POSITIVE SEMIDEFINITE ELEMENTS AND SUM OF SQUARES IN LOCAL HENSELIAN RINGS

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Abstract. A classical problem in Real Geometry (which goes back to Hilbert's 17th problem) concerns the representation of positive semidefinite elements (psd) of a ring A as sums of squares (sos) of elements of A. If $A = K$ is a field and $\frac{1}{2} \in A$, this problem has always a positive solution (psd=sos) due mainly to Artin-Scheier theory of (formally) real fields, which was developed with a view towards obtaining a solution to Hilbert's 17th problem. However, the situation is in general radically different when A is a (commutative unital) ring (with non-empty real spectrum), because strong dimensional restrictions appear and , apart from (formally) real fields, there are few real rings with the property psd=sos. This makes this type of rings special.

If A is an excellent ring of real dimension $\geqslant 3$, it is already known that it contains positive semidefinite elements that cannot be represented as sums of squares in A. In addition, if A is a local henselian noetherian ring (with non-empty real spectrum) such that every positive definite element of A is a sum of squares in A, then A is real reduced and consequently its real and Krull dimensions coincide. Thus, local excellent henselian rings with the property psd=sos have Krull dimension $\leqslant 2$.

In this work we determine all local excellent henselian rings A of embedding dimension $\leqslant 3$ such that $\frac{1}{2} \in A$ and have the property that every positive semidefinite element of A is a sum of squares of elements of A. We also present several families of examples (both irreducible and reducible) in higher embedding dimension that have the property psd=sos, a global-local criterion for the 2-dimensional analytic setting and an improvement of Scheiderer's result concerning principal saturated preorderings of low order.

Contents

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Part 1. Introduction

s1 1. Background and state of the art

The interest of Hilbert in the representation of positive semidefinite polynomials as sums of squares of polynomials goes back to Minkowski's thesis defense in 1885. Minkowski made the following assertion: "It is not likely that every nonnegative form can be represented by a sum of squares of forms" [H3, p.342]. Hilbert was at first sceptical, but later worked on this question and showed the existence of nonnegative forms that are not sums of squares of forms [H1]. In 1893 Hilbert proved his famous Nullstellensatz for algebraic closed fields in [H2, p.320] (see also [H4, Ch.4, p.234]), which concerns the computation of the ideal of all polynomials vanishing identically on an algebraic set $X \subset C^n$ by means of the radical ideal of any ideal of polynomials of $C[\mathtt{x}_1, \ldots, \mathtt{x}_n]$ that describes X, where C denotes an algebraically closed field. The next natural step was to obtain a Nullstellensatz for *real closed fields* R, that is, R is a not algebraically closed field such that $R[\sqrt{-1}]$ is an algebraically closed field. A real closed field R admits a unique ordering (compatible with its operations), whose set of non-negative elements coincides with the set R^2 of its squares. An initial natural obstruction for a real Nullstellensatz concerned sum of squares of polynomials and more generally positive semidefinite polynomials. The real Nullstellensatz was proved by Dubois and Risler in [D, R] and involves the use of the real radical ideal (1.2), see also [St]. The problem of representing positive semidefinite polynomials as sum of squares of rational forms (Hilbert's 17th Problem) was definitely posed in the 1900 International Congress of Mathematicians in Paris. It was solved by E. Artin in 1927 [Ar] using as a key tool Artin-Scheier's theory of (formally) real fields [AS1, AS2].

1.1. **The qualitative problem.** Hilbert's 17th Problem can be generalized in the natural way to rings of functions of any type (taking values in a real closed field). More generally, it can be stated for an arbitrary (commutative unital) ring A, but we have to establish before which elements of A are positive semidefinite. Such elements are defined by means of the theory of the real spectrum $\operatorname{Sper}(A)$ of A, as follows: An element $f \in A$ is *positive semidefinite* (psd) if $f \geqslant_\alpha 0$ for every prime cone $\alpha \in \operatorname{Sper}(A)$. Recall that a *prime cone* α can be understood as a pair $\alpha := (\mathfrak{p}_\alpha, \leqslant_\alpha)$ where $\mathfrak{p}_\alpha$ is a prime ideal of A (called the *support* $\operatorname{supp}(\alpha)$ of α) and $\leqslant_\alpha$ is an ordering of the quotient field $\operatorname{qf}(A/\mathfrak{p}_\alpha)$ (compatible with its operations). Alternatively, if $\mathfrak{R}(\alpha)$ is the *real closure* of the ordered field $(\operatorname{qf}(A/\mathfrak{p}_\alpha), \leqslant_\alpha)$ (that is, the smallest real closed field that contains $\operatorname{qf}(A/\mathfrak{p}_\alpha)$ as an ordered subfield), we identify α with the preimage under the canonical homomorphism $A \to \mathfrak{R}(\alpha)$ of the set $\mathfrak{R}(\alpha)^2$ of squares of $\mathfrak{R}(\alpha)$. We refer the reader to [BCR, §4.3] for further details. By [BCR, Thm.4.3.7] $\operatorname{Sper}(A) = \varnothing$ if and only if -1 is a finite sum of squares in A. In particular, if $A = K$ is a field, recall that K is *(formally) real* if its real spectrum $\operatorname{Sper}(K)$ is non-empty, that is, -1 is not a finite sum of squares in K.

Qualitative problem. To determine whether every positive semidefinite element of A is a sum of squares of elements of A.

We denote the set of positive semidefinite elements of A with $\mathcal{P}(A)$ and the set of all (finite) sums of squares (sos) of A with ΣA^2. The *qualitative problem* consists of determining under which conditions the equality $\mathcal{P}(A) = \Sigma A^2$ holds (that is, when psd=sos for A). If case $\operatorname{Sper}(A) = \varnothing$, we assume that all the elements of A are positive semidefinite and the problem consists of determining whether $A = \Sigma A^2$ or not.

Examples 1.1 (Real algebraic examples). Some remarkable examples of rings A with the property $\mathcal{P}(A) = \Sigma A^2$ in the algebraic setting over $\mathbb{R}$ are ring of polynomials of: (1) rational non-singular curves or compact irreducible curves whose real singular points are ordinary multiple points with independent tangents [Sch5, Thm.4.18] and (2) compact non-singular surfaces [Sch4, Cor.3.4]. The irreducible real algebraic curves with the property psd=sos are essentially the previous ones [Sch5, Thm.4.18], whereas the full characterization of all (possible reducible) real algebraic curves C over $\mathbb{R}$ with the property psd=sos was collected in [Pl]. More precisely, a real algebraic curve with the property psd=sos satisfies: (1) *all real singularities of C are ordinary multiple points with independent tangents*, (2) *the intersection points between the irreducible components of C are real*, (3) *the irreducible components of C that have no non-constant bounded polynomial functions are rational non-singular curves* and (4) *the configuration of the later irreducible components of C contains no loops.* On the other hand, by [FRS1, Main Thm.1.1] there exists no real algebraic set of dimension $\geqslant 3$ such that psd=sos. ▪

1.2. **Reduction to the case of (formally) real residue fields.** Assume first $\operatorname{Sper}(A) = \varnothing$ (or equivalently, -1 is a finite sum of squares in A). If in addition $\frac{1}{2} \in A$, then each $a \in A$ is a sum of squares in view of the well-known relation

$$a = \Big(\frac{a+1}{2}\Big)^2 + (-1)\Big(\frac{a-1}{2}\Big)^2. \tag{1.1}$$

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This happens for instance if A is a ring of odd characteristic n (because 2 is a unit, as $2\cdot(\frac{n+1}{2}) = 1$, and $-1 = 1^2 + \overset{(n-1)}{\cdots} + 1^2$).

If A is a ring of characteristic 2, sums of squares in A are squares in A (because double products are 0). Each element $a \in A$ is a square in A if and only if the Frobenius homeomorphism $\varphi : A \to A,\ x \mapsto x^2$ is surjective. The kernel of φ is the ideal of elements of A whose square is 0. Thus, if A is a finite field of characteristic 2, then the Frobenius map is surjective because it is injective (as its kernel is the zero ideal). If A is infinite, this is not necessarily true any more. Consider for instance $A = \mathbb{F}_2(\mathtt{x})$ and check that $\mathtt{x}$ is not a square.

1.2.1. *Artin's approximation.* To represent an element $a \in A$ as a sum of squares in A one has to find a solution in A^p for the equation $a = \mathtt{X}_1^2 + \cdots + \mathtt{X}_p^2$ for some $p \geqslant 1$. Thus, we have to solve in A^p polynomial equations in p variables with coefficients in A. In general it is easier to solve polynomial equations in complete rings, so it seems a good choice to work in the framework of Artin's approximation (see §2.1). Recall that a local Noetherian ring $(A, \mathfrak{m})$ satisfies *Artin's approximation* (AP) if every finite system of polynomial equations with coefficients in A has a solution in A if (and only if) it has a solution in the $\mathfrak{m}$-adic completion $\widehat{A}$ of A. Rotthaus proved in [Rt2] that a local Noetherian ring whose residue field has characteristic zero satisfies AP if and only if it is excellent and henselian. Thus, we will focus on *local excellent henselian rings.*

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1.2.2. *Local henselian rings.* Let us relate next the non-emptiness of the real spectrum of a local henselian ring $(A, \mathfrak{m})$ with the non-emptiness of the real spectrum of its residue field $\kappa := A/\mathfrak{m}$: *Let $(A, \mathfrak{m})$ be a local henselian ring. Then $\frac{1}{2} \in A$ and* $\operatorname{Sper}(A) \neq \varnothing$ *if and only if* $\operatorname{Sper}(\kappa) \neq \varnothing$, *or equivalently, if κ is a (formally) real field.*

Proof. Assume first $\frac{1}{2} \in A$ (that is, $2 \notin \mathfrak{m}$) and $\operatorname{Sper}(A) \neq \varnothing$. Let us check: $\operatorname{Sper}(\kappa) \neq \varnothing$.

Suppose $\operatorname{Sper}(\kappa) = \varnothing$, that is, $-1 \in \Sigma\kappa^2$. Thus, there exist elements $a_1, \ldots, a_q \in A\backslash\mathfrak{m}$ such that $-1 = a_1^2 + \cdots + a_q^2$ module $\mathfrak{m}$. As $2 \notin \mathfrak{m}$, we deduce a_1 is a simple root of the polynomial equation $\mathtt{t}^2 + a_2^2 + \cdots + a_q^2 + 1 = 0$ module $\mathfrak{m}$. As A is a henselian ring, we may assume that a_1 is a root of $\mathtt{t}^2 + a_2^2 + \cdots + a_q^2 + 1 = 0$ in A, so $-1 = a_1^2 + \cdots + a_q^2$ in A, which means that $\operatorname{Sper}(A) = \varnothing$, against our assumption.

Suppose now $\operatorname{Sper}(\kappa) \neq \varnothing$. Thus, κ is a (formally) real field and in particular it has characteristic zero, so $2 \notin \mathfrak{m}$. The natural map $\operatorname{Sper}(\kappa) \to \operatorname{Sper}(A)$ induced by the homomorphism $A \to \kappa,\ a \mapsto a + \mathfrak{m}$ implies that $\operatorname{Sper}(A) \neq \varnothing$, as required. □

Remark 1.2. If $(A, \mathfrak{m})$ is a local henselian ring and its residue field $\kappa := A/\mathfrak{m}$ has characteristic $q \geqslant 3$, then $2 \notin \mathfrak{m}$ and $-1 = 1^2 + \overset{(q-1)}{\cdots} + 1^2 \in \Sigma\kappa^2$, so $\operatorname{Sper}(\kappa) = \varnothing$. As $2 \notin \mathfrak{m}$ and A is a local ring, 2 is a unit in A, that is, $\frac{1}{2} \in A$. Summarizing we have both $\operatorname{Sper}(\kappa) = \varnothing$ and $\frac{1}{2} \in A$. We deduce (by §1.2.2) $\operatorname{Sper}(A) = \varnothing$, so using (1.1) $\mathcal{P}(A) = A = \Sigma A^2$. ▪

Again the situation is different if $\kappa := A/\mathfrak{m}$ is a field of characteristic 2. As usual, if $f \in \kappa[[\mathtt{x}]] := \kappa[[\mathtt{x}_1, \ldots, \mathtt{x}_n]]$, we denote the *order of* $f := \sum_{\nu:=(\nu_1,\ldots,\nu_n)} a_\nu \mathtt{x}^\nu$ with $\omega(f) := \inf\{|\nu| := \nu_1 + \cdots + \nu_n :\ a_\nu \neq 0\}$.

Example 1.3. If $A := \kappa[[\mathtt{x}_1, \ldots, \mathtt{x}_n]]/\mathfrak{a}$ is a formal ring such that $\mathfrak{a} \neq \mathfrak{m}$ and κ is a field of characteristic 2, the Frobenius map is not surjective, so $\mathcal{P}(A) = A \neq A^2 = \Sigma A^2$. ▪

Proof. We eliminate variables subsequently using the elements of $\mathfrak{a}$ of order 1 (by means of Weierstrass' division theorem [ZS, Ch.7.§.1.Thm.5]), so we may assume $\omega(\mathfrak{a}) := \inf\{\omega(f) :\ f \in \mathfrak{a}\} \geqslant 2$. As $\mathfrak{a} \neq \mathfrak{m}$, the number of variables $n \geqslant 1$. As $\omega(\mathtt{x}_1) = 1 < 2 = \omega(\mathfrak{a})$, we have $\mathtt{x}_1 \notin \mathfrak{a}$ and $\mathtt{x}_1 + \mathfrak{a} \in A = \mathcal{P}(A)$. If $\mathtt{x}_1 + \mathfrak{a} \in \Sigma A^2 = A^2$, there exist $a \in \kappa[[\mathtt{x}_1, \ldots, \mathtt{x}_n]]$ and $b \in \mathfrak{a}$ such that $\mathtt{x}_1 = a^2 + b$. As $\omega(b) \geqslant 2$, we have $1 = \omega(\mathtt{x}_1) = 2\omega(a)$, which is a contradiction. □

In the following we assume that the residue field κ of the local henselian ring $(A, \mathfrak{m})$ is (formally) real (but without further restrictions). This implies, in particular, as A is henselian that: *A contains a copy of the field $\mathbb{Q}$ of rational numbers.*

Proof. As A has characteristic zero, $\mathbb{Z} \hookrightarrow A$. Pick $m \in \mathbb{Z}\backslash\{0\}$ and consider the polynomial equation $m\mathtt{t} - 1$, which has a root in the residue field $\kappa := A/\mathfrak{m}$, because it is (formally) real. As the previous equation has degree 1 and A is henselian, there exists $b \in A$ such that $mb - 1 = 0$, that is, $\frac{1}{m} = b \in A$. Thus, $\mathbb{Q} \hookrightarrow A$, as required. □

1.2.3. *Real ideals and real (reduced) rings.* The concepts of real ideal and real (reduced) ring will be crucial along this work. Recall that an ideal $\mathfrak{a} \subset A$ is *real* if for each (finite) collection $a_1, \dots, a_r \in A$ such that $a_1^2 + \cdots + a_r^2 \in \mathfrak{a}$, we have $a_i \in \mathfrak{a}$ for $i = 1, \dots, r$. The *real-radical* of an ideal $\mathfrak{a} \subset A$ is the smallest real ideal $\sqrt[r]{\mathfrak{a}}$ of A that contains $\mathfrak{a}$. By [BCR, Prop.4.1.7]

$$\sqrt[r]{\mathfrak{a}} = \{a \in A : \ \exists\ a_1, \dots, a_r \in A, m \geqslant 1 \text{ such that } a^{2m} + a_1^2 + \cdots + a_r^2 \in \mathfrak{a}\}. \quad (1.2)$$ rri

A ring A is *real* (reduced) if its zero ideal is real. This implies in particular that if A is real (reduced), then $\operatorname{Sper}(A) \neq \varnothing$. The converse is not true in general (if A is not a field), but if in addition $\mathcal{P}(A) = \Sigma A^2$ holds and A is a *connected noetherian ring* (which happens for instance if A is a local noetherian ring), then A is real (reduced) [Sch2, Lem.6.3]. The *real reduction* of A is the quotient $A/\sqrt[r]{(0)}$. In case $A = K$ is a field, it is (formally) real (that is, $\operatorname{Sper}(K) \neq \varnothing$) if and only it is real (reduced). Thus, a prime ideal $\mathfrak{p}$ of a ring A is the support of a prime cone α if and only if it is a real (prime) ideal.

