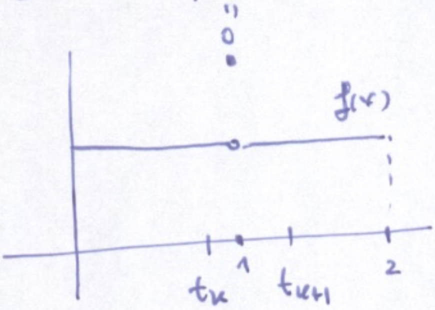
6.1. Para las funciones siguientes, determina si existe su integral y en su caso calcularla usando la definición de integral

a) $f(x) = \begin{cases} 1 & \text{si } x \in [0,2] \setminus \{1\} \\ 2 & \text{si } x = 1 \end{cases}$

Sea $P = \{ \underset{0}{\overset{\shortparallel}{t_0}} < t_1 < \dots < t_n = 2 \}$ una partición del intervalo $[0,2]$



Sumas inferiores $\underline{S}(f,P) = \sum_{i=1}^{n} \overset{1}{\overset{\shortparallel}{\min\{f(x) : x \in [t_{i-1}, t_i]\}}} \cdot (t_i - t_{i-1}) =$

$$= \sum_{i=1}^{n} 1 \cdot (t_i - t_{i-1}) = \sum_{i=1}^{n} (t_i - t_{i-1}) =$$

$$= t_n - t_{n-1} + t_{n-1} - t_{n-2} + \dots + t_1 - t_0 =$$

$$= t_n - t_0 = 2 - 0 = 2$$

Sumas superiores $\overline{S}(f,P) = \sum_{i=1}^{n} \max\{f(x) : x \in [t_{i-1}, t_i]\} (t_i - t_{i-1})$

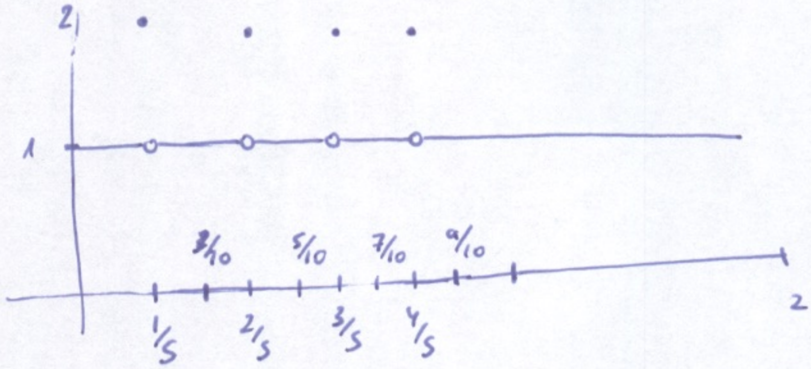
$$= \sum_{\substack{i=1 \\ i \neq k+1}}^{n} 1 \cdot (t_i - t_{i-1}) + 2(t_{k+1} - t_k) =$$

$$= \sum_{i=1}^{n} (t_i - t_{i-1}) + (t_{k+1} - t_k) =$$

$$= \underbrace{2}_{\text{Visto en el caso anterior}} + (t_{k+1} - t_k)$$

Haciendo las particiones muy finas concluimos que la función es integrable y que $\int_0^2 f = 2$.

b) $f(x) = \begin{cases} 1 & \text{si } x \in [0,2] \setminus \{1/5, 2/5, 3/5, 4/5\} \\ 2 & \text{si } x = 1/5, 2/5, 3/5, 4/5 \end{cases}$



$$\int_0^1 f = \int_0^{3/10} f + \int_{3/10}^{5/10} f + \int_{5/10}^{7/10} f + \int_{7/10}^{9/10} f + \int_{9/10}^{2} f$$

Por el caso anterior, tenemos que cada una de las integrals anteriores existe y vale 1·(longitud del intervalo). Así que

$$\int_0^1 f = \frac{3}{10} + \frac{2}{10} + \frac{2}{10} + \frac{2}{10} + \frac{11}{10} = \frac{20}{10} = 2.$$

6.3. Supongamos que $f \geq 0$ y continua en $[a,b]$. Si $\int_a^b f = 0$, prueba que $f(x) = 0$, para todo $x \in [a,b]$

Supongamos que $f(x_0) > 0$ para algún $x_0 \in [a,b]$ y sea $\delta > 0$ tal que $f(x)$ en $[x_0-\delta, x_0+\delta] \subset [a,b]$

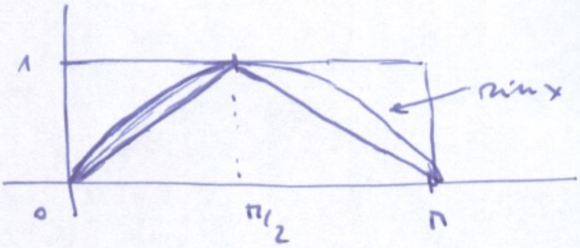
$$0 = \int_a^b f = \underbrace{\int_a^{x_0-\delta} f}_{f\geq 0 \;\geq 0} + \underbrace{\int_{x_0+\delta}^{x_0+\delta} f}_{= f(c)\cdot(x_0+\delta-(x_0-\delta)) = f(c)\cdot 2\delta} + \underbrace{\int_{x_0+\delta}^{b} f}_{\geq 0 \Leftarrow f \geq 0}$$

Teorema del valor medio del cálculo integral
$\exists\, c \in [x_0-\delta, x_0+\delta]$ tal que $\int_{x_0-\delta}^{x_0+\delta} f = f(c)\cdot(x_0+\delta-(x_0-\delta))$

Por tanto $0 = \int_a^b f \geq f(c)\cdot 2\delta > 0$!!

Así que $f(x) = 0 \quad \forall x \in [a,b]$.

