Moral hazard, risk sharing, and the optimal pool size*

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Abstract

In risk pools, the effective share of the own loss borne is the sum of the direct share (the coinsurance rate) and the indirect share borne by the residual claimant. In a model with n identical risk-averse individuals and moral hazard, we derive two main results: First, for all mixed risk averse utility functions, the effective share required for incentive compatibility for the high effort increases in the pool size. This is a downside of larger pools as it, ceteris paribus, reduces risk sharing. Second, however, we find that the optimal pool size nevertheless converges to infinity as the positive effect from better diversification of the risk borne by other individuals always outweighs the negative effect from the larger risk of own damage. In our basic model, we restrict attention to two effort levels, but we show that our results extend to a model with continuous effort choice.

Keywords: risk sharing, moral hazard, mutuals, partnerships

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1 Introduction

Formal risk pools such as mutual insurance arrangements, as well as partnerships of lawyers, farmers, and physicians benefit from risk sharing, but are encumbered by free riding (moral hazard). In contrast to traditional insurance arrangements where risks are transferred to an insurance company and its stockholders, the members of risk pools are the residual claimants of the transferred risks. Our paper addresses the following question: in a world with independent risks, risk-averse participants, moral hazard and perfect enforceability of contracts, does the optimal size of the risk pool converge to infinity if the risk sharing arrangement is properly designed?

As a first intuition, the answer would be a straightforward 'yes'. With independent risks, risk-averse individuals clearly benefit from better risk sharing in a larger pool. However, with moral hazard and considering a broad class of utility functions displaying risk aversion, the answer is more involved. As in any moral hazard setting, incentive compatibility for effort provision requires that pool members bear part of their own loss themselves. In risk pools, a share of the own loss is borne directly as a coinsurance rate, and an additional share indirectly as residual claimant. We will refer to the sum of these two parts as the effective share. As the residual claimant share of the own loss is decreasing in the pool size, the coinsurance rate needs to increase in the pool size to keep effort incentives constant.

In our model, we consider general mixed risk averse utility functions, for which the derivatives are (weakly) alternating in sign (see, e.g., Caballé and Pomansky, 1996) and which include most of the commonly used von Neumann-Morgenstern utility functions (see Eeckhoudt and Schlesinger, 2006). Our first research question is how the minimum effective share required for incentive compatibility changes when the pool size increases. For the special case with linear marginal utility ($u''' = 0$) such as quadratic utility functions, we show that it suffices to increase the coinsurance rate to an extent that keeps the effective share constant. As will be detailed in our model, the reason is that higher-order risk preferences do not matter for those individuals. By contrast, for utility functions where the third derivative is strictly positive ("prudence", see Kimball, 1990) and where higher order derivatives weakly alternate in sign, the issue turns out to be more intricate. For those utility functions, we find that the effective share required for incentive compatibility increases in the pool size. Ceteris paribus, this is bad news from the perspective of risk sharing: the larger the pool size, the higher is the part of the own effective share that needs to be borne in order to satisfy the incentive compatibility constraint.

Given this countervailing effect, we address our second research question regarding the optimal pool size. We start again with the special case where higher order risk preferences do not matter, so that the effective share required for incentive compatibility is constant in the pool size. We can then easily show that the optimal pool size converges to infinity due to improved risk sharing. For utility functions where the third derivative is strictly positive and higher-order derivatives weakly alternate in sign, however, there are two countervailing effects. On the one hand, recall from above that risk sharing is reduced as each individual needs to bear a higher effective share of their own loss in order to ensure incentive compatibility. On the other hand, the risk borne as residual claimant shrinks due to the diversification effect and because the other members also have to bear a higher effective share of their own losses. Our main result is that these positive effects always outweigh the utility loss from the higher effective share of the own loss. Thus, the optimal pool size converges to infinity for *all* mixed risk averse utility functions with derivatives that weakly alternate in sign.

We first derive our results for the special case of two effort levels, and then show that they carry over to the case of continuous effort. For an intuition of our results, let us first discuss why the effective share of the own loss required for incentive compatibility needs to increase in the pool size when individuals are prudent ($u''' > 0$): When the pool size increases, the risk from the potential losses of other pool members decreases due to the diversification effect. As a consequence, extreme income levels become less likely. Due to the curvature of the utility function, prudent individuals are characterized by downside risk aversion (see Menezes et al., 1980), implying that their incentive to avoid own losses is, ceteris paribus, lower when low income levels become less likely. This incentive-reducing impact of larger pools needs to be countervailed by a higher effective share of the own loss.

Our main result that the benefit of a better diversification in larger pools always dominates follows the same logic: Consider again the risk each individual faces from the potential losses of other pool members. This risk is decreasing in the pool size (mean preserving contraction) and in the effective share. Thus, whenever an individual faces no own loss, her utility increases in the pool size and her own share. By contrast, her utility decreases in the own share if she faces a loss. We show that, in expectation, the first effect always dominates. With two effort levels only, the (binding) incentive compatibility constraint just requires that the utility difference without own loss and with own loss is constant in the pool size. It follows that, whenever the utility level in the case without own loss increases, the utility level with own loss needs to increase at the same degree. Therefore, the negative effect of a higher effective share in the own loss-case can never outweigh the positive diversification effect of a larger pool size.

Our result, however, is not limited to two effort levels but carries over to the case of continuous effort. The intuition follows from the fact that our finding does not hinge on the optimality of effort. In fact, whenever the same effort level is implemented for two pools of different size, then utility is higher for the larger pool. Thus, even if different pools have different optimal effort levels, as will typically be the case with continuous effort, the larger pool could (suboptimally) implement the effort level that is optimal for the smaller pool, and still have a higher utility. Then, by definition, utility of the larger pool would even be higher for the optimal effort level.

Summing up, we show that the downside of larger pools is that, for prudent individuals with mixed risk averse utility functions, the minimum effective share increases in pool size. But this increase arises because prudent individuals largely benefit from the reduction in the downside risk of larger pools, which reduces ceteris paribus their effort incentives. Thus, it is precisely the benefit from larger pools that calls for the higher effective share and this explains why this negative effect can never dominate.

In our paper, the optimality of larger risk pools arises because pool members are able to adjust the coinsurance rate optimally to larger pools, thereby taking the impact on effort incentives into account. If the coinsurance rate is kept constant, effort incentives decrease in the pool size, and the optimal pool size would be reached when the marginal benefits from better risk sharing are equal to the marginal costs from lower effort (see the simulations in Lee and Ligon, 2001). For second-best optimal coinsurance rates, however, we show that, in the absence of transaction costs, there is no interior solution.

In this paper, we focus on formal risk pooling organizations where losses are observable and risk transfers of each member can be specified in an explicit and perfectly enforceable contract.[1] Formal risk pools can take different forms. Partnerships are risk pools for a limited number of members, where the partners of a common profession (e.g. lawyers, farmers, physicians) cooperate to advance their mutual interests and share their business risks. In these cases, risk pooling takes the form of a joint business operation. Gaynor and Gertler (1995) as well as Lang and Gordon (1995) consider moral hazard problems in partnerships. In order to structure their empirical contributions on efficiency and risk sharing in a tractable way, they restrict attention to quadratic utility functions, and neither consider higher-order utility effects nor the optimal pool size. Examples for more specialized risk pools are Risk Retention Groups (RRG) and mutual insurance companies. RRG were formed in the U.S. during the liability insurance crises in the 1970s and 1980s, in order to share the liability risks of organizations and persons operating in similar businesses.[2] One important advantage of the RRG are lower regulation standards (Leverty, 2011). Mutual insurance companies are formal risk pools where the policyholders are both customers and owners of the insurance company. This organizational form for insurance providers can be observed world-wide. In the U.S., mutual insurers' relative importance in the life insurance market declined over time (Zanjani, 2007), but mutuals are major market players in property-casualty insurance.[3]

An important question that has been examined in the literature is why mutual insurance companies and stock insurers coexist in the same insurance markets. The

[1] In the absence of any formal insurance markets, risk pools are an important risk sharing arrangement in developing countries. Informal risk pools in developing countries face serious information problems, and the enforceability of ex ante agreements might be difficult or impossible (Bold, 2009). Many articles on informal partnerships confirm that moral hazard increases in the pool size and conclude that stable pools might hence be of limited size (see, e.g., Genicot and Ray, 2003 and Bramoullé and Kranton, 2007).

[2] In 1981, the U.S. Congress passed the Products Liability Risk Retention Act to allow a new type of insurance vehicle, the Risk Retention Groups, to cover product liability exposures. In 1986, the Act was expanded to allow the RRG to cover all casualty risks except workers compensation.

[3] According to the Federal Insurance Office, four mutual insurance groups (State Farm Mutual Automobile Insurance (1), Liberty Mutual Insurance (2), Nationwide Mutual Group (6) and USAA Insurance (10), were ranked under the top ten property-casualty insurance providers in the U.S. in 2014.

two main differences between both forms concern risk bearing and governance. Stock insurers generally offer contracts in which policyholders transfer risks for a fixed premium, whereas the companies' stockholders completely bear the transferred risks. By contrast, policies offered by mutual insurance companies usually have a participating nature, since policyholders are owners and thus residual claimants of the insurance pool. While this can be a serious downside of small mutuals, Smith and Stutzer (1995) show that mutuals can be superior to stock insurers in case of economywide aggregate risk and moral hazard problems. Mayers and Smith (2013) point out that mutuals may have limited access to capital markets, which may diminish the control of the management by owners. Laux and Mürmann (2010) identify a countervailing effect, since mutual insurance companies may have comparative advantages in raising external capital when stock insurers face free rider and commitment problems. Amongst others empirical studies, Cummins et al. (1999) provide evidence for the theoretical finding that mutual insurance companies are more successful in personal lines that require less managerial discretion with respect to individualized pricing and underwriting (Mayers and Smith, 1988). Thus, whether mutuals or stock insurers are preferable depends on the specific market and firm situation.

Ligon and Thistle (2008) show that mutual insurance arrangements are equivalent to a fairly priced stock insurance policy with the same coverage plus a zero mean background risk. As the background risk disappears when the pool size converges to infinity, insurance contracts of a mutual converge to contracts offered by a stock insurer[4] Thus, neglecting all other issues such as transaction costs and heterogeneity of policyholders, stock insurers and infinitely large mutuals are equivalent and both superior to smaller mutuals.

While our paper shows that the optimal pool size with moral hazard is infinite if pool members are able to adjust the coinsurance rate optimally, empirical evidence shows that mutuals are often relatively small compared to stock insurers. Comparing the size distributions of stock and mutual property-liability insurers, Ligon and Thistle (2005) show that the proportion of small mutuals is much greater than the proportion of small stock insurers. In their theoretical model, Ligon and Thistle (2005) consider

[4]We are grateful to an anonymous referee for pointing out this equivalence.

adverse selection as an explanation. In a separating equilibrium, mutuals attract low-risk consumers and offer higher expected indemnities, but are smaller in size than stock insurers in order to be unattractive for high-risk consumers. Another reason for limited pool sizes are transaction costs, which are neglected in our model. In particular, adjusting coinsurance rates optimally may be expensive when types are heterogenous (see e.g. Murgai et al., 2002). For the points we wish to make, transaction costs would not add much to the existing literature: Taking transaction costs into account, the optimal pool size would be reached when, after accounting for the required increase in the effective share, the marginal benefit of improved risk diversification is equal to marginal transaction costs.

The remainder of the paper is organized as follows: Section 2 introduces the model for two effort levels. Section 3 derives the individual effort choices, and section 4 the optimal effective shares. Section 5 analyzes the impact of the pool size on incentive compatibility. The optimal pool size is derived in section 6. Section 7 extends our analysis to the case of continuous effort and section 8 concludes.

2 The model for two effort levels

There are n identical risk-averse individuals with initial wealth W_0. Each individual i faces the risk of a loss $L < W_0$ and can exert unobservable effort $x_i \in \{0, 1\}$ at cost $C(x_i) = cx_i$ where $c > 0$. Choosing effort $x_i = 1$ reduces the loss probability from p_0 (associated with the effort $x_i = 0$) to p_1 where $0 < p_1 < p_0 < 1$.

Individual losses are assumed to be independent, which is a reasonable assumption for many risk pools such as risk sharing for accidents in mutual insurance arrangements, but also for liability for medical malpractice or sharing contracts in law firms. We consider a strictly increasing and strictly concave analytic utility function $u(W)$ where $(-1)^l u^l \leq 0$ for all $l \geq 3$. The set of functions satisfying these conditions (also called "mixed risk aversion", see Caballé and Pomansky, 1996) contains the usually used utility functions for risk aversion with $u' > 0$ and $u'' < 0$, including quadratic utility functions, logarithmic functions, and exponential utility functions. Effort costs are additively separable and W denotes the individual's final wealth.

We consider the following game: At stage 0, each individual decides upon whether to join the risk pool or not. If an individual does not join the pool, no other insurance is available for the type of risk considered in this paper. At stage 1, the n individuals who joined the pool agree cooperatively on a coinsurance rate $\alpha_n \in [0,1)$ which maximizes an objective function that aggregates the expected utilities of all pool members.[5] As all individuals and the coinsurance rates are identical, this also maximizes the utility of each pool member. We restrict attention to coinsurance rates that are independent of the number of losses (linear sharing rules). A coinsurance rate α_n in stage 1 is feasible if and only if it is in the core, i.e. if there is no coalition $\widetilde{n} < n$ that yields a higher payoff for all $\widetilde{n}$ members of the (sub)coalition.[6]

When setting α_n, the pool members take into account that, in stage 2, each pool member will choose the effort level that maximizes her individual utility; depending on n, on α_n and the expected effort choices of all other pool members. Thus, the effort is chosen non-cooperatively and, due to the unobservability of the effort choice, the pool members cannot sign a contract that is contingent on effort. It follows that, if the pool members aim at implementing a specific effort level vector, they need to take *incentive compatibility* into account.

