $\mathbb{R}^n$ and $F \in \mathcal{N}(W)$ such that $\mathcal{Z}_W(F) \cap S \subset C$. Then, there exist an open semialgebraic neighborhood $V \subset U$ of S in $\mathbb{R}^n$ and $G \in \mathcal{N}(V)$ such that $\mathcal{Z}_S(G) = \mathcal{Z}_C(F)$.*

Proof of Lemma 4.7. By Lemma 2.4 there exists $h \in \mathfrak{a}$ such that $T = \mathcal{Z}_S(h)$. Let $U, W \subset \mathbb{R}^n$ be respective open semialgebraic neighborhoods of S, T in $\mathbb{R}^n$ such that $W \subset U$ and there exist respective Nash extensions $H \in \mathcal{N}(U)$ and $F \in \mathcal{N}(W)$ of h and f. Since $\mathcal{Z}_W(H|_W) \cap S = \mathcal{Z}_S(h) = T$, we deduce

$$\begin{aligned}\mathcal{Z}_W(F^2 + (H|_W)^2) \cap S &= \mathcal{Z}_W(F) \cap \mathcal{Z}_W(H|_W) \cap S\\ &= \mathcal{Z}_W(F) \cap T\\ &= \mathcal{Z}_W(F^2 + (H|_W)^2) \cap T.\end{aligned}$$

By Lemma 4.8, there exist an open semialgebraic neighborhood $V \subset U$ of S in $\mathbb{R}^n$ and $G \in \mathcal{N}(V)$ such that $\mathcal{Z}_V(G) \cap S = \mathcal{Z}_W(F^2 + (H|_W)^2) \cap T$. Thus, the function $g = G|_S \in \mathcal{N}(S)$ satisfies

$$\mathcal{Z}_S(g) = \mathcal{Z}_V(G) \cap S = \mathcal{Z}_W(F^2 + (H|_W)^2) \cap T = \mathcal{Z}_W(F) \cap T = \mathcal{Z}_T(f),$$

and we are done. $\square$

Before proving Lemma 4.8 we recall a stratification of a semialgebraic set $S \subset \mathbb{R}^n$ already introduced in [11, 5.17]. Namely,

(4.9) Dismantling of a semialgebraic set into locally compact pieces. Let $S \subset \mathbb{R}^n$ be a semialgebraic set. We define

$$\rho_0(S) = \mathrm{Cl}_{\mathbb{R}^n}(S)\setminus S \quad \text{and} \quad \rho_1(S) = \rho_0(\rho_0(S)) = \mathrm{Cl}_{\mathbb{R}^n}(\rho_0(S)) \cap S.$$

The following properties hold true:

(i) S is locally compact if and only if $\rho_1(S)$ is empty.
(ii) The semialgebraic set $S_{\mathrm{lc}} = S\setminus\rho_1(S) = \mathrm{Cl}_{\mathbb{R}^n}(S)\setminus \mathrm{Cl}_{\mathbb{R}^n}(\rho_0(S))$ is the largest locally compact and dense subset of S. Moreover, S_{lc} equals the set of points of S having a compact neighborhood in S.

We construct the family $\mathcal{P}_S = \{\mathcal{P}_i(S)\}_{i\geq 1}$ of *maximal locally compact pieces* of S as follows: Consider $N_1 = S$ and $N_{i+1} = \rho_1(N_i)$ for $i \geq 1$ and define $\mathcal{P}_i(S) = N_i\setminus N_{i+1}$ for $i \geq 1$. By [4, 2.8.13], and the definition of ρ_1 it follows that $\dim N_{i+1} < \dim N_i - 1$. In particular, the family (of nonempty elements of) $\mathcal{P}_S$ is finite. Moreover, $\mathcal{P}_i(S)$ is the largest locally compact dense subset of N_i. This together with the equality $\mathcal{P}_1(S) = S\setminus\rho_1(S)$ justify the name of these sets associated to S. Furthermore, $\mathcal{P}_i(S) = N_i\setminus N_{i+1}$ is an open and dense subset of N_i. Proceeding inductively one realizes that each N_i is closed in S and $\mathrm{Cl}_S(\mathcal{P}_i(S)) = N_i$ for $i \geq 1$.

Proof of Lemma 4.8. We begin by proving that we are reduced to check the following property.

Property H. There exists an open semialgebraic neighborhood $A \subset W$ of C in $\mathbb{R}^n$ such that $\mathrm{Cl}_{\mathbb{R}^n}(\mathcal{Z}_A(F)) \cap S \subset W$.

Assume Property H proved for a while and let us prove that $\rho_0(\mathcal{Z}_A(F)) \cap S = \varnothing$. Of course, it is enough to check the inclusion $\mathrm{Cl}_{\mathbb{R}^n}(\mathcal{Z}_A(F)) \cap S \subset \mathcal{Z}_A(F) \cap S$.

Indeed, since $C \subset A$ and by hypothesis $\mathcal{Z}_W(F) \cap S \subset C$, we have

$$\begin{aligned}\mathrm{Cl}_{\mathbb{R}^n}(\mathcal{Z}_A(F)) \cap S &= \mathrm{Cl}_{\mathbb{R}^n}(\mathcal{Z}_A(F)) \cap W \cap S \subset \mathrm{Cl}_{\mathbb{R}^n}(\mathcal{Z}_W(F)) \cap W \cap S\\ &= \mathrm{Cl}_W(\mathcal{Z}_W(F)) \cap S = \mathcal{Z}_W(F) \cap S = \mathcal{Z}_W(F) \cap C\\ &= \mathcal{Z}_W(F) \cap C \cap A = \mathcal{Z}_W(F) \cap S \cap A = \mathcal{Z}_A(F) \cap S.\end{aligned}$$

Now, since $\mathcal{Z}_A(F)$ is locally compact and the intersection $\rho_0(\mathcal{Z}_A(F)) \cap S$ is empty, $V = U \backslash \rho_0(\mathcal{Z}_A(F))$ is an open semialgebraic neighborhood of S in $\mathbb{R}^n$. Moreover, since $\mathcal{Z}_A(F)$ is closed in V, we deduce, by (2.12), that $\mathcal{Z}_A(F)$ is a Nash subset of V, that is, there is a Nash function $G \in \mathcal{N}(V)$ such that $\mathcal{Z}_V(G) = \mathcal{Z}_A(F)$. Hence,

$$\begin{aligned}\mathcal{Z}_S(G) &= \mathcal{Z}_V(G) \cap S = \mathcal{Z}_A(F) \cap S = \mathcal{Z}_W(F) \cap S \cap A\\ &= \mathcal{Z}_W(F) \cap C \cap A = \mathcal{Z}_C(F)\end{aligned}$$

as wanted. Thus, it only remains to prove Property H.

Indeed, with the notations of (4.9), let $N_1 = C$ and $N_{i+1} = \rho_1(N_i) \subset N_i$ for $i \geq 1$. This way the family of maximal locally compact pieces of C is defined by

$$\mathcal{P}_C = \{\mathcal{P}_i(C) = N_i \backslash N_{i+1} : 1 \leq i \leq r\};$$

this means that $N_{r+1} = \varnothing$, and so N_r is locally compact. Recall that $N_{i+1} \subset C$ is a closed subset of N_i and $\mathrm{Cl}_C(\mathcal{P}_i(C)) = N_i$ for $i \geq 1$. Moreover, since C is closed in S, each N_i is closed in S. Consider, for $i = 1, \ldots, r$, the closed semialgebraic sets in $\mathbb{R}^n$:

$$T_{i+1} = \rho_0(\mathcal{P}_i(C)) = \mathrm{Cl}_{\mathbb{R}^n}(N_i \backslash N_{i+1}) \backslash (N_i \backslash N_{i+1}) = \rho_0(N_i) \cup N_{i+1}.$$

Since each N_i is closed in S we have $\rho_0(N_i) \cap S = \varnothing$, and so

$$T_{i+1} \cap S = (\rho_0(N_i) \cap S) \cup N_{i+1} = N_{i+1} = \bigcup_{j=i+1}^{r} \mathcal{P}_j(C) \subset C. \qquad (☺)$$

Define $U_i = U \backslash (T_{i+1} \cup \rho_0(N_r)) = U \backslash (T_{i+1} \cup T_{r+1})$ which is an open semialgebraic subset of $\mathbb{R}^n$ for $i = 1, \ldots, r$, because T_{i+1} is a closed semialgebraic subset of $\mathbb{R}^n$. Observe that $U_i = U_r \backslash T_{i+1} \subset U_r = U \backslash \rho_0(N_r)$ and that U_r is an open semialgebraic neighborhood of S in $\mathbb{R}^n$, because $\rho_0(N_r)$ does not intersect S since N_r is closed in S. Moreover, since $\mathrm{Cl}_{\mathbb{R}^n}(N_i) = N_i \sqcup \rho_0(N_i)$ and $T_{i+1} = \rho_0(N_i) \cup N_{i+1}$, the semialgebraic set

$$\mathcal{P}_i(C) = N_i \backslash N_{i+1} = (N_i \backslash N_{i+1}) \cap U_r = (\mathrm{Cl}_{\mathbb{R}^n}(N_i) \backslash T_{i+1}) \cap U_r = \mathrm{Cl}_{\mathbb{R}^n}(N_i) \cap U_i$$

is closed in U_i. For each $i = 1, \ldots, r$ the open semialgebraic set $W_i = U_i \cap W$ contains $\mathcal{P}_i(C)$ as a closed subset. Let A_i be an open semialgebraic set in $\mathbb{R}^n$ satisfying

$$\mathcal{P}_i(C) \subset A_i \subset \mathrm{Cl}_{U_i}(A_i) \subset W_i.$$

Since $U_i = U_r \backslash T_{i+1}$, we have

$$\mathrm{Cl}_{U_r}(A_i) \subset \mathrm{Cl}_{U_i}(A_i) \cup (\mathrm{Cl}_{U_r}(A_i) \cap T_{i+1}) \subset \mathrm{Cl}_{U_i}(A_i) \cup T_{i+1} \subset W_i \cup T_{i+1}. \qquad (☻)$$

Consider the open semialgebraic set

$$A := \bigcup_{i=1}^{r} A_i \subset \bigcup_{i=1}^{r} W_i \subset W \cap \bigcup_{i=1}^{r} U_i = W \cap U_r$$

of $\mathbb{R}^n$ which contains $C = \bigcup_{i=1}^{r} \mathcal{P}_i(C)$, and let us see that $\mathrm{Cl}_{\mathbb{R}^n}(\mathcal{Z}_A(F)) \cap S \subset W$. Indeed, since $S \subset U_r$ and using (☺) and (☻), we have

$$\begin{aligned}\mathrm{Cl}_{\mathbb{R}^n}(\mathcal{Z}_A(F)) \cap S &= \mathrm{Cl}_{\mathbb{R}^n}(\mathcal{Z}_A(F)) \cap S \cap U_r = \mathrm{Cl}_{U_r}(\mathcal{Z}_A(F)) \cap S \subset \mathrm{Cl}_{U_r}(A) \cap S \\ &= \bigcup_{i=1}^{r} \mathrm{Cl}_{U_r}(A_i) \cap S \subset \bigcup_{i=1}^{r}(W_i \cap S) \cup \bigcup_{i=1}^{r}(T_{i+1} \cap S) \subset W \cup C = W,\end{aligned}$$

and we are done. □

(4.10) Adapted semialgebraic neighborhoods. We end this section by proving the existence of basis of open semialgebraic neighborhoods of S in $\mathbb{R}^n$ on which the Nash closure behaves neatly with respect to the irreducible components of S. Namely,

Definitions 4.11. Let $S \subset \mathbb{R}^n$ be a semialgebraic set and let $S_1, \ldots, S_\ell$ be the irreducible components of S. An open semialgebraic neighborhood U of S in $\mathbb{R}^n$ is a *neighborhood adapted to* S if for each $i = 1, \ldots, \ell$ there exist $F_{i1}, \ldots, F_{ir} \in \mathcal{N}(U)$ whose restrictions to S constitute a system of generators of the ideal $\mathcal{J}_S(S_i)$.

