A CONVERSE TO CARTAN'S THEOREM B:

THE EXTENSION PROPERTY FOR REAL ANALYTIC AND NASH SETS

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Introduction

Analytic case. Let $\Omega \subset \mathbb{R}^n$ be an open set. A subset $X \subset \Omega$ has the *analytic extension property* if each analytic function $f : X \to \mathbb{R}$ extends to an analytic function on Ω.

Main Problem I. Which sets $X \subset \Omega$ do have the analytic extension property?

Nash case. Let $\Omega \subset \mathbb{R}^n$ be an open semialgebraic set. A subset $X \subset \Omega$ has the *Nash extension property* if each local Nash function $f : X \to \mathbb{R}$ extends to a Nash function on Ω.

Main Problem II. Which sets $X \subset \Omega$ do have the Nash extension property?

Necessary condition

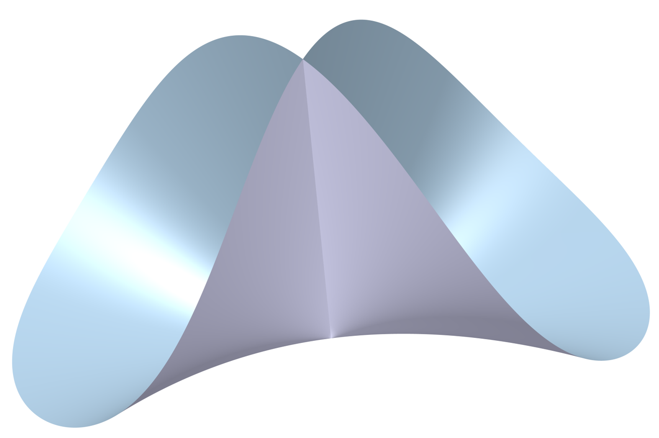
Analytic case. X is the zero set of an analytic function (C-analytic set).

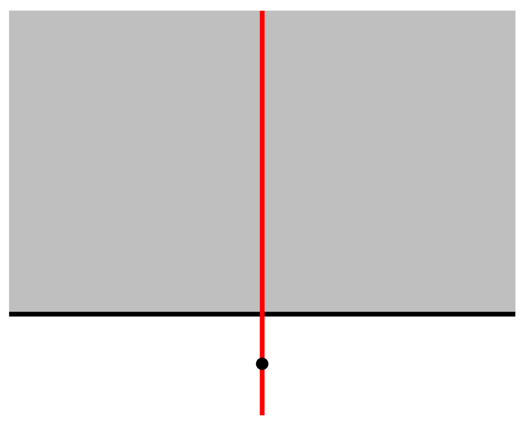
Nash case. X is the zero set of a Nash function (Nash set).

Classical example. The necessary condition is not sufficient. Consider Whitney's umbrella $W := \{y^2 - zx^2 = 0\} \subset \mathbb{R}^3$ and

$$f : W \to \mathbb{R},\ (x, y, z) \mapsto \begin{cases} \frac{x}{z+1} & \text{if } (x, y, z) \neq (0, 0, -1), \\ 0 & \text{otherwise,} \end{cases}$$

which is analytic (and Nash) on W, but it does not extend analytically to $\mathbb{R}^3$.





Coherence & Cartan's Theorem B

A **sufficient condition** is provided by coherence and Cartan's Theorem B.

Coherence. A C-analytic set X is *coherent* if its local equations at each point $x \in X$ are generated by its global equations.

$$\mathcal{J}_{X,x} := \{f_x \in \mathcal{O}_{\mathbb{R}^n,x} :\ X_x \subset \mathcal{Z}(f_x)\} \quad \text{and} \quad \mathcal{I}(X) := \{f \in \mathcal{O}(\mathbb{R}^n) :\ X \subset \mathcal{Z}(f)\}$$

$$X \text{ is coherent } \iff \mathcal{J}_{X,x} = \mathcal{I}_{X,x} := \mathcal{I}(X)\mathcal{O}_{\mathbb{R}^n,x}\ \forall x \in X$$

Cartan's Theorem B (1957) $\Longrightarrow$ If $X \subset \Omega$ is a coherent C-analytic set, X has the analytic extension property.