In stage 3, verifiable losses occur, and transfers are made through the risk pool in stage 4. Figure 1 summarizes the timeline of the game which we solve by backward induction.

Suppose for the moment that low effort maximizes utility even for $n = 1$, i.e. for the no-insurance case. Then, low effort is a fortiori also optimal for larger pools, because the risk is maximum in the no-insurance case.[7] But then, there is no incentive problem

[5]We write $\alpha_n \in [0,1)$ in order to allow for the full-insurance case, but we exclude no-insurance ($\alpha_n = 1$) as this is identical to the case where an individual does not participate in the pool.

[6]Observe that, when the members of the pool agree on α_n, there is no private information as all pool members are identical, and because effort costs $C(x_i)$ and loss probabilities depending on the effort chosen are common knowledge. Furthermore, α_n depends only on losses and not (directly) of efforts chosen; i.e. private information does not matter for the enforcement of the sharing rule agreed upon. Therefore, we can apply the basic concept of a core of a coalitional game with transferable payoffs (see e.g. Osborne and Rubinstein, 1994, pp. 258

[7]Formally, this is implied by the proof of Lemma 1 below. Thus, for all pool sizes, incentives for high effort are maximum for $\alpha_n = 1$. But as this is identical to $n = 1$, there is no equilibrium with high effort for any n if low effort is optimal for $n = 1$ (note that there are no externalities for $\alpha = 1$, so that the individually rational behavior is also socially optimal).

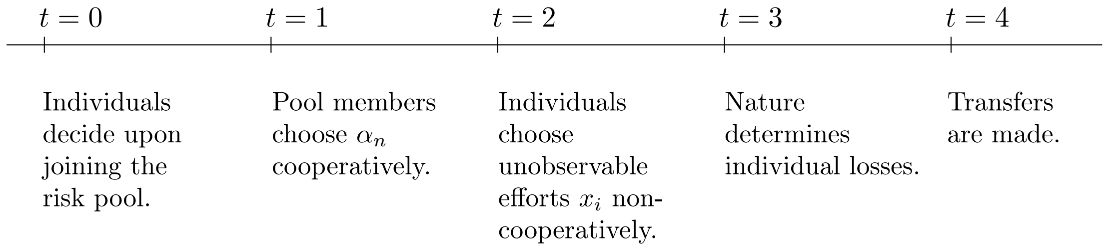

Figure 1: Timeline

at all, and irrespective of the size of the pool, members will optimally agree on $\alpha_n = 0$ to maximize risk sharing. Larger pools are then superior due to the pure insurance effect, and the whole problem we are interested in disappears. To exclude this trivial solution where low effort is always optimal, we introduce the following assumption:

Assumption 1. *High effort maximizes utility in the no-insurance case, i.e.*

$$(p_0 - p_1)\left[u\left(W_0\right) - u\left(W_0 - L\right)\right] > c.$$

Note that Assumption 1 does not exclude that low effort is optimal for larger pools, since losses can then be divided among all pool members. Thus, Assumption 1 is compatible with a setting where high effort is optimal for small pools, while high or low effort is optimal for large pools.

To streamline the analysis, we first assume that the members of the pool *always* want to implement high effort. We prove later that large pools are also superior when it maximizes the pool members' utility to implement low effort for large pools.

3 Stage 2: Individual effort choices

Following backward induction, we start with stage 2 on the pool members' effort choices. Each participant's incentive compatibility constraint for choosing high effort in a symmetric Nash Equilibrium is

$$\begin{aligned} \mathbb{E}\left[u\left(\alpha_n, x_i = 1, \mathbf{x}_{-i} = \mathbf{1}\right)\right] \quad &\geq \quad \mathbb{E}\left[u\left(\alpha_n, x_i = 0, \mathbf{x}_{-i} = \mathbf{1}\right)\right] && (1) \\ \text{and } \alpha_n \quad &\in \quad [0, 1)\,, && (2) \end{aligned}$$

where the effort level vector $\mathbf{x}_{-i} = \mathbf{1}$ means that all but i choose high effort.[8]

With α_n being the share of the own loss directly borne by each individual, the remaining part $1-\alpha_n$ is equally shared among all members of the pool. The *effective share* of the own loss ultimately borne by each individual is then the sum of the coinsurance rate and the share indirectly borne via the redistribution in the pool, $\beta_n = \alpha_n + \frac{1-\alpha_n}{n}$. In the following, we focus our exposition mostly on the effective share β_n, because one important insight of our analysis is that the effective share β_n required to implement high effort increases in the pool size if and only if $u''' > 0$ (recall that we restrict attention to mixed risk aversion, i.e. we assume that $(-1)^l u^l \leq 0$ for all $l \geq 3$).

Making use of β_n, we can write the incentive compatibility constraint (ICC) as

$$\begin{aligned} & p_1 \sum_{k=0}^{n-1} b(k; n-1, p_1) u \left(W_0 - \beta_n L - \left(\frac{1-\beta_n}{n-1} \right) kL \right) \\ & + (1-p_1) \sum_{k=0}^{n-1} b(k; n-1, p_1) u \left(W_0 - \left(\frac{1-\beta_n}{n-1} \right) kL \right) - c \\ \geq \; & p_0 \sum_{k=0}^{n-1} b(k; n-1, p_1) u \left(W_0 - \beta_n L - \left(\frac{1-\beta_n}{n-1} \right) kL \right) \\ & + (1-p_0) \sum_{k=0}^{n-1} b(k; n-1, p_1) u \left(W_0 - \left(\frac{1-\beta_n}{n-1} \right) kL \right). \end{aligned} \tag{3}$$

Lemma 1. *Suppose that high effort is a Nash Equilibrium for some $\widetilde{\beta}_n$. Then, any $\beta_n > \widetilde{\beta}_n$ also implements high effort as a Nash Equilibrium.*

Proof. See Appendix.

Lemma 1 expresses that the incentive to choose the high effort increases in the part of the own loss that is effectively borne via the coinsurance rate α_n and the redistribution in the pool. Thus, if high effort is incentive compatible for some $\widetilde{\beta}_n$, it will also be incentive compatible for any β_n above.

[8] Bold letters denote vectors.

4 Stage 1: Optimal choice of the effective share β_n

Suppose that n individuals have joined the pool, and assume that the pool members' utility is maximized when they implement high effort. Then, the pool's optimization problem in stage 1 boils down to maximizing expected

$$\max_{\beta_n} \mathbb{E}\left[u\left(\beta_n, \mathbf{x} = \mathbf{1}\right)\right] \tag{4}$$

subject to incentive compatibility

$$\mathbb{E}\left[u\left(\beta_n, x_i = 1, \mathbf{x}_{-i} = \mathbf{1}\right)\right] \geq \mathbb{E}\left[u\left(\beta_n, x_i = 0, \mathbf{x}_{-i} = \mathbf{1}\right)\right]. \tag{5}$$

From Lemma 1, we know that there may be several effective shares β_n that implement high effort for pool size n. Define $\beta_n^{\min}$ as the minimum effective share required for incentive compatibility. Lemma 2 expresses that the pool members will always maximize risk sharing, thereby taking incentive compatibility into account:

Lemma 2. *Suppose high effort is optimal for pool size n. Then, subject to incentive compatibility, the pool members' expected utility is maximized for $\beta_n^{\min}$.*

Proof. See Appendix.

Lemma 2 states that, when implementing high effort, the pool members agree upon the lowest possible coinsurance rate $\alpha_n^{\min}$, and hence also the lowest possible effective share $\beta_n^{\min} = \alpha_n^{\min} + \frac{1-\alpha_n^{\min}}{n}$ that only just ensures incentive compatibility.[9] This implies that the ICC is binding whenever $\alpha_n^{\min} > 0$. In the following, we will first restrict attention to this case. Given that the participation constraint and the ICC are fulfilled, existence of the symmetric Nash Equilibrium with high effort in stage 2 is ensured. For stage 1, we assume cooperative behavior, so that the utility maximizing coinsurance rate $\alpha_n^{\min}$ is chosen. The case in which all pool members choose to exert low effort is discussed in Corollary 1. The case where $\alpha_n^{\min} = 0$ is discussed in Corollary 2.

In the next section, we analyze the impact of n on $\beta_n^{\min}$, that is, we consider to what extent the effective share of the own loss that just ensures incentive compatibility depends on the pool size. Subsequently, we analyze the impact of the pool size on

[9]Note that the existence of a $\alpha_n^{\min} \in [0, 1)$ is ensured by Assumption 1.

expected utility.

5 The impact of the pool size on incentive compatibility

To illustrate the importance of the higher derivatives of the utility function, we start with the special case of a quadratic utility function $u = W - AW^2 - cx_i$, where $A < \frac{1}{2W_0}$ ensures that the marginal utility of wealth is positive in the relevant wealth range. Obviously, $u''' = 0$ in this example. Quadratic utility functions are convenient as the expected utility of the final wealth can be represented by the expected value and the variance of this wealth (Tobin, 1958; Baron, 1977):

$$\mathbb{E}[u] = \mathbb{E}[W] - A\left(\mathbb{E}[W]^2 + Var[W]\right) - cx_i. \tag{6}$$

Recall from Lemma 2 that, when the pool members want to implement high effort as a Nash Equilibrium, they will choose the effective share such that the incentive compatibility constraint (3) is binding. In our case, the expected final wealth is composed of the initial wealth, reduced by two possible sources for losses: the part borne of the expected own loss, $p\beta_n L$, and the expected loss of all other members in the risk pool that has to be borne by re-distribution, $p(1-\beta_n)L$. *Given* that high effort is chosen, the expected payoff of each pool member is

$$\mathbb{E}[W] = W_0 - p_1\beta_n L - p_1(1-\beta_n)L - c = W_0 - p_1 L - c, \tag{7}$$

and hence independent of β_n. Similarly, as the individual risks are independent, the variance of the final wealth is composed of the variances of the two possible sources for losses:

$$\begin{aligned}\mathrm{Var}[W] &= p_1(1-p_1)\beta_n^2 L^2 + p_1(1-p_1)(n-1)\left(\frac{1-\beta_n}{n-1}\right)^2 L^2 \\ &= p_1(1-p_1)L^2\left(\beta_n^2 + \frac{(1-\beta_n)^2}{n-1}\right).\end{aligned} \tag{8}$$

For a constant effective share β_n, we get $\frac{\partial \mathrm{Var}[\cdot]}{\partial n} = -\left(p_1 - p_1^2\right) L^2 \frac{(1-\beta_n)^2}{(n-1)^2} < 0$. This expresses the benefit of larger pools that the variance, and hence the risk borne by each individual, ceteris paribus decreases in the pool size.

The crucial question is then how the effective share β_n needs to be adjusted via the coinsurance rate α_n, when the pool size n increases. As part of the proof of Proposition 1, we will show that, for a quadratic utility function, the minimum effective share $\beta_n^{\min}$ required for incentive compatibility is independent of the pool size. Thus, the coinsurance rate $\alpha_n^{\min}$ required for incentive compatibility increases in the pool size, but just to the extent that the effective share β_n remains the same. The intuition is that, for quadratic utility functions, the marginal utility decreases at a constant rate when the income decreases, so that the changes in marginal utilities in response to changes in wealth differences are identical at all levels of wealth. As a consequence, the fact that the expected levels of wealth differ in the pool size has no impact for quadratic utility functions, and $\beta_n^{\min}$ is thus constant in n.

We now extend to the general case with $u''' \geq 0$. We obtain the following Proposition:

Proposition 1. *Assume a strictly increasing and strictly concave analytic utility function $u(W)$, where $u' > 0$, $u'' < 0$ and $(-1)^l u^l \leq 0$ for all $l \geq 3$. Suppose $\alpha_n^{\min}$ is strictly positive for all n. Then, (i) $\frac{\partial \beta_n^{\min}}{\partial n} = 0$ for $u''' = 0$, and (ii) $\frac{\partial \beta_n^{\min}}{\partial n} > 0$ for $u''' > 0$.*

Proof. See Appendix.

Proposition 1 says that the effective share of the own loss borne required for incentive compatibility increases in the pool size for utility functions $u''' > 0$. This implies that it is not straightforward that larger pools lead to higher utility as the increase in $\beta_n^{\min}$ leads to a countervailing negative impact of larger pools on risk sharing.

The intuition is as follows: As discussed in the introduction, individuals with $u''' > 0$ have a particular aversion to downside risk (Menezes et al., 1980). Due to the diversification effect from independent risks, the risk from the losses caused by other pool members decreases in n, and extremely low income levels become less likely. Therefore, the expected loss in utility when choosing the low effort instead of the high effort also decreases, because the additional loss occurs less often in situations where the income level is already low. In order to ensure incentive compatibility, $\beta_n^{\min}$ needs to increase in n, and this adjustment reduces the risk sharing benefit of larger pools. Of course, prudence neutral individuals (with $u''' = 0$) where higher-order risk preferences do not matter also have a general aversion to risk, but no special aversion to downside risk, so that $\beta_n^{\min}$ is constant in the pool size n.

6 The impact of pool size on expected utility

We have just seen that $\beta_n^{\min}$ increases in the pool size for $u''' > 0$, which reduces the benefits of larger pools. Nevertheless, we show that the benefits from risk sharing in larger pools always dominate:

Proposition 2. *Assume a strictly increasing and strictly concave analytic utility function $u(W)$ where $u' > 0$, $u'' < 0$ and $(-1)^l u^l \leq 0$ for all $l \geq 3$. Suppose high effort is optimal for all n and $\beta_n^{\min}$ is chosen for n. Then, the effort vector implemented by $\beta_n^{\min}$ is in the core, all individuals join the pool, and the utility of each pool member is strictly increasing in n.*

Proof. See Appendix.

