ANALYTIC GEOMETRY:

HOLOMORPHIC FUNCTIONS AND SINGULARITIES

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Preface

These notes grew out of a course given by the second author at Universidad Complutense de Madrid. The first author, who was in the audience sixteen years ago, helped to improve the teaching material in an important way and we decided to spend some time to write down a kind of text-book in "Analytic Geometry". We hope that the final product could be used by graduate students and by us and some of our colleagues to lecture courses on this subject.

Our goal is to provide an elementary presentation of what we consider the basic results that an student should know to have a reasonable background in Local Analytic Geometry. We have gone beyond important results on this matter, like the equivalence between the two possible presentations of the local models for reduced analytic spaces, but without using sheaf theory. Just in the Appendix where we study the irreducible components of an analytic set we attack questions of a global nature.

Chapters I to IV are of algebraic character. We summarize them by saying that we present Mather's Finiteness Theorem in Chapter III, its main algebraic consequences in Chapters III and IV, like the Jacobian Criteria, Newton-Puiseux Theorem and the Nulstellensatz, and the required preliminary results in Chapters I and II: essentially Weierstrass's Preparation and Division Theorems.

Chapters V to VII have a topological-geometric flavor. At a first sight they could be understood as the translation to a geometric language of the precedent algebraic results. But we should remark that the chosen presentation allows the reader to understand how two different approaches of the local study of complex analytic sets converge in an unified way.

In the 50's of the past century Hans Grauert and Reinhold Remmert obtained very strong results on analytic geometry that are out of the scope of these notes, but the first steps consist of showing that analytic sets look locally as tame ramified coverings of open subsets of $\mathbb{C}^d$. This way they invented complex analytic spaces. On the other hand, Henri Cartan and their pupils (among them Jean Pierre Serre and some colleagues) studied the same objects with a different point of view, which pays special attention to the equations defining locally (and globally) analytic sets. In the local case this approach allowed the use of powerful results in Commutative Algebra, and to attack problems of global nature they introduced the cohomology of coherent sheaves. Referring to the enormous strength of such Commutative Algebra methods and those in sheaf theory to approach the global case Remmert said: *French have tanks, we have only bows and arrows.* Chapter VII is devoted to highlight at an elementary level the confluence of these two presentations of the local models for reduced analytic spaces.

The book is not self-contained. We provide at the end two bibliographical lists. The first one (called 'Basic bibliography') is constituted by texts where the reader will find proofs for the auxiliary results in Linear and Commutative Algebra, Real and Complex Analysis and Topology used along the text. The second one (called 'Further recommended references') is a selection of classical books on Complex and Real Analytic Geometry. Most of them present more advanced topics than the ones treated in this book, but we are sure that they cover the appetite of curious readers.

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CHAPTER I

Formal and convergent power series

In this chapter we study formal and convergent power series. We start by recalling the notion of convergence and its main properties: absolute convergence and reordering of iterated sums of convergent series. We then introduce formal and convergent series and discuss the operations with them: sum, multiplication, substitution and derivation. We obtain the Identity Principle for functions associated to convergent power series. Finally we state and prove some essential results of the theory: Rückert's Division Theorem, Weierstrass's Preparation Theorem, Implicit Function Theorem and Inverse Function Theorem.

The books [RF, Sh, S] are good references where the non-expert reader can learn about basic facts in analysis, with emphasis on questions related to convergence of numerical series. In [K] one finds Baire's theorem, which is needed to prove the existence of a simultaneous change of coordinates for countable many series. We need the Banach fixed point theorem [P]. As in most chapters some elementary linear algebra is needed, we refer to [L].

1 Series of Real and Complex Numbers

We denote by $\sum_{\nu\geq 1} a_\nu = \sum_\nu a_\nu$ a series of elements of the field $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$, that is, a function $a : \mathbb{N}^n \to \mathbb{K}$, $\nu \mapsto a_\nu$. The multi-indices $\nu := (\nu_1, \ldots, \nu_n)$ are elements of $\mathbb{N}^n$, where $\mathbb{N} := \{0, 1, 2, \ldots\}$. We will use the standard notations $|\nu| := \nu_1 + \cdots + \nu_n$ and $\nu! := \nu_1! \cdots \nu_n!$. Given a finite subset $I \subset \mathbb{N}^n$ we denote $\mathrm{Sat}(I) := \{\nu \in \mathbb{N}^n : \ |\nu| \leq \max\{|\mu| : \ \mu \in I\}\}$.

Definition I.1.1 The series $\sum_\nu a_\nu$ *converges to the element* $c \in \mathbb{K}$ if for each real number $\varepsilon > 0$ there is a finite set $I_\varepsilon \subset \mathbb{N}^n$ such that $|\sum_{\nu\in I} a_\nu - c| < \varepsilon$ for each finite set of indices $I \supset I_\varepsilon$. In that case we say that c is the *sum of the series* and we write $c := \sum_\nu a_\nu$.

Remarks I.1.2 (i) If each $a_\nu \in \mathbb{R}$ and $\sum_\nu a_\nu = c \in \mathbb{C}$, then $c \in \mathbb{R}$.

(ii) If $\sum_\nu a_\nu$ converges to $c \in \mathbb{K}$, then $c = \lim_{p\to\infty} \sum_{|\nu|\leq p} a_\nu$.

(iii) Let $\sum_\nu a_\nu,\ \sum_\nu b_\nu$ converge to $c, d \in \mathbb{K}$ respectively and let $\lambda, \mu \in \mathbb{K}$. Then $\sum_\nu (\lambda a_\nu + \mu b_\nu)$ converges to $\lambda c + \mu d$.

(iv) Let $\sum_\nu a_\nu,\ \sum_\nu b_\nu$ be two convergent series of real numbers such that $\sum_{\nu\in I} a_\nu \leq \sum_{\nu\in I} b_\nu$ for each finite set of indices I. Then $\sum_\nu a_\nu \leq \sum_\nu b_\nu$.

(v) If $0 \leq a_\nu \leq b_\nu$ and $\sum_\nu b_\nu$ is convergent, then $\sum_\nu a_\nu$ is also convergent.

(vi) For series with indices in $\mathbb{N}$ the convergence in the sense of Definition I.1.1 implies the classical one, but not conversely: $\sum_k (-1)^k/k$ does not converge according to Definition I.1.1; however the limit $\lim_{p\to\infty} \sum_{1\leq k\leq p} (-1)^k/k$ exist.

1.a Convergence criteria.

We present next some convergence criteria that will be useful along the sequel.

Proposition I.1.3 *Let $\sum_\nu a_\nu$ be a series of non-negative real numbers. The following assertions are equivalent:*

(i) *$\sum_\nu a_\nu$ is convergent.*

(ii) *There exists $M > 0$ such that $\sum_{\nu\in I} a_\nu < M$ for each finite $I \subset \mathbb{N}^n$.*

In this case $\sum_\nu a_\nu = \sup\{\sum_{\nu\in I} a_\nu :\ I \subset \mathbb{N}^n$ is finite$\}$.

Proof. Assume first that such an M exists. Then

$$c := \sup\Big\{\sum_{\nu\in I} a_\nu :\ I \subset \mathbb{N}^n \text{ is finite}\Big\} < +\infty.$$

We claim: *the series $\sum_\nu a_\nu$ converges to c.*

For every $\varepsilon > 0$ there exists $I_\varepsilon \subset \mathbb{N}^n$ such that $c - \varepsilon < \sum_{\nu\in I_\varepsilon} a_\nu$ and for each finite set $I \supset I_\varepsilon$ we have

$$\sum_{\nu\in I_\varepsilon} a_\nu \leq \sum_{\nu\in I} a_\nu \leq c.$$

Consequently, if $I \supset I_\varepsilon$,

$$0 \leq c - \sum_{\nu\in I} a_\nu \leq c - \sum_{\nu\in I_\varepsilon} a_\nu < \varepsilon.$$

Conversely, suppose that $\sum_\nu a_\nu$ converges to c. By Remark I.1.2 we have

$$c = \lim_{p\to\infty} \sum_{|\nu|\le p} a_\nu.$$

Notice that $M := c+1 > 0$, as each $a_\nu \ge 0$. If $I \subset \mathbb{N}^n$ is finite and $p := \sup\{|\nu| : \nu \in I\}$, it holds $\sum_{\nu\in I} a_\nu \le \sum_{\nu\in \mathrm{Sat}(I)} a_\nu = \sum_{|\nu|\le p} a_\nu \le c < M$, as required. □

Proposition I.1.4 *Let $\sum_\nu a_\nu$ be a series. The following assertions are equivalent:*

(i) *The series of complex numbers $\sum_\nu a_\nu$ is convergent.*

(ii) *The series of real numbers $\sum_\nu |a_\nu|$ is convergent.*

In addition, we have $|\sum_\nu a_\nu| \le \sum_\nu |a_\nu|$.

Proof. We distinguish two cases:

Case 1. Assume first that each $a_\nu \in \mathbb{R}$. For every ν we set

$$p_\nu := \max\{a_\nu, 0\},\ q_\nu := \max\{-a_\nu, 0\}.$$

Observe that

- $p_\nu + q_\nu = |a_\nu|$ and $p_\nu - q_\nu = a_\nu$.
- $0 \le p_\nu, q_\nu \le |a_\nu|$.

If the series $\sum_\nu |a_\nu|$ converges, the same happens with the series $\sum_\nu p_\nu$ and $\sum_\nu q_\nu$ by Proposition I.1.3. Thus, also the series $\sum_\nu a_\nu = \sum(p_\nu - q_\nu)$ converges by Remark I.1.2(iii).

Suppose now that $\sum_\nu a_\nu$ converges to c. Take $\varepsilon = 1$ and consider the set of indices I_1 provided by the definition of convergence. For every $L \subset \mathbb{N}^n$ write

$$L^+ := \{\nu \in L : a_\nu > 0\} \quad \text{and} \quad L^- := \{\nu \in L : a_\nu \le 0\}.$$

For every finite set of indices I we put $J := I^+ \cup I_1$. As each $p_\nu \geq 0$, $J \supset I_1$ and $J^- = I_1^-$, we have

$$\begin{aligned}\sum_{\nu\in I} p_\nu &= \sum_{\nu\in I^+} p_\nu \leq \sum_{\nu\in J} p_\nu = \sum_{\nu\in J^+} a_\nu = \sum_{\nu\in J} a_\nu - c + c - \sum_{\nu\in J^-} a_\nu \\ &\leq \Big|\sum_{\nu\in J} a_\nu - c\Big| + \Big|\sum_{\nu\in J^-} a_\nu\Big| + |c| < 1 + \Big|\sum_{\nu\in I_1^-} a_\nu\Big| + |c| =: M.\end{aligned}$$

By Proposition I.1.3 the series $\sum_\nu p_\nu$ converges. Analogously, $\sum_\nu q_\nu$ converges. By Remark I.1.2 (iii) the series $\sum_\nu |a_\nu| = \sum_\nu (p_\nu + q_\nu)$ is also convergent. The inequality $|\sum_\nu a_\nu| \leq \sum_\nu |a_\nu|$ follows from Remark I.1.2 (iv).

Case 2. The general case: each $a_\nu \in \mathbb{C}$. Denote

$$u_\nu := \Re(a_\nu) = \tfrac{1}{2}(a_\nu + \overline{a}_\nu) \quad \text{and} \quad v_\nu := \Im(a_\nu) = \tfrac{1}{2\sqrt{-1}}(a_\nu - \overline{a}_\nu)$$

the real and the imaginary parts of a_ν. We have:

$$a_\nu = u_\nu + \sqrt{-1}v_\nu,\ \ |a_\nu| \leq |u_\nu| + |v_\nu|,\ \ |u_\nu| \leq |a_\nu| \quad \text{and} \quad |v_\nu| \leq |a_\nu|.$$

Assume first that $\sum_\nu |a_\nu|$ is convergent. Then $\sum_\nu |u_\nu|$ and $\sum |v_\nu|$ are convergent. As each $u_\nu, v_\nu \in \mathbb{R}$, the series $\sum_\nu u_\nu$ and $\sum_\nu v_\nu$ are convergent by the Case 1. Consequently, the same happens with $\sum_\nu a_\nu = \sum_\nu u_\nu + \sqrt{-1}\sum_\nu v_\nu$.

Assume now that $\sum_\nu a_\nu$ converges to c, so $\sum_\nu \overline{a}_\nu$ converges to $\overline{c}$ and

- $\sum_\nu u_\nu = \sum_\nu \frac{1}{2}(a_\nu + \overline{a}_\nu)$ converges to $\Re(c)$.
- $\sum_\nu v_\nu = \sum_\nu \frac{1}{2\sqrt{-1}}(a_\nu - \overline{a}_\nu)$ converges to $\Im(c)$.

Consequently, $\sum_\nu |u_\nu|$ and $\sum_\nu |v_\nu|$ are convergent, so $\sum_\nu(|u_\nu| + |v_\nu|)$ is convergent. As $|a_\nu| \leq |u_\nu| + |v_\nu|$, the series $\sum_\nu |a_\nu|$ is also convergent.

For each finite set $I \subset \mathbb{N}^n$, we have $|\sum_{\nu\in I} a_\nu| \leq \sum_{\nu\in I} |a_\nu|$, so

$$\Big|\sum_\nu a_\nu\Big| \leq \sup\Big\{\Big|\sum_{\nu\in I} a_\nu\Big| :\ I \subset \mathbb{N}^n\Big\} \leq \sup\Big\{\sum_{\nu\in I} |a_\nu| :\ I \subset \mathbb{N}^n\Big\} = \sum_\nu |a_\nu|,$$

as required. $\square$

From the previous results we conclude the following.

Corollary I.1.5 *Let $\sum_\nu a_\nu$ be a series of complex terms. The following assertions are equivalent:*

(i) *The series of complex numbers $\sum_\nu a_\nu$ is convergent.*

(ii) *The two series $\sum_\nu \Re(a_\nu)$ and $\sum_\nu \Im(a_\nu)$ are convergent.*

(iii) *The two series $\sum_\nu |\Re(a_\nu)|$ and $\sum_\nu |\Im(a_\nu)|$ are convergent.*

(iv) *There exists $M > 0$ such that $\sum_{\nu\in I} |a_\nu| < M$ for each finite set of indices $I \subset \mathbb{N}^n$.*

(v) *The limit* $\lim_{p\to+\infty} \sum_{|\nu|\leq p} |a_\nu|$ *exists.*

(vi) *The limit* $\lim_{p\to+\infty} \sum_{\nu\in\{0,1,\dots,p\}^n} |a_\nu|$ *exists.*

Corollary I.1.6 *Let $\sum_\nu a_\nu$ and $\sum_\nu b_\nu$ be two series such that $\sum_\nu a_\nu$ converges and $|b_\nu| \leq |a_\nu|$ for all ν. Then $\sum_\nu b_\nu$ converges.*

Proposition I.1.7 (Iterated series) *Let $\sum_\nu a_\nu$ be a convergent series and consider a permutation τ of the set of indices $\{1,\dots,n\}$. Then the sum of the* iterated series $\sum_{\nu_{\tau(1)}} \cdots \sum_{\nu_{\tau(n)}} a_\nu$ *converges and coincides with $\sum_\nu a_\nu$.*

Proof. Suppose for simplicity that τ is the identity and let us argue by induction on n. For $n = 1$ the assertion is trivial, so we assume $n > 1$ and the result is true for series with indices in $\mathbb{N}^{n-1}$.

1.a.1 We claim: *For every $\nu_1 \in \mathbb{N}$ the series $\sum_{\nu'} a_{(\nu_1,\nu')}$ converges.* To that end, we check: *$\sum_{\nu'} |a_{(\nu_1,\nu')}|$ converges.*

Consider a finite set of indices $I' \subset \mathbb{N}^{n-1}$. We have

$$\sum_{\nu'\in I'} |a_{(\nu_1,\nu')}| = \sum_{\nu\in\{\nu_1\}\times I'} |a_\nu|. \tag{1.1}$$

By Proposition I.1.4 the series of non-negative real numbers $\sum_\nu |a_\nu|$ converges, so it verifies the criterion of Proposition I.1.3. By (1.1) the series $\sum_{\nu'} |a_{(\nu_1,\nu')}|$ also verifies that criterion, so it converges.

1.a.2 Consequently, $\sum_{\nu'} a_{(\nu_1,\nu')}$ converges and we write $b_{\nu_1} := \sum_{\nu'} a_{(\nu_1,\nu')}$. Let us show: $\sum_{\nu_1} b_{\nu_1} = \sum_{\nu} a_\nu$.

Write $c := \sum_\nu a_\nu$ and let $\varepsilon > 0$. Let $I_{\varepsilon/2} \subset \mathbb{N}^n$ be a finite set such that $|\sum_{\nu\in I} a_\nu - c| < \varepsilon/2$ for each finite set $I \supset I_{\varepsilon/2}$. Let $J_{\varepsilon/2} := \pi_1(I_{\varepsilon/2})$ where $\pi_1 : \mathbb{N}^n \to \mathbb{N}, \nu := (\nu_1, \dots, \nu_n) \to \nu_1$ is the projection onto the first factor. Fix a finite set $J \supset J_{\varepsilon/2}$ with k elements and choose a finite set $I' \subset \mathbb{N}^{n-1}$ such that $I_{\varepsilon/2} \subset J \times I'$ and

$$\Big|\sum_{\nu'\in I'} a_{(\nu_1,\nu')} - b_{\nu_1}\Big| < \frac{1}{2k}\varepsilon$$

for each $\nu_1 \in J$. We have:

$$\begin{aligned}\Big|\sum_{\nu_1\in J} b_{\nu_1} - c\Big| &= \Big|\sum_{\nu_1\in J} b_{\nu_1} - \sum_{\nu_1\in J}\sum_{\nu'\in I'} a_{(\nu_1,\nu')} + \sum_{\nu_1\in J}\sum_{\nu'\in I'} a_{(\nu_1,\nu')} - c\Big|\\ &\le \sum_{\nu_1\in J} |b_{\nu_1} - \sum_{\nu'\in I'} a_{(\nu_1,\nu')}| + \Big|\sum_{\nu\in J\times I'} a_\nu - c\Big| < k\frac{\varepsilon}{2k} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This shows that the series $\sum_{\nu_1} b_{\nu_1}$ converges to c.

1.a.3 In addition, b_{ν_1} is by the induction hypothesis the sum of the iterated series $\sum_{\nu_2} \cdots \sum_{\nu_n} a_{(\nu_1,\nu_2,\dots,\nu_n)}$, so $\sum_{\nu_1} \cdots \sum_{\nu_n} a_\nu$ converges and coincides with

$$\sum_{\nu_1} \cdots \sum_{\nu_n} a_\nu = \sum_{\nu_1} b_{\nu_1} = c,$$

as required. $\square$

Examples I.1.8 (i) Let $\lambda_1, \dots, \lambda_n$ be complex numbers such that $|\lambda_j| < 1$. Then the series $\sum_\nu \lambda^\nu := \sum_\nu \lambda_1^{\nu_1} \cdots \lambda_n^{\nu_n}$ converges to $\prod_{j=1}^n (\frac{1}{1-\lambda_j})$.

We prove first that the series converges. To that end, we show that

$$\begin{aligned}\lim_{p\to\infty} \sum_{\nu\in\{0,1,\dots,p\}^n} |\lambda_1^{\nu_1} \cdots \lambda_n^{\nu_n}| &= \lim_{p\to\infty} \prod_{j=1}^n \Big(\sum_{\nu_j=0}^p |\lambda_j|^{\nu_j}\Big)\\ &= \lim_{p\to\infty} \prod_{j=1}^n \Big(\frac{1-|\lambda_j|^{p+1}}{1-|\lambda_j|}\Big) = \prod_{j=1}^n \Big(\frac{1}{1-|\lambda_j|}\Big)\end{aligned}$$

Now, the sum is computed using iterated series and we have

$$\begin{aligned}\sum_{\nu}\lambda^{\nu}&:=\sum_{\nu}\lambda_1^{\nu_1}\cdots\lambda_n^{\nu_n}=\sum_{\nu_1}\cdots\sum_{\nu_n}\lambda_1^{\nu_1}\cdots\lambda_n^{\nu_n}=\prod_{j=1}^{n}\sum_{\nu_j}\lambda_j^{\nu_j}\\&=\prod_{j=1}^{n}\lim_{p_j\to+\infty}\sum_{\nu_j=0}^{p_j}\lambda_j^{\nu_j}=\prod_{j=1}^{n}\lim_{p_j\to+\infty}\Big(\frac{1-\lambda_j^{p_j+1}}{1-\lambda_j}\Big)=\prod_{j=1}^{n}\Big(\frac{1}{1-\lambda_j}\Big).\end{aligned}$$

(ii) For each $\nu_1\geq 1$ define $a_{\nu_1}:=\lambda^{\nu_1}\cdot\nu_1!$. Then the series $\sum_{\nu_1\geq 1}a_{\nu_1}$ is not convergent. As the series has positive terms, its convergence coincides with the classical convergence. Thus, we can apply the quotient criterion:

$$\lim_{\nu_1\to+\infty}\frac{a_{\nu_1+1}}{a_{\nu_1}}=\lim_{\nu_1\to+\infty}\frac{\lambda^{\nu_1+1}\cdot(\nu_1+1)!}{\lambda^{\nu_1}\cdot\nu_1!}=\lambda\lim_{\nu_1\to+\infty}(\nu_1+1)=+\infty.$$

Consequently, $\sum_{\nu_1\geq 1}a_{\nu_1}$ is not convergent.

2 Formal power series

As before let us denote $\mathbb{K}:=\mathbb{R}$ or $\mathbb{C}$.

Definitions I.2.1 A *formal power series* in the indeterminates $\mathtt{x}_1,\dots,\mathtt{x}_n$ is an expression $f:=\sum_{\nu\in\mathbb{N}^n}a_\nu\mathtt{x}_1^{\nu_1}\cdots\mathtt{x}_n^{\nu_n}$, shortly written $\sum_\nu a_\nu\mathtt{x}^\nu$, where $a_\nu\in\mathbb{K}$ for each ν. The values a_ν are the *coefficients* of $\sum a_\nu\mathtt{x}^\nu$ and the first of them $a_{(0,\dots,0)}$ is denoted by $f(0)$. The set of all these formal power series will be denoted $\mathcal{F}_n$, $\mathbb{K}[[\mathtt{x}_1,\dots,\mathtt{x}_n]]$ or $\mathbb{K}[[\mathtt{x}]]$.

The series whose coefficients are all 0 is the zero series and will be denoted with 0. The *order* of a non-zero formal power series $\sum a_\nu\mathtt{x}^\nu\in\mathcal{F}_n$, denoted $\omega(f)$, is the smallest integer $p\geq 0$ such that $a_\nu\neq 0$ for some ν with $|\nu|=p$. Otherwise, that is, if $f=0$, we write $\omega(f)=+\infty$. For each $p\in\mathbb{N}$ the homogeneous polynomial $f_p=\sum_{|\nu|=p}a_\nu\mathtt{x}^\nu$, which has either degree p or it is identically zero, is the *homogeneous component of degree p of f*. We have $\omega(f)=\inf\{p\in\mathbb{N}:\ f_p\neq 0\}$. If $\omega(f)=k$, then the homogeneous component f_k of f is called the *initial form of f*.

2.a Operations with formal power series. Before proving that $\mathcal{F}_n$ has a natural structure of unital commutative ring, we introduce the concept of summable family.

Definitions I.2.2 (Summable family) A family $\{f_\lambda : \lambda \in \Lambda\}$ of formal power series

$$f_\lambda = \sum a_{\lambda\nu}\mathtt{x}^\nu$$

is called *summable* if for each integer $p \geq 0$ the subfamily Λ_p of the series f_λ of order $\leq p$ is finite.

For every $\nu \in \mathbb{N}^n$ the set C_ν of all $\lambda \in \Lambda$ with $a_{\lambda\nu} \neq 0$ is finite (if $a_{\lambda\nu} \neq 0$ then $\omega(f_\lambda) \leq |\nu|$) and the sum $\sum_{\lambda\in\Lambda} a_{\lambda\nu} = \sum_{\lambda\in C_\nu} a_{\lambda\nu} \in \mathbb{K}$ (because it is a finite sum). Consequently, the formal power series $\sum_\nu \left(\sum_{\lambda\in\Lambda} a_{\lambda\nu}\right)\mathtt{x}^\nu$ is well-defined and it is called the *sum* of the family $\{f_\lambda : \lambda \in L\}$ and denoted with $\sum_\lambda f_\lambda$.

Definition and Lemma I.2.3 *Let $f = \sum_\nu a_\nu \mathtt{x}^\nu$ and $g = \sum_\nu b_\nu \mathtt{x}^\nu$ be two formal power series.* We define their *sum* by

$$f + g := \sum_\nu (a_\nu + b_\nu)\mathtt{x}^\nu$$

and their *product* by

$$fg := \sum_\nu \Big(\sum_{\lambda+\mu=\nu} a_\lambda b_\mu\Big)\mathtt{x}^\nu.$$

We have:

(i) $f + g, fg \in \mathfrak{F}_n$.

(ii) $\omega(f+g) \geq \min\{\omega(f), \omega(g)\}$ *and* $\omega(fg) = \omega(f) + \omega(g)$. *If the initial forms of f, g share no monomial, then* $\omega(f+g) = \min\{\omega(f), \omega(g)\}$.

(iii) *If $f \neq 0$ and $g \neq 0$, then $fg \neq 0$.*

(iv) *$\mathfrak{F}_n$ is a commutative ring with unit, which contains the field of coefficients $\mathbb{K}$, so it is a $\mathbb{K}$-algebra. In addition, $\mathfrak{F}_n$ is an integral domain.*

Proof. (i) Observe that $f + g$ is the sum of the summable family $\{f, g\}$ and fg is the the sum of the summable family $\{(\sum_{\lambda+\mu=\nu} a_\lambda b_\mu)\mathtt{x}^\nu\}_{\nu\in\mathbb{N}^n}$. Thus, $f + g, fg \in \mathfrak{F}_n$.

(ii) Let $\omega(f) = p$ and $\omega(g) = q$ and write $f := \sum_k f_k$ and $g := \sum_\ell g_\ell$ as the sum of its homogeneous components. We have $f_p, g_q \neq 0$ and $f + g = \sum_{k\geq\min\{p,q\}}(f_k + g_k)$, so $\omega(f+g) \geq \min\{p, q\} = \min\{\omega(f), \omega(g)\}$. In addition, if f_p and g_q share no monomial, then the initial form of $f+g$ is $f_{\min\{p,q\}}+g_{\min\{p,q\}}$ and $\omega(f+g)=\min\{\omega(f),\omega(g)\}$.

If $fg=\sum_{k+\ell\geq p+q} f_k g_\ell$ and $(fg)_{p+q}=f_p g_q\neq 0$, so $\omega(fg)=p+q=\omega(f)+\omega(g)$.

If $f=0$, then $\omega(f+g)=\omega(g)=\min\{\omega(f),\omega(g)\}$ and $\omega(fg)=\omega(0)=\omega(f)+\omega(g)$.

(iii) If $f,g\neq 0$, then $\omega(fg)=\omega(f)+\omega(g)<+\infty$, so $fg\neq 0$.

(iv) The map $\mathbb{K}\hookrightarrow\mathcal{F}_n,\ a\mapsto f_a$ where $f_a:=\sum_\nu a_\nu \mathrm{x}^\nu$ and

$$a_\nu := \begin{cases} a & \text{if } \nu=(0,\dots,0),\\ 0 & \text{otherwise,}\end{cases}$$

is a monomorphism, so $\mathcal{F}_n$ is a $\mathbb{K}$-algebra. In addition, $\mathcal{F}_n$ is an integral domain by (iii). $\square$

As one can expect, finite combinations and finite products of summable families are still a summable family. We leave the concrete details to the reader.

Lemma I.2.4 *Let $\{f_\lambda:\ \lambda\in\Lambda\}$ and $\{g_\lambda:\ \lambda\in\Lambda\}$ be two summable families and let u,v be two formal power series. Then*

(i) *The family $\{uf_\lambda+vg_\lambda:\ \lambda\in\Lambda\}$ is summable and its sum is the series $u\sum_\lambda f_\lambda+v\sum_\lambda g_\lambda$.*

(ii) *The family $\{f_\lambda g_\mu:\ (\lambda,\mu)\in\Lambda\times\Lambda\}$ is summable and its sum is*

$$\sum_{(\lambda,\mu)\in\Lambda\times\Lambda} f_\lambda g_\mu=\Big(\sum_\lambda f_\lambda\Big)\Big(\sum_\mu g_\mu\Big).$$

2.b Substitution. Let $f:=\sum_\nu a_\nu \mathrm{x}^\nu\in\mathcal{F}_n$ and $g_1,\dots,g_n\in\mathcal{F}_m$ be formal power series with orders $\omega(g_1)\geq 1,\dots,\omega(g_n)\geq 1$. Then for each ν the order

$$\omega(a_\nu g_1^{\nu_1}\cdots g_n^{\nu_n})\geq \nu_1\omega(g_1)+\cdots+\nu_n\omega(g_n)\geq|\nu|,$$

so the family $\{a_\nu g_1^{\nu_1}\cdots g_n^{\nu_n}:\ \nu\in\mathbb{N}^n\}$ is summable. The sum of this family is called the *substitution of* $g_1,\dots,g_n$ *in* f and denoted with $f(g_1,\dots,g_n)$.

2.b.1 It is a straightforward computation to check that for any other formal power series h it holds

(i) $(h+f)(g_1,\dots,g_n) = h(g_1,\dots,g_n) + f(g_1,\dots,g_n)$ and

(ii) $(hf)(g_1,\dots,g_n) = h(g_1,\dots,g_n)f(g_1,\dots,g_n)$.

2.b.2 As an application of substitution, consider the equality

$$(1-\mathtt{x}_1)\sum_{\nu_1}\mathtt{x}_1^{\nu_1} = 1.$$

Then for each $f \in \mathcal{F}_n$ with $f(0) = a \neq 0$, we have

$$1 = \Big(1 - \Big(1 - \frac{1}{a}f\Big)\Big)\sum_{\nu_1}\Big(1 - \frac{1}{a}f\Big)^{\nu_1},$$

and consequently there exists the formal power series

$$\frac{1}{f} = \frac{1}{a}\sum_{\nu_1}\Big(1 - \frac{1}{a}f\Big)^{\nu_1}.$$

Conversely, if $f \in \mathcal{F}_n$ is a unit, there exists $g \in \mathcal{F}_n$ such that $fg = 1$, so $\omega(f) + \omega(g) = \omega(fg) = \omega(1) = 0$ and $\omega(f) = 0$.

2.b.3 It follows that $\mathcal{F}_n$ is a local ring whose unique maximal ideal $\widehat{\mathfrak{m}}_n$ consists of the formal power series with order ≥ 1. Clearly this ideal is generated by the indeterminates: $\widehat{\mathfrak{m}}_n = \{\mathtt{x}_1,\dots,\mathtt{x}_n\}\mathcal{F}_n = \{f \in \mathcal{F}_n : \ \omega(f) \geq 1\}$.

Indeed, if $f = \sum_\nu a_\nu \mathtt{x}^\nu \in \mathcal{F}_n$ has $\omega(f) \geq 1$, then

$$f = \sum_{i=1}^n \mathtt{x}_i\Big(\sum_{\nu_1=0,\dots,\nu_{i-1}=0,\nu_i\geq 1} a_\nu \mathtt{x}_i^{\nu_i-1}\mathtt{x}_{i+1}^{\nu_{i+1}}\cdots\mathtt{x}_n^{\nu_n}\Big).$$

On the other hand, if $f \in \{\mathtt{x}_1,\dots,\mathtt{x}_n\}\mathcal{F}_n$, then $f = \mathtt{x}_1f_1 + \cdots + \mathtt{x}_nf_n$ where $f_i \in \mathcal{F}_n$, so $\omega(f) \geq \min\{\omega(\mathtt{x}_i) + \omega(f_i) : \ i = 1,\dots,n\} \geq 1$.

2.c Derivatives and Taylor expansion. Let $1 \leq i \leq n$. The *derivative with respect to* $\mathtt{x}_i$ of a formal power series $f := \sum a_\nu \mathtt{x}^\nu$ is the formal power series

$$\frac{\partial f}{\partial \mathtt{x}_i} := \sum_{\nu_i>0}\nu_i a_\nu \mathtt{x}_1^{\nu_1}\cdots\mathtt{x}_i^{\nu_i-1}\cdots\mathtt{x}_n^{\nu_n} = \sum_\nu(\nu_i+1)a_{\nu+\mathtt{e}_i}x^\nu.$$

where $\mathtt{e}_i := (0,\dots,0,\overset{(i)}{1},0,\dots,0)$. Observe that $\omega(\frac{\partial f}{\partial \mathtt{x}_i}) \geq \omega(f) - 1$.

2.c.1 If $\{f_\lambda : \lambda \in \Lambda\}$ is a summable family of formal power series, the family $\{\frac{\partial_\lambda f_\lambda}{\partial \mathtt{x}_i} : \lambda \in \Lambda\}$ is also summable and its sum is $\frac{\partial(\sum_\lambda f_\lambda)}{\partial \mathtt{x}_i}$.

2.c.2 In the same way, the usual properties of derivatives hold true in this formal setting.

(i) The *Leibniz Formula*:

$$\frac{\partial(fg)}{\partial \mathtt{x}_i} = f\frac{\partial g}{\partial \mathtt{x}_i} + g\frac{\partial f}{\partial \mathtt{x}_i}.$$

(ii) The *Chain Rule*:

$$\frac{\partial(f(g_1,\ldots,g_n))}{\partial \mathtt{x}_i} = \sum_{1\leq j\leq n} \frac{\partial f}{\partial \mathtt{x}_j}(g_1,\ldots,g_n)\frac{\partial g_j}{\partial \mathtt{x}_i}.$$

(iii) The *Schwartz Rule*:

$$\frac{\partial}{\partial \mathtt{x}_i}\Big(\frac{\partial f}{\partial \mathtt{x}_j}\Big) = \frac{\partial}{\partial \mathtt{x}_j}\Big(\frac{\partial f}{\partial \mathtt{x}_i}\Big).$$

This last formula provides the definition by induction of the *derivatives of higher order* $\frac{\partial^{|\alpha|} f}{\partial \mathtt{x}^\alpha}$ where $\partial \mathtt{x}^\alpha$ stands for $\partial \mathtt{x}_1^{\alpha_1}\cdots\partial \mathtt{x}_n^{\alpha_n}$. Namely, if $f = \sum_\nu a_\nu \mathtt{x}^\nu$, then $\frac{\partial^{|\alpha|} f}{\partial \mathtt{x}^\alpha} = \sum_\nu a_{\nu+\alpha}\frac{(\nu+\alpha)!}{\nu!}\mathtt{x}^\nu$ where $\nu! = \nu_1!\cdots\nu_n!$. We deduce in particular that $\frac{\partial^{|\alpha|} f}{\partial \mathtt{x}^\alpha}(0) = a_\alpha \alpha!$.

2.c.3 It follows from 2.c.2 that can be written as:

$$f = \sum \frac{1}{\nu!}\frac{\partial^{|\nu|} f}{\partial \mathtt{x}^\nu}(0)\mathtt{x}^\nu.$$

This expression is the *Taylor expansion* of the formal power series f. In addition, if $\mathtt{x} := (\mathtt{x}_1,\ldots,\mathtt{x}_n), \mathtt{y} := (\mathtt{y}_1,\ldots,\mathtt{y}_m)$ and $f \in \mathbb{K}[[\mathtt{x},\mathtt{y}]]$,

$$f(\mathtt{x},\mathtt{y}) = \sum_\nu \frac{1}{\nu!}\frac{\partial^{|\nu|} f}{\partial \mathtt{x}^\nu}(0,\mathtt{y})\mathtt{x}^\nu. \tag{2.2}$$

Let us prove (2.2). The family $\{h_\nu := \frac{1}{\nu!}\frac{\partial^{|\nu|} f}{\partial \mathtt{x}^\nu}(0,\mathtt{y})\mathtt{x}^\nu : \nu \in \mathbb{N}^n\}$ is summable and we denote

$$h := \sum_\nu h_\nu = \sum_\nu \frac{1}{\nu!}\frac{\partial^{|\nu|} f}{\partial \mathtt{x}^\nu}(0,\mathtt{y})\mathtt{x}^\nu$$

its sum. Write $g_\nu(\mathtt{y}) := \frac{\partial^{|\nu|} f}{\partial \mathtt{x}^\nu}(0, \mathtt{y})$, whose Taylor expansion is

$$g_\nu(\mathtt{y}) = \sum_\mu \frac{1}{\mu!} \frac{\partial^{|\mu|} g_\nu}{\partial \mathtt{y}^\mu}(0)\mathtt{y}^\mu.$$

Consequently,

$$\begin{aligned} h = \sum_\nu \frac{1}{\nu!}\Big(\sum_\mu \frac{1}{\mu!} \frac{\partial^{|\mu|} g_\nu}{\partial \mathtt{y}^\mu}(0)\mathtt{y}^\mu\Big)\mathtt{x}^\nu &= \sum_{(\nu,\mu)} \frac{1}{\nu!}\frac{1}{\mu!} \frac{\partial^{|\mu|} g_\nu}{\partial \mathtt{y}^\mu}(0)\mathtt{x}^\nu\mathtt{y}^\mu \\ &= \sum_{(\nu,\mu)} \frac{1}{(\nu,\mu)!} \frac{\partial^{|(\nu,\mu)|}}{\mathtt{x}^\nu\mathtt{y}^\mu} f(0)\mathtt{x}^\nu\mathtt{y}^\mu = f, \end{aligned}$$

as required.

3 Convergent power series

In this section we study convergent power series. We will see that many of the properties already seen for formal power series are still valid for convergent power series, also called *analytic power series.*

Definitions I.3.1 Let $f := \sum_\nu a_\nu \mathtt{x}^\nu \in \mathcal{F}_n$ be a formal power series. If $x \in \mathbb{K}^n$ and the series $\sum a_\nu x^\nu$ of elements of $\mathbb{K}$ converges to $c \in \mathbb{K}$, we say that f *converges at x to c* and write $f(x) = c$. Now let $A \subset \mathbb{K}^n$. We say that f *converges uniformly on A* if:

(i) f converges at each point of A and

(ii) for each positive real number ε there exists a finite set $I_\varepsilon \subset \mathbb{N}^n$ such that $|\sum_{\nu\in I_\varepsilon} a_\nu x^\nu - f(x)| < \varepsilon$ for each finite set of indices $I \supset I_\varepsilon$ and each point $x \in A$.

The *open domain* $D(f)$ of f is the interior of the set of points $C(f)$ at which f converges. The series f is called *convergent* if $D(f) \neq \varnothing$. We also consider the set $C^*(f) := C(f) \cap \{\mathtt{x}_1 \neq 0, \dots, \mathtt{x}_n \neq 0\}$.

The set of convergent power series is denoted $\mathcal{O}_n$, $\mathbb{K}\{\mathtt{x}_1, \dots, \mathtt{x}_n\}$ or $\mathbb{K}\{\mathtt{x}\}$.

Remark I.3.2 If $f \in \mathcal{F}_n$ converges uniformly on $A_1, \dots, A_r \subset C(f)$, then f converges uniformly on $\bigcup_{i=1}^r A_i$.

Examples I.3.3 (i) If $f \in \mathcal{F}_n$, then $0 \in C(f)$ and $f(0)$ is the independent term $a_{(0,\dots,0)}$ of f.

(ii) Let $f := \sum_{\nu_1 \geq 0} \nu_1! \mathtt{x}_1^{\nu_1} \in \mathcal{F}_1$. Then $C(f) = \{0\}$ and $f \in \mathcal{F}_1 \setminus \mathcal{O}_1$.

If $x \in \mathbb{K} \setminus \{0\}$, then by Example I.1.8(ii) the series

$$\sum_{\nu_1 \geq 0} |\nu_1! x^{\nu_1}| = \sum_{\nu_1 \geq 0} \nu_1! |x|^{\nu_1}$$

does not converge, so the same happens with the series $\sum_{\nu_1 \geq 0} \nu_1! x^{\nu_1}$.

(iii) Let $f \in \mathcal{F}_n$, $0 < \lambda < 1$, $x \in C^*(f)$ and $x' \in D(f)$. Then $\lambda x \in C^*(f)$ and $\lambda x' \in D(f)$.

3.a Domain of a convergent power series. We endow the affine space $\mathbb{K}^n$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with the usual euclidean topology. Every point $x_0 := (x_{01}, \dots, x_{0n}) \in \mathbb{K}^n$ has a basis of open neighborhoods that consists of the polycylinders $\Delta_\rho(x_0)$ of center x_0 and polyradius $\rho := (\rho_1, \dots, \rho_n)$ where $\rho_i > 0$. Namely,

$$\Delta_\rho(x_0) := \{x \in \mathbb{K}^n : |x_i - x_{0i}| < \rho_i \text{ for } 1 \leq i \leq n\} = \prod_{i=1}^n D_{\rho_i}(x_{0i})$$

where $D_{\rho_i}(x_{0i}) := \{x_i \in \mathbb{K} : \ |x_i - x_{0i}| < \rho_i\}$ is the disc of center x_{0i} and radius $\rho_i > 0$. We use the notation: Given a point $z := (z_1, \dots, z_n) \in \mathbb{K}^n$ whose coordinates are all non-zero, $\Delta_{|z|}$ will stand for the polycylinder of polyradius $|z| := (|z_1|, \dots, |z_n|)$ centered at the origin.

Proposition I.3.4 *Let $f = \sum_\nu a_\nu \mathtt{x}^\nu \in \mathcal{F}_n$. We have:*

(i) *If $x \in C^*(f)$ and $0 < \lambda < 1$, then f converges uniformly in $\Delta_{|\lambda x|}$.*

(ii) *The set $D(f)$ is the union of the $\Delta_{|x|}$ where $x \in C^*(f)$.*

(iii) *f converges uniformly on every compact subset of $D(f)$.*

(iv) *$f \in \mathcal{O}_n$ if and only if $C^*(f) \neq \varnothing$.*

(v) *If $f \in \mathcal{O}_n$, then $D(f)$ is a connected open neighborhood of the origin.*

Proof. (i) As $\sum a_\nu x^\nu$ converges, there exists by Corollary I.1.5 $M > 0$ such that $|a_\nu x^\nu| < M$ for all $\nu \in \mathbb{N}^n$. Now, if $z \in \Delta_{|\lambda x|}$ and $I \subset \mathbb{N}^n$ is a finite subset, we have

$$\sum_{\nu\in I} |a_\nu z^\nu| \le \sum_{\nu\in I} |a_\nu x^\nu| \lambda^{|\nu|} < M \sum_{\nu\in I} \lambda^{|\nu|}. \tag{3.3}$$

By Example I.1.8

$$\sum_{\nu\in I} \lambda^{|\nu|} \le \sum_{\nu} \lambda^{|\nu|} = \Big(\sum_{\nu_1} \lambda^{\nu_1}\Big)\cdots\Big(\sum_{\nu_n} \lambda^{\nu_n}\Big) = \Big(\frac{1}{1-\lambda}\Big)^n =: \Lambda.$$

We conclude that the series $\sum_\nu a_\nu z^\nu$ converges.

Let us check that the convergence on $\Delta_{|\lambda x|}$ is uniform. Fix $\varepsilon > 0$ and a finite set $I_\varepsilon \subset \mathbb{N}^n$ such that for each finite set of indices $I \supset I_\varepsilon$

$$\sum_{\nu\notin I} \lambda^{|\nu|} = \Big|\sum_{\nu\in I} \lambda^{|\nu|} - \Lambda\Big| \le \Big|\sum_{\nu\in I_\varepsilon} \lambda^{|\nu|} - \Lambda\Big| = \sum_{\nu\notin I_\varepsilon} \lambda^{|\nu|} < \frac{\varepsilon}{M}$$

By (3.3) we deduce

$$\Big|\sum_{\nu\in I} a_\nu z^\nu - f(z)\Big| \le \sum_{\nu\notin I} |a_\nu z^\nu| \le M \sum_{\nu\notin I} \lambda^{|\nu|} < \varepsilon,$$

as required.

(ii) We have $\Delta_{|x|} = \bigcup_{0<\lambda<1} \Delta_{|\lambda x|} \subset D(f)$. For the converse inclusion, let $z \in D(f) \setminus \{0\}$. Pick $\eta > 0$ such that $\Delta_\eta(z) \subset D(f)$ and let us choose $x := (x_1, \dots, x_n) \in C^*(f) \cap \Delta_\eta(z)$ given by

$$x_i := \begin{cases} z_i + \frac{\eta}{2} & \text{if } \Re(z_i) \ge 0, \\ z_i - \frac{\eta}{2} & \text{if } \Re(z_i) < 0. \end{cases}$$

This way each coordinate $x_i \neq 0$. We have $z \in \Delta_{|\lambda x|}$ where

$$0 < \lambda := \sup\Big\{\frac{|z_i|}{|x_i|} : i = 1, \dots, n\Big\} < 1, \quad \text{so} \quad D(f) \subset \bigcup_{x\in C^*(f)} \Delta_{|\lambda x|}.$$

(iii) If $K \subset D(f)$ is a compact set, then K can be covered by finitely many $\Delta_{|\lambda_i x_i|}$ where $0 < \lambda_i < 1$ and $x_i \in C^*(f)$. By (i) we know that f converges uniformly on $\Delta_{|\lambda_i x_i|}$, so it converges uniformly on $K \subset \bigcup_i \Delta_{|\lambda_i x_i|}$, as required.

(iv) If $f \in \mathcal{O}_n$, then $D(f) = \bigcup_{x \in C^*(f)} \Delta_{|x|} \neq \varnothing$, so $C^*(f) \neq \varnothing$. Conversely, if $C^*(f) \neq \varnothing$, then $D(f) \neq \varnothing$ and $f \in \mathcal{O}_n$.

(v) The connectedness of the set $\Delta_{|x|}$ for each $x \in C^*(f)$ and the fact that $0 \in \Delta_{|x|}$, guarantees that $D(f) = \bigcup_{x \in C^*(f)} \Delta_{|x|}$ is a connected open neighborhood of the origin. □

Remark I.3.5 (i) Note that if $f := \sum_\nu a_\nu \mathtt{x}^\nu \in \mathcal{O}_n$ and $J \subset \mathbb{N}^n$, then the series $g := \sum_{\nu \in J} a_\nu \mathtt{x}^\nu \in \mathcal{O}_n$ and $D(g) \subset D(f)$. Observe that g is obtained from f making 0 all the coefficients a_ν corresponding to $\nu \notin J$.

(ii) Let $f \in \mathcal{O}_n$ and let $\mu \in \mathbb{N}^n$ be such that $f = \mathtt{x}^\mu g$ for some $g \in \mathcal{F}_n$. Then $g \in \mathcal{O}_n$ and $D(f) \subset D(g)$.

3.b Function associated to a convergent power series.

For every pair $\nu, \mu \in \mathbb{N}^n$ we define the monomial

$$\rho_{\nu\mu}(\mathtt{x}) := \begin{cases} \frac{\nu!}{\mu!(\nu-\mu)!}\mathtt{x}^{\nu-\mu} & \text{if each } \mu_i \leq \nu_i, \\ 0 & \text{otherwise.} \end{cases}$$

Consider new indeterminates $\mathtt{y}_1, \dots, \mathtt{y}_n$. By Newton's binomial the following formula holds

$$(\mathtt{x}+\mathtt{y})^\nu := \prod_{i=1}^n (\mathtt{x}_i + \mathtt{y}_i)^{\nu_i} = \sum_{0 \leq \mu_i \leq \nu_i} \frac{\nu!}{\mu!(\nu-\mu)!}\mathtt{x}^{\nu-\mu}\mathtt{y}^\mu = \sum_\mu \rho_{\nu\mu}(\mathtt{x})\mathtt{y}^\mu. \quad (3.4)$$

Proposition and Definition I.3.6 *Let $f = \sum a_\nu \mathtt{x}^\nu$ be a convergent power series. Then the* associated function

$$\widehat{f} : D(f) \to \mathbb{K},\ x \mapsto f(x)$$

is continuous and for each $x_0 := (x_{01}, \dots, x_{0n}) \in D(f)$ it holds:

(i) *For every $\mu \in \mathbb{N}^n$ the series $\sum_\nu a_\nu \rho_{\nu\mu}(x_0)$ converges to some $b_\mu \in \mathbb{K}$.*

(ii) *The power series $g := \sum_\mu b_\mu \mathtt{y}^\mu$ is convergent and $g(x - x_0) = f(x)$ for x close enough to x_0.*

Proof. To prove that $\widehat{f}$ is continuous it is enough to check that it is locally the uniform limit of a sequence of polynomials. For every integer $p \geq 0$ denote

$f_p := \sum_{|\nu|\leq p} a_\nu \mathtt{x}^\nu$. Fix $y \in D(f)$ and let $0 < \lambda < 1$ and $x \in C^*(f)$ be such that $y \in \Delta_{|\lambda x|}$. By Proposition I.3.4 the sequence of polynomials $\{f_p\}_{p\geq 1}$ converges uniformly to $\widehat{f}|_{\Delta_{|\lambda x|}}$ on $\Delta_{|\lambda x|}$, so $\widehat{f}|_{\Delta_{|\lambda x|}}$ is continuous and consequently $\widehat{f}$ is continuous at y.

We proceed to prove (i) and (ii). If $y := (y_1, \dots, y_n) \in \mathbb{K}^n$ is close enough to the origin, the point $|x_0| + |y| := (|x_{01}| + |y_1|, \dots, |x_{0n}| + |y_n|)$ belongs to the open set $D(f)$.

(i) Let us prove that the series $\sum_{(\nu,\mu)} a_\nu \rho_{\nu\mu}(x_0) y^\nu$ converges.

If $I, J \subset \mathbb{N}^n$ are finite sets we have

$$\sum_{(\nu,\mu)\in I\times J} |a_\nu \rho_{\nu\mu}(x_0) y^\mu| \leq \sum_{\nu\in I} |a_\nu| \sum_\mu \rho_{\nu\mu}(|x_0|)|y^\mu| = \sum_{\nu\in I} |a_\nu|(|x_0| + |y|)^\nu$$

and our assertion follows from the convergence of f at $|x_0|+|y|$ and Proposition I.1.4. By Proposition I.1.7 the iterated series

$$\sum_\mu \Big(\sum_\nu a_\nu \rho_{\nu\mu}(x_0)\Big) y^\mu$$

exists. In particular, each series $\sum_\nu a_\nu \rho_{\nu\mu}(x_0)$ converges to some $b_\mu \in \mathbb{K}$ and statement (i) follows.

(ii) If x is close to x_0, then $y = x - x_0$ is close to the origin. By Proposition I.1.7 and (3.4)

$$\begin{aligned}\sum_\mu b_\mu y^\mu = \sum_{(\nu,\mu)} a_\nu \rho_{\nu\mu}(x_0) y^\mu &= \sum_\nu \sum_\mu a_\nu \rho_{\nu\mu}(x_0) y^\mu \\ &= \sum_\nu a_\nu \Big(\sum_\mu \rho_{\nu\mu}(x_0) y^\mu\Big) = \sum_\nu a_\nu (x_0 + y)^\nu. \quad (3.5)\end{aligned}$$

We deduce by (3.5)

$$g(x - x_0) = g(y) = \sum_\mu b_\mu y^\mu = \sum_\nu a_\nu (x_0 + y)^\nu = f(x_0 + y) = f(x),$$

as required. □

3.c Operations with convergent power series.

If f and g converge at the point $x \in \mathbb{K}^n$, then $f + g$ and fg converge at that point and

$$(f + g)(x) = f(x) + g(x), \quad (fg)(x) = f(x)g(x).$$

In addition, if $f, g \in \mathcal{O}_n$, then

$$\varnothing \neq D(f) \cap D(g) \subset D(f+g) \cap D(fg),$$

so $f+g, fg \in \mathcal{O}_n$.

In this way, $\mathcal{O}_n$ is a commutative subring with unit of the integral domain $\mathcal{F}_n$, so $\mathcal{O}_n$ is also an integral domain. The map $\mathbb{K} \to \mathcal{O}_n,\ a \mapsto f_a := \sum_\nu a_\nu \mathtt{x}^\nu$ where

$$a_\nu := \begin{cases} a & \text{if } \nu = (0, \ldots, 0), \\ 0 & \text{otherwise,} \end{cases}$$

is a monomorphism of rings, so $\mathcal{O}_n$ is a $\mathbb{K}$-subalgebra of $\mathcal{F}_n$.

3.d Substitution. Concerning substitution of convergent power series we have the following result.

Proposition I.3.7 *Let $f \in \mathcal{O}_n$ and $g_1, \ldots, g_n \in \mathcal{O}_m$ be such that $\omega(g_j) \geq 1$. Pick a point $z \in \mathbb{K}^m$ close to the origin. Then*

(i) *The power series $g_1, \ldots, g_n$ and $f(g_1, \ldots, g_n)$ converge at z.*

(ii) *The power series f converges at $(g_1(z), \ldots, g_n(z))$ and*

$$f(g_1, \ldots, g_n)(z) = f(g_1(z), \ldots, g_n(z)).$$

(iii) $f(g_1, \ldots, g_n) \in \mathcal{O}_m$.

Proof. The proof is conducted in several steps:

3.d.1 Write $f := \sum_\nu a_\nu \mathtt{x}^\nu$ and $g_i := \sum_\mu b_{i\mu} \mathtt{z}^\mu$ for $1 \leq i \leq n$, where

$$\mathtt{x} := (\mathtt{x}_1, \ldots, \mathtt{x}_n) \quad \text{and} \quad \mathtt{z} := (\mathtt{z}_1, \ldots, \mathtt{z}_m).$$

Denote $g_i^* := \sum_\mu |b_{i\mu}| \mathtt{z}^\mu$ for $1 \leq i \leq n$. As each g_i converge at $z := (z_1, \ldots, z_m)$ close to the origin, the series $g_1^*, \ldots, g_n^*$ converge at $|z| := (|z_1|, \ldots, |z_m|)$. As $g_i^*(0) = 0$ (because $\omega(g_j) \geq 1$) and each function $\widehat{g_i^*}$ is continuous, the values $y_i := g_i^*(|z|)$ are close to 0 for z close to the origin. As f is convergent, we may assume that $y := (y_1, \ldots, y_n) \in D(f)$. As $|g_i(z)| \leq g_i^*(|z|) = y_i$ for $i = 1, \ldots, n$, we conclude by Proposition I.3.4 that $g(z) := (g_1(z), \ldots, g_n(z)) \in D(f)$.

3.d.2 We prove next that $f(g_1,\dots,g_n)$ converges at z to $f(g(z))$ for $z\in\mathbb{K}^m$ close to zero. For each integer $p\geq 0$ we consider the polynomial

$$f_p := \sum_{|\nu|\leq p} a_\nu \mathtt{x}^\nu$$

and the power series $h_p := f_p(g_1,\dots,g_n)$. We have

$$h_p(z) = f_p(g_1(z),\dots,g_n(z)) = f_p(g(z))$$

for $z\in\mathbb{K}^m$ close to 0. As $f(x)=\lim_{p\to\infty} f_p(x)$ for all $x\in\mathbb{K}^n$ close to zero, we deduce

$$f(g(z)) = \lim_{p\to\infty} f_p(g(z)) = \lim_{p\to\infty} h_p(z) \tag{3.6}$$

for $z\in\mathbb{K}^m$ close to 0.

3.d.3 We claim: *the series $f(g_1,\dots,g_n)$ converges at z to $f(g(z))$.*

Observe that

$$\begin{aligned} f(g_1,\dots,g_n) - h_p &= (f-f_p)(g_1,\dots,g_n) \\ &= \sum_{|\nu|>p} a_\nu g_1^{\nu_1}\cdots g_n^{\nu_n} = \sum_\nu c_{p\nu}\mathtt{z}^\nu \in \mathcal{F}_m. \end{aligned}$$

3.d.4 Let us prove: *$\sum_\nu c_{p\nu}\mathtt{z}^\nu$ converges at z and*

$$|f(g_1,\dots,g_n)(z) - h_p(z)| = \Big|\sum_\nu c_{p\nu} z^\nu\Big| \leq \sum_\nu |c_{p\nu}||z^\nu| \leq \sum_{|\nu|>p} |a_\nu y^\nu|$$

where $y := (g_1^*(|z|),\dots,g_n^*(|z|))$.

Let $I\subset\mathbb{N}^n$ be a finite set and define $q:=\max\{|\nu|:\ \nu\in I\}$. It holds

$$\begin{aligned} \sum_{\nu\in I} |c_{p\nu}||z^\nu| &\leq \sum_{p<|\nu|\leq q} |a_\nu|\Big(\sum_{|\mu|\leq q} |b_{1\mu}||z^\mu|\Big)^{\nu_1}\cdots\Big(\sum_{|\mu|\leq q} |b_{n\mu}||z^\mu|\Big)^{\nu_n} \\ &\leq \sum_{p<|\nu|\leq q} |a_\nu|\Big(\sum_{\mu} |b_{1\mu}||z^\mu|\Big)^{\nu_1}\cdots\Big(\sum_{\mu} |b_{n\mu}||z^\mu|\Big)^{\nu_n} \\ &= \sum_{p<|\nu|\leq q} |a_\nu||y^\nu| \leq \sum_{p<|\nu|} |a_\nu y^\nu|. \end{aligned}$$

By Proposition I.1.3 the series $\sum_\nu |c_{p\nu}\mathtt{z}^\nu|$ converges at z because the series $\sum_\nu |a_\nu y^\nu|$ converges by Proposition I.1.4. Consequently, by Proposition I.1.4 the series $\sum_\nu c_{p\nu}\mathtt{z}^\nu$ converges at z. In addition,

$$|f(g_1,\dots,g_n)(z) - h_p(z)| = \Big|\sum_\nu c_{p\nu} z^\nu\Big| \leq \sum_\nu |c_{p\nu}||z^\nu| \leq \sum_{|\nu|>p} |a_\nu y^\nu|.$$

3.d.5 For $p = 0$, we have $f(g_1, \dots, g_n) = a_0 + \sum_\nu c_{0\nu}\mathtt{z}^\nu$ is convergent at z.

3.d.6 As the series $\sum_\nu |a_\nu y^\nu|$ is convergent, $\lim_{p\to+\infty} \sum_{|\nu|>p} |a_\nu y^\nu| = 0$ and by (3.6) we conclude

$$0 = \lim_{p\to+\infty} (f(g_1, \dots, g_n)(z) - h_p(z)) = f(g_1, \dots, g_n)(z) - f(g(z)),$$

that is, $f(g_1, \dots, g_n)(z) = f(g(z))$ and $f(g_1, \dots, g_n) \in \mathcal{O}_m$, as required. $\square$

Remarks I.3.8 (i) A first consequence of the preceding result is the following: *if f is a convergent power series with $f(0) = a \neq 0$, then the formal power series*

$$\frac{1}{f} = \frac{1}{a}\sum_{\nu_1}\Big(1 - \frac{1}{a}f\Big)^{\nu_1}$$

is convergent, because the series $h(\mathtt{x}_1) := \sum_{\nu_1} \mathtt{x}_1^{\nu_1}$ is convergent (observe for instance that it converges for $x_1 = \frac{1}{2}$).

(ii) Thus, $f \in \mathcal{O}_n$ is a unit if and only if $\omega(f) = 0$.

(iii) Consequently, $\mathcal{O}_n$ is a local ring whose maximal ideal $\mathfrak{m}_n$ consists of the convergent power series with order ≥ 1. As in the formal case, this ideal is generated by the indeterminates: $\mathfrak{m}_n = \{\mathtt{x}_1, \dots, \mathtt{x}_n\}\mathcal{O}_n$.

3.e Derivatives and Taylor expansion.

We are ready to prove that the function associated to convergent power series is an analytic function.

Proposition I.3.9 *Let f be a convergent power series. Then the associated function $\widehat{f} : D(f) \to \mathbb{K}$ is* analytic*, that is, $\widehat{f}$ is smooth and for each point $x_0 \in D(f)$:*

(i) *The series $\frac{\partial^{|\nu|} f}{\partial \mathtt{x}^\nu} \in \mathcal{F}_n$ converges at x_0 to $\frac{\partial^{|\nu|}\widehat{f}}{\partial \mathtt{x}^\nu}(x_0)$ and*

(ii) *The power series*

$$T_{x_0} f := \sum_\nu \frac{1}{\nu!}\frac{\partial^{|\nu|} f}{\partial \mathtt{x}^\nu}(x_0)\mathtt{x}^\nu$$

is convergent and for x close enough to x_0 it holds $f(x) = T_{x_0} f(x - x_0)$.

Proof. As partial derivatives are defined recursively, it is enough to prove (i) for the partial derivatives with indices ν such that $|\nu| = 1$. Without loss of generality we consider the case $\nu = \mathtt{e}_1 := (1, 0, \dots, 0)$, that is, we prove: *the partial derivative* $\frac{\partial \widehat{f}}{\partial \mathtt{x}_1}(x_0)$ *exists and* $\frac{\partial f}{\partial \mathtt{x}_1}$ *converges at* x_0 *to that derivative.*

Write $f = \sum_\nu a_\nu \mathtt{x}^\nu$. Following the notations of Proposition I.3.6 for each $\mu := (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$ we write $b_\mu := \sum_\nu a_\nu \rho_{\nu\mu}(x_0) \in \mathbb{K}$ and consider the series $g := \sum_\mu b_\mu \mathtt{y}^\mu \in \mathcal{O}_n$. We know that $f(x) = g(x - x_0)$ for x close to x_0. Thus, $\frac{\partial \widehat{f}}{\partial \mathtt{x}_1}(x_0) = \frac{\partial \widehat{g}}{\partial \mathtt{x}_1}(0)$.

We have $g = b_{\mathtt{e}_1}\mathtt{y}_1 + \mathtt{y}_1 g_1 + g_2$ where

$$g_1 := \sum_{\mu_1 \geq 1, \mu \neq \mathtt{e}_1} b_\mu \mathtt{y}^{\mu - \mathtt{e}_1} \in \mathcal{O}_n, \text{ and } g_2 := \sum_{\mu_1 = 0} b_\mu \mathtt{y}^\mu \in \mathbb{K}\{\mathtt{y}_2, \dots, \mathtt{y}_n\} \subset \mathcal{O}_n.$$

Note that $\mathtt{y}_1 g_1, g_2 \in \mathcal{O}_n$ because they are parts of the expansion of g and consequently both are convergent. Thus, $g_1 \in \mathcal{O}_n$ and observe in addition that $g_1(0) = 0$. For $t \neq 0$ small enough

$$\widehat{g}(t, 0, \dots, 0) = t(b_{\mathtt{e}_1} + \widehat{g}_1(t, 0, \dots, 0)) + \widehat{g}_2(0) = t(b_{\mathtt{e}_1} + \widehat{g}_1(t, 0, \dots, 0)) + \widehat{g}(0).$$

As $\widehat{g}_1$ is continuous and $g_1(0) = 0$, the derivative

$$\frac{\partial \widehat{g}}{\partial \mathtt{x}_1}(0) = \lim_{t \to 0}\Big(\frac{\widehat{g}(t, 0, \dots, 0) - \widehat{g}(0)}{t}\Big) = \lim_{t \to 0}(b_{\mathtt{e}_1} + \widehat{g}_1(t, 0, \dots, 0)) = b_{\mathtt{e}_1}.$$

In addition, we have

$$\frac{\partial f}{\partial \mathtt{x}_1} = \sum_{\nu_1 \geq 1} a_\nu \nu_1 \mathtt{x}^{\nu - \mathtt{e}_1} = \sum_{\nu_1 \geq 1} a_\nu \frac{\nu!}{(\nu - \mathtt{e}_1)!\mathtt{e}_1!}\mathtt{x}^{\nu - \mathtt{e}_1} = \sum_\nu a_\nu \rho_{\nu \mathtt{e}_1}(\mathtt{x}).$$

By Proposition I.3.6 the series $\frac{\partial f}{\partial \mathtt{x}_1}$ is convergent at x_0 and $\sum a_\nu \rho_{\nu, \mathtt{e}_1}(x_0) = b_{\mathtt{e}_1}$. Thus,

$$\frac{\partial f}{\partial \mathtt{x}_1}(x_0) = b_{\mathtt{e}_1} = \frac{\partial \widehat{g}}{\partial \mathtt{x}_1}(0) = \frac{\partial \widehat{f}}{\partial \mathtt{x}_1}(x_0).$$

We conclude that $D(f) \subset D(\frac{\partial^{|\nu|} f}{\partial \mathtt{x}^\nu})$ for each $\nu \in \mathbb{N}^n$, so $\frac{\partial^{|\nu|} f}{\partial \mathtt{x}^\nu} \in \mathcal{O}_n$. In addition, for each $x_0 \in D(f)$ and $\nu \in \mathbb{N}^n$ we have $\frac{\partial^{|\nu|} \widehat{f}}{\partial \mathtt{x}^\nu}(x_0) = \frac{\partial^{|\nu|} f}{\partial \mathtt{x}^\nu}(x_0)$, so $\widehat{f}$ is smooth and the assertion (i) in the statement follows.

To prove (ii) recall that $g \in \mathcal{O}_n$ by Proposition I.3.6 and $f(x) = g(x - x_0)$ for x close to x_0. Let us check that $T_{x_0} f = g$, that is, we have to prove that

$b_\mu = \frac{1}{\mu!}\frac{\partial^{|\mu|} f}{\partial \mathtt{x}^\mu}(x_0)$ for each $\mu \in \mathbb{N}^n$. Indeed,

$$\begin{aligned}\frac{1}{\mu!}\frac{\partial^{|\mu|} f}{\partial \mathtt{x}^\mu}(x_0) = \frac{1}{\mu!}\sum_\nu a_{\nu+\mu}\frac{(\nu+\mu)!}{\nu!}x_0^\nu = \sum_\nu a_{\nu+\mu}\frac{(\nu+\mu)!}{\mu!\nu!}x_0^\nu \\ = \sum_\alpha a_\alpha \frac{\alpha!}{\mu!(\mu-\alpha)!}x_0^{\alpha-\mu} = \sum_\alpha a_\alpha \rho_{\alpha\mu}(x_0) = b_\mu.\end{aligned}$$

Consequently, $f(x) = g(x - x_0) = T_{x_0}f(x - x_0)$ for x close to x_0, as required. □

The previous result allows us to call *analytic series* the convergent power series of the ring $\mathcal{O}_n$.

3.f Identity Principle. We prove next a crucial result in analytic geometry concerning the "chromosomal" behavior of analytic series.

Proposition I.3.10 (Identity Principle) *Let $f \in \mathcal{O}_n$. The following assertions are equivalent:*

(i) $f = 0$.

(ii) $\widehat{f}$ *vanishes on a non-empty open subset of* $D(f)$.

(iii) f *and all its derivatives of all orders vanish at some point* $x_0 \in D(f)$.

Proof. The chain of implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is straightforward. Let $A := \{x \in D(f) : T_x f = 0\}$. Observe that by Proposition I.3.9(ii) the sets A and $D(f) \setminus A$ are open subsets of $D(f)$. As $D(f)$ is by Proposition I.3.4 connected and $x_0 \in A \neq \varnothing$, we deduce that $D(f) = A$, so $f = T_0 f = 0$, as required. □

The previous result generalizes the fact that *the points in the zero set of an analytic function in one variable are all isolated*, a straightforward exercise that is left to the reader.

Corollary I.3.11 *Let $f \in \mathbb{K}[\mathtt{x}_1, \ldots, \mathtt{x}_n]$ be a non-zero polynomial. Then the open set $\Omega := \{x \in \mathbb{K}^n : f(x) \neq 0\}$ is a dense subset of $\mathbb{K}^n$.*

Proof. Observe that $f \in \mathcal{O}_n$ and $D(f) = \mathbb{K}^n$. By Proposition I.3.10 the function $\widehat{f}$ does not vanish on any non-empty open subset of $D(f)$. Thus, the open set $\Omega = \widehat{f}^{-1}(\mathbb{K} \setminus \{0\})$ is dense in $\mathbb{K}^n$, as claimed. □

4 Rückert's and Weierstrass's Theorems

The purpose of this section is to prove Rückert's Division and Weierstrass's Preparation Theorems for the ring $\mathcal{O}_n$. As an application, we will prove in next section Implicit and Inverse Function Theorem. In order to apply the Fixed Point Theorem for complete metric spaces (X, d) we need some preliminaries on Banach spaces. Recall that a map $T : (X, d) \to (X, d)$ is *contractive* if $d(T(x), T(y)) < cd(x, y)$ for some real constant $0 < c < 1$.

Theorem I.4.1 (Fixed Point Theorem) *Let X be a complete metric space and $T : X \to X$ a contractive map. Then T has a unique fixed point.*

4.a Preliminaries on Banach algebras. For $f := \sum a_\nu \mathtt{x}^\nu \in \mathcal{F}_n$ we denote $f^* := \sum_\nu |a_\nu| \mathtt{x}^\nu$. Let $\rho := (\rho_1, \ldots, \rho_n) \in \mathbb{R}^n$ be such that each $\rho_i > 0$ and define $\mathcal{B}_\rho := \{f \in \mathcal{F}_n : \ \rho \in C(f)\} \subset \mathcal{O}_n$. Consider the function

$$\| \cdot \|_\rho : \mathcal{B}_\rho \to \mathbb{R}, \ f \mapsto f^*(\rho) = \sum_\nu |a_\nu| \rho^\nu.$$

Given $\eta := (\eta_1, \ldots, \eta_n)$, $\rho := (\rho_1, \ldots, \rho_n) \in \mathbb{R}^n$, we say that $\eta < \rho$ if each $\eta_i < \rho_i$. In particular, $\rho > 0$ means that each $\rho_i > 0$.

Lemma I.4.2 *The following statements hold:*

(i) *$\mathcal{B}_\rho$ is a Banach $\mathbb{K}$-algebra for each $\rho > 0$ and $\|fg\|_\rho \leq \|f\|_\rho \cdot \|g\|_\rho$ for each pair $f, g \in \mathcal{B}_\rho$.*

(ii) *Given $\eta, \rho \in \mathbb{R}^n$ such that $0 < \eta < \rho$, it holds $\mathcal{B}_\rho \subset \mathcal{B}_\eta$.*

(iii) *$\mathcal{O}_n = \bigcup_\rho \mathcal{B}_\rho$ and given $f_1, \ldots, f_k \in \mathcal{O}_n$ there exists $\rho := (\rho_1, \ldots, \rho_n) \in \mathbb{R}^n$ such that each $\rho_i > 0$ and $f_1, \ldots, f_k \in \mathcal{B}_\rho$.*

(iv) *If $a_0, \ldots, a_{p-1} \in \mathcal{F}_{n-1} \cap \mathcal{B}_\rho$ and $f := \mathtt{x}_n^p + \sum_{j=0}^{p-1} a_j \mathtt{x}_n^j \in \mathcal{F}_n$, then*

$$\|f\|_\rho = \rho_n^p + \sum_{j=0}^{p-1} \|a_j\|_\rho \rho_n^j.$$

(v) $\mathbb{K}[\mathtt{x}] \subset \mathcal{B}_\rho$ *for* $\rho > 0$.

(vi) *If* $\mathtt{x}_n f \in \mathcal{B}_\rho$ *for some* $f \in \mathcal{O}_n$, *then* $f \in \mathcal{B}_\rho$.

Proof. (i) We have to prove that $(\mathcal{B}_\rho, \|\cdot\|_\rho)$ is a normed linear space, which is complete and, in addition, a ring that contains $\mathbb{K}$.

4.a.1 Pick $f, g \in \mathcal{B}_\rho$. This means that $\rho \in C(f) \cap C(g) \subset C(f+g) \cap C(fg)$, so $f+g, fg \in \mathcal{B}_\rho$. Write $f := \sum_\nu a_\nu \mathtt{x}^\nu$ and $g := \sum_\nu b_\nu \mathtt{x}^\nu$ and observe that

$$\|f+g\|_\rho = \sum_\nu |a_\nu + b_\nu|\rho^\nu \leq \sum_\nu (|a_\nu| + |b_\nu|)\rho^\nu = \|f\|_\rho + \|g\|_\rho.$$

$$\begin{aligned}\|fg\|_\rho = (fg)^*(\rho) &= \sum_\nu \Big| \sum_{\alpha+\beta=\nu} a_\alpha \beta_\beta \Big| \rho^\nu \leq \sum_\nu \Big(\sum_{\alpha+\beta=\nu} |a_\alpha||\beta_\beta| \Big) \rho^\nu \\ &= f^*(\rho) g^*(\rho) = \|f\|_\rho \cdot \|g\|_\rho.\end{aligned}$$

In addition, $\|f\|_\rho = 0$ if and only if $|a_\nu| = 0$ for each $\nu \in \mathbb{N}^n$, that is, if and only if $f = 0$. Observe that $\mathbb{K} \subset \mathcal{B}_\rho$ and for each $\lambda \in \mathbb{K}$

$$\|\lambda f\|_\rho = \sum_\nu |\lambda a_\nu|\rho^\nu = |\lambda| \sum_\nu |a_\nu|\rho^\nu = |\lambda| \cdot \|f\|_\rho.$$

Consequently, $(\mathcal{B}_\rho, \|\cdot\|_\rho)$ is a normed $\mathbb{K}$-algebra.

It only remains to check that $(\mathcal{B}_\rho, \|\cdot\|_\rho)$ is complete.

4.a.2 Let $\{f_k\}_{k \geq 1}$ be a Cauchy sequence of $\mathcal{B}_\rho$ and write $f_k := \sum a_{k\nu} \mathtt{x}^\nu$ for $k \geq 1$. We claim: *every sequence* $\{a_{k\nu}\}_{k\geq 1}$ *is a Cauchy sequence of* $\mathbb{K}$.

Indeed, fix a multi-index $\nu \in \mathbb{N}^n$ and let $\varepsilon > 0$. Let $k_0 \in \mathbb{N}$ be such that if $k, \ell \geq k_0$ then $\|f_k - f_\ell\|_\rho = \sum_\mu |a_{k\mu} - a_{\ell\mu}|\rho^\mu < \varepsilon\rho^\nu$. In particular, $|a_{k\nu} - a_{\ell\nu}|\rho^\nu < \varepsilon\rho^\nu$ and $|a_{k\nu} - a_{\ell\nu}| < \varepsilon$. This means that $\{a_{k\nu}\}_{k\geq 1}$ is a Cauchy sequence of $\mathbb{K}$.

4.a.3 As $\mathbb{K}$ is complete, there exists $\lim_{k\to+\infty} a_{k\nu} =: a_\nu \in \mathbb{K}$. We claim: $f := \sum_\nu a_\nu \mathtt{x}^\nu \in \mathcal{B}_\rho$ and $f = \lim_{k\to\infty} f_k$.

Fix $\varepsilon > 0$ and let us check that for k large enough

$$\|f - f_k\|_\rho = \sum_\nu |a_\nu - a_{k\nu}|\rho^\nu < \varepsilon. \tag{4.7}$$

4.a.4 To that end, we prove first that if $I \subset \mathbb{N}^n$ is finite, then for k large enough

$$\sum_{\nu \in I} |a_\nu - a_{k\nu}|\rho^\nu \leq \frac{\varepsilon}{2}.$$

As $\{f_k\}_{k\geq 1}$ is a Cauchy sequence, for k, ℓ large enough, say $k, \ell \geq k_0$,

$$\sum_{\nu} |a_{k\nu} - a_{\ell\nu}|\rho^\nu < \frac{\varepsilon}{2}.$$

Consequently, for $k, \ell \geq k_0$ we have

$$\sum_{\nu \in I} |a_{k\nu} - a_{\ell\nu}|\rho^\nu < \frac{\varepsilon}{2}$$

and taking the limit in this finite sum as $\ell \to \infty$,

$$\sum_{\nu \in I} |a_{k\nu} - a_\nu|\rho^\nu \leq \frac{\varepsilon}{2}. \tag{4.8}$$

4.a.5 For each $I \subset \mathbb{N}^n$ finite and $k \geq k_0$ inequality (4.8) holds, so

$$\|f_k - f\|_\rho = \sum_{\nu} |a_{k\nu} - a_\nu|\rho^\nu \leq \frac{\varepsilon}{2} < \varepsilon \tag{4.9}$$

and we deduce inequality (4.7). In particular, $f_{k_0} - f \in \mathcal{B}_\rho$, so

$$f = f_{k_0} - (f - f_{k_0}) \in \mathcal{B}_\rho.$$

In addition, (4.7) implies $\lim_{k\to+\infty} f_k = f$.

Statements (ii) and (iii) are straightforward applications of Proposition I.3.4 and (iv), (v) and (vi) are consequences of direct computations left to the reader as an exercise. □

4.b Rückert's Division and Weierstrass's Preparation Theorems.

A formal power series $f \in \mathcal{F}_n = \mathbb{K}[[\mathtt{x}]] = \mathbb{K}[[\mathtt{x}_1, \dots, \mathtt{x}_n]]$ is called *regular of order p with respect to the variable* $\mathtt{x}_n$ if $f(0, \dots, 0, \mathtt{x}_n) = \mathtt{x}_n^p g(\mathtt{x}_n)$ for some $g \in \mathbb{K}[[\mathtt{x}_n]]$ with $g(0) \neq 0$. More generally, we say that f is *regular with respect to the variable* $\mathtt{x}_n$ if it is regular with respect to the variable $\mathtt{x}_n$ of some order. A polynomial $\mathtt{x}_n^p + a_{p-1}\mathtt{x}_{n-1}^{p-1} + \cdots + a_0$ with coefficients $a_0, \dots, a_{p-1} \in \mathcal{F}_{n-1}$ is called *distinguished with respect to* $\mathtt{x}_n$ if it is a regular series of order p with respect to $\mathtt{x}_n$ or, equivalently, if $a_1(0) = \cdots = a_p(0) = 0$.

Remarks I.4.3 (i) If $f \in \mathbb{K}[[\mathtt{x}_1]] \setminus \{0\}$, then f is regular of order $\omega(f)$.

(ii) A series $f \in \mathcal{F}_n$ is regular with respect to $\mathtt{x}_n$ if and only if $f(0, \mathtt{x}_n) \neq 0$.

(iii) If $f, g \in \mathcal{F}_n$ are regular with respect to $\mathtt{x}_n$, then fg is also regular with respect to $\mathtt{x}_n$.

(iv) If $f \in \mathcal{F}_n$ is regular of order p with respect to $\mathtt{x}_n$, there exist $c \in \mathbb{K} \setminus \{0\}$, $a_0, \ldots, a_{p-1} \in \mathcal{F}_{n-1}$ with $a_i(0) = 0$ and $b \in \mathcal{F}_n$ such that $\omega(b) \geq 1$ and

$$f = \mathtt{x}_n^p b + c\mathtt{x}_n^p + \sum_{j=0}^{p-1} a_j \mathtt{x}_n^j.$$

In addition, $\omega(f) \leq p$ and if $f \in \mathcal{O}_n$, then $a_0, \ldots, a_{p-1} \in \mathcal{O}_{n-1}$ and $b \in \mathcal{O}_n$.

The following result will be often useful:

Lemma I.4.4 *Let $f \in \mathcal{F}_n \setminus \{0\}$. Then after a linear change of coordinates, f becomes regular of order $\omega(f)$ with respect to $\mathtt{x}_n$.*

Proof. Denote $f := \sum_\nu a_\nu \mathtt{x}^\nu$ and $p := \omega(f)$. Then $f_p := \sum_{|\nu|=p} a_\nu \mathtt{x}^\nu \neq 0$ is the initial form of f. Denote $\mathtt{x}' := (\mathtt{x}_1, \ldots, \mathtt{x}_{n-1})$ and observe that the difference $\mathbb{K}^{n-1} \setminus \{f_p(\mathtt{x}', 1) = 0\}$ is non-empty. Thus, there exists $c := (c_1, \ldots, c_{n-1}) \in \mathbb{K}^{n-1}$ such that $b := f_p(c_1, \ldots, c_{n-1}, 1) \neq 0$. Consider the linear change of coordinates

$$(\mathtt{y}_1, \ldots, \mathtt{y}_n) \mapsto (\mathtt{x}_1, \ldots, \mathtt{x}_n) := (\mathtt{y}_1, \ldots, \mathtt{y}_n) + (c_1\mathtt{y}_n, \ldots, c_{n-1}\mathtt{y}_n, 0)$$

and define $g(\mathtt{y}) := f(\mathtt{y}_1 + c_1\mathtt{y}_n, \ldots, \mathtt{y}_{n-1} + c_{n-1}\mathtt{y}_n, \mathtt{y}_n)$. We have

$$g(0, \ldots, 0, \mathtt{y}_n) = f(c_1\mathtt{y}_n, \ldots, c_{n-1}\mathtt{y}_n, \mathtt{y}_n) = b\mathtt{y}_n^p + \text{ terms of higher degree},$$

so g is regular of order $p = \omega(f)$ with respect to $\mathtt{x}_n$, as required. □

Theorem I.4.5 (Rückert's Division) *Let $\Phi \in \mathcal{O}_n$ be a convergent power series, regular of order p with respect to $\mathtt{x}_n$. For every $f \in \mathcal{O}_n$ there exist $Q \in \mathcal{O}_n$ and $R \in \mathcal{O}_{n-1}[\mathtt{x}_n]$ with $\deg_{\mathtt{x}_n}(R) < p$ such that $f = Q\Phi + R$. This conditions determine Q and R uniquely. Furthermore, if Φ is a distinguished polynomial in $\mathtt{x}_n$ and $f \in \mathcal{O}_{n-1}[\mathtt{x}_n]$, then $Q \in \mathcal{O}_{n-1}[\mathtt{x}_n]$.*

Proof. If $p = 0$, we take $Q = f\Phi^{-1}$ and $R = 0$, so we may assume $p \geq 1$. As Φ is regular of order p with respect to $\mathtt{x}_n$, we can write

$$\Phi := \varphi + c\mathtt{x}_n^p \quad \text{and} \quad \varphi := \sum_{i=0}^{p-1} a_i(\mathtt{x}')\mathtt{x}_n^i + \mathtt{x}_n^p b(\mathtt{x}), \tag{4.10}$$

where $a_0, \ldots, a_{p-1} \in \mathcal{O}_{n-1} = \mathbb{K}\{\mathtt{x}'\}$, $\mathtt{x}' = (\mathtt{x}_1, \ldots, \mathtt{x}_{n-1})$, $b \in \mathcal{O}_n$ with $\omega(b) \geq 1$ and $c \in \mathbb{K} \setminus \{0\}$. Up to multiplication by $\frac{1}{c}$ we assume $c = 1$.

4.b.1 Suppose the result proved for a while. Then there exist $Q \in \mathcal{O}_n$ and $R \in \mathcal{O}_{n-1}[\mathtt{x}_n]$ with $\deg_{\mathtt{x}_n}(R) < p$ such that

$$\begin{aligned} f = Q\Phi + R = Q \cdot (\Phi - \mathtt{x}_n^p) + \mathtt{x}_n^p Q + R = \varphi Q + \mathtt{x}_n^p Q + R \\ \iff f - \varphi Q = \mathtt{x}_n^p Q + R. \end{aligned}$$

4.b.2 Observe that Q is a fixed point for the map

$$T : \mathcal{O}_n \to \mathcal{O}_n, \ h \mapsto T(h),$$

where $T(h) := \sum_{\nu_n \geq p} b_\nu \mathtt{x}^{\nu - p\mathtt{e}_n}$ is the convergent series that satisfies

$$f - \varphi h = \sum_{\nu} b_\nu \mathtt{x}^\nu = R_h + \mathtt{x}_n^p T(h)$$

and $R_h := \sum_{\nu_n < p} b_\nu \mathtt{x}^\nu \in \mathcal{O}_n$. In addition, by Remark I.3.5

$$D(f) \cap D(\varphi) \cap D(h) \subset D(f - \varphi h) = D(R_h + \mathtt{x}_n^p T(h)) \subset D(R_h) \cap D(T(h)).$$

4.b.3 Consequently, if $f, \varphi, h \in \mathcal{B}_\rho := \{g \in \mathcal{O}_n : \ \|g\|_\rho < +\infty\}$ for some $\rho \in \mathbb{R}^n$ with $\rho > 0$, then $T_\rho := T|_{\mathcal{B}_\rho} : \mathcal{B}_\rho \to \mathcal{B}_\rho$ by Lemma I.4.2. If we prove that T *is a contractive map for a suitable* ρ, it will have by Theorem I.4.1 a unique fixed point $Q \in \mathcal{B}_\rho \subset \mathcal{O}_n$. Observe that also $R_Q := f - \Phi Q \in \mathcal{O}_{n-1}[\mathtt{x}_n]$ with $\deg_{\mathtt{x}_n}(R) < p$ is uniquely determined by Q and the uniqueness of Q, R_Q will follow. Thus, we are reduced to prove that for $\rho > 0$ small enough the map T_ρ *is contractive.*

4.b.4 Let $h, g \in \mathcal{O}_n$ and let $T(h), T(g) \in \mathcal{O}_n$. They satisfy

$$f - \varphi h = R_h + \mathtt{x}_n^p T(h) \quad \text{and} \quad f - \varphi g = R_g + \mathtt{x}_n^p T(g).$$

We have

$$\varphi(g - h) = R_h - R_g + \mathtt{x}_n^p (T(h) - T(g)).$$

Computing norms for a suitable $\rho > 0$ in the previous equality and taking into account that $R_h - R_g \in \mathcal{O}_{n-1}[\mathtt{x}_n]$ and $\deg_{\mathtt{x}_n}(R_h - R_g) < p$, we deduce

$$\begin{aligned}\|\mathtt{x}_n^p(T(h) - T(g))\|_\rho &\leq \|R_h - R_g\|_\rho + \|\mathtt{x}_n^p(T(h) - T(g))\|_\rho \\ &= (R_h - R_g)^*(\rho) + \rho_n^p(T(h) - T(g))^*(\rho) \\ &= (R_h - R_g + \mathtt{x}_n^p(T(h) - T(g)))^*(\rho) \\ &= \|R_h - R_g + \mathtt{x}_n^p(T(h) - T(g))\|_\rho \\ &= \|\varphi(h-g)\|_\rho \leq \|\varphi\|_\rho \|h-g\|_\rho.\end{aligned}$$

As $\|\mathtt{x}_n^p(T(h) - T(g))\|_\rho = \rho_n^p \|T(h) - T(g)\|_\rho$, we have

$$\|T(h) - T(g)\|_\rho \leq \frac{\|\varphi\|_\rho}{\rho_n^p}\|h-g\|_\rho.$$

4.b.5 Thus, it is enough to find $\rho > 0$ such that $\frac{\|\varphi\|_\rho}{\rho_n^p} < \frac{1}{2}$ to have that T_ρ is a contractive map with contractive constant $0 < c \leq \frac{1}{2}$.

Pick $\eta := (\eta_1, \dots, \eta_n) > 0$ such that $f, a_0, \dots, a_{p-1}, b \in \mathcal{B}_\eta$. We have

$$\frac{\|\varphi\|_\eta}{\eta_n^p} \leq \frac{1}{\eta_n^p}\Big(\sum_{i=0}^{p-1}\|a_i\|_\eta \eta_n^i + \eta_n^p\|b\|_\eta\Big) = \|b\|_\eta + \frac{1}{\eta_n^p}\sum_{i=0}^{p-1}\|a_i\|_\eta \eta_n^i.$$

Denote $\eta' := (\eta_1, \dots, \eta_{n-1})$. As $b \in \mathcal{B}_\eta$ and $\omega(b) \geq 1$, we may take $0 < \rho < \eta$ small enough such that $\|b\|_\rho < \frac{1}{4}$. Fix $\rho_n > 0$ such that there exists $0 < \rho' < \eta'$ satisfying $\|b\|_{(\rho', \rho_n)} < \frac{1}{4}$. As each $a_i(0) = 0$, the function

$$0 \leq \alpha := (\alpha_1, \dots, \alpha_{n-1}) \mapsto \frac{1}{\rho_n^p}\sum_{i=0}^{p-1}\|a_i\|_\alpha \rho_n^i$$

is continuous and values 0 for $\alpha = 0$. Pick $0 < \rho' := (\rho_1, \dots, \rho_{n-1}) \leq \eta'$ small enough such that also $\frac{1}{\rho_n^p}\sum_{i=0}^{p-1}\|a_i\|_{\rho'}\rho_n^i < \frac{1}{4}$. Then

$$\frac{\|\varphi\|_\rho}{\rho_n^p} \leq \|b\|_\rho + \frac{1}{\rho_n^p}\sum_{i=0}^{p-1}\|a_i\|_\rho \rho_n^i < \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

as required.

4.b.6 If $\Phi \in \mathcal{O}_{n-1}[\mathtt{x}_n]$ is a distinguished polynomial of degree p with respect to $\mathtt{x}_n$ and $f \in \mathcal{O}_{n-1}[\mathtt{x}_n]$, then applying the division in $\mathcal{O}_{n-1}[\mathtt{x}_n]$ (note that Φ is monic with respect to $\mathtt{x}_n$) there exist $Q, R \in \mathcal{O}_{n-1}[\mathtt{x}_n]$ with $\deg_{\mathtt{x}_n}(R) < p$ such that $f = \Phi Q + R$. By the uniqueness proved above, the previous division is the one provided by Rückert's Division Theorem and we are done. $\square$

Remark I.4.6 The previous result also holds when substituting $\mathcal{O}_n$ by $\mathcal{F}_n$ and $\mathcal{O}_{n-1}$ by $\mathcal{F}_{n-1}$. The proof in the formal case is similar. The Banach space used in such case is $\mathcal{F}_n$ with the norm $\|f\| := \exp^{-v(f)}$, where $v(f)$ stands for the greatest integer $m \geq 0$ such that $f \in \{\mathtt{x}_1, \dots, \mathtt{x}_{n-1}\}^m \mathcal{F}_n$. See Exercises I.17 and I.18.

Theorem I.4.7 (Weierstrass's Preparation) *Let $\Phi \in \mathcal{O}_n$ be a regular series of order p with respect to $\mathtt{x}_n$. Then there exist a distinguished polynomial $P \in \mathcal{O}_{n-1}[\mathtt{x}_n]$ of degree p and a unit $U \in \mathcal{O}_n$ such that $\Phi = PU$. These conditions determine P and U uniquely.*

Proof. If $p = 0$, we take $P = 1$ and $U = \Phi$, so we may assume $p \geq 1$. The proof is conducted in two steps:

4.b.7 Assume the result proved and let $P := \mathtt{x}_n^p + \sum_{i=0}^{p-1} a_i(\mathtt{x}')\mathtt{x}_n^i \in \mathcal{O}_{n-1}[\mathtt{x}_n]$ be a distinguished polynomial of degree p and let $U \in \mathcal{O}_n$ be a unit such that $\Phi = PU$. Observe that $R := \mathtt{x}_n^p - P \in \mathcal{O}_{n-1}[\mathtt{x}_n]$ and has degree $\leq p-1$ with respect to $\mathtt{x}_n$. Write $Q := U^{-1} \in \mathcal{O}_n$ and observe that

$$\Phi = PU \iff \Phi U^{-1} = P = P - \mathtt{x}_n^p + \mathtt{x}_n^p = -R + \mathtt{x}_n^p \iff \mathtt{x}_n^p = \Phi Q + R.$$

4.b.8 By Rückert's Division Theorem I.4.5 there exist series $Q \in \mathcal{O}_n$ and $a_0, \dots, a_{p-1} \in \mathcal{O}_{n-1}$ such that

$$\mathtt{x}_n^p = Q\Phi - \sum_{i=0}^{p-1} a_i \mathtt{x}_n^i.$$

As Φ is regular of order p with respect to $\mathtt{x}_n$, we have $\Phi(0, \mathtt{x}_n) = \mathtt{x}_n^p g(\mathtt{x}_n)$ with $g(0) \neq 0$ and consequently

$$\mathtt{x}_n^p = Q(0, \mathtt{x}_n)\mathtt{x}_n^p g(\mathtt{x}_n) - \sum_{i=1}^{p} a_i(0)\mathtt{x}_n^i.$$

Thus, $a_1(0) = \cdots = a_p(0) = 0$ and $Q(0,0) \neq 0$, so

$$P := \mathtt{x}_n^p + a_{p-1}(\mathtt{x}')\mathtt{x}_n^{p-1} + \cdots + a_0(\mathtt{x}')$$

is the distinguished polynomial we sought and Q is a unit of $\mathcal{O}_n$. Define $U := Q^{-1}$ and observe that $\Phi = PU$. The uniqueness of the division and the previous step imply the uniqueness of P and U, as required. □

Remark I.4.8 The same result holds true when substituting $\mathcal{O}_n$ by $\mathcal{F}_n$ and $\mathcal{O}_{n-1}$ by $\mathcal{F}_{n-1}$. The proof in the formal case is the same, once we have proved the corresponding Division Theorem.

5 Applications: Implicit and Inverse Function Theorems.

We present next the Implicit and Inverse Function Theorems as applications of Rückert's Division Theorem. Afterwards as a by-product we also prove the Rank Theorem.

5.a Implicit Function Theorem. We write $\mathcal{A}_{n+m} := \mathbb{K}\langle\langle \mathtt{x}, \mathtt{y} \rangle\rangle$, where $\mathtt{x} := (\mathtt{x}_1, \ldots, \mathtt{x}_n)$ and $\mathtt{y} := (\mathtt{y}_1, \ldots, \mathtt{y}_m)$, to refer indistinctly to the rings $\mathbb{K}\{\mathtt{x}, \mathtt{y}\}$ or $\mathbb{K}[[\mathtt{x}, \mathtt{y}]]$. Analogously, $\mathcal{A}_n := \mathbb{K}\langle\langle \mathtt{x} \rangle\rangle$ denotes indistinctly the rings $\mathbb{K}\{\mathtt{x}\}$ or $\mathbb{K}[[\mathtt{x}]]$.

Theorem I.5.1 (Implicit Function Theorem) *Let $f_1, \ldots, f_m \in \mathbb{K}\langle\langle \mathtt{x}, \mathtt{y} \rangle\rangle$ be power series such that $f_i(0,0) = 0$ and*

$$\det\Big(\frac{\partial f_i}{\partial \mathtt{y}_j}(0,0)\Big)_{1 \le i,j \le m} \neq 0.$$

Then there exist unique power series $g_1, \ldots, g_m \in \mathbb{K}\langle\langle \mathtt{x} \rangle\rangle$ such that $g_j(0) = 0$ and $f_i(\mathtt{x}, g_1, \ldots, g_m) = 0$ for $i = 1, \ldots, m$.

Proof. The proof is conducted by induction on m.

5.a.1 Assume first $m = 1$. Then the conditions we have are: $f(0,0) = 0$ and $\frac{\partial f}{\partial \mathtt{y}}(0,0) \neq 0$. By 2.c.3

$$f(\mathtt{x}, \mathtt{y}) = \sum_{k \ge 0} \frac{1}{k!} \frac{\partial^k f}{\partial \mathtt{y}^k}(\mathtt{x}, 0)\mathtt{y}^k,$$

and making $\mathtt{x} = 0$ we have

$$f(0,\mathtt{y}) = \sum_{k\geq 0} \frac{1}{k!}\frac{\partial^k f}{\partial \mathtt{y}^k}(0,0)\mathtt{y}^k = \frac{\partial f}{\partial \mathtt{y}}(0,0)\mathtt{y} + \sum_{k\geq 2} \frac{1}{k!}\frac{\partial^k f}{\partial \mathtt{y}^k}(0,0)\mathtt{y}^k = \mathtt{y}F(\mathtt{y}),$$

where $F(0) \neq 0$, so f is regular of order 1 with respect to $\mathtt{y}$. By Weierstrass's Preparation Theorem I.4.7 we can write $f = (\mathtt{y} - g(\mathtt{x}))u(\mathtt{x},\mathtt{y})$ where $g \in \mathbb{K}\langle\langle \mathtt{x}\rangle\rangle$, $g(0) = 0$ (because $\mathtt{y} - g(\mathtt{x})$ is a distinguished polynomial with respect to $\mathtt{y}$) and u is a unit of $\mathbb{K}\langle\langle \mathtt{x},\mathtt{y}\rangle\rangle$. Notice that $g(x)$ is the unique solution of the equation $f(\mathtt{x},\mathtt{y}) = 0$ such that $g(0) = 0$.

5.a.2 Assume the result true for $m-1$ equations and let us check that it is also true for m. As

$$\det\Big(\frac{\partial f_i}{\partial y_j}(0,0)\Big)_{1\leq i,j\leq m} \neq 0,$$

we may assume that $\frac{\partial f_m}{\partial \mathtt{y}_m}(0,0) \neq 0$. As we have seen in the case $m = 1$, the series f_m is regular of order 1 with respect to $\mathtt{y}_m$. By Rückert's Division Theorem I.4.5 we can write

$$f_i(\mathtt{x},\mathtt{y}) = Q_i(\mathtt{x},\mathtt{y})f_m(\mathtt{x},\mathtt{y}',\mathtt{y}_m) + R_i(\mathtt{x},\mathtt{y}') \tag{5.11}$$

where $\mathtt{y}' := (\mathtt{y}_1,\dots,\mathtt{y}_{m-1})$, $Q_i \in \mathbb{K}\langle\langle \mathtt{x},\mathtt{y}\rangle\rangle$ and $R_i \in \mathbb{K}\langle\langle \mathtt{x},\mathtt{y}'\rangle\rangle$ is a polynomial of degree 0 with respect to $\mathtt{y}_m$. As each $f_i(0,0) = 0$, we have $R_i(0,0) = 0$. In addition,

$$\frac{\partial f_i}{\partial \mathtt{y}_j}(0,0) = Q_i(0,0)\frac{\partial f_m}{\partial \mathtt{y}_j}(0,0) + \frac{\partial R_i}{\partial \mathtt{y}_j}(0,0).$$

As $\frac{\partial R_i}{\partial \mathtt{y}_m} = 0$ (because R_i does not depend on $\mathtt{y}_m$), we have

$$\begin{aligned}
\det&\Big(\frac{\partial f_i}{\partial y_j}(0,0)\Big)_{1\leq i,j\leq m}\\
&= \det\left(\begin{array}{c|c} \Big(Q_i(0,0)\frac{\partial f_m}{\partial \mathtt{y}_j}(0,0) + \frac{\partial R_i}{\partial \mathtt{y}_j}(0,0)\Big)_{1\leq i,j\leq m-1} & \Big(Q_i(0,0)\frac{\partial f_m}{\partial \mathtt{y}_m}(0,0)\Big)_{1\leq i\leq m-1} \\ \hline \Big(\frac{\partial f_m}{\partial \mathtt{y}_j}(0,0)\Big)_{1\leq j\leq m-1} & \frac{\partial f_m}{\partial \mathtt{y}_m}(0,0)\end{array}\right)\\
&= \det\left(\begin{array}{c|c} \Big(\frac{\partial R_i}{\partial \mathtt{y}_j}(0,0)\Big)_{1\leq i,j\leq m-1} & (0)_{1\leq i\leq m-1} \\ \hline \Big(\frac{\partial f_m}{\partial \mathtt{y}_j}(0,0)\Big)_{1\leq j\leq m-1} & \frac{\partial f_m}{\partial \mathtt{y}_m}(0,0)\end{array}\right) = \frac{\partial f_m}{\partial \mathtt{y}_m}(0,0)\det\Big(\frac{\partial R_i}{\partial y_j}(0,0)\Big)_{1\leq i,j\leq m-1}.
\end{aligned}$$

As $\det(\frac{\partial f_i}{\partial y_j}(0,0))_{1\leq i,j\leq m} \neq 0$, we have $\det(\frac{\partial R_i}{\partial y_j}(0,0))_{1\leq i,j\leq m-1} \neq 0$. By induction hypothesis there exist unique series $g_1,\dots,g_{m-1} \in \mathbb{K}\langle\langle \mathtt{x}\rangle\rangle$ such that

$g_i(0) = 0$ and $R_i(\mathtt{x}, g'(\mathtt{x})) = 0$ for $i = 1, \dots, m-1$, where $g' := (g_1, \dots, g_{m-1})$. Define $F_m := f_m(\mathtt{x}, g'(\mathtt{x}), \mathtt{y}_m) \in \mathbb{K}\langle\langle \mathtt{x}, \mathtt{y}_m \rangle\rangle$ and observe that

$$F_m(0,0) = f_m(0, g'(0), 0) = 0,$$
$$\frac{\partial F_m}{\partial \mathtt{y}_m}(0,0) = \frac{\partial f_m}{\partial \mathtt{y}_m}(0, g'(0), 0) = \frac{\partial f_m}{\partial \mathtt{y}_m}(0,0) \neq 0.$$

By the case $m = 1$ there exists a unique series $g_m \in \mathbb{K}\langle\langle \mathtt{x} \rangle\rangle$ such that

$$f_m(\mathtt{x}, g'(\mathtt{x}), g_m(\mathtt{x})) = F_m(\mathtt{x}, g_m(\mathtt{x})) = 0$$

and $g_m(0) = 0$. Write $g := (g_1, \dots, g_m)$ and observe that for $i = 1, \dots, m-1$

$$f_i(\mathtt{x}, g(\mathtt{x})) = Q_i(\mathtt{x}, g(\mathtt{x})) f_m(\mathtt{x}, g(\mathtt{x})) + R_i(\mathtt{x}, g'(\mathtt{x})) = 0$$

because $f_m(\mathtt{x}, g(\mathtt{x})) = 0$ and $R_i(\mathtt{x}, g'(\mathtt{x})) = 0$. This proves the existencial part of the statement.

5.a.3 Observe that if $h_1, \dots, h_m \in \mathbb{K}\langle\langle \mathtt{x} \rangle\rangle$ are solutions of

$$f_1(\mathtt{x}, \mathtt{y}) = 0, \dots, f_m(\mathtt{x}, \mathtt{y}) = 0,$$

the series $h_1, \dots, h_{m-1}$ are by (5.11) solutions of

$$R_1(\mathtt{x}, \mathtt{y}') = 0, \dots, R_{m-1}(\mathtt{x}, \mathtt{y}') = 0,$$

so $h_i = g_i$ for $i = 1, \dots, m-1$. Now, $h_m = g_m$ because both are solutions of the equation $F_m(\mathtt{x}, \mathtt{y}_m) = 0$, as required. □

5.b Consequences of the Implicit Function Theorem.

We prove next as a by-product of the Implicit Function Theorem the Inverse Function Theorem and the Rank Theorem.

Corollary I.5.2 (Inverse Function Theorem) *Let $f_1, \dots, f_n \in \mathbb{K}\langle\langle \mathtt{x} \rangle\rangle$ be power series such that $f_i(0) = 0$ and*

$$\det\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\Big)_{1 \le i,j \le n} \neq 0.$$

Then there exist unique power series $g_1, \dots, g_n \in \mathbb{K}\langle\langle \mathtt{y} \rangle\rangle$ such that $g_j(0) = 0$ and $f_i(g_1, \dots, g_n) = \mathtt{y}_i$ for $i = 1, \dots, n$. In addition, $g_i(f_1, \dots, f_n) = \mathtt{x}_i$ for $i = 1, \dots, n$.

Proof. Let $\xi_i := f_i(\mathtt{x})-\mathtt{y}_i \in \mathbb{K}\langle\langle \mathtt{x}, \mathtt{y}\rangle\rangle$ for $i = 1, \ldots, n$. Observe that $\xi_i(0,0) = 0$ and

$$\det\Big(\frac{\partial \xi_i}{\partial \mathtt{x}_j}(0)\Big)_{1\leq i,j\leq n} \neq 0.$$

By the Implicit Function Theorem there exist unique series $g_1, \ldots, g_n \in \mathbb{K}\langle\langle \mathtt{y}\rangle\rangle$ such that $g_j(0) = 0$ and $\xi_i(g_1, \ldots, g_n) = f_i(g_1, \ldots, g_n)-\mathtt{y}_i = 0$ for $i = 1, \ldots, n$. Consequently, $f(g_1, \ldots, g_n) = (\mathtt{y}_1, \ldots, \mathtt{y}_n)$ where $f := (f_1, \ldots, f_n)$. By the chain rule we have

$$\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\Big)_{1\leq i,j\leq n}\Big(\frac{\partial g_j}{\partial \mathtt{y}_i}(0)\Big)_{1\leq i,j\leq n} = I_n.$$

Thus, we deduce

$$\det\Big(\frac{\partial g_j}{\partial \mathtt{y}_i}(0)\Big)_{1\leq i,j\leq n} \neq 0.$$

Let $\zeta_j := g_j(\mathtt{y}) - \mathtt{x}_j \in \mathbb{K}\langle\langle \mathtt{x}, \mathtt{y}\rangle\rangle$ for $j = 1, \ldots, n$. Observe that $\zeta_j(0,0) = 0$ and

$$\det\Big(\frac{\partial \zeta_j}{\partial \mathtt{x}_i}(0)\Big)_{1\leq i,j\leq n} \neq 0.$$

By the Implicit Function Theorem there exist unique series $h_1, \ldots, h_n \in \mathbb{K}\langle\langle \mathtt{x}\rangle\rangle$ such that $h_i(0) = 0$ and

$$\zeta_j(h_1, \ldots, h_n) = g_j(h_1, \ldots, h_n) - \mathtt{x}_j = 0 \quad \text{for} \quad j = 1, \ldots, n.$$

Thus, $g(h_1, \ldots, h_n) = (\mathtt{x}_1, \ldots, \mathtt{x}_n)$ where $g := (g_1, \ldots, g_n)$. We have

$$f(g(h_1, \ldots, h_n)) = (f_1, \ldots, f_n).$$

As $f(g_1, \ldots, g_n) = (\mathtt{y}_1, \ldots, \mathtt{y}_n)$, we deduce that $(h_1, \ldots, h_n) = (f_1, \ldots, f_n)$, so $g(f_1, \ldots, f_n) = (\mathtt{x}_1, \ldots, \mathtt{x}_n)$, as required. □

Denote $\mathbb{K}\langle\langle \mathtt{x}\rangle\rangle := \mathbb{K}\langle\langle \mathtt{x}_1, \ldots, \mathtt{x}_n\rangle\rangle$, $\mathbb{K}\langle\langle \mathtt{y}\rangle\rangle := \mathbb{K}\langle\langle \mathtt{y}_1, \ldots, \mathtt{y}_n\rangle\rangle$ and $\mathbb{K}\langle\langle \mathtt{z}\rangle\rangle := \mathbb{K}\langle\langle \mathtt{z}_1, \ldots, \mathtt{z}_m\rangle\rangle$. We prove next the Rank theorem as a consequence of the Implicit Function Theorem. Let $f_1 \ldots, f_m \in \mathbb{K}\langle\langle \mathtt{x}\rangle\rangle$. The rank of the matrix

$$M := \Big(\frac{\partial f_j}{\partial \mathtt{x}_i}\Big)_{\substack{1\leq j\leq m\\ 1\leq i\leq n}}$$

is the largest $r \geq 0$ such that there exists a non-zero minor of the matrix M of order r.

Corollary I.5.3 (Rank Theorem) *Let $f_1 \dots, f_m \in \mathbb{K}\langle\langle \mathtt{x} \rangle\rangle$ be power series such that $f_i(0) = 0$ and*

$$\operatorname{rk}\Big(\frac{\partial f_j}{\partial \mathtt{x}_i}\Big)_{\substack{1\le j\le m\\ 1\le i\le n}} = \operatorname{rk}\Big(\frac{\partial f_j}{\partial \mathtt{x}_i}(0)\Big)_{\substack{1\le j\le m\\ 1\le i\le n}} = r.$$

Denote $f := (f_1, \dots, f_m)$. Then there exist power series $g_1, \dots, g_n \in \mathbb{K}\langle\langle \mathtt{y} \rangle\rangle$ and $h_1, \dots, h_m \in \mathbb{K}\langle\langle \mathtt{z} \rangle\rangle$ such that

$$\det\Big(\frac{\partial g_i}{\partial \mathtt{y}_k}(0)\Big)_{1\le i,k\le n} \neq 0, \quad \det\Big(\frac{\partial h_j}{\partial \mathtt{z}_\ell}(0)\Big)_{1\le j,\ell\le m} \neq 0$$

and $h(f(g_1, \dots, g_n)) = (\mathtt{y}_1, \dots, \mathtt{y}_r, 0, \dots, 0)$, where $h := (h_1, \dots, h_m)$.