Remark 4.12. Observe that, by Theorem 2.9, each open semialgebraic neighborhood of S in $\mathbb{R}^n$ contains a neighborhood adapted to S. Moreover, each open semialgebraic neighborhood of S contained in a neighborhood adapted to S is also a neighborhood adapted to S.

By means of diagram (∗) in (2.7) and the noetherianity of the rings of Nash functions on a semialgebraic set, one proves standardly the following properties concerning neighborhoods adapted to a semialgebraic set. We leave most of the concrete details to the reader.

Lemma 4.13. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set with irreducible components $S_1, \ldots, S_\ell$ and let U be a neighborhood adapted to S. Then,*

(i) $\overline{S_i}_U^{\mathrm{an}} \cap S = S_i$,
(ii) $\mathcal{J}_U(S_i) = \mathcal{J}_S(S_i) \cap \mathcal{N}(U)$,
(iii) $\overline{S_1}_U^{\mathrm{an}}, \ldots, \overline{S_\ell}_U^{\mathrm{an}}$ *are the irreducible components of* $\overline{S}_U^{\mathrm{an}}$,
(iv) $\dim(\mathrm{Reg}(\overline{S_i}_U^{\mathrm{an}}) \cap \mathrm{Cl}_{\mathbb{R}^n}(S_j)) < \dim S_i$ *for* $1 \le i, j \le \ell$ *with* $i \ne j$.

Proof. We just prove (iv). By (iii) and the Identity Principle, we deduce that

$$\dim(\overline{S_i}^{\mathrm{an}}_U \cap \overline{S_j}^{\mathrm{an}}_U) < \min\{\dim(\overline{S_i}^{\mathrm{an}}_U), \dim(\overline{S_j}^{\mathrm{an}}_U)\} = \min\{\dim S_i, \dim S_j\}$$

if $i \neq j$. Now, since

$$\begin{aligned}\operatorname{Reg}(\overline{S_i}^{\mathrm{an}}_U) \cap \operatorname{Cl}_{\mathbb{R}^n}(S_j) &= \operatorname{Reg}(\overline{S_i}^{\mathrm{an}}_U) \cap U \cap \operatorname{Cl}_{\mathbb{R}^n}(S_j)\\ &= \operatorname{Reg}(\overline{S_i}^{\mathrm{an}}_U) \cap \operatorname{Cl}_U(S_j) \subset \overline{S_i}^{\mathrm{an}}_U \cap \overline{S_j}^{\mathrm{an}}_U,\end{aligned}$$

we are done. □

Example 4.14. Note that the previous result is false if U is not adapted to S. Consider $S = S_1 \cup S_2$ where $S_k = \{(t, (-1)^k t\sqrt{1+t}) : -1 < t < 2\}$ for $k = 1, 2$ and $U = \mathbb{R}^2$. Observe that S_1, S_2 are the irreducible components of S and $S_k \subsetneq \overline{S}^{\mathrm{an}}_U \cap S = \overline{S_k}^{\mathrm{an}}_U \cap S = S$ where $\overline{S_k}^{\mathrm{an}}_U = \overline{S}^{\mathrm{an}}_U = \mathcal{Z}_{\mathbb{R}^2}(y^2 - x^2 - x^3)$ for $k = 1, 2$.

The following result, that we include without proof, "compares" the irreducible components of the Nash closure of a semialgebraic set in two neighborhoods adapted to itself.

Lemma 4.15. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set with irreducible components $S_1, \ldots, S_\ell$. Let U be a neighborhood adapted to S and let $V_1, V_2 \subset U$ be open semialgebraic neighborhoods of S in $\mathbb{R}^n$. Then, there is an open semialgebraic neighborhood $W \subset V_1 \cap V_2$ of S in $\mathbb{R}^n$ such that $\overline{S_i}^{\mathrm{an}}_{V_1} \cap W = \overline{S_i}^{\mathrm{an}}_{V_2} \cap W$ for $i = 1, \ldots, \ell$. In particular, $\overline{S}^{\mathrm{an}}_{V_1} \cap W = \overline{S}^{\mathrm{an}}_{V_2} \cap W$.*

Corollary 4.16. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set with irreducible components $S_1, \ldots, S_\ell$. Let U be a neighborhood adapted to S and for $i = 1, \ldots, \ell$ let $U_i \subset U$ be a neighborhood adapted to S_i. Then, there is an open semialgebraic neighborhood $W \subset U$ of S in $\mathbb{R}^n$ such that $\overline{S_i}^{\mathrm{an}}_U \cap W = \overline{S_i}^{\mathrm{an}}_{U_i} \cap W$, for each $i = 1, \ldots, \ell$.*

Proof. The inclusion $\overline{S_i}^{\mathrm{an}}_{U_i} \subset \overline{S_i}^{\mathrm{an}}_U$ is clear. Next, we check that $(\operatorname{Cl}_{\mathbb{R}^n}(\overline{S_i}^{\mathrm{an}}_{U_i}) \setminus \overline{S_i}^{\mathrm{an}}_{U_i}) \cap S = \varnothing$; for that, we prove the equality $\operatorname{Cl}_{\mathbb{R}^n}(\overline{S_i}^{\mathrm{an}}_{U_i}) \cap S = S_i$. Indeed, by Lemma 4.13, we have

$$S_i \subset \operatorname{Cl}_{\mathbb{R}^n}(\overline{S_i}^{\mathrm{an}}_{U_i}) \cap S \subset \operatorname{Cl}_{\mathbb{R}^n}(\overline{S_i}^{\mathrm{an}}_U) \cap S \cap U = \operatorname{Cl}_U(\overline{S_i}^{\mathrm{an}}_U) \cap S = \overline{S_i}^{\mathrm{an}}_U \cap S = S_i.$$

Moreover, since $\overline{S_i}^{\mathrm{an}}_{U_i}$ is locally compact, $V_i = \mathbb{R}^n \setminus (\operatorname{Cl}_{\mathbb{R}^n}(\overline{S_i}^{\mathrm{an}}_{U_i}) \setminus \overline{S_i}^{\mathrm{an}}_{U_i})$ is an open semialgebraic neighborhood of S in $\mathbb{R}^n$; hence, $V = U \cap \bigcap_{i=1}^{\ell} V_i$ is an open semialgebraic neighborhood of S in $\mathbb{R}^n$. Since $Z_i = \overline{S_i}^{\mathrm{an}}_{U_i} \cap V$ is closed in V, we deduce, by (2.12), that Z_i is a Nash subset of V. Since $S_i \subset Z_i$, we have $\overline{S_i}^{\mathrm{an}}_V \subset Z_i$. Now, there is, by Lemma 4.15, an open semialgebraic neighborhood $W \subset V$ of S in $\mathbb{R}^n$ such that

$$\overline{S_i}^{\mathrm{an}}_U \cap W = \overline{S_i}^{\mathrm{an}}_V \cap W \subset Z_i \cap W = \overline{S_i}^{\mathrm{an}}_{U_i} \cap V \cap W = \overline{S_i}^{\mathrm{an}}_{U_i} \cap W \subset \overline{S_i}^{\mathrm{an}}_U \cap W,$$

for each $i = 1, \ldots, \ell$, and we are done. □

5. Some Generalizations of the Notion of Nash Set

In this section, we present the classes of semialgebraic sets for which the most significant classical problems in Real Geometry admit a satisfactory solution (see Sec. 6).

5.1. *Nash sets*

We begin with a detailed analysis of Nash sets (see (2.12)).

Lemma 5.1. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set and U a neighborhood adapted to S. Then, S is a Nash set if and only if $S_x = \overline{S}^{\mathrm{an}}_{U,x}$ for all $x \in S$.*

Proof. Assume first that S is a Nash set. By Lemma 4.15 there is an open semialgebraic neighborhood $W \subset U$ of S in $\mathbb{R}^n$ such that $S = \overline{S}^{\mathrm{an}}_U \cap W$; hence, $S_x = \overline{S}^{\mathrm{an}}_{U,x}$ for all $x \in S$.

Conversely, let $F_1, \ldots, F_r \in \mathcal{N}(U)$ be a system of generators of $\mathcal{J}_U(S)$. The equality $S_x = \overline{S}^{\mathrm{an}}_{U,x}$ for all $x \in S$ implies that S is locally compact, because $\overline{S}^{\mathrm{an}}_U$ is so. Thus, $\mathrm{Cl}_{\mathbb{R}^n}(S) \setminus S$ is a closed subset of $\mathbb{R}^n$. Hence, $U_0 = \mathbb{R}^n \setminus (\mathrm{Cl}_{\mathbb{R}^n}(S) \setminus S)$ is an open semialgebraic subset of $\mathbb{R}^n$ which contains S as a closed subset.