For an intuition of the Proposition, let us distinguish two cases: First, let us stick to our assumption that high effort is optimal for all pool sizes. Afterwards, we consider the case where low effort is optimal for all pool sizes larger than some pool size $\widehat{n}$. This case emerges as a simple Corollary to Proposition 2.

When high effort is optimal for all pool sizes, the downside of larger pools is that, for prudent individuals with mixed risk aversion, $\beta_n^{\min}$ increases in n. This increase in the effective share required to implement high effort as a Nash Equilibrium is due to the fact that prudent individuals largely benefit from the reduction in the downside risk, which ceteris paribus reduces their effort incentive. Thus, it is precisely the benefit from larger pools itself that induces the negative side effect of an increase in the required effective share, which explains why the side effect can never dominate. Observe that the fact that the utility of all pool members increases in n implies that the allocation is in the core: The best that any coalition $\widetilde{n} < n$ could do is to agree on $\beta_{\widetilde{n}}^{\min}$ and it follows from Proposition 2 that this leads to lower utility.

To further sharpen the intuition for the superiority of larger pools, it is instructive to consider the situation from a more formal point of view. When the high effort level is implemented for two pool sizes $n+1$ and n, then the utility comparison for these two pool sizes can be written as follows:

$$p_1\mathbb{E}[u_L(\beta_{n+1}^{\min})] + (1-p_1)\mathbb{E}[u_0(\beta_{n+1}^{\min})] > p_1\mathbb{E}[u_L(\beta_n^{\min})] + (1-p_1)\mathbb{E}[u_0(\beta_n^{\min})], \tag{9}$$

where $\mathbb{E}[u_L(\beta_n^{\min})]$ denotes the expected utility in case of own loss in a pool of n participants, and $\mathbb{E}[u_0(\beta_n^{\min})]$ denotes the expected utility in case without own loss. For later reference, note that the utility comparison for the two pool sizes holds whenever the two effort levels implemented via the effective share are identical, i.e. in what follows, we do not make use of the fact that high effort is efficient.

Rearranging gives

$$\mathbb{E}[u_0(\beta_{n+1}^{\min})] - p_1\underbrace{\mathbb{E}[u_0(\beta_{n+1}^{\min}) - u_L(\beta_{n+1}^{\min})]}_{\Delta u(\beta_{n+1}^{\min})} > \mathbb{E}[u_0(\beta_n^{\min})] - p_1\underbrace{\mathbb{E}[u_0(\beta_n^{\min}) - u_L(\beta_n^{\min})]}_{\Delta u(\beta_n^{\min})}. \tag{10}$$

In this expression, the first part on either side of the utility comparison is the expected utility in the case without own loss, while the second parts consisting of $\Delta u(\beta_{n+1}^{\min})$ and $\Delta u(\beta_n^{\min})$ denotes the differences in expected utility without own loss and with own loss for risk pool sizes $n+1$ and n, respectively. The advantage of this representation is that the binding ICC can be written as $(p_0-p_1)\Delta u(\beta_{n+1}^{\min}) = c$ for pool size $n+1$ and $(p_0-p_1)\Delta u(\beta_n^{\min}) = c$ for pool size n. Thereby $(p_0-p_1)\Delta u(\beta_{n+1}^{\min})$ and $(p_0-p_1)\Delta u(\beta_n^{\min})$, capture the marginal benefits from high effort, which always equal the marginal costs c in case of binding ICCs. As a consequence, the marginal benefit is constant in n so that the utility comparison becomes:

$$\mathbb{E}[u_0(\beta_{n+1}^{\min})] > \mathbb{E}[u_0(\beta_n^{\min})], \tag{11}$$

meaning that it is only the difference in the expected utilities from the share of the losses from *other* pool members that matters for the utility comparison of the two pool sizes. This shows immediately that the utility in larger pools increases both for $u''' = 0$ *and* for $u''' > 0$: For $u''' = 0$, β_n remains constant, and it is only the better diversification of the risk from the losses of the other pool members that matters. For $u''' > 0$, in addition β_n increases in the pool size (recall Proposition 1), and this decreases the share of the losses borne from other pool members and hence also increases expected utility. Note that the optimality of larger pools implies that the allocation implemented via $\beta_n^{\min}$ is in the core - the maximum utility any coalition $\widetilde{n} < n$ can reach is by agreeing on $\beta_{\widetilde{n}}^{\min}$, and we know from Proposition 2 that this utility is lower than the one for the grand coalition of all n pool members.

So far, we have assumed that implementing the high effort is always optimal; irrespective of the pool size. The following Corollary expresses that our result on the optimality of larger pools is independent of this assumption:

Corollary 1. *Suppose that high effort is optimal for some pool sizes, but low effort for other pool sizes. Then, the utility of each pool member is still strictly increasing in n.*

Proof. See Appendix.

The intuition for Corollary 1 is simple. To see the point, recall that the whole argument for the superiority of larger pools that follows Proposition 2 is entirely independent of whether high effort is efficient or not - it extends to *all* cases where the high effort is implemented for two pools of different size. First, we show that the argument also holds when comparing two pools that both implement low effort. Second, and a little more intricate, Proposition 2 implies that, whenever the high effort is implemented for some pool size $n_1 < \widehat{n}$ and some pool size $n_2 \geq \widehat{n}$, then the utility of each pool member is higher for pool size n_2. Importantly, this is even the case if the low effort were optimal for the pool of size n_2. With low effort, the pool members would clearly agree on $\alpha_n = 0$, as this maximizes risk-sharing. Thus, when there are pool sizes where $\alpha_n = 0$ and low effort is optimal, the comparison of the utilities from two pool sizes is as follows:

$$\begin{aligned}\mathbb{E}\left[u\left(n = n_2, \alpha_{n_2} = 0, \mathbf{x} = \mathbf{0}\right)\right] &> \mathbb{E}\left[u\left(n = n_2, \alpha_{n_2}^{\min}, \mathbf{x} = \mathbf{1}\right)\right] \\ &> \mathbb{E}\left[u\left(n = n_1, \alpha_{n_1}^{\min}, \mathbf{x} = \mathbf{1}\right)\right].\end{aligned}$$

In other words, given that the larger pool is superior even when the pool members (suboptimally) implement the high effort, it follows by definition of optimality that the larger pool is also superior if the low effort is implemented. Finally, given that Proposition 2 carries over to the case of low effort, the same kind of argument applies if low effort is efficient for the smaller pool, and high effort for the larger pool.

So far, we have assumed that $\alpha_n^{\min} > 0$, so that the ICC is binding. The following Corollary covers the case of full insurance where $\alpha_n^{\min} = 0$:

Corollary 2. *Suppose that $\alpha_n^{\min} = 0$ (full-insurance) is optimal for some n. Then, the utility of each pool member is still strictly increasing in n.*

Proof. See Appendix.

The proofs of Propositions 1 and 2 rely on the assumption that effort incentives are kept constant when increasing the pool size. In particular, the ICCs are binding for pool sizes n and $n+1$. If the ICC is binding for $\alpha_n^{\min} = 0$, the optimality of increasing n follows directly from these Propositions. If the ICC is slack for $\alpha_n^{\min} = 0$, the pool members can still implement the same effort incentives when increasing the pool size, irrespective of whether this is optimal or suboptimal. The resulting expected utility will be higher for the larger pool according to Proposition 2. If this is suboptimal for the larger pool, expected utility would be even higher for optimal effort incentives.

7 Continuous effort

We now show that our main result of the superiority of larger pools carries over to the case with continuous effort. Assume that effort costs $c(x)$ are convex in effort x; $c'(x) > 0$ and $c''(x) > 0$. The loss probability $p(x) > 0$ is decreasing in effort at a decreasing rate; $p'(x) < 0$ and $p''(x) > 0$, where $p(0) < 1$. Furthermore, following e.g. Ligon and Thistle (2008), suppose that $\lim_{x \downarrow 0} p'(x) = -\infty$. These assumptions ensure that effort chosen by each individual is always positive.

Expected utility is given as

$$\begin{aligned}\mathbb{E}[u(\beta_n, x_i, \mathbf{x}_{-i})] = {} & p(x_i) \sum_{k=0}^{n-1} b(k; n-1, p(x_{-i})) \Big[u\Big(W_0 - \beta_n L - \Big(\frac{1-\beta_n}{n-1}\Big) kL \Big) \\ & + (1 - p(x_i)) \sum_{k=0}^{n-1} b(k; n-1, p(x_{-i})) \Big[u\Big(W_0 - \Big(\frac{1-\beta_n}{n-1}\Big) kL \Big) \Big] - c(x_i).\end{aligned} \tag{12}$$

Denote $x_i^*(\beta_n, \mathbf{x}_{-i})$ as the utility maximizing effort of individual i given the pool size

n, the effective share β_n and the vector of efforts exerted by all other members, denoted by $\mathbf{x}_{-i}$:

$$x_i^* := \underset{x_i}{\operatorname{argmax}}\ \mathbb{E}[u(\beta_n, x_i, \mathbf{x}_{-i})]. \tag{13}$$

We proceed in several steps which jointly prove our result. In the *first step*, we prove that the utility maximizing effort level of individual i is always positive and increasing in the effective share β_n:

Lemma 3. *(i) $x_i^* > 0$ $\forall n$, β_n, and $\mathbf{x}_{-i}$. (ii) $\frac{\partial x_i^*}{\partial \beta_n} > 0$ $\forall n$, β_n, and $\mathbf{x}_{-i}$.*

Proof. See Appendix.

Part (i) of the Lemma follows directly from our standard assumptions on effort costs and loss probabilities. *Part (ii)* expresses that the utility maximizing effort increases in the effective share β_n. Thus, the members of the pool can increase the incentive compatible efforts, i.e., the efforts chosen in the Nash Equilibrium, by agreeing on a higher coinsurance rate α. Note that Lemma 3 implies that the *highest* implementable effort is bounded above by the effort chosen in the no-insurance case ($\alpha = \beta = 1$). For further reference, denote $x_n^{\max}$ as the effort level in the Nash Equilibrium for $\beta_n \to 1$. In the following, we restrict attention to symmetric equilibria, and denote the incentive compatible equilibrium vector as $\mathbf{x}_n^* (\beta_n)$.

In the *second step*, we show that, for any pool size, there is an interior solution for the utility maximizing effective share:[10]

$$\beta_n^* := \underset{\beta_n}{\operatorname{argmax}}\ \mathbb{E}[u(\beta_n, \mathbf{x}_n^* (\beta_n))]. \tag{14}$$

Proposition 3. *For all pool sizes n, there exists an expected utility maximizing effective share $\beta_n^* \in (\frac{1}{n}, 1)$.*

Proof. See Appendix.

Proposition 3 expresses that, for any pool size, neither no-insurance ($\alpha = \beta = 1$) nor maximum insurance ($\beta = \frac{1}{n}$, i.e., $\alpha = 0$) can be optimal. Maximum insurance is dominated because of the incentive effect, and no-insurance due to the benefit of

[10]As this result is interesting in its own, we label it as a Proposition rather than as a Lemma.

risk-sharing.

The next Lemma says that the lowest implementable effort, which we denote by $x_n^{\min}$, decreases in the pool size:

Lemma 4. $\frac{\partial x_n^{\min}}{\partial n} < 0$.

Proof. See Lee and Ligon (2001), pp. 182-184. ■

Lemma 4 has already been proven by Lee and Ligon (2001),[11] so that we do not need to repeat the proof, which is technically challenging, but which has a straightforward intuition: For any pool size, the minimum implementable effort level $x_n^{\min}$ is reached by maximizing risk-sharing. As risk-sharing is maximized for $\alpha = 0$, the effective share $\beta_n = \alpha_n + \frac{1-\alpha_n}{n}$ boils down to $\beta_n = \frac{1}{n}$, and is thus decreasing in n. This is the effect pointed out by Lee and Ligon (2001) - the larger the pool, the lower is the impact of the redistribution on the effort choice.

In our *third step*, we prove that any effort level that is implementable for small pool sizes is also implementable for larger pools:

Lemma 5. *Let* $x_{n_1} = x_{n_1}^*(\beta_{n_1})$*, where* $\beta_{n_1} \in (\frac{1}{n_1}, 1)$*. Let* $n_2 > n_1$*. Then there exists* $\beta_{n_2} \in (\frac{1}{n_2}, 1)$ *such that* $x_{n_1} = x_{n_2}^*(\beta_{n_2})$*.*

Proof. See Appendix.

The fact that our main result carries over to the case with continuous effort follows from Lemma 5. For our discrete model, we have shown that, whenever the same effort level is implemented for pool size n_1 and any pool size $n_2 > n_1$ by adjusting the effective share β_n accordingly, the utility of each pool member is strictly higher for pool sine n_2. The proof was independent of whether the high effort is efficient or not. Lemma 5 states that, in the continuous case, any effort level that is implementable for pool size n_1 is also implementable for any pool size $n_2 > n_1$. Thus, our proof still applies.

Recall that in the discrete version, we have made use of the fact that $\beta_{n+1}^{\min}$ and $\beta_n^{\min}$ only just implement the high effort for pool sizes $n+1$ and n, respectively. As the ICCs are hence binding for those minimum effective shares, we could conclude that

[11] Lee and Ligon (2001) restrict their attention to $\alpha = 0$, but this is the case to be analyzed when considering the lowest implementable effort.