6.4. Prueba que $\frac{\pi}{2} \leq \int_0^\pi \operatorname{sen}(x)\,dx \leq \pi$



Área del triángulo de base $[0,\pi]$, isósceles y de altura 1 $= \frac{\pi \cdot 1}{2}$ $\leq \int_0^\pi \operatorname{sen}(x)\,dx =$ área bajo la gráfica de la curva $y = \operatorname{sen}(x)$ $\leq$ Área del rectángulo $[0,\pi]\times[0,1]$ $= \pi$

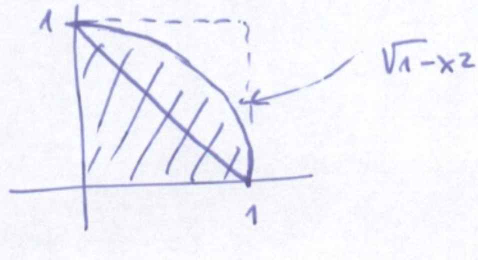
Concluimos que $\frac{\pi}{2} \leq \int_0^\pi \operatorname{sen}(x)\,dx \leq \pi$.

Encuentra cotas superiores e inferiores para las siguientes integrales

a) $\int_0^\pi \operatorname{sen}^8 x\,dx$

$$0 \leq \int_0^\pi \operatorname{sen}^8 x\,dx \underset{\substack{\uparrow \\ \operatorname{sen}^8(x) \leq \operatorname{sen}(x) \text{ en } [0,\pi]}}{\leq} \int_0^\pi \operatorname{sen}(x)\,dx = \left[-\cos(x)\right]_0^\pi = 1 + 1 = 2$$

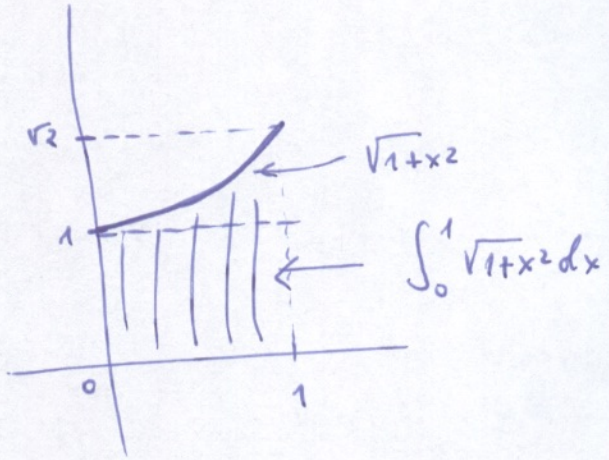
b) $\int_0^1 \sqrt{1-x^2}\,dx$



$\frac{1\cdot 1}{2} =$ Área triángulo rectángulo de catetos 1, 1 $\leq \int_0^1 \sqrt{1-x^2}\,dx \leq$ Área del cuadrado de lado 1 $= 1\cdot 1$

Así que $\frac{1}{2} \leq \int_0^1 \sqrt{1-x^2}\,dx \leq 1$.

c) $\int_0^1 \sqrt{1+x^2}\,dx$



$$1 \leq \begin{matrix}\text{Área del}\\ \text{cuadrado}\\ [0,1]\times[0,1]\end{matrix} \leq \int_0^1 \sqrt{1+x^2}\,dx \leq \begin{matrix}\text{Área del rectángulo}\\ [0,1]\times[0,\sqrt{2}]\end{matrix} = \sqrt{2}$$

Así que $1 \leq \int_0^1 \sqrt{1+x^2}\,dx \leq \sqrt{2}$

<u>6.10.</u> Si la función f es continua en $[0,1]$, prueba que $\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} f(k/n)\right) = \int_0^1 f$.

Si f es continua entonces $\int_0^1 f$ existe y se puede calcular usando cualquier familia de particiones tal que la distancia entre dos puntos consecutivos de la partición se haga tan pequeña como queramos. Así que elegimos las particiones

$$P_n = \{0 = t_0,\ t_1 = \tfrac{1}{n},\ t_2 = \tfrac{2}{n}, \dots, t_k = \tfrac{k}{n}, \dots, t_n = \tfrac{n}{n} = 1\}$$

Además, podemos elegir en vez de el máximo o el mínimo de f en el intervalo $[t_{k-1}, t_k]$ el valor de f en cualquier punto de dicho intervalo. Por ejemplo en t_k. Así que consideramos la sumas

$$\sum_{k=1}^{n} f(t_k)(t_k - t_{k-1}) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n} = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \longrightarrow \int_0^1 f$$

6.9. Representa las gráficas de las funciones

a) $F(x) = \int_0^x (-3t^2+24t-45)\,dt = \left[-t^3+12t^2-45t\right]_0^x =$

$= -x^3+12x^2-45x$

- $\mathrm{Dom}\, F = \mathbb{R}$
- Como F es un polinomio su gráfica no tiene asíntotas
- Crecimiento y decrecimiento

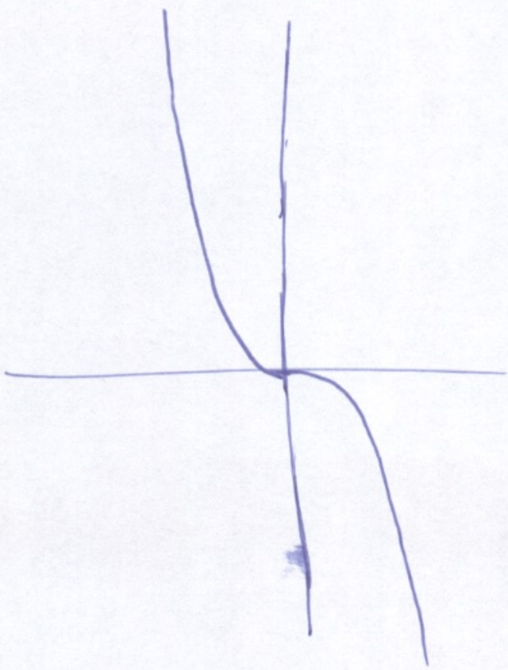
$$F'(x) = -3x^2+12x-45 = -3(x^2-4x+15) = -3(x^2-4x+4+11) =$$
$$= -3((x-2)^2+11) < 0 \quad \forall x \in \mathbb{R}$$