Proof. The proof is conducted in several steps:

5.b.1 After reordering the series $f_1, \dots, f_m$ and the variables $\mathtt{x}_1, \dots, \mathtt{x}_n$ we may assume that

$$\det\Big(\frac{\partial f_j}{\partial \mathtt{x}_i}(0)\Big)_{1\le i,j\le r} \neq 0.$$

5.b.2 *We may assume $f_j = \mathtt{x}_j$ for $j = 1, \dots, r$ and $f_j \in \mathbb{K}\langle\langle \mathtt{x}_1, \dots, \mathtt{x}_r \rangle\rangle$ for $j = r+1, \dots, m$.*

Set $\xi_j := f_j(\mathtt{x}) - \mathtt{y}_j \in \mathbb{K}\langle\langle \mathtt{x}, \mathtt{y} \rangle\rangle$ for $j = 1, \dots, r$. We have $\xi_j(0,0) = 0$ and

$$\det\Big(\frac{\partial \xi_j}{\partial \mathtt{x}_i}(0)\Big)_{1\le i,j\le r} \neq 0.$$

By the Implicit Function Theorem there exist power series

$$g_1, \dots, g_r \in \mathbb{K}\langle\langle \mathtt{y}_1, \dots, \mathtt{y}_r \rangle\rangle$$

such that $g_i(0) = 0$ and

$$\xi_j(g_1, \dots, g_r, \mathtt{x}_{r+1}, \dots, \mathtt{x}_n) = f_j(g_1, \dots, g_r, \mathtt{x}_{r+1}, \dots, \mathtt{x}_n) - \mathtt{y}_j = 0$$

for $j = 1, \dots, r$. Define $g_i := \mathtt{y}_i$ for $i = r+1, \dots, n$. We have

$$f(g_1, \dots, g_n) = (\mathtt{y}_1, \dots, \mathtt{y}_r, a_{r+1}(\mathtt{y}), \dots, a_m(\mathtt{y}))$$

for some $a_i \in \mathbb{K}\langle\langle \mathtt{y} \rangle\rangle$. Write $a_j := \mathtt{y}_i$ for $i = 1, \dots, r$. By the chain rule we have

$$\Big(\frac{\partial a_j}{\partial \mathtt{y}_k}\Big)_{\substack{1\le j\le m\\ 1\le k\le n}} = \Big(\frac{\partial f_j}{\partial \mathtt{x}_i}(g_1, \dots, g_n)\Big)_{\substack{1\le j\le m\\ 1\le i\le n}} \Big(\frac{\partial g_i}{\partial \mathtt{y}_k}\Big)_{1\le i,k\le n}.$$

Consequently,

$$\mathrm{rk}\Big(\frac{\partial a_j}{\partial \mathtt{y}_k}\Big)_{\substack{1\le j\le m\\ 1\le k\le n}} = r.$$

Observe that

$$\Big(\frac{\partial a_j}{\partial \mathtt{y}_k}\Big)_{\substack{1\le j\le r\\ 1\le k\le n}} = (I_r\,|\,0)$$

has rank r, so $\frac{\partial a_j}{\partial \mathtt{y}_k} = 0$ for $j = r+1,\dots,m$ and $k = r+1,\dots,n$. Thus, by 2.c.3 we have $a_j \in \mathbb{K}\langle\langle \mathtt{y}_1,\dots,\mathtt{y}_r\rangle\rangle$ for $j = r+1,\dots,m$. As $g_i(0) = 0$,

$$I_r = \Big(\frac{\partial f_j}{\partial \mathtt{x}_i}(0)\Big)_{1\le i,j\le r}\Big(\frac{\partial g_i}{\partial \mathtt{y}_k}(0)\Big)_{1\le i,k\le r} \quad\rightsquigarrow\quad \det\Big(\frac{\partial g_i}{\partial \mathtt{y}_k}(0)\Big)_{1\le i,k\le r} \neq 0.$$

Thus, as $g_i := \mathtt{y}_i$ for $i = r+1,\dots,n$, we conclude

$$\det\Big(\frac{\partial g_i}{\partial \mathtt{y}_k}(0)\Big)_{1\le i,k\le n} = \det\Big(\frac{\partial g_i}{\partial \mathtt{y}_k}(0)\Big)_{1\le i,k\le r} \neq 0.$$

5.b.3 We assume $f_j := \mathtt{x}_j$ for $j = 1,\dots,r$ and $f_j \in \mathbb{K}\langle\langle \mathtt{x}_1,\dots,\mathtt{x}_r\rangle\rangle$ for each $j = r+1,\dots,m$. Define $\mathtt{t} := (\mathtt{t}_1,\dots,\mathtt{t}_m)$ and

$$F_j(\mathtt{t}) := \begin{cases} \mathtt{t}_j & \text{for } j = 1,\dots,r,\\ \mathtt{t}_j + f_j(\mathtt{t}_1,\dots,\mathtt{t}_r) & \text{for } j = r+1,\dots,m.\end{cases}$$

We have

$$\det\Big(\frac{\partial F_j}{\partial \mathtt{t}_\ell}(0)\Big)_{1\le j,\ell\le m} \neq 0.$$

By the Inverse Function Theorem there exist $h_1,\dots,h_m \in \mathbb{K}\langle\langle \mathtt{z}\rangle\rangle$ such that $F_j(h_1,\dots,h_m) = \mathtt{z}_j$ for $j = 1,\dots,m$ and $h_\ell(F_1,\dots,F_m) = \mathtt{t}_\ell$ for $\ell = 1,\dots,m$. We have

$$I_m = \Big(\frac{\partial F_j}{\partial \mathtt{t}_s}(0)\Big)_{1\le j,s\le m}\Big(\frac{\partial h_s}{\partial \mathtt{z}_\ell}(0)\Big)_{1\le s,\ell\le m} \quad\rightsquigarrow\quad \det\Big(\frac{\partial h_s}{\partial \mathtt{z}_\ell}(0)\Big)_{1\le s,\ell\le m} \neq 0.$$

In addition,

$$\mathtt{z}_j = \begin{cases} F_j(h_1,\dots,h_m) = h_j & \text{for } j = 1,\dots,r,\\ F_j(h_1,\dots,h_m) = h_j + f_j(\mathtt{z}_1,\dots,\mathtt{z}_r) & \text{for } j = r+1,\dots,m.\end{cases}$$

Consequently,

$$h_j = \begin{cases} \mathtt{z}_j & \text{for } j = 1,\dots,r,\\ \mathtt{z}_j - f_j(\mathtt{z}_1,\dots,\mathtt{z}_r) & \text{for } j = r+1,\dots,m.\end{cases}$$

Set $h := (h_1, \ldots, h_m)$ and observe that

$$h(f_1, \ldots, f_m) = (\mathtt{x}_1, \ldots, \mathtt{x}_r, 0, \ldots, 0),$$

as required. $\square$

Exercises

Number I.1 Prove using only the definition of convergence that the series $\sum_{\nu \geq 1} \frac{(-1)^\nu}{\nu}$ is not convergent.

Number I.2 For each $\nu := (\nu_1, \nu_2) \in \mathbb{N}^2$ define $a_\nu := \frac{\sin(\nu_1)+\cos(\nu_2)}{(\nu_1+\nu_2+1)!}$. Determine the convergence of the series $\sum_\nu a_\nu$.

Number I.3 Let $\sum_\nu a_\nu$ be a series such that each $a_\nu \in \mathbb{C}$. Show that $\sum_\nu a_\nu = c \in \mathbb{C}$ if and only if $\sum_\nu \overline{a}_\nu = \overline{c} \in \mathbb{C}$.

Number I.4 Let $\sum_\nu a_\nu$ be a series and let $J \subset \mathbb{N}^n$ be a finite set and define

$$b_\nu := \begin{cases} 0 & \text{if } \nu \in J, \\ a_\nu & \text{if } \nu \in \mathbb{N}^n \setminus J. \end{cases}$$

Denote $\sum_{\nu \in J} a_\nu = a$. Show that the series $\sum_\nu a_\nu$ converges to c if and only if the series $\sum_\nu b_\nu$ converges to $c - a$. We write $\sum_{\nu \notin J} a_\nu = c - a$.

Number I.5 Prove that in the ring of formal power series the following equality holds $(1 - \mathtt{x}_1) \cdots (1 - \mathtt{x}_n) \sum_{|\nu| \geq 0} \mathtt{x}^\nu = 1$.

Number I.6 Let $f := \sum_\nu a_\nu \mathtt{x}^\nu \in \mathcal{F}_n$ and $f^* := \sum_\nu |a_\nu| \mathtt{x}^\nu \in \mathcal{F}_n^*$ be formal power series with complex coefficients. Pick $a := (a_1, \ldots, a_n) \in \mathbb{C}^n$ and denote $|a| := (|a_1|, \ldots, |a_n|) \in \mathbb{R}^n$. Prove that $D(f) = D(f^*)$ and $a \in D(f)$ if and only if $|a| \in D(f^*)$

Number I.7 Let $f, g \in \mathcal{F}_n$. Show that $C(f) \cap C(g) \subset C(f+g) \cap C(fg)$ and that for all $z \in C(f) \cap C(g)$

$$(f+g)(z) = f(z) + g(z) \quad \text{and} \quad (fg)(z) = f(z)g(z).$$

Number I.8 Let $\ell := (\ell_1, \ldots, \ell_n) : \mathbb{K}^m \to \mathbb{K}^n$ be a linear map.

(i) Show that the map $\ell^* : \mathcal{O}_n \to \mathcal{O}_m,\ f \mapsto f(\ell_1, \ldots, \ell_n)$ is a ring homomorphism. Show that for each $f \in \mathcal{O}_m$, the order $\omega(\ell^*(f)) \geq \omega(f)$ and the equality holds for each $f \in \mathcal{O}_n$ if ℓ is surjective.

(ii) Let $k : \mathbb{K}^n \to \mathbb{K}^p$ be another linear map. Check that $(k \circ \ell) = \ell^* \circ k^*$.

Number I.9 Let $\mathfrak{m}_n$ be the maximal ideal of the local ring $\mathcal{A}_n := \mathcal{F}_n$ or $\mathcal{O}_n$. Show that $\mathcal{A}_n/\mathfrak{m}_n \cong \mathbb{K}$.

Number I.10 Let $f, g \in \mathcal{O}_n$ and $a \in D(f) \cap D(g)$. Show that $T_a(fg) = T_a(f) \cdot T_a(g)$.

Number I.11 Consider the function $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(t) := \begin{cases} 0 & \text{if } t \leq 0, \\ e^{-\frac{1}{t}} & \text{if } t > 0. \end{cases}$$

Prove that f is a smooth function on $\mathbb{R}$. Is there a series $f \in \mathcal{O}_1$ such that $\widehat{f} = F$?

Number I.12 Let F be the field of fractions of $\mathbb{R}\{\mathtt{x}\}$. Show that for each $f \in \mathbb{C}\{\mathtt{x}\}$ there exist $g, h \in \mathbb{R}\{\mathtt{x}\}$ such that $f = g + \sqrt{-1}h$ and that g, h are unique with this property. In addition, $\mathbb{R}\{\mathtt{x}\} = F \cap \mathbb{C}\{\mathtt{x}\}$.

Number I.13 Let $\mathfrak{m}_2$ be the maximal ideal of $\mathcal{A}_2 := \mathcal{F}_2$ or $\mathcal{O}_2$. Show that $\mathfrak{m}_2^2$ cannot be generated by only two elements of $\mathfrak{m}_2^2$.

Number I.14 Find two series $f, g \in \mathcal{B}_\rho \subset \mathcal{O}_n$ such that $\|fg\|_\rho < \|f\|_\rho \|g\|_\rho$.

Number I.15 Let $\{f_k\}_{k \geq 1}$ be a sequence in $\mathcal{F}_n$ such that each $f_m \neq 0$. Show that there exist $a_1, \dots, a_{n-1} \in \mathbb{K}$ such that all the series $h_m := f_m(\mathtt{x}_1 + a_1\mathtt{x}_n, \dots, \mathtt{x}_{n-1} + a_{n-1}\mathtt{x}_n, \mathtt{x}_n)$ are regular with respect to $\mathtt{x}_n$ of order $\omega(f_m)$. (*Hint*: Use Baire's Theorem).

Number I.16 Consider the series $f := \sum_{(\nu_1,\nu_2) \neq 0} \mathtt{x}_1^{\nu_1}\mathtt{x}_2^{\nu_2} \in \mathbb{K}\{x_1, \mathtt{x}_2\}$. Show that there exists a series $\varphi \in \mathbb{K}\{\mathtt{x}\}$ such that $f(\mathtt{x}, \varphi(\mathtt{x})) = 0$. Compute $\frac{\partial^{\nu_1}\varphi}{\partial \mathtt{x}^{\nu_1}}(0)$ for each $\nu_1 \geq 0$.

Number I.17 Let $n \geq 2$ be an integer. For each $\nu' := (\nu_1, \dots, \nu_{n-1}) \in \mathbb{N}^{n-1}$ consider the monomial $\mathtt{x}'^{\nu'} := \mathtt{x}_1^{\nu_1} \cdots \mathtt{x}_{n-1}^{\nu_{n-1}}$. For each $r \geq 0$ consider the ideals $\mathfrak{a}_r := \{\mathtt{x}'^{\nu'} : |\nu'| = r\}\mathcal{F}_n$ and $\mathfrak{b}_r := \mathfrak{a}_r \cap \mathcal{F}_{n-1}$. Prove the following:

(i) $f \in \mathfrak{a}_r$ if and only if $\omega(f) \geq r$.

(ii) $\mathfrak{a}_{r+1} \subset \mathfrak{a}_r$ for all $r \geq 0$.

(iii) If $f := \sum_{\nu_n \geq 0} a_{\nu_n} \mathtt{x}_n^{\nu_n}$ where $a_{\nu_n} \in \mathcal{F}_{n-1}$, the following two sets coincide:

$$\{r \in \mathbb{N} : \ f \in \mathfrak{a}_r\} \quad \text{and} \quad \{r \in \mathbb{N} : \ a_j \in \mathfrak{b}_r \ \forall\, j \in \mathbb{N}\}.$$

(iv) For each $f \in \mathcal{F}_n$ denote $v(f) := \sup\{r \in \mathbb{N} : \ f \in \mathfrak{a}_r\}$. Then the map

$$d : \mathcal{F}_n \times \mathcal{F}_n \to \mathbb{R}, \ (f, g) \mapsto e^{-v(f-g)}$$

is a distance and $(\mathcal{F}_n, d)$ is a complete metric space.

(v) Define $|\cdot|: \mathcal{F}_n \to \mathbb{R},\ f \mapsto d(f,0)$. Fix $f, g \in \mathcal{F}_n$. Show that:

(v.1) If $g \in \mathcal{F}_n \setminus \mathfrak{m}_n$, then $|fg| = |f|$.

(v.2) $|\mathtt{x}_n^p f| = |f|$.

(v.3) $|fg| \leq |f||g|$.

(v.4) $|\cdot|$ is not a norm on $\mathcal{F}_n$.

Number I.18 Let $\Phi \in \mathcal{F}_n$ be a formal power series, regular of order p with respect to $\mathtt{x}_n$. Show that for each $f \in \mathcal{F}_n$ there exist $Q \in \mathcal{F}_n$ and $R \in \mathcal{F}_{n-1}[\mathtt{x}_n]$ with $\deg_{\mathtt{x}_n}(R) < p$ such that $f = Q\Phi + R$. Show that these conditions determine Q and R uniquely. Prove in addition that if Φ is distinguished in $\mathtt{x}_n$ and $f \in \mathcal{F}_{n-1}[\mathtt{x}_n]$, then $Q \in \mathcal{F}_{n-1}[\mathtt{x}_n]$.

Number I.19 Let $\Phi \in \mathcal{O}_{n-1}[\mathtt{x}_n]$ be regular of order p with respect to $\mathtt{x}_n$, $P \in \mathcal{O}_{n-1}[\mathtt{x}_n]$ a distinguished polynomial of degree p with respect to $\mathtt{x}_n$ and $U \in \mathcal{O}_n$ a unit such that $\Phi = PU$. Prove that $U \in \mathcal{O}_{n-1}[\mathtt{x}_n]$.

Number I.20 Let $p \in \mathbb{N}$ be a positive integer and let $u \in \mathbb{C}\{\mathtt{t}\}$ be such that $u(0) \neq 0$. Show that there exists $v \in \mathbb{C}\{\mathtt{t}\}$ such that $v^p = u$. Is the result still true if $u(0) = 0$? For which values of $u(0)$ is the result true if we change the field $\mathbb{C}$ by $\mathbb{R}$?

Number I.21 (Hensel's Lemma) Let $Q \in \mathcal{A}_n[\mathtt{z}]$ be a monic polynomial and $a \in \mathbb{K}$ a root of multiplicity p of $Q(0, \mathtt{z}) \in \mathbb{K}[\mathtt{z}]$. Show that there exist monic polynomials $P, U \in \mathcal{A}_n[\mathtt{z}]$ such that $Q = PU$ and

(i) P has degree p and $P(0, \mathtt{z}) = (\mathtt{z} - a)^p$;

(ii) $U(0, a) \neq 0$.

Prove that these conditions determine P and U uniquely.

Noether's normalization and Local parameterization

We devote this chapter to describe the categories of analytic and formal rings over $\mathbb{R}$ and $\mathbb{C}$. We prove first that $\mathcal{O}_n$ and $\mathcal{F}_n$ are noetherian unique factorization domains. Next we prove Noether's Projection Lemma and as a consequence we approach a fundamental construction: local parameterization theorem. We do not prove most of the necessary prerequisites on Algebra and we refer the reader to the classical books [H, L, AM].

1 Preliminaries on analytic and formal rings

Again we set $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We write $\mathcal{A}_n$ to refer indistinctly to either $\mathcal{O}_n$ or $\mathcal{F}_n$. The first part of this section is devoted to prove certain algebraic properties of the ring $\mathcal{A}_n$ that will be useful along the sequel

1.a Basic properties of $\mathcal{A}_n$. We begin by proving that $\mathcal{A}_n$ is a noetherian ring. This result follows from the following lemma, whose proof is straightforward, and Rückert's Division Theorem I.4.5. Let A be a ring. An A-module M is *noetherian* if all its A-submodules are finitely generated. In particular, the ring A is *noetherian* if it is a noetherian A-module, that is, if all its ideals are finitely generated. Recall that if A is a noetherian ring and M is a finitely generated A-module, then M is a noetherian A-module.

Lemma II.1.1 *Let $\ell := (\ell_1, \dots, \ell_n) : \mathbb{K}^n \to \mathbb{K}^n$ be a linear isomorphism whose inverse map is denoted ℓ^{-1}. Then the map $\ell^* : \mathcal{O}_n \to \mathcal{O}_n,\ f \mapsto f(\ell_1, \dots, \ell_n)$ is an isomorphism and its inverse is $(\ell^{-1})^*$.*

Theorem II.1.2 (Notherianity) *The ring $\mathcal{A}_n$ is noetherian.*

Proof. We argue by induction on n. If $n = 0$ the result is trivial because

$\mathcal{A}_0 = \mathbb{K}$, so we assume $n > 0$ and let $\mathfrak{a}$ be a non-zero ideal of $\mathcal{A}_n$. Pick $\Phi \in \mathfrak{a} \setminus \{0\}$. By Lemmas I.4.4 and II.1.1 we may assume that Φ is regular of order p with respect to $\mathtt{x}_n$. By Rückert's Division Theorem the ring $\mathcal{A}_n/\Phi\mathcal{A}_n$ is generated, as an $\mathcal{A}_{n-1}$-module, by the classes of $1, \mathtt{x}_n, \ldots, \mathtt{x}_n^{p-1}$ modulo the ideal $\Phi\mathcal{A}_n$. This is so because each $f \in \mathcal{A}_n$ can be written as

$$f = \Phi Q + \sum_{j=0}^{p-1} a_j \mathtt{x}_n^j$$

where $Q \in \mathcal{A}_n$ and $a_j \in \mathcal{A}_{n-1}$ for $j = 0, \ldots, p-1$. As $\mathcal{A}_{n-1}$ is a noetherian ring by induction hypothesis, the quotient $\mathcal{A}_n/\Phi\mathcal{A}_n$ is a noetherian $\mathcal{A}_{n-1}$-module (because it is a finitely generated module over the noetherian ring $\mathcal{A}_{n-1}$). Thus, $\mathfrak{a}/\Phi\mathcal{A}_n$ is a noetherian $\mathcal{A}_{n-1}$-module, because it is a $\mathcal{A}_{n-1}$-submodule of the finitely generated $\mathcal{A}_{n-1}$-module $\mathcal{A}_n/\Phi\mathcal{A}_n$. Let $f_1, \ldots, f_s \in \mathfrak{a}$ whose classes modulo $\Phi\mathcal{A}_n$ constitute a system of generators of $\mathfrak{a}/\Phi\mathcal{A}_n$ as a $\mathcal{A}_{n-1}$-module. Consequently, $f_1, \ldots, f_s, \Phi$ generate $\mathfrak{a}$, so $\mathfrak{a}$ is finitely generated and $\mathcal{A}_n$ is a noetherian ring. □

Next, as a consequence of Rückert's Division Theorem I.4.5 and Weierstrass's Preparation Theorem I.4.7, we show that the ring $\mathcal{A}_n$ is a unique factorization domain.

Theorem II.1.3 (Factoriality) *The ring $\mathcal{A}_n$ is a unique factorization domain.*

Proof. We have proved in Lemma I.2.3 and Theorem II.1.2 that $\mathcal{A}_n$ is a noetherian integral domain. Thus, to prove that $\mathcal{A}_n$ is a unique factorization domain, it is enough to check that *each irreducible element $\Phi \in \mathcal{A}_n$ is a prime element*, that is, *$\Phi\mathcal{A}_n$ is a prime ideal.*

1.a.1 We proceed by induction on n. If $n = 0$ the assertion is trivial because $\mathcal{A}_0 = \mathbb{K}$, so we assume $n > 0$ and the result proved for less than n indeterminates. After a linear change of coordinates we may assume by Lemmas I.4.4 and II.1.1 that Φ is regular of order p with respect to $\mathtt{x}_n$. By Weierstrass's Preparation Theorem I.4.7 we may write $\Phi = PU$ where $P \in \mathcal{A}_{n-1}[\mathtt{x}_n]$ is a distinguished polynomial with respect to $\mathtt{x}_n$. Observe that $\Phi\mathcal{A}_n = P\mathcal{A}_n$, so we may assume that $\Phi = P$.

1.a.2 By induction hypothesis the ring $\mathcal{A}_{n-1}$ is a unique factorization domain and by Gauss's Lemma, the ring $\mathcal{A}_{n-1}[\mathtt{x}_n]$ is also a unique factorization domain. We claim: *P is irreducible in $\mathcal{A}_{n-1}[\mathtt{x}_n]$.*

Suppose $P = P_1P_2$ for some $P_1, P_2 \in \mathcal{A}_{n-1}[\mathtt{x}_n]$. As P is irreducible in $\mathcal{A}_n$, we may assume that P_1 is a unit in $\mathcal{A}_n$ and $P_2 = PP_1^{-1}$ in $\mathcal{A}_n$. As $P \in \mathcal{A}_{n-1}[\mathtt{x}_n]$ is a distinguished polynomial and $P_2 \in \mathcal{A}_{n-1}[\mathtt{x}_n]$ a polynomial, the quotient P_1^{-1} of the division $P_2 = PP_1^{-1} + 0$ is by Rückert's Division Theorem I.4.5 an element of $\mathcal{A}_{n-1}[\mathtt{x}_n]$. As $P_1, P_1^{-1} \in \mathcal{A}_{n-1}[\mathtt{x}_n]$, we deduce that $P_1 \in \mathcal{A}_{n-1}$ is a unit, so P is irreducible in $\mathcal{A}_{n-1}[\mathtt{x}_n]$. Thus, $P\mathcal{A}_{n-1}[\mathtt{x}_n]$ is a prime ideal.

1.a.3 Let us check now: *$P\mathcal{A}_n$ is a prime ideal.*

Suppose that P divides the product fg of two power series $f, g \in \mathcal{A}_n$, that is, $fg = Ph$ for some $h \in \mathcal{A}_n$. By Rückert's Division Theorem we can write

$$f = QP + R \quad \text{and} \quad g = Q'P + R',$$

where $Q, Q' \in \mathcal{A}_n$ and $R, R' \in \mathcal{A}_{n-1}[\mathtt{x}_n]$ with $\deg(R), \deg(R') < p$. Hence,

$$\begin{aligned} RR' = (f - QP)(g - Q'P) &= fg - QgP - Q'fP + QQ'P^2 \\ &= (h - Qg - Q'f + QQ'P)P = Q''P + 0. \end{aligned}$$

By Rückert's Division Theorem we deduce that $Q'' \in \mathcal{A}_{n-1}[\mathtt{x}_n]$. As $P\mathcal{A}_{n-1}[\mathtt{x}_n]$ is a prime ideal, we may assume that P divides $R = f - QP$ in $\mathcal{A}_{n-1}[\mathtt{x}_n] \subset \mathcal{A}_n$, so P divides f in $\mathcal{A}_n$. Consequently, $f \in P\mathcal{A}_n$ and $P\mathcal{A}_n$ is a prime ideal, as required. □

Lemma II.1.4 (Factorization of distinguished polynomials) Each distinguished polynomial $P \in \mathcal{A}_{n-1}[\mathtt{x}_n]$ has a unique factorization of the type $P = P_1^{\alpha_1} \cdots P_s^{\alpha_s}$, where each $\alpha_i \geq 1$ and each P_i is a distinguished polynomial and it is irreducible both in $\mathcal{A}_{n-1}[\mathtt{x}_n]$ and $\mathcal{A}_n$.

Proof. As P is a distinguished polynomial of degree p, we have $P(0, \mathtt{x}_n) = \mathtt{x}_n^p$. As $\mathcal{A}_{n-1}[\mathtt{x}_n]$ is a unique factorization domain, there exist irreducible polynomials $P_i \in \mathcal{A}_{n-1}[\mathtt{x}_n]$ of degree p_i and integers $\alpha_i \geq 1$ such that $P = P_1^{\alpha_1} \cdots P_s^{\alpha_s}$. As P is monic, we may assume that each polynomial P_i is also monic. Observe that $p = \deg_{\mathtt{x}_n}(P) = \sum_{i=1}^s \alpha_i p_i$. Note that

$$\mathtt{x}_n^p = P(0, \mathtt{x}_n) = P_1(0, \mathtt{x}_n)^{\alpha_1} \cdots P_s(0, \mathtt{x}_n)^{\alpha_s}.$$

As each P_i is monic, we deduce that $P_i(0, \mathtt{x}_n) = \mathtt{x}_n^{p_i}$, so P_i is distinguished. By 1.a.3 we deduce that each distinguished polynomial P_i is irreducible in $\mathcal{A}_n$, as required. □

Recall that an integral domain A is *integrally closed* if each element q in its field of fractions $\mathrm{qf}(A)$ that is a root of a monic polynomial with coefficients in A belongs to A.

Lemma II.1.5 *Let A be a unique factorization domain. Then A is integrally closed.*

Proof. Let $q := \frac{a}{b} \in \mathrm{qf}(A)$ be such that $\gcd(a,b) = 1$. Suppose that q is a root of the polynomial $\mathtt{t}^n + a_{n-1}\mathtt{t}^{n-1} + \cdots + a_0 \in A[\mathtt{t}]$. Then

$$\Big(\frac{a}{b}\Big)^n + a_{n-1}\Big(\frac{a}{b}\Big)^{n-1} + \cdots + a_0 = 0.$$

Multiplying the previous equality by b^n, we have

$$a^n = -b(a_{n-1}a^{n-1} + \cdots + a_0 b^{n-1}).$$

As $\gcd(a,b) = 1$ and $b|a^n$, we deduce that b is a unit in A, so $q = ab^{-1} \in A$, as required. □

Corollary II.1.6 *The domain $\mathcal{A}_n$ is integrally closed.*

1.b Analytic and formal rings. An *analytic* (resp. a *formal*) *ring* (over the coefficient field $\mathbb{K}$) is a ring isomorphic to $\mathcal{O}_n/\mathfrak{a}$ (resp. $\mathcal{F}_n/\mathfrak{a}$) where $\mathfrak{a}$ is an ideal of $\mathcal{A}_n$. We will refer to both type of rings indistinctly as $\mathcal{A}_n/\mathfrak{a}$. If A, B are two analytic (resp. formal) rings, an *analytic* (resp. a *formal*) *homomorphism* $A \to B$ is a homomorphism of $\mathbb{K}$-algebras.

Given a ring A and a prime ideal $\mathfrak{p}$ of A, we denote $\kappa(\mathfrak{p})$ the quotient field of the ring $A/\mathfrak{p}$. If A is local, $\mathfrak{m}_A$ is its maximal ideal. We recall here Krull's Theorem. We provide an elementary proof of this result in Appendix B.

Theorem II.1.7 (Krull) *Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$ and let $\mathfrak{a}$ be an ideal of A. Then $\bigcap_{k\geq 1}(\mathfrak{a} + \mathfrak{m}^k) = \mathfrak{a}$.*

Proposition II.1.8 *Let $\mathfrak{a}$ be a proper ideal of $\mathcal{A}_n$ and denote $A := \mathcal{A}_n/\mathfrak{a}$. The canonical homomorphism $\mathbb{K} \to A/\mathfrak{m}_A$ is an isomorphism and $A = \mathbb{K} \oplus \mathfrak{m}_A$.*

Proof. As $A := \mathcal{A}_n/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of $\mathcal{A}_n$ and $\mathcal{A}_n$ is by Theorem II.1.2 a noetherian ring, we deduce that A is also a noetherian ring. As $\mathcal{A}_n$ is a local ring, the same happens with A. Let $\mathfrak{m}_n$ be the maximal ideal of $\mathcal{A}_n$. By the correspondence theorem for ideals $\mathfrak{m}_A = \mathfrak{m}_n/\mathfrak{a}$. By the third isomorphism theorem $A/\mathfrak{m}_A \cong \mathcal{A}_n/\mathfrak{m}_n$. Consider the homomorphism

$$\varphi : \mathcal{A}_n \to \mathbb{K},\ f \mapsto f(0)$$

whose kernel is $\ker(\varphi) = \{f \in \mathcal{A}_n :\ \omega(f) \geq 1\} = \mathfrak{m}_n$. Thus, $\mathcal{A}_n/\mathfrak{m}_n \cong \mathbb{K}$ via the quotient map $\overline{\varphi}$. Observe that $(\overline{\varphi})^{-1}$ is the homomorphism $\pi \circ \mathtt{j}$ where $\pi : \mathcal{A}_n \to \mathcal{A}_n/\mathfrak{a}$ is the canonical projection and $\mathtt{j} : \mathbb{K} \hookrightarrow \mathcal{A}_n$ is the canonical inclusion.

Finally, $f = f(0)+(f-f(0))$ for $f \in A$, where $f(0) \in \mathbb{K}$ and $f-f(0) \in \mathfrak{m}_A$. In addition, $\mathbb{K} \cap \mathfrak{m}_A = \{0\}$, so $A = \mathbb{K} \oplus \mathfrak{m}_A$. □

Theorem II.1.9 (Substitution) *Let $\mathfrak{a} \subset \mathcal{A}_n$ and $\mathfrak{b} \subset \mathcal{A}_m$ be ideals and the quotients $A := \mathcal{A}_n/\mathfrak{a}$ and $B := \mathcal{A}_m/\mathfrak{b}$. Every homomorphism $\varphi : A \to B$ is local, that is, $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$. In addition,*

(i) *φ is completely determined by the images $\varphi(\mathtt{x}_i + \mathfrak{a})$ for $i = 1, \ldots, n$.*

(ii) *If $\mathfrak{a} = (0)$ and $b_1, \ldots, b_n \in \mathfrak{m}_B$, there exists a unique homomorphism $\varphi : \mathcal{A}_n \to B$ such that $\varphi(\mathtt{x}_i) = b_i$ for $i = 1, \ldots, n$.*

Proof. If $\varphi(\mathfrak{m}_A) \not\subset \mathfrak{m}_B$, there exists $f \in \mathfrak{m}_A$ such that $\varphi(f) = a + g$ where $a \in \mathbb{K} \setminus \{0\}$ and $g \in \mathfrak{m}_B$. Then $\varphi(f-a) = g$ is not a unit in B, while $f - a$ is a unit in A, which is a contradiction.

(i) We have to show that: *if $\phi, \varphi : A \to B$ are two homomorphisms such that $\phi(\mathtt{x}_i + \mathfrak{a}) = \varphi(\mathtt{x}_i + \mathfrak{a})$ for $1 \leq i \leq n$, then $\phi = \varphi$.*

Note that each $f \in \mathcal{A}_n$ can be written for each $k \geq 0$ as

$$f = g + h \quad \text{where } g \in \mathbb{K}[\mathtt{x}] := \mathbb{K}[\mathtt{x}_1, \ldots, \mathtt{x}_n] \text{ and } \omega(h) \geq k.$$

In particular, $h + \mathfrak{a} \in \mathfrak{m}_A^k$. We claim:

$$\phi(f+\mathfrak{a}) - \varphi(f+\mathfrak{a}) = \phi(h+\mathfrak{a}) - \varphi(h+\mathfrak{a}) \in \mathfrak{m}_B^k.$$

Indeed, as ϕ, φ are $\mathbb{K}$-algebra homomorphisms and $\phi(\mathtt{x}_i + \mathfrak{a}) = \varphi(\mathtt{x}_i + \mathfrak{a})$ for $1 \leq i \leq n$, we deduce that $\phi(g+\mathfrak{a}) = \varphi(g+\mathfrak{a})$ for each $g \in \mathbb{K}[\mathtt{x}]$. As φ is local, $\varphi(\mathfrak{m}_A^k) \subset \mathfrak{m}_B^k$ and $\phi(\mathfrak{m}_A^k) \subset \mathfrak{m}_B^k$, so the claim follows.

As $\phi(f+\mathfrak{a})-\varphi(f+\mathfrak{a}) \in \mathfrak{m}_B^k$ for each $k \geq 1$, we conclude by Krull's Theorem II.1.7 that

$$\phi(f+\mathfrak{a})-\varphi(f+\mathfrak{a}) \in \bigcap_{k\geq 1} \mathfrak{m}_B^k = (0),$$

so $\phi = \varphi$.

(ii) Suppose now that $\mathfrak{a} := (0)$ and let $b_1, \dots, b_n \in \mathfrak{m}_B$. Denote the canonical projection $\pi : \mathcal{A}_m \to B := \mathcal{A}_m/\mathfrak{b}$ and let $g_1, \dots, g_n \in \mathfrak{m}_m$ be such that $\pi(g_i) = b_i$ for $1 \leq i \leq n$. Consider the $\mathbb{K}$-algebra homomorphism $\varphi : \mathcal{A}_n \to B$ given by:

$$f \mapsto f(g_1, \dots, g_n) \mapsto \varphi(f) := \pi(f(g_1, \dots, g_n)) = f(g_1, \dots, g_n) + \mathfrak{b}.$$

We have $\varphi(\mathtt{x}_i) = b_i$ for $i = 1, \dots, n$, as required. □

Corollary II.1.10 *Let $\varphi : \mathcal{A}_n \to \mathcal{A}_m$ be a $\mathbb{K}$-algebra homomorphism and write $f_i := \varphi(\mathtt{x}_i)$ for $i = 1, \dots, n$. We have*

(i) *φ is an isomorphism if and only if $n = m$ and*

$$\det\Big(\frac{\partial f_i}{\partial \mathtt{y}_j}(0)\Big)_{1\leq i,j\leq n} \neq 0.$$

(ii) *φ is surjective if and only if $n \geq m$ and*

$$\operatorname{rk}\Big(\frac{\partial f_i}{\partial \mathtt{y}_j}(0)\Big)_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} = m.$$

Proof. (i) The if part is a straightforward consequence of the Inverse Function Theorem I.5.2 and Substitution Theorem II.1.9. If φ is an isomorphism, there exists $\psi : \mathcal{A}_m \to \mathcal{A}_n$ such that $\psi \circ \varphi = \mathrm{id}_{\mathcal{A}_n}$ and $\varphi \circ \psi = \mathrm{id}_{\mathcal{A}_m}$. Let $g_j := \psi(\mathtt{y}_j)$ for $j = 1, \dots, m$ and write $f := (f_1, \dots, f_n)$ and $g := (g_1, \dots, g_m)$, we have

$$f(g_1, \dots, g_m) = (\mathtt{y}_1, \dots, \mathtt{y}_n) \quad \text{and} \quad g(f_1, \dots, f_n) = (\mathtt{x}_1, \dots, \mathtt{x}_m).$$

Recall that φ and ψ are local homomorphisms, so $f_i(0) = 0$ and $g_j(0) = 0$. By the chain rule

$$\Big(\frac{\partial f_i}{\partial \mathtt{y}_j}(0)\Big)_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} \cdot \Big(\frac{\partial g_j}{\partial \mathtt{x}_i}(0)\Big)_{\substack{1\leq j\leq m\\ 1\leq i\leq n}} = I_n,$$

$$\Big(\frac{\partial g_j}{\partial \mathtt{x}_i}(0)\Big)_{\substack{1\leq j\leq m\\ 1\leq i\leq n}} \cdot \Big(\frac{\partial f_i}{\partial \mathtt{y}_j}(0)\Big)_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} = I_m.$$

Consequently, $n = m$ and

$$\det\Big(\frac{\partial f_i}{\partial \mathtt{y}_j}(0)\Big)_{1\leq i,j\leq n} \neq 0,$$

as required.

(ii) If φ is surjective there exist, for each $j = 1, \dots, m$, a series $h_j \in \mathcal{A}_n$ such that $\varphi(h_j) = \mathtt{y}_j$. Write $h := (h_1, \dots, h_m)$ and observe that

$$h(f_1, \dots, f_n) = (\mathtt{y}_1, \dots, \mathtt{y}_m).$$

Recall that φ is a local homomorphism, so $f_i(0) = 0$ and $h_j(0) = 0$. By the chain rule

$$\Big(\frac{\partial h_j}{\partial \mathtt{x}_i}(0)\Big)_{\substack{1\leq j\leq m\\ 1\leq i\leq n}} \cdot \Big(\frac{\partial f_i}{\partial \mathtt{y}_j}(0)\Big)_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} = I_m.$$

Consequently, $n \geq m$ and

$$\mathrm{rk}\Big(\frac{\partial f_i}{\partial \mathtt{y}_j}(0)\Big)_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} = m.$$

Conversely, we may assume that

$$\det\Big(\frac{\partial f_i}{\partial \mathtt{y}_j}(0)\Big)_{1\leq i,j\leq m} \neq 0.$$

Let $\mathtt{j} : \mathcal{A}_m \hookrightarrow \mathcal{A}_n$ be the natural inclusion such that $\mathtt{y}_j \mapsto \mathtt{x}_j$ for $j = 1, \dots, m$ and let $\psi := \varphi \circ \mathtt{j}$. Observe that $\psi(\mathtt{y}_j) = f_j$ for $j = 1, \dots, m$. As

$$\det\Big(\frac{\partial f_i}{\partial \mathtt{y}_j}(0)\Big)_{1\leq i,j\leq m} \neq 0,$$

we deduce by (i) that ψ is an isomorphism, so φ is surjective, as required. $\square$

Example II.1.11 Consider the homomorphism $\varphi : \mathcal{A}_1 \to \mathcal{A}_1,\ f(\mathtt{t}) \mapsto f(\mathtt{t}^2)$. Observe that φ is injective but $\frac{\partial \mathtt{t}^2}{\partial \mathtt{t}}(0) = 0$.

Corollary II.1.12 *Let $f_1, \dots, f_n \in \mathcal{A}_n$ be such that $f_i(0) = 0$. The following assertions are equivalent:*

(i) $\mathfrak{m}_n = \{f_1, \dots, f_n\}\mathcal{A}_n$.

(ii) $\det \left(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\right)_{1\leq i,j\leq n} \neq 0.$

Proof. (ii) $\Longrightarrow$ (i) The homomorphism $\varphi : \mathcal{A}_n \to \mathcal{A}_n,\ g \mapsto g(f_1,\dots,f_n)$ is by Corollary II.1.10 an isomorphism. Observe that

$$\mathfrak{m}_n = \varphi(\mathfrak{m}_n) = \{\varphi(\mathtt{x}_1),\dots,\varphi(\mathtt{x}_n)\}\mathcal{A}_n = \{f_1,\dots,f_n\}\mathcal{A}_n.$$

(i) $\Longrightarrow$ (ii) As $\mathfrak{m}_n = \{f_1,\dots,f_n\}\mathcal{A}_n$, there exist $g_{ij} \in \mathfrak{m}_n$ such that

$$\mathtt{x}_k = \sum_{i=1}^n g_{ki} f_i. \tag{1.1}$$

As $f_i(0) = 0$, the initial form of f_i is $\sum_{j=1}^n \frac{\partial f_i}{\partial \mathtt{x}_j}(0)\mathtt{x}_j$. Write $\lambda_{ki} := g_{ki}(0)$. Equaling the initial forms in equation (1.1) we have

$$\mathtt{x}_k = \sum_{i=1}^n \lambda_{ki} \sum_{j=1}^n \frac{\partial f_i}{\partial \mathtt{x}_j}(0)\mathtt{x}_j.$$

We rewrite the previous information using matrices and get

$$I_n \begin{pmatrix} \mathtt{x}_1 \\ \vdots \\ \mathtt{x}_n \end{pmatrix} = \big(\lambda_{ki}\big)_{1\leq k,i\leq n} \cdot \Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\Big)_{1\leq i,j\leq n} \begin{pmatrix} \mathtt{x}_1 \\ \vdots \\ \mathtt{x}_n \end{pmatrix}.$$

Thus, we conclude

$$I_n = \big(\lambda_{ki}\big)_{1\leq k,i\leq n} \cdot \Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\Big)_{1\leq i,j\leq n},$$

so $\det\big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\big)_{1\leq i,j\leq n} \neq 0$, as required. □

2 Noether's normalization lemma

We begin this section with a summary of well-known results in Commutative Algebra.

2.a Preliminaries on heights and dimensions. The *height* $\mathrm{ht}(\mathfrak{p})$ of a prime ideal $\mathfrak{p}$ of a ring A is the maximal length of a chain of distinct prime ideals contained in $\mathfrak{p}$. If A is noetherian this height is always finite. The *Krull's dimension* $\dim(A)$ of A is the supremum of the heights of the prime ideals of A. For an arbitrary ideal $\mathfrak{a}$ the height $\mathrm{ht}(\mathfrak{a})$ is the minimum of the heights of all the prime ideals of A that contain $\mathfrak{a}$. In particular, as $\sqrt{\mathfrak{a}}$ is the intersection of all the prime ideals that contain $\mathfrak{a}$, we have $\mathrm{ht}(\mathfrak{a}) = \mathrm{ht}(\sqrt{\mathfrak{a}})$. If A is a local ring, the dimension of A coincides with the height of its unique maximal ideal.

Recall that if A is noetherian, then the radical ideal $\sqrt{\mathfrak{a}}$ is a finite intersection of prime ideals: $\sqrt{\mathfrak{a}} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$. We have

$$\mathrm{ht}(\mathfrak{a}) = \mathrm{ht}(\sqrt{\mathfrak{a}}) = \min\{\mathrm{ht}(\mathfrak{p}_1), \dots, \mathrm{ht}(\mathfrak{p}_r)\}.$$

Note also that if $\mathfrak{p} \subset \mathfrak{a}$ and $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{a})$, then $\mathfrak{p} = \mathfrak{a}$. If A is an integral domain, then (0) is a prime ideal of A and a prime ideal $\mathfrak{p}$ has height 0 if and only if $\mathfrak{p} = (0)$. In general, if $\mathfrak{a}$ is an ideal of A, we have as a straightforward consequence of the correspondence theorem for ideals,

$$\dim(A/\mathfrak{a}) + \mathrm{ht}(\mathfrak{a}) \leq \dim(A). \tag{2.2}$$

We will see that if $A = \mathcal{A}_n$, then the equality always holds.

Remark II.2.1 Let A be a unique factorization domain and $f \in A$. Let $f := f_1^{\alpha_1} \cdots f_r^{\alpha_r}$ be the factorization of f as a product of powers of distinct irreducible factors. Write $\mathfrak{a} := fA$ and $\mathfrak{p}_i := f_iA$. Then $\sqrt{\mathfrak{a}} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ and $\mathrm{ht}(\mathfrak{p}_i) = 1$ for $i = 1, \dots, r$.

Lemma II.2.2 *Let A be a ring and let $\mathfrak{a}_1, \dots, \mathfrak{a}_r, \mathfrak{q}$ be ideals of A such that $\mathfrak{q}$ is prime and $\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_r \subset \mathfrak{q}$. Then $\mathfrak{a}_i \subset \mathfrak{q}$ for some $i = 1, \dots, r$*

Proof. Suppose that $\mathfrak{a}_i \not\subset \mathfrak{q}$ for $i = 1, \dots, r$. Then there exists $x_i \in \mathfrak{a}_i \setminus \mathfrak{q}$ for $i = 1, \dots, r$. Then $x_1 \cdots x_r \in \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_r \subset \mathfrak{q}$, so $x_j \in \mathfrak{q}$ for some $j = 1, \dots, r$, which is a contradiction. Consequently, there exists an index $i = 1, \dots, r$ such that $\mathfrak{a}_i \subset \mathfrak{q}$, as required. □

Lemma II.2.3 *Let A be a ring and let $\mathfrak{a} \subset \mathfrak{b}$ be two ideals. Suppose that $\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ and $\mathrm{ht}(\mathfrak{a}) = \mathrm{ht}(\mathfrak{b})$. Then $\mathfrak{b} \subset \mathfrak{p}_i$ for some $i = 1, \dots, r$.*

Proof. Let $\mathfrak{q}$ be a prime ideal of A such that $\mathfrak{b} \subset \mathfrak{q}$ and $\mathrm{ht}(\mathfrak{b}) = \mathrm{ht}(\mathfrak{q})$. Observe that

$$\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r = \mathfrak{a} \subset \mathfrak{b} \subset \mathfrak{q}.$$

By Lemma II.2.2 there exists an index $i = 1, \ldots, r$ such that $\mathfrak{p}_i \subset \mathfrak{q}$. Thus,

$$\mathrm{ht}(\mathfrak{a}) \leq \mathrm{ht}(\mathfrak{p}_i) \leq \mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{b}) = \mathrm{ht}(\mathfrak{a}),$$

so $\mathrm{ht}(\mathfrak{p}_i) = \mathrm{ht}(\mathfrak{q})$ and $\mathfrak{p}_i \subset \mathfrak{q}$. We conclude $\mathfrak{p}_i = \mathfrak{q}$ that contains $\mathfrak{b}$, as required. $\square$

Definition II.2.4 Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$. For each ideal $\mathfrak{a}$ of A with $\sqrt{\mathfrak{a}} = \mathfrak{m}$ we define $\delta(\mathfrak{a})$ as the minimum of the cardinals of the finite systems of generators of $\mathfrak{a}$. The *Chevalley's dimension* of A is $\delta(A) := \min\{\delta(\mathfrak{a}) : \sqrt{\mathfrak{a}} = \mathfrak{m}\}$.

Theorem II.2.5 *Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$. Then $\dim(A) = \delta(A) \leq \delta(\mathfrak{m})$.*

The previous result suggests the following definition.

Definition II.2.6 A local noetherian ring A with maximal ideal $\mathfrak{m}$ is a *regular local ring* if $\dim(A) = \delta(\mathfrak{m})$.

An important example of regular local rings is $\mathcal{A}_n$.

Theorem II.2.7 *The ring $\mathcal{A}_n$ is a regular local ring of dimension n.*

Proof. We have already proved that $\mathcal{A}_n$ is a local noetherian ring with maximal ideal $\mathfrak{m}_n = \{\mathtt{x}_1, \ldots, \mathtt{x}_n\}\mathcal{A}_n$. Thus, $\dim(\mathcal{A}_n) \leq \delta(\mathfrak{m}_n) \leq n$. We claim: $n \leq \dim(\mathcal{A}_n)$ and so $n \leq \dim(\mathcal{A}_n) \leq \delta(\mathfrak{m}_n) \leq n$. Thus, $\dim(\mathcal{A}_n) = \delta(\mathfrak{m}_n) = n$ and $\mathcal{A}_n$ is a regular local ring of dimension n.

For simplicity denote $\mathcal{A}_{n-i} := \mathbb{K}\{\mathtt{x}_{i+1}, \ldots, \mathtt{x}_n\}$ or $\mathbb{K}[[\mathtt{x}_{i+1}, \ldots, \mathtt{x}_n]]$ for $i = 1, \ldots, n$. Consider the homomorphism of rings

$$\psi_i : \mathcal{A}_n \to \mathcal{A}_{n-i},\ f \mapsto f(0, \ldots, 0, \mathtt{x}_{i+1}, \ldots, \mathtt{x}_n).$$

Observe that ψ_i is surjective, so $\mathcal{A}_n/\ker(\psi_i) \cong \mathcal{A}_{n-i}$. As $\mathcal{A}_{n-i}$ is an integral domain, the ideal $\ker(\psi_i)$ is prime. Observe that

$$\mathfrak{p}_i := \{\mathtt{x}_1, \dots, \mathtt{x}_i\}\mathcal{A}_n \subset \ker(\psi_i),$$

but also the converse equality holds.

Pick $f \in \ker(\psi_i)$. By Rückert's Division Theorem there exist $Q_1 \in \mathcal{A}_n$ and $R_1 \in \mathcal{A}_{n-1}[\mathtt{x}_1]$ with $\deg_{\mathtt{x}_1}(R_1) = 0$ such that $f = Q_1\mathtt{x}_1 + R_1$. In particular, $R_1 \in \mathcal{A}_{n-1}$. Divide now R_1 by $\mathtt{x}_2$ and we get $Q_2 \in \mathcal{A}_{n-1}$ and $R_2 \in \mathcal{A}_{n-2}[\mathtt{x}_2]$ with $\deg_{\mathtt{x}_2}(R_2) = 0$ such that $R_1 = Q_2\mathtt{x}_2 + R_2$. In particular, $R_2 \in \mathcal{A}_{n-2}$. We proceed recursively until we get $R_{i-1} = Q_i\mathtt{x}_i + R_i$ where $Q_i \in \mathcal{A}_{n-i+1}$ and $R_i \in \mathcal{A}_{n-i}[\mathtt{x}_i]$ with $\deg_{\mathtt{x}_i}(R_i) = 0$, so $R_i \in \mathcal{A}_{n-i}$. Consequently,

$$f = \mathtt{x}_1Q_1 + \cdots + \mathtt{x}_iQ_i + R_i.$$

As $R_i \in \mathcal{A}_{n-i}$,

$$0 = \psi_i(f) = \psi_i(\mathtt{x}_1Q_1 + \cdots + \mathtt{x}_iQ_i + R_i) = \psi_i(R_i) = R_i,$$

so $f = \mathtt{x}_1Q_1 + \cdots + \mathtt{x}_iQ_i \in \{\mathtt{x}_1, \dots, \mathtt{x}_i\}\mathcal{A}_n$.

Consequently, $\mathfrak{p}_i := \{\mathtt{x}_1, \dots, \mathtt{x}_i\}\mathcal{A}_n$ is a prime ideal of $\mathcal{A}_n$ and $\mathtt{x}_{i+1} \notin \mathfrak{p}_i$ because $\psi_i(\mathtt{x}_{i+1}) = \mathtt{x}_{i+1} \neq 0$. We have constructed a chain of distinct prime ideals $(0) \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ in the integral domain $\mathcal{A}_n$. Thus, $n \leq \dim(\mathcal{A}_n)$, as required. $\square$

Let $A \subset B$ be two commutative and unitary rings such that A is a subring of B. An element $b \in B$ is *integral over A* if there exists a monic polynomial $P \in A[\mathtt{x}]$ such that $P(b) = 0$. This property is equivalent to the fact that the subring $A[b]$ is a finitely generated A-module. The ring B is *integral over A* if each element $b \in B$ is integral over A. Recall that if B is a *finitely generated A-module*, that is, there exist $b_1, \dots, b_r \in B$ such that $B = b_1A + \cdots + b_rA$, then B is integral over A. In addition, to be "finitely generated over a ring" is a transitive property: *if $A \subset B \subset C$ is a chain of subrings, B is a finitely generated A-module and C is a finitely generated B-module, then C is a finitely generated A-module*. Consequently, if B is integral over A and C is integral over B, then C is integral over A. Thus, if $x \in C$ is integral over B and B is integral over A, then x is integral over A. The *integral closure $\overline{A}$ of the ring A in B* is the collection of all elements of B that are integral over A. It holds that $\overline{A}$ is a subring of B that contains A. With this terminology an integral domain A is integrally closed if and only if it coincides with its integral closure in its quotient field.

Theorem II.2.8 (Going-up) *Let $A \subset B$ be two commutative and unitary rings such that A is a subring of B. Suppose that B is integral over A. Then:*

(i) *If $\mathfrak{p}$ is a prime ideal of A, there exists a prime ideal $\mathfrak{q}$ of B such that $\mathfrak{q} \cap A = \mathfrak{p}$.*

(ii) *If $\mathfrak{q}_1 \subset \mathfrak{q}_2$ are prime ideals of B such that $\mathfrak{q}_1 \cap A = \mathfrak{q}_2 \cap A$, then $\mathfrak{q}_1 = \mathfrak{q}_2$.*

(iii) *If $\mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ is a chain of prime ideals in A and $\mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_s$ is a chain of prime ideals in B such that $s < r$ and $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for $i = 1, \ldots, s$, there exist prime ideals $\mathfrak{q}_{s+1} \subsetneq \cdots \subsetneq \mathfrak{q}_r$ such that $\mathfrak{q}_s \subsetneq \mathfrak{q}_{s+1}$ and $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for $i = s+1, \ldots, r$.*

As a consequence of the going-up theorem, we have the following result.

Corollary II.2.9 *Let $A \subset B$ be two commutative and unitary rings such that A is a subring of B and B is integral over A. Then* $\dim(A) = \dim(B)$.

Proof. Observe that (i) and (iii) in Theorem II.2.8 imply $\dim(A) \leq \dim(B)$ and (ii) implies $\dim(B) \leq \dim(A)$, so $\dim(A) = \dim(B)$, as required. $\square$

Theorem II.2.10 (Going-down) *Let $A \subset B$ be commutative and unitary integral domains such that A is a subring of B. Suppose that B is integral over A and A is integrally closed. Let $\mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ be a chain of prime ideals in A and $\mathfrak{q}_{s+1} \subsetneq \cdots \subsetneq \mathfrak{q}_r$ is a chain of prime ideals in B such that $1 \leq s < r$ and $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for $i = s, \ldots, r$. Then there exists a chain of prime ideals $\mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_s$ in B such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for $i = 1, \ldots, s$.*

2.b Weierstrass's preparation for ideals. We prove next a kind of Weierstrass's preparation result for ideals of $\mathcal{A}_n$ that will be the key to prove the local parameterization theorem. We need the following preliminary result, which is a consequence of Krull's Theorem II.1.7.

Lemma II.2.11 *The following assertions hold:*

(i) *Let $\mathfrak{a}$ be an ideal of $\mathcal{A}_{n-1}$ and let $\mathfrak{b} := \mathfrak{a}\mathcal{A}_n$. Pick $f \in \mathcal{A}_n$ and write $f := \sum_{k \geq 0} a_k \mathtt{x}_n^k$ with $a_k \in \mathcal{A}_{n-1}$. Then $f \in \mathfrak{b}$ if and only if each $a_k \in \mathfrak{a}$.*

(ii) *If $\mathfrak{p}$ is a prime ideal of $\mathcal{A}_{n-1}$, then $\mathfrak{p}\mathcal{A}_n$ is a prime ideal of $\mathcal{A}_n$.*

(iii) *Let $\mathfrak{a}$ be an ideal of $\mathcal{A}_{n-1}$ and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be prime ideals of $\mathcal{A}_{n-1}$ such that $\sqrt{\mathfrak{a}} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$. Denote $\mathfrak{b} := \mathfrak{a}\mathcal{A}_n$. Then*

$$\sqrt{\mathfrak{b}} = \sqrt{\mathfrak{a}}\mathcal{A}_n = \bigcap_{i=1}^{r} (\mathfrak{p}_i \mathcal{A}_n).$$

Proof. (i) Assume $f \in \mathfrak{b}$. Then $f = f_1 g_1 + \cdots + f_r g_r$ where $f_j \in \mathfrak{a}$ and $g_j := \sum_{k \geq 0} b_{jk} \mathtt{x}_n^k \in \mathcal{A}_n$ with $b_{jk} \in \mathcal{A}_{n-1}$. Thus,

$$\sum_{k \geq 0} a_k \mathtt{x}_n^k = f = \sum_{j=1}^{r} f_j g_j = \sum_{j=1}^{r} f_j \Big(\sum_{k \geq 0} b_{jk} \mathtt{x}_n^k \Big) = \sum_{k \geq 0} \Big(\sum_{j=1}^{r} f_j b_{jk} \Big) \mathtt{x}_n^k,$$

so $a_k = \sum_{j=1}^{r} f_j b_{jk} \in \mathfrak{a}$.

Assume now that $a_k \in \mathfrak{a}$ for each $k \geq 0$ and write $f_\ell := \sum_{k=0}^{\ell-1} a_k \mathtt{x}_n^k \in \mathfrak{a}\mathcal{A}_n$. Thus,

$$f - f_\ell = \sum_{k \geq \ell} a_k \mathtt{x}_n^k = \mathtt{x}_n^\ell \sum_{k \geq \ell} a_k \mathtt{x}_n^{k-\ell} \in \mathfrak{m}_n^\ell,$$

so $f = f_\ell + (f - f_\ell) \in \mathfrak{a}\mathcal{A}_n + \mathfrak{m}_n^\ell$ for each $\ell \geq 1$. By Krull's Theorem II.1.7

$$f \in \bigcap_{\ell \geq 1} (\mathfrak{a}\mathcal{A}_n + \mathfrak{m}_n^\ell) = \mathfrak{a}\mathcal{A}_n = \mathfrak{b}.$$

(ii) Suppose by contradiction that there exist $f, g \in \mathcal{A}_n \setminus \mathfrak{p}\mathcal{A}_n$ such that $fg \in \mathfrak{p}\mathcal{A}_n$. Write $f := \sum_{k \geq 0} a_k \mathtt{x}_n^k$ and $g := \sum_{k \geq 0} b_k \mathtt{x}_n^k$ with $a_k, b_k \in \mathcal{A}_{n-1}$. As $f \notin \mathfrak{p}\mathcal{A}_n$, there exists $k_0 \geq 0$ such that $a_{k_0} \notin \mathfrak{p}$ but $a_0, \ldots, a_{k_0-1} \in \mathfrak{p}$. Analogously, there exists $k_1 \geq 0$ such that $b_{k_1} \notin \mathfrak{p}$ but $b_0, \ldots, b_{k_1-1} \in \mathfrak{p}$. As $\mathfrak{p}$ is prime, $a_{k_0} b_{k_1} \notin \mathfrak{p}$. Write $fg := \sum_{k \geq 0} c_k \mathtt{x}_n^k$ with $c_k \in \mathcal{A}_{n-1}$. As $fg \in \mathfrak{p}\mathcal{A}_n$, we know by (i) that each $c_k \in \mathfrak{p}$. We have

$$c_{k_0+k_1} := \sum_{j=0}^{k_0+k_1} a_j b_{k_0+k_1-j} = a_{k_0} b_{k_1} + \sum_{j=0}^{k_0-1} a_j b_{k_0+k_1-j} + \sum_{\ell=0}^{k_1-1} a_{k_0+k_1-\ell} b_\ell$$

and consequently

$$a_{k_0} b_{k_1} = c_{k_0+k_1} - \sum_{j=0}^{k_0-1} a_j b_{k_0+k_1-j} - \sum_{\ell=0}^{k_1-1} a_{k_0+k_1-\ell} b_\ell \in \mathfrak{p},$$

which is a contradiction because $a_{k_0} b_{k_1} \notin \mathfrak{p}$.

(iii) First, $\sqrt{\mathfrak{a}}\mathcal{A}_n = (\bigcap_{i=1}^r \mathfrak{p}_i)\mathcal{A}_n \subset \bigcap_{i=1}^r(\mathfrak{p}_i\mathcal{A}_n)$. Pick $f \in \bigcap_{i=1}^r(\mathfrak{p}_i\mathcal{A}_n)$ and write $f := \sum_{k\geq 0} a_k\mathtt{x}_n^k$ where $a_k \in \mathcal{A}_{n-1}$ for all $k \geq 0$. By (i) each $a_k \in \mathfrak{p}_i$ for $i = 1, \dots, r$, so each $a_k \in \bigcap_{i=1}^r \mathfrak{p}_i = \sqrt{\mathfrak{a}}$ and by (i) $f \in \sqrt{\mathfrak{a}}\mathcal{A}_n$. We have proved

$$\sqrt{\mathfrak{a}}\mathcal{A}_n = \bigcap_{i=1}^r(\mathfrak{p}_i\mathcal{A}_n).$$

Observe that $\mathfrak{a}\mathcal{A}_n \subset \sqrt{\mathfrak{a}}\mathcal{A}_n = \bigcap_{i=1}^r(\mathfrak{p}_i\mathcal{A}_n)$, so $\sqrt{\mathfrak{b}} \subset \bigcap_{i=1}^r(\mathfrak{p}_i\mathcal{A}_n)$. As $\mathcal{A}_n$ is a noetherian ring, there exists $\ell \geq 1$ such that $(\sqrt{\mathfrak{a}})^\ell \subset \mathfrak{a}$, so

$$(\sqrt{\mathfrak{a}}\mathcal{A}_n)^\ell = (\sqrt{\mathfrak{a}})^\ell\mathcal{A}_n \subset \mathfrak{a}\mathcal{A}_n = \mathfrak{b} \subset \sqrt{\mathfrak{b}}$$

and $\sqrt{\mathfrak{a}}\mathcal{A}_n \subset \sqrt{\mathfrak{b}} \subset \bigcap_{i=1}^r(\mathfrak{p}_i\mathcal{A}_n) = \sqrt{\mathfrak{a}}\mathcal{A}_n$, as required. □

Lemma II.2.12 (Weierstrass's preparation) *Let $\mathfrak{a}$ be a non-zero ideal of $\mathcal{A}_n$. After a linear change of coordinates there exist $r \geq 1$ and distinguished polynomials $P_i \in \mathfrak{a} \cap \mathcal{A}_{n-i}[\mathtt{x}_{n-i+1}]$ for $1 \leq i \leq r$, whose degrees coincide with their orders as power series and such that $\mathfrak{a} \cap \mathcal{A}_{n-r} = (0)$. In addition*

(i) *After the linear change above $\mathcal{A}_n/\mathfrak{a}$ is a finite $\mathcal{A}_{n-r}$-module.*

(ii) $\operatorname{ht}(\mathfrak{a}) = r$ *and* $\dim(\mathcal{A}_n/\mathfrak{a}) + \operatorname{ht}(\mathfrak{a}) = \dim(\mathcal{A}_n)$.

Proof. We construct the linear change recursively. First pick $f_1 \in \mathfrak{a}\setminus(0)$. After a linear change provided by Lemma I.4.4 and an application of Weierstrass's Preparation Theorem $f_1 = P_1U_1$ where $P_1 \in \mathcal{A}_{n-1}[\mathtt{x}_n]$ is a distinguished polynomial with respect to $\mathtt{x}_n$ of degree $\deg_{\mathtt{x}_n}(P_1) = \omega(P_1)$ and $U_1 \in \mathcal{A}_n$ is a unit. Thus, $P_1 \in \mathfrak{a}\cap\mathcal{A}_{n-1}[\mathtt{x}_n]$ is the first polynomial we seek. Assume we have chosen distinguished polynomials P_i with respect to $\mathtt{x}_{n-i+1}$ and $\deg_{\mathtt{x}_{n-i+1}}(P_i) = \omega(P_i)$ for $1 \leq i < j$. If $\mathfrak{a} \cap \mathcal{A}_{n-j+1} \neq (0)$, we choose $f_j \in \mathfrak{a} \cap \mathcal{A}_{n-j+1} \setminus (0)$. We use again Lemma I.4.4 to make f_j regular with respect to $\mathtt{x}_{n-j+1}$. Note that the used linear change only involves the indeterminates $\mathtt{x}_1, \dots, \mathtt{x}_{n-j+1}$. Consequently, it does not affect the property that the already constructed polynomials $P_1, \dots, P_{j-1}$ are distinguished with respect to the other variables. We then apply Weierstrass's Preparation Theorem to write $f_j = P_jU_j$ where $P_j \in \mathcal{A}_{n-j}[\mathtt{x}_{n-j+1}]$ is a distinguished polynomial with respect to $\mathtt{x}_{n-j+1}$ of degree $\deg_{\mathtt{x}_{n-j+1}}(P_j) = \omega(P_j)$ and $U_j \in \mathcal{A}_{n-j+1}$ is a unit. After finitely many steps, say r, we have the desired polynomials $P_1, \dots, P_r$ and $\mathfrak{a} \cap \mathcal{A}_{n-r} = (0)$, as required.

Next we prove (i) and (ii).

(i) Let p_i denote the degree of P_i with respect to $\mathtt{x}_n$ for $1 \le i \le r$. Each $f \in \mathcal{A}_n$ can be successively divided by $P_1, \dots, P_r$ using Rückert's Division Theorem as follows:

$$\begin{aligned} f &= Q_1P_1 + \sum_{\nu_n=0}^{p_1-1} a_{\nu_n}\mathtt{x}_n^{\nu_n} \\ &= Q_1P_1 + \sum_{\nu_n=0}^{p_1-1}\Big(H_{\nu_n}P_2 + \sum_{\nu_{n-1}=0}^{p_2-1} b_{\nu_{n-1}\nu_n}\mathtt{x}_{n-1}^{\nu_{n-1}}\Big)\mathtt{x}_n^{\nu_n} \\ &= Q_1P_1 + Q_2P_2 + \sum_{\nu_{n-1},\nu_n} b_{\nu_{n-1}\nu_n}\mathtt{x}_{n-1}^{\nu_{n-1}}\mathtt{x}_n^{\nu_n} = \cdots \end{aligned}$$

Proceeding recursively f is written as an element $\sum_{i=1}^r Q_iP_i \in \mathfrak{a}$ plus a polynomial $Q \in \mathcal{A}_{n-r}[\mathtt{x}_{n-r+1}, \dots, \mathtt{x}_n]$ with $\deg_{\mathtt{x}_{n-i+1}}(Q) \le p_i$ for $i = 1, \dots, r$.

(ii) As $\mathcal{A}_{n-r} \cap \mathfrak{a} = (0)$, the homomorphism $\mathcal{A}_{n-r} \hookrightarrow \mathcal{A}_n \to \mathcal{A}_n/\mathfrak{a}$ is injective and from (i) we deduce that $\mathcal{A}_n/\mathfrak{a}$ is integral over $\mathcal{A}_{n-r}$. By Corollary II.2.9

$$\dim(\mathcal{A}_n/\mathfrak{a}) = \dim(\mathcal{A}_{n-r}) = n - r.$$

Consequently, by (2.2)

$$\operatorname{ht}(\mathfrak{a}) \le \dim(\mathcal{A}_n) - \dim(\mathcal{A}_n/\mathfrak{a}) \le r.$$

2.b.1 Let us prove that $\operatorname{ht}(\mathfrak{a}) \ge r$. Once we have proved this, we conclude $\operatorname{ht}(\mathfrak{a}) = \dim(\mathcal{A}_n) - \dim(\mathcal{A}_n/\mathfrak{a})$, so $\dim(\mathcal{A}_n) = \operatorname{ht}(\mathfrak{a}) + \dim(\mathcal{A}_n/\mathfrak{a})$. It is enough to show:

$$\operatorname{ht}(\{P_1, \dots, P_r\}\mathcal{A}_n) \ge r$$

and to that end we prove

$$\operatorname{ht}(\{P_j, \dots, P_r\}\mathcal{A}_n) > \operatorname{ht}(\{P_{j-1}, \dots, P_r\}\mathcal{A}_n)$$

for $1 \le j < r$. For $j = r$ we have $\operatorname{ht}(P_r\mathcal{A}_n) > 0$ because $P_r \ne 0$.

2.b.2 For simplicity we approach the case $j = 1$, that is,

$$\operatorname{ht}(\{P_1, \dots, P_r\}\mathcal{A}_n) > \operatorname{ht}(\{P_2, \dots, P_r\}\mathcal{A}_n).$$

Consider the ideal

$$\mathfrak{b} := \sqrt{\{P_2, \dots, P_r\}\mathcal{A}_{n-1}}.$$

As $\mathcal{A}_{n-1}$ is noetherian, we can write $\mathfrak{b} := \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$ where each $\mathfrak{p}_i$ is a prime ideal of $\mathcal{A}_{n-1}$. Denote $\mathfrak{q}_i := \mathfrak{p}_i\mathcal{A}_n$. By Lemma II.2.11 $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ are prime ideals of $\mathcal{A}_n$ and $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s = \sqrt{\{P_2, \dots, P_r\}\mathcal{A}_n}$.

2.b.3 Suppose now that

$$\begin{aligned}\operatorname{ht}(\sqrt{\{P_1,\dots,P_r\}\mathcal{A}_n}) &= \operatorname{ht}(\{P_1,\dots,P_r\}\mathcal{A}_n)\\ &= \operatorname{ht}(\{P_2,\dots,P_r\}\mathcal{A}_n) = \operatorname{ht}(\sqrt{\{P_2,\dots,P_r\}\mathcal{A}_n}).\end{aligned}$$

By Lemma II.2.3 we may assume that $\sqrt{\{P_1,\dots,P_r\}\mathcal{A}_n} \subset \mathfrak{q}_1 = \mathfrak{p}_1\mathcal{A}_n$, so $P_1 = \mathtt{x}_n^{p_1} + \sum_{j=0}^{p_1-1} b_j\mathtt{x}_n^j \in \mathfrak{p}_1\mathcal{A}_n$ where each $b_j \in \mathcal{A}_{n-1}$. By Lemma II.2.11 we deduce that $1, b_0, \dots, b_{p_1-1} \in \mathfrak{p}_1$, which is a contradiction because $1 \notin \mathfrak{p}_1$. Consequently,

$$\operatorname{ht}(\{P_2,\dots,P_r\}\mathcal{A}_n) < \operatorname{ht}(\{P_1,\dots,P_r\}\mathcal{A}_n),$$

as required. □

We can generalize the formula $\dim(\mathcal{A}_n/\mathfrak{a}) + \operatorname{ht}(\mathfrak{a}) = \dim(\mathcal{A}_n)$ for arbitrary analytic and formal integral domains.

Corollary II.2.13 *Let $\mathfrak{p}$ be a prime ideal of $\mathcal{A}_n$ and let $\mathfrak{A}$ be an ideal of the quotient $A := \mathcal{A}_n/\mathfrak{p}$. Then*

$$\operatorname{ht}(\mathfrak{A}) + \dim(A/\mathfrak{A}) = \dim(A).$$

Proof. Assume first $\mathfrak{A} = \mathfrak{Q}$ is a prime ideal. By Lemma II.2.12 the inclusion $\mathcal{A}_d \hookrightarrow A$, where $d = \dim(\mathcal{A}_n/\mathfrak{p}) = \dim(\mathcal{A}_d)$, makes A a finitely generated $\mathcal{A}_d$-module. By Corollary II.2.9 and the Going-down Theorem II.2.10 we have

$$\operatorname{ht}(\mathfrak{Q}) = \operatorname{ht}(\mathfrak{Q} \cap \mathcal{A}_d) \quad \text{and} \quad \dim(A/\mathfrak{Q}) = \dim(\mathcal{A}_d/(\mathfrak{Q} \cap \mathcal{A}_d)).$$

As the formula of the statement holds for $\mathcal{A}_d$, it also holds for A, as required.

We approach next the general case. Let $\mathfrak{Q}$ be a prime ideal of A such that $\operatorname{ht}(\mathfrak{Q}) = \operatorname{ht}(\mathfrak{A})$. By the correspondence theorem for ideals we can write $\mathfrak{Q} := \mathfrak{q}/\mathfrak{p}$ and $\mathfrak{A} := \mathfrak{a}/\mathfrak{p}$ where $\mathfrak{a}$ is an ideal that contains $\mathfrak{p}$ and $\mathfrak{q}$ is a prime ideal that contains $\mathfrak{a}$ and satisfies $\operatorname{ht}(\mathfrak{q}) = \operatorname{ht}(\mathfrak{a})$. To guarantee the latter equality we use in addition Corollary II.2.15 (that only uses the present result for the prime case). By the third isomorphism theorem for rings

$$\begin{aligned}\operatorname{ht}(\mathfrak{q}/\mathfrak{p}) &= \operatorname{ht}(\mathfrak{Q}) = \dim(\mathcal{A}_n/\mathfrak{p}) - \dim(A/\mathfrak{Q})\\ &= \dim(\mathcal{A}_n/\mathfrak{p}) - \dim(\mathcal{A}_n/\mathfrak{q}) = n - \operatorname{ht}(\mathfrak{p}) - n + \operatorname{ht}(\mathfrak{q}) = \operatorname{ht}(\mathfrak{q}) - \operatorname{ht}(\mathfrak{p}).\end{aligned}$$

Consequently,

$$\begin{aligned}\operatorname{ht}(\mathfrak{A}) = \operatorname{ht}(\mathfrak{q}) - \operatorname{ht}(\mathfrak{p}) &= \operatorname{ht}(\mathfrak{a}) - \operatorname{ht}(\mathfrak{p}) = n - \operatorname{ht}(\mathfrak{p}) - (n - \operatorname{ht}(\mathfrak{a}))\\ &= \dim(\mathcal{A}_n/\mathfrak{p}) - \dim(\mathcal{A}_n/\mathfrak{a}) = \dim(A) - \dim(A/\mathfrak{A}),\end{aligned}$$

so $\operatorname{ht}(\mathfrak{A}) + \dim(A/\mathfrak{A}) = \dim(A)$, as required. □

The previous result is not true in general if A is not an integral domain.

Example II.2.14 Let $\mathfrak{p}_1 := \{\mathtt{x}_1, \mathtt{x}_2\}\mathcal{A}_3$ and $\mathfrak{p}_2 := \{\mathtt{x}_3\}\mathcal{A}_3$ and $\mathfrak{a} := \mathfrak{p}_1 \cap \mathfrak{p}_2$. Define $\mathfrak{P} := \mathfrak{p}_1/\mathfrak{a}$ and $A := \mathcal{A}_3/\mathfrak{a}$. Then $\dim(A/\mathfrak{P}) + \operatorname{ht}(\mathfrak{P}) < \dim(A)$.

Consider the surjective homomorphism

$$\varphi : \mathcal{A}_3 \to \mathcal{A}_1 := \mathbb{K}\langle\langle \mathtt{x}_3 \rangle\rangle, \ f \mapsto f(0, 0, \mathtt{x}_3).$$

The kernel of φ is $\mathfrak{p}_1$. To prove this fact, apply Rückert's Division Theorem and show that each $f \in \mathcal{A}_3$ can be written as $f = \mathtt{x}_1 Q_1 + \mathtt{x}_2 Q_2 + R(\mathtt{x}_3)$ where $Q_1, Q_2 \in \mathcal{A}_3$ and $R(\mathtt{x}_3) \in \mathcal{A}_1$. Thus,

$$\dim(A/\mathfrak{P}) = \dim(\mathcal{A}_3/\mathfrak{p}_1) = \dim(\mathcal{A}_1) = 1$$

and $\operatorname{ht}(\mathfrak{p}_1) = \dim(\mathcal{A}_3) - \dim(\mathcal{A}_3/\mathfrak{p}_1) = 3 - 1 = 2$. Analogously, the ideal $\mathfrak{p}_2$ is the kernel of the surjective homomorphism

$$\psi : \mathcal{A}_3 \to \mathcal{A}_2 := \mathbb{K}\langle\langle \mathtt{x}_1, \mathtt{x}_2 \rangle\rangle, \ f \mapsto f(\mathtt{x}_1, \mathtt{x}_2, 0).$$

Consequently,

$$2 = \dim(\mathcal{A}_2) = \dim(\mathcal{A}_3/\mathfrak{p}_2) = \dim(\mathcal{A}_3) - \operatorname{ht}(\mathfrak{p}_2) = 3 - \operatorname{ht}(\mathfrak{p}_2),$$

so $\operatorname{ht}(\mathfrak{p}_2) = 1$. In addition,

$$\begin{aligned}\dim(A) = \dim(\mathcal{A}_3/\mathfrak{a}) &= \dim(\mathcal{A}_3) - \operatorname{ht}(\mathfrak{a})\\ &= 3 - \min\{\operatorname{ht}(\mathfrak{p}_1), \operatorname{ht}(\mathfrak{p}_2)\} = 3 - \min\{2, 1\} = 2.\end{aligned}$$

Let us prove that $\operatorname{ht}(\mathfrak{P}) = 0$. Suppose by contradiction that there exists a prime ideal $\mathfrak{Q} \subsetneq \mathfrak{P}$. By the correspondence theorem for ideals, there exists a prime ideal $\mathfrak{q}$ of $\mathcal{A}_3$ such that $\mathfrak{a} \subset \mathfrak{q} \subsetneq \mathfrak{p}_1$. Let $f \in \mathfrak{p}_1 \setminus \mathfrak{q}$. Then $f\mathtt{x}_3 \in \mathfrak{a} \subset \mathfrak{q}$, so $\mathtt{x}_3 \in \mathfrak{q} \subset \mathfrak{p}_1$ because $\mathfrak{q}$ is prime. As $\mathtt{x}_3 \notin \mathfrak{p}_1$, we achieve a contradiction. Consequently, $\operatorname{ht}(\mathfrak{P}) = 0$.

We conclude $\dim(A/\mathfrak{P}) + \operatorname{ht}(\mathfrak{P}) = 1 + 0 < 2 = \dim(A)$. □

Corollary II.2.15 *Let $\mathfrak{P} \subset \mathfrak{P}'$ be two prime ideals of a ring $A := \mathcal{A}_n/\mathfrak{a}$ where $\mathfrak{p}$ is a prime ideal of $\mathcal{A}_n$. Then all unrefinable chains of prime ideals in between $\mathfrak{P}$ and $\mathfrak{P}'$ have the same length, which is equal to $\operatorname{ht}(\mathfrak{P}') - \operatorname{ht}(\mathfrak{P})$.*

Proof. Let $\mathfrak{P} \subsetneq \mathfrak{P}_1 \subsetneq \cdots \subsetneq \mathfrak{P}_{r-1} \subsetneq \mathfrak{P}'$ be a non-refinable chain. By the third isomorphism theorem for rings

$$\begin{aligned}1 = \operatorname{ht}(\mathfrak{P}'/\mathfrak{P}_{r-1}) &= \dim(A/\mathfrak{P}_{r-1}) - \dim\left(A/\mathfrak{P}_{r-1}/\mathfrak{P}'/\mathfrak{P}_{r-1}\right)\\ &= \dim(A) - \operatorname{ht}(\mathfrak{P}_{r-1}) - \dim(A/\mathfrak{P}')\\ &= \dim(A) - \operatorname{ht}(\mathfrak{P}_{r-1}) - (\dim(A) - \operatorname{ht}(\mathfrak{P}'))\\ &= \operatorname{ht}(\mathfrak{P}') - \operatorname{ht}(\mathfrak{P}_{r-1}).\end{aligned}$$

Thus, $\operatorname{ht}(\mathfrak{P}') = \operatorname{ht}(\mathfrak{P}_{r-1}) + 1$. Proceeding recursively $\operatorname{ht}(\mathfrak{P}') = r + \operatorname{ht}(\mathfrak{P})$ or equivalently $r = \operatorname{ht}(\mathfrak{P}') - \operatorname{ht}(\mathfrak{P})$, so all unrefinable chains of prime ideals in between $\mathfrak{P}$ and $\mathfrak{P}'$ have the same length r, as required. □

2.c Noether's Normalization Lemma.

We present next the main result of this section. Before that we recall the Primitive Element Theorem.

Theorem II.2.16 (Primitive Element Theorem) *Let $L|K$ be a field extension of characteristic zero and let $C \subset K$ be an infinite subset. If $a_1, \dots, a_r$ are algebraic elements over K, there exist $\lambda_2, \dots, \lambda_r \in C$ such that*

$$K(a_1, \dots, a_r) = K(a_1 + \lambda_2 a_2 + \cdots + \lambda_r a_r).$$

Let A be a commutative ring with unity that is an integral domain. The discriminant $\Delta \in A$ of a monic polynomial $P \in A[\mathtt{t}]$ of degree d is $(-1)^{\binom{d}{2}}$ times the resultant of P and its derivative $\frac{\partial P}{\partial \mathtt{t}}$. Recall that $\Delta = 0$ if and only if P has multiple roots in an algebraic closure of $\operatorname{qf}(A)$. In particular, if A is a unique factorization domain of characteristic zero, irreducible polynomials of $A[\mathtt{t}]$ have non-zero discriminant as they have no multiple roots.

Theorem II.2.17 (Noether's Normalization Lemma) *Let $\mathfrak{a} \neq (0)$ be an ideal of heigth r of $\mathcal{A}_n$ and denote $d := n - r$. We have:*

(i) *After a linear change of coordinates A is a finitely generated $\mathcal{A}_d$-module via the canonical homomorphism $\mathcal{A}_d \hookrightarrow A := \mathcal{A}_m/\mathfrak{a}$, that is injective.*

(ii) *$A = \mathcal{A}_d[\theta_{d+1}, \dots, \theta_n]$ where $\theta_j := \mathtt{x}_j + \mathfrak{a}$ for $d + 1 \leq j \leq n$.*

(iii) *Assume in what follows* $\mathfrak{a} = \mathfrak{p}$ *is prime. Let* $K := \mathrm{qf}(\mathcal{A}_d)$ *and* $L := \mathrm{qf}(A)$. *After an additional linear change of coordinates involving only the last* r *indeterminates* $\mathtt{x}_{d+1}, \dots, \mathtt{x}_n$, *the element* θ_{d+1} *is a primitive element of* L *over* K, *that is,* $L = K[\theta_{d+1}]$.

(iv) *The irreducible polynomial* P *of each* $\theta \in \mathfrak{m}_A$ *over* K *has coefficients in* $\mathcal{A}_d$ *and it is a distinguished polynomial* $P(\mathtt{x}, \mathtt{t}) \in \mathcal{A}_d[\mathtt{t}]$ *with respect to* $\mathtt{t}$ *whose discriminant* $\Delta \in \mathcal{A}_d \setminus \{0\}$.

(v) *Assume* $\theta \in \mathfrak{m}_A$ *is a primitive element of the extension* $L|K$ *and denote* $\overline{A}$ *the integral closure of* A *in* L. *Then* $\Delta\overline{A} \subset \mathcal{A}_d + \mathcal{A}_d\theta + \cdots + \mathcal{A}_d\theta^{p-1}$ *where* $p := [L : K]$ *is the degree of* L *over* K.

(vi) *The ring* $\overline{A}$ *is a finitely generated* $\mathcal{A}_d$*-module, so* $\overline{A}$ *is a finitely generated* A*-module.*

The linear changes in the statement can be done simultaneously for any given finite family of ideals of the same height.

Proof. The first two statements (i) and (ii) follow from Lemma II.2.12, which can be applied to finitely many ideals simultaneously. Assume in what follows that $\mathfrak{a} = \mathfrak{p}$ is prime.

(iii) Let $\lambda_{d+1} := 1, \lambda_{d+2}, \dots, \lambda_n \in \mathbb{K}$ such that

$$\theta := \sum_{k=d+1}^{n} \lambda_k \mathtt{x}_k + \mathfrak{a}$$

is a primitive element of L over K. Consider the linear change

$$\mathtt{y}_i := \mathtt{x}_i, \quad \text{for } i \neq d+1 \quad \text{and} \quad \mathtt{y}_{d+1} := \sum_{k=d+1}^{n} \lambda_i \mathtt{x}_k.$$

This linear change of coordinates can be done simultaneously for finitely many ideals as a consequence of the proof of the Primitive Element Theorem.

(iv) Fix $\theta \in \mathfrak{m}_A$ and let $P := \mathtt{t}^p + a_{p-1}\mathtt{t}^{p-1} + \cdots + a_0 \in K[\mathtt{t}]$ be the irreducible polynomial of θ over K. Let F be the splitting field of P over L and let $\theta := \alpha_1, \alpha_2, \dots, \alpha_p \in F$ be the roots of P in F. All the roots α_ℓ are different because P is irreducible and we are working in zero characteristic. As A is a finitely generated $\mathcal{A}_d$-module and $\mathcal{A}_d \subset A$, all the elements of A are integral over $\mathcal{A}_d$, so θ is a root of a monic polynomial $Q \in \mathcal{A}_d[\mathtt{t}]$. Observe that

P divides Q in $K[\mathtt{t}]$, so $\alpha_1, \alpha_2, \ldots, \alpha_p$ are roots of Q and $\alpha_1, \alpha_2, \ldots, \alpha_p$ are integral over $\mathcal{A}_d$. Using Cardano's-Viète's identities,

$$a_k = (-1)^{p-k} \sum_{1 \leq i_1 < \cdots < i_k \leq p-k} \alpha_{i_1} \cdots \alpha_{i_k},$$

so $a_k \in K$ is integral over $\mathcal{A}_d$. As $\mathcal{A}_d$ is by Corollary II.1.6 integrally closed, we deduce that each $a_k \in \mathcal{A}_d$ and $P \in \mathcal{A}_d[\mathtt{t}]$.

Denote $\mathfrak{m}_d$ the maximal ideal of $\mathcal{A}_d$ and recall that $\theta \in \mathfrak{m}_A$. By the Going-up theorem II.2.8 and Corollary II.2.9

$$\mathrm{ht}(\mathfrak{m}_d) = \dim(\mathcal{A}_d) = \dim(A) = \mathrm{ht}(\mathfrak{m}_A) \leq \mathrm{ht}(\mathfrak{m}_A \cap \mathcal{A}_d) \leq \mathrm{ht}(\mathfrak{m}_d).$$

As $\mathfrak{m}_A \cap \mathcal{A}_d \subset \mathfrak{m}_d$, we deduce $\mathfrak{m}_A \cap \mathcal{A}_d = \mathfrak{m}_d$. Consequently,

$$a_0 = -(\theta^p + a_{p-1}\theta^{p-1} + \cdots + a_1\theta) \in \mathfrak{m}_A \cap \mathcal{A}_d = \mathfrak{m}_d.$$

As $P(0,0) = a_0(0) = 0$, we deduce that the polynomial P is a regular series with respect to $\mathtt{t}$ of some order $1 \leq s \leq p$. By Weierstrass's Preparation Theorem $P = P_1 U$, where $P_1 \in \mathcal{A}_d[\mathtt{t}]$ is a distinguished polynomial of degree s and $U \in \mathcal{A}_{d+1}$ is a unit. By Rückert's Division Theorem U is a polynomial of $\mathcal{A}_d[\mathtt{t}]$ (because $P = P_1 U + 0$) and both P_1 and U are monic polynomials. Write $\mathtt{x} := (\mathtt{x}_1, \ldots, \mathtt{x}_d)$. It follows $P_1(\mathtt{x}, \theta) \cdot U(\mathtt{x}, \theta) = P(\mathtt{x}, \theta) = 0$ and, since A is an integral domain, either $P_1(\mathtt{x}, \theta) = 0$ or $U(\mathtt{x}, \theta) = 0$. As P is the irreducible polynomial of θ and it is not a unit in $\mathcal{A}_{d+1}$ (because $P(0,0) = 0$), we conclude $P = P_1$ is a distinguished polynomial with respect to $\mathtt{t}$.

As $P \in \mathcal{A}_d[\mathtt{t}]$ is an irreducible polynomial, its discriminant $\Delta \in \mathcal{A}_d \setminus \{0\}$.

(v) Let $y_1 \in \overline{A} \subset L$ and we keep the notations in (iv). As θ is a primitive element of the extension $L|K$,

$$y_1 = b_0 + b_1\theta + \cdots + b_{p-1}\theta^{p-1}, \tag{2.3}$$

where each $b_i \in K$. As $F|K$ is a Galois extension, there exist K-automorphisms $\sigma_1 := \mathrm{id}_F, \sigma_2, \ldots, \sigma_p$ of $F|K$ such that $\sigma_i(\theta) = \alpha_i$ for $i = 1, \ldots, p$.

Applying the K-automorphisms σ_i to the equation (2.3), we get

$$y_i := \sigma_i(y_1) = b_0 + b_1\alpha_i + \cdots + b_{p-1}\alpha_i^{p-1} \quad \text{for } 1 \leq i \leq p,$$

which we look at as a system of linear equations on the unknowns $b_i \in \mathbb{K}$. We have

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix} = \begin{pmatrix} 1 & \alpha_1 & \cdots & \alpha_1^{p-1} \\ 1 & \alpha_2 & \cdots & \alpha_2^{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_p & \cdots & \alpha_p^{p-1} \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-1} \end{pmatrix}$$

and we summarize it as $y^t = Mb^t$. We multiply both members by $\mathrm{Adj}(M^t)$. Recall that $\mathrm{Adj}(M^t) \cdot M = \det(M) I_p$ where $\det(M)$ is the Vandermonde determinant, whose value is $\delta := \prod_{1\leq i<\leq j}(\alpha_i - \alpha_j) \neq 0$. Thus,

$$\det(M)^2 = \prod_{1\leq i<j\leq p} (\alpha_i - \alpha_j)^2 = \Delta.$$

so $\det(M)\mathrm{Adj}(M^t) \cdot y^t = (\det(M))^2 I_p b^t = \Delta b^t$. Consequently,

$$\Delta b_j = [\delta y \mathrm{Adj}(M)]_j \in \mathbb{Z}[\alpha_1, \ldots, \alpha_p, y_1, \ldots, y_p].$$

We have proved in (iv) that each α_k is integral over $\mathcal{A}_d$. As $y_1 \in \overline{A}$ and A is integral over $\mathcal{A}_d$, we deduce that y_1 is integral over $\mathcal{A}_d$. As $y_i = \sigma_i(y_1)$ and $\sigma_i \in \mathrm{Gal}(L : K)$, we deduce that each y_i is a root of the irreducible polynomial of y_1 over K. The same proof of the first part of (iv) shows that $y_2, \ldots, y_p$ are integral over $\mathcal{A}_d$. Thus, $\Delta b_i \in K$ is integral over $\mathcal{A}_d$ for $i = 0, \ldots, p-1$. As $\mathcal{A}_d$ is integrally closed, $\Delta b_i \in \mathcal{A}_d$ for $i = 0, \ldots, p-1$. Consequently,

$$\Delta y_1 = \Delta b_0 + \Delta b_1 \theta + \cdots + \Delta b_{p-1}\theta^{p-1} \in \mathcal{A}_d + \mathcal{A}_d\theta + \cdots + \mathcal{A}_d\theta^{p-1}.$$

(vi) As a consequence of (v), $\overline{A}$ is a $\mathcal{A}_d$-submodule of

$$M := \mathcal{A}_d + \mathcal{A}_d\frac{\theta}{\Delta} + \cdots + \mathcal{A}_d\frac{\theta^{p-1}}{\Delta},$$

which is finitely generated over $\mathcal{A}_d$. As $\mathcal{A}_d$ is a noetherian ring, M is a noetherian module, so $\overline{A}$ is finitely generated over $\mathcal{A}_d$. As $\mathcal{A}_d \subset A$, we conclude that $\overline{A}$ is finitely generated over A. □

3 Local parameterization theorem

Now we come back to the situation of Noether's Projection Lemma to prove local parameterization theorem. We recall some results concerning *localization of rings*. A subset S of a commutative ring with unity A is *multiplicatively closed* if $0 \notin S$, $1 \in S$ and $xy \in S$ when $x, y \in S$. The set

$$S^{-1}A := \{\frac{a}{s} : a \in A,\ s \in S\}$$

is the *localization of A with respect to S*. Recall that $\frac{a_1}{s_1}, \frac{a_2}{s_2} \in S^{-1}A$ are equal if there exists $t \in S$ such that $t(s_2 a_1 - s_1 a_2) = 0$. We endow $S^{-1}A$ with the natural structure of ring. The prime ideals of $S^{-1}A$ are in one-to-one correspondence ($\mathfrak{q} \to S^{-1}\mathfrak{q}$) with the prime ideals of A that do not meet S. If $\mathfrak{p}$ is a prime ideal of A, then $S := A \setminus \mathfrak{p}$ is a multiplicatively closed set and $A_\mathfrak{p} := S^{-1}A$ is the *localization of A at* $\mathfrak{p}$. The map $A \to A_\mathfrak{p}$, $a \mapsto \frac{a}{1}$ is a homomorphism, that is injective if A is an integral domain and in this case $A \subset A_\mathfrak{p} \subset K := \mathrm{qf}(A)$. The ring $A_\mathfrak{p}$ is local and its unique maximal ideal is $\mathfrak{p}A_\mathfrak{p}$. In addition, $\dim(A_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}A_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p})$ and there exists a bijection between the set of prime ideals of $A_\mathfrak{p}$ and the set of prime ideals of A contained in $\mathfrak{p}$. In particular, if $\mathfrak{q} \subset \mathfrak{p}$ is a prime ideal of A, then $\mathrm{ht}(\mathfrak{q}A_\mathfrak{p}) = \mathrm{ht}(\mathfrak{q})$. On the other hand, given a non-nilpotent $f \in A$, we denote with $A_f := S^{-1}A$ where S is the multiplicatively closed set $\{f^k : \ k \geq 0\}$.

An ideal $\mathfrak{q}$ of A is *primary* if the zero divisors of $A/\mathfrak{q}$ are nilpotent. Recall that each non-zero ideal $\mathfrak{a}$ of a noetherian ring A can be written as a finite intersection $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$, where each $\mathfrak{q}_i$ is primary. An intersection of the previous type is called a *primary decomposition of* $\mathfrak{a}$. Such a decomposition is *redundant* if some of the primary ideals $\mathfrak{q}_i$ contains the intersection of the others. We only work with (irredundant) primary decompositions. Even in this case a primary decomposition of an ideal is not unique in general. However, the collection $\{\sqrt{\mathfrak{q}_1}, \ldots, \sqrt{\mathfrak{q}_r}\}$ is uniquely determined by $\mathfrak{a}$.

Theorem II.3.1 (Local Parameterization) *Let $\mathfrak{p}$ be a prime ideal of $\mathcal{A}_n$ of height r and denote $d := n - r$. After a linear change of coordinates the following conditions hold:*

(i) *The canonical homomorphism $\mathcal{A}_d \hookrightarrow A := \mathcal{A}_n/\mathfrak{p}$ is injective and A is a finitely generated $\mathcal{A}_d$-module.*

(ii) *The irreducible polynomial over K of $\theta_j := \mathtt{x}_j + \mathfrak{p} \in \mathfrak{m}_A$ is a distinguished polynomial $P_j \in \mathcal{A}_d[\mathtt{x}_j]$ of degree p_j for $d + 1 \leq j \leq n$.*

(iii) *Let $L := \mathrm{qf}(A)$ and $K := \mathrm{qf}(\mathcal{A}_d)$. Then θ_{d+1} is a primitive element of the field extension $L|K$ and the discriminant $\Delta \in \mathcal{A}_d \setminus \{0\}$ of P_{d+1} does not belong to $\mathfrak{p}$.*

(iv) *For each $j = d + 2, \ldots, n$ there exists a polynomial $Q_j \in \mathcal{A}_d[\mathtt{x}_{d+1}]$ of degree $< p := p_{d+1}$ such that $\Delta\mathtt{x}_j - Q_j \in \mathfrak{p}$.*

(v) *Consider the ideal $\mathfrak{a} := \{P_{d+1}, \Delta\mathtt{x}_{d+2} - Q_{d+2}, \ldots, \Delta\mathtt{x}_n - Q_n\}\mathcal{A}_n$ and the positive integer $q_0 := \sum_{j=d+2}^{n} p_j$. We have $\Delta^q\mathfrak{p} \subset \mathfrak{a} \subset \mathfrak{p}$ for each and $\mathfrak{p} = [\mathfrak{a} : \Delta^q]$ for $q \geq q_0$.*

(vi) *There exists an ideal* $\mathfrak{b}$ *of* $\mathcal{A}_n$ *such that* $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{p}$ *and* $\Delta \in \sqrt{\mathfrak{b}} \setminus \mathfrak{p}$.

Proof. After a linear change of coordinates statements (i) to (iii) hold by Theorem II.2.17. To prove that $\Delta \notin \mathfrak{p}$ note that $\Delta \in \mathcal{A}_d \setminus \{0\}$ and $\mathcal{A}_d \cap \mathfrak{p} = (0)$. Next, we prove the remaining statements.

(iv) Write $\theta := \theta_{d+1}$. Each $\theta_j \in A \subset \overline{A}$ for $j = d+2, \dots, n$. By Theorem II.2.17 $\Delta \overline{A} \subset \mathcal{A}_d + \mathcal{A}_d \theta + \cdots + \mathcal{A}_d \theta^{p-1}$. Thus, there exists a polynomial $Q_j \in \mathcal{A}_d[\mathtt{x}_{d+1}]$ of degree $< p$ such that $\Delta \mathtt{x}_j - Q_j \in \mathfrak{p}$.

(v) Observe that $\mathfrak{a} \subset \mathfrak{p}$ because $P_{d+1} \in \mathfrak{p}$. We prove next: $\Delta^q \mathfrak{p} \subset \mathfrak{a}$ *for each* $q \geq q_0$.

Pick $q \geq q_0$ and let us check that $\Delta^q f \in \mathfrak{a}$ for each $f \in \mathfrak{p}$. Applying Rückert's Division Theorem successively we can write

$$f = \sum_{j=d+2}^{n} P_j A_j + \sum_{\substack{\nu_j < p_j \\ d+2 \leq j \leq n}} a_\nu \mathtt{x}_{d+2}^{\nu_{d+2}} \cdots \mathtt{x}_n^{\nu_n}$$

where $A_j \in \mathcal{A}_n$ and $a_\nu \in \mathcal{A}_{d+1}$. As $q \geq \sum_{j=p+2}^{n} (p_j - 1)$,

$$\Delta^q f = \sum_{j=d+2}^{n} \Delta^q P_j A_j + \sum_{\substack{\nu_j < p_j \\ d+2 \leq j \leq n}} \Delta^{q_\nu} a_\nu (\Delta \mathtt{x}_{d+2})^{\nu_{d+2}} \cdots (\Delta \mathtt{x}_n)^{\nu_n} \tag{3.4}$$

where $q_\nu = q - |\nu|$ for each ν in the sum above. Write $\mathtt{x}' := (\mathtt{x}_1, \dots, \mathtt{x}_d)$. As $q \geq p_j$, we have

$$P_j^*(\mathtt{x}', \mathtt{x}_{d+1}) := \Delta^q P_j\Big(\mathtt{x}', \frac{Q_j(\mathtt{x}', \mathtt{x}_{d+1})}{\Delta}\Big) \in \mathcal{A}_d[\mathtt{x}_{d+1}] \tag{3.5}$$

for $j = d+2, \dots, n$. As $P_j(\mathtt{x}', \theta_j) = 0$ and $\Delta \theta_j = Q_j(\mathtt{x}', \theta)$, we deduce that $P_j^*(\mathtt{x}', \theta) = 0$ for $j = d+2, \dots, n$. As $P_{d+1} \in \mathcal{A}_d[\mathtt{x}_{d+1}]$ is the irreducible polynomial of θ_{d+1} over K, there exists $G_j \in \mathcal{A}_d[\mathtt{x}_{d+1}]$ such that

$$P_j^*(\mathtt{x}', \mathtt{x}_{d+1}) = G_j(\mathtt{x}', \mathtt{x}_{d+1}) P_{d+1}(\mathtt{x}', \mathtt{x}_{d+1}).$$

In addition, by (3.5) and since $q \geq p_j$, there exists $H_j \in \mathcal{A}_d[\mathtt{x}_{d+1}, \mathtt{x}_j]$ such that

$$\Delta^q P_j(\mathtt{x}', \mathtt{x}_j) - P_j^*(\mathtt{x}', \mathtt{x}_{d+1}) = (\Delta \mathtt{x}_j - Q_j(\mathtt{x}', \mathtt{x}_{d+1})) H_j$$

for $j = d+2, \dots, n$. Consequently, if

$$R := \sum_{\substack{\nu_j < p_j \\ d+2 \leq j \leq n}} \Delta^{q_\nu} a_\nu (\mathtt{y}_{d+2})^{\nu_{d+2}} \cdots (\mathtt{y}_n)^{\nu_n} \in \mathcal{A}_d[\mathtt{y}_{d+2}, \dots, \mathtt{y}_n],$$

we deduce from (3.4)

$$\Delta^q f - R(\Delta\mathtt{x}_{d+2},\dots,\Delta\mathtt{x}_n) = \sum_{j=d+2}^{n} A_j(G_jP_{d+1} + (\Delta\mathtt{x}_j - Q_j)H_j) \in \mathfrak{a}. \quad (3.6)$$

Using Taylor expansion, we write

$$R(\mathtt{y}_{d+2}+\mathtt{z}_{d+2},\dots,\mathtt{y}_n+\mathtt{z}_n) = R(\mathtt{y}_{d+2},\dots,\mathtt{y}_n) + \sum_{j=d+2}^{n} \mathtt{z}_j S_j(\mathtt{y},\mathtt{z})$$

where $S_j \in \mathcal{A}_d[\mathtt{y},\mathtt{z}] := \mathcal{A}_d[\mathtt{y}_{d+2},\dots,\mathtt{y}_n,\mathtt{z}_{d+2},\dots,\mathtt{z}_n]$.

As we can write $\Delta_j\mathtt{x}_j = Q_j + (\Delta_j\mathtt{x}_j - Q_j)$, we deduce

$$\begin{aligned} R(\Delta\mathtt{x}_{d+2},\dots,\Delta\mathtt{x}_n) &- R(Q_{d+2},\dots,Q_n) \\ &= \sum_{j=d+2}^{n} (\Delta_j\mathtt{x}_j - Q_j)S_j(Q_j,(\Delta_j\mathtt{x}_j - Q_j)) \in \mathfrak{a}. \quad (3.7)\end{aligned}$$

Consequently, by (3.6) and (3.7)

$$\begin{aligned}\Delta^q f - R(Q_{d+2},\dots,Q_n) &= \Delta^q f - R(\Delta\mathtt{x}_{d+2},\dots,\Delta\mathtt{x}_n) \\ &\quad + R(\Delta\mathtt{x}_{d+2},\dots,\Delta\mathtt{x}_n) - R(Q_{d+2},\dots,Q_n) \in \mathfrak{a}.\end{aligned}$$

Observe that $R(Q_{d+2},\dots,Q_n) \in \mathcal{A}_d[\mathtt{x}_{d+1}]$. We divide $R(Q_{d+2},\dots,Q_n)$ by P_{d+1} and obtain $A_{d+1}, R^* \in \mathcal{A}_d[\mathtt{x}_{d+1}]$ such that $\deg_{\mathtt{x}_{d+1}}(R^*) < p$ and

$$R(Q_{d+2},\dots,Q_n) = P_{d+1}A_{d+1} + R^*.$$

Thus, $\Delta^q f - R^* \in \mathfrak{a} \subset \mathfrak{p}$. As $f \in \mathfrak{p}$, also $R^* \in \mathfrak{p}$, so $R^*(\mathtt{x}',\theta) = 0$. As $\deg_{\mathtt{x}_{d+1}}(R^*) < p = \deg_{\mathtt{x}_{d+1}}(P_{d+1})$ and P_{d+1} is the irreducible polynomial of θ over K, we deduce that $R^* = 0$, so $\Delta^q f \in \mathfrak{a}$.

Recall that $[\mathfrak{a} : \Delta^q] = \{f \in \mathcal{A}_n : \Delta^q f \in \mathfrak{a}\}$. As $\Delta^q\mathfrak{p} \subset \mathfrak{a}$, we have $\mathfrak{p} \subset [\mathfrak{a} : \Delta^q]$. Now, pick $f \in [\mathfrak{a} : \Delta^q]$. Then $f\Delta^q \in \mathfrak{a} \subset \mathfrak{p}$. As $\mathfrak{p}$ is a prime ideal and $\Delta \notin \mathfrak{p}$, we have $f \in \mathfrak{p}$. Consequently, $\mathfrak{p} = [\mathfrak{a} : \Delta^q]$.

(vi) Let $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ be the primary decomposition of $\mathfrak{a}$. Define

$$\mathfrak{b} := \bigcap_{\Delta\in\sqrt{\mathfrak{q}_i}} \mathfrak{q}_i \quad \text{and} \quad \mathfrak{b}' := \bigcap_{\Delta\notin\sqrt{\mathfrak{q}_i}} \mathfrak{q}_i.$$

Let us prove that $\mathfrak{p} \subset \mathfrak{b}'$, that is, $\mathfrak{p} \subset \mathfrak{q}_i$ for each index i such that $\Delta \notin \sqrt{\mathfrak{q}_i}$. As $\Delta^q\mathfrak{p} \subset \mathfrak{a}$, we have $\Delta^q\mathfrak{p} \subset \mathfrak{q}_i$. As each $\mathfrak{q}_i$ is a primary ideal, either $\Delta \in \sqrt{\mathfrak{q}_i}$

or $\mathfrak{p} \subset \mathfrak{q}_i$, so $\mathfrak{p} \subset \mathfrak{b}'$. Thus, $\mathfrak{b} \cap \mathfrak{p} \subset \mathfrak{b} \cap \mathfrak{b}' = \mathfrak{a} \subset \mathfrak{b} \cap \mathfrak{p}$, so $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{p}$. In addition, $\Delta \in \sqrt{\mathfrak{b}} \setminus \mathfrak{p}$. □

Remarks II.3.2 (i) By Rückert's Division Theorem we have a canonical isomorphism

$$\mathcal{A}_d[\mathtt{x}_{d+1}]/(P_{d+1}\mathcal{A}_d[\mathtt{x}_{d+1}]) \to A' := \mathcal{A}_{d+1}/(P_{d+1}\mathcal{A}_{d+1})$$

induced by the surjective homomorphism

$$\mathcal{A}_d[\mathtt{x}_{d+1}] \hookrightarrow \mathcal{A}_{d+1} \to \mathcal{A}_{d+1}/(P_{d+1}\mathcal{A}_{d+1}),$$

whose kernel is the ideal $P_{d+1}\mathcal{A}_d[\mathtt{x}_{d+1}]$.

(ii) The homomorphism $A' \to A := \mathcal{A}_n/\mathfrak{p}$ is an inclusion.

To that end we have to prove that $\mathcal{A}_{d+1} \cap \mathfrak{p} = P_{d+1}\mathcal{A}_{d+1}$. The inclusion $P_{d+1}\mathcal{A}_{d+1} \subset \mathcal{A}_{d+1} \cap \mathfrak{p}$ is clear. To prove the converse pick $f \in \mathcal{A}_{d+1} \cap \mathfrak{p}$. By Rückert's Division Theorem there exist $Q \in \mathcal{A}_{d+1}$ and $R \in \mathcal{A}_d[\mathtt{x}_{d+1}]$ with $\deg_{\mathtt{x}_{d+1}}(R) < \deg_{\mathtt{x}_{d+1}}(P_{d+1})$ such that $f = QP_{d+1} + R$. As $f \in \mathfrak{p}$, we deduce that $R(\mathtt{x}', \theta) = 0$, so $R = 0$ because P_{d+1} is the irreducible polynomial of θ over K. Thus, $f \in P_{d+1}\mathcal{A}_{d+1}$ and $\mathcal{A}_{d+1} \cap \mathfrak{p} = P_{d+1}\mathcal{A}_{d+1}$.

(iii) The inclusion of integral domains $A' \hookrightarrow A$ induces a new inclusion $A'_\Delta \hookrightarrow A_\Delta$, which is in fact and isomorphism. To that end, we have to prove that its image contains A.

Pick $f \in A$. By Theorem II.2.17 (v) $\Delta f \in \mathcal{A}_d + \mathcal{A}_d\theta + \cdots + \mathcal{A}_d\theta^{p-1}$, so $\Delta f + \mathfrak{p} = P(\mathtt{x}_{d+1}) + \mathfrak{p}$ for some $P \in \mathcal{A}_d[\mathtt{x}_{d+1}]$ and

$$f = \frac{P}{\Delta} \in A'_\Delta,$$

as required.

As a consequence of the local parameterization theorem, we obtain:

Theorem II.3.3 (Abhyankar) *Let $\mathfrak{p}$ be a prime ideal of $\mathcal{A}_n$. Then the local ring $(\mathcal{A}_n)_\mathfrak{p}$ is a regular local ring.*

Proof. We use the notations of Theorem II.3.1. After a linear change of coordinates the ideal $\mathfrak{a}$ generates $\mathfrak{p}(\mathcal{A}_n)_\mathfrak{p}$, because $\Delta \notin \mathfrak{p}$ and consequently it is a unit in $(\mathcal{A}_n)_\mathfrak{p}$. Thus,

$$P_{d+1}, \Delta\mathtt{x}_{d+2} - Q_{d+2}, \ldots, \Delta\mathtt{x}_n - Q_n$$

is a set of $n-d$ generators of the maximal ideal of $(\mathcal{A}_n)_\mathfrak{p}$, so $\delta((\mathcal{A}_n)_\mathfrak{p}) \leq n-d$. As

$$\delta((\mathcal{A}_n)_\mathfrak{p}) \leq n-d = r = \mathrm{ht}(\mathfrak{p}) = \dim((\mathcal{A}_n)_\mathfrak{p}) \leq \delta((\mathcal{A}_n)_\mathfrak{p}),$$

the local ring $(\mathcal{A}_n)_\mathfrak{p}$ is regular by Definition II.2.6, as required. □

Exercises

Number II.1 Show that the ring $\mathcal{O}(\mathbb{C})$ of analytic functions over $\mathbb{C}$ is not noetherian.

Number II.2 Let A be a unique factorization domain and $\mathfrak{p}$ a non-zero prime ideal of A. Show that the following assertions are equivalent:

(i) $\mathfrak{p}$ has height one.

(ii) $\mathfrak{p}$ is a principal ideal.

Number II.3 Find a noetherian ring A and a non-zero prime ideal $\mathfrak{p}$ of A of height zero.

Number II.4 Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $K = \mathrm{qf}(\mathcal{A}_n)$. Does there exist $\xi \in K$ such that $P(\xi) = 0$, where $P := \mathtt{t}^n + \sum_{j=1}^n \mathtt{x}_j \mathtt{t}^{n-j}$?

Number II.5 Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $P \in \mathbb{K}[\mathtt{x}_1, \ldots, \mathtt{x}_n]$ be an irreducible polynomial such that $P(0) = 0$. Show that the ideal $P\mathcal{A}_n$ is radical.

Number II.6 Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be pairwise disjoint prime ideals of $\mathcal{A}_n$. Denote $\mathfrak{a} := \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ and $A := \mathcal{A}_n/\mathfrak{a}$. Show that the the following assertions are equivalent:

(i) $\mathrm{ht}(\mathfrak{p}_i) = \mathrm{ht}(\mathfrak{p}_j)$ for $1 \leq i, j \leq r$.

(ii) For each prime ideal $\mathfrak{P}$ in A, $\dim(A) = \dim(A/\mathfrak{P}) + \mathrm{ht}(\mathfrak{P})$.

Number II.7 Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\varphi : \mathcal{A}_1 \to \mathcal{A}_1$ a homomorphism of $\mathbb{K}$-algebras. Show that φ is an isomorphism of $\mathbb{K}$-algebras if and only if $\omega(\varphi(\mathtt{t})) = 1$.

Number II.8 Let $\mathbb{K} = \mathbb{C}$ and define $A := \mathcal{A}_n/\mathfrak{a}$ where $\mathfrak{a}$ is an ideal of $\mathcal{A}_n$. Let $\varphi : A \to \mathcal{A}_1$ be a $\mathbb{K}$-algebra homomorphism. Let $f \in A$ be such that $p := \omega(\varphi(f)) > 0$. Show that there exists a $\mathbb{K}$-algebra homomorphism $\psi : \mathcal{A}_1 \to \mathcal{A}_1$ such that $(\psi \circ \varphi)(f) = \mathtt{t}^p$.

Number II.9 Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and denote $\mathbb{Q}\{\mathtt{t}\} := \mathbb{K}\{\mathtt{t}\} \cap \mathbb{Q}[[\mathtt{t}]]$. Denote $E := \mathrm{qf}(\mathbb{Q}\{\mathtt{t}\})$ and $K := \mathrm{qf}(\mathbb{K}\{\mathtt{t}\})$. Show that:

(i) $\mathrm{trans\,deg}(E|\mathbb{Q}(\mathtt{t})) = +\infty$ and $\mathrm{trans\,deg}(K|\mathbb{K}(\mathtt{t})) = +\infty$.