$\Delta u(\beta_{n+1}^{\min}) = \Delta u(\beta_n^{\min})$. While any $\beta_n > \beta_n^{\min}$ a fortiori implements high effort in the discrete version, any effort $x \in (x_n^{\min}, x_n^{\max})$ in the continuous version is implemented by a unique β_n. Thus, for any effort $x^* \in (x_n^{\min}, x_n^{\max})$, there exists a unique β_n that implements x^* for pool size n, and a unique β_{n+1} that implements x^* for pool size $n+1$, as given by the first order condition for both pool sizes. When comparing the utilities with different pool sizes, we can again make use of the fact that β_{n+1} and β_n implement a given effort x^* for pool sizes $n+1$ and n, respectively. From the ICCs, we know that $\Delta u(\beta_{n+1}) = -\frac{c'(x^*)}{p'(x^*)}$ and $\Delta u(\beta_n) = -\frac{c'(x^*)}{p'(x^*)}$. From the identity of the right hand sides, it then follows immediately that $\Delta u(\beta_{n+1}) = \Delta u(\beta_n)$, and the utility comparison again reduces to $\mathbb{E}[u_0(\beta_{n+1})] > \mathbb{E}[u_0(\beta_n)]$. We can hence proceed to our *fourth step*:

Proposition 4. *Suppose the effective share is chosen to maximize the pool members' utility; taking incentive compatibility into account. Then, with continuous effort, increasing the pool size increases the pool members' expected utility, i.e.* $\frac{\partial \mathbb{E}[u(\beta_n^*, x_n^*(\beta_n^*), n]}{\partial n} > 0$.

Proof. See Appendix.

Note that the optimal effort level will generally be different for different pool sizes - intuitively, avoiding risk becomes less important when a higher pool size allows for better risk sharing.[12] However, to prove our claim, it is sufficient to show that a larger pool size would lead to a higher utility *even when the members of the pool suboptimally* agreed on an effective share $\widehat{\beta}_n$ that implements the effort which is optimal for the lower pool size $\widetilde{n}$, but not for the pool size n considered. But if the larger pool size leads to higher utility even in this case, it a fortiori leads to higher utility when the utility-maximizing effective share β_n is chosen instead. Note that the utility maximizing effective share β_n^* the grand coalition of all members agree upon is always in the core, i.e., there is no coalition that has an incentive to block the grand coalition. The reason is that the best any coalition $\widetilde{n} < n$ can do is to agree on the utility maximizing $\beta_{\widetilde{n}}^*$, and we know from Proposition 4 that this leads to a lower utility.

[12] While this may lead to the result that low effort is optimal for large pools in the model with discrete effort choice, our assumption $\lim_{x \downarrow 0} p'(x) = -\infty$ ensures that positive effort is always optimal in the continuous case. See Lemma 3.

8 Conclusion

We extend the literature on risk pools, such as partnerships and mutual insurance arrangements, to the optimal pool size in case of moral hazard. We assume that n individuals with mixed risk averse utility functions, for which the derivatives are (weakly) alternating in sign, agree on the coinsurance rate that maximizes their utility, thereby taking the incentive compatibility constraint for the effort choice into account. Our main result is that, neglecting transaction costs, the optimal pool size converges to infinity. This holds both for binary and for continuous effort. In reality, transaction costs may increase in pool sizes, and these costs are neglected in our model. Thus, from a practical perspective, our result shows that the optimal pool size equilibrates the benefits from better risk sharing with transaction costs at the margin, whereas the residual claimant principle has no impact on the optimal pool size if the coinsurance rate is optimally adjusted.

Starting with binary effort choices, we first consider quadratic utility functions. For this special case, incentive compatibility for the choice of the high effort requires the same *effective* share $\beta_n^{\min}$ of the own loss for all pool sizes, where the effective share is defined as the sum of the coinsurance rate and the share of the own loss borne as residual claimant. The fact that $\beta_n^{\min}$ is constant in n implies that the risk from the losses of the other pool members has no impact on effort incentives. Then, the pool size only influences the degree of risk sharing, and each individual's expected utility is strictly increasing in the number of policyholders.

For prudent individuals with mixed risk averse utility functions, however, the incentive to choose high effort is lower for larger pools, even when the effective share β_n is the same. The reason is that individuals with these utility functions are avers to downside risk, and the probability of particularly low income levels decreases in the pool size. As a consequence, we find that the minimum effective share for incentive compatibility increases in n. While this is an interesting insight in itself, our main result is that the benefit from larger pools always dominates the negative impact of a higher effective share. Thus, given that coinsurance rates are optimally adjusted and contracts are enforceable, the optimal pool size is infinite, even in the case with moral hazard and prudent individuals.

For the continuous effort case, we can prove that an optimal coinsurance rate strictly between 0 and 1 always exists (i.e. neither full nor no-insurance can be optimal). In the Appendix, we provide a sufficient condition for the uniqueness of the optimal coinsurance. Observe, however, that we do not assume this Condition to hold as our proof that each participant's utility increases in the pool size is entirely independent of whether the optimal coinsurance rate is unique or not. To see this, recall first that we have shown that, for any coinsurance rate, the effort vector chosen by the participants is unique. Now suppose that there are multiple coinsurance rates which lead to different effort levels, but still to the same expected utility - more risk-sharing with lower effort may be equally good as less risk-sharing with higher effort. As each of this multiple optimal coinsurance rates leads to the same utility by definition of optimality, it does not matter for our argument which of these coinsurance rates the participants agree upon in the cooperative game at stage 1[13] - each of these coinsurance rates leads to a higher utility for larger compared to smaller pools. This follows from the proof that any effort vector that can be implemented for n can also be implemented for $\widetilde{n} > n$, and that this effort vector then leads to higher utility even when it is suboptimal for $\widetilde{n}$.

As we are interested in the impact of the pool size on utility when the coinsurance rate is optimally adjusted, we neglect other important issues such as heterogeneous individuals, transaction costs and externalities. If there are externalities of the pool members' activities on third parties which cannot be internalized, then it may well be socially optimal to restrict the maximum pool size. The same result arises when the externality depends on the pool members' precaution effort: When the effort the pool members coordinate upon via the coinsurance rate decreases in the pool size, then it may be socially optimal to restrict the pool size in order to increase the effort. Furthermore, it needs to be mentioned that we restrict attention to a monopolistic risk pool. The

[13] The proof of the superiority of larger pools implies that each of these coinsurance rates is in the core. Without loss of generality, we could e.g. assume that they agree upon the lowest coinsurance rate that maximizes utility.

optimality of an infinite pool size in our model highlights the benefits of a monopoly for risk pooling, but various frictions may allow for competition among pools. For instance, some risk pools may prefer to attract only low-risk consumers and/or certain profession members (partnerships and RRG). In addition, risk pools may have limited sales force resources or other capacity constraints preventing them from covering the entire market. Additionally, some consumer groups may be inert due to significant switching costs, and may hence be reluctant to leave their actual pool. While it is thus interesting to analyze competition between mutual insurers or different risk pools, this is beyond the scope of this paper and therefore left to future research.

Appendix

Proof of Lemma 1

Proof. First, define the function $f(\beta_n)$ as

$$f(\beta_n) := \sum_{k=0}^{n-1} b(k; n-1, p_1)\Big[u\Big(W_0 - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big) - u\Big(W_0 - \beta_n L - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\Big] - \frac{c}{p_0 - p_1}, \quad (15)$$

so that the (ICC) can be written as $f(\beta_n) \geq 0$. By definition of $\widetilde{\beta}_n$, $f(\widetilde{\beta}_n) \geq 0$ holds. The first derivative of f with respect to β_n is:

$$f'(\beta_n) := L \sum_{k=0}^{n-1} b(k; n-1, p_1)\Big[u'\Big(W_0 - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\frac{k}{n-1} + u'\Big(W_0 - \beta_n L - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\Big(1 - \frac{k}{n-1}\Big)\Big], \quad (16)$$

which is strictly positive as $u'(W) > 0$ for each W. Therefore, for any $\beta_n > \widetilde{\beta}_n$ we have $f(\beta_n) > f(\widetilde{\beta}_n) \geq 0$, so that the ICC holds. ■

Proof of Lemma 2

Proof. For the case where high effort is optimal, we need to prove that the minimum β_n, which ensures incentive compatibility, maximizes expected utility. Expected utility for any individual i with pool size n is:

$$\begin{aligned}\mathbb{E}[u(\beta_n, x_i)] = p \sum_{k=0}^{n-1} b(k; n-1, p) u\Big(W_0 - \beta_n L - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big) \\ + (1-p) \sum_{k=0}^{n-1} b(k; n-1, p) u\Big(W_0 - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big) - cx_i.\end{aligned} \quad (17)$$

We prove that $\frac{\partial \mathbb{E}[u(\beta_n, x_i)]}{\partial \beta_n} < 0$ in the relevant range. For the relevant range, recall that $\beta_n = \alpha_n + \frac{1-\alpha_n}{n}$, where $\alpha_n \in [0, 1)$. Thus, $\beta_n \in [\frac{1}{n}, 1)$. We show that $\frac{\partial \mathbb{E}[u(\beta_n, x_i)]}{\partial \beta_n} = 0$ for $\beta_n = \frac{1}{n}$, and $\frac{\partial \mathbb{E}[u(\beta_n, x_i)]}{\partial \beta_n} < 0 \; \forall \; \beta_n > \frac{1}{n}$. The first and second derivatives of the expected utility with respect to β_n are

$$
\begin{aligned}
\frac{\partial \mathbb{E}[u(\beta_n, x_i)]}{\partial \beta_n} &= L\Big(p\sum_{k=0}^{n-1} b(k;n-1,p)u'\Big(W_0-\beta_n L-\Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\Big(-1+\frac{k}{n-1}\Big)\\
&\quad + (1-p)\sum_{k=0}^{n-1} b(k;n-1,p)u'\Big(W_0-\Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\Big(\frac{k}{n-1}\Big)\Big)
\end{aligned} \tag{18}
$$

and

$$
\begin{aligned}
\frac{\partial^2 \mathbb{E}[u(\beta_n, x_i)]}{\partial \beta_n^2} &= L^2\Big(p\sum_{k=0}^{n-1} b(k;n-1,p)u''\Big(W_0-\beta_n L-\Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\Big(-1+\frac{k}{n-1}\Big)^2\\
&\quad + (1-p)\sum_{k=0}^{n-1} b(k;n-1,p)u''\Big(W_0-\Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\Big(\frac{k}{n-1}\Big)^2\Big),
\end{aligned} \tag{19}
$$

respectively. As $u_i'' < 0$, $\frac{\partial^2 \mathbb{E}[u(\beta_n, x_i)]}{\partial \beta_n^2} < 0$, i.e. $\frac{\partial \mathbb{E}[u(\beta_n, x_i)]}{\partial \beta_n}$ is strictly decreasing.

Next, we show that $\frac{\partial \mathbb{E}[u(\frac{1}{n}, x_i)]}{\partial \beta_n} = 0$. Substituting β_n for $\frac{1}{n}$ in equation (18), we get

$$
\begin{aligned}
\frac{\partial \mathbb{E}[u(\frac{1}{n}, x_i)]}{\partial \beta_n} &= L\Big(p\sum_{k=0}^{n-1} b(k;n-1,p)u'\Big(W_0-\frac{k+1}{n}L\Big)\Big(-1+\frac{k}{n-1}\Big)\\
&\quad + (1-p)\sum_{k=0}^{n-1} b(k;n-1,p)u'\Big(W_0-\frac{k}{n}L\Big)\Big(\frac{k}{n-1}\Big)\Big)\\
&= L\Big(-\sum_{k=0}^{n-1}\binom{n-1}{k}p^{k+1}(1-p)^{n-1-k}u'\Big(W_0-\frac{k+1}{n}L\Big)\Big(\frac{n-1-k}{n-1}\Big)\\
&\quad + \sum_{k=0}^{n-1}\binom{n-1}{k}p^{k}(1-p)^{n-k}u'\Big(W_0-\frac{k}{n}L\Big)\Big(\frac{k}{n-1}\Big)\Big).
\end{aligned} \tag{20}
$$

Observe that $\frac{n-1-k}{n-1} = 0$ for $k = n-1$ and $\frac{k}{n-1} = 0$ for $k = 0$. Thus,

$$
\begin{aligned}
\frac{\partial \mathbb{E}[u(\frac{1}{n}, x_i)]}{\partial \beta_n} &= L\Big(- \sum_{k=0}^{n-2} \frac{(n-1)!}{(n-1-k)!k!} p^{k+1}(1-p)^{n-1-k} u'\Big(W_0 - \frac{k+1}{n}L\Big)\Big(\frac{n-1-k}{n-1}\Big) \\
&+ \sum_{k=1}^{n-1} \frac{(n-1)!}{(n-1-k)!k!} p^{k}(1-p)^{n-k} u'\Big(W_0 - \frac{k}{n}L\Big)\Big(\frac{k}{n-1}\Big)\Big) \\
&= \frac{L}{n-1}\Big(- \sum_{k=0}^{n-2} \frac{(n-1)!}{(n-1-k-1)!k!} p^{k+1}(1-p)^{n-1-k} u'\Big(W_0 - \frac{k+1}{n}L\Big) \\
&+ \sum_{k=1}^{n-1} \frac{(n-1)!}{(n-1-k)!(k-1)!} p^{k}(1-p)^{n-k} u'\Big(W_0 - \frac{k}{n}L\Big)\Big) \\
&\underset{\text{in the first addend}}{\overset{(k=\ell-1)}{=}} \frac{L}{n-1}\Big(- \sum_{\ell=1}^{n-1} \frac{(n-1)!}{(n-1-\ell)!(\ell-1)!} p^{\ell}(1-p)^{n-\ell} u'\Big(W_0 - \frac{\ell}{n}L\Big) \\
&+ \sum_{k=1}^{n-1} \frac{(n-1)!}{(n-1-k)!(k-1)!} p^{k}(1-p)^{n-k} u'\Big(W_0 - \frac{k}{n}L\Big)\Big) = 0,
\end{aligned}
\tag{21}
$$

where, in the last step, we substitute k for $\ell - 1$ in the first addend in order to demonstrate equality between the two addends. As $\frac{\partial \mathbb{E}[u(\beta_n, x_i)]}{\partial \beta_n}$ is strictly decreasing, it follows that $\frac{\partial \mathbb{E}[u(\beta_n, x_i)]}{\partial \beta_n} < 0 \ \forall \beta_n > \frac{1}{n}$. As the expected utility is decreasing in β_n for a given effort, pool members will agree on $\beta_n^{\min}$. ■

Proof of Proposition 1

Proof. We prove that $\frac{\partial \beta_n^{\min}}{\partial n} = 0$ if $u''' = 0$, while $\frac{\partial \beta_n^{\min}}{\partial n} > 0$ if $u''' > 0$. For ease of notation, we here use β_n instead of $\beta_n^{\min}$. We proceed as follows.