Moreover, each point $x \in S$ admits an open semialgebraic neighborhood $V^x \subset U_0 \cap U$ in $\mathbb{R}^n$ such that $S \cap V^x = \overline{S}^{\mathrm{an}}_U \cap V^x$. Define $U_1 = \bigcup_{x \in S} V^x \subset U_0 \cap U$, which is an open (non-necessarily semialgebraic) neighborhood of S in $\mathbb{R}^n$, that satisfies $S = \overline{S}^{\mathrm{an}}_U \cap U_1$. Consider the coherent analytic sheaf of ideals $\mathcal{F}$ on U_0 whose fibers are

$$\mathcal{F}_x = \begin{cases} (F_{1,x}, \ldots, F_{r,x})\mathcal{O}_{U_0,x} & \text{if } x \in U_1, \\ \mathcal{O}_{U_0,x} & \text{if } x \in U_0 \setminus S, \end{cases}$$

which is well defined because $S = \overline{S}^{\mathrm{an}}_U \cap U_1 = \mathcal{Z}_{U_1}(F_1, \ldots, F_r)$ and it is closed in U_0. By [6], the sheaf $\mathcal{F}$ is globally generated by finitely many analytic sections $G_1, \ldots, G_s \in \mathcal{O}(U_0)$. In particular $S = \mathcal{Z}_{U_0}(G_1, \ldots, G_s)$ is a global analytic subset of U_0. Now, $S = \overline{S}^{\mathrm{an}}_{U_0}$ is, by (2.10), a Nash subset of U_0; hence, a Nash set. □

Remark 5.2. In the previous result the hypothesis that U is adapted to S is not superfluous. Indeed, consider $S = \{(t, t\sqrt{1+t}) : -1 < t < 2\}$, which is a Nash set, and let $U = \mathbb{R}^2$; hence, $\overline{S}^{\mathrm{an}}_U = \{y^2 - x^2 - x^3 = 0\}$ and so $S_0 \neq \overline{S}^{\mathrm{an}}_{U,0}$.

Proposition 5.3. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set and let $S_1, \ldots, S_\ell$ be its irreducible components. Then, S is a Nash set if and only if each S_i is a Nash set.*

Proof. First, if S is a Nash set, let U be a neighborhood adapted to S such that $S = \overline{S}^{\mathrm{an}}_U$. The irreducible components of $\overline{S}^{\mathrm{an}}_U$ are, by Lemma 4.13(iii), $\overline{S_1}^{\mathrm{an}}_U, \ldots, \overline{S_\ell}^{\mathrm{an}}_U$ and so $S_i = \overline{S_i}^{\mathrm{an}}_U$ for $i = 1, \ldots, \ell$, that is, each S_i is a Nash set.

Conversely, if each S_i is a Nash set, then each S_i is a Nash subset of $U_i = \mathbb{R}^n \setminus (\mathrm{Cl}_{\mathbb{R}^n}(S_i) \setminus S_i)$ of $\mathbb{R}^n$. Moreover, $S \subset V = \bigcap_{i=1}^{\ell} U_i$ because each S_i is closed in S; hence, since $S = \bigcup_{i=1}^{\ell} S_i$ is a finite union of Nash subsets of V, it is a Nash set. □

5.2. w-*Nash sets*

Definition 5.4. Let $S \subset \mathbb{R}^n$ be a semialgebraic set and let $S_1, \ldots, S_\ell$ be its irreducible components. We say that S is a w-Nash set *(or just S is w-Nash) if there exists an open semialgebraic neighborhood U of S in $\mathbb{R}^n$ such that* $\mathrm{Reg}(\overline{S_i}_U^{\mathrm{an}}) \subset \mathrm{Cl}_{\mathbb{R}^n}(S_i)$ *for $i = 1, \ldots, \ell$. It is straightforward to check that each Nash set is also a* w-*Nash set.*

Remark 5.5. One can check straightforwardly that: *If $V \subset U$ are open semialgebraic neighborhoods of S in $\mathbb{R}^n$ such that* $\mathrm{Reg}(\overline{S_i}_U^{\mathrm{an}}) \cap V \subset \mathrm{Cl}_{\mathbb{R}^n}(S_i)$ *for each i, then* $\mathrm{Reg}(\overline{S_i}_V^{\mathrm{an}}) \subset \mathrm{Cl}_{\mathbb{R}^n}(S_i)$ *for $i = 1, \ldots, \ell$.* Thus, we may always assume that the open semialgebraic neighborhood U in Definition 5.4 is adapted to S and it is contained in a fixed open semialgebraic neighborhood W of S.

As for Nash sets, we present now a local characterization of w-Nash sets.

Lemma 5.6. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set and let $S_1, \ldots, S_\ell$ be its irreducible components. Then, S is a* w-*Nash set if and only if there is an open semialgebraic neighborhood U of S in $\mathbb{R}^n$ such that* $(\mathrm{Reg}(\overline{S_i}_U^{\mathrm{an}}))_x \subset (\mathrm{Cl}_{\mathbb{R}^n}(S_i))_x$ *for $i = 1, \ldots, \ell$ and all $x \in S$.*

Proof. Just, the "if" implication requires some comment. Consider the open semialgebraic set $V = U \setminus C$, where $C = \mathrm{Cl}_{\mathbb{R}^n}(\bigcup_{i=1}^{\ell} \mathrm{Reg}(\overline{S_i}_U^{\mathrm{an}}) \setminus \mathrm{Cl}_{\mathbb{R}^n}(S_i))$ and let us see that $S \subset V$; we have to check that $S \cap C = \varnothing$. Fix $x \in S$ and note that the germ $(\bigcup_{i=1}^{\ell} \mathrm{Reg}(\overline{S_i}_U^{\mathrm{an}}) \setminus \mathrm{Cl}_{\mathbb{R}^n}(S_i))_x = \varnothing$. Hence, there exists an open semialgebraic neighborhood W^x of x in $\mathbb{R}^n$ such that $(\bigcup_{i=1}^{\ell} \mathrm{Reg}(\overline{S_i}_U^{\mathrm{an}}) \setminus \mathrm{Cl}_{\mathbb{R}^n}(S_i)) \cap W^x = \varnothing$, that is, $x \notin C$.

Next, fix an index $i = 1, \ldots, \ell$ and let us see that $\mathrm{Reg}(\overline{S_i}_U^{\mathrm{an}}) \cap V \subset \mathrm{Cl}_{\mathbb{R}^n}(S_i)$. Indeed,

$$\begin{aligned}\mathrm{Reg}(\overline{S_i}_U^{\mathrm{an}}) \cap V = \mathrm{Reg}(\overline{S_i}_U^{\mathrm{an}}) \setminus C \subset \mathrm{Reg}(\overline{S_i}_U^{\mathrm{an}}) \setminus \left(\bigcup_{j=1}^{\ell} \mathrm{Reg}(\overline{S_j}_U^{\mathrm{an}}) \setminus \mathrm{Cl}_{\mathbb{R}^n}(S_j) \right) \\ \subset \mathrm{Reg}(\overline{S_i}_U^{\mathrm{an}}) \setminus (\mathrm{Reg}(\overline{S_i}_U^{\mathrm{an}}) \setminus \mathrm{Cl}_{\mathbb{R}^n}(S_i)) \subset \mathrm{Cl}_{\mathbb{R}^n}(S_i).\end{aligned}$$

Thus, by Remark 5.5, S is a w-Nash set. □

Proposition 5.7. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set and let $S_1, \ldots, S_\ell$ be its irreducible components. Then, S is a* w-*Nash set if and only if each S_i is a* w-*Nash set.*

Proof. Let U be a neighborhood adapted to S and let $U_i \subset U$ be a neighborhood adapted to S_i for $i = 1, \dots, \ell$ such that $\operatorname{Reg}(\overline{S_i}_U^{\mathrm{an}}) \subset \operatorname{Cl}_{\mathbb{R}^n}(S_i)$ if S is a w-Nash set and $\operatorname{Reg}(\overline{S_i}_{U_i}^{\mathrm{an}}) \subset \operatorname{Cl}_{\mathbb{R}^n}(S_i)$ if each S_i is a w-Nash set. By Corollary 4.16, there exists an open semialgebraic neighborhood $W \subset U$ of S in $\mathbb{R}^n$ such that $\overline{S_i}_{U_i}^{\mathrm{an}} \cap W = \overline{S_i}_U^{\mathrm{an}} \cap W$ for $i = 1, \dots, \ell$. Moreover,

$$\operatorname{Reg}(\overline{S_i}_U^{\mathrm{an}}) \cap W = \operatorname{Reg}(\overline{S_i}_U^{\mathrm{an}} \cap W) = \operatorname{Reg}(\overline{S_i}_{U_i}^{\mathrm{an}} \cap W) = \operatorname{Reg}(\overline{S_i}_{U_i}^{\mathrm{an}}) \cap W, \qquad (*)$$

for $i = 1, \dots, \ell$. Now, the statement follows from Remark 5.5 and the equality $(*)$. $\square$

5.3. q-*Nash sets*

Definition 5.8. Let $S \subset \mathbb{R}^n$ be a semialgebraic set. We say that S is a q-Nash set *(or just S is q-Nash) if for each Nash path $\gamma : (-1, 1) \to \mathbb{R}^n$ such that $\gamma(0) \in S$, $\gamma((-1, 0)) \cap \gamma((0, 1)) = \varnothing$ and $(\operatorname{im}\gamma)_{\gamma(0)} \subset \overline{S_{\gamma(0)}}^{\mathrm{an}}$, we have $\dim(S \cap \operatorname{im}\gamma) = 1$. Of course, each Nash set is a q-Nash set.*

Lemma 5.9. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set. The next assertions are equivalent*:

(i) *S is a q-Nash semialgebraic set.*
(ii) *For each open semialgebraic neighborhood U of S in $\mathbb{R}^n$, there is another open semialgebraic neighborhood $V \subset U$ of S in $\mathbb{R}^n$ and finitely many injective* (*continuous*) *semialgebraic paths $\alpha_1, \dots, \alpha_s : [-1, 1] \to \mathbb{R}^n$, whose restrictions $\alpha_i : (-1, 1) \to V$ are Nash paths, such that*
   (1) $\overline{S}_V^{\mathrm{an}} \setminus S = \bigcup_{i=1}^s \alpha_i((-1, 0))$,
   (2) $\alpha_i([-1, 0]) \cap S = \{\alpha_i(0)\}$ *for* $i = 1, \dots, s$,
   (3) $\alpha_i([0, 1)) \subset S$ *for* $i = 1, \dots, s$.

Proof. (i) $\Longrightarrow$ (ii) We may assume that U is a neighborhood adapted to S and define inductively the following semialgebraic sets: $T_0 = \overline{S}_U^{\mathrm{an}}$, $T_k = T_{k-1} \setminus Z_k$ and $Z_k = \operatorname{Reg}(T_{k-1})$ for $k \geq 1$. Note that $\dim T_k < \dim T_{k-1}$ for all $k \geq 1$ and so there is a positive integer $r \geq 1$ such that $T_k = Z_{k+1} = \varnothing$ for all $k \geq r$ but $Z_r \neq \varnothing$. Moreover, each nonempty Z_k is an affine Nash manifold and $\overline{S}_U^{\mathrm{an}} = \bigsqcup_{k=1}^r Z_k$.