(ii) For each $n \geq 1$ there exists an injective $\mathbb{K}$-algebra homomorphism $\varphi : \mathcal{O}_n \to \mathcal{O}_2$.

(iii) If $\varphi : A \to \mathbb{K}\{\mathtt{t}\}$ is an injective $\mathbb{K}$-algebra homomorphism, then $\dim(A) \leq 1$.

Number II.10 Show that $e^{\mathtt{t}} := \sum_{k\geq 0} \frac{1}{k!}\mathtt{t}^k \in \mathbb{R}\{\mathtt{t}\}$ is not algebraic over $\mathbb{R}(\mathtt{t})$.

Number II.11 Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\mathfrak{a} := (\mathtt{x}_1^2 - \mathtt{x}_2^3)\mathcal{A}_2$, $\mathfrak{b} := (\mathtt{x}_2 - \mathtt{x}_1^2)\mathcal{A}_2$, $A := \mathcal{A}_2/\mathfrak{a}$ and $B := \mathcal{A}_2/\mathfrak{b}$. Determine if A and B are regular local rings.

Number II.12 Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\varphi : \mathcal{A}_4 \to \mathcal{A}_2$ the $\mathbb{K}$-algebra homomorphism given by $\varphi(f) := f(\mathtt{y}_1^3, \mathtt{y}_1^2\mathtt{y}_2, \mathtt{y}_1\mathtt{y}_2^2, \mathtt{y}_2^3)$ and $\mathfrak{p} := \ker(\varphi)$. Show that:

(i) $\mathfrak{p}$ is a prime ideal of $\mathcal{A}_4$ of height 2.

(ii) $\mathfrak{p} = \{\mathtt{x}_1\mathtt{x}_3 - \mathtt{x}_2^2, \mathtt{x}_1\mathtt{x}_4 - \mathtt{x}_2\mathtt{x}_3, \mathtt{x}_2\mathtt{x}_4 - \mathtt{x}_3^2\}\mathcal{A}_4$.

(iii) $\mathfrak{p}$ does not admit a system of generators of cardinality ≤ 2.

Number II.13 Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\varphi : \mathcal{A}_3 \to \mathcal{A}_1$, $f \mapsto f(\mathtt{t}^3, \mathtt{t}^4, \mathtt{t}^5)$. Find a system of generators of the ideal $\mathfrak{p} := \ker(\varphi)$. Show that $\mathfrak{p}$ is a prime ideal, compute $r := \mathrm{ht}(\mathfrak{p})$ and prove that $\mathfrak{p}$ does not admit a system of generators of cardinal r.

Mather's Finitennes Theorem and Jacobian Criteria

We devote the first part of this chapter to study finite $\mathbb{K}$-algebra homomorphisms in our setting and to prove Mather's Finiteness Theorem for categories of analytic and formal rings over $\mathbb{R}$ and $\mathbb{C}$. This result allows us to characterize the finiteness of $\mathbb{K}$-algebra homomorphisms when working with modules over analytic and formal rings. In the second part of this chapter we afford some Jacobian criteria. As in the precedent chapter, [H, L, AM] contain the needed background.

1 Mather's Finitennes Theorem

Once more we set $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$ and $\mathcal{A}_n := \mathcal{O}_n$ or $\mathcal{F}_n$.

Definitions III.1.1 Let $\varphi : A \to B$ be a local homomorphism of local rings.

(i) φ is said to be *quasifinite* if the quotient $B/\varphi(\mathfrak{m}_A)B$ has finite dimension as linear space over $\mathbb{K} = A/\mathfrak{m}_A$ (via φ).

(ii) φ is said to be *finite* if B is a finitely generated A-module (via φ).

1.a Equivalence of finiteness and quasifiniteness. We will see that both concepts are equivalent in our setting as a consequence of the following result:

Theorem III.1.2 (Mather's Finiteness Theorem) *Let $\mathfrak{a}$ and $\mathfrak{b}$ be respective ideals of $\mathcal{A}_n$ and $\mathcal{A}_m$. Denote $A := \mathcal{A}_n/\mathfrak{a}$ and $B := \mathcal{A}_m/\mathfrak{b}$. Let $\varphi : A \to B$ be a $\mathbb{K}$-algebra homomorphism. If M is a finitely generated B-module and $M/\varphi(\mathfrak{m}_A)M$ has finite dimension as linear space over $\mathbb{K} = A/\mathfrak{m}_A$, then M is a finitely generated A-module.*

Proof. The proof is conducted in several steps.

1.a.1 Assume first that $A := \mathcal{A}_n$, $B := \mathcal{A}_{n+1}$ and $\varphi : \mathcal{A}_n \hookrightarrow \mathcal{A}_{n+1}$ is the inclusion. Let M be a finitely generated B-module such that $M/\mathfrak{m}_n M$ is a finitely generated linear space over $\mathbb{K} = \mathcal{A}_n/\mathfrak{m}_n$. Let $m_1, \dots, m_s$ be a system of generators of M as $\mathcal{A}_{n+1}$-module such that the classes $m_1 + \mathfrak{m}_n M, \dots, m_s + \mathfrak{m}_n M$ generate the finitely generated $\mathbb{K}$-linear space $M/\mathfrak{m}_n M$. Write $\mathtt{y} := \mathtt{x}_{n+1}$ and consider the element $\mathtt{y} m_i + \mathfrak{m}_n M$ of $M/\mathfrak{m}_n M$. As $m_1, \dots, m_s$ is a system of generators of M as $\mathcal{A}_{n+1}$-module, each element $h \in \mathfrak{m}_n M$ can be written as $h = \sum_{j=1}^s h_j m_j$ where $h_j \in \mathfrak{m}_n \mathcal{A}_{n+1}$. Consequently, there exist $c_{ij} \in \mathbb{K}$ and $h_{ij} \in \mathfrak{m}_n \mathcal{A}_{n+1}$ such that

$$\mathtt{y} m_i = -\sum_{j=1}^s c_{ij} m_j - \sum_{j=1}^s h_{ij} m_j$$

for $i = 1, \dots, s$. Thus,

$$\mathtt{y} m_i + \sum_{j=1}^s (c_{ij} + h_{ij}) m_j = 0$$

and we write

$$\begin{pmatrix} \mathtt{y} + c_{11} + h_{11} & c_{12} + h_{12} & \cdots & c_{1s} + h_{1s} \\ c_{21} + h_{21} & \mathtt{y} + c_{22} + h_{22} & \cdots & c_{2s} + h_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{s1} + h_{s1} & c_{s2} + h_{s2} & \cdots & \mathtt{y} + c_{ss} + h_{ss} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Write $C := (c_{ij})_{1 \le i,j \le s}$, $H := (h_{ij})_{1 \le i,j \le s}$ and $m := (m_1, \dots, m_s)$. We summarize the previous system as $(\mathtt{y} I_s + C + H) m^t = 0$. We multiply the previous system by $\operatorname{Adj}((\mathtt{y} I_s + C + H)^t)$ and we obtain $\det(\mathtt{y} I_s + C + H) m^t = 0$. Write $\Phi := \det(\mathtt{y} I_s + C + H) \in \mathcal{A}_{n+1}$, so we have $\Phi m_i = 0$ for $i = 1, \dots, s$. As $h_{ij}(0, \mathtt{y}) = 0$ for $1 \le i, j \le s$, we have $\Phi(0, \mathtt{y}) = \det(\mathtt{y} I_s + C) = P(-\mathtt{y})$ where $P \in \mathbb{K}[\mathtt{t}]$ is the characteristic polynomial of C. Thus, $\Phi \neq 0$ is regular of order $p \le s$ with respect to $\mathtt{y}$. Pick $m \in M$ and let $f_1, \dots, f_s \in \mathcal{A}_{n+1}$ be such that $m = \sum_{i=1}^s f_i m_i$. By Rückert's Division Theorem there exist $Q_i \in \mathcal{A}_{n+1}$ and $R_i := \sum_{j=0}^{p-1} b_{ij} \mathtt{y}^j \in \mathcal{A}_n[\mathtt{y}]$ with $b_{ij} \in \mathcal{A}_n$ and $f_i = Q_i \Phi + R_i$. As $\Phi m_i = 0$,

$$m = \sum_{i=1}^s f_i m_i = \sum_{i=1}^s (Q_i \Phi + R_i) m_i = \sum_{i=1}^s R_i m_i = \sum_{i=1}^s \sum_{j=0}^{p-1} b_{ij} \mathtt{y}^j m_i.$$

Thus, $\{\mathtt{y}^j m_i : \ i = 1, \dots, s, \ j = 0, \dots, p-1\}$ is a finite system of generators of M as $\mathcal{A}_n$-module.

1.a.2 We approach next the general case: $A := \mathcal{A}_n/\mathfrak{a}$ and $B := \mathcal{A}_m/\mathfrak{b}$. As usual $\mathtt{x} := (\mathtt{x}_1, \ldots, \mathtt{x}_n)$ and $\mathtt{y} := (\mathtt{y}_1, \ldots, \mathtt{y}_m)$. Let $\mathcal{A}_n := \mathbb{K}\langle\langle \mathtt{x} \rangle\rangle$, $\mathcal{A}_m := \mathbb{K}\langle\langle \mathtt{y} \rangle\rangle$ and $\mathcal{A}_{n+j} := \mathbb{K}\langle\langle \mathtt{x}, \mathtt{y}_1, \ldots, \mathtt{y}_j \rangle\rangle$ and $j = 1, \ldots, m$. Denote

$$\mathtt{j}_j : \mathcal{A}_{n+j-1} \hookrightarrow \mathcal{A}_{n+j}$$

the inclusion. Let $\pi_1 : \mathcal{A}_n \to \mathcal{A}_n/\mathfrak{a}$ and $\pi_2 : \mathcal{A}_m \to \mathcal{A}_m/\mathfrak{b}$ be the canonical projections. Let $g_i \in \mathfrak{m}_m$ be such that $\varphi(\mathtt{x}_i + \mathfrak{a}) = g_i + \mathfrak{b}$ for $i = 1, \ldots, n$. Define $\psi : \mathcal{A}_{n+m} \to \mathcal{A}_m,\ f \mapsto f(g_1, \ldots, g_n, \mathtt{y}_1, \ldots, \mathtt{y}_m)$ which is a surjective homomorphism such that $\varphi \circ \pi_1 = \pi_2 \circ \psi \circ \mathtt{j}_m \circ \cdots \circ \mathtt{j}_1$. We have the commutative diagram:

$$\begin{tikzcd}
\mathcal{A}_n \arrow[r, hook, "\mathtt{j}_1"] \arrow[d, "\pi_1"'] & \cdots \arrow[r, hook, "\mathtt{j}_{j-1}"] & \mathcal{A}_{n+j-1} \arrow[r, hook, "\mathtt{j}_j"] & \mathcal{A}_{n+j} \arrow[r, hook, "\mathtt{j}_{j+1}"] & \cdots \arrow[r, hook, "\mathtt{j}_m"] & \mathcal{A}_{n+m} \arrow[r, "\psi"] \arrow[dr, "\pi_2 \circ \psi"'] & \mathcal{A}_m \arrow[d, "\pi_2"] \\
\mathcal{A}_n/\mathfrak{a} \arrow[rrrrrr, "\varphi"] & & & & & & \mathcal{A}_m/\mathfrak{b}
\end{tikzcd}$$

where $\pi_2 \circ \psi$ is a surjective homomorphism. Let M be a finitely generated B-module. As $\pi_2 \circ \psi$ is a surjective homomorphism, M is a finitely generated $\mathcal{A}_{n+m}$-module.

1.a.3 We claim: $M/(\pi_2 \circ \psi \circ \mathtt{j}_m \circ \cdots \circ \mathtt{j}_j)(\mathfrak{m}_{n+j-1})M$ *is a finitely generated* $\mathbb{K}$*-linear space for* $j = 1, \ldots, m$.

Observe that $\varphi(\mathfrak{m}_A) = \varphi \circ \pi_1(\mathfrak{m}_n) \subset (\pi_2 \circ \psi \circ \mathtt{j}_m \circ \cdots \circ \mathtt{j}_j)(\mathfrak{m}_{n+j-1})$ for $j = 1, \ldots, m$. Thus, the map

$$\begin{aligned} M/\varphi(\mathfrak{m}_A)M &\to M/(\pi_2 \circ \psi \circ \mathtt{j}_m \circ \cdots \circ \mathtt{j}_j)(\mathfrak{m}_{n+j-1})M, \\ m + \varphi(\mathfrak{m}_A)M &\mapsto m + (\pi_2 \circ \psi \circ \mathtt{j}_m \circ \cdots \circ \mathtt{j}_j)(\mathfrak{m}_{n+j-1})M \end{aligned}$$

is a well-defined surjective homomorphism. As $M/\varphi(\mathfrak{m}_A)M$ is a finitely generated $\mathbb{K}$-linear space, so is $M/(\pi_2 \circ \psi \circ \mathtt{j}_m \circ \cdots \circ \mathtt{j}_j)(\mathfrak{m}_{n+j-1})M$ for $j = 1, \ldots, m$.

1.a.4 As M is a finitely generated $\mathcal{A}_{n+m}$-module, we deduce recursively using 1.a.1 and 1.a.3 that M is a finitely generated $\mathcal{A}_n$-module. This structure is induced by $\varphi \circ \pi_1$, so M is a finitely generated A-module, as required. □

Corollary III.1.3 *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be respective ideals of* $\mathcal{A}_n$ *and* $\mathcal{A}_m$*. Denote* $A := \mathcal{A}_n/\mathfrak{a}$ *and* $B := \mathcal{A}_m/\mathfrak{b}$*. Let* $\varphi : A \to B$ *be a* $\mathbb{K}$*-algebra homomorphism. The following assertions are equivalent:*

(i) φ *is finite.*

(ii) φ *is quasi-finite.*

Proof. If φ is quasi-finite, then by Mather's Finiteness Theorem B is a finitely generated A-module and φ is finite. Assume now that φ is finite. Then there exist $b_1, \dots, b_m \in B$ such that $B = b_1\varphi(A) + \cdots + b_m\varphi(A)$. Fix $b \in B$. As $A = \mathbb{K} \oplus \mathfrak{m}_A$, there exist $\lambda_i \in \mathbb{K}$ and $f_i \in \mathfrak{m}_A$ such that

$$b = \sum_{j=1}^{m} b_j(\lambda_j + \varphi(f_j)) = \sum_{j=1}^{m} \lambda_j b_j + \sum_{j=1}^{m} b_j\varphi(f_j).$$

Thus, $b+\varphi(\mathfrak{m}_A)B = \sum_{j=1}^{m} \lambda_j(b_j+\varphi(\mathfrak{m}_A)B)$ and $\{b_j+\varphi(\mathfrak{m}_A)B : \ j = 1, \dots, m\}$ generates $B/(\varphi(\mathfrak{m}_A)B)$ as a $\mathbb{K}$-linear space. Consequently,

$$\dim_K(B/\varphi(\mathfrak{m}_A)B) < +\infty$$

and φ is quasi-finite, as required. $\square$

1.b Consequences of Mather's Finiteness Theorem.

Let us present some consequences of Mather's Finiteness Theorem. We begin with a useful finiteness criterion that follows from Nakayama's Lemma, which we recall next.

Lemma III.1.4 (Nakayama) *Let $(A, \mathfrak{m})$ be a local ring and let M be a finitely generated A-module. We have:*

(i) *If $M = \mathfrak{m}M$, then $M = 0$.*

(ii) *If N is an A-submodule of M and $M = N + \mathfrak{m}M$, then $M = N$.*

Lemma III.1.5 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be respective ideals of $\mathcal{A}_n$ and $\mathcal{A}_m$. Let $\mathfrak{m}_A$ and $\mathfrak{m}_B$ be the respective maximal ideals of the analytic algebras $A := \mathcal{A}_n/\mathfrak{a}$ and $B := \mathcal{A}_m/\mathfrak{b}$. Let $\varphi : A \to B$ be a $\mathbb{K}$-algebra homomorphism. The following assertions are equivalent:*

(i) φ *is finite.*

(ii) $\mathfrak{m}_B = \sqrt{\mathfrak{B}}$ *where* $\mathfrak{B} := \varphi(\mathfrak{m}_A)B$.

Proof. Assume φ is finite. Then φ is quasifinite, so $\dim_{\mathbb{K}}(B/\mathfrak{B}) < +\infty$ and the chain of B-modules

$$\mathfrak{B} \subset \cdots \subset \mathfrak{m}_B^{k+1} + \mathfrak{B} \subset \mathfrak{m}_B^k + \mathfrak{B} \subset \cdots \subset \mathfrak{m}_B + \mathfrak{B} \subset B$$

has to be finite. Thus, there exists $k \geq 1$ such that $\mathfrak{m}_B^{k+1} + \mathfrak{B} = \mathfrak{m}_B^k + \mathfrak{B}$, so $\mathfrak{m}_B^k + \mathfrak{B} = \mathfrak{m}_B(\mathfrak{m}_B^k + \mathfrak{B}) + \mathfrak{B}$. As the ideal $\mathfrak{m}_B^k + \mathfrak{B}$ is a finitely generated B-module, $\mathfrak{B} = \mathfrak{m}_B^k + \mathfrak{B}$ by Nakayama's Lemma. Consequently, $\mathfrak{m}_B^k \subset \mathfrak{B}$ and $\mathfrak{m}_B \subset \sqrt{\mathfrak{B}}$. Thus, statement (ii) follows.

Suppose now that the equality $\mathfrak{m}_B = \sqrt{\mathfrak{B}}$ holds. As the ring B is noetherian, $\mathfrak{m}_B^k \subset \mathfrak{B}$ for some k. Thus,

$$\dim(B/\mathfrak{B}) \leq \dim(B/\mathfrak{m}_B^k). \tag{1.1}$$

The canonical projection $\pi : \mathcal{A}_m \to B$ is surjective. Notice that $\pi(\mathfrak{m}_m^k) = \mathfrak{m}_B^k$ and by the third isomorphism theorem

$$B/\mathfrak{m}_B^k \cong (\mathcal{A}_m/\mathfrak{b})/((\mathfrak{m}_m^k + \mathfrak{b})/\mathfrak{b}) \cong \mathcal{A}_m/(\mathfrak{m}_m^k + \mathfrak{b}).$$

As $\mathfrak{m}_m^k \subset \mathfrak{m}_m^k + \mathfrak{b}$, there exists a surjective homomorphism

$$\mathcal{A}_m/\mathfrak{m}_m^k \to \mathcal{A}_m/(\mathfrak{m}_m^k + \mathfrak{b}) \cong B/\mathfrak{m}_B^k.$$

Thus, $\dim_{\mathbb{K}}(B/\mathfrak{m}_B^k) \leq \dim_{\mathbb{K}}(\mathcal{A}_m/\mathfrak{m}_m^k) < +\infty$ because $\mathcal{A}_m/\mathfrak{m}_m^k$ is isomorphic to the $\mathbb{K}$-linear space $\mathcal{P}_{m,k}(\mathbb{K})$ of polynomials of degree $< k$ in m variables with coefficients in $\mathbb{K}$, which has finite dimension over $\mathbb{K}$. By (1.1) we conclude that φ is quasifinite, so φ is finite, as required. $\square$

Corollary III.1.6 *Let $\mathfrak{a}$ be an ideal of $\mathcal{A}_n$ and denote $A := \mathcal{A}_n/\mathfrak{a}$. A nontrivial $\mathbb{K}$-algebra homomorphism $\varphi : A \to \mathcal{A}_1$ is finite.*

Proof. By Lemma III.1.5 we have to prove that $\sqrt{\varphi(\mathfrak{m}_A)\mathcal{A}_1} = \mathfrak{m}_1$. Let $f \in \varphi(\mathfrak{m}_A) \setminus \{0\}$ and write $\omega(f) = d < +\infty$. Thus, we can write $f = \mathtt{t}^d u(\mathtt{t})$, where $u \in \mathcal{A}_1$ is a unit. Thus, $\mathtt{t}^d \in \varphi(\mathfrak{m}_A)\mathcal{A}_1$, so $\mathfrak{m}_1 = \{\mathtt{t}\}\mathcal{A}_1 \subset \sqrt{\varphi(\mathfrak{m}_A)\mathcal{A}_1}$, as required. $\square$

The following result follows from Lemma III.1.5 and Nakayama's Lemma and generalizes Corollary II.1.10(ii).

Corollary III.1.7 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be respective ideals of $\mathcal{A}_n$ and $\mathcal{A}_m$. Denote $A := \mathcal{A}_n/\mathfrak{a}$ and $B := \mathcal{A}_m/\mathfrak{b}$ and let $\mathfrak{m}_A$ and $\mathfrak{m}_B$ be their respective maximal ideals. Let $\varphi : A \to B$ be a $\mathbb{K}$-algebra homomorphism. The following assertions are equivalent:*

(i) *φ is surjective*

(ii) *$\mathfrak{m}_B = \varphi(\mathfrak{m}_A)B$.*

Proof. If φ is surjective then $\varphi(\mathfrak{m}_A) = \mathfrak{m}_B$ because φ is local (so $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$) and $\varphi(A \setminus \mathfrak{m}_A) \subset B \setminus \mathfrak{m}_B$.

If $\mathfrak{m}_B = \varphi(\mathfrak{m}_A)B$, then by Lemma III.1.5 φ is a finite $\mathbb{K}$-algebra homomorphism, so B is a finitely generated A-module. We have $B = \varphi(A) + \varphi(\mathfrak{m}_A)B$ because

$$B = \mathbb{K} + \mathfrak{m}_B \subset \varphi(A) + \varphi(\mathfrak{m}_A)B \subset B$$

and by Nakayama's Lemma $\varphi(A) = B$, as required. $\square$

The following result characterizes regular local rings of the type $\mathcal{A}_n/\mathfrak{a}$.

Lemma III.1.8 *A quotient $A := \mathcal{A}_n/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal of $\mathcal{A}_n$, is a regular local ring of dimension d if and only if it is isomorphic to $\mathcal{A}_d$.*

Proof. We have already seen in Theorem II.2.7 that $\mathcal{A}_d$ is a regular local ring of dimension d. Assume now that A is a regular local ring of dimension d. Then $\mathfrak{m}_A$ is generated by d elements $f_1, \dots, f_d$. Consider the unique $\mathbb{K}$-algebra homomorphism $\varphi : \mathcal{A}_d \to A$ such that $\varphi(\mathtt{x}_i) = f_i$ for $i = 1, \dots, d$. By Corollary III.1.7 the homomorphism φ is surjective. To show that φ is injective observe that by the first isomorphy theorem $A \cong \mathcal{A}_d/\ker(\varphi)$. Thus, $d = \dim(A) = \dim(\mathcal{A}_d/\ker(\varphi)) = d - \operatorname{ht}(\ker(\varphi))$, so $\operatorname{ht}(\ker(\varphi)) = 0$ and $\ker(\varphi) = (0)$ because $\mathcal{A}_d$ is an integral domain. Consequently, φ is also injective, as required. $\square$

2 Jacobian Criteria

In this section we present some Jacobian criteria to determine when a quotient $\mathcal{A}_n/\mathfrak{a}$ is a regular local ring.

Remark III.2.1 Let $\mathfrak{a}$ be an ideal of $\mathcal{A}_n$ such that the quotient $\mathcal{A}_n/\mathfrak{a}$ is a regular local ring of dimension d. By Lemma III.1.8 $\mathcal{A}_n/\mathfrak{a} \cong \mathcal{A}_d$, which is an integral domain. Thus, $\mathfrak{a}$ is a prime ideal.

Lemma III.2.2 *Let $f_1, \ldots, f_n \in \mathcal{A}_n$ be a system of generators of $\mathfrak{m}_n$ and let $r \leq n$. Define $\mathfrak{p} := \{f_1, \ldots, f_r\}\mathcal{A}_n$. Then $\mathfrak{p}$ is a prime ideal of height r and the quotient $\mathcal{A}_n/\mathfrak{p}$ is a regular local ring of dimension $n - r$.*

Proof. As $\mathfrak{m}_n = \{f_1, \ldots, f_n\}\mathcal{A}_n$, we deduce by Corollary II.1.12 that

$$\det\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\Big)_{1\leq i,j\leq n} \neq 0.$$

By Corollary II.1.10(i) the $\mathbb{K}$-algebra homomorphism

$$\varphi : \mathcal{A}_n \to \mathcal{A}_n,\ g \mapsto g(f_1, \ldots, f_n)$$

is an isomorphism. Write $\mathfrak{q} := \{\mathtt{x}_1, \ldots, \mathtt{x}_r\}\mathcal{A}_n$ and observe that $\varphi(\mathfrak{q}) = \mathfrak{p}$. Thus, it is enough to prove the statement for $\mathfrak{q}$. Inside the proof of Theorem II.2.7 we have shown that $\mathfrak{q}$ is a prime ideal and $\mathcal{A}_n/\mathfrak{q} \cong \mathcal{A}_{n-r}$, so

$$\mathrm{ht}(\mathfrak{q}) = \dim(\mathcal{A}_n) - \dim(\mathcal{A}_n/\mathfrak{q}) = n - (n - r) = r$$

and the statement holds. $\square$

Corollary III.2.3 *Let $f_1, \ldots, f_r \in \mathfrak{m}_n$ be such that* $\det(\frac{\partial f_i}{\partial \mathtt{x}_j}(0))_{1\leq i,j\leq r} \neq 0$. *Then $\mathfrak{p} := \{f_1, \ldots, f_r\}\mathcal{A}_n$ is a prime ideal of height r and $A := \mathcal{A}_n/\mathfrak{p}$ is a regular local ring of dimension $d := n - r$.*

Proof. Write $f_{r+1} := \mathtt{x}_{r+1}, \ldots, f_n := \mathtt{x}_n$ and observe that

$$\det\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\Big)_{1\leq i,j\leq n} \neq 0.$$

By Corollary II.1.12 $\mathfrak{m}_n = \{f_1, \ldots, f_n\}\mathcal{A}_n$. By Lemma III.2.2 the statement follows. $\square$

Let $(A, \mathfrak{m})$ be a local noetherian ring. The *embedding dimension* $\mathrm{edim}(A)$ *of* A is defined by $\mathrm{edim}(A) := \dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$.

Lemma III.2.4 *We have:*

(i) *The cardinal of a minimal system of generators of* $\mathfrak{m}$ *is* $e := \operatorname{edim}(A)$.

(ii) *If* $\{h_1, \dots, h_m\}$ *is a system of generators of* $\mathfrak{m}$*, we may assume that* $\{h_1, \dots, h_e\}$ *is a system of generators of* $\mathfrak{m}$.

Proof. (i) Let $\{f_1, \dots, f_s\}$ be a system of generators of $\mathfrak{m}$ and let $f \in \mathfrak{m}$. Then there exist $a_1, \dots, a_s \in A$ such that $f = a_1 f_1 + \dots + a_s f_s$. Thus,

$$f + \mathfrak{m}^2 = (a_1 + \mathfrak{m})(f_1 + \mathfrak{m}^2) + \dots + (a_s + \mathfrak{m})(f_s + \mathfrak{m}^2),$$

so $\{f_1 + \mathfrak{m}^2, \dots, f_s + \mathfrak{m}^2\}$ is a system of generators of $\mathfrak{m}/\mathfrak{m}^2$ as $A/\mathfrak{m}$-linear space and $s \geq \operatorname{edim}(A)$. Conversely, let now $g_1, \dots, g_e \in \mathfrak{m}$ be such that $\{g_1 + \mathfrak{m}^2, \dots, g_e + \mathfrak{m}^2\}$ is a basis of $\mathfrak{m}/\mathfrak{m}^2$ as $A/\mathfrak{m}$-linear space. Then each $f \in \mathfrak{m}$ can be written as $f = g_1 a_1 + \ldots + g_e a_e + h$ where $a_1, \dots, a_e \in A$ and $h \in \mathfrak{m}^2$. Thus, $\mathfrak{m} = \{g_1, \dots, g_e\}A + \mathfrak{m}\mathfrak{m}$. By Nakayama's Lemma $\mathfrak{m} = \{g_1, \dots, g_e\}A$, so $\{g_1, \dots, g_e\}$ is a minimal system of generators of A.

(ii) If $\{h_1, \dots, h_m\}$ is a system of generators of $\mathfrak{m}$, $\{h_1 + \mathfrak{m}^2, \dots, h_m + \mathfrak{m}^2\}$ is a system of generators of $\mathfrak{m}/\mathfrak{m}^2$ as $A/\mathfrak{m}$-linear space. As $\dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = e$, we may assume $\{h_1 + \mathfrak{m}^2, \dots, h_e + \mathfrak{m}^2\}$ is a basis of $\mathfrak{m}/\mathfrak{m}^2$ as $A/\mathfrak{m}$-linear space. As we have seen in (i), $\{h_1, \dots, h_e\}$ is a system of generators of $\mathfrak{m}$, as required. □

Corollary III.2.5 *Let* $(A, \mathfrak{m})$ *be local ring. Then* A *is a regular local ring if and only if* $\operatorname{edim}(A) = \dim(A)$.

As a consequence of Corollary II.1.12 and Lemma III.2.4, we can compute the embedding dimension of $\mathcal{A}_n$ and a criterion to decide if $f_1, \dots, f_r \in \mathfrak{m}_n$ constitute a system of generators.

Lemma III.2.6 *Let* $\mathfrak{m}_n$ *be the maximal ideal of* $\mathcal{A}_n$. *Then* $\mathfrak{m}_n/\mathfrak{m}_n^2$ *is a* $\mathbb{K}$-*linear space of dimension* n *and it is naturally isomorphic to the dual* $\mathbb{K}$-*linear space* $\mathbb{K}^{n,*}$.

Corollary III.2.7 *Let* $f_1, \dots, f_r \in \mathcal{A}_n$ *be a system of generators of* $\mathfrak{m}_n$. *Then*

$$\operatorname{rk}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\Big)_{\substack{1\leq i\leq r\\ 1\leq j\leq n}} = n.$$

We prove next the converse of Lemma III.2.2.

Lemma III.2.8 *Let $\mathfrak{p}$ be a prime ideal of $\mathcal{A}_n$ such that $\mathcal{A}_n/\mathfrak{p}$ is a regular local ring of dimension $d := n - r$. Then there exists a system of generators $f_1, \dots, f_n$ of $\mathfrak{m}_n$ such that $\mathfrak{p} = \{f_1, \dots, f_r\}\mathcal{A}_n$.*

Proof. Let us denote $A := \mathcal{A}_n/\mathfrak{p}$ and $\mathfrak{m}_A := \mathfrak{m}_n/\mathfrak{p}$ its maximal ideal. We have $\mathfrak{m}_A^2 = (\mathfrak{m}_n^2 + \mathfrak{p})/\mathfrak{p}$, so $\mathfrak{m}_A/\mathfrak{m}_A^2$ is by the third isomorphy theorem isomorphic to $\mathfrak{m}_n/(\mathfrak{m}_n^2+\mathfrak{p})$. In addition, A is isomorphic to $\mathcal{A}_d$ because A is a regular local ring of dimension d. Consequently, $\mathfrak{m}_A/\mathfrak{m}_A^2$ and $\mathfrak{m}_d/\mathfrak{m}_d^2$ are isomorphic as $\mathbb{K}$-linear spaces. As $\mathfrak{m}_d/\mathfrak{m}_d^2$ has by Lemma III.2.6 dimension d, the quotient $\mathfrak{m}_A/\mathfrak{m}_A^2$ is a $\mathbb{K}$-linear space of dimension d, so the same happens with $\mathfrak{m}_n/(\mathfrak{m}_n^2 + \mathfrak{p})$. By the third isomorphy theorem $\mathfrak{m}_n/(\mathfrak{m}_n^2 + \mathfrak{p})$ and $(\mathfrak{m}_n/\mathfrak{m}_n^2)/((\mathfrak{m}_n^2 + \mathfrak{p})/\mathfrak{m}_n^2)$ are isomorphic. Thus, by Lemma III.2.6

$$\begin{aligned}d &= \dim_{\mathbb{K}}((\mathfrak{m}_n/\mathfrak{m}_n^2)/((\mathfrak{m}_n^2 + \mathfrak{p})/\mathfrak{m}_n^2))\\ &= \dim_{\mathbb{K}}((\mathfrak{m}_n/\mathfrak{m}_n^2)) - \dim_{\mathbb{K}}((\mathfrak{m}_n^2 + \mathfrak{p})/\mathfrak{m}_n^2) = n - \dim_{\mathbb{K}}((\mathfrak{m}_n^2 + \mathfrak{p})/\mathfrak{m}_n^2),\end{aligned}$$

so $\dim_{\mathbb{K}}((\mathfrak{m}_n^2 + \mathfrak{p})/\mathfrak{m}_n^2) = n - d = r$. Let $f_1, \dots, f_r \in \mathfrak{p} \subset \mathfrak{m}_n$ whose classes module $\mathfrak{m}_n^2$ constitute a basis of $(\mathfrak{m}_n^2 + \mathfrak{p})/\mathfrak{m}_n^2$ as $\mathbb{K}$-linear subspace of $\mathfrak{m}_n/\mathfrak{m}_n^2$. Let $f_{r+1}, \dots, f_n \in \mathfrak{m}_n$ such that the classes of $f_1, \dots, f_n$ module $\mathfrak{m}_n^2$ constitute a basis of $\mathfrak{m}_n/\mathfrak{m}_n^2$. Thus, $\det(\frac{\partial f_i}{\partial \mathtt{x}_j}(0))_{1\le i,j\le n} \neq 0$ and by Corollary II.1.12 $\mathfrak{m}_n = \{f_1, \dots, f_n\}\mathcal{A}_n$. By Lemma III.2.2 the ideal $\mathfrak{q} := \{f_1, \dots, f_r\}\mathcal{A}_n$ is prime and has height r. As $\operatorname{ht}(\mathfrak{p}) = \dim(\mathcal{A}_n) - \dim(\mathcal{A}_n/\mathfrak{p}) = n - (n - r) = r = \operatorname{ht}(\mathfrak{q})$ and $\mathfrak{q} \subset \mathfrak{p}$, we have $\mathfrak{q} = \mathfrak{p}$. Thus, $\mathfrak{p} = \{f_1, \dots, f_r\}\mathcal{A}_n$ and $\mathfrak{m}_n = \{f_1, \dots, f_n\}\mathcal{A}_n$, as required. □

We present now the Jacobian criterion.

Theorem III.2.9 (Jacobian criterion) *Let $\mathfrak{a} := \{f_1, \dots, f_s\}\mathcal{A}_n$ for some $f_1, \dots, f_s \in \mathcal{A}_n$. Denote*

$$r := \operatorname{rk}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\Big)_{\substack{1\le i\le s\\ 1\le j\le n}}.$$

Then $\operatorname{edim}(\mathcal{A}_n/\mathfrak{a}) + r = n$.

Proof. Write $A := \mathcal{A}_n/\mathfrak{a}$ and $e := \operatorname{edim}(A)$. Let $g_1, \dots, g_e \in \mathfrak{m}_n$ whose classes modulo $\mathfrak{a}$ generate $\mathfrak{m}_A = \mathfrak{m}_n/\mathfrak{a}$. Then $f_1, \dots, f_s, f_{s+1} := g_1, \dots, f_{s+e} := g_e$ is a system of generators of $\mathfrak{m}_n$. By Corollary III.2.7

$$\operatorname{rk}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\Big)_{\substack{1\leq i\leq s+e\\ 1\leq j\leq n}} = n$$

and consequently

$$r = \operatorname{rk}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\Big)_{\substack{1\leq i\leq s\\ 1\leq j\leq n}} \geq n-e.$$

We may assume that

$$\operatorname{rk}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\Big)_{\substack{1\leq i\leq r\\ 1\leq j\leq n}} = r.$$

By Corollary III.2.3 the ideal $\mathfrak{p} := \{f_1,\ldots,f_r\}\mathcal{A}_n$ is prime of height r and the quotient $\mathcal{A}_n/\mathfrak{p}$ is a regular local ring of dimension $d := n-r$. As $\mathfrak{p}\subset\mathfrak{a}$, the map

$$\varphi : B := \mathcal{A}_n/\mathfrak{p} \to A := \mathcal{A}_n/\mathfrak{a},\ \ f+\mathfrak{p}\mapsto f+\mathfrak{a}$$

is a well-defined surjective homomorphism. Thus, $\varphi(\mathfrak{m}_B) = \mathfrak{m}_A$ and $\operatorname{edim}(B)\geq \operatorname{edim}(A) = e$. As $B\cong\mathcal{A}_{n-r}$, by Lemma III.2.6 $e\leq\operatorname{edim}(B) = n-r$, so $r\leq n-e$. Consequently, $r = n-e$, as required. □

Corollary III.2.10 *Let $f_1,\ldots,f_s\in\mathcal{A}_n$ and let $\mathfrak{a} := \{f_1,\ldots,f_s\}\mathcal{A}_n$. Denote*

$$r := \operatorname{rk}\Big(\frac{\partial f_i}{\partial \mathtt{y}_j}(0)\Big)_{\substack{1\leq i\leq s\\ 1\leq j\leq n}}.$$

Then $\mathcal{A}_n/\mathfrak{a}$ is a regular local ring if and only if $\operatorname{ht}(\mathfrak{a}) = r$.

Proof. The quotient $\mathcal{A}_n/\mathfrak{a}$ is a regular local ring if and only if $\operatorname{edim}(\mathcal{A}_n/\mathfrak{a}) = \dim(\mathcal{A}_n/\mathfrak{a})$. By the Jacobian criterion III.2.9 we have $\operatorname{edim}(\mathcal{A}_n/\mathfrak{a}) = n-r$. As $\dim(\mathcal{A}_n/\mathfrak{a}) = n-\operatorname{ht}(\mathfrak{a})$, we deduce that $\operatorname{edim}(\mathcal{A}_n/\mathfrak{a}) = \dim(\mathcal{A}_n/\mathfrak{a})$ if and only if $\operatorname{ht}(\mathfrak{a}) = r$, as required. □

2.a Generalized Jacobian criterion. Our next purpose is to provide a Jacobian criterion to determine when the localization at a prime ideal of an integral domain of the type $\mathcal{A}_n/\mathfrak{a}$ is a regular local ring. This result can be understood as the natural generalization of Abhyankar's Theorem II.3.3. For its proof we need some general results for arbitrary regular local rings whose proofs are more demanding than the corresponding ones for $\mathcal{A}_n$. A preliminary key result is Krull's Principal Ideal Theorem.

Theorem III.2.11 (Krull's Principal Ideal) *Let A be a noetherian ring and $f \in A$. Consider the quotient A/fA and let $\mathfrak{p}$ be an ideal of A such that $f \in \mathfrak{p}$ and $\mathfrak{p}/fA$ is a minimal prime ideal of A/fA. Then* $\mathrm{ht}(\mathfrak{p}) \leq 1$. *If in addition f does not belong to any minimal prime ideal of A, then* $\mathrm{ht}(\mathfrak{p}) = 1$ *and* $\dim(A/fA) \leq \dim(A) - 1$.

Proof. Let $fA = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ be an irredundant primary decomposition of fA and assume that $\sqrt{\mathfrak{q}_1} = \mathfrak{p}$. As the previous primary decomposition is irredundant and $\mathfrak{p}/fA$ is a minimal prime ideal of A/fA, we have $\mathfrak{p}_i \setminus \mathfrak{p} \neq \varnothing$ for $i = 2, \ldots, r$. Consider the local ring $A_\mathfrak{p}$, whose maximal ideal is $\mathfrak{p}A_\mathfrak{p}$. As $\mathfrak{p}_i \setminus \mathfrak{p} \neq \varnothing$ for $i = 2, \ldots, r$, we have $fA_\mathfrak{p} = \mathfrak{q}_1 A_\mathfrak{p}$ and $\sqrt{fA_\mathfrak{p}} = \mathfrak{p}A_\mathfrak{p}$. Thus, $\mathrm{ht}(\mathfrak{p}) = \dim(A_\mathfrak{p}) \leq \delta(fA_\mathfrak{p}) \leq 1$.

If f does not belong to any minimal prime ideal of A, then $\mathrm{ht}(\mathfrak{p}) > 0$, so $\mathrm{ht}(\mathfrak{p}) = 1$, as required. □

Lemma III.2.12 (Prime avoidance) *Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension ≥ 1 and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \subset A$ be prime ideals of A. Suppose that $\mathfrak{p}_i \neq \mathfrak{m}$ for $i = 1, \ldots, r$. Then $\mathfrak{m} \not\subset \mathfrak{m}^2 \cup \bigcup_{i=1}^r \mathfrak{p}_i$.*

Proof. We proceed by induction on r. If $r = 1$ and $\mathfrak{m} \subset \mathfrak{p}_1 \cup \mathfrak{m}^2 \subset \mathfrak{p}_1 + \mathfrak{m}^2$, then by Nakayama's Lemma $\mathfrak{m} \subset \mathfrak{p}_1$, which is a contradiction.

Suppose the result true for $r-1$ prime ideals and let us check that it is also true for r. Pick $x_i \in \mathfrak{m} \setminus (\mathfrak{m}^2 \cup \bigcup_{j \neq i} \mathfrak{p}_j)$. If $x_i \notin \mathfrak{p}_i$ for some i we are done. Otherwise, $x_i \in \mathfrak{p}_i$ for $i = 1, \ldots, r$ and define $x := x_1 + (x_2 \cdots x_r)^2 \in \mathfrak{m} \setminus \mathfrak{m}^2$ and let us check that $x \notin \mathfrak{p}_i$ for $i = 1, \ldots, r$. If $x \in \mathfrak{p}_1$, then $x_2 \cdots x_r \in \mathfrak{p}_1$, which is a contradiction. If $x \in \mathfrak{p}_i$ for $i = 2, \ldots, r$, then $x_1 \in \mathfrak{p}_i$, which is a contradiction. Consequently, $x \in \mathfrak{m} \setminus (\mathfrak{m}^2 \cup \bigcup_{i=1}^r \mathfrak{p}_i)$, as required. □

Recall that the minimal prime ideals of a noetherian ring correspond to the radical ideals occurring in an irredundant primary decomposition of the zero ideal.

Corollary III.2.13 *Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension ≥ 1 and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \subset A$ be the minimal prime ideals of A. Let $x_1 \in \mathfrak{m} \setminus \bigcup_{i=1}^r \mathfrak{p}_i$. Then there exist $x_2, \ldots, x_r \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that* $\dim(A/\mathfrak{a}_j) < \dim(A/\mathfrak{a}_{j-1})$ *for*

$j = 1, \dots, r$ *where*

$$\mathfrak{a}_j := \begin{cases} \{x_1, \dots, x_j\}A & \textit{if } j = 1, \dots, r, \\ 0 & \textit{if } j = 0 \end{cases}$$

and $\dim(A/\mathfrak{a}_r) = 0$.

Proof. We proceed by induction on the dimension n of A. If $\dim(A) = 1$, then by Krull's Principal Ideal Theorem $0 \le \dim(A/x_1A) \le \dim(A) - 1 = 0$ and we are done. Suppose the result true for dimension $< n$ and let us check that it is true for dimension $n \ge 1$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r \subset A$ be the minimal prime ideals of A. By Krull's Principal Ideal Theorem III.2.11 $\dim(A/x_1A) \le \dim(A) - 1$. Denote $B := A/x_1A$ and let $\mathfrak{m}_B = \mathfrak{m}/x_1A$ be its unique maximal ideal. If $\dim(B) = 0$, we are done. So we assume $\dim(B) \ge 1$. Note that $\mathfrak{m}_B^2 = (\mathfrak{m}^2 + x_1A)/x_1A$. By Lemma III.2.12 there exists $x_2 \in \mathfrak{m} \setminus (\mathfrak{m}^2 + x_1A)$ such that $x_2 + x_1A$ does not belong to any minimal prime ideal of B. As $\dim(B) \le n-1$, there exist by induction hypothesis $x_3, \dots, x_r \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $\dim(B/\mathfrak{b}_j) < \dim(B/\mathfrak{b}_{j-1})$ for $j = 2, \dots, r$ where

$$\mathfrak{b}_j := \begin{cases} \{x_1, \dots, x_j\}B & \text{if } j = 2, \dots, r, \\ 0 & \text{if } j = 0 \end{cases}$$

and $\dim(B/\mathfrak{b}_r) = 0$. By third isomorphy theorem the result follows. □

Lemma III.2.14 *Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension $n \ge 1$ and let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal prime ideals of A. For each $x_1 \in \mathfrak{m} \setminus \bigcup_{i=1}^r \mathfrak{p}_i$ we have* $\dim(A/x_1A) = \dim(A) - 1$.

Proof. By Lemma III.2.13 there exist $x_2, \dots, x_r \in \mathfrak{m}$ such that for $j = 1, \dots, r$ we have $\dim(A/\mathfrak{a}_j) < \dim(A/\mathfrak{a}_{j-1})$ where

$$\mathfrak{a}_j := \begin{cases} \{x_1, \dots, x_j\}A & \text{if } j = 1, \dots, r, \\ 0 & \text{if } j = 0 \end{cases}$$

and $\dim(A/\mathfrak{a}_r) = 0$. Thus, $\sqrt{\mathfrak{a}_r} = \mathfrak{m}$, so $n = \dim(A) \le \delta(\mathfrak{a}_r) \le r$. In addition,

$$0 = \dim(A/\mathfrak{a}_r) < \dim(A/\mathfrak{a}_{r-1}) < \cdots < \dim(A/\mathfrak{a}_1) < \dim(A) = n,$$

so $n \ge r$. Hence, $r = n$ and $\dim(A/\mathfrak{a}_1) = \dim(A) - 1$, as required. □

Corollary III.2.15 *Let* $(A, \mathfrak{m})$ *be a noetherian local integral domain of dimension* $n \geq 1$. *For each* $x \in \mathfrak{m} \setminus \{0\}$ *we have* $\dim(A/xA) = \dim(A) - 1$.

Theorem III.2.16 *Let* $(A, \mathfrak{m})$ *be a regular local ring. Then* A *is an integral domain.*

Proof. We work by induction on the Krull's dimension n of A. If $\dim(A) = 0$ then $\operatorname{edim}(A) = 0$, so $\mathfrak{m} = 0$ and $A \cong A/\mathfrak{m}$ is a field, which is an integral domain. Assume the result if $0 \leq \dim(A) < n$ and let us check that the result is also true for dimension n. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal prime ideals of A. If we prove that $\mathfrak{p}_i = (0)$ for some $i = 1, \ldots, r$, we have $i = 1 = r$ (because of minimality), so $A \cong A/\mathfrak{p}_1 = A/(0)$ is an integral domain.

As $\dim(A) > 0$, there exists by Lemma III.2.12 $x \in \mathfrak{m} \setminus (\mathfrak{m}^2 \cup \bigcup_{i=1}^r \mathfrak{p}_i)$. By Lemma III.2.14 $\dim(A/xA) = \dim(A) - 1$. Observe that

$$\begin{aligned}&\dim_{A/\mathfrak{m}}(\mathfrak{m}/(\mathfrak{m}^2 + xA))\\ &\qquad = \dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) - \dim_{A/\mathfrak{m}}((\mathfrak{m}^2 + xA)/\mathfrak{m}^2) = \dim(A) - 1.\end{aligned}$$

Thus, $A/(xA)$ is a regular local ring and by induction hypothesis it is an integral domain. Consequently, $\mathfrak{q} := xA$ is a prime ideal and it is not a minimal prime ideal because $x \notin \bigcup_{i=1}^r \mathfrak{p}_i$. We may assume that $\mathfrak{p}_1 \subsetneq \mathfrak{q}$. Let $y \in \mathfrak{p}_1$ and let $a \in A$ such that $y = ax$. As $\mathfrak{p}_1$ is prime and $x \notin \mathfrak{p}_1$, we have $a \in \mathfrak{p}_1$. Thus, $\mathfrak{p}_1 \subset x\mathfrak{p}_1 \subset \mathfrak{m}\mathfrak{p}_1 \subset \mathfrak{p}_1$, so $\mathfrak{m}\mathfrak{p}_1 = \mathfrak{p}_1$. By Nakayama's Lemma $\mathfrak{p}_1 = (0)$, as required. □

Corollary III.2.17 *Let* $(A, \mathfrak{m})$ *be a regular local ring of dimension* n *and* $x_1, \ldots, x_r \in \mathfrak{m}$. *The following conditions are equivalent:*

(i) *There exist* $x_{r+1}, \ldots, x_n \in A$ *such that* $\mathfrak{m} = \{x_1, \ldots, x_n\}A$.

(ii) *The classes of* $x_1, \ldots, x_r$ *in* $\mathfrak{m}/\mathfrak{m}^2$ *are linearly independent over* $A/\mathfrak{m}$.

(iii) $A/(\{x_1, \ldots, x_r\}A)$ *is a regular local ring of dimension* $n - r$.

Proof. (i) $\Longrightarrow$ (ii) The classes of $x_1, \ldots, x_n$ generate $\mathfrak{m}/\mathfrak{m}^2$ as $A/\mathfrak{m}$-linear space. As $\dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = \operatorname{edim}(A) = \dim(A) = n$, the previous classes are linearly independent over $A/\mathfrak{m}$, so the classes of $x_1, \ldots, x_r$ in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over $A/\mathfrak{m}$.

(ii) $\Longrightarrow$ (iii) Let $x_{r+1}, \dots, x_n \in \mathfrak{m}$ be such that the classes $x_1, \dots, x_n$ constitute a basis of $\mathfrak{m}/\mathfrak{m}^2$ as $A/\mathfrak{m}$-linear space. Then $\mathfrak{m} = \{x_1, \dots, x_n\}A$ by Nakayama's Lemma.

As A is an integral domain, then $\dim(A/x_1A) = \dim(A) - 1$ by Corollary III.2.15. The classes of $x_2, \dots, x_n$ generate the maximal ideal of A/x_1A, so A/x_1A is a regular local ring of dimension $n-1$. If we proceed recursively, $A/(\{x_1, \dots, x_r\}A)$ is a regular local ring of dimension $n-r$.

(iii) $\Longrightarrow$ (i) Let $\mathfrak{m}/(\{x_1, \dots, x_r\}A)$ be generated by the classes of certain elements $x_{r+1}, \dots, x_n \in \mathfrak{m}$. Then $\mathfrak{m} = \{x_1, \dots, x_n\}A$, as required. □

Corollary III.2.18 *Let $(A, \mathfrak{m})$ be a regular local ring of dimension n and let $\mathfrak{p}$ be a non-zero prime ideal of A. We have:*

(i) *If $\mathfrak{p} \subset \mathfrak{m}^2$, then $A/\mathfrak{p}$ is not a regular local ring.*

(ii) *If the quotient $A/\mathfrak{p}$ is a regular local ring of dimension $n-r$ there exist $x_1, \dots, x_r \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $\mathfrak{p} = \{x_1, \dots, x_r\}A$.*

Proof. (i) Write $\mathfrak{m} := \{x_1, \dots, x_n\}A$. As $\mathfrak{p}$ is a non-zero prime ideal and A is an integral domain, $\dim(B) < \dim(A)$ where $B := A/\mathfrak{p}$. In addition, its maximal ideal is $\mathfrak{m}_B = \mathfrak{m}/\mathfrak{p}$ and $\mathfrak{m}_B^2 = (\mathfrak{m}^2 + \mathfrak{p})/\mathfrak{p} = \mathfrak{m}^2/\mathfrak{p}$, so $B/\mathfrak{m}_B \cong A/\mathfrak{m}$ and $\mathfrak{m}_B/\mathfrak{m}_B^2 \cong \mathfrak{m}/\mathfrak{m}^2$ by the third isomorphy theorem. Consequently B is not a regular local ring because,

$$\dim_{B/\mathfrak{m}_B}(\mathfrak{m}_B/\mathfrak{m}_B^2) = \dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = \dim(A) > \dim(B).$$

(ii) Denote $B := A/\mathfrak{p}$ and $\mathfrak{m}_B := \mathfrak{m}/\mathfrak{p}$. As $\mathfrak{m}_B^2 = (\mathfrak{m}^2 + \mathfrak{p})/\mathfrak{p}$, we have by the third isomorphy theorem $B/\mathfrak{m}_B \cong A/\mathfrak{m}$ and

$$\mathfrak{m}_B/\mathfrak{m}_B^2 \cong \mathfrak{m}/(\mathfrak{m}^2 + \mathfrak{p}) \cong (\mathfrak{m}/\mathfrak{m}^2)/((\mathfrak{m}^2 + \mathfrak{p})/\mathfrak{m}^2).$$

Henceforth,

$$n - r = \dim(B) = \dim_{B/\mathfrak{m}_B}(\mathfrak{m}_B/\mathfrak{m}_B^2) = \dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) - \dim((\mathfrak{m}^2 + \mathfrak{p})/\mathfrak{m}^2).$$

Thus, $\dim((\mathfrak{m}^2 + \mathfrak{p})/\mathfrak{m}^2) = r$ and there exist $x_1, \dots, x_r \in \mathfrak{m} \setminus \mathfrak{m}^2$ whose classes constitute a basis of $(\mathfrak{m}^2 + \mathfrak{p})/\mathfrak{m}^2$ as $A/\mathfrak{m}$-linear space. By Corollary III.2.17 the quotient $A/(\{x_1, \dots, x_r\}A)$ is a regular local ring of dimension $n-r$. Thus, $\mathfrak{q} := \{x_1, \dots, x_r\}A \subset \mathfrak{p}$ is a prime ideal of A such that $A/\mathfrak{q}$ is a local ring of dimension $n-r$. The homomorphism $\varphi : A/\mathfrak{q} \to A/\mathfrak{p},\ a+\mathfrak{q} \mapsto a+\mathfrak{p}$ is surjective. Then $\ker(\varphi) = \mathfrak{p}/\mathfrak{q}$ and $(A/\mathfrak{q})/(\mathfrak{p}/\mathfrak{q}) \cong A/\mathfrak{p}$. Consequently,

$$n-r = \dim(A/\mathfrak{q}) \geq \dim(A/\mathfrak{p}) + \mathrm{ht}(\mathfrak{p}/\mathfrak{q}) = n-r+\mathrm{ht}(\mathfrak{p}/\mathfrak{q}).$$

Thus, $\mathrm{ht}(\mathfrak{p}/\mathfrak{q}) = 0$, so $\mathfrak{p} = \mathfrak{q} = \{x_1, \ldots, x_r\}A$, as required. □

Lemma III.2.19 *Let $\mathfrak{a} := \{f_1, \ldots, f_m\}\mathcal{A}_n \subset \mathfrak{q}$ be ideals of $\mathcal{A}_n$ such that $\mathfrak{q}$ is prime. Let $g_1, \ldots, g_r \in \mathcal{A}_n$ be such that $\mathfrak{a}\mathcal{A}_{n,\mathfrak{q}} = \{g_1, \ldots, g_r\}\mathcal{A}_{n,\mathfrak{q}}$. Then*

$$\mathrm{rk}_{\mathcal{A}_n/\mathfrak{q}}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j} + \mathfrak{q}\Big)_{\substack{1\leq i\leq m,\\ 1\leq j\leq n}} = \mathrm{rk}_{\mathcal{A}_n/\mathfrak{q}}\Big(\frac{\partial g_k}{\partial \mathtt{x}_j} + \mathfrak{q}\Big)_{\substack{1\leq k\leq r,\\ 1\leq j\leq n}}.$$

Proof. As $g_1, \ldots, g_r$ are elements of $\mathfrak{a}\mathcal{A}_{n,\mathfrak{q}}$, there exist $b_1, \ldots, b_r \in \mathcal{A}_n \setminus \mathfrak{q}$ such that $b_k g_k \in \mathfrak{a}$. Thus, each $g_k \in \mathfrak{q}$ and $b_k g_k = \sum_{i=1}^m a_{ik} f_i$ with $a_{ik} \in \mathcal{A}_n$, so

$$\frac{\partial b_k}{\partial \mathtt{x}_j} g_k + b_k \frac{\partial g_k}{\partial \mathtt{x}_j} = \sum_{i=1}^m \Big(\frac{\partial a_{ik}}{\partial \mathtt{x}_j} f_i + a_{ik}\frac{\partial f_i}{\partial \mathtt{x}_j}\Big).$$

Hence, taking classes modulo $\mathfrak{q}$, we have

$$b_k \frac{\partial g_k}{\partial \mathtt{x}_j} + \mathfrak{q} = \sum_{i=1}^m \Big(a_{ik}\frac{\partial f_i}{\partial \mathtt{x}_j}\Big) + \mathfrak{q}.$$

As each $b_k \notin \mathfrak{q}$, we deduce that

$$\begin{aligned}
&\mathrm{rk}_{\mathcal{A}_n/\mathfrak{q}}\bigg(\Big(\frac{\partial f_i}{\partial \mathtt{x}_j} + \mathfrak{q}\Big)_{\substack{1\leq i\leq m,\\ 1\leq j\leq n}}\bigg|\Big(\frac{\partial g_k}{\partial \mathtt{x}_j} + \mathfrak{q}\Big)_{\substack{1\leq k\leq r,\\ 1\leq j\leq n}}\bigg)\\
&\qquad = \mathrm{rk}_{\mathcal{A}_n/\mathfrak{q}}\bigg(\Big(\frac{\partial f_i}{\partial \mathtt{x}_j} + \mathfrak{q}\Big)_{\substack{1\leq i\leq m,\\ 1\leq j\leq n}}\bigg|\Big(b_k\frac{\partial g_k}{\partial \mathtt{x}_j} + \mathfrak{q}\Big)_{\substack{1\leq k\leq r,\\ 1\leq j\leq n}}\bigg)\\
&\qquad = \mathrm{rk}_{\mathcal{A}_n/\mathfrak{q}}\bigg(\Big(\frac{\partial f_i}{\partial \mathtt{x}_j} + \mathfrak{q}\Big)_{\substack{1\leq i\leq m,\\ 1\leq j\leq n}}\bigg|\Big(\sum_{i=1}^m\Big(a_{ik}\frac{\partial f_i}{\partial \mathtt{x}_j}\Big) + \mathfrak{q}\Big)_{\substack{1\leq k\leq r,\\ 1\leq j\leq n}}\bigg)\\
&\qquad = \mathrm{rk}_{\mathcal{A}_n/\mathfrak{q}}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j} + \mathfrak{q}\Big)_{\substack{1\leq i\leq m,\\ 1\leq j\leq n}}.
\end{aligned}$$

Consequently,

$$\mathrm{rk}_{\mathcal{A}_n/\mathfrak{q}}\Big(\frac{\partial g_k}{\partial \mathtt{x}_j} + \mathfrak{q}\Big)_{\substack{1\leq k\leq r,\\ 1\leq j\leq n}} \leq \mathrm{rk}_{\mathcal{A}_n/\mathfrak{q}}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j} + \mathfrak{q}\Big)_{\substack{1\leq i\leq m,\\ 1\leq j\leq n}}.$$

Let us prove the converse inequality. Note that $f_i \in \mathfrak{a}_q = \{g_1, \dots, g_r\}\mathcal{A}_{n,\mathfrak{q}}$, so there exist $c_i \in \mathcal{A}_n \setminus \mathfrak{q}$ such that $c_i f_i \in \{g_1, \dots, g_r\}\mathcal{A}_n$. Consequently, $c_i f_i = \sum_{k=1}^m d_{ki} g_k$ where $d_{ki} \in \mathcal{A}_n$. Repeating the previous argument one shows that

$$\operatorname{rk}_{\mathcal{A}_n/\mathfrak{q}}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j} + \mathfrak{q}\Big)_{\substack{1\leq i\leq m,\\ 1\leq j\leq n}} \leq \operatorname{rk}_{\mathcal{A}_n/\mathfrak{q}}\Big(\frac{\partial g_k}{\partial \mathtt{x}_j} + \mathfrak{q}\Big)_{\substack{1\leq k\leq r,\\ 1\leq j\leq n}},$$

as required. □

Lemma III.2.20 *Let $\mathfrak{a}$ be an ideal of $\mathcal{A}_n$ and let $f \in \mathfrak{a}^2$. Then $\frac{\partial f}{\partial \mathtt{x}_j} \in \mathfrak{a}$ for $j = 1, \dots, n$.*

Proof. Write $f := g_1 h_1 + \cdots + g_r h_r$ where $g_i, h_i \in \mathfrak{a}$. We have

$$\frac{\partial f}{\partial \mathtt{x}_j} = \sum_{i=1}^r \Big(\frac{\partial g_i}{\partial \mathtt{x}_j} h_i + g_i \frac{\partial h_i}{\partial \mathtt{x}_j}\Big) \in \mathfrak{a},$$

as required. □

Recall that if $\mathfrak{a}$ is an ideal of a noetherian ring A the *associated prime ideals of* $\mathfrak{a}$ are those prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ of A that contain $\mathfrak{a}$ such that $\mathfrak{p}_1/\mathfrak{a}, \dots, \mathfrak{p}_r/\mathfrak{a}$ are the minimal prime ideals of $A/\mathfrak{a}$. After all this preparation, we are ready to prove the generalized Jacobian criterion.

Theorem III.2.21 (Generalized Jacobian criterion) *Let $f_1, \dots, f_m \in \mathfrak{m}_n$ and denote $\mathfrak{a} := \{f_1, \dots, f_m\}\mathcal{A}_n$. Let $\mathfrak{p} \subset \mathcal{A}_n$ be a prime ideal associated to $\mathfrak{a}$ and let $\mathfrak{q} \subset \mathcal{A}_n$ be a prime ideal that contains $\mathfrak{p}$. The following assertions are equivalent:*

(i) *The ring $\mathcal{A}_{n,\mathfrak{q}}/\mathfrak{a}\mathcal{A}_{n,\mathfrak{q}}$ is a regular local ring.*

(ii) $\operatorname{rk}_{\mathcal{A}_n/\mathfrak{q}}(\frac{\partial f_i}{\partial \mathtt{x}_j} + \mathfrak{q})_{1\leq i\leq m,\, 1\leq j\leq n} = \operatorname{ht}(\mathfrak{p})$.

Proof. The proof is conducted in several steps:

2.a.1 We consider first the case $\mathfrak{a} = \mathfrak{p} = \mathfrak{q}$.

The ideal $\mathfrak{p}\mathcal{A}_{n,\mathfrak{p}}$ is the maximal ideal of the ring $\mathcal{A}_{n,\mathfrak{p}}$. Consequently, the quotient $\mathcal{A}_{n,\mathfrak{p}}/\mathfrak{p}\mathcal{A}_{n,\mathfrak{p}}$ is a field, so it is a regular local ring of dimension 0. We have to prove

$$r := \mathrm{rk}_{\mathcal{A}_n/\mathfrak{p}}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j} + \mathfrak{p}\Big)_{1\leq i\leq m,\, 1\leq j\leq n} = \mathrm{ht}(\mathfrak{p}).$$

By Local Parameterization Theorem II.3.1 we may assume, after a linear change of coordinates, that $\mathcal{A}_d \hookrightarrow \mathcal{A}_n/\mathfrak{p}$ where $d := n - \mathrm{ht}(\mathfrak{p})$. We have to show that $r = n - d$. We will keep the notations introduced in Local Parameterization Theorem II.3.1. We have

$$\Delta^q\mathfrak{p} \subset \{P_{d+1}, \Delta\mathtt{x}_{d+2} - Q_{d+2}, \ldots, \Delta\mathtt{x}_n - Q_n\} \subset \mathfrak{p}$$

where $q \geq 0$ is large enough and $\Delta \in \mathcal{A}_d \setminus \{0\}$ is the discriminant of P_{d+1}. Recall that $\Delta \notin \mathfrak{p}$. Thus,

$$\mathfrak{p}A_{\mathfrak{p}} = \{P_{d+1}, \Delta\mathtt{x}_{d+2} - Q_{d+2}, \ldots, \Delta\mathtt{x}_n - Q_n\}A_{\mathfrak{p}}.$$

By Lemma III.2.19 we have

$$\begin{aligned} r &= \mathrm{rk}_{\mathcal{A}_n/\mathfrak{q}}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j} + \mathfrak{q}\Big)_{1\leq i\leq m,\, 1\leq j\leq n} \\ &= \mathrm{rk}_{\mathcal{A}_n/\mathfrak{q}}\begin{pmatrix} \frac{\partial P_{d+1}}{\partial \mathtt{x}_1} + \mathfrak{q} & \cdots & \frac{\partial P_{d+1}}{\partial \mathtt{x}_{d+1}} + \mathfrak{q} & 0 & \cdots & 0 \\ -\frac{\partial Q_{d+2}}{\partial \mathtt{x}_1} + \mathfrak{q} & \cdots & \frac{\partial Q_{d+2}}{\partial \mathtt{x}_{d+1}} + \mathfrak{q} & \Delta + \mathfrak{q} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial Q_n}{\partial \mathtt{x}_1} + \mathfrak{q} & \cdots & \frac{\partial Q_n}{\partial \mathtt{x}_{d+1}} + \mathfrak{q} & 0 & \cdots & \Delta + \mathfrak{q} \end{pmatrix}. \end{aligned}$$

As P_{d+1} is the irreducible polynomial of the primitive element θ of the field extension $\mathrm{qf}(\mathcal{A}_n/\mathfrak{p})|\,\mathrm{qf}(\mathcal{A}_d)$ and $\mathfrak{q} = \mathfrak{p}$, the derivative $\frac{\partial P_{d+1}}{\partial \mathtt{x}_{d+1}} \notin \mathfrak{q}$, so the matrix in the right has rank $n - d$. Thus, $r = n - d$, as required.

2.a.2 Suppose $\mathcal{A}_{n,\mathfrak{q}}/\mathfrak{a}\mathcal{A}_{n,\mathfrak{q}}$ is a regular local ring. We claim: $\mathfrak{a}\mathcal{A}_{n,\mathfrak{q}} = \mathfrak{p}\mathcal{A}_{n,\mathfrak{q}}$.

As $\mathcal{A}_{n,\mathfrak{q}}/\mathfrak{a}\mathcal{A}_{n,\mathfrak{q}}$ is a regular local ring, it is an integral domain, so $\mathfrak{a}\mathcal{A}_{n,\mathfrak{q}}$ is a prime ideal. As $\mathfrak{p}$ is a minimal prime ideal associated to $\mathfrak{a}$, we conclude that $\mathrm{ht}(\mathfrak{a}\mathcal{A}_{n,\mathfrak{q}}) = \mathrm{ht}(\mathfrak{p}\mathcal{A}_{n,\mathfrak{q}})$, so $\mathfrak{a}\mathcal{A}_{n,\mathfrak{q}} = \mathfrak{p}\mathcal{A}_{n,\mathfrak{q}}$.

2.a.3 Thus, we may assume for the proof that $\mathfrak{a} = \mathfrak{p}$. Write $r := \mathrm{ht}(\mathfrak{p})$ and $s := \mathrm{ht}(\mathfrak{q})$. By Corollary II.2.15 $\dim(\mathcal{A}_{n,\mathfrak{q}}/\mathfrak{p}\mathcal{A}_{n,\mathfrak{q}}) = \mathrm{ht}(\mathfrak{q}) - \mathrm{ht}(\mathfrak{p}) = s - r$. Recall that by Corollary II.3.3 $\mathcal{A}_{n,\mathfrak{q}}$ is a regular local ring. By Corollary III.2.18 the ideal $\mathfrak{p}\mathcal{A}_{n,\mathfrak{q}}$ can be generated by elements $g_1, \ldots, g_r \in \mathfrak{q} \setminus \mathfrak{q}^2$. By

Corollary III.2.17 there exist elements $g_{r+1},\dots,g_s \in \mathfrak{q}\setminus\mathfrak{q}^2$ such that $\mathfrak{q}\mathcal{A}_{n,\mathfrak{q}} = (g_1,\dots,g_s)\mathcal{A}_{n,\mathfrak{q}}$. By 2.a.1 we have

$$\mathrm{rk}_{\mathcal{A}_n/\mathfrak{q}}\Big(\frac{\partial g_k}{\partial \mathtt{x}_j}+\mathfrak{q}\Big)_{1\le k\le s,\,1\le j\le n} = \mathrm{ht}(\mathfrak{q}) = s,$$

which is the maximal possible rank. Deleting the columns corresponding to $g_{r+1},\dots,g_s$ we get

$$\mathrm{rk}_{\mathcal{A}_n/\mathfrak{q}}\Big(\frac{\partial g_k}{\partial \mathtt{x}_j}+\mathfrak{q}\Big)_{1\le k\le r,\,1\le j\le n} = r = \mathrm{ht}(\mathfrak{p}).$$

By Lemma III.2.19 we conclude

$$\mathrm{rk}_{\mathcal{A}_n/\mathfrak{q}}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}+\mathfrak{q}\Big)_{1\le i\le m,\,1\le j\le n} = \mathrm{rk}_{\mathcal{A}_n/\mathfrak{q}}\Big(\frac{\partial g_k}{\partial \mathtt{x}_j}+\mathfrak{q}\Big)_{1\le k\le r,\,1\le j\le n} = \mathrm{ht}(\mathfrak{p}).$$

2.a.4 Assume now $\mathrm{rk}_{\mathcal{A}_n/\mathfrak{q}}(\frac{\partial f_i}{\partial \mathtt{x}_j}+\mathfrak{q})_{1\le i\le m,\,1\le j\le n} = \mathrm{ht}(\mathfrak{p}) = r$. We can suppose

$$\mathrm{rk}_{\mathcal{A}_n/\mathfrak{q}}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}+\mathfrak{q}\Big)_{1\le i\le r,\,1\le j\le r} = r.$$

Let us check that the classes of $f_1,\dots,f_r$ in $\mathfrak{q}\mathcal{A}_{n,\mathfrak{q}}/\mathfrak{q}^2\mathcal{A}_{n,\mathfrak{q}}$ are linearly independent over the field $(\mathcal{A}_{n,\mathfrak{q}})/(\mathfrak{q}\mathcal{A}_{n,\mathfrak{q}})$. Let $a_1,\dots,a_r\in\mathcal{A}_n$ be such that $a_1f_1+\cdots+a_rf_r\in\mathfrak{q}^2$. We have to prove: $a_1,\dots,a_r\in\mathfrak{q}$.

For each $j=1,\dots,n$ we have by Lemma III.2.20

$$0+\mathfrak{q} = \frac{\partial}{\partial \mathtt{x}_j}\Big(\sum_{i=1}^r a_if_i\Big)+\mathfrak{q} = \sum_{i=1}^r\Big(\frac{\partial a_i}{\partial \mathtt{x}_j}f_i + a_i\frac{\partial f_i}{\partial \mathtt{x}_j}\Big)+\mathfrak{q} = \sum_{i=1}^r(a_i+\mathfrak{q})\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}+\mathfrak{q}\Big).$$

Consequently, we can write

$$\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}+\mathfrak{q}\Big)_{1\le i\le r,\,1\le j\le r}(a_i+\mathfrak{q})^t_{1\le i\le r} = (0+\mathfrak{q})^t_{1\le i\le r}.$$

As $\mathrm{rk}_{\mathcal{A}_n/\mathfrak{q}}(\frac{\partial f_i}{\partial \mathtt{x}_j}+\mathfrak{q})_{1\le i\le r,\,1\le j\le r} = r$, we have $a_i+\mathfrak{q}=0+\mathfrak{q}$ for $i=1,\dots,r$. By Corollary II.3.3 and Corollary III.2.17 the quotient

$$\mathcal{A}_{n,\mathfrak{q}}/(\{f_1,\dots,f_r\}\mathcal{A}_{n,\mathfrak{q}})$$

is a regular local ring. In particular, $\{f_1,\dots,f_r\}\mathcal{A}_{n,\mathfrak{q}}\subset\mathfrak{p}\mathcal{A}_{n,\mathfrak{q}}$ is a prime ideal of height $r=\mathrm{ht}(\mathfrak{p})=\mathrm{ht}(\mathfrak{p}\mathcal{A}_{n,\mathfrak{q}})$. Consequently, $\{f_1,\dots,f_r\}\mathcal{A}_{n,\mathfrak{q}}=\mathfrak{p}\mathcal{A}_{n,\mathfrak{q}}$ and $\mathcal{A}_{n,\mathfrak{q}}/\mathfrak{p}\mathcal{A}_{n,\mathfrak{q}}$ is a regular local ring, as required. $\square$

Corollary III.2.22 *Let $\mathfrak{a}$ be an ideal of $\mathcal{A}_n$ and let $\mathfrak{p}$ be an associated prime of $\mathcal{A}_n$ to $\mathfrak{a}$. Let $\mathfrak{q}$ be a prime ideal of $\mathcal{A}_n$ such that $\mathfrak{p} \subset \mathfrak{q}$ and $\mathcal{A}_{n,\mathfrak{q}}/\mathfrak{a}\mathcal{A}_{n,\mathfrak{q}}$ is a regular local ring. Then $\mathcal{A}_{n,\mathfrak{p}}/\mathfrak{a}\mathcal{A}_{n,\mathfrak{p}} = \mathcal{A}_{n,\mathfrak{p}}/\mathfrak{p}\mathcal{A}_{n,\mathfrak{p}} = \mathrm{qf}(\mathcal{A}_n/\mathfrak{p})$.*

Proof. As $\mathcal{A}_{n,\mathfrak{q}}/\mathfrak{a}\mathcal{A}_{n,\mathfrak{q}}$ is a regular local ring, then by 2.a.2 $\mathfrak{a}\mathcal{A}_{n,\mathfrak{q}} = \mathfrak{p}\mathcal{A}_{n,\mathfrak{q}}$. As $\mathfrak{p} \subset \mathfrak{q}$, we have $\mathcal{A}_n \setminus \mathfrak{p} \subset \mathcal{A}_n \setminus \mathfrak{p}$, so $\mathfrak{a}\mathcal{A}_{n,\mathfrak{p}} = \mathfrak{p}\mathcal{A}_{n,\mathfrak{p}}$. Consequently, $\mathcal{A}_{n,\mathfrak{p}}/\mathfrak{a}\mathcal{A}_{n,\mathfrak{p}} = \mathcal{A}_{n,\mathfrak{p}}/\mathfrak{p}\mathcal{A}_{n,\mathfrak{p}} = \mathrm{qf}(\mathcal{A}_n/\mathfrak{p})$, as required. □

We apply the generalized Jacobian criterion to the situation of Local Parameterization Theorem II.3.1 to obtain its final formulation in our setting. Let $S, T \subset A$ be two multiplicatively closed sets of a ring A and let U be the image of T in $S^{-1}A$. Then $ST := \{st : \ s \in S, \ t \in T\}$ is a multiplicatively closed set that contains S and T and $(ST)^{-1}A$ and $U^{-1}(S^{-1}A)$ are isomorphic. In addition, if $f : A \to B$ is a surjective homomorphism such that B is an integral domain and $S \subset A$ is a multiplicatively closed set, then $f(S) \subset B$ is a multiplicatively closed set and $(f(S))^{-1}B$ is isomorphic to $S^{-1}A/(S^{-1}\ker(f))$.

Proposition III.2.23 (Local Parameterization Theorem) *Let $\mathfrak{p}$ be a prime ideal of height r of $\mathcal{A}_n$ and denote $d := n - r$. After a linear change of coordinates we have:*

(i) *The canonical homomorphism $\mathcal{A}_d \to A := \mathcal{A}_n/\mathfrak{p}$ is injective and finite.*

(ii) *The class $\theta := \mathtt{x}_{d+1} + \mathfrak{p}$ is a primitive element of the field extension $L|K$ where $L := \mathrm{qf}(\mathcal{A}_n/\mathfrak{p})$ and $K := \mathrm{qf}(\mathcal{A}_d)$.*

(iii) *The irreducible polynomial over K of the primitive element θ is a distinguished polynomial $P_{d+1} \in \mathcal{A}_d[\mathtt{x}_{d+1}]$, whose discriminant is $\Delta \in \mathcal{A}_d \setminus \{0\}$.*

(iv) *Write $A' := \mathcal{A}_{d+1}/(P_{d+1}\mathcal{A}_{d+1})$. The canonical homomorphism $A'_\Delta \to A_\Delta$ is an isomorphism.*

(v) *If $\mathfrak{Q}$ is a prime ideal of A_Δ, then $(A_\Delta)_\mathfrak{Q}$ is a regular local ring.*

Proof. The first four statements have been already proved in Theorem II.3.1 and Remarks II.3.2. The last one will follow from the Generalized Jacobian Criterion.

Observe that $\mathfrak{Q} := (\mathfrak{q}/\mathfrak{p})_\Delta$ for some ideal $\mathfrak{q}$ of $\mathcal{A}_n$ that contains $\mathfrak{p}$ and $\Delta \notin \mathfrak{q}$. Denote $\mathfrak{Q}_0 := \mathfrak{q}/\mathfrak{p}$. Consider the multiplicatively closed sets

$$S := \{\Delta^k : \ k \geq 0\}, \quad T_0 := \mathcal{A}_n \setminus \mathfrak{q}, \quad T := A \setminus \mathfrak{Q}_0 \quad \text{and} \quad U := A_\Delta \setminus \mathfrak{Q}.$$

Observe that $ST = T$ because $\Delta \notin \mathfrak{q}$, so $(A_\Delta)_{\mathfrak{Q}} \cong A_{\mathfrak{Q}_0}$. As $\mathfrak{p} \subset \mathfrak{q}$, we have $A_{\mathfrak{Q}_0} \cong \mathcal{A}_{n,\mathfrak{q}}/\mathfrak{p}\mathcal{A}_{n,\mathfrak{q}}$. Thus, we have to prove: $\mathcal{A}_{n,\mathfrak{q}}/\mathfrak{p}\mathcal{A}_{n,\mathfrak{q}}$ *is a regular local ring.*

By Corollary III.2.19 and the Generalized Jacobian criterion it is enough to prove that there exist $f_1, \dots, f_r \in \mathfrak{p}$ such that $\mathfrak{p}\mathcal{A}_{n,\mathfrak{q}} = \{f_1, \dots, f_r\}\mathcal{A}_{n,\mathfrak{q}}$ and $\operatorname{rk}_{\mathcal{A}_n/\mathfrak{q}}(\frac{\partial f_i}{\partial \mathtt{x}_j} + \mathfrak{q})_{1\le i\le r, 1\le j\le n} = \operatorname{ht}(\mathfrak{p}) = r$. We will keep all the notations introduced in Local Parameterization Theorem II.3.1. We have

$$\Delta^q\mathfrak{p} \subset \{P_{d+1}, \Delta\mathtt{x}_{d+2} - Q_{d+2}, \dots, \Delta\mathtt{x}_n - Q_n\} \subset \mathfrak{p}$$

where $q \ge 0$ is large enough and $\Delta \in \mathcal{A}_d \setminus \{0\}$ is the discriminant of P_{d+1}. As $\Delta \notin \mathfrak{q}$,

$$\mathfrak{p}A_\mathfrak{q} = \{f_1 := P_{d+1}, f_2 := \Delta\mathtt{x}_{d+2} - Q_{d+2}, \dots, f_r := \Delta\mathtt{x}_n - Q_n\}A_\mathfrak{q}.$$

In Corollary IV.2.4 we will show that if a series $f \in \mathcal{A}_n$, then

$$f \in \sqrt{\Big\{\frac{\partial f}{\partial \mathtt{x}_1}, \dots, \frac{\partial f}{\partial \mathtt{x}_n}\Big\}\mathcal{A}_n}.$$

Using this we deduce that if $\frac{\partial P_{d+1}}{\partial \mathtt{x}_1}, \dots, \frac{\partial P_{d+1}}{\partial \mathtt{x}_d}, \frac{\partial P_{d+1}}{\partial \mathtt{x}_{d+1}} \in \mathfrak{q}$, then

$$P_{d+1} \in \sqrt{\Big\{\frac{\partial P_{d+1}}{\partial \mathtt{x}_1}, \dots, \frac{\partial P_{d+1}}{\partial \mathtt{x}_{d+1}}\Big\}\mathcal{A}_n} \subset \mathfrak{q}.$$

In addition, by the properties of the resultant of two polynomials there exists polynomials $F_1, F_2 \in \mathcal{A}_d[\mathtt{x}_{d+1}]$ such that

$$\Delta = \pm\operatorname{Res}_{\mathtt{x}_{d+1}}\Big(P_{d+1}, \frac{\partial P_{d+1}}{\partial \mathtt{x}_{d+1}}\Big) = F_1 P_{d+1} + F_2\frac{\partial P_{d+1}}{\partial \mathtt{x}_{d+1}} \in \mathfrak{q},$$

which is a contradiction because $\Delta \notin \mathfrak{q}$. Consequently, there exist $1 \le j \le d+1$ such that $\frac{\partial P_{d+1}}{\partial \mathtt{x}_j} \notin \mathfrak{q}$. Using again that $\Delta \notin \mathfrak{q}$ the matrix

$$\Big(\frac{\partial f_i}{\partial \mathtt{x}_j} + \mathfrak{q}\Big)_{\substack{1\le i\le m,\\ 1\le j\le n}} = \begin{pmatrix} \frac{\partial P_{d+1}}{\partial \mathtt{x}_1} + \mathfrak{q} & \cdots & \frac{\partial P_{d+1}}{\partial \mathtt{x}_{d+1}} + \mathfrak{q} & 0 & \cdots & 0 \\ -\frac{\partial Q_{d+2}}{\partial \mathtt{x}_1} + \mathfrak{q} & \cdots & \frac{\partial Q_{d+2}}{\partial \mathtt{x}_{d+1}} + \mathfrak{q} & \Delta + \mathfrak{q} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial Q_n}{\partial \mathtt{x}_1} + \mathfrak{q} & \cdots & \frac{\partial Q_n}{\partial \mathtt{x}_{d+1}} + \mathfrak{q} & 0 & \cdots & \Delta + \mathfrak{q} \end{pmatrix}$$

has rank $r = \operatorname{ht}(\mathfrak{p})$, as required. □

Exercises

Number III.1 Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $A := \mathcal{A}_n/\mathfrak{a}$ and $B := \mathcal{A}_m/\mathfrak{b}$ where $\mathfrak{a}$ is an ideal of $\mathcal{A}_n$ and $\mathfrak{b}$ is an ideal of $\mathcal{A}_m$. Assume that A, B are integral domains of the same dimension. Let $\varphi : A \to B$ be a homomorphism of $\mathbb{K}$-algebras. Show that:

(i) If φ is surjective, then φ is an isomorphism of $\mathbb{K}$-algebras.

(ii) If φ is finite, then φ is injective.

Number III.2 Prove that the map $d_n : \mathcal{A}_n \times \mathcal{A}_n \to \mathbb{R},\ (f, g) \mapsto e^{-\omega(f-g)}$ defines a metric on $\mathcal{A}_n$. Let $\varphi : \mathcal{A}_n \to \mathcal{A}_m$ be a $\mathbb{K}$-algebra homomorphism. Show that

(i) A sequence $\{f_k\}_{k\geq 1} \subset \mathcal{A}_n$ converges to $f \in \mathcal{A}_n$ in $(\mathcal{A}_n, d_n)$ if and only if for each $\ell \geq 1$ there exists $k_0 \in \mathbb{N}$ such that if $k \geq k_0$, then $\omega(f - f_k) \geq \ell$.

(ii) $\varphi : (\mathcal{A}_n, d_n) \to (\mathcal{A}_m, d_m)$ is continuous.

(iii) For each $f \in \mathcal{A}_n$, we have $\varphi(f) = f(\varphi(\mathtt{x}_1), \ldots, \varphi(\mathtt{x}_n))$.

Number III.3 Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and consider the $\mathbb{K}$-algebra homomorphism $\varphi : \mathcal{A}_3 \to \mathcal{A}_2$ given by $\varphi(f) = f(\mathtt{y}_1\mathtt{y}_2, \mathtt{y}_2^2, \mathtt{y}_1^2)$. Let $B \subset \mathcal{A}_2$ be the set of all series $\sum_{\nu:=(\nu_1,\nu_2)} a_\nu \mathtt{y}_1^{\nu_1}\mathtt{y}_2^{\nu_2}$ such that $a_\nu = 0$ if $\nu_1 - \nu_2$ is odd. Show that:

(i) φ is finite and $\operatorname{im}(\varphi) = B$.

(ii) $\mathfrak{p} := \ker(\varphi)$ is a principal prime ideal of $\mathcal{A}_3$ and find a generator of $\mathfrak{p}$.

(iii) The ring $A := \mathcal{A}_3/\mathfrak{p}$ is integrally closed, it has no nilpotent elements but it is not a unique factorization domain.

Number III.4 Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $A := \mathcal{A}_n/\mathfrak{a}$ and $B := \mathcal{A}_m/\mathfrak{b}$ where $\mathfrak{a}$ is an ideal of $\mathcal{A}_n$ and $\mathfrak{b}$ is an ideal of $\mathcal{A}_m$. Denote $\mathfrak{m}_A$ and $\mathfrak{m}_B$ the maximal ideals of A and B. Show that:

(i) The quotient $\mathfrak{m}_A/\mathfrak{m}_A^2$ has structure of $\mathbb{K}$-linear space.

(ii) If $\varphi : A \to B$ is a $\mathbb{K}$-algebra homomorphism, the map $\overline{\varphi} : \mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ given by $\overline{\varphi}(f + \mathfrak{m}_A^2) = \varphi(f) + \mathfrak{m}_B^2$ is a well-defined homomorphism of $\mathbb{K}$-linear spaces.

(iii) Let $C := B/(\varphi(\mathfrak{m}_A)B)$ and let $\pi : B \to C$ be the canonical projection. Show that the sequence of $\mathbb{K}$-homomorphisms $\mathfrak{m}_A/\mathfrak{m}_A^2 \xrightarrow{\overline{\varphi}} \mathfrak{m}_B/\mathfrak{m}_B^2 \xrightarrow{\overline{\pi}} \mathfrak{m}_C \to 0$ is exact.

(iv) If $\mathfrak{a} := \{f_1, \ldots, f_m\}\mathcal{A}_n$ and $A := \mathcal{A}_n/\mathfrak{a}$, then

$$\dim(A) \leq \dim(\mathfrak{m}_A/\mathfrak{m}_A^2) = n - \operatorname{rk}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\Big)_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}.$$

(v) φ is surjective if and only if $\overline{\varphi}$ is surjective.

(vi) Assume $B = \mathcal{A}_m$. The homomorphism $\varphi : A \to \mathcal{A}_m$ is an isomorphism if and only if so is $\overline{\varphi}$.

(vii) Find a $\mathbb{K}$-algebra homomorphism $\varphi : A \to B$, which is not an isomorphism but $\overline{\varphi} : \mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is an isomorphism.

Number III.5 Let $\mathfrak{p}$ be a non-zero prime ideal of $\mathcal{A}_n$. Show that there exist $f_1, \ldots, f_r \in \mathfrak{p}$ such that $\operatorname{rk}_{\mathcal{A}_n/\mathfrak{p}}(\frac{\partial f_i}{\partial \mathtt{x}_j} + \mathfrak{p})_{1\leq i\leq m,\, 1\leq j\leq n} = \operatorname{ht}(\mathfrak{p})$.

CHAPTER IV

Newton-Puiseux Theorem and Rückert's Nullstellensatz

The first part of this chapter is devoted to study the algebraic closure of the quotient field $\mathbb{C}(\langle \mathtt{t} \rangle)$ of the ring $\mathbb{C}\langle\langle \mathtt{t} \rangle\rangle$ of power series in one variable. This involves the proof of Newton-Puiseux's Theorem. In the second part of the chapter we prove Rückerts's Nullstellensatz as a consequence of Local Parameterization Theorem and Newton-Puiseux's Theorem. We end the chapter with a section devoted to introduce the concept of isolated singularity and to study some properties of hypersurface isolated singularities, like the Milnor number. Once again [H, L, AM] are useful to afford the non-proved prerequisites.

1 Newton-Puiseux Theorem

We start by defining power series with rational exponents over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definitions and Notations IV.1.1 A *formal Puiseux series* in the indeterminate $\mathtt{t}$ is an expression $f := \sum_{m\geq 0} a_m \mathtt{t}^{m/p}$ where each *coefficient* $a_m \in \mathbb{K}$ and p is a positive integer. We denote $f(0) := a_0$.

Fix $p \geq 1$ and let $\mathbb{K}[[\mathtt{t}^{1/p}]]$ be the set consisting of all Puiseux series of the form $\sum a_m \mathtt{t}^{m/p}$. Let $\mathtt{s}$ be another indeterminate. For every $p \geq 1$ we have a canonical bijection

$$\tau_p : \mathbb{K}[[\mathtt{s}]] \to \mathbb{K}[[\mathtt{t}^{1/p}]],\ f \mapsto f(\mathtt{t}^{1/p})$$

that induces a natural structure of $\mathbb{K}$-algebra on $\mathbb{K}[[\mathtt{t}^{1/p}]]$. With such structure $\mathbb{K}[[\mathtt{t}^{1/p}]]$ is isomorphic to $\mathbb{K}[[\mathtt{s}]]$ as a $\mathbb{K}$-algebra. If $q = pr$ there exists a $\mathbb{K}$-algebra monomorphism

$$\tau_{pq} : \mathbb{K}[[\mathtt{t}^{1/p}]] \hookrightarrow \mathbb{K}[[\mathtt{t}^{1/q}]],$$
$$f := \sum_{m\geq 0} a_m \mathtt{t}^{m/p} \mapsto f = \sum_{m\geq 0} a_m \mathtt{t}^{mr/pr} = \sum_{m\geq 0} a_m \mathtt{t}^{mr/q}.$$

The previous monomorphisms τ_{pq} and isomorphisms τ_p allow us to reduce any problem concerning finitely many Puiseux series to a problem concerning finitely many (ordinary) power series. Define:

$$\mathbb{K}[[\mathtt{t}^*]] := \bigcup_{p\geq 1} \mathbb{K}[[\mathtt{t}^{1/p}]].$$

Let $f := \sum a_m \mathtt{t}^{m/p} \in \mathbb{K}[[\mathtt{t}^*]]$ and $g := \sum b_m \mathtt{t}^{m/q} \in \mathbb{K}[[\mathtt{t}^*]]$ be two formal Puiseux series. Using the $\mathbb{K}$-algebra monomorphisms above $\tau_{p,pq}$ and $\tau_{q,pq}$, we may assume that $p = q$ and we define $f + g$ and fg in $\mathbb{K}[[\mathtt{t}^*]]$ as we do in $\mathbb{K}[[\mathtt{t}^{1/p}]]$. We can understand $\mathbb{K}[[\mathtt{t}^*]]$ as the direct limit of the directed family

$$\{\mathbb{K}[[\mathtt{t}^{1/p}]], \tau_{pq}\}_{p,q\geq 1}.$$

For $p = 1$ we obtain the ring $\mathbb{K}[[\mathtt{t}]]$ of (ordinary) formal power series (in one variable). It holds that $\mathbb{K}[[\mathtt{t}^*]]$ is an integral domain and its quotient field $\mathbb{K}((\mathtt{t}^*))$ can be described as

$$\mathbb{K}((\mathtt{t}^*)) = \bigcup_{p\geq 1} \mathbb{K}((\mathtt{t}^{1/p})),$$

where $\mathbb{K}((\mathtt{t}^{1/p}))$ denotes the quotient field of $\mathbb{K}[[\mathtt{t}^{1/p}]]$. Again, we can understand $\mathbb{K}((\mathtt{t}^*))$ as the direct limit of the obvious directed family of fields.

Definition and Remark IV.1.2 The isomorphisms τ_p provide an easy way to introduce convergence. A formal Puiseux series $f := \sum_{m\geq 0} a_m \mathtt{t}^{m/p}$ is called *convergent* if the power series $\tau_p^{-1}(f) = \sum_{m\geq 0} a_m \mathtt{s}^m$ is convergent. The convergent Puiseux series form a subring $\mathbb{K}\{\mathtt{t}^*\}$ of the ring $\mathbb{K}[[\mathtt{t}^*]]$. We write $\mathbb{K}(\{\mathtt{t}^*\}) := \operatorname{qf}(\mathbb{K}\{\mathtt{t}^*\})$. We have formulae similar to the ones above:

$$\mathbb{K}\{\mathtt{t}^*\} := \bigcup_{p\geq 1} \mathbb{K}\{\mathtt{t}^{1/p}\} \quad \text{and} \quad \mathbb{K}(\{\mathtt{t}^*\}) = \bigcup_{p\geq 1} \mathbb{K}(\{\mathtt{t}^{1/p}\}),$$

that allow us to understand $\mathbb{K}\{\mathtt{t}^*\}$ and $\mathbb{K}(\{\mathtt{t}^*\})$ as direct limits of suitable directed families. As one can expect, for $p = 1$ we obtain (ordinary) convergent power series (in one variable).

We will use the notation $\mathbb{K}\langle\!\langle \mathtt{t}^* \rangle\!\rangle$ to refer indistinctly to $\mathbb{K}[[\mathtt{t}^*]]$ and $\mathbb{K}\{\mathtt{t}^*\}$. We write $\mathbb{K}(\!\langle \mathtt{t}^* \rangle\!) := \operatorname{qf}(\mathbb{K}\langle\!\langle \mathtt{t}^* \rangle\!\rangle)$.

Lemma IV.1.3 *The following properties hold:*

(i) *Every* $f \in \mathbb{K}\langle\langle \mathtt{t}^* \rangle\rangle$ *can be uniquely written as* $f = \mathtt{t}^{m/p}u$ *for some rational number* $m/p \geq 0$ *and some unit* u *of* $\mathbb{K}\langle\langle \mathtt{t}^* \rangle\rangle$.

(ii) *The set of all* $f \in \mathbb{K}\langle\langle \mathtt{t}^* \rangle\rangle$ *with* $f(0) = 0$ *is the unique prime ideal* $\mathfrak{m}^* \neq (0)$ *of* $\mathbb{K}\langle\langle \mathtt{t}^* \rangle\rangle$.

(iii) $\mathbb{K}\langle\langle \mathtt{t}^* \rangle\rangle$ *is integrally closed in its quotient field.*

(iv) $\mathbb{K}\langle\langle \mathtt{t}^* \rangle\rangle$ *is integral over* $\mathbb{K}\langle\langle \mathtt{t} \rangle\rangle$.

Proof. (i) There exists p such that $f \in \mathbb{K}\{\mathtt{t}^{1/p}\}$ and write $\tau_p^{-1}(f) = \mathtt{s}^m v(\mathtt{s})$ where $v \in \mathbb{K}\langle\langle \mathtt{s} \rangle\rangle$. Then $f = \mathtt{t}^{m/p}u$ where $u := \tau_p(v)$ is a unit of $\mathbb{K}\langle\langle \mathtt{t}^{1/p} \rangle\rangle$. If we write $f = \mathtt{t}^{n/q}w$ for some $q \geq 1$ and a unit $w \in \mathbb{K}\langle\langle \mathtt{t}^{1/pq} \rangle\rangle$, we have that $\mathtt{t}^{mq/pq}, \mathtt{t}^{np/pq}, u, w \in \mathbb{K}\{\mathtt{t}^{1/pq}\}$ and using the isomorphism τ_{pq} we can work in $\mathbb{K}\{\mathtt{x}\}$ where $\mathtt{x} := \mathtt{t}^{1/pq}$. We conclude that $m/p = n/q$ and $u = v$.

(ii) The condition in the statement defines a non-zero ideal $\mathfrak{m}^*$. In addition, if $f(0) \neq 0$, then by (i) $f = u$ is a unit. This shows that $\mathfrak{m}^*$ is the unique maximal ideal of $\mathbb{K}\{\mathtt{t}^*\}$. Let us see that it is also the unique non-zero prime ideal of $\mathbb{K}\{\mathtt{t}^*\}$.

Let $\mathfrak{p} \neq (0)$ be a prime ideal of $\mathbb{K}\langle\langle \mathtt{t}^* \rangle\rangle$. Take $f \in \mathfrak{p} \setminus \{0\}$ and by (i) write $f = \mathtt{t}^{m/p}u$ for some unit u. Then $\mathtt{t}^m = (u^{-1}f)^p \in \mathfrak{p}$ and as $\mathfrak{p}$ is prime, $\mathtt{t} \in \mathfrak{p}$. Fix $g \in \mathfrak{m}^*$ and write $g := \mathtt{t}^{n/q}v$ where $n, q \geq 1$ and $v \in \mathbb{K}\langle\langle \mathtt{t}^{1/q} \rangle\rangle$, so

$$g^q = \mathtt{t}^n v^q = (\mathtt{t}^{n-1}v^q)\mathtt{t} \in \mathfrak{p}.$$

As $\mathfrak{p}$ is prime, $g \in \mathfrak{p}$. Thus, $\mathfrak{m}^* \subset \mathfrak{p}$ and the equality follows because $\mathfrak{m}^*$ is a maximal ideal.

(iii) Let $h \in \mathbb{K}(\langle \mathtt{t}^* \rangle)$ be integral over $\mathbb{K}\langle\langle \mathtt{t}^* \rangle\rangle$ and let $a_0, \ldots, a_{s-1} \in \mathbb{K}\langle\langle \mathtt{t}^* \rangle\rangle$ such that

$$h^s + a_{s-1}h^{s-1} + \cdots + a_0 = 0.$$

Let $p \geq 1$ be such that $h \in \mathbb{K}\langle\langle \mathtt{t}^{1/p} \rangle\rangle$ and each $a_k \in \mathbb{K}\langle\langle \mathtt{t}^{1/p} \rangle\rangle$. Using the isomorphism τ_p, we may assume that $p = 1$. As $\mathbb{K}\langle\langle \mathtt{t} \rangle\rangle$ is integrally closed in its quotient field, $h \in \mathbb{K}\langle\langle \mathtt{t}^* \rangle\rangle$.

(iv) Given $\xi \in \mathbb{K}\langle\langle \mathtt{t}^* \rangle\rangle$ there exist $p \geq 1$ and $f \in \mathbb{K}\langle\langle \mathtt{s} \rangle\rangle$ with $\xi = f(\mathtt{t}^{1/p})$. It is enough to find a monic polynomial $P(\mathtt{x}, \mathtt{y}) \in \mathbb{K}\langle\langle \mathtt{x} \rangle\rangle[\mathtt{y}]$ with $P(\mathtt{s}^p, f(\mathtt{s})) = 0$.

Consider the $\mathbb{K}$-algebra homomorphism

$$\varphi : \mathbb{K}\langle\langle \mathtt{x}, \mathtt{y} \rangle\rangle \to \mathbb{K}\langle\langle \mathtt{s} \rangle\rangle, \ g(\mathtt{x}, \mathtt{y}) \mapsto g(\mathtt{s}^p, f(\mathtt{s})),$$

which is finite by Corollary III.1.6. As $\dim(\mathbb{K}\langle\langle \mathtt{s}\rangle\rangle) = 1 < 2 = \dim(\mathbb{K}\langle\langle \mathtt{x}, \mathtt{y}\rangle\rangle)$, we deduce by Corollary II.2.9 that φ is not injective. Consequently, there exists $H \in \ker(\varphi) \setminus \{0\}$ such that $H(\mathtt{s}^p, f(\mathtt{s})) = 0$. Write $H = \mathtt{x}^n Q$ with $Q(0, \mathtt{y}) \neq 0$, so Q is regular with respect to $\mathtt{y}$. By Weierstrass's Preparation Theorem there exist a distinguished polynomial $P \in \mathbb{K}\langle\langle \mathtt{x}\rangle\rangle[\mathtt{y}]$ and a unit $U \in \mathbb{K}\langle\langle \mathtt{x}, \mathtt{y}\rangle\rangle$ such that $Q = PU$. As $\ker(\varphi)$ is a prime ideal and $\mathtt{x}^n U \notin \ker(\varphi)$, we have $P(\mathtt{s}^p, f(\mathtt{s})) = 0$, as required. $\square$

Remark IV.1.4 The ring $\mathbb{K}\langle\langle \mathtt{t}^*\rangle\rangle$ is not noetherian. To prove this consider for instance the ideal $\mathfrak{a} := \{\mathtt{t}^{1/n} :\ n \geq 1\}\mathbb{K}\langle\langle \mathtt{t}^*\rangle\rangle$, which is not finitely generated.

We are ready to state Newton-Puiseux's Theorem, although its proof requires some preliminary work.

Theorem IV.1.5 (Newton-Puiseux) *The field $\mathbb{C}(\langle \mathtt{t}^*\rangle)$ of Puiseux series is algebraically closed.*

1.a Preliminary results.

Before proving Theorem IV.1.5, we prove some preliminary ones.

Lemma IV.1.6 *Let A be a noetherian local integral domain of dimension 1. The following assertions are equivalent:*

(i) *A is integrally closed.*

(ii) *The maximal ideal $\mathfrak{m}$ of A is principal.*

(iii) *A is a regular local ring.*

Proof. The proof is conducted in several steps:

1.a.1 As A is a local ring, $\mathrm{ht}(\mathfrak{m}) = \dim(A) = 1$. Thus, A will be a regular local ring if and only if $\mathfrak{m}$ is a principal ideal, so conditions (ii) and (iii) are equivalent.

1.a.2 Assume that A is integrally closed and let us check that $\mathfrak{m}$ is principal. As $\mathfrak{m} \neq (0)$ (because otherwise A would have dimension 0), we pick $a \in \mathfrak{m} \setminus \{0\}$ and consider the ideal $\mathfrak{a} := aA$. As A is noetherian, $\sqrt{\mathfrak{a}}$ is the intersection of finitely many prime ideals. But $\mathfrak{m}$ is the unique non-zero prime ideal of A

because $\mathrm{ht}(\mathfrak{m}) = 1$. Thus, $\sqrt{\mathfrak{a}} = \mathfrak{m}$, so there exists $q \geq 1$ such that $\mathfrak{m}^q \subset \mathfrak{a}$. Let p be the smallest integer such that $\mathfrak{m}^p \subset \mathfrak{a}$, so $\mathfrak{m}^{p-1} \not\subset \mathfrak{a}$. If $p = 1$, then $\mathfrak{m} = \mathfrak{a}$ is principal. So let us assume $p > 1$. Let $b \in \mathfrak{m}^{p-1} \setminus \mathfrak{a}$ and let $x := \frac{a}{b} \in \mathrm{qf}(A) \setminus \{0\}$. We claim: $x^{-1}\mathfrak{m}$ *is an ideal of* A *not contained in* $\mathfrak{m}$. Once we have proved this, we conclude $x^{-1}\mathfrak{m} = A$, so $\mathfrak{m} = x(x^{-1}\mathfrak{m}) = xA$ and $\mathfrak{m}$ is principal.

We prove first that $x^{-1}\mathfrak{m} \subset A$. Let $z \in \mathfrak{m}$, so $bz \in \mathfrak{m}^p \subset \mathfrak{a} = aA$, so there exists $c \in A$ such that $bz = ac$ and $x^{-1}z = c \in A$. Once this have been proved, it is immediate to show that $x^{-1}\mathfrak{m}$ is an ideal of A.

We check next that $x^{-1}\mathfrak{m} \not\subset \mathfrak{m}$. Assume by contradiction that $x^{-1}\mathfrak{m} \subset \mathfrak{m}$ and consider the homomorphism of A-modules $\phi : \mathfrak{m} \to \mathfrak{m},\ z \mapsto x^{-1}z$. As A is noetherian, $\mathfrak{m} = \{m_1, \ldots, m_r\}A$. We write

$$x^{-1}m_i = \phi(m_i) = a_{i1}m_1 + \cdots + a_{ir}m_r.$$

Consequently,

$$\begin{pmatrix} a_{11} - x^{-1} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} - x^{-1} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} - x^{-1} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We summarize the previous identity as $(M - x^{-1}I_r)m^t = 0$. Multiplying it by $\mathrm{Adj}((M - x^{-1}I_r)^t)$, we get $\det(M - x^{-1}I_r)m^t = 0$. As A is an integral domain and some $m_i \neq 0$, we deduce that $\det(M - x^{-1}I_r) = 0$. Observe that $\det(M - x^{-1}I_r)$ is the substitution of x^{-1} in the characteristic polynomial of M, which is a monic polynomial with coefficients in A up to multiplication by $(-1)^r$. Thus, x^{-1} is integral over A. As A is integrally closed, $x^{-1} \in A$, so $b = ax^{-1} \in aA = \mathfrak{a}$, against our choice.

1.a.3 To finish let us prove that if $\mathfrak{m} := yA$ is a principal ideal, then A is integrally closed. Observe first that by Krull's Theorem II.1.7 for each non-zero element $a \in A$ there exists $\omega(a) := \min\{q \geq 1 :\ a \in \mathfrak{m}\}$. Thus, each element $a \in A$ can be written as $a = y^{\omega(a)}u$ where u is a unit in A. Pick an integral element $\frac{a}{b} \in \mathrm{qf}(A)$ over A. Write $a = y^{\omega(a)}u$ and $b = y^{\omega(b)}v$ where u, v are units in A. Thus, $\frac{a}{b} = \frac{u}{v}y^{\omega(a)-\omega(b)}$ and we have to show that $\omega(a) - \omega(b) \geq 0$. As $\frac{a}{b} \in \mathrm{qf}(A)$ is integral over A, we deduce that $y^{\omega(a)-\omega(b)}$ is integral over A.

If $\ell := \omega(a) - \omega(b) < 0$, then $\frac{1}{y^\ell}$ is integral over A and the same happens with $\frac{1}{y}$. Thus, there exists a monic polynomial $P \in A[\mathtt{t}]$ such that $P(\frac{1}{y}) = 0$, so $y^{\deg(P)}P(\frac{1}{y}) = 0$. As $0, y \in \mathfrak{m}$, we conclude that $1 \in \mathfrak{m}$, which is a contradiction. Consequently, $\omega(a) - \omega(b) \geq 0$, as required. □

Lemma IV.1.7 (Roots of a unit) *Let $\mathbb{K} = \mathbb{C}$ and let $u \in \mathcal{A}_n$ be a unit and let $p \geq 1$. Let $c \in \mathbb{C}$ be a p-th root of $a := u(0)$. Then there exists a unique unit $v \in \mathcal{A}_n$ such that $v(0) = c$ and $v^p = u$.*

Proof. Consider $f := (\mathtt{y} + c)^p - u \in \mathcal{A}_{n+1}$. Observe that $f(0,0) = 0$, $\frac{\partial f}{\partial \mathtt{y}} = p(\mathtt{y} + c)^{p-1}$ and

$$\frac{\partial f}{\partial \mathtt{y}}(0,0) = pc^{p-1} \neq 0.$$

By the Implicit Function Theorem there exists a unique unit $v := c + g(\mathtt{x}) \in \mathcal{A}_n$ such that $v^p = u$ and $v(0) = c + g(0) = c$, as required. □

Corollary IV.1.8 *Let $f \in \mathbb{C}\langle\langle \mathtt{t} \rangle\rangle$ be a power series of order p. Then there exists an automorphism of $\mathbb{C}\langle\langle \mathtt{t} \rangle\rangle$ that maps f to $\mathtt{t}^p$.*

Proof. By Lemma IV.1.7 we can write $f(\mathtt{t}) = \mathtt{t}^p u = (\mathtt{t}v)^p$ where $u, v \in \mathbb{C}\langle\langle \mathtt{t} \rangle\rangle$ are units. The $\mathbb{C}$-algebra homomorphism $\varphi : \mathbb{C}\langle\langle \mathtt{t} \rangle\rangle \to \mathbb{C}\langle\langle \mathtt{t} \rangle\rangle,\ g(\mathtt{t}) \mapsto g(\mathtt{t}v)$ is an isomorphism by Corollary II.1.10 and $\varphi^{-1}(f) = \mathtt{t}^p$, as required. □

Lemma IV.1.9 *Let $\mathbb{K} = \mathbb{C}$ and let $A := \mathcal{A}_n/\mathfrak{a}$ for an ideal $\mathfrak{a}$ of $\mathcal{A}_n$. Let B be an integral domain such that A is a subring of B and B is a finitely generated A-module. Then there exists $m \geq 1$ such that $B = \mathcal{A}_m/\mathfrak{q}$ for some prime ideal $\mathfrak{q}$ of $\mathcal{A}_m$.*

Proof. The proof is conducted in several steps:

1.a.4 We may assume: $A = \mathcal{A}_d$.

By Noether's Normalization Lemma II.2.17, we can suppose that there exists an injective homomorphism $\mathcal{A}_d \hookrightarrow A$ and A is a finitely generated $\mathcal{A}_d$-module. Thus, B is a finitely generated $\mathcal{A}_d$-module.

1.a.5 It holds: *B is a local ring.* In addition, *if $\mathfrak{m}$ is the maximal ideal of B, for each $b \in \mathfrak{m}$ there exists a distinguished polynomial $P \in \mathcal{A}_d[\mathtt{t}]$ such that $P(b) = 0$.*

As there exists an injective homomorphism $\mathcal{A}_d \hookrightarrow B$ and B is a finitely generated $\mathcal{A}_d$-module (that is, B is integral over $\mathcal{A}_d$), $\dim(B) = \dim(\mathcal{A}_d) = d$.

Let $\mathfrak{m}$ be a maximal ideal of B of height d. By Going-up Theorem II.2.8(ii) we have $\mathfrak{m} \cap \mathcal{A}_d = \mathfrak{m}_d$ is the maximal ideal of $\mathcal{A}_d$.

Let $b \in B$ and let $Q \in \mathcal{A}_d[\mathtt{t}]$ be a monic polynomial such that $Q(b) = 0$. Observe that Q is regular with respect to $\mathtt{t}$. By the Preparation and Division Theorems we have $Q = P \cdot U$, where $P \in \mathcal{A}_d[\mathtt{t}]$ is a distinguished polynomial with respect to $\mathtt{t}$ and $U \in \mathcal{A}_d[\mathtt{t}]$ is a unit in $\mathcal{A}_{d+1}$. This means

$$P := a_0 + \cdots + a_{p-1}\mathtt{t}^{p-1} + \mathtt{t}^p \in \mathcal{A}_d[\mathtt{t}], \quad \text{where } a_0, \ldots, a_{p-1} \in \mathfrak{m}_d$$
$$U := u_0 + \cdots + u_{q-1}\mathtt{t}^{q-1} + \mathtt{t}^q \in \mathcal{A}_d[\mathtt{t}], \quad \text{where } u_0 \notin \mathfrak{m}_d.$$

As B is an integral domain and $P(b)U(b) = Q(b) = 0$, there are two possibilities:

- If $P(b) = 0$, then $b^p = -(a_0 + \cdots + a_1 b^{p-1}) \in \mathfrak{m}_d B \subset \mathfrak{m}$, so $b \in \mathfrak{m}$.
- If $U(b) = 0$, then $bu_0^{-1}(-u_1 + \cdots - u_{q-1}b^{q-2} - b^{q-1}) = 1$, so b is a unit in B.