Theorem (FGh, 2025) $X \subset \Omega$ has the analytic extension property $\iff$ X is a coherent analytic set. If X is not coherent, there are 'many' failing functions!

Nash Theorem B (Coste-Ruiz-Shiota, 2000) $\Longrightarrow$ If $X \subset \Omega$ is a coherent Nash set, X has the Nash extension property.

Theorem (FGh, 2025) $X \subset \Omega$ has the Nash extension property $\iff$ X is a coherent Nash set. If X is not coherent, there are 'many' failing functions!

Distinguished sets: Sets of 'tails' and points of non-coherence

Let $X \subset \Omega$ be a C-analytic set.

$T(X) := \{x \in X : \mathcal{J}_{X,x} \neq \mathcal{I}_{X,x}\} \subset \operatorname{Sing}(X)$ is C-semianalytic & $\dim(T(X)) < \dim(X)$.

$N(X) := \{x \in X : \mathcal{J}_X$ is not of finite type at $x\}$ is closed, C-semianalytic & $\dim(N(X)) \leq \dim(X) - 2$.

Properties of 'tails' and non-coherence

(1) $\operatorname{Cl}(T(X)) = \operatorname{Cl}(T(X) \setminus N(X)) = T(X) \cup N(X)$.

(2) X is coherent $\iff T(X) = \varnothing \iff N(X) = \varnothing$

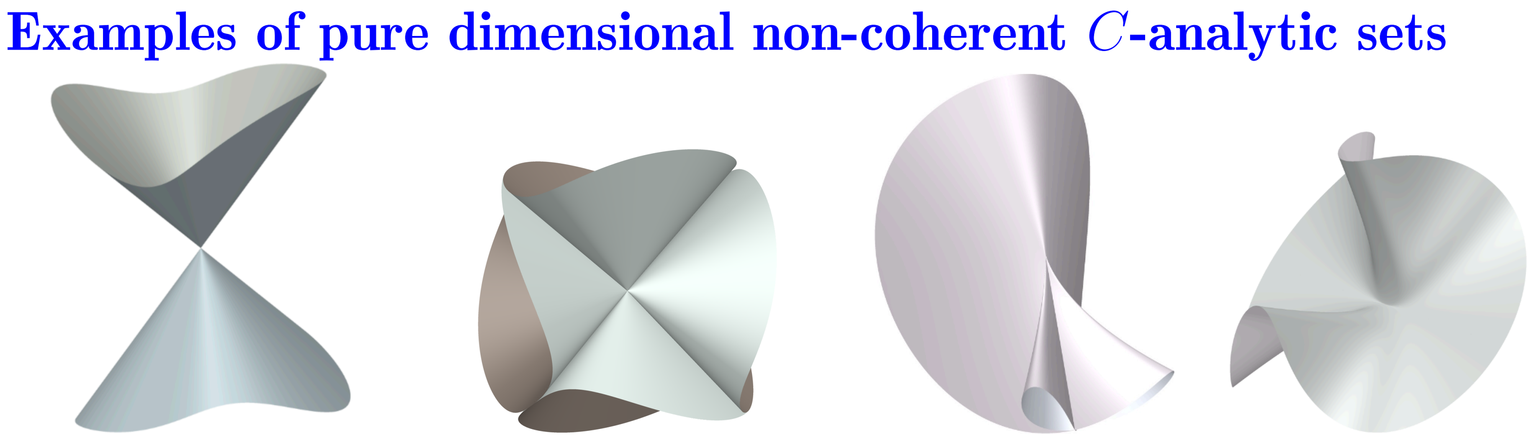
(3) If S is a connected component of $\operatorname{Cl}(T(X))$, then $S \cap N(X) \neq \varnothing$.