Por tanto F es siempre decreciente

- Curvatura

$$F''(x) = -6x+12 = -6(x-2) \begin{cases} >0 & \text{si } x<2 \rightarrow F \text{ convexa} \\ =0 & \text{si } x=2 \rightarrow F \text{ tiene un punto de inflexión en } x=2 \\ <0 & \text{si } x>2 \rightarrow F \text{ cóncava} \end{cases}$$

- $F(x)=0$ en $x=0$



b) $F(x) = \int_0^{\ln(x+1)} e^{t^2}\,dt \qquad x \geq 0$

$F = G \circ h \; : \quad h(x) = \ln(x+1) \;, \quad G(y) = \int_0^{y} e^{t^2}\,dt$

$F'(x) = G'(h(x)) \cdot h'(x) = e^{(\ln(x+1))^2} \cdot \frac{1}{x+1}$

• $\text{Dom } F = \{ x \in \mathbb{R} : x > -1 \}$

• Crecimiento y decrecimiento

$$F'(x) = \underbrace{e^{(\ln(x+1))^2}}_{>0} \cdot \underbrace{\frac{1}{x+1}}_{>0 \;\leftarrow\; x>-1} > 0 \Rightarrow$$ la función es estrictamente ~~de~~creciente

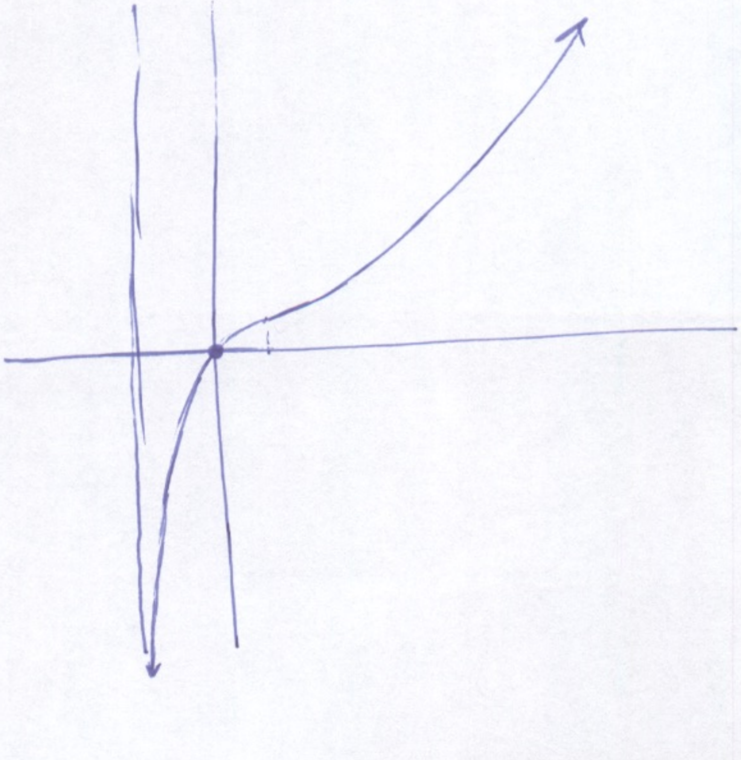
• Curvatura

$$F''(x) = 2\ln(x+1) \cdot e^{(\ln(x+1))^2} \cdot \frac{1}{x+1} - e^{(\ln(x+1))^2} \cdot \frac{1}{(x+1)^2}$$

cambia de signo cerca de $x = 1/2$

$$= \underbrace{\left(2\ln(x+1) - \frac{1}{x+1} \right)}_{?} \cdot \underbrace{e^{(\ln(x+1))^2}}_{>0} \cdot \underbrace{\frac{1}{x+1}}_{>0}$$

Así que $\text{signo}(F''(x)) = \text{signo}\left(2\ln(x+1) - \frac{1}{x+1} \right)$



$$\lim_{x \to \infty} \int_0^{\ln(x+1)} e^{t^2}\,dt = +\infty$$

$$\lim_{x \to -1^+} \int_0^{\ln(x+1)} e^{t^2}\,dt = -\infty$$

$$F(0) = \int_0^0 e^{t^2}\,dt = 0$$

c) $F(x) = \int_x^{2x} \operatorname{sen}^8(t)\, dt \qquad x \in [0, \pi]$

$F(x) = -F_1(x) + F_2(x)$: $\qquad F_1(x) = \int_0^{x} \operatorname{sen}^8(t)\, dt$

$\qquad\qquad F_2(x) = \int_0^{2x} \operatorname{sen}^8(t)\, dt$

$F_2(x) = (G_2 \circ h)(x) \qquad G_2(y) = \int_0^{y} \operatorname{sen}^8(t)\, dt$

$\qquad\qquad h(x) = 2x$

$F_2'(x) = G_2'(h(x)) \cdot h'(x)$

$= \operatorname{sen}^8(2x) \cdot 2$

$F'(x) = -F_1'(x) + F_2'(x) = -\operatorname{sen}^8(x) + 2\operatorname{sen}^8(2x) =$

$= -\operatorname{sen}^8(x) + 2 \cdot (2\operatorname{sen}(x)\cos(x))^8 =$

$= (-1 + 2^9 \cos^8(x)) \operatorname{sen}^8(x)$

Como es muy complicada ya la derivada no pintamos la gráfica.