This shows that B is a local ring, $\mathfrak{m}$ is its maximal ideal and for each $b \in \mathfrak{m}$, there exists a distinguished polynomial $P \in \mathcal{A}_d[\mathtt{t}]$ such that $P(b) = 0$.

1.a.6 *The residual field of B is isomorphic to $\mathbb{C}$.*

As B is integral over $\mathcal{A}_d$ and $\mathfrak{m} \cap \mathcal{A}_d = \mathfrak{m}_d$, the inclusion $\mathbb{C} \cong \mathcal{A}_d/\mathfrak{m}_d \hookrightarrow B/\mathfrak{m}$ is an algebraic field extension, so $B/\mathfrak{m} \cong \mathcal{A}_d/\mathfrak{m}_d \cong \mathbb{C}$ because $\mathbb{C}$ is algebraically closed.

1.a.7 *It holds: $B = \mathbb{C} \oplus \mathfrak{m}$.*

As $\mathbb{C} \hookrightarrow \mathcal{A}_d \hookrightarrow B$ and all the elements of $\mathbb{C} \setminus \{0\}$ are units, we deduce that $\mathbb{C} \cap \mathfrak{m} = \{0\}$. Let $b \in B$ and let $c \in \mathbb{C}$ such that $b + \mathfrak{m} = c + \mathfrak{m}$. Thus, $b - c = f \in \mathfrak{m}$ and $b = c + f \in \mathbb{C} + \mathfrak{m}$.

1.a.8 *There exist elements $b_1, \ldots, b_m \in \mathfrak{m}$ such that $B = \mathcal{A}_d[b_1, \ldots, b_m]$.*

As B is a finitely generated $\mathcal{A}_d$-module, there exist $f_1, \ldots, f_m \in B$ such that $B = \sum_{j=1}^m \mathcal{A}_d f_j = \mathcal{A}_d[f_1, \ldots, f_m]$. As $f_i = c_i + b_i$ where $c_i \in \mathbb{C}$ and $b_i \in \mathfrak{m}$, we have $B = \mathcal{A}_d[b_1, \ldots, b_m]$.

1.a.9 *Let $\mathcal{A}_{n+m} := \mathbb{C}\langle\langle \mathtt{x}_1, \ldots, \mathtt{x}_n, \mathtt{y}_1, \ldots, \mathtt{y}_m \rangle\rangle$. Then there exists a $\mathbb{K}$-algebra epimorphism $\Phi : \mathcal{A}_{n+m} \to B$.* Consequently, by the first isomorphy theorem $B = \operatorname{im}(\Phi) \cong \mathcal{A}_{n+m}/\ker(\Phi)$, as required.

Write $\mathtt{y} := (\mathtt{y}_1, \dots, \mathtt{y}_m)$. Then there exists a surjective $\mathbb{K}$-algebra homomorphism

$$\varphi : \mathcal{A}_d[\mathtt{y}] \to B = \mathcal{A}_d[b_1, \dots, b_m],\ f \mapsto f(b_1, \dots, b_m).$$

Let us extend φ to a $\mathbb{K}$-algebra epimorphism $\Phi : \mathcal{A}_{n+m} \to B$.

By 1.a.4 for each $b_i \in \mathfrak{m}$ there exists a distinguished polynomial $P_i \in \mathcal{A}_d[\mathtt{y}_i]$ such that $P_i(b_i) = 0$. After division by the polynomials P_i, we write each $f \in \mathcal{A}_{n+m}$ as:

$$f = g_1P_1 + \cdots + g_mP_m + R,$$

where $R \in \mathcal{A}_d[\mathtt{y}]$ and each $g_j \in \mathcal{A}_{n+m}$. We define $\Phi(f) := \varphi(R)$.

1.a.10 Let us check: Φ *is a well-defined homomorphism.* Once this is done, as $\Phi|_{\mathcal{A}_d[\mathtt{y}]} = \varphi$ and φ is surjective, we conclude that Φ is the $\mathbb{K}$-algebra epimorphism we sought.

1.a.11 To prove 1.a.10 it is enough to show: *if* $R \in \mathcal{A}_d[\mathtt{y}] \cap \{P_1, \dots, P_m\}\mathcal{A}_{n+m}$, *then* $\varphi(R) = 0$. Once this is done, observe that if $f_1, f_2 \in \mathcal{A}_{n+m}$, we write $f_i = g_{i1}P_1 + \cdots + g_{im}P_m + R_i$ where $R_i \in \mathcal{A}_d[\mathtt{y}]$ and each $g_{ij} \in \mathcal{A}_{n+m}$. Then

$$\begin{aligned} f_1 + f_2 &= R_1 + R_2 + \sum_{j=1}^{m} (g_{1j} + g_{2j})P_j, \\ f_1f_2 &= R_1R_2 + \sum_{j=1}^{m} (g_{1j}f_2 + R_1g_{2j})P_j. \end{aligned}$$

Consequently,

$$\begin{aligned} &\Phi(f_1 + f_2) = \varphi(R_1 + R_2) = \varphi(R_1) + \varphi(R_2) = \Phi(f_1) + \Phi(f_2), \\ &\Phi(f_1f_2) = \varphi(R_1R_2) = \varphi(R_1)\varphi(R_2) = \Phi(f_1)\Phi(f_2), \\ &\Phi(c) = c \quad \text{for each } c \in \mathbb{C}, \end{aligned}$$

so Φ is a $\mathbb{K}$-algebra homomorphism, as required.

1.a.12 To finish we prove 1.a.11. Pick $R \in \mathcal{A}_d[\mathtt{y}] \cap \{P_1, \dots, P_m\}\mathcal{A}_{n+m}$. Thus, there exist $g_1, \dots, g_m \in \mathcal{A}_{n+m}$ such that $R = g_1P_1 + \cdots + g_mP_m$. We can write $g_i = g_i^{(r)} + g_i'$ where $g_i^{(r)} \in \mathcal{A}_d[\mathtt{y}]$ and $g_i' \in \mathfrak{q}^r$ where $\mathfrak{q} := \{\mathtt{y}_1, \dots, \mathtt{y}_m\}\mathcal{A}_{n+m}$. Thus,

$$R = g_1^{(r)}P_1 + \cdots + g_m^{(r)}P_m + h_r$$

where $h_r \in \mathfrak{q}^r$. Thus,

$$h_r = R - (g_1^{(r)}P_1 + \cdots + g_m^{(r)}P_m) \in \mathfrak{q}^r \cap \mathcal{A}_d[\mathtt{y}] = (\mathfrak{q} \cap \mathcal{A}_d[\mathtt{y}])^r.$$

Note that $\mathfrak{q}_0 := \mathfrak{q} \cap \mathcal{A}_d[\mathtt{y}] = \{\mathtt{y}_1, \dots, \mathtt{y}_m\}\mathcal{A}_d[\mathtt{y}]$. Consider the ideals

$$\mathfrak{a} := \{P_1, \dots, P_m\}\mathcal{A}_d[\mathtt{y}] \subset \{\mathtt{x}_1, \dots, \mathtt{x}_d, \mathtt{y}_1, \dots, \mathtt{y}_m\}\mathcal{A}_d[\mathtt{y}] =: \mathfrak{p}$$

and observe that $\mathfrak{p}$ is a prime ideal of $\mathcal{A}_d[\mathtt{y}]$ that contains $\mathfrak{q} \cap \mathcal{A}_d[\mathtt{y}]$. As $R \in \bigcap_{r \geq 0}(\mathfrak{a} + \mathfrak{p}^r)$, we deduce by Krull's Theorem applied to the local ring $(\mathcal{A}_d[\mathtt{y}]_{\mathfrak{p}}, \mathfrak{p}\mathcal{A}_d[\mathtt{y}]_{\mathfrak{p}})$ that

$$R \in \bigcap_{r \geq 0}(\mathfrak{a}\mathcal{A}_d[\mathtt{y}]_{\mathfrak{p}} + (\mathfrak{p}\mathcal{A}_d[\mathtt{y}]_{\mathfrak{p}})^r) = \mathfrak{a}\mathcal{A}_d[\mathtt{y}]_{\mathfrak{p}}.$$

Consequently, there exist $a_0 \in \mathcal{A}_d[\mathtt{y}] \setminus \mathfrak{p}$ and $a_1, \dots, a_m \in \mathcal{A}_d[\mathtt{y}]$ such that

$$\begin{aligned} &a_0 R = a_1 P_1 + \cdots + a_m P_m \\ &\qquad\qquad \rightsquigarrow \quad \varphi(a_0)\varphi(R) = \varphi(a_1)\varphi(P_1) + \cdots + \varphi(a_m)\varphi(P_m) = 0. \end{aligned}$$

1.a.13 As B is an integral domain, it only remains to check: $\varphi(a_0) \neq 0$.

Write $a_0 = c_0 + c_1$ where $c_0 \in \mathcal{A}_d$ and $c_1 \in \mathfrak{q}_0$. Observe that

$$\varphi(a_0) = \varphi(c_0) + \varphi(c_1) = c_0 + \varphi(c_1)$$

where $\varphi(c_1) \in \mathfrak{m}$. If $\varphi(a_0) = 0$, we have $c_0 \in \mathfrak{m} \cap \mathcal{A}_d = \mathfrak{m}_d$, so

$$a_0 = c_0 + c_1 \in \{\mathtt{x}_1, \dots, \mathtt{x}_d, \mathtt{y}_1, \dots, \mathtt{y}_m\}\mathcal{A}_d[\mathtt{y}] = \mathfrak{p},$$

which is a contradiction. Consequently, $\varphi(a_0) \neq 0$, as required. □

We are ready to prove Theorem IV.1.5.

Proof of Theorem IV.1.5. Let us prove that each polynomial $P \in \mathbb{C}(\langle \mathtt{t}^* \rangle)[\mathtt{y}]$ of degree $n \geq 1$ has some root in $\mathbb{C}(\langle \mathtt{t}^* \rangle)$. The proof is conducted in several steps:

1.a.14 *Reduction to the case of an irreducible distinguished polynomial* $P \in \mathbb{C}\langle\langle \mathtt{x} \rangle\rangle[\mathtt{y}]$. We may assume that $P \in \mathbb{C}\langle\langle \mathtt{t}^* \rangle\rangle[\mathtt{y}]$ and let $a \in \mathbb{C}\langle\langle \mathtt{t}^* \rangle\rangle \setminus \{0\}$ be the leading coefficient of P. After changing, if necessary, P by the polynomial $a^{n-1}P(\frac{\mathtt{y}}{a}) \in \mathbb{C}\langle\langle \mathtt{t}^* \rangle\rangle \setminus \{0\}$, we may assume in addition that P is monic. Write $n := \deg_{\mathtt{y}}(P)$ and

$$P = \mathtt{y}^n + a_{n-1}(\mathtt{t}^{1/q})\mathtt{y}^{n-1} + \cdots + a_0(\mathtt{t}^{1/q}),$$

where $a_0, \dots, a_{n-1} \in \mathbb{C}(\langle \mathtt{x} \rangle)$ and $q \geq 1$. Denote

$$P^* = \mathtt{y}^n + a_{n-1}(\mathtt{x})\mathtt{y}^{n-1} + \cdots + a_0(\mathtt{x}).$$

If we substitute P by P^*, we may assume $P \in \mathbb{C}\langle\langle \mathtt{x} \rangle\rangle[\mathtt{y}]$. Let $c \in \mathbb{C}$ be a root of $P(0,\mathtt{y}) \in \mathbb{C}[\mathtt{y}]$. After the change $\mathtt{y} := \mathtt{t} + c$ we may assume $c = 0$. Thus, $P(0,0) = 0$, so $P \in \mathbb{C}\langle\langle \mathtt{x} \rangle\rangle[\mathtt{y}]$ is a regular series of order ≥ 1.

By Weierstrass's Preparation Theorem I.4.7 and Lemma II.1.4 there exist $U \in \mathbb{C}\langle\langle \mathtt{x},\mathtt{y} \rangle\rangle$ and irreducible distinguished polynomials $P_1, \ldots, P_r \in \mathbb{C}\langle\langle \mathtt{x} \rangle\rangle[\mathtt{y}]$ such that $P = U \cdot P_1 \cdots P_r$. After Substituting P by P_1 we may assume that $P \in \mathbb{C}\langle\langle \mathtt{x} \rangle\rangle[\mathtt{y}]$ is an irreducible distinguished polynomial.

1.a.15 *Proof of the statement for an irreducible distinguished polynomial* $P \in \mathbb{C}\langle\langle \mathtt{x} \rangle\rangle[\mathtt{y}]$. By I.1.a.3 P is irreducible in $\mathbb{C}\langle\langle \mathtt{x},\mathtt{y} \rangle\rangle$. Consequently,

$$A := \mathbb{C}\langle\langle \mathtt{x},\mathtt{y} \rangle\rangle/(P\mathbb{C}\langle\langle \mathtt{x},\mathtt{y} \rangle\rangle)$$

is an integral domain. By Lemma III.2.14 A has dimension 1 (use alternatively Exercise number II.2 and Lemma II.2.12). The integral closure $\overline{A}$ of A in $\mathrm{qf}(A)$ has $\dim(\overline{A}) = \dim(A) = 1$. By Noether's Normalization Lemma II.2.17 the ring $\overline{A}$ is a finitely generated A-module. By Lemma IV.1.9 there exist $m \geq 1$ and a prime ideal $\mathfrak{q}$ of $\mathcal{A}_m$ such that $\overline{A} \cong \mathcal{A}_m/\mathfrak{q}$. In particular, $\overline{A}$ is a noetherian local integral domain of dimension 1 that is in addition integrally closed. By Lemma IV.1.6 $\overline{A} \cong \mathcal{A}_m/\mathfrak{q}$ is a regular local ring of dimension 1. By Lemma III.1.8 $\overline{A} \overset{\phi}{\cong} \mathbb{C}\langle\langle \mathtt{x} \rangle\rangle$.

Consider the sequence of homomorphisms

$$\mathbb{C}\langle\langle \mathtt{x},\mathtt{y} \rangle\rangle \overset{\pi}{\to} A := \mathbb{C}\langle\langle \mathtt{x},\mathtt{y} \rangle\rangle/(P\mathbb{C}\langle\langle \mathtt{x},\mathtt{y} \rangle\rangle) \overset{\mathtt{i}}{\hookrightarrow} \overline{A} \overset{\phi}{\to} \mathbb{C}\langle\langle \mathtt{x} \rangle\rangle$$

and denote $\varphi := \phi \circ \mathtt{i} \circ \pi$. Then $\varphi(P) = 0$ while $\varphi(\mathtt{x}) \neq 0$, because otherwise $\mathtt{x}$ would belong to $P\mathbb{C}\langle\langle \mathtt{x},\mathtt{y} \rangle\rangle$ and P would divide $\mathtt{x}$, which is a contradiction. Thus, $\varphi(\mathtt{x}) \in \mathbb{C}\langle\langle \mathtt{x} \rangle\rangle$ is a series of order $p > 0$. By Corollary IV.1.8 there exists an isomorphism $\psi : \mathbb{C}\langle\langle \mathtt{x} \rangle\rangle \to \mathbb{C}\langle\langle \mathtt{x} \rangle\rangle$ such that $\psi(\varphi(\mathtt{x})) = \mathtt{x}^p$. Define $\xi := \psi(\varphi(\mathtt{y})) \in \mathbb{C}\langle\langle \mathtt{x} \rangle\rangle$ and observe that

$$0 = \psi(\varphi(P)) = P(\psi(\varphi(\mathtt{x})), \psi(\varphi(\mathtt{y}))) = P(\mathtt{x}^p, \xi).$$

Consequently, $P(\mathtt{t}, \xi(\mathtt{t}^{1/p})) = 0$ where $\xi(\mathtt{t}^{1/p}) \in \mathbb{C}\langle\langle \mathtt{t}^* \rangle\rangle$, as required. □

The preceding proof can be examined more carefully to get further interesting information.

Lemma IV.1.10 *Let* $p \geq 1$ *and let* $\xi := e^{2\pi i/p}$. *Then* $\mathbb{C}(\langle \mathtt{t}^{1/p} \rangle) = \mathbb{C}(\langle \mathtt{t} \rangle)[\mathtt{t}^{1/p}]$ *and the field extension* $\mathbb{C}(\langle \mathtt{t}^{1/p} \rangle)|\mathbb{C}(\langle \mathtt{t} \rangle)$ *is a Galois extension of degree* p. *In* *addition, the Galois group* $G(\mathbb{C}(\langle \mathtt{t}^{1/p}\rangle) : \mathbb{C}(\langle \mathtt{t}\rangle))$ *is cyclic of order* p *and it is generated by the automorphism*

$$\sigma : \mathbb{C}(\langle \mathtt{t}^{1/p}\rangle) \to \mathbb{C}(\langle \mathtt{t}^{1/p}\rangle),\ f(\mathtt{t}^{1/p}) \mapsto f(\xi \mathtt{t}^{1/p}).$$

Proof. Consider the polynomial $P := \mathtt{y}^p - \mathtt{t} \in \mathbb{C}\langle\langle \mathtt{t}\rangle\rangle[\mathtt{y}]$, which is irreducible in $\mathbb{C}\langle\langle \mathtt{t}\rangle\rangle[\mathtt{y}]$ by Einsenstein's criterion because $\mathtt{t}$ is an irreducible element of the unique factorization domain $\mathbb{C}\langle\langle \mathtt{t}\rangle\rangle$. Thus, P is also irreducible in $\mathbb{C}(\langle \mathtt{t}\rangle)[\mathtt{y}]$. As $\mathtt{t}^{1/p}$ is a root of P in $\mathbb{C}(\langle \mathtt{t}^{1/p}\rangle)$, we deduce that P is the irreducible polynomial of the algebraic element $\mathtt{t}^{1/p}$ of $\mathbb{C}(\langle \mathtt{t}^{1/p}\rangle)$ over $\mathbb{C}(\langle \mathtt{t}\rangle)$. As the roots of P are $\xi^k \mathtt{t}^{1/p}$ for $k = 0, \ldots, p-1$ and $\xi \in \mathbb{C}$, the field extension $\mathbb{C}(\langle \mathtt{t}\rangle)[\mathtt{t}^{1/p}]|\mathbb{C}(\langle \mathtt{t}\rangle)$ is a Galois extension of degree p.

Let $h \in \mathbb{C}(\langle \mathtt{t}^{1/p}\rangle)$. There exist $m \in \mathbb{Z}$ and a unit u in $\mathbb{C}\langle\langle \mathtt{t}^{1/p}\rangle\rangle$ such that $h := \mathtt{t}^{m/p}u$. Write $u := v(\mathtt{t}^{1/p})$ where $v \in \mathbb{C}\langle\langle \mathtt{y}\rangle\rangle$ is a unit. By Weierstrass's Division Theorem $v(\mathtt{y}) = (\mathtt{y}^p - \mathtt{t})Q(\mathtt{y}, \mathtt{t}) + R(\mathtt{y}, \mathtt{t})$ where $Q \in \mathbb{C}\langle\langle \mathtt{t}, \mathtt{y}\rangle\rangle$ and $R \in \mathbb{C}\langle\langle \mathtt{t}\rangle\rangle[\mathtt{y}]$ has $\deg_{\mathtt{y}}(R) < p$. Thus, $u = v(\mathtt{t}^{1/p}) = R(\mathtt{t}^{1/p}, \mathtt{t}) \in \mathbb{C}(\langle \mathtt{t}\rangle)[\mathtt{t}^{1/p}]$, hence $h = \mathtt{t}^{m/p}u \in \mathbb{C}(\langle \mathtt{t}\rangle)[\mathtt{t}^{1/p}]$ and $\mathbb{C}(\langle \mathtt{t}^{1/p}\rangle) = \mathbb{C}(\langle \mathtt{t}\rangle)[\mathtt{t}^{1/p}]$. Consequently, $\mathbb{C}(\langle \mathtt{t}^{1/p}\rangle)|\mathbb{C}(\langle \mathtt{t}\rangle)$ is a Galois extension of degree p and $G(\mathbb{C}(\langle \mathtt{t}^{1/p}\rangle) : \mathbb{C}(\langle \mathtt{t}\rangle))$ has order p.

As σ is a $\mathbb{C}(\langle \mathtt{t}\rangle)$-automorphism and $\sigma^k \neq \sigma^j$ for $0 \leq k < j \leq p-1$, we conclude that $G(\mathbb{C}(\langle \mathtt{t}^{1/p}\rangle) : \mathbb{C}(\langle \mathtt{t}\rangle))$ is a cyclic group of order p and it is generated by σ, as required. □

As $\mathbb{C}(\langle \mathtt{t}^*\rangle) = \bigcup_{p\geq 1} \mathbb{C}(\langle \mathtt{t}^{1/p}\rangle)$, we conclude the following.

Corollary IV.1.11 *The field* $\mathbb{C}(\langle \mathtt{t}^*\rangle)$ *is the algebraic closure of* $\mathbb{C}(\langle \mathtt{t}\rangle)$.

Corollary IV.1.12 *Let* $P \in \mathbb{C}\langle\langle \mathtt{t}\rangle\rangle[\mathtt{x}]$ *be an irreducible distinguished polynomial. Then:*

(i) $\deg_{\mathtt{x}}(P)$ *is the smallest integer* p *such that* P *has a root* $h \in \mathbb{C}\langle\langle \mathtt{t}^{1/p}\rangle\rangle$.

(ii) *If* $\xi := e^{2\pi i/p}$, *then* $P = \prod_{k=1}^{p}(\mathtt{x} - f(\xi^k \mathtt{t}^{1/p}))$, *where* $f \in \mathbb{C}\langle\langle \mathtt{x}\rangle\rangle$ *satisfies* $h = f(\mathtt{t}^{1/p})$, *and the splitting field of* P *over* $\mathbb{C}(\langle \mathtt{t}\rangle)$ *is* $\mathbb{C}(\langle \mathtt{t}^{1/p}\rangle)$.

Proof. Let $h \in \mathbb{C}(\langle \mathtt{t}^*\rangle)$ be a root of P and let $p \geq 1$ be the smallest positive integer such that $h \in \mathbb{C}(\langle \mathtt{t}^{1/p}\rangle)$. As $P \in \mathbb{C}\langle\langle \mathtt{t}\rangle\rangle[\mathtt{x}]$ is monic and $\mathbb{C}(\langle \mathtt{t}^{1/p}\rangle)$ is integrally closed, $h \in \mathbb{C}\langle\langle \mathtt{t}^{1/p}\rangle\rangle$.

Recall that the Galois group of the Galois extension $\mathbb{C}(\langle \mathtt{t}^{1/p}\rangle)|\mathbb{C}(\langle \mathtt{t}\rangle)$ is the cyclic group of order p generated by the automorphism

$$\sigma : \mathbb{C}(\langle \mathtt{t}^{1/p}\rangle) \to \mathbb{C}(\langle \mathtt{t}^{1/p}\rangle),\ g(\mathtt{t}^{1/p}) \mapsto g(\xi \mathtt{t}^{1/p}).$$

Write $h = f(\mathtt{t}^{1/p})$ where $f \in \mathbb{C}\langle\langle \mathtt{x}\rangle\rangle$. The image of h under σ^k is $f(\xi^k \mathtt{t}^{1/p})$ for $k = 1, \dots, p$. By Cardano-Viète formulae $Q := \prod_{k=1}^{p}(\mathtt{x} - f(\xi^k \mathtt{t}^{1/p})) \in \mathbb{C}\langle\langle \mathtt{t}\rangle\rangle[\mathtt{x}]$ because its coefficients are fixed under σ. Thus, P divides Q because P is the irreducible polynomial of h over $\mathbb{C}(\langle \mathtt{t}^*\rangle)$.

The roots of P are those $f(\xi^k \mathtt{t}^{1/p})$ for $k = 1, \dots, p$ that are pairwise different because $\mathbb{C}(\langle \mathtt{t}^{1/p}\rangle)|\mathbb{C}(\langle \mathtt{t}\rangle)$ is a Galois extension and its Galois group is $\langle\sigma\rangle$. If we prove that the elements $f(\xi^k \mathtt{t}^{1/p})$ are pairwise different, we will conclude that $P = Q$ and (i) and (ii) will follow because both extensions $\mathbb{C}(\langle \mathtt{t}^{1/p}\rangle)|\mathbb{C}(\langle \mathtt{t}\rangle)$ and $\mathbb{C}(\langle \mathtt{t}\rangle)[h]|\mathbb{C}(\langle \mathtt{t}\rangle)$ have degree p and $h \in \mathbb{C}(\langle \mathtt{t}^{1/p}\rangle)$.

By Weierstrass's Division Theorem $f(\mathtt{x}) = (\mathtt{x}^p - \mathtt{t})Q(\mathtt{x}, \mathtt{t}) + R(\mathtt{x}, \mathtt{t})$ where $Q \in \mathbb{C}\langle\langle \mathtt{t}, \mathtt{x}\rangle\rangle$ and $R \in \mathbb{C}\langle\langle \mathtt{t}\rangle\rangle[\mathtt{x}]$ is a polynomial with $\deg_{\mathtt{x}}(R) < p$. Thus, $h := f(\mathtt{t}^{1/p}) = R(\mathtt{t}^{1/p}, \mathtt{t})$. Write $R := a_0 + a_1\mathtt{x} + \cdots + a_{p-1}\mathtt{x}^{p-1}$ where each $a_i \in \mathbb{C}\langle\langle \mathtt{t}\rangle\rangle$. Denote

$$\mathcal{F} := \{j = 1, \dots, p-1 :\ a_j \neq 0\} := \{j_1, \dots, j_s\}.$$

Notice that, as p is the smallest positive integer such that $h \in \mathbb{C}(\langle \mathtt{t}^{1/p}\rangle)$, we have $\gcd(p, \mathcal{F}) = 1$. If $f(\xi^k \mathtt{t}^{1/p}) = f(\mathtt{t}^{1/p})$ for some $1 \leq k \leq p$, we deduce $\xi^{kj_\ell} = 1$ for each $\ell = 1, \dots, s$. As $\gcd(p, \mathcal{F}) = 1$, there exist integers $m_0, m_1, \dots, m_s$ such that $1 = m_0 p + m_1 j_1 + \cdots + m_s j_s$. Thus,

$$\xi^k = (\xi^k)^{m_0 p + m_1 j_1 + \cdots + m_s j_s} = (\xi^p)^{k m_0} \prod_{\ell=1}^{s} (\xi^{k j_\ell})^{m_j} = 1,$$

so $k = p$. If $f(\xi^{k_1} \mathtt{t}^{1/p}) = f(\xi^{k_2} \mathtt{t}^{1/p})$ for some $1 \leq k_1 \leq k_2 \leq p$, we have

$$f(\mathtt{t}^{1/p}) = \sigma^{-k_1}(f(\xi^{k_1} \mathtt{t}^{1/p})) = \sigma^{-k_1}(f(\xi^{k_2} \mathtt{t}^{1/p})) = f(\xi^{k_2 - k_1} \mathtt{t}^{1/p}),$$

so $k_2 = k_1$. We conclude that the elements $f(\xi^k \mathtt{t}^{1/p})$ are pairwise different for $k = 1, \dots, p$, as required. $\square$

2 Rückert's Nullstellensatz

Before presenting the algebraic version of Rückert's Nullstellensatz we recall Hilbert's Nullstellensatz in order to motivate its statement. Let $\mathfrak{a}$ be an ideal of $\mathbb{C}[\mathtt{x}] := \mathbb{C}[\mathtt{x}_1, \ldots, \mathtt{x}_n]$ and denote $Z(\mathfrak{a}) := \{x \in \mathbb{C}^n : f(x) = 0 \; \forall f \in \mathfrak{a}\}$. Consider the set $\mathcal{F}(\mathfrak{a})$ of all the $\mathbb{C}$-algebra homomorphisms $\varphi : \mathbb{C}[\mathtt{x}] \to \mathbb{C}$ such that $\mathfrak{a} \subset \ker(\varphi)$. A $\mathbb{C}$-algebra homomorphism $\varphi : \mathbb{C}[\mathtt{x}] \to \mathbb{C}$ is surjective, so $\ker(\varphi)$ is a maximal ideal. In addition, $\varphi(f) = f(p)$ where $p := (\varphi(\mathtt{x}_1), \ldots, \varphi(\mathtt{x}_n)) \in \mathbb{C}^n$, so $\ker(\varphi) = \mathfrak{m}_p := \{f \in \mathbb{C}[\mathtt{x}] : \; f(p) = 0\}$ is the maximal ideal of $\mathbb{C}[\mathtt{x}]$ associated to p. We can write $\varphi := \varphi_p$ and have $p \in Z(\mathfrak{a})$. Given $Z \subset \mathbb{C}^n$ denote $I(Z) := \{f \in \mathbb{C}[\mathtt{x}] : \; f(x) = 0 \; \forall x \in \mathbb{C}^n\}$.

Hilbert's Nullstellensatz states $\sqrt{\mathfrak{a}} = I(Z(\mathfrak{a}))$. In addition, it follows from our previous comments:

$$I(Z(\mathfrak{a})) = \bigcap_{\varphi \in \mathcal{F}(\mathfrak{a})} \ker(\varphi) = \bigcap_{p \in Z(\mathfrak{a})} \ker(\varphi_p).$$

The purpose of this section is to prove Rückert's Nullstellensatz in its algebraic version. Given an ideal $\mathfrak{a}$ of $\mathcal{A}_n$ we denote $\mathcal{F}(\mathfrak{a})$ the set of all the $\mathbb{C}$-algebra homomorphisms $\varphi : \mathcal{A}_n \to \mathbb{C}\langle\langle \mathtt{t}^* \rangle\rangle$ such that $\mathfrak{a} \subset \ker(\varphi)$.

Theorem IV.2.1 (Rückert's Nullstellensatz) *If $\mathfrak{a}$ is an ideal of $\mathcal{A}_n$, then*

$$\sqrt{\mathfrak{a}} = \bigcap_{\varphi \in \mathcal{F}(\mathfrak{a})} \ker(\varphi).$$

2.a Preliminary results. Before proving this theorem we would like to analyze the behaviour of the $\mathbb{C}$-algebra homomorphisms $\varphi : \mathcal{A}_n \to \mathbb{C}\langle\langle \mathtt{t}^* \rangle\rangle$.

Lemma IV.2.2 *Let $\varphi : \mathcal{A}_n \to \mathbb{C}\langle\langle \mathtt{t}^* \rangle\rangle$ be a $\mathbb{C}$-algebra homomorphism. Then φ is a local homomorphism, it is defined for each $f \in \mathcal{A}_n$ by the substitution*

$$\varphi(f) = f(\xi_1, \ldots, \xi_n)$$

where $\xi_i := \varphi(\mathtt{x}_i) \in \mathfrak{m}^*$ for $1 \leq i \leq n$ and $\varphi(\mathcal{A}_n) \subset \mathbb{C}\langle\langle \mathtt{t}^{1/p} \rangle\rangle$ for some $p \geq 1$ large enough.

Proof. If $\varphi(\mathfrak{m}_n) \not\subset \mathfrak{m}^*$, there exists $f \in \mathfrak{m}_n$ such that $\varphi(f) = a + g$ where $a \in \mathbb{C} \setminus \{0\}$ and $g \in \mathfrak{m}^*$. Then $\varphi(f - a) = g$ is not a unit in $\mathbb{C}\langle\langle \mathtt{t}^* \rangle\rangle$, while $f - a$ is a unit in $\mathcal{A}_n$, which is a contradiction.

Write $\xi_i := \varphi(\mathtt{x}_i) \in \mathfrak{m}^*$ for $1 \leq i \leq n$ and consider the homomorphism

$$\phi : \mathcal{A}_n \to \mathbb{C}\langle\langle \mathtt{t}^* \rangle\rangle, \; f \mapsto f(\xi_1, \ldots, \xi_n).$$

As $\xi_1,\dots,\xi_n \in \mathbb{C}\langle\langle \mathtt{t}^{1/p}\rangle\rangle \cap \mathfrak{m}^*$ for p large enough, the previous $\mathbb{C}$-algebra homomorphism is well-defined by Theorem II.1.9 and Definitions IV.1.1. We will show that $\varphi = \phi$. This will show in addition that $\varphi(\mathcal{A}_n) \subset \mathbb{C}\langle\langle \mathtt{t}^{1/p}\rangle\rangle$.

As $\mathbb{C}\langle\langle \mathtt{t}^*\rangle\rangle$ is not noetherian, we cannot apply Krull's Theorem as we did in Theorem II.1.9. Write $\xi_i = u_i \mathtt{t}^{m_i/q_i}$ where $u_i \in \mathbb{C}\langle\langle \mathtt{t}^*\rangle\rangle$ is a unit for $i = 1,\dots,n$ and

$$\frac{m_0}{q_0} := \min\left\{\frac{m_1}{q_1},\dots,\frac{m_n}{q_n}\right\}.$$

Suppose by contradiction that there exists $f \in \mathcal{A}_n$ such that $\varphi(f) \neq \phi(f)$ and write $\varphi(f) - \phi(f) = u\mathtt{t}^{m/q}$ where $u \in \mathbb{C}\langle\langle \mathtt{t}^*\rangle\rangle$ is a unit. Let $s \geq 1$ be an integer such that $s\frac{m_0}{q_0} \geq 1 + \frac{m}{q}$. Denote

$$f = g + \sum_{|\nu|=s} h_\nu \mathtt{x}_1^{\nu_1}\cdots \mathtt{x}_n^{\nu_n}$$

where $g \in \mathbb{C}[\mathtt{x}_1,\dots,\mathtt{x}_n]$ and $h_\nu \in \mathcal{A}_n$ for $|\nu| = s$. As φ and ϕ are homomorphisms of $\mathbb{C}$-algebras, $\varphi(g) = \phi(g)$ and

$$\begin{aligned} u\mathtt{t}^{\frac{m}{p}} &= \varphi(f) - \phi(f) = \\ &= \sum_{|\nu|=s} (\varphi(h_\nu) - \phi(h_\nu))\xi_1^{\nu_1}\cdots\xi_n^{\nu_n} = \sum_{|\nu|=s} \zeta_\nu \mathtt{t}^{\nu_1\frac{m_1}{q_1}+\cdots+\nu_n\frac{m_n}{q_n}} = \\ &= \sum_{|\nu|=s} \theta_\nu \mathtt{t}^{(\nu_1+\cdots+\nu_n)\frac{m_0}{q_0}} = \Big(\sum_{|\nu|=s}\theta_\nu\Big)\mathtt{t}^{s\frac{m_0}{q_0}} = \eta \mathtt{t}^{1+\frac{m}{p}}. \end{aligned}$$

where $\zeta_\nu, \theta_\nu, \eta \in \mathbb{C}\langle\langle \mathtt{t}^*\rangle\rangle$. Thus, $u = \eta\mathtt{t}$, which is a contradiction because u is a unit, as required. □

Remark IV.2.3 Let $\mathcal{G}(\mathfrak{a})$ denote the set of those $\mathbb{C}$-algebra homomorphisms $\varphi : \mathcal{A}_n \to \mathbb{C}\langle\langle \mathtt{t}\rangle\rangle$ such that $\mathfrak{a} \subset \ker(\varphi)$. As $\mathbb{C}\langle\langle \mathtt{t}\rangle\rangle$ and $\mathbb{C}\langle\langle \mathtt{t}^{1/p}\rangle\rangle$ are isomorphic as $\mathbb{C}$-algebras, we deduce by Lemma IV.2.2 that

$$\bigcap_{\varphi\in\mathcal{F}(\mathfrak{a})} \ker(\varphi) = \bigcap_{\varphi\in\mathcal{G}(\mathfrak{a})} \ker(\varphi).$$

2.b Proof of Rückert's Nullstellensatz. We are ready to prove Rückert's Nullstellensatz.

Proof of Theorem IV.2.1. By Remark IV.2.3 we have to prove

$$\sqrt{\mathfrak{a}} = \bigcap_{\varphi\in\mathcal{G}(\mathfrak{a})} \ker(\varphi). \tag{2.1}$$

2.b.1 *Reduction to the prime case.* As $\mathbb{C}\langle\langle \mathtt{t} \rangle\rangle$ is an integral domain, $\ker(\varphi)$ is a prime ideal for each $\varphi \in \mathcal{G}(\mathfrak{a})$, so $\mathcal{G}(\mathfrak{a}) = \mathcal{G}(\sqrt{\mathfrak{a}})$. As $\mathcal{A}_n$ is a noetherian ring, there exist prime ideals $\mathfrak{p}_i$ of $\mathcal{A}_n$ with $i, = 1 \dots, r$ such that $\sqrt{\mathfrak{a}} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$. We claim: $\mathcal{G}(\sqrt{\mathfrak{a}}) = \bigcup_{i=1}^r \mathcal{G}(\mathfrak{p}_i)$.

If $\varphi \in \bigcup_{i=1}^r \mathcal{G}(\mathfrak{p}_i)$, then $\varphi \in \mathcal{G}(\mathfrak{p}_i)$ for some $1 \leq i \leq r$, so $\sqrt{\mathfrak{a}} \subset \mathfrak{p}_i \subset \ker(\varphi)$ and $\varphi \in \mathcal{G}(\sqrt{\mathfrak{a}})$. Conversely, if $\varphi \in \mathcal{G}(\sqrt{\mathfrak{a}})$, then $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r = \sqrt{\mathfrak{a}} \subset \ker(\varphi)$ and by Lemma II.2.2 there exists $1 \leq j \leq r$ such that $\mathfrak{p}_j \subset \ker(\varphi)$. Thus, $\varphi \in \mathcal{G}(\mathfrak{p}_j) \subset \bigcup_{i=1}^r \mathcal{G}(\mathfrak{p}_i)$. Consequently,

$$\bigcap_{\varphi \in \mathcal{G}(\mathfrak{a})} \ker(\varphi) = \bigcap_{\varphi \in \bigcup_{i=1}^r \mathcal{G}(\mathfrak{p}_i)} \ker(\varphi) = \bigcap_{i=1}^r \bigcap_{\varphi \in \mathcal{G}(\mathfrak{p}_i)} \ker(\varphi),$$

and it is enough to prove: *if $\mathfrak{p}$ is a prime ideal of $\mathcal{A}_n$, then*

$$\mathfrak{p} = \bigcap_{\varphi \in \mathcal{G}(\mathfrak{p})} \ker(\varphi). \tag{2.2}$$

2.b.2 *Proof for the prime case.* We have to show: *For every $f \notin \mathfrak{p}$ there exists $\varphi \in \mathcal{G}(\mathfrak{p})$ such that $\varphi(f) \neq 0$.*

Write $d := n - \mathrm{ht}(\mathfrak{p})$. After a linear change of coordinates, we may assume by Local Parameterization Theorem II.3.1 and Remark II.3.2:

(i) *The canonical homomorphism $\mathcal{A}_d \hookrightarrow A := \mathcal{A}_n/\mathfrak{p}$ is finite and injective.*

(ii) *There exists an irreducible distinguished polynomial $P \in \mathcal{A}_d[\mathtt{x}_{d+1}]$ with respect to $\mathtt{x}_{d+1}$ whose discriminant $\Delta \in \mathcal{A}_d \setminus \mathfrak{p}$ has the property that the canonical homomorphism $A'_\Delta \to A_\Delta$, where $A' := \mathcal{A}_d[\mathtt{x}_{d+1}]/P\mathcal{A}_d[\mathtt{x}_{d+1}]$, is an isomorphism.*

As $\mathcal{A}_d \hookrightarrow A$ is finite, $\alpha := f + \mathfrak{p}$ is integral over $\mathcal{A}_d$ and there exists a polynomial monic equation

$$\alpha^m + b_{m-1}\alpha^{m-1} + \cdots + b_0 = 0, \tag{2.3}$$

of minimal degree m, where each $b_i \in \mathcal{A}_d$. This minimality implies that $b_0 \neq 0$ because otherwise we can divide (2.3) by α to get another equation of smaller degree (recall that A is an integral domain).

Let $\alpha' := -(\alpha^{m-1} + b_{m-1}\alpha^{m-2} + \cdots + b_1)$ and note that $\alpha\alpha' = b_0 \in \mathcal{A}_d \setminus \{0\}$. Consider the non-zero element $\Delta b_0 \in \mathcal{A}_d$, which is a power series of order $p \geq 0$ and write $\Delta b_0 = \sum_{\ell \geq k} B_\ell \in \mathcal{A}_d$ as the sum of its homogeneous components.

Its initial form $B_k \in \mathbb{C}[\mathtt{x}_1, \ldots, \mathtt{x}_d]$ is a non-zero homogeneous polynomial of degree p and we choose a tuple $z \in \mathbb{C}^d$ such that $B_k(z) \neq 0$. Consider the $\mathbb{C}$-algebra homomorphism $\psi : \mathcal{A}_d \to \mathbb{C}\langle\langle \mathtt{t} \rangle\rangle$, $g \mapsto g(z\mathtt{t})$ and observe that $\psi(\Delta b_0) = \sum_{\ell \geq k} B_\ell(z)\mathtt{t}^\ell \neq 0$, because its initial form is $B_k(z)\mathtt{t}^k \neq 0$.

Write $P := \mathtt{x}_{d+1}^p + a_{p-1}\mathtt{x}_{d+1}^{p-1} + \cdots + a_0 \in \mathcal{A}_d[\mathtt{x}_{d+1}]$. By Newton-Puiseux's Theorem IV.1.5 the polynomial

$$Q := \mathtt{x}_{d+1}^p + \psi(a_{p-1})\mathtt{x}_{d+1}^{p-1} + \cdots + \psi(a_0) \in \mathbb{C}\langle\langle \mathtt{t} \rangle\rangle[\mathtt{x}_{d+1}]$$

has some root $\eta_{d+1} \in \mathbb{C}((\mathtt{t}^*))$. As Q is monic and $\mathbb{C}\langle\langle \mathtt{t}^* \rangle\rangle$ is integrally closed, $\eta_{d+1} \in \mathbb{C}\langle\langle \mathtt{t}^* \rangle\rangle$ and let $p \geq 1$ be such that $\eta_{d+1} \in \mathbb{C}\langle\langle \mathtt{t}^{1/p} \rangle\rangle$. The $\mathbb{C}$-algebra homomorphism

$$\Psi : A' := \mathcal{A}_d[\mathtt{x}_{d+1}]/P\mathcal{A}_d[\mathtt{x}_{d+1}] \to \mathbb{C}\langle\langle \mathtt{t}^{1/p} \rangle\rangle,\ R + P\mathcal{A}_d[\mathtt{x}_{d+1}] \mapsto R(z\mathtt{t}, \eta_{d+1}),$$

is well-defined because $Q(\eta_{d+1}) = 0$, and extends ψ. As $\Psi(\Delta) = \psi(\Delta) \neq 0$, the $\mathbb{C}$-algebra homomorphism Ψ extends naturally to a $\mathbb{C}$-algebra homomorphism

$$\Psi_\Delta : A'_\Delta \to \mathbb{C}((\mathtt{t}^{1/p})),\ \frac{R + P\mathcal{A}_d[\mathtt{x}_{d+1}]}{\Delta^r} \mapsto \frac{\Psi(R)}{\psi(\Delta)^r}.$$

Denote $\theta_p : \mathbb{C}\langle\langle \mathtt{t}^{1/p} \rangle\rangle \to \mathbb{C}\langle\langle \mathtt{s} \rangle\rangle$ the inverse of the isomorphism

$$\tau_p : \mathbb{C}\langle\langle \mathtt{s} \rangle\rangle \to \mathbb{C}\langle\langle \mathtt{t}^{1/p} \rangle\rangle,\ h \mapsto h(t^{1/p})$$

and denote $\Theta_p : \mathbb{C}((\mathtt{t}^{1/p})) \to \mathbb{C}((\mathtt{s}))$ its natural extension to the quotient fields. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{A}_n \arrow[r, "\pi"] & A \arrow[r, hook, "\mathtt{i}"] & A_\Delta \arrow[r, leftrightarrow, "\rho", "\cong"'] & A'_\Delta \arrow[r, "\Psi_\Delta"] & \mathbb{C}((\mathtt{t}^{1/p})) \arrow[r, "\Theta_p"] & \mathbb{C}((\mathtt{s})) \\
 & & & A' \arrow[u, hook] \arrow[ull, hook] \arrow[r, "\Psi"] & \mathbb{C}\langle\langle \mathtt{t}^{1/p} \rangle\rangle \arrow[u, hook] \arrow[r, "\theta_p"] & \mathbb{C}\langle\langle \mathtt{s} \rangle\rangle \arrow[u, hook] \\
 & & & \mathcal{A}_d \arrow[u, hook] \arrow[uull, hook] \arrow[r, "\psi"] & \mathbb{C}\langle\langle \mathtt{t}^{1/p} \rangle\rangle \arrow[u, leftrightarrow, "\mathrm{id}"] \arrow[r, "\theta_p"] & \mathbb{C}\langle\langle \mathtt{s} \rangle\rangle \arrow[u, leftrightarrow, "\mathrm{id}"']
\end{tikzcd}$$

Denote $\phi := \Theta_p \circ \Psi_\Delta \circ \rho \circ \mathtt{i} : A \to \mathbb{C}((\mathtt{s}))$. As A is integral over $\mathcal{A}_d$, the ring $\phi(A)$ is integral over the ring $(\theta_p \circ \psi)(\mathcal{A}_d) \subset \mathbb{C}\langle\langle \mathtt{s} \rangle\rangle$. As $\mathbb{C}\langle\langle \mathtt{s} \rangle\rangle$ is integrally closed in its quotient field $\mathbb{C}((\mathtt{s}))$, we conclude that $\phi(A) \subset \mathbb{C}\langle\langle \mathtt{s} \rangle\rangle$. Thus, we have a $\mathbb{C}$-algebra homomorphism of $\mathbb{C}$-algebras $\phi : A \to \mathbb{C}\langle\langle \mathtt{s} \rangle\rangle$.

As $\phi(\alpha\alpha') = \phi(b_0) = (\theta_p \circ \psi)(b_0) \neq 0$, we have $\phi(f + \mathfrak{p}) = \phi(\alpha) \neq 0$. Consider the $\mathbb{C}$-algebra homomorphism $\varphi := \phi \circ \pi : \mathcal{A}_n \to \mathbb{C}\langle\langle \mathtt{s} \rangle\rangle$, which satisfies $\mathfrak{p} \subset \ker(\varphi)$ (so $\varphi \in \mathcal{G}(\mathfrak{p})$) and $\varphi(f) \neq 0$, as required. □

Corollary IV.2.4 *If $f \in \mathcal{A}_n$, then $f \in \sqrt{\{\frac{\partial f}{\partial \mathtt{x}_1}, \dots, \frac{\partial f}{\partial \mathtt{x}_n}\}\mathcal{A}_n}$.*

Proof. Denote $J(f) := \sqrt{\{\frac{\partial f}{\partial \mathtt{x}_1}, \dots, \frac{\partial f}{\partial \mathtt{x}_n}\}\mathcal{A}_n}$ and let $\varphi : \mathcal{A}_n \to \mathbb{C}\langle\langle \mathtt{t} \rangle\rangle$ be a $\mathbb{C}$-algebra homomorphism such that $J(f) \subset \ker(\varphi)$ and denote $g_i := \varphi(\mathtt{x})$. As $\varphi(f) = f(g_1, \dots, g_n)$,

$$\frac{\partial \varphi(f)}{\partial \mathtt{t}} = \sum_{j=1}^{n} \frac{\partial f}{\partial \mathtt{x}_j}(g_1, \dots, g_n)\frac{\partial g_j}{\partial \mathtt{t}} = \sum_{j=1}^{n} \varphi\Big(\frac{\partial f}{\partial \mathtt{x}_j}\Big)\frac{\partial g_j}{\partial \mathtt{t}} = 0.$$

Thus, $\varphi(f) \in \mathbb{C}$ and, as φ is a local homomorphism, $\varphi(f) = 0$. By Rückert's Nullstellensatz IV.2.1 $f \in \sqrt{J(f)}$, as required. □

Corollary IV.2.5 *Let $\mathfrak{a}$ be an ideal of $\mathcal{A}_m$ and let $\varphi : \mathcal{A}_n \to \mathcal{A}_m/\mathfrak{a}$ be a $\mathbb{C}$-algebra homomorphism. Denote $\varphi(\mathtt{x}_i) := h_i + \mathfrak{a}$ for $i = 1, \dots, n$. The following assertions are equivalent:*

(i) *φ is finite.*

(ii) *$\mathcal{G}(\{h_1, \dots, h_n\}\mathcal{A}_m + \mathfrak{a})$ contains only the $\mathbb{C}$-algebra homomorphism*

$$\varphi : \mathcal{A}_m \to \mathbb{C}\langle\langle \mathtt{t} \rangle\rangle,\ f \mapsto f(0).$$

Proof. The maximal ideal of $\mathcal{A}_m/\mathfrak{a}$ is $\mathfrak{m}_m/\mathfrak{a}$. By Lemma III.1.5 φ is finite if and only if $\mathfrak{m}_m/\mathfrak{a} = \sqrt{\varphi(\mathfrak{m}_n)\mathcal{A}_m/\mathfrak{a}}$. By Rückert's Nullstellensatz condition (ii) is equivalent to $\sqrt{\{h_1, \dots, h_n\}\mathcal{A}_m + \mathfrak{a}} = \mathfrak{m}_m$ or equivalently

$$\mathfrak{m}_m/\mathfrak{a} = \sqrt{\varphi(\mathfrak{m}_n)\mathcal{A}_m/\mathfrak{a}}.$$

Thus, conditions (i) and (ii) are equivalent, as required. □

2.c Alternative formulation of Rückert's Nullstellensatz. We finish this section with an alternative formulation of Rückert's Nullstellensatz. Before that we need a characterization of prime ideals of $\mathcal{A}_n$ of height $\geq n-1$.

Lemma IV.2.6 *Let $\varphi : \mathcal{A}_n \to \mathbb{C}\langle\langle \mathtt{t} \rangle\rangle$ be a $\mathbb{C}$-algebra homomorphism. We have:*

(i) $\operatorname{ht}(\ker(\varphi)) = n$ *if and only if* $\ker(\varphi) = \mathfrak{m}_n$.

(ii) $\ker(\varphi)$ *is a prime ideal of height* $\geq n-1$.

Proof. (i) By the first isomorphy theorem

$$\mathcal{A}_n / \ker(\varphi) \cong \operatorname{im}(\varphi) \subset \mathbb{C}\langle\langle \mathtt{t} \rangle\rangle.$$

As $\mathbb{C}\langle\langle \mathtt{t} \rangle\rangle$ is an integral domain, $\ker(\varphi)$ is a prime ideal. Thus, $\operatorname{ht}(\ker(\varphi)) = n$ if and only if $\ker(\varphi) = \mathfrak{m}_n$.

(ii) Let us assume that $\operatorname{ht}(\ker(\varphi)) < n$. By Corollary III.1.6 φ is a finite homomorphism, so $\dim(\mathcal{A}_n / \ker(\varphi)) = \dim(\mathbb{C}\langle\langle \mathtt{t} \rangle\rangle) = 1$. Thus, $\operatorname{ht}(\ker(\varphi)) = n - \dim(\mathcal{A}_n / \ker(\varphi)) = n-1$, as required. □

Lemma IV.2.7 *Let $\mathfrak{p}$ be a prime ideal of $\mathcal{A}_n$ of height $\geq n-1$. Then there exists a $\mathbb{C}$-algebra homomorphism $\varphi : \mathcal{A}_n \to \mathbb{C}\langle\langle \mathtt{t} \rangle\rangle$ such that $\mathfrak{p} = \ker(\varphi)$.*

Proof. If $\operatorname{ht}(\mathfrak{p}) = n$, then $\mathfrak{p} = \mathfrak{m}_n$. So $\varphi : \mathcal{A}_n \to \mathbb{C}\langle\langle \mathtt{t} \rangle\rangle,\ f \mapsto f(0)$ is a $\mathbb{C}$-algebra homomorphism that satisfies $\ker(\varphi) = \mathfrak{m}_n$.

Thus, we assume that $\operatorname{ht}(\mathfrak{p}) = n-1$. Denote $A := \mathcal{A}_n/\mathfrak{p}$ and observe that $\dim(A) = 1$. The integral closure $\overline{A}$ of A in $\operatorname{qf}(A)$ has $\dim(\overline{A}) = \dim(A) = 1$. By Noether's Normalization Lemma II.2.17 the ring $\overline{A}$ is a finitely generated A-module. By Lemma IV.1.9 there exist $m \geq 1$ and a prime ideal $\mathfrak{q}$ of $\mathcal{A}_m$ such that $\overline{A} \cong \mathcal{A}_m/\mathfrak{q}$. In particular, $\overline{A}$ is a noetherian local integral domain of dimension 1 that is in addition integrally closed. By Lemma IV.1.6 $\overline{A} \cong \mathcal{A}_m/\mathfrak{q}$ is a regular local ring of dimension 1. By Lemma III.1.8 $\overline{A} \overset{\phi}{\cong} \mathbb{C}\langle\langle \mathtt{t} \rangle\rangle$. Consider the $\mathbb{C}$-algebra homomorphism $\varphi : \mathcal{A}_n \overset{\pi}{\to} A \hookrightarrow \overline{A} \overset{\phi}{\cong} \mathbb{C}\langle\langle \mathtt{t} \rangle\rangle$ and observe that $\ker(\varphi) = \mathfrak{p}$, as required. □

As a consequence of Lemmas IV.2.6 and IV.2.7 and Rückert's Nullstellensatz IV.2.1, we deduce the following.

Corollary IV.2.8 (Rückert's Nullstellensatz) *Let $\mathfrak{a}$ be an ideal of $\mathcal{A}_n$. Then*

$$\sqrt{\mathfrak{a}} = \mathfrak{m}_n \cap \bigcap_{\substack{\mathfrak{a}\subset\mathfrak{p},\\ \operatorname{ht}(\mathfrak{p})=n-1}} \mathfrak{p}.$$

3 Isolated singularities

The purpose of this section is to understand the concept of isolated singularity. Before that we need some preliminary work. We assume in this section that $\mathbb{K} = \mathbb{C}$ as we need Rückert's Nullstellensatz. Given an ideal $\mathfrak{a}$ of $\mathcal{A}_n$, the quotient $\mathcal{A}_n/\mathfrak{a}$ is a regular local ring if and only if there exist $f_1, \dots, f_r \in \mathfrak{a}$ such that

$$\operatorname{rk}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\Big)_{\substack{1\le i\le r\\ 1\le j\le n}} = \operatorname{ht}(\mathfrak{a}),$$

see Corollary III.2.10. In particular, the ideal $\mathfrak{a}$ is prime. Theorem III.2.21 guarantees that all the localizations $(\mathcal{A}_n/\mathfrak{a})_{\mathfrak{q}/\mathfrak{a}} \cong \mathcal{A}_{n,\mathfrak{q}}/\mathfrak{a}\mathcal{A}_{n,\mathfrak{q}}$ are regular local rings for each prime ideal $\mathfrak{q}$ of $\mathcal{A}_n$ that contains $\mathfrak{a}$.

As $\mathcal{A}_n$ is a local ring, the localization $\mathcal{A}_{n,\mathfrak{m}_n}$ coincides with $\mathcal{A}_n$, so the simplex type of "singularity" arises when asking the following condition. An ideal $\mathfrak{a}$ of $\mathcal{A}_n$ has an *isolated singularity* if $\mathcal{A}_n/\mathfrak{a}$ is not a regular ring but the localizations $\mathcal{A}_{n,\mathfrak{q}}/\mathfrak{a}\mathcal{A}_{n,\mathfrak{q}}$ are regular local rings for each prime ideal $\mathfrak{q}$ of $\mathcal{A}_n$ such that $\mathfrak{a} \subset \mathfrak{q} \subsetneq \mathfrak{m}_n$.

Lemma IV.3.1 *An ideal $\mathfrak{a}$ of $\mathcal{A}_n$ has an isolated singularity if and only if it is not a regular ring but the localizations $\mathcal{A}_{n,\mathfrak{p}}/\mathfrak{a}\mathcal{A}_{n,\mathfrak{p}}$ are regular local rings for each prime ideal $\mathfrak{p}$ of $\mathcal{A}_n$ such that $\mathfrak{a} \subset \mathfrak{p}$ and* $\operatorname{ht}(\mathfrak{p}) = n-1$.

Proof. The only if implication is clear, so let us prove the if implication. Write $\mathfrak{a} := \{f_1, \dots, f_m\}\mathcal{A}_n$. Let $\mathfrak{q} \neq \mathfrak{m}_n$ be a prime ideal that contains $\mathfrak{a}$ and let $\mathfrak{p}_1$ be a prime ideal associated to $\mathfrak{a}$. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{A}_n$ of height $n-1$ such that $\mathfrak{q} \subset \mathfrak{p}$. The existence of $\mathfrak{p}$ follows from Corollary II.2.15 applied to the couple $\mathfrak{q}/\mathfrak{a} \subset \mathfrak{m}_n/\mathfrak{a}$. By hypothesis $\mathcal{A}_{n,\mathfrak{p}}/\mathfrak{a}\mathcal{A}_{n,\mathfrak{p}}$ is a regular local ring and by Corollary III.2.22 $\mathcal{A}_{n,\mathfrak{p}}/\mathfrak{a}\mathcal{A}_{n,\mathfrak{p}} = \mathcal{A}_{n,\mathfrak{p}_1}/\mathfrak{p}_1\mathcal{A}_{n,\mathfrak{p}_1}$ is a field, so a regular local ring. By the generalized Jacobian criterion III.2.21

$$\operatorname{rk}_{\mathcal{A}_n/\mathfrak{p}_1}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j} + \mathfrak{p}_1\Big)_{\substack{1\le i\le m,\\ 1\le j\le n}} = \operatorname{rk}_{\mathcal{A}_n/\mathfrak{p}}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j} + \mathfrak{p}\Big)_{\substack{1\le i\le m,\\ 1\le j\le n}} = \operatorname{ht}(\mathfrak{p}_1).$$

Consequently,

$$\begin{aligned}\operatorname{ht}(\mathfrak{p}_1) = \operatorname{rk}_{\mathcal{A}_n/\mathfrak{p}}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j} + \mathfrak{p}\Big)_{\substack{1\le i\le m,\\ 1\le j\le n}} &\le \operatorname{rk}_{\mathcal{A}_n/\mathfrak{q}}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j} + \mathfrak{q}\Big)_{\substack{1\le i\le m,\\ 1\le j\le n}}\\ &\le \operatorname{rk}_{\mathcal{A}_n/\mathfrak{p}_1}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j} + \mathfrak{p}_1\Big)_{\substack{1\le i\le m,\\ 1\le j\le n}} = \operatorname{ht}(\mathfrak{p}_1)\end{aligned}$$

and by the generalized Jacobian criterion III.2.21 $\mathcal{A}_{n,\mathfrak{q}}/\mathfrak{a}\mathcal{A}_{n,\mathfrak{q}}$ is a regular local ring, as required. □

Consequently, to determine if an ideal $\mathfrak{a}$ of $\mathcal{A}_n$ has an isolated singularity we only have to take care about prime ideals of $\mathcal{A}_n$ of height $n-1$. By the generalized Jacobian criterion III.2.21 we have the following.

Corollary IV.3.2 *Let $\mathfrak{a} := \{f_1, \dots, f_m\}\mathcal{A}_n$ be an ideal of $\mathcal{A}_n$ and let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the associated prime ideals of $\mathcal{A}_n$ to $\mathfrak{a}$. The following conditions are equivalent*

(i) $\mathfrak{a}$ *has an isolated singularity.*

(ii) $\operatorname{rk}(\frac{\partial f_i}{\partial \mathtt{x}_j}(0))_{\substack{1\le i\le m,\\ 1\le j\le n}} \neq \operatorname{ht}(\mathfrak{p}_\ell)$ *for some $1 \le \ell \le r$ and for each prime ideal $\mathfrak{p}$ of $\mathcal{A}_n$ of height $n-1$ that contains $\mathfrak{a}$ the equalities*

$$\operatorname{rk}_{\mathcal{A}_n/\mathfrak{p}}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j} + \mathfrak{p}\Big)_{\substack{1\le i\le m,\\ 1\le j\le n}} = \operatorname{ht}(\mathfrak{p}_\ell)$$

hold for each $1 \le \ell \le r$ such that $\mathfrak{p}_\ell \subset \mathfrak{p}$.

3.a Hypersurface singularities.

In what follows we will focus on hypersurface singularities, that is, principal ideals of $\mathcal{A}_n$. Let us determine when a hypersurface singularity is isolated. Let $f \in \mathfrak{m}_n$ be a non-zero power series and let $f = f_1^{\alpha_1} \cdots f_r^{\alpha_r}$ be the factorization of f as a product of powers of distinct irreducible factors. By Remark II.2.1 the associated primes of $\mathcal{A}_n$ to $f\mathcal{A}_n$ are $\mathfrak{p}_i = f_i\mathcal{A}_n$ and $\operatorname{ht}(\mathfrak{p}_i) = 1$ for $i = 1, \dots, r$. We denote $J(f) := \{\frac{\partial f}{\partial \mathtt{x}_1}, \dots, \frac{\partial f}{\partial \mathtt{x}_n}\}\mathcal{A}_n$ the *Jacobian ideal* of f.

Lemma IV.3.3 *The ideal $f\mathcal{A}_n$ is an isolated hypersurface singularity if and only if $\sqrt{J(f)} = \mathfrak{m}_n$.*

Proof. The associated prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ of $f\mathcal{A}_n$ have height one. By Corollary IV.3.2 $f\mathcal{A}_n$ is an isolated singularity if and only if $\operatorname{rk}(\frac{\partial f}{\partial \mathtt{x}_j}(0))_{1\leq j\leq n} = 0$ and for each prime ideal $\mathfrak{p}$ of $\mathcal{A}_n$ of height $n-1$ that contains f

$$\operatorname{rk}_{\mathcal{A}_n/\mathfrak{p}}\Big(\frac{\partial f}{\partial \mathtt{x}_j} + \mathfrak{p}\Big)_{1\leq j\leq n} = \operatorname{ht}(\mathfrak{p}_\ell) = 1$$

for each $1 \leq \ell \leq r$ such that $f \in \mathfrak{p}_\ell$. This means that $f \in \mathfrak{m}_n^2$ and for each prime ideal $\mathfrak{p}$ of $\mathcal{A}_n$ of height $n-1$ that contains f there exists an index $j = 1, \dots, n$ such that $\frac{\partial f}{\partial \mathtt{x}_j} \notin \mathfrak{p}$. Thus, no prime ideal $\mathfrak{p}$ of $\mathcal{A}_n$ of height $n-1$ that contains f contains the Jacobian ideal $J(f)$. By Corollaries IV.2.4 and IV.2.8 we have

$$\sqrt{J(f)} = \sqrt{f\mathcal{A}_n + J(f)} = \mathfrak{m}_n,$$

as required. □

Corollary IV.3.4 *Let $f \in \mathfrak{m}_n^2$. The following assertions are equivalent:*

(i) *$f\mathcal{A}_n$ has an isolated singularity.*

(ii) *There exists $k \geq 1$ such that $\mathfrak{m}_n^k \subset J(f)$.*

(iii) $\dim_{\mathbb{C}}(\mathcal{A}_n/J(f)) < +\infty$.

Proof. By Lemma IV.3.3 statements (i) and (ii) are equivalent. Assume now that (ii) holds. Then

$$\dim_{\mathbb{C}}(\mathcal{A}_n/J(f)) \leq \dim_{\mathbb{C}}(\mathcal{A}_n/\mathfrak{m}_n^k) < +\infty$$

because $\mathcal{A}_n/\mathfrak{m}_n^k$ is $\mathbb{C}$-linear isomorphic to the set of polynomials in the variables $\mathtt{x}_1, \dots, \mathtt{x}_n$ of (total) degree $\leq k-1$ and (iii) follows. Let us assume now that $\dim_{\mathbb{C}}(\mathcal{A}_n/J(f)) < +\infty$. Consider the $\mathbb{C}$-algebra homomorphism

$$\varphi : \mathcal{A}_n \to \mathcal{A}_n,\ g \mapsto g\Big(\frac{\partial f}{\partial \mathtt{x}_1}, \dots, \frac{\partial f}{\partial \mathtt{x}_n}\Big).$$

As $\dim_{\mathbb{C}}(\mathcal{A}_n/J(f)) < +\infty$, we deduce by Corollary III.1.3 that φ is a finite homomorphism. By Lemma III.1.5 $\mathfrak{m}_n = \sqrt{J(f)}$ and by Lemma IV.3.3 $f\mathcal{A}_n$ has an isolated singularity and (i) follows, as required. □

Recall that $\mu_n(f) := \dim_{\mathbb{C}}(\mathcal{A}_n/J(f)) < +\infty$ is the *Milnor number* of $f\mathcal{A}_n$. We say that a power series $f \in \mathcal{A}_n$ is *not square-free* if there exist $g \in \mathcal{A}_n$ and $h \in \mathfrak{m}_n$ such that $f = gh^2$. Otherwise we say that f is *square-free.*

Lemma IV.3.5 *Let $f \in \mathfrak{m}_n^2$ and assume that $f\mathcal{A}_n$ has an isolated singularity. Then f is square-free in $\mathcal{A}_n$. The converse is true if $n = 2$.*

Proof. Let $f \in \mathcal{A}_n$ have a factorization $f = gh^2$ where $g \in \mathcal{A}_n$ and $h \in \mathfrak{m}_n$. Then h divides all the derivatives $\frac{\partial f}{\partial \mathtt{x}_i}$, so $\sqrt{J(f)} \subset \sqrt{h\mathcal{A}_n} \subsetneq \mathfrak{m}_n$.

Conversely, let $f \in \mathfrak{m}_2^2$ be a non-zero square-free power series. Let us prove that $\mathfrak{m}_2 \subset \sqrt{J(f)}$. The prime ideals of $\mathcal{A}_2$ different from $\mathfrak{m}_2$ and (0) have height 1. As $\mathcal{A}_2$ is a unique factorization domain, its prime ideals of height 1 are principal and generated by an irreducible power series. If $\mathfrak{m}_2 \not\subset \sqrt{J(f)}$, then there exists a prime ideal of height 1 that contains $\sqrt{J(f)}$, so by Corollary IV.2.4 there exists an irreducible power series $a \in \mathfrak{m}_2$ that divides f and $\frac{\partial f}{\partial \mathtt{x}_j}$ for $j = 1, \dots, n$. Write $f = ab$ where $b \in \mathcal{A}_2$ and observe that

$$\frac{\partial f}{\partial \mathtt{x}_i} = a\frac{\partial b}{\partial \mathtt{x}_i} + b\frac{\partial a}{\partial \mathtt{x}_i}.$$

As f is square-free, a does not divide b, so a divides all its derivatives, which is impossible. Consequently, $\sqrt{J(f)} = \mathfrak{m}_2$, as required. □

Given a power series $f \in \mathfrak{m}_n^2$ we define the *Hessian matrix* of f as

$$H(f) := \left(\frac{\partial^2 f}{\partial \mathtt{x}_i \partial \mathtt{x}_j}(0)\right)_{1 \le i,j \le n}.$$

The initial form of f is $\frac{1}{2}\mathtt{x}H(f)\mathtt{x}^t$. An isolated hypersurface singularity corresponding to a series $f \in \mathcal{A}_n$ is called *a Morse singularity* if its Milnor number is 1. This is equivalent to the fact that $J(f) = \mathfrak{m}_n$ or by Corollary II.1.12 to $\operatorname{rk}(H(f)) = n$. Let us find next a suitable isomorphism of $\mathcal{A}_n$ to obtain a suitable representation of a power series $f \in \mathfrak{m}_n^2$.

Theorem IV.3.6 *Let $f \in \mathfrak{m}_n^2$ and $r := \operatorname{rk}(H(f))$. Write $\mathtt{x}' := (\mathtt{x}_1, \dots, \mathtt{x}_r)$, $\mathtt{x}'' := (\mathtt{x}_{r+1}, \dots, \mathtt{x}_n)$ and $\|\mathtt{x}'\|^2 := \mathtt{x}_1^2 + \cdots + \mathtt{x}_r^2$. We have:*

(i) *There exists an isomorphism $\varphi : \mathcal{A}_n \to \mathcal{A}_n$ such that $\varphi(f) = \|\mathtt{x}'\|^2 + g(\mathtt{x}'')$ where $\omega(g) \ge 3$. In particular, if $r = n$, we have $\varphi(f) = \mathtt{x}_1^2 + \cdots + \mathtt{x}_n^2$.*

(ii) *If $\phi : \mathcal{A}_n \to \mathcal{A}_n$ is an isomorphism such that*

$$\phi(f) = \mathtt{x}_1^2 + \cdots + \mathtt{x}_{r'}^2 + g'(\mathtt{x}_{r'+1}, \dots, \mathtt{x}_n)$$

where $\omega(g') \ge 3$, then $r = r'$ and there exists a $\mathbb{C}$-algebra isomorphism $\psi : \mathbb{C}\langle\langle \mathtt{x}'' \rangle\rangle \to \mathbb{C}\langle\langle \mathtt{x}'' \rangle\rangle$ such that $\psi(g) = g'$.

Proof. (i) The proof of this item is conducted in several steps:

3.a.1 After the isomorphism induced by a suitable linear change of coordinates in $\mathbb{C}^n$ that diagonalizes the symmetric matrix $H(f)$, we may assume that the initial form of f is $\|\mathtt{x}'\|^2$.

3.a.2 In addition, we may assume: $f(\mathtt{x}) = f(0,\mathtt{x}'') + \sum_{i,j=1}^{r} h_{ij}(\mathtt{x})\mathtt{x}_i\mathtt{x}_j$ *where* $h_{ij} = h_{ji} \in \mathcal{A}_n$ *for* $1 \leq i,j \leq r$. Observe that $\omega(f(0,\mathtt{x}'')) \geq 3$ because the initial form of f is $\|\mathtt{x}'\|^2$ and $h_{ij}(0) = \delta_{ij}$ for $1 \leq i,j \leq r$.

Consider the system of equations

$$h_j(\mathtt{x}',\mathtt{x}'') := \frac{\partial f}{\partial \mathtt{x}_j}(\mathtt{x}',\mathtt{x}'') = 0$$

for $j = 1,\dots,r$. We have $(\frac{\partial h_j}{\partial x_k}(0,0))_{1\leq j,k\leq r} = 2I_r$. By the Implicit Function Theorem there exists $\zeta_j \in \mathbb{C}\langle\!\langle \mathtt{x}''\rangle\!\rangle$ with $\zeta_j(0) = 0$ such that $h_j(\zeta(\mathtt{x}''),\mathtt{x}'') = 0$, where $\zeta := (\zeta_1,\dots,\zeta_r)$, for $j = 1,\dots,r$. After applying the isomorphism

$$\varphi_0 : \mathcal{A}_n \to \mathcal{A}_n,\ g \mapsto g(\mathtt{x}' + \zeta(\mathtt{x}''),\mathtt{x}'')$$

we may assume $\frac{\partial f}{\partial \mathtt{x}_j}(0,\mathtt{x}'') = 0$ for $j = 1,\dots,r$. Indeed, let $f' := \varphi_0(f)$ and observe

$$\frac{\partial f'}{\partial \mathtt{x}_j}(0,\mathtt{x}'') = \frac{\partial f}{\partial \mathtt{x}_j}(\zeta(\mathtt{x}''),\mathtt{x}'') = 0$$

for $j = 1,\dots,r$, as claimed.

If we consider the Taylor expansion of f with respect to $\mathtt{x}'$, we may write

$$f(\mathtt{x}) = \sum_{\nu\in\mathbb{N}^r} \frac{1}{\nu!}\frac{f^{|\nu|}}{\partial \mathtt{x}^\nu}(0,\mathtt{x}'')\mathtt{x}'^\nu = f(0,\mathtt{x}'') + \sum_{i,j=1}^{r} h_{ij}(\mathtt{x})\mathtt{x}_i\mathtt{x}_j$$

because $\frac{\partial f}{\partial \mathtt{x}_j}(0,\mathtt{x}'') = 0$ for $j = 1,\dots,r$. Changing both h_{ij} and h_{ji} by $\frac{h_{ij}+h_{ji}}{2}$ we may assume $h_{ij} = h_{ji}$ for $1 \leq i,j \leq r$.

3.a.3 *Construction of a* $\mathbb{C}$*-algebra isomorphism* $\varphi : \mathcal{A}_n \to \mathcal{A}_n$ *such that* $\varphi(f) = \|\mathtt{x}'\|^2 + f(0,\mathtt{x}'')$.

The matrix $Q := (h_{ij})_{1\leq i,j\leq r}$ is symmetric and $Q(0,0) = I_r$. Let us diagonalize Q over $\mathcal{A}_n$. We construct inductively a lower triangular square matrix U of order r with coefficients in $\mathcal{A}_n$ such that $U(0,0) = I_r$ and such that UQU^t is a diagonal matrix of order r. In particular, $U(0,0)Q(0,0)U^t(0,0) = I_rI_rI_r = I_r$. The rows u_i of U are computed inductively as follows:

$$\begin{cases} u_1 := (1,0,\dots,0) \\ u_i := (u_{i1},\dots,u_{i,i-1},1,0,\dots,0) \quad \text{where} \quad u_{ij}(0) = 0. \end{cases}$$

Consider the matrices

$$U_{i-1} := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ u_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ u_{i-1,1} & u_{i-1,2} & \cdots & 1 \end{pmatrix}, \quad H_{i-1} := \begin{pmatrix} h_{11} & \cdots & h_{1,i-1} \\ h_{21} & \cdots & h_{2,i-1} \\ \vdots & & \vdots \\ h_{i-1,1} & \cdots & h_{i-1,i-1} \end{pmatrix}$$

$$\text{and} \quad H^*_{i-1} := \begin{pmatrix} h_{11} & \cdots & h_{1,i-1} & h_{1i} \\ h_{21} & \cdots & h_{2,i-1} & h_{2i} \\ \vdots & & \vdots & \vdots \\ h_{i-1,1} & \cdots & h_{i-1,i-1} & h_{i-1\,i} \end{pmatrix}$$

for $i = 2, \dots, r$. The elements $u_{ij} \in \mathcal{A}_n$ are the components of the solutions of the square linear system of order $i-1$ given by

$$U_{i-1}H^*_{i-1} \begin{pmatrix} u_{i1} \\ \vdots \\ u_{i,i-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.4}$$

The coefficient matrix of the linear system (3.4) is $U_{i-1}H_{i-1}$, whose determinant Δ_{i-1} is a unit of $\mathcal{A}_n$ because $\Delta_{i-1}(0) = \det(I_{i-1}I_{i-1}) = 1 \neq 0$. Thus, the system (3.4) admits a unique solution over $\mathcal{A}_n$ and $u_{ij}(0) = -h_{ji}(0) = 0$ for $j = 1, \dots, i-1$. Hence, U is a square lower triangular matrix of order r with coefficients in $\mathcal{A}_n$ such that the coefficients in its diagonal are all 1, $U(0,0) = I_r$ and whose determinant is 1. Let $V = U^{-1}$ be the inverse of U, which has coefficients in $\mathcal{A}_n$. Consequently, $D := UQU^t$ is a diagonal matrix such that

$$D(0,0) = U(0,0)Q(0,0)U^t(0,0) = I_rI_rI_r = I_r.$$

Write the diagonal matrix $D := \langle d_i \rangle_{1 \le i \le r}$, where $d_i \in \mathcal{A}_n$ is a power series such that $d_i(0) = 1$ for $i = 1, \dots, r$. Let $a_i \in \mathcal{A}_n$ be such that $a_i^2 = d_i$ for $i = 1, \dots, r$ and consider the diagonal matrix $A := \langle a_i \rangle_{1 \le i \le r}$. Observe that $A^2 = D$. As $D = UQU^t$, we have $Q = VDV^t$ and

$$\begin{aligned} \sum_{i,j=1}^{r} h_{ij}(\mathtt{x})\mathtt{x}_i\mathtt{x}_j &= \mathtt{x}'Q\mathtt{x}'^t = \mathtt{x}'VDV^t\mathtt{x}'^t \\ &= \mathtt{x}'VAAV^t\mathtt{x}'^t = (\mathtt{x}'VA)(\mathtt{x}'VA)^t = \sum_{i=1}^{r} \mathtt{y}_i^2, \end{aligned}$$

where $\mathtt{y} := (\mathtt{y}_1, \ldots, \mathtt{y}_r)$ and $V(0,0)A(0,0) = I_r I_r = I_r$. Consider the isomorphism

$$\varphi_1 : \mathcal{A}_n := \mathbb{C}\langle\langle \mathtt{y}, \mathtt{x}'' \rangle\rangle \to \mathcal{A}_n := \mathbb{C}\langle\langle \mathtt{x}', \mathtt{x}'' \rangle\rangle,\ g \mapsto g(\mathtt{x}'VA, \mathtt{x}''),$$

which satisfies

$$\varphi_1^{-1}(f) = f(0, \mathtt{x}'') + \sum_{i=1}^{r} \mathtt{y}_i^2.$$

(ii) Let $\phi : \mathcal{A}_n \to \mathcal{A}_n$ be an isomorphism such that

$$\phi(f) = \mathtt{x}_1^2 + \cdots + \mathtt{x}_{r'}^2 + g(\mathtt{x}_{r'+1}, \ldots, \mathtt{x}_n)$$

where $\omega(g) \geq 3$.

3.a.4 Let us prove first that $r = r'$. Denote $h_i := \phi(\mathtt{x}_i)$ for $i = 1, \ldots, n$. By Corollary II.1.10 we have

$$\det \Big(\frac{\partial h_i}{\partial \mathtt{x}_j}(0) \Big)_{1 \leq i,j \leq n} \neq 0. \tag{3.5}$$

Consider the initial form $\ell_i := \sum_{j=1}^{n} \frac{\partial h_i}{\partial \mathtt{x}_j}(0)\mathtt{x}_j$ of h_i and the linear isomorphism $\ell := (\ell_1, \ldots, \ell_n) : \mathbb{C}^n \to \mathbb{C}^n$, see (II.1.10). As $\phi(f) = f(h_1, \ldots, h_n)$, the initial form of $\phi(f)$ is $H(f)(\ell) = \mathtt{x}_1^2 + \cdots + \mathtt{x}_{r'}^2$. Consequently, $H(f)$ and $\mathtt{x}_1^2 + \cdots + \mathtt{x}_{r'}^2$ are congruent quadratic forms, so they have the same rank and $r = r'$.

3.a.5 To finish it is enough to show: *Let us write $f := \|\mathtt{x}'\|^2 + g(\mathtt{x}'') \in \mathfrak{m}_n^2$ where $\omega(g) \geq 3$ and let $\phi : \mathcal{A}_n \to \mathcal{A}_n$ be a $\mathbb{C}$-algebra isomorphism such that $\phi(f) = \|\mathtt{x}'\|^2 + g'(\mathtt{x}'')$ and $\omega(g') \geq 3$. Then there exists a $\mathbb{C}$-algebra isomorphism $\psi : \mathbb{C}\langle\langle \mathtt{x}' \rangle\rangle \to \mathbb{C}\langle\langle \mathtt{x}' \rangle\rangle$ such that $\psi(h) = g$.*

Write $h_i := \phi(\mathtt{x}_i)$ for $i = 1, \ldots, n$. We have

$$h_1^2 + \ldots + h_r^2 + g(h_{r+1}, \ldots, h_n) = \|\mathtt{x}'\|^2 + g'(\mathtt{x}'').$$

Again by Corollary II.1.10 the linear map $\ell := (\ell_1, \ldots, \ell_n) : \mathbb{C}^n \to \mathbb{C}^n$, where ℓ_i is the initial form of h_i, is an isomorphism. Write $\ell_i(\mathtt{x}) := \ell_i'(\mathtt{x}') + \ell_i''(\mathtt{x}'')$ where $\mathtt{x}' := (\mathtt{x}_1, \ldots, \mathtt{x}_r)$ and $\mathtt{x}'' := (\mathtt{x}_{r+1}, \ldots, \mathtt{x}_n)$. Thus

$$\sum_{i=1}^{r} \ell_i'(\mathtt{x}')^2 + 2\sum_{i=1}^{r} \ell_i'(\mathtt{x}')\ell_i''(\mathtt{x}'') + \sum_{i=1}^{r} \ell_i''(\mathtt{x}'')^2 = \|\mathtt{x}'\|^2 = \sum_{i=1}^{r} \mathtt{x}_i^2.$$

We deduce

$$\sum_{i=1}^{r} \ell_i'(\mathtt{x}')^2 = \sum_{i=1}^{r} \mathtt{x}_i^2 \quad \text{and} \quad 2\sum_{i=1}^{r} \ell_i'(\mathtt{x}')\ell_i''(\mathtt{x}'') + \sum_{i=1}^{r} \ell_i''(\mathtt{x}'')^2 = 0.$$

From the first equality we deduce that $\{\ell_1', \ldots, \ell_r'\}$ are linearly independent. Making $\mathtt{x}' = 0$ in the second equality, we have $\sum_{i=1}^{r} \ell_i''(\mathtt{x}'')^2 = 0$, so $\ell_i'' = 0$ for $i = 1, \ldots, r$. Thus, after a linear change of coordinates, we may assume that $\ell_i' := \mathtt{x}_i$ for $i = 1, \ldots, r$, so $h_i := \mathtt{x}_i + a_i(\mathtt{x})$ where $\omega(a_i) \geq 2$ for $i = 1, \ldots, r$. Consequently,

$$\begin{aligned} \varphi(\|\mathtt{x}'\|^2) &= \|\mathtt{x}'\|^2 + \sum_{i=1}^{r}(2\mathtt{x}_i + a_i)a_i, \\ \varphi(g) &= g(h_{r+1}, \ldots, h_n) - \sum_{i=1}^{r}(2\mathtt{x}_i + a_i)a_i. \end{aligned} \tag{3.6}$$

As $\varphi(\|\mathtt{x}'\|^2 + g(\mathtt{x}'')) = \|\mathtt{x}'\|^2 + g'(\mathtt{x}'')$,

$$g'(\mathtt{x}'') = \varphi(g(\mathtt{x}'')) + \sum_{i=1}^{r}(2\mathtt{x}_i + a_i(\mathtt{x}', \mathtt{x}''))a_i(\mathtt{x}', \mathtt{x}''). \tag{3.7}$$

Consider the system of equations in the indeterminantes $\mathtt{x}'$ given by

$$g_i(\mathtt{x}', \mathtt{x}'') := 2\mathtt{x}_i + a_i(\mathtt{x}', \mathtt{x}'') = 0 \quad \text{for } i = 1, \ldots, r.$$

Observe that $g_i(0,0) = 0$ and $(\frac{\partial g_i}{\partial \mathtt{x}_j})_{1 \leq i,j \leq r} = 2I_r$. By the Implicit Function Theorem there exist $\xi_1, \ldots, \xi_r \in \mathbb{C}\langle\langle \mathtt{x}'' \rangle\rangle$ such that $\xi_i(0) = 0$ and

$$g_i(\xi, \mathtt{x}'') = 0 \qquad \text{for } i = 1, \ldots, r, \tag{3.8}$$

where $\xi := (\xi_1, \ldots, \xi_r)$. As $\omega(a_i) \geq 2$ and $2\xi_i = -a(\xi, \mathtt{x}'')$, we have $\omega(\xi_i) \geq 2$.

3.a.6 Consider the $\mathbb{C}$-algebra homomorphism

$$\psi : \mathbb{C}\langle\langle \mathtt{x}' \rangle\rangle \to \mathbb{C}\langle\langle \mathtt{x}' \rangle\rangle, \ a \mapsto a(h_{r+1}(\xi, \mathtt{x}''), \ldots, h_n(\xi, \mathtt{x}'')).$$

By (3.6) and (3.8)

$$\begin{aligned} \psi(g) &= g(h_{r+1}(\xi, \mathtt{x}''), \ldots, h_n(\xi, \mathtt{x}'')) \\ &= g(h_{r+1}(\xi, \mathtt{x}''), \ldots, h_n(\xi, \mathtt{x}'')) - \sum_{i=1}^{r} g_i(\xi, \mathtt{x}''))a_i(\xi, \mathtt{x}'')) = \varphi(g)(\xi, \mathtt{x}''). \end{aligned}$$

In addition, by (3.7) and (3.8)

$$g'(\mathtt{x}'') = \varphi(g)(\xi, \mathtt{x}'') + \sum_{i=1}^{r} g_i(\xi, \mathtt{x}'')a_i(\xi, \mathtt{x}'') = \varphi(g)(\xi, \mathtt{x}'') = \psi(g),$$

that is, $\psi(g(\mathtt{x}'')) = g'(\mathtt{x}'')$.

3.a.7 To finish we have to prove: *ψ is a $\mathbb{C}$-algebra isomorphism.* Denote $h'_i := h_i(\xi, \mathtt{x}'')$ for $i = r+1, \dots, n$. By Corollary II.1.10 we must show

$$\det\Big(\frac{\partial h'_i}{\partial \mathtt{x}_j}(0)\Big)_{r+1\le i,j\le n} \neq 0.$$

As $\omega(\xi_i) \geq 2$, we have $\omega(\frac{\partial \xi_i}{\partial \mathtt{x}_j}) \geq 1$. As

$$\frac{\partial h'_i}{\partial \mathtt{x}_j} = \frac{\partial h_i}{\partial \mathtt{x}_j}(\xi, \mathtt{x}'') + \sum_{k=1}^{r} \frac{\partial h_i}{\partial \mathtt{x}_k}(\xi, \mathtt{x}'')\frac{\partial \xi_k}{\partial \mathtt{x}_j}$$

for $j = r+1, \dots, n$, we deduce

$$\frac{\partial h'_i}{\partial \mathtt{x}_j}(0) = \frac{\partial h_i}{\partial \mathtt{x}_j}(0,0)$$

for $j = r+1, \dots, n$. As $h_i = \mathtt{x}_i + a_i$ with $\omega(a_i) \geq 2$ for $i = 1, \dots, r$, we conclude by (3.5)

$$\det\Big(\frac{\partial h'_i}{\partial \mathtt{x}_j}(0)\Big)_{r+1\le i,j\le n} = \det\Big(\frac{\partial h_i}{\partial \mathtt{x}_j}(0,0)\Big)_{1\le i,j\le n} \neq 0,$$

as required. □

Remarks IV.3.7 (i) Let $f \in \mathfrak{m}_n^2$ and let $\varphi : \mathcal{A}_n \to \mathcal{A}_n$ be an isomorphism such that $\varphi(f) = \mathtt{x}_1^2 + \cdots + \mathtt{x}_r^2 + g(\mathtt{x}_{r+1}, \dots, \mathtt{x}_n)$ where $\omega(g) \geq 3$. Then

$$\mu_n(f) = \dim_{\mathbb{C}}(\mathcal{A}_n/J(f)) = \dim_{\mathbb{C}}(\mathcal{A}_n/J(\varphi(f))) = \mu_n(\varphi(f)).$$

As $J(\varphi(f)) = \{\mathtt{x}_1, \dots, \mathtt{x}_r, \frac{\partial g}{\partial \mathtt{x}_{r+1}}, \dots, \frac{\partial g}{\partial \mathtt{x}_n}\}\mathcal{A}_n$, we have

$$\dim_{\mathbb{C}}(\mathcal{A}_n/J(\varphi(f))) = \dim_{\mathbb{C}}(\mathbb{C}\langle\!\langle \mathtt{x}_{r+1}, \dots, \mathtt{x}_n \rangle\!\rangle/J(g)) = \mu_{n-r}(g).$$

(ii) A power series $f \in \mathfrak{m}_n^2$ has $\mu_n(f) = 1$ if and only if $\mathrm{rk}(H(f)) = n$, that is, if there exists an isomorphism $\varphi : \mathcal{A}_n \to \mathcal{A}_n$ such that $\varphi(f) = \mathtt{x}_1^2 + \cdots + \mathtt{x}_n^2$.

Examples IV.3.8 The isomorphism classes of some square-free power series $f \in \mathfrak{m}_n^2$ have special names. We present next the A-D-E singularities:

- $A_k : \mathtt{x}_1^2+\cdots+\mathtt{x}_{n-1}^2+\mathtt{x}_n^{k+1}$ where $k \geq 1$. We have $\mu_n(A_k) = \mu_1(\mathtt{x}_n^{k+1}) = k$.
- $D_k : \mathtt{x}_1^2 + \cdots + \mathtt{x}_{n-2}^2 + \mathtt{x}_{n-1}^2\mathtt{x}_n + \mathtt{x}_n^k$ where $k \geq 3$. We have
$$\mu_n(D_k) = \mu_2(\mathtt{x}_{n-1}^2\mathtt{x}_n + \mathtt{x}_n^k) = k + 1.$$
- $E_6 : \mathtt{x}_1^2 + \cdots + \mathtt{x}_{n-2}^2 + \mathtt{x}_{n-1}^3 + \mathtt{x}_n^4$. We have $\mu_n(E_6) = \mu_2(\mathtt{x}_{n-1}^3 + \mathtt{x}_n^4) = 6$.
- $E_7 : \mathtt{x}_1^2 + \cdots + \mathtt{x}_{n-2}^2 + \mathtt{x}_{n-1}^3 + \mathtt{x}_{n-1}\mathtt{x}_n^3$. We have
$$\mu_n(E_7) = \mu_2(\mathtt{x}_{n-1}^3 + \mathtt{x}_{n-1}\mathtt{x}_n^3) = 7.$$
- $E_8 : \mathtt{x}_1^2 + \cdots + \mathtt{x}_{n-2}^2 + \mathtt{x}_{n-1}^3 + \mathtt{x}_n^5$. We have $\mu_n(E_8) = \mu_2(\mathtt{x}_{n-1}^3 + \mathtt{x}_n^5) = 8$.

It holds that all the previous singularities define different isomorphism classes. We leave the concrete details to the reader.

Exercises

Number IV.1 Let $\mathbb{C}\langle\langle \mathtt{t}^* \rangle\rangle$ be the ring of Puiseux series in one variable and let $\mathbb{C}((\mathtt{t}^*))$ be its quotient field. Show that

(i) $\mathbb{C}\langle\langle \mathtt{t}^* \rangle\rangle$ is a non-noetherian local ring of dimension 1.
(ii) $\mathbb{C}\langle\langle \mathtt{t}^* \rangle\rangle$ is the integral closure of $\mathbb{C}\langle\langle \mathtt{t} \rangle\rangle$ in $\mathbb{C}((\mathtt{t}^*))$.

Number IV.2 Show that for each $n \geq 1$ there exists an injective $\mathbb{C}$-algebra homomorphism $\varphi : \mathbb{C}\{\mathtt{x}_1, \ldots, \mathtt{x}_n, \mathtt{t}\} \to \mathbb{C}[[\mathtt{t}]]$ such that $\varphi(\mathtt{t}) = \mathtt{t}$.

Number IV.3 Let $n \geq 2$. Is there an injective $\mathbb{C}$-algebra homomorphism $\varphi : \mathcal{A}_n \to \mathbb{C}\langle\langle \mathtt{t}^* \rangle\rangle$?

Number IV.4 Let $\mathbb{C}\langle\langle \mathtt{t}^* \rangle\rangle$ be the ring of Puiseux series in one variable, $q \geq 1$ an integer, $f := \sum_{k\geq 0} a_k \mathtt{t}^k \in \mathbb{C}\langle\langle \mathtt{t} \rangle\rangle$, $M := \{k \geq 0 : \; a_k \neq 0\}$ and $\xi := f(\mathtt{t}^{1/q})$. Show that the following assertions are equivalent:

(i) $q := \min\{n \geq 1 : \; \xi \in \mathbb{C}\langle\langle \mathtt{t}^{1/n} \rangle\rangle\}$.
(ii) $\gcd(\{q\} \cup M) = 1$.

Number IV.5 Let $P \in \mathbb{C}\langle\langle \mathtt{t} \rangle\rangle[\mathtt{x}]$ be an irreducible monic polynomial. Show that there exist a unit $U \in \mathbb{C}\langle\langle \mathtt{t}, \mathtt{x} \rangle\rangle$, an integer $q \geq 1$ and $g \in \mathbb{C}\langle\langle \mathtt{t} \rangle\rangle$ such that $P = \prod_{k=1}^{q}(\mathtt{x} - g(\xi^k \mathtt{t}^{1/p}))U$ where $\xi := e^{2\pi i/q}$.

Number IV.6 Show that if $f \in \mathbb{C}\langle\langle \mathtt{x}, \mathtt{y}\rangle\rangle$ is irreducible, then its initial form is the power of a linear form. Is the result still true if we change $\mathbb{C}$ by $\mathbb{R}$?

Number IV.7 Find an irreducible polynomial $P \in \mathbb{C}[\mathtt{x}, \mathtt{y}]$, which is reducible but not a unit in $\mathbb{C}\langle\langle \mathtt{x}, \mathtt{y}\rangle\rangle$.

Number IV.8 Let $P \in \mathbb{C}[\mathtt{x}, \mathtt{y}]$ be a polynomial.

(i) Use Hensel's Lemma (see Exercise number I.21) to find a factorization of P in $\mathbb{C}(\langle \mathtt{x}\rangle)[\mathtt{y}]$ as a product of irreducible factors.

(ii) Show that if P is monic, then the previous factorization holds in $\mathbb{C}\langle\langle \mathtt{x}\rangle\rangle[\mathtt{y}]$.

(iii) What can we say about the previous factorization if P is irreducible in $\mathbb{C}[\mathtt{x}, \mathtt{y}]$?

Number IV.9 Find a counterexample to Rückert's Nullstellensatz if we change $\mathbb{C}$ by $\mathbb{R}$.

Number IV.10 Find a polynomial $f \in \mathbb{C}[\mathtt{x}] := \mathbb{C}[\mathtt{x}_1, \dots, \mathtt{x}_n]$ such that

$$f \notin \sqrt{\Big\{\frac{\partial f}{\partial \mathtt{x}_1}, \dots, \frac{\partial f}{\partial \mathtt{x}_n}\Big\}\mathbb{C}[\mathtt{x}]}.$$

Show that if f is homogeneous of degree $d \geq 1$, then $f \in \{\frac{\partial f}{\partial \mathtt{x}_1}, \dots, \frac{\partial f}{\partial \mathtt{x}_n}\}\mathbb{C}[\mathtt{x}]$.

Number IV.11 Show that the Milnor number of the singularities A-D-E are:

- $\mu_n(A_k) = k$ for $k \geq 1$.
- $\mu_n(D_k) = k + 1$ for $k \geq 3$.
- $\mu_n(E_k) = k$ for $k = 6, 7, 8$.

Compute basis of the $\mathbb{C}$-linear space $\mathcal{A}_n/J(f)$ in each case.

Number IV.12 Let $f \in \mathfrak{m}_n^2$ be an isolated singularity and let $\varphi : \mathcal{A}_n \to \mathcal{A}_n$ be an isomorphism. Denote $g := \varphi(f)$. Show that:

(i) The $\mathbb{K}$-linear space $\mathcal{A}_n/(\mathfrak{m}^k + J(f))$ have finite dimension for each $k \geq 1$.

(ii) The $\mathbb{K}$-linear spaces $\mathcal{A}_n/(\mathfrak{m}^k + J(f))$ and $\mathcal{A}_n/(\mathfrak{m}^k + J(g))$ have finite dimension for each $k \geq 1$.

(iii) Show that the A-D-E singularities define different isomorphism classes.

Analytic submanifolds and Riemann extension theorem

We begin this chapter adapting many of the results in Chapter I to the geometric setting. We introduce next the concept of analytic set and study some of its basic properties. After this we analyze the analytic submanifolds and the set of smooth points of an analytic set. We finish this chapter with some crucial results in complex analytic geometry: Maximum modulus principle and Riemann's Extension Theorem.

The Implicit Function, Inverse Function and Rank Theorems presented in the first section have well-known precedents for real and complex differentiable maps. We have assumed that the reader is familiar with them, but in any case [Pr, RF, Sh] provide the reader all what he needs concerning these and deeper results in analysis. In particular, the basics on integration and differentiation of functions in one complex variable can be found in [Sh], although our text is almost self-contained on these topics. Of course some elementary notions on general topology are needed and we refer the reader to [K] for further reading.

1 Preliminaries.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For each $a \in \mathbb{K}^n$ and $A \subset \mathbb{K}^n$ we write $a+A := \{a+x : x \in A\}$ and denote $\tau_a : \mathbb{K}^n \to \mathbb{K}^n,\ x \mapsto x + \vec{a}$ the translation of vector $\vec{a}$. To ease notation we will identify an analytic power series $f \in \mathcal{O}_n$ with the associated function $\widehat{f} : D(f) \to \mathbb{K}$.

Definition V.1.1 Let $\Omega \subset \mathbb{K}^n$ be an open set and let $f : \Omega \to \mathbb{C}$ be a continuous function. We say that f is an *analytic function* if for each $a \in \Omega$ there exists an analytic power series $F_a \in \mathcal{O}_n$ such that $f(x) = F_a(x-a)$ for x close to a. We define the *order of f at a* as $\nu_a(f) := \omega(F_a)$.

Remarks V.1.2 (i) Let $f : \Omega \to \mathbb{K}$ be an analytic function. Then f is a smooth function and for each $a \in \Omega$, the series F_a coincides with the Taylor series $T_af := \sum_\nu \frac{\partial^{|\nu|}f}{\partial \mathtt{x}^\nu}(a)\mathtt{x}^\nu$ (see Proposition I.3.9). Thus, $\nu_a(f) := \omega(T_a(f))$. If $0 \in \Omega$, we denote for simplicity $f \in \mathcal{O}_n$ the Taylor expansion of f at the origin.

(ii) Let $f \in \mathcal{O}_n$, $\Omega := D(f)$ and $f : \Omega \to \mathbb{K}$ the function associated to f. Then f is by Proposition I.3.6 an analytic function.

Lemma V.1.3 *Let $\mathcal{O}(\Omega)$ be the set of all analytic functions on Ω. Then the pointwise sum and product of functions endow $\mathcal{O}(\Omega)$ with a structure of ring. In addition, if $f \in \mathcal{O}(\Omega)$ and $f(x) \neq 0$ for all $x \in \Omega$, then $\frac{1}{f} \in \mathcal{O}(\Omega)$.*

Proof. The first part follows from the fact that given $f, g \in \mathcal{O}(\Omega)$ and $a \in \Omega$, then $T_a(f+g) = T_af + T_a(g)$ and $T_a(fg) = T_a(f) \cdot T_a(g)$ (see Exercise I.10).

Let $f \in \mathcal{O}(\Omega)$ be such that $f(x) \neq 0$ for all $x \in \Omega$. Then the function $\frac{1}{f}$ is continuous. Pick $a \in \Omega$ and let $F := T_a(f) \in \mathcal{O}_n$. As $F(0) \neq 0$, there exists $G \in \mathcal{O}_n$ such that $FG = 1$. As $f(x) = F(x-a)$ for x close to a, we have $\frac{1}{f(x)} = G(x-a)$ for x close to a. Thus, f is analytic, as required. $\square$

Proposition V.1.4 (Identity Principle) *Let $\Omega \subset \mathbb{K}^n$ be a non-empty connected open set and $f \in \mathcal{O}(\Omega)$. The following assertions are equivalent:*

(i) $f = 0$.

(ii) *The set $\{x \in \Omega : f(x) = 0\}$ has non-empty interior.*

(iii) *There exists $a \in \Omega$ such that $\frac{\partial^{|\nu|}f}{\partial \mathtt{x}^\nu}(a) = 0$ for all $\nu \in \mathbb{N}^n$.*

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are clear, so let us prove that (iii) $\Longrightarrow$ (i). For each $x \in \Omega$ let $U^x \subset (x + D(T_xf)) \cap \Omega$ be an open neighborhood of x such that $f(y) = T_xf(y-x)$ for each $y \in U^x$. The family $\{U^x\}_{x\in\Omega}$ is an open covering of Ω. By Proposition I.3.10 $f|_{U^a} = 0$. Pick $x \in \Omega$ and let us check that $f|_{U^x} = 0$.

As Ω is connected, there exist points $x_0 := a, x_1, \dots, x_r := x$ such that $U^{x_{i-1}} \cap U^{x_i} \neq \varnothing$ for $i = 1, \dots, r$. We prove by induction on r that $f(x) = 0$. The case $r = 0$ is trivially true because we already know that $f|_{U^{x_0}} = 0$ is identically zero. As $U^{x_0} \cap U^{x_1} \neq \varnothing$ and $f|_{U^{x_0}}$ is identically zero, we deduce again by Proposition I.3.10 that $f|_{U^{x_1}} = 0$. By the induction hypothesis applied to $U^{x_1}, \dots, U^{x_r}$, we conclude that $f|_{U^x} = 0$. Thus, $f = 0$, as required. □

Remark V.1.5 The previous result reflects the 'chromosomal' behavior of analytic functions over a connected open set $\Omega \subset \mathbb{K}^n$. If two analytic functions coincide in a small neighborhood of a point $a \in \Omega$, then such analytic functions are equal. This represents a great difference with the behavior of smooth functions. A great disadvantage with respect to smooth functions is that partitions of unity cannot exist in the analytic setting, so it is more difficult to paste information as we can easily do in the smooth case.

Definition V.1.6 Let $\Omega \subset \mathbb{K}^n$ be an open set. A map $f := (f_1, \dots, f_m) : \Omega \to \mathbb{K}^m$ is *analytic* if each component $f_i \in \mathcal{O}(\Omega)$.

As a consequence of Proposition I.3.7 and the Chain rule, we have:

Lemma V.1.7 *Let $\Omega \subset \mathbb{K}^n$ and $\Omega' \subset \mathbb{K}^m$ be open sets and let $f \in \mathcal{O}(\Omega)$ and let $g := (g_1, \dots, g_n) : \Omega' \to \Omega$ be an analytic map. Then $f \circ g \in \mathcal{O}(\Omega')$ and*

$$\frac{\partial (f \circ g)}{\partial \mathtt{y}_j}(b) = \sum_{i=1}^{m} \frac{\partial f}{\partial \mathtt{x}_i}(g(b)) \frac{\partial g_i}{\partial \mathtt{y}_j}(b)$$

for $j = 1, \dots, m$ and $b \in \Omega'$.

1.a Implicit Function, Inverse Function and Rank Theorems. Let us reformulate the Implicit Function, Inverse Function and Rank Theorems introduced in Section I.5 for analytic series in our current setting. We write $\mathtt{x} := (\mathtt{x}_1, \dots, \mathtt{x}_n)$ and $\mathtt{y} := (\mathtt{y}_1, \dots, \mathtt{y}_m)$. Let $\Omega_1, \Omega_2 \subset \mathbb{K}^n$ be open sets and let $\mathtt{j}_i : \Omega_i \hookrightarrow \mathbb{K}^n$ be the inclusion. An analytic map $f : \Omega_1 \to \Omega_2$ is an *analytic equivalence* if $\mathtt{j}_2 \circ f : \Omega_1 \to \mathbb{K}^n$ is an analytic map, f is a homeomorphism and $\mathtt{j}_1 \circ f^{-1} : \Omega_2 \to \mathbb{K}^n$ is an analytic map.

Theorem V.1.8 (Inverse Function Theorem) *Let $\Omega \subset \mathbb{K}^n$ be an open neighborhood of the origin and let $f := (f_1, \dots, f_n) : \Omega \to \mathbb{K}^n$ be an analytic map such that $f(0) = 0$ and*

$$\det\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\Big)_{1 \le i,j \le n} \neq 0.$$

Then there exist open neighborhoods $U_1 \subset \Omega$ and $U_2 \subset \mathbb{K}^n$ of the origin such that $f|_{U_1} : U_1 \to U_2$ is an analytic equivalence.

Proof. By Corollary I.5.2 there exist $g_1, \ldots, g_n \in \mathcal{O}_n$ such that

$$f_i(g_1, \ldots, g_n) = \mathtt{y}_i \quad \text{and} \quad g_i(f_1, \ldots, f_n) = \mathtt{x}_i \quad \text{for} \quad i = 1, \ldots, n.$$

Let $V_2 := \bigcap_{i=1}^n D(g_i)$ and denote $g := (g_1, \ldots, g_n) : V_2 \to \mathbb{K}^n$. Denote $U_1 := \bigcap_{i=1}^n D(f_i) \cap \Omega_1 \cap f^{-1}(V_2)$, which is an open neighborhood of the origin and $U_2 := f(U_1)$. Observe that $g|_{U_2} \circ f|_{U_1} = \mathrm{id}_{U_1}$ and $f|_{U_1} \circ g|_{U_2} = \mathrm{id}_{U_2}$, so $f|_{U_1} : U_1 \to U_2$ is a homeomorphism whose inverse is an analytic map. By the Invariance of domain theorem we conclude that $U_2 \subset \mathbb{K}^n$ is an open neighborhood of the origin, as required. □

Theorem V.1.9 (Implicit Function Theorem) *Let n, m be positive integers and let $\Omega \subset \mathbb{K}^n \times \mathbb{K}^m$ be an open set such that $(0,0) \in \Omega$. Let $f := (f_1, \ldots, f_m) : \Omega \to \mathbb{K}^m$ be an analytic map such that $f(0,0) = 0$ and*

$$\det\Big(\frac{\partial f_i}{\partial \mathtt{y}_j}(0,0)\Big)_{1 \le i,j \le m} \neq 0.$$

Then there exist

(i) *an open neighborhood $\Delta \subset \Omega$ of $(0,0)$,*

(ii) *an open neighborhood $U \subset \mathbb{K}^n$ of 0 and*

(iii) *an analytic map $g : U \to \mathbb{K}^m$*

such that $g(0) = 0$ and the following equality holds

$$\{(x,y) \in \Delta : \ f(x,y) = 0\} = \{(x, g(x)) : \ x \in U\}.$$

Proof. Consider the map

$$F := (\mathtt{x}_1, \ldots, \mathtt{x}_n, f_1, \ldots, f_m) : \Omega \to \mathbb{K}^n \times \mathbb{K}^m, \ (x,y) \mapsto (x, f(x,y)).$$

Observe that $F(0,0) = (0,0)$ and

$$\det\left(\begin{array}{c|c} I_n & 0 \\ \hline \Big(\frac{\partial f_i}{\partial \mathtt{x}_k}(0,0)\Big)_{\substack{1 \le i \le m \\ 1 \le k \le n}} & \Big(\frac{\partial f_i}{\partial \mathtt{y}_j}(0,0)\Big)_{1 \le i,j \le m} \end{array}\right) = \det\Big(\frac{\partial f_i}{\partial \mathtt{y}_j}(0,0)\Big)_{1 \le i,j \le m} \neq 0.$$

By the Inverse Function Theorem V.1.8 there exist open neighborhoods $V_1 \subset \Omega$ and $V_2 \subset \mathbb{K}^n \times \mathbb{K}^m$ of the origin such that $F|_{V_1} : V_1 \to V_2$ is an analytic equivalence. Shrinking V_1 if necessary we may assume that $V_1 := U_1 \times W_1$ where $U_1 \subset \mathbb{K}^n$ is an open neighborhood of the origin and $W_1 \subset \mathbb{K}^m$ is an open neighborhood of the origin.

Let $h := (h_1, \dots, h_n, u_1, \dots, u_m) : V_2 \to V_1$ be the inverse of $F|_{V_1}$, which is an analytic map. As $F|_{V_1} \circ h = \mathrm{id}_{V_2}$, we have $h_k = \mathtt{x}_k$ for $k = 1, \dots, n$ and

$$f_i(\mathtt{x}_1, \dots, \mathtt{x}_n, u_1(\mathtt{x}, \mathtt{y}), \dots, u_m(\mathtt{x}, \mathtt{y})) = \mathtt{y}_i$$

for $i = 1, \dots, m$. As $F|_{V_1}$ is an analytic equivalence onto its image, $(x, z) \in V_1$ satisfies $f(x, z) = 0$ if and only if $z_i = u_i(x, 0)$ for $i = 1, \dots, m$. Thus, it is enough to define $\Delta := V_1$, $U := U_1$, $g_i := u_i|_{U_i \times \{0\}}$ for $i = 1, \dots, m$ and $g := (g_1, \dots, g_m) : U \to \mathbb{K}^m$. $\square$

Remark V.1.10 With the notations in Theorem V.1.9 observe that if $V \subset \mathbb{K}^n$ is an open neighborhood of the origin and $h : V \to \mathbb{K}^m$ is another function (with no hypothesis concerning it) such that $f(x, h(x)) = 0$ for each $x \in V$, then h and g coincide in $U \cap V$.

Theorem V.1.11 (Rank Theorem) *Let $\Omega \subset \mathbb{K}^n$ be an open neighborhood of the origin and let $f := (f_1, \dots, f_m) : \Omega \to \mathbb{K}^m$ be an analytic map such that $f(0) = 0$. Assume that*

$$\operatorname{rk}\Big(\frac{\partial f_j}{\partial \mathtt{x}_i}(x)\Big)_{\substack{1 \le j \le m \\ 1 \le i \le n}} = r$$

for each $x \in \Omega$. Then there exist

(i) *connected open neighborhoods $U_1, U_2 \subset \mathbb{K}^n$ of the origin,*

(ii) *connected open neighborhoods $V_1, V_2 \subset \mathbb{K}^m$ of the origin,*

(iii) *analytic equivalences $g : U_1 \to U_2$ and $h : V_1 \to V_2$*

such that $f(U_2) \subset V_1$ and

$$h \circ f \circ g : U_1 \to V_2, \ (x_1, \dots, x_n) \mapsto (x_1, \dots, x_r, 0, \dots, 0).$$

Proof. As the rank

$$\operatorname{rk}\Big(\frac{\partial f_j}{\partial \mathtt{x}_i}(x)\Big)_{\substack{1 \le j \le m \\ 1 \le i \le n}}$$

equals r for each $x \in \Omega$, we have

$$\operatorname{rk}\Big(\frac{\partial f_j}{\partial \mathtt{x}_i}\Big)_{\substack{1\le j\le m\\ 1\le i\le n}} = r.$$

By Corollary I.5.3 there exist $g_1, \dots, g_n \in \mathbb{K}\{\mathtt{y}_1, \dots, \mathtt{y}_n\}$ and $h_1, \dots, h_m \in \mathbb{K}\{\mathtt{z}_1, \dots, \mathtt{z}_m\}$ such that

$$\det\Big(\frac{\partial g_i}{\partial \mathtt{x}_k}(0)\Big)_{1\le i,k\le n} \neq 0, \quad \det\Big(\frac{\partial h_j}{\partial \mathtt{z}_\ell}(0)\Big)_{1\le j,\ell\le m} \neq 0$$

and $h(f(g_1, \dots, g_n)) = (\mathtt{y}_1, \dots, \mathtt{y}_r, 0, \dots, 0)$, where $h := (h_1, \dots, h_m)$. Denote $g := (g_1, \dots, g_n)$. By the Inverse Function Theorem V.1.8 there exist connected open neighborhoods $U_1, U_2 \subset \mathbb{K}^n$ of the origin and connected open neighborhoods $V_1, V_2 \subset \mathbb{K}^m$ of the origin such that $g : U_1 \to U_2$ and $h : V_1 \to V_2$ are analytic equivalences. Shrinking U_1 if necessary we may assume in addition $f(U_2) \subset V_1$. By construction and the Identity Principle,

$$h \circ f \circ g : U_1 \to V_2, \ (x_1, \dots, x_n) \mapsto (x_1, \dots, x_r, 0, \dots, 0),$$

as required. $\square$

2 Analytic sets.

Let $U \subset \Omega \subset \mathbb{K}^n$ be open sets and let $f_1, \dots, f_r \in \mathcal{O}(\Omega)$ be analytic functions. The *zero set of* $f_1, \dots, f_r$ *in* U is

$$Z_U(f_1, \dots, f_r) := \{x \in U : \ f_1(x) = 0, \dots, f_r(x) = 0\}.$$

Definitions V.2.1 Let $X \subset \Omega$. We say that X is *locally analytic* if for each $x \in X$ there exist an open neighborhood $U \subset \Omega$ of x and analytic functions $f_1, \dots, f_r \in \mathcal{O}(U)$ such that $X \cap U = Z_U(f_1, \dots, f_r)$. If X is locally analytic and closed in Ω, then X is an *analytic subset of* Ω.