(4) A general idea in Real Geometry is that non-coherence arises when the irreducible components of the objects are not pure dimensional.

C-semianalytic set: A locally finite union in Ω of *basic C-semianalytic subsets* $\{f = 0, g_1 > 0, \ldots, g_r > 0\}$ where $r \geq 1$ and $f, g_i \in \mathcal{O}(\mathbb{R}^n)$.

Main results

Examples of pure dimensional non-coherent C-analytic sets



$z(x+y)(x^2+y^2) - x^4 = 0$ $\quad$ $z^2(x+y)^2(x^2+y^2) - x^6 = 0$ $\quad$ $(x^2 + zy^2)x - y^4 = 0$ $\quad$ $(x^2 + z^2y^2)x - y^4 = 0$

Obstructing set of a meromorphic function

Let $\zeta := \frac{f}{g} \in \mathcal{M}(X)$ be a well-defined meromorphic function on a C-analytic set X.

$$\frac{f}{g} = -a|_X\colon\ a \in \mathcal{O}(\mathbb{R}^n) \iff f_x \in g_x\mathcal{O}_{\mathbb{R}^n,x} + \mathcal{I}(X)\mathcal{O}_{\mathbb{R}^n,x} = g_x\mathcal{O}_{\mathbb{R}^n,x} + \mathcal{I}_{X,x}\ \forall x \in X$$

Obstructing set: $\mathsf{O}(\zeta) := \{x \in X :\ f_x \notin g_x\mathcal{O}_{\mathbb{R}^n,x} + \mathcal{I}_{X,x}\}$ (closed subset of X).

Main Theorem (FGh). Let $X \subset \Omega$ be a C-analytic set with $N(X) \neq \varnothing$. Let

- $Y \subset X$ be a C-analytic subset that contains no irreducible component of X and meets $T(X)$,
- $U_0 \subset \mathbb{R}^n$ be an open neighborhood of Y,
- $h \in H^0(U_0, \mathcal{J}_X)$ be such that $h_y \in \mathcal{J}_{X,y} \setminus \mathcal{I}_{X,y}$ for each $y \in Y \cap T(X)$.

There exist $\zeta \in (\mathcal{M}(X) \cap H^0(X, \mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X)) \setminus \mathcal{O}(\mathbb{R}^n)$ such that $\mathsf{O}(\zeta) = Y \cap T(X)$.

Remarks (1) Analogous result in the Nash case.
(2) If $Y \cap N(X) = \varnothing$ or $\dim(Y) = 0$, $\exists h \in H^0(U_0, \mathcal{J}_X)$ (where $U_0 := \Omega \setminus N(X)$): $h_y \in \mathcal{J}_{X,y} \setminus \mathcal{I}_{X,y}\ \forall y \in Y \cap T(X)$. In addition, $\mathsf{O}(\zeta) = Y \cap T(X)$.

An application: Smooth semialgebraic functions vs Nash functions

Let S be a semialgebraic set.

$$\mathcal{S}^{(\infty)}(S) := \bigcap_{p \geq 0} \mathcal{S}^p(S) :\ \ \mathcal{S}^p(S) := \{\text{semialgebraic} + \mathcal{C}^p \text{ functions on } S\} \quad \forall p \geq 0$$

$$\mathcal{N}(S) = H^0(S, (\mathcal{N}_{\mathbb{R}^n})|_S) = \varinjlim \mathcal{N}(V)|_S :\ V \text{ open semialgebraic neighborhood of } S$$

Problem III. For which semialgebraic sets $\mathcal{S}^{(\infty)}(S) = \mathcal{N}(S)$?