Step 1. We prove Lemma 6 that is required for the proofs of Propositions 1 and 2, as well as Corollary 1.

Step 2. We rewrite the ICC under the assumption that u is an analytic function.

Step 3. We compare the ICCs for pool sizes n and $n+1$ for $u''' = 0$ and prove part (i) of Proposition 1.

Step 4. We compare the ICCs for pool sizes n and $n+1$ for $u''' > 0$ and prove part (ii) of Proposition 1.

Step 1. Proof of Lemma 6

The following Lemma states a property of binomial distributions that will be used in later proofs.

Lemma 6. *Define* $\mathbb{E}[B(n,p)^\ell]$ *as the* ℓ*-th moment (about* 0*) of the binomial distribution* $B(n,p)$*. Then:*

$$\left(\frac{n-1}{n}\right)^\ell \leq \frac{\mathbb{E}[B(n-1,p)^\ell]}{\mathbb{E}[B(n,p)^\ell]} \tag{22}$$

for all $\ell \geq 1$*, where the condition holds with equality for* $\ell = 1$ *and the RHS is strictly larger than the LHS for* $\ell > 1$*.*

Proof. As $\mathbb{E}[B(n,p)^\ell]$ is the ℓ-th moment (about 0) of $B(n,p)$, we can write:

$$\begin{aligned} n^\ell \mathbb{E}[B(n-1,p)^\ell] &= \sum_{j=0}^{n-1} \binom{n-1}{j} n^\ell j^\ell p^j q^{n-1-j} = \sum_{j=1}^{n-1} \binom{n-1}{j} n^\ell j^\ell p^j q^{n-1-j} \\ (n-1)^\ell \mathbb{E}[B(n,p)^\ell] &= \sum_{k=0}^{n} \binom{n}{k} (n-1)^\ell k^\ell p^k q^{n-k} = \sum_{k=1}^{n} \binom{n}{k} (n-1)^\ell k^\ell p^k q^{n-k}, \end{aligned}$$

where $q := 1-p$. Thus, condition (22) is equivalent to

$$\begin{aligned} \Gamma_\ell(n) &:= n^\ell \mathbb{E}[B(n-1,p)^\ell] - (n-1)^\ell \mathbb{E}[B(n,p)^\ell] \\ &= \sum_{j=1}^{n-1} \binom{n-1}{j} n^\ell j^\ell p^j q^{n-1-j} - \sum_{k=1}^{n} \binom{n}{k} (n-1)^\ell k^\ell p^k q^{n-k} \geq 0. \end{aligned} \tag{23}$$

For simplicity, we denote $a_j := \binom{n-1}{j} n^\ell j^\ell$ and $b_k := \binom{n}{k}(n-1)^\ell k^\ell$, so we have

$$\Gamma_\ell(n) = \sum_{j=1}^{n-1} a_j p^j q^{n-1-j} - \sum_{k=1}^{n} b_k p^k q^{n-k}.$$

As $p + q = 1$,

$$a_j p^j q^{n-1-j} = a_j (p+q) p^j q^{n-1-j} = a_j p^{j+1} q^{n-1-j} + a_j p^j q^{n-j} = a_j p^{j+1} q^{n-(j+1)} + a_j p^j q^{n-j}$$

and we deduce

$$\begin{aligned} \Gamma_\ell(n) = \sum_{j=1}^{n-1} a_j p^j q^{n-1-j} - \sum_{k=1}^{n} b_k p^k q^{n-k} &= \sum_{j=1}^{n-1} (a_j p^{j+1} q^{n-(j+1)} + a_j p^j q^{n-j}) - \sum_{k=1}^{n} b_k p^k q^{n-k} \\ &= (a_1 - b_1) p q^{n-1} + \sum_{k=2}^{n-1} (a_{k-1} + a_k - b_k) p^k q^{n-k} + (a_{n-1} - b_n) p^n. \end{aligned}$$

To prove that $\Gamma_\ell(n) \geq 0$ it is enough to show that each coefficient

$$a_1 - b_1 \geq 0, \quad a_{n-1} - b_n \geq 0, \quad a_{k-1} + a_k - b_k \geq 0 \quad \text{for } k = 2, \ldots, n-1,$$

while some coefficient is strictly larger than 0. First, we approach the simpler coefficients:

$$\begin{aligned} a_1 - b_1 &= \binom{n-1}{1} n^\ell - \binom{n}{1}(n-1)^\ell \\ &= (n-1)n^\ell - n(n-1)^\ell = (n-1)n(n^{\ell-1} - (n-1)^{\ell-1}) \geq 0, \\ a_{n-1} - b_n &= \binom{n-1}{n-1} n^\ell (n-1)^\ell - \binom{n}{n}(n-1)^\ell n^\ell = n^\ell (n-1)^\ell - (n-1)^\ell n^\ell = 0, \end{aligned}$$

where $\ell \geq 1$. The difference between a_1 and b_1 is equal to zero for $\ell = 1$ and strictly positive for $\ell > 1$. Thus, it only remains to determine what happens with $a_{k-1} + a_k - b_k$ for $k = 2, \ldots, n-1$. We expand them in a clearer way

$$a_{k-1} + a_k - b_k = \binom{n-1}{k-1} n^\ell (k-1)^\ell + \binom{n-1}{k} n^\ell k^\ell - \binom{n}{k}(n-1)^\ell k^\ell.$$

As $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, we deduce that

$$\begin{aligned} a_{k-1} + a_k - b_k =& \binom{n-1}{k-1}(n^\ell (k-1)^\ell - (n-1)^\ell k^\ell) + \binom{n-1}{k}(n^\ell k^\ell - (n-1)^\ell k^\ell) \\ =& \frac{(n-1)!}{(k-1)!(n-1-(k-1))!}(n^\ell (k-1)^\ell - (n-1)^\ell k^\ell) \\ &+ \frac{(n-1)!}{k!(n-1-k)!}(n^\ell k^\ell - (n-1)^\ell k^\ell) \\ =& \frac{(n-1)!}{k!(n-k)!}\big(k(n^\ell (k-1)^\ell - (n-1)^\ell k^\ell) + (n-k)(n^\ell k^\ell - (n-1)^\ell k^\ell)\big). \end{aligned}$$

Consequently, to prove that $a_{k-1} + a_k - b_k \geq 0$ we must show that

$$k(n^\ell (k-1)^\ell - (n-1)^\ell k^\ell) + (n-k)(n^\ell k^\ell - (n-1)^\ell k^\ell) \geq 0. \tag{24}$$

Or equivalently that

$$n^\ell k(k-1)^\ell + n^{\ell+1} k^\ell + (n-1)^\ell k^{\ell+1} - (n-1)^\ell k^{\ell+1} - n(n-1)^\ell k^\ell - n^\ell k^{\ell+1} \geq 0.$$

The previous inequality can be simplified as

$$k^{\ell-1}(n^\ell - (n-1)^\ell) - n^{\ell-1}(k^\ell - (k-1)^\ell) \geq 0. \tag{25}$$

For $\ell = 1$ this is again, equal to zero. Thus, we are reduced to prove equation (25) for $k = 2, \ldots, n-1$. We will successively reduce equation (25) until we achieve a family of inequalities easy to be checked. We begin by using the well-known identity

$$x^\ell - y^\ell = (x-y)\sum_{j=0}^{\ell-1} x^j y^{\ell-1-j}$$

to write

$$n^\ell - (n-1)^\ell = \sum_{j=0}^{\ell-1} n^j (n-1)^{\ell-1-j} \quad \text{and} \quad k^\ell - (k-1)^\ell = \sum_{j=0}^{\ell-1} k^j (k-1)^{\ell-1-j}.$$

Thus, equation (25) reduces to prove the following

$$\begin{aligned} k^{\ell-1}\Big(\sum_{j=0}^{\ell-1} n^j (n-1)^{\ell-1-j}\Big) &- n^{\ell-1}\Big(\sum_{j=0}^{\ell-1} k^j (k-1)^{\ell-1-j}\Big) \\ &= \sum_{j=0}^{\ell-1}(k^{\ell-1} n^j (n-1)^{\ell-1-j} - n^{\ell-1} k^j (k-1)^{\ell-1-j}) \geq 0. \end{aligned} \tag{26}$$

It is sufficient to show that each addend of the previous sum is ≥ 0. Namely,

$$k^{\ell-1} n^j (n-1)^{\ell-1-j} - n^{\ell-1} k^j (k-1)^{\ell-1-j} \geq 0 \quad \forall\, j = 0, \ldots, \ell-1. \tag{27}$$

Let us rewrite these addends in a clearer way

$$\begin{aligned} &(k^{\ell-1} n^j (n-1)^{\ell-1-j} - n^{\ell-1} k^j (k-1)^{\ell-1-j}) \\ &\qquad = k^j n^j (k^{\ell-1-j} (n-1)^{\ell-1-j} - n^{\ell-1-j} (k-1)^{\ell-1-j}) \\ &\qquad = k^j n^j ((k(n-1))^{\ell-1-j} - (n(k-1))^{\ell-1-j}). \end{aligned} \tag{28}$$

As $0 \leq j \leq \ell - 1$, to prove that inequalities (27) hold, it is enough to check that

$$\begin{aligned} &(k(n-1))^{\ell-1-j} \geq (n(k-1))^{\ell-1-j} \quad \forall\, k = 2, \ldots, n-1 \\ &\qquad \iff k(n-1) \geq n(k-1) \quad \forall\, k = 2, \ldots, n-1. \end{aligned} \tag{29}$$

But his is trivially true because

$$k(n-1) - n(k-1) = kn - k - nk + n = n - k \geq 0 \quad \forall\, k = 2, \ldots, n-1.$$

Going backwards and putting all the reductions together we conclude that equation

(25) holds and consequently also equation (23) (or equivalently (22)) holds, as wanted.

Step 2. Rewriting the ICC under the analytic assumption

We assume that u is an analytic function (that is, it coincides with its Taylor expansion as a series centered at W_0):

$$u(W) = \sum_{\ell=0}^{\infty} \frac{u^{(\ell)}(W_0)}{\ell!}(W - W_0)^{\ell}. \tag{30}$$

Observe that

$$u(W_0 - \beta L) = \sum_{\ell=0}^{\infty} \frac{u^{(\ell)}(W_0)}{\ell!}(W_0 - \beta L - W_0)^{\ell} = \sum_{\ell=0}^{\infty} \frac{u^{(\ell)}(W_0)(-1)^{\ell} L^{\ell}}{\ell!}\beta^{\ell}. \tag{31}$$

We use (31) to obtain an alternative expression for the ICC (3) in terms of the Taylor expansion of u. For simplicity we denote temporarily $z_{k,n} := (\frac{1-\beta_n}{n-1})k$ and $w_{k,n} := \beta_n + (\frac{1-\beta_n}{n-1})k$ and we rewrite the ICC as follows

$$\frac{c}{p_0 - p_1} = \sum_{k=0}^{n-1} b(k; n-1, p_1)\Big[u\Big(W_0 - z_{k,n}L\Big) - u\Big(W_0 - w_{k,n}L\Big)\Big] \tag{32}$$

and we obtain using (31)

$$\frac{c}{p_0 - p_1} = \sum_{\ell=0}^{\infty} \frac{u^{(\ell)}(W_0)(-1)^{\ell} L^{\ell}}{\ell!}\Big[\sum_{k=0}^{n-1} b(k; n-1, p_1)(z_{k,n}^{\ell} - w_{k,n}^{\ell})\Big]. \tag{33}$$

Observe that $z_{k,n}^0 - w_{k,n}^0 = 0$ and for $\ell \geq 1$

$$z_{k,n}^{\ell} - w_{k,n}^{\ell} = \Big(\frac{1-\beta_n}{n-1}\Big)^{\ell} k^{\ell} - \Big(\beta_n + \Big(\frac{1-\beta_n}{n-1}\Big)k\Big)^{\ell} = -\sum_{j=0}^{\ell-1} \binom{\ell}{j} \beta_n^{\ell-j} \Big(\frac{1-\beta_n}{n-1}\Big)^{j} k^{j}.$$