6.12. Calcula las siguientes primitivas elementales.

a) $\int x\,dx = \frac{x^2}{2} + C$

b) $\int x^3\,dx = \frac{x^4}{4} + C$

c) $\int (3x^5 + 2x^3 + 7)\,dx = \frac{3x^6}{6} + \frac{2x^4}{4} + 7x + C = \frac{x^6}{2} + \frac{x^4}{2} + 7x + C$

d) $\int (x-2)^2\,dx = \frac{(x-2)^3}{3} + C$

e) $\int \cos x\,dx = \operatorname{sen} x + C$

f) $\int \operatorname{sen}\,dx = -\cos x + C$

g) $\int (3\cos x + 2\operatorname{sen} x)\,dx = 3\operatorname{sen} x - 2\cos x + C$

h) $\int 2x\cos(x^2)\,dx = \operatorname{sen}(x^2) + C$

i) $\int \frac{1}{x}\,dx = \ln|x| + C$

j) $\int \frac{1}{x^k}\,dx = \int x^{-k}\,dx = \frac{x^{-k+1}}{-k+1} + C = \frac{1}{(1-k)x^{1-k}} + C$ si $k \geq 2$

k) $\int e^x\,dx = e^x + C$

l) $\int \frac{2x}{(x^2-1)^3}\,dx = \frac{1}{-2(x^2-1)^2} + C$

m) $\int \cosh(x)\,dx = \operatorname{senh}(x) + C$

n) $\int \operatorname{senh}(x)\,dx = \cosh(x) + C$

ñ) $\int (3\cosh(x) + 2\operatorname{senh}(x))\,dx = 3\operatorname{senh}(x) + 2\cosh(x) + C$

o) $\int \cosh(x)\cdot\cosh(\operatorname{senh}(x))\,dx = \operatorname{senh}(\operatorname{senh}(x)) + C$

p) $\int \frac{1}{x-1}\,dx = \ln|x-1| + C$

q) $\int \frac{1}{x+1}\,dx = \ln|x+1| + C$

r) $\int \frac{1}{x^2+1}\,dx = \operatorname{arctg}(x) + C$

s) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \operatorname{arcsen}(x) + C$

t) $\int \frac{1}{\sqrt{1+x^2}}\,dx = \operatorname{arcsenh}(x) + C$

6.13. Calcula las siguientes primitivas usando la Regla de integración por partes:

1) $\int x e^x dx = xe^x - \int e^x dx = xe^x - e^x + C$

$u = x,\ du = dx$

$dv = e^x dx, v = e^x$

2) $\int x \operatorname{sen}(x)\,dx = -x\cos(x) + \int \cos(x)\,dx = -\cancel{x}\cos(x) + \operatorname{sen}(x) + C$

$u = x \quad du = dx$

$dv = \operatorname{sen}(x)\,dx \quad v = -\cos(x).$

3) $\int x\cos(x)\,dx = x\operatorname{sen}(x) - \int \operatorname{sen}(x)\,dx = x\operatorname{sen}(x) + \cos(x)$

$u = x \quad du = dx$

$dv = \cos(x)\,dx \quad v = \operatorname{sen}(x)$

4) $\int \frac{x}{e^x}\,dx = \int x e^{-x} dx = -xe^{-x} + \int e^{-x} dx = -xe^{-x} - e^{-x} + C$

$u = x \quad du = dx$

$dv = e^{-x} dx \quad v = -e^{-x} dx$

5) $\int \frac{\log(x)}{x^3}\,dx = -\frac{1}{2x^2}\log(x) + \int \frac{1}{2x^3}\,dx = -\frac{1}{2x^2}\log(x) - \frac{1}{4x^2} + C$

$u = \log(x) \quad du = \frac{1}{x}\,dx$

$dv = \frac{1}{x^3}\,dx \quad v = -\frac{1}{2x^2}$

6) $\int e^x \operatorname{sen} x \, dx = -e^x \cos x + \int e^x \cos x \, dx = -e^x \cos x + e^x \operatorname{sen} x - \int e^x \operatorname{sen} x \, dx$

$u = e^x \quad du = e^x dx$

$dv = \operatorname{sen} x \, dx \quad v = -\cos x$

$u = e^x \quad du = e^x dx$

$dv = \cos x \, dx \quad v = \operatorname{sen} x$

Por tanto, $2 \int e^x \operatorname{sen} x \, dx = -e^x \cos x + e^x \operatorname{sen} x$

Por tanto, $\int e^x \operatorname{sen} x \, dx = \dfrac{-e^x \cos x + e^x \operatorname{sen} x}{2} + C$

7) $\int \operatorname{arctg}(x) \, dx = x \operatorname{arctg}(x) - \int \frac{x}{1+x^2} dx = x \operatorname{arctg}(x) - \frac{1}{2} \ln(1+x^2) + C$

$u = \operatorname{arctg}(x) \quad du = \dfrac{1}{1+x^2} dx$

$dv = dx \quad v = x$

8) $\int \operatorname{arcsen}(x) \, dx = x \operatorname{arcsen}(x) - \int \frac{x}{\sqrt{1-x^2}} dx = x \operatorname{arcsen}(x) + \sqrt{1-x^2} + C$

$u = \operatorname{arcsen}(x) \quad du = \dfrac{1}{\sqrt{1-x^2}} dx$

$dv = dx \quad v = x$

6.14. Demuestra las siguientes fórmulas de reducción

1) $$\int sen^n(x)\,dx = -\frac{1}{n} sen^{n-1}(x)\cdot\cos(x) + \frac{n-1}{n}\int sen^{n-2}(x)\,dx \quad n>2 \text{ y par}$$

$$u=(sen(x))^{n-1} \qquad du=(n-1)\,sen(x)^{n-2}\cos(x)\,dx$$

$$dv = sen\,x\,dx \qquad v=-\cos(x)$$

$$\int sen^n(x)\,dx = -(sen(x))^{n-1}\cos(x) + (n-1)\int (sen(x))^{n-2}\cdot(\cos(x))^2\,dx =$$

$$= -(sen(x))^{n-1}\cos(x) + (n-1)\int (sen(x))^{n-2}(1-(sen(x))^2)\,dx =$$

$$= -(sen(x))^{n-1}\cos(x) + (n-1)\int sen(x)^{n-2}\,dx - (n-1)\int (sen(x))^n\,dx$$

Por tanto, despejando

$$n\int sen^n(x)\,dx = -(sen(x))^{n-1}\cos(x) + (n-1)\int sen(x)^{n-2}\,dx$$

Así que

$$\int sen^n(x)\,dx = -\frac{1}{n}(sen(x))^{n-1}\cos(x) + \frac{n-1}{n}\int sen(x)^{n-2}\,dx$$