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1.3. **Dimensional restrictions for the qualitative problem.** A noetherian local ring A such that $\operatorname{Sper}(A) \neq \varnothing$ and $\mathcal{P}(A) = \Sigma A^2$ is a real (reduced) ring [Sch2, Lem.6.3]. In addition, strong dimensional restrictions appear in general even under mild hypotheses [Sch2, Cor.1.3]. In case we focus on excellent rings, we have the following result [FRS1, Main Thm.1.1] (see also [Fe3, Thm.1.2]).

mfrs **Theorem 1.4** ([FRS1, Main Thm.1.1])**.** *Let A be an excellent ring of real dimension $\geqslant 3$. Then $\mathcal{P}(A) \neq \Sigma A^2$.*

The previous result involves the concept of *real dimension.* Given two prime cones $\alpha, \beta \in \operatorname{Sper}(A)$, we say that α is a *specialization* of β (written $\beta \to \alpha$) if $f >_\alpha 0$ implies $f >_\beta 0$ for each $f \in A$. This implies $\mathfrak{q} := \operatorname{supp}(\beta) \subset \operatorname{supp}(\alpha) =: \mathfrak{p}$. We set $\dim(\beta \to \alpha) := \dim(A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p})$, and define the *real dimension* of A as

$$\dim_r(A) := \sup\{\dim(\beta \to \alpha)\colon\ \alpha,\ \beta \in \operatorname{Sper}(A),\ \beta \to \alpha\}.$$

Therefore, $\dim_r(A) \leqslant \dim(A)$.

dimr=dim

1.3.1. Let us show: *If $(A, \mathfrak{m})$ is a real (reduced) local henselian noetherian ring, then* $\dim_r(A) = \dim(A)$.

Proof. By [ABR, Prop.II.2.4] each non-refinable specialization chain finishes on a prime cone α whose support is $\mathfrak{m}$, so $A_{\operatorname{supp}(\alpha)} = A_\mathfrak{m} = A$. As A is real (reduced), all its minimal prime ideals are real [BCR, Lem.4.1.5]. Thus,

$$\begin{aligned}\dim_r(A) &= \sup\{\dim(A/\operatorname{supp}(\beta)) :\ \beta \in \operatorname{Sper}(A)\}\\ &= \max\{\dim(A/\mathfrak{p}) :\ \mathfrak{p} \text{ is a minimal prime ideal of } A\} = \dim(A),\end{aligned}$$

as required. □

Consequently, if A is a local excellent henselian ring with the property $\mathcal{P}(A) = \Sigma A^2$, then it is real (reduced) [Sch2, Lem.6.3] and by Theorem 1.4 we deduce $\dim(A) = \dim_r(A) \leqslant 2$.

1.3.2. *The case of dimension 0.* If A is a real (reduced) local noetherian ring of dimension 0, then it is a (formally) real field. The property $\mathcal{P}(K) = \Sigma K^2$ is true for fields K of characteristic different from 2. If the field K is not (formally) real (and has characteristic $q \geqslant 3$), we have already shown that $K = \Sigma K^2$. If K is (formally) real, $\mathcal{P}(K) = \Sigma K^2$ by the general theory of Artin-Schreier for (formally) real fields [BCR, §1]. The property $\mathcal{P}(Q(A)) = \Sigma Q(A)^2$ extends to the total ring of fractions $Q(A)$ of a real (reduced) Noetherian ring A using that $Q(A) = \prod_{i=1}^r Q(A/\mathfrak{p}_i)$ (where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the minimal primes of A) and that $\mathcal{P}(Q(A/\mathfrak{p}_i)) = \Sigma Q(A/\mathfrak{p}_i)^2$ for $i = 1, \ldots, r$, because each $Q(A/\mathfrak{p}_i)$ is by [BCR, Lem.4.1.5] a (formally) real field. The property $\mathcal{P}(Q(A)) = \Sigma Q(A)^2$ is still true by (1.1) when A is a reduced Noetherian ring, if $2 \notin \mathfrak{p}_i$ for $i = 1, \ldots, r$.

Thus, we will focus on real (reduced) local excellent henselian rings A of dimension either 1 or 2. In case A is a real (reduced) complete ring of finite dimension, then $A = \kappa[[\mathtt{x}_1, \ldots, \mathtt{x}_n]]/\mathfrak{a}$, where $\mathfrak{a}$ is a real ideal

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1.4. **The quantitative problem and Pythagoras numbers.** An element a of a ring A is a sum of squares in A if and only if there exists $q_a \geqslant 1$ such that the polynomial equation $\mathtt{X}_1^2 + \cdots + \mathtt{X}_{q_a}^2 - a = 0$ has a solution in A^{q_a}. In case $a \in \Sigma A^2$ it would be desirable to predict (or at least to estimate) the integer q_a (needed to represent a as a sum of squares) before trying to represent the element a as a sum of squares. This will allow to deal only with one equation $\mathtt{X}_1^2 + \cdots + \mathtt{X}_{q_a}^2 - a = 0$ for the element a that one wants to represent as sums of squares. A related invariant when dealing with sums of squares of a ring A is the *Pythagoras number* $p(A)$ of the ring A, which is the smallest integer $p \geqslant 1$ such that every sum of squares of A is a sum of p squares. We write $p(A) = +\infty$ if such an integer does not exist. If we denote the set of elements of A that are sums of p squares in A with $\Sigma_p A^2$, then $p(A) = \inf\{p \geqslant 1 : \Sigma A^2 = \Sigma_p A^2\}$. The Pyhtagoras number of a ring is a very delicate invariant whose estimation (*quantitative problem*) has deserved a lot of attention from specialists in number theory, quadratic forms, real algebra and real geometry [BCR, CDLR, CLRR, L, Pf1, Sch2, Sch3]. Again in this case strong dimensional restrictions appear. We proved in [FRS1] the following:

mfrs2 **Theorem 1.5** ([FRS1, Main Thm.1.1])**.** *Let A be an excellent ring of real dimension $\geqslant 3$. Then $p(A) = +\infty$.*

In particular, a real (reduced) local excellent henselian ring of dimension $\geqslant 3$ (and consequently of real dimension $\geqslant 3$) has an infinite Pythagoras number. In this article we focus on the qualitative problem and no new results concerning Pythagoras number appear. However, Pythagoras numbers will be a crucial invariant to apply Strong Artin's Approximation Theorem 5.6 along this article and we will survey below the main results we will need along this work.

1.5. **Background for dimension $\leqslant 2$.** We recall next the main known results concerning the qualitative and the quantitative problems for local excellent henselian rings of dimension $\leqslant 2$.

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1.5.1. *The* 2*-dimensional case.* The qualitative problem in this case was solved in [Fe8] when the residue field $\kappa := A/\mathfrak{m}$ admits a unique ordering (for instance, if $\kappa = \mathbb{Q}$ or $\kappa = R$ is a real closed field), $(A, \mathfrak{m})$ has embedding dimension $\leqslant 3$ and the Pythagoras number of A is finite.

unique **Theorem 1.6** ([Fe8, Cor.1.9])**.** *Let $(A, \mathfrak{m})$ be a local henselian excellent ring of dimension* 2 *and embedding dimension* 3*. Suppose that the residue field κ admits a unique ordering and $p(A) < +\infty$. Then $\mathcal{P}(A) = \Sigma A^2$ if and only if the completion $\widehat{A}$ is isomorphic to:*

(1) $\kappa[[\mathtt{x}, \mathtt{y}]]$ *or*
(2) $\kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/(\mathtt{zx}, \mathtt{zy})$ *or*
(3) $\kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/(\mathtt{z}^2 - F(\mathtt{x}, \mathtt{y}))$,

where $F \in \kappa[[\mathtt{x}, \mathtt{y}]]$ is one of the series in the following list:

(i) $a\mathtt{x}^2 + b\mathtt{y}^{2k}$ *such that* $a, b > 0$ *and* $k \geqslant 1$,
(ii) $a\mathtt{x}^2 + \mathtt{y}^{2k+1}$ *where* $a > 0$ *and* $k \geqslant 1$,
(iii) $a\mathtt{x}^2$ *where* $a > 0$,
(iv) $\mathtt{x}^2\mathtt{y} + (-1)^k a\mathtt{y}^k$ *where* $a > 0$ *and* $k \geqslant 3$,
(v) $\mathtt{x}^2\mathtt{y}$,
(vi) $\mathtt{x}^3 + a\mathtt{x}\mathtt{y}^2 + b\mathtt{y}^3$ *irreducible*,
(vii) $\mathtt{x}^3 + a\mathtt{y}^4$ *where* $a > 0$,
(viii) $\mathtt{x}^3 + \mathtt{x}\mathtt{y}^3$,
(ix) $\mathtt{x}^3 + \mathtt{y}^5$.

The previous theorem should be understood as a natural generalization of the results in [Fe2, Fe4, Fe5, Fe7, FR1, Rz2] concerning the case of analytic rings A (with coefficients in the field $\mathbb{R}$ of real numbers) of dimension 2 and embedding dimension $\leqslant 3$ that have the property $\mathcal{P}(A) = \Sigma A^2$. Such results were also adapted in [Fe6] to the framework of local henselian excellent rings with real closed residue field. The list in Theorem 1.6 (see also List 2.1 below) has a large history when the residue field is real closed. In [Fe4, Fe5] we proved that the local henselian excellent rings of dimension 2, embedding dimension 3, with real closed residue field and Pythagoras number 2 coincide essentially with those in the List in Theorem 1.6 (after taking into account that positive elements of a real closed field admit positive ℓth roots for each $\ell \geqslant 2$). In [Fe7] we showed 'positive extension properties' for the elements of the previous List when the residue field is real closed, whereas in [FR2] we analyze relations between the Pythagoras numbers of real analytic germs.

Concerning the quantitative problem we showed in [Fe1] that a local henselian excellent ring of dimension $\leqslant 2$ and real closed residue field κ has finite Pythagoras number. In [FRS2] we strongly improved the previous result and proved the following.

pyth **Theorem 1.7** ([FRS2, Prop.2.7 & Thm.2.9])**.** *Let $(A, \mathfrak{m})$ be a local henselian excellent ring of dimension 2 such that its residue field κ satisfies $p(\kappa[\mathtt{t}]) < +\infty$. Let m be the number of generators of the completion $\widehat{A}$ as a $\kappa[[\mathtt{x}_1, \mathtt{x}_2]]$-module. Then $p(A) \leqslant 2p(\kappa[\mathtt{t}])m$.*

We cannot assure that if a local henselian excellent ring A of dimension 2 with residue field κ has finite Pythagoras number, then $p(\kappa[\mathtt{t}])$ is also finite. However, as a kind of converse of Theorem 1.7, we proved in [FRS2, Prop.2.11] that if A has finite Pythagoras number, then κ has finite Pythagoras number and there exists a finite (formally) real extension $\kappa'|\kappa$ such that $p(\kappa'[\mathtt{t}])$ is also finite.

1.5.2. *The 1-dimensional case.* The qualitative problem for this case was solved by Scheiderer in [Sch3, §3] when the residue field $\kappa := A/\mathfrak{m}$ is real closed.

1dims **Theorem 1.8** ([Sch3, Thm.3.9])**.** *Let $(A, \mathfrak{m})$ be a one-dimensional local excellent henselian ring with real closed residue field κ and completion $\widehat{A}$. The following conditions are equivalent:*

(i) $\mathcal{P}(A) = \Sigma A^2$.
(ii) $\mathcal{P}(\widehat{A}) = \Sigma \widehat{A}^2$.
(iii) *There is $n \geqslant 1$ such that $\widehat{A} \cong \kappa[[\mathtt{x}_1, \ldots, \mathtt{x}_n]]/(\mathtt{x}_i\mathtt{x}_j : \ i < j)$.*

Concerning the quantitative problem in [FRS2, Rmk.2.1 & Lem.2.3] we proved that if $(A, \mathfrak{m})$ is a one-dimensional local excellent henselian ring, its Pythagoras number is finite if and only if its residue field $\kappa := A/\mathfrak{m}$ has finite Pythagoras number. In fact, if $\widehat{A}$ is the completion of A, then $p(A) = p(\widehat{A}) \leqslant p(\kappa)m$ ([FRS2, Rmk.2.1 & Lem.2.8]), where m is the (finite) minimal number of generators of $\widehat{A}$ as a $\kappa[[\mathtt{t}]]$-module (Noether's normalization).

