7.1. Obtén, mediante un cambio de variable, una primitiva en los casos siguientes:

(1) $$\int e^x \operatorname{sen}(e^x)\,dx = \int \operatorname{sen}(t)\,dt = -\cos(t) + C = -\cos(e^x) + C$$

$t = e^x \quad dt = e^x dx$

(2) $$\int x e^{-x^2} dx = \int \frac{1}{2} e^{-t} dt = -\frac{1}{2} e^{-t} + C = -\frac{1}{2} e^{-x^2} + C$$

$t = x^2 \quad dt = 2x\,dx$

(3) $$\int \frac{\log x}{x}\,dx = \int t\,dt = \frac{t^2}{2} + C = \frac{(\log x)^2}{2} + C$$

$t = \log x \quad dt = \frac{1}{x} dx$

(4) $$\int \frac{e^x}{e^{2x} + 2e^x + 1}\,dx = \int \frac{dt}{(t+1)^2} = \int (t+1)^{-2} dt = \frac{(t+1)^{-2+1}}{-2+1} + C = -\frac{1}{t+1} + C =$$

$t = e^x \quad dt = e^x dx$

$$= -\frac{1}{e^x + 1} + C$$

(5) $$\int e^{e^x} e^x dx = \int e^t dt = e^t + C = e^{e^x} + C$$

$t = e^x \quad dt = e^x dx$

(6) $$\int \frac{x}{\sqrt{1-x^4}}\,dx = \int \frac{1}{2} \cdot \frac{dt}{\sqrt{1-t^2}} = \frac{1}{2} \arcsin t + C = \frac{1}{2} \arcsin x^2 + C$$

$t = x^2 \quad dt = 2x\,dx$

(7) $$\int \frac{e^{x^{1/2}}}{\sqrt{x}}\,dx = \int 2e^t dt = 2e^t + C = 2e^{\sqrt{x}} + C$$

$t = \sqrt{x} \quad dt = \frac{1}{2\sqrt{x}} dx$

(8) $$\int x\sqrt{1-x^2}\,dx = \int \frac{1}{2}\sqrt{1-t}\,dt = \frac{1}{2} \frac{(1-t)^{1/2+1}}{1/2+1} + C = \frac{1}{3}(1-t)^{3/2} + C = \frac{1}{3}(1-x^2)^{3/2} + C$$

$t = x^2 \quad dt = 2x\,dx$

7.2. $\displaystyle\int_2^3 \frac{\operatorname{sen}(x^2)}{x}\,dx = \int_4^9 \frac{\operatorname{sen}(y)}{\sqrt{y}}\cdot\frac{dy}{2\sqrt{y}} = \int_4^9 \frac{\operatorname{sen}(y)}{2y}\,dy$. Por tanto la respuesta correcta es la c)

$y = x^2 \quad dy = 2x\,dx$

$x \quad y=x^2$

$3 \to 9$

$2 \to 4$

7.3. Demuestra que $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$$\int_a^b f(a+b-x)\,dx = \int_b^a f(t)\cdot(-dt) = -\int_b^a f(t)\,dt = \int_a^b f(t)\,dt$$

$t = a+b-x \quad dt = -dx$

$x \quad t = a+b-x$

$a \longrightarrow b$

$b \longrightarrow a$

7.4. Sea $f:\mathbb{R}\to\mathbb{R}$ una función periódica de periodo p. Demuestra la igualdad

$$\int_a^{a+p} f(t)\,dt = \int_0^p f(t)\,dt$$

Como f tiene periodo p, entonces $f(t) = f(t+p)$. Por tanto

$$\int_0^a f(t)\,dt = \int_p^{a+p} f(x-p)\,dx = \int_p^{a+p} f(x)\,dx$$

$\uparrow$

$f(x) = f(x-p)$

Cambio $x = t+p$

$dx = dt$

$t \to x = t+p$

$a \longrightarrow a+p$

$0 \longrightarrow p$

Ahora

$$\int_a^{a+p} f(t)\,dt = \int_a^p f(t)\,dt + \int_p^{a+p} f(t)\,dt = \int_a^p f(t)\,dt + \int_0^a f(t)\,dt =$$

$$= \int_0^a f(t)\,dt + \int_a^p f(t)\,dt = \int_0^p f(t)\,dt$$

7.5 Calcula las siguientes integrales de funciones racionales:

(a) $$\int \frac{4}{x^4-1}\,dx = \int \frac{1}{x-1} + \frac{-1}{x+1} + \frac{-2}{x^2+1}\,dx = \ln(x-1) - \ln(x+1) - 2\,\mathrm{arctg}(x) + C$$

$$\frac{4}{x^4-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} = \frac{1}{x-1} + \frac{-1}{x+1} + \frac{-2}{x^2+1}$$

$$4 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)$$

③ $0 = A + B + C$ $\qquad A+B=0,\ C=0$

② $0 = A - B + D$ $\qquad A-B=2,\ D=-2$

① $0 = A + B - C$ $\qquad A=1,\ B=-1$

⓪ $4 = A - B - D$

b) $$\int \frac{x-3}{x^3+x^2+1}\,dx = \int \frac{-3}{x} + \frac{3x+4}{(x+1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx = (*)$$

$$\frac{x-3}{x^3+x^2+1} = \frac{A}{x} + \frac{Bx+C}{(x+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{-3}{x} + \frac{3x+4}{(x+1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$x-3 = A(x^2+x+1) + Bx^2 + Cx$$