Remarks V.2.2 (i) Recall that a subset C of a topological space Y is closed if an only if for each $y \in Y$ there exists an open neighborhood V^y of y such that $V^y \cap C$ is closed in V^y. Using that $Z_U(1) = \varnothing$ for each $U \subset \mathbb{K}^n$, one deduces: *a subset* $X \subset \Omega$ *is an analytic subset of* Ω *if and only if for each* $x \in \Omega$ *there* *exist an open neighborhood* $U \subset \Omega$ *and analytic functions* $f_1, \dots, f_r \in \mathcal{O}(U)$ *such that* $X \cap U = Z_U(f_1, \dots, f_r)$.

(ii) Let X be an analytic subset of Ω and let $\Omega_1 \subset \Omega$ be an open subset that contains X. Then X is analytic in Ω_1. Indeed, given $x \in \Omega_1 \subset \Omega$ there exist an open subset $U \subset \Omega$ and $f_1, \dots, f_r \in \mathcal{O}(U)$ such that $X \cap U = Z_U(f_1, \dots, f_r)$. Therefore $U_1 := U \cap \Omega_1$ is an open subset of Ω_1 and the analytic functions $g_j := f_j|_{U_1} \in \mathcal{O}(U_1)$ satisfy $X \cap U_1 = Z_{U_1}(g_1, \dots, g_r)$.

(iii) If X is a closed and discrete subset of Ω, then X is an analytic subset of Ω. As X is closed, we have to check that it is locally analytic. As X is discrete, for each $a := (a_1, \dots, a_n) \in X$ there exists an open neighborhood $U \subset \Omega$ of a such that $X \cap U = \{a\} = Z_U(\mathtt{x}_1 - a_1, \dots, \mathtt{x}_n - a_n)$, so X is locally analytic, as required.

(iv) If X is locally analytic in Ω, then X is locally closed in Ω, so there exists an open subset $\Theta \subset \Omega$ such that X is closed in Θ. Consequently, X is analytic in Θ.

(v) If X is locally analytic in Ω and $Y \subset X$ is an open subset, then Y is locally analytic in Ω.

(vi) If $X \subset \Omega$ is analytic in Ω and $\Theta \subset \Omega$ is an open subset, then $X \cap \Theta$ is analytic in Θ.

(vii) The intersection of two analytic subsets of Ω is an analytic subset of Ω. This is based on the fact that

$$Z_U(f_1, \dots, f_r) \cap Z_U(g_1, \dots, g_s) = Z_U(f_1, \dots, f_r, g_1, \dots, g_s)$$

if $f_i, g_j \in \mathcal{O}(\Omega)$.

Concerning unions of analytic sets we achieve the same obstructions we have for closed sets. We need to ask for local finiteness.

Lemma V.2.3 *Let* $\{X_i\}_{i \in I}$ *be a locally finite family of analytic subsets of an open subset* $\Omega \subset \mathbb{K}^n$. *Then* $X := \bigcup_{i \in I} X_i$ *is an analytic subset of* Ω.

Proof. As the union of the members of a locally finite family of closed sets is a closed set, it is enough to prove: X *is locally analytic*.

Pick a point $a \in X$. As $\{X_i\}_{i \in I}$ is a locally finite family, there exists an open neighborhood $U \subset \Omega$ of a such that the set $J := \{i \in I : X_i \cap U \neq \varnothing\}$ is a finite set. For simplicity we write $J := \{1, \ldots, r\}$ and we may assume that $a \in X_i$ for $i = 1, \ldots, r$. Shrinking U if necessary we may assume $X_i \cap U = Z_U(f_{i1}, \ldots, f_{is})$ for $f_{ij} \in \mathcal{O}(U)$. Thus,

$$X \cap U = \bigcup_{i=1}^{r} (X_i \cap U) = \bigcup_{i=1}^{r} Z_U(f_{i1}, \ldots, f_{is}).$$

Consider all the products $\prod_{i=1}^{r} f_{ij_i}$ where $j_i \in \{1, \ldots, s\}$. We denote such products $g_1, \ldots, g_\ell \in \mathcal{O}(U)$. We claim: $X \cap U = Z_U(g_1, \ldots, g_\ell)$.

Observe that $X_i \cap U = Z_U(f_{i1}, \ldots, f_{is}) \subset Z_U(f_{ij_i})$ for each $j_i \in \{1, \ldots, s\}$. Thus, $X_i \cap U \subset Z_U(g_k)$ for each $k = 1, \ldots, \ell$, so $X \cap U \subset Z_U(g_1, \ldots, g_\ell)$. To prove the converse inclusion, pick $x \in Z_U(g_1, \ldots, g_\ell)$. If $x \notin X \cap U$, then $x \notin X_i \cap U$ for $i = 1, \ldots, r$, so we may assume that $f_{i1}(x) \neq 0$ for $i = 1, \ldots, r$. Thus, $\prod_{i=1}^{r} f_{i1}(x) \neq 0$, so one of the g_i does not vanish at x, which is a contradiction. We conclude $X \cap U = Z_U(g_1, \ldots, g_\ell)$, as required. $\square$

Theorem V.2.4 *Let X be an analytic subset of an open set $\Omega \subset \mathbb{K}^n$. Suppose that X does not contain any connected component of Ω. Then $\Omega \setminus X$ is dense in Ω.*

Proof. As $\mathbb{K}^n$ is locally connected, the connected components of an open subset of $\mathbb{K}^n$ are open sets. Consequently, we may assume Ω is connected. Suppose that $\Omega \setminus X$ is not dense in Ω. Then there exists a non-empty open set $\Theta \subset \Omega$ such that $(\Omega \setminus X) \cap \Theta = \varnothing$, so $\Theta \subset X$ and $\operatorname{Int}(X) \neq \varnothing$. If $\operatorname{Int}(X)$ is closed in Ω, then $\operatorname{Int}(X)$ is closed and open in Ω, so $X = \Omega$ because Ω is connected and X is non-empty.

Thus, we may assume there exists $a \in \operatorname{Cl}(\operatorname{Int}(X)) \setminus \operatorname{Int}(X) \subset X$. Let $U \subset \Omega$ be an open neighborhood of a and $f_1, \ldots, f_r \in \mathcal{O}(U)$ such that $X \cap U = Z_U(f_1, \ldots, f_r)$. In addition, we may assume by the local connectedness of $\mathbb{K}^n$ that U is connected. As $a \in \operatorname{Cl}(\operatorname{Int}(X))$, we deduce that $f_1, \ldots, f_r$ vanish identically on the non-empty open set $\operatorname{Int}(X) \cap U$, so by the Identity Principle $f_1, \ldots, f_r$ are identically zero on U. Thus, $a \in U = X \cap U \subset X$, so $a \in \operatorname{Int}(X)$, which is a contradiction. Hence, $X = \Omega$, as required. $\square$

Examples V.2.5 (i) The set $X := \{\frac{1}{n}\}_{n \geq 1}$ is an analytic subset of the open set $\Omega := \mathbb{K} \setminus \{0\}$ because it is closed and discrete in Ω. However, X is not analytic in $\mathbb{K}$ because it is not closed in $\mathbb{K}$.

(ii) The interval $[-1,1] \subset \mathbb{R}$ is not locally analytic in any of its neighborhoods in $\mathbb{R}$. It is enough to prove that for each $\varepsilon > 0$ the set $[-1,-1+\varepsilon)$ is not an analytic subset of $U := (-1-\varepsilon,-1+\varepsilon)$. If $f \in \mathcal{O}(U)$ vanishes identically on $[-1,-1+\varepsilon)$, then f is identically zero on U by the Identity Principle. Thus, it is not posible to write $[-1,-1+\varepsilon) = Z_U(f_1,\dots,f_r)$ where $f_i \in \mathcal{O}(U)$ and $[-1,-1+\varepsilon)$ is not an analytic subset of U, as required.

(iii) Let $\Omega \subset \mathbb{K}^n$ be an open set and let $f := (f_1,\dots,f_m) : \Omega \to \mathbb{K}^m$ be an analytic map. The fibers of f are analytic subsets of Ω, because given $p := (p_1,\dots,p_m) \in \mathbb{K}^m$ the functions $g_i : \Omega \to \mathbb{K}^m$, $x \mapsto f_i(x) - p_i$ are analytic and $f^{-1}(p) = Z_\Omega(g_1,\dots,g_m) \subset \Omega$ is analytic.

3 Analytic submanifolds

We introduce next the notion of analytic submanifold of $\mathbb{K}^n$. Recall that each point $a := (a_1,\dots,a_n) \in \mathbb{K}^n$ has a basis of open neighborhoods that consists of the polycylinders $\Delta_\rho(a)$ of center a and polyradius $\rho := (\rho_1,\dots,\rho_n)$ where $\rho_i > 0$. Namely,

$$\Delta_\rho(a) := \{x \in \mathbb{K}^n : |x_i - a_i| < \rho_i \text{ for } 1 \le i \le n\} = \prod_{i=1}^n D_{\rho_i}(a_i)$$

where $D_{\rho_i}(a_i) := \{x \in \mathbb{K} : |x_i - a_i| < \rho_i\}$ is an interval centered at a_i if $\mathbb{K} = \mathbb{R}$ and the disc of center a_i and radius $\rho_i > 0$ if $\mathbb{K} = \mathbb{C}$.

Definition V.3.1 A subset $M \subset \mathbb{K}^n$ is an *analytic submanifold of* $\mathbb{K}^n$ *of dimension* d if for each point $a \in M$ there exists an analytic equivalence $u := (u_1,\dots,u_n) : U \to \Delta_\rho(0)$ between an open neighborhood $U \subset \mathbb{K}^n$ of a and a polycylinder $\Delta_\rho(0) \subset \mathbb{K}^n$ of center the origin such that $u(a) = 0$ and $X \cap U = Z_U(u_{d+1} = 0,\dots,u_n = 0)$.

An analytic submanifold M of $\mathbb{K}^n$ is a locally analytic subset of each of its open neighborhoods. In addition, M is locally compact and locally connected. Consequently, $\mathrm{Cl}(M) \setminus M$ is a closed subset of $\mathbb{K}^n$ (so M is a closed subset of the open set $\Omega := \mathbb{K}^n \setminus (\mathrm{Cl}(M) \setminus M)$) and the connected components of M are open subsets of M. The open subsets of an analytic submanifold of dimension d are still analytic submanifolds of its same dimension. Thus, the connected components of an analytic submanifold of dimension d are analytic submanifolds of dimension d. The simplest example of analytic submanifold of $\mathbb{K}^n$ is a non-empty open subset of $\mathbb{K}^n$, which is an analytic submanifold of $\mathbb{K}^n$ of dimension n.

3.a Characterization of analytic submanifolds. Let $\Theta \subset \mathbb{K}^d$ be an open set. An analytic map $f : \Theta \to \mathbb{K}^n$ is called an *analytic immersion* if f is a homeomorphism onto its image $f(\Theta)$ and

$$\operatorname{rk}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(x)\Big)_{\substack{1\le i\le n\\ 1\le j\le d}} = d \quad \forall\, x \in \Theta. \tag{3.1}$$

Example V.3.2 (Lemniscate) The analytic map

$$f : \mathbb{R} \to \mathbb{R}^2,\ t \mapsto \Big(\frac{t}{1+t^4}, \frac{t^3}{1+t^4}\Big)$$

satisfies the rank condition (3.1) because

$$f'(t) = \frac{1}{(1+t^4)^2}(1-3t^4, t^2(3-t^4)) \neq (0,0)$$

for each $t \in \mathbb{R}$. In addition, $f : \mathbb{R} \to f(\mathbb{R})$ is a continuous bijection.

If $f(t) = f(s)$ for $t \neq s$, we have $t \neq 0$, $s \neq 0$,

$$\frac{t}{1+t^4} = \frac{s}{1+s^4} \quad \text{and} \quad \frac{t^3}{1+t^4} = \frac{s^3}{1+s^4}.$$

After diving the latter expressions, we have $t^2 = s^2$, so $1+t^4 = 1+s^4$. Thus, $s = t$, which is a contradiction.

However, $f : \mathbb{R} \to f(\mathbb{R})$ is not an analytic immersion because it is not a homeomorphism. Observe that $\mathbb{R}$ is not compact but $f(\mathbb{R}) \subset [-1,1] \times [-1,1]$ is a compact subset of $\mathbb{R}^2$.

Lemma V.3.3 *A subset $M \subset \mathbb{K}^n$ is an analytic submanifold of $\mathbb{K}^n$ of dimension d if and only if for each point $a \in M$ there exist an open neighborhood $\Theta \subset \mathbb{K}^d$ of the origin and an analytic immersion $f := (f_1, \dots, f_n) : \Theta \to \mathbb{K}^n$ such that $f(0) = a$ and $f(\Theta)$ is an open neighborhood of a in M.*

Proof. We prove both implications:

3.a.1 Suppose that $M \subset \mathbb{K}^n$ is an analytic submanifold of $\mathbb{K}^n$ and pick a point $a \in M$. There exists an analytic equivalence $u := (u_1, \dots, u_n) : U \to \Delta_\rho(0)$ between an open neighborhood $U \subset \mathbb{K}^n$ of a and $\Delta_\rho(0) \subset \mathbb{K}^n$ with $u(a) = 0$ and $M \cap U = Z_U(u_{d+1} = 0, \dots, u_n = 0)$. Note that $u(M \cap U) = \Delta_{\rho'}(0) \times \{0\}$ where $\rho' := (\rho_1, \dots, \rho_d)$ and $u|_{M\cap U} : M \cap U \to \Delta_{\rho'}(0) \times \{0\}$ is a homeomorphism. Consider the analytic equivalence $\mathtt{j} : \Delta_{\rho'}(0) \times \{0\} \to \Delta_{\rho'}(0) \times \{0\},\ x \mapsto (x, 0)$ and the analytic map $f := u^{-1} \circ \mathtt{j} : \Delta_{\rho'}(0) \to \mathbb{K}^n$. Observe that f is a homeomorphism onto its image $M \cap U$, which is an open subset of M, because it is a composition of homeomorphisms. Let us check that

$$\operatorname{rk}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(x)\Big)_{\substack{1\le i\le n\\ 1\le j\le d}} = d \quad \forall\, x \in \Delta_{\rho'}(0).$$

To that end write $u^{-1} := v := (v_1, \dots, v_n)$ and observe that

$$\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(x)\Big)_{\substack{1\le i\le n\\ 1\le j\le d}} = \Big(\frac{\partial v_i}{\partial \mathtt{x}_j}(x, 0)\Big)_{1\le i,k\le n} \cdot \left(\frac{I_d}{0}\right),$$

which has rank d because v is an analytic equivalence.

3.a.2 Conversely, pick a point $a \in M$ and let $\Theta \subset \mathbb{K}^d$ be an open neighborhood of the origin and $f := (f_1, \dots, f_n) : \Theta \to \mathbb{K}^n$ an analytic immersion such that $f(0) = a$ and $f(\Theta)$ is an open neighborhood of a in M. As

$$\operatorname{rk}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(x)\Big)_{\substack{1\le i\le n\\ 1\le j\le d}} = d \quad \forall\, x \in \Theta,$$

we may assume

$$\det\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(x)\Big)_{1\le i,j\le d} \neq 0.$$

Write $x' := (x_1, \dots, x_d)$ and $x'' := (x_{d+1}, \dots, x_n)$. Consider the analytic map

$$g := (g_1, \dots, g_n) : \Theta \times \mathbb{K}^{n-d} \to \mathbb{K}^n,\ (x', x'') \mapsto f(x') + (0, x'').$$

Observe that $g(0, 0) = f(0) = a$ and

$$\det\Big(\frac{\partial g_i}{\partial \mathtt{x}_j}(0,0)\Big)_{1\le i,j\le n} = \det\left(\begin{array}{c|c} \Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0,0)\Big)_{1\le i,j\le d} & (0)_{\substack{1\le i\le d\\ d+1\le j\le n}} \\ \hline \Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0,0)\Big)_{\substack{d+1\le i\le n\\ 1\le j\le d}} & I_{n-d} \end{array}\right)$$
$$= \det\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\Big)_{1\le i,j\le d} \neq 0.$$

By the Inverse Function Theorem V.1.8 there exist an open neighborhood $U_1 \subset \Theta \times \mathbb{K}^{n-d}$ of the origin and an open neighborhood $U_2 \subset \mathbb{K}^n$ of a such that $g|_{U_1} : U_1 \to U_2$ is an analytic equivalence. We may assume that $U_1 = \Delta_\rho(0)$ for some $\rho := (\rho_1, \ldots, \rho_n) > 0$ and $M \cap U_2 \subset f(\Theta)$. As $g(\Delta_{\rho'}(0) \times \{0\}) = M \cap U_2$ where $\rho' := (\rho_1, \ldots, \rho_d)$, the map $u := g^{-1} : U_2 \to \Delta_\rho(0)$ is the analytic equivalence we sought. Consequently, M is an analytic submanifold of $\mathbb{K}^n$ of dimension d, as required. $\square$

Remark V.3.4 Let $M \subset \mathbb{K}^n$ be a connected subset such that for each $a \in M$ there exists an integer $d(a)$, an open neighborhood $U \subset \mathbb{K}^n$ of a, a polycylinder $\Delta_\rho(0) \subset \mathbb{K}^n$ and an analytic equivalence $u := (u_1, \ldots, u_n) : U \to \Delta_\rho(0)$ such that $u(a) = 0$ and $X \cap U = Z_U(u_{d(a)+1}, \ldots, u_n)$. Then for each pair of points $a, b \in M$ we have $d(a) = d(b)$ and M is an analytic submanifold of dimension $d := d(a)$.

By Invariance of domain theorem we deduce that $d(a)$ is univocally determined by a and does not depend on the analytic equivalence u. For each point $a \in M$ fix an analytic equivalence $u := (u_1, \ldots, u_n) : U^a \to \Delta_\rho(0)$ between an open neighborhood $U^a \subset \mathbb{K}^n$ of a and a polycylinder $\Delta_\rho(0) \subset \mathbb{K}^n$ such that $u(a) = 0$ and $X \cap U = Z_U(u_{d(a)+1}, \ldots, u_n)$. Again by Invariance of domain theorem we deduce that $d(a) = d(p)$ for each $p \in U^a$. Fix $a \in M$ and let $b \in M$. We claim: $d(a) = d(b)$.

As M is connected, there exist points $x_0 := b, x_1, \ldots, x_r := a$ such that $U^{x_{i-1}} \cap U^{x_i} \neq \varnothing$ for $i = 1, \ldots, r$. We proceed by induction on r. If $r = 0$, there is nothing to prove. Assume the result true for $r - 1$ and let us check that it is also true for r. Pick a point $p \in U^{x_{r-1}} \cap U^a$. Then $d(a) = d(p) = d(x_{r-1})$. By induction hypothesis we have $d(b) = d(x_{r-1}) = d(a)$, as required.

3.b Identity principle for analytic submanifolds. Let us prove next the Identity Principle for connected analytic submanifolds.

Theorem V.3.5 (Identity Principle) *Let $M \subset \mathbb{K}^n$ be a connected analytic submanifold and $\Omega \subset \mathbb{K}^n$ an open neighborhood of M. Let $f \in \mathcal{O}(\Omega)$ that vanishes identically on a non-empty open subset W of M. Then $f|_M$ is identically zero.*

Proof. The proof is conducted in two steps:

3.b.1 Pick a point $b \in M$. By Lemma V.3.3 there exist a connected open neighborhood $\Theta_b \subset \mathbb{K}^d$ of the origin and an analytic immersion $g : \Theta \to \mathbb{K}^n$ such that $g(0) = b$ and $U^b := g(\Theta)$ is an open neighborhood of b in M. Suppose that f vanishes identically on a non-empty open subset of U^b. Then $f \circ g$ vanishes identically on a non-empty open subset of the connected open set Θ. By the Identity Principle $f \circ g = 0$, so $f|_{U^b} = 0$.

3.b.2 For each point $b \in M$ let U^b be the open neighborhood of a constructed in 3.b.1. Fix points $a \in W$ and $b \in M$ and let us prove that $f(b) = 0$. By 3.b.1 $f|_{U^a} = 0$. As M is connected, there exist points $x_0 := b, x_1, \ldots, x_r := a$ such that $U^{x_{i-1}} \cap U^{x_i} \neq \varnothing$ for $i = 1, \ldots, r$. Let us prove by induction on r that $f(b) = 0$. If $r = 0$, there is nothing to prove. Assume the result true for $r - 1$ and let us check that it is also true for r. As $f|_{U^a} = 0$, we deduce that f vanishes identically on the non-empty open set $U^{x_{r-1}} \cap U^a$. By 3.b.1 $f|_{U^{x_{r-1}}} = 0$ and by induction hypothesis $f(b) = 0$, as required. $\square$

Corollary V.3.6 *Let $\Omega \subset \mathbb{K}^n$ be an open set and let $X, M \subset \Omega$ be, respectively, an analytic subset of Ω and a connected analytic submanifold. Suppose that X contains a non-empty open subset of M. Then $M \subset X$.*

Proof. Suppose that $M \cap X \subsetneq M$. As X contains a non-empty subset of M, we have $\mathrm{Int}_M(M \cap X) \neq \varnothing$. As M is connected, $\mathrm{Int}_M(M \cap X) \neq M$ and $\mathrm{Int}_M(M \cap X)$ is open in M, there exists a point

$$a \in \mathrm{Cl}_M(\mathrm{Int}_M(M \cap X)) \setminus \mathrm{Int}_M(M \cap X).$$

As X is a closed subset of Ω, we deduce $a \in \mathrm{Cl}_M(\mathrm{Int}_M(M \cap X)) \subset X$. Then there exist an open neighborhood $U \subset \Omega$ of a and analytic functions $f_1, \ldots, f_r \in \mathcal{O}(U)$ such that $X \cap U = Z_U(f_1, \ldots, f_r)$. Shrinking U if necessary we may assume that $M \cap U$ is connected, so $M \cap U$ is a connected analytic submanifold.

As $a \in \mathrm{Cl}_M(\mathrm{Int}_M(M \cap X))$ and U is an open neighborhood of a,

$$\varnothing \neq \mathrm{Int}_M(M \cap X) \cap U \subset M \cap U \cap X.$$

As $U \cap \mathrm{Int}_M(M \cap X)$ is a non-empty open subset of $M \cap U$ contained in $X \cap U = Z_U(f_1, \ldots, f_r)$, we deduce by the Identity Principle that $f_1, \ldots, f_r$ are identically zero on $M \cap U$, so $M \cap U \subset X \cap U$. Thus, $a \in M \cap U \subset M \cap X$

and $M \cap U$ is an open subset of M, so $a \in \mathrm{Int}_M(M \cap X)$, which is a contradiction. Thus, $M \cap X = M$, as required. $\square$

Corollary V.3.6 above is frequently used in the following equivalent form.

Corollary V.3.7 *Let $M \subset \Omega$ be a connected analytic submanifold and let $X \subset \Omega$ be an analytic set such that $M \not\subset X$. Then $\mathrm{Cl}_M(M \setminus X) = M$.*

Proof. As $M \not\subset X$, it follows from Corollary V.3.6 that $X \cap M$ does not contain any non-empty open subset of M, so $\mathrm{Int}_M(M \cap X) = \varnothing$. Thus,

$$\mathrm{Cl}_M(M \setminus X) = M \setminus \mathrm{Int}_M(M \cap X) = M,$$

as required. $\square$

3.c Smooth points of an analytic set.

Let us obtain now a useful partition of an analytic set.

Definition V.3.8 Let $\Omega \subset \mathbb{K}^n$ be an open set and let $X \subset \Omega$ be an analytic subset of Ω. For each $d \geq 1$ we define the set $\mathrm{Smooth}_d(X)$ of *smooth points of X of dimension* d as the set of points $x \in X$ that admit an open neighborhood $U \subset X$ of x that is an analytic submanifold of dimension d.

Remarks V.3.9 (i) $\mathrm{Smooth}_d(X)$ is an open subset of X for each $d \geq 1$.

(ii) If $\mathrm{Smooth}_d(X) \neq \varnothing$, then $\mathrm{Smooth}_d(X)$ is an analytic submanifold of dimension d.

(iii) $\mathrm{Smooth}_d(X) \cap \mathrm{Smooth}_e(X) = \varnothing$ if $d \neq e$.

Definition V.3.10 Let $\Omega \subset \mathbb{K}^n$ be an open set and let $X \subset \Omega$ be an analytic subset of Ω. We define the *set of smooth points of X* as the open subset $\mathrm{Smooth}(X) := \bigsqcup_{d \geq 0} \mathrm{Smooth}_d(X)$ of X. The *set of non-smooth points of X* is the closed subset $\mathrm{Non\text{-}smooth}(X) := X \setminus \mathrm{Smooth}(X)$ of X. We write

$$X = \mathrm{Non\text{-}smooth}(X) \sqcup \bigsqcup_{d \geq 0} \mathrm{Smooth}_d(X).$$

Lemma V.3.11 *Let $\Omega \subset \mathbb{K}^n$ be an open set and let $f := (f_1, \dots, f_r) : \Omega \to \mathbb{K}^r$ be an analytic map such that*

$$\operatorname{rk}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(x)\Big)_{\substack{1\leq i\leq r\\ 1\leq j\leq n}} = r \quad \forall\, x \in M := Z_\Omega(f_1, \dots, f_r).$$

Then M is an analytic submanifold of dimension $d := n - r$.

Proof. Fix a point $a := (a_1, \dots, a_n) \in M$. As

$$\operatorname{rk}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(a)\Big)_{\substack{1\leq i\leq r\\ 1\leq j\leq n}} = r,$$

we may assume

$$\det\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(a)\Big)_{1\leq i,j\leq r} \neq 0.$$

Denote $x'' := (x_{r+1}, \dots, x_n)$ and $a'' := (a_{r+1}, \dots, a_n)$. Consider the analytic map

$$h := (h_1, \dots, h_n) : \Omega \to \mathbb{K}^n,\ x \mapsto (f(x), x'' - a'').$$

Observe that $h(a) = (f(a), a'' - a'') = 0$ and

$$\det\Big(\frac{\partial h_i}{\partial \mathtt{x}_j}(a)\Big)_{1\leq i,j\leq n} = \det\left(\begin{array}{c|c} \Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(a)\Big)_{1\leq i,j\leq r} & \Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(a)\Big)_{\substack{1\leq i\leq r\\ r+1\leq j\leq n}} \\ \hline (0)_{\substack{r+1\leq i\leq n\\ 1\leq j\leq r}} & I_{n-r} \end{array}\right)$$

$$= \det\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(a)\Big)_{1\leq i,j\leq r} \neq 0.$$

By the Inverse Function Theorem V.1.8 there exist open neighborhoods $U_1 \subset \Omega$ of a and $U_2 \subset \mathbb{K}^n$ of the origin such that $h|_{U_1} : U_1 \to U_2$ is an analytic equivalence. We may assume that $U_2 = \Delta_\rho(0)$ for some $\rho := (\rho_1, \dots, \rho_n) > 0$ and $M \cap U_1 = Z_{U_1}(h_1, \dots, h_r)$. Consequently, M is an analytic submanifold of $\mathbb{K}^n$ of dimension d, as required. $\square$

Examples V.3.12 (i) The analytic subset $X := Z_{\mathbb{K}^2}(\mathtt{y} - \mathtt{x}^2)$ of $\mathbb{K}^2$ is an analytic submanifold of dimension 1 because $\operatorname{rk}(-2x,\, 1) = 1$ for each point $(x, y) \in X$.

(ii) The analytic subset $X := Z_{\mathbb{K}^2}(\mathtt{x}^2 + \mathtt{y}^2 - 1)$ of $\mathbb{K}^2$ is an analytic submanifold of dimension 1 because $\operatorname{rk}(2x,\, 2y) = 1$ for each $(x, y) \in X$.

(iii) The analytic subset $X := Z_{\mathbb{R}^2}(\mathtt{x}^2(1-\mathtt{x})+\mathtt{y}^2)$ of $\mathbb{R}^2$ has no non-smooth points but it is not an analytic submanifold of dimension 1.

Observe that $\operatorname{rk}(2x-3x^2,\ 2y) = 1$ if and only if $(x,y) \in \mathbb{R}^2 \setminus \{(0,0),(\frac{2}{3},0)\}$. Observe that $(\frac{2}{3},0) \notin X$, so $X \setminus \{(0,0)\}$ is an analytic submanifold of dimension 1 and $X \setminus \{(0,0)\} \subset \operatorname{Smooth}_1(X)$. The origin is an isolated point of X, so $(0,0) \in \operatorname{Smooth}_0(X)$. Consequently,

$$X = \operatorname{Smooth}(X) = \operatorname{Smooth}_1(X) \cup \operatorname{Smooth}_0(X)$$

and $\operatorname{Smooth}_1(X) = X \setminus \{(0,0)\}$ and $\operatorname{Smooth}_0(X) = \{(0,0)\}$.

(iv) Consider the analytic subset $X := Z_{\mathbb{R}^2}(\mathtt{y}^2 - \mathtt{x}^3)$ of $\mathbb{K}^2$. Then

$$\operatorname{Smooth}_1(X) = X \setminus \{(0,0)\} \quad \text{and} \quad \text{Non-smooth}(X) = \{(0,0)\}.$$

We have $X \setminus \{(0,0)\} \subset \operatorname{Smooth}_1(X)$ because $\operatorname{rk}(-3x^2,\ 2y) = 1$ for each point $(x,y) \in \mathbb{K}^2 \setminus \{(0,0)\}$. Now we have to check that $(0,0) \in \text{Non-smooth}(X)$.

As the function $\mathtt{y}^2 - \mathtt{x}^3$ does not vanish identically on any non-empty open subset of $\mathbb{K}^2$, we deduce $\operatorname{Smooth}_2(X) = \varnothing$. As $(0,0)$ is not an isolated point of X, either $(0,0) \in \operatorname{Smooth}_1(X)$ or $(0,0) \in \text{Non-smooth}(X)$. Suppose by contradiction that $(0,0) \in \operatorname{Smooth}_1(X)$. Then there exists an open subset $\Omega \subset \mathbb{K}$ that contains 0 and an analytic map $f := (f_1, f_2) : \Omega \to \mathbb{K}^2$ such that $f(0) = (0,0)$, $f(\Omega) \subset X$ and $\operatorname{rk}(f_1'(0), f_2'(0)) = 1$. As $f_1(0) = f_2(0) = 0$, the analytic series f_1, f_2 belong to the maximal ideal of $\mathcal{O}_2$. As $\operatorname{rk}(f_1'(0), f_2'(0)) = 1$, either $\omega(f_1) = 1$ or $\omega(f_2) = 1$. As $f_1^3 - f_2^2 = 0$, we have $3\omega(f_1) = 2\omega(f_2)$, so either

- $\omega(f_1) = \frac{2}{3}$ and $\omega(f_2) = 1$ or
- $\omega(f_1) = 1$ and $\omega(f_2) = \frac{3}{2}$,

which is a contradiction. Consequently, $(0,0) \in \text{Non-smooth}(X)$, as required.

4 Holomorphic functions

In this section we consider $\mathbb{K} = \mathbb{C}$.

Definition V.4.1 Let $\Omega \subset \mathbb{C}^n$ be an open subset. A continuous function $f : \Omega \to \mathbb{C}$ is *holomorphic* if for each $a \in \Omega$ there exists $\frac{\partial f}{\partial \mathtt{x}_j}(a)$ for $j = 1, \ldots, n$.

Remark V.4.2 It is clear that if $f \in \mathcal{O}(\Omega)$, then f is a holomorphic function.

Recall the following basic result that will be useful in the sequel.

Lemma V.4.3 *Let $f : [a, b] \to \mathbb{C}$ be a continuous function. Then*

$$\Big| \int_a^b f(t)dt \Big| \leq \int_a^b |f(t)|dt.$$

Proof. Write $\int_a^b f(t)dt = \rho e^{i\theta}$ where $\rho > 0$ and $\theta \in [0, 2\pi)$. We have

$$\Big| \int_a^b f(t)dt \Big| = e^{-i\theta} \int_a^b f(t)dt = \int_a^b e^{-i\theta} f(t)dt.$$

As $| \int_a^b f(t)dt|$ is a real number,

$$\begin{aligned} \Big| \int_a^b f(t)dt \Big| = \Re\Big(\int_a^b e^{-i\theta} f(t)dt \Big) &= \int_a^b \Re(e^{-i\theta} f(t))dt \\ &\leq \int_a^b |e^{-i\theta} f(t)|dt = \int_a^b |f(t)|dt, \end{aligned}$$

as required. □

4.a Cauchy's function.

Let $a := (a_1, \ldots, a_n) \in \mathbb{C}^n$ be a point and let $\rho := (\rho_1, \ldots, \rho_n) > 0$. Let $\Delta := \Delta_\rho(a) = \prod_{j=1}^n D_{\rho_j}(a_j)$ and let $\Upsilon := \Upsilon_\rho(a) = \prod_{j=1}^n \partial D_{\rho_j}(a_j)$. We assume that $\partial D_{\rho_j}(a_j)$ is oriented counterclockwise. Let $f : \Upsilon \to \mathbb{C}$ be a continuous function and denote $i := \sqrt{-1}$. Then the function

$$F : \Delta \times \Upsilon,\ (x, \xi) \mapsto \frac{f(\xi)}{(\xi - x)} := \frac{f(\xi)}{\prod_{i=1}^n (\xi_i - x_i)}$$

is also continuous and we define the *Cauchy function of f* as

$$\mathrm{Ch}(f) : \Delta \to \mathbb{C},\ x \mapsto \frac{1}{(2\pi i)^n} \int_\Upsilon \frac{f(\xi)d\xi}{\xi - x}$$

Observe that

$$\begin{aligned} \mathrm{Ch}(f)(x) &= \frac{1}{(2\pi i)^n} \int_\Upsilon \frac{f(\xi)d\xi}{\xi - x} \\ &= \frac{1}{(2\pi i)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{f(a + (\rho_1 e^{it_1}, \ldots, \rho_n e^{it_n}))}{\prod_{j=1}^n (a_j + \rho_j e^{it_j} - x_j)} i^n \Big(\prod_{j=1}^n \rho_j \Big) e^{i \sum_{j=1}^n t_j} dt_1 \cdots dt_n. \end{aligned}$$

4.a.1 Let $\mathsf{e} := (1, \ldots, 1)$ and $\nu \in \mathbb{N}^n$. Define

$$a_\nu := \frac{1}{(2\pi i)^n} \int_\Upsilon \frac{f(\xi) d\xi}{(\xi - a)^{\nu + \mathsf{e}}}$$

and consider the formal power series $F := \sum_\nu a_\nu \mathrm{x}^\nu \in \mathcal{F}_n$. Then:

(i) $F \in \mathcal{O}_n$ and $\Delta - a \subset D(F)$.

(ii) $\mathrm{Ch}(f)$ is an analytic function on Δ and $\mathrm{Ch}(f)(x) = F(x - a)$ for each $x \in \Delta$.

(i) We may assume for simplicity that $a = 0$. We have to check that $\Delta \subset D(F)$. Denote $\Upsilon_i := \partial D_{\rho_i}(0)$. Pick $x := (x_1, \ldots, x_n) \in \Delta$. Then $|x_j| < \rho_j$ for $j = 1, \ldots, n$ and there exists $0 < q < 1$ such that $\frac{|x_j|}{\rho_j} < q$ for $j = 1, \ldots, n$. Denote $M := \sup_{\xi \in \Upsilon}\{|f(\xi)|\}$. By Lemma V.4.3

$$\begin{aligned}
|a_\nu| &= \Big|\frac{1}{(2\pi i)^n} \int_\Upsilon \frac{f(\xi) d\xi}{\xi^{\nu + \mathsf{e}}}\Big| \\
&= \Big|\frac{1}{(2\pi i)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{f(\rho_1 e^{it_1}, \ldots, \rho_n e^{it_n})}{\rho^{\nu + \mathsf{e}} e^{i \sum_{j=1}^n (\nu_j + 1) t_j}} i^n \rho^{\mathsf{e}} e^{i \sum_{j=1}^n t_j} dt_1 \cdots dt_n\Big| \\
&= \Big|\frac{1}{(2\pi)^n} \frac{1}{\rho^\nu} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{f(\rho_1 e^{it_1}, \ldots, \rho_n e^{it_n})}{e^{i \sum_{j=1}^n \nu_j t_j}} dt_1 \cdots dt_n\Big| \\
&\leq \frac{1}{(2\pi)^n} \frac{1}{\rho^\nu} \int_0^{2\pi} \cdots \int_0^{2\pi} \Big|\frac{f(\rho_1 e^{it_1}, \ldots, \rho_n e^{it_n})}{e^{i \sum_{j=1}^n \nu_j t_j}}\Big| dt_1 \cdots dt_n \leq \frac{M}{\rho^\nu}.
\end{aligned}$$

Thus, $|a_\nu x^\nu| \leq M(\frac{x}{\rho})^\nu \leq M q^{|\nu|}$ for $\nu \in \mathbb{N}^n$. As $\sum_\nu M q^{|\nu|} = M(\frac{1}{1-q})^n$ and $0 < q < 1$, the series $\sum_\nu |a_\nu x^\nu|$ is convergent, so $\sum_\nu a_\nu x^\nu$ is convergent too and $\Delta \subset D(F)$.

(ii) Let $x \in \Delta$ and let $I \subset \mathbb{N}^n$ be a finite set. Then

$$\begin{aligned}
\Big|\mathrm{Ch}(f)(x) - \sum_{\nu \in I} a_\nu x^\nu\Big| &= \Big|\frac{1}{(2\pi i)^n} \int_\Upsilon \frac{f(\xi) d\xi}{\xi - x} - \frac{1}{(2\pi i)^n} \sum_{\nu \in I} \Big(\int_\Upsilon \frac{f(\xi) d\xi}{\xi^{\nu + \mathsf{e}}}\Big) x^\nu\Big| \\
&= \Big|\frac{1}{(2\pi i)^n} \int_\Upsilon f(\xi) \Big(\frac{1}{\xi - x} - \sum_{\nu \in I} \frac{x^\nu}{\xi^\nu \xi^{\mathsf{e}}}\Big) d\xi\Big| \\
&= \Big|\frac{1}{(2\pi i)^n} \int_\Upsilon \frac{f(\xi)}{\xi^{\mathsf{e}}} \Big(\frac{1}{1 - \frac{x}{\xi}} - \sum_{\nu \in I} \frac{x^\nu}{\xi^\nu}\Big) d\xi\Big|.
\end{aligned}$$

Define $h_I(\xi) := \frac{1}{1-\frac{x}{\xi}} - \sum_{\nu\in I} \frac{x^\nu}{\xi^\nu}$. Denote

$$M := \sup_{\xi\in\Upsilon}\{|f(\xi)|\} \quad \text{and} \quad N_I := \sup_{\xi\in\Upsilon}\{h_I(\xi)\}.$$

By Lemma V.4.3

$$\begin{aligned}
&\Big|\mathrm{Ch}(f)(x) - \sum_{\nu\in I} a_\nu x^\nu\Big| \\
&= \Big|\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} \frac{f(\rho_1 e^{it_1},\dots,\rho_n e^{it_n})}{\rho^{\mathsf{e}} e^{i\sum_{j=1}^n t_j}} h_I(\rho_1 e^{it_1},\dots,\rho_n e^{it_n})\rho^{\mathsf{e}} e^{i\sum_{j=1}^n t_j} dt_1\cdots dt_n\Big| \\
&= \Big|\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} f(\rho_1 e^{it_1},\dots,\rho_n e^{it_n}) h_I(\rho_1 e^{it_1},\dots,\rho_n e^{it_n}) dt_1\cdots dt_n\Big| \le MN_I.
\end{aligned}$$

The series $\sum_\nu z^\nu$ converges uniformly to $\prod_{i=1}^n \frac{1}{1-z_i}$ on the compact subsets of $\Delta_1(0)$. Thus, it converges uniformly on $\mathrm{Cl}(\Delta_q(0))$ because $0 < q < 1$. Fix $\varepsilon > 0$ and let $I_{\varepsilon/M} \subset \mathbb{N}^n$ be a finite set such that

$$\Big|\prod_{i=1}^n \frac{1}{1-z_i} - \sum_{\nu\in I} z^\nu\Big| < \frac{\varepsilon}{M}$$

for each $z \in \mathrm{Cl}(\Delta_q(0))$ and each finite set $I \subset \mathbb{N}^n$ that contains $I_{\varepsilon/M}$. As $\frac{x}{\xi} \in \mathrm{Cl}(\Delta_q(0))$ for each $\xi \in \Upsilon$, we have $N_I < \frac{\varepsilon}{M}$ for each finite set $I \subset \mathbb{N}^n$ that contains $I_{\varepsilon/M}$. Consequently,

$$\Big|\mathrm{Ch}(f)(x) - \sum_{\nu\in I} a_\nu x^\nu\Big| \le MN_I < \varepsilon.$$

Thus, $\sum_\nu a_\nu x^\nu$ converges to $\mathrm{Ch}(f)(x)$ for each $x \in \Delta$, as required.

4.b Osgood's Theorem. Let us prove next that complex analytic functions coincide with holomorphic functions. We recall first 'Cauchy integral formula'.

Theorem V.4.4 (Cauchy integral formula) *Let $\Omega \subset \mathbb{C}$ be an open set and let $a \in \Omega$. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function and fix $\rho > 0$ such that the closed disc $D_\rho(a)$ is contained in Ω. Denote $\Upsilon := \partial D_\rho(a)$. Then*

$$f(z) = \frac{1}{2\pi i}\int_\Upsilon \frac{f(\xi)d\xi}{\xi - z} = \mathrm{Ch}(f|_\Upsilon)(z)$$

for each $z \in D_\rho(a)$.

Proof. The proof is conducted in several steps:

4.b.1 *Cauchy-Riemann equalities.* Pick a point $b \in \mathbb{C}$ and let $g : \Theta \to \mathbb{C}$ be a holomorphic function on an open neighborhood Θ of b. As g is holomorphic, there exists

$$g'(b) = \lim_{z \to b} \frac{g(z) - g(b)}{z - b}.$$

Write $\mathtt{z} := \mathtt{x} + i\mathtt{y}$ and $g(\mathtt{x} + i\mathtt{y}) := g_1(\mathtt{x}, \mathtt{y}) + ig_2(\mathtt{x}, \mathtt{y})$ where g_1, g_2 are $\mathcal{C}^1$ real functions in two variables. Observe that for $h \in \mathbb{R}$

$$\begin{aligned}
\frac{\partial g_1}{\partial \mathtt{x}}(b) + i\frac{\partial g_2}{\partial \mathtt{x}}(b) &= \lim_{h \to 0} \frac{g(b+h) - g(b)}{h} = g'(b), \\
\frac{\partial g_1}{\partial \mathtt{y}}(b) + i\frac{\partial g_2}{\partial \mathtt{y}}(b) &= i \lim_{h \to 0} \frac{g(b+ih) - g(b)}{ih} = ig'(b).
\end{aligned}$$

Consequently, we obtain the Cauchy-Riemann equalities

$$\begin{cases} \frac{\partial g_1}{\partial \mathtt{x}}(b) = \frac{\partial g_2}{\partial \mathtt{y}}(b), \\ \frac{\partial g_2}{\partial \mathtt{x}}(b) = -\frac{\partial g_1}{\partial \mathtt{y}}(b). \end{cases}$$

4.b.2 *Integral of a holomorphic function over properly oriented curves.* Let $C_1, C_2 \subset \mathbb{C}$ be two smooth compact connected curves such that C_1 is contained in the interior of the bounded closed subset of $\mathbb{C}$ whose boundary is C_2. Assume C_i^+ is the counterclockwise orientation of C_i and C_i^- is the clockwise orientation of C_i. Let S be the bounded closed subset of $\mathbb{C}$ whose boundary is $C_1 \cup C_2$. Let $\Theta \subset \mathbb{C}$ be an open neighborhood of S and let $g : \Theta \to \mathbb{C}$ be a holomorphic function. By Green's Theorem and Cauchy-Riemann equalities

$$\begin{aligned}
\int_{C_2^+ \cup C_1^-} f(\mathtt{z}) d\mathtt{z} &= \int_{C_2^+ \cup C_1^-} (f_1(\mathtt{x}, \mathtt{y}) + if_2(\mathtt{x}, \mathtt{y}))(d\mathtt{x} + id\mathtt{y}) \\
&= \int_{C_2^+ \cup C_1^-} (f_1(\mathtt{x}, \mathtt{y}) d\mathtt{x} - f_2(\mathtt{x}, \mathtt{y}) d\mathtt{y}) \\
&\quad + i \int_{C_2^+ \cup C_1^-} (f_1(\mathtt{x}, \mathtt{y}) d\mathtt{y} + f_2(\mathtt{x}, \mathtt{y}) d\mathtt{x}) \\
&= \int_{\Theta} \Big(-\frac{\partial f_2}{\partial \mathtt{x}}(\mathtt{x}, \mathtt{y}) - \frac{\partial f_1}{\partial \mathtt{y}}(\mathtt{x}, \mathtt{y}) \Big) d\mathtt{x} d\mathtt{y} \\
&\quad + i \int_{\Theta} \Big(\frac{\partial f_1}{\partial \mathtt{x}}(\mathtt{x}, \mathtt{y}) - \frac{\partial f_2}{\partial \mathtt{y}}(\mathtt{x}, \mathtt{y}) \Big) d\mathtt{x} d\mathtt{y} = 0.
\end{aligned}$$

4.b.3 Let Υ_0 be the circumference centered in $z \in \mathbb{C}$ of radius $\rho_0 > 0$. Then

$$\int_{\Upsilon_0} \frac{d\xi}{\xi - z} = \int_0^{2\pi} \frac{\rho_0 i e^{it} dt}{z + \rho_0 e^{it} - z} = 2\pi i.$$

4.b.4 Fix $z \in D_\rho(a)$ and $\varepsilon > 0$. As f is continuous, there exists $\delta > 0$ such that $D_\delta(z) \subset D_\rho(a)$ and if $|\xi - z| \leq \delta$, then $|f(\xi) - f(z)| < \varepsilon$. Let Υ_δ be the circumference centered in z of radius $\delta > 0$. Let S be the bounded closed region of $\mathbb{C}$ whose boundary is $\Upsilon \cup \Upsilon_\delta$. Let $\Theta := \Omega \setminus \{z\}$, which is an open neighborhood of S. Observe that the function

$$g : \Theta \to \mathbb{C},\ \xi \mapsto g(\xi) := \frac{f(\xi) - f(z)}{\xi - z}$$

is holomorphic. Then by Lemma V.4.3, 4.b.2 and 4.b.3

$$\begin{aligned}
&\Big|\Big(\frac{1}{2\pi i}\int_\Upsilon \frac{f(\xi)d\xi}{\xi - z}\Big) - f(z)\Big| = \Big|\frac{1}{2\pi i}\int_\Upsilon \frac{f(\xi) - f(z)}{\xi - z}d\xi\Big| \\
&=\frac{1}{2\pi}\Big|\int_{\Upsilon^+ \cup \Upsilon_{\delta-}} \frac{f(\xi) - f(z)}{\xi - z}d\xi + \int_{\Upsilon_\delta^+} \frac{f(\xi) - f(z)}{\xi - z}d\xi\Big| \\
&=\frac{1}{2\pi}\Big|\int_{\Upsilon_\delta^+} \frac{f(\xi) - f(z)}{\xi - a}d\xi\Big| = \frac{1}{2\pi}\Big|\int_0^{2\pi} \frac{f(a + \delta e^{it}) - f(a)}{a + \delta e^{it} - a} i\delta e^{it} dt\Big| \\
&=\frac{1}{2\pi}\Big|\int_0^{2\pi} (f(a + \delta e^{it}) - f(a))dt\Big| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(a + \delta e^{it}) - f(a)|dt \leq \varepsilon.
\end{aligned}$$

As this holds for each $\varepsilon > 0$, we conclude

$$f(z) = \frac{1}{2\pi i}\int_\Upsilon \frac{f(\xi)d\xi}{\xi - z} = \mathrm{Ch}(f|_\Upsilon)(z),$$

as required. □

Corollary V.4.5 (Riemman's Extension I) *Let $\Omega \subset \mathbb{C}$ be an open set and let $X \subset \Omega$ be a finite set. Let $f : \Omega \setminus X \to \mathbb{C}$ be a bounded analytic function. Then there exists an analytic function $F \in \mathcal{O}(\Omega)$ such that $F|_{\Omega \setminus X} = f$.*

Proof. As analyticity is a local concept, we may assume that $X = \{0\}$ is a singleton and Ω is a disc centered at the origin. As f is bounded on $\Omega \setminus X$: *the function*

$$h(z) = \begin{cases} z^2 f(z) & \textit{if } z \in \Omega \setminus X \\ 0 & \textit{if } z = 0 \end{cases}$$

is holomorphic in Ω *and* $h'(0) = 0$.

As f is holomorphic on $\Omega \setminus X$, then also h is holomorphic on $\Omega \setminus X$, so we only have to check that h is holomorphic at the origin. We have

$$h'(0) = \lim_{z \to 0} \frac{z^2 f(z) - 0}{z - 0} = \lim_{z \to 0}(zf(z)) = 0$$

because f is bounded on $\Omega \setminus X$.

By 4.a.1(ii) and Cauchy's integral formula V.4.4 we deduce that h is an analytic function in Ω. As $h(0) = h'(0) = 0$, we deduce $h(z) = z^2 F(z)$ where $F \in \mathcal{O}(\Omega)$. Observe that F and f coincide on $\Omega \setminus X$, so F is the extension we sought. $\square$

Theorem V.4.6 (Osgood) *Let* $\Omega \subset \mathbb{C}^n$ *be an open set and let* $f : \Omega \to \mathbb{C}$ *be a holomorphic function. Let* $a \in \Omega$ *and let* $\rho > 0$ *be a polyradius such that* $\operatorname{Cl}(\Delta_\rho(a)) \subset \Omega$. *Denote* $\Upsilon := \Upsilon_\rho(a)$. *Then* f *is analytic on* $\Delta_\rho(a)$ *and* $f(x) = \operatorname{Ch}(f|_\Upsilon)(x)$ *for all* $x \in \Delta_\rho(a)$.

Proof. As $\operatorname{Ch}(f|_\Upsilon)$ is by Lemma 4.a.1 an analytic function on $\Delta_\rho(a)$, it is enough to check that

$$f(x) = \operatorname{Ch}(f|_\Upsilon)(x) \quad \forall\, x \in \Delta_\rho(a). \tag{4.2}$$

We prove the previous equality by induction on n. For $n = 1$ the equality (4.2) holds by Cauchy's integral formula V.4.4 and 4.b.2. Assume the result true for $n-1$ variables and let us check that it is also true for n variables. Choose a polyradius $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_n) > 0$ such that $\Delta_{\rho+\varepsilon}(a) \subset \Omega$. Pick a point $x := (x_1, \ldots, x_n) \in \Delta_\rho(a)$ and consider the holomorphic function in one variable

$$g : D_{\rho_1+\varepsilon_1}(a_1) \to \mathbb{C},\ \xi_1 \mapsto f(\xi_1, x_2, \ldots, x_n).$$

Denote $\Upsilon_j := \partial D_{\rho_j}(a_j)$ for $j = 1, \ldots, n$. Then by Cauchy's integral formula V.4.4 and 4.b.2 we have

$$f(x) = g(x_1) = \frac{1}{2\pi i}\int_{\Upsilon_1} \frac{g(\xi_1)}{\xi_1 - x_1} d\xi_1 = \frac{1}{2\pi i}\int_{\Upsilon_1} \frac{f(\xi_1, x_2, \ldots, x_n)}{\xi_1 - x_1} d\xi_1.$$

The function

$$h : \prod_{j=2}^{n} D_{\rho_j+\varepsilon_j}(a_j) \to \mathbb{C},\ (\xi_2, \ldots, \xi_n) \to \frac{1}{2\pi i}\int_{\Upsilon_1} \frac{f(\xi_1, \xi_2, \ldots, \xi_n)}{\xi_1 - x_1} d\xi_1$$

is holomorphic and has $n-1$ variables, so by induction hypothesis and Fubini's Theorem

$$\begin{aligned} f(x) &= h(x_2, \dots, x_n) \\ &= \frac{1}{(2\pi i)^{n-1}} \int_{\Upsilon_2} \cdots \int_{\Upsilon_n} \Big(\frac{1}{2\pi i} \int_{\Upsilon_1} \frac{f(\xi_1, \xi_2, \dots, \xi_n)}{\xi_1 - x_1} d\xi_1 \Big) \frac{d\xi_2 \cdots d\xi_n}{(\xi_2 - x_2) \cdots (\xi_n - x_n)} \\ &= \frac{1}{(2\pi i)^n} \int_{\Upsilon_1} \cdots \int_{\Upsilon_n} \frac{f(\xi_1, \dots, \xi_n)}{(\xi_1 - x_1) \cdots (\xi_n - x_n)} d\xi_1 \cdots d\xi_n \\ &= \frac{1}{(2\pi i)^n} \int_{\Upsilon} \frac{f(\xi) d\xi}{\xi - x} = \mathrm{Ch}(f|_{\Upsilon})(x), \end{aligned}$$

as required. □

4.c Maximum modulus principle. We prove next that compact complex analytic submanifolds are finite sets. To that end, we prove first the Maximum modulus principle.

Theorem V.4.7 (Maximum modulus principle) *Let $\Omega \subset \mathbb{C}^n$ be an open set and let $f \in \mathcal{O}(\Omega)$. Let $M \subset \Omega$ be a connected analytic submanifold and suppose that $|f|_M|$ has a local maximum. Then $f|_M$ is constant.*

Proof. The proof is conducted in several steps:

4.c.1 *First reduction.* We may assume that there exists $\rho' := (\rho'_1, \dots, \rho'_n) > 0$ such that $M = \Omega = \Delta_{\rho'}(0)$ and $|f|$ has a global maximum at the origin.

Let $a \in M$ be a point at which $|f|_M|$ attains a local maximum and denote $d := \dim(M)$. As M is an analytic submanifold, there exist a polyradius $\rho' := (\rho'_1, \dots, \rho'_n) > 0$ and an analytic immersion $g : \Delta_{\rho'}(0) \subset \mathbb{C}^d \to \mathbb{C}^n$ such that $g(\Delta_{\rho'}(0)) \subset M$ is an open neighborhood of $g(0) = a$ and $|f \circ g|$ has a global maximum at the origin. Thus, it is enough to substitute M by $\Delta_{\rho'}(0)$ and f by $f \circ g$.

4.c.2 *Second reduction.* We may assume $f(0) = |f(0)| > 0$.

If $|f(0)| = 0$, then $|f(x)| \leq 0$ for all $x \in \Omega$ and f is identically zero, so f is constant. Thus, we can suppose that $|f(0)| > 0$ and we substitute f by $g := \frac{\overline{f(0)}}{|f(0)|} f$. Observe that $g(0) = \frac{\overline{f(0)}}{|f(0)|} f(0) = |f(0)|$ and $g(0) = |f(0)| = |g(0)| > 0$. If we prove that g is constant, then f is also constant.

4.c.3 We claim: *If $f : \Delta_{\rho'}(0) \to \mathbb{C}$ is an analytic function and $f(0) \geq |f(z)|$ for all $z \in \Delta_\rho(0)$ for some $\rho < \rho'$, then $f \equiv f(0)$.*

4.c.4 We prove first the following: *Let* $0 < \rho := (\rho_1, \dots, \rho_n) < \rho'$ *and let* $\Upsilon := \prod_{j=1}^n \partial D_{\rho_j}(0)$. *Then* $f(0) = f(\xi)$ *for all* $\xi \in \Upsilon$.

Denote $\mathsf{e} := (1, \dots, 1)$. By Osgood's Theorem V.4.6

$$\begin{aligned} f(0) &= \frac{1}{(2\pi i)^n} \int_\Upsilon \frac{f(\xi)}{\xi} d\xi \\ &= \frac{1}{(2\pi i)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{f(\rho_1 e^{it_1}, \dots, \rho_n e^{it_n})}{\rho^{\mathsf{e}} e^{i\sum_{j=1}^n t_j}} i^n \rho^{\mathsf{e}} e^{i\sum_{j=1}^n t_j} dt_1 \cdots dt_n \\ &= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\rho_1 e^{it_1}, \dots, \rho_n e^{it_n}) dt_1 \cdots dt_n. \end{aligned}$$

As $f(0) \in \mathbb{R}$, we have

$$\begin{aligned} \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \Re(f(\rho_1 e^{it_1}, \dots, \rho_n e^{it_n})) dt_1 \cdots dt_n = f(0), \\ \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \Im(f(\rho_1 e^{it_1}, \dots, \rho_n e^{it_n})) dt_1 \cdots dt_n = 0. \end{aligned}$$

As $\Re(f(\xi)) \leq |f(\xi)| \leq |f(0)| = f(0)$ for $\xi \in \Upsilon$, we have $f(0) - \Re(f(\xi)) \geq 0$ for $\xi \in \Upsilon$. As

$$\frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} (f(0) - \Re(f(\rho_1 e^{it_1}, \dots, \rho_n e^{it_n}))) dt_1 \cdots dt_n = 0,$$

we deduce $f(0) = \Re(f(\xi))$ for $\xi \in \Upsilon$. Thus,

$$\Re(f(\xi)) \leq |f(\xi)| \leq |f(0)| = f(0) = \Re(f(\xi)),$$

so $|f(\xi)| = \Re(f(\xi))$ for $\xi \in \Upsilon$. As

$$\Re(f(\xi))^2 = |f(\xi)|^2 = \Re(f(\xi))^2 + \Im(f(\xi))^2,$$

we conclude $\Im(f(\xi)) = 0$, so $f(\xi) = \Re(f(\xi)) = f(0)$ for $\xi \in \Upsilon$.

4.c.5 Pick now $x := (x_1, \dots, x_n) \in \Delta_{\rho'}(0)$ such that $x_1 \cdots x_n \neq 0$. Then $\rho_i := |x_i| < \rho'_i$ for $i = 1, \dots, n$, so by 4.c.4 $f(x) = f(0)$. Thus, $f - f(0)$ is identically zero on $\Delta_{\rho'}(0) \cap \{\mathtt{x}_1 \cdots \mathtt{x}_n \neq 0\}$ and by the Identity Principle, we conclude $f \equiv f(0)$, as required. □

Corollary V.4.8 *Let* $M \subset \mathbb{C}^n$ *be a compact analytic submanifold. Then* M *is a finite set.*

Proof. As M is an analytic submanifold, its connected components are open subsets of M. As M is compact and M is the union of its connected components, M has finitely many connected components. Consequently, to prove the statement we may assume that M is connected. Let us prove: *M is a singleton.*

Consider the projections $\pi_i : \mathbb{C}^n \to \mathbb{C},\ (x_1, \ldots, x_n) \mapsto x_i$. As M is compact, $\pi_i|_M$ attains a maximum on M. As M is connected, we deduce by the Maximum modulus principle that $\pi_i|_M$ is constant. As this happens with all the projections of M, we conclude that M is a singleton, as required. □

5 Riemann's Extension Theorem

The proof of Riemann's Extension Theorem is quite involved and requires some preliminary work.

5.a Preliminaries on roots of polynomials. Let us see next some key results concerning roots of polynomials.

Lemma V.5.1 *Let $P := \mathtt{t}^p + a_{p-1}\mathtt{t}^{p-1} + \cdots + a_0 \in \mathbb{C}[\mathtt{t}]$ and let $\alpha \in \mathbb{C}$ be a root of P. Then $|\alpha| < 1 + \sum_{i=0}^{p-1} |a_i|$.*

Proof. If $|\alpha| < 1$, then $|\alpha| < 1 + \sum_{i=0}^{p-1} |a_i|$, so we may assume $|\alpha| \geq 1$. As $P(\alpha) = 0$, then

$$\alpha^p = -\sum_{k=0}^{p-1} a_k \alpha^k \quad \rightsquigarrow \quad \alpha = -\sum_{k=0}^{p-1} a_k \alpha^{k-p}$$

The modulus of both members of the last equality are equal and since $|\alpha| \geq 1$,

$$|\alpha| = \Big|\sum_{k=0}^{p-1} a_k \alpha^{k-p}\Big| \leq \sum_{k=0}^{p-1} |a_k| \cdot |\alpha|^{k-p} < 1 + \sum_{k=0}^{p-1} |a_k|,$$

as required. □

Lemma V.5.2 (Continuity of roots) *For each $a := (a_0, \dots, a_{n-1}) \in \mathbb{C}^p$ consider the polynomial $P_a(\mathtt{t}) := \mathtt{t}^p + a_{p-1}\mathtt{t}^{p-1} + \dots + a_0 \in \mathbb{C}[\mathtt{t}]$. For each $\varepsilon > 0$ there exists $\delta > 0$ such that if $|a_i| < \delta$ for $i = 0, \dots, p-1$, then the modulus of each root of P_a is strictly smaller than ε.*

Proof. Pick $a := (a_0, \dots, a_{n-1}) \in \mathbb{C}^p$ such that $|a_i| \leq 1$ for $i = 0, \dots, p-1$. By Lemma V.5.1 the modulus of each root of P_a is strictly smaller than $1+p$. Fix $0 < \varepsilon < p+1$ and let $K_\varepsilon := \mathrm{Cl}(D_{1+p}(0)) \setminus D_\varepsilon(0)$. Consider the compact set $C := \prod_{j=0}^{p-1} \mathrm{Cl}(D_1(0))$ and the continuous function

$$\varphi : C \times K_\varepsilon,\ (a_0, \dots, a_{p-1}, z) \mapsto z^p + a_{p-1}z^{p-1} + \dots + a_0.$$

Observe that $\varphi(0, z) = z^p \neq 0$ for all $z \in K_\varepsilon$. As K_ε is a compact set, there exists $\rho := \min_{z \in K_\varepsilon}\{|\varphi(0, z)|\} > 0$. As φ is continuous over the compact set $C \times K_\varepsilon$, it is uniformly continuous and there exist $\delta > 0$ such that if $a := (a_0, \dots, a_{p-1}), b := (b_0, \dots, b_{p-1}) \in C$, $|a_i - b_i| < \delta$ for $i = 0, \dots, p-1$ and $|z - w| < \delta$, then

$$|\varphi(a, z) - \varphi(b, w)| < \frac{\rho}{2}.$$

If $a := (a_0, \dots, a_{p-1}) \in C$, $|a_i| < \delta$ for $i = 0, \dots, p-1$ and $z \in K_\varepsilon$, then

$$\rho - |\varphi(a, z)| \leq |\varphi(0, z)| - |\varphi(a, z)| \leq |\varphi(0, z) - \varphi(a, z)| < \frac{\rho}{2},$$

so $|\varphi(a, z)| > \frac{\rho}{2}$ and z is not a root of the polynomial $P_a(\mathtt{t})$. Thus, the roots of $P_a(\mathtt{t})$ belong to $D_{1+p}(0) \setminus K_\varepsilon = D_\varepsilon(0)$ if $|a_i| < \delta$ for $i = 0, \dots, p-1$, as required. □

5.b Proof of Riemann's Extension Theorem. Let $\Omega \subset \mathbb{C}^n$ be an open set and let $a \in \Omega$. Let $S \subset \Omega$ and let $f : S \to \mathbb{C}$ be a function. We say that f is *locally bounded at a* if there exists an open neighborhood $V \subset \Omega$ of a such that $f(S \cap V)$ is a bounded subset of $\mathbb{C}$. The function f is *locally bounded on* Ω if it is locally bounded at each of its points. Observe that if f is continuous at a, then f is locally bounded at a.

Theorem V.5.3 (Riemann's Extension) *Let $\Omega \subset \mathbb{C}^n$ be an open set and let $X \subset \Omega$ be a closed subset such that for each $a \in \Omega$ there exists a connected open neighborhood $V \subset \Omega$ of a and a non-zero analytic function $g_a \in \mathcal{O}(V)$ satisfying $X \cap V \subset Z_V(g_a)$. Let $f : \Omega \setminus X \to \mathbb{C}$ be a locally bounded analytic function. Then there exists an analytic function F on Ω such that $F|_{\Omega \setminus X} = f$.*

Proof. The proof is conducted in several steps:

5.b.1 It is enough to prove: *For each $a \in X$ there exists a polycylinder $\Delta_\rho(a)$ and an analytic extension $F_a \in \mathcal{O}(\Delta_\rho(a))$ of $f|_{\Delta_\rho(a)\setminus X}$.*

Once this is proved we define

$$F : \Omega \to \mathbb{C},\ x \mapsto \begin{cases} f(x) & \text{if } x \in \Omega \setminus X, \\ F_x(x) & \text{if } x \in X \end{cases}$$

and we claim that F is well-defined. Pick $z \in \Delta_{\rho_1}(a_1) \cap \Delta_{\rho_2}(a_2)$. As $\Delta_{\rho_i}(a_i) \setminus X$ is by Theorem V.2.4 dense in $\Delta_{\rho_i}(a_i)$ and open because X is closed in Ω, the intersection

$$(\Delta_{\rho_1}(a_1) \setminus X) \cap (\Delta_{\rho_2}(a_2) \setminus X)$$

is a non-empty open subset of $\Delta_{\rho_1}(a_1) \cap \Delta_{\rho_2}$. As both F_{a_1} and F_{a_2} coincide with f on $(\Delta_{\rho_1}(a_1) \setminus X) \cap (\Delta_{\rho_2}(a_2) \setminus X)$, we deduce that both F_{a_1} and F_{a_2} coincide by the Identity Principle on $\Delta_{\rho_1}(a_1) \cap \Delta_{\rho_2}(a_2)$. Thus, $F_{a_1}(z) = F_{a_2}(z)$ and F is well-defined.

5.b.2 We may assume that a is the origin and our statement is reduced to prove: *There exists a polycylinder Δ centered at the origin and an analytic extension $F \in \mathcal{O}(\Delta)$ of $f|_{\Delta\setminus X}$.*

5.b.3 We are focused on the origin and $X \cap V \subset Z(g_0)$ for a suitable neighborhood V of the origin. Denote $\mathtt{x}' := (\mathtt{x}_1, \ldots, \mathtt{x}_{n-1})$. We may assume: *$g_0 := P := \sum_{j=0}^{p-1} a_j(\mathtt{x}')[\mathtt{x}_n]$ is a distinguished polynomial with respect to $\mathtt{x}_n$ whose order coincides with its degree with respect to $\mathtt{x}_n$.*

As $g_0 \in \mathcal{O}_n$, we can suppose after a linear change of coordinates that g_0 is a regular series with respect to $\mathtt{x}_n$ of order $\omega(g_0)$. By Weierstrass's preparation theorem there exists a distinguished polynomial $P \in \mathcal{O}_{n-1}[\mathtt{x}_n]$ with respect to $\mathtt{x}_n$ of degree $\omega(g_0)$ and a unit $U \in \mathcal{O}_n$ such that $g_0 = PU$. We may assume that $\Delta \subset D(P) \cap D(U)$ and $Z_\Delta(U) = \varnothing$ (because $U(0) \neq 0$), so $Z_\Delta(g_0) = Z_\Delta(P)$.

5.b.4 Write $(x', x_n) \in \mathbb{C}^n$ with $x' := (x_1, \ldots, x_{n-1})$. Let $\rho := (\rho_1, \ldots, \rho_n) > 0$ be such that $f|_{\Delta_p(0)\setminus X}$ is a bounded function. For $\frac{\rho_n}{2} > 0$ there exists by Lemma V.5.2 $\delta_0 > 0$ such that if $x' \in \prod_{j=1}^{n-1} D_{\rho_j}(0)$ and $|a_j(x')| < \delta_0$ for $j = 0, \ldots, p-1$, the modulus of the roots of $P(x', \mathtt{t})$ is smaller than $\frac{\rho_n}{2}$. As the functions a_j are continuous, there exists a positive real number $\delta > 0$ smaller than $\min\{\rho_i : i = 1, \ldots, n-1\}$ such that if $|x_k| < \delta$ for $k = 1, \ldots, n-1$, then $|a_j(x')| < \delta_0$ for $j = 0, \ldots, p-1$. Thus, the modulus of each root of $P(x', \mathtt{t})$ is smaller than $\frac{\rho_n}{2}$. Define $\Delta' := (\prod_{k=1}^{n-1} D_\delta(0)) \times D_{\frac{2\rho_n}{3}}(0)$ and $\Upsilon_n := \partial D_{\frac{2\rho_n}{3}}(0)$. Consider

$$F : \Delta' \to \mathbb{C},\ x := (x', x_n) \mapsto \frac{1}{2\pi i} \int_{\Upsilon_n} \frac{f(x', \xi_n)}{\xi_n - x_n} d\xi_n$$

and let us check: *F is the function we sought.*

5.b.5 *F is well-defined.* We have to check: *For each $x' \in \prod_{k=1}^{n-1} D_\delta(0)$ the function $f|_{\{x'\}\times\Upsilon_n}$ is continuous.*

We have to prove that for each point $x' \in \prod_{k=1}^{n-1} D_\delta(0)$ the intersection $(\{x'\}\times\Upsilon_n)\cap X = \varnothing$. Suppose by contradiction that $(x', x_n) \in (\{x'\}\times\Upsilon_n)\cap X$. Then $P(x', x_n) = 0$ and $|x_n| = \frac{2\rho_n}{3}$. But all roots of the polynomial $P(x', \mathtt{t})$ and, in particular x_n, belong to the disc $D_{\frac{\rho_n}{2}}(0)$, which a contradiction.

5.b.6 *F is holomorphic* and consequently by Osgood's Theorem F *is an analytic function.*

By the Theorem of derivation under the integral sign the derivatives $\frac{\partial F}{\partial \mathtt{x}_k}$ exist for $k = 1, \dots, n-1$. Fix $x := (x', x) \in \Delta'$ and define

$$G : D_{\frac{2\rho_n}{3}}(0) \to \mathbb{C},\ z \mapsto F(x', z) = \frac{1}{2\pi i}\int_{\Upsilon_n} \frac{f(x', \xi_n)}{\xi_n - z} d\xi_n = \mathrm{Ch}(f|_{\{x'\}\times\Upsilon_n})(z),$$

which is an analytic function on $D_{\frac{2\rho_n}{3}}(0)$ by Lemma 4.a.1. Thus, there exists $\frac{\partial F}{\partial \mathtt{x}_n}(x', x_n) = G'(x_n)$. Consequently, F is holomorphic on Δ'.

5.b.7 $F|_{\Delta'\setminus X} = f|_{\Delta'\setminus X}$.

Pick a point $x := (x', x_n) \in \Delta' \setminus X$. Consider the finite set

$$Y := \{z \in D_{\rho_n}(0) :\ P(x', z) = 0\} \subset D_{\frac{\rho_n}{2}}(0),$$

where the last inclusion follows from 5.b.4. Consider the bounded univariate holomorphic function

$$h : D_{\rho_n}(0) \setminus Y \to \mathbb{C},\ z \mapsto f(x', z).$$

By Corollary V.4.5 there exists an analytic function $H : D_{\rho_n}(0) \to \mathbb{C}$ such that $H|_{D_{\rho_n}(0)\setminus Y} = h$. Observe that $f(x', x_n) = h(x_n)$ and $x_n \in D_{\frac{2\rho_n}{3}}(0) \setminus Y$. By Cauchy's integral formula V.4.4

$$f(x', x_n) = h(x_n) = \frac{1}{2\pi i}\int_{\Upsilon_n} \frac{h(\xi_n)}{\xi_n - x_n} d\xi_n = \frac{1}{2\pi i}\int_{\Upsilon_n} \frac{f(x', \xi_n)}{\xi_n - x_n} d\xi_n = F(x', x_n),$$

as required. □

Remark V.5.4 If $\Omega \subset \mathbb{C}^n$ is an open set and $X \subset \Omega$ is a proper analytic subset that contains no connected component of Ω then by Corollary V.3.6 and Riemann's Extension Theorem V.5.3 each locally bounded analytic function $f \in \mathcal{O}(\Omega \setminus X)$ can be extended to an analytic function on Ω.

5.c Thin sets. Let $\Omega \subset \mathbb{C}^n$ be an open set and let $X \subset \Omega$ be a closed subset. We say $X \subset \Omega$ is a *thin subset of* Ω if $\Omega \setminus X$ is dense in Ω and for each open subset $\Theta \subset \Omega$ and each locally bounded analytic function $f \in \mathcal{O}(\Theta \setminus X)$ there exists $F \in \mathcal{O}(\Theta)$ such that $F|_{\Theta \setminus X} = f$.

Remark V.5.5 If $\Omega \subset \mathbb{C}^n$ is a connected open set and $X \subset \Omega$ is a thin set, then $\Omega \setminus X$ is connected.

Suppose by contradiction that $\Omega \setminus X$ is not connected. As $\Omega \setminus X$ is open, there exist non-empty disjoint open sets $\Theta_1, \Theta_2 \subset \mathbb{C}^n$ such that $\Omega \setminus X = \Theta_1 \cup \Theta_2$. Consider the analytic function

$$f : \Omega \setminus X \to \mathbb{C},\ z \mapsto \begin{cases} 0 & \text{if } z \in \Theta_1, \\ 1 & \text{if } z \in \Theta_2. \end{cases}$$

As f is locally bounded, there exists $F \in \mathcal{O}(\Omega)$ such that $F|_{\Omega \setminus X} = f$. As Ω is connected and Θ_1, Θ_2 are non-empty, we achieve a contradiction by the Identity Principle. Consequently, $\Omega \setminus X$ is connected.

Corollary V.5.6 *Let $\Omega \subset \mathbb{C}^n$ and let $X \subset \Omega$ be a proper analytic subset. Then X is a thin subset of Ω.*

Proof. As X is an analytic subset of Ω, then it is closed in Ω and by Theorem V.2.4 $\Omega \setminus X$ is a dense subset of Ω. Let $\Omega' \subset \Omega$ be an open set and let $f \in \mathcal{O}(\Omega' \setminus X)$ be a locally bounded analytic function. As $X \subset \Omega$ is a proper analytic subset, X contains no connected component of Ω' by Corollary V.3.6. By Remark V.5.4 f extends to an analytic function on Ω'. Consequently, X is a thin subset of Ω, as required. $\square$

Corollary V.5.7 *Let $\Omega \subset \mathbb{C}^n$ be an open set and let $X \subset \Omega$ be an analytic subset. Let $M \subset \Omega$ be a connected analytic submanifold that contains X. Then $M \setminus X$ is connected and dense in M.*

Proof. Suppose by contradiction that $M \setminus X$ is not connected. Then there exist disjoint non-empty open subsets N_1, N_2 of $M \setminus X$ such that $M \setminus X = N_1 \cup N_2$. By Corollary V.3.6 $M \setminus X$ is dense in M, so $M = \mathrm{Cl}(N_1) \cup \mathrm{Cl}(N_2)$. As M is connected, there exists a point $a \in \mathrm{Cl}(N_1) \cap \mathrm{Cl}(N_2)$. Let $d := \dim(M)$ and let $\Omega \subset \mathbb{C}^d$ be a connected open neighborhood of the origin and $\varphi : \Omega \to \mathbb{C}^n$ be an analytic immersion such that $\varphi(\Omega)$ is an open neighborhood of a in M and $\varphi(0) = a$. Then $Y := \varphi^{-1}(X) \subset \Omega$ is analytic and $\Omega \setminus Y = \varphi^{-1}(N_1) \cup \varphi^{-1}(N_2)$. Note that $\varphi^{-1}(N_1)$ and $\varphi^{-1}(N_2)$ are non-empty disjoint open subsets of $\Omega \setminus Y$, so $\Omega \setminus Y$ is not connected, against Remark V.5.5 and Corollary V.5.6. Thus, $M \setminus X$ is connected. □

Exercises

Number V.1 Let $\Omega \subset \mathbb{K}^m$ be an open set and let $f := (f_1, \dots, f_n) : \Omega \to \mathbb{K}^n$ be an analytic map with finite fibers. Show that

$$m = \max\Big\{\mathrm{rk}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(x)\Big)_{\substack{1\le i\le n\\ 1\le j\le m}} \,:\, x \in \Omega\Big\} \le n.$$

Number V.2 Let $\Omega \subset \mathbb{C}^n$ be a connected open set and let $f : \Omega \to \mathbb{C}$ be a non-constant analytic function. Prove that f is an open map. Is the result still true if we consider a non-constant analytic map $f : \Omega \to \mathbb{C}^m$?

Number V.3 Let $f : \Omega \to \Theta$ be an analytic map between open subsets $\Omega \subset \mathbb{K}^m$ and $\Theta \subset \mathbb{K}^n$. Let $X \subset \Theta$ be an analytic subset. Prove that $f^{-1}(X)$ is an analytic subset of Ω.

Number V.4 Is the set $X := \{t \ge 0\}$ an analytic subset of $\mathbb{R}$?

Number V.5 Let $\Omega \subset \mathbb{K}^n$ be a connected open set. Show that $\mathcal{O}(\Omega)$ is an integral domain. Is the ring of continuous functions on Ω taking values on $\mathbb{K}$ an integral domain?

Number V.6 Let M be a connected analytic submanifold of $\mathbb{K}^n$ and let $\Omega \subset \mathbb{K}^n$ be an open neighborhood of M. Let $f \in \mathcal{O}(\Omega)$ be an analytic function such that $\frac{\partial f}{\partial \mathtt{x}_j}|_M = 0$ for $j = 1, \dots, n$. Prove that $f|_M$ is constant.

Number V.7 Let $\Omega \subset \mathbb{C}$ be an open neighborhood of the origin and let $p := \omega(f - f(0))$. Prove that for $\varepsilon > 0$ small enough there exists $\delta > 0$ such that for each $z \in \mathbb{C}$ satisfying $0 < |z - f(0)| < \delta$ the fiber $f^{-1}(z) \cap D_\varepsilon(0)$ has p elements.

Number V.8 Determine if the set $X := \{\mathtt{xy} = 0\} \subset \mathbb{K}^2$ is an analytic submanifold of $\mathbb{K}^2$.

Number V.9 Show that every analytic submanifold of $\mathbb{K}^n$ is a locally closed subset of $\mathbb{K}^n$.

Number V.10 Show that each analytic function $f : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}$ can be extended to an analytic function on $\mathbb{C}^2$. Is the result still true if we change $\mathbb{C}$ by $\mathbb{R}$?

Number V.11 A subset $X \subset \Omega$ of an open subset $\Omega \subset \mathbb{K}^n$ is *globally analytic* if there exist analytic functions $f_1, \dots, f_r \in \mathcal{O}(\Omega)$ such that $X = Z_\Omega(f_1, \dots, f_r)$. Find an open subset $\Omega \subset \mathbb{C}^2$ and an analytic subset X of Ω that is not globally analytic in Ω.

Number V.12 Let $X \subsetneq \mathbb{C}^n$ be an analytic set and let $f : \mathbb{C}^n \setminus X \to \mathbb{C}$ be an analytic function such that $f(\mathbb{C}^n \setminus X)$ is a bounded subset of $\mathbb{C}$. Show that f is constant.

Number V.13 Let $f \in \mathcal{O}(\mathbb{C}^n)$ be an analytic function such that $|f(z)| \leq M|z|^{|\nu|}$ for each z outside a polydisc for some $M > 0$ and some $\nu \in \mathbb{N}^n$. Prove that f is a polynomial of degree smaller than or equal to $|\nu|$.

Number V.14 Prove that the Maximum modulus principle is false if we change $\mathbb{C}$ by $\mathbb{R}$. Show also that there are compact analytic submanifolds of $\mathbb{R}^n$ that are not finite sets.

Number V.15 Prove that Riemann extension theorem is not true in the real case.

Number V.16 Find a proper analytic subset X of $\mathbb{R}$ such that $\mathbb{R} \setminus X$ is not connected.

Number V.17 Are the functions $\Re : \mathbb{C} \to \mathbb{C}$ and $\Im : \mathbb{C} \to \mathbb{C}$ analytic functions on $\mathbb{C}$?

Number V.18 Let $\Omega \subset \mathbb{C}^n$ be a connected open set and let $f : \Omega \to \mathbb{C}$ be a continuous function such that there exists a non-zero polynomial $P \in \mathbb{C}[\mathtt{x}_1, \dots, \mathtt{x}_n, \mathtt{y}]$ such that $P(x, f(x)) = 0$ for all $x \in \Omega$. Show that $f \in \mathcal{O}(\Omega)$. Is the result true if we change $\mathbb{C}$ by $\mathbb{R}$?

Number V.19 Let $\Omega \subset \mathbb{C}^n$ be an open set and $X \subset \Omega$ a closed set. Show that:

(i) If X is a thin subset of an open subset of Ω, then X is a thin subset of Ω.
(ii) If X is a subset of a thin subset of Ω, then X is a thin subset of Ω.
(iii) If X is a thin subset of Ω and Y is a thin subset of $\Omega \setminus X$, then $X \cup Y$ is a thin subset of Ω.
(iv) If X and Y are thin subsets of Ω, then $X \cup Y$ and $X \cap Y$ are thin subsets of Ω.
(v) If X is a thin subset of Ω and $\Theta \subset \Omega$ is open then $X \cap \Theta$ is a thin subset of Θ.

Number V.20 Let $\{X_i\}_{i \in I}$ be a locally finite family of thin subsets of an open subset Ω of $\mathbb{C}^n$. Show that $X := \bigcup_{i \in I} X_i$ is a thin subset of Ω.

Number V.21 Let $\Theta \subset \Omega \subset \mathbb{C}^n$ be open sets and let $X \subset \Omega$ be a closed subset such that $X \cap \mathrm{Cl}(\Theta)$ is a thin subset of Ω. Prove that $X \cap \Theta$ is a thin subset of Θ.

Number V.22 Find a thin subset of $\mathbb{C}$ that it is not an analytic subset of $\mathbb{C}$.

Number V.23 Let $\Omega \subset \mathbb{C}^n$ be an open set and let $X \subset \Omega$. We say that X is a *locally thin subset of* Ω if for each $a \in X$ there exists an open neighborhood $U \subset \Omega$ of a such that $X \cap U$ is a thin subset of U. Show that X is a thin subset of Ω if and only if X is closed and locally thin in Ω.

CHAPTER VI

Complex analytic set and function germs

In the first part of this chapter we introduce the notions of germ set and germ function, with special emphasis in analytic germs and the properties of their irreducible components. This part finishes with a Nullstellensatz for complex analytic germs, which is a geometric interpretation of the algebraic Nullstellensatz studied in Chapter IV. The second section is devoted to analyze the dimension of an analytic germ set and to obtain a Jacobian Criterion that provides an algebraic characterization of the points of an analytic set that have an small open neighborhood inside this set that is an analytic submanifold.

This and the next chapter can be understood as the natural geometrization of many results in Chapters II, II and IV. Some knowledge of basic analysis and topology is required. The already quoted references [K, RF] are much more than needed. We recall without proof the existence of a Nullstellensatz for real analytic germs. The interested reader can study its first proof in [R].

1 Generalities on germ sets

As in the previous Chapters, we denote $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$. Fix an open subset Ω of $\mathbb{K}^n$ and a point $a \in \Omega$. When writing *U is a neighborhood of a* we understand that U is a neighborhood of a in $\mathbb{K}^n$.

Definitions VI.1.1 We define in the set $\mathcal{P}(\Omega)$ of subsets of Ω the following equivalence relation: $X \equiv Y$ *if there exists a neighborhood U of a such that* $X \cap U = Y \cap U$. The equivalence class X_a of $X \in \mathcal{P}(\Omega)$ is the *germ of X at the point a*. The germ at a of the empty set is denoted by $\varnothing$.

Remarks VI.1.2 (i) If $X \in \mathcal{P}(\Omega)$ and U is an open subset of $\mathbb{K}^n$ containing a, then $(X \cap U)_a = X_a$.

(ii) Let $X_{1,a}, \dots, X_{r,a}$ be germs at the point a and let $Y_{i,1}, \dots, Y_{i,k}$ be representatives of $X_{i,a}$ for $i = 1, \dots, r$. Then there exists an open neighborhood U of a such that $Y_{i,j} \cap U = Y_{i,\ell} \cap U$ for $i = 1, \dots, r$ and $j = 1, \dots, k$.

For fixed i, j, ℓ there exists an open neighborhood $U_{i,j,\ell}$ of a such that $Y_{i,j} \cap U_{i,j,\ell} = Y_{i,\ell} \cap U_{i,j,\ell}$, so $U := \bigcap_{i,j,\ell} U_{i,j,\ell}$ fits the situation.

(iii) Notice that $X_a = \varnothing$ if and only if $a \notin \mathrm{Cl}(X)$.

Indeed, $X_a = \varnothing$ means that there exists an open neighborhood U of a such that $X \cap U = \varnothing$, that is, $a \notin \mathrm{Cl}(X)$.

We introduce next the boolean operations with germs: intersection, union and difference of germs. They are defined in the obvious way, using representatives. They do not depend on the chosen representatives and enjoy the usual properties of the corresponding operations with sets.

Definition and Remarks VI.1.3 Given germs X_a and Y_a we define:

$$X_a \cap Y_a := (X \cap Y)_a, \quad X_a \cup Y_a := (X \cup Y)_a \quad \text{and} \quad X_a \setminus Y_a := (X \setminus Y)_a.$$

It is checked straightforwardly that these definitions are consistent. In addition, the following conditions are equivalent:

(i) There exists an open neighborhood U of a such that $X \cap U \subset Y \cap U$.

(ii) $X_a \cap Y_a = X_a$.

(iii) $X_a \cup Y_a = Y_a$.

(iv) There exist representatives X_1 and Y_1 of, respectively, X_a and Y_a, such that $X_1 \subset Y_1$.

If this is the case we write $X_a \subset Y_a$.

Corollary VI.1.4 *Given subsets X and Y of Ω we have:*

(i) $X_a = Y_a$ *if and only if* $X_a \subset Y_a \subset X_a$.

(ii) $X_a \setminus Y_a = \varnothing$ *if and only if* $X_a \subset Y_a$.

(iii) $X_a \cap Y_a \subset X_a \subset X_a \cup Y_a$.

Using the properties stated in the Corollary above the reader can check the following.

Proposition VI.1.5 *Let X, Y and Z be subsets of Ω. Then:*

(i) *If $X_a \subset Y_a \subset Z_a$, then $X_a \subset Z_a$.*

(ii) $X_a \cup Y_a \subset Z_a$ *if and only if* $X_a \subset Z_a$ *and* $Y_a \subset Z_a$.

(iii) $Z_a \subset X_a \cap Y_a$ *if and only if* $Z_a \subset X_a$ *and* $Z_a \subset Y_a$.

(iv) $X_a \cap (Y_a \cup Z_a) = (X_a \cap Y_a) \cup (X_a \cap Z_a)$.

(v) $X_a \cup (Y_a \cap Z_a) = (X_a \cup Y_a) \cap (X_a \cup Z_a)$.

Next we introduce the basic topological notions about germs: openness, closedness, density and so on. Again they are defined by using representatives and it is straightforward to check its consistency.

Definitions VI.1.6 Let X and Y be subsets of Ω such that $X_a \subset Y_a$. Then:

(i) The germ X_a is *open* in Y_a if there exists an open neighborhood U of a such that $X \cap U$ is open in $Y \cap U$.

(ii) The germ X_a is *closed* in Y_a if there exists an open neighborhood U of a such that $X \cap U$ is closed in $Y \cap U$.

(iii) The germ X_a is *dense* in Y_a if there exists an open neighborhood U of a such that $X \cap U$ is dense in $Y \cap U$.

(iv) The germ X_a is *open,* (resp. *closed, dense*) if it is open, (resp. closed, dense) in Ω_a.

(v) The *closure* of the germ X_a, denoted $\mathrm{Cl}(X_a)$, is the germ $(\mathrm{Cl}(X))_a$.

Example VI.1.7 Let $\mathbb{K} = \mathbb{R} = \Omega$, $a = 1$ and $X := [1, 2)$. The germ X_a is closed because there exists an open neighborhood $U := (\frac{1}{2}, \frac{3}{2})$ of a in Ω such that $X \cap U = [1, \frac{3}{2}) = [1, 2] \cap U$ is closed in U.

The following pure set theoretical topological property will be useful. In fact, the definition of the closure of a germ is consistent as an straightforward consequence of the following result.

Lemma VI.1.8 *Let Z be a topological space and let $X, U \subset Z$ be such that U is open. Then* $\mathrm{Cl}(X) \cap U = \mathrm{Cl}(X \cap U) \cap U$.

Proof. To prove the non-obvious inclusion pick $x \in \mathrm{Cl}(X) \cap U$ and let $W \subset Z$ be a neighborhood of x. Also $U_1 := W \cap U$ is a neighborhood of x in Z, so $U_1 \cap X \neq \varnothing$ and $W \cap (X \cap U) \neq \varnothing$. Thus, $x \in \mathrm{Cl}(X \cap U) \cap U$. □

Remarks VI.1.9 (i) Let $X, Y \subset \Omega$. Assume $a \in X$, $X_a \subset Y_a$ and X_a is an open germ in Y_a. Then $X_a = Y_a$.

Indeed, there exists an open neighborhood U of a such that $X \cap U$ is open in $Y \cap U$, so $X \cap U = (Y \cap U) \cap U_1$ for some open subset U_1 of $\mathbb{K}^n$. As $a \in X \cap U \subset U_1$, the set $W := U \cap U_1$ is a neighborhood of a. Thus, $X_a = (X \cap U)_a = (Y \cap W)_a = Y_a$.

(ii) The germ X_a is closed in Y_a if and only if $X_a = \mathrm{Cl}(X_a) \cap Y_a$.

(iii) A closed germ Y_a that contains the germ X_a contains its closure $\mathrm{Cl}(X_a)$.

As the germ Y_a contains X_a, there exists a neighborhood U_1 of a such that $X \cap U_1 \subset Y \cap U_1$. As the germ $(Y \cap U_1)_a = Y_a$ is closed, there exists a neighborhood U_2 of a such that $(Y \cap U_1) \cap U_2$ is closed in U_2. Write $U := U_1 \cap U_2$. Then $Y \cap U$ is closed in U and $X \cap U \subset Y \cap U$, so

$$\mathrm{Cl}(X \cap U) \cap U \subset \mathrm{Cl}(Y \cap U) \cap U = Y \cap U$$

and this implies $\mathrm{Cl}(X_a) \subset Y_a$.

(iv) The germ X_a is open in Y_a if and only if $Y_a \setminus X_a$ is closed in Y_a.

Suppose first that X_a is open in Y_a. Then there exists an open neighborhood U of a such that $X \cap U$ is open in $Y \cap U$, so $(Y \setminus X) \cap U = (Y \cap U) \setminus (X \cap U)$ is closed in $Y \cap U$. Thus, $Y_a \setminus X_a = (Y \setminus X)_a$ is closed in Y_a. Conversely, suppose that $(Y \setminus X)_a = Y_a \setminus X_a$ is closed in Y_a. Then there exists an open neighborhood U of a such that $(Y \setminus X) \cap U$ is closed in $Y \cap U$, so $(Y \cap U) \setminus (Y \setminus X) \cap U$ is open in $Y \cap U$. Thus, $X \cap U$ is open in $Y \cap U$ or, equivalently, the germ X_a is open in Y_a.

(v) The germ X_a is dense in Y_a if and only if $Y_a \subset \mathrm{Cl}(X_a)$.

Next we introduce the notions of connected germ and connected components of a germ.

Definitions VI.1.10 Let X and Y be subsets of Ω such that $X_a \subset Y_a$. The germ X_a is *connected* if for each neighborhood U of a there exists a neighborhood $U_1 \subset U$ of a such that $X \cap U_1$ is connected. In other words, the germ X_a is connected if X is a locally connected topological space around a.

The germ X_a is a *connected component* of the germ Y_a if for each neighborhood U of a there exists a neighborhood $U_1 \subset U$ of a such that $X \cap U_1 \subset Y \cap U_1$ and $X \cap U_1$ is a connected component of $Y \cap U_1$. In particular, the germ X_a is a connected germ.

Remarks VI.1.11 (i) The definition of connected germ is consistent.

If X, Z are representatives of the germ X_a there exists a neighborhood W of a such that $X \cap W = Z \cap W$. Thus, for each neighborhood U of a the intersection $U \cap W$ is a neighborhood of a, so it contains a neighborhood U_1 of a such that $X \cap U_1$ is connected. We have

$$Z \cap U_1 = Z \cap (W \cap U) \cap U_1 = (Z \cap W) \cap U \cap U_1 = (X \cap W) \cap U \cap U_1 = X \cap U_1,$$

so $Z \cap U_1$ is connected.

(ii) Consider the circle $\mathbb{S}^1 := \{\mathtt{x}^2 + \mathtt{y}^2 = 1\} \subset \mathbb{R}^2$ and $X := \mathbb{S}^1 \setminus \{a\}$ where $a := (0, 1)$. Note that X is connected but the germ X_a is not connected because $X \cap U$ is not connected for each neighborhood U of a in $\mathbb{R}^2$ contained in a small open disc centered at a.

(iii) The argument presented in (i) shows that the definition of connected component of a germ is consistent.

(iv) With the notations of (ii) the connected components of X_a are Y_a and Z_a, where $Y := X \cap \{\mathtt{x} < 0\}$ and $Z := Y \cap \{\mathtt{x} > 0\}$.

We collect next some elementary properties about connectedness of germs. They are the formulation in terms of germs of some basic properties about connectedness of topological spaces.

Proposition VI.1.12 *Let $X, Y, Z \subset \Omega$ be such that X_a is a connected component of Y_a. Suppose that $Z_a \subset Y_a$ and $X_a \cap Z_a \neq \varnothing$. Then:*

(i) *If the germ Z_a is connected, then $Z_a \subset X_a$.*

(ii) *If Z_a is open and closed in Y_a, then $X_a \subset Z_a$.*

Proof. (i) As $Z_a \subset Y_a$, there exists a neighborhood U_1 of a such that

$$Z \cap U_1 \subset Y \cap U_1.$$

As X_a is a connected component of Y_a, there exists a neighborhood $U_2 \subset U_1$ of a such that $X \cap U_2$ is a connected component of $Y \cap U_2$. By the connectedness of Z_a there exists a neighborhood $U_3 \subset U_2$ of a such that $Z \cap U_3$ is connected. We have

$$Z \cap U_3 \subset Z \cap U_2 = (Z \cap U_1) \cap U_2 \subset (Y \cap U_1) \cap U_2 = Y \cap U_2.$$

On the other hand,

$$(X \cap Z \cap U_3)_a = (X \cap Z)_a = X_a \cap Z_a \neq \varnothing.$$

In particular, $X \cap Z \cap U_3 \neq \varnothing$, so $(X \cap U_2) \cap (Z \cap U_3) \neq \varnothing$. Thus, $Z \cap U_3$ is a connected subset of $Y \cap U_2$ that meets the connected component $X \cap U_2$ of $Y \cap U_2$, so $Z \cap U_3 \subset X \cap U_2$. Consequently, $Z_a = (Z \cap U_3)_a \subset (X \cap U_2)_a = X_a$.

(ii) As Z_a is open and closed in Y_a, there exist neighborhoods W_1 and W_2 of a such that $Z \cap W_1$ is open in $Y \cap W_1$ and $Z \cap W_2$ is closed in $Y \cap W_2$. Define $W := W_1 \cap W_2$. Then $Z \cap W$ is open and closed in $Y \cap W$. As X_a is a connected component of Y_a, there exists a neighborhood $U \subset W$ of a such that $X \cap U$ is a connected component of $Y \cap U$. Note that $Z \cap U$ is open and closed in $Y \cap U$, because $U \subset W$, and $(Z \cap U) \cap (X \cap U) \neq \varnothing$ because $Z_a \cap X_a \neq \varnothing$. Thus, $Z \cap U$ is an open and closed subset of $Y \cap U$ that meets the connected component $X \cap U$ of $Y \cap U$, so $X \cap U \subset Z \cap U$. Consequently, $X_a \subset Z_a$. □

Corollary VI.1.13 *Let $Z \subset \Omega$. Then Z_a is connected if and only if it is the unique connected component of Z_a.*

Proof. If Z_a is the unique connected component of Z_a, then Z_a is connected. Suppose now that Z_a is connected and let X_a be a connected component of Z_a. Then $X_a \cap Z_a = X_a \neq \varnothing$ and Z_a is connected. By Proposition VI.1.12 $Z_a \subset X_a$, so $X_a = Z_a$, as required. □

2 Analytic germ sets and functions. Nullstellensatz

Next we introduce the notion of germ of an analytic function and study its more elementary properties.

Definition VI.2.1 We introduce in the set

$$\mathcal{F}_a := \{(U, f) : U \text{ is an open neighborhood of } a \text{ and } f \in \mathcal{O}(U)\}$$

the equivalence relation: $(U_1, f_1) \sim (U_2, f_2)$ if there exists an open neighborhood $U \subset U_1 \cap U_2$ of a such that $f - g$ vanishes identically on U. The equivalence class of (U, f) is denoted f_a and it is called the *germ of f at a*. The quotient $\mathcal{O}_a := \mathcal{F}_a / \sim$ is the set of *analytic function germs at the point a*.

Remarks VI.2.2 (i) If U is an open subset of $\mathbb{K}^n$ and $f \in \mathcal{O}(U)$ is an analytic function, then $(U_1, f) \sim (U_2, f)$ for each pair of open neighborhoods $U_1, U_2 \subset U$ of a. In other words, all pairs $(W, f) \in \mathcal{F}_a$ are representatives of the germ f_a and we will say that *f is a representative of f_a*.

(ii) Given $f_a \in \mathcal{O}_a$ we define $f_a(a) := f(a)$ where (U, f) is an arbitrary representative of f_a.

(iii) Let $(U_i, f_i) \in \mathcal{F}_a$ for $i = 1, \ldots, r$ be representatives of the germs $f_{1,a}, \ldots, f_{r,a} \in \mathcal{O}_a$. The open neighborhood $U := \bigcap_{i=1}^r U_i$ of a satisfies the equalities $f_{i,a} = (U, f_i)_a$ for $i = 1, \ldots, r$.

Definition VI.2.3 Let (U, f) and (W, g) be representatives of $f_a, g_a \in \mathcal{O}_a$. The sum and product of f_a and g_a are defined in the obvious way: $f_a + g_a$ and $f_a \cdot g_a$ are, respectively, the germs having $(U \cap W, f + g)$ and $(U \cap W, f \cdot g)$ as representatives. One can check that the definition is consistent and $(\mathcal{O}_a, +, \cdot)$ is a commutative ring whose unity is the germ of the function with constant value 1. The neutral element for addition is the germ of the function of constant value 0. We denote these elements 1 and 0.

Proposition VI.2.4 *$\mathcal{O}_a$ is a local ring whose maximal ideal is*

$$\mathfrak{m}_a := \{f_a \in \mathcal{O}_a : f_a(a) = 0\}.$$

Proof. It is enough to see that each $f_a \in \mathcal{O}_a \setminus \mathfrak{m}_a$ is a unit. Indeed, if $f_a(a) \neq 0$ and (U, f) is a representative of f_a, the set $W := \{x \in U : f(x) \neq 0\}$ is an open neighborhood of a and the function $g : W \to \mathbb{K}$, $x \mapsto \frac{1}{f(x)}$ is analytic. As $f_a \cdot g_a = 1$, the germ f_a is a unit in $\mathcal{O}_a$, as required. $\square$

Proposition VI.2.5 (i) *The map $T_a : \mathcal{O}_a \to \mathcal{O}_n$, $f_a \mapsto T_a f$ that maps the germ f_a to the Taylor expansion at a of an arbitrary representative f of f_a is a well-defined isomorphism of local rings. In particular,* $\operatorname{ht}(\mathfrak{m}_a) = \operatorname{ht}(\mathfrak{m}_n) = n$.

(ii) $\mathcal{O}_a$ *is a noetherian regular local ring of dimension* n *and a unique factorization domain.*

Proof. (i) Given $(U_1, f) \sim (U_2, g)$ the difference $f - g$ vanishes on a neighborhood $U \subset U_1 \cap U_2$ of a. Then the derivatives of all orders at the point a of f and g coincide, so $T_a f = T_a g$ and T_a is a well-defined map. It holds $T_a(f+g) = T_a f + T_a g$ and let us prove: $T_a(f \cdot g) = T_a f \cdot T_a g$.

Write

$$T_a f = \sum_\alpha a_\alpha \mathtt{x}^\alpha, \quad T_a g = \sum_\beta b_\beta \mathtt{x}^\beta \quad \text{and} \quad T_a(fg) = \sum_\nu d_\nu \mathtt{x}^\nu.$$

We have $T_a f \cdot T_a g = \sum_\nu c_\nu \mathtt{x}^\nu$ where $c_\nu = \sum_{\alpha+\beta=\nu} a_\alpha b_\beta$. In addition,

$$a_\alpha = \frac{1}{\alpha!}\frac{\partial^{|\alpha|} f}{\partial \mathtt{x}^\alpha}(a), \quad b_\beta = \frac{1}{\beta!}\frac{\partial^{|\beta|} g}{\partial \mathtt{x}^\beta}(a) \quad \text{and} \quad d_\nu = \frac{1}{\nu!}\frac{\partial^{|\nu|}(fg)}{\partial \mathtt{x}^\nu}(a).$$

We must prove that $c_\nu = d_\nu$, that is,

$$\frac{1}{\nu!}\frac{\partial^{|\nu|}(fg)}{\partial \mathtt{x}^\nu}(a) = \sum_{\alpha+\beta=\nu} \frac{1}{\alpha! \cdot \beta!}\frac{\partial^{|\alpha|} f}{\partial \mathtt{x}^\alpha}(a)\frac{\partial^{|\beta|} g}{\partial \mathtt{x}^\beta}(a).$$

We proceed by induction on $|\nu|$. For $|\nu| = 1$ the equality follows from Leibniz Formula. If $|\nu| > 1$ we may assume $\nu_n \geq 1$. If $\mathsf{e}_n := (0, \dots, 0, 1)$,

$$\begin{aligned}
\frac{1}{\nu!}\frac{\partial^{|\nu|}(fg)}{\partial \mathtt{x}^\nu}(a) &= \frac{1}{\nu_n} \cdot \frac{\partial}{\partial \mathtt{x}_n}\Big(\frac{1}{(\nu - \mathsf{e}_n)!} \cdot \frac{\partial^{|\nu|-1}(fg)}{\partial \mathtt{x}^{\nu-\mathsf{e}_n}}(a)\Big) \\
&= \frac{1}{\nu_n} \cdot \frac{\partial}{\partial \mathtt{x}_n}\Big(\sum_{\gamma+\delta=\nu-\mathsf{e}_n} \frac{1}{\gamma! \cdot \delta!}\frac{\partial^{|\gamma|} f}{\partial \mathtt{x}^\gamma}(a) \cdot \frac{\partial^{|\delta|} g}{\partial \mathtt{x}^\delta}(a)\Big) \\
&= \frac{1}{\nu_n}\Big(\sum_{\gamma+\delta=\nu-\mathsf{e}_n} \frac{1}{\gamma! \cdot \delta!} \cdot \frac{\partial^{|\gamma|+1} f}{\partial \mathtt{x}^{\gamma+\mathsf{e}_n}}(a) \cdot \frac{\partial^{|\delta|} g}{\partial \mathtt{x}^\delta}(a) \\
&\quad + \sum_{\gamma+\delta=\nu-\mathsf{e}_n} \frac{1}{\gamma! \cdot \delta!} \cdot \frac{\partial^{|\gamma|} f}{\partial \mathtt{x}^\gamma}(a) \cdot \frac{\partial^{|\delta|+1} g}{\partial \mathtt{x}^{\delta+\mathsf{e}_n}}(a)\Big).
\end{aligned}$$

Write $\alpha := \gamma + \mathsf{e}_n$ and $\beta := \delta$ in the first addend and $\alpha := \gamma$ and $\beta := \delta + \mathsf{e}_n$ in the second one. As $\nu_n = \alpha_n + \beta_n$,

$$d_\nu = \frac{1}{\nu_n}\Big(\sum_{\alpha+\beta=\nu} \frac{\alpha_n + \beta_n}{\alpha! \cdot \beta!} \cdot \frac{\partial^{|\alpha|} f}{\partial \mathtt{x}^\alpha} \cdot \frac{\partial^{|\beta|} g}{\partial \mathtt{x}^\beta}\Big) = \sum_{\alpha+\beta=\nu} \frac{1}{\alpha! \cdot \beta!} \cdot \frac{\partial^{|\alpha|} f}{\partial \mathtt{x}^\alpha} \cdot \frac{\partial^{|\beta|} g}{\partial \mathtt{x}^\beta} = c_\nu.$$

Thus, T_a is a ring homomorphism. It is injective because $T_a f = 0$ means that $f_a = (U, T_a f(x-a)) = (U, 0) = 0$ for a suitable neighborhood U of a. It is surjective because given $g \in \mathcal{O}_n$ there exist an open neighborhood $U := a + D(g)$ of a and an analytic function $f : U \to \mathbb{K}$, $x \mapsto \widehat{g}(x-a)$ such that $T_a f = g$.

We show next $T_a(\mathfrak{m}_a) = \mathfrak{m}_n$.

Pick $h \in \mathfrak{m}_n$. As T_a is surjective, there exists $f_a \in \mathcal{O}_a$ such that $T_a(f_a) = h$. If $f_a \notin \mathfrak{m}_a$ there exists $u_a \in \mathcal{O}_a$ such that $f_a \cdot u_a = 1$, so

$$h \cdot T_a(u_a) = T_a(f_a) \cdot T_a(u_a) = T_a(f_a \cdot u_a) = T_a(1) = 1$$

and h is a unit in $\mathcal{O}_n$, which is a contradiction. Thus, $\mathfrak{m}_n \subset T_a(\mathfrak{m}_a)$. As $\mathfrak{m}_n$ is a maximal ideal and $T(\mathfrak{m}_a) \neq \mathcal{O}_n$ (because $1 \notin T_a(\mathfrak{m}_a)$), we conclude $T_a(\mathfrak{m}_a) = \mathfrak{m}_n$.

(ii) This follows from part (i) because $\mathcal{O}_n$ enjoys the correspondent properties. $\square$

2.a Zero-set germs and ideals.

In order to provide a geometric meaning to the algebraic version of Rückert's Nullstellensatz we need some preliminary results and notations.

Definitions VI.2.6 Let $f_1, \ldots, f_r \in \mathcal{O}(U)$ be analytic functions on an open subset U of $\mathbb{K}^n$. Recall that

$$Z_U(f_1, \ldots, f_r) := \{x \in U : f_1(x) = 0, \ldots, f_r(x) = 0\}.$$

The *germ of zeros of* $f_a \in \mathcal{O}_a$ is $Z(f_a) := [Z_U(f)]_a$ where $(U, f) \in \mathcal{F}_a$ is a representative of f_a.

Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_a$. As $\mathcal{O}_a$ is noetherian, there exist $f_{1,a}, \ldots, f_{r,a} \in \mathfrak{a}$ such that $\mathfrak{a} = \{f_{1,a}, \ldots, f_{r,a}\}\mathcal{O}_a$. We define the *germ of zeros* of $\mathfrak{a}$ as

$$Z(\mathfrak{a}) := \bigcap_{i=1}^{r} Z(f_{i,a}).$$

Remarks VI.2.7 (i) The Identity Principle guarantees that the germ of zeros of $f_a \in \mathcal{O}_a$ is well-defined

(ii) The definition of $Z(\mathfrak{a})$ for an ideal $\mathfrak{a}$ of $\mathcal{O}_a$ is consistent.

Let $f_{1,a},\dots,f_{r,a},g_{1,a},\dots,g_{s,a}\in\mathcal{O}_a$ be such that

$$\mathfrak{a}=\{f_{1,a},\dots,f_{r,a}\}\mathcal{O}_a=\{g_{1,a},\dots,g_{s,a}\}\mathcal{O}_a.$$

Let U be an open neighborhood of a such that there exist representatives (U,f_i) and (U,g_j) of the involved germs. As the $f_{i,a}$ are combinations of the $g_{j,a}$ in $\mathcal{O}_a$ and viceversa, we may assume after shrinking U if necessary that such combinations still hold in $\mathcal{O}(U)$. Consequently,

$$Z_U(f_1,\dots,f_r)\subset Z_U(g_1,\dots,g_s)\subset Z_U(f_1,\dots,f_r),$$

so $\bigcap_{i=1}^r Z(f_{i,a})=\bigcap_{j=1}^s Z(g_{j,a})$.

(iii) As $Z_U(f)$ is a closed subset of U for each $f\in\mathcal{O}(U)$, the germs $Z(f_a)$ and $Z(\mathfrak{a})$ are closed.

The next Proposition collects some basic properties that express the relationship between ideals of $\mathcal{O}_a$ and their zero sets. Its routine proof is left to the reader.

Proposition VI.2.8 *Let $f_a,g_a\in\mathcal{O}_a$ and let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of $\mathcal{O}_a$.*

(i) $Z(f_a\cdot g_a)=Z(f_a)\cup Z(g_a)$.

(ii) *If $\mathfrak{a}\subset\mathfrak{b}$, then $Z(\mathfrak{b})\subset Z(\mathfrak{a})$.*

(iii) $Z(\mathfrak{a}\cdot\mathfrak{b})=Z(\mathfrak{a}\cap\mathfrak{b})=Z(\mathfrak{a})\cup Z(\mathfrak{b})$.