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2. Statements of the main results for the 2-dimensional case

In [Fe8, Thm.1.5] we provided the list of candidates of local henselian excellent rings A with (formally) real residue field that have the property $\mathcal{P}(A) = \Sigma A^2$. We recall here such list of complete rings of embedding dimension $\leqslant 3$ (which generalizes the list provided in Theorem 1.6) for future quotation, see also [Li, §24-25].

lista **List 2.1** (General list of candidates)**.** For each (formally) real field consider the following list of complete rings:

(1) $\kappa[[\mathtt{x},\mathtt{y}]]$,
(2) $\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{zx},\mathtt{zy})$,
(3) $\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2 - F(\mathtt{x},\mathtt{y}))$

where $F \in \kappa[[\mathtt{x},\mathtt{y}]]$ is one of the series in the following list:

(i) $a\mathtt{x}^2 + b\mathtt{y}^{2k}$ where $a \notin -\Sigma\kappa^2$, $b \neq 0$ and $k \geqslant 1 \rightsquigarrow (A_{2k-1})$,
(ii) $a\mathtt{x}^2 + \mathtt{y}^{2k+1}$ where $a \notin -\Sigma\kappa^2$ and $k \geqslant 1 \rightsquigarrow (A_{2k})$,
(iii) $a\mathtt{x}^2$ where $a \notin -\Sigma\kappa^2 \rightsquigarrow (A_\infty)$,
(iv) $\mathtt{x}^2\mathtt{y} + (-1)^k a\mathtt{y}^k$ where $a \notin -\Sigma\kappa^2$ and $k \geqslant 3 \rightsquigarrow (D_{k+1})$,
(v) $\mathtt{x}^2\mathtt{y} \rightsquigarrow (D_\infty)$,
(vi) $\mathtt{x}^3 + a\mathtt{x}\mathtt{y}^2 + b\mathtt{y}^3$ irreducible $\rightsquigarrow (D_4)$,
(vii) $\mathtt{x}^3 + a\mathtt{y}^4$ where $a \notin -\Sigma\kappa^2 \rightsquigarrow (E_6)$,
(viii) $\mathtt{x}^3 + \mathtt{x}\mathtt{y}^3 \rightsquigarrow (E_7)$,
(ix) $\mathtt{x}^3 + \mathtt{y}^5 \rightsquigarrow (E_8)$.

In [Fe8, Thm.1.5] we proved the following result.

gp=s **Theorem 2.2** (Necessary condition, [Fe8, Thm.1.5])**.** *Let $(A,\mathfrak{m})$ be a local henselian excellent ring of dimension* 2 *and embedding dimension* 3 *such that its residue field $\kappa := A/\mathfrak{m}$ is (formally) real. If $\mathcal{P}(A) = \Sigma A^2$, then the completion $\widehat{A}$ is isomorphic to one of the complete rings appearing in List* 2.1*.*

One main purpose of this article is to prove the converse of Theorem 2.2. Namely,

list2 **Theorem 2.3** (Sufficient condition)**.** *Let $(A,\mathfrak{m})$ be a local henselian excellent ring of dimension* 2 *and embedding dimension* 3 *such that its residue field $\kappa := A/\mathfrak{m}$ is (formally) real. Assume that the completion $\widehat{A}$ is isomorphic to one of the rings in List* 2.1*. Then $\mathcal{P}(A) = \Sigma A^2$.*

Summarizing we have the following:

Main Theorem 2.4 (Full characterization)**.** *Let $(A,\mathfrak{m})$ be a local henselian excellent ring of dimension* 2 *and embedding dimension* 3 *such that its residue field $\kappa := A/\mathfrak{m}$ is (formally) real. Then, $\mathcal{P}(A) = \Sigma A^2$ if and only if the completion $\widehat{A}$ is isomorphic to one of the complete rings appearing in List* 2.1*.*

In [Sch3, Thm.4.1] it was shown Theorem 2.3 for the case (1) of List 2.1. In [Fe8, Thm.3.11] we proved Theorem 2.3 for the case (2) of List 2.1, whereas in [Fe8, Thm.5.9] we proved Theorem 2.3 for the cases (3.iv-3.ix) of List 2.1 (the order 3 cases) when $p(A) < +\infty$. A main step of the current work is to reduce the proof of Theorem 2.3(3) to the situation when the Pythagoras number $p(A)$ of A is finite. Under this assumption we approach also the remaining cases (3.i-3.iii) of List 2.1. We improve in Theorem 8.6 the key result [Fe8, Thm.5.5] to prove cases (3.iv-3.ix) of List 2.1 and we use it also to prove the remaining case (3.ii) in Theorem 9.8. Once this is done we approach cases (3.i) and (3.iii) by means of Theorems 8.7, 9.10 and 9.11.

The proof of Theorem 2.3 is quite involved and requires an extensive use of Rotthaus results [Rt1] on Artin's approximation Theorem. This forces us to work in the framework of local henselian excellent rings $(A,\mathfrak{m})$ that contain a copy of $\mathbb{Q}$. This holds under our hypothesis, because the residue field κ is (formally) real (recall that $\operatorname{Sper}(A)\neq\varnothing$, $\frac{1}{2}\in A$ and see §1.2.2), so $\mathbb{Z}\subset A\setminus\mathfrak{m}$ and as A is local, $\mathbb{Q}\subset A\setminus\mathfrak{m}$.

sap

2.1. **Strong Artin's approximation and bounded Pythagoras numbers.** To prove the property $\mathcal{P}(A)=\Sigma A^2$ for the rings $A=\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F)$ in List 2.1 we will reduce the general situation to the case of local henselian excellent rings with finite Pythagoras number. This is due to the use of Strong Artin's approximation (Theorem 5.6) we make: *To represent $f+\mathtt{z}g\in\mathcal{P}(A)$ (where $f,g\in\kappa[[\mathtt{x},y]]$) as a sum of squares in A we find for each $n\geqslant 1$ elements $f_n+\mathtt{z}g_n\in\Sigma A^2$ (where $f_n,g_n\in\kappa[[\mathtt{x},y]]$) such that $\omega(f-f_n),\omega(g-g_n)\geqslant n$.* Thus, there exist $a_{ni},b_{ni},q_n\in\kappa[[\mathtt{x},\mathtt{y}]]$ for $i=1,\dots,p_n$ such that

$$f_n+\mathtt{z}g_n=\sum_{i=1}^{p_n}(a_{ni}+\mathtt{z}b_{ni})^2+(\mathtt{z}^2-F)q_n.$$

Consequently, the polynomial equation

$$f+\mathtt{z}g=(\mathtt{X}_1+\mathtt{z}\mathtt{Y}_1)^2+\cdots+(\mathtt{X}_{p_n}+\mathtt{z}\mathtt{Y}_{p_n})^2+(\mathtt{z}^2-F)\mathtt{Z}$$

has a solution $\bmod\ \mathfrak{m}_2^n$ in $\kappa[[\mathtt{x},\mathtt{y}]]$ for each $n\geqslant 1$. To apply Strong Artin's approximation we need that the sequence of positive integers $\{p_n\}_{n\geqslant 1}$ is bounded by some positive integer $p<+\infty$. This fact implies that the polynomial equation

$$f+\mathtt{z}g=(\mathtt{X}_1+\mathtt{z}\mathtt{Y}_1)^2+\cdots+(\mathtt{X}_p+\mathtt{z}\mathtt{Y}_p)^2+(\mathtt{z}^2-F)\mathtt{Z}$$

has a solution $\bmod\ \mathfrak{m}_2^n$ in $\kappa[[\mathtt{x},\mathtt{y}]]$ for each $n\geqslant 1$. Now, by Strong Artin's approximation there exist $a_i,b_i,q\in\kappa[[\mathtt{x},\mathtt{y}]]$ such that

$$f+\mathtt{z}g=(a_1+\mathtt{z}b_1)^2+\cdots+(a_p+\mathtt{z}b_p)^2+(\mathtt{z}^2-F)q,$$

so $f+\mathtt{z}g\in\Sigma_pA^2\subset\Sigma A^2$.

To guarantee that the sequence of positive integers $\{p_n\}_{n\geqslant 1}$ associated to each $f+\mathtt{z}g\in\mathcal{P}(A)$ is bounded, a sufficient condition is to ask that the Pythagoras number $p(A)$ of the ring A is finite. The main purpose of Section 6 is to reduce the general case to the case when $p(A)$ is finite in order to get advantage of the shown strength of Strong Artin's approximation.

taue

2.2. **Pythagoras numbers and τ-invariant.** We have seen in Theorem 1.7 that to bound the Pythagoras number of a local henselian excellent ring A of dimension 2 with (formally) real residue field κ, it is almost mandatory to bound the Pythagoras number $p(\kappa[\mathtt{t}])$, or equivalently, by a celebrated result of Cassels [C] to bound $p(\kappa(\mathtt{t}))$. The invariant $p(\kappa[\mathtt{t}])$ has been bounded by Scheiderer in [Sch3] following [Pf1]. For a (formally) real field κ define the invariant

$$\tau(\kappa):=\sup\{s(F):\ F|\kappa \text{ finite, non (formally) real}\},$$

where $s(F)$ denotes the *level of* F, that is, the minimum number of elements of F needed to represent -1 as a sum of squares in F, which is always a power of 2, see [L, Pfister's Thm.XI.2.2]. Scheiderer proposed in [Sch3, Prop.5.17] (using in an essential way Pfister's results [Pf1]) the following chain of inequalities:

$$1+\tau(\kappa)\leqslant p(\kappa[\mathrm{y}])=p(\kappa(\mathrm{y}))\leqslant p(\kappa[[\mathrm{x},\mathrm{y}]])\leqslant p(\kappa[[\mathrm{x}]][\mathrm{y}])\leqslant 2\tau(\kappa),\tag{2.1}$$ p2

see also Appendix B. In addition, $\tau(\kappa)=\tau(\kappa((\mathrm{x})))$, see [Sch3, Lem.5.13]. The Pythagoras number $p(\kappa[[\mathrm{x},\mathrm{y}]])$ has been also studied by Hu in [Hu, §3] where he showed that

$$\begin{aligned}p(\kappa[[\mathrm{x},\mathrm{y}]])&=p(\kappa[\mathrm{x}][[\mathrm{y}]])=p(\kappa[[\mathrm{x}]][\mathrm{y}])=p(\kappa((\mathrm{x},\mathrm{y})))=p(\mathrm{qf}(\kappa[\mathrm{x}][[\mathrm{y}]]))\\&=p(\kappa((\mathrm{x}))(y))=\sup\{p(K(\mathrm{x})):\ K|\kappa \text{ is a finite field extension}\}.\end{aligned}$$

The last equality is due to Becher-Grimm-Van Geel [BGV]. It is conjectured in [BGV, Conj.4.16] that the inequality $p(\kappa[\mathrm{y}])\leqslant p(\kappa((\mathtt{x}))(\mathtt{y}))$ is in fact an equality or, equivalently, that $p(K(\mathtt{x}))\leqslant p(\kappa(\mathtt{x}))$ for each finite extension $K|\kappa$. However, it is still an open problem whether $p(\kappa)<+\infty$

implies $p(\kappa(\mathtt{t}))<+\infty$ (or equivalently, $\tau(\kappa)<+\infty$) for every (formally) real field κ. Recall for instance that $4=p(\mathbb{Q})\leqslant p(\mathbb{Q}[\mathtt{y}])=p(\mathbb{Q}((\mathtt{x},\mathtt{y})))=p(\mathbb{Q}[[\mathtt{x},\mathtt{y}]])=5$ (see [P] and [Hu, Rem.3.5]), so $\tau(\mathbb{Q})=4$ (use (2.1)).

mconj

2.2.1. *Pythagoras numbers and τ-invariants of fields with finite transcendence degree.* We recall next some bounds concerning $p(\kappa)$ and $\tau(\kappa)$, when κ is a field of finite transcendence degree over either $\mathbb{Q}$ or a real closed field R. Let κ be a field and denote the *virtual cohomological 2-dimension of* κ with $e:=\operatorname{vcd}_2(\kappa)=\operatorname{cd}_2(\kappa(\sqrt{-1}))$, see [S, §3] and [Sch1, §7 & §9] for further details concerning the cohomological 2-dimension cd_2 of a field κ. Milnor's conjecture (proved by Voevodsky [V1, V2]) together with Pfister's result [Pf2, Chap.6, Thm.3.3] imply that $p(\kappa)\leqslant 2^e$, see [Hu, §5] or [Sch3, Rem.5.14] for a more detailed explanation. If $\kappa|\mathbb{Q}$ is an extension of transcendence degree $d\geqslant 0$ (over $\mathbb{Q}$), we have by [Pf3, §4, p.37] $e\leqslant d+2$, so $p(\kappa)\leqslant 2^{d+2}$ (see also [Pf3, pp. 37–38]). Consequently, $1+\tau(\kappa)\leqslant p(\kappa(\mathtt{t}))\leqslant 2^{d+3}$, so $\tau(\kappa)\leqslant 2^{d+2}$, because $\tau(\kappa)$ is a power of 2.

Next, if $\kappa|R$ is a finitely generated extension of transcendence degree d over a real closed field R, we have by [AGV, Ch.X.Thm.2.1] $e:=\operatorname{cd}_2(\kappa(\sqrt{-1}))\leqslant\operatorname{cd}_2(R(\sqrt{-1}))+d=d$. Recall here that $\operatorname{cd}_2(R(\sqrt{-1}))=0$, because $R(\sqrt{-1})$ is an algebraically closed field and its total Galois group is trivial. Thus, $p(\kappa)\leqslant 2^d$ and $1+\tau(\kappa)\leqslant p(\kappa(\mathtt{t}))\leqslant 2^{d+1}$, so $\tau(\kappa)\leqslant 2^d$. For alternative proofs of these bounds see original Pfister's Theorem [Pf1] or [L, XI.Thm.4.10].

If $\kappa|\mathbb{Q}$ is a field extension of transcendence degree d, the bound $p(\kappa)\leqslant 2^{d+2}$ for $d\geqslant 2$ was improved by Jannsen [J2, Cor.0.7]. He proved

$$p(\kappa)\leqslant 2^{d+1}\quad\text{for all } d\geqslant 2, \tag{2.2}$$ subound

using Kato's Hasse principle for Galois cohomology of degree $d+2$ of κ with coefficients $\mathbb{Z}_2$ (see [CJ, J1, J2, K]). In Colliot-Thélène's appendix to [K] it is shown that $p(\kappa)\leqslant 4$ if $d=0$ and $p(\kappa)\leqslant 7$ if $d=1$. The sharper bound $p(\kappa)\leqslant 6$ for $d=1$ is contained in an unpublished preprint of Pop, according to [Pf2, p.100]).

Consequently, if $d\geqslant 1$, we have $1+\tau(\kappa)\leqslant p(\kappa(\mathtt{t}))\leqslant 2^{d+2}$, so $\tau(\kappa)\leqslant 2^{d+1}$. In case $d=0$ (that is, κ is contained in the field of real algebraic numbers $\overline{\mathbb{Q}}^r$), then $\tau(\kappa)+1\leqslant p(\kappa[\mathtt{t}])=p(\kappa(\mathtt{t}))\leqslant 6$, so $\tau(\kappa)\leqslant 4$, that is, either $\tau(\kappa)=1,2$ or 4 (and all the values are valid). Namely, as $\overline{\mathbb{Q}}^r$ is a real closed field, $\tau(\overline{\mathbb{Q}}^r)=1$. Let $\kappa\subset\overline{\mathbb{Q}}^r$ be a subfield of finite degree over $\mathbb{Q}$. If $p(\kappa)=3$, then $p(\kappa(\mathtt{t}))=4$ by [Pf2, Ch.7, Thm.1.9] and $1+\tau(\kappa)\leqslant 4\leqslant 2\tau(\kappa)$, so $\tau(\kappa)=2$ (which happens for instance if $\kappa=\mathbb{Q}[\sqrt{5}]$ by [Pf2, Ch.7, Ex.1.4(3)]). Analogously, if $p(\kappa)=4$, we have by [Pf2, Ch.7, Thm.1.9] $1+\tau(\kappa)\leqslant p(\kappa(\mathtt{t}))=5\leqslant 2\tau(\kappa)$, so $\tau(\kappa)=4$ (which happens for instance if $\kappa=\mathbb{Q}$).