② $0 = A+B$ $\qquad A=-3$

① $1 = A+C$ $\qquad B=3$

⓪ $-3 = A$ $\qquad C=4$

$$(*) = -3\log(x) + \frac{3}{2}\int \frac{2x+1}{(x+1/2)^2+3/4}\,dx + \frac{5}{2}\int \frac{1}{(x+1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx =$$

$$= -3\log(x) + \frac{3}{2}\log((x+1/2)^2+3/4) + 5/2\int \frac{1/(\sqrt{3}/2)^2}{\left(\frac{x+1/2}{\sqrt{3}/2}\right)^2+1}\,dx$$

$$= -3\log(x) + \frac{3}{2}\log((x+1/2)^2+3/4) + \frac{5/2}{\sqrt{3}/2}\,\mathrm{arctg}\left(\frac{x+1/2}{\sqrt{3}/2}\right) + C$$

$$= -3\log(x) + \frac{3}{2}\log(x^2+x+1) + 5/\sqrt{3}\,\mathrm{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

7.6.- a) Calcula $\int \arcsin(x)\,dx$.

$$\int \arcsin(x)\,dx = x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx = (*)$$

$u = \arcsin(x)$, $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$

$dv = dx \quad v = x$

$$(*) = x\arcsin(x) + \frac{1}{2}\int -2x\,(1-x^2)^{-1/2}\,dx = x\arcsin(x) + \frac{1}{2}\cdot\frac{(1-x^2)^{-1/2+1}}{-1/2+1} + C$$

$$= x\arcsin(x) + \sqrt{1-x^2} + C$$

b) Análogamente, prueba que $\int f^{-1}(x)\,dx = x\,f^{-1}(x) - F(f^{-1}(x))$

donde $F = \int f(x)\,dx$

$$\int f^{-1}(x)\,dx = x\,f^{-1}(x) - \int x\left(f^{-1}(x)\right)'dx = (*)$$

$u = f^{-1}(x) \quad du = \left(f^{-1}(x)\right)'dx$

$dv = dx \quad v = x$

$$(*) = x\,f^{-1}(x) - \int f(t)\,dt = x\,f^{-1}(x) - F(t) + C = x\,f^{-1}(x) - F(f^{-1}(x)) + C$$

$t = f^{-1}(x) \quad dt = \left(f^{-1}(x)\right)'dx$

$x = f(t)$

7.7. Calcula una primitiva en los siguientes casos:

1), 2) y 3) Resueltos en clase

4) $$\int \frac{dx}{\sqrt{1+e^x}} = \int \frac{1}{t}\cdot\frac{2t}{t^2-1}\,dt = \int \frac{2}{t^2-1}\,dt = (*)$$

$\sqrt{1+e^x} = t$

$1+e^x = t^2 \rightsquigarrow e^x = t^2-1$

$e^x dx = 2t\,dt \rightarrow dx = \dfrac{2t}{t^2-1}\,dt$

$$\frac{2}{t^2-1} = \frac{A}{t-1} + \frac{B}{t+1} = \frac{A(t+1) + B(t-1)}{t^2-1} = \frac{(A+B)t + A - B}{t^2-1}$$

$\begin{array}{l} A+B=0 \\ A-B=2 \end{array} \rightarrow A=1,\ B=1$

$$(*) = \int\left(\frac{1}{t-1} - \frac{1}{t+1}\right)dt = \log(t-1) - \log(t+1) + C =$$

$= \log(\sqrt{1+e^x}-1) - \log(\sqrt{1+e^x}+1) + C = \log\left(\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right) + C$

5) $\int \frac{dx}{2+tg(x)} = \int \frac{1}{2+t} \cdot \frac{1}{1+t^2}\, dt = (*)$

$t = tg\,x$

$arctg(t) = x$

$\frac{1}{1+t^2}\, dt = dx$

$\frac{1}{(2+t)(1+t^2)} = \frac{A}{2+t} + \frac{B\cancel{t}+C}{t^2+1} = \frac{A(t^2+1)+(Bt+C)(2+t)}{(2+t)(t^2+1)}$

t^2 $\quad 0 = A+B$ $\qquad B = -1/5$

t $\quad 0 = 2B+C \qquad 0 = A+B \qquad A = 1/5 \quad C = 2/5$

1 $\quad 1 = A+2C \qquad 1 = A-4B$

$(*) = \int \left(\frac{1/5}{t+2} + \frac{-1/5\,\cancel{t}+2/5}{t^2+1}\right) dt = 1/5 \log(t+2) - \frac{1}{5}\int \frac{t-2}{t^2+1}\, dt$