Therefore, for $\ell \geq 1$

$$\sum_{k=0}^{n-1} b(k; n-1, p_1)(z_{k,n}^{\ell} - w_{k,n}^{\ell}) = -\sum_{j=0}^{\ell-1} \binom{\ell}{j} \beta_n^{\ell-j} \Big(\frac{1-\beta_n}{n-1}\Big)^{j} \mathbb{E}[B(n-1, p_1)^{j}],$$

where $B(n-1, p_1)$ is the involved binomial distribution. We rewrite the ICC as:

$$\frac{c}{p_0 - p_1} = \sum_{\ell=1}^{\infty} \frac{u^{(\ell)}(W_0)(-1)^{\ell+1} L^{\ell}}{\ell!}\Big[\sum_{j=0}^{\ell-1} \binom{\ell}{j} \beta_n^{\ell-j} (1-\beta_n)^{j} \frac{\mathbb{E}[B(n-1, p_1)^{j}]}{(n-1)^{j}}\Big], \tag{34}$$

which will be our reference in what follows in this section. By the ICC (34), it holds that β_n is a positive root of the equation

$$\sum_{\ell=1}^{\infty}\frac{u^{(\ell)}(W_0)(-1)^{\ell+1}L^{\ell}}{\ell!}\Big[\sum_{j=0}^{\ell-1}\binom{\ell}{j}\beta_n^{\ell-j}(1-\beta_n)^j\frac{\mathbb{E}[B(n-1,p_1)^j]}{(n-1)^j}\Big]-\frac{c}{p_0-p_1}=0. \tag{35}$$

Step 3. Comparing the ICCs (35) for β_n and β_{n+1} for $u'''=0$ to prove part (i) of Proposition 1

We compare the ICCs for pool sizes n and $n+1$ for $u'''=0$. As the ICC (35) is binding for β_n in case of a pool size of n, and also binding for β_{n+1} in case of a pool size of $n+1$, we deduce that

$$\begin{aligned}\sum_{\ell=1}^{\infty}\frac{u^{(\ell)}(W_0)(-1)^{\ell+1}L^{\ell}}{\ell!}\Big[\sum_{j=0}^{\ell-1}\binom{\ell}{j}\Big(\beta_n^{\ell-j}(1-\beta_n)^j\frac{\mathbb{E}[B(n-1,p_1)^j]}{(n-1)^j}&\\ -\beta_{n+1}^{\ell-j}(1-\beta_{n+1})^j\frac{\mathbb{E}[B(n,p_1)^j]}{n^j}\Big)\Big]&=0.\end{aligned} \tag{36}$$

Consider a utility function, where the derivatives $u^{(\ell)}$ are equal to zero for $\ell>2$ (such as a quadratic utility function), then for $\ell>2$ the addends of the sum are zero. For $\ell=1$ and $\ell=2$, $\frac{\mathbb{E}[B(n-1,p_1)^{\ell-1}]}{(n-1)^{\ell-1}}=\frac{\mathbb{E}[B(n,p_1)^{\ell-1}]}{n^{\ell-1}}$ according to the proof of Lemma 6 (Step 1), and it is easy to see, that the LHS is equal to zero, if $\beta_n=\beta_{n+1}$. This proves part (i) of Proposition 1.

Step 4. Comparing the ICCs (35) for β_n and β_{n+1} for $u'''>0$ to prove part (ii) of Proposition 1

In the following, we prove the desired inequality for Proposition 1:

$$\beta_n<\beta_{n+1}, \tag{37}$$

for a function u with higher order derivative $u^{\ell}\neq 0$, for some $\ell>2$. Assume by contradiction that $\beta_n\geq\beta_{n+1}$. As $u'(W_0)>0$, we know that the first addend $u'(W_0)L(\beta_n-\beta_{n+1})$ of the sum (36) is weakly positive, as well as its second addend $\frac{-u''(W_0)L^2}{2}(\beta_n^2-\beta_{n+1}^2+2\beta_n(1-\beta_n)p_1-2\beta_{n+1}(1-\beta_{n+1})p_1)$. As $\frac{u^{(\ell)}(W_0)(-1)^{\ell+1}L^{\ell}}{\ell!}\geq 0$ for all $\ell \geq 1$ and the inequality is strictly positive for some $\ell > 2$, there exists some $\ell > 2$ such that

$$\sum_{j=0}^{\ell-1} \binom{\ell}{j} \Big(\beta_n^{\ell-j} (1-\beta_n)^j \frac{\mathbb{E}[B(n-1,p_1)^j]}{(n-1)^j} - \beta_{n+1}^{\ell-j} (1-\beta_{n+1})^j \frac{\mathbb{E}[B(n,p_1)^j]}{n^j} \Big) < 0 \tag{38}$$

because otherwise equality (36) does not hold. Rewrite (38) as

$$\sum_{j=0}^{\ell-1} \binom{\ell}{j} \Big(\beta_n^{\ell-j} (1-\beta_n)^j \frac{\mathbb{E}[B(n-1,p_1)^j]}{(n-1)^j} \Big) < \sum_{j=0}^{\ell-1} \binom{\ell}{j} \Big(\beta_{n+1}^{\ell-j} (1-\beta_{n+1})^j \frac{\mathbb{E}[B(n,p_1)^j]}{n^j} \Big).$$

By Lemma 6 (Step 1) we know that

$$\frac{\mathbb{E}[B(n,p_1)^j]}{n^j} < \frac{\mathbb{E}[B(n-1,p_1)^j]}{(n-1)^j}$$

for all $n \geq 1$ and all $j > 1$. For $\ell > 2$ we deduce

$$\begin{aligned}
\sum_{j=0}^{\ell-1} \binom{\ell}{j} \Big(\beta_n^{\ell-j} (1-\beta_n)^j \frac{\mathbb{E}[B(n,p_1)^j]}{n^j} \Big) &< \sum_{j=0}^{\ell-1} \binom{\ell}{j} \Big(\beta_n^{\ell-j} (1-\beta_n)^j \frac{\mathbb{E}[B(n-1,p_1)^j]}{(n-1)^j} \Big) \\
&\leq \sum_{j=0}^{\ell-1} \binom{\ell}{j} \Big(\beta_{n+1}^{\ell-j} (1-\beta_{n+1})^j \frac{\mathbb{E}[B(n,p_1)^j]}{n^j} \Big).
\end{aligned} \tag{39}$$

As $\beta_n \geq \beta_{n+1}$, we have $1-\beta_n \leq 1-\beta_{n+1}$, so

$$(1-\beta_n)^\ell \frac{\mathbb{E}[B(n,p_1)^\ell]}{n^\ell} \leq (1-\beta_{n+1})^\ell \frac{\mathbb{E}[B(n,p_1)^\ell]}{n^\ell}. \tag{40}$$

By (39) and (40) it holds

$$\mathbb{E}\Big[\sum_{j=0}^{\ell} \binom{\ell}{j} \Big(\beta_n^{\ell-j} (1-\beta_n)^j \frac{B(n,p_1)^j}{n^j} \Big) \Big] < \mathbb{E}\Big[\sum_{j=0}^{\ell} \binom{\ell}{j} \Big(\beta_{n+1}^{\ell-j} (1-\beta_{n+1})^j \frac{B(n,p_1)^j}{n^j} \Big) \Big].$$

Thus, we conclude

$$\mathbb{E}\Big[\Big(\beta_n + \frac{(1-\beta_n)B(n,p_1)}{n} \Big)^\ell \Big] < \mathbb{E}\Big[\Big(\beta_{n+1} + \frac{(1-\beta_{n+1})B(n,p_1)}{n} \Big)^\ell \Big]. \tag{41}$$

We rewrite the previous inequality as

$$\sum_{k=0}^{n} b(k;n,p_1) \Big(\beta_n + \frac{(1-\beta_n)k}{n} \Big)^\ell < \sum_{k=0}^{n} b(k;n,p_1) \Big(\beta_{n+1} + \frac{(1-\beta_{n+1})k}{n} \Big)^\ell. \tag{42}$$

For $k = n$ it holds

$$b(n; n, p_1)\Big(\beta_n + \frac{(1-\beta_n)n}{n}\Big)^\ell = p_1^n = b(n; n, p_1)\Big(\beta_{n+1} + \frac{(1-\beta_{n+1})n}{n}\Big)^\ell,$$

so (42) is equivalent to

$$\sum_{k=0}^{n-1} b(k; n, p_1)\Big(\beta_n + \frac{(1-\beta_n)k}{n}\Big)^\ell < \sum_{k=0}^{n-1} b(k; n, p_1)\Big(\beta_{n+1} + \frac{(1-\beta_{n+1})k}{n}\Big)^\ell. \tag{43}$$

Necessarily, there exists at least one $k = 0, \ldots, n-1$ such that

$$\Big(\beta_n + \frac{(1-\beta_n)k}{n}\Big)^\ell < \Big(\beta_{n+1} + \frac{(1-\beta_{n+1})k}{n}\Big)^\ell \quad \rightsquigarrow \quad \beta_n + \frac{(1-\beta_n)k}{n} < \beta_{n+1} + \frac{(1-\beta_{n+1})k}{n}.$$

because otherwise strict inequality (43) does not hold. Thus,

$$(\beta_n - \beta_{n+1}) + \frac{(\beta_{n+1} - \beta_n)k}{n} = (\beta_n - \beta_{n+1})\Big(1 - \frac{k}{n}\Big) < 0.$$

As $k = 0, \ldots, n-1$, we conclude that $\beta_n < \beta_{n+1}$, which contradicts our assumption that $\beta_n \geq \beta_{n+1}$.

Thus, it holds that $\beta_n < \beta_{n+1}$ for all $n \geq 1$ and functions u with higher order derivatives $u^\ell \neq 0$, for $\ell > 2$. Together with Step 3 above that shows that $\beta_n = \beta_{n+1}$ for $u''' = 0$, this proves Proposition 1. ■

Proof of Proposition 2

Proof. We prove that each pool member's utility increases in the pool size, when high effort is always implemented. First, we demonstrate that the expected utility for pool size $n+1$ is higher than for pool size n. Then, we prove that the expected utility profile for pool size n is in the core.

Expected utility increasing in n

According to Lemma 2, $\beta_n^{\min}$ will be chosen for pool size n, while $\beta_{n+1}^{\min}$ will be chosen for pool size $n+1$. For ease of notation, we again use β_n instead of $\beta_n^{\min}$. As ICC is binding for both pool sizes, it suffices to prove that the expected utility without own loss is increasing in n, following equation (10):

$$\sum_{k=0}^{n} b(k;n,p_1)u\left(W_0-\Big(\frac{1-\beta_{n+1}}{n}\Big)kL\right) > \sum_{k=0}^{n-1} b(k;n-1,p_1)u\left(W_0-\Big(\frac{1-\beta_n}{n-1}\Big)kL\right) \tag{44}$$

We first rewrite the utility comparison under the analytic assumption. Using the Taylor expansion of u, we know:

$$\begin{aligned}
&\sum_{k=0}^{n-1} b(k;n-1,p_1)u\Big(W_0-\Big(\frac{1-\beta_n}{n-1}\Big)kL\Big) \\
&=\sum_{k=0}^{n-1} b(k;n-1,p_1)\sum_{\ell=0}^{\infty}\frac{u^{(\ell)}(W_0)(-1)^{\ell}L^{\ell}}{\ell!}\Big(\frac{1-\beta_n}{n-1}\Big)^{\ell}k^{\ell} \\
&=\sum_{\ell=0}^{\infty}\frac{u^{(\ell)}(W_0)(-1)^{\ell}L^{\ell}}{\ell!}\Big(\frac{1-\beta_n}{n-1}\Big)^{\ell}\sum_{k=0}^{n-1} b(k;n-1,p_1)k^{\ell} \\
&=\sum_{\ell=0}^{\infty}\frac{u^{(\ell)}(W_0)(-1)^{\ell}L^{\ell}}{\ell!}(1-\beta_n)^{\ell}\frac{\mathbb{E}[B(n-1,p_1)^{\ell}]}{(n-1)^{\ell}},
\end{aligned} \tag{45}$$

where $B(n-1,p_1)$ is the involved binomial distribution. Thus, the utility comparison is equivalent to the following condition (which will be our reference in what follows in this section):

$$\sum_{\ell=1}^{\infty}\frac{u^{(\ell)}(W_0)(-1)^{\ell}L^{\ell}}{\ell!}\Big((1-\beta_{n+1})^{\ell}\frac{\mathbb{E}[B(n,p_1)^{\ell}]}{n^{\ell}}-(1-\beta_n)^{\ell}\frac{\mathbb{E}[B(n-1,p_1)^{\ell}]}{(n-1)^{\ell}}\Big)>0. \tag{46}$$

A sufficient condition under which inequality (46) holds for all $n\geq 1$ is that each addend in the infinite sum (46) is ≥ 0 and at least one addend is strictly larger than 0, that is,

$$\frac{u^{(\ell)}(W_0)(-1)^{\ell}L^{\ell}}{\ell!}\Big((1-\beta_{n+1})^{\ell}\frac{\mathbb{E}[B(n,p_1)^{\ell}]}{n^{\ell}}-(1-\beta_n)^{\ell}\frac{\mathbb{E}[B(n-1,p_1)^{\ell}]}{(n-1)^{\ell}}\Big)\geq 0 \tag{47}$$

for all $\ell\geq 1$, where the inequality is strict for some $\ell\geq 1$.

According to the assumption of mixed-risk aversion, $\frac{u^{(\ell)}(W_0)(-1)^{\ell}L^{\ell}}{\ell!}\leq 0$ for all $\ell\geq 1$, and inequalities (47) are equivalent to

$$(1-\beta_{n+1})^{\ell}\frac{\mathbb{E}[B(n,p_1)^{\ell}]}{n^{\ell}}\leq(1-\beta_n)^{\ell}\frac{\mathbb{E}[B(n-1,p_1)^{\ell}]}{(n-1)^{\ell}}\quad\forall\ell\geq 1. \tag{48}$$

Clearly, the previous inequalities are equivalent to the following one

$$\frac{\mathbb{E}[B(n-1,p_1)^\ell]}{\mathbb{E}[B(n,p_1)^\ell]} \geq \Big(\frac{n-1}{n}\Big)^\ell \Big(\frac{1-\beta_{n+1}}{1-\beta_n}\Big)^\ell \quad \forall \ell \geq 1. \tag{49}$$

Observe that if $\ell = 1$, we deduce the necessary condition $\beta_n \leq \beta_{n+1}$, so $(\frac{1-\beta_{n+1}}{1-\beta_n})^\ell \leq 1$ for $\ell \geq 1$. Then, condition (49) is satisfied as $\left(\frac{n-1}{n}\right)^\ell \leq \frac{\mathbb{E}[B(n-1,p_1)^\ell]}{\mathbb{E}[B(n,p_1)^\ell]}$ for all $\ell \geq 1$ (where the inequality is strict for $\ell > 1$) according to Lemma 6 (see Step 1 of the proof of Proposition 1).