Now, since S is a q-Nash set, we deduce, by [2, VII.4.2], that if $\dim(Z_k \setminus S) \geq 2$, then $\operatorname{Cl}_{\mathbb{R}^n}(Z_k \setminus S)) \cap S = \varnothing$. Moreover, if $\dim(Z_k \setminus S) = 0$, then $Z_k \setminus S$ is a finite subset of U and so its closure $Z_k \setminus S$ does not intersect S. Let

$$\mathcal{F} = \{k = 1, \dots, r : \dim(Z_k \setminus S) = 1 \quad \& \quad \operatorname{Cl}_{\mathbb{R}^n}(Z_k \setminus S) \cap S \neq \varnothing\}.$$

If $\mathcal{F} = \varnothing$, then $V = U \setminus \bigcup_{k=1}^r \operatorname{Cl}_{\mathbb{R}^n}(Z_k \setminus S)$ is an open semialgebraic neighborhood of S in $\mathbb{R}^n$ and $S = \overline{S}_U^{\mathrm{an}} \cap V = \overline{S}_V^{\mathrm{an}}$. Indeed,

$$S \subset \overline{S}_V^{\mathrm{an}} \subset \overline{S}_U^{\mathrm{an}} \cap V \subset \bigsqcup_{k=1}^r (Z_k \setminus \operatorname{Cl}_{\mathbb{R}^n}(Z_k \setminus S)) \subset \bigsqcup_{k=1}^r (Z_k \setminus (Z_k \setminus S)) \subset S,$$

and so, $S = \overline{S}_V^{\mathrm{an}}$. Thus, the choice of no Nash path α_i guarantees that conditions (1) to (3) are fulfilled; hence, we may assume that $\mathcal{F} \neq \varnothing$. Moreover, for all $k \in \mathcal{F}$

we have

$$\varnothing \neq \mathrm{Cl}_{\mathbb{R}^n}(Z_k \backslash S) \cap S = \mathrm{Cl}_{\mathbb{R}^n}(Z_k \backslash S) \backslash (\mathbb{R}^n \backslash S) \subset \mathrm{Cl}_{\mathbb{R}^n}(Z_k \backslash S) \backslash (Z_k \backslash S),$$

and so, by [4, 2.8.13], $\mathrm{Cl}_{\mathbb{R}^n}(Z_k \backslash S) \cap S$ is a finite set. Write

$$\bigcup_{k \in \mathcal{F}} \mathrm{Cl}_{\mathbb{R}^n}(Z_k \backslash S) \cap S = \{a_1, \ldots, a_p\}$$

and consider the one dimensional Nash germs $\overline{(Z_k \backslash S)_{a_j}}^{\mathrm{an}}$ for $k \in \mathcal{F}$ and $j = 1, \ldots, p$. By Abyhankar–Rückert local parametrization theorem, there are pairwise disjoint open semialgebraic neighborhoods $B_j \subset W = U \backslash \bigcup_{k \notin \mathcal{F}} \mathrm{Cl}_{\mathbb{R}^n}(Z_k \backslash S)$ of a_j in $\mathbb{R}^n$ and injective (continuous) semialgebraic paths $\alpha_i : [-1, 1] \to \mathbb{R}^n$ for $i = 1, \ldots, s$, whose restrictions $\alpha_i : (-1, 1) \to B_{j(i)}$, with $1 \leq j(i) \leq p$, are Nash paths, such that:

- $\alpha_i(0) = a_{j(i)}$,
- $\alpha_i((-1, 1))$ is an irreducible Nash subset of $B_{j(i)}$,
- $\alpha_i([-1, 0]) \cap S = \{\alpha_i(0)\}$ and $\alpha_i([0, 1]) \subset S$,
- $\bigsqcup_{k \in \mathcal{F}} \overline{((Z_k \backslash S) \cap B)_B}^{\mathrm{an}} = \bigcup_{i=1}^{s} \alpha_i((-1, 1))$, where $B = \bigcup_{j=1}^{p} B_j \subset W$.

Consider the open semialgebraic sets $V_0 = U \backslash \bigcup_{k=1}^{r} \mathrm{Cl}_{\mathbb{R}^n}(Z_k \backslash S)$ and $V = V_0 \cup B$. Notice that $V_0 \cap S = S \backslash \{a_1, \ldots, a_p\}$ and so $S \subset V$. Moreover, since $\alpha_i((0, 1)) \subset S \subset \overline{S}_V^{\mathrm{an}}$ and $\alpha_i((-1, 1)) \subset V$, we have $\alpha_i((-1, 1)) \subset \overline{S}_V^{\mathrm{an}}$. Thus,

$$\overline{S}_V^{\mathrm{an}} \backslash S \subset (\overline{S}_U^{\mathrm{an}} \backslash S) \cap V = \bigsqcup_{k \in \mathcal{F}} (Z_k \backslash S) \cap V = \bigsqcup_{k \in \mathcal{F}} (Z_k \backslash S) \cap B$$

$$\subset \bigsqcup_{k \in \mathcal{F}} \left(\overline{((Z_k \backslash S) \cap B)_B}^{\mathrm{an}} \right) \backslash S = \bigcup_{i=1}^{s} \alpha_i((-1, 1)) \backslash S = \bigcup_{i=1}^{s} \alpha_i((-1, 0)) \subset \overline{S}_V^{\mathrm{an}} \backslash S.$$

The neighborhood V and the paths α_i's are the ones we sought.

(ii) $\Longrightarrow$ (i) Let $\gamma : (-1, 1) \to \mathbb{R}^n$ be a Nash path with $\gamma(0) \in S$, $\gamma((-1, 0)) \cap \gamma((0, 1)) = \varnothing$ and $(\mathrm{im}\,\gamma)_{\gamma(0)} \subset \overline{S_{\gamma(0)}}^{\mathrm{an}}$. Let $V \subset U$ and $\alpha_1, \ldots, \alpha_s$ be as in the statement of (ii) for $U = \mathbb{R}^n$. Suppose, by way of contradiction, that $\dim(S \cap \mathrm{im}\,\gamma) = 0$; then, we may assume that $S \cap \mathrm{im}\,\gamma = \{\gamma(0)\}$ and $\mathrm{im}\,\gamma \subset \overline{S}_V^{\mathrm{an}}$. Thus,

$$\gamma(0) \in \mathrm{Cl}_{\mathbb{R}^n}(\overline{S}_V^{\mathrm{an}} \backslash S) \cap S \subset \bigcup_{i=1}^{s} \mathrm{Cl}_{\mathbb{R}^n}(\alpha_i((-1, 0))) \cap S$$

$$\subset \bigcup_{i=1}^{s} \alpha_i([-1, 0]) \cap S = \{\alpha_1(0), \ldots, \alpha_s(0)\}.$$

Since $\mathrm{im}\,\gamma \backslash \{\gamma(0)\} \subset \overline{S}_V^{\mathrm{an}} \backslash S$, it follows that $\mathrm{im}\,\gamma \backslash \{\gamma(0)\} \subset \bigcup_{i=1}^{s} \alpha_i((-1, 0))$. Thus, we may assume that $\gamma(0) = \alpha_1(0)$ and that the irreducible Nash germs $(\mathrm{im}\,\gamma)_{\gamma(0)}$ and $(\mathrm{im}\,\alpha_1)_{\alpha_1(0)}$ coincide. But, this is impossible because $\alpha_1((0, 1)) \subset S$ and $S \cap \mathrm{im}\,\gamma = \{\gamma(0)\}$; hence, $\dim(S \cap \mathrm{im}\,\gamma) = 1$, and we are done. □

Proposition 5.10. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set and let $S_1, \ldots, S_\ell$ be its irreducible components. Then, S is a* q*-Nash set if and only if each S_i is a* q*-Nash set.*

Proof. The "if" part is straightforward. For the converse, let U be a neighborhood adapted to S and let us prove that S_1 is a q-Nash set. Indeed, let $\gamma\colon(-1,1) \to \overline{S_1}^{\text{an}}_U$ be a Nash path such that $\gamma(0) \in S_1$ and $\gamma((-1,0)) \cap \gamma((0,1)) = \varnothing$ and so

$$\gamma((-1,1)) \cap S = \gamma((-1,1)) \cap \overline{S_1}^{\text{an}}_U \cap S = \gamma((-1,1)) \cap S_1.$$

Since S is a q-Nash set, $\dim(\gamma((-1,1)) \cap S_1) = \dim(\gamma((-1,1)) \cap S) = 1$; hence, S_1 is a q-Nash set. □

Examples 5.11. (i) $S_1 = \{y \neq 0\} \cup \{(0,0)\} \subset \mathbb{R}^2$ is a w-Nash set, but it is neither a Nash set nor a q-Nash set.

(ii) $S_2 = \{x \geq 0\} \subset \mathbb{R}$ is a q-Nash set, but it is neither a Nash set nor a w-Nash set.

(iii) $S_3 = \{x^2 - zy^2 = 0, z \geq 0\} \subset \mathbb{R}^3$ is q-Nash and w-Nash set, but not a Nash set.

(iv) $S_4 = \{y > 0\} \cup \{(0,0)\} \subset \mathbb{R}^2$ is neither a q-Nash set nor a w-Nash set.

Next, let us see that "almost" each q-Nash set is a w-Nash set. Namely,

Corollary 5.12. *Let $S \subset \mathbb{R}^n$ be a* q*-Nash set. Then S is a* w*-Nash set if and only if the one dimensional irreducible components of S, if any, are* w*-Nash.*

Proof. By Propositions 5.7 and 5.10, it is enough to prove that each irreducible q-Nash set $S \subset \mathbb{R}^n$ of dimension ≥ 2 is a w-Nash set. Indeed, let V and $\alpha_1, \ldots, \alpha_s$ be as in Lemma 5.9. Then, $\overline{S}^{\text{an}}_V = S \cup \bigcup_{i=1}^s \alpha_i((-1,0))$ and so $\operatorname{Reg}(\overline{S}^{\text{an}}_V) \setminus \bigcup_{i=1}^s \alpha_i((-1,0)) \subset S$. Since $\operatorname{Reg}(\overline{S}^{\text{an}}_V)$ is pure dimensional of dimension ≥ 2, we deduce $\operatorname{Reg}(\overline{S}^{\text{an}}_V) \subset \operatorname{Cl}_{\mathbb{R}^n}(S)$ and so S is a w-Nash set, as wanted. □

5.4. *The 1-dimensional case*

Proposition 5.13. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set of dimension* 1*. Then,*

(i) *S is a* q*-Nash set.*

(ii) *S is a Nash set if and only if S is a* w*-Nash set.*

Proof. Statement (i) follows almost straighforwardly from Lemma 5.9. Concerning part (ii) just the "if" part requires some explanation. By Propositions 5.3 and 5.7, we may assume that S is irreducible. Since S is one dimensional, it is locally compact and so closed in $U_0 = \mathbb{R}^n \setminus \rho_0(S)$. Since S is a w-Nash set, there is an open semialgebraic neighborhood $U \subset U_0$ of S in $\mathbb{R}^n$ such that $\operatorname{Reg}(\overline{S}^{\text{an}}_U) \subset \operatorname{Cl}_{\mathbb{R}^n}(S)$.