(iv) $Z(\mathfrak{a}+\mathfrak{b})=Z(\mathfrak{a})\cap Z(\mathfrak{b})$.

Definition VI.2.9 Given a subset $X\subset\Omega$ and a point $a\in\Omega$ the *ideal of the germ* X_a is $\mathcal{J}(X_a):=\{f_a\in\mathcal{O}_a : X_a\subset Z(f_a)\}$.

Remarks VI.2.10 (i) Let us see that $\mathcal{J}(X_a)$ is an ideal.

Given $f_a,g_a\in\mathcal{J}(X_a)$, we have $X_a\subset Z(f_a)\cap Z(g_a)\subset Z(f_a+g_a)$, so $f_a+g_a\in\mathcal{J}(X_a)$. In addition, given $h_a\in\mathcal{O}_a$ we have $X_a\subset Z(f_a)\subset Z(h_af_a)$, so $h_af_a\in\mathcal{J}(X_a)$.

(ii) $\mathcal{J}(X_a)$ is a radical ideal because $Z_U(f)=Z_U(f^k)$ for each $(U,f)\in\mathcal{F}_a$ and every positive integer k.

(iii) Let $Y\subset\Omega$. Then $\mathcal{J}(X_a\cup Y_a)=\mathcal{J}(X_a)\cap\mathcal{J}(Y_a)$.

Indeed, $f_a \in \mathcal{J}(X_a \cup Y_a)$ if and only if $X_a \cup Y_a \subset Z(f_a)$, that is, $X_a \subset Z(f_a)$ and $Y_a \subset Z(f_a)$ or, equivalently, $f_a \in \mathcal{J}(X_a) \cap \mathcal{J}(Y_a)$.

(iv) Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_a$. Then $\mathfrak{a} \subset \mathcal{J}(Z(\mathfrak{a}))$.

If $f_a \in \mathfrak{a}$ then $f_a\mathcal{O}_a \subset \mathfrak{a}$, so $Z(\mathfrak{a}) \subset Z(f_a)$ and $f_a \in \mathcal{J}(Z(\mathfrak{a}))$.

(v) The inclusion $\mathfrak{a} \subset \mathcal{J}(Z(\mathfrak{a}))$ implies that $\sqrt{\mathfrak{a}} \subset \mathcal{J}(Z(\mathfrak{a}))$ because $\mathcal{J}(X_a)$ is a radical ideal.

(vi) $X_a \subset \mathrm{Cl}(X_a) \subset Z(\mathcal{J}(X_a))$.

Let $\mathcal{J}(X_a) = \{f_{1,a}, \dots, f_{r,a}\}\mathcal{O}_a$. As each germ $f_{j,a} \in \mathcal{J}(X_a)$, we have $X_a \subset Z(f_{j,a})$, so

$$X_a \subset Z(f_{1,a}) \cap \cdots \cap Z(f_{r,a}) = Z(\mathcal{J}(X_a)).$$

But $Z(\mathcal{J}(X_a))$ is a closed germ. Thus, $\mathrm{Cl}(X_a) \subset Z(\mathcal{J}(X_a))$ by Remark VI.1.9.

2.b Analytic germs. Irreducible components. Next, we introduce the class of analytic germs and present some elementary properties about these germs. The germ X_a of a set $X \subset \Omega$ is *analytic* if there exists an ideal $\mathfrak{a}$ in $\mathcal{O}_a$ such that $X_a = Z(\mathfrak{a})$.

Proposition VI.2.11 *The following conditions are equivalent:*

(i) *X_a is an analytic germ.*

(ii) *$X_a = Z(\mathcal{J}(X_a))$.*

(iii) *There exist an open neighborhood U of a in $\mathbb{K}^n$ and an analytic set $Y \subset U$ in U such that $X_a = Y_a$.*

Proof. (i) $\Longrightarrow$ (ii) Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_a$ such that $X_a = Z(\mathfrak{a})$. Then

$$\mathfrak{a} \subset \mathcal{J}(Z(\mathfrak{a})) = \mathcal{J}(X_a) \quad \rightsquigarrow \quad X_a \subset Z(\mathcal{J}(X_a)) \subset Z(\mathfrak{a}) = X_a.$$

(ii) $\Longrightarrow$ (iii) Write $\mathcal{J}(X_a) = \{f_{1,a}, \dots, f_{r,a}\}$. Then $X_a = \bigcap_{i=1}^n Z(f_{i,a})$, so there exist an open neighborhood U of a and representatives $f_j \in \mathcal{O}(U)$ of $f_{j,a}$ for $j = 1, \dots, r$ such that $X \cap U = Z_U(f_1, \dots, f_r)$. Then $Y := X \cap U$ is an analytic subset of U and $X_a = Y_a$.

(iii) $\Longrightarrow$ (i) Let U be an open neighborhood of a and let $Y \subset U$ an analytic subset of U such that $X_a = Y_a$. As Y is analytic in U and $a \in U$, there exist an open neighborhood $W \subset U$ of a and $f_1, \ldots, f_r \in \mathcal{O}(W)$ such that $Y \cap W = Z_W(f_1, \ldots, f_r)$. Define $\mathfrak{a} := \{f_{1,a}, \ldots, f_{r,a}\}\mathcal{O}_a$. Then X_a is an analytic germ because $X_a = Y_a = (Y \cap W)_a = Z(\mathfrak{a})$, as required. □

Example VI.2.12 By Example V.2.5 (ii) the interval $[-1, 1] \subset \mathbb{R}$ is not an analytic subset of $\mathbb{R}$ but its germ at the origin is analytic because $X_0 = U_0$ where $U := (-1, 1)$ is an analytic subset of U.

Definition VI.2.13 An analytic germ X_a is *reducible* if there exist analytic germs Y_a and Z_a strictly contained in X_a such that $X_a = Y_a \cup Z_a$. The germ X_a is *irreducible* if it is not reducible.

Proposition VI.2.14 (i) *The germ X_a is irreducible if and only if $\mathcal{J}(X_a)$ is a prime ideal of $\mathcal{O}_a$.*

(ii) *Suppose that the germ X_a is irreducible and let $X_{1,a}, \ldots, X_{r,a}$ be analytic germs such that $X_a \subset \bigcup_{i=1}^r X_{i,a}$. Then $X_a \subset X_{k,a}$ for some $k = 1, \ldots, r$.*

Proof. (i) If $\mathcal{J}(X_a)$ is not a prime ideal, there exist $f_a, g_a \in \mathcal{O}_a \setminus \mathcal{J}(X_a)$ whose product $f_a \cdot g_a \in \mathcal{J}(X_a)$. Therefore, $X_a \not\subset Z(f_a)$ and $X_a \not\subset Z(g_a)$, but $X_a \subset Z(f_a \cdot g_a) = Z(f_a) \cup Z(g_a)$. Thus, $X_a = Y_a \cup Z_a$ where

$$Y_a := X_a \cap Z(f_a) \quad \text{and} \quad Z_a := X_a \cap Z(g_a).$$

Consequently, $Y_a \neq X_a$ and $Z_a \neq X_a$, so X_a is reducible.

Suppose now that X_a is a reducible analytic germ. Then there exist analytic germs Y_a and Z_a different from X_a such that $X_a = Y_a \cup Z_a$. We have $\mathcal{J}(X_a) \subsetneq \mathcal{J}(Y_a)$. Otherwise, we would have $X_a = Z(\mathcal{J}(X_a)) = Z(\mathcal{J}(Y_a)) = Y_a$ because both X_a and Y_a are analytic germs. Analogously, $\mathcal{J}(X_a) \subsetneq \mathcal{J}(Z_a)$. Thus, there exist $f_a \in \mathcal{J}(Y_a) \setminus \mathcal{J}(X_a)$ and $g_a \in \mathcal{J}(Z_a) \setminus \mathcal{J}(X_a)$ such that

$$X_a = Y_a \cup Z_a \subset Z(f_a) \cup Z(g_a) = Z(f_a \cdot g_a).$$

Consequently, $f_a \cdot g_a \in \mathcal{J}(X_a)$, so $\mathcal{J}(X_a)$ is not a prime ideal.

(ii) As $X_a \subset \bigcup_{i=1}^r X_{i,a}$, we have

$$\bigcap_{i=1}^r \mathcal{J}(X_{i,a}) = \mathcal{J}\Big(\bigcup_{i=1}^r X_{i,a}\Big) \subset \mathcal{J}(X_a).$$

As X_a is irreducible, the ideal $\mathcal{J}(X_a)$ is prime. By Lemma II.2.2 there exists $1 \leq k \leq r$ such that $\mathcal{J}(X_{k,a}) \subset \mathcal{J}(X_a)$, so

$$X_a = Z(\mathcal{J}(X_a)) \subset Z(\mathcal{J}(X_{k,a})) = X_{k,a}$$

as required. □

Theorem VI.2.15 (Irreducible components) *Let X_a be an analytic germ. Then there exist irreducible analytic germs $X_{1,a}, \dots, X_{r,a}$ satisfying the following conditions:*

$$X_a = \bigcup_{i=1}^{r} X_{i,a} \quad \textit{and} \quad X_{i,a} \not\subset X_{j,a} \quad \textit{for} \quad i \neq j.$$

The family $\{X_{i,a}\}_{i=1}^r$ satisfying these conditions is unique and it is called the family of the irreducible components of X_a.

Proof. We prove first the existential part of the statement. As $\mathcal{J}(X_a)$ is a radical ideal of the noetherian ring $\mathcal{O}_a$, there exist prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ in $\mathcal{O}_a$ such that $\mathcal{J}(X_a) = \bigcap_{i=1}^r \mathfrak{p}_i$ and $\mathfrak{p}_j \not\subset \mathfrak{p}_i$ for $i \neq j$. Define the analytic germ $X_{i,a} := Z(\mathfrak{p}_i)$ and observe that

$$X_a = Z(\mathcal{J}(X_a)) = Z\Big(\bigcap_{i=1}^{r} \mathfrak{p}_i\Big) = \bigcup_{i=1}^{r} Z(\mathfrak{p}_i) = \bigcup_{i=1}^{r} X_{i,a}.$$

To prove that each $X_{i,a}$ is irreducible it is enough to check that $\mathcal{J}(X_{i,a}) = \mathfrak{p}_i$ because the latter is a prime ideal. If the inclusion $\mathfrak{p}_i \subset \mathcal{J}(Z(\mathfrak{p}_i)) = \mathcal{J}(X_{i,a})$ were strict, there would exist $f_{i,a} \in \mathcal{J}(X_{i,a}) \setminus \mathfrak{p}_i$. As $\mathfrak{p}_j \not\subset \mathfrak{p}_i$ for $i \neq j$, there exists $f_{j,a} \in \mathfrak{p}_j \setminus \mathfrak{p}_i$ for each $j \neq i$. Thus, $f_{j,a} \in \mathfrak{p}_j \subset \mathcal{J}(Z(\mathfrak{p}_j)) = \mathcal{J}(X_{j,a})$ for $i \neq j$ and $f_{i,a} \in \mathcal{J}(X_{i,a})$. Define $f_a := \prod_{k=1}^r f_{k,a}$. We have

$$X_a = \bigcup_{i=1}^{r} X_{i,a} = \bigcup_{i=1}^{r} Z(\mathcal{J}(X_{i,a})) \subset \bigcup_{k=1}^{r} Z(f_{k,a}) = Z(f_a),$$

so $f_a \in \mathcal{J}(X_a) \subset \mathfrak{p}_i$. As $\mathfrak{p}_i$ is prime, some $f_{k,a} \in \mathfrak{p}_i$, against our choice. This proves that $\mathcal{J}(X_{i,a})$ is a prime ideal, so $X_{i,a}$ is irreducible. In addition, suppose that $X_{i,a} \subset X_{j,a}$ for some indices $i \neq j$. Then $\mathfrak{p}_j = \mathcal{J}(X_{j,a}) \subset \mathcal{J}(X_{i,a}) = \mathfrak{p}_i$, which is false.

To prove the uniqueness we proceed as follows. Let $\{Y_{j,a}\}_{j=1}^{s}$ be a family of irreducible analytic germs such that

$$X_a = \bigcup_{j=1}^{s} Y_{j,a} \quad \text{and} \quad Y_{i,a} \not\subset Y_{j,a} \quad \text{for} \quad i \neq j.$$

Each $X_{i,a} \subset \bigcup_{j=1}^{s} Y_{j,a}$. By Proposition VI.2.14 there exists $j = 1, \ldots, s$ such that $X_{i,a} \subset Y_{j,a}$. As $Y_{j,a} \subset \bigcup_{k=1}^{r} X_{k,a}$, we have $Y_{j,a} \subset X_{k,a}$ for some $k = 1, \ldots, r$. Thus, $X_{i,a} \subset X_{k,a}$, so $i = k$ and $X_{i,a} = Y_{j,a}$. Consequently, $r \leq s$ and after changing the roles of the $X_{i,a}$ and the $Y_{j,a}$, we conclude $r = s$ and, after reordering, $X_{i,a} = Y_{i,a}$ for $i = 1, \ldots, r$. $\square$

The algebraic results obtained in Chapters I-IV for the ring $\mathcal{O}_n$ of convergent series admit an immediate translation to the geometric language of analytic germs at the origin of $\mathbb{K}^n$. To translate that for germs at an arbitrary point $a \in \mathbb{K}^n$ we need the following technical result. Once it is proved, we can always suppose we are working with germs at the origin.

Lemma VI.2.16 *Let Ω_1 and Ω_2 be open subsets of $\mathbb{K}^n$ and let $u : \Omega_1 \to \Omega_2$ be an analytic equivalence. Let $a \in \Omega_1$ and $b := u(a) \in \Omega_2$. Denote $\mathcal{G}_a$ and $\mathcal{G}_b$, respectively, the families of set germs at the points a and b. For each germ $Z_a \in \mathcal{G}_a$ denote Z one of its representatives. Define*

$$u_* : \mathcal{G}_a \to \mathcal{G}_b,\ X_a \mapsto (u(X \cap \Omega_1))_b \quad \textit{and}$$
$$u^* : \mathcal{O}_b \to \mathcal{O}_a,\ (W, g)_b \mapsto (u^{-1}(W), g \circ u)_a.$$

Let $X_a, Y_a \in \mathcal{G}_a$. Then the following properties hold:

(i) *The map u_* is bijective.*

(ii) $u_*(X_a \cap Y_a) = u_*(X_a) \cap u_*(Y_a)$.

(iii) $u_*(X_a \cup Y_a) = u_*(X_a) \cup u_*(Y_a)$.

(iv) $u_*(X_a) \subset u_*(Y_a)$ *if and only if* $X_a \subset Y_a$.

(v) $u_*(\mathrm{Cl}(X_a)) = \mathrm{Cl}(u_*(X_a))$.

(vi) *The germ $X_a \in \mathcal{G}_a$ is closed (resp. open, dense) in Y_a if and only if the germ $u_*(X_a) \in \mathcal{G}_b$ is closed (resp. open, dense) in $u_*(Y_a) \in \mathcal{G}_b$.*

(vii) *The map u^* is an isomorphism of local rings.*

(viii) $u_*(Z(u^*(\mathfrak{a}))) = Z(\mathfrak{a})$ *for each ideal* $\mathfrak{a}$ *in* $\mathcal{O}_b$.

(ix) $u^*(\mathcal{J}(u_*(X_a))) = \mathcal{J}(X_a)$.

Proof. Let $v : \Omega_2 \to \Omega_1$ be the inverse of u, which is an analytic equivalence. To prove (i) note that

$$\begin{aligned}(v_* \circ u_*)(X_a) &= v_*((u(X \cap \Omega_1))_b) = (v(u(X \cap \Omega_1) \cap \Omega_2))_a \\ &= ((v \circ u)(X \cap \Omega_1) \cap v(\Omega_2))_a = (X \cap \Omega_1 \cap \Omega_1)_a = X_a.\end{aligned}$$

Thus, $v_* \circ u_* = \mathrm{id}$ and, analogously, $u_* \circ v_* = \mathrm{id}$. This shows that v_* and u_* are mutually inverse, so u_* is a bijection. To prove (ii) note that

$$\begin{aligned}u_*(X_a \cap Y_a) &= u_*((X \cap Y)_a) = (u(X \cap Y \cap \Omega_1))_b \\ &= (u(X \cap \Omega_1) \cap u(Y \cap \Omega_1))_b \\ &= (u(X \cap \Omega_1))_b \cap (u(Y \cap \Omega_1))_b = u_*(X_a) \cap u_*(Y_a).\end{aligned}$$

(iii) The proof is the same as the one above changing $\cap$ by $\cup$.

(iv) Notice that $u_*(X_a) \subset u_*(Y_a)$ if and only if there exists a neighborhood $W \subset \Omega_2$ of b such that $u(X \cap \Omega_1) \cap W \subset u(Y \cap \Omega_1) \cap W$. As v is bijective, this is equivalent to $v(u(X \cap \Omega_1) \cap W) \subset v(u(Y \cap \Omega_1) \cap W)$, that is,

$$X \cap \Omega_1 \cap v(W) \subset Y \cap \Omega_1 \cap v(W).$$

As $U := v(W) \cap \Omega_1$ is a neighborhood of a because v is a homeomorphism, the last inclusion, that can be written as $X \cap U \subset Y \cap U$, implies $X_a \subset Y_a$.

(v) We have

$$\begin{aligned}u_*(\mathrm{Cl}(X_a)) &= (u(\mathrm{Cl}(X) \cap \Omega_1))_b = (u(\mathrm{Cl}(X \cap \Omega_1) \cap \Omega_1))_b \\ &= (\mathrm{Cl}(u(X \cap \Omega_1)) \cap \Omega_2)_b = (\mathrm{Cl}(u(X \cap \Omega_1)))_b \\ &= \mathrm{Cl}(u(X \cap \Omega_1)_b) = \mathrm{Cl}(u_*(X_a)).\end{aligned}$$

(vi.1) X_a is closed in Y_a if and only if $X_a = \mathrm{Cl}(X_a) \cap Y_a$ and this is equivalent by (ii) and (v) to $u_*(X_a) = u_*(\mathrm{Cl}(X_a) \cap Y_a) = \mathrm{Cl}(u_*(X_a)) \cap u_*(Y_a)$, that is, $u_*(X_a)$ is closed in $u_*(Y_a)$.

(vi.2) X_a is dense in Y_a if and only if $Y_a \subset \mathrm{Cl}(X_a)$ and this is equivalent by (iv) and (v) to $u_*(Y_a) \subset u_*(\mathrm{Cl}(X_a)) = \mathrm{Cl}(u_*(X_a))$, that is, $u_*(X_a)$ is dense in $u_*(Y_a)$.

(vi.3) X_a is open in Y_a if and only if $Y_a \setminus X_a$ is closed in Y_a. This is equivalent by (vi.1) to $u_*(Y_a) \setminus u_*(X_a) = u_*(Y_a \setminus X_a)$ is closed in $u_*(Y_a)$ or, equivalently, $u_*(X_a)$ is open in $u_*(Y_a)$.

(vii) Let $g_b := (W, g)_b \in \mathcal{O}_b$ and $f_a := (U, f)_a \in \mathcal{O}_a$. As $u \circ v = \mathrm{id}$ and $v \circ u = \mathrm{id}$,

$$\begin{aligned}(u^* \circ v^*)(f_a) &= u^*(v^*(U, f)_a) = u^*((v^{-1}(U), f \circ v)_b) \\ &= (u^{-1}(v^{-1}(U)), f \circ v \circ u)_a = (U, f)_a = f_a.\end{aligned}$$

Analogously, $(v^* \circ u^*)(g_b) = g_b$. Thus, u^* and v^* are mutually inverse isomorphisms.

(viii) If $\mathfrak{a} := \{g_{1,b}, \dots, g_{r,b}\}\mathcal{O}_b$ then $u^*(\mathfrak{a}) = \{u^*(g_{1,b}), \dots, u^*(g_{r,b})\}\mathcal{O}_a$ because u^* is an isomorphism. There exist a neighborhood $W \subset \Omega_2$ of b and representatives $g_j \in \mathcal{O}(W)$ of $g_{j,b}$ for $j = 1, \dots, r$. Then each $f_j := g_j \circ u \in \mathcal{O}(U)$ where $U := v(W) \subset \Omega_1$ and the analytic set $X := Z_U(f_1, \dots, f_r)$ in U satisfies the equality $Z(u^*(\mathfrak{a})) = X_a$. Note that $X_a = \bigcap_{j=1}^r Z(f_{j,a})$ and by (ii)

$$u_*(Z(u^*(\mathfrak{a}))) = u_*(X_a) = \bigcap_{j=1}^r u_*(Z(f_{j,a})).$$

Note that $u(Z_U(f_j)) = Z_{u(U)}(f_j \circ u^{-1}|_{u(U)}) = Z_W(g_j)$, so $u_*(Z(f_{j,a})) = Z(g_{j,b})$ for $j = 1, \dots, r$. Consequently,

$$u_*(Z(u^*(\mathfrak{a}))) = \bigcap_{j=1}^r u_*(Z(f_{j,a})) = \bigcap_{j=1}^r Z(g_{j,b}) = Z(\mathfrak{a}).$$

(ix) For each function germ $f_a \in \mathcal{O}_a$ the condition $f_a \in u^*(\mathcal{J}(u_*(X_a)))$ is equivalent to $v^*(f_a) \in v^*(u^*(\mathcal{J}(u_*(X_a))))$ and, since u^* and v^* are mutually inverse isomorphisms, we have $v^*(f_a) \in \mathcal{J}(u_*(X_a))$. Let $U \subset \Omega_1$ be an open neighborhood of a and choose a representative $X \subset U$ of the germ X_a and a representative $f \in \mathcal{O}(U)$ of f_a. Then $v^*(f_a) = v^*((U, f)_a) = (v^{-1}(U), f \circ v)_b$ and $v^*(f_a) \in \mathcal{J}(u_*(X_a))$ if and only if

$$(u(X \cap U))_b = u_*(X_a) \subset Z(f \circ v|_{v^{-1}(U)})_b = Z(f \circ v|_{u(U)})_b.$$

This means that $u(X \cap U) \cap W \subset Z_W(f \circ v)$ for a suitable open neighborhood $W \subset u(U)$ of b. As $v : \Omega_2 \to \Omega_1$ is a homeomorphism and $v(W) \subset U$, the previous condition is equivalent to

$$X \cap v(W) = X \cap U \cap v(W) = v(u(X \cap U) \cap W) \subset v(Z_W(f \circ v)) = Z_{v(W)}(f)$$

or equivalently $X_a \subset Z(f_a)$, so $f_a \in \mathcal{J}(X_a)$, as required. □

Theorem VI.2.17 (Nullstellensatz) *Let $\mathbb{K} = \mathbb{C}$ and let $\mathfrak{a}$ be an ideal of $\mathcal{O}_a$. Then $\mathcal{J}(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*

Proof. The translation $u : \mathbb{C}^n \to \mathbb{C}^n$, $z \mapsto z - a$ is an analytic equivalence with $u(a) = 0$ and inverse $v : \mathbb{C}^n \to \mathbb{C}^n$, $z \mapsto z + a$. By Lemma VI.2.16 (ix) we may assume $a := 0$. In addition, the isomorphism $T_0 : \mathcal{O}_0 \to \mathcal{O}_n$, $f_0 \mapsto T_0 f = f$ introduced in Proposition VI.2.5 allows us to suppose $\mathcal{O}_a = \mathcal{O}_n$. To ease notation we denote $f, g, \ldots$ the elements in $\mathcal{O}_a$. Let

$$\mathcal{F}_{\mathfrak{a}} := \{\varphi : \mathcal{O}_n \to \mathbb{C}\{\mathtt{t}\}, \quad \varphi \text{ is a } \mathbb{C}\text{-algebra homomorphism and } \mathfrak{a} \subset \ker(\varphi)\}.$$

By Rückert's Nullstellensatz IV.2.1 $\sqrt{\mathfrak{a}} = \bigcap_{\varphi \in \mathcal{F}_{\mathfrak{a}}} \ker(\varphi)$. We will use this to show $\mathcal{J}(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

The inclusion $\sqrt{\mathfrak{a}} \subset \mathcal{J}(Z(\mathfrak{a}))$ is clear. We prove the converse inclusion by contradiction. Suppose there exists $f \in \mathcal{J}(Z(\mathfrak{a})) \setminus \sqrt{\mathfrak{a}}$. Let $\varphi \in \mathcal{F}_{\mathfrak{a}}$ be such that $\varphi(f) \neq 0$. Write $\sqrt{\mathfrak{a}} := \{f_1, \ldots, f_r\}\mathcal{O}_n$ and notice that

$$Z(f_1) \cap \cdots \cap Z(f_r) = Z(\sqrt{\mathfrak{a}}) = Z(\mathfrak{a}) \subset Z(f)$$

because $f \in \mathcal{J}(Z(\mathfrak{a}))$. Let $W \subset U := D(f) \cap \bigcap_{j=1}^r D(f_j)$ be an open neighborhood of 0 such that $\bigcap_{j=1}^r Z_W(f_j) \subset Z_W(f)$. Denote $g_i := \varphi(\mathtt{x}_i) \in \mathbb{C}\{\mathtt{t}\}$ for $i = 1, \ldots, n$. Let $\varepsilon > 0$ be small enough to guarantee that the map

$$g : \Delta := \{t \in \mathbb{C} : |t| < \varepsilon\} \to \mathbb{C}^n, \, t \mapsto (g_1(t), \ldots, g_n(t))$$

is a well-defined analytic map, $g(\Delta) \subset W$ and

$$\begin{aligned} f(g_1(t), \ldots, g_n(t)) &= f(g_1, \ldots, g_n)(t) \\ f_j(g_1(t), \ldots, g_n(t)) &= f_j(g_1, \ldots, g_n)(t) \end{aligned}$$

for $t \in \Delta$ and $1 \leq j \leq r$. By Theorem II.1.9

$$f(g_1, \ldots, g_n) = f(\varphi(\mathtt{x}_1), \ldots, \varphi(\mathtt{x}_n)) = \varphi(f) \neq 0.$$

Thus, there exists $t_0 \in \Delta$ such that $x_0 := (g_1(t_0), \ldots, g_n(t_0)) \in W$ satisfies $f(x_0) \neq 0$. However, $\varphi(f_j) = 0$ for each $j = 1, \ldots, r$, so

$$\begin{aligned} f_j(x_0) = f_j(g_1(t_0), \ldots, g_n(t_0)) &= f_j(g_1, \ldots, g_n)(t_0) \\ &= f_j(\varphi(\mathtt{x}_1), \ldots, \varphi(\mathtt{x}_n))(t_0) = \varphi(f_j)(t_0) = 0, \end{aligned}$$

and $x_0 \in \bigcap_{j=1}^r Z_W(f_j) \subset Z_W(f)$, which is a contradiction. We conclude $\mathcal{J}(Z(\mathfrak{a})) \subset \sqrt{\mathfrak{a}}$, as required. $\square$

Remarks VI.2.18 (i) The next example shows that Nullstellensatz VI.2.17 does not hold for $\mathbb{K} = \mathbb{R}$. Let f denote the germ at the origin of the analytic function $\mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x^2 + y^2$. As $\mathcal{O}_0$ is a unique factorization domain and f is irreducible in $\mathcal{O}_0$, the ideal $\mathfrak{a} := f\mathcal{O}_0$ is prime, so it is a radical ideal. Let g be the germ of the map $\mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x$. We have $g \in \mathcal{J}(Z(\mathfrak{a}))$. However $g \notin \mathfrak{a} = \sqrt{\mathfrak{a}}$ because $\omega(g) = 1$ and $\omega(f) = 2$.

(ii) There exists a Nullstellensatz in the real case. The statement is the following. Given a commutative ring with unity A and an ideal $\mathfrak{a}$ in A, its *real radical* is defined as

$$\sqrt[R]{\mathfrak{a}} := \Big\{ f \in A : \exists k \in \mathbb{Z}^+,\ n \in \mathbb{N} \text{ and } f_1, \ldots, f_n \in A \text{ with } f^{2k} + \sum_{j=1}^n f_j^2 \in \mathfrak{a} \Big\}.$$

The real Nullstellensatz states $\mathcal{J}(Z(\mathfrak{a})) = \sqrt[R]{\mathfrak{a}}$ for each ideal $\mathfrak{a}$ in $\mathcal{O}_a$.

3 Dimension Theory

We fix an open set $\Omega \subset \mathbb{K}^n$, an analytic subset $X \subset \Omega$ and a point $a \in \Omega$.

Definitions VI.3.1 The quotient $\mathcal{O}_{X,a} := \mathcal{O}_a/\mathcal{J}(X_a)$ is the *local analytic ring* of X at the point a and the *dimension* of X_a is defined as

$$\dim(X_a) := \begin{cases} -1 & \text{if } X_a = \varnothing, \\ \dim(\mathcal{O}_{X,a}) & \text{otherwise.} \end{cases}$$

Remarks VI.3.2 (i) $\dim(X_a) = -1$ if and only if $\mathcal{J}(X_a) = \mathcal{O}_{X,a}$.

(ii) Suppose that $a \in X$. Then $\dim(X_a) = \dim(\mathcal{O}_a) - \operatorname{ht}(\mathcal{J}(X_a))$ and, since $\dim(\mathcal{O}_a) = n$ and $\operatorname{ht}(\mathcal{J}(X_a)) \geq 0$, we have $0 \leq \dim(X_a) \leq n$.

(iii) The *dimension of* X is $\dim(X) := \max\{\dim(X_a) : a \in X\} \leq n$.

(iv) $\dim(X_a) = 0$ if and only if a is an isolated point of X.

Denote $a := (a_1, \ldots, a_n)$ and for $i = 1, \ldots, n$ consider the analytic map

$$f_i : \mathbb{K}^n \to \mathbb{K},\ z := (z_1, \ldots, z_n) \mapsto z_i - a_i.$$

By Proposition VI.2.5 $\mathfrak{m}_a = \{f_{1,a}, \ldots, f_{n,a}\}\mathcal{O}_a$. Suppose first that a is an isolated point of X. Then there exists a neighborhood U of a with $X \cap U = \{a\}$, so $X_a \subset Z(f_{i,a})$ for $i = 1, \ldots, n$. Thus, $\mathfrak{m}_a = \{f_{1,a}, \ldots, f_{n,a}\}\mathcal{O}_a \subset \mathcal{J}(X_a)$. As $\mathfrak{m}_a$ is a maximal ideal, $\mathfrak{m}_a = \mathcal{J}(X_a)$ and $\dim(X_a) = \dim(\mathcal{O}_a) - \mathrm{ht}(\mathcal{J}(X_a)) = 0$.

Conversely, suppose $\dim(X_a) = 0$. The radical ideal $\mathcal{J}(X_a)$ of the noetherian ring $\mathcal{O}_a$ is a finite intersection $\mathcal{J}(X_a) = \bigcap_{j=1}^r \mathfrak{p}_j$ of prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ of $\mathcal{O}_a$. Then for $j = 1, \ldots, r$, we have

$$\begin{aligned} n = n - \dim(X_a) = n - (\dim(\mathcal{O}_a) - \mathrm{ht}(\mathcal{J}(X_a))) \\ = \mathrm{ht}(\mathcal{J}(X_a)) \leq \mathrm{ht}(\mathfrak{p}_j) \leq \mathrm{ht}(\mathfrak{m}_a) = n, \end{aligned}$$

so $\mathrm{ht}(\mathfrak{p}_j) = \mathrm{ht}(\mathfrak{m}_a)$ for $j = 1, \ldots, n$. Thus, each prime ideal $\mathfrak{p}_j = \mathfrak{m}_a$ and $\mathcal{J}(X_a) = \mathfrak{m}_a = \{f_{1,a}, \ldots, f_{n,a}\}\mathcal{O}_a$. We conclude

$$X_a = Z(\mathcal{J}(X_a)) = \bigcap_{j=1}^r Z(f_{j,a}) = \{a\}_a,$$

so a is an isolated point of X.

We collect some elementary properties of the dimension of analytic germs.

Proposition VI.3.3 *Let X and Y be analytic subsets of Ω such that the germs X_a and Y_a are non-empty.*

(i) *If $X_a \subset Y_a$, then* $\dim(X_a) \leq \dim(Y_a)$.

(ii) *If $X_a \subsetneq Y_a$ and Y_a is irreducible, then* $\dim(X_a) < \dim(Y_a)$.

(iii) *Let $X_{1,a}, \ldots, X_{r,a}$ be the irreducible components of X_a. Then*

$$\dim(X_a) = \max\{\dim(X_{j,a}) : j = 1, \ldots, r\}.$$

(iv) $\dim(X_a \cup Y_a) = \max\{\dim(X_a), \dim(Y_a)\}$.

(v) *If $X \subset Y$, then* $\dim(X) \leq \dim(Y)$.

(vi) $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.

Proof. (i) We have $\mathcal{J}(Y_a) \subset \mathcal{J}(X_a)$, so $\mathrm{ht}(\mathcal{J}(Y_a)) \leq \mathrm{ht}(\mathcal{J}(X_a))$. Thus,

$$\dim(X_a) = n - \mathrm{ht}(\mathcal{J}(X_a)) \leq n - \mathrm{ht}(\mathcal{J}(Y_a)) = \dim(Y_a).$$

(ii) Let $\mathfrak{p}$ be a prime ideal containing $\mathcal{J}(X_a)$ such that $\mathrm{ht}(\mathcal{J}(X_a)) = \mathrm{ht}(\mathfrak{p})$. Suppose by contradiction that $\dim(X_a) = \dim(Y_a)$. Then

$$\mathrm{ht}(\mathcal{J}(Y_a)) = \mathrm{ht}(\mathcal{J}(X_a)) = \mathrm{ht}(\mathfrak{p}) \quad \text{and} \quad \mathcal{J}(Y_a) \subset \mathcal{J}(X_a) \subset \mathfrak{p}.$$

As $\mathcal{J}(Y_a)$ and $\mathfrak{p}$ are prime ideals of $\mathcal{O}_a$, we have $\mathcal{J}(Y_a) = \mathfrak{p}$. Consequently, $\mathfrak{p} = \mathcal{J}(Y_a) \subset \mathcal{J}(X_a) \subset \mathfrak{p}$, so $\mathcal{J}(Y_a) = \mathcal{J}(X_a)$. As X_a and Y_a are analytic germs, we have

$$X_a = Z(\mathcal{J}(X_a)) = Z(\mathcal{J}(Y_a)) = Y_a,$$

which is a contradiction.

(iii) $\mathcal{J}(X_a) = \bigcap_{j=1}^r \mathcal{J}(X_{j,a})$ and each $\mathcal{J}(X_{j,a})$ is a prime ideal of $\mathcal{O}_a$. Thus, $\mathrm{ht}(\mathcal{J}(X_a)) = \min\{\mathrm{ht}(\mathcal{J}(X_{j,a}))\}_{j=1}^r$. Therefore,

$$\dim(X_a) = n - \mathrm{ht}(\mathcal{J}(X_a)) = n - \min\{\mathrm{ht}(\mathcal{J}(X_{j,a}))\}_{j=1}^r = \max\{\dim(X_{j,a})\}_{j=1}^r.$$

(iv) Let $X_{1,a}, \dots, X_{r,a}$ and $Y_{1,a}, \dots, Y_{s,a}$ be the irreducible components of X_a and Y_a. Then

$$\mathcal{J}(X_a) = \bigcap_{i=1}^r \mathcal{J}(X_{i,a}), \quad \mathcal{J}(Y_a) = \bigcap_{j=1}^s \mathcal{J}(Y_{j,a}),$$
$$\mathcal{J}(X_a \cup Y_a) = \mathcal{J}(X_a) \cap \mathcal{J}(Y_a) = \bigcap_{i=1}^r \mathcal{J}(X_{i,a}) \cap \bigcap_{j=1}^s \mathcal{J}(Y_{j,a}),$$

where each $\mathcal{J}(X_{i,a})$ and each $\mathcal{J}(Y_{j,a})$ are prime ideals. Consequently,

$$\begin{aligned}
\dim(X_a \cup Y_a) &= n - \mathrm{ht}(\mathcal{J}(X_a \cup Y_a)) = n - \mathrm{ht}\left(\bigcap_{i=1}^r \mathcal{J}(X_{i,a}) \cap \bigcap_{j=1}^s \mathcal{J}(Y_{j,a})\right) \\
&= n - \min\{\mathrm{ht}(\mathcal{J}(X_{i,a})), \mathrm{ht}(\mathcal{J}(Y_{j,a})) : 1 \le i \le r,\ 1 \le j \le s\} \\
&= \max\{\dim(X_{j,a}),\ \dim(Y_{i,a}) : 1 \le j \le r,\ 1 \le i \le s\} \\
&= \max\{\dim(X_a), \dim(Y_a)\}.
\end{aligned}$$

(v) $\dim(X) = \max\{\dim(X_a) : a \in X\} = \dim(X_b)$ for some point $b \in X$. As $X \subset Y$, we have $X_b \subset Y_b$ and by part (i)

$$\dim(X) = \dim(X_b) \le \dim(Y_b) \le \dim(Y).$$

(vi) By (v) $\max\{\dim(X), \dim(Y)\} \le \dim(X \cup Y)$. Conversely, there exists $a \in X \cup Y$ such that $\dim(X \cup Y) = \dim(X \cup Y)_a$ and by (iv)

$$\begin{aligned}
\dim(X \cup Y) = \dim(X \cup Y)_a &= \max\{\dim(X_a),\ \dim(Y_a\} \\
&\le \max\{\dim(X), \dim(Y)\},
\end{aligned}$$

as required. □

4 Jacobian Criteria

The next result explains the relationship, in the complex case, between the algebraic notion of regular local ring and the geometric concept of smooth point. To ease the notations we identify function germs at a point with their representatives. We fix an open set $\Omega \subset \mathbb{C}^n$, an analytic subset $X \subset \Omega$ and a point $a \in \Omega$.

Definition VI.4.1 Let $X \subset \Omega$ be such that $a \in \mathrm{Cl}(X)$. We say that X_a *is a d-dimensional smooth germ* if there exists a neighborhood U of a such that $X \cap U$ is a smooth submanifold of $\mathbb{K}^n$ of dimension d.

Remarks VI.4.2 (i) The set $\mathrm{Smooth}_d(X)$ of smooth points of X of dimension d introduced in Definition V.3.8 coincides with the set of points $a \in X$ such that X_a is a d-dimensional smooth germ. Recall the definitions of the *set of smooth points of* X as the union $\mathrm{Smooth}(X) := \bigcup_{d\geq 0} \mathrm{Smooth}_d(X)$ and the *set of non-smooth points of* X as the complement $\text{Non-smooth}(X) := X \setminus \mathrm{Smooth}(X)$.

(ii) Note that in the conditions of the definition above the intersection $X \cap U$ is an analytic subset of U, so X_a is an analytic germ.

Theorem VI.4.3 (Jacobian Criterion) *Let $X \subset \Omega$ be an analytic subset of Ω that contains the point a and let d be a non-negative integer. The following conditions are equivalent.*

(i) $\dim(X_a) \geq d$ *and there exist* $f_1, \dots, f_s \in \mathcal{O}_a$ *such that*

$$\mathcal{J}(X_a) = \{f_1, \dots, f_s\}\mathcal{O}_a \quad \textit{and} \quad \mathrm{rk}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(a)\Big)_{\substack{1\leq i\leq s\\ 1\leq j\leq n}} = n - d.$$

(ii) $\mathcal{O}_{X,a}$ *is a regular local ring of dimension* d.

(iii) *There exist* $g_1, \dots, g_{n-d} \in \mathcal{O}_a$ *such that* $\mathcal{J}(X_a) = \{g_1, \dots, g_{n-d}\}\mathcal{O}_a$ *and*

$$\mathrm{rk}\Big(\frac{\partial g_i}{\partial \mathtt{x}_j}(a)\Big)_{\substack{1\leq i\leq n-d\\ 1\leq j\leq n}} = n - d.$$

(iv) *X_a is a smooth germ of dimension d.*

(v) $\dim(X_a) = d$ *and there exist* $g_1, \dots, g_{n-d} \in \mathcal{J}(X_a)$ *such that*

$$\operatorname{rk}\Big(\frac{\partial g_i}{\partial \mathtt{x}_j}(a)\Big)_{\substack{1\le i\le n-d\\ 1\le j\le n}} = n-d.$$

Proof. (i) $\Longrightarrow$ (ii) After reordering the variables and the generators of $\mathcal{J}(X_a)$ we may assume that

$$\det\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(a)\Big)_{1\le i,j\le n-d} \neq 0.$$

Denote $\mathfrak{p} := \{f_1, \dots, f_{n-d}\}\mathcal{O}_a$. By Lemmas III.2.2 and VI.2.16 the quotient $\mathcal{O}_a/\mathfrak{p}$ is a regular local ring of dimension d. In particular, $\mathfrak{p}$ is a prime ideal, $\operatorname{ht}(\mathfrak{p}) = n-d$ and $\operatorname{ht}(\mathcal{J}(X_a)) = n - \dim(X_a) \le n-d = \operatorname{ht}(\mathfrak{p})$. Let $\mathfrak{q}$ be a prime ideal of $\mathcal{O}_a$ containing $\mathcal{J}(X_a)$ such that $\operatorname{ht}(\mathcal{J}(X_a)) = \operatorname{ht}(\mathfrak{q})$. As $\mathfrak{p} \subset \mathcal{J}(X_a) \subset \mathfrak{q}$ and $\operatorname{ht}(\mathfrak{q}) \le \operatorname{ht}(\mathfrak{p})$, we have $\mathfrak{p} = \mathfrak{q}$, so $\mathcal{J}(X_a) = \mathfrak{p}$. Thus, $\mathcal{O}_{X,a} = \mathcal{O}_a/\mathcal{J}(X_a) = \mathcal{O}_a/\mathfrak{p}$ is a regular local ring of dimension d.

(ii) $\Longrightarrow$ (iii) As $\mathcal{O}_{X,a} = \mathcal{O}_a/\mathcal{J}(X_a)$ is a regular local ring of dimension d, there exist by Lemma III.2.8 analytic germs $g_1, \dots, g_n \in \mathcal{O}_a$ such that $\mathfrak{m}_a = \{g_1, \dots, g_n\}\mathcal{O}_a$ and $\mathcal{J}(X_a) = \{g_1, \dots, g_{n-d}\}\mathcal{O}_a$. By Corollary III.2.7 and Lemma VI.2.16,

$$\det\Big(\frac{\partial g_i}{\partial \mathtt{x}_j}(a)\Big)_{1\le i,j\le n} \neq 0 \quad \rightsquigarrow \quad \operatorname{rk}\Big(\frac{\partial g_i}{\partial \mathtt{x}_j}(a)\Big)_{\substack{1\le i\le n-d\\ 1\le j\le n}} = n-d.$$

(iii) $\Longrightarrow$ (iv) As $\mathcal{J}(X_a) = \{g_1, \dots, g_{n-d}\}\mathcal{O}_a$ and

$$\operatorname{rk}\Big(\frac{\partial g_i}{\partial \mathtt{x}_j}(a)\Big)_{\substack{1\le i\le n-d\\ 1\le j\le n}} = n-d,$$

there exists an open neighborhood U of a contained in a common domain of $g_1, \dots, g_{n-d}$ such that

$$X \cap U = Z_U(g_1, \dots, g_{n-d}) \quad \text{and} \quad \operatorname{rk}\Big(\frac{\partial g_i}{\partial \mathtt{x}_j}(p)\Big)_{\substack{1\le i\le n-d\\ 1\le j\le n}} = n-d$$

for each point $p \in U$. Consequently, $X \cap U$ is a d-dimensional smooth submanifold, so X_a is a smooth germ of dimension d.

(iv) $\Longrightarrow$ (v) Let U be an open neighborhood of a such that $X \cap U$ is a d-dimensional smooth submanifold of $\mathbb{K}^n$. After shrinking U if necessary we may assume that there exist $g_1, \ldots, g_{n-d} \in \mathcal{O}(U)$ such that

$$X \cap U = Z_U(g_1, \ldots, g_{n-d}) \quad \text{and} \quad \operatorname{rk}\Big(\frac{\partial g_i}{\partial \mathtt{x}_j}(a)\Big)_{\substack{1\leq i\leq n-d\\ 1\leq j\leq n}} = n-d.$$

By Corollary III.2.3, $\mathfrak{p} := \{g_{1,a}, \ldots, g_{n-d,a}\}\mathcal{O}_a$ is a prime ideal of height $n-d$ and $\mathcal{O}_a/\mathfrak{p}$ is a regular local ring of dimension d and $X_a = \bigcap_{i=1}^{n-d} Z(g_{i,a}) = Z(\mathfrak{p})$. By the Nullstellensatz VI.2.17

$$\mathfrak{p} = \sqrt{\mathfrak{p}} = \mathcal{J}(Z(\mathfrak{p})) = \mathcal{J}(X_a).$$

Consequently, $\dim(X_a) = \dim(\mathcal{O}_a/\mathcal{J}(X_a)) = \dim(\mathcal{O}_a/\mathfrak{p}) = d$.

(v) $\Longrightarrow$ (i) The condition

$$\operatorname{rk}\Big(\frac{\partial g_i}{\partial \mathtt{x}_j}(a)\Big)_{\substack{1\leq i\leq n-d\\ 1\leq j\leq n}} = n-d$$

implies by Corollary III.2.3 that $\mathfrak{p} := \{g_1, \ldots, g_{n-d}\}\mathcal{O}_a \subset \mathcal{J}(X_a)$ is a prime ideal of $\mathcal{O}_a$ with $\operatorname{ht}(\mathfrak{p}) = n-d$. Let $\mathfrak{q}$ be a prime ideal of $\mathcal{O}_a$ containing $\mathcal{J}(X_a)$ such that $\operatorname{ht}(\mathfrak{q}) = \mathcal{J}(X_a)$. Then $\mathfrak{p} \subset \mathfrak{q}$ and

$$\operatorname{ht}(\mathfrak{p}) = n-d = n-\dim(X_a) = n-\dim(\mathcal{O}_a/\mathcal{J}(X_a)) = \operatorname{ht}(\mathcal{J}(X_a)) = \operatorname{ht}(\mathfrak{p}).$$

Therefore, $\mathfrak{p} = \mathfrak{q}$. As $\mathfrak{p} \subset \mathcal{J}(X_a) \subset \mathfrak{q}$, we conclude

$$\mathcal{J}(X_a) = \mathfrak{p} = \{g_1, \ldots, g_{n-d}\}\mathcal{O}_a,$$

as required. $\square$

Definition VI.4.4 Let $X \subset \Omega$ be an analytic set and let $a \in X$. We have proved that a is a d-dimensional smooth point if and only if $\mathcal{O}_{X,a}$ is a d-dimensional regular local ring. This is why the smooth points are called *regular points* in the complex analytic case. We denote $\operatorname{Reg}_d(X) := \operatorname{Smooth}_d(X)$ and recall $\operatorname{Reg}_d(X) \cap \operatorname{Reg}_e(X) = \varnothing$ if $e \neq d$. The set of *regular points of* X is

$$\operatorname{Reg}(X) := \bigsqcup_{d\geq 0} \operatorname{Reg}_d(X) = \bigsqcup_{d\geq 0} \operatorname{Smooth}_d(X) = \operatorname{Smooth}(X).$$

Its complement is the set $\operatorname{Sing}(X) := X \setminus \operatorname{Reg}(X) = \text{Non-smooth}(X)$ of *singular points* of X. It follows from V.3.10 that $\operatorname{Reg}_d(X)$ is an open subset of X for each $d \geq 0$, so also $\operatorname{Reg}(X) = \bigcup_{d\geq 0} \operatorname{Reg}_d(X)$ is open in X.

Remarks VI.4.5 (i) Let $X, Y \subset \Omega$ be analytic sets such that $X_a = Y_a$ and let d be a non-negative integer. Then $(\mathrm{Reg}_d(X))_a = (\mathrm{Reg}_d(Y))_a$.

As $X_a = Y_a$, there exists an open neighborhood U of a such that $X \cap U = Y \cap U$. As

$$\mathrm{Reg}_d(X) \cap U = \mathrm{Reg}_d(X \cap U) = \mathrm{Reg}_d(Y \cap U) = \mathrm{Reg}_d(Y) \cap U,$$

we conclude $(\mathrm{Reg}_d(X))_a = (\mathrm{Reg}_d(Y))_a$.

(ii) We have denoted $\mathrm{Reg}_d(X_a) := (\mathrm{Reg}_d(X))_a$ to emphasize that the germ $\mathrm{Reg}_d(X_a)$ depends only on the germ X_a.

(iii) If $X_a = Y_a$, then $(\mathrm{Reg}(X))_a = (\mathrm{Reg}(Y))_a$.

Indeed,

$$\begin{aligned}(\mathrm{Reg}(X))_a &= \Big(\bigsqcup_{d\geq 0} \mathrm{Reg}_d(X)\Big)_a = \bigsqcup_{d\geq 0} (\mathrm{Reg}_d(X))_a \\ &= \bigsqcup_{d\geq 0} (\mathrm{Reg}_d(Y))_a = \Big(\bigsqcup_{d\geq 0} \mathrm{Reg}_d(Y)\Big)_a = (\mathrm{Reg}(Y))_a.\end{aligned}$$

We denote $\mathrm{Reg}(X_a) := (\mathrm{Reg}(X))_a$. This germ depends only on the germ X_a and not on the analytic set X.

(iv) $\mathrm{Reg}_e(X)$ is an open and closed subset of $\mathrm{Reg}(X)$ for each $e \geq 0$.

The equality $\mathrm{Reg}_e(X) = \mathrm{Smooth}_e(X)$ shows that $\mathrm{Reg}_e(X)$ is an open subset of $\mathrm{Reg}(X)$. In addition,

$$\mathrm{Reg}_e(X) = \mathrm{Reg}(X) \setminus \bigsqcup_{k\neq e} \mathrm{Reg}_k(X),$$

so $\mathrm{Reg}_e(X)$ is closed in $\mathrm{Reg}(X)$.

Corollary VI.4.6 *Let $X \subset \Omega$ be an analytic set and $a \in \mathrm{Reg}_d(X)$ for some integer $d \geq 0$. Then X_a is an irreducible germ.*

Proof. By the Jacobian Criterion X_a is a d-dimensional regular germ. Thus, $\mathcal{O}_a/\mathcal{J}(X_a)$ is a regular local ring of dimension d and by Lemma III.1.8 $\mathcal{O}_a/\mathcal{J}(X_a)$ is isomorphic to $\mathcal{O}_d$, which is an integral domain. Consequently, $\mathcal{J}(X_a)$ is a prime ideal, so X_a is an irreducible germ. □

Remark VI.4.7 Let $f \in \mathcal{O}_a$ be a complex function germ and denote

$$J(f) := \Big\{\frac{\partial f}{\partial \mathtt{x}_1}(a), \ldots, \frac{\partial f}{\partial \mathtt{x}_n}(a)\Big\}\mathcal{O}_a.$$

By Lemma IV.3.3 the ideal $f\mathcal{O}_a$ has an isolated singularity if and only if $\sqrt{J(f)} = \mathfrak{m}_a$. This means that $Z(J(f)) = \{a\}$ and provides a geometrical explanation of the name *isolated singularity.*

4.a Analytic maps with discrete fibers. The next result provides a useful characterization of those complex analytic maps with discrete fibers.

Theorem VI.4.8 *Let $\Omega \subset \mathbb{C}^n$ be an open set and let $X \subset \Omega$ be an analytic subset of Ω that contains the origin. Let*

$$g : \Omega \to \mathbb{C}^d,\ x \mapsto (g_1(x), \ldots, g_d(x))$$

be an analytic map such that $g(0) = 0$ and let $\varphi := \pi \circ g^*$, where

$$g^* : \mathcal{O}_d \to \mathcal{O}_n,\ f \mapsto f(g_1, \ldots, g_d)$$
$$\pi : \mathcal{O}_n \to \mathcal{O}_{X,0} := \mathcal{O}_n/\mathcal{J}(X_0),\ f \mapsto f + \mathcal{J}(X_0).$$

The following conditions are equivalent:

(i) *The origin $0 \in \mathbb{C}^n$ is an isolated point of $X \cap g^{-1}(0)$.*

(ii) *The analytic homomorphism φ is finite.*

Proof. The proof is conducted in several steps:

4.a.1 *Initial preparation.* We identify both rings $\mathcal{O}_0$ and $\mathcal{O}_n$ via the isomorphism $T : \mathcal{O}_0 \to \mathcal{O}_n$, $f_0 \mapsto T_0 f$ provided by Taylor's expansion VI.2.5. Let $f_1, \ldots, f_r$ be analytic functions defined in an open neighborhood of $0 \in \mathbb{C}^n$ such that $\mathcal{J}(X_0) = \{f_{1,0}, \ldots, f_{r,0}\}\mathcal{O}_0$. Consequently,

$$(X \cap g^{-1}(0))_0 = Z(f_{1,0}, \ldots, f_{r,0}, g_{1,0}, \ldots, g_{d,0})$$

and by the Nullstellensatz VI.2.17

$$\begin{aligned}\mathcal{J}((X \cap g^{-1}(0))_0) &= \mathcal{J}(Z(f_{1,0}, \ldots, f_{r,0}, g_{1,0}, \ldots, g_{d,0}))\\ &= \sqrt{\{f_{1,0}, \ldots, f_{r,0}, g_{1,0}, \ldots, g_{d,0}\}\mathcal{O}_0} = \sqrt{\mathcal{J}(X_0) + \{g_{1,0}, \ldots, g_{d,0}\}\mathcal{O}_0}.\end{aligned}$$

Now, 0 is an isolated point of $X \cap g^{-1}(0)$ if and only if $(X \cap g^{-1}(0))_0 = \{0\}_0$, or equivalently, $\mathcal{J}((X \cap g^{-1}(0)))_0 = \mathcal{J}(\{0\}_0) = \mathfrak{m}_0$, that is,

$$\sqrt{\mathcal{J}(X_0) + \{g_{1,0}, \ldots, g_{d,0}\}\mathcal{O}_0} = \mathfrak{m}_0. \tag{4.1}$$

Let $\mathfrak{m}_A$ be the maximal ideal of $A := \mathcal{O}_{X,0}$. Then (4.1) is equivalent to

$$\sqrt{\mathcal{J}(X_0) + \{g_{1,0}, \ldots, g_{d,0}\}\mathcal{O}_0}\Big/\mathcal{J}(X_0) = \mathfrak{m}_0/\mathcal{J}(X_0) = \mathfrak{m}_A.$$

If $\mathfrak{m}_d := \{\mathtt{x}_1, \ldots, \mathtt{x}_d\}$ denotes the maximal ideal of $\mathcal{O}_d$, we have

$$\begin{aligned}\varphi(\mathfrak{m}_d)\mathcal{O}_{X,0} &= \{\varphi(\mathtt{x}_1), \ldots, \varphi(\mathtt{x}_d)\}\mathcal{O}_{X,0} \\ &= \{g_{1,0} + \mathcal{J}(X_0), \ldots, g_{d,0} + \mathcal{J}(X_0)\}\mathcal{O}_{X,0} \\ &= (\{g_{1,0}, \ldots, g_{d,0}\}\mathcal{O}_0 + \mathcal{J}(X_0))/\mathcal{J}(X_0).\end{aligned}$$

4.a.2 Let us prove (i) $\Longrightarrow$ (ii). As $0 \in \mathbb{C}^n$ is an isolated point of $X \cap g^{-1}(0)$,

$$\mathfrak{m}_A = \sqrt{\mathcal{J}(X_0) + \{g_{1,0}, \ldots, g_{d,0}\}\mathcal{O}_0}\Big/\mathcal{J}(X_0) = \sqrt{\varphi(\mathfrak{m}_d)\mathcal{O}_{X,0}}$$

and this implies by Lemma III.1.5 that φ is a finite homomorphism.

4.a.3 To prove (ii) $\Longrightarrow$ (i) suppose that φ is a finite homomorphism. By Lemma III.1.5

$$\sqrt{(\{g_{1,0}, \ldots, g_{d,0}\}\mathcal{O}_0 + \mathcal{J}(X_0))/\mathcal{J}(X_0)} = \sqrt{\varphi(\mathfrak{m}_d)\mathcal{O}_{X,0}} = \mathfrak{m}_A = \mathfrak{m}_0/\mathcal{J}(X_0).$$

We conclude

$$\sqrt{\mathcal{J}(X_0) + \{g_{1,0}, \ldots, g_{d,0}\}\mathcal{O}_0} = \mathfrak{m}_0,$$

so 0 is an isolated point of $X \cap g^{-1}(0)$. □

Remark VI.4.9 The implication (i) $\Longrightarrow$ (ii) is not true if $\mathbb{K} = \mathbb{R}$. Consider $\Omega := X := \mathbb{R}^2$ and the analytic function $g : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x^2 + y^2$. Notice that $g(0, 0) = 0$ and $g^{-1}(0) = \{(0, 0)\}$. Thus, the origin $(0, 0)$ in $\mathbb{R}^2$ is an isolated point of $X \cap g^{-1}(g(0, 0))$ and $\mathcal{J}(X_0) = \{0\}$. The homomorphism $\varphi : \mathcal{O}_1 := \mathbb{R}\{\mathtt{t}\} \to \mathcal{O}_2 := \mathbb{R}\{\mathtt{x}, \mathtt{y}\}$ satisfies $\varphi(\mathtt{t}) = g_0$, so $\varphi(\mathfrak{m}_1)\mathcal{O}_2 = g\mathcal{O}_2$. The analytic series $\mathtt{x}^2 + \mathtt{y}^2$ is irreducible in $\mathbb{R}\{\mathtt{x}, \mathtt{y}\}$ because $\mathbb{R}\{\mathtt{x}, \mathtt{y}\}$ and $\mathbb{C}\{\mathtt{x}, \mathtt{y}\}$ are unique factorization domains and the factorization of $\mathtt{x}^2 + \mathtt{y}^2$ as a product of irreducible factors in $\mathbb{C}\{\mathtt{x}, \mathtt{y}\}$ is

$$\mathtt{x}^2 + \mathtt{y}^2 = (\mathtt{x} + \sqrt{-1}\mathtt{y})(\mathtt{x} - \sqrt{-1}\mathtt{y}).$$

Thus, $g\mathcal{O}_2$ is a prime ideal, so it is a radical ideal. If the homomorphism φ is finite, $\{\mathtt{x},\mathtt{y}\}\mathcal{O}_2 = \mathfrak{m}_2 = \sqrt{g_0\mathcal{O}_2} = g_0\mathcal{O}_2$. In particular, $\mathtt{x} = (\mathtt{x}^2+\mathtt{y}^2)f$ for some $f \in \mathcal{O}_2$, which is impossible because $\omega(\mathtt{x}) = 1$ and $\omega((\mathtt{x}^2+\mathtt{y}^2)f) \geq 2$.

Exercises

Number VI.1 Check that the definitions of union, intersection and difference of germ sets are consistent.

Number VI.2 Let $a \in \mathbb{K}^n$ and let $\mathfrak{a}$ be an ideal of $\mathcal{O}_a$. Prove that the germs $Z(\mathfrak{a})$ and $Z(\sqrt{\mathfrak{a}})$ coincide.

Number VI.3 Let $\Omega \subset \mathbb{K}^n$ be an open subset and let $f \in \mathcal{O}(\Omega)$. Prove that for each point $a \in Z_\Omega(f)$ there exists an open neighborhood $U \subset \Omega$ of a such that

$$Z_U\big(\tfrac{\partial f}{\partial \mathtt{x}_1},\dots,\tfrac{\partial f}{\partial \mathtt{x}_n}\big) \subset Z_U(f).$$

Number VI.4 Let $\Omega \subset \mathbb{C}^n$ be an open neighborhood of the origin of $\mathbb{C}^n$ and let $f, g \in \mathcal{O}(\Omega)$ be such that $Z(f_0) \subset Z(g_0)$. Prove that there exist a compact neighborhood $K \subset \Omega$ of the origin, a positive integer m and a positive real number c such that $|g(x)^m| \leq c \cdot |f(x)|$ for each point $x \in K$.

Number VI.5 Let $X_0 \subset \mathbb{C}^n_0$ be an analytic germ and let $f \in \mathcal{O}_0 \setminus \mathcal{J}(X_0)$. Prove that there exist an open neighborhood $\Omega \subset \mathbb{C}^n$ of the origin and $h \in \mathcal{O}(\Omega)$ such that $h(0) = 0$ and if $Y := h(\Omega)$, the following conditions hold: $Y_0 \subset X_0$ and $Y_0 \not\subset Z(f_0)$.

Number VI.6 Determine the values of $\varepsilon \in \mathbb{C}$ such that the analytic germ X_a is irreducible, where $a := (0,0,\varepsilon)$ and $X := \{(x,y,z) \in \mathbb{C}^3 : x^2 = zy^2\}$.

Number VI.7 Prove that $\mathbb{K}^n$ is the unique n-dimensional analytic subset of $\mathbb{K}^n$.

Number VI.8 Let $a \in \mathbb{K}^n$ and let Y_a be a connected smooth d-dimensional analytic germ. Let X_a be an analytic germ such that $Y_a \not\subset X_a$. Prove that $Y_a \setminus X_a$ is dense in Y_a.

Number VI.9 Let $a \in \mathbb{C}^n$ and let X_a be an analytic germ. Prove that the following statements are equivalent:

(i) There exists $f_a \in \mathcal{O}_a \setminus \{0\}$ such that $f_a(a) = 0$ and $X_a = Z(f_a)$.

(ii) The irreducible components of X_a are $(n-1)$-dimensional.

Is it true that (i) implies (ii) in the real case? What happens with the converse implication (ii) $\implies$ (i)?

Number VI.10 Let $\Omega \subset \mathbb{C}^n$ be an open neighborhood of the origin and let $X \subset \Omega$ be an analytic subset of Ω such that $0 \in X$.

(i) For every linear subspace H of $\mathbb{C}^n$ let us denote $\dim_{\mathbb{C}}(H)$ the dimension of H as a $\mathbb{C}$-linear space. Prove that for each linear subspace H of $\mathbb{C}^n$ such that the origin is an isolated point of $X \cap H$, we have $\dim(X_0) \leq n - \dim_{\mathbb{C}}(H)$.

(ii) Show that if $d := \dim(X_0)$, there exists a linear subspace H of $\mathbb{C}^n$ such that 0 is an isolated point of $X \cap H$ and $\dim_{\mathbb{C}}(H) = n - d$.

Number VI.11 Let $f := \mathtt{x}^2 + \mathtt{y}^2 - \mathtt{z}^2$, $g := \mathtt{x}^3 - \mathtt{yz}$ and $h := \mathtt{xy} - \mathtt{z}$. Consider the analytic sets $X := Z_{\mathbb{C}^3}(f)$ and $Y := Z_{\mathbb{C}^3}(g, h)$. Compute $\dim(X_0)$ and $\dim(Y_0)$.

Number VI.12 Let $H \subset \mathbb{K}^n$ be a linear subspace. Prove that H_a is for each point $a \in H$ a d-dimensional smooth germ where d is the dimension of H as a linear space. Prove that H_a is an irreducible germ.

Number VI.13 Let $Y := \{\mathtt{x} = 0\} \subset \mathbb{R}^2$ and let $X_n := \{\mathtt{y} = n\mathtt{x}\} \subset \mathbb{R}^2$ for $n \geq 0$. Define

$$X := Y \cup \bigcup_{n \geq 0} X_n, \quad \Omega := \mathbb{R}^2 \setminus Y \quad \text{and} \quad Z := X \cap \Omega.$$

Is X an analytic subset of $\mathbb{R}^2$? Is Z an analytic subset of Ω?

Number VI.14 Let $f \in \mathbb{C}[\mathtt{x}_1, \ldots, \mathtt{x}_n]$ a polynomial without multiple factors and define $X := Z_{\mathbb{C}^n}(f)$. Prove that:

(i) $\mathcal{J}(X_a) = f_a \mathcal{O}_a$ for each point $a \in X$.

(ii) $\operatorname{Sing}(X) = \left\{a \in X : \frac{\partial f}{\partial \mathtt{x}_j}(a) = 0 \ \text{ for } j = 1, \ldots, n\right\}$.

Number VI.15 (i) Determine the set of singular points of the complex analytic set

$$X := \{\mathtt{x}^2 + \mathtt{x}^3 = \mathtt{y}^2\} \cup \{\mathtt{x} = 1\} \subset \mathbb{C}^2.$$

(ii) Prove that $\operatorname{Sing}(X)$ is an analytic subset of $\mathbb{C}^2$ and determine $\operatorname{Sing}(\operatorname{Sing}(X))$.

Number VI.16 Let Ω be an open subset of $\mathbb{C}^n$ and let $X \subset \Omega$ be an analytic subset of Ω. Let $f : \Omega \to \mathbb{C}^r$ be an analytic map and let $a \in X$ be an isolated point of $X \cap f^{-1}(f(a))$. Prove that there exist basis of neighborhoods $\{U_n\}_{n \in \mathbb{N}}$ of a and $\{W_n\}_{n \in \mathbb{N}}$ of $f(a)$ such that each restriction $f|_{X \cap U_n} : X \cap U_n \to W_n$ is a proper map.

CHAPTER VII

Analytic coverings and Local parameterization

The first section of this chapter has a purely topological nature. It is devoted to study topological coverings with finitely many sheets. This leads to introduce quasianalytic and analytic coverings in Sections 2 and 3. Roughly speaking they are topological coverings outside a thin subset of the target space and in the optimal situation its regular locus is dense and connected. This is the case when the cover space is an small enough neighborhood of a point a of a complex analytic set X if the germ X_a is irreducible. In addition, if $d = \dim(X_a)$, the covering projection can be chosen as a linear projection onto $\mathbb{C}^d$. If X_a is not irreducible but equidimensional, a less demanding result is obtained: only the connection of the regular locus of the covering is lost. This is the main result developed in Sections 4 and 5 and it is usually known as Local parameterization theorem.

Among the consequences of the Local parameterization it is worthwhile mentioning that every complex analytic set is locally connected, its subset of regular points is dense and the intersection of an arbitrary family of analytic sets is again an analytic set. We prove in addition a particular case of the Open mapping Theorem: *those analytic maps* $\Omega \subset \mathbb{C}^n \to \mathbb{C}^n$ *whose fibers are discrete sets are open maps.* We refer the reader to Exercise number V.2 for the one-dimensional case.

The last section is mainly devoted to study the dimension of the intersection of complex analytic germs. The best result we obtain states that given analytic subsets X and Y of an open subset of $\mathbb{C}^n$ such that the germs X_a and Y_a are equidimensional, the dimension of each irreducible component of $X_a \cap Y_a$ is greater than or equal to $\dim(X_a) + \dim(Y_a) - n$.

The prerequisites have topological nature and we recommend [K].

1 Topological coverings

We begin by recalling the definition of topological covering.

Definition VII.1.1 Let $\pi : X \to Y$ be a continuous map between topological spaces X and Y. The triple (X, π, Y) is a *topological covering* if for each point $y \in Y$ there exist an open neighborhood $W \subset Y$ of y, an integer $s := s(y) \geq 1$ and disjoint open subsets $U_1, \ldots, U_s$ of X such that $\pi^{-1}(W) = \bigsqcup_{i=1}^s U_i$ and $\pi|_{U_i} : U_i \to W$ is a homeomorphism for $i = 1, \ldots, s$. If $s : W \to \mathbb{Z}^+$, $y \mapsto s(y)$ is a constant function of value s, the triple (X, π, Y) is an *s-sheeted topological covering.* If each connected component X_i of X is homeomorphic to Y via the corresponding restriction of $\pi|_{X_i} : X_i \to Y$, we say that the topological covering is *trivial.*

Remarks VII.1.2 Let (X, π, Y) be a topological covering.

(i) The function $\mathtt{s} : Y \to \mathbb{Z}$, $y \mapsto \#(\pi^{-1}(y))$ is locally constant and, consequently, continuous.

Pick a point $y_0 \in Y$. There exist an open neighborhood $W \subset Y$ of y_0 and open subsets $U_1, \ldots, U_s$ in X, where $s := \mathtt{s}(y_0)$, such that $\pi^{-1}(W) = \bigsqcup_{i=1}^s U_i$ and $\pi|_{U_i} : U_i \to W$ is a homeomorphism. Thus, $\pi^{-1}(y) \subset \pi^{-1}(W) = \bigsqcup_{j=1}^s U_j$ for each $y \in W$. As $\pi|_{U_j} : U_j \to W$ is a bijection, the intersection $U_j \cap \pi^{-1}(y)$ is a singleton. Thus, $\mathtt{s}(y) = s = \mathtt{s}(y_0)$ for all $y \in W$, so $\mathtt{s}|_W$ is constant. As continuity is a local property, $\mathtt{s}$ is continuous.

(ii) If Y is connected then the cardinality of the fibers of π is constant.

As Y is connected and $\mathtt{s}$ is a continuous function, the image $\mathtt{s}(Y) \subset \mathbb{Z}$ is connected. Thus, $\mathtt{s}$ is constant, as claimed.

1.a Basic properties of topological coverings.

We prove next some basic properties of topological coverings that will be useful in the sequel.

Proposition VII.1.3 *Let (X, π, Y) be a topological covering, let $\Omega \subset Y$ be an open subset of Y and let $Z \subset X$ be an open and closed subset of X. Then*

(i) *The fibers of π are finite and non-empty. Moreover, π is a local homeomorphism and proper map. In particular it is an open map.*

(ii) *The triple $(\pi^{-1}(\Omega), \pi|_{\pi^{-1}(\Omega)}, \Omega)$ is a topological covering.*

(iii) *The triple* $(Z, \pi|_Z, \pi(Z))$ *is a topological covering and, if* Y *is connected, then* $\pi(Z) = Y$.

(iv) *Suppose that* X*is locally connected and* Y *is connected. Then the set of connected components of* X *is finite and not larger than the number of sheets of the covering.*

Proof. (i) We prove first: *the fibers of* π *are finite.*

Pick a point $y \in Y$. With the notations of Definition VII.1.1 $\pi^{-1}(y)$ is contained in $\pi^{-1}(W) = \bigsqcup_{i=1}^{s(y)} U_i$. As $\pi|_{U_i} : U_i \to W$ is bijective, each intersection $\pi^{-1}(y) \cap U_i$ consists of a unique point. Thus, $\pi^{-1}(y)$ consists on $s(y)$ points.

We check now: π *is a local homeomorphism* and consequently an open map.

Let $x \in X$ and denote $y := \pi(x)$. With the notations of VII.1.1 there exists an index $j = 1, \ldots, s(y)$ such that $x \in U_j$ and $\pi|_{U_j} \to W$ is a homeomorphism, so π is a local homeomorphism.

As π is a continuous map with finite fibers, to prove that it is proper it is enough to show: π *is a closed map.* Let $C \subset X$ be a closed subset of X. We claim: $Y \setminus \pi(C)$ *is an open subset of* Y.

Pick $y \in Y \setminus \pi(C)$ and let $W \subset Y$ be an open neighborhood of y such that there exist an integer $s \geq 1$ and disjoint open subsets $U_1, \ldots, U_s$ of X satisfying $\pi^{-1}(W) = \bigsqcup_{i=1}^{s} U_i$ and $\pi|_{U_i} : U_i \to W$ is a homeomorphism for $i = 1, \ldots, s$. Each difference $U_j \setminus C$ is an open subset of U_j, so $\pi(U_j \setminus C)$ is an open subset of Y (because π is an open map). Thus, the intersection $V := \bigcap_{j=1}^{s} \pi(U_j \setminus C)$ is an open subset of Y. All reduces to check: $y \in V \subset Y \setminus \pi(C)$.

If $y \notin V$, there exists an index $j = 1, \ldots, s$ such that $y \notin \pi(U_j \setminus C)$. Then the unique point x_j in $\pi^{-1}(y) \cap U_j$ does not belong to $U_j \setminus C$, so $x_j \in C$ and $y = \pi(x_j) \in \pi(C)$, which is false.

We check now: $V \subset Y \setminus \pi(C)$. For each point $q \in V \subset W$ consider its fiber $\pi^{-1}(q) \subset \pi^{-1}(W) = \bigsqcup_{j=1}^{s} U_j$ and let $p_j := \pi^{-1}(q) \cap U_j$. Therefore $\pi^{-1}(q) = \{p_1, \ldots, p_s\}$. Suppose by contradiction that $q \in \pi(C)$. Then $p_j \in C$ for some $j = 1, \ldots, s$. As $\pi^{-1}(q) \cap U_j = \{p_j\}$, we conclude $q \notin \pi(U_j \setminus C)$, which is false because $q \in V$.

(ii) Let $y \in \Omega$ and, with the notations of VII.1.1, define $W_1 := W \cap \Omega$, which is an open neighborhood of y in Ω. The disjoint open subsets $V_j := U_j \cap \pi^{-1}(\Omega)$ of $\pi^{-1}(\Omega)$ satisfy $\pi^{-1}(W_1) = \bigsqcup_{i=j}^{s} V_j$ and each restriction map $\pi|_{V_j} : V_j \to W_1$

is a homeomorphism because it is obviously bijective, continuous and open. Thus, $(\pi^{-1}(\Omega), \pi|_{\pi^{-1}(\Omega)}, \Omega)$ is a topological covering.

(iii) As Z is open and π is an open map, $T := \pi(Z)$ is an open subset of Y. Pick a point $y \in T$ and let $s := s(y)$ as in VII.1.1. Let $W \subset T$ be an open neighborhood of y whose preimage $\pi^{-1}(W) = \bigsqcup_{j=1}^{s} U_j$ for some open subsets $U_1, \dots, U_s$ of X such that the restrictions $\pi|_{U_k} : U_k \to W$ are homeomorphisms. Write $\pi^{-1}(y) := \{x_1, \dots, x_s\}$ where $x_j \in U_j$. As $y \in \pi(Z)$, we can suppose, after reordering the indices, that $x_1, \dots, x_r \in Z$ and $x_{r+1}, \dots, x_s \in X \setminus Z$ for some integer $r \geq 1$. Define

$$V := \bigcap_{i=1}^{r} \pi(Z \cap U_i) \cap \Big(W \setminus \bigcup_{j=r+1}^{s} \pi(Z \cap U_j)\Big),$$

which is an open subset of W, because each intersection $Z \cap U_k$ is open and closed in U_k and $\pi|_{U_k} : U_k \to W$ a homeomorphism.

Note that $y = \pi(x_1) \in V \subset \pi(Z \cap U_1) \subset \pi(Z) \subset T$, so V is an open neighborhood of y in T. For $i = 1, \dots, r$ the set $V_i := Z \cap U_i \cap \pi^{-1}(V)$ is an open subset of Z and if $i \neq j$ we have $V_i \cap V_j \subset U_i \cap U_j = \varnothing$.

1.a.1 It is enough to prove: $(\pi|_Z)^{-1}(V) = \bigsqcup_{i=1}^{r} V_i$ *and* $\pi|_{V_i} : V_i \to V$ *is a homeomorphism for each index* $i = 1, \dots, r$.

For the non-obvious inclusion pick $z \in (\pi|_Z)^{-1}(V)$. Then

$$\pi(z) \in Y \setminus \pi(Z \cap U_j)$$

for $j = r+1, \dots, s$, so $z \notin \bigcup_{j=r+1}^{s}(Z \cap U_j)$. In addition, $\pi(z) \in W$, so $z \in \bigcup_{i=1}^{s}(Z \cap U_i)$ and there exists $i = 1, \dots, r$ such that $z \in Z \cap U_i$. Thus, $z \in Z \cap U_i \cap \pi^{-1}(V) = V_i$.

Each restriction $\pi|_{V_i} : V_i \to V$ is an injective and continuous map because $\pi|_{U_i} : U_i \to W$ is so. In addition, V_i is open in U_i because $Z \cap \pi^{-1}(V)$ is open in X. Thus, $\pi|_{V_i} : V_i \to V$ is an injective, continuous, open map and let us check that it is surjective. For each $y \in V \subset \pi(Z \cap U_i)$, there exists $z' \in Z \cap U_i$ such that $\pi(z') = y$. As $z' \in \pi^{-1}(V)$, we conclude $\pi(V_i) = V$.

For the second part of the statement note that $\pi(Z)$ is open and closed in Y because Z is open and closed in X and $\pi : X \to Y$ is an open and closed map. Hence, $\pi(Z) = Y$ whenever Y is connected.

(iv) As X is locally connected, its connected components $\{X_i\}_{i \in I}$ are open and closed subsets of X. As Y is connected, it follows from (iii) that $\pi(X_i) = Y$

for each $i \in I$. Pick a point $y \in Y$ and let $s := \#(\pi^{-1}(y))$. As $\pi(X_i) = Y$, the point y has at least one preimage in each X_i, so $\#(I) \leq s$. Thus, the number of connected components of X is less than or equal to the number of sheets of the covering. $\square$

Proposition VII.1.4 *Let (X, π, W) be an s-sheeted topological covering such that W is a connected open subset of $\mathbb{C}^n$. Let A be a thin subset of W. Then $X \setminus \pi^{-1}(A)$ is dense in X and, if X is connected, then $X \setminus \pi^{-1}(A)$ is connected.*

Proof. We prove first: $X \setminus \pi^{-1}(A)$ *is dense in* X.

Let $U \subset X$ be a non-empty open subset. As $\pi : X \to W$ is by Proposition VII.1.3 an open map, $\pi(U)$ is a non-empty open subset of W. As $W \setminus A$ is dense in W, we have $\pi(U) \cap (W \setminus A) \neq \varnothing$, so $U \cap (X \setminus \pi^{-1}(A)) \neq \varnothing$.

Suppose now by contradiction: X is connected but $X \setminus \pi^{-1}(A)$ in not connected.

Then $X \setminus \pi^{-1}(A) = U_1 \sqcup U_2$, where U_1 and U_2 are non-empty disjoint open subsets of $X \setminus \pi^{-1}(A)$. Consequently,

$$X = \mathrm{Cl}_X(X \setminus \pi^{-1}(A)) = \mathrm{Cl}_X(U_1 \cup U_2) = \mathrm{Cl}_X(U_1) \cup \mathrm{Cl}_X(U_2).$$

As X is connected, $\mathrm{Cl}_X(U_1) \cap \mathrm{Cl}_X(U_2) \neq \varnothing$. Pick $x_0 \in \mathrm{Cl}_X(U_1) \cap \mathrm{Cl}_X(U_2)$. As π is a local homeomorphism and W is locally connected, there exists a connected open neighborhood V of x_0 in X such that $\pi(V)$ is an open subset of W and $\pi|_V : V \to \pi(V)$ is a homeomorphism. As A is a thin subset of W, the intersection $\pi(V) \cap A$ is a thin subset of $\pi(V)$. Thus, the difference

$$\pi(V) \setminus A = \pi(V) \setminus (\pi(V) \cap A)$$

is by Remark V.5.5 connected. As $\pi|_V : V \to \pi(V)$ is a homeomorphism, also $V \setminus \pi^{-1}(A) = \pi^{-1}(\pi(V) \setminus A) \cap V$ is connected. However, $V \cap U_i$ is an open subset of $V \setminus \pi^{-1}(A)$,

$$V \setminus \pi^{-1}(A) = V \cap (X \setminus \pi^{-1}(A)) = (V \cap U_1) \sqcup (V \cap U_2)$$

and $V \cap U_i \neq \varnothing$ because $x_0 \in \mathrm{Cl}_X(U_i)$. So $V \setminus \pi^{-1}(A)$ is not connected, which is a contradiction. $\square$

Next, we state a purely topological criterion to determine that certain families of neighborhoods of a point constitute a basis of neighborhoods of such point.

Lemma VII.1.5 *Let Y be a compact topological space and let $\{U_m\}_{m\in\mathbb{N}}$ be a family of open sets in Y such that $\mathrm{Cl}(U_{m+1}) \subset U_m$ and $\bigcap_{m\in\mathbb{N}} U_m = \{y\}$. Then $\{U_m\}_{m\in\mathbb{N}}$ is a basis of neighborhoods of the point y.*

Proof. Suppose by contradiction that there exists an open neighborhood $W \subset Y$ of y such that $U_m \setminus W$ is non-empty for each $m \in \mathbb{N}$. Thus, also $\mathrm{Cl}(U_m) \setminus W \neq \varnothing$. The family of closed subsets $\mathcal{C} := \{\mathrm{Cl}(U_m) \setminus W\}_{m\in\mathbb{N}}$ of Y has the finite-intersection property:

$$\bigcap_{m=1}^{r} \mathrm{Cl}(U_m) \setminus W = \mathrm{Cl}(U_r) \setminus W \neq \varnothing.$$

As Y is compact and $\mathrm{Cl}(U_{m+1}) \subset U_m$,

$$\varnothing \neq \bigcap_{m\in\mathbb{N}} \mathrm{Cl}(U_m) \setminus W \subset \bigcap_{m\in\mathbb{N}} \mathrm{Cl}(U_m) = \bigcap_{m\in\mathbb{N}} U_m = \{y\},$$

so $y \notin W$, which is a contradiction. □

2 Quasianalytic coverings

Let us recall the concept of quasianalytic covering.

Definitions VII.2.1 A *quasianalytic covering* is a triple (X, π, W) satisfying the following properties:

(i) X is a Hausdorff and locally compact topological space.

(ii) W is a connected subset of $\mathbb{C}^n$ for some $n \geq 1$.

(iii) The map $\pi : X \to W$ is proper, surjective and has finite fibers.

(iv) There exists a thin subset $A \subset W$ such that the triple

$$(X \setminus \pi^{-1}(A), \pi|_{X\setminus\pi^{-1}(A)}, W \setminus A)$$

is an s-sheeted topological covering for some $s \in \mathbb{N}$. The number s is called the *number of sheets* of (X, π, W).

We say that A is *a critical set* of the quasicovering (X, π, W) and the difference $X^\circ := X \backslash \pi^{-1}(A)$ is its associated *regular locus*. In addition, if X° is connected, the quasianalytic covering (X, π, W) is *irreducible*, while if X° is dense in X, the quasianalytic covering (X, π, W) is called *analytic*.

Example VII.2.2 Consider $X := \{(\mathtt{x}+\mathtt{y})(\mathtt{x}-\mathtt{y}) = 0\} \subset \mathbb{C}^2$ and denote $\pi : X \to \mathbb{C}$, $(x, y) \mapsto x$ the projection onto the first coordinate. We claim: $(X, \pi, \mathbb{C})$ *is an analytic covering.*

Define $Y := \{\mathtt{x} \neq 0, \mathtt{y} = -1\} \subset \mathbb{C}^2$ and $Z := \{\mathtt{x} \neq 0, \mathtt{y} = 1\} \subset \mathbb{C}^2$.

(i) The topological spaces $X \setminus \{(0,0)\}$ and $Y \sqcup Z$ are homeomorphic via the map

$$\varphi : X \setminus \{(0,0)\} \to Y \sqcup Z,\ (x, y) \mapsto \big(x, \tfrac{y}{x}\big)$$

whose inverse is $\varphi^{-1} : Y \sqcup Z \to X \setminus \{(0,0)\}$, $(u, v) \mapsto (u, uv)$. Let

$$\rho : Y \sqcup Z \to \mathbb{C} \setminus \{0\},\ (u, v) \mapsto u.$$

Then $(Y \sqcup Z, \rho, \mathbb{C} \setminus \{0\})$ is a two-sheeted topological covering. As $\rho \circ \varphi = \pi$, the triple $(X \setminus \{(0,0)\}, \pi|_{X \setminus \{p\}}, \mathbb{C} \setminus \{0\})$ is a two-sheeted topological covering.

(ii) As $A := \{0\}$ is a proper analytic subset of $\mathbb{C}$, it is by Corollary V.5.6 a thin subset of $\mathbb{C}$. Also, $X \setminus \pi^{-1}(A) = X \setminus \{(0,0)\}$ is a dense subset of X.

(iii) To finish we check that $\pi : X \to \mathbb{C}$, which is clearly a continuous surjective map with finite fibers, is proper. It is enough to show: π *is closed.*

Write $X_\varepsilon := \{\mathtt{y} = \varepsilon\mathtt{x}\}$ for $\varepsilon = \pm 1$. We have $X = X_1 \cup X_{-1}$ and the restriction maps $\pi_\varepsilon := \pi|_{X_\varepsilon} : X_\varepsilon \to \mathbb{C}$, $(x, y) \mapsto x$ are homeomorphisms. Let C be a closed subset of X. We want to check: $\pi(C)$ is a closed subset of $\mathbb{C}$.

As $C_\varepsilon := C \cap X_\varepsilon$ is a closed subset of X_ε, its image $\pi_\varepsilon(C_\varepsilon)$ is a closed subset of $\mathbb{C}$. Consequently, $\pi(C) = \pi_{-1}(C_{-1}) \cup \pi_1(C_1)$ is a closed subset of $\mathbb{C}$, as required.

2.a Basic properties of quasi-analytic and analytic coverings. We have used the expression A *is a critical set* in the definition of quasianalytic covering and not A is *the critical* set of the covering. The reason is that we can modify suitably any given critical set. As the definitions of irreducible quasicovering and analytic covering depend on the chosen critical set A, we must prove that the connectedness and the density of the difference $X \setminus \pi^{-1}(A)$ are independent of A.

Proposition VII.2.3 *Let (X, π, W) be an s-sheeted analytic covering and let $B := \{y \in W : \#(\pi^{-1}(y)) \neq s\}$. We have:*

(i) *If $A \subset W$ is a critical set of (X, π, W), then $B \subset A$.*

(ii) *$\#(\pi^{-1}(y)) < s$ for each point $y \in B$. In addition, $\#(\pi^{-1}(y)) \leq s$ for each point $y \in W$.*

(iii) *B is a thin subset of W.*

(iv) *If $A \subset W$ is a critical set of (X, π, W) then $A = B \cup C$ for some thin subset C of $W \setminus B$.*

(v) *The triple $(X \setminus \pi^{-1}(B), \pi|_{X\setminus\pi^{-1}(B)}, W \setminus B)$ is an s-sheeted topological covering.*

Proof. (i) The triple $(X\setminus\pi^{-1}(A), \pi|_{X\setminus\pi^{-1}(A)}, W\setminus A)$ is an s-sheeted topological covering, so $s = \#(\pi^{-1}(y))$ for each point $y \in W \setminus A$. Thus, $(W \setminus A) \cap B = \varnothing$, so $B \subset A$.

2.a.1 Before proving (ii) and (iii) we need the following construction. As the fibers of π are finite, for each point $y \in W$ let us denote $m := \#(\pi^{-1}(y))$ and write $\pi^{-1}(y) := \{x_1, \dots, x_m\}$. As X is a Hausdorff space, there exist pairwise disjoint open neighborhoods $U_j \subset X$ of x_j for $j = 1, \dots, m$. Therefore $C := X \setminus \bigcup_{j=1}^m U_j$ is a closed subset of X. As π is proper, $\pi(C)$ is a closed subset of W, which does not contain y. Thus, $V_0 := W \setminus \pi(C) \subset W$ is an open neighborhood of y such that

$$\pi^{-1}(V_0) = X \setminus \pi^{-1}(\pi(C)) \subset X \setminus C = \bigcup_{j=1}^{m} U_j.$$

Pick the connected component V of V_0 that contains y and consider the restriction map $\pi|_{\pi^{-1}(V)} : \pi^{-1}(V) \to V$. Then $\pi|_{\pi^{-1}(V\setminus A)} : \pi^{-1}(V \setminus A) \to V \setminus A$ is by Proposition VII.1.3 an s-sheeted topological covering. As $x_j \in \pi^{-1}(V) \cap U_j$, we deduce that $U_j' := \pi^{-1}(V) \cap U_j$ is a non-empty open and closed subset of $\pi^{-1}(V)$. Hence, $U_j' \setminus \pi^{-1}(A)$ is an open and closed subset of $\pi^{-1}(V) \setminus \pi^{-1}(A)$. As $\mathrm{Cl}_X(X \setminus \pi^{-1}(A)) = X$, we conclude $U_j' \setminus \pi^{-1}(A) \neq \varnothing$ for $j = 1, \dots, m$. By Proposition VII.1.3 the restriction map $\pi|_{U_j'\setminus\pi^{-1}(A)} : U_j' \setminus \pi^{-1}(A) \to V \setminus A$ is a topological covering. We are now ready to prove:

(ii) Suppose that there exists $y \in B$ such that $m > s$. The number s of sheets of the map $\pi|_{\pi^{-1}(V\setminus A)} : \pi^{-1}(V \setminus A) \to V \setminus A$ is greater than or equal to the number of sets U_j', which is m, because

$$\pi(U_j' \setminus \pi^{-1}(A)) = V \setminus A \quad \text{and} \quad \pi^{-1}(V \setminus A) = \bigcup_{j=1}^{m} (U_j' \setminus \pi^{-1}(A)).$$

As this is impossible, we conclude $\#(\pi^{-1}(y)) < s$.

(iii) It is enough to check: *B is closed in W.* Once this is proved, we deduce that B is a thin subset of W because $B \subset A$ and A is a thin subset of W.

2.a.2 We will prove something stronger: *For each $y \in W \setminus B$ there exists an open neighborhood $V \subset W$ of y such that $\pi|_{\pi^{-1}(V)} : \pi^{-1}(V) \to V$ is a trivial s-sheeted topological covering.* In particular, $W \setminus B$ is open in W, so B is closed in W.

Let $y \in W \setminus B$. Then $m = \#(\pi^{-1}(y)) = s$. With the notations in 2.a.1 and since $\pi(U_j' \setminus \pi^{-1}(A)) = V \setminus A$, the number s of sheets of

$$\pi|_{\pi^{-1}(V \setminus A)} : \pi^{-1}(V \setminus A) \to V \setminus A$$

is greater than or equal to the number of sets U_j', which is s. Thus,

$$\pi|_{U_j' \setminus \pi^{-1}(A)} : U_j' \setminus \pi^{-1}(A) \to V \setminus A$$

is a 1-sheeted topological covering for $j = 1, \dots, s$. As $\pi|_{\pi^{-1}(V)} : \pi^{-1}(V) \to V$ is proper and $\pi(U_j' \setminus \pi^{-1}(A)) = V \setminus A$, we have $\pi(U_j') = V$ because $V \setminus A$ is dense in V. As $U_i \cap U_j = \varnothing$ for $i \neq j$ and $\#(\pi^{-1}(y)) \leq s$ for each point $y \in W$, we conclude that $\pi|_{U_j'} : U_j' \to V$ is bijective and proper, so it is a homeomorphism. As $\pi^{-1}(V) = \bigcup_{j=1}^{s} U_j'$, the restriction map $\pi|_{\pi^{-1}(V)} : \pi^{-1}(V) \to V$ is a trivial s-sheeted topological covering.

(iv) Clearly $C := A \cap (W \setminus B)$ satisfies $A = B \cup C$ and C is a thin subset of $W \setminus B$ because A is a thin subset of W and C is a closed subset of A.

(v) This statement follows straightforwardly from 2.a.2. $\square$

Proposition VII.2.4 *Let (X, π, W) be an s-sheeted quasianalytic covering and let $\mathcal{F}$ be the family of all thin subsets A in W such that the triple*

$$(X \setminus \pi^{-1}(A), \pi|_{X \setminus \pi^{-1}(A)}, W \setminus A)$$

is a topological covering. Then

(i) *The set* $B := \bigcap_{A\in\mathcal{F}} A$ *is a thin subset of* W *and for each* $A \in \mathcal{F}$ *there exists a thin subset* C *in* $W \setminus B$ *such that* $A = B \cup C$.

(ii) *The triple* $(X \setminus \pi^{-1}(B), \pi|_{X\setminus\pi^{-1}(B)}, W \setminus B)$ *is an* s*-sheeted topological covering.*

Proof. (i) B is a thin subset of W because it is an intersection of thin subsets of W. Let $A \in \mathcal{F}$. As B is closed in W and A is a thin subset of W, it follows that $C := A \setminus B = A \cap (W \setminus B)$ is a thin subset of $W \setminus B$ and $A = B \cup (A \setminus B)$.

(ii) Pick $y \in W \setminus B = \bigcup_{A\in\mathcal{F}}(W \setminus A)$. Then there exists $A \in \mathcal{F}$ such that $y \in W \setminus A$. As $(X \setminus \pi^{-1}(A), \pi|_{X\setminus\pi^{-1}(A)}, W \setminus A)$ is a topological covering, there exist an open neighborhood $V \subset W \setminus A \subset W \setminus B$ of y and pairwise disjoint open subsets $U_1, \dots, U_s$ of $X \setminus \pi^{-1}(A)$ such that $\pi^{-1}(V) = \bigsqcup_{j=1}^{s} U_j$ and $\pi|_{U_j} : U_j \to V$ is a homeomorphism for $j = 1, \dots, s$. Note that V is an open subset of $W \setminus B$ because it is open in $W \setminus A$ and the latter is an open subset of W. Consequently, $(X \setminus \pi^{-1}(B), \pi|_{X\setminus\pi^{-1}(B)}, W \setminus B)$ is an s-sheeted topological covering, as required. □

Remark VII.2.5 With the notations of Proposition VII.2.4 it holds by Proposition VII.2.3 that if (X, π, W) be an s-sheeted analytic covering then

$$\bigcap_{A\in\mathcal{F}} A = \{y \in W : \#(\pi^{-1}(y)) \neq s\}.$$

Corollary VII.2.6 *Let* (X, π, W) *be a quasianalytic covering and suppose that there exists a critical set* A_0 *of the covering such that* $X \setminus \pi^{-1}(A_0)$ *is connected. Then for each critical set* A *of* (X, π, W) *the difference* $X \setminus \pi^{-1}(A)$ *is connected. Thus, the irreducibility of* (X, π, W) *does not depend on the chosen critical set.*

Proof. Let $\mathcal{F}$ be the family of all thin subsets A in W such that the triple

$$(X \setminus \pi^{-1}(A), \pi|_{X\setminus\pi^{-1}(A)}, W \setminus A)$$

is a topological covering and define $B := \bigcap_{A\in\mathcal{F}} A$. As $B \subset A_0$,

$$X \setminus \pi^{-1}(A_0) \subset X \setminus \pi^{-1}(B).$$

By Proposition VII.2.4 the triple $(X \setminus \pi^{-1}(B), \pi|_{X\setminus\pi^{-1}(B)}, W \setminus B)$ is a topological covering and $A_0 \setminus B$ is a thin subset of $W \setminus B$ because A_0 is a thin subset of W. By Proposition VII.1.4 $X \setminus \pi^{-1}(A_0)$ is a dense subset of $X \setminus \pi^{-1}(B)$. Therefore,

$$X \setminus \pi^{-1}(A_0) \subset X \setminus \pi^{-1}(B) \subset \mathrm{Cl}_X(X \setminus \pi^{-1}(A_0)).$$

The connectedness of $X \setminus \pi^{-1}(A_0)$ implies that $X \setminus \pi^{-1}(B)$ is connected. Now, let $A \in \mathcal{F}$. By Proposition VII.2.4 there exists a thin subset C in $W \setminus B$ such that $A = B \cup C$. In addition, $X \setminus \pi^{-1}(B)$ is connected, C is a thin subset of $W \setminus B$ and $(X \setminus \pi^{-1}(B), \pi|_{X \setminus \pi^{-1}(B)}, W \setminus B)$ is a topological covering. By Proposition VII.1.4 $X \setminus \pi^{-1}(A)$ is connected because

$$X \setminus \pi^{-1}(A) = (X \setminus \pi^{-1}(B)) \setminus \pi^{-1}(C),$$

as required. □

Definition and Proposition VII.2.7 Let (X, π, W) be an s-sheeted quasi-analytic covering and let $\mathcal{F}$ be the set of all thin subsets A in W such that

$$(X \setminus \pi^{-1}(A), \pi|_{X \setminus \pi^{-1}(A)}, W \setminus A)$$

is a topological covering. Define $B := \bigcap_{A \in \mathcal{F}} A$ and $\widehat{X} := \mathrm{Cl}_X(X \setminus \pi^{-1}(B))$. Then $(\widehat{X}, \pi|_{\widehat{X}}, W)$ *is an s-sheeted analytic covering called the* analytic locus of (X, π, W). The set $H_{(X,\pi,W)} := \mathrm{Int}_X(\pi^{-1}(B)) = \pi^{-1}(B) \setminus \widehat{X}$ is called the *failing open set* of the covering (X, π, W).

Proof. As $\pi : X \to W$ is a proper map with finite fibers and $\widehat{X}$ is a closed subset of X, also the restriction $\pi|_{\widehat{X}} : \widehat{X} \to W$ is proper with finite fibers. Let us check: *$\pi|_{\widehat{X}}$ is surjective.*

As π is proper,

$$W = \mathrm{Cl}_W(W \setminus B) = \mathrm{Cl}_W(\pi(X \setminus \pi^{-1}(B))) = \pi(\mathrm{Cl}_X(X \setminus \pi^{-1}(B))) = \pi(\widehat{X}).$$

In addition, $\widehat{X} \setminus \pi^{-1}(B) = X \setminus \pi^{-1}(B)$. Thus, $(\widehat{X} \setminus \pi^{-1}(B), \pi|_{\widehat{X} \setminus \pi^{-1}(B)}, W \setminus B)$ is an s-sheeted topological covering and recall that B is a thin subset of W. To show that $(\widehat{X}, \pi|_{\widehat{X}}, W)$ is an analytic covering we have to prove: $\widehat{X} \setminus \pi^{-1}(B)$ *is dense in $\widehat{X}$.*

This follows from the definition of $\widehat{X}$ because

$$\widehat{X} = \mathrm{Cl}_X(X \setminus \pi^{-1}(B)) = \mathrm{Cl}_X(X \setminus \pi^{-1}(B)) \cap \widehat{X} = \mathrm{Cl}_{\widehat{X}}(\widehat{X} \setminus \pi^{-1}(B)),$$

as required. □

Remark VII.2.8 (i) With the notations of Corollary VII.2.6 and Proposition VII.2.7 let A be a critical set of the quasianalytic covering (X, π, W). Then $A \setminus B$ is a thin subset of $W \setminus B$ because A is a thin subset of W. By Proposition VII.1.4 $X \setminus \pi^{-1}(A)$ is a dense subset of $X \setminus \pi^{-1}(B)$. Therefore,

$$X \setminus \pi^{-1}(A) \subset X \setminus \pi^{-1}(B) \subset \mathrm{Cl}_X(X \setminus \pi^{-1}(A)),$$

so $\mathrm{Cl}_X(X \setminus \pi^{-1}(A)) = \mathrm{Cl}_X(X \setminus \pi^{-1}(B))$. Thus, the failing open set of (X, π, W) can be defined as $H_{(X,\pi,W)} = \mathrm{Int}_X(\pi^{-1}(A))$ where A is any critical set of the covering (X, π, W).

(i) It is clear that a quasianalytic covering (X, π, W) is an analytic covering if and only if $H_{(X,\pi,W)} = \varnothing$.

3 Analytic coverings

In this section we study some main properties of analytic coverings and we introduce the concept of ramification index. We begin by analyzing the local nature of analytic coverings.

3.a Local structure of analytic coverings. The next result explains the local behavior of analytic coverings and plays a key role along the section.

Theorem VII.3.1 *Let (X, π, W) be an s-sheeted analytic covering with critical set $A \subset W \subset \mathbb{C}^n$ and let $a \in X$. Then there exists a basis of neighborhoods $\{V_m\}_{m \in \mathbb{N}}$ of the point a in X such that:*

(i) *V_m is connected and $\mathrm{Cl}_X(V_m)$ is compact for each $m \in \mathbb{N}$.*

(ii) *$\pi^{-1}(\pi(a)) \cap V_m = \{a\}$ for each $m \in \mathbb{N}$.*

(iii) *$W_m := \pi(V_m)$ is an open subset of $\mathbb{C}^n$ and the triple $(V_m, \pi|_{V_m}, W_m)$ is an analytic covering with critical set $A \cap W_m$.*

Proof. Let $X^\circ := X \setminus \pi^{-1}(A)$ be the regular locus of the covering. The fiber $\pi^{-1}(\pi(a)) = \{a_1 := a, a_2, \dots, a_r\}$ is finite and, as X is a Hausdorff space, there exist pairwise disjoint open neighborhoods $U_j \subset X$ of a_j for $j = 1, \dots, r$. As X is locally compact, there exists an open neighborhood U of a such that $\mathrm{Cl}_X(U)$ is compact and $U \subset \mathrm{Cl}_X(U) \subset U_1$. Note that $\mathrm{Cl}_X(U) \cap \pi^{-1}(\pi(a)) = \{a\}$, so $(\mathrm{Cl}_X(U) \setminus U) \cap \pi^{-1}(\pi(a)) = \varnothing$ and $\pi(a) \in W \setminus \pi(\mathrm{Cl}_X(U) \setminus U)$.

The difference $\mathrm{Cl}_X(U) \setminus U$ is compact because it is a closed subset of the compact set $\mathrm{Cl}_X(U)$. Therefore, $\pi(\mathrm{Cl}_X(U) \setminus U)$ is compact, so it is a closed subset of W. As $\pi(a) \notin \pi(\mathrm{Cl}_X(U) \setminus U)$ and W is locally connected, there exists a basis of connected open neighborhoods $\{W_m\}_{m\in\mathbb{N}}$ of $\pi(a)$ in W such that $\mathrm{Cl}(W_{m+1}) \subset W_m$ and $W_m \cap \pi(\mathrm{Cl}_X(U) \setminus U) = \varnothing$. Define $V_m := U \cap \pi^{-1}(W_m)$, which is an open neighborhood of a in X such that $\mathrm{Cl}_X(V_m)$ is compact. We have:

3.a.1 *The map $\pi|_{V_m} : V_m \to W_m$ is proper.*

The restriction map $\pi|_{\mathrm{Cl}_X(U)} \to W$ is a proper map because it is continuous, $\mathrm{Cl}_X(U)$ is compact and W is Hausdorff. Consequently, also

$$\pi|_{\mathrm{Cl}_X(U)\cap\pi^{-1}(W_m)} : \mathrm{Cl}_X(U) \cap \pi^{-1}(W_m) \to W_m$$

is proper. Let us check: $\mathrm{Cl}_X(U) \cap \pi^{-1}(W_m) = U \cap \pi^{-1}(W_m) = V_m$.

As $W_m \cap \pi(\mathrm{Cl}_X(U) \setminus U) = \varnothing$, we have $(\mathrm{Cl}_X(U) \setminus U) \cap \pi^{-1}(W_m) = \varnothing$, so

$$\mathrm{Cl}_X(U) \cap \pi^{-1}(W_m) = U \cap \pi^{-1}(W_m) = V_m. \tag{3.1}$$

3.a.2 *The fibers of each restriction map $\pi|_{V_m} : V_m \to W_m$ are finite* because the fibers of $\pi : X \to W$ are finite.

3.a.3 *The map $\pi|_{V_m} : V_m \to W_m$ is surjective.*

By 3.a.1 $\pi(V_m)$ is a closed subset of W_m. As X° is dense in X, the intersection $V_m^\circ := V_m \cap X^\circ = V_m \setminus \pi^{-1}(A)$ is non-empty and consequently $\pi(V_m^\circ) \neq \varnothing$. In addition, $\pi(V_m^\circ) = \pi(V_m) \setminus A$ is a closed subset of $W_m \setminus A$. But $W_m \cap A$ is a thin subset of the connected set W_m, so $W_m \setminus A = W_m \setminus (W_m \cap A)$ is connected. If we prove that $\pi(V_m^\circ)$ is an open subset of $W_m \setminus A$, we will have that $\pi(V_m^\circ) = W_m \setminus A$, because $\pi(V_m^\circ)$ is a non-empty closed and open subset of $W_m \setminus A$. As $\pi|_{V_m} : V_m \to W_m$ is proper and $W_m \cap A$ is a thin subset of W_m, we deduce $\pi(\mathrm{Cl}_{V_m}(V_m^\circ)) = \mathrm{Cl}_{W_m}(\pi(V_m^\circ)) = \mathrm{Cl}_{W_m}(W_m \setminus A) = W_m$ because $W_m \setminus A$ is dense in W_m.

3.a.4 *$\pi(V_m^\circ)$ is an open subset of $W_m \setminus A$.*

The triple $(X^\circ, \pi|_{X^\circ}, W \setminus A)$ is a topological covering, so the restriction map $\pi|_{X^\circ} : X^\circ \to W \setminus A$ is an open map (because it is a local homeomorphism). As $V_m^\circ = V_m \cap X^\circ$ is an open subset of X°, its image $\pi(V_m^\circ)$ is an open subset of $W \setminus A = \pi(X^\circ)$. Thus, $\pi(V_m^\circ) \subset W_m \setminus A$ is an open subset of $W_m \setminus A$.

3.a.5 *The triple $(V_m^\circ, \pi|_{V_m^\circ}, W_m \setminus A)$ is a topological covering and the number of preimages of each point of $W_m \setminus A$ under the map $\pi|_{V_m^\circ}$ is constant.*

As $(X \setminus \pi^{-1}(A), \pi|_{X\setminus\pi^{-1}(A)}, W \setminus A)$ is a topological covering, also the triple

$$(\pi^{-1}(W_m) \setminus \pi^{-1}(A), \pi|_{\pi^{-1}(W_m)\setminus\pi^{-1}(A)}, W_m \setminus A)$$

is a topological covering. Let us prove: *V_m° is open and closed in the difference $\pi^{-1}(W_m) \setminus \pi^{-1}(A)$.*

The equality

$$V_m^\circ = (U \cap \pi^{-1}(W_m)) \setminus \pi^{-1}(A) = U \cap \pi^{-1}(W_m \setminus \pi^{-1}(A))$$

proves that V_m° is open in $\pi^{-1}(W_m) \setminus \pi^{-1}(A)$. In addition, by (3.1)

$$V_m^\circ = (\mathrm{Cl}_X(U) \cap \pi^{-1}(W_m)) \setminus \pi^{-1}(A) = \mathrm{Cl}_X(U) \cap (\pi^{-1}(W_m) \setminus \pi^{-1}(A)),$$

so V_m° is closed in $\pi^{-1}(W_m) \setminus \pi^{-1}(A)$. By Proposition VII.1.3 the triple $(V_m^\circ, \pi|_{V_m^\circ}, W_m \setminus A)$ is a topological covering. As $W_m \setminus A$ is a connected set, the cardinality of the fibers of the previous topological covering is constant by Remark VII.1.2.

3.a.6 We have proved that $(V_m, \pi|_{V_m}, W_m)$ is a quasianalytic covering. Let us check: *$(V_m, \pi|_{V_m}, W_m)$ is an analytic covering with critical set $W_m \cap A$* because: *$V_m \setminus \pi^{-1}(A)$ is dense in V_m.*

Let $E \subset V_m$ be a non-empty subset of V_m. Thus, E is an open subset of X, so $E \cap (X \setminus \pi^{-1}(A)) \neq \varnothing$. Hence,

$$E \cap (V_m \setminus \pi^{-1}(A)) = E \cap (X \setminus \pi^{-1}(A)) \neq \varnothing.$$

We conclude $V_m \setminus \pi^{-1}(A)$ is dense in V_m

3.a.7 Next we check: *$\{V_m\}_{m\in\mathbb{N}}$ is a basis of open neighborhoods of a.*

As $\mathrm{Cl}(W_{m+1}) \subset W_m$, we have by Lemma VI.1.8

$$\begin{aligned}\mathrm{Cl}_X(V_{m+1}) = \mathrm{Cl}_X(U \cap \pi^{-1}(W_{m+1})) &= \mathrm{Cl}_X(U) \cap \mathrm{Cl}_X(\pi^{-1}(W_{m+1})) \\ &\subset \mathrm{Cl}_X(U) \cap \pi^{-1}(\mathrm{Cl}(W_{m+1})) \subset \mathrm{Cl}_X(U) \cap \pi^{-1}(W_m) = V_m.\end{aligned}$$

In addition, by (3.1)

$$\bigcap_{m\in\mathbb{N}} V_m = \mathrm{Cl}_X(U) \cap \bigcap_{m\in\mathbb{N}} \pi^{-1}(W_m) = \mathrm{Cl}_X(U) \cap \pi^{-1}(\pi(a)) = \{a\}.$$

By Lemma VII.1.5 we deduce $\{V_m\}_{m\in\mathbb{N}}$ is a basis of open neighborhoods of a.

3.a.8 To finish we prove: *V_m is connected for each $m \in \mathbb{N}$.*

Note that $W_m \setminus A$ is connected and locally connected. Consequently, also $V_m \setminus \pi^{-1}(A)$ is locally connected because it is locally homeomorphic to $W_m \setminus A$ via the local homeomorphism $\pi|_{V_m \setminus \pi^{-1}(A)} : V_m \setminus \pi^{-1}(A) \to W_m \setminus A$. By Proposition VII.1.3 $V_m \setminus \pi^{-1}(A)$ has finitely many connected components, say $C_1, \dots, C_r$. Let us prove: *$a \in \mathrm{Cl}_X(C_j)$ for $j = 1, \dots, r$.*

As C_j is open and closed in $V_m \setminus \pi^{-1}(A)$ and $W_m \setminus A$ is connected, it follows by Proposition VII.1.3 that $(C_j, \pi|_{C_j}, \pi(C_j) = W_m \setminus A)$ is a topological covering. As π is a proper map,

$$W_m = \mathrm{Cl}_{W_m}(W_m \setminus A) = \mathrm{Cl}_{W_m}(\pi(C_j)) \subset \pi(\mathrm{Cl}_X(C_j)),$$

so $\pi(a) \in W_m \subset \pi(\mathrm{Cl}_X(C_j))$ for $j = 1, \dots, r$. As $C_j \subset V_m \subset U$,

$$\varnothing \neq \pi^{-1}(\pi(a)) \cap \mathrm{Cl}_X(C_j) \subset \pi^{-1}(\pi(a)) \cap \mathrm{Cl}_X(U) = \{a\},$$

that is, $a \in \mathrm{Cl}_X(C_j)$ and $a \in \mathrm{Cl}_X(C_j) \cap V_m$.