2) $$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\cdot sen(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx \quad n>2 \text{ y par}$$

$$u=\cos^{(n-1)}(x) \qquad du=(n-1)\cos^{n-2}(x)\cdot(-sen(x))\,dx$$

$$dv=\cos(x)\,dx,\ v=sen(x)$$

$$\int \cos^n(x)\,dx = \cos^{(n-1)}(x)\cdot sen(x) + (n-1)\int \cos^{n-2}(x)\,sen^2(x)\,dx$$

$$= \cos^{(n-1)}(x)\,sen(x) + (n-1)\int (\cos(x))^{n-2}\cdot(1-(\cos(x))^2)\,dx =$$

$$= \cos^{(n-1)}(x)\,sen(x) + (n-1)\int (\cos(x))^{n-2}\,dx - (n-1)\int(\cos(x))^n\,dx$$

Por tanto, despejando

$$n\int \cos^n(x)\,dx = \cos^{(n-1)}(x)\,sen(x) + (n-1)\int(\cos(x))^{n-2}\,dx.$$ Así que

$$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{(n-1)}(x)\,sen(x) + \frac{n-1}{n}\int(\cos(x))^{n-2}\,dx$$

3) $\int \frac{dx}{(x^2+1)^n} = \int \frac{1+x^2-x^2}{(x^2+1)^n}\,dx = \int \frac{1}{(x^2+1)^{n-1}}\,dx - \frac{1}{2}\int x\,\frac{2x}{(x^2+1)^n}\,dx =$

$u = x \qquad du = dx$

$dv = \frac{2x}{(x^2+1)^n} \qquad v = \frac{(x^2+1)^{-n+1}}{-n+1} = \frac{1}{(1-n)(x^2+1)^{n-1}}$

$= \int \frac{1}{(x^2+1)^{n-1}}\,dx - \frac{1}{2}\left[\frac{x}{(1-n)(x^2+1)^{n-1}} - \int \frac{1}{(1-n)(x^2+1)^{n-1}}\,dx\right] =$

$= \left(1 - \frac{1}{2(n-1)}\right)\int \frac{1}{(x^2+1)^{n-1}}\,dx + \frac{1}{2}\,\frac{x}{(n-1)(x^2+1)^{n-1}} =$

$= \frac{x}{2(n-1)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{1}{(x^2+1)^{n-1}}\,dx$

6.16 Obtén una primitiva en los siguientes casos:

1) $\int x\sqrt[5]{5-x^2}\,dx = \int \frac{1}{2}\sqrt[5]{5-u}\,du = \frac{1}{2}\,\frac{(5-u)^{1/5+1}}{1/5+1} + C = \frac{5}{12}(5-u)^{6/5} + C =$

$u = x^2 \qquad du = 2x\,dx$

$= \frac{5}{12}(5-x^2)^{6/5} + C$

2) $\int x e^{-x^2}\,dx = \int -\frac{1}{2}e^t\,dt = -\frac{1}{2}e^t + C = -\frac{1}{2}e^{-x^2} + C$

$t = -x^2 \qquad dt = -2x\,dx$

3) $\int \frac{dx}{\sqrt{e^x}} = \int \frac{2}{t}\cdot\frac{1}{t}\cdot\frac{1}{2\ln(t)}\,dt = \int \frac{1}{t\ln(t)}\,dt = \ln|\ln|t|| + C =$

$t = \sqrt{e^x} \to x = \ln(t^2) = 2\ln(t) \to dx = \frac{2}{t}\,dt$

$= \ln|\ln$

3) $\displaystyle\int \frac{dx}{x\sqrt{\ln x}} = \int \frac{2t\,dt}{t} = \int 2dt = 2t + C = 2\sqrt{\ln x} + C$

$t = \sqrt{\ln x} \rightarrow t^2 = \ln x \rightarrow 2t\,dt = \frac{1}{x}\,dx$

4) $\displaystyle\int \frac{1-\operatorname{sen} x}{x+\cos x}\,dx = \int \frac{dt}{t} = \ln|t| + c = \ln|x+\cos x| + C$

$t = x + \cos x \quad dt = (1-\operatorname{sen} x)\,dx$

5) $\displaystyle\int \frac{x^2}{\sqrt[3]{x^3+1}}\,dx = \int t\,dt = \frac{t^2}{2} + C = \frac{\left(\sqrt[3]{x^3+1}\right)^2}{2} + C$

$t = \sqrt[3]{x^3+1} \quad dt = \frac{1}{3}(x^3+1)^{-2/3} \cdot 3x^2\,dx = \frac{x^2}{\left(\sqrt[3]{x^3+1}\right)^2}\,dx$

6) $\displaystyle\int \frac{\arccos\left(\frac{x}{2}\right)}{\sqrt{4-x^2}}\,dx = \int t\,dt = \frac{t^2}{2} + C = \left(\arccos\left(\frac{x}{2}\right)\right)^2 + C$

$t = \arccos\left(\frac{x}{2}\right) \quad dt = \frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \cdot \frac{1}{2}\,dx = \frac{-dx}{\sqrt{4-x^2}}$

7) $\displaystyle\int \frac{\sqrt[3]{1+\ln x}}{x}\,dx = \int t \cdot 3t^2\,dt = \frac{3}{4}t^4 + C = \frac{3}{4}\sqrt[3]{1+\ln x} + C$

$t = \sqrt[3]{1+\ln x} = (1+\ln x)^{1/3} \quad dt = \frac{1}{3}(1+\ln x)^{-2/3} \cdot \frac{1}{x}\,dx =$

$= \frac{1}{3x} \cdot \frac{1}{t^2}\,dx \rightarrow \frac{1}{x}\,dx = 3t^2\,dt$

8) $\displaystyle\int \tan\left(\sqrt{x-1}\right) \cdot \frac{dx}{\sqrt{x-1}} = 2\int \tan(t)\,dt = 2\int \frac{\operatorname{sen}(t)}{\cos(t)}\,dt =$

$t = \sqrt{x-1} \quad dt = \frac{1}{2\sqrt{x-1}}\,dx$

$= -2\ln(\cos(t)) + C = -2\ln\left(\cos\left(\sqrt{x-1}\right)\right) + C$

9) $\int x^2 \cosh(x^3+3)\,dx = \int \frac{1}{3}\cosh(t)\,dt = \frac{1}{3}\operatorname{senh}(t)+C = \frac{1}{3}\operatorname{senh}(x^3+3)+C$