2.3. **Reduction via Tougeron's Implicit Function Theorem.** To reduce the proof of Theorem 2.3 to the case when A has finite Pythagoras number, we will make a sophisticated use in Section 6 of Tougeron's Implicit Function Theorem 7.1: We essentially show (under some mild conditions) that if $f+\mathtt{z}g\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F))$, there exists a finitely generated extension of fields $\kappa_1|\mathbb{Q}$ (subextension of $\kappa|\mathbb{Q}$) such that after a change of coordinates (and after multiplication by suitable units of $\kappa[[\mathtt{x},\mathtt{y}]]$) we may assume $f,g,F\in\kappa_1[[\mathtt{x},\mathtt{y}]]$ and $f+\mathtt{z}g\in\mathcal{P}(\kappa_1[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F))$. As $\kappa_1|\mathbb{Q}$ is a finitely generated extension, say by d elements, it has transcendence degree $\leqslant d$ and $p(\kappa_1(\mathtt{t}))\leqslant 2\tau(\kappa_1)\leqslant 2^{\max\{d+2,3\}}$. By Theorem 1.7 the Pythagoras number $p(\kappa_1[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F))\leqslant 2^{\max\{d+4,5\}}$ is finite! and we will be able to use Strong Artin's approximation as we have announced in §2.1.

As one can expect, the field κ_1 depends on each pair $(F,f+\mathtt{z}g)$, so we do not have further information concerning the Pythagoras number of the ring $\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F)$, which may be infinite. As commented above, we only know how to bound such Pythagoras number if the field $\kappa(\mathtt{t})$ has finite Pythagoras number, that is, if $\tau(\kappa)<+\infty$ (see §2.2 and §2.2.1).

2.4. **Higher embedding dimension and further applications.** A full classification of all local excellent henselian rings of dimension 2 with the property psd=sos (for every embedding dimension) is out of our scope. The strategy we have followed is *ad hoc* for embedding dimension $\leqslant 3$ and lies strongly to the fact that most of them are described using a single equation. In fact, the 1-dimensional case of embedding dimension 3 (where the corresponding 'zero ideals' are not principal) show the difficulty of dealing with the general situation without understanding the deep meaning of having the property psd=sos. The existence of single equations to describe unmixed local excellent henselian rings of dimension 2 and embedding dimension $\leqslant 3$ is crucial also to reduce the general case to the case of finite Pythagoras number (as we will see in Section 7). However, we present in Section 15 several families of examples of dimension 2 with the property psd=sos in higher embedding dimension and in Section 16 some obstructions to have the property psd=sos in higher embedding dimension (Corollary 16.1).

We also present in Section 17 a global local property (Theorem 17.1) that provides information about the completions of the local rings of an excellent ring that has the property psd=sos (see also Theorem 17.9 and its application to embedding dimensions 3 and 4 using Main Theorem 3.1 below). As a byproduct we also obtain an improvement of [Sch1, Cor.4.6] (see Corollary 17.7). The global local property (Theorem 17.1) proposed in this article is a generalization of the one proposed in [Fe7, Thm.1.5] to prove one of its main results [Fe7, Thm.1.6] (Theorem 17.8).

As a further application we study in Section 18 principal saturated preorderings of low order and we generalize some results of Scheiderer [Sch6] for general (formally) real fields κ with $\tau(\kappa) < +\infty$, which are not necessarily real closed fields like in [Sch6]. The results we present here (Corollary 18.2) improve also a weaker version of Corollary 18.2 proposed in [Fe8, §6] where we additional needed to impose that κ has a unique ordering.

We also in Section 18 with the transference of sums of squares when extending coefficients to a real closed field and afterwards descending. This is somehow related to a question formulated by Sturmfels to Scheiderer concerning polynomial with rational coefficients that are sums of squares of polynomials with real coefficients [Sch6]. A weaker approach to this problem in our setting was already presented in [Fe8, Rem.5.10(iii)].

s3 3. Statements of the main results for the 1-dimensional case

The one dimensional case is straighforwardly studied when the embedding dimension of the local excellent henselian ring A with the property $\mathcal{P}(A) = \Sigma A^2$ is $\leqslant 2$. If the embedding dimension is 1, the completion $\widehat{A} \cong \kappa[[\mathtt{t}]]$, whereas if it is 2, then $\widehat{A} \cong \kappa[[\mathtt{x},\mathtt{y}]]/(\mathtt{y}^2 - a\mathtt{x}^2)$ where $a \notin -\Sigma\kappa^2$. The situation turns out more complicated if the embedding of A is equal to 3. In this case $\widehat{A} \cong \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/\mathfrak{a}$ for some real radical ideal $\mathfrak{a}$ of $\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ (recall [Sch2, Lem.6.3]). A priori there is no bound for the number of generators of $\mathfrak{a}$ and no clue concerning their shape (we approach this fact in Theorem 3.5). The techniques used by Scheiderer to prove Theorem 1.8 rely strongly on the fact that the residue field is real closed and cannot be generalized to arbitrary (formally) real fields.

maindim1gk **Main Theorem 3.1** (Paddington's Theorem)**.** *Let $(A,\mathfrak{m})$ be a 1-dimensional local excellent henselian ring of embedding dimension $n \leqslant 3$ and denote $\mathtt{x} := (\mathtt{x}_1,\dots,\mathtt{x}_n)$. Denote $\kappa := A/\mathfrak{m}$ its residue field and assume that κ is (formally) real, denote $\overline{\kappa}$ the algebraic closure of κ and let $\widehat{A} \cong \kappa[[\mathtt{x}]]/\mathfrak{a}$ be the completion of A. Then $\mathcal{P}(A) = \Sigma A^2$ if and only if either:*

(i) (Expected cases:) *A is real (reduced) and $\widehat{A} \otimes_{\kappa[[\mathtt{x}]]} \overline{\kappa}[[\mathtt{x}]] = \overline{\kappa}[[\mathtt{x}]]/\mathfrak{a}\overline{\kappa}[[\mathtt{x}]]$ is isomorphic to $\overline{\kappa}[[\mathtt{w}_1,\dots,\mathtt{w}_n]]/(\mathtt{w}_i\mathtt{w}_j : \ i \neq j)$, or*
(ii) (Unexpected case:) *$A \cong \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/\mathfrak{a}$ where $\mathfrak{a} := (\mathtt{z}^2 - \mathtt{x}\mathtt{y}, \mathtt{y}^2 - 2b\mathtt{y}\mathtt{x} - 4c^2\mathtt{z}\mathtt{x} + d\mathtt{x}^2)$ and $P := \mathtt{t}^4 - 2b\mathtt{t}^2 - 4c^2\mathtt{t} + d \in \kappa[\mathtt{t}]$ is a chimeric polynomial.*

An irreducible polynomial $P \in \kappa[\mathtt{t}]$ is *chimeric (over κ)* if $\kappa[\mathtt{t}]/(P)$ is a (formally) real field and for each $Q := a_3\mathtt{t}^3 + a_2\mathtt{t}^2 + a_1\mathtt{t} + a_0 \in \mathcal{P}(\kappa[\mathtt{t}]/(P))$ (of degree $\leqslant 3$) there exists $\mu \in \Sigma\kappa^2$ such that $Q + \mu P \in \mathcal{P}(\kappa[\mathtt{t}])$. By Lemma 14.1 $P(\eta) \in \Sigma\kappa^2$ for each $\eta \in \kappa$ and by Lemma 14.8 P has either zero or four roots in $\mathfrak{R}(\alpha)$ for each $\alpha \in \operatorname{Sper}(\kappa)$.

We will prove in Sections 9 to 12 the following equivalent version of Theorem 3.1.

maindim1 **Theorem 3.2.** *Let $(A, \mathfrak{m})$ be a 1-dimensional local excellent henselian ring of embedding dimension $\leqslant 3$, denote $\kappa := A/\mathfrak{m}$ its residue field, assume that it is (formally) real and let $\widehat{A}$ be the completion of A. Then $\mathcal{P}(A) = \Sigma A^2$ if and only if $\widehat{A}$ is isomorphic to $\kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/\mathfrak{a}$ where $\mathfrak{a}$ is one of the following ideals:*

(i) $\mathfrak{a} := (\mathtt{y}, \mathtt{z})$.
(ii) $\mathfrak{a} := (\mathtt{y}^2 - a\mathtt{x}^2, \mathtt{z})$ *for some* $a \notin -\Sigma\kappa^2$.
(iii) $\mathfrak{a} := (\mathtt{y}^2 - a\mathtt{x}^2, \mathtt{xz}, \mathtt{yz})$ *for some* $a \notin -\Sigma\kappa^2$.
(iv) $\mathfrak{a} := (\mathtt{y}^2 - \mathtt{xz}, \mathtt{yz} + p\mathtt{yx} + q\mathtt{x}^2, q\mathtt{xy} + p\mathtt{xz} + \mathtt{z}^2)$ *where* $P := \mathtt{t}^3 + p\mathtt{t} + q \in \kappa[\mathtt{t}]$ *is irreducible.*
(v) $\mathfrak{a} := (\mathtt{z}^2 - \mathtt{xy}, \mathtt{y}^2 - 2b\mathtt{yx} - 4c^2\mathtt{zx} + d\mathtt{x}^2)$ *where* $P := \mathtt{t}^4 - 2b\mathtt{t}^2 - 4c^2\mathtt{t} + d \in \kappa[\mathtt{t}]$ *is a chimeric polynomial.*

Remark 3.3. As we already know from Theorem 1.8, cases (iv) and (v) do not appear when $\kappa = R$ is a real closed field, because irreducible polynomials in $R[\mathtt{t}]$ have degree $\leqslant 2$. ▪

The difficult point to prove Theorem 3.2 is its 'only if' part. To that end, one has to show that for each 1-dimensional local excellent henselian ring A of embedding dimension $\leqslant 3$ such that $\widehat{A}$ is not isomorphic to one of the complete rings quoted above there exists $f \in \mathcal{P}(\widehat{A})\backslash\Sigma A^2$. In all cases we have found such an f of order $\leqslant 2$ (contrast this fact with Lemma 10.3). In many cases there are few elements in the difference $\mathcal{P}(\widehat{A})\backslash\Sigma A^2$ and we have felt as 'finding needles in a haystack'.

3.1. Minimal systems of generators of zero ideals for low embedding dimension.

Initial steps to prove Theorem 3.2 concern the finding of minimal systems of generators of zero ideals:

generators2 **Lemma 3.4.** *Let $(A, \mathfrak{m})$ be a 1-dimensional local excellent henselian ring of embedding dimension $\leqslant 2$, let $\kappa := A/\mathfrak{m}$ and let $\widehat{A}$ be its completion. Suppose $\mathcal{P}(A) = \Sigma A^2$. Then either $\widehat{A} \cong \kappa[[\mathtt{x}]]$ or $\widehat{A} \cong \kappa[[\mathtt{x}, \mathtt{y}]]/(\mathtt{y}^2 - a\mathtt{x}^2)$ for some $a \notin -\Sigma\kappa^2$.*

generators **Theorem 3.5** (Minimal system of generators)**.** *Let $(A, \mathfrak{m})$ be a 1-dimensional local excellent henselian ring of embedding dimension 3, let $\kappa := A/\mathfrak{m}$ be its (formally) real residue field and let $\widehat{A} = \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/\mathfrak{a}$ be its completion. Suppose $\mathcal{P}(A) = \Sigma A^2$. Then (after a change of coordinates) either:*

(i) $\mathfrak{a} := (\mathtt{z}^2 - \mathtt{xy}, F + \mathtt{z}G)$ *for some* $F, G \in \kappa[[\mathtt{x}, \mathtt{y}]]$ *with* $\omega(F) = 2$ *and* $\omega(G) \geqslant 1$, *or*
(ii) $\mathfrak{a} := (\mathtt{z}^2 - \mathtt{xy}, (\mathtt{z} + F_1)G_2 - F_2G_1, (\mathtt{z} - F_1)G_1 - F_3G_2)$ *for some series* $F_i, G_j \in \kappa[[\mathtt{x}, \mathtt{y}]]$ *such that* $\omega(F_2) = \omega(F_3) = 1$, $\mathtt{xy} = F_1^2 + F_2F_3$ *and either* $\omega(G_1) = 1$ *or* $\omega(G_2) = 1$.

In both cases $\mathfrak{a} \cap \kappa[[\mathtt{x}, \mathtt{y}]] = H\kappa[[\mathtt{x}, \mathtt{y}]]$ is a principal ideal. In addition, in case (i)

$$H = \begin{cases} F & \text{if } F \text{ divides } G, \\ (F_*^2 - \mathtt{xy}G_*^2)D & \text{if } D = \gcd(F, G),\ \omega(D) \leqslant 1,\ F = DF_* \text{ and } G = DG_*, \end{cases}$$

whereas in case (ii)

$$H = \begin{cases} 2F_1G_1Q_2 - F_2G_1 + F_3G_1Q_2^2 & \text{if } \gcd(G_1, G_2) = G_1 \text{ and } G_2 = G_1Q_2, \\ 2F_1Q_1G_2 - F_2Q_1^2G_2 + F_3G_2 & \text{if } \gcd(G_1, G_2) = G_2 \text{ and } G_1 = Q_1G_2, \\ 2F_1G_1G_2 - F_2G_1^2 + F_3G_2^2 & \text{if } G_1, G_2 \text{ are relatively prime,} \end{cases}$$

where H divides the series $P := 2F_1G_1G_2 - F_2G_1^2 + F_3G_2^2 \in \mathfrak{a} \cap \kappa[[\mathtt{x},\mathtt{y}]]$ of order 3 *and P divides $F^2 - \mathtt{xy}G^2$ as well.*

3.2. **Higher embedding dimension.** Again obtaining a full satisfactory result for higher embedding dimension for 1-dimensional local excellent henselian rings is out of our scope. In Corollary 11.4 we prove that if $A := \kappa[[\mathtt{x}_1,\dots,\mathtt{x}_n]]/\mathfrak{a}$ has the property $\mathcal{P}(A) = \Sigma A^2$ and embedding dimension n, then the ideal $\mathfrak{a}$ contains (at least) $n-1$ elements of order 2 whose leading forms are κ-linearly independent. Having in mind Scheiderer's Theorem 1.8, Main Theorem 3.2 and Conjecture 3.7 we conjecture that for arbitrary embedding dimension n it holds a kind of following statement:

conj2 *Conjecture* 3.6. Let $(A,\mathfrak{m})$ be a 1-dimensional local excellent henselian ring of embedding dimension n, denote $\kappa := A/\mathfrak{m}$ its residue field, $\overline{\kappa}$ its algebraic closure, $\mathtt{x} := (\mathtt{x}_1,\dots,\mathtt{x}_n)$ and let $\widehat{A} \cong \kappa[[\mathtt{x}]]/\mathfrak{a}$ be the completion of A. Then $\mathcal{P}(A) = \Sigma A^2$ if and only if A is real (reduced) and $\widehat{A} \otimes_{\kappa[[\mathtt{x}]]} \overline{\kappa}[[\mathtt{x}]] \cong \overline{\kappa}[[\mathtt{x}]]/\mathfrak{a}\overline{\kappa}[[\mathtt{x}]] \cong \overline{\kappa}[[\mathtt{w}_1,\dots,\mathtt{w}_n]]/(\mathtt{w}_i\mathtt{w}_j :\ i \neq j)$.