Expected utility profile in the core

It remains to be shown that the expected utility profile $\mathbb{E}[u(\beta_n^{\min}, \mathbf{x}_n)]$ is in the core. This requires that there is no coalition $\widetilde{n} < n$ such that $\mathbb{E}[u(\beta_{\widetilde{n}}, \mathbf{x}_{\widetilde{n}})] > \mathbb{E}[u(\beta_n^{\min}, \mathbf{x}_n)]$. Suppose $\mathbb{E}[u(\beta_{\widetilde{n}}, \mathbf{x}_{\widetilde{n}})] > \mathbb{E}[u(\beta_n^{\min}, \mathbf{x}_n)]$ exists. From Lemma 2, we know that $\mathbb{E}[u(\beta_{\widetilde{n}}, \mathbf{x}_{\widetilde{n}})]$ is maximized for $\beta_{\widetilde{n}} = \beta_{\widetilde{n}}^{\min}$. However, we have just proved that $\mathbb{E}[u(\beta_n^{\min}, \mathbf{x}_n)] > \mathbb{E}[u(\beta_{\widetilde{n}}^{\min}, \mathbf{x}_{\widetilde{n}})]$ $\forall \widetilde{n} < n$. Thus, a contradiction, which proves that the expected utility profile $\mathbb{E}[u(\beta_n^{\min}, \mathbf{x}_n)]$ is in the core. Observe that $\mathbb{E}[u(\beta_{\widetilde{n}}, x_{\widetilde{n}})] < \mathbb{E}[u(\beta_n^{\min}, x_n)]$ is sufficient to ensure that all individuals join the pool. ■

Proof of Corollary 1

Proof. We have to consider three cases. First, consider the case where high effort is optimal for some pool size $n_1 < \widehat{n}$ and low effort is optimal for some pool size $n_2 \geq \widehat{n}$. As argued in the main text, Propostion 2, that considers the case of high effort, directly applies because members of the pool with size n_2 could decide to suboptimally implement the high effort. According to Propostion 2 they would still be better off compared to the members of the pool with size n_1. But then, by definition, their utility would be even higher when they implement the optimal low effort.

Second, we have to consider the case where both pools want to implement the low effort, that is not covered by Proposition 2. We prove that a larger pool size also increases utility when low effort is optimal. Suppose low effort is optimal for pool size n. In this case, $\mathbb{E}[u(\alpha_n = 0, x_i = 0, \mathbf{x}_{-i} = \mathbf{0})] > \mathbb{E}[u(\alpha, x_i = 1, \mathbf{x}_{-i} = \mathbf{1})]$ for all $\alpha \in [0, 1]$.

Recall from Lemma 2 that, for given effort, expected utility is strictly decreasing in β. Thus, if low effort is implemented, then $\alpha_n = 0$ is optimal.

In case of $\alpha_n = 0$, for every loss an individual's wealth shrinks by $\frac{L}{n}$, irrespectively of whether the loss was incurred by individual i or any other member of the risk pool. Therefore, a pool member's expected utility, given low effort, can be written as:

$$\begin{aligned}\mathbb{E}\left[u\left(\alpha_n = 0, x_i = 0, \mathbf{x}_{-i} = \mathbf{0}\right)\right] =& p_0 \sum_{k=0}^{n-1} b(k; n-1, p_0) u\left(W_0 - \frac{1+k}{n} L\right) \\ &+ (1-p_0) \sum_{k=0}^{n-1} b(k; n-1, p_0) u\left(W_0 - \frac{k}{n} L\right) \\ =& \sum_{k=0}^{n} b(k; n, p_0) u\left(W_0 - \frac{k}{n} L\right).\end{aligned} \tag{50}$$

Rewriting the utility comparison for pool sizes $n+1$ and n under the analytic assumption gives

$$\sum_{\ell=1}^{\infty} \frac{u^{(\ell)}(W_0)(-1)^{\ell} L^{\ell}}{\ell!} \left(\left(\frac{1}{n+1}\right)^{\ell} \mathbb{E}[B(n+1, p_0)^{\ell}] - \left(\frac{1}{n}\right)^{\ell} \mathbb{E}[B(n, p_0)^{\ell}]\right) > 0. \tag{51}$$

As $\frac{u^{(\ell)}(W_0)(-1)^{\ell} L^{\ell}}{\ell!}$ is (weakly) negative, the infinite sum is positive, if

$$\left(\frac{1}{n+1}\right)^{\ell} \mathbb{E}[B(n+1, p_0)^{\ell}] \leq \left(\frac{1}{n}\right)^{\ell} \mathbb{E}[B(n, p_0)^{\ell}] \quad \forall \ell \geq 1, \tag{52}$$

while the inequality is strict for some $\ell \geq 1$. Note that this condition can be rearranged to:

$$\left(\frac{n}{n+1}\right)^{\ell} \leq \frac{\mathbb{E}[B(n, p_0)^{\ell}]}{\mathbb{E}[B(n+1, p_0)^{\ell}]}, \tag{53}$$

which is true according to Lemma 6 (see Step 1 of the proof of Proposition 1).

Third, consider the case where low effort is optimal for some pool size $n_1 < \widehat{n}$ and high effort is optimal for some pool size $n_2 \geq \widehat{n}$. Given that we have just shown that Proposition 2 carries over to the case of low effort, the same argument as in the first case applies: Members of the larger pool could suboptimally implement the low effort and would still be better off compared to the members of the smaller pool. But then, their utility would be even higher when they implement the optimal high effort.

Analogously to the proof of Proposition 2, the expected utility profile is in the core for all three cases considered here. ■

Proof of Corollary 2

Proof. So far, we have assumed that there exists a $\beta_n^{\min} \in (\frac{1}{n}, 1)$ such that the ICC is binding. Note that $\beta_n^{\min} < 1$ is ensured by Assumption 1. We will now consider the case of $\alpha_n^{\min} = 0$, which is equivalent to $\beta_n^{\min} = \frac{1}{n}$, and prove that each pool member's expected utility is still increasing in n. Recall that our previous proofs do not rely on the fact that $\beta_n^{\min} > \frac{1}{n}$, but it is assumed that the effort incentives are kept constant in n. Therefore, we will show that if the ICC is binding for $\beta_n^{\min} = \frac{1}{n}$, the optimality of increasing the pool size follows directly from Propositions 1 and 2.

The case to be analyzed here is the one where $\beta_n^{\min} = \frac{1}{n}$ and the ICC is slack:

$$\mathbb{E}\left[u\left(\beta_n = \frac{1}{n}, x_i = 1, \mathbf{x}_{-i} = \mathbf{1}\right)\right] > \mathbb{E}\left[u\left(\beta_n = \frac{1}{n}, x_i = 0, \mathbf{x}_{-i} = \mathbf{1}\right)\right], \tag{54}$$

which can be rearranged to:

$$f\left(\beta_n = \frac{1}{n}\right) = \sum_{k=0}^{n-1} b(k; n-1, p_1)\left[u\left(W_0 - \frac{k}{n}L\right) - u\left(W_0 - \frac{1+k}{n}L\right)\right] - \frac{c}{p_0 - p_1} > 0. \tag{55}$$

Now, we have to consider two cases when increasing the pool size from n to $n+1$, as the effort incentives may change. If $\beta_{n+1}^{\min} = \frac{1}{n+1}$, the high effort is efficient for both pool sizes and the optimality of the larger pool follows from the proof of Corollary 1, where we just have to substitute p_1 for p_0. If, on the other hand, the ICC is violated for $\beta_{n+1} = \frac{1}{n+1}$, we have to compare the expected utilities from pool size n and $\beta_n^{\min}$, and from pool size $n+1$ and $\beta_{n+1}^{\min} > \frac{1}{n+1}$.

From the proof of Lemma 1 we know that $f(\beta_n = 1) > f(\beta_n = \frac{1}{n}) > 0$. Furthermore, $\beta_n = 1$ is equal to no-insurance and $f(\beta = 1)$ is independent of n. Hence, $f(\beta_{n+1} = 1) > f(\beta_n = \frac{1}{n}) > 0 = f(\beta_{n+1} = \beta_{n+1}^{\min})$. Therefore, there exists a $\widehat{\beta}_{n+1}$ such that $f(\beta_{n+1} = \widehat{\beta}_{n+1}) = f(\beta_n = \frac{1}{n})$, where $\frac{1}{n+1} < \widehat{\beta}_{n+1} < 1$. In other words, when increasing the pool size to $n+1$, $\widehat{\beta}_{n+1}$ keeps the effort incentives constant. It follows from the proof of Proposition 1 that $\widehat{\beta}_{n+1} \geq \beta_n^{\min} = \frac{1}{n}$.

From the proof of Proposition 2 we know that $\mathbb{E}[u(\beta_{n+1} = \widehat{\beta}_{n+1}, x_i = 1, \mathbf{x}_{-i} = \mathbf{1})] >$

$\mathbb{E}[u(\beta_n = \frac{1}{n}, x_i = 1, \mathbf{x}_{-i} = \mathbf{1})]$. Finally, it follows from Lemma 2 that $\mathbb{E}[u(\beta_{n+1} = \beta_{n+1}^{\min}, x_i = 1, \mathbf{x}_{-i} = \mathbf{1})] > \mathbb{E}[u(\beta_{n+1} = \widehat{\beta}_{n+1}, x_i = 1, \mathbf{x}_{-i} = \mathbf{1})]$. Therefore, we have:

$$\mathbb{E}[u(\beta_{n+1} = \beta_{n+1}^{\min}, x_i = 1, \mathbf{x}_{-i} = \mathbf{1})] > \mathbb{E}[u(\beta_n = \frac{1}{n}, x_i = 1, \mathbf{x}_{-i} = \mathbf{1})] \tag{56}$$

and the expected utility is increasing in n. In accordance with the proof of Proposition 2, the expected utility profile is in the core. ■

Proof of Lemma 3

Part (i)

Proof. We show that, in the continuous version, our standard assumptions ensure that individuals will always exert a strictly positive effort. Given $\mathbf{x}_{-i}$ and n members of the risk pool, the expected utility of individual i is:

$$\begin{aligned}\mathbb{E}[u(\beta_n, x_i, \mathbf{x}_{-i})] = p(x_i) \sum_{k=0}^{n-1} b(k; n-1, p(x_{-i})) \Big[u\Big(W_0 - \beta_n L - \Big(\frac{1-\beta_n}{n-1}\Big) kL\Big) \\ + (1 - p(x_i)) \sum_{k=0}^{n-1} b(k; n-1, p(x_{-i})) \Big[u\Big(W_0 - \Big(\frac{1-\beta_n}{n-1}\Big) kL\Big)\Big] - c(x_i)\end{aligned}$$

Define $V(x_i) := \mathbb{E}[u(\beta_n, x_i, \mathbf{x}_{-i})]$ as the expected utility as a function of x_i. The first order condition for choosing the utility maximizing effort level by individual i is $V'(x_i) = 0$, i.e.

$$\sum_{k=0}^{n-1} b(k; n-1, p(x_{-i})) \Big[u\Big(W_0 - \Big(\frac{1-\beta_n}{n-1}\Big) kL\Big) - u\Big(W_0 - \beta_n L - \Big(\frac{1-\beta_n}{n-1}\Big) kL\Big)\Big] + \frac{c'(x_i)}{p'(x_i)} = 0. \tag{57}$$

As $\lim_{x \downarrow 0} \frac{c'(x_i)}{p'(x_i)} = 0$ and $\beta_n > 0$, we have $V'(x_i = 0) > 0$. Next, note that x_i appears on $V'(x_i)$ only in the part $\frac{c'(x_i)}{p'(x_i)}$, so that $V''(x_i) = \frac{c''(x_i)p'(x_i) - c'(x_i)p''(x_i)}{p'(x_i)^2} < 0$, where the sign follows from $c''(x_i) > 0$, $p'(x_i) < 0$, $c'(x_i) \geq 0$ and $p''(x_i) < 0$. Thus, there is an interior solution for x_i^* where $x_i^* > 0$. ■

Part (ii)

Proof. We prove that, for a given pool size, effort is strictly increasing in β_n. Write $F := V'(x_i)$ to get

$$F(x_i,\beta_n) = \sum_{k=0}^{n-1} b(k;n-1,p(x_{-i}))\Big[u\Big(W_0 - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big) - u\Big(W_0 - \beta_n L - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\Big] + \frac{c'(x_i)}{p'(x_i)}, \quad (58)$$

where the incentive compatible effort x_i^* is implicitly defined by $F(x_i^*,\beta_n) = 0$. From the implicit function theorem, $\frac{dx_i^*}{d\beta_n} = -\frac{\frac{\partial F}{\partial \beta_n}}{\frac{\partial F}{\partial x_i^*}}$, where

$$\begin{aligned}\frac{\partial F}{\partial \beta_n} =& L\sum_{k=0}^{n-1} b(k;n-1,p(x_{-i}))\Big[u'\Big(W_0 - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\frac{k}{n-1} \\ &+ u'\Big(W_0 - \beta_n L - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\Big(1-\frac{k}{n-1}\Big)\Big]\end{aligned} \quad (59)$$

and

$$\frac{\partial F}{\partial x_i^*} = \frac{c''(x_i^*)p'(x_i^*) - c'(x_i^*)p''(x_i^*)}{p'(x_i^*)^2}. \quad (60)$$