$t = x^3+3 \quad dt = 3x^2\,dx$

6.17. Calcule las siguientes primitivas con el cambio de variable que se indica.

a) $\int \frac{dx}{x(1-x)} = \int \frac{2\operatorname{sen}t\cos t\,dt}{\operatorname{sen}^2 t\cdot(1-\operatorname{sen}^2 t)} = \int \frac{2\operatorname{sen}t\cos t\,dt}{\operatorname{sen}^2 t\cdot\cos^2 t} = \int \frac{2}{\operatorname{sen}t\cos t}\,dt =$

$x = \operatorname{sen}^2 t, \quad dx = 2\operatorname{sen}t\cdot\cos t\,dt$

$= \int \frac{4}{\operatorname{sen}(2t)}\,dt = 2\int \frac{1}{\operatorname{sen}(z)}\,dz \underset{6.15}{=} 2\ln\left|\tan\left(\frac{z}{2}\right)\right| + C =$

$2t = z \quad 2dt = dz$

$= 2\ln|\tan(t)| + C = 2\ln|\tan(\underbrace{\cancel{\operatorname{sen}^2 t}}_{\operatorname{arcsen}(\sqrt{x})})| + C$

b) $\int \frac{dx}{\sqrt{x^2-2}} = \int \frac{\sqrt{2}\operatorname{senh}(u)\,du}{\sqrt{2\cosh^2 u - 2}} = \int \frac{\operatorname{senh}(u)}{\sqrt{\cosh^2 u - 1}}\,du =$

$x = \sqrt{2}\cosh u \quad dx = \sqrt{2}\operatorname{senh}u\,du \quad u = \operatorname{arcsenh}\left(\frac{x}{\sqrt{2}}\right)$

$\underset{\cosh^2 u - \operatorname{senh}^2 u = 1}{=} \int \frac{\operatorname{senh}(u)}{\operatorname{senh}(u)}\,du = \int du = u + C = \operatorname{arcsenh}\left(\frac{x}{\sqrt{2}}\right) + C$

c) $\int \frac{dx}{e^x+1} = \int -\frac{1}{t}\cdot\frac{1}{1/t+1}\,dt = \int -\frac{1}{t}\cdot\frac{t}{t+1}\,dt = -\int \frac{1}{t+1}\,dt =$

$x = -\ln t \rightarrow dx = -\frac{1}{t}\,dt, \quad t = e^{-x}$

$= -\ln|t+1| + C = -\ln|e^{-x}+1| + C$

d) $\displaystyle\int \frac{x\,dx}{\sqrt{x+1}} = \int \frac{(t^2-1)\,2t\,dt}{t} = \int 2(t^2-1)\,dt = 2\left(\frac{t^3}{3} - t\right) + C =$

$t = \sqrt{x+1}, \quad t^2 = x+1, \quad x = t^2 - 1, \quad dx = 2t\,dt$

$\displaystyle = 2\left(\frac{(\sqrt{x+1})^3}{3} - \sqrt{x+1}\right) + C$

e) $\displaystyle\int \frac{\sqrt{x^2+1}}{x^2}\,dx = \int \frac{\cosh^2(t)}{\operatorname{senh}^2(t)}\,dt = \int \frac{\operatorname{senh}^2(t)+1}{\operatorname{senh}^2(t)} = \int\left(1 + \frac{1}{\operatorname{senh}^2(t)}\right)dt =$

$x = \operatorname{senh}(t) \qquad dx = \cosh(t)\,dt \qquad \sqrt{\operatorname{senh}^2(t)+1} = \cosh(t)$

$\displaystyle = t - \frac{\cosh(t)}{\operatorname{senh}(t)} + C = \operatorname{arcsenh}(x) - \frac{\cosh(\operatorname{arcsenh}(x))}{x}$

f) $\displaystyle\int \sqrt{a^2+x^2}\,dx = \int \sqrt{a^2 + a^2\operatorname{senh}^2 t}\cdot a\cosh(t)\,dt = a^2 \int \cosh^2(t)\,dt = \otimes$

$x = a\operatorname{senh} t \qquad dx = a\cosh(t)\,dt$

$$\begin{aligned} \cosh(2t) &= \cosh^2(t) + \operatorname{senh}^2(t) \\ 1 &= \cosh^2(t) - \operatorname{senh}^2(t) \\ \hline 1 + \cosh(2t) &= 2\cosh^2(t) \end{aligned}$$

$\displaystyle \otimes = a^2 \int \frac{1+\cosh(2t)}{2}\,dt = a^2\left(\frac{t}{2} + \frac{\operatorname{senh}(2t)}{4}\right) + C =$

$\displaystyle = a^2\left(\frac{\operatorname{arcsenh}\left(\frac{x}{a}\right)}{2} + \frac{\operatorname{senh}\left(2\operatorname{arcsenh}\left(\frac{x}{a}\right)\right)}{4}\right) + C$

6.18 Calcular las siguientes primitivas

e) $\displaystyle\int \frac{1}{1+\operatorname{sen}(x)}\,dx = \int \frac{1}{1+\operatorname{sen}(x)} \cdot \frac{1-\operatorname{sen}(x)}{1-\operatorname{sen}(x)}\,dx =$

$$\int \frac{1-\operatorname{sen}x}{1-\operatorname{sen}^2 x}\,dx = \int \frac{1-\operatorname{sen}x}{\cos^2 x}\,dx = \int\left(\frac{1}{\cos^2 x} - \frac{\operatorname{sen}x}{\cos^2 x}\right)dx =$$

$$= \operatorname{tg}x - \frac{1}{\cos x} + C$$

6.21 Sea $f:\mathbb{R}\to\mathbb{R}$ una función continua y de período p.

Demuestra la igualdad

$$\int_a^{a+p} f(t)\,dt = \int_0^p f(t)\,dt$$

Como f es periódica de período $p \Rightarrow f(t+p) = f(t) \quad \forall t\in\mathbb{R}$