We refer the reader to Example 10.4 in relation with Conjecture 3.6(i).

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3.3. **Chimeric polynomials.** Despite our strong efforts, we have found no (formally) real field κ such that $\kappa[\mathtt{t}]$ contains chimeric polynomials and in fact we conjecture:

conj3 *Conjecture* 3.7. There exists no chimeric polynomial over any (formally) real field.

Equivalently, this would mean that case (ii) in Main Theorem 3.1 and case (v) in Theorem 3.2 do not occur. The proof of Conjecture 3.7 for a general (formally) real field seems to us a very difficult task. However, we expose next our evidences to propose such conjecture:

(i) Define $\mathfrak{D}_\kappa$ the set of orderings α of κ such that κ is dense in $\mathfrak{R}(\alpha)$. If $\mathfrak{D}_\kappa$ is dense in $\operatorname{Sper}(\kappa)$, there exists by Lemma 14.11 no chimeric polynomial $P \in \kappa[\mathtt{t}]$. If κ is the quotient field of a finitely generated $\mathbb{Q}$-algebra that is a real integral domain, $\mathfrak{D}_\kappa$ is dense in $\mathfrak{R}(\alpha)$ (Lemma 14.12). Examples of (formally) real fields κ such that $\mathfrak{D}_\kappa = \operatorname{Sper}(\kappa)$ are: fields such that all its orderings are Archimedean, real closed fields, pseudo real closed fields [Pr1], pseudo classically closed fields [Pop] and RC_π-fields [Er]. See [ADF] for further details concerning this type of fields.

(ii) If all the orderings of a (formally) real field κ are Archimedean, we show in Theorem 14.14 (whose proof is quite involved and postponed until Appendix D) that there exists no chimeric polynomial over the field of iterated Laurent power series $\kappa((\mathtt{x}_1))\cdots((\mathtt{x}_n))$ in n variables with coefficients in κ. For instance, if κ is a (formally) real algebraic extension of $\mathbb{Q}$ or $\kappa = \mathbb{R}$, then there exists no chimeric polynomial over $\kappa((\mathtt{x}_1))\cdots((\mathtt{x}_n))$. We refer the reader to [Schl] for further details concerning fields with only Archimedean orderings.

(iii) Using Corollaries 14.25 and 14.26 (based on elementary equivalence of fields) and the (formally) real fields described in (i) and (ii) we can constructs many additional fields with no chimeric polynomials over them.

In view of [Fo] (concerning the embedding of henselian fields of characteristic zero in a field of generalized power series (possibly, with a factor set)) and [Pr2, §8] the previous results reinforce the idea that there exists no chimeric polynomial over any (formally) real field.

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4. Structure of the manuscript

This manuscript is organized as follows. In Section 4 we recall (mainly without proofs, but providing precise references) basic tools to deal with formal rings (like Weierstrass' Theorems, Finite Determinacy, Strong Artin's Approximation, Tougeron's Implicit Function Theorem or

Curve selection Lemma). After that the article is divided in three parts. The first part concerns the 2-dimensional case (and it is constituted by Sections 5, 6, 7, 8 and 9), the second part refers to the 1-dimensional case (and it consists of Sections 10, 11, 12, 13 and 14), whereas in the third part we explore some 2-dimensional examples of higher embedding dimension and some applications of the main results of this manuscript (it consists of Sections 15, 16, 17 and 18).

In Section 6 we present some preliminary results concerning positive semidefinite and positive definite elements to approach the 2-dimensional case. In Section 7 we reduce (making a sophisticated use of Tougeron's Implicit Function Theorem) the proof of Theorem 2.3 to the case when the involved formal ring has finite Pythagoras number. In Section 8 we improve Elephant's Theorem [Fe8, Thm.5.5] and we extend its use to prove the property $\mathcal{P}(A) = \Sigma A^2$ for all the formal rings in List 2.1 that are finite determined (provided that $p(A)$ is finite). In Section 9 we prove the main Theorem 2.3. In Section 10 we show (Theorem 10.2) the 'if part' of Theorem 3.2, whereas in Section 11 we prove Lemma 3.4 and Theorem 3.5. The 'only if' part of Theorem 3.2 is proved in Sections 12 and 13. In Section 14 we study chimeric polynomials and provide several families of (formally) real fields κ with no chimeric polynomials over them (see §3.3). In Section 15 we present several families of examples of 2-dimensional local excellent rings A_n (both irreducible and reducible, Example 15.2, Example 15.3 and Theorem 15.17) with the property $\mathcal{P}(A_n) = \Sigma A_n^2$ and embedding dimension n, whereas in Section 16 we propose further obstructions for 2-dimensional local excellent henselian rings A of higher embedding dimension such that $\mathcal{P}(A) = \Sigma A^2$. Section 17 approach a Global-local criterion, which provides information concerning the global analytic case from the local case in some particular (although enough general) situations (see Theorems 17.8 and 17.12). In Section 18 we present two further applications: and improvement of Scheiderer's Theorem [Sch6, Sch7] concerning principal saturated preorderings of low order (§18.1) and a result concerning transference of sums of squares (§18.2) inspired by [Sch8].

The article finishes with several Appendices. In Appendix A we recall two results concerning rings and fields of Puiseux series (when we are dealing with an arbitrary field of characteristic zero). In Appendix B we present a (leveled) proof of the formula $1 + \tau(\kappa) \leqslant p(\kappa[\mathtt{t}]) \leqslant 2\tau(\kappa)$ for each (formally) real field κ. This formula is useful to develop some of the results in §14.4. In Appendix C we provide an additional curious property that satisfy chimeric polynomials and with an Appendix where we present some examples and further obstructions for 2-dimensional local excellent henselian rings A of higher embedding dimension such that $\mathcal{P}(A) = \Sigma A^2$. Finally in Appendix D we prove that there exists no chimeric polynomial over $\kappa((\mathtt{x}_1))\cdots((\mathtt{x}_n))$ for each $n \geqslant 1$ if κ is an Archimedean fields (Theorem 14.14).

Part 2. **The 2-dimensional case**

s5 5. Basic tools when dealing with formal rings

The purpose of this section is to fix some notations and nomenclature and to recall some of the main tools that we will use along this work.

5.1. **Formal rings.** A *formal ring* over a field κ is a ring $A := \kappa[[\mathtt{x}]]/\mathfrak{a}$ where $\mathfrak{a}$ is an ideal of the ring $\kappa[[\mathtt{x}]]$ of (formal power) series in the variables $\mathtt{x} := (\mathtt{x}_1, \ldots, \mathtt{x}_n)$. If $f \in \kappa[[\mathtt{x}]]$, we write $f := \sum_{k=0}^{\infty} f_k$ where $f_k \in \kappa[\mathtt{x}]$ is (either 0 or) an homogeneous polynomial of degree k. Denote the *order of* f with $\omega(f) := \inf\{k : \ f_k \neq 0\}$ and the *leading form of* f with $\ell(f) := f_{\omega(f)}$. We write $\mathfrak{m}_n := (\mathtt{x}_1, \ldots, \mathtt{x}_n)\kappa[[\mathtt{x}]]$ to refer to the maximal ideal of $\kappa[[\mathtt{x}]]$ and $\|\mathtt{x}\|^2 := \mathtt{x}_1^2 + \cdots + \mathtt{x}_n^2$. In addition, $\kappa((\mathtt{x}))$ stands for the quotient field of $\kappa[[\mathtt{x}]]$. Recall that a series $f \in \kappa[[\mathtt{x}_1, \ldots, \mathtt{x}_n]]$ is *regular with respect to* $\mathtt{x}_n$ *(of order* d*)* if $f(0, \ldots, 0, \mathtt{x}_n) = \sum_{k \geqslant d} a_k \mathtt{x}_n^k$ with $a_d \neq 0$. Every non-zero series $f \in \kappa[[\mathtt{x}]]$ is regular with respect to $\mathtt{x}_n$ after a change of coordinates. Let $\mathtt{x}' := (\mathtt{x}_1, \ldots, \mathtt{x}_{n-1})$. A *Weierstrass polynomial* $P \in \kappa[[\mathtt{x}']][\mathtt{x}_n]$ is a monic polynomial of degree

d such that $P(0,\mathtt{x}_n)=\mathtt{x}_n^d$. We recall next the main tools concerning formal rings that we will use freely along this article.

5.1.1. *Weierstrass Theorems.* By [ZS, Ch.7.§.1.Thm.5 & Cor.1] the ring $\kappa[[\mathtt{x}]]$ enjoys Weierstrass division and preparation theorems, which are fundamental tools when dealing with rings of series with coefficients in a field κ.

Theorem 5.1 (Weierstrass division theorem [ZS, Ch.7.§.1.Thm.5]). *Let $f,g\in\kappa[[\mathtt{x}]]$ be series such that f is a regular series with respect to $\mathtt{x}_n$ of order d. Then there exist $Q\in\kappa[[\mathtt{x}]]$ and a polynomial $R\in\kappa[[\mathtt{x}']][\mathtt{x}_n]$ of degree $\leqslant d-1$ such that $f=gQ+R$.*

Theorem 5.2 (Weierstrass preparation theorem [ZS, Ch.7.§.1.Cor.1]). *Let $f\in\kappa[[\mathtt{x}]]$ be a regular series with respect to $\mathtt{x}_n$ of order d. Then there exist a Weierstrass polynomial $P\in\kappa[[\mathtt{x}']][\mathtt{x}_n]$ of degree d and a unit $U\in\kappa[[\mathtt{x}]]$ such that $f=PU$.*

5.1.2. *Implicit and Inverse Function Theorems.* As a consequence of the previous results, Implicit and Inverse Function Theorems arise standardly.

ift **Theorem 5.3** (Implicit Function Theorem). *Let $f_1,\dots,f_m\in\kappa[[\mathtt{x},\mathtt{y}]]$ be such that $f_i(0,0)=0$ and $\det(\frac{\partial f_i}{\partial \mathtt{y}_j}(0,0))_{1\leqslant i,j\leqslant m}\neq 0$. Then there exist unique series $g_1,\dots,g_m\in\kappa[[\mathtt{x}]]$ such that $g_j(0)=0$ and $f_i(\mathtt{x},g_1,\dots,g_m)=0$ for $i=1,\dots,m$.*

ift2 **Theorem 5.4** (Inverse Function Theorem). *Let $f_1,\dots,f_n\in\kappa[[\mathtt{x}]]$ be series such that $f_i(0)=0$ and $\det(\frac{\partial f_i}{\partial \mathtt{x}_j}(0))_{1\leqslant i,j\leqslant n}\neq 0$. Then there exist unique series $g_1,\dots,g_n\in\kappa[[\mathtt{y}]]$ such that $g_i(0)=0$ and $f_i(g_1,\dots,g_n)=\mathtt{y}_i$ and $g_i(f_1,\dots,f_n)=\mathtt{x}_i$ for $i=1,\dots,n$.*

1v *Remarks* 5.5. (i) Let $U\in\kappa[[\mathtt{x}]]$ be a unit such that $U(0)\in\Sigma_p\kappa^2$. By the Implicit Function Theorem the quotient $\frac{U}{U(0)}$ is a square in $\kappa[[\mathtt{x}]]$, so $U\in\Sigma_p\kappa[[\mathtt{x}]]^2$.

(ii) Let $U\in\kappa[[\mathtt{x}]]$ be a unit and let $\mathfrak{a}\subset\kappa[[\mathtt{x}]]$ be a real ideal such that $U\in\mathcal{P}(\kappa[[\mathtt{x}]]/\mathfrak{a})$. As $\operatorname{Sper}(\kappa)$ correspond to the prime cones of $\operatorname{Sper}(\kappa[[\mathtt{x}]]/\mathfrak{a})$ whose support is the maximal ideal of $\kappa[[\mathtt{x}]]/\mathfrak{a}$, we deduce that $U(0)\in\mathcal{P}(\kappa)=\Sigma\kappa^2$. By (i) we conclude $U\in\Sigma(\kappa[[\mathtt{x}]]/\mathfrak{a})^2$.

(iii) Let $b:=\sum_{k\geqslant\ell}b_k\mathtt{t}^k\in\kappa[[\mathtt{t}]]$ with $b_\ell\neq 0$.

(iii.1) $b\in\Sigma\kappa((\mathtt{t}))^2$ *if and only if ℓ is even and* $b_\ell\in\Sigma\kappa^2$. If such is the case, $b\in\Sigma\kappa[[\mathtt{t}]]^2$.

Write $b:=b_\ell\mathtt{t}^\ell U$ where $U\in\kappa[[\mathtt{x}]]$ is a unit such that $U(0)=1$. By (i) U is a square in $\kappa[[\mathtt{x}]]$, so $b\in\Sigma\kappa((\mathtt{t}))^2$ (resp. $b\in\Sigma\kappa[[\mathtt{t}]]^2$) if and only if $b_\ell\mathtt{t}^\ell\in\Sigma\kappa((\mathtt{t}))^2$ (resp. $b_\ell\mathtt{t}^\ell\in\Sigma\kappa[[\mathtt{t}]]^2$). Now the statement follows readily.