As C_j is connected and $C_j \subset \mathrm{Cl}_X(C_j) \cap V_m \subset \mathrm{Cl}_X(C_j)$, each intersection $\mathrm{Cl}_X(C_j) \cap V_m$ is connected and contains the point a. Consequently, the union

$$\Big(\bigcup_{j=1}^r \mathrm{Cl}_X(C_j)\Big) \cap V_m$$

is connected. Thus,

$$V_m = \mathrm{Cl}_X(V_m \setminus \pi^{-1}(A)) \cap V_m = \mathrm{Cl}_X\Big(\bigcup_{j=1}^r C_j\Big) \cap V_m = \Big(\bigcup_{j=1}^r \mathrm{Cl}_X(C_j)\Big) \cap V_m$$

is connected, as required. □

Corollary VII.3.2 *Let (X, π, W) be an s-sheeted analytic covering and let $A \subset W \subset \mathbb{C}^n$ be a critical set. Then the following properties hold:*

(i) *X is locally connected and a first countable space.*

(ii) *The map $\pi : X \to W$ is open.*

(iii) *Both X and its regular locus $X^\circ := X \setminus \pi^{-1}(A)$ have finitely many connected components.*

(iv) *Let C be a connected component of X. Then the triple $(C, \pi|_C, W)$ is an analytic covering having A as a critical set.*

Proof. (i) This statement has been proved in Theorem VII.3.1.

(ii) Let U be an open subset of X and let $a \in U$. By Theorem VII.3.1 there exists an open neighborhood $V_m \subset U$ of a in X such that $W_m = \pi(V_m)$ is an open subset of W, that is open in $\mathbb{C}^n$. Thus, $\pi(U)$ is a neighborhood of each of its points, so π is an open map.

(iii) As $(X^\circ, \pi|_{X^\circ}, W \setminus A)$ is a topological covering, $W \setminus A$ is connected and X° is locally connected (because it is locally homeomorphic to $W \setminus A$), the regular locus X° has by Proposition VII.1.3 finitely many connected components, say $C_1, \dots, C_r$. Thus, X is a finite union of connected sets:

$$X = \mathrm{Cl}_X(X^\circ) = \bigcup_{i=1}^{r} \mathrm{Cl}_X(C_i),$$

so it has finitely many connected components.

(iv.1) By part (iii) C is an open and closed subset of X. As $\pi : X \to W$ is an open and closed map, $\pi(C)$ is a non-empty open and closed subset of the connected set W, so $\pi(C) = W$. As $\pi : X \to W$ is proper with finite fibers and C is closed in X, also $\pi|_C : C \to W$ is a surjective and proper map whose fibers are finite.

(iv.2) As $C \setminus \pi^{-1}(A) = C \cap X^\circ$ is open and closed in X° and $\pi(C) = W \setminus A$, the triple

$$(C \setminus \pi^{-1}(A), \pi|_{C \setminus \pi^{-1}(A)}, W \setminus A)$$

is by Proposition VII.1.3 a topological covering.

(iv.3) To finish we check: $C \setminus \pi^{-1}(A)$ *is dense in* C.

Let $U \subset C$ be a non-empty open subset. As C is open in X, then U is also open in X. Thus, $X^\circ \cap U \neq \varnothing$, so

$$(C \setminus \pi^{-1}(A)) \cap U = X^\circ \cap (U \cap C) = X^\circ \cap U \neq \varnothing,$$

as required. □

Definition VII.3.3 Given an analytic covering (X, π, W) and a point $a \in X$ there exists by Theorem VII.3.1 a connected open neighborhood $V \subset X$ of a such that

(i) $\pi^{-1}(\pi(a)) \cap V = \{a\}$.

(ii) $(V, \pi|_V, \pi(V))$ is an analytic covering.

The *ramification index of* (X, π, W) *at the point* a is the number $\mathtt{r}(a)$ of sheets of the covering $(V, \pi|_V, \pi(V))$.

Lemma VII.3.4 *Let* (X, π, W) *be an* s*-sheeted analytic covering and* $a \in X$. *We have:*

(i) *The ramification index of* (X, π, W) *at* a *is well-defined.*

(ii) *Let* $b \in W$ *with* $\pi^{-1}(b) := \{a_1, \dots, a_r\}$. *Then*

$$\sum_{j=1}^{r} \mathtt{r}(a_j) = s.$$

Proof. (i) Let U and V be connected open neighborhoods of a such that

$$\pi^{-1}(\pi(a)) \cap V = \pi^{-1}(\pi(a)) \cap U = \{a\}$$

and $(V, \pi|_V, \pi(V))$ and $(U, \pi|_U, \pi(U)))$ are analytic coverings. We claim: *the number* $\mathtt{r}_V(a)$ *of sheets of* $(V, \pi|_V, \pi(V))$ *coincides with the number* $\mathtt{r}_U(a)$ *of sheets of* $(U, \pi|_U, \pi(U))$.

By Theorem VII.3.1 we may assume that $V \subset U$. Thus, $\mathtt{r}_V(a) \leq \mathtt{r}_U(a)$ and let us prove that we have in fact an equality.

3.a.9 Let $\{W_m\}_{m\in\mathbb{N}}$ be a basis of open neighborhoods of $\pi(a)$ in $\pi(U)$. Then $\{\pi^{-1}(W_m) \cap U\}_{m\in\mathbb{N}}$ *is a basis of open neighborhoods of* a *in* U.

Let $U' \subset U$ be an open neighborhood of a. As $U \setminus U'$ is closed in U and $\pi : U \to \pi(U)$ is an open and closed map, $\pi(U) \setminus \pi(U \setminus U')$ is an open subset of $\pi(U)$ that contains $\pi(a)$ because $\pi^{-1}(\pi(a)) \cap U = \{a\}$. Consequently, $\pi(a) \in W_m \subset \pi(U) \setminus \pi(U \setminus U')$ for some $m \in \mathbb{N}$, so $a \in \pi^{-1}(W_m) \cap U \subset U'$.

3.a.10 Let $B_U := \{y \in \pi(U) : \#(\pi^{-1}(y)) < \mathtt{r}_U(a)\}$. Then B_U is by Proposition VII.2.3 a thin subset of $\pi(U)$. Thus, $\pi(U) \setminus B_U$ is a dense subset of $\pi(U)$. Therefore $W_m \cap (\pi(U) \setminus B_U) \neq \varnothing$ for each $m \in \mathbb{N}$. Notice that $\#((\pi^{-1}(y) \cap U) = \mathtt{r}_U(a)$ for each $y \in W_m \setminus B_U$. As V is a neighborhood of a, there exists $m \in \mathbb{N}$ such that $\pi^{-1}(W_m) \cap U \subset V$. Thus, for each point $y \in W_m \setminus B_U$ there exist pairwise distinct points $x_1, \dots, x_{\mathtt{r}_U(a)}$ such that

$$\pi^{-1}(y) \cap U = \{x_1, \dots, x_{\mathtt{r}_U(a)}\} \subset \pi^{-1}(W_m) \cap U \subset V.$$

This provides the equality $\mathtt{r}_V(a) = \mathtt{r}_U(a)$ because

$$\mathtt{r}_U(a) = \#(\pi^{-1}(y) \cap U) \leq \max\{\#(\pi^{-1}(z)) : z \in \pi(V)\} = \mathtt{r}_V(a).$$

(ii) By Theorem VII.3.1 there exist pairwise disjoint connected open neighborhoods $V_j \subset X$ of a_j such that $(V_j, \pi|_{V_j}, \pi(V_j))$ is an analytic covering for $j = 1, \dots, r$. Define $B := \{z \in W : \#(\pi^{-1}(z)) < s\}$. By Proposition VII.2.3 B is a thin subset of W, so the difference $W \setminus B$ is dense in W. In addition, $\bigcap_{i=1}^r \pi(V_j)$ is an open neighborhood of b because π is an open map. Thus, there exists a sequence

$$\{b_m\}_{m\in\mathbb{N}} \subset (W \setminus B) \cap \bigcap_{i=1}^r \pi(V_j)$$

that converges to b. For every $m \in \mathbb{N}$ the fibre $\pi^{-1}(b_m)$ consists of s points and we write $\pi^{-1}(b_m) := \{a_{1,m}, \dots, a_{s,m}\}$. As

$$\{a_{i,m} : i = 1, \dots, s, \, m \in \mathbb{N}\} \subset \bigsqcup_{j=1}^r V_j,$$

there exists $j = 1, \dots, r$ such that infinitely many points $a_{i,m}$ belong to V_j. Thus, for each index $i = 1, \dots, s$ we may assume that there exists an index j such that $a_{i,m} \in V_j$ for each $m \in \mathbb{N}$. As $K := \{b\} \cup \{b_m\} \subset W$ is compact and the map $\pi : X \to W$ is proper, the inverse image

$$\pi^{-1}(K) = \{a_1, \dots, a_r\} \cup \bigcup_{m\in\mathbb{N}} \{a_{1,m}, \dots, a_{s,m}\}$$

is compact. Thus, we may assume that each sequence $\{a_{i,m}\}_{m\in\mathbb{N}}$ converges to a point $c_i \in K$ for $i = 1, \dots, s$. Therefore,

$$\pi(c_i) = \pi(\lim_{m\to\infty} \{a_{i,m}\}) = \lim_{m\to\infty} \{\pi(a_{i,m})\} = \lim_{m\to\infty} \{b_m\}_{m\in\mathbb{N}} = b,$$

so each $c_i \in \pi^{-1}(b) := \{a_1, \dots, a_r\}$. Hence, for each index $i = 1, \dots, s$ there exists $j = 1, \dots, r$ such that $c_i = a_j$.

As $\{b_m\}_{m\in\mathbb{N}} \subset (W \setminus B) \cap \bigcap_{i=1}^r \pi(V_j)$, for each $j = 1, \dots, r$ there exists $i = 1, \dots, s$ such that $\{a_{i,m} : m \in \mathbb{N}\} \subset V_j$, so $c_i \in V_j$. Consequently,

$$\pi^{-1}(b_m) = \bigsqcup_{j=1}^r \pi^{-1}(b_m) \cap V_j,$$

so the number s of sheets of (X, π, W) equals the sum of the number of sheets of the triple $(V_j, \pi|_{V_j}, \pi(V_j))$ for $j = 1, \dots, r$, as required. □

Lemma VII.3.5 *Let A be a critical set of an analytic covering (X, π, W) and let $B \subset W \setminus A$ be a thin subset of $W \setminus A$. Then (X, π, W) is an analytic covering with critical set $A \cup B$.*

Proof. By Exercise number V.19 $A \cup B$ is a thin subset of W. The triple $(X \setminus \pi^{-1}(A \cup B), \pi|_{X \setminus \pi^{-1}(A \cup B)}, W \setminus (A \cup B))$ is a topological covering because $(X \setminus \pi^{-1}(A), \pi|_{X \setminus \pi^{-1}(A)}, W \setminus A)$ is by Proposition VII.1.3 a topological covering,

$$W \setminus (A \cup B) = (W \setminus A) \setminus B \quad \text{and} \quad \pi^{-1}(W \setminus (A \cup B)) = X \setminus \pi^{-1}(A \cup B).$$

To prove the analyticity of the covering let us check next: $X \setminus \pi^{-1}(A \cup B)$ *is dense in* X.

Let $U \subset X$ be a non-empty open subset. As $\pi : X \to W$ is an open map, $\pi(U)$ is an open subset of W. As $A \cup B$ is a thin subset of W, we have $\pi(U) \cap (W \setminus (A \cup B)) \neq \varnothing$, so $U \cap (X \setminus \pi^{-1}(A \cup B)) \neq \varnothing$. Thus, $X \setminus \pi^{-1}(A \cup B)$ is dense in X, as required. $\square$

3.b Analytic functions with respect to a covering.

We study next the concept of analytic function with respect to a covering.

Definition VII.3.6 Let (X, π, W) be an analytic covering with regular locus X°. A continuous function $f : X \to \mathbb{C}$ is said to be *analytic with respect to the covering* if for each point $a \in X^\circ$ there exists an open neighborhood $U \subset X^\circ$ of a such that:

(i) The restriction $\pi|_U : U \to \pi(U)$ is a homeomorphism.

(ii) The composition $f \circ (\pi|_U)^{-1} : \pi(U) \to \mathbb{C}$ is an analytic function.

Example VII.3.7 We observed in Example VII.2.2 that $(X, \pi, \mathbb{C})$, where

$$X := \{(\mathtt{x}+\mathtt{y})(\mathtt{x}-\mathtt{y}) = 0\} \subset \mathbb{C}^2 \quad \text{and} \quad \pi : X \to \mathbb{C},\ (x, y) \mapsto x$$

is an analytic covering, $A := \{0\}$ is a critical set and $X^\circ := X \setminus \{(0,0)\}$ a regular locus. We claim: $f : X \to \mathbb{C}$, $(x, y) \mapsto x + y$ *is an analytic function with respect to* $(X, \pi, \mathbb{C})$.

Consider the open neighborhood $U := \{(x, x) \in X,\ x \neq 0\} \subset X^\circ$ of a point $p := (a, a) \in X^\circ$. Then $f \circ (\pi|_U)^{-1} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $x \mapsto 2x$ is analytic. Consider the open neighborhood $V := \{(x, -x) \in X,\ x \neq 0\} \subset X^\circ$ of $q := (a, -a) \in X^\circ$. The function $f \circ (\pi|_V)^{-1} : \mathbb{C} \setminus \{0\} \to \mathbb{C},\ x \mapsto 0$ is analytic, so $f : X \to \mathbb{C},\ (x, y) \mapsto x + y$ is an analytic function with respect to $(X, \pi, \mathbb{C})$.

Theorem VII.3.8 *Let $f : X \to \mathbb{C}$ be an analytic function with respect to an s-sheeted analytic covering (X, π, W) with critical set A. Let $S \in \mathbb{C}[\mathtt{x}_1, \ldots, \mathtt{x}_s]$ be a symmetric polynomial. Then*

(i) *The function*

$$S(f) : W \setminus A \to \mathbb{C},\ y \mapsto S(f(x_1), \ldots, f(x_s)),$$

where $\pi^{-1}(y) = \{x_1, \ldots, x_s\}$, is an analytic function.

(ii) *The function $S(f) : W \setminus A \to \mathbb{C}$ is locally bounded.*

(iii) *The function $S(f)$ can be extended analytically to W and it satisfies*

$$S(f)(y) = S(f(x_1), \overset{\mathtt{r}(x_1)}{\ldots}, f(x_1), \ldots, f(x_r), \overset{\mathtt{r}(x_r)}{\ldots}, f(x_r))$$

for each $y \in W$, where $\pi^{-1}(y) = \{x_1, \ldots, x_r\}$ and $\mathtt{r}(x_j)$ is the ramification index *at the point x_j of (X, π, W).*

Proof. (i) Let $y \in W \setminus A$ and write $X^\circ := X \setminus \pi^{-1}(A)$. As $(X^\circ, \pi|_{X^\circ}, W \setminus A)$ is a topological covering, there exist an open neighborhood W_0 of y in $W \setminus A$ and pairwise disjoint open subsets $U_j \subset X^\circ$ such that $\pi^{-1}(W_0) = \bigsqcup_{i=1}^s U_i$ and the restriction $\pi|_{U_i} : U_i \to W_0$ is a homeomorphism for $i = 1, \ldots, s$. Let x_i be the unique point of the singleton $U_i \cap \pi^{-1}(y)$ for $i = 1, \ldots, s$. As f is analytic with respect to the covering and $x_1, \ldots, x_s \in X^\circ$, there exist open subsets $V_i \subset X^\circ$ such that $x_i \in V_i \subset U_i$, the restriction map $\pi|_{V_i} : V_i \to \pi(V_i)$ is a homeomorphism and $f \circ (\pi|_{V_i})^{-1} : \pi(V_i) \to \mathbb{C}$ is analytic for $i = 1, \ldots, s$. As $\pi : X^\circ \to W \setminus A$ is an open map, $W_1^y := \bigcap_{i=1}^s \pi(V_i)$ is an open neighborhood of y in $W \setminus A$. Define $V_i' := V_i \cap \pi^{-1}(W_1^y)$, which is an open neighborhood of x_i in X. Note that $\pi^{-1}(W_1^y) = \bigsqcup_{i=1}^s V_i'$, $\pi|_{V_i'} : V_i' \to W_1^y$ is a homeomorphism and $f \circ (\pi|_{V_i'})^{-1} : W_1^y \to \mathbb{C}$ is analytic. The function

$$F_y : W_1^y \to \mathbb{C},\ z \mapsto S(f \circ (\pi|_{V_1'})^{-1}(z), \ldots, f \circ (\pi|_{V_s'})^{-1}(z))$$

is well-defined because S is a symmetric polynomial. In addition, F_y is analytic because it is a sum of products of analytic functions. We have proved that for each $y \in W \setminus A$ there exist an open neighborhood W_1^y of y in $W \setminus A$ and an analytic function $F_y : W_1^y \to \mathbb{C}$ such that $S(f)|_{W_1^y} = F_y$. Consequently, $S(f)$ is an analytic function.

(ii) Pick a point $y \in W$ and let $K \subset W$ be a compact neighborhood of y. As $\pi : X \to W$ is a proper map, $\pi^{-1}(K)$ is a compact subset of X. As $f : X \to \mathbb{C}$ is continuous, $f(\pi^{-1}(K))$ is compact. Thus,

$$K_0 := f(\pi^{-1}(K)) \times \overset{(s)}{\cdots} \times f(\pi^{-1}(K)) \subset \mathbb{C}^s$$

is compact, so also $S(K_0)$ is compact. For each point $z \in K \setminus A$ the fiber $\pi^{-1}(z) := \{\zeta_1, \dots, \zeta_s\}$ is contained in $\pi^{-1}(K)$, so K_0 contains each permutation of the s-uple $(f(\zeta_1), \dots, f(\zeta_s))$. Consequently,

$$S(f)(z) = S(f(\zeta_1), \dots, f(\zeta_s)) \in S(K_0),$$

so $S(f)(K \setminus A) \subset S(K_0)$ is a bounded set.

(iii) As A is a thin subset of W and $S(f) : W \setminus A \to \mathbb{C}$ is a locally bounded function, $S(f)$ admits an analytic extension that we denote for simplicity $S(f) : W \to \mathbb{C}$. Such extension is unique, because $W \setminus A$ is a dense subset of W. Let us check that this extension satisfies the condition in the statement. For each point $y \in A \subset \operatorname{Cl}(W \setminus A)$

$$\begin{aligned} S(f)(y) &= S(f)(\lim_{m\to\infty} \{y_m\}) = \lim_{m\to\infty} \{S(f)(y_m)\} \\ &= \lim_{m\to\infty} \{S(f(x_{1,m}), \dots, f(x_{s,m}))\} = S\big(\lim_{m\to\infty} \{(f(x_{1,m}), \dots, f(x_{s,m}))\}\big) \\ &= S(f(x_1), \overset{\mathtt{r}(x_1)}{\cdots}, f(x_1), \dots, f(x_r), \overset{\mathtt{r}(x_r)}{\cdots}, f(x_r)) \end{aligned}$$

where $\{y_m\}_m \subset W \setminus A$ is a sequence that converges to y and $\{x_{i,m}\}_m \subset X^\circ$ is a sequence that converges to x_j for $\sum_{k=1}^{i-1} \mathtt{r}(x_k) \le j \le \sum_{k=1}^{i} \mathtt{r}(x_k)$ and $i = 1, \dots, r$. For the construction of these sequences $\{x_{i,m}\}_m$ see the proof of Lemma VII.3.4 (ii). □

Notation VII.3.9 Denote

$$\sigma_j(\mathtt{x}_1, \dots, \mathtt{x}_s) := (-1)^j \sum_{1 \le i_1 < \cdots < i_j \le s} \mathtt{x}_{i_1} \cdots \mathtt{x}_{i_j} \in \mathbb{Z}[\mathtt{x}_1, \dots, \mathtt{x}_s]$$

for $1 \le j \le s$. The polynomials $\sigma_1, \dots, \sigma_s$ are *the elementary symmetric forms in s variables.*

Construction VII.3.10 (i) Let $f : X \to \mathbb{C}$ be an analytic function with respect to an s-sheeted analytic covering (X, π, W). By Theorem VII.3.8 there exist analytic functions $\sigma_j(f) \in \mathcal{O}(W)$ such that for each point $y \in W$ there exist $x_1, \dots, x_s \in \pi^{-1}(y)$ with $\sigma_j(f)(y) = \sigma_j(f(x_1), \dots, f(x_s))$. The *annihilating polynomial of* f is defined as

$$P_f(\mathtt{t}) := \mathtt{t}^s + \sum_{j=1}^{s} \sigma_j(f)\mathtt{t}^{s-j} \in \mathcal{O}(W)[\mathtt{t}].$$

For each point $w \in W$ we denote

$$P_f(w, \mathtt{t}) := \mathtt{t}^s + \sum_{j=1}^{s} \sigma_j(f)(w)\mathtt{t}^{s-j} \in \mathbb{C}[\mathtt{t}].$$

After evaluating at $t \in \mathbb{C}$ we have

$$P_f(w, t) := t^s + \sum_{j=1}^{s} \sigma_j(f)(w)t^{s-j} \quad \text{for each} \quad (w, t) \in W \times \mathbb{C}.$$

(ii) For each point $x \in X$ we have: $P_f(\pi(x), f(x)) = 0$. This suggests the name of annihilating polynomial for P_f.

Let $x \in X^\circ := X \setminus \pi^{-1}(A)$ and write $\pi^{-1}(\pi(x)) := \{x := x_1, \dots, x_s\}$. By Cardano-Viète formulae

$$\begin{aligned} P_f(\pi(x), \mathtt{t}) &= \mathtt{t}^s + \sum_{j=1}^{s} \sigma_j(f)(\pi(x))\mathtt{t}^{s-j} \\ &= \mathtt{t}^s + \sum_{j=1}^{s} \sigma_j(f(x_1), \dots, f(x_s))\mathtt{t}^{s-j} = \prod_{j=1}^{s}(\mathtt{t} - f(x_j)). \end{aligned}$$

As $f(x) = f(x_1)$, we have $P_f(\pi(x), f(x)) = \prod_{j=1}^{s}(f(x) - f(x_j)) = 0$. Thus, the continuous function

$$P_f(\pi, f) : X \to \mathbb{C},\ x \mapsto P_f(\pi(x), f(x))$$

vanishes on the dense subset X° of X, so $P_f(\pi(x), f(x)) = 0$ for each $x \in X$.

Corollary VII.3.11 (Maximum modulus principle) *Let $f : X \to \mathbb{C}$ be an analytic function with respect to an s-sheeted analytic covering (X, π, W) such that X is connected. Suppose that the function $|f| : X \to \mathbb{R}$, $x \mapsto |f(x)|$ attains its maximum value in X. Then f is constant.*

Proof. Let $x_0 \in X$ be such that $|f(x)| \leq |f(x_0)|$ for each point $x \in X$ and consider the non-empty closed subset $M := f^{-1}(f(x_0))$ of X. Let us prove that M is open in X. Once this is done $M = X$ because X is connected, so f is the constant function that attains the value $f(x_0)$ at each point of X.

Pick $a \in M$ and let $U \subset X$ be an open neighborhood of a such that $\pi^{-1}(\pi(a)) \cap U = \{a\}$ and the restriction $\pi|_U : U \to \pi(U)$ is an r-sheeted analytic covering. Note that $h := f|_U : U \to \mathbb{C}$ is an analytic function with respect to the covering $(U, \pi|_U, \pi(U))$. Let $\sigma_1, \dots, \sigma_r$ be the elementary symmetric forms in r variables.

3.b.1 We claim: *The function $\sigma_j(h) : \pi(U) \to \mathbb{C}$ is constant for $j = 1, \dots, r$.*

As U is open and connected and π is an open map, $\pi(U)$ is a connected open subset of $\mathbb{C}$ and $\sigma_j(h) : \pi(U) \to \mathbb{C}$ is an analytic function. By the Maximum modulus principle V.4.7 it is enough to prove: $|\sigma_j(h)| : \pi(U) \to \mathbb{R}$ *attains at the point $\pi(a)$ its maximum on $\pi(U)$.*

Let $y \in \pi(U)$. Then there exist non-necessarily pairwise distinct points $x_1, \dots, x_r \in \pi^{-1}(y)$ such that $\sigma_j(h)(y) = \sigma_j(h(x_1), \dots, h(x_r))$. As $a \in M$, we have $|h(x_j)| = |f(x_j)| \leq |f(a)|$, so

$$\begin{aligned}
|\sigma_j(h)(y)| = |\sigma_j(h(x_1), \dots, h(x_r))| &= \Big|(-1)^j \sum_{1 \leq i_1 < \cdots < i_j \leq r} h(x_{i_1}) \cdots h(x_{i_r})\Big| \\
&\leq \sum_{1 \leq i_1 < \cdots < i_j \leq r} |f(x_{i_1})| \cdots |f(x_{i_r})| \leq \sum_{1 \leq i_1 < \cdots < i_j \leq r} |f(a)| \cdots |f(a)| \\
&= \sigma_j(|f(a)|, \dots, |f(a)|) = \Big|(-1)^j \binom{r}{j} |f(a)|^j\Big| = \Big|(-1)^j \binom{r}{j} f(a)^j\Big| \\
&= |\sigma_j(f(a), \dots, f(a))| = |\sigma_j(h)(\pi(a))|.
\end{aligned}$$

Thus, the function $|\sigma_j(h)| : \pi(U) \to \mathbb{R}$ attains at $\pi(a)$ its maximum value on $\pi(U)$, so 3.b.1 holds.

3.b.2 Let $P_h \in \mathcal{O}(\pi(U))[\mathtt{t}]$ be the annihilating polynomial of h:

$$P_h(\mathtt{t}) := \mathtt{t}^r + \sum_{j=1}^{r} \sigma_j(h) \mathtt{t}^{r-j}.$$

By 3.b.1 $\sigma_j(h)(\pi(x)) = \sigma_j(h)(\pi(a))$ for $x \in U$. As $(\pi|_U)^{-1}(\pi(a)) = \{a\}$,

$$P_h(\pi(x), \mathtt{t}) = P_h(\pi(a), \mathtt{t}) = (\mathtt{t} - f(a))^r.$$

As $h(x) = f(x)$ for each $x \in U$, we deduce by VII.3.10 that

$$0 = P_h(\pi(x), h(x)) = (h(x) - f(a))^r = (f(x) - f(a))^r,$$

so $f(x) = f(a)$. Therefore, $a \in U \subset M$, so M is an open subset of X, as required. $\square$

Example VII.3.12 The conclusion in Corollary VII.3.11 does not hold if we substitute the hypothesis: *the function* $|f|$ *attains its maximum value in* X, by: $|f|$ *attains a local maximum in* X.

Consider the complex lines $X_\varepsilon := \{\mathtt{y} = \varepsilon\mathtt{x}\} \subset \mathbb{C}^2$ where $\varepsilon = \pm 1$, its union $X := X_{-1} \cup X_1$, the projection $\pi : X \to \mathbb{C}$, $(x, y) \mapsto x$ and the function $f : X \to \mathbb{C}$, $(x, y) \mapsto x + y$, which is analytic with respect to the analytic covering $(X, \pi, \mathbb{C})$. Let $a := (1, -1) \in X$ and $U := \mathbb{C}^2 \setminus X_1$, which is an open neighborhood of a in $\mathbb{C}^2$ such that $U \cap X = U \cap X_{-1}$ and $f|_{U \cap X} \equiv 0$. Therefore $|f|$ has a local maximum at the point a, but f is not constant because $f(1, 1) = 2 \neq f(a)$.

Theorem VII.3.13 (Identity principle) *Let* $f : X \to \mathbb{C}$ *be an analytic function with respect to an irreducible analytic covering* (X, π, W) *that vanishes on a non-empty open subset of* X. *Then* $f \equiv 0$.

Proof. Let $A \subset W$ be a critical set of the covering. As (X, π, W) is an irreducible analytic covering, the triple $(X^\circ, \pi|_{X^\circ}, W \setminus A)$ is a topological covering and $X^\circ := X \setminus \pi^{-1}(A)$ is connected and dense in X. Let $U \subset X$ be a non-empty subset of X on which f is identically zero. As X° is dense in X, there exists a point $a \in U \cap X^\circ$. Let $B \subset X^\circ$ be the set of points $x \in X^\circ$ such that there exist $r \geq 1$ connected open subsets $U_1, \dots, U_r$ of X° such that $a \in U_1$, $x \in U_r$, $U_i \cap U_{i+1} \neq \varnothing$ for $i = 1, \dots, r-1$, $\pi|_{U_i} : U_i \to \pi(U_i)$ is a homeomorphism and $f \circ (\pi|_{U_i})^{-1} : \pi(U_i) \to \mathbb{C}$ is analytic. Observe that $a \in B$.

3.b.3 We claim: *The function* f *vanishes identically on* B.

Pick a point $x \in B$. Let us prove by induction on r that $f(x) = 0$. If $r = 1$ the function $f \circ (\pi|_{U_1})^{-1} : \pi(U_1) \to \mathbb{C}$ is analytic on the connected open set $\pi(U_1) \subset W$ and $f \circ (\pi|_{U_1})^{-1}$ vanishes on the non-empty open subset $\pi(U \cap U_1)$ of $\pi(U_1)$. By the Identity Principle V.1.4 $f \circ (\pi|_{U_1})^{-1} \equiv 0$ on $\pi(U_1)$, so $f|_{U_1} \equiv 0$. Assume the result true for $r-1$ connected open subsets U_j and let us check that it is also true for r. We have proved that $f|_{U_1} \equiv 0$. Observe that

$f \circ (\pi|_{U_2})^{-1}$ is analytic on the connected open subset $\pi(U_2)$ and vanishes on the non-empty open subset $\pi(U_1 \cap U_2) \subset \pi(U_2)$. By the Identity Principle V.1.4 $f \circ (\pi|_{U_2})^{-1}$ vanishes on $\pi(U_2)$, so $f|_{U_2} \equiv 0$. By induction on r we conclude that $f|_{U_r} \equiv 0$, so in particular $f(x) = 0$.

3.b.4 Let us prove: $X^\circ = B$.

As the analytic covering (X, π, W) is irreducible, its regular locus X° is connected, so it is enough to prove that the non-empty set B is open and closed in X°.

With the above notation, given $x \in B$ we have $x \in U_r \subset B$, so B is open in X°. Let $z \in X^\circ \setminus B$. As X is locally connected, there exists a connected open neighborhood $V \subset X^\circ$ of z such that $\pi|_V : V \to \pi(V)$ is a homeomorphism and $f \circ (\pi|_V)^{-1} : \pi(V) \to \mathbb{C}$ is analytic. Suppose that $B \cap V \neq \varnothing$ and let $x \in B \cap V$. Define $U_{r+1} := V$ and observe that $x \in U_r \cap U_{r+1}$ and $z \in U_{r+1}$. Thus, $z \in B$, which is a contradiction. Consequently, $B \cap V = \varnothing$, so $z \notin \mathrm{Cl}_{X^\circ}(B)$. We conclude B is open and closed in X°, as required. □

Corollary VII.3.14 (Maximum modulus principle II) *Let (X, π, W) be an irreducible analytic covering and let $f : X \to \mathbb{C}$ be an analytic function with respect to (X, π, W). Suppose that $|f| : X \to \mathbb{R}$ has a local maximum. Then f is constant.*

Proof. Let $a \in X$ be a point at which $|f| : X \to \mathbb{R}$ attains a local maximum. Then there exists an open neighborhood $U_1 \subset X$ of a such that $|f(x)| \leq |f(a)|$ for each $x \in U_1$. By Theorem VII.3.1 there exists a connected open neighborhood $U \subset U_1$ such that $(U, \pi|_U, \pi(U))$ is an analytic covering. The restriction map $|f||_U : U \to \mathbb{R}$ attains at $a \in U$ an absolute maximum, so by Corollary VII.3.11 $f|_U$ is constant. By Theorem VII.3.13 we conclude that also f is constant, as required. □

4 Weak local parameterization theorem

Along the whole section we fix a non-empty open subset Ω of $\mathbb{C}^n$. Our purpose in this section and the next one is to provide geometric content to the Local

Parameterization Theorem II.3.1. The main tool will be the use of analytic coverings.

4.a Proof of the weak local parameterization theorem. We state and prove next the weak local parameterization theorem. This geometric interpretation of Theorem II.3.1 will be enough to deduce some important consequences concerning analytic sets: complex analytic sets are locally connected, their subset of regular points are dense and the intersection of an arbitrary family if analytic sets is again an analytic set.

Given an analytic map $f := (f_1, \dots, f_n) : \Omega \to \mathbb{C}^n$ where $\Omega \subset \mathbb{C}^d$ we define the *Jacobian matrix of f at a point $a \in \Omega$* as

$$\operatorname{Jac}(f) := \Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(a)\Big)_{\substack{1\le i\le n\\ 1\le j\le d}}.$$

Theorem VII.4.1 (Weak local parameterization theorem) *Let $X \subset \Omega$ be an analytic subset of Ω such that $0 \in X$ and X_0 is a d-dimensional irreducible germ. Then there exist a linear projection $\pi : \mathbb{C}^n \to \mathbb{C}^d$ and a connected open neighborhood $U \subset \Omega$ of the origin, whose image under π is denoted $W := \pi(U)$, that satisfy the following properties:*

(i) *There exists $\Delta \in \mathcal{O}(W) \setminus \{0\}$ such that $(X \cap U, \pi|_{X\cap U}, W)$ is a quasianalytic covering with critical set $A := Z_W(\Delta)$.*

(ii) *The regular locus $X^\circ := (X \cap U) \setminus \pi^{-1}(A)$ is a d-dimensional analytic submanifold with finitely many connected components.*

(iii) *Let C be a connected component of the regular locus X°. Then $\pi(C) = W \setminus A$ and $0 \in \operatorname{Cl}_{X\cap U}(C)$.*

Proof. The proof is conducted in several steps:

4.a.1 *Initial preparation.* Write $\mathtt{x}' := (\mathtt{x}_1, \dots, \mathtt{x}_d)$. As X_0 is an irreducible germ, $\mathfrak{p} := \mathcal{J}(X_0)$ is a prime ideal of $\mathcal{O}_0 \cong \mathcal{O}_n$. By Theorem II.3.1 and Lemma VI.2.16 applied to $\mathfrak{p}$, after a linear change of coordinates, the following properties hold:

(i) There exist distinguished polynomials $P_{d+1}, \dots, P_n \in \mathcal{O}_d[\mathtt{t}]$ such that $P_i(\mathtt{x}', \mathtt{x}_i) \in \mathfrak{p}$ for $i = d+1, \dots, n$.

(ii) Denote $P := P_{d+1}(\mathtt{x}', \mathtt{x}_{d+1})$ and let Δ be its discriminant. Then $\Delta \notin \mathfrak{p}$.

(iii) There exist polynomials $Q_j \in \mathcal{O}_d[\mathtt{t}]$ for $j = d+2, \dots, n$ such that the ideal

$$\mathfrak{a} := \{P,\, \Delta(\mathtt{x}')\mathtt{x}_{d+2} - Q_{d+2}(\mathtt{x}', \mathtt{x}_{d+1}), \dots, \Delta(\mathtt{x}')\mathtt{x}_n - Q_n(\mathtt{x}', \mathtt{x}_{d+1})\}\mathcal{O}_n$$

satisfies $\Delta^q\mathfrak{p} \subset \mathfrak{a} \subset \mathfrak{p}$ for some $q \geq 1$ and

$$\Delta^q \cdot P_j\Big(\mathtt{x}', \frac{Q_j(\mathtt{x}', \mathtt{x}_{d+1}}{\Delta}\Big) \in \mathfrak{a} \quad \text{for} \quad j = d+2, \dots, n.$$

Denote $\pi : \mathbb{C}^n \to \mathbb{C}^d$, $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_d)$ the projection onto the first d coordinates. It is enough to prove: *there exists an open neighborhood $U \subset \Omega$ of the origin such that the following conditions hold:*

(1) *The triple $(X \cap U, \pi|_{X\cap U}, W := \pi(U))$ is a quasianalytic covering with critical set $A := Z_W(\Delta)$.*

(2) *The regular locus $X^\circ = (X \cap U) \setminus \pi^{-1}(U)$ of the covering has finitely many connected components C_j. In addition, each C_j is a d-dimensional analytic submanifold whose closure contains the origin and $\pi(C_j) = W$.*

4.a.2 *Let us find suitable equations of a representative of the germ $X_0 \setminus Z(\Delta)$.*

As $\Delta^q\mathfrak{p} \subset \mathfrak{a} \subset \mathfrak{p}$, we have

$$X_0 = Z(\mathcal{J}(X_0)) = Z(\mathfrak{p}) \subset Z(\mathfrak{a}) \subset Z(\Delta^q\mathfrak{p}) = Z(\Delta) \cup Z(\mathfrak{p}).$$

Consequently,

$$\begin{aligned} X_0 \setminus Z(\Delta) &= Z(\mathfrak{a}) \setminus Z(\Delta) \\ &= Z(P,\, \Delta\mathtt{x}_j - Q_j(\mathtt{x}', \mathtt{x}_{d+1}),\, j = d+2, \dots, n) \setminus Z(\Delta). \end{aligned}$$

To ease notation we denote with the same symbol a function germ at the origin an a representative of such a germ in an small enough neighborhood of the origin. Denote $x' := (x_1, \dots, x_d)$ the points in $\mathbb{C}^d$ and $x'' := (x_{d+1}, \dots, x_n)$ the points in $\mathbb{C}^{n-d}$. It holds $X_0 = Z(\mathfrak{p}) \subset Z(P_{d+1}, \dots, P_n)$. Thus, there exists an open neighborhood $U_1 \subset \Omega$ of the origin such that

$$X \cap U_1 \subset Z_{U_1}(P_{d+1}, \dots, P_n)$$

and the following equality holds

$$\begin{aligned} X \cap U_1 \setminus Z_{U_1}(\Delta) = \Big\{&(x', x'') \in U_1,\, \Delta(x') \neq 0,\, P(x', x_{d+1}) = 0, \\ &x_j = \frac{Q_j(x', x_{d+1})}{\Delta(x')} \quad \text{for } j = d+2, \dots, n\Big\}. \end{aligned}$$

Let $\rho_0 := (\rho, \dots, \rho) > 0$ be such that U_1 contains $\Delta_{\rho_0}(0) := \prod_{i=1}^n D_\rho(0)$, where $D_\rho(0)$ is the disc of $\mathbb{C}$ centered at $0 \in \mathbb{C}$ and radius ρ. Write each distinguished polynomial P_j as

$$P_j(\mathtt{x}', \mathtt{t}) := \sum_{i=0}^{d_j-1} a_{i,j}(\mathtt{x}')\mathtt{t}^i + \mathtt{t}^{d_j} \quad \text{where } a_{i,j} \in \mathfrak{m}_d. \tag{4.2}$$

By Lemma V.5.2 there exists $\varepsilon > 0$ such that for $x' := (x_1, \dots, x_d) \in \mathbb{C}^d$ with each $|a_{ij}(x')| < \varepsilon$ all the roots of the polynomials $P_j(x', \mathtt{t}) \in \mathbb{C}[\mathtt{t}]$ belong to $D_{\frac{\rho}{2}}(0)$. As the analytic functions a_{ij} are continuous, there exists $\delta_0 > 0$ such that each $|a_{ij}(x')| < \varepsilon$ whenever $\|x'\|_1 := \sum_{j=1}^d |x'_j| < \delta_0$. Let $\delta := \frac{\delta_0}{d} > 0$. If $x' \in D_\delta(0) \times \overset{d}{\cdots} \times D_\delta(0)$, then $\|x'\|_1 < \frac{\delta_0}{d} \cdot d = \delta_0$. Thus, all the roots of the polynomials $P_j(x', \mathtt{t}) \in \mathbb{C}[\mathtt{t}]$ belong to $D_{\frac{\rho}{2}}(0)$. Let $\eta := \min\{\rho, \delta\} > 0$ and define the open subsets

$$W := D_\eta(0) \times \overset{d}{\cdots} \times D_\eta(0) \subset \mathbb{C}^d \quad \text{and} \quad U := W \times D_\rho(0) \times \overset{n-d}{\cdots} \times D_\rho(0) \subset \mathbb{C}^n.$$

We have $\pi(U) = D$. In addition, as $U \subset U_1$,

$$X \cap U \setminus Z_U(\Delta) = \Big\{(x', x'') \in U,\ \Delta(x') \neq 0,\ P(x', x_{d+1}) = 0, \\ x_j = \frac{Q_j(x', x_{d+1})}{\Delta(x')} \quad \text{for } j = d+2, \dots, n\Big\}$$

and $X \cap U \subset Z_U(P_{d+1}, \dots, P_n)$.

4.a.3 Define $X_1 := X \cap U$ and $X_1^\circ := X_1 \setminus Z_U(\Delta)$. We claim: *the fibers of the map $\pi|_{X_1^\circ} : X_1^\circ \to W \setminus Z_W(\Delta)$ consist on $s := \deg(P)$ points.*

If $a' \in W \setminus Z_W(\Delta)$, the polynomial $P(a', \mathtt{t}) \in \mathbb{C}[\mathtt{t}]$ has s distinct complex roots $a_{1,d+1}, \dots, a_{s,d+1}$. For $j = d+2, \dots, n$ and $k = 1, \dots, s$ we denote

$$a_{k,j} := \frac{Q_j(a', a_{k,d+1})}{\Delta(a')} \in \mathbb{C} \quad \text{and} \quad a_k := (a', a_{k,d+1}, \dots, a_{k,n}) \in \mathbb{C}^n.$$

For $j = d+2, \dots, n$ we have

$$P_j(a_k) = P_j(a', a_{k,j}) = P_j\Big(a', \frac{Q_j(a', a_{k,d+1})}{\Delta(a')}\Big) = 0.$$

Thus, the fiber $(\pi|_{X_1^\circ})^{-1}(a') = \{a_1, \dots, a_s\}$ has s points.

4.a.4 It holds: *the triple* $(X_1^\circ, \pi|_{X_1^\circ}, W \setminus Z_W(\Delta))$ *is an s-sheeted trivial topological covering.*

As the roots of $P(a', \mathtt{t})$ are simple because $\Delta(a') \neq 0$, we have

$$\tfrac{\partial P}{\partial \mathtt{t}}(a', a_{k,d+1}) \neq 0$$

for each $a' \in W \setminus Z_W(\Delta)$ and $k = 1, \ldots, s$. By the Implicit function theorem, there exist open disjoint subsets $\Omega_k \subset \mathbb{C}$, open subsets $W_k \subset W \setminus Z_W(\Delta)$ and analytic functions $\psi_k : W_k \to \Omega_k$ such that $a' \in W_k$, $\Psi_k(a') = a_{k,d+1} \in \Omega_k$ for $k = 1, \ldots, s$ and $P(x', \psi_k(x')) = 0$ for all $x' \in W_k$. Define the open neighborhood $W^{a'} := W_1 \cap \cdots \cap W_s \subset W$ of a' and the sets

$$V_k := \Big\{x \in \pi^{-1}(W^{a'}) : x_{d+1} = \psi_k(x'),\ x_j = \frac{Q_j(x', x_{d+1})}{\Delta(x')},\ j = d+2, \ldots, n\Big\}$$

for $k = 1, \ldots, s$. We have $(\pi|_{X_1^\circ})^{-1}(W^{a'}) = \bigsqcup_{k=1}^s V_k$. Each set V_k is a closed subset of $\pi^{-1}(W^{a'})$, so it is also a closed subset of $(\pi|_{X_1^\circ})^{-1}(W^{a'})$. Thus,

$$V_k = (\pi|_{X_1^\circ})^{-1}(W^{a'}) \setminus \bigsqcup_{i \neq k} V_i$$

is an open subset of X_1°. As each restriction $\pi|_{V_k} \to W^{a'}$ is a homeomorphism whose inverse is the continuous map

$$\varphi_k : W^{a'} \to V_k,\ x' \mapsto \Big(x', \psi_k(x'), \frac{Q_j(x', \psi_k(x'))}{\Delta(x')}, j = d+2, \ldots, n\Big),$$

statement 4.a.4 follows.

4.a.5 Let us prove: X_1° *is an analytic d-dimensional submanifold with finitely many connected components.*

Let $a \in X_1^\circ$ and let $a' := \pi(a) \in W \setminus Z_W(\Delta)$ be its projection. Thus, there exist a root $a_{k,d+1}$ of the polynomial $P(a', \mathtt{t})$ such that

$$a = \Big(a', a_{k,d+1}, \frac{Q_{d+2}(a', a_{k,d+1})}{\Delta(a')}, \ldots, \frac{Q_n(a', a_{k,d+1})}{\Delta(a')}\Big),$$

an open neighborhood $W^{a'} \subset \mathbb{C}^d$, an open subset V_k of X_1° and an analytic diffeomorphism $\varphi_k : W^{a'} \to V_k$ with Jacobian matrix $\operatorname{Jac}(\varphi_k)(x') = (I_d \,|\, *)$ at each $x' \in W^{a'}$. Therefore, $\operatorname{Jac}(\varphi_k)(x')$ has constant rank d on $W^{a'}$. Consequently, X_1° is an analytic submanifold.

As W is a connected open subset and $Z_W(\Delta) \neq W$ because $\Delta \neq 0$, the difference $W \setminus Z_W(\Delta)$ is by Corollary V.5.6 connected. By Proposition VII.1.3 and 4.a.4 X_1° has at most s connected components. Such connected components are open subsets of the d-dimensional analytic submanifold X_1°, so they are d-dimensional submanifolds.

4.a.6 We have: $\pi|_{X_1} : X_1 \to W$ *is proper, surjective and its fibers are finite.*

As $Z_W(\Delta)$ is by Corollary V.5.6 a thin subset of W, once we prove 4.a.6 we conclude that $(X_1, \pi|_{X_1}, W)$ is a quasianalytic covering.

Let us prove first: *the fibers of* $\pi|_{X_1}$ *are finite.*

Given $a' \in W$ and a point $x \in X_1$ with $\pi(x) = a'$ we have

$$x = (a', x_{d+1}, \dots, x_n) \in X_1 \subset Z_U(P_{d+1}, \dots, P_n).$$

Therefore, $P_j(a', x_j) = 0$ for $d+1 \leq j \leq n$ and this implies that

$$\#(\pi^{-1}(a')) \leq \prod_{j=d+1}^{n} \deg(P_j),$$

so $\#(\pi^{-1}(a'))$ is a finite set.

Secondly, we show: $\pi|_{X_1} : X_1 \to W$ *is a proper map.*

As both X_1 and W are locally compact and Hausdorff topological spaces, it is enough to check that $(\pi|_{X_1})^{-1}(K)$ is compact for each compact subset K of W. As K is closed in $\mathbb{C}^d$, the preimage $\pi^{-1}(K)$ is a closed subset of $\mathbb{C}^n$, so $(\pi|_{X_1})^{-1}(K) = \pi^{-1}(K) \cap X_1$ is a closed subset of X_1 and the latter set is closed in U. Thus, $(\pi|_{X_1})^{-1}(K)$ is closed in U. Denote $\overline{D}_{\frac{\varrho}{2}}(0) \subset \mathbb{C}$ the closed disc of radius $\rho/2$ centered at the origin and consider the compact set

$$\Sigma := K \times \overline{D}_{\frac{\varrho}{2}}(0) \times \overset{n-d}{\cdots} \times \overline{D}_{\frac{\varrho}{2}}(0).$$

As the roots of the polynomials $P_j(x', \mathtt{t}) \in \mathbb{C}[\mathtt{t}]$ are contained in $D_{\frac{\varrho}{2}}(0)$ for each $x' \in K$ and $j = d+1, \dots, n$, we deduce $(\pi|_{X_1})^{-1}(K) \subset \Sigma \subset U$. As $(\pi|_{X_1})^{-1}(K)$ is a closed subset of the compact set Σ, we conclude $(\pi|_{X_1})^{-1}(K)$ is compact.

To finish the proof of 4.a.6 we show: $\pi|_{X_1} : X_1 \to W$ *is a surjective map.*

As $\pi|_{X_1}$ is a closed map and $\pi(X_1^\circ) = W \setminus Z_W(\Delta)$ is a dense subset of W,

$$\begin{aligned} W = \mathrm{Cl}_W(W \setminus Z_W(\Delta)) &= \mathrm{Cl}_W(\pi(X_1^\circ)) \\ &= \pi(\mathrm{Cl}_U(X_1^\circ)) \subset \pi(X_1) \subset \pi(U) = W. \end{aligned}$$

4.a.7 To finish we prove: *the intersection $X_1 \cap \mathrm{Cl}(C)$ contains the origin for each connected component C of X°.*

As the origin $0 \in W = W \cap \mathrm{Cl}(W \setminus Z_W(\Delta))$, there exists a sequence $\{y_m\}_{m\in\mathbb{N}} \subset W \setminus Z_W(\Delta)$ such that $\lim_{m\to\infty}\{y_m\} = 0$. As the projection $\pi|_{X_1} : X_1 \to W$ is a proper map and $K := \{0\} \cup \{y_m\}_{m\in\mathbb{N}}$ is a compact set, its preimage $X_1 \cap \pi^{-1}(K)$ is a compact space. By Proposition VII.1.3 the triple $(C, \pi|_C, W \setminus Z_W(\Delta))$ is a topological covering and for each $m \in \mathbb{N}$ there exists $x_m \in C$ such that $\pi(x_m) = y_m$. As $X_1 \cap \pi^{-1}(K)$ is compact, we may assume $\{x_m\}_{m\in\mathbb{N}} \subset C$ converges to some point $x \in X_1 \cap \pi^{-1}(K) \cap \mathrm{Cl}(C)$. Thus,

$$0 = \lim_{m\to\infty}\{y_m\} = \lim_{m\to\infty}\{\pi(x_m)\} = \pi(\lim_{m\to\infty}\{y_m\}) = \pi(x).$$

Consequently,

$$x := (0, x_{d+1}, \ldots, x_n) \in \pi^{-1}(0) \cap X_1 \subset Z_U(P_{d+1}, \ldots, P_n).$$

By Equation (4.2) we have $x_j^{d_j} = P_j(0, x_j) = 0$ for $j = d+1, \ldots, n$, so each $x_j = 0$. Thus, $x = 0 \in X_1 \cap \mathrm{Cl}(C)$, as required. □

Remarks VII.4.2 (i) With the notations of Theorem VII.4.1,

$$\dim(X_0 \cap Z(\Delta)) < \dim(X_0).$$

As $\Delta \notin \mathfrak{p} = \mathcal{J}(X_0)$, we have $X_0 \not\subset Z(\Delta)$, so $X_0 \cap Z(\Delta) \subsetneq X_0$. As X_0 is an irreducible germ, $\dim(X_0 \cap Z(\Delta)) < \dim(X_0)$ by Proposition VI.3.3.

(ii) Let $g : \mathbb{C}^n \to \mathbb{C}^d$, $z \mapsto (g_1(z), \ldots, g_d(z))$ be an analytic map such that $g(0) = 0$ and denote

$$g^* : \mathcal{O}_d \to \mathcal{O}_n,\ f \mapsto f(g_1, \ldots, g_d) \quad \text{and} \quad \overline{g^*} : \mathcal{O}_d \to \mathcal{O}_n/\mathcal{J}(X_0)$$

the composition of g^* with the canonical epimorphism $\mathcal{O}_n \to \mathcal{O}_n/\mathcal{J}(X_0)$. The linear map π in the statement of Theorem VII.4.1 can be replaced by the analytic map g if $\overline{g^*}$ is an injective and finite homomorphism. This is so because these are the only properties of the homomorphism $\mathcal{O}_d \to \mathcal{O}_n/\mathcal{J}(X_0)$ induced by π that we have used in the previous proof.

4.b Applications of the weak local parameterization theorem. Let us prove some remarkable consequences of the weak local parameterization theorem. Fix an open subset $\Omega \subset \mathbb{C}^n$.

Corollary VII.4.3 *Let $X \subset \Omega$ be an analytic set. Then for each point $a \in X$ there exists an open neighborhood $U \subset \Omega$ of a such that for each analytic subset $Y \subset \Omega$ satisfying $X_a \subset Y_a$ the inclusion $X \cap U \subset Y$ holds.*

Proof. We proceed by induction on $d := \dim(X_a)$. If $d = 0$ there exists an open neighborhood $U \subset \Omega$ of a such that $X \cap U = \{a\}$. If $Y \subset \Omega$ is an analytic set satisfying $X_a \subset Y_a$, we have $X \cap U = \{a\} \subset Y \cap U \subset Y$.

Suppose $d := \dim(X_a) \geq 1$ and the result is proved for germs of dimension $e \leq d - 1$. We distinguish two cases:

Case 1. *Irreducible case.* Assume first that the germ X_a is irreducible. By Theorem VII.4.1 there exist an open neighborhood $V \subset \Omega$ of a and an analytic function $\Delta \in \mathcal{O}(V) \setminus \{0\}$ such that $X \cap V \setminus Z_V(\Delta)$ is a d-dimensional analytic submanifold whose connected components $C_1, \dots, C_m$ are d-dimensional analytic submanifolds satisfying $a \in \mathrm{Cl}(C_j)$ for $1 \leq j \leq m$.

By Remark VII.4.2 $\dim(Z(\Delta) \cap X_a) < d$ and by the inductive hypothesis there exists an open neighborhood $U \subset V$ of a such that for each analytic set $Z \subset V$ satisfying $Z(\Delta) \cap X_a \subset Z_a$ we have

$$Z_U(\Delta) \cap X = Z_V(\Delta) \cap X \cap U \subset Z.$$

Let $Y \subset \Omega$ be an analytic subset such that $X_a \subset Y_a$. Thus, $Z(\Delta) \cap X_a \subset Y_a$, so $Z_U(\Delta) \cap X \subset Y$. As $X_a \subset Y_a$, there exists an open neighborhood $V_1 \subset V$ of a such that $X \cap V_1 \subset Y \cap V_1$. As $a \in \mathrm{Cl}(C_i)$ for each connected component C_i of $X \cap U \setminus Z_U(\Delta)$, we have

$$\varnothing \neq C_i \cap V_1 \subset X \cap V_1 \subset Y \cap V_1 \subset Y.$$

Thus, Y contains the non-empty subset $C_i \cap V_1$ of the connected analytic submanifold C_i. By Corollary V.3.6 $C_i \subset Y$, so

$$X \cap U = (Z_U(\Delta) \cap X) \cup (X \cap U \setminus Z_U(\Delta)) \subset (Z_U(\Delta) \cap X) \cup \bigcup_{i=1}^{m} C_i \subset Y.$$

Case 2. *General case.* Let $X_{1,a}, \dots, X_{r,a}$ be the irreducible components of the germ X_a and let $X_1, \dots, X_r$ be representatives of these germs that are analytic subsets of an open subset $\Omega' \subset \Omega$. By Case 1 there exists an open neighborhood $U \subset \Omega$ of a such that for each analytic subset $Y \subset \Omega$ satisfying $X_{i,a} \subset Y_a$ for $i = 1, \dots, r$ the inclusion $X_i \cap U \subset Y$ holds for $i = 1, \dots, r$.

Let $Y \subset \Omega$ be an analytic set such that $X_a \subset Y_a$. Then $X_{i,a} \subset Y_a$ for $i = 1, \dots, r$, so $X_i \cap U \subset Y$ for $i = 1, \dots, r$. After shrinking U if necessary,

$$X \cap U = \Big(\bigcup_{i=1}^{r} X_i \Big) \cap U = \bigcup_{i=1}^{r} (X_i \cap U) \subset Y,$$

as required. $\square$

Corollary VII.4.4 *Let $\{X_i\}_{i \in I}$ be a family of analytic subsets of Ω. Then $X := \bigcap_{i \in I} X_i$ is an analytic subset of Ω.*

Proof. The set X is closed in Ω because each X_i is closed. Thus, it is enough to prove that X is locally analytic. Let $a \in X$ and consider the family $\mathcal{F}$ of ideals $\sum_{i \in J} \mathcal{J}(X_{i,a})$ in $\mathcal{O}_a$, where $J \subset I$ is a finite set. As $\mathcal{O}_a$ is a noetherian ring, there exists a finite set $H \subset I$ such that $\mathfrak{a} := \sum_{i \in H} \mathcal{J}(X_{i,a})$ is a maximal element in the family $\mathcal{F}$. Let $\Omega_1 \subset \Omega$ be an open neighborhood of a such that there exists an analytic subset $Y \subset \Omega_1$ satisfying $Y_a = Z(\mathfrak{a})$. As $\mathfrak{a}$ is a maximal element in $\mathcal{F}$, we have $\mathcal{J}(X_{i,a}) \subset \mathfrak{a}$ for each $i \in I$, so $Y_a = Z(\mathfrak{a}) \subset Z(\mathcal{J}(X_{i,a})) = X_{i,a}$ for each $i \in I$. By Corollary VII.4.3 there exists an open neighborhood U_1 of a such that $Y \cap U_1 \subset X_i \cap U_1$ for each $i \in I$, so $Y \cap U_1 \subset X \cap U_1$. On the other hand,

$$Z(\mathfrak{a}) = Z\Big(\sum_{i \in H} \mathcal{J}(X_{i,a}) \Big) = \bigcap_{i \in H} Z(\mathcal{J}(X_{i,a})) = \bigcap_{i \in H} X_{i,a},$$

so there exists an open neighborhood $U_2 \subset U_1$ of a in $\mathbb{C}^n$ such that

$$\Big(\bigcap_{i \in H} X_i \Big) \cap U_2 = Y \cap U_2.$$

Consequently,

$$\begin{aligned} Y \cap U_2 = (Y \cap U_1) \cap U_2 \subset (X \cap U_1) \cap U_2 & \\ = X \cap U_2 \subset \Big(\bigcap_{i \in H} X_i \Big) \cap U_2 = Y \cap U_2. \end{aligned}$$

Thus, $X \cap U_2 = Y \cap U_2$ and, as $Y \subset \Omega_1$ is analytic, there exist an open neighborhood $U \subset U_2$ of a and $f_1, \dots, f_r \in \mathcal{O}(U)$ such that $Y \cap U = Z_U(f_1, \dots, f_r)$. Consequently, $X \cap U = Y \cap U = Z_U(f_1, \dots, f_r)$, as required. $\square$

Corollary VII.4.5 *Every analytic set is locally connected.*

Proof. Let $X \subset \Omega$ be an analytic set and let $a \in X$. It is enough to prove by induction on $d := \dim(X_a)$ that there exists a connected neighborhood $V \subset X$ of a. If $d = 0$ there exists an open subset $U \subset \Omega$ such that $X \cap U = \{a\}$, so $V := \{a\}$ is a connected neighborhood of a in X. Let $d \geq 1$ and assume the result is proved for analytic sets whose germ at the point a has dimension strictly smaller than d.

Case 1. *X_a is an irreducible germ.* By Theorem VII.4.1 there exist an open neighborhood U of a and a non-zero analytic function $\Delta \in \mathcal{O}(U)$ such that the connected components $C_1, \dots, C_m$ of $X \cap U \setminus Z_U(\Delta)$ satisfy $a \in \mathrm{Cl}(C_j)$. By Remark VII.4.2 $\dim(X \cap Z_U(\Delta)) < d$. If $a \notin Z_U(\Delta)$, we may assume by Theorem VII.4.1 that $X \cap U = C_1$ is a connected analytic submanifold. If $a \in Z_U(\Delta)$, by the induction hypothesis there exists an open neighborhood $W \subset U$ of a such that $X \cap Z_W(\Delta)$ is a connected set that contains a. Also $\mathrm{Cl}_U(C_i)$ is a connected set and $a \in \mathrm{Cl}_U(C_i) \cap \mathrm{Cl}_U(C_j)$ for each pair of indices i, j. Thus, the union

$$V := (X \cap Z_W(\Delta)) \cup \bigcup_{i=1}^{m} \mathrm{Cl}_U(C_i) \subset X$$

is connected and

$$\begin{aligned} a \in X \cap W = X \cap U \cap W &= ((X \cap Z_U(\Delta)) \cup (X \cap U \setminus Z_U(\Delta))) \cap W \\ &= (X \cap Z_W(\Delta)) \cup ((X \cap U \setminus Z_U(\Delta)) \cap W) \\ &\subset (X \cap Z_W(\Delta)) \cup \bigcup_{i=1}^{m} \mathrm{Cl}_U(C_i) = V. \end{aligned}$$

This proves that $V \subset X$ is a connected neighborhood of a.

Case 2. In the general case let $X_{1,a}, \dots, X_{r,a}$ be the irreducible components of the germ X_a. Let $W \subset X$ be an open neighborhood of a such that there exist analytic subsets $X_1, \dots, X_r$ of W that are representatives of the germs $X_{1,a}, \dots, X_{r,a}$ and $X \cap W = X_1 \cup \cdots \cup X_r$. As $X_{1,a}, \dots, X_{r,a}$ are irreducible germs, there exists a connected neighborhood $V_i \subset X_i$ of a such that $X_i \cap V_i$ is connected for $i = 1, \dots, r$. The union $V := \bigcup_{i=1}^{r} V_i$ is connected because $a \in V_i$ for each $i = 1, \dots, r$. Let $U \subset W$ be an open neighborhood of a such that $X_i \cap U \subset V_i$ for $i = 1, \dots, r$. Thus,

$$X \cap U = (X \cap W) \cap U = \Big(\bigcup_{i=1}^{r} X_i\Big) \cap U = \bigcup_{i=1}^{r} X_i \cap U \subset \bigcup_{i=1}^{r} V_i = V,$$

as required. □

Proposition VII.4.6 *Let $X \subset \Omega$ be an analytic set. Let $a \in X$ and denote $d := \dim(X_a)$. Then*

(i) $a \in \mathrm{Cl}_X(\mathrm{Reg}_d(X))$.

(ii) *There exists an open neighborhood $U \subset \Omega$ of a such that $\dim(X_b) \leq d$ for each point $b \in X \cap U$.*

Proof. (i) Let $V \subset \Omega$ be an open neighborhood of a such that there exist analytic subsets $X_1, \ldots, X_r$ of V whose germs $X_{1,a}, \ldots, X_{r,a}$ are the irreducible components of X_a and satisfy $X \cap V = X_1 \cup \cdots \cup X_r$. By Proposition VI.3.3 we may assume $\dim(X_{1,a}) = d$. As the germ $X_{1,a}$ is irreducible, there exists by Theorem VII.4.1 an open neighborhood $U \subset V$ of a and an analytic function $\Delta \in \mathcal{O}(U) \setminus \{0\}$ such that the difference $(X_1 \cap U) \setminus Z_U(\Delta)$ is a d-dimensional analytic submanifold.

4.b.1 We claim: $(X_1 \cap W) \setminus Z_W(\Delta) \not\subset \bigcup_{i=2}^r X_i$ *for each open neighborhood $W \subset U$ of a.*

Otherwise, there exists an open neighborhood $W \subset U$ of a such that

$$X_1 \cap W = (X_1 \cap Z_W(\Delta)) \cup ((X_1 \cap W) \setminus Z_W(\Delta)) \subset (X_1 \cap Z_W(\Delta)) \cup \bigcup_{i=2}^r X_i,$$

so $X_{1,a} = (X_1 \cap W)_a \subset (X_{1,a} \cap Z(\Delta)) \cup \bigcup_{i=2}^r X_{i,a}$. As $X_{1,a}$ is irreducible, we deduce by Proposition VI.2.14 that either $X_{1,a} \subset X_{1,a} \cap Z(\Delta)$ or $X_{1,a} \subset X_{k,a}$ for some index $k = 2, \ldots, r$. As $\dim(X_{1,a} \cap Z(\Delta)) < d = \dim(X_{1,a})$, we may assume $X_{1,a} \subset X_{2,a}$, which is a contradiction.

Thus, $W_1 := W \setminus (X_2 \cup \cdots \cup X_r \cup Z_W(\Delta))$ is an open subset of $\mathbb{C}^n$ and $X_1 \cap W_1$ is a non-empty open subset of X_1. Pick a point $b \in X_1 \cap W_1$ and observe that

$$X_b = ((X_1 \cup \cdots \cup X_r) \cap W_1)_b = X_{1,b}.$$

As $b \notin Z_W(\Delta)$, we have $b \in \mathrm{Reg}_d(X) \cap W$. Thus, every open neighborhood $W \subset U$ of a meets the set $\mathrm{Reg}_d(X)$ of d-dimensional regular points of X, so $a \in \mathrm{Cl}_X(\mathrm{Reg}_d(X))$.

(ii) Assume first that the germ X_a is irreducible. Suppose by contradiction that for each open neighborhood $V \subset \Omega$ of a there exists a point $b \in X \cap V$

such that $\dim(X_b) > d$. By Theorem VII.4.1 there exist an open neighborhood $U \subset \Omega$ of a, a function $\Delta \in \mathcal{O}(U) \setminus \{0\}$ and an analytic map $\pi : \mathbb{C}^n \to \mathbb{C}^d$ such that $(X \cap U, \pi|_{X\cap U}, \pi(X \cap U))$ is a quasianalytic covering whose regular locus $(X \cap U)^\circ$ is a d-dimensional analytic submanifold and whose critical set is $Z_U(\Delta)$. Let $b \in X \cap U$ be such that $\dim(X_b) = e > d$. By part (i) there exists a point $p \in \operatorname{Reg}_e(X) \cap U$, so there exists an open neighborhood $W \subset U$ of p such that $X \cap W$ is an e-dimensional analytic submanifold. Thus,

$$(X \cap U)^\circ \subset \operatorname{Reg}_d(X), \quad X \cap W \subset \operatorname{Reg}_e(X) \quad \text{and} \quad \operatorname{Reg}_d(X) \cap \operatorname{Reg}_e(X) = \varnothing.$$

Consequently, $(X\cap U)^\circ \cap X\cap W = \varnothing$, which is a contradiction because $(X\cap U)^\circ$ is a dense subset of $X \cap U$ and $X \cap W$ is a non-empty open subset of $X \cap U$.

In the general case, let $V \subset \Omega$ be an open neighborhood of a such that there exist analytic subsets $X_1, \dots, X_r$ of V that are representatives of the irreducible components $X_{1,a}, \dots, X_{r,a}$ of X_a and $X \cap V = X_1 \cup \cdots \cup X_r$. Let $U \subset V$ be an open neighborhood of a such that $\dim(X_{i,b}) \le \dim(X_{i,a})$ for each $b \in X \cap U$ and $i = 1, \dots, r$. For each $b \in U$ we have

$$X_b = (X \cap U)_b = (X_1 \cap U)_b \cup \cdots \cup (X_r \cap U)_b = X_{1,b} \cup \cdots \cup X_{r,b}.$$

By Proposition VI.3.3

$$\dim(X_b) = \max\{\dim(X_{i,b})\}_{i=1}^r \le \max\{\dim(X_{i,a})\}_{i=1}^r = \dim(X_a)$$

as required. □

Corollary VII.4.7 *Let $X \subset \Omega$ be an analytic subset and $a \in X$. Then*

(i) $\operatorname{Reg}(X)$ *is an open and dense subset of* X.

(ii) $\dim(X_a) = \max\{e \in \mathbb{N} : a \in \operatorname{Cl}_X(\operatorname{Reg}_e(X))\}$.

(iii) $\dim(X) = \max\{e \in \mathbb{N} : \operatorname{Reg}_e(X) \neq \varnothing\}$.

Proof. (i) As $\operatorname{Reg}(X) = \bigsqcup_{e\ge 0} \operatorname{Reg}_e(X)$, to prove that $\operatorname{Reg}(X)$ is an open subset of X it is enough to show: *each set* $\operatorname{Reg}_e(X)$ *is open in* X.

Pick a point $a \in \operatorname{Reg}_e(X)$. Then X_a is an e-dimensional smooth germ, so there exists an open neighborhood $U \subset \mathbb{C}^n$ of a such that $X \cap U$ is an e-dimensional smooth submanifold of $\mathbb{C}^n$. Hence, $X \cap U$ is an open subset of X contained in $\operatorname{Reg}_e(X)$, so $\operatorname{Reg}_e(X)$ is open in X.

In addition, let $d := \dim(X)$ and for $0 \le e \le d$ define

$$X_{(e)} := \{x \in X : \dim(X_a) = e\}.$$

Then $X = \bigcup_{e=0}^{d} X_{(e)}$ and by Proposition VII.4.6 $X_{(e)} \subset \mathrm{Cl}_X(\mathrm{Reg}_e(X))$. Thus,

$$X = \bigcup_{e=0}^{d} X_{(e)} \subset \bigcup_{e=0}^{d} \mathrm{Cl}_X(\mathrm{Reg}_e(X)) = \mathrm{Cl}_X\Big(\bigcup_{e=0}^{d} \mathrm{Reg}_e(X)\Big) = \mathrm{Cl}_X(\mathrm{Reg}(X)).$$

(ii) By Proposition VII.4.6 there exists an open neighborhood $U \subset \Omega$ of a such that $\dim(X_b) \le \dim(X_a)$ for each point $b \in U \cap X$.

Let $e \ge 0$ be such that $a \in \mathrm{Cl}_X(\mathrm{Reg}_e(X))$. Then $U \cap \mathrm{Reg}_e(X)$ is non-empty and let $b \in U \cap \mathrm{Reg}_e(X) \subset U \cap X$. By the Jacobian Criterion VI.4.3 $e = \dim(X_b) \le \dim(X_a)$, so

$$\max\{e \in \mathbb{N} : a \in \mathrm{Cl}_X(\mathrm{Reg}_e(X))\} \le \dim(X_a).$$

By Proposition VII.4.6 we have $a \in \mathrm{Cl}_X(\mathrm{Reg}_d(X))$ where $d := \dim(X_a)$.

(iii) Let $a \in X$ be such that $\dim(X) = \dim(X_a)$. By part (ii)

$$\begin{aligned}\dim(X) = \dim(X_a) = \max\{e \in \mathbb{N} : a \in \mathrm{Cl}_X(\mathrm{Reg}_e(X))\}& \\ \le \max\{e \in \mathbb{N} : \mathrm{Reg}_e(X) \ne \varnothing\}&.\end{aligned}$$

For the converse inequality, let $e \ge 0$ be such that $\mathrm{Reg}_e(X) \ne \varnothing$. Then there exists a point $b \in \mathrm{Reg}_e(X)$, so $e = \dim(X_b) \le \dim(X)$. $\square$

5 Strong local parameterization theorem

In this section we go deeper in our purpose of giving geometric meaning to Local Parametrization Theorem II.3.1. We will show that each irreducible analytic set germ 'can be understood' as an irreducible analytic covering. Conversely, we show that each analytic covering defines an analytic set. This goes deep into the 'equivalence' between the models for equidimensional analytic germs and analytic coverings.

5.a Preliminary results. Before presenting the strong version of local parameterization theorem we need some preliminary results that have interest by their own.

Theorem VII.5.1 *Let $f := (f_1, \dots, f_n) : \Omega \to D$ be an analytic map that is a homeomorphism between open subsets Ω and D of $\mathbb{C}^n$. Then f is an analytic equivalence.*

Proof. As f is a homeomorphism, the map $f^{-1} : D \to \Omega$ is well-defined and continuous. Consider the analytic function

$$h : \Omega \to \mathbb{C},\ z \mapsto \det\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(z)\Big)_{1 \le i,j \le n}$$

and the analytic subset $X := \{x \in \Omega : h(x) = 0\}$. By the Inverse Function Theorem the restriction $f^{-1}|_{D \setminus f(X)} : D \setminus f(X) \to \Omega$ is analytic, so it is enough to prove: X is empty.

5.a.1 Let us consider the following statements:

(i) *For each point $x \in X$ we have $\frac{\partial f_i}{\partial \mathtt{x}_j}(x) = 0$ for $1 \le i, j \le n$.*

(ii) *For each $x \in X$ the germ X_x has dimension ≤ 0.*

(iii) *The set X is empty.*

We will prove that (i) implies (ii) and that the latter implies (iii). After that, we will see that (i) holds and this way we will be done.

5.a.2 The implication (i) $\Longrightarrow$ (ii) is evident if $X = \varnothing$. Thus, we suppose X is not empty, so by Corollary VII.4.7 $\operatorname{Reg}(X) \neq \varnothing$. It is enough to prove: $\operatorname{Reg}(X) = \operatorname{Reg}_0(X)$. Assume this for a while. By VI.4.4 $\operatorname{Reg}_d(X) = \varnothing$ for each $d > 0$. By Corollary VII.4.7 we conclude $\dim(X) = 0$, so (ii) holds.

Let $x \in \operatorname{Reg}(X)$. Then there exists an open neighborhood $U \subset \mathbb{C}^n$ of x such that $X \cap U$ is a connected analytic submanifold. As each partial derivative $\frac{\partial f_i}{\partial \mathtt{x}_j}|_{X \cap U}$ vanishes identically, it follows from Exercise number V.6 that the map $f|_{X \cap U}$ is constant. As f is injective, $X \cap U = \{x\}$, so $\dim(X_x) = 0$ and by the Jacobian Criterion VI.4.3 $x \in \operatorname{Reg}_0(X)$.

5.a.3 Let us prove (ii) $\Longrightarrow$ (iii). By Remark VI.3.2 (iv) X is a discrete subset of Ω because each $x \in X$ is an isolated point of X. As $f : \Omega \to D$ is a homeomorphism, the set $f(X)$ is discrete, so $f(X)$ is an analytic set with empty interior. In addition, the function

$$g := f^{-1}|_{D \setminus f(X)} : D \setminus f(X) \to \Omega$$

is locally bounded, because $f^{-1} : D \to \Omega$ is continuous. By Riemann's extension theorem V.5.3 applied to each coordinate of g there exists an analytic map $\widetilde{g} : D \to \mathbb{C}^n$ such that $\widetilde{g}|_{D \setminus f(X)} = g = f^{-1}|_{D \setminus f(X)}$. As both g and f^{-1} are continuous maps that coincide in the dense subset $D \setminus f(X)$ of D, we deduce $\widetilde{g} = f^{-1}$. Thus, $X = \varnothing$ because for each $x \in \Omega$ we have $h(x) \neq 0$. To prove that $h(x) \neq 0$ observe

$$1 = \det(d_x \mathrm{id}_\Omega) = \det(d_x(g \circ f)) = \det(d_{f(x)} g \circ d_x f) = \det(d_{f(x)} g) \cdot h(x).$$

5.a.4 Now, we prove (i) by induction on n.

If $n = 1$, we have $h = \frac{\partial f}{\partial \mathtt{x}_1}$, so (i) holds trivially. Let $n > 1$ and suppose that $f(0) = 0$ and $x := 0 \in X$. We will prove that $\frac{\partial f_i}{\partial \mathtt{x}_j}(0)$ for each $i, j = 1, \dots, n$. To that end we show by contradiction: *the rank*

$$r := \operatorname{rk}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\Big)_{1 \leq i,j \leq n}$$

equals zero.

Otherwise, $r > 0$ and we may assume

$$\det\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\Big)_{1 \leq i,j \leq r} \neq 0.$$

As $0 \in X$, we have $h(0) = 0$, so $r < n$. The analytic map

$$F : \Omega \to \mathbb{C}^n,\ z \mapsto (f_1(z), \dots, f_r(z), z_{r+1}, \dots, z_n)$$

satisfies the equalities $F(0) = 0$ and

$$\det\Big(\frac{\partial F_i}{\partial \mathtt{x}_j}(0)\Big)_{1 \leq i,j \leq n} = \det\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(0)\Big)_{1 \leq i,j \leq r} \neq 0.$$

By the Inverse Function Theorem there exist open neighborhoods $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \Omega$ of the origin such that $F|_{\Omega_2} : \Omega_2 \to \Omega_1$ is an analytic equivalence.

In addition, $\Theta_1 = f(\Omega_2)$ is an open subset of $\mathbb{C}^n$ because $f : \Omega \to D$ is a homeomorphism and the composition

$$H := (H_1, \dots, H_n) := f|_{\Omega_2} \circ F^{-1} : \Omega_1 \to \Theta_1$$

is a homeomorphism. We claim: *$H_i(x) = x_i$ for each $x := (x_1, \dots, x_n) \in \Omega_1$ and $i = 1, \dots, r$.*

Let $z := (z_1, \dots, z_n) \in \Omega_2$ such that $x = F(z)$. Then

$$\begin{aligned}(H_1(x), \dots, H_n(x)) &= H(x) = f(F^{-1}(x)) = f(z)\\ &= (f_1(z), \dots, f_r(z), f_{r+1}(z), \dots, f_n(z))\\ &= (F_1(z), \dots, F_r(z), f_{r+1}(z), \dots, f_n(z))\\ &= (x_1, \dots, x_r, f_{r+1}(z), \dots, f_n(z)),\end{aligned}$$

as claimed.

5.a.5 Consider the open subsets Ω_1^* and Ω_2^* of $\mathbb{C}^{n-r}$ defined as

$$\Omega_1^* := \{u \in \mathbb{C}^{n-r} : (0, u) \in \Omega_1\} \quad \text{and} \quad \Theta_1^* := \{u \in \mathbb{C}^{n-r} : (0, u) \in \Theta_1\}.$$

The restriction map

$$H|_{\{0\}\times\Omega_1^*} : \{0\} \times \Omega_1^* \to \{0\} \times \Theta_1^*,\ (0, u) \mapsto (0, H_{r+1}(0, u), \dots, H_n(0, u))$$

is a homeomorphism. Consequently, the well-defined analytic map

$$H^* : \Omega_1^* \to \Theta_1^*,\ u \mapsto (H_{r+1}(0, u), \dots, H_n(0, u))$$

is a homeomorphism. Let us denote

$$h^* : \Omega_1^* \to \mathbb{C},\ u \mapsto \det\Big(\frac{\partial H_i}{\partial \mathtt{x}_j}(0, u)\Big)_{r+1\le i,j\le n}.$$

Consider the analytic set $Z := \{u \in \Omega_1^* :\ h^*(u) = 0\} \subset \Omega_1^*$. By the induction hypothesis $\frac{\partial H_i}{\partial \mathtt{x}_j}(0, u) = 0$ for each $u \in Z$ and $r + 1 \le i, j \le n$. As (i) implies (iii), we deduce that $Z = \varnothing$. As $0 \in X$, we have $\operatorname{Jac}(f)(0) = 0$, so

$$\operatorname{Jac}(H)(0) = \operatorname{Jac}(f)(0) \cdot \operatorname{Jac}(F^{-1})(0) = 0.$$

Consequently,

$$\begin{aligned}0 = \det\Big(\frac{\partial H_i}{\partial \mathtt{x}_j}(z)\Big)_{1\le i,j\le n} &= \det\left(\begin{array}{c|c} I_r & * \\ \hline 0 & \big(\frac{\partial H_i}{\partial \mathtt{x}_j}(0)\big)_{r+1\le i,j\le n}\end{array}\right)\\ &= \det\Big(\frac{\partial H_i}{\partial \mathtt{x}_j}(z)\Big)_{r+1\le i,j\le n} = h^*(0),\end{aligned}$$

so $0 \in Z$, which is a contradiction. Therefore $r = 0$, as required. $\square$

Remark VII.5.2 The statement of Theorem VII.5.1 is false in the real case. The analytic map $f : \mathbb{R} \to \mathbb{R}$, $t \mapsto t^3$ is a homeomorphism, but it is not an analytic equivalence. Otherwise, its inverse $g : \mathbb{R} \to \mathbb{R}$ would be a differentiable function and $1 = f'(0) \cdot g'(0)$. Thus, $f'(0) \neq 0$, which is false.

Lemma VII.5.3 *Let $\Omega \subset \mathbb{C}^n$ and $W \subset \mathbb{C}^d$ be open subsets and let $\pi : \Omega \to W$ be an analytic map. Let $X \subset \Omega$ be a subset such that the triple $(X, \pi|_X, W)$ is an analytic covering whose regular locus is a d-dimensional analytic submanifold. Then the restriction $h|_X : X \to \mathbb{C}$ of each analytic function $h \in \mathcal{O}(\Omega)$ is an analytic function with respect to the covering $(X, \pi|_X, W)$.*

Proof. The regular locus X° of $(X, \pi|_X, W)$ is an analytic submanifold of dimension d. Thus, for each point $a \in X^\circ$ there exists an analytic immersion $f : U \to \mathbb{C}^n$ where $U \subset \mathbb{C}^d$ is an open neighborhood of 0 and $V := f(U) \subset X^\circ$ is an open neighborhood of a.

As $(X^\circ, \pi|_{X^\circ}, W)$ is a topological covering, the map $\pi|_{X^\circ} : X^\circ \to \pi(X^\circ)$ is a local homeomorphism. Thus, there exists an open neighborhood $V_1 \subset V$ of a such that $\pi|_{V_1} : V_1 \to \Omega_1 := \pi(V_1)$ is a homeomorphism. Observe that Ω_1 is an open subset of W because $\pi|_{X^\circ} : X^\circ \to \pi(X^\circ)$ is an open map and $\pi(X^\circ)$ is an open subset of W. In addition, $\Omega_0 := f^{-1}(V_1)$ is an open neighborhood of 0 in $\mathbb{C}^d$ because $f : U \to V$ is a continuous map.

Then $(\pi \circ f)|_{\Omega_0} : \Omega_0 \to \Omega_1$ is an analytic homeomorphism because both $f|_{\Omega_0} : \Omega_0 \to V_1$ and $\pi|_{V_1} : V_1 \to \Omega_1$ are analytic homeomorphisms. By Theorem VII.5.1 the composition $(\pi \circ f)|_{\Omega_0} : \Omega_0 \to \Omega_1$ is an analytic equivalence, so its inverse $(f|_{\Omega_0})^{-1} \circ (\pi|_{V_1})^{-1} : \Omega_1 \to \Omega_0$ is an analytic map. Thus,

$$h \circ (\pi|_{V_1})^{-1} = (h \circ f|_{\Omega_0}) \circ (f|_{\Omega_0})^{-1} \circ (\pi|_{V_1})^{-1} : \Omega_1 \to \mathbb{C}$$

is an analytic function. Consequently, $h|_X$ is an analytic function with respect to the covering $(X, \pi|_X, W)$, as required. □

Theorem VII.5.4 *Let $\pi : \Omega \to W$ be an analytic map between open subsets $\Omega \subset \mathbb{C}^n$ and $W \subset \mathbb{C}^d$. Let X be a subset of Ω such that the triple $(X, \pi|_X, W)$ is an analytic covering whose regular locus is a d-dimensional analytic submanifold. Then X is an analytic subset of Ω.*

Proof. It is enough to prove the following: *For each point $x \in \Omega \setminus X$ there exists $h^x := h \in \mathcal{O}(\Omega)$ such that $h(x) \neq 0$ and $X \subset Z_\Omega(h)$.*

Once this is proved $X = \bigcap_{x \in \Omega \setminus X} Z_\Omega(h^x)$ is by Corollary VII.4.4 an analytic subset of Ω.

Let $x := (x_1, \dots, x_n) \in \Omega \setminus X$. As the fibers of $\pi|_X$ are finite,

$$(\pi|_X)^{-1}(\pi(x)) := \{y_1, \dots, y_r\} \subset X$$

and we write $y_j := (y_{1,j}, \dots, y_{n,j}) \in \mathbb{C}^n$. Note that $x \neq y_j$ for $j = 1, \dots, r$, so there exists $\tau(j) \in \{1, \dots, n\}$ such that $y_{\tau(j),j} \neq x_{\tau(j)}$. Consider the polynomial

$$f(\mathtt{x}) := \prod_{j=1}^{r} (\mathtt{x}_{\tau(j)} - y_{\tau(j),j}) \in \mathbb{C}[\mathtt{x}] := \mathbb{C}[\mathtt{x}_1, \dots, \mathtt{x}_n].$$

We have $f(x) \neq 0$ and $f(x_j) = 0$ for $j = 1, \dots, r$. By Lemma VII.5.3 the restriction map $f|_X : X \to \mathbb{C}$ is an analytic function with respect to the covering $(X, \pi|_X, W)$. Let $P_f \in \mathcal{O}(W)[\mathtt{t}]$ be the annihilating polynomial of the analytic function $f|_X$:

$$P_f(\mathtt{t}) := \mathtt{t}^s + \sum_{i=1}^{s} \sigma_i(f|_X)\mathtt{t}^{s-i} \in \mathcal{O}(W)[\mathtt{t}],$$

where s is the number of sheets of the covering and $\sigma_1, \dots, \sigma_s$ are the elementary symmetric forms. The analytic function we are looking for is

$$h : \Omega \to \mathbb{C},\ z \mapsto P_f(\pi(z), f(z)) = f(z)^s + \sum_{i=1}^{s} \sigma_i(f|_X)(\pi(z))f(z)^{s-j}.$$

By Construction VII.3.10 (ii) $X \subset Z_\Omega(h)$, so let us check: $h(x) \neq 0$.

For $i = 1, \dots, s$

$$\sigma_i(f|_X)(\pi(x)) = \sigma_i(f(y_1), \overset{\mathtt{r}(y_1)}{\dots}, f(y_1), \dots, f(y_r), \overset{\mathtt{r}(y_r)}{\dots}, f(y_r))$$

where $\mathtt{r}(y_j)$ is the ramification index of $(X, \pi|_X, W)$ at the point y_j.

As $f(y_j) = 0$ for $j = 1, \dots, r$, we have $\sigma_i(f|_X)(\pi(x)) = \sigma_i(0, \dots, 0) = 0$ for $i = 1, \dots, s$. Consequently, $h(x) = f(x)^s \neq 0$, as required. $\square$

5.b Proof of strong local parameterization theorem. We are ready to state and prove the strong local parameterization theorem.

Theorem VII.5.5 (Strong local parameterization theorem) *Let Ω be an open subset of $\mathbb{C}^n$ and let $X \subset \Omega$ be an analytic set. Let $a \in X$ be such that X_a is a d-dimensional irreducible germ. We have:*

(i) *There exist a connected open neighborhood $U \subset \Omega$ of a and a linear map $\pi : \mathbb{C}^n \to \mathbb{C}^d$ such that if $W := \pi(U)$ then $(X \cap U, \pi|_{X \cap U}, W)$ is an irreducible analytic covering with critical set $A := Z_W(\Delta)$ for an analytic function $\Delta \in \mathcal{O}(W) \setminus \{0\}$ and its regular locus $(X \cap U) \setminus \pi^{-1}(A)$ is a d-dimensional connected analytic submanifold.*

(ii) *Let $Z \subset X$ be an analytic subset of Ω such that $Z_a \subsetneq X_a$. Then the triple $(X \cap U, \pi|_{X \cap U}, W)$ is an irreducible analytic covering with critical set $B := A \cup \pi(Z \cap U)$.*

Proof. (i) We may assume $a = 0$. The proof of this statement is conducted in several steps:

5.b.1 By Theorem VII.4.1 there exist a connected open neighborhood V_1 of the origin and a linear map $\pi : \mathbb{C}^n \to \mathbb{C}^d$ such that $(X \cap V_1, \pi|_{X \cap V_1}, \pi(V_1))$ is a quasianalytic covering. Denote $X_1 := X \cap V_1$, $\pi_1 := \pi|_{X_1}$ and $W_1 := \pi(V_1) = \pi(X_1)$. In addition, there exists $\Delta \in \mathcal{O}(W_1) \setminus \{0\}$ whose zero set $A := Z_{W_1}(\Delta)$ is a critical set of the covering and if $d = \dim(X_0)$, the regular locus $X_1^\circ := X \cap V_1 \setminus \pi^{-1}(A)$ of (X_1, π_1, W_1) is an analytic submanifold of dimension d with finitely many connected components, whose closures contain the origin. It is enough to prove: *there exists an open neighborhood $V_2 \subset V_1$ of the origin such that $X_2^\circ := X \cap V_2 \setminus \pi^{-1}(A)$ is dense in $X_2 := X \cap V_2$ and connected.*

5.b.2 We claim: *Let C be a connected component of X_1°. Then $\operatorname{Cl}_{X_1}(C)$ is an analytic subset of V_1.*

As C is closed in X_1°,

$$C = \operatorname{Cl}_{X_1^\circ}(C) = \operatorname{Cl}_{X_1}(C) \cap X_1^\circ = \operatorname{Cl}_{X_1}(C) \setminus \pi^{-1}(A).$$

As X_1° has finitely many connected components, C is open in X_1°, so it is also a d-dimensional analytic submanifold. It holds: $(\operatorname{Cl}_{X_1}(C), \pi|_{\operatorname{Cl}_{X_1}(C)}, W_1)$ *is an analytic covering that has A as critical set.*

As C is an open and closed subset of X_1°, the triple $(C, \pi|_C, \pi(C))$ is a topological covering. In addition, by Theorem VII.4.1 $\pi(C) = W_1 \setminus A$. As $\operatorname{Cl}_{X_1}(C) \setminus \pi^{-1}(A) = \operatorname{Cl}_{X_1}(C) \cap X_1^\circ = C$, we conclude that

$$(\operatorname{Cl}_{X_1}(C) \setminus \pi^{-1}(A), \pi_1|_{\operatorname{Cl}_{X_1}(C)\setminus\pi^{-1}(A)}, W_1 \setminus A)$$

is a topological covering. Observe that $\mathrm{Cl}_{X_1}(C)$ is locally compact and Hausdorff and $\pi_1|_{\mathrm{Cl}_{X_1}(C)}$ is proper because $\mathrm{Cl}_{X_1}(C)$ is a closed subset of X_1. In addition, $\pi_1|_{\mathrm{Cl}_{X_1}(C)}$ has finite fibers because π_1 has finite fibers. As $\pi_1|_{\mathrm{Cl}_{X_1}(C)}$ is proper and $\pi(C) = W_1 \setminus A$ is a dense subset of W_1, we conclude that $\pi(\mathrm{Cl}_{X_1}(C)) = W_1$, so $\pi_1|_{\mathrm{Cl}_{X_1}(C)}$ is surjective. Thus, $(\mathrm{Cl}_{X_1}(C), \pi|_{\mathrm{Cl}_{X_1}(C)}, W_1)$ is an analytic covering that has A as critical set. As $\mathrm{Cl}_{X_1}(C)^\circ = C$ is an analytic submanifold of dimension d, the set $\mathrm{Cl}_{X_1}(C)$ is by Theorem VII.5.4 an analytic subset of V_1.

5.b.3 Let us prove: *X_1° is connected.*

Recall that $X_1^\circ := X_1 \setminus \pi^{-1}(A)$ has finitely many connected components $C_1, \dots, C_r$ and $X_1 \cap \pi^{-1}(A) = X \cap Z_{V_1}(\Delta)$. Then

$$\begin{aligned}\bigcup_{j=1}^r \mathrm{Cl}_{X_1}(C_j) \subset X_1 &= (X_1 \setminus \pi^{-1}(A)) \cup (X_1 \cap \pi^{-1}(A)) \\ &= X_1^\circ \cup (X_1 \cap \pi^{-1}(A)) = \bigcup_{j=1}^r C_j \cup (X_1 \cap \pi^{-1}(A)) \\ &\subset \bigcup_{j=1}^r \mathrm{Cl}_{X_1}(C_j) \cup (X \cap Z_{V_1}(\Delta)).\end{aligned}$$

Thus, $X_1 = \bigcup_{j=1}^r \mathrm{Cl}_{X_1}(C_j) \cup (X \cap Z_{V_1}(\Delta))$, so

$$X_0 = X_{1,0} = \bigcup_{j=1}^r (\mathrm{Cl}_{X_1}(C_j))_0 \cup (X_0 \cap Z(\Delta)). \tag{5.3}$$

By Remark VII.4.2 (i) $\dim(X_0 \cap Z(\Delta)) < \dim(X_0)$, so $X_0 \not\subset X_0 \cap Z(\Delta)$. By 5.b.2 each $(\mathrm{Cl}_{X_1}(C_j))_0$ is an analytic germ. As X_0 is an irreducible germ, we may assume by Proposition VI.2.14 and Equation (5.3) that $X_0 \subset (\mathrm{Cl}_{X_1}(C_1))_0$. As $\mathrm{Cl}_{X_1}(C_1) \subset X_1 \subset X$, we have $(\mathrm{Cl}_{X_1}(C_1))_0 \subset X_0$, so

$$(\mathrm{Cl}_{X_1}(C_1))_0 = X_0. \tag{5.4}$$

In particular, each germ $(\mathrm{Cl}_{X_1}(C_j))_0 \subset (\mathrm{Cl}_{X_1}(C_1))_0$. Fix $j = 2, \dots, r$ and let $U_j \subset V_1$ be an open neighborhood of 0 such that $\mathrm{Cl}_{X_1}(C_j) \cap U_j \subset \mathrm{Cl}_{X_1}(C_1) \cap U_j$. Then

$$C_j \cap U_j = \mathrm{Cl}_{X_1}(C_j) \cap X_1^\circ \cap U_j \subset \mathrm{Cl}_{X_1}(C_1) \cap X_1^\circ \cap U_j = C_1 \cap U_j.$$

As $0 \in \mathrm{Cl}_{X_1^\circ}(C_j)$, there exists $x \in C_j \cap U_j \subset C_1 \cap U_j$. Then $x \in C_1 \cap C_j$ and this is false. Consequently, C_1 is the unique connected component of X_1°, so X_1° is connected.

5.b.4 We claim: $X_2^\circ := X_2 \setminus \pi^{-1}(A)$ *is dense in* $X_2 := X \cap V_2$ *for a suitable neighborhood* V_2 *of the origin.*

By Equation (5.4)

$$X_0 = (\mathrm{Cl}_{X_1}(C_1))_0 = (\mathrm{Cl}_{X_1}(X_1^\circ))_0,$$

so there exists an open neighborhood $V_2 \subset V_1$ of the origin such that

$$\mathrm{Cl}_{X_2}(X_2^\circ) = \mathrm{Cl}_{X_1}(X_1^\circ \cap V_2) \cap V_2 = \mathrm{Cl}_{X_1}(X_1^\circ) \cap V_2 = X \cap V_2.$$

Thus, $X_2^\circ := X_2 \setminus \pi^{-1}(A)$ is dense in X_2.

5.b.5 Applying 5.b.1, 5.b.2 and 5.b.3 to the analytic subset $X_2 = X \cap V_2$ of V_2 and using Remark VII.4.2 (ii) we find a connected open neighborhood $U \subset V_2$ of a such that if $W = \pi(U) \subset W_1$ then $(X \cap U, \pi|_{X\cap U}, W)$ is a quasianalytic covering with critical set $A := Z_W(\Delta)$ and whose regular locus $(X \cap U) \setminus \pi^{-1}(A)$ is a d-dimensional connected analytic submanifold. As $X_2^\circ := X_2 \setminus \pi^{-1}(A)$ is by 5.b.4 dense in $X_2 = X \cap V_2$, we deduce that $X \cap U \setminus \pi^{-1}(A)$ is dense in $X \cap U$. Thus, $(X \cap U, \pi|_{X\cap U}, W)$ is an irreducible analytic covering with critical set $A := Z_W(\Delta)$.

(ii) By Lemma VII.3.5 it is enough to prove that $\pi(Z \cap U) \setminus A$ is a thin subset of $W \setminus A$. Denote $X_1 := X \cap U$ and $X_1^\circ := X_1 \setminus \pi^{-1}(A)$.

5.b.6 Let us show: $\pi(Z \cap U) \setminus A$ *is analytic in* $W \setminus A$.

As Z is closed in Ω, it is a closed subset of X, so $Z \cap U$ is closed in $X \cap U$. The map $\pi|_{X\cap U} \to W$ is proper, so $\pi(Z \cap U)$ is closed in W. Hence, $\pi(Z \cap U) \setminus A$ is closed in $W \setminus A$, so it is enough to prove: $\pi(Z \cap U) \setminus A$ *is locally analytic.*

Pick a point $a \in \pi(Z \cap U) \setminus A$. As $\pi|_{X_1^\circ} : X_1^\circ \to W \setminus A$ is a topological covering and $a \in W \setminus A$, there exist an open neighborhood V of a in $W \setminus A$ and open subsets $B_1, \dots, B_s$ of X_1° such that $\pi^{-1}(V) \cap X_1^\circ = \bigsqcup_{j=1}^s B_j$ and each restriction map $\pi|_{B_j} : B_j \to V$ is a homeomorphism.

Note that $Z \cap U \setminus \pi^{-1}(A)$ is analytic in $U \setminus \pi^{-1}(A)$ because Z is analytic in Ω. After shrinking V and B_j if necessary, there exist analytic functions $f_{j,k} \in \mathcal{O}(B_j)$ such that $Z \cap B_j = Z_{B_j}(f_{j,1}, \dots, f_{j,\ell})$ for $j = 1, \dots, s$. As X_1° is an analytic submanifold it is locally connected, so we may assume that each B_j is connected. After shrinking V and B_j once more we may suppose that each composition $h_{j,k} := f_{j,k} \circ (\pi|_{B_j})^{-1} : V \to \mathbb{C}$ is analytic. To prove 5.b.6 it is enough to check:

$$(\pi(Z \cap U) \setminus A) \cap V = \bigcup_{j=1}^{s} Z_V(h_{j,1}, \dots, h_{j,\ell}). \tag{5.5}$$

As $\pi(B_j) = W$, we deduce

$$Z_V(h_{j,1},\dots,h_{j,\ell}) = Z_V(f_{j,1}\circ(\pi|_{B_j})^{-1},\dots,f_{j,\ell}\circ(\pi|_{B_j})^{-1})$$
$$= \pi(Z_{B_j}(f_{j,1},\dots,f_{j,\ell})) = \pi(Z\cap B_j). \quad (5.6)$$

As $Z\cap\pi^{-1}(V)\subset Z\cap U$ and $V\subset W\setminus A$, we have by (5.6)

$$\bigcup_{j=1}^{s} Z_V(h_{j,1},\dots,h_{j,\ell}) = \bigcup_{j=1}^{s}\pi(Z\cap B_j) = \pi\Big(Z\cap\bigcup_{j=1}^{s}B_j\Big) = \pi(Z\cap\pi^{-1}(V))$$
$$= \pi(Z\cap U)\cap\pi(\pi^{-1}(V)) = \pi(Z\cap U)\cap V = (\pi(Z\cap U)\setminus A)\cap V,$$

so (5.5) holds.

5.b.7 Recall that $W\setminus A$ is connected and $\pi(Z\cap U)\setminus A$ is analytic in $W\setminus A$. Thus, to prove that $\pi(Z\cap U)\setminus A$ is a thin subset of $W\setminus A$ it is enough to show: $\pi(Z\cap U)\setminus A\neq W\setminus A$.

Otherwise, $\pi(Z\cap U)\cap V = V$. Consequently $V=\bigcup_{j=1}^{s}Z_V(h_{j,1},\dots,h_{j,\ell})$ and $V_a=\bigcup_{j=1}^{s}Z(h_{j,1,a},\dots,h_{j,\ell,a})$. As the germ V_a is irreducible, we may assume $V_a = Z(h_{1,1,a},\dots,h_{1,\ell,a})$, so $h_{1,k}\equiv 0$ for $k=1,\dots,\ell$. Then each $f_{1,k}|_{B_j}\equiv 0$ and this implies

$$Z\cap B_j = Z_{B_j}(f_{j,1},\dots,f_{j,\ell}) = B_j.$$

Thus, Z contains the non-empty open subset B_j of the connected analytic submanifold X_1°. By Theorem V.3.5, $X_1^\circ\subset Z$. As X_1° is a dense subset of X and Z is a closed subset of Ω, we have $X=\mathrm{Cl}_\Omega(X_1^\circ)\subset\mathrm{Cl}_\Omega(Z)=Z$, so $X_a\subset Z_a$, which is a contradiction. Consequently, $\pi(Z\cap U)\setminus A\neq W\setminus A$, as required.

$\square$

5.c Local parameterization for equidimensional germs. The weak and the strong local parameterization theorems only deal with irreducible analytic germs. However, we can obtain a less demanding statement (we lose the irreducibility of the covering) for a wider class of analytic germs: those whose irreducible components have the same dimension. These analytic germs are called *equidimensional*. The following result together with Theorem VII.5.4 shows the 'equivalence' between analytic coverings and the models for equidimensional analytic germs.

Theorem VII.5.6 (Equidimensional local parametrization) *Let Ω be an open subset of $\mathbb{C}^n$ and let $X \subset \Omega$ be an analytic set. Let $a \in X$ be such that the germ X_a is equidimensional and let $d := \dim(X_a)$. Then there exists an open neighborhood $U \subset \Omega$ of a and a linear projection $\pi : \mathbb{C}^n \to \mathbb{C}^d$ such that the triple $(X \cap U, \pi|_{X\cap U}, W := \pi(U))$ is an analytic covering whose regular locus is a d-dimensional analytic submanifold.*

Proof. The proof is conducted in several steps

5.c.1 Assume for simplicity that a is the origin. Let $\Omega_1 \subset \Omega$ be an open neighborhood of $0 \in \mathbb{C}^n$ and let $X_1, \dots, X_s$ be analytic subsets of Ω_1 such that $X_{1,0}, \dots, X_{s,0}$ are the irreducible components of X_0 and $X \cap \Omega_1 = \bigcup_{j=1}^s X_j$. Note that $\mathfrak{p}_j := \mathcal{J}(X_{j,0})$ is a prime ideal of $\mathcal{O}_n$ for $j = 1, \dots, s$ and by the equidimensionality of X_0 $\operatorname{ht}(\mathfrak{p}_j) = n - \dim(X_{j,0}) = n - d := r$. By Weierstrass's preparation for ideals II.2.12, Noether's Normalization Theorem II.2.17 and the proof of the Primitive Element Theorem we may assume that the $\mathbb{C}$-algebra homomorphisms $\mathcal{O}_d \to B_j := \mathcal{O}_n/\mathfrak{p}_j$, $v \mapsto v + \mathfrak{p}_j$ are injective and finite for $j = 1, \dots, s$. Let $\pi : \mathbb{C}^n \to \mathbb{C}^d$ be the projection onto the first d coordinates.

5.c.2 By 4.a.2 in the proof of Theorem VII.4.1 there exist open neighborhoods $U \subset \Omega_1$ of the origin of $\mathbb{C}^n$ and W of the origin in $\mathbb{C}^d$ and analytic functions $\Delta_1, \dots, \Delta_s \in \mathcal{O}(W) \setminus \{0\}$ such that $\pi(U) = W$ and $(X_j \cap U, \pi|_{X_j \cap U}, W)$ is an irreducible analytic covering for $j = 1, \dots, s$ with critical set $A_j := Z_W(\Delta_j)$ and regular locus $X_j \cap U \setminus \pi^{-1}(A_j) = X_j \cap U \setminus Z_U(\Delta_j)$. In addition, each regular locus is a d-dimensional analytic submanifold. By Remark VII.4.2 (i) $\dim(X_{j,0} \cap Z(\Delta_j)) < \dim(X_{j,0})$. For each $j = 1, \dots, s$ define

$$Z_j := (Z_U(\Delta_j) \cap X_j) \cup \Big(U \cap X_j \cap \bigcup_{i \neq j} X_i\Big) \subset U \cap X_j \tag{5.7}$$

which is an analytic subset of U. We claim: $Z_{j,0} \subsetneq X_{j,0}$.

Otherwise, there would exist an open neighborhood $U' \subset U$ of the origin satisfying

$$X_j \cap U' = Z_j \cap U' = (Z_{U'}(\Delta_j) \cap X_j) \cup \bigcup_{i \neq j} (X_j \cap X_i \cap U').$$

Consequently,

$$X_{j,0} = (Z(\Delta_{j,0}) \cap X_{j,0}) \cup \bigcup_{i \neq j} (X_{j,0} \cap X_{i,0}).$$

As $X_{j,0}$ is an irreducible germ and $\dim(X_{j,0} \cap Z(\Delta_{j,0})) < \dim(X_{j,0})$, there exists an index $i \neq j$ such that $X_{j,0} = X_{j,0} \cap X_{i,0} \subset X_{i,0}$, which is a contradiction.

5.c.3 By Theorem VII.5.5 (ii) each triple $(X_j \cap U, \pi|_{X_j \cap U}, W)$ is an analytic covering with critical set $A'_j := A_j \cup \pi(Z_j)$ and its regular locus is an analytic submanifold of dimension d. In fact, $A'_j = \pi(Z_j)$ because

$$A_j = Z_W(\Delta_j) = \pi(Z_U(\Delta_j) \cap X_j) \subset \pi(Z_j).$$

Define $Z := Z_1 \cup \cdots \cup Z_s$ and $A := \pi(Z) = \bigcup_{j=1}^{s} A'_j \subset \pi(U) = W$. As each A'_j is a thin subset of W, also A is by Exercise number V.20 a thin subset of W. By Lemma VII.3.5 the triple $(X_j \cap U, \pi|_{X_j \cap U}, W)$ is an analytic covering with critical set A and its regular locus is an analytic submanifold of dimension d.

5.c.4 We claim: *the triple $(X \cap U, \pi|_{X \cap U}, W)$ is an analytic covering with critical set A and its regular locus $(X \cap U) \setminus \pi^{-1}(A)$ is an analytic submanifold of dimension d.*

5.c.4.1 The fibers of $\pi|_{X \cap U} \to W$ are finite because $X = \bigcup_{j=1}^{s} X_j \cap U$ and the fibers of each restriction $\pi|_{X_j \cap U} : X_j \cap U \to W$ are finite.

5.c.4.2 To prove that $\pi|_{X \cap U} : X \cap U \to W$ is proper it is enough to show that it is a closed map.

Let C be a closed subset of $X \cap U$. Then $C \cap (X_j \cap U)$ is closed in $X_j \cap U$. As $\pi|_{X_j \cap U} : X_j \cap U \to W$ is a closed map, the set $\pi(C \cap X_j \cap U)$ is closed in W, so $\pi(C)$ is closed in W because

$$\pi(C) = \bigcup_{j=1}^{s} \pi(C \cap X_j \cap U).$$

5.c.4.3 Let us prove: $(X \cap U \setminus \pi^{-1}(A), \pi|_{X \cap U \setminus \pi^{-1}(A)}, W \setminus A)$ *is a topological covering.*

By 5.c.3 each triple

$$(X_j \cap U \setminus \pi^{-1}(A), \pi|_{X_j \cap U \setminus \pi^{-1}(A)}, W \setminus A)$$

is a topological covering. Let us prove: *the family $\{X_j \cap U \setminus \pi^{-1}(A)\}_{j=1}^{s}$ is a cover of $X \cap U \setminus \pi^{-1}(A)$ by pairwise disjoint open subsets.*

Given $j \neq k$ we have

$$\begin{aligned}(X_j \cap U \setminus \pi^{-1}(A)) &\cap (X_k \cap U \setminus \pi^{-1}(A)) \\ &= X_j \cap X_k \cap U \setminus \pi^{-1}(A) \subset X_j \cap X_k \cap U \setminus Z_j = \varnothing\end{aligned}$$

because $Z_j \subset Z \subset \pi^{-1}(A)$ and by Equation (5.7) $X_j \cap X_k \cap U \subset Z_j$.

5.c.4.4 To prove that $(X \cap U, \pi|_{X\cap U}, W)$ is an analytic covering it is enough to show: *the regular part $X \cap U \setminus \pi^{-1}(A)$ is dense in $X \cap U$.*

Notice that the difference $X_j \cap U \setminus \pi^{-1}(A)$ is a dense subset of $X_j \cap U$ because $(X_j \cap U, \pi|_{X_j\cap U}, W)$ is by 5.c.3 an analytic covering with critical set A. Consequently,

$$
\begin{aligned}
X \cap U &= \bigcup_{j=1}^{s} X_j \cap U \subset \bigcup_{j=1}^{s} \mathrm{Cl}_{X\cap U}(X_j \cap U \setminus \pi^{-1}(A)) \\
&= \mathrm{Cl}_{X\cap U}\Big(\bigcup_{j=1}^{s}(X_j \cap U \setminus \pi^{-1}(A)\Big) = \mathrm{Cl}_{X\cap U}(X \cap U \setminus \pi^{-1}(A)) \subset X \cap U,
\end{aligned}
$$

as required. □

5.d Applications of the strong local parameterization theorem.

Let us prove some remarkable consequences of the strong local parameterization theorem.

Proposition VII.5.7 (Maximum modulus principle) *Let $\Omega \subset \mathbb{C}^n$ be an open set, $f \in \mathcal{O}(\Omega)$ and $X \subset \Omega$ a connected analytic subset of Ω such that the restriction to X of the function $|f| : \Omega \to \mathbb{R}$, $x \mapsto |f(x)|$ attains its maximum value in X. Then $f|_X$ is constant.*

Proof. The proof is conducted in several steps:

5.d.1 Fix a point $a \in X$. Then there exist an open neighborhood $V \subset \Omega$ of a and analytic subsets $X_1, \dots, X_r$ in V such that $X \cap V = X_1 \cup \cdots \cup X_r$ and $X_{1,a}, \dots, X_{r,a}$ are the irreducible components of the germ X_a. As each germ $X_{i,a}$ is irreducible, there exist by Theorem VII.5.5 an open neighborhood $U_i \subset V$ of the point a and a linear map $\pi_i : \mathbb{C}^n \to \mathbb{C}^{d_i}$ where $d_i := \dim(X_{i,a})$ such that the triple $(X_i \cap U_i, \pi_i|_{X_i\cap U_i}, W_i)$ is an irreducible analytic covering and $W_i = \pi_i(U_i)$.

Note that $f|_{X_i\cap U_i} : X_i \cap U_i \to \mathbb{C}$ is an analytic function with respect to the covering $(X_i \cap U_i, \pi_i|_{X_i\cap U_i}, W_i)$ and consider the open neighborhood $U^a := \bigcap_{i=1}^r U_i \subset V \subset \Omega$ of a.

5.d.2 Assume that $|f|$ attains its maximum value in X at a point $c \in X \cap U^a$. We claim: *f is constant on $X \cap U^a$.*

With the notations in 5.d.1 we may assume that $c \in X_1 \cap U^a \subset X_1 \cap U_1$. Then $|f|$ attains at c its maximum value on $X_1 \cap U_1$. By Corollary VII.3.11 f is constant on $X_1 \cap U_1$. Thus, $f \equiv f(a)$ on $X_1 \cap U_1$. As $a \in X_i \cap U_i$ for $i = 1, \dots, r$ and $|f|$ attains its maximum value at a, it follows that f is constant on $X_i \cap U_i$ for $i = 1, \dots, r$, so f is constant on $X \cap U^a$.

5.d.3 Let $a_0 \in X$ be such that the function $|f| : X \to \mathbb{R}$ attains at a_0 its maximum value on X. By 5.d.2 it follows that $f|_{X \cap U^{a_0}}$ is constant. Let us check: $f(a) = f(a_0)$ for each $a \in X$.

As X is connected and $\{X \cap U^a\}_{a \in X}$ is an open covering of X, there exist points $a_1, \dots, a_s := a \in X$ such that $(X \cap U^{a_{i-1}}) \cap (X \cap U^{a_i}) \neq \varnothing$ for $i = 1, \dots, s$. Let us prove by induction on s that: *if f is constant on $X \cap U^{a_0}$ and $f|_{X \cap U^{a_0}} = f(a_0)$, then it is constant on $X \cap \bigcup_{i=0}^{s} U^{a_i}$ and it values $f(a_0)$ at all its points*. In particular, $f(a) = f(a_0)$.