We have: $\frac{\partial F}{\partial \beta_n} > 0$ because $u'(W) > 0$ for each W and $\frac{k}{n-1} \in [0,1]$. In addition, $\frac{\partial F}{\partial x_i^*} < 0$ because $c'(x_i) \geq 0$, $p''(x_i) > 0$, $c''(x_i) > 0$ and $p'(x_i) < 0$ for each x_i. Thus, $\frac{dx_i^*}{d\beta_n} > 0$. ■

Proof of Proposition 3

Proof. A sufficient condition that the expected utility achieves a maximum at $\beta_n^* \in (\frac{1}{n}, 1)$ is that the total derivative of the expected utility with respect to β_n is strictly negative at $\beta_n = 1$, i.e. $\frac{d\mathbb{E}[u(\beta_n, x_n^*(\beta_n))]}{d\beta_n}|_{\beta_n=1} < 0$, and strictly positive at $\beta_n = \frac{1}{n}$, i.e. $\frac{d\mathbb{E}[u(\beta_n, x_n^*(\beta_n))]}{d\beta_n}|_{\beta_n=\frac{1}{n}} > 0$. For ease of notation we continue to use x instead of $x_n^*(\beta_n)$. The total derivative of the expected utility with respect to β_n can be written as:

$$\frac{d\mathbb{E}[u(\beta_n,x)]}{d\beta_n} = \frac{\partial\mathbb{E}[u(\beta_n,x)]}{\partial\beta_n} + \frac{\partial\mathbb{E}[u(\beta_n,x)]}{\partial x^*}\frac{dx}{d\beta_n} \quad (61)$$

First, we will demonstrate that $\frac{d\mathbb{E}[u(\beta_n,x)]}{d\beta_n} < 0$ at $\beta_n = 1$. Then, we will prove that $\frac{d\mathbb{E}[u(\beta_n,x)]}{d\beta_n} > 0$ at $\beta_n = \frac{1}{n}$.

Analysis of the first derivative at $\beta_n = 1$

From Lemma 3 part (ii), we know that $\frac{dx}{d\beta_n} > 0$. Furthermore,

$$\begin{aligned} \frac{\partial \mathbb{E}[u(\beta_n, x)]}{\partial \beta_n}|_{\beta_n=1} =& L\Big[p(x)\sum_{k=0}^{n-1} b(k; n-1, p(x))u'(W_0 - L)(\frac{k}{n-1} - 1)] \\ &+ (1-p(x))\sum_{k=0}^{n-1} b(k; n-1, p(x))u'(W_0)\frac{k}{n-1}\Big] < 0 \end{aligned} \tag{62}$$

(cf. proof of Lemma 2). When considering the effect of adjusting β_n, the pool members will anticipate that in equilibrium all individuals will choose the same effort. Therefore, as opposed to the derivation of the ICC (from the perspective of a single individual) we need to consider all efforts when calculating the derivative of the expected utility with respect to the effort (chosen by all members of the risk pool in equilibrium). However, at $\beta_n = 1$ the effect of the other $n-1$ members of the pool does not matter, as all individuals cover their potential losses themselves. Therefore, according to the ICC:

$$\frac{\partial \mathbb{E}[u(\beta_n, x)]}{\partial x}|_{\beta_n=1} = -p'(x)[u(W_0) - u(W_0 - L)] - c'(x) = 0 \tag{63}$$

Hence, $\frac{d\mathbb{E}[u(\beta_n,x)]}{d\beta_n}|_{\beta_n=1} < 0$ and there exists an expected utility maximizing $\beta_n^* < 1$.

Analysis of the first derivative at $\beta_n = \frac{1}{n}$

From the proof of Lemma 2 we know that $\frac{\partial \mathbb{E}[u(\beta_n,x)]}{\partial \beta_n}|_{\beta_n=\frac{1}{n}} = 0$. Therefore, as $\frac{dx}{d\beta_n} > 0$, the sign of the total derivative of the expected utility with respect to β_n at $\beta_n = \frac{1}{n}$ is determined by the partial derivative with respect to x:

$$\begin{aligned} \frac{\partial \mathbb{E}[u(\beta_n, x)]}{\partial x} =& p'(x)\Big[(1-p(x))\sum_{k=0}^{n-1} b'(k; n-1, p(x))u\Big(W_0 - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big) \\ &+ p(x)\sum_{k=0}^{n-1} b'(k; n-1, p(x))u\Big(W_0 - \beta_n L - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\Big] + h(\beta_n, x), \end{aligned} \tag{64}$$

where from the ICC we know that x is implicitly defined by:

$$h(\beta_n, x) := p'(x) \sum_{k=0}^{n-1} b(k; n-1, p(x)) \Big[u\Big(W_0 - \beta_n L - \Big(\frac{1-\beta_n}{n-1}\Big) kL \Big) \\ - u\Big(W_0 - \Big(\frac{1-\beta_n}{n-1}\Big) kL \Big) \Big] - c'(x) = 0. \quad (65)$$

Therefore, at $\beta_n = \frac{1}{n}$, equation (64) reduces to:

$$\frac{\partial \mathbb{E}[u(\beta_n, x)]}{\partial x}|_{\beta_n = \frac{1}{n}} = p'(x) \sum_{k=0}^{n} b'(k; n, p(x)) u\Big(W_0 - \frac{kL}{n} \Big) \quad (66)$$

As $p'(x) < 0$, we have to show that

$$\sum_{k=0}^{n} b'(k; n, p(x)) u\Big(W_0 - \frac{k}{n} L \Big) < 0, \quad (67)$$

where

$$b'(k; n, p) = \binom{n}{k} p^{k-1} (1-p)^{n-k-1} (k - np). \quad (68)$$

Observe that $\mu = np$ is the mean of the binomial distribution $X = B(n, p)$. So we can rewrite inequality (67) as

$$\frac{1}{p(1-p)} \sum_{k=0}^{n} b(k; n, p)(k - \mu) u\Big(W_0 - \frac{k}{n} L \Big) = \frac{1}{p(1-p)} \mathbb{E}\Big[(X - \mu) u\Big(W_0 - \frac{X}{n} L \Big) \Big] < 0.$$

As u is an analytic function,

$$\mathbb{E}\Big[(X - \mu) u\Big(W_0 - \frac{X}{n} L \Big) \Big] = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell u^{(\ell)}(W_0) L^\ell}{\ell! n^\ell} \mathbb{E}[(X - \mu) X^\ell].$$

For $\ell = 0$, we have $\mathbb{E}[(X - \mu)] = 0$, so we have to prove

$$\sum_{\ell=1}^{\infty} \frac{(-1)^\ell u^{(\ell)}(W_0) L^\ell}{\ell! n^\ell} \mathbb{E}[(X - \mu) X^\ell] < 0$$

As $(-1)^\ell u^{(\ell)}(W_0) < 0$ for each $\ell \geq 1$, it is enough to show that $\mathbb{E}[(X - \mu) X^\ell] > 0$ for each $\ell \geq 1$. According to Knoblauch (Thm. 2.2), the ℓ-th raw moment of X can be written as

$$\mathbb{E}[X^\ell] = \sum_{i=0}^{\ell} b_{\ell,i} p^i n^{\underline{i}} \tag{69}$$

where $n^{\underline{i}} := \binom{n}{i} i! = n(n-1)\cdots(n-i+1)$ and $b_{\ell,i}$ is defined recursively as follows:

$$\begin{aligned} b_{0,i} &:= \delta_{i0}, \\ b_{\ell,i} &:= i b_{\ell-1,i} + b_{\ell-1,i-1}, \end{aligned} \tag{70}$$

where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ otherwise, as well as $b_{\ell,i} = 0$ for $\ell < 0, i < 0$ or $i > \ell$. In particular $b_{\ell,\ell} = 1$ for $\ell \geq 0$. Observe that $n \cdot n^{\underline{i}} = n^{\underline{i+1}} + i n^{\underline{i}}$. Consequently, for $\ell \geq 1$

$$\begin{aligned} \mathbb{E}[(X-\mu)X^\ell] = \mathbb{E}[X^{\ell+1}] - np\mathbb{E}[X^\ell] &= \sum_{j=0}^{\ell+1} b_{\ell+1,j} p^j n^{\underline{j}} - np \sum_{i=0}^{\ell} b_{\ell,i} p^i n^{\underline{i}} \\ &= \sum_{j=0}^{\ell+1} (j b_{\ell,j} + b_{\ell,j-1}) p^j n^{\underline{j}} - \sum_{i=0}^{\ell} b_{\ell,i} p^{i+1} n \cdot n^{\underline{i}} \\ &= \sum_{j=0}^{\ell} j b_{\ell,j} p^j n^{\underline{j}} + \sum_{j=1}^{\ell+1} b_{\ell,j-1} p^j n^{\underline{j}} - \sum_{i=0}^{\ell} b_{\ell,i} p^{i+1} n^{\underline{i+1}} - \sum_{i=0}^{\ell} b_{\ell,i} p^{i+1} i n^{\underline{i}} \\ &= \sum_{j=1}^{\ell} j b_{\ell,j} p^j (1-p) n^{\underline{j}} > 0. \end{aligned} \tag{71}$$

We conclude that $\frac{d\mathbb{E}[u(\beta_n,x)]}{d\beta_n}|_{\beta_n=\frac{1}{n}} > 0$ and there exists an expected utility maximizing $\beta_n^* > \frac{1}{n}$. ■

Proof of Lemma 5

Proof. Let $x_{n_1} = x_{n_1}^*(\beta_{n_1})$ be the incentive compatible effort level for $\beta_{n_1} \in (\frac{1}{n_1}, 1)$. Let $n_2 > n_1$. From *part (ii)* of Lemma 3, we know that the effort is strictly increasing in β_n. From Lemma 4, we know that $x_{n_1}^*(\frac{1}{n_1}) > x_{n_2}^*(\frac{1}{n_2})$. As $x_{n_1}^*(1) = x_{n_2}^*(1)$ and $\beta_{n_1} < 1$, there exists a $\beta_{n_2} \in (\frac{1}{n_2}, 1)$ such that $x_1 = x_{n_2}^*(\beta_{n_2})$. ■

Proof of Proposition 4

Proof. Let $\beta^*_{n_1}$ be the optimal effective share for pool size n_1 and let $x_{n_1} = x^*_{n_1}(\beta^*_{n_1})$ be the corresponding incentive compatible effort level. From Lemma 5 we know that for $n_2 > n_1$ there exists β_{n_2} such that $x_{n_1} = x^*_{n_2}(\beta_{n_2})$. Observe that β_{n_2} is not necessarily optimal for pool size n_2. From the proof of Proposition 2 for our binary model, we know that the utility for each pool member is strictly increasing in the pool size whenever the *same* effort is implemented. Thus, $\mathbb{E}[u(\beta_{n_2}, x_{n_1}, n_2)] > \mathbb{E}[u(\beta^*_{n_1}, x_{n_1}, n_1)]$ for each $n_2 > n_1$. Finally, from the definition of optimality, it follows that $\mathbb{E}[u(\beta^*_{n_2}, x^*_{n_2}(\beta^*_{n_2}), n_2)] \geq \mathbb{E}[u(\beta_{n_2}, x_{n_1}, n_2)] > \mathbb{E}[u(\beta^*_{n_1}, x_{n_1}, n_1)] = \mathbb{E}[u(\beta^*_{n_1}, x^*_{n_1}(\beta^*_{n_1}), n_1)]$. It immediately follows that the expected utility profile $\mathbb{E}[u(\beta^*_{n_2}, x^*_{n_2}(\beta^*_{n_2}), n_2)]$ is in the core. ■

Uniqueness of β^*_n

For a sufficient condition of uniqueness of the optimal coinsurance rate, define $p[x^*_n(\beta_n)]$ as the accident probability for the incentive compatible effort implemented via the effective share β_n. We can then prove that a sufficient condition for uniqueness (proof available on request) is that

(i) $\frac{d^2 p[x^*_n(\beta_n)]}{d\beta^2} = \frac{\partial^2 p}{\partial (x^*_n)^2}\left(\frac{\partial x^*_n}{\partial \beta_n}\right)^2 + \frac{\partial p}{\partial x^*_n}\frac{\partial^2 x^*_n}{\partial \beta^2_n} \geq 0$ and

(ii) $\frac{u'(W_0-L)}{u'(W_0)} - \frac{2-p[x^*_n(\beta_n=1)]}{1-p[x^*_n(\beta_n=1)]} \geq 0.$

Condition *(i)* expresses that the accident probability that is implemented via β_n decreases at a (weakly) decreasing rate. As $\frac{\partial^2 p}{\partial (x^*_n)^2} > 0$, $\left(\frac{\partial x^*_n}{\partial \beta^*_n}\right)^2 > 0$ and $\frac{\partial p}{\partial x^*_n} < 0$, a sufficient condition is that $\frac{\partial^2 x^*_n}{\partial \beta^2_n} \leq 0$, i.e. that the incentive compatible effort increases at a decreasing rate in β_n.

Intuitively, *(ii)* holds if the marginal utility of income with loss is large compared to the marginal utility without loss (i.e. if $\frac{u'(W_0-L)}{u'(W_0)}$ is high), and if the accident probability in case the effort incentive is maximized, $p[x^*_n(\beta_n = 1)]$ is low (i.e. if $\frac{2-p[x^*_n(\beta_n=1)]}{1-p[x^*_n(\beta_n=1)]}$ is low).

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