$= 1/5 \log(t+2) - \frac{1}{2\cdot 5}\int \frac{2t}{t^2+1}\, dt + \frac{2}{5}\int \frac{1}{t^2+1}\, dt =$

$= 1/5 \log(t+2) - \frac{1}{10}\log(t^2+1) + \frac{2}{5}\cdot arctg(t) + C$

$= 1/5 \log(t+2) - \frac{1}{10}\log((tg\,x)^2+1) + \frac{2}{5}x + C$

$= \frac{1}{5}\log(tg\,x+2) - \frac{1}{10}$

6) $\int sen^3 x \cos^4 x\, dx = \int (sen\,x)^3(\cos x)^4\, dx = \int sen\,x\,(sen\,x)^2(\cos x)^4\, dx$

$= \int sen\,x\,(1-(\cos x)^2)(\cos x)^4\, dx = \int sen\,x\cdot(\cos x)^4\, dx - \int sen\,x\,(\cos x)^6\, dx =$

$= -\frac{(\cos x)^5}{5} + \frac{(\cos x)^7}{7} + C$

7) $\int \frac{dx}{\sqrt{\sqrt{x}+1}} = \int \frac{2(t^2-1)\cdot 2t\, dt}{t} = \int 4(t^2-1)\, dt = \frac{4t^3}{3} - 4t + C = (*)$

$\sqrt{\sqrt{x}+1} = t$

$\sqrt{x}+1 = t^2$

$\sqrt{x} = t^2-1$

$x = (t^2-1)^2 \quad dx = 2(t^2-1)\cdot 2t\, dt$

$(*) = \frac{4(\sqrt{\sqrt{x}+1})^3}{3} - 4\sqrt{\sqrt{x}+1} + C$

8) $\int \frac{arctg(x)}{1+x^2}\, dx = \frac{(arctg\, x)^2}{2} + C$

9) $\int \frac{x^2-1}{x^2+1}\,dx = \int \frac{x^2+1-2}{x^2+1}\,dx = \int 1 - \frac{2}{x^2+1}\,dx = x - 2\,\mathrm{arctg}(x) + C$

10) $\int \mathrm{arcsen}(\sqrt{x})\,dx = x\,\mathrm{arcsen}(\sqrt{x}) - \int \frac{x}{2\sqrt{1-x}\,\sqrt{x}}\,dx = (*)$

$u = \mathrm{arcsen}(\sqrt{x}) \quad du = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}\,dx$

$dv = dx \quad v = x$

$(*) = x\,\mathrm{arcsen}(\sqrt{x}) - \frac{1}{2}\int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx = x\,\mathrm{arcsen}(\sqrt{x}) - \frac{1}{2}\int \frac{\sqrt{1-t^2}}{t}\cdot(-2t\,dt) = (**)$

$\sqrt{1-x} = t$

$1-x = t^2 \rightarrow \sqrt{x} = \sqrt{1-t^2}$

$-dx = 2t\,dt$

$(**) = x\,\mathrm{arcsen}(\sqrt{x}) + \int \sqrt{1-t^2}\,dt = x\,\mathrm{arcsen}(\sqrt{x}) + \int \sqrt{1-\mathrm{sen}^2 z}\,\cdot$

$\boxed{\begin{array}{l} t = \mathrm{sen}\,z \\ dt = \cos z\,dz \end{array}}$

$\cdot \cos z\,dz = x\,\mathrm{arcsen}(\sqrt{x}) + \int \cos^2 z\,dz = (\circ)$

$\left.\begin{array}{l} \cos^2 z - \mathrm{sen}^2 z = \cos(2z) \\ \cos^2 z + \mathrm{sen}^2 z = 1 \end{array}\right\} \rightarrow \cos^2 z = \frac{1}{2}(1+\cos(2z))$

$(\circ) = x\,\mathrm{arcsen}(\sqrt{x}) + \int \frac{1}{2} + \frac{1}{2}\cos(2z)\,dz =$

$= x\,\mathrm{arcsen}(\sqrt{x}) + \frac{1}{2}z + \frac{\mathrm{sen}(2z)}{4} + C =$

$= x\,\mathrm{arcsen}(\sqrt{x}) + \frac{1}{2}\mathrm{arcsen}(t) + \frac{\mathrm{sen}(2\,\mathrm{arcsen}(t))}{4} + C =$

$= x\,\mathrm{arcsen}(\sqrt{x}) + \frac{1}{2}\mathrm{arcsen}(\sqrt{1-x}) + \mathrm{sen}\left(\frac{2\,\mathrm{arcsen}(\sqrt{1-x})}{4}\right) + C$

11) $\int \left(\mathrm{sen}\,x \int_0^x \mathrm{sen}\,t\,dt\right) dx = \int (\mathrm{sen}\,x\,[-\cos t]_0^x\,dx =$

$= \int \mathrm{sen}\,x\,[-\cos x + 1]\,dx = \frac{(\cos x)^2}{2} - \cos x + C$