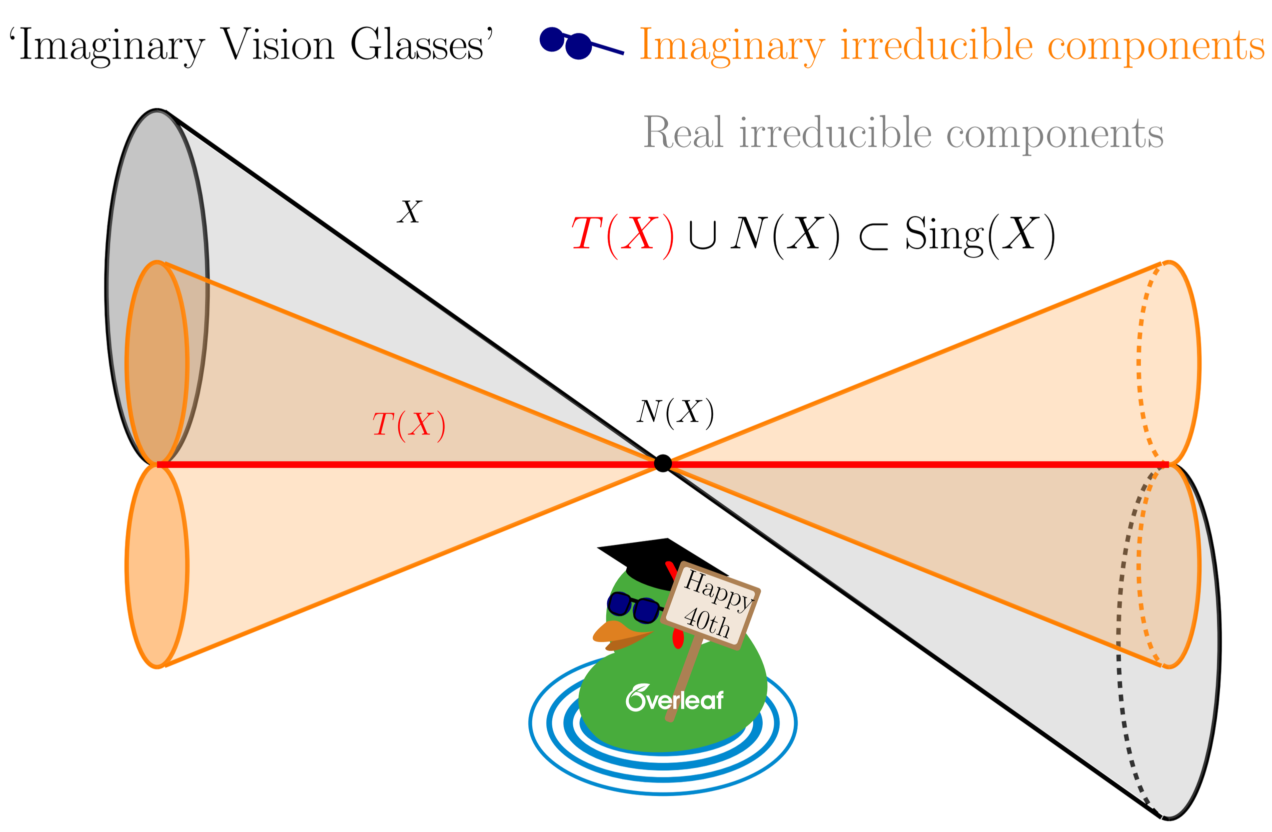
Define

$$\mathcal{J}^{\bullet}_{S,x} := \{f_x \in \mathcal{N}_{\mathbb{R}^n,x} :\ S_x \subset \mathcal{Z}(f_x)\} \quad \text{and} \quad A(S) := \{x \in S :\ \mathcal{I}^{\bullet}_{X,x} \neq \mathcal{J}^{\bullet}_{S,x}\}.$$

where X is the Nash closure of S in a 'suitable' semialgebraic neighborhood of S in $\mathbb{R}^n$.

Theorem (FGh, 2025) $\mathcal{N}(S) = \mathcal{S}^{(\infty)}(S) \iff A(S) = \varnothing$.

Tails and non-coherence points



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