(iii.2) Consequently, $b\notin\Sigma\kappa((\mathtt{t}))^2$ if and only if either ℓ is odd or ℓ is even and $b_\ell\notin\Sigma\kappa^2$. ▪

5.1.3. *Strong Artin's Approximation Theorem.* In this work we make an extended use of this celebrated result, so we recall here the precise statement.

saat **Theorem 5.6** (Strong Artin's Approximation, [PP, Po]). *Let $F\in(\kappa[[\mathtt{x}]][[\mathtt{y}]][\mathtt{z}])^r$ be a finite system of equations where $\mathtt{y}:=(\mathtt{y}_1,\dots,\mathtt{y}_m)$ and $\mathtt{z}:=(\mathtt{z}_1,\dots,\mathtt{z}_p)$. Then there exists a function $\beta:\mathbb{N}\to\mathbb{N}$ with the following property: For each $c\in\mathbb{N}$, each $y\in\mathfrak{m}_n^m$ and each $z\in\kappa[[\mathtt{x}]]^p$ such that $F(y,z)\in\mathfrak{m}_n^{\beta(c)}$ there exist $\widetilde{z}\in\mathfrak{m}_n^m$ and $\widetilde{z}\in\kappa[[\mathtt{x}]]^p$ such that $F(\widetilde{y},\widetilde{z})=0$ and $\widetilde{y}-y,\widetilde{z}-z\in\mathfrak{m}_n^c$.*

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5.1.4. *Equivalence of series and finite determinacy tools.* Given two series $f,g\in\mathfrak{m}_n$ we say that f is *right equivalent to* g if there exists an automorphism $\Phi:\kappa[[\mathtt{x}]]\to\kappa[[\mathtt{x}]]$ such that $\Phi(f)=g$. A series $f\in\mathfrak{m}_n$ is k-determined if each $g\in\kappa[[\mathtt{x}]]$ with $f-g\in\mathfrak{m}_n^{k+1}$ is right equivalent to f. If f is k-determined for some $k\geqslant 1$, then f is right equivalent to a polynomial of $\kappa[\mathtt{x}]$.

fdqd **Theorem 5.7** (Finite Determinacy Theorem). *Let $f\in\mathfrak{m}_n^2\subset\kappa[[\mathtt{x}]]$. Suppose that $\mathfrak{m}_n^k\subset\mathfrak{m}_nJ(f)$ where $J(f):=(\frac{\partial f}{\partial\mathtt{x}_1},\dots,\frac{\partial f}{\partial\mathtt{x}_n})$. Then f is k-determined.*

The previous result is proved in [Fe8, Appendix B] for a general field κ of characteristic zero. In [Fe8, Ex.2.6] we recall (using Theorem 5.7) the finite determinacy of some series of $\kappa[[\mathtt{x},\mathtt{y}]]$ that includes the singularities (3.i-3.ii), (3.iv) and (3.vi-3.viii) in List 2.1.

examples *Examples* 5.8. Let κ be a field of characteristic zero, $a,b\in\kappa\backslash\{0\}$ and write $\mathfrak{m}_2:=(\mathtt{x},\mathtt{y})\kappa[[\mathtt{x},\mathtt{y}]]$. We have:

(i) If $F:=a\mathtt{x}^2+b\mathtt{y}^\ell$ where $\ell\geqslant 2$, then F is ℓ-determined.

(ii) If $F:=a\mathtt{x}^2\mathtt{y}+b\mathtt{y}^\ell$ with $\ell\geqslant 3$, then F is ℓ-determined.

(iii) If $F:=\mathtt{x}^3+a\mathtt{x}\mathtt{y}^2+b\mathtt{y}^3$, then F is 3-determined if its discriminant $4a^3+27b^2\neq 0$.

(iv) If $F:=\mathtt{x}^3+\mathtt{x}\mathtt{y}^3$, then F is 5-determined.

(v) If $F:=\mathtt{x}^3+a\mathtt{y}^k$ where $k\geqslant 3$, then F is k-determined. ■

5.2. **Formal rings such that $\mathcal{P}(A)\neq\Sigma A^2$.** In [Fe8] we presented an interesting tool to bound series using powers of the square of the norm $\|\mathtt{x}'\|^2:=\mathtt{x}_1^2+\cdots+\mathtt{x}_{n-1}^2$.

genlist0 **Lemma 5.9** ([Fe8, Lem.3.1])**.** *Let κ be a (formally) real field and $f\in\kappa[[\mathtt{x}]]$ a series of order $\geqslant 2s$. Then there exists $M\in\Sigma\kappa^2$ such that $M^2\|\mathtt{x}\|^{2s}-f\in\mathcal{P}(\kappa[[\mathtt{x}]])$.*

Inside the proof of [Fe8, Lem.3.2] (Claim [3.ii]) we showed the following:

gen22 **Lemma 5.10.** *Let $A:=\kappa[[\mathtt{x}_1,\dots,\mathtt{x}_n]]/\mathfrak{a}$ be a formal ring and suppose*

$$P:=\mathtt{x}_m^r+\sum_{j=0}^{r-1}a_j(\mathtt{x}_1,\dots,\mathtt{x}_d)\mathtt{x}_m^j\in\mathfrak{a}\cap\kappa[[\mathtt{x}_1,\dots,\mathtt{x}_d]][\mathtt{x}_m]$$

for some $d+1\leqslant m\leqslant n$ and $\omega(a_j)\geqslant r-j$ for $0\leqslant j\leqslant r-1$. Then there exists $N\in\Sigma\kappa^2$ such that the quadratic form $(N^2r^2+1)^2(\mathtt{x}_1^2+\cdots+\mathtt{x}_d^2)-\mathtt{x}_n^2\in\mathcal{P}(A)$.

genlist **Corollary 5.11.** *Let $A:=\kappa[[\mathtt{x}_1,\dots,\mathtt{x}_n]]/\mathfrak{a}$ be a formal ring such that $\mathcal{P}(A)=\Sigma A^2$. Then $\omega(\mathfrak{a})\leqslant 2$.*

Proof. After a change of coordinates there exists a Weierstrass polynomial

$$P\in\mathfrak{a}\cap\kappa[[\mathtt{x}_1,\dots,\mathtt{x}_{n-1}]][\mathtt{x}_{n-1}].$$

By Lemma 5.10 there exists $M\in\kappa\backslash\{0\}$ such that $q:=M^2(\mathtt{x}_1^2+\cdots+\mathtt{x}_{n-1}^2)-\mathtt{x}_n^2\in\mathcal{P}(A)=\Sigma A^2$. As q is not a positive semidefinite quadratic form, $q\notin\Sigma\kappa[\mathtt{x}_1,\dots,\mathtt{x}_n]^2$, so $\omega(\mathfrak{a})\leqslant 2$, as required. □

As a consequence of Lemma 5.9, we provide next a large class of formal rings A such that $\mathcal{P}(A)\neq\Sigma A^2$.

sosnicht **Lemma 5.12.** *Let $\mathfrak{a}$ be a real ideal of $\kappa[[\mathtt{x}]]$ of order $\omega(\mathfrak{a})\geqslant 2$ and denote $A:=\kappa[[\mathtt{x}_1,\dots,\mathtt{x}_n]]/\mathfrak{a}$. If $h+\mathfrak{a}\in\mathcal{P}(A)$ satisfies $\omega(h)=1$, then $h+\mathfrak{a}\notin\Sigma A^2$*

Proof. Suppose $h\in\Sigma A^2$. Then there exists $a\in\mathfrak{a}$ such that $h+a\in\Sigma\kappa[[\mathtt{x}]]^2$. Thus, $\omega(h+a)$ is even. As $\omega(a)\geqslant 2$ (because $\omega(\mathfrak{a})\geqslant 2$) and $\omega(h)=1$, we deduce $1=\omega(h+a)$, which is a contradiction. Consequently, $h\notin\Sigma A^2$, as required. □

rest4 **Lemma 5.13.** *Let $A:=\kappa[[\mathtt{x}]]/\mathfrak{a}$ for some ideal $\mathfrak{a}$ such that $\omega(\mathfrak{a}):=\min\{\omega(f):\ f\in\mathfrak{a}\}\geqslant 2$ and denote $\mathtt{x}':=(\mathtt{x}_1,\dots,\mathtt{x}_d)$ and $\mathtt{x}'':=(\mathtt{x}_{d+1},\dots,\mathtt{x}_n)$. Assume that $h(\mathtt{x}')-q(\mathtt{x}'')\in\mathfrak{a}$ for some $h\in\mathfrak{m}_d^4$ and some non-zero positive semidefinite quadratic form $q\in\kappa[\mathtt{x}'']$. After a change of coordinates there exists $M\in\kappa$ such that $M^2\|\mathtt{x}'\|^2+\mathtt{x}_n\in\mathcal{P}(A)\backslash\Sigma A^2$.*

Proof. By Lemma 5.9 there exists $M\in\kappa$ such that $M^4\|\mathtt{x}'\|^4-h(\mathtt{x}')\in\mathcal{P}(\kappa[[\mathtt{x}']])$. As $q\in\mathcal{P}(\kappa[\mathtt{x}''])$ is a quadratic form, $q\in\Sigma\kappa[\mathtt{x}'']^2$. After a change of coordinates that involves only variables

$\mathtt{x}'' := (\mathtt{x}_{d+1}, \ldots, \mathtt{x}_n)$ we may assume $q(\mathtt{x}'') = \mathtt{x}_n^2 + \ell_1^2 + \cdots + \ell_p^2$ where $\ell_k \in \kappa[\mathtt{x}'']$ are linear forms. In addition,

$$(M^2\|\mathtt{x}'\|^2 + \mathtt{x}_n) + (M^2\|\mathtt{x}'\|^2 - \mathtt{x}_n) = 2M^2\|\mathtt{x}'\|^2 \in \mathcal{P}(\kappa[[\mathtt{x}']])$$

Consequently,

$$M^4\|\mathtt{x}'\|^4 - \mathtt{x}_n^2 = M^4\|\mathtt{x}'\|^4 - h(\mathtt{x}') + h(\mathtt{x}') - q(\mathtt{x}'') + \ell_1^2 + \cdots + \ell_p^2 \in \mathcal{P}(A)$$

and $(M^2\|\mathtt{x}'\|^2 + \mathtt{x}_n) + (M^2\|\mathtt{x}'\|^2 - \mathtt{x}_n) \in \mathcal{P}(A)$, so $M^2\|\mathtt{x}'\|^2 + \mathtt{x}_n \in \mathcal{P}(A)\backslash\Sigma A^2$ (because $\omega(\mathfrak{a}) \geqslant 2$), as required. $\square$

5.3. **Exchanging positiveness.** Along this work we freely use the following facts proved in [Fe8, §2.3].

trans **Lemma 5.14** ([Fe8, Lem.2.7])**.** *Let $\varphi : A \to B$ be a homomorphism of rings. We have:*

(i) *For each $\beta \in \operatorname{Sper}(B)$ there exists $\alpha \in \operatorname{Sper}(A)$ such that $\operatorname{sign}_\alpha(g) = \operatorname{sign}_\beta(\varphi(g))$ for each $g \in A$.*
(ii) *If $f \in \mathcal{P}(A)$, then $\varphi(f) \in \mathcal{P}(B)$.*

The following result will be useful when dealing with blow-ups in the proof of Theorem 2.3.

blowup **Lemma 5.15** ([Fe8, Lem.2.8])**.** *Let A, B be two rings and $g \in B$ a non-nilpotent element. Denote the localization of B at the multiplicative set $S := \{g^n : n \geqslant 0\}$ with B_g. Let $\varphi : A \to B_g$ be a homomorphism of rings and $f \in \mathcal{P}(A)$. Write $\varphi(f) = \frac{b}{g^\ell}$ for some $\ell \geqslant 1$ and assume g does not divide b. Define*

$$k := \begin{cases} 1 & \textit{if } \ell \textit{ is odd,} \\ 2 & \textit{if } \ell \textit{ is even.} \end{cases}$$

Then $g^k b \in \mathcal{P}(B)$.

minimalp **Corollary 5.16.** *Let A be a notherian ring, let $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_s\}$ be a collection of ideals of A such that $\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_s \subset \sqrt{(0)}$ and let $f \in A$. Then $f \in \mathcal{P}(A)$ if and only if $f \in \mathcal{P}(A/\mathfrak{p}_i)$ for $i = 1, \ldots, n$.*

Proof. Pick $f \in \mathcal{P}(A)$. By Lemma 5.14 we have $f \in \mathcal{P}(A/\mathfrak{a}_i)$ for $i = 1, \ldots, s$. Let $f \in A$ such that $f \in \mathcal{P}(A/\mathfrak{a}_i)$ for $i = 1, \ldots, s$. Pick $\alpha \in \operatorname{Sper}(A)$. As $\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_s \subset \sqrt{(0)} \subset \operatorname{supp}(\alpha)$, we may assume by [AM, Prop.1.11] $\mathfrak{a}_1 \subset \operatorname{supp}(\alpha)$. Consider the homomorphism $\theta : A/\mathfrak{a}_1 \to A/\operatorname{supp}(\alpha) \cong (A/\mathfrak{a}_1)/(\operatorname{supp}(\alpha)/\mathfrak{a}_1)$. As $f \in \mathcal{P}(A/\mathfrak{a}_1)$, we deduce by Lemma 5.14 $f \in \mathcal{P}(A/\operatorname{supp}(\alpha))$, so in particular $f \in \alpha$. Thus, $f \in \alpha$ for each $\alpha \in \operatorname{Sper}(A)$, that is, $f \in \mathcal{P}(A)$, as required. $\square$

minimalr *Remark* 5.17. The previous result applies to the family $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ of the minimal prime ideals of A, but also to $\{\mathfrak{a}_{I_1}, \ldots, \mathfrak{a}_{I_s}\}$ where $\mathfrak{a}_{I_j} = \bigcap_{i \in I_j} \mathfrak{p}_i$, $I_j \subset \{1, \ldots, r\}$ and $\bigcup_{j=1}^s I_j = \{1, \ldots, r\}$.

intrad **Lemma 5.18.** *Let A noetherian ring of dimension n and let $\mathfrak{a}$ be a non-zero radical ideal of A of height $n-1$. Let $\mathfrak{a}_1, \mathfrak{a}_2$ be non-zero ideals of A such that all the prime ideals associated to $\mathfrak{a}_i$ have height $n-1$ and $\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2$. Then $\mathfrak{a}_i$ is a radical ideal for $i = 1, 2$.*