~~Hacemos el cambio~~

Escribimos $\displaystyle\int_0^p f(t)\,dt = \int_0^{a+p} f(t)\,dt + \int_{a+p}^p f(t)\,dt$

$$\text{①} \quad \int_{a+p}^p f(t)\,dt = \int_a^0 f(x+p)\,dx \underset{\substack{\uparrow \\ f \text{ periódica}}}{=} \int_a^0 f(x)\,dx$$

$$x = t-p \qquad dx = dt \qquad p\to 0,\ a+p \to a$$

Por tanto,

$$\int_0^p f(t)\,dt = \int_0^{a+p} f(t)\,dt + \int_{a+p}^p f(t)\,dt \underset{①}{=}$$

$$= \int_0^{a+p} f(t)\,dt + \int_a^0 f(x)\,dx = \int_0^{a+p} f(t)\,dt + \int_a^0 f(t)\,dt =$$

$$= \int_a^0 f(t)\,dt + \int_0^{a+p} f(t)\,dt = \int_a^{a+p} f(t)\,dt.$$

6.22 a) Calcula $\int \arcsin x \, dx$

La vamos a hacer por partes

$u = \arcsin x \qquad du = \frac{1}{\sqrt{1-x^2}} dx$

$dv = dx \qquad v = x$

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx = x \arcsin x - \sqrt{1-x^2} + C$$

b) Prueba que si $F = \int f$, entonces

$$\int f^{-1}(x) \, dx = x f^{-1}(x) - F(f^{-1}(x))$$

$u = f^{-1}(x) \qquad du = (f^{-1}(x))' dx$

$dv = dx \qquad v = x$

$$\int f^{-1}(x) dx = x f^{-1}(x) - \int x \cdot (f^{-1}(x))' dx = x f^{-1}(x) - \int f(f^{-1}(x)) \cdot$$

$$\cdot (f^{-1}(x))' dx \underset{\uparrow}{=} x f^{-1}(x) - F(f^{-1}(x))$$

$$(F(f^{-1}(x)))' = F'(f^{-1}(x)) \cdot (f^{-1}(x))' =$$

$$= f(f^{-1}(x)) \cdot (f^{-1}(x))' =$$

$$= x \cdot (f^{-1}(x))'$$

c) Calcular $\int \sqrt{x^2-1} \, dx = x\sqrt{x^2-1} - \int \frac{x^2}{\sqrt{x^2-1}} dx =$

$u = \sqrt{x^2-1} \qquad du = \frac{x}{\sqrt{x^2-1}}$

$dv = dx \qquad v = x$

$$= x\sqrt{x^2-1} - \left(\int \frac{x^2-1}{\sqrt{x^2-1}} dx + \int \frac{1}{\sqrt{x^2-1}} dx \right) = x\sqrt{x^2-1} - \int \sqrt{x^2-1} \, dx -$$

$$- \int \frac{1}{\sqrt{x^2-1}} dx = x\sqrt{x^2-1} - \operatorname{arccosh}(x) + C - \int \sqrt{x^2-1} \, dx$$

Pasando al otro miembro $\int \sqrt{x^2-1}\,dx$, nos queda

$$2\int \sqrt{x^2-1}\,dx = x\sqrt{x^2-1} - \operatorname{arccosh}(x) + C$$

$$\int \sqrt{x^2-1}\,dx = \frac{1}{2}\left(x\sqrt{x^2-1} - \operatorname{arccosh}(x)\right) + C'$$

6.23. Calcula una primitiva en los siguientes casos:

1) $$\int \frac{dx}{1+\sqrt{1+x}} = \int \frac{2t\,dt}{1+t} = \int \frac{2t+2-2}{1+t}\,dt = \int \left(2 - \frac{2}{1+t}\right)dt =$$

$$t=\sqrt{1+x} \to t^2 = 1+x \to x = t^2-1 \to dx = 2t\,dt$$

$$= 2t - 2\ln|1+t| + C = 2\sqrt{1+x} - 2\ln|1+\sqrt{1+x}| + C$$

2) $\int \frac{dx}{1+e^x}$ está resuelto en el 6.17(c)

3) $$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} = \int \frac{dx}{x^{1/2}+x^{1/3}} = \int \frac{dx}{x^{3/6}+x^{2/6}} = \int \frac{6t^5\,dt}{t^3+t^2} =$$

$$t = x^{1/6} \to t^6 = x \to 6t^5\,dt = dx$$

$$= \int \frac{6t^3\,dt}{t+1} = 6\int \frac{t^3}{t+1}\,dt = 6\int \left(t^2 - t + 1 - \frac{1}{1+t}\right)dt = ⊗$$

$$\begin{array}{l|l} t^3 & t+1 \\ \hline -t^3-t^2 & t^2-t+1 \\ \quad +t^2+t & \\ \qquad -t-1 & \\ \qquad\ \ -1 & \end{array}$$

$$⊗ = 6\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \ln(1+t)\right] + C = 6\left(\frac{x^{3/6}}{3} - \frac{x^{2/6}}{2} + x^{1/6} - \ln(1+x^{1/6})\right) + C.$$

4) $\displaystyle\int \frac{dx}{\sqrt{1+e^x}} = \int \frac{2t\,dt}{t(t^2-1)} = \int \frac{2}{1-t^2}\,dt = \circledast$

$t = \sqrt{1+e^x} \rightarrow t^2 = 1+e^x \rightarrow 2t\,dt = e^x dx \;;\; e^x = t^2-1$

$$\Downarrow$$

$$dx = \frac{2t\,dt}{t^2-1}$$

$$\frac{2}{1-t^2} = \frac{A}{1-t} + \frac{B}{1+t} = \frac{A(1+t)+B(1-t)}{(1+t)(1-t)}$$