Now, since S is an irreducible 1-dimensional semialgebraic set, $\overline{S}^{\rm an}_U$ is pure dimensional, and so

$$S \subset \overline{S}^{\rm an}_U = \mathrm{Cl}_U(\mathrm{Reg}(\overline{S}^{\rm an}_U)) \subset \mathrm{Cl}_{\mathbb{R}^n}(S) \cap U = \mathrm{Cl}_U(S) = S,$$

hence, $S = \overline{S}^{\rm an}_U$ and, by (2.12), S is a Nash set, as wanted. □

6. Classical Problems of Real Geometry for the Ring of Nash Functions on a Semialgebraic Set

In this section we approach Substitution Theorem, Positivstellensätze, 17th Hilbert Problem and real Nullstellensatz for arbitrary semialgebraic sets and we characterize the families of those for which the previous problems have a solution.

6.1. *Substitution theorem*

Let K be a real closed field extension of $\mathbb{R}$. For every semialgebraic set $S \subset \mathbb{R}^n$ there is by [4, 5.1.2], a semialgebraic subset $S_K \subset K^n$ called the *extension of S to K*, which satisfies $S = S_K \cap \mathbb{R}^n$. Moreover, given another semialgebraic set $T \subset \mathbb{R}^m$ and a (continuous) semialgebraic map $f : S \to T$ there is, by [4, 5.3.2], a (continuous) semialgebraic map $f_K : S_K \to T_K$ called the *extension of f to K*, which fulfills $f_K|_S = f$.

If $S = M \subset \mathbb{R}^n$ is an affine Nash manifold, the classical Substitution Theorem due to Efroymson [9] and Bochnak–Efroymson [5] says that for every $\mathbb{R}$-algebras homomorphism $\varphi : \mathcal{N}(M) \to K$ the point $\varphi(\mathtt{x}) = (\varphi(\mathtt{x}_1), \ldots, \varphi(\mathtt{x}_n)) \in K^n$ belongs to M_K and $\varphi(f) = f_K(\varphi(\mathtt{x}))$ for every $f \in \mathcal{N}(M)$. Next, we prove that this result is not longer true if we substitute M by an arbitrary semialgebraic set S, and determine the subclass of those which enjoy this property.

Definition 6.1. Given a semialgebraic set $S \subset \mathbb{R}^n$ we denote by $\mathcal{F}_S$ the family of all open semialgebraic neighborhoods of S in $\mathbb{R}^n$ and we write $\mathcal{X}(S) = \bigcap_{U \in \mathcal{F}_S} U_K \subset K^n$. Moreover, for every ideal $\mathfrak{A}$ of $\mathcal{N}(\mathbb{R}^n_S)$ we define the *envelope of S_K with respect to $\mathfrak{A}$* as

$$\mathcal{E}(S, \mathfrak{A}) = \mathcal{X}(S) \cap \bigcap_{F_{U,S} \in \mathfrak{A}} (\mathcal{Z}_U(F))_K \subset K^n.$$

Although the set $(\mathcal{Z}_U(F))_K = \mathcal{Z}_{U_K}(F_K)$ depends on U, and not only on the equivalence class $F_{U,S} \in \mathcal{N}(\mathbb{R}^n_S)$, the intersection in the right-hand side of the equality above just depends of the ideal $\mathfrak{A}$, and not on the representatives of its elements used to describe it. Moreover, $S_K \subset \mathcal{E}(S, \mathcal{J}(S))$ and this justifies to call $\mathcal{E}$ the envelope of S_K.

Lemma 6.2. *Let K be a real closed field extension of $\mathbb{R}$. Let $S \subset \mathbb{R}^n$ be a semialgebraic set and let $\mathfrak{A}$ be an ideal of $\mathcal{N}(\mathbb{R}^n_S)$. Fix a point $p \in \mathcal{E}(S, \mathfrak{A})$. Then, the* *map*

$$\mathrm{ev}_p : \mathcal{N}(\mathbb{R}^n_S) \to K,\ G_S \mapsto G_K(p),$$

is a well defined $\mathbb{R}$*-algebras homomorphism whose kernel contains the ideal* $\mathfrak{A}$.

Proof. Let $H_S = 0_S \in \mathcal{N}(\mathbb{R}^n_S)$. Then, H vanishes identically on an open semialgebraic neighborhood W of S in $\mathbb{R}^n$, that is, $W = \mathcal{Z}_W(H|_W)$. Since $p \in \mathfrak{X}(S)$ we have

$$p \in W_K = (\mathcal{Z}_W(H|_W))_K = \mathcal{Z}_{W_K}(H_K|_{W_K}),$$

and so $H_K(p) = 0$. Now, it is straightforward to check that ev_p is an $\mathbb{R}$-algebras homomorphism. Finally, since $p \in \mathcal{E}(S, \mathfrak{A})$ it follows that $p \in (\mathcal{Z}_U(F))_K$ for each $F_{U,S} \in \mathfrak{A}$, that is, $\mathrm{ev}_p(F_{U,S}) = F_K(p) = 0$. This way, $\mathfrak{A} \subset \ker \mathrm{ev}_p$. $\square$

We state a Substitution Theorem for $\mathcal{N}(\mathbb{R}^n_S)$ valid for arbitrary semialgebraic sets in terms of the envelope of S_K with respect to an ideal of $\mathcal{N}(\mathbb{R}^n_S)$.

Proposition 6.3 (Substitution Theorem). *Let* $S \subset \mathbb{R}^n$ *be a semialgebraic set and let* $\mathfrak{A}$ *be an ideal of the ring* $\mathcal{N}(\mathbb{R}^n_S)$. *Let* K *be a real closed extension of* $\mathbb{R}$ *and let* $\varphi : \mathcal{N}(\mathbb{R}^n_S) \to K$ *be an* $\mathbb{R}$*-algebras homomorphism whose kernel contains* $\mathfrak{A}$. *Then,* $\varphi(\mathtt{x}) = (\varphi(\mathtt{x}_1), \ldots, \varphi(\mathtt{x}_n)) \in \mathcal{E}(S, \mathfrak{A})$ *and* $\varphi(F_S) = F_K(\varphi(\mathtt{x}))$ *for each* $F_S \in \mathcal{N}(\mathbb{R}^n_S)$.

Proof. We must prove first that $\varphi(\mathtt{x}) \in U_K$ for every open semialgebraic neighborhood U of S in $\mathbb{R}^n$. Consider the natural homomorphism $\rho_{U,S} : \mathcal{N}(U) \to \mathcal{N}(\mathbb{R}^n_S)$, $G \mapsto G_{U,S}$. The composition $\psi_U = \varphi \circ \rho_{U,S} : \mathcal{N}(U) \to K$ is an $\mathbb{R}$-algebras homomorphism and, by [4, 8.5.2], $\varphi(\mathtt{x}) = \psi_U(\mathtt{x}) \in U_K$. On the other hand, let $F_{U,S} \in \mathfrak{A}$ and let us check that $\varphi(\mathtt{x}) \in (\mathcal{Z}_U(F))_K$. By [4, 8.5.2], applied to ψ_U, we deduce that

$$F_K(\varphi(\mathtt{x})) = F_K(\psi_U(\mathtt{x})) = \psi_U(F) = \varphi(F_{U,S}) = 0,$$

because $F_{U,S} \in \mathfrak{A} \subset \ker \varphi$. Thus, $\varphi(\mathtt{x}) \in \mathcal{Z}_{U_K}(F_K) = (\mathcal{Z}_U(F))_K$. Putting all together, it follows that $\varphi(\mathtt{x}) \in \mathcal{E}(S, \mathfrak{A})$.

For the second part, let $F_{U,S} \in \mathcal{N}(\mathbb{R}^n_S)$. Now, we apply [4, 8.5.2], to ψ_U and this way we have $\varphi(F_{U,S}) = \varphi(\rho_{U,S}(F)) = \psi_U(F) = F_K(\psi_U(\mathtt{x})) = F_K(\varphi(\mathtt{x}))$, as wanted. $\square$

Lemma 6.4. *If* $S \subset \mathbb{R}^n$ *is a Nash set, then* $\mathcal{E}(S, \mathcal{J}(S)) = S_K$.

Proof. Indeed, we already know that $S_K \subset \mathcal{E}(S, \mathcal{J}(S))$. Conversely, let $p \in \mathcal{E}(S, \mathcal{J}(S))$. By Lemma 6.2, $\mathrm{ev}_p : \mathcal{N}(\mathbb{R}^n_S) \to K$, $F_{U,S} \mapsto F_K(p)$ is well-defined and factorizes through $\mathcal{N}(S)$; denote by $\varphi : \mathcal{N}(S) \to K$ the induced homomorphism. Since S is a Nash set, there is an open semialgebraic neighborhood U of S in $\mathbb{R}^n$ such that $S = \mathcal{Z}_U(G)$ for some $G \in \mathcal{N}(U)$. By [4, 8.5.2], applied to $\varphi \circ \rho_{U,S} : \mathcal{N}(U) \to \mathcal{N}(S) \to K$, we deduce that $\varphi(\mathtt{x}) \in U_K$ and $G_K(\varphi(\mathtt{x})) = \varphi(G|_S) = 0$. Thus, $\varphi(\mathtt{x}) \in (\mathcal{Z}_U(G))_K = S_K$. $\square$

Corollary 6.5. *Let $S \subset \mathbb{R}^n$ be a Nash set. Let K be a real closed extension of $\mathbb{R}$ and let $\varphi : \mathcal{N}(S) \to K$ be an $\mathbb{R}$-algebras homomorphism. Then, $\varphi(\mathtt{x}) = (\varphi(\mathtt{x}_1), \dots, \varphi(\mathtt{x}_n)) \in S_K$ and $\varphi(f) = f_K(\varphi(\mathtt{x}))$ for all $f \in \mathcal{N}(S)$.*

As we see next the Substitution Theorem in its classical formulation is not true for semialgebraic sets which are not Nash. Recall that $\mathbb{R}(\{\mathtt{t}^*\})_{\mathrm{alg}}$ denotes the real closed field of algebraic Puiseux series endowed with the unique ordering making $\mathtt{t} > 0$.