Proof. As $\mathfrak{a}$ is a radical ideal, the minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ associated to $\mathfrak{a}$ provide the unique irredundant primary decomposition of $\mathfrak{a}$. Let $\mathfrak{q}_{1i}, \ldots, \mathfrak{q}_{is_i}$ be primary ideals of A that constitute a primary decomposition of $\mathfrak{a}_i$ for $i = 1, 2$. We have $\mathfrak{a} = \bigcap_{j=1}^{s_1} \mathfrak{q}_{1j} \cap \bigcap_{j=1}^{s_2} \mathfrak{q}_{2j}$. For each $k = 1, \ldots, r$ there exists $(j_k, i_k) \in \{(j, i) : 1 \leqslant i \leqslant 2,\ 1 \leqslant j \leqslant s_i\}$ such that $\mathfrak{q}_{j_k, i_k} \subset \mathfrak{p}_k$. Thus, $\mathfrak{a} \subset \bigcap_{k=1}^r \mathfrak{q}_{j_k, i_k} \subset \bigcap_{k=1}^r \mathfrak{p}_k = \mathfrak{a}$. Consequently, $\mathfrak{q}_{j_k, i_k} = \mathfrak{p}_k$ for $k = 1, \ldots, r$ and all the minimal prime ideals of $\mathfrak{a}$ have height $n-1$ (because this happens for $\mathfrak{a}_1$ and $\mathfrak{a}_2$). Let $\mathfrak{q}_{ji}$ be a primary ideal that is not between the $\mathfrak{q}_{j_k, i_k}$ for $k = 1, \ldots, r$. Then $\sqrt{\mathfrak{q}_{ji}}$ is a prime ideal of height $n-1$ (because by hypothesis the prime ideals associated to $\mathfrak{a}_i$ have height $n-1$), so it is a minimal prime ideal associated to $\mathfrak{a}_i$. As the minimal prime ideals associated to $\mathfrak{a}$, $\mathfrak{a}_1$ and $\mathfrak{a}_2$ have all the same height $n-1$, the ideals $\mathfrak{a}$, $\mathfrak{a}_1$ and $\mathfrak{a}_2$ have no immersed components. Thus, all the prime ideals $\sqrt{\mathfrak{q}_{ji}}$ correspond to a prime ideal $\mathfrak{p}_k = \mathfrak{q}_{j_k,i_k}$. As the primary decompositions are irredundant, $\mathfrak{q}_{ji} = \mathfrak{q}_{j_k,i_k} = \mathfrak{p}_k$ for some $k = 1,\dots,r$ and both $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are radical ideals, as required. $\square$

5.4. **Constructible sets and curve selection lemma.** A *constructible subset* of $\kappa[[\mathtt{x}]]$ is a finite boolean combination of sets of the type $\{\alpha \in \operatorname{Sper}(\kappa[[\mathtt{x}]]) :\ f >_\alpha 0\}$ for some $f \in \kappa[[\mathtt{x}]]$. Fix a constructible set C of $\kappa[[\mathtt{x}]]$ and let $f \in \kappa[[\mathtt{x}]]$. We say that f is *positive semidefinite on* C if $f \geqslant_\alpha 0$ for each $\alpha \in C$. We denote $\mathcal{P}(C)$ the collection of all $f \in \kappa[[\mathtt{x}]]$ that are positive semidefinite on C. As usual we identify $\operatorname{Sper}(\kappa)$ with the subspace of $\operatorname{Sper}(\kappa[[\mathtt{x}]])$ consisting of all prime cones of $\kappa[[\mathtt{x}]]$ with support $\mathfrak{m}_n$. If $\alpha \in \operatorname{Sper}(\kappa[[\mathtt{x}]])$ and $\operatorname{ht}(\operatorname{supp}(\alpha)) \geqslant n-1$, then either $\operatorname{supp}(\alpha) = \mathfrak{m}_n$ (that is, $\alpha \in \operatorname{Sper}(\kappa)$) or there exists $\eta \in \mathfrak{R}(\alpha)[[\mathtt{t}]]^n$ such that $\operatorname{supp}(\alpha) = \{f \in \kappa[[\mathtt{x}]] :\ f(\eta) = 0\}$, where $\mathfrak{R}(\alpha)$ is the real closure of κ endowed with the ordering induced by α. After changing $\mathtt{t}$ by $-\mathtt{t}$ we may assume $\alpha = \psi^{-1}(\alpha)$, where $\psi : \kappa[[\mathtt{x}]] \to \mathfrak{R}(\alpha)[[\mathtt{t}]]$ is the evaluation in η and α_0 is the unique ordering of $\mathfrak{R}(\alpha)[[\mathtt{t}]]$ in which $\mathtt{t}$ is positive.

reduction **Lemma 5.19.** *Let $C \subset \operatorname{Sper}(\kappa[[\mathtt{x}]])$ be a constructible set and let $f \in \kappa[[\mathtt{x}]]$. Then $f \in \mathcal{P}(C)$ if and only if $f \geqslant_\alpha 0$ for each $\alpha \in C$ such that* $\operatorname{ht}(\operatorname{supp}(\alpha)) \geqslant n-1$.

Proof. The only if implication is clear by definition of $\mathcal{P}(C)$. Suppose now that $f \geqslant_\alpha 0$ for each $\alpha \in C$ such that $\operatorname{ht}(\operatorname{supp}(\alpha)) \geqslant n-1$, whereas $f \notin \mathcal{P}(C)$. Thus, there exists $\beta \in C$ such that $f <_\beta 0$. We may assume that $d := \dim(\kappa[[\mathtt{x}]]/\operatorname{supp}(\beta))$ is maximal between those $\gamma \in C$ such that $f <_\gamma 0$. In particular, $\operatorname{ht}(\operatorname{supp}(\beta)) \leqslant n-2$. We have $d = n - \operatorname{ht}(\operatorname{supp}(\beta)) \geqslant n-(n-2) = 2$ and let α_0 be the ordering of κ induced by β. As $\operatorname{supp}(\alpha_0) = \mathfrak{m}_n$, we have

$$\dim(\beta \to \alpha_0) = \dim(\kappa[[\mathtt{x}]]_{\mathfrak{m}_n}/\operatorname{supp}(\beta)) = \dim(\kappa[[\mathtt{x}]]/\operatorname{supp}(\beta)) = d \geqslant 2.$$

Consider the non-empty constructible set $C' := C \cap \{f <_\beta 0\}$ and observe that $\dim_{\alpha_0}(C') = d \geqslant 2$, because $\beta \in C$. By [ABR, Thm.VII.4.2] there exists $\eta \subset \mathfrak{R}(\alpha)[[\mathtt{t}]]^n$ such that both prime cones induced by η in $\kappa[[\mathtt{x}]]$ are contained in C', against our assumption. We conclude $f \in \mathcal{P}(C)$, as required. $\square$

redr *Remarks* 5.20. (i) The previous result corrects two imprecisions in [Fe8, pages 28 & 40], where it is said: "by [ABR, Prop.VII.5.1] there exists $\beta_1 \in \operatorname{Sper}(\kappa[[\mathtt{x},\mathtt{y}]])$ such that $\beta \to \beta_1 \to \alpha$ and $\operatorname{supp}(\beta_1)$ is a real prime ideal of height 1, so we may assume $\operatorname{ht}(\operatorname{supp}(\beta)) = 1$". The first part is not correct, but the assumption it is so by Lemma 5.19. It also corrects non-rigorous uses of [ABR, Prop.VII.5.1] in [Fe8, pages 17 & 51].

(ii) If $f \in \mathfrak{m}_n$ in the statement of Lemma 5.19, we only have to care of prime cones of C such that $\operatorname{ht}(\operatorname{supp}(\alpha)) = n-1$ (because $f(0) = 0$). The same happens if C is an open constructible subset of $\operatorname{Sper}(\kappa[[\mathtt{x}]])$.

Let $\alpha \in C$ and let $\beta \in \operatorname{Sper}(\kappa[[\mathtt{x}]])$ be such that $\beta \to \alpha$. Then $\alpha \in \operatorname{Cl}(\beta)$. If $\beta \notin C$, then $\operatorname{Cl}(\beta) \cap C = \varnothing$ (because C is open), so $\alpha \notin C$, which is a contradiction. Thus, if $\alpha \in C$ has as support $\mathfrak{m}_n$, it corresponds to an ordering of κ. Let $\mathfrak{R}(\alpha)$ be the real closure of $(\kappa, \leqslant_\alpha)$. The evaluation homomorphism $\varphi_\eta : \kappa[[\mathtt{x}]] \to \mathfrak{R}(\alpha)[[\mathtt{t}]],\ f \to f(\eta)$ induces an ordering β_η of $\kappa[[\mathtt{x}]]$ for each $\eta \in \mathfrak{R}(\alpha)[[\mathtt{t}]]^n$, whose support has height $n-1$, such that $\beta_\eta \to \alpha$, so $\beta_\eta \in C$ (because $\alpha \in \operatorname{Cl}(\beta_\eta)$). Thus we only have to care about prime cones of C with support of height $n-1$ to determine the set $\mathcal{P}(C)$.

(iii) Consequently, if $n = 2$ and $f \in \mathfrak{m}_2$ in the statement of Lemma 5.19, we only have to care of prime cones of C such that $\operatorname{ht}(\operatorname{supp}(\alpha)) = 1$. ■

We generalize [Fe8, §2.5] to obtain the following version of the curve selection lemma proposed in [Fe8, §2.5], which characterizes the prime cones of $\operatorname{Sper}(\kappa[[\mathtt{x}]])$ whose support has height 1.

dimen1 **Lemma 5.21** (Curve selection lemma). *Let κ be a (formally) real field and $\beta \in \operatorname{Sper}(\kappa[[\mathtt{x}]])$ a prime cone such that $\dim(\kappa[[\mathtt{x}]]/\operatorname{supp}(\beta)) = 1$, $\kappa[[\mathtt{x}_1]] \hookrightarrow \kappa[[\mathtt{x}]]/\operatorname{supp}(\beta)$ and $\mathtt{x}_1 >_\beta 0$. Let $\alpha \in \operatorname{Sper}(\kappa)$ be such that $\beta \to \alpha$ and let $\mathfrak{R}(\alpha)$ be the real closure of $(\kappa, \leqslant_\alpha)$. Then there exists a homomorphism $\phi : \kappa[[\mathtt{x}]] \to \mathfrak{R}(\alpha)[[\mathtt{t}]]$ and $\varepsilon \in \{-1, +1\}$ such that $\phi(\mathtt{x}_1) = \varepsilon\mathtt{t}^q$ for some $q \geqslant 1$ and $f \geqslant_\beta 0$ if and only if $\phi(f) = f(\mathtt{t}^q, \phi(\mathtt{x}_2), \dots, \phi(\mathtt{x}_n)) \geqslant 0$. In addition, $f \in \operatorname{supp}(\beta)$ if and only if $\phi(f) = 0$.*

Proof. As $\operatorname{ht}(\operatorname{supp}(\beta)) = 1$, by Rückert's parametrization [Rz1, Prop.3.4] (that works over κ, because it is a consequence of Weiertrass' Division and Preparation Theorems) there exist (after a κ-linear change of coordinates) an irreducible polynomial $P_k \in \kappa[[\mathtt{x}_1]][\mathtt{x}_k]$ of degree p_k for $k = 2, \dots, n$ and polynomials $Q_3, \dots, Q_n \in \kappa[[\mathtt{x}_1]][\mathtt{x}_2]$ of degree $< p_2$ such that $\Delta\mathtt{x}_j - Q_j \in \operatorname{supp}(\beta)$ for $j = 3, \dots, n$ and Δ is the discriminant of P. In addition, for $k \geqslant 1$ large enough $\Delta^k \operatorname{supp}(\beta) \subset \{P_2, \Delta\mathtt{x}_3 - Q_3, \dots, \Delta\mathtt{x}_n - Q_n\}\kappa[[\mathtt{x}]] \subset \operatorname{supp}(\beta)$. Consider the homomorphism $\theta : \kappa[[\mathtt{x}_1, \mathtt{x}_2]] \hookrightarrow \kappa[[\mathtt{x}]] \to \kappa[[\mathtt{x}]]/\operatorname{supp}(\beta)$. As $\kappa[[\mathtt{x}]]/\operatorname{supp}(\beta)$ is an integral domain, $\ker(\theta)$ is a prime ideal that contains P_2. We claim: $\ker(\theta) = \operatorname{supp}(\beta) \cap \kappa[[\mathtt{x}_1, \mathtt{x}_2]] = (P_2)\kappa[[\mathtt{x}_1, \mathtt{x}_2]]$.

Suppose there exists $H \in \ker(\theta) \setminus \operatorname{supp}(\beta)$. By Weierstrass Division Theorem we may assume $H \in \kappa[[\mathtt{x}_1]][\mathtt{x}_2]$ (and it is not constant, because $\ker(\theta)$ is a prime ideal). The resultant $\operatorname{Res}_{\mathtt{x}_2}(P_2, H) \in \kappa[[\mathtt{x}_1]]$, belongs to $\ker(\theta)$, because there exists polynomials $A_1, A_2 \in \kappa[[\mathtt{x}_1]][\mathtt{x}_2]$ such that $\operatorname{Res}_{\mathtt{x}_2}(P_2, H) = A_1 H + A_2 P_2 \in \ker(\theta)$. As P_2 is irreducible and $\deg_{\mathtt{x}_2}(H) < \deg_{\mathtt{x}_2}(P_2)$, we deduce $\operatorname{Res}_{\mathtt{x}_2}(P_2, H) \neq 0$, which is a contradiction, because $\operatorname{supp}(\beta) \cap \kappa[[\mathtt{x}_1]] = (0)$. Consequently, $\ker(\theta) = (P_2)\kappa[[\mathtt{x}_1, \mathtt{x}_2]]$.