Luego $2 = A(1+t) + B(1-t)$

Para $t=1$, deducimos $2A = 2 \rightarrow A = 1$

Para $t=-1$, deducimos $2B = 2 \rightarrow B = 1$

Así que $\dfrac{2}{1-t^2} = \dfrac{1}{1-t} + \dfrac{1}{1+t}$

$$\circledast = \int\left(\frac{1}{1-t} + \frac{1}{1+t}\right)dt = -\ln|1-t| + \ln|1+t| + C =$$

$$= -\ln\left|1-\sqrt{1+e^x}\right| + \ln\left|1+\sqrt{1+e^x}\right| + C$$

5) $\displaystyle\int \frac{dx}{2+\tan x} = \int \frac{1}{2+t}\cdot\frac{1}{1+t^2}\,dt = \circledast$

$t = \tan(x) \quad dt = 1+(\tan(x))^2\,dx = (1+t^2)\,dx \rightarrow dx = \dfrac{dt}{1+t^2}$

$$\frac{1}{2+t}\cdot\frac{1}{1+t^2} = \frac{A}{2+t} + \frac{Bt+C}{1+t^2} = \frac{A(1+t^2)+(Bt+C)(2+t)}{(2+t)(1+t^2)}$$

Luego $1 = A(1+t^2) + (Bt+C)(2+t)$

Para $t=-2$ deducimos que $1 = 5A \rightarrow A = 1/5$

$t=0$ tenemos $1 = A + 2C \rightarrow 2C = 4/5 \rightarrow C = 2/5$

$t=1$ tenemos $1 = 2A + (B+C)\cdot 3$

$1 = \frac{2}{5} + \left(B+\frac{2}{5}\right)\cdot 3 \rightarrow \frac{3}{5} = 3\left(B+\frac{2}{5}\right) \rightarrow B+\frac{2}{5} = \frac{1}{5}$

Así que $B = -1/5$

Por tanto, $\frac{1}{2+t}\cdot\frac{1}{1+t^2} = \frac{1/5}{2+t} + \frac{-1/5\,t + 2/5}{1+t^2} =$

$$= 1/5\left(\frac{1}{2+t} + \frac{2-t}{1+t^2}\right)$$

$$\circledast = \frac{1}{5}\int\left(\left(\frac{1}{2+t}\right) + \frac{2}{1+t^2} - \frac{t}{1+t^2}\right)dt =$$

$$= \frac{1}{5}\left[\ln|t+2| + 2\,\text{arctg}(t) - \frac{1}{2}\ln|1+t^2|\right] + C =$$

$$= \frac{1}{5}\left[\ln|2+\tan(x)| + 2x - \frac{1}{2}\ln|1+(\tan(x))^2|\right] + C.$$

6) $\int \text{sen}^3 x \cos^4 x \, dx$ Resuelto en 6.8 (b)

7) $\int \frac{dx}{\sqrt{\sqrt{x}+1}} = \int \frac{2t\,dt}{\sqrt{t+1}} = \int \frac{2(u^2-1)\,2u\,du}{u} = \int 4(u^2-1)\,du =$

$t=\sqrt{x} \quad t^2 = x \quad 2t\,dt = dx$

$u=\sqrt{t+1} \quad u^2 = t+1 \quad 2u\,du = dt$

$$= 4\left(\frac{u^3}{3} - u\right) + C = 4\left(\frac{(\sqrt{\sqrt{x}+1})^3}{3} - \sqrt{\sqrt{x}+1}\right) + C$$

8) $\int \frac{\text{arctg}(x)}{1+x^2}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{(\text{arctg}(x))^2}{2} + C$

$u = \text{arctg}(x) \qquad du = \frac{1}{1+x^2}\,dx$

9) $\int \frac{x^2-1}{x^2+1}\,dx = \int \frac{x^2+1-2}{x^2+1}\,dx = \int 1 - \frac{2}{x^2+1}\,dx = x - 2\,\text{arctg}(x) + C$

10) $\int \text{arcsen}\sqrt{x}\,dx = x\,\text{arcsen}\sqrt{x} - \int \frac{\sqrt{x}}{2\sqrt{\cancel{1-x}}}\,dx = x\,\text{arcsen}\sqrt{x} - \frac{1}{2}\int \frac{\ }{\sqrt{\frac{x}{1-x}}}\,dx$

$u = \text{arcsen}\sqrt{x} \qquad du = \frac{1}{\sqrt{x-1}}\cdot\frac{1}{2\sqrt{x}}$

$dv = dx \qquad v = x$

$$\int \sqrt{\frac{x}{1-x}}\, dx = -\int u \cdot \frac{2u}{(1+u^2)^2}\, du = \otimes$$

$$u = \sqrt{\frac{x}{1-x}}\, dx \qquad u^2 = \frac{x}{1-x} = -1 + \frac{1}{1-x} \qquad 2u\, du = -\frac{1}{(1-x)^2}\, dx$$

Por tanto, $dx = -2u\,(1-x)^2 du = -2u \cdot \frac{1}{(1+u^2)^2}\, du$

$$\otimes = -\frac{u}{1+u^2} + \int \frac{1}{1+u^2}\, du = -\frac{u}{1+u^2} + \operatorname{arctg}(u) + C =$$

$$u_1 = u \qquad du_1 = du$$

$$dv = \frac{-2u}{(1+u^2)^2}\, du \quad v = \frac{1}{1+u^2}$$

$$= -\sqrt{\frac{x}{1-x}} \cdot (1-x) + \operatorname{arctg}\left(\sqrt{\frac{x}{1-x}}\right) + C$$

Por tanto $\int \operatorname{arcsen}(\sqrt{x}) = x \operatorname{arcsen}(\sqrt{x}) + \frac{1}{2}\sqrt{x}\sqrt{1-x} - \frac{1}{2}\operatorname{arctg}\left(\sqrt{\frac{x}{1-x}}\right) + C$

11) $\int \left(\operatorname{sen} x \int_0^x \operatorname{sen} t\, dt\right) dx = \int \left(\operatorname{sen} x \left[-\cos t\right]_0^x\right) dx =$

$$= \int \operatorname{sen} x\,(-\cos(x) + 1)\, dx = \int \left(\left(-\operatorname{sen}(x) \cdot \cos(x)\right) + \operatorname{sen} x\right) dx =$$

$$= \frac{\cos^2(x)}{2} - \cos(x) + C$$