Proposition 6.6. *Let $S \subset \mathbb{R}^n$ be a connected semialgebraic set which is not Nash. Then, there is an $\mathbb{R}$-algebras homomorphism $\psi : \mathcal{N}(S) \to K = \mathbb{R}(\{\mathtt{t}^*\})_{\mathrm{alg}}$ such that $\psi(\mathtt{x}) = (\psi(\mathtt{x}_1), \dots, \psi(\mathtt{x}_n)) \notin S_K$.*

Proof. Let U be a neighborhood adapted to S. By Lemma 5.1 there exists a point $p \in S$ such that $S_p \subsetneq \overline{S}^{\mathrm{an}}_{U,p}$, and so $p \in \mathrm{Cl}_{\mathbb{R}^n}(\overline{S}^{\mathrm{an}}_U \setminus S)$.

By the Nash Curve Selection Lemma [4, 8.1.13], there is a Nash path $\gamma : (-1,1) \to U$ such that $\gamma(0) = p$ and $\gamma((0,1)) \subset \overline{S}^{\mathrm{an}}_U \setminus S$. Denote by Γ_p the germ at p of $\Gamma = \operatorname{im}\gamma$, and consider the point $\gamma(\mathtt{t}) = (\gamma_1(\mathtt{t}), \dots, \gamma_n(\mathtt{t})) \in K^n$. Observe that, by the Identity Principle, $\Gamma \subset \overline{S}^{\mathrm{an}}_U$. Next, let us check that $\gamma(\mathtt{t})$ occurs in the envelope $\mathcal{E}(S, \mathcal{J}(S))$ of S_K.

Indeed, note first that if $A \subset \mathbb{R}^n$ is an open semialgebraic neighborhood of S in $\mathbb{R}^n$, then $\gamma(\mathtt{t}) \in A_K$ because $\Gamma_p \subset \overline{S}^{\mathrm{an}}_{U,p} \subset A_p$. Hence, $\gamma(\mathtt{t}) \in \mathcal{X}(S)$.

Next, let $F_{V,S} \in \mathcal{J}(S)$ and note that, by Lemma 4.15, there is an open semialgebraic neighborhood $W \subset U \cap V$ of S in $\mathbb{R}^n$ such that $\overline{S}^{\mathrm{an}}_U \cap W = \overline{S}^{\mathrm{an}}_{U\cap V} \cap W \subset \mathcal{Z}_W(F)$; hence,

$$\Gamma_p \subset \overline{S}^{\mathrm{an}}_{U,p} = (\overline{S}^{\mathrm{an}}_U \cap W)_p \subset (\mathcal{Z}_W(F))_p,$$

that is, $\gamma(\mathtt{t}) \in (\mathcal{Z}_W(F))_K$ and so $\gamma(\mathtt{t}) \in \mathcal{E}(S, \mathcal{J}(S))$. By Lemma 6.2, we have

$$\mathrm{ev}_{\gamma(\mathtt{t})} : \mathcal{N}(\mathbb{R}^n_S) \to \mathcal{N}(S) \equiv \mathcal{N}(\mathbb{R}^n_S)/\mathcal{J}(S) \xrightarrow{\psi} K, \quad F_S \mapsto F|_S \overset{\psi}{\mapsto} F_K(\gamma(\mathtt{t})).$$

The homomorphism ψ satisfies $\psi(\mathtt{x}) = \gamma(\mathtt{t}) \notin S_K$, because $\gamma((0,1)) \subset \overline{S}^{\mathrm{an}}_U \setminus S$ and $\mathbb{R}(\{\mathtt{t}^*\})_{\mathrm{alg}}$ is endowed with the unique ordering making $\mathtt{t} > 0$. □

6.2. *Positivstellensätze*

Recall that the *cone* $P[a_1, \dots, a_r]$ generated by a finite subset $\{a_1, \dots, a_r\}$ of a commutative ring with unity A is the set

$$P[a_1, \dots, a_r] = \{p + q_1 b_1 + \cdots + q_s b_s : s \in \mathbb{N}\}$$

where $p, q_1, \dots, q_s$ are sums of squares in A and $b_1, \dots, b_s$ are products of the a_i's.

In case $S = M$ is an affine Nash manifold, the Positivstellensätze are immediate consequences of the classical Substitution Theorem. The same happens in the more general setting of Nash sets, just adapting the proof of [4, 8.5.5].

Proposition 6.7 (Positivstellensätze). *Let $S \subset \mathbb{R}^n$ be a Nash set, let $f, g_1, \ldots, g_r \in \mathcal{N}(S)$ and let $W = \{x \in S : g_1(x) \geq 0, \ldots, g_r(x) \geq 0\}$. Then:*

(i) $\forall\, x \in W\;\; f(x) \geq 0 \Leftrightarrow \exists\, m \in \mathbb{N}\; \exists\, g, h \in P[g_1, \ldots, g_r]$ *such that* $fg = f^{2m} + h$.
(ii) $\forall\, x \in W\;\; f(x) > 0 \Leftrightarrow \exists\, g, h \in P[g_1, \ldots, g_r]$ *such that* $fg = 1 + h$.
(iii) $\forall\, x \in W\;\; f(x) = 0 \Leftrightarrow \exists\, m \in \mathbb{N}\; \exists\, g \in P[g_1, \ldots, g_r]$ *such that* $f^{2m} + g = 0$.

The next result shows that in case the semialgebraic set S is not Nash, the ring $\mathcal{N}(S)$ does not enjoy neither the Artin-Lang Property ([4, 4.1.2 and 4.4.1]) nor the Positivstellensätze.

Corollary 6.8. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set which is not Nash. Then, there are finitely many polynomials $g_1, \ldots, g_r, f_0 \in \mathbb{R}[\mathtt{x}]$ and an $\mathbb{R}$-algebras homomorphism $\varphi : \mathcal{N}(S) \to K = \mathbb{R}(\{\mathtt{t}^*\})_{\mathrm{alg}}$ such that the intersection*

$$S \cap \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_r(x) \geq 0, f_0(x) \neq 0\}$$

is empty but $\varphi(f_0) \neq 0$ and $\varphi(g_i) \geq 0$ for each $i = 1, \ldots, r$. In particular, $f = -f_0^2 \geq 0$ on $W = \{x \in S : g_1(x) \geq 0, \ldots, g_r(x) \geq 0\}$ but there does not exist an expression of the type $fp = f^{2m} + q$ for any $m \geq 1$ and $p, q \in P[g_1, \ldots, g_r]$.

Proof. First, we write $S = \bigcup_{i=1}^{\ell} T_i$ as a union of basic semialgebraic sets, where

$$T_i = \{x \in \mathbb{R}^n : a_i(x) = 0, b_{1i}(x) > 0, \ldots, b_{si}(x) > 0\}$$

for some nonzero polynomials $a_i, b_{ji} \in \mathbb{R}[\mathtt{x}]$. By Proposition 6.6, there exists an $\mathbb{R}$-algebras homomorphism $\varphi : \mathcal{N}(S) \to K$ such that $\varphi(\mathtt{x}) \notin S_K = \bigcup_{i=1}^{\ell} T_{i,K}$, that is,

$$\varphi(\mathtt{x}) \in \bigcap_{i=1}^{\ell} (K^n \setminus T_{i,K}) = \bigcap_{i=1}^{\ell} (\mathbb{R}^n \setminus T_i)_K.$$

Therefore, for each index $i = 1, \ldots, \ell$ there is a polynomial c_i among the polynomials $a_i, b_{1i}, \ldots, b_{si}$ such that the sign of $c_i(\varphi(\mathtt{x}))$ is different to the constant sign of $c_i|_{T_i}$. We may assume that $c_i(\varphi(\mathtt{x})) \leq 0$ just for $i = 1, \ldots, r$. Write $g_i = -c_i$ for $i = 1, \ldots, r$ and $f_0 = \prod_{i \in I} c_i$, where $I = \{i = 1, \ldots, \ell : c_i(\varphi(\mathtt{x})) \neq 0\}$. By construction, the intersection

$$S \cap \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_r(x) \geq 0, f_0(x) \neq 0\} = \varnothing$$

and, $f_0, g_1, \ldots, g_r$ being polynomials, $\varphi(f_0) = f_0(\varphi(\mathtt{x})) \neq 0$ and $\varphi(g_i) = g_i(\varphi(\mathtt{x})) \geq 0$ for $i = 1, \ldots, r$.

Suppose next that there exist an integer $m \geq 1$ and $p, q \in P[g_1, \ldots, g_r]$ such that $fp = f^{2m} + q$. Then, $f_0^{4m} + q + pf_0^2 = 0$. Thus,

$$\varphi(f_0)^{4m} + \varphi(q) + \varphi(p)\varphi(f_0)^2 = \varphi(f_0^{4m} + q + pf_0^2) = 0,$$

against the fact that $\varphi(f_0) \neq 0$ and $\varphi(g_i) \geq 0$ for each $i = 1, \ldots, r$. $\square$

6.3. *17th Hilbert problem*

Next, we determine the class of semialgebraic sets $S \subset \mathbb{R}^n$ whose ring $\mathcal{N}(S)$ of Nash functions admits a positive answer to the 17th Hilbert Problem. We provide also quantitative information about the number of squares needed to represent a positive semidefinite Nash function on S.

Proposition 6.9 (17th Hilbert Problem for Nash functions). *Let $S \subset \mathbb{R}^n$ be a d-dimensional* w*-Nash semialgebraic set and let $f \in \mathcal{N}(S)$ be such that $f(x) \geq 0$ for each $x \in S$. Then, f is a sum of 2^d squares in the total ring of fractions $\mathcal{K}$ of $\mathcal{N}(S)$.*

Proof. We may assume that $f \neq 0$. Let $S_1, \ldots, S_\ell$ be the irreducible components of S. Since S is a w-Nash set, there exists, by Remark 5.5, a neighborhood U adapted to S such that for each index $1 \leq i \leq l$,

(i) $\overline{S_{iU}}^{\mathrm{an}} \cap S = S_i$,
(ii) $\mathrm{Reg}(\overline{S_{iU}}^{\mathrm{an}}) \subset \mathrm{Cl}_{\mathbb{R}^n}(S_i)$, and
(iii) f admits a Nash extension $F \in \mathcal{N}(U)$.

Recall that $\overline{S_{1U}}^{\mathrm{an}}, \ldots, \overline{S_{\ell U}}^{\mathrm{an}}$ are, by Lemma 4.13(iii), the irreducible components of $\overline{S}_U^{\mathrm{an}}$. Let $Z_i = \mathrm{Sing}(\overline{S_{iU}}^{\mathrm{an}})$ and let $P_i \in \mathcal{N}(U)$ be a Nash equation of Z_i for $i = 1, \ldots, \ell$. As one can check each $p_i = P_i|_S$ is a nonzero divisor in $\mathcal{N}(S)$ and so, also $p = p_1 \cdots p_\ell$ is a nonzero divisor in $\mathcal{N}(S)$.