If $s = 0$, there is nothing to prove, so let us assume the result true for $s - 1$ and let us prove that the result is also true for s. As f is constant on $X \cap U^{a_0}$, it is also constant on $(X \cap U^{a_0}) \cap (X \cap U^{a_1})$ and this constant value is $f(a_0)$. Thus, the restriction of $|f|$ to $X \cap U^{a_1}$ attains its maximum value $|f(a_0)|$ at each point $b \in (X \cap U^{a_0}) \cap (X \cap U^{a_1})$. By 5.d.2 $f|_{X \cap U^{a_1}}$ is constant. By induction hypothesis f is constant on $X \cap \bigcup_{i=1}^{s} U^{a_i}$ and it values $f(a_0)$ at all its points and the same happens on $X \cap \bigcup_{i=0}^{s} U^{a_i}$, as required. $\square$

Corollary VII.5.8 *Let $X \subset \Omega$ be a non-empty compact analytic subset of an open subset Ω of $\mathbb{C}^n$. Then X is a finite set. In particular, if X is connected then it is a singleton.*

Proof. The second part is an straightforward consequence of the first one. To prove this, let $\pi_j : \Omega \to \mathbb{C}, x := (x_1, \dots, x_n) \mapsto x_j$ be the projection onto the jth coordinate for $j = 1, \dots, n$. As X is by Corollary VII.4.5 locally connected, its connected components $\{C_i\}_{i \in I}$ are open and closed subsets of X. Thus, they constitute an open cover of the compact set X, so I is finite.

In addition each C_i is compact because it is a closed subset of the compact set X. Hence, the continuous function $|\pi_j|_{C_i}| : C_i \to \mathbb{R}$ attains its maximum value on C_i. By Proposition VII.5.7 each projection $\pi_j|_{C_i}$ is constant, so C_i is a singleton. As I is finite, X is a finite set. $\square$

Remark VII.5.9 Both Proposition VII.5.7 and Corollary VII.5.8 are false in the real case. The unit circumference $\mathbb{S}^1 := \{\mathtt{x}^2 + \mathtt{y}^2 = 1\} \subset \mathbb{R}^2$ is a compact analytic set with infinitely many points. The projection $\pi : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x$ is analytic and the restriction to $\mathbb{S}^1$ of $|\pi|$ attains it maximum value on $\mathbb{S}^1$.

Example VII.5.10 We analyze the strong local parameterization theorem over a two-dimensional example. We work simultaneously over $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$. Define $f : \mathbb{K}^3 \to \mathbb{K}$, $(x, y, z) \mapsto z^2 - xy^2$, which is an analytic map and whose zero set is $X := Z_{\mathbb{K}^3}(f)$, which is an analytic subset of $\mathbb{K}^3$. Let us check first: $\mathcal{J}(X_0) = f_0\mathcal{O}_0$ *is a prime ideal*, so X_0 *is a two-dimensional irreducible germ.*

Observe that $f_0 = \mathtt{z}^2 - \mathtt{x}\mathtt{y}^2$ is a distinguished polynomial in $\mathbb{K}\{\mathtt{x}, \mathtt{y}\}[\mathtt{z}]$ with respect to the indeterminate $\mathtt{z}$. By Eisenstein's Criterion it is irreducible in $\mathbb{K}\{\mathtt{x}, \mathtt{y}\}[\mathtt{z}]$ because $\mathtt{x}$ is irreducible in $\mathbb{K}\{\mathtt{x}, \mathtt{y}\}$ and $\mathtt{x}^2$ does not divide $\mathtt{x}\mathtt{y}^2$. By Lemma II.1.4 f_0 is irreducible in $\mathbb{K}\{\mathtt{x}, \mathtt{y}, \mathtt{z}\}$. As $\mathcal{O}_0 \cong \mathbb{K}\{\mathtt{x}, \mathtt{y}, \mathtt{z}\}$ is by Theorem II.1.3 a unique factorization domain, $f_0\mathcal{O}_0$ is a prime ideal. As $X_0 = Z(f_0)$, we deduce $f_0\mathcal{O}_0 \subset \mathcal{J}(X_0)$.

Observe that $f(u^2, v, uv) \equiv 0$. Thus, X contains the image of the analytic map $\mathbb{K}^2 \to \mathbb{K}^3$, $(u, v) \mapsto (u^2, v, uv)$. Therefore, $\mathcal{J}(X_0) \subset \ker(\varphi)$ where

$$\varphi : \mathbb{K}\{\mathtt{x}, \mathtt{y}, \mathtt{z}\} \to \mathbb{K}\{\mathtt{u}, \mathtt{v}\},\ h \mapsto h(\mathtt{u}^2, \mathtt{v}, \mathtt{u}\mathtt{v}).$$

Let $g_0 \in \ker(\varphi)$. Then $\varphi(f_0) = \varphi(g_0) = 0$ and by Rückert's Division Theorem I.4.5 there exist $Q \in \mathbb{K}\{\mathtt{x}, \mathtt{y}, \mathtt{z}\}$ and $a, b \in \mathbb{K}\{\mathtt{x}, \mathtt{y}\}$ such that $g_0 = f_0 \cdot Q + a\mathtt{z} + b$. Hence,

$$\begin{aligned} 0 = g_0(\mathtt{u}^2, \mathtt{v}, \mathtt{u}\mathtt{v}) = f_0(\mathtt{u}^2, \mathtt{v}, \mathtt{u}\mathtt{v})&Q(\mathtt{u}^2, \mathtt{v}, \mathtt{u}\mathtt{v}) \\ &+ a(\mathtt{u}^2, \mathtt{v})\mathtt{u}\mathtt{v} + b(\mathtt{u}^2, \mathtt{v}) = a(\mathtt{u}^2, \mathtt{v})\mathtt{u}\mathtt{v} + b(\mathtt{u}^2, \mathtt{v}). \end{aligned}$$

Write the series $a, b \in \mathbb{K}\{\mathtt{x}, \mathtt{y}\}$ as follows:

$$a := \sum_{\nu := (\nu_1, \nu_2)} a_\nu \mathtt{x}^{\nu_1}\mathtt{y}^{\nu_2} \quad \text{and} \quad b := \sum_{\nu := (\nu_1, \nu_2)} b_\nu \mathtt{x}^{\nu_1}\mathtt{y}^{\nu_2}.$$

After substituting $\mathtt{x} = \mathtt{u}^2$ and $\mathtt{y} = \mathtt{v}$ we have

$$a(\mathtt{u}^2, \mathtt{v}) = \sum_{\nu := (\nu_1, \nu_2)} a_\nu \mathtt{u}^{2\nu_1}\mathtt{v}^{\nu_2} \quad \text{and} \quad b(\mathtt{u}^2, \mathtt{v}) = \sum_{\nu := (\nu_1, \nu_2)} b_\nu \mathtt{u}^{2\nu_1}\mathtt{v}^{\nu_2}.$$

As $a(\mathtt{u}^2, \mathtt{v})\mathtt{u}\mathtt{v} = -b(\mathtt{u}^2, \mathtt{v})$,

$$\sum_{\nu := (\nu_1, \nu_2)} a_\nu \mathtt{u}^{2\nu_1+1}\mathtt{v}^{\nu_2+1} = -\sum_{\nu := (\nu_1, \nu_2)} b_\nu \mathtt{u}^{2\nu_1}\mathtt{v}^{\nu_2}.$$

All exponents of the indeterminate $\mathtt{u}$ in the left hand side are odd, while the exponents of $\mathtt{u}$ in the right hand side are even. Therefore, $a_\nu = b_\nu = 0$ for each $\nu := (\nu_1, \nu_2)$, so $a = b = 0$ and $g_0 = f_0 \cdot Q \in f_0\mathcal{O}_0$. Thus,

$$f_0\mathcal{O}_0 \subset \mathcal{J}(X_0) \subset \ker(\varphi) \subset f_0\mathcal{O}_0,$$

so $\mathcal{J}(X_0) = f_0\mathcal{O}_0$ is a prime ideal.

As $\mathcal{J}(X_0)$ is a principal ideal of the unique factorization domain $\mathcal{O}_0$, we have $\operatorname{ht}(\mathcal{J}(X_0)) = 1$. Consequently,

$$\begin{aligned}\dim(X_0) = \dim \mathcal{O}_{X,a} &= \dim(\mathcal{O}_0/\mathcal{J}(X_0)) \\ &= \dim(\mathbb{K}\{\mathtt{x},\mathtt{y},\mathtt{z}\}) - \operatorname{ht}(\mathcal{J}(X_0)) = 3 - 1 = 2.\end{aligned}$$

Next we show that the canonical projection $\pi : \mathbb{K}^3 \to \mathbb{K}^2$, $(x, y, z) \mapsto (x, y)$ can be chosen to be the covering projection in Theorems VII.4.1 and VII.5.5. To that end, let us show that the induced homomorphism

$$\pi^* : \mathbb{K}\{\mathtt{x},\mathtt{y}\} \to \mathbb{K}\{\mathtt{x},\mathtt{y},\mathtt{z}\}/\mathcal{J}(X_0) := A,\ h \mapsto h + \mathcal{J}(X_0)$$

is injective and finite. By Lemma III.1.5 π^* is finite if and only if the maximal ideal $\mathfrak{m}_A$ of A satisfies $\mathfrak{m}_A = \sqrt{\pi^*(\mathfrak{m}_2)A}$, where $\mathfrak{m}_2 := \{\mathtt{x},\mathtt{y}\}\mathbb{K}\{\mathtt{x},\mathtt{y}\}$ is the maximal ideal of $\mathbb{K}\{\mathtt{x},\mathtt{y}\}$. Note that

$$\begin{aligned}\pi^*(\mathfrak{m}_2)A &= \{\mathtt{x} + \mathcal{J}(X_0), \mathtt{y} + \mathcal{J}(X_0)\}A, \\ (\mathtt{z} + \mathcal{J}(X_0))^2 &= \mathtt{x}\mathtt{y}^2 + \mathcal{J}(X_0) \in \pi^*(\mathfrak{m}_2)A,\end{aligned}$$

so $\mathfrak{m}_A = \sqrt{\pi^*(\mathfrak{m}_2)A}$. As $\mathbb{K}\{\mathtt{x},\mathtt{y}\}$ is an integral domain and

$$\begin{aligned}\operatorname{ht}(\ker(\pi^*)) &= 2 - \dim(\mathbb{K}\{\mathtt{x},\mathtt{y}\}/\operatorname{ht}(\ker(\pi^*))) \\ &= 2 - \dim(\operatorname{im}\pi^*) = 2 - \dim(\mathcal{O}_0/\mathcal{J}(X_0)) = 0,\end{aligned}$$

we conclude that π^* is injective. Then for $\mathbb{K} = \mathbb{C}$ the canonical projection $\pi : \mathbb{K}^3 \to \mathbb{K}^2$ satisfies the hypotheses of the strong local parameterization theorem at the origin for $d = \dim(X_0) = 2$. The discriminant of the polynomial $P_3(\mathtt{x},\mathtt{y},\mathtt{z}) = \mathtt{z}^2 - \mathtt{x}\mathtt{y}^2$ is

$$\Delta := \operatorname{Res}\left(P_3, \frac{\partial P_3}{\partial \mathtt{z}}\right) = \det\begin{pmatrix} -\mathtt{x}\mathtt{y}^2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = -4\mathtt{x}\mathtt{y}^2.$$

Therefore, there exist positive real numbers ρ, η such that if we denote

$$U := D(0,\rho) \times D(0,\rho) \times D(0,\eta) \quad \text{and} \quad W := D(0,\rho) \times D(0,\rho),$$

the triple $(X \cap U, \pi|_{X\cap U}, W)$ is an irreducible analytic covering with critical set

$$A := Z_W(\Delta) = \{(x,y) \in \mathbb{C}^2 : |x| < \rho,\ |y| < \rho, x \neq 0, y \neq 0\}.$$

The situation changes drastically if $\mathbb{K} = \mathbb{R}$. We claim: *the map*

$$\pi : X \cap U := X \cap (-\rho,\rho)^2 \times (-\eta,\eta) \to (-\rho,\rho)^2,\ (x,y,z) \mapsto (x,y)$$

is not surjective.

Given a point $(x,y,z) \in X \cap (-\rho,\rho)^2 \times (-\eta,\eta)$ we have $z^2 = xy^2$, so if $x < 0$, then $y = 0$ and $z = 0$. Consequently, $\pi(X \cap U)$ does not contain the points $(x,y) \in (-\rho,\rho)^2$ with $x < 0$ and $y \neq 0$.

In addition, $X^\circ := X \cap U \setminus \pi^{-1}(A)$ is not connected and it has four connected components. Besides, X° is not dense in $X \cap U$ because

$$X \cap U \setminus \mathrm{Cl}_{X\cap U}(X^\circ) = \{(x,y,z) \in \mathbb{R}^3 :\ -\rho < x < 0,\ y = 0,\ z = 0\}.$$

6 Applications. Dimension of intersections

Let us present some applications of the previous results of this chapter. All through this section we fix an open subset $\Omega \subset \mathbb{C}^n$.

6.a Analytic maps with discrete fibers. A classical result in the theory of functions in one complex variable states that non-constant holomorphic functions are open maps. We state next the most immediate generalization of this result to analytic maps $\Omega \subset \mathbb{C}^n \to \mathbb{C}^n$ whose fibers are discrete.

Theorem VII.6.1 *Let $f := (f_1, \dots, f_n) : \Omega \to \mathbb{C}^n$ an analytic map whose fibers are discrete sets. Then f is an open map.*

Proof. Let $\Gamma := \{(x,y) \in \Omega \times \mathbb{C}^n : y = f(x)\}$ and denote

$$\pi_1 : \Omega \times \mathbb{C}^n \to \mathbb{C}^n,\ (x,y) \mapsto x \quad \text{and} \quad \pi_2 : \Omega \times \mathbb{C}^n \to \mathbb{C}^n,\ (x,y) \mapsto y$$

the canonical projections. It is enough to prove: $\varpi := \pi_2|_\Gamma : \Gamma \to \mathbb{C}^n$ *is an open map.* Suppose this is proved for a while and let $U \subset \Omega$ be an open subset. Then $\Gamma \cap (U \times \mathbb{C}^n)$ is open in Γ, so $f(U) = \varpi(\Gamma \cap (U \times \mathbb{C}^n))$ is an open subset of $\mathbb{C}^n$, as required.

6.a.1 We claim: Γ *is an* n*-dimensional analytic submanifold of* $\Omega \times \mathbb{C}^n$. In particular, Γ_a *is an* n*-dimensional irreducible analytic germ for each point* $a \in \Gamma$.

The continuous map $\varphi : \Omega \to \Gamma$, $x \mapsto (x, f(x))$ is a homeomorphism because its inverse $\pi_1|_\Gamma : \Gamma \to \Omega$, $(x, y) \mapsto x$ is continuous. In addition, φ is an analytic map and its Jacobian matrix is

$$\Big(\frac{\partial \varphi_i}{\partial \mathtt{x}_j}(x)\Big)_{\substack{1\le i\le 2n\\ 1\le j\le n}} = \left(\begin{array}{c} I_n \\ \hline \Big(\frac{\partial f_j}{\partial \mathtt{x}_i}(x)\Big)_{1\le i,j\le n}\end{array}\right),$$

which has rank n at each point $x \in \Omega$. By Lemma V.3.3 Γ is an n-dimensional analytic submanifold of $\Omega \times \mathbb{C}^n$.

6.a.2 Let us show: $\varpi : \Gamma \to \mathbb{C}^n$ *is an open map*. It is enough to prove: *if* $V \subset \Gamma$ *is an open subset of* Γ *that contains the origin* $(0, 0) \in \Gamma \subset \mathbb{C}^n \times \mathbb{C}^n$, *then* $\varpi(V)$ *is a neighborhood of* $0 \in \mathbb{C}^n$.

As $f^{-1}(0)$ is a discrete subset of Ω and $\varphi : \Omega \to \Gamma$, $x \mapsto (x, f(x))$ is a homeomorphism, we deduce that $\varphi(f^{-1}(0)) = \varpi^{-1}(0)$ is a discrete subset of Γ. Thus, $(0, 0)$ is an isolated point of $\varpi^{-1}(0) = \Gamma \cap \pi_2^{-1}(0)$. Let $\mathtt{x} := (\mathtt{x}_1, \dots, \mathtt{x}_n)$ and $\mathtt{y} := (\mathtt{y}_1, \dots, \mathtt{y}_n)$. By Theorem VI.4.8 the analytic homomorphism

$$\varphi : \mathbb{C}\{\mathtt{y}\} \to \mathbb{C}\{\mathtt{x}, \mathtt{y}\}/\mathcal{J}(\Gamma_0),\ g(\mathtt{y}) \mapsto g(\mathtt{y}) + \mathcal{J}(\Gamma_0)$$

is finite and injective. Let $\Omega_1 \subset \mathbb{C}^{2n}$ be an open neighborhood of the origin such that $V = \Omega_1 \cap \Gamma$. By Remark VII.4.2 there exists an open neighborhood $U \subset \Omega_1$ of $(0, 0) \in \mathbb{C}^n$ such that the triple $(\Gamma \cap U, \pi_2|_{\Gamma\cap U}, W := \pi_2(U))$ is a quasianalytic covering. In particular,

$$0 \in W = \pi_2(U) = \pi_2(\Gamma \cap U) \subset \pi_2(\Gamma \cap \Omega_1) = \pi_2(V) = \varpi(V),$$

so $\varpi(V)$ is a neighborhood of the origin in $\mathbb{C}^n$, as required. $\square$

The dimension of a linear subspace X of $\mathbb{K}^n$ can be defined as the minimum number of hyperplanes $H_1, \dots, H_d$ passing through the origin in $\mathbb{K}^n$ such that $X \cap H_1 \cap \dots \cap H_d = \{0\}$. This is the idea behind the following result.

Proposition VII.6.2 *Let* $X \subset \Omega$ *be an analytic set and let* $a \in X$. *Then*

$$\dim(X_a) = \min\{k \in \mathbb{N} : \exists f_1, \dots, f_k \in \mathcal{O}_a \text{ with } X_a \cap Z(f_1, \dots, f_k) = \{a\}\}.$$

Proof. Assume that $a = 0$ and let $d := \dim(X_0)$. As $d = \dim(\mathcal{O}_{X,0})$, we may assume by Local Parameterization Theorem II.3.1 that the homomorphism

$$\varphi : \mathcal{O}_d \to \mathcal{O}_{X,0} := \mathcal{O}_n/\mathcal{J}(X_0),\ f \mapsto f + \mathcal{J}(X_0)$$

is injective and finite. Denote $\pi := (\pi_1, \dots, \pi_d) : \mathbb{C}^n \to \mathbb{C}^d$ the projection onto the first d coordinates. By Theorem VI.4.8 the origin is an isolated point of $X \cap \pi^{-1}(0)$, so $X_0 \cap Z(\pi_{1,0} \dots, \pi_{d,0}) = \{0\}$. Consequently,

$$\min\{k \in \mathbb{N} : \exists f_1, \dots, f_k \in \mathcal{O}_0 \text{ with } X_0 \cap Z(f_1, \dots, f_k) = \{0\}\} \leq \dim(X_0).$$

To prove the converse inequality, let $f_{1,0}, \dots, f_{k,0} \in \mathcal{O}_0$ be such that

$$X_0 \cap Z(f_{1,0}, \dots, f_{k,0}) = \{0\}. \tag{6.8}$$

Let $U \subset \Omega$ be an open neighborhood of 0 and representatives $f_1, \dots, f_k \in \mathcal{O}(U)$ of the germs $f_{1,0}, \dots, f_{k,0}$. By (6.8) the analytic map

$$f : U \mapsto \mathbb{C}^k,\ x \mapsto (f_1(x), \dots, f_k(x))$$

satisfies $f(0) = 0$ and 0 is an isolated point of $X \cap f^{-1}(0)$. By Theorem VI.4.8 the analytic homomorphism

$$\varphi : \mathcal{O}_k \to \mathcal{O}_{X,0},\ g \mapsto g(f_1, \dots, f_k) + \mathcal{J}(X_0)$$

is finite, so the induced homomorphism

$$\overline{\varphi} : \mathcal{O}_k/\ker(\varphi) \to \mathcal{O}_{X,0},\ g + \ker(\varphi) \mapsto g(f_1, \dots, f_k) + \mathcal{J}(X_0)$$

is finite and injective. Thus,

$$\dim(X_0) = \dim(\mathcal{O}_{X,0}) = \dim(\mathcal{O}_k/\ker(\varphi)) = k - \operatorname{ht}(\ker(\varphi)) \leq k,$$

as required. □

Remark VII.6.3 We have proved in Proposition VII.6.2 that if $\dim(X_0) = d$ there exist linear forms $f_1, \dots, f_d$ such that $X_0 \cap Z(f_{1,0}, \dots, f_{d,0}) = \{0\}$.

Lemma VII.6.4 *Let $X \subset \Omega$ be an analytic set and $a \in X$ such that the germ X_a is equidimensional and let $\dim(X_a) = d$. Then there exists an open neighborhood $U \subset \Omega$ of a such that the germ X_b is equidimensional of dimension d for each $b \in X \cap U$.*

Proof. By Proposition VII.4.6 and Theorem VII.5.6 there exist an open neighborhood $U \subset \Omega$ of a and an analytic map $\pi : U \to \mathbb{C}^d$ such that the triple $(X \cap U, \pi|_{X\cap U}, W := \pi(U))$ is an analytic covering whose regular part $(X \cap U)^\circ$ is a d-dimensional analytic submanifold and $\dim(X_b) \leq d$ for each $b \in X \cap U$. We deduce

$$(X \cap U)^\circ \subset \mathrm{Reg}_d(X) \cap U \quad \text{and} \quad X \cap U = \mathrm{Cl}_{X\cap U}(X \cap U)^\circ. \tag{6.9}$$

6.a.3 We claim: $\mathrm{Reg}(X) \cap U = \mathrm{Reg}_d(X) \cap U$.

Fix $e \neq d$ and observe that by (6.9)

$$\mathrm{Reg}_e(X) \cap U \subset X \cap U = \mathrm{Cl}_{X\cap U}(X \cap U)^\circ \subset \mathrm{Cl}_{X\cap U}(\mathrm{Reg}_d(X) \cap U).$$

By Remarks VI.4.5 $\mathrm{Reg}_d(X)$ is closed in $\mathrm{Reg}(X)$. Thus,

$$\mathrm{Reg}_e(X) \cap U \subset (\mathrm{Reg}(X) \cap U) \cap \mathrm{Cl}_{X\cap U}(\mathrm{Reg}_d(X) \cap U) = \mathrm{Reg}_d(X) \cap U.$$

Consequently, $\mathrm{Reg}_e(X) \cap U = \varnothing$ because $\mathrm{Reg}_e(X) \cap \mathrm{Reg}_d(X) = \varnothing$ if $e \neq d$.

6.a.4 Suppose by contradiction that there exists a point $b \in X \cap U$ such that the germ X_b is not equidimensional of dimension d. Let $V \subset U$ be an open neighborhood of b such that there exist analytic sets $X_1, \ldots, X_r \subset V$ that are representatives of the irreducible components $X_{1,b}, \ldots, X_{r,b}$ of the analytic germ X_b and $X \cap V = X_1 \cup \cdots \cup X_r$. As X_b is not equidimensional of dimension d, we may suppose that $e := \dim(X_{1,b}) < d$.

6.a.5 We claim: $b \in \mathrm{Cl}_V(X_1 \setminus \bigcup_{j=2}^r X_j)$.

Otherwise, there exists an open neighborhood $V_1 \subset V$ of b such that

$$X_1 \cap V_1 \subset \bigcup_{j=2}^r X_j \quad \rightsquigarrow \quad X_{1,b} = (X_1 \cap V_1)_b \subset \Big(\bigcup_{j=2}^r X_j\Big)_b = \bigcup_{j=2}^r X_{j,b}.$$

By Proposition VI.2.14 $X_{1,b} \subset X_{j,b}$ for some $j = 2, \ldots, r$, which is false. This proves the claim.

6.a.6 In addition, $X \cap V \setminus \bigcup_{j=2}^r X_j = X_1 \setminus \bigcup_{j=2}^r X_j$. Consequently,

$$\mathrm{Reg}(X) \cap V \setminus \bigcup_{j=2}^r X_j = \mathrm{Reg}\Big(X \cap V \setminus \bigcup_{j=2}^r X_j\Big) = \mathrm{Reg}\Big(X_1 \setminus \bigcup_{j=2}^r X_j\Big). \tag{6.10}$$

As $b \in \mathrm{Cl}_V(X_1 \setminus \bigcup_{j=2}^r X_j)$, the open subset $V_2 := V \setminus \bigcup_{j=2}^r X_j$ of $\mathbb{C}^n$ is non-empty and $Y := X_1 \setminus \bigcup_{j=2}^r X_j$ is an analytic subset of V_2. In addition,

$\dim(Y_b) = e$, so by Proposition VII.4.6 there exists a point $p \in \operatorname{Reg}(Y)$ with $\dim(Y_p) = e$. As

$$Y = X_1 \cap V_2 = X_1 \cap \Big(V \setminus \bigcup_{j=2}^{r} X_j\Big) = X \cap \Big(V \setminus \bigcup_{j=2}^{r} X_j\Big),$$

we deduce $X_p = Y_p$. Thus, $p \in \operatorname{Reg}(X)$ and $\dim(X_p) = \dim(Y_p) = e$, so $p \in \operatorname{Reg}_e(X)$, which is a contradiction because $\operatorname{Reg}_e(X) \cap U = \varnothing$ by 6.a.3.

We conclude that the germ X_b is equidimensional of dimension d for each $b \in X \cap U$, as required. □

Remark VII.6.5 The previous result does not hold if we substitute $\mathbb{K} = \mathbb{C}$ by $\mathbb{K} = \mathbb{R}$. Consider Whitney's umbrella $X := \{\mathtt{z}^2 - \mathtt{x}\mathtt{y}^2\}$. We proved in Example VII.5.10 that X_0 is an irreducible germ of dimension 2, so it is an equidimensional germ of dimension 2. However, for each open neighborhood U of the origin in $\mathbb{R}^3$ there exists $\varepsilon > 0$ such that the point $b := (-2\varepsilon, 0, 0) \in U$. If $B \subset \mathbb{R}^3$ is the open ball centered at b with radius ε,

$$X \cap B = \{\mathtt{y} = 0, \mathtt{z} = 0\} \cap B,$$

so $\dim(X_b) = 1$.

Lemma VII.6.6 *Let $U \subset \mathbb{C}^n$ be an open subset and $a \in U$. Let $X, X_1, \dots, X_r$ be analytic subsets of U such that $X = X_1 \cup \cdots \cup X_r$ and $X_{1,a}, \dots, X_{r,a}$ are the irreducible components of the germ X_a. Then for each open neighborhood $V \subset U$ of a there exists a point $b \in V$ such that $X_b = X_{1,b}$ is a regular germ.*

Proof. Each X_j is a closed subset of U, so $U \setminus \bigcup_{j=2}^r X_j$ is an open subset of U. We claim: *$X_1 \cap V \setminus \bigcup_{j=2}^r X_j$ is a non-empty open subset of $X_1 \cap V$.*

Otherwise,

$$X_{1,a} = (X_1 \cap V)_a \subset \Big(\bigcup_{j=2}^{r} X_j\Big)_a = \bigcup_{j=2}^{r} X_{j,a},$$

but this is false because $X_{1,a}, \dots, X_{r,a}$ are the irreducible components of X_a. By Corollary VII.4.7 $a \in \operatorname{Cl}_{X_1}(\operatorname{Reg}(X_1))$, so there exists a point

$$b \in \operatorname{Reg}(X_1) \cap \Big(X_1 \cap V \setminus \bigcup_{j=2}^{r} X_j\Big).$$

Thus, $X_{1,b}$ is a regular germ and $X_b = X_{1,b}$. $\square$

Proposition VII.6.7 *Let $X \subset \Omega$ be an analytic subset such that $a \in X$ and X_a is an equidimensional germ of dimension d. Let $f_1, \ldots, f_s \in \mathcal{O}_a$ such that $f_i(a) = 0$ for $i = 1, \ldots, s$. Then the dimension of each irreducible component of the germ $Y_a := X_a \cap Z(f_1, \ldots, f_s)$ is greater than or equal to $d - s$.*

Proof. The proof is conducted in two steps:

6.a.7 We claim: $\dim(Y_a) \geq d - s$.

Let $e := \dim(Y_a)$. By Proposition VII.6.2 there exist $g_1, \ldots, g_e \in \mathcal{O}_a$ such that $Y_a \cap Z(g_1, \ldots, g_e) = \{0\}$. Thus,

$$X_a \cap Z(f_1, \ldots, f_s, g_1, \ldots, g_e) = \{0\}.$$

By Proposition VII.6.2 we have $d = \dim(X_a) \leq s + e$.

6.a.8 By Lemma VII.6.4 there exists an open neighborhood $U_1 \subset \Omega$ of a such that X_b is an equidimensional germ of dimension d for each $b \in X \cap U_1$. After shrinking U_1 we may assume that there exist representatives $f_i \in \mathcal{O}(U_1)$ of the germs f_i that we denote with the same symbols. Then $Y := X \cap Z_{U_1}(f_1, \ldots, f_s)$ is an analytic subset of U_1 that represents the germ Y_a. Let $U_2 \subset U_1$ be an open neighborhood of a such that there exist analytic subsets $Y_1, \ldots, Y_r$ of U_2 such that $Y \cap U_2 = \bigcup_{i=1}^r Y_i$ and $Y_{1,a}, \ldots, Y_{r,a}$ are the irreducible components of the analytic germ Y_a. Let us prove: $\dim(Y_{i,a}) \geq d - s$ *for* $i = 1, \ldots, r$. It is enough to show: $\dim(Y_{1,a}) \geq d - s$.

By Proposition VII.4.6 there exists an open neighborhood $U \subset U_2$ such that $\dim(Y_{1,x}) \leq \dim(Y_{1,a})$ for each $x \in Y_1 \cap U$. Note also that

$$\bigcup_{i=1}^{r} Y_i \cap U = Y \cap U = X \cap Z_U(f_1, \ldots, f_s).$$

By Lemma VII.6.6 there exists $b \in U$ such that $Y_b = Y_{1,b}$ is a regular germ, so it is by Corollary VI.4.6 irreducible. In addition, $Y_b = X_b \cap Z(f_1, \ldots, f_s)$. As $\dim(X_b) = d$, it follows from 6.a.7

$$\dim(Y_{1,a}) \geq \dim(Y_{1,b}) = \dim(Y_b) = \dim(X_b \cap Z(f_1, \ldots, f_s)) \geq d - s,$$

as required. $\square$

Example VII.6.8 The previous result VII.6.7 does not hold if $\mathbb{K} = \mathbb{R}$. As $X := \{\mathtt{z} = 0\} \subset \mathbb{R}^3$ is a two-dimensional analytic submanifold, X_0 is an irreducible germ of dimension $d = 2$. In particular it is equidimensional. Let $f = \mathtt{x}^2 + \mathtt{y}^2 + \mathtt{z}^2 - 2\mathtt{z} \in \mathcal{O}(\mathbb{R}^3)$ and note that X is the affine tangent plane to $Z := Z_{\mathbb{R}^3}(f)$ at the origin. Observe that $Y := X \cap Z = \{0\}$ and the dimension of $Z_0 := X_0 \cap Y_0$ equals 0, while $2 - 1 = 1 > 0$.

Lemma VII.6.9 (Weak form of the semicontinuity lemma) *Let Λ be a topological space and let $a \in \Omega$. Let $f_i : \Lambda \times \Omega \to \mathbb{C}$ be continuous functions for $i = 1, \dots, r$ and denote*

$$F : \Lambda \times \Omega \to \mathbb{C}^s,\ (\lambda, z) \mapsto (f_1(\lambda, z), \dots, f_s(\lambda, z)).$$

Suppose that for each $\lambda \in \Lambda$ the function $f_{j,\lambda} : \Omega \to \mathbb{C}$, $z \mapsto f_j(\lambda, z)$ is analytic for $j = 1, \dots, s$ and denote $\Sigma_\lambda := Z_\Omega(f_{1,\lambda}, \dots, f_{s,\lambda})$. Then each $\lambda_0 \in \Lambda$ has an open neighborhood $\Xi \subset \Lambda$ such that $\dim(\Sigma_{\lambda,a}) \leq \dim(\Sigma_{\lambda_0,a})$ *for each $\lambda \in \Xi$.*

Proof. We can suppose $a := 0$ and let $d := \dim(\Sigma_{\lambda_0,0})$. By Remark VII.6.3 we may assume $\Sigma_{\lambda_0,0} \cap H_0 = \{0\}$ where

$$H := \{\mathtt{x}_1 = 0, \dots, \mathtt{x}_d = 0\} \subset \mathbb{C}^n.$$

Thus, there exists $\varepsilon > 0$ such that the open neighborhood

$$V := \{(x_1, \dots, x_n) \in \mathbb{C}^n : |x_i| < \varepsilon \quad \text{for } i = 1, \dots, n\}$$

satisfies $\mathrm{Cl}(V) \subset \Omega$ and $\mathrm{Cl}(V) \cap H \cap \Sigma_{\lambda_0} = \{0\}$. Notice that the subset $K := \mathrm{Cl}(V) \cap H \cap \Sigma_{\lambda_0} \setminus V$ of $\mathbb{C}^n$ is compact and $0 \notin K$. In addition, $F(\lambda_0, x) \neq 0$ *for each $x \in K$.*

Otherwise, there exists $x \in K$ such that

$$f_1(\lambda_0, x) = 0, \dots, f_s(\lambda_0, x) = 0,$$

so $x \in \mathrm{Cl}(V) \cap H \cap \Sigma_{\lambda_0} = \{0\}$, which is a contradiction because $0 \notin K$.

6.a.9 We claim: *There exists an open neighborhood $\Xi \subset \Lambda$ of λ_0 such that $F(\lambda, x) \neq 0$ for each $(\lambda, x) \in \Xi \times K$.* Thus, $\Sigma_\lambda \cap K = \varnothing$ *for each $\lambda \in \Xi$.*

As $F(\lambda_0, x) \neq 0$ for each $x \in K$, there exist by the continuity of the map $F : \Lambda \times \Omega \to \mathbb{C}^s$ open neighborhoods $V_{\lambda_0,x} \subset \Lambda$ of λ_0 and $W_x \subset \Omega$ of x such that $F(\lambda, y) \neq 0$ for each $(\lambda, y) \in V_{\lambda_0,x} \times W_x$. As $\{W_x\}_{x \in K}$ is an open cover of the compact space K, there exist $x_1, \dots, x_m \in K$ such that $K \subset W_{x_1} \cup \cdots \cup W_{x_m}$. Consider the open neighborhood $\Xi := V_{\lambda_0,x_1} \cap \cdots \cap V_{\lambda_0,x_m} \subset \Lambda$ of λ_0. Given $(\lambda, x) \in V \times K$ there exists $j = 1, \dots, m$ with $x \in W_{x_j}$, so $(\lambda, x) \in V_{\lambda_0,x_j} \times W_{x_j}$. Thus, $F(\lambda, x) \neq 0$.

6.a.10 As each $g_{\lambda,j} \in \mathcal{O}(\Omega)$, the set $\Sigma_\lambda \cap H$ is analytic in Ω, so $\Sigma_\lambda \cap V \cap H$ is an analytic subset of V for each $\lambda \in \Lambda$. As $\Sigma_\lambda \cap K = \varnothing$ for each $\lambda \in \Xi$,

$$\Sigma_\lambda \cap V \cap H = \Sigma_\lambda \cap \mathrm{Cl}(V) \cap H$$

is a compact analytic subset of V for each $\lambda \in \Xi$. By Corollary VII.5.8 $\Sigma_\lambda \cap V \cap H$ is a finite set, so either $(\Sigma_\lambda \cap V \cap H)_0 = \varnothing$ or $(\Sigma_\lambda \cap V \cap H)_0 = \{0\}$. By Proposition VII.6.2 $\dim(\Sigma_{\lambda,0}) \leq d = \dim(\Sigma_{\lambda_0,0})$ for each $\lambda \in \Xi$, as required. $\square$

Theorem VII.6.10 *Let X_a and Y_a be equidimensional analytic germs for some point $a \in \Omega$. Let $d := \dim(X_a)$ and $e := \dim(Y_a)$. Then the dimension of every irreducible component of the germ $Z_a := X_a \cap Y_a$ is greater than or equal to $d + e - n$.*

Proof. We may assume $a := 0$. We distinguish two cases:

6.a.11 *Irreducible case.* Suppose first that Z_0 is irreducible. Let $V \subset \Omega$ be an open neighborhood of the origin and let $f_1, \ldots, f_s, g_1, \ldots, g_r \in \mathcal{O}(V)$ be such that

$$X \cap V = Z_U(f_1, \ldots, f_s) \quad \text{and} \quad Y \cap U = Z_U(g_1, \ldots, g_r).$$

For each $\rho > 0$ let $\mathcal{B}_\rho := \{x \in \mathbb{C}^n : \|x\| < \rho\}$ and choose $\varepsilon > 0$ such that $\mathcal{B}_{2\varepsilon} \subset V$. Note that given $x, y \in \mathcal{B}_\varepsilon$ we have $\|x + y\| \leq \|x\| + \|y\| < 2\varepsilon$, so $x + y \in \mathcal{B}_{2\varepsilon} \subset V$. Consider the well-defined analytic map

$$\psi : \mathcal{B}_\varepsilon \times \mathcal{B}_\varepsilon \to \mathbb{C}^{r+s}, (u, x) \mapsto (g_1(x+u), \ldots, g_r(x+u), f_1(x+u), \ldots, f_s(x+u)).$$

For each $u \in \mathcal{B}_\varepsilon$ consider the analytic subset of $\mathcal{B}_\varepsilon$ defined by

$$\Sigma_u := \{x \in \mathcal{B}_\varepsilon : g_1(x + u) = 0, \ldots, g_r(x + u) = 0, \\ f_1(x + u) = 0, \ldots, f_s(x + u) = 0\}.$$

The germ of Σ_0 at the origin is

$$\Sigma_{0,0} = Z(g_1, \ldots, g_s, f_1, \ldots, f_r) = X_0 \cap Y_0 = Z_0.$$

By Lemma VII.6.9, there exists an open neighborhood $U_1 \subset \mathcal{B}_\varepsilon$ of $0 \in \mathbb{C}^n$ such that $\dim(\Sigma_{u,0}) \leq \dim(\Sigma_{0,0})$ for each $u \in U_1$. By Lemma VII.6.4 there exists an open neighborhood $U \subset U_1$ of $0 \in \mathbb{C}^n$ such that for each point $u \in U$ the analytic germ Y_u is equidimensional of dimension $e = \dim(Y_0)$.

As $\dim(X_0) = d$, we deduce by Proposition VII.4.6 that $0 \in \mathrm{Cl}_X(\mathrm{Reg}_d(X))$. Consequently, there exists a point $u \in \mathrm{Reg}_d(X) \cap U$. Consider the translation $\tau : \mathbb{C}^n \to \mathbb{C}^n$, $v \mapsto v-u$. Notice that $\tau(u) = 0$ and with the notations of Lemma VI.2.16 let $\tau_* : \mathcal{G}_u \to \mathcal{G}_0,\ T_u \mapsto (\tau(T \cap U))_0$. We have $\tau_*(X_u \cap Y_u) = \Sigma_{u,0}$.

As X_u is a d-dimensional regular germ, there exist by the Jacobian Criterion VI.4.3 $h_1, \dots, h_{n-d} \in \mathcal{O}_u$ such that $X_u = Z(h_1, \dots, h_{n-d})$. As Y_u is an equidimensional germ, this implies by Proposition VII.6.7 that

$$\begin{aligned}\dim(Y_u \cap X_u) &= \dim(Y_u \cap Z(h_1, \dots, h_{n-d})) \\ &\geq \dim(Y_u) - (n-d) = e + d - n.\end{aligned}$$

By Lemma VI.2.16

$$\begin{aligned}\dim(Z_0) &= \dim(\Sigma_{0,0}) \geq \dim(\Sigma_{u,0}) \\ &= \dim(\tau_*(X_u \cap Y_u)) = \dim(X_u \cap Y_u) \geq e + d - n.\end{aligned}$$

6.a.12 *General case.* As the germs X_0 and Y_0 are equidimensional, there exist by Proposition VII.4.6 and Lemma VII.6.4 an open neighborhood $U \subset \Omega$ of the origin and analytic sets $X, Y, Z, Z_1, \dots, Z_m$ of U such that

- X, Y, Z are representatives of the germs X_0, Y_0, Z_0 and $Z = X \cap Y$.
- $Z = Z_1 \cup \dots \cup Z_m$ and $Z_1, \dots, Z_m$ are representatives of the irreducible components $Z_{1,0}, \dots, Z_{m,0}$ of the germ Z_0.
- For each $x \in X \cap Y$ both X_x and Y_x are equidimensional germs of dimensions $\dim(X_x) = d$ and $\dim(Y_x) = e$.
- For each $x \in U$ and $j = 1, \dots, m$ we have $\dim(Z_{j,x}) \leq \dim(Z_{j,0})$.

It is enough to prove that $\dim(Z_{1,0}) \geq d + e - n$. By Lemma VII.6.6 there exists a point $a \in U$ such that $Z_a = Z_{1,a}$ is a regular germ, which is irreducible by Corollary VI.4.6. As $Z_a = X_a \cap Y_a$ where Z_a is irreducible and X_a and Y_a are equidimensional, it follows from Case 1 that

$$\dim(Z_{1,0}) \geq \dim(Z_{1,a}) = \dim(Z_a) \geq \dim(X_a) + \dim(Y_a) - n = d + e - n,$$

as required. □

Corollary VII.6.11 *Let $a \in \Omega$ and let X_a and Y_a be two analytic germs such that* $\dim(X_a) \geq d$ *and* $\dim(Y_a) \geq e$. *Then* $\dim(X_a \cap Y_a) \geq d + e - n$.

Proof. Let $X_{1,a}$ and $Y_{1,a}$ be irreducible components of X_a and Y_a such that $\dim(X_a) = \dim(X_{1,a})$ and $\dim(Y_a) = \dim(Y_{1,a})$. As $X_{1,a}$ and $Y_{1,a}$ are equidimensional germs, we have by Theorem VII.6.10

$$\dim(X_{1,a} \cap Y_{1,a}) \geq \dim(X_{1,a}) + \dim(Y_{1,a}) - n.$$

As $X_{1,a} \cap Y_{1,a} \subset X_a \cap Y_a$, we get

$$\begin{aligned}\dim(X_a \cap Y_a) \geq \dim(X_{1,a} \cap Y_{1,a}) &\geq \dim(X_{1,a}) + \dim(Y_{1,a}) - n \\ &= \dim(X_a) + \dim(Y_a) - n \geq d + e - n,\end{aligned}$$

as required. □

Example VII.6.12 The latter result does not hold in the real case. Consider the analytic subsets of $\mathbb{R}^3$ defined as

$$X := \{\mathtt{z} = 0\} \subset \mathbb{R}^3 \quad \text{and} \quad Y := \{\mathtt{z} - \mathtt{x}^2 - \mathtt{y}^2 = 0\} \subset \mathbb{R}^3.$$

It follows from Lemma V.3.11 that both are two-dimensional analytic submanifolds of $\mathbb{R}^3$ because the jacobian matrices of the functions $f := \mathtt{z}$ and $g := \mathtt{z} - \mathtt{x}^2 - \mathtt{y}^2$ at a point $a := (x, y, z) \in \mathbb{R}^3$ have rank 1:

$$d_a f = (0, 0, 1) \quad \text{and} \quad d_a g = (-2x, -2y, 1).$$

As $X \cap Y = \{(0, 0, 0)\}$, we have $\dim(X_0 \cap Y_0) = 0 < \dim(X_0) + \dim(Y_0) - 3$.

Exercises

Number VII.1 Let $\pi : X \to Y$ be a continuous, proper and surjective map that is a local homeomorphism between the Hausdorff and locally compact topological spaces X and Y. Prove that (X, π, Y) is a topological covering.

Number VII.2 For each real number $\varepsilon > 0$ denote $D_\varepsilon^* := \{0 < |z| < \varepsilon\} \subset \mathbb{C}$. Let $s \geq 1$ be an integer and $\eta > 0$ a real number such that $\eta^s = \varepsilon$. Let $\pi : D_\eta^* \to D_\varepsilon^*$, $z \mapsto z^s$. Prove that $(D_\eta^*, \pi, D_\varepsilon^*)$ is an s-sheeted topological covering.

Number VII.3 Let $X := \{\mathtt{x}^2 - \mathtt{y}^2 = 0\} \subset \mathbb{C}$ and consider the functions

$$\pi : X \to \mathbb{C},\ (x, y) \mapsto x \quad \text{and} \quad f : \mathbb{C}^2 \to \mathbb{C},\ (x, y) \mapsto x + y.$$

Calculate the annihilating polynomial of the analytic function $f : \mathbb{C}^2 \to \mathbb{C}$ with respect to the covering $(X, \pi, \mathbb{C})$.

Number VII.4 Find an analytic covering (X, π, W) such that X is connected but the Identity Principle does not hold.

Number VII.5 A triple (X, π, W) is an analytic *pseudo-covering* if it satisfies all the conditions in the definition of analytic covering except for the finiteness of the fibers of $\pi : X \to W$. Suppose that $(\mathbb{C}^n, \pi, \mathbb{C}^d)$ is an analytic pseudo-covering, where

$$\pi : \mathbb{C}^n \to \mathbb{C}^d,\ (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_d).$$

Suppose that the regular part X° of $(\mathbb{C}^n, \pi, \mathbb{C}^d)$ is a d-dimensional analytic submanifold. Prove that the fibers of π are finite.

Number VII.6 Let $\Omega \subset \mathbb{C}^n$ be an open set, $X \subset \Omega$ a non-empty subset, $W \subset \mathbb{C}^d$ an open subset and $\pi : \Omega \to W := \pi(\Omega)$ an analytic map such that $(X, \pi|_X, W)$ is an analytic covering whose regular part is a d-dimensional analytic submanifold. Let $\mathcal{A}(X)$ be the set of those functions $f : X \to \mathbb{C}$ that are analytic with respect to the covering $(X, \pi|_X, W)$.

(i) Prove that $\mathcal{A}(X)$ is a ring with the sum and product defined pointwise.

(ii) Find an injective ring homomorphism $\varphi : \mathcal{O}(W) \to \mathcal{A}(X)$ such that $\mathcal{A}(X)$ is integral over $\mathcal{O}(W)$ via φ.

Number VII.7 (i) Is compact the complex circle $C := \{\mathtt{x}^2 + \mathtt{y}^2 = 1\} \subset \mathbb{C}^2$?

(ii) Is an analytic subset of $\mathbb{C}^n$ the set $X := \{x \in \mathbb{C}^n : \|x\| = 1\}$?

Number VII.8 Let $\Omega \subset \mathbb{C}^n$ be an open subset and let $X \subset \Omega$ be an analytic set. Let $a \in X$ be a point such that the germ X_a is irreducible. Prove that there exists an open neighborhood $U \subset \Omega$ of a satisfying the following condition: *if $Z \subset \Omega$ is an analytic set and $X_a \not\subset Z_a$, then $X_x \not\subset Z_x$ for each point $x \in X \cap U$.*

Number VII.9 Let $\Omega \subset \mathbb{C}^n$ be an open subset, let $X \subset \Omega$ be an analytic set and let d be a non-negative integer. Prove that the following statements are equivalent:

(i) X is a pure dimensional set of dimension d, that is, $\dim(X_x) = d$ for each $x \in X$.

(ii) For each point $a \in X$ there exist an open neighborhood $U \subset \Omega$ of a and an analytic map $\pi : U \to \mathbb{C}^d$ such that the triple $(X \cap U, \pi|_{X \cap U}, \pi(U))$ is an analytic covering with $\pi(U)$ an open subset of $\mathbb{C}^d$ and whose regular part is a d-dimensional analytic submanifold.

Number VII.10 Let $\Omega \subset \mathbb{C}^n$ be an open subset, let $X \subset \Omega$ be an analytic set and let d be a non-negative integer. Prove that the following statements are equivalent:

(i) For each point $x \in X$ the germ X_x is equidimensional of dimension d.

(ii) The set X is pure dimensional of dimension d.

Number VII.11 Let $\Omega := \{x := (x_0, \ldots, x_n) \in \mathbb{C}^{n+1} : x_0 \neq 0\}$. Prove that

$$f : \Omega \to \mathbb{C}^n,\ (x_0, \ldots, x_n) \mapsto \Big(\frac{x_1}{x_0}, \ldots, \frac{x_1}{x_0}\Big)$$

is an open map.

Number VII.12 What is the dimension of the analytic germ X_0 where

$$X := \{\mathtt{x}^2 - \mathtt{z}\mathtt{y}^2 = 0,\ \mathtt{x}^2 + \mathtt{y}^2 - \mathtt{z}^2 = 0\} \subset \mathbb{C}^3?$$

Number VII.13 Let $\Omega \subset \mathbb{C}^n$ be an open set and let $f : \Omega \to \mathbb{C}^m$, $z \mapsto (f_1(z), \ldots, f_m(z))$ be an analytic map. Let $X \subset \Omega$ be an analytic subset of Ω and $a \in X$. Prove that there exists an open neighborhood $U \subset \Omega$ of the point a such that

$$\dim(X \cap f^{-1}(f(x)))_x \leq \dim(X \cap f^{-1}(f(a)))_a$$

for each point $x \in X \cap U$.

Number VII.14 Let $\Omega \subset \mathbb{C}^n$ be an open set, let $X \subset \Omega$ be an analytic set and $a \in X$. Let $f_1, \ldots, f_r \in \mathcal{O}_a$ be such that $\mathcal{J}(X_a) = \sum_{j=1}^r f_j \mathcal{O}_a$. Let

$$T_aX := \Big\{u := (u_1, \ldots, u_n) \in \mathbb{C}^n : \sum_{k=1}^n \frac{\partial f_j}{\partial \mathtt{x}_j}(a) u_j = 0\Big\}.$$

(i) Prove that T_aX only depends on the germ X_a and not on the chosen generators of $\mathcal{J}(X_a)$.

(ii) Prove that T_aX is a $\mathbb{C}$-linear subspace of $\mathbb{C}^n$.

(iii) Prove that X_a is a regular germ if and only if $\dim(X_a) = \dim(T_aX)$.

APPENDIX A

Global irreducible components

Almost all results of global nature in analytic geometry require the use of sheaf theory. One exception is the characterization in purely topological terms of the notion of irreducible complex analytic set and the decomposition of each complex analytic set as a locally finite union of irreducible analytic sets. We devote this appendix to prove these results and we begin by proving its local counterpart, which states that the irreducible components of an analytic germ X_a are the closures of the connected components of the germ $\mathrm{Reg}(X_a)$ of regular points. The prerequisites needed have topological nature and we recommend [K].

A.1 Irreducible components of analytic germs

To relate the connected components of $\mathrm{Reg}(X_a)$ with the irreducible components of X_a we need a preliminary result. We fix an open set $\Omega \subset \mathbb{C}^n$.

Lemma A.1.1 *Let $X \subset \Omega$ be an analytic set and let $a \in X$ be a point such that the germ X_a is irreducible. Then there exists an open neighborhood $U \subset \Omega$ of a satisfying the following condition:* if $Z \subset U$ is an analytic set and $X_a \not\subset Z_a$, then $X_x \not\subset Z_x$ for each point $x \in X \cap U$.

Proof. Denote $d := \dim(X_a)$. By the Strong Local Parameterization Theorem VII.5.5 there exist an open neighborhood $U \subset \Omega$ of a and a linear map $\pi : \mathbb{C}^n \to \mathbb{C}^d$ such that $(X \cap U, \pi|_{X\cap U}, W := \pi(U))$ is an irreducible analytic covering whose regular part $(X \cap U)^\circ$ is a connected analytic submanifold of dimension d. Suppose by contradiction that there exist an analytic subset $Z \subset U$ and a point $x \in X \cap U$ such that $X_a \not\subset Z_a$ while $X_x \subset Z_x$. Let $V \subset U$ be an open neighborhood of x such that $X \cap V \subset Z$. Thus,

$$M := (X \cap U)^\circ \cap V \subset X \cap V \subset Z \qquad \text{(A.1.1)}$$

and M is non-empty because $(X \cap U)^\circ$ is dense in $X \cap U$. By Corollary V.3.6 and (A.1.1) we have $(X \cap U)^\circ \subset Z$, so $X \cap U = \mathrm{Cl}_U((X \cap U)^\circ) \subset Z$. Consequently, $X_a = (X \cap U)_a \subset Z_a$, which is a contradiction. $\square$

A.1.a Connected components of the regular locus.

We prove next that the closures of the connected components of the regular locus of analytic set germ provide its irreducible compnents

Lemma A.1.2 *Let $a \in \Omega$ and let X_a be an analytic germ. Then* $\mathrm{Reg}(X_a)$ *has finitely many connected components and their closures are the irreducible components of the germ X_a.*

Proof. By Lemma A.1.1 there exist an open neighborhood $V \subset \Omega$ of a and analytic sets $X_1, \dots, X_r \subset V$ that are representatives of the irreducible components $X_{1,a}, \dots, X_{r,a}$ of the germ X_a such that: if $x \in V \cap X_i \cap X_j$ and $i \neq j$, then $X_{i,x} \not\subset X_{j,x}$. In addition, $X = \bigcup_{k=1}^r X_k$ is a representative of X_a. For each $i = 1, \dots, r$ consider the analytic subset $Z_i := \bigcup_{j \neq i} X_j$ of V. As $X_{i,a} \not\subset X_{j,a}$ if $i \neq j$, we have by Proposition VI.2.14 $X_{i,a} \not\subset Z_{i,a}$ for $i = 1, \dots, r$. Thus, $Y_i := X_i \cap Z_i \subset V$ is an analytic set and $Y_{i,a} \subsetneq X_{i,a}$.

Denote $d_i := \dim(X_{i,a})$. By Theorem VII.5.5 there exist an open neighborhood $U \subset V$ of a and linear maps $\pi_i : \mathbb{C}^n \to \mathbb{C}^{d_i}$ such that the triples $(X_i \cap U, \pi_i|_{X_i \cap U}, \pi_i(U))$ are irreducible analytic coverings and the critical set A_i of each covering $(X_i \cap U, \pi_i|_{X_i \cap U}, \pi_i(U))$ contains $\pi_i(Y_i)$. Define $M_i := \mathrm{Reg}(X_i) \setminus Z_i$.

A.1.a.1 $\mathrm{Reg}(X) = \bigcup_{i=1}^r M_i$.

As $M_i \subset X_i \setminus Z_i$, we have $Z_{i,x} = \varnothing$ for each $x \in M_i$. As $X = X_i \cup Z_i$, it holds $X_x = X_{i,x}$ for each $x \in M_i$. Thus, $\mathcal{J}(X_x) = \mathcal{J}(X_{i,x})$ and

$$\mathcal{O}_{X,x} = \mathcal{O}_x/\mathcal{J}(X_x) = \mathcal{O}_x/\mathcal{J}(X_{i,x}) = \mathcal{O}_{X_i,x}$$

is a regular local ring because $x \in \mathrm{Reg}(X_i)$, so $x \in \mathrm{Reg}(X)$.

Conversely, for each $x \in \mathrm{Reg}(X)$ the germ X_x is by Corollary VI.4.6 irreducible. As $X_x = \bigcup_{k=1}^r X_{k,x}$, there exists an index $i = 1, \dots, r$ such that $X_x = X_{i,x}$, so $\mathcal{J}(X_x) = \mathcal{J}(X_{i,x})$ and

$$\mathcal{O}_{X_i,x} = \mathcal{O}_x/\mathcal{J}(X_{i,x}) = \mathcal{O}_x/\mathcal{J}(X_x) = \mathcal{O}_{X,x}$$

is a regular local ring because $x \in \mathrm{Reg}(X)$. The choice of the analytic sets X_k implies $X_{j,x} = \varnothing$ if $j \neq i$, so $Z_{i,x} = \varnothing$ and $x \notin Z_i$. Thus, $x \in \mathrm{Reg}(X_i) \setminus Z_i = M_i$.

A.1.a.2 $M_i \cap M_j = \varnothing$ if $i \neq j$.

We have

$$M_i \subset X_i \setminus Z_i = X_i \setminus \bigcup_{k\neq i} X_k \subset X_i \setminus X_j \subset X_i \setminus M_j,$$

hence $M_i \cap M_j = \varnothing$ if $i \neq j$.

A.1.a.3 *Each intersection $M_i \cap U$ is a connected open and closed subset of* $\operatorname{Reg}(X) \cap U$. Consequently, *the germs $M_{1,a}, \dots, M_{r,a}$ are the connected components of* $\operatorname{Reg}(X_a)$.

Notice that $M_i = \operatorname{Reg}(X_i) \setminus Z_i$ is an open subset of X because $\operatorname{Reg}(X_i)$ is open in X_i and $X \setminus Z_i = X_i \setminus Z_i$ is open in X. In addition, by A.1.a.2

$$M_i = \operatorname{Reg}(X) \setminus \bigsqcup_{j\neq i} M_j.$$

so M_i is closed in $\operatorname{Reg}(X)$. Thus, $M_i \cap U$ is open and closed in in $\operatorname{Reg}(X) \cap U$.

Let us prove that $M_i \cap U$ is connected. As $(X_i \cap U)^\circ := (X_i \cap U) \setminus \pi_i^{-1}(A_i)$ is an analytic submanifold contained in X_i, it is also contained in $\operatorname{Reg}(X_i)$. As $\pi_i(Y_i) \subset A_i$, we have $(X_i \cap U)^\circ \subset \operatorname{Reg}(X_i) \setminus Y_i = M_i$, so

$$(X_i \cap U)^\circ \subset M_i \cap U \subset X_i \cap U. \tag{A.1.2}$$

As $(X_i \cap U)^\circ$ is connected and dense in $X_i \cap U$, we conclude that $M_i \cap U$ is connected.

A.1.a.4 To finish let us check: $X_{i,a} = \operatorname{Cl}(M_{i,a})$ *for* $i = 1, \dots, r$.

As $(X_i \cap U)^\circ$ is dense in $X_i \cap U$, we have using (A.1.2):

$$X_i \cap U = \operatorname{Cl}_U((X_i \cap U)^\circ) \subset \operatorname{Cl}_U(M_i \cap U) \subset X_i \cap U,$$

so $\operatorname{Cl}_{X_i \cap U}(M_i \cap U) = X_i \cap U$. Thus, $\operatorname{Cl}(M_{i,a}) = X_{i,a}$, as required. □

A.2 Irreducible components of analytic sets

An analytic set $X \subset \Omega$ is *reducible* if there exist analytic sets $Y, Z \subset \Omega$ such that $X = Y \cup Z$, $Y \subsetneq X$ and $Z \subsetneq X$. Otherwise X is *irreducible.* Let us check next that the irreducibilty of X depends only on X and not on Ω. We will say that X *is an irreducible analytic set* without any explicit reference to the open subset Ω in which X is analytic.

Remarks A.2.1 (i) Let $\Omega_1 \subset \Omega_2 \subset \mathbb{C}^n$ be open sets and let $Y \subset X \subset \Omega_1$. Suppose that X is analytic in Ω_2. Then Y is analytic in Ω_1 if and only if Y is analytic in Ω_2.

By Remark V.2.2 if Y is analytic in Ω_2, then it is also analytic in Ω_1. Suppose now that Y is analytic in Ω_1. In particular, Y is locally analytic. As Y is closed in Ω_1 and $X \subset \Omega_1$, we deduce that Y is closed in X. As X is closed in Ω_2, we deduce that Y is closed in Ω_2, so Y is an analytic subset of Ω_2.

(ii) Let $\Omega_1 \subset \Omega_2 \subset \mathbb{C}^n$ be open sets and let $X \subset \Omega_1$ be analytic in Ω_2. Then X is reducible in Ω_1 if and only if X is reducible in Ω_2.

If X is reducible in Ω_i there exist analytic subsets Y and Z in Ω_i such that $Y \subsetneq X$, $Z \subsetneq X$ and $X = Y \cup Z$. By part (i) both Y and Z are analytic subsets of Ω_j, so X is reducible in Ω_j.

(iii) Let $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ be open sets and let $X \subset \Omega_1$ be analytic subset. Then X is reducible in Ω_1 if and only if it is reducible in Ω_2.

This follows from (ii) because both statements are equivalent to the reducibility of X in $\Omega_1 \cap \Omega_2$.

Examples A.2.2 (i) $\mathbb{C}^n$ is an irreducible analytic set.

Otherwise, there exist analytic sets $X \subsetneq \mathbb{C}^n$ and $Y \subsetneq \mathbb{C}^n$ such that $\mathbb{C}^n = X \cup Y$. By Theorem V.2.4 $\mathbb{C}^n \setminus X$ is a dense subset of $\mathbb{C}^n$. In addition, $\mathbb{C}^n \setminus Y$ is an open subset of $\mathbb{C}^n$. Thus, $(\mathbb{C}^n \setminus X) \cap (\mathbb{C}^n \setminus Y) \neq \varnothing$, so $\mathbb{C}^n \neq X \cup Y$, which is a contradiction.

(ii) The hyperplane $H := \{\mathtt{x}_n = 0\} \subset \mathbb{C}^n$ is an irreducible analytic set.

Otherwise, there exist analytic sets $X \subsetneq H$ and $Y \subsetneq H$ in $\mathbb{C}^n$ such that $H = X \cup Y$. As the map $\varphi : \mathbb{C}^{n-1} \to H,\ x' \mapsto (x', 0)$ is an analytic diffeomorphism, the sets

$$X_1 := \{x' \in \mathbb{C}^{n-1} : (x', 0) \in X\} \quad \text{and} \quad Y_1 := \{x' \in \mathbb{C}^{n-1} : (x', 0) \in Y\}$$

are analytic in $\mathbb{C}^{n-1}$. In addtion, $X_1 \subsetneq \mathbb{C}^{n-1}$, $Y_1 \subsetneq \mathbb{C}^{n-1}$ and $\mathbb{C}^{n-1} = X_1 \cup Y_1$, which is a contradiction because $\mathbb{C}^{n-1}$ is irreducible.

A.2.a Irreducible components.

The main result of this appendix is the following, which characterizes the irreducible components of a complex analytic set X in terms of the closures of the connected components of the regular locus of X.

Theorem A.2.3 *Let $X \subset \Omega$ be an analytic set. Let $\{M_i\}_{i\in I}$ be the family of connected components of* $\operatorname{Reg}(X)$ *and denote $X_i := \operatorname{Cl}(M_i)$. We have:*

(i) *$X = \bigcup_{i\in I} X_i$, the family $\{X_i\}_{i\in I}$ is locally finite in Ω and $X_i \not\subset X_k$ if $i \neq k$.*

(ii) *Each $X_i \subset \Omega$ is an irreducible analytic set.*

(iii) *Let $\{Y_j\}_{j\in J}$ be a locally finite family of analytic subsets of Ω such that $X = \bigcup_{j\in I} Y_j$ and $Y_j \not\subset Y_k$ if $j \neq k$. Then there exists a bijection $\sigma : J \to I$ such that $Y_j = X_{\sigma(j)}$ for each $j \in J$.*

The collection $\{X_i\}_{i\in I}$ is the family of *irreducible components of X*.

Proof. (i) Fix a point $a \in \Omega$. By Lemma A.1.2 the germ $\operatorname{Reg}(X_a)$ has finitely many connected components. Thus, there exist an open neighborhood $U \subset \Omega$ of a such that $\operatorname{Reg}(X)\cap U$ has finitely many connected components $N_1, \ldots, N_r$.

By Corollary VII.4.5 X is locally connected. Thus, its open subset $\operatorname{Reg}(X)$ is locally connected and each connected component M_i of $\operatorname{Reg}(X)$ is open and closed in $\operatorname{Reg}(X)$. Consequently, $M_i \cap U$ is open and closed in $\operatorname{Reg}(X)\cap U$ and it is the union of those connected components N_j of $\operatorname{Reg}(X) \cap U$ that meet $M_i \cap U$. Define

$$\Lambda(i) := \{j = 1, \ldots, r : M_i \cap U \cap N_j \neq \varnothing\},$$

for each $i \in I$ and it holds $M_i \cap U = \bigcup_{j\in\Lambda(i)} N_j$.

A.2.a.1 $\Lambda(i) \cap \Lambda(k) = \varnothing$ *if $i \neq j$.*

Otherwise, there exist $\ell \in \Lambda(i) \cap \Lambda(k)$ and

$$N_\ell \subset (M_i \cap U) \cap (M_k \cap U) = \varnothing,$$

which is a contradiction.

A.2.a.2 *There exists a finite subset $F \subset I$ such that $\Lambda(i) = \varnothing$ for each $i \in I \setminus F$.*

By A.2.a.1 we write $\{1, \ldots, r\} = \bigsqcup_{i\in I} \Lambda(i)$. As $\{1, \ldots, r\}$ is finite and the previous union is pairwise disjoint, there exists a finite subset $F \subset I$ such that $\Lambda(i) = \varnothing$ for each $i \in I \setminus F$.

A.2.a.3 *The family $\Sigma := \{X_i\}_{i\in I}$ is locally finite.* It is enough to check: $X_i \cap U = \varnothing$ for each $i \in I \setminus F$.

If $\Lambda(i) = \varnothing$, then $M_i \cap U = \bigcup_{j\in\Lambda(i)} N_j = \varnothing$ and

$$X_i \cap U = \mathrm{Cl}(M_i) \cap U = \mathrm{Cl}(M_i \cap U) \cap U = \varnothing.$$

A.2.a.4 $X = \bigcup_{i\in I} X_i$.

As the family Σ is locally finite, the family $\{M_i\}_{i\in I}$ is locally finite too. By Corollary VII.4.7 and since X is closed in Ω, we have

$$X = \mathrm{Cl}(\mathrm{Reg}(X)) = \mathrm{Cl}\Big(\bigcup_{i\in I} M_i\Big) = \bigcup_{i\in I} \mathrm{Cl}(M_i) = \bigcup_{i\in I} X_i.$$

A.2.a.5 To finish the proof of (i) we have to check: $X_i \not\subset X_k$ if $i \neq k$.

Suppose by contradiction that $M_i \subset X_i \subset X_k$. As M_i is an open subset of $\mathrm{Reg}(X)$ and $\mathrm{Reg}(X)$ is an open subset of X, the set M_i is open in X. Thus, M_i is a non-empty open subset of X_k.

As M_k is a dense subset of $X_k = \mathrm{Cl}(M_k)$, we have $M_i \cap M_k \neq \varnothing$, which is a contradiction.

(ii) We prove first that $X_i \subset \Omega$ is an analytic set. As X_i is closed in Ω, it is enough to show that X_i is locally analytic.

A.2.a.6 To that end, let us check: *For each $x \in X_i$ the germ $X_{i,x}$ is analytic.*

As we have commented above, each connected component M_i of $\mathrm{Reg}(X)$ is an open and closed subset of $\mathrm{Reg}(X)$. Thus, $M_{i,x}$ is an open and closed germ in $\mathrm{Reg}(X_x)$, so it is a union of connected components of $\mathrm{Reg}(X_x)$. This germ has, by Lemma A.1.2, finitely many connected components. Therefore, there exist connected components $N_{1,x}, \dots, N_{\ell,x}$ of $\mathrm{Reg}(X_x)$ such that $M_{i,x} = N_{1,x} \cup \dots \cup N_{\ell,x}$. Consequently,

$$X_{i,x} = (\mathrm{Cl}(M_i))_x = \mathrm{Cl}(M_{i,x}) = \mathrm{Cl}(N_{1,x} \cup \dots \cup N_{\ell,x}) = \mathrm{Cl}(N_{1,x}) \cup \dots \cup \mathrm{Cl}(N_{\ell,x}).$$

By Lemma A.1.2 each $\mathrm{Cl}(N_{j,x})$ is an irreducible component of the analytic germ X_x, so $\mathrm{Cl}(N_{j,x})$ is an analytic germ. By Remark VI.2.10

$$\mathcal{J}(X_{i,x}) = \mathcal{J}(\mathrm{Cl}(N_{1,x}) \cup \dots \cup \mathrm{Cl}(N_{\ell,x})) = \mathcal{J}(\mathrm{Cl}(N_{1,x})) \cap \dots \cap \mathcal{J}(\mathrm{Cl}(N_{\ell,x})).$$

By Proposition VI.2.8

$$\begin{aligned} Z(\mathcal{J}(X_{i,x})) &= Z(\mathcal{J}(\mathrm{Cl}(N_{1,x})) \cap \cdots \cap \mathcal{J}(\mathrm{Cl}(N_{\ell,x}))) \\ &= Z(\mathcal{J}(\mathrm{Cl}(N_{1,x}))) \cup \cdots \cup Z(\mathcal{J}(\mathrm{Cl}(N_{\ell,x}))) \\ &= \mathrm{Cl}(N_{1,x}) \cup \cdots \cup \mathrm{Cl}(N_{\ell,x}) = X_{i,x}. \end{aligned}$$

Thus, $X_{i,x}$ is an analytic germ and we conclude that $X_i \subset \Omega$ is an analytic set.

To finish the proof of part (ii) we must see that X_i is irreducible.

A.2.a.7 We prove first: $M_i \subset \mathrm{Reg}(X_i)$ *for each* $i \in I$.

Pick a point $a \in M_i$. As Σ is a locally finite family, there exist an open neighborhood $U \subset \Omega$ of a and a finite set $F \subset I$ such that $X_k \cap U = \varnothing$ for each $k \in I \setminus F$. Thus, $X \cap U = \big(\bigcup_{k\in F} X_k\big) \cap U$, so $X_a = \bigcup_{k\in F} X_{k,a}$. As F is finite and X_a is by Corollary VI.4.6 an irreducible germ because $a \in M_i \subset \mathrm{Reg}(X)$, we deduce by Proposition VI.2.14 that $X_a = X_{j,a}$ for some index $j \in I$. In particular, $X_{i,a} \subset X_a = X_{j,a}$, so there exists an open neighborhood $V \subset U$ of a such that $X_i \cap V \subset X_j$. In addition, $M_i \cap V$ is open in $\mathrm{Reg}(X)$ and $\mathrm{Reg}(X)$ is open in X, hence $M_i \cap V$ is open in X and it is contained in $X_i \cap V \subset X_j$. Consequently, $M_i \cap V$ is a non-empty open subset of X_j. As M_j is dense in X_j, it follows that $M_i \cap V \cap M_j \neq \varnothing$. Then $i = j$, so $X_a = X_{i,a}$. As $a \in M_i \subset \mathrm{Reg}(X)$, the germ $X_{i,a} = X_a$ is regular. Thus, $a \in \mathrm{Reg}(X_i)$ and the inclusion $M_i \subset \mathrm{Reg}(X_i)$ holds.

A.2.a.8 $\mathrm{Reg}(X_i)$ *is a connected, dense and open subset of* X_i.

As $M_i \subset \mathrm{Reg}(X_i) \subset X_i = \mathrm{Cl}(M_i)$ and M_i is connected, $\mathrm{Reg}(X_i)$ is also connected.

A.2.a.9 We claim: X_i *is an irreducible analytic set.*

Let $Y, Z \subset \Omega$ be analytic sets such that $Y \subsetneq X_i$ and $X_i = Y \cup Z$. As Y is closed in Ω, it is closed in X_i, so $X_i \setminus Y$ is a non-empty open subset of X_i. As $\mathrm{Reg}(X_i)$ is a dense subset of X_i,

$$A := \mathrm{Reg}(X_i) \setminus Y = \mathrm{Reg}(X_i) \cap (X_i \setminus Y) \subset Z$$

is a non-empty open subset of the connected analytic submanifold $\mathrm{Reg}(X_i)$. By Corollary V.3.6 $\mathrm{Reg}(X_i) \subset Z$, so $X_i = \mathrm{Cl}(\mathrm{Reg}(X_i)) \subset Z$ and we conclude that X_i is an irreducible analytic subset of Ω.

(iii) We prove now the uniqueness of the family of irreducible components of the analytic set X.

Let $j \in J$ and pick a point $x \in Y_j$. As Σ is a locally finite family, there exists a finite subset $F \subset I$ such that $x \notin Z := \bigcup_{i \in I \setminus F} X_i$. As each subset $X_i \subset \Omega$ is analytic, it follows from Lemma V.2.3 that $Z \subset \Omega$ is an analytic set. In addition, $Y_j \not\subset Y_j \cap Z$ because $x \in Y_j \setminus Z$. The equality

$$Y_j = (Y_j \cap Z) \cup \bigcup_{i \in F} (X_i \cap Y_j)$$

and the irreducibility of Y_j implies that $Y_j \subset X_i \cap Y_j \subset X_i$ for some index $i \in F$. Changing the roles of both families, there exists an index $k \in J$ such that $X_i \subset Y_k$. Thus, $Y_j \subset X_i \subset Y_k$, so $j = k$ and $X_i = Y_j$. One concludes that there exists a bijection $\sigma : J \to I$, $j \mapsto i$, as required. □

Remarks A.2.4 (i) Let $\Sigma := \{X_i\}_{i \in I}$ be the family of irreducible components of an analytic set $X = \bigcup_{i \in I} X_i \subset \Omega$ and let $a \in X$. Then there exists a finite set $F \subset I$ such that $X_a = \bigcup_{i \in F} X_{i,a}$.

As Σ is locally finite there exists an open neighborhood $U \subset \Omega$ of a such that $X_i \cap U = \varnothing$ for each $i \in I \setminus F$. Thus, $X \cap U = \bigcup_{i \in F} (X_i \cap U)$ and the equality in the statement follows.

(ii) In the proof of A.2.a.7 in Theorem A.2.3 we proved that if X_a is irreducible for some point $a \in \Omega$ then there exists an index $i \in I$ such that $X_a = X_{i,a}$.

A.2.b Properties of the irreducible components.

We prove next that the family of irreducible components of an analytic set is countable.

Corollary A.2.5 *The family of irreducible components of an analytic set is countable.*

Proof. Let $\Sigma := \{X_i\}_{i \in I}$ be the family of irreducible components of an analytic set $X \subset \Omega$. To prove that I is a countable set let us show that I is a countable union of finite sets.

Let $\{U_m\}_{m \in \mathbb{N}}$ be a countable basis of open subsets of Ω and let $x \in \Omega$. As Σ is locally finite, there exist $m(x) \in \mathbb{N}$ and a finite set $F_{m(x)} \subset I$ such that $x \in U_{m(x)}$ and $X_i \cap U_{m(x)} = \varnothing$ for each index $i \in I \setminus F_{m(x)}$. As the set $P := \{m(x) : x \in \Omega\} \subset \mathbb{N}$ is countable and $I = \bigcup_{m(x) \in P} F_{m(x)}$ we conclude that I is countable, as required. □

Corollary A.2.6 *Let $\{X_i\}_{i\in I}$ be a locally finite family of analytic subsets of Ω. Then $X := \bigcup_{i\in I} X_i \subset \Omega$ is an analytic set and*

$$\dim(X) = \sup\{\dim(X_i) : i \in I\}.$$

In particular, the dimension of an analytic set is the supremum of the dimensions of its irreducible components.

Proof. The analyticity of X was proved in Lemma V.2.3. By Proposition VI.3.3 $\dim(X_i) \le \dim(X)$ for each $i \in I$. Conversely, let $d := \dim(X)$ and let $a \in X$ be such that $\dim(X) = \dim(X_a)$. By Remark A.2.4 there exists a finite set $F \subset I$ such that $X_a = \bigcup_{i\in F} X_{i,a}$. This implies by Proposition VI.3.3 that

$$\begin{aligned}\dim(X) = d = \dim(X_a) &= \max\big\{\dim(X_{i,a}) : i \in F\big\}\\ &\le \max\{\dim(X_i) : i \in F\} \le \sup\{\dim(X_i) : i \in I\},\end{aligned}$$

as required. □

Corollary A.2.7 *Let $X \subset \Omega$ be a zero-dimensional analytic set. Then X is a countable set.*

Proof. Let $\{X_i\}_{i\in I}$ be the family of irreducible components of X. By Corollary A.2.5 the set I is countable and by Corollary A.2.6 each $\dim(X_i) = 0$. We claim that each X_i is a singleton.

Pick a point $a \in X_i$. Then $\dim(X_{i,a}) = 0$, so there exists an open neighborhood $U \subset \Omega$ of a such that $X_i \cap U = \{a\}$. This means that X_i is a discrete subset of Ω. Thus, given a point $a_i \in X_i$ the difference $Y_i := X_i \setminus \{a_i\}$ is a discrete subset of Ω. In particular $Y_i \subset \Omega$ is analytic and $X_i = \{a_i\} \cup Y_i$. As X_i is irreducible, $Y_i = \varnothing$, so $X_i = \{a_i\}$, as required. □

Lemma A.2.8 *Let $\{X_i\}_{i\in I}$ be the family of irreducible components of an analytic set $X \subset \Omega$. Let $i_0 \in I$ and define $Y := \bigcup_{i\neq i_0} X_i$. Then $Y \subset \Omega$ is an analytic set, $X_{i_0} \not\subset Y$ and $\operatorname{Reg}(X_{i_0}) \setminus Y$ is a connected and dense subset of X_{i_0}.*

Proof. Let $\{M_i\}_i$ be the family of connected components of $\operatorname{Reg}(X)$. By Theorem A.2.3 we may assume $X_i = \operatorname{Cl}(M_i)$ for each $i \in I$. As the family

$\{X_i\}_{i\neq i_0}$ is locally finite, the union $Y := \bigcup_{i\neq i_0} X_i \subset \Omega$ is by Lemma V.2.3 an analytic set.

Suppose by contradiction that $X_{i_0} \subset Y$. Then $X = Y$ and $M_{i_0} \subset X_{i_0} \subset Y$. By Corollary VII.4.5 X is locally connected. Thus, its open subset $\operatorname{Reg}(X)$ is locally connected too, so the connected component M_{i_0} of $\operatorname{Reg}(X)$ is open in $\operatorname{Reg}(X)$. Therefore M_{i_0} is open in Y. As the family $\{M_i\}_{i\neq i_0}$ is locally finite,

$$\operatorname{Cl}\Big(\bigcup_{i\neq i_0} M_i\Big) = \bigcup_{i\neq i_0} \operatorname{Cl}(M_i) = \bigcup_{i\neq i_0} X_i = Y.$$

Thus, $\bigcup_{i\neq i_0} M_i$ is a dense subset of Z and consequently it meets the non-empty subset M_i of Y, which is a contradiction. Consequently, $Y \subsetneq X$ and $X_{i_0} \not\subset Y$.

For the second part note first that $\operatorname{Reg}(X_{i_0}) \not\subset Y$ because otherwise $X_{i_0} = \operatorname{Cl}(\operatorname{Reg}(X_{i_0})) \subset Y$. As $\operatorname{Reg}(X_{i_0})$ is a connected analytic submanifold and $Y \subset \Omega$ is an analytic set such that $\operatorname{Reg}(X_{i_0}) \not\subset Y$, it follows from Theorem V.3.7 and Corollary V.5.7 that $\operatorname{Reg}(X_{i_0}) \setminus Y$ is a connected and dense subset of $\operatorname{Reg}(X_{i_0})$. As $\operatorname{Reg}(X_{i_0})$ is dense in X_{i_0}, we conclude $\operatorname{Reg}(X_{i_0}) \setminus Y$ is a dense subset of X_{i_0}, as required. □

We end this section proving that an analytic set is irreducible if and only if its regular locus is connected.

Corollary A.2.9 *An analytic set $X \subset \Omega$ is irreducible if and only if its subset* $\operatorname{Reg}(X)$ *of regular points is connected.*

Proof. Let $\{M_i\}_{i\in I}$ be the family of connected components of $\operatorname{Reg}(X)$ and define $X_i := \operatorname{Cl}(M_i)$. We proved in Theorem A.2.3 that $\{X_i\}_{i\in I}$ is the family of irreducible components of X.

If $\operatorname{Reg}(X)$ is connected, then $I = \{i_0\}$ and $X = X_{i_0}$ is irreducible. Suppose now that $\operatorname{Reg}(X)$ is not connected. Pick $i \in I$ and denote $J := I \setminus \{i\} \neq \varnothing$. By Lemma A.2.8 $Y := \bigcup_{j\in J} X_j$ is an analytic subset of Ω such that $X_i \not\subset Y$, so $Y \subsetneq X$. As $J \neq \varnothing$, we deduce $X_i \subsetneq X$. In addition, $X = X_i \cup Y$, so X is reducible, as required. □

A.3 Some consequences

We finish this appendix with some useful properties of irreducible analytic sets.

A.3.a Analytic functions on irreducible analytic sets.

We present next some properties of the analytic functions and maps on irreducible analytic sets.

Corollary A.3.1 *Let $f : \Omega \to \mathbb{C}$ be an analytic function and let $X \subset \Omega$ be an irreducible analytic set. Then:*

(i) *If there exists a non-empty open subset M of X such that $M \subset Z_\Omega(f)$ then $f|_X = 0$.*

(ii) *If the function $X \to \mathbb{R}$, $x \mapsto |f(x)|$ attains a local maximum on X then $f|_X$ is constant.*

Proof. (i) By Corollary VII.4.7 $\operatorname{Reg}(X)$ is a dense subset of X. Thus, $A := M \cap \operatorname{Reg}(X)$ is a non-empty open subset of the connected analytic submanifold $\operatorname{Reg}(X)$. As $A \subset Z_\Omega(f)$, it follows by the Identity Principle V.3.5 that $\operatorname{Reg}(X) \subset Z_\Omega(f)$. Thus, $X = \operatorname{Cl}(\operatorname{Reg}(X)) \subset \operatorname{Cl}(Z_\Omega(f)) = Z_\Omega(f)$, so $f|_X = 0$.

(ii) Let $a \in X$ and let $U \subset \Omega$ be an open neighborhood of a such that $|f(x)| \leq |f(a)|$ for each $x \in X \cap U$. By Corollary VII.4.5 X is locally connected, so there exists an open neighborhood $V \subset U$ of a such that $X \cap V$ is connected. As $|f|_{X\cap V} : X \cap V \to \mathbb{R}$ attains at the point a an absolute maximum and $X \cap V$ is an analytic subset of V, we deduce by the Maximum modulus principle VII.5.7 that $f|_{X\cap V}$ is constant. By part (i) f is constant on X, as required. □

Corollary A.3.2 *Let $X \subset \Omega$ be an irreducible analytic set and let $Y \subset \Omega$ be an analytic set with $X \not\subset Y$. Then $Z := X \setminus Y$ is a connected irreducible analytic subset of $\Theta := \Omega \setminus Y$ and $\operatorname{Cl}_X(Z) = X$.*

Proof. As Y is a closed subset of Ω, it holds that $Z \subset \Theta$ is an analytic set. In addition, $\operatorname{Reg}(X) \setminus Y \subset \operatorname{Reg}(Z) \subset Z$ and $\operatorname{Reg}(X) \setminus Y$ is by Corollaries A.2.9 and V.5.7 connected and dense in $\operatorname{Reg}(X)$. As $\operatorname{Reg}(X)$ is dense in X, we conclude that $\operatorname{Reg}(X) \setminus Y$ is dense in X. Thus, $\operatorname{Reg}(X) \setminus Y$ is dense in Z. As $\operatorname{Reg}(X) \setminus Y$ is connected, $\operatorname{Reg}(Z)$ and Z are connected. Finally, by Corollary A.2.9 Z is irreducible, as required. □

Corollary A.3.3 *Let $X \subset \Omega$ be an irreducible analytic set and $f : \Omega \to \mathbb{C}^m$ be an analytic map whose restriction $f|_X : X \to \mathbb{C}^m$ is a non-constant closed map. Then $f|_X$ is a proper map.*

Proof. As Ω is locally compact and satisfies the second countably axiom, its closed subset X satisfies both properties. Thus, X is a locally compact and Lindelöf space. Thus, there exists an exhaustion $\{K_\ell\}_{\ell\in\mathbb{N}}$ of X by compact subsets, that is, $X = \bigcup_{\ell\geq 1} K_\ell$ and $K_\ell \subset \operatorname{Int}_X(K_{\ell+1})$ for each $\ell \geq 1$. To prove that the closed map $f|_X : X \to \mathbb{C}^m$ is proper it is enough to prove that its fibers are compact.

A.3.a.1 Suppose by contradiction that there exists a point $b \in \mathbb{C}^m$ such that its fiber $Y := (f|_X)^{-1}(b) = X \cap f^{-1}(b)$ is not compact. Note that $Y \subset \Omega$ is an analytic set and $Y \subsetneq X$ because $f|_X$ is non-constant. As X is irreducible, $X \setminus Y$ is by Corollary A.3.2 a dense subset of X. Let $\{C_\ell\}_{\ell\in\mathbb{N}}$ be a countable basis of closed neighborhoods of b in $\mathbb{C}^m$. As Y is a closed non-compact subset of X, we have $Y \not\subset K_\ell$ for each $\ell \geq 1$. In addition, $Y \subset X \cap f^{-1}(\operatorname{Int}(C_\ell))$ because $b \in \operatorname{Int}(C_\ell)$ for each $\ell \geq 1$. Thus, $X \cap f^{-1}(\operatorname{Int}(C_\ell)) \not\subset K_\ell$ for each $\ell \geq 1$, hence $U_\ell := X \cap f^{-1}(\operatorname{Int}(C_\ell)) \setminus K_\ell$ is a non-empty open subset of X for each $\ell \geq 1$. As $X \setminus Y$ is dense in X, we have $M_\ell := (X \setminus Y) \cap U_\ell \neq \varnothing$. Pick a point $x_\ell \in M_\ell$ for each $\ell \geq 1$.

A.3.a.2 We claim: $C := \{x_\ell : \ell \in \mathbb{N}\}$ *is a closed subset of* X.

Otherwise, there exists a point $a \in \operatorname{Cl}_X(C) \setminus C$. As $X = \bigcup_{\ell\in\mathbb{N}} K_\ell$, there exists $k \in \mathbb{N}$ such that $a \in K_k \subset \operatorname{Int}_X(K_{k+1})$. In addition, $a \notin C$, so $a \neq x_\ell$ for $\ell = 1, \dots, k$. Thus, $W := \operatorname{Int}_X(K_{k+1}) \setminus \{x_1, \dots, x_k\} \subset X$ is an open neighborhood of $a \in \operatorname{Cl}_X(C)$. Therefore $W \cap C \neq \varnothing$, hence there exists an index $\ell > k$ such that $x_\ell \in \operatorname{Int}_X(K_{k+1}) \subset K_{k+1} \subset K_\ell$. Consequently,

$$x_\ell \in K_\ell \cap M_\ell \subset K_\ell \cap U_\ell = \varnothing,$$

which is a contradiction. Thus, C is a closed subset of X.

A.3.a.3 As $x_\ell \in M_\ell \subset X \setminus Y$, we have $x_\ell \notin f^{-1}(b)$ for each $\ell \geq 1$. As C is a closed subset of X, we deduce that $f(C)$ is a closed subset of $\mathbb{C}^m$. Thus, $b \notin f(C)$. Since each point

$$x_\ell \in M_\ell \subset U_\ell \subset f^{-1}(\operatorname{Int}(C_\ell))$$

we have $f(x_\ell) \in C_\ell$, so $b = \lim_{\ell\to\infty} f(x_\ell)$. As each $f(x_\ell) \in f(C)$, we get $b = \lim_{\ell\to\infty} f(x_\ell) \in f(C)$, which is a contradiction. Consequently, the fibers of f are compact, as required. $\square$

Corollary A.3.4 *Let $\Omega \subset \mathbb{C}^n$ be a connected and open set and let $f : \Omega \to \mathbb{C}^n$ be a closed and non-constant analytic map. Then f is open and surjective.*

Proof. As Ω is connected, it is an irreducible analytic set because $\operatorname{Reg}(\Omega) = \Omega$. By Corollary A.3.3 f is a proper map. Thus, each fiber $f^{-1}(p) \subset \Omega$ is by Example V.2.5(iii) a compact analytic set. By Corollary VII.5.8 $f^{-1}(p)$ is a finite set. By Theorem VII.6.1 f is an open map. As $f(\Omega)$ is a non-empty closed and open subset of $\mathbb{C}^n$, we conclude that f is surjective, as required. □

Remark A.3.5 The previous result does not hold in the real case even if $n = 1$. The non-constant analytic map $f : \mathbb{R} \to \mathbb{R}$, $t \mapsto t^2$ is closed but it is neither open nor surjective because $f(\mathbb{R}) = [0, +\infty)$. To prove that f is closed, let $C \subset \mathbb{R}$ be a closed set and denote $C_1 := C \cap (-\infty, 0]$ and $C_2 := C \cap [0, +\infty)$. As the restrictions

$$f|_{(-\infty,0]} : (-\infty, 0] \to [0, +\infty) \quad \text{and} \quad f|_{[0,+\infty)} : [0, +\infty) \to [0, +\infty)$$

are homeomorphisms, both $f(C_1)$ and $f(C_2)$ are closed subsets of $[0, +\infty)$, hence they are closed in $\mathbb{R}$. Consequently, $f(C) = f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$ is closed in $\mathbb{R}$ and f is a closed map.

A.3.b Local irreducible versus global irreducibility. We begin analyzing some significant examples to contrast the irreducibility of analytic sets and germs.

Examples A.3.6 (i) The affine subspaces X of $\mathbb{C}^n$ are irreducible analytic sets because they are connected and $X = \operatorname{Reg}(X)$.

(ii) Let $f := \mathtt{x}^2 - \mathtt{z}\mathtt{y}^2 \in \mathbb{C}[\mathtt{x}, \mathtt{y}, \mathtt{z}]$. Let us prove: *the analytic set $X := Z_{\mathbb{C}^3}(f)$ is irreducible.* However, if $w \in \mathbb{C} \setminus \{0\}$ and $a := (0, 0, w)$, *the germ X_a is reducible.* To prove the irreducibility of X we show: $\operatorname{Reg}(X)$ *is connected.*

Let $\Omega := \mathbb{C}^3 \setminus \{\mathtt{x} = 0, \mathtt{y} = 0\} \subset \mathbb{C}^3$ and consider the analytic map $f : \Omega \to \mathbb{C}$, $(x, y, z) \mapsto x^2 - zy^2$. For each point $p := (x, y, z) \in \Omega$ we have

$$\operatorname{rk}(\operatorname{Jac} f(p)) = \operatorname{rk}(2x, -2yz, -y^2) = 1$$

and by Lemma V.3.11 $M := Z_\Omega(f) \subset X$ is an analytic submanifold of dimension $3 - 1 = 2$. Consider the analytic map

$$\varphi : \mathbb{C}^2 \to X,\ (u, v) \mapsto (uv, v, u^2)$$

and let us check: *φ is surjective and $\varphi(U) = M$ where $U := \mathbb{C}^2 \setminus \{\mathtt{v} = 0\}$.*

Given $p := (x, y, z) \in X$ suppose first that $x = 0$. If $y = 0$, then $p = (0, 0, z) = \varphi(u, 0)$ where $u^2 = z$. If $y \neq 0$, we have $z = 0$, so $\varphi(0, y) = p$. Suppose now $x \neq 0$, define $u := \frac{x}{y} \in \mathbb{C} \setminus \{0\}$ and take $v \in \mathbb{C} \setminus \{0\}$ such that $v^2 = u$. Then $(u, y) \in U$ and $z = \frac{x^2}{y^2} = u^2$. Thus, $\varphi(u, y) = (uy, y, u^2) = (x, y, z) = p$. In addition, if $(u, v) \in U$, then $\varphi(U) \subset X \cap \{\mathtt{y} \neq 0\} = M$. On the other hand, $\varphi(u, 0) \in \{\mathtt{x} = 0, \mathtt{y} = 0\}$ for each $u \in \mathbb{C}$, so $\varphi(U) = M$.

Thus, $M \subset \operatorname{Reg}(X) \subset X = \varphi(\operatorname{Cl}(U)) \subset \operatorname{Cl}(M)$ and $M = \varphi(U)$ is connected because U is a connected open subset of $\mathbb{C}^2$. We deduce $\operatorname{Reg}(X)$ is connected and X is irreducible.

Next, we prove that each germ X_a is reducible. Consider the polynomial $g := \mathtt{x}^2 - (\mathtt{z} - \varepsilon)\mathtt{y}^2$, the analytic set $Y := Z_{\mathbb{C}^3}(g)$ and the analytic equivalence $u : \mathbb{C}^3 \to \mathbb{C}^3$, $(x, y, z) \mapsto (x, y, z - \varepsilon)$. Notice that $u(a) = 0$ and, with the notations in Lemma VI.2.16, $X_a = u_*(Y_0)$. Thus, it is enough to show that Y_0 is a reducible germ. As

$$g(\mathtt{x}, \mathtt{y}, \mathtt{z}) = \mathtt{x}^2 - (\mathtt{z} - \varepsilon)\mathtt{y}^2 = (\mathtt{x} - \sqrt{\mathtt{z} - \varepsilon}\,\mathtt{y}) \cdot (\mathtt{x} + \sqrt{\mathtt{z} - \varepsilon}\,\mathtt{y}),$$

we have $Y_0 := T_{1,0} \cup T_{2,0}$ where

$$T_1 := \{\mathtt{x} - \sqrt{\mathtt{z} - \varepsilon}\,\mathtt{y} = 0\} \quad \& \quad T_2 := \{\mathtt{x} + \sqrt{\mathtt{z} - \varepsilon}\,\mathtt{y} = 0\},$$

so the germ X_a is reducible, as required. □

Next, we show that local irreducibility of a connected analytic set provides global irreducibility.

Corollary A.3.7 *Let $X \subset \Omega$ be a connected analytic set such that the germ X_a is irreducible for each point $a \in X$. Then X is irreducible.*

Proof. Suppose that X is reducible and let $\{X_i\}_{i \in I}$ be the family of irreducible components of X.

A.3.b.1 We claim: *There exist $a \in X$ and $k \in I$ such that $a \in X_k$ and $X_{k,a} \subset \big(\bigcup_{i \neq k} X_i\big)_a$.*

Pick $i_0 \in I$ and notice that $Y := \bigcup_{i \neq i_0} X_i \subset \Omega$ is by Lemma V.2.3 an analytic subset. In particular, X_{i_0} and Y are closed subsets of X and $X =$

$X_{i_0} \cup Y$. As X is connected, $X_{i_0} \cap Y \neq \varnothing$. Pick a point $a \in X_{i_0} \cap Y$. Then $X_{i_0,a} \neq \varnothing$, $Y_a \neq \varnothing$ and $X_a = X_{i_0,a} \cup Y_a$. As the germ X_a is irreducible, either $X_a = Y_a$ or $X_a = X_{i_0}$.

If $X_a = Y_a$, then $X_{i_0,a} \subset X_a = Y_a = \big(\bigcup_{i \neq i_0} X_i\big)_a$ and we take $k = i_0$. If $X_a = X_{i_0}$ and since $a \in Y$, there exists $k \neq i_0$ such that $a \in X_k$. In addition, $X_{k,a} \subset X_a = X_{i_0,a} \subset \big(\bigcup_{i \neq k} X_i\big)_a$, as claimed.

A.3.b.2 The union $Z := \bigcup_{i \neq k} X_i \subset \Omega$ is by Lemma V.2.3 an analytic set. As $X_{k,a} \subset Z_a$, there exists an open neigboorhood $U \subset \Omega$ of a such that $X_k \cap U \subset Z$. As X_k is irreducible, $\operatorname{Reg}(X_k)$ is by Corollary A.2.9 a connected analytic submanifold and by Corollary VII.4.7 it is dense in X_k. In particular, it meets the non-empty open subset $X_k \cap U$ of X_k. Thus, $M := \operatorname{Reg}(X_k) \cap U$ is a non-empty open subset of the connected submanifold $\operatorname{Reg}(X_k)$ and

$$M = \operatorname{Reg}(X_k) \cap U \subset X_k \cap U \subset Z.$$

By Corollary V.3.6 $\operatorname{Reg}(X_k) \subset Z$, so $X_k \subset Z$, which is a contradiction by Lemma A.2.8. We conclude that X is irreducible, as required. □

We prove next that irreducible analytic sets are pure dimensional.

Corollary A.3.8 *Let $X \subset \Omega$ be an irreducible analytic set of dimension d. Then $\dim(X_a) = d$ and X_a is pure dimensional for each point $a \in X$.*

Proof. By Corollary A.2.9 $\operatorname{Reg}(X)$ is connected. By VI.4.4 and Remark VI.4.5 we know that $\operatorname{Reg}(X) = \bigsqcup_{e=0}^{d} \operatorname{Reg}_e(X)$ and the set $\operatorname{Reg}_e(X)$ of e-dimensional regular points of X is an open subset of $\operatorname{Reg}(X)$. For each $k = 0, \dots, d$

$$\operatorname{Reg}_k(X) = \operatorname{Reg}(X) \setminus \bigsqcup_{e \neq k} \operatorname{Reg}_e(X)$$

is an open and closed subset of the connected set $\operatorname{Reg}(X)$. As $\dim(X) = d$, we deduce $\operatorname{Reg}_d(X) \neq \varnothing$, so $\operatorname{Reg}(X) = \operatorname{Reg}_d(X)$ and $\operatorname{Reg}_k(X) = \varnothing$ for $k \neq d$. Thus, $a \in \operatorname{Cl}(\operatorname{Reg}_d(X))$ and $a \notin \operatorname{Cl}_X(\operatorname{Reg}_k(X))$ for $k \neq d$. Thus, the germ X_a is pure dimensional and by Corollary VII.4.7 $\dim(X_a) = d$, as required. □

We confront next two analytic sets one of which is irreducible.

Corollary A.3.9 *Let $X \subset \Omega$ and $Y \subset \Omega$ be two analytic sets such that X is irreducible and $X \not\subset Y$. Then for each point $a \in \Omega$ the analytic germs X_a and Y_a do not share any irreducible component.*

Proof. Suppose by contradiction that X_a and Y_a share an irreducible component for some point $a \in \Omega$. Then there exist an open neighborhood $U \subset \Omega$ of a and analytic subsets $Z, X_1, \dots, X_r, Y_1, \dots, Y_s$ in U such that

$$X \cap U = Z \cup \bigcup_{i=1}^{r} X_i, \quad Y \cap U = Z \cup \bigcup_{j=1}^{s} Y_j$$

and $\{Z_a, X_{1,a}, \dots X_{r,a}\}$ and $\{Z_a, Y_{1,a}, \dots Y_{s,a}\}$ are the families of the irreducible components of X_a and Y_a. Notice that

$$M := Z \setminus \bigcup_{i=1}^{r} X_i = (X \cap U) \setminus \bigcup_{i=1}^{r} X_i$$

is a non-empty open subset of $X \cap U$. In addition, $X \setminus Y$ is by Corollary A.3.2 a dense subset of X, hence $M \cap (X \setminus Y) \neq \varnothing$, which is a contradiction because $M \subset Z \subset Y$. Thus, the analytic germs X_a and Y_a do not share any irreducible component, as required. □

Corollary A.3.10 *Let $X \subset \Omega$ and $Y \subset \Omega$ be analytic sets such that $Y \subsetneq X$ and X is irreducible. Then* $\dim(Y_a) < \dim(X_a)$ *for each point $a \in Y$.*

Proof. Let $a \in Y$ and $d := \dim(Y_a)$. By Proposition VI.3.3 and since $Y_a \subset X_a$, we have $\dim(Y_a) \leq \dim(X_a)$. Suppose by contradiction $\dim(X_a) = d$. Let Z_a be an irreducible component of Y_a of dimension d and let $X_{1,a}, \dots, X_{r,a}$ be the irreducible components of X_a. Then,

$$Z_a \subset Y_a \subset X_a = X_{1,a} \cup \dots \cup X_{r,a}$$

and by Proposition VI.2.14 there exists $i = 1, \dots, r$ such that $Z_a \subset X_{i,a}$. In addition,

$$d = \dim(Z_a) \leq \dim(X_{i,a}) \leq \dim(X_a) = d.$$

As $X_{i,a}$ is an irreducible germ, we deduce by Proposition VI.3.3 that $Z_a = X_{i,a}$. Thus, the germs X_a and Y_a share an irreducible component, which is a contradiction by Corollary A.3.9 because $Y \subsetneq X$. Consequently, $\dim(Y_a) < \dim(X_a)$,

as required. □

Given an analytic set X we end this appendix analyzing the set of points of X of dimension strictly greater than a fixed value d.

Corollary A.3.11 *Let $X \subset \Omega$ be an analytic set and for each non-negative integer d let $\Gamma_d(X) := \{x \in X : \dim(X_x) > d\}$. Then:*

(i) *Each $\Gamma_d(X)$ is an analytic subset of Ω.*

(ii) $\dim(\Gamma_d(X)_a) = \dim(X_a)$ *for each $a \in \Gamma_d(X)$.*

Proof. (i) Let $\Sigma := \{X_i\}_{i \in I}$ be the family of irreducible components of X and let $d(i) := \dim(X_i)$. By Corollary A.3.8 $\dim(X_{i,a}) = d(i)$ for each point $a \in X_i$. Let $\Lambda(d) := \{i \in I : d(i) > d\}$. We claim: $\Gamma_d(X) = \bigcup_{i \in \Lambda(d)} X_i$.

As Σ is a locally finite family, for each $x \in \Gamma_d(X) \subset X$ there exists a finite subset $F \subset I$ such that $X_x = \bigcup_{i \in F} X_{i,x}$. By Proposition VI.3.3

$$d < \dim(X_x) = \max\{\dim(X_{i,x}) : i \in F\}$$

and there exists an index $i \in F$ with $d < \dim(X_{i,x}) = d(i)$. Thus, $x \in \bigcup_{i \in \Lambda(d)} X_i$. Conversely, let $x \in \bigcup_{i \in \Lambda(d)} X_i$. Then there exists $i \in I$ such that $d(i) > d$ and $x \in X_i$, so $d < d(i) = \dim(X_{i,x}) \leq \dim(X_x)$ and $x \in \Gamma_d(X)$, as claimed.

By Lemma V.2.3 and Theorem A.2.3 we conclude that $\Gamma_d(X) = \bigcup_{i \in \Lambda(d)} X_i$ is an analytic subset of Ω.

(ii) Given $a \in \Gamma_d(X) \subset X$ we have $(\Gamma_d(X))_a \subset X_a$. Therefore,

$$\dim(\Gamma_d(X)_a) \leq \dim(X_a).$$

Conversely, let $F \subset I$ be a finite set such that $X_a = \bigcup_{i \in F} X_i$. Then

$$d < \dim(X_a) = \max\{\dim(X_{i,a}) : i \in F\} = \max\{d(i) : i \in F\} = d(k)$$

for some $k \in F$. Thus, $k \in \Lambda(d)$, so $X_k \subset \Gamma_d(X)$ and this implies

$$\dim(X_a) = d(k) = \dim(X_{k,a}) \leq \dim(\Gamma_d(X)_a),$$

as required. □

APPENDIX B

An elementary proof of Krull's Theorem

We present an elementary proof of Krull's Intersection Theorem II.1.7 without use of the $\mathfrak{m}$-adic topology.

Theorem B.1 (Krull's intersection) *Let $(A, \mathfrak{m})$ be a local noetherian ring, let M be a finitely generated A-module and let N be a submodule of M. Then $N = \bigcap_{k \geq 0}(N + \mathfrak{m}^k M)$. In particular, $\mathfrak{a} = \bigcap_{k \geq 0}(\mathfrak{a} + \mathfrak{m}^k)$ for each ideal $\mathfrak{a}$ of A.*

Proof. The second part of the statement follows from the first if we take $M = A$ and $N = \mathfrak{a}$. The proof of the first part of the statement is conducted in several steps:

B.1.a First reduction. Let M be a finitely generated A-module and denote $E_M := \bigcap_{k \geq 0} \mathfrak{m}^k M$. It is enough to prove: *the A-module $E_M = 0$.*

Assume B.1.a proved and consider the finitely generated A-module M/N. We have $E_{M/N} := \bigcap_{k \geq 0} \mathfrak{m}^k(M/N) = 0$. Suppose there exists

$$x \in \bigcap_{k \geq 0}(N + \mathfrak{m}^k M) \setminus N.$$

Then $x + N \neq 0 + N$, so there exists $\ell \geq 0$ such that $x + N \notin \mathfrak{m}^\ell(M/N)$. As $x \in N + \mathfrak{m}^\ell M$, there exist $m_1, \ldots, m_r \in M$ and $a_1, \ldots, a_r \in \mathfrak{m}^\ell$ such that

$$x + N = \sum_{i=1}^{r} a_i m_i + N \in \mathfrak{m}^\ell(M/N),$$

which is a contradiction. Consequently, $N = \bigcap_{k \geq 0}(N + \mathfrak{m}^k M)$.

B.1.b Second reduction. We may assume: $\mathfrak{m}E_M = 0$.

The product $\mathfrak{m}E_M$ is a submodule of M and the quotient $Q := M/\mathfrak{m}E_M$ is a finitely generated A-module. We claim: $\mathfrak{m}E_Q = 0$.

Pick $q \in E_Q$ and $m \in M$ be such that $q := m + \mathfrak{m}E_M$. Fix $k \geq 1$. As $q \in E_Q \subset \mathfrak{m}^k Q$, there exist $m_1, \ldots, m_s \in M$ and $b_1, \ldots, b_s \in \mathfrak{m}^k$ such that

$$m + \mathfrak{m}E_M = q = \sum_{j=1}^{s} b_j(m_j + \mathfrak{m}E_M) = \Big(\sum_{j=1}^{s} b_j m_j\Big) + \mathfrak{m}E_M.$$

Consequently,

$$m = \Big(m - \sum_{j=1}^{s} b_j m_j\Big) + \sum_{j=1}^{s} b_j m_j \in \mathfrak{m}E_M + \mathfrak{m}^k M \subset \mathfrak{m}^k M.$$

Thus, $m \in \mathfrak{m}^k M$ for each $k \geq 1$, so $m \in E_M$. Now, if $a \in \mathfrak{m}$, then $am \in \mathfrak{m}E_M$, so $aq = a(m + \mathfrak{m}E_M) = 0 + \mathfrak{m}E_M$ and $\mathfrak{m}E_Q = 0$.

B.1.b.1 Let us check: *if* $E_Q = 0$*, then* $E_M = 0$.

Observe that $E_M = \bigcap_{k\geq 0} \mathfrak{m}^k M \subset \bigcap_{k\geq 0}(\mathfrak{m}E_M + \mathfrak{m}^k M)$. Let us show: $\bigcap_{k\geq 0}(\mathfrak{m}E_M + \mathfrak{m}^k M) \subset \mathfrak{m}E_M$.

Pick $x \in \mathfrak{m}E_M + \mathfrak{m}^k M$ for each $k \geq 0$. Then

$$x + \mathfrak{m}E_M \in \bigcap_{k\geq 0} \mathfrak{m}^k Q = E_Q = 0,$$

so $x \in \mathfrak{m}E_m$.

Consequently, $E_M = \mathfrak{m}E_M$. By Nakayama's Lemma III.1.4 we conclude $E_M = 0$, as required.

B.1.c Proof of the reduced version.

We have to prove: *if* $\mathfrak{m}E_M = 0$*, then* $E_M = 0$.

Consider the non-empty family

$$\mathfrak{F} := \{F \text{ is a submodule of } M \text{ and } F \cap E = \{0\}\}$$

of submodules of E (notice that $\{0\} \in \mathfrak{F}$). As M is a finitely generated A-module and A is noetherian, the A-module M is noetherian, so the family $\mathfrak{F}$ has a maximal element F_0.

B.1.c.1 We claim: *for each $a \in \mathfrak{m}$ there exists $k \geq 0$ such that $a^k M \subset F_0$.*

Denote $M_j := \{m \in M : a^j m \in F_0\}$ for each $j \geq 0$ and consider the family of submodules $\{M_j\}_{j \geq 0}$ of M, which is a chain because $M_j \subset M_{j+1}$ for each $j \geq 0$. As M is a noetherian A-module, there exists $k \geq 0$ such that $M_k = M_{k+1}$. Let us check: $(a^k M + F_0) \cap E_M = \{0\}$, or equivalently, $a^k M + F_0 \in \mathfrak{F}$.

Let $x \in (a^k M + F_0) \cap E_M$. Then $x := a^k m + y \in E_M$ for some $m \in M$ and some $y \in F_0$. Thus, $a^{k+1} m + ay = ax \in \mathfrak{m} E_M = 0$, so $a^{k+1} m = -ay \in F_0$ and $m \in M_{k+1} = M_k$. Hence $a^k m \in F_0$, so

$$x = a^k m + y \in F_0 \cap E_M = \{0\} \quad \rightsquigarrow \quad a^k M + F_0 \in \mathfrak{F}.$$

Consequently, $F_0 \subset a^k M + F_0 \in \mathfrak{F}$. As F_0 is a maximal element in $\mathfrak{F}$, we have $F_0 = a^k M + F_0$, so $a^k M \subset F_0$.

B.1.c.2 Let $a_1, \ldots, a_p \in \mathfrak{m}$ be such that $\mathfrak{m} = \{a_1, \ldots, a_p\} A$. By B.1.c.1 there exists $k \geq 0$ such that $a_j^k M \subset F_0$ for $j = 1, \ldots, p$. Let us check: *$E_M \subset \mathfrak{m}^\ell M \subset F_0$ where $\ell := kp$.*

Pick $a \in \mathfrak{m}^\ell$. Write $a = \sum_{|\nu|=\ell} b_\nu a_1^{\nu_1} \cdots a_p^{\nu_p}$ where $\nu := (\nu_1, \ldots, \nu_p)$, $|\nu| := \nu_1 + \cdots + \nu_p$ and each $b_\nu \in A$. For each ν with $|\nu| = \ell = kp$, there exists $j = 1, \ldots, p$ such that $\nu_j \geq k$, so $b_\nu a_1^{\nu_1} \cdots a_p^{\nu_p} = b_\nu a_j^{\nu_j - k} a_j^k \prod_{i \neq j} a_i^{\nu_i}$. If $m \in M$,

$$am = \sum_{|\nu|=\ell} b_\nu a_1^{\nu_1} \cdots a_p^{\nu_p} m = \sum_{|\nu|=\ell} \Big(b_\nu a_j^{\nu_j - k} \prod_{i \neq j} a_i^{\nu_i} \Big) a_j^k m \in F_0$$

because $a_j^k m \in F_0$. Thus, $E_M \subset \mathfrak{m}^\ell M \subset F_0$.

B.1.c.3 As $F_0 \in \mathfrak{F}$, we conclude $E_M = E_M \cap F_0 = 0$, as required. □

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