Observe now that the Nash function $F_0 = (P_1 \cdots P_\ell)^2 F \in \mathcal{N}(U)$ is positive semidefinite on the Nash set $\overline{S}_U^{\mathrm{an}} = \bigcup_{i=1}^{\ell} \overline{S_{iU}}^{\mathrm{an}}$, because $\mathrm{Reg}(\overline{S_{iU}}^{\mathrm{an}}) \subset \mathrm{Cl}_U(S_i)$, each P_i is identically zero on $\mathrm{Sing}(\overline{S_{iU}}^{\mathrm{an}})$ and F is positive semidefinite on S. Since $\overline{S}_U^{\mathrm{an}}$ is a Nash set, applying Proposition 6.7(i) to $f_0 = F_0|_{\overline{S}_U^{\mathrm{an}}}$ and $g_1 = \cdots = g_r = 0$, there exist two sums of squares g, h in $\mathcal{N}(\overline{S}_U^{\mathrm{an}})$ such that $f_0 g = f_0^{2m} + h$. Hence,

$$f_0(f_0^{2m} + h)^2 = f_0^2 g(f_0^{2m} + h). \tag{6.1}$$

Now, we distinguish two cases accordingly to either f is a zero divisor or not. If f is a nonzero divisor in $\mathcal{N}(S)$ the same holds for $f_0|_S = p^2 f$ and so also for $f_0^{2m}|_S + h|_S$; hence, the formula (6.1) allows us to represent f as a sum of squares in $\mathcal{K}$.

Suppose next that $f \neq 0$ is a zero divisor in $\mathcal{N}(S)$. We may assume that $f \in \bigcap_{i=1}^{k} \mathcal{J}_S(S_i)$ for some $1 \leq k < \ell$ and $f \notin \bigcup_{i=k+1}^{\ell} \mathcal{J}_S(S_i)$. Note that $p^2 f \in \bigcap_{i=1}^{k} \mathcal{J}_S(S_i)$ and $p^2 f \notin \bigcup_{i=k+1}^{\ell} \mathcal{J}_S(S_i)$. Let $Q \in \mathcal{N}(U)$ be a Nash equation of $\bigcup_{i=k+1}^{\ell} \overline{S_{iU}}^{\mathrm{an}}$. Notice that $q = Q|_S \notin \bigcup_{i=1}^{k} \mathcal{J}_S(S_i)$, $q \in \bigcap_{i=k+1}^{\ell} \mathcal{J}_S(S_i)$ and $(Q^2 F_0)|_{\overline{S}_U^{\mathrm{an}}} = 0$. Moreover, $(p^2 f)^{2m} + h + q^2$ is not a zero divisor in $\mathcal{N}(S)$. Next, we rewrite equation (6.1) as follows:

$$\begin{aligned} f_0(f_0^{2m} + h + (Q|_{\overline{S}_U^{\mathrm{an}}})^2)^2 &= f_0^2 g(f_0^{2m} + h) \\ &\quad + f_0(Q|_{\overline{S}_U^{\mathrm{an}}})^2((Q|_{\overline{S}_U^{\mathrm{an}}})^2 + 2(f_0^{2m} + h)) = f_0^2 g(f_0^{2m} + h), \end{aligned}$$

and this new formula provides a representation of f as a sum of squares in $\mathcal{K}$.

Next, we prove that 2^d squares are enough. Since $\mathcal{K} \cong \bigoplus_{i=1}^{\ell} \operatorname{qf}(\mathcal{N}(S)/\mathcal{J}_S(S_i))$ (see (2.3)) and $\operatorname{qf}(\mathcal{N}(S)/\mathcal{J}_S(S_i))$ is a subfield of $\operatorname{qf}(\mathcal{N}(S_i))$, for our purposes it is enough to approach the irreducible case. Therefore, assume S irreducible and let $x \in \operatorname{Reg}(S)$. Denote by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{N}(\mathbb{R}^n_S)$ associated to the point x and let $\mathfrak{n}_x = \mathfrak{m}_x/\mathcal{J}(S)$. Consider the chain of inclusions

$$\begin{aligned}\mathcal{N}(S) = \mathcal{N}(\mathbb{R}^n_S)/\mathcal{J}(S) &\hookrightarrow (\mathcal{N}(\mathbb{R}^n_S)/\mathcal{J}(S))_{\mathfrak{n}_x}\\ &\cong \mathcal{N}(\mathbb{R}^n_S)_{\mathfrak{m}_x}/(\mathcal{J}(S)\mathcal{N}(\mathbb{R}^n_S)_{\mathfrak{m}_x}) \hookrightarrow \mathcal{N}_x/\mathcal{J}(S_x).\end{aligned}$$

Thus, $\operatorname{tr.deg}_{\mathbb{R}} \mathcal{K} \leq \operatorname{tr.deg}_{\mathbb{R}} \operatorname{qf}(\mathcal{N}_x/\mathcal{J}(S_x))$. By Noether's normalization theorem, we have

$$\operatorname{tr.deg}_{\mathbb{R}} \operatorname{qf}(\mathcal{N}_x/\mathcal{J}(S_x)) = \operatorname{tr.deg}_{\mathbb{R}} \mathbb{R}((\mathtt{x}_1, \ldots, \mathtt{x}_d))_{\mathrm{alg}} = d,$$

since $\dim S_x = \dim S = d$. Now, by [4, 6.3.15], the number of squares needed to represent a sum of squares in $\mathcal{K}$ is upperly bounded by 2^d, and we are done. $\square$

The next result shows that to be w-Nash is a necessary condition to guarantee that 17th Hilbert Problem has a positive answer for the ring $\mathcal{N}(S)$.

Proposition 6.10. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set which is not* w*-Nash. Then, there exists a positive semidefinite Nash function on S which is not a sum of squares in the total ring of fractions $\mathcal{K}$ of $\mathcal{N}(S)$.*

Proof. Let $S_1, \ldots, S_\ell$ be the irreducible components of S and let U be a neighborhood adapted to S. Let us check that we may assume that there is $x \in S$ such that the germ $(\operatorname{Reg}(\overline{S_1}^{\mathrm{an}}_U)\setminus \operatorname{Cl}_{\mathbb{R}^n}(S))_x \neq \varnothing$. By Lemma 5.6, we may assume that there is $x \in S$ such that $(\operatorname{Reg}(\overline{S_1}^{\mathrm{an}}_U))_x \not\subset (\operatorname{Cl}_{\mathbb{R}^n}(S_1))_x$; hence, $(\operatorname{Reg}(\overline{S_1}^{\mathrm{an}}_U)\setminus \operatorname{Cl}_{\mathbb{R}^n}(S_1))_x \neq \varnothing$ has dimension $d_1 = \dim S_1$. Moreover, by Lemma 4.13(iv), the semialgebraic set $\operatorname{Reg}(\overline{S_1}^{\mathrm{an}}_U) \cap \bigcup_{j=2}^{\ell} \operatorname{Cl}_{\mathbb{R}^n}(S_j)$ has dimension $\leq d_1 - 1$ and so $(\operatorname{Reg}(\overline{S_1}^{\mathrm{an}}_U)\setminus \operatorname{Cl}_{\mathbb{R}^n}(S))_x \neq \varnothing$.

Clearly, $x \in \operatorname{Cl}_{\mathbb{R}^n}(\operatorname{Reg}(\overline{S_1}^{\mathrm{an}}_U)\setminus \operatorname{Cl}_{\mathbb{R}^n}(S))$ and, by the Nash Curve Selection Lemma [4, 8.1.13], there is a Nash path $\alpha\colon(-1,1) \to \mathbb{R}^n$ such that $\alpha(0) = x$ and $Z = \alpha((-1,0)) \subset \operatorname{Reg}(\overline{S_1}^{\mathrm{an}}_U)\setminus \operatorname{Cl}_{\mathbb{R}^n}(S)$. By [19, Theorem 1], there exists a polynomial $H \in \mathbb{R}[\mathtt{x}]$ such that $H|_{\operatorname{Cl}_{\mathbb{R}^n}(S)\setminus\{x\}} > 0$ and $H_x|_{Z_x\setminus\{x\}} < 0$. Thus, $h = H|_S$ is a positive semidefinite Nash function on S and we claim that it is not a sum of squares in $\mathcal{K}$. Otherwise, there would exist $g_0, g_1, \ldots, g_p \in \mathcal{N}(S)$ such that g_0 is a nonzero divisor in $\mathcal{N}(S)$ and $g_0^2 h = g_1^2 + \cdots + g_p^2$. We will prove that $g_0|_{S_1} \equiv 0$ which contradicts the fact that g_0 is a nonzero divisor. Since U is a neighborhood adapted to S, there exist an open semialgebraic neighborhood $V \subset U$ of S and Nash functions $G_i \in \mathcal{N}(V)$ such that $G_i|_S = g_i$ and

$$(G_0|_{\overline{S}^{\mathrm{an}}_V})^2 H|_{\overline{S}^{\mathrm{an}}_V} = (G_1|_{\overline{S}^{\mathrm{an}}_V})^2 + \cdots + (G_p|_{\overline{S}^{\mathrm{an}}_V})^2.$$

For our purposes, it suffices to show that $G_0|_{\overline{S_1}^{\mathrm{an}}_V} \equiv 0$ and in fact, since $\overline{S_1}^{\mathrm{an}}_V$ is an irreducible Nash subset of V, it is enough to check that G_0 vanishes identically on a semialgebraic subset of $\overline{S_1}_V^{\mathrm{an}}$ of its same dimension. Observe that G_0 is identically zero on the semialgebraic set $T = \{H < 0\} \cap \overline{S_1}_V^{\mathrm{an}}$ and all reduces to see that $\dim T = d_1$. By Lemma 4.15, there is an open semialgebraic neighborhood $W \subset V$ of S in $\mathbb{R}^n$ such that $\overline{S_1}_V^{\mathrm{an}} \cap W = \overline{S_1}_U^{\mathrm{an}} \cap W$ and so T contains the open semialgebraic subset

$$\{H < 0\} \cap \mathrm{Reg}(\overline{S_1}_U^{\mathrm{an}}) \cap W = \{H < 0\} \cap \mathrm{Reg}(\overline{S_1}_V^{\mathrm{an}}) \cap W,$$

of the d_1-dimensional manifold $\mathrm{Reg}(\overline{S_1}_V^{\mathrm{an}})$, which is nonempty because the germ $Z_x \subset (\{H < 0\} \cap \mathrm{Reg}(\overline{S_1}_U^{\mathrm{an}}))_x$. Consequently, $\dim T = d_1$, and we are done. □

6.4. *Real Nullstellensatz*

Recall that an ideal $\mathfrak{a}$ of a commutative ring with unity is said to be *real* if $\sum_{i=1}^r f_i^2 \in \mathfrak{a} \Longrightarrow f_i \in \mathfrak{a}$ for $i = 1, \dots, r$. The smallest real ideal of A containing a given ideal $\mathfrak{a}$ of A is its *real radical*, which is defined as

$$\sqrt[R]{\mathfrak{a}} = \{f \in A : \exists\, m \in \mathbb{N},\ \exists\, g_1, \dots, g_p \in A \ \text{ such that } \ f^{2m} + g_1^2 + \cdots + g_p^2 \in \mathfrak{a}\}.$$

By its definition, $\sqrt[R]{\mathfrak{a}}$ is a radical ideal (see [4, 4.1.7]).

Proposition 6.11 (Real Nullstellensatz). *Let $S \subset \mathbb{R}^n$ be a $\mathfrak{q}$-Nash set and let $\mathfrak{a}$ be an ideal of the ring $\mathcal{N}(S)$. Then, $\mathcal{J}_S(\mathcal{Z}_S(\mathfrak{a})) = \sqrt[R]{\mathfrak{a}}$.*

Proof. As usual, since $\mathcal{N}(S)$ is a noetherian ring, it is enough to prove the statement in case $\mathfrak{a} = \mathfrak{p}$ is a real prime ideal, and so $\sqrt[R]{\mathfrak{p}} = \mathfrak{p}$. The inclusion $\mathfrak{p} \subset \mathcal{J}_S(\mathcal{Z}_S(\mathfrak{p}))$ is clear. Conversely, let $f \in \mathcal{J}_S(\mathcal{Z}_S(\mathfrak{p}))$ and $f_1, \dots, f_r \in \mathfrak{p}$ be a system of generators of $\mathfrak{p}$. Let $V \subset \mathbb{R}^n$ be an open semialgebraic neighborhood of S in $\mathbb{R}^n$ and $F, F_1, \dots, F_r \in \mathcal{N}(V)$ be Nash functions such that $F|_S = f$ and $F_i|_S = f_i$.

By Lemma 5.9, there are an open semialgebraic neighborhood $W \subset V$ of S in $\mathbb{R}^n$ and finitely many injective (continuous) semialgebraic paths $\alpha_1, \dots, \alpha_s : [-1, 1] \to \mathbb{R}^n$, whose restrictions $\alpha_j : (-1, 1) \to W$ are Nash paths, such that

(1) $\overline{S}_W^{\mathrm{an}} \setminus S = \bigcup_{j=1}^s \alpha_j((-1, 0))$,
(2) $\alpha_j([-1, 0]) \cap S = \{\alpha_j(0)\}$ for $j = 1, \dots, s$,
(3) $\alpha_j([0, 1)) \subset S$ for $j = 1, \dots, s$.

Property I. Moreover, shrinking W if necessary, we may assume that: *For each $j = 1, \dots, s$ either $\alpha_j((-1, 1)) \subset \mathcal{Z}_W(\mathfrak{p} \cap \mathcal{N}(W))$ or $\mathcal{Z}_W(\mathfrak{p} \cap \mathcal{N}(W)) \cap \alpha_j((-1, 0)) = \varnothing$.*

Indeed, we can suppose that $F_1|_W, \dots, F_r|_W$ generate the ideal $\mathfrak{p} \cap \mathcal{N}(W)$. Let $G = (F_1|_W)^2 + \cdots + (F_r|_W)^2$ and note that $\mathcal{Z}_W(\mathfrak{p} \cap \mathcal{N}(W)) = \mathcal{Z}_W(G)$. After relabeling, we may assume that $G \circ \alpha_j|_{(-1,1)} \not\equiv 0$ exactly for $j = r + 1, \dots, s$. We choose $\varepsilon > 0$ such that $\alpha_j([-\varepsilon, 0)) \cap \mathcal{Z}_W(G|_W) = \varnothing$ for $j = r + 1, \dots, s$ and $\alpha_j([-\varepsilon, \varepsilon]) \cap \alpha_k([-\varepsilon, \varepsilon])$ is either empty or equals $\{\alpha_j(0) = \alpha_k(0)\}$ for $1 \le j < k \le s$.

Then, taking $W\setminus\bigcup_{j=1}^{s}\mathrm{Cl}_{\mathbb{R}^n}(\alpha_j([-1,-\varepsilon]))$ instead of W and reparametrizing the injective (continuous) semialgebraic paths $\alpha_j|_{[-\varepsilon,\varepsilon]}$, we achieve our assumption.

Property J. By the real Nullstellensatz for the ring $\mathcal{N}(W)$, see [4, 8.6.5], we have $\mathcal{J}_W(\mathcal{Z}_W(\mathfrak{p}\cap\mathcal{N}(W)))=\mathfrak{p}\cap\mathcal{N}(W)$; hence, to show that $f\in\mathfrak{p}$ it is enough to check that $\mathcal{Z}_W(\mathfrak{p}\cap\mathcal{N}(W))\subset\mathcal{Z}_W(F|_W)$.

Since $f_i=(F_i|_W)|_S$ and $\mathfrak{p}=(f_1,\dots,f_r)\mathcal{N}(S)$ it follows that $\mathcal{Z}_S(\mathfrak{p}\cap\mathcal{N}(W))=\mathcal{Z}_S(\mathfrak{p})$. Moreover, let $H\in\mathcal{N}(W)$ such that $\overline{S}_W^{\mathrm{an}}=\mathcal{Z}_W(H)$. Note that $H\in\mathfrak{p}\cap\mathcal{N}(W)$ because $H|_S=0$. Therefore,

$$\mathcal{Z}_W(\mathfrak{p}\cap\mathcal{N}(W))=\mathcal{Z}_{\overline{S}_W^{\mathrm{an}}}(\mathfrak{p}\cap\mathcal{N}(W))=\mathcal{Z}_S(\mathfrak{p})\cup\mathcal{Z}_{\overline{S}_W^{\mathrm{an}}\setminus S}(\mathfrak{p}\cap\mathcal{N}(W)).$$

Notice that $\alpha_j((-1,1))\subset\mathcal{Z}_W(\mathfrak{p}\cap\mathcal{N}(W))$ if and only if $\alpha_j([0,1))\subset\mathcal{Z}_S(\mathfrak{p})$. Thus, since $f\in\mathcal{J}_S(\mathcal{Z}_S(\mathfrak{p}))$, we have $F\circ\alpha_j\equiv 0$ for all $j\in J=\{j=1,\dots,s:\alpha_j([0,1))\subset\mathcal{Z}_S(\mathfrak{p})\}$. Moreover, by Property I, $\mathcal{Z}_W(\mathfrak{p}\cap\mathcal{N}(W))\cap\alpha_j((-1,0))=\varnothing$ for $j\notin J$; hence,

$$\mathcal{Z}_{\overline{S}_W^{\mathrm{an}}\setminus S}(\mathfrak{p}\cap\mathcal{N}(W))=\bigcup_{j=1}^{s}\alpha_j((-1,0))\cap\mathcal{Z}_W(\mathfrak{p}\cap\mathcal{N}(W))\subset\bigcup_{j\in J}\alpha_j((-1,1))\subset\mathcal{Z}_W(F|_W).$$

Putting all together, we have

$$\mathcal{Z}_W(\mathfrak{p}\cap\mathcal{N}(W))=\mathcal{Z}_S(\mathfrak{p})\cup\mathcal{Z}_{\overline{S}_W^{\mathrm{an}}\setminus S}(\mathfrak{p}\cap\mathcal{N}(W))\subset\mathcal{Z}_W(F|_W),$$

as wanted. □

Now, we prove that $\mathfrak{q}$-Nash sets S are exactly those for which the ring $\mathcal{N}(S)$ admits a classical real Nullstellensatz.

Proposition 6.12. *Let $S\subset\mathbb{R}^n$ be a semialgebraic set which is not $\mathfrak{q}$-Nash. Then, there exists a real prime ideal $\mathfrak{p}$ of $\mathcal{N}(S)$ such that $\mathfrak{p}\neq\mathcal{J}_S(\mathcal{Z}_S(\mathfrak{p}))$.*

Proof. Let U be a neighborhood adapted to S. Since S is not $\mathfrak{q}$-Nash, we may find a Nash path $\gamma:(-1,1)\to\overline{S}_U^{\mathrm{an}}$ such that $\gamma((-1,0))\cap\gamma((0,1))=\varnothing$ and $S\cap\operatorname{im}\gamma=\{\gamma(0)\}$.

Consider the prime ideal $\mathfrak{p}$ of $\mathcal{N}(S)$ consisting of all functions $f\in\mathcal{N}(S)$ such that there exist $\varepsilon>0$ and a Nash extension F of f to an open semialgebraic neighborhood $V\subset U$ of S in $\mathbb{R}^n$ so that $\gamma((-\varepsilon,\varepsilon))\subset V$ and $F\circ\gamma|_{(-\varepsilon,\varepsilon)}\equiv 0$.

Let us check that $\mathcal{Z}_S(\mathfrak{p})=\{\gamma(0)\}$. Indeed, the germ $(\operatorname{im}\gamma)_{\gamma(0)}$ is irreducible and so there is an open semialgebraic neighborhood $B\subset V$ of $\gamma(0)$ in $\mathbb{R}^n$ such that $\Gamma=B\cap\operatorname{im}\gamma$ is an irreducible Nash set contained in the compact set $\gamma([-1/2,1/2])$. Since Γ is locally compact, the semialgebraic set $T=\mathrm{Cl}_{\mathbb{R}^n}(\Gamma)\setminus\Gamma$ is closed; moreover, $T\cap S=\varnothing$ because $T\subset\gamma([-1/2,1/2])\setminus\{\gamma(0)\}$. Now, since Γ is closed in the open semialgebraic neighborhood $W=V\setminus T$ of S in $\mathbb{R}^n$, we deduce by (2.12) that Γ is a Nash subset of W. Thus, there is $G\in\mathcal{N}(W)$ whose zeroset is Γ. Note that $g=G|_S\in\mathfrak{p}$ and $\mathcal{Z}_S(g)=\Gamma\cap S=\{\gamma(0)\}$; hence, $\mathcal{Z}_S(\mathfrak{p})=\{\gamma(0)\}$.

This way the polynomial function $\|\mathtt{x}-\gamma(0)\|^2\in\mathcal{J}_S(\mathcal{Z}_S(\mathfrak{p}))\setminus\mathfrak{p}$, and we are done. □

6.5. *The 1-dimensional case*

To finish, we remark the following consequence of Sec. 5.4 and the main results of this section.

Proposition 6.13. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set of dimension* 1. *Then*:

(i) *The real Nullstellensatz holds true for $\mathcal{N}(S)$.*
(ii) *The Substitution Theorem, the Positivstellensätze and/or the 17th Hilbert problem have a positive answer for $\mathcal{N}(S)$ if and only if S is a* w*-Nash set.*

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