Changing $\mathtt{x}_1$ by $-\mathtt{x}_1$, we may assume $\mathtt{x}_1 >_\beta 0$ (this possible need to change the chain of $\mathtt{x}_1$ corresponds to the $\varepsilon \in \{-1, +1\}$ that appears in the statement). Consider the chain of homomorphisms

$$\mathtt{j} : \kappa[[\mathtt{x}_1]] \hookrightarrow \kappa[[\mathtt{x}_1]][\mathtt{x}_2]/(P_2) \to \kappa[[\mathtt{x}]]/\operatorname{supp}(\beta).$$

As $P_2, \dots, P_n, \Delta\mathtt{x}_3 - Q_3, \dots, \Delta\mathtt{x}_n - Q_n \in \operatorname{supp}(\beta)$, we have

$$(\kappa[[\mathtt{x}_1]][\mathtt{x}_2]/(P_2))_\Delta \cong (\kappa[[\mathtt{x}]]/\operatorname{supp}(\beta))_\Delta.$$

As $\Delta = \mathtt{x}_1^\ell w$ where $\ell \geqslant 1$ and $w \in \kappa[[\mathtt{x}_1]]$, we have $\kappa[[\mathtt{x}_1]]_\Delta = \kappa((\mathtt{x}_1))$. Consequently,

$$(\kappa[[\mathtt{x}]]/\operatorname{supp}(\beta))_\Delta \cong (\kappa[[\mathtt{x}_1]][\mathtt{x}_2]/(P_2))_\Delta \cong \kappa[[\mathtt{x}_1]]_\Delta[\mathtt{x}_2]/(P_2) \cong \kappa((\mathtt{x}_1))[\mathtt{x}_2]/(P_2),$$

which is the quotient field of $\kappa[[\mathtt{x}_1]][\mathtt{x}_2]/(P_2)$, so it is also the quotient field of $\kappa[[\mathtt{x}]]/\operatorname{supp}(\beta)$. Write $\alpha_1 := \mathtt{j}^{-1}(\beta)$. By [ABR, Ex.II.3.13] the real closure of $(\kappa((\mathtt{x}_1)), \leqslant_{\alpha_1})$ is $\mathfrak{R}(\alpha)((\mathtt{x}_1^*))$. As $(\kappa((\mathtt{x}_1))[\mathtt{x}_2]/(P_2), \leqslant_\beta)$ is a finite (formally) real algebraic extension of $(\kappa((\mathtt{x}_1)), \leqslant_{\alpha_1})$ and $\alpha_1 = \mathtt{j}^{-1}(\beta)$, we deduce

$$\varphi : \operatorname{qf}(\kappa[[\mathtt{x}]]/\operatorname{supp}(\beta)) \cong \kappa((\mathtt{x}_1))[\mathtt{x}_2]/(P_2) \hookrightarrow \mathfrak{R}(\alpha)((\mathtt{x}_1^*)).$$

As $P_k \in \kappa[[\mathtt{x}_1]][\mathtt{x}_k] \cap \operatorname{supp}(\beta)$ is a monic polynomial for $k = 2, \dots, n$, there exist $q \geqslant 1$ and $\zeta_2, \dots, \zeta_n \in \mathfrak{R}(\alpha)[[\mathtt{t}]]$ such that $\varphi(\mathtt{x}_k) = \zeta_k(\mathtt{x}_1^{1/q})$ for $k = 2, \dots, n$. Define

$$\phi : \kappa[[\mathtt{x}]] \to \mathfrak{R}(\alpha)[[\mathtt{t}]],\ f \mapsto f(\mathtt{t}^q, \zeta_2, \dots, \zeta_n)$$

and observe that $f \in \kappa[[\mathtt{x}]]$ satisfies $f \geqslant_\beta 0$ if and only if $\phi(f) \geqslant 0$. Analogously, $f \in \operatorname{supp}(\beta)$ if and only if $\phi(f) \geqslant 0$, as required. □

dimen1b **Corollary 5.22.** *Let κ be a (formally) real field and let $P \in \kappa[[\mathtt{x}]][\mathtt{y}]$ be an irreducible Weierstrass polynomial of degree n. Assume that $L := \kappa((\mathtt{x}))[\mathtt{y}]/(P)$ is a formally real field and let $\alpha \in \operatorname{Sper}(L)$ be such that $\mathtt{x} >_\alpha 0$. Let $\zeta := \sum_{k \geqslant 0} b_k \mathtt{x}^{k/n} \in \mathfrak{R}(\alpha)[[\mathtt{x}^{1/n}]] \subset \overline{\kappa}[[\mathtt{x}^{1/n}]]$ be a root of P and assume that there exists $k_0 \geqslant 1$ such that $b_{k_0} \neq 0$ and $\gcd(n, k_0) = 1$. Then there exists $\theta \in \overline{\kappa} \setminus \{0\}$ and $f \in \kappa[[\mathtt{t}]]$ such that: $\theta^n \in \kappa \cap \alpha$, $\zeta = f(\theta\mathtt{x}^{1/n})$ and $H \in \kappa[[\mathtt{x}]][\mathtt{y}]$ satisfies $H \geqslant_\alpha 0$ if and only if $H(\frac{\mathtt{t}^n}{\theta^n}, f(\mathtt{t})) \geqslant 0$ in $\kappa[[\mathtt{t}]]$ with respect to its unique ordering in which $\mathtt{t} > 0$ and κ is endowed with the ordering induced by α.*

Proof. As P is an irreducible Weierstrass polynomial, all its roots in $\overline{\kappa}((\mathtt{x}^*))$ belong to $\overline{\kappa}[[\mathtt{x}^*]]$. Consequently, if $\zeta \in \overline{\kappa}[[\mathtt{x}^*]]$ is a root of P, there exists $n \geqslant 1$ such that $\zeta \in \overline{\kappa}[[\mathtt{x}^{1/n}]]$. As L is a formally real field, there exists a root $\zeta := \sum_{k\geqslant 0} b_k \mathtt{x}^{k/n} \in \mathfrak{R}(\alpha)[[\mathtt{x}^{1/n}]]$ of P.

By Corollary A.2 there exists $\theta \in \overline{\kappa}\setminus\{0\}$ such that $\theta^n \in \kappa$ and $f \in \kappa[[\mathtt{t}]]$ such that $\zeta = f(\theta \mathtt{x}^{1/n})$. As $b_{k_0}\theta^{k_0} \in \kappa$ and $b_{k_0} \in \mathfrak{R}(\alpha)$, we deduce $\theta^{k_0} \in \mathfrak{R}(\alpha)\setminus\{0\}$. As $\gcd(n, k_0) = 1$, there exists $p, q \in \mathbb{Z}$ such that $1 = np + k_0 q$, so $\theta = \theta^{np+k_0 q} = (\theta^n)^p(\theta^{k_0})^q \in \mathfrak{R}(\alpha)$.

If n is even, $\theta^n >_\alpha 0$ (because $\theta \in \mathfrak{R}(\alpha)$). Suppose next n is odd. If $\theta^n >_\alpha 0$, we do nothing. Otherwise, we change θ by $\theta^* := -\theta >_\alpha 0$ and $f(\mathtt{t})$ by $f^*(\mathtt{t}) := f(-\mathtt{t})$ and we have $\zeta = f(\theta \mathtt{x}^{1/n}) = f(-(-\theta)\mathtt{x}^{1/n}) = f^*(\theta^* \mathtt{x}^{1/n})$. Observe that $(\theta^*)^n = -\theta^n >_\alpha 0$.

As $L \cong \kappa((\mathtt{x}))[\zeta] \subset \mathfrak{R}(\alpha)((\mathtt{x}^*))$ and the ordering of $\kappa((\mathtt{x}))[\zeta]$ is the one induced by $\mathfrak{R}(\alpha)((\mathtt{x}^*))$, we deduce that $H \in \kappa[[\mathtt{x}]][\mathtt{y}]$ satisfies $H \geqslant_\alpha 0$ if and only if $H(\mathtt{x}, \zeta) \geqslant 0$ in $\mathfrak{R}(\alpha)((\mathtt{x}^*))$. We substitute $\mathtt{x}$ by $\mathtt{t}^n/\theta^n >_\alpha 0$ if $\mathtt{t} > 0$ (both if n is either even other odd) and obtain $H(\mathtt{x}, \zeta) \geqslant 0$ in $\mathfrak{R}(\alpha)((\mathtt{x}^*))$ if and only if $H(\frac{\mathtt{t}^n}{\theta^n}, f(\mathtt{t})) \geqslant 0$ in $\kappa[[\mathtt{t}]]$ with respect to its unique ordering in which $\mathtt{t} > 0$ and κ is endowed with the ordering induced by α, as required. □

dimen1br *Remark* 5.23. Under the previous hypothesis if we consider an ordering $\alpha \in \operatorname{Sper}(L)$ such that $\mathtt{x} <_\alpha 0$ (so $-\mathtt{x} >_\alpha 0$), we change $\mathtt{x}$ by $-\mathtt{x}$ (that is, we consider the isomorphism of $\theta : L \to L,\ h(\mathtt{x}, \mathtt{y}) \mapsto h(-\mathtt{x}, \mathtt{y})$) and we then can apply the previous result to the new situation. ■

6. Basic tools to approach the 2-dimensional case

s6

In this section we study some main properties of the set $\mathcal{P}(A)$ of positive semidefinite elements of a ring $A := \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/(\mathtt{z}^2 - F)$ where κ is a (formally) real field and $F \in \mathfrak{m}_2 \subset \kappa[[\mathtt{x}, \mathtt{y}]]$.

chpsd

6.1. **Characterization of positive semidefinite elements.** Let $F \in \mathfrak{m}_2 \subset \kappa[[\mathtt{x}, \mathtt{y}]]$ where κ is a (formally) real field and let $A := \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/(\mathtt{z}^2 - F)$. The ring A is a rank 2 free module over $\kappa[[\mathtt{x}, \mathtt{y}]]$. This means that the elements of A are uniquely represented in the form $f + \mathtt{z}g$ where $f, g \in \kappa[[\mathtt{x}, \mathtt{y}]]$. The elements of A are operated using the relation $\mathtt{z}^2 = F(\mathtt{x}, \mathtt{y})$. Consider the inclusion $\mathtt{i} : \kappa[[\mathtt{x}, \mathtt{y}]] \hookrightarrow A$. Recall that

$$\mathcal{P}(\{F \geqslant 0\}) := \{f \in \kappa[[\mathtt{x}, \mathtt{y}]] :\ f \geqslant_\alpha 0,\ \forall \alpha \in \operatorname{Sper}(\kappa[[\mathtt{x}, \mathtt{y}]]) \text{ with } F \geqslant_\alpha 0\},$$

the maximal ideal of $\kappa[[\mathtt{x}, \mathtt{y}]]$ is denoted with $\mathfrak{m}_2$ and the maximal ideal of A with $\mathfrak{m}_A$. By [Fe8, Cor.3.5]

$$\mathcal{P}(A) = \{f + \mathtt{z}g \in A :\ f \in \mathcal{P}(\{F \geqslant 0\}) \text{ and } f^2 - Fg^2 \in \mathcal{P}(\kappa[[\mathtt{x}, \mathtt{y}]])\}. \tag{6.1}$$ psd3

We prove next that the positive semidefinite units of A are sums of squares in A. Thus, in the following when we test the equality $\mathcal{P}(A) = \Sigma A^2$, we focus on elements $f + \mathtt{z}g \in \mathcal{P}(A)$ such that $\omega(f) \geqslant 1$.

order0 **Lemma 6.1.** *Let $F \in \mathfrak{m}_2\setminus\{0\}$ and let $A := \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/(\mathtt{z}^2 - F)$. We have:*

(i) $\operatorname{Sper}(\kappa) \subset \mathcal{P}(\{F \geqslant 0\})$.
(ii) *If $f + \mathtt{z}g \in \mathcal{P}(A)$ and $\omega(f) = 0$, then $f + \mathtt{z}g \in \Sigma A^2$.*

Proof. (i) Let $\alpha \in \operatorname{Sper}(\kappa)$ be an ordering of κ. Then α corresponds to a prime cone α' of $\operatorname{Sper}(A)$ such that $\operatorname{supp}(\alpha') = \mathfrak{m}_A$. As $F \in \mathfrak{m}_2 \subset \mathfrak{m}_A = \operatorname{supp}(\alpha')$, we conclude $\alpha \equiv \alpha' \in \mathcal{P}(\{F \geqslant 0\})$.

(ii) As $f + \mathtt{z}g \in \mathcal{P}(A)$ and $\omega(f) = 0$, we have $f(0,0) \in \alpha\setminus\{0\}$ for each ordering α of κ, so $f(0,0) \in \mathcal{P}(\kappa) = \Sigma\kappa^2 \subset \Sigma A^2$. Let $u := \frac{f+\mathtt{z}g}{f(0,0)}$, which satisfies $h(0,0) = 1$. Thus, there exists a unit $v \in A$ such that $v^2 = u$, so $f + \mathtt{z}g = f(0,0)v^2 \in \Sigma A^2$, as required. □

The following result borrowed from [Fe8] allows to take advantage in Section 7 of the previous description $\mathcal{P}(A)$ if we take either $C = \operatorname{Sper}(\kappa[[\mathtt{x}, \mathtt{y}]])$ or $C = \{F \geqslant 0\}$.

orderings **Corollary 6.2** ([Fe8, Cor.13]). *Let $f \in \mathfrak{m}_2 \subset \kappa[[\mathtt{x},\mathtt{y}]]$, $n \geqslant 1$ and $M \in \kappa\setminus\{0\}$. Let $C \subset$* Sper$(\kappa[[\mathtt{x},\mathtt{y}]])$ *be a constructible set and assume $f \in \mathcal{P}(C)$. We have:*

(i) *There exists $r \geqslant 1$ such that if $g \in \kappa[[\mathtt{x},\mathtt{y}]]$ and $f-g \in \mathfrak{m}_2^r$, then $g+M^2(\mathtt{x}^2+\mathtt{y}^2)^n \in \mathcal{P}(C)$.*
(ii) *If in addition $f(\mathtt{x},0) \neq 0$, there exists $r \geqslant 1$ such that if $g \in \kappa[[\mathtt{x},\mathtt{y}]]$ and $f-g \in \mathfrak{m}_2^r$, then $g+M^2\mathtt{y}^{2n} \in \mathcal{P}(C)$.*

We prove next that under some mild conditions the positive definite elements $f+\mathtt{z}g \in \mathcal{P}(A)$ satisfy $f \neq 0$. Before proving that we need a preliminary result.

nonzero **Lemma 6.3.** *Let $L|\kappa$ be an extension of fields. Let $f \in \kappa[[\mathtt{x}]]\setminus\{0\}$ and let $\eta \in L[[\mathtt{t}]]^n$ be such that $\eta(0)=0$. For each $k \geqslant 1$ the series $f(\eta+\mathtt{t}^k\mathtt{x}) \in L[[\mathtt{t},\mathtt{x}]]$ is non-zero.*

Proof. The maps $\phi : L[[\mathtt{t},\mathtt{x}]] \to L[[\mathtt{t},\mathtt{x}]],\ h(\mathtt{t},\mathtt{x}) \mapsto h(\mathtt{t},\eta+\mathtt{x})$ and

$$\psi : L[[\mathtt{t},\mathtt{x}]] \to L[[\mathtt{t},\mathtt{x}]],\ h(\mathtt{t},\mathtt{x}) \mapsto h(\mathtt{t},-\eta+\mathtt{x})$$

are well-defined isomorphisms mutually inverse. Thus, if $f \in \kappa[[\mathtt{x}]]\setminus\{0\}$, then $F(\mathtt{t},\mathtt{x}) := f(\eta+\mathtt{x}) \in L[[\mathtt{t},\mathtt{x}]]\setminus\{0\}$. Let us check that: $F(\mathtt{t},\mathtt{t}^k\mathtt{x}) \neq 0$.

Consider $F(\mathtt{t},\mathtt{x})$ as an element of $L[[\mathtt{t}]][[\mathtt{x}]]$ and write $F := \sum_{\ell\geqslant 0} F_\ell(\mathtt{t},\mathtt{x})$ where each $F_\ell \in L[[\mathtt{t}]][\mathtt{x}]$ is a homogeneous polynomial of degree ℓ. Let s be the smallest ℓ such that $F_\ell \neq 0$. Then

$$F(\mathtt{t},\mathtt{t}^k\mathtt{x}) = \sum_{\ell\geqslant s} F_\ell(\mathtt{t},\mathtt{t}^k\mathtt{x}) = \sum_{\ell\geqslant s} \mathtt{t}^{k\ell}F_\ell(\mathtt{t},\mathtt{x}).$$

As $\mathtt{t}^{k\ell}F_\ell(\mathtt{t},\mathtt{x}) \in L[[\mathtt{t}]][\mathtt{x}]$ is a homogeneous polynomial of degree ℓ and $F_s \neq 0$, we conclude $f(\eta+\mathtt{t}^k\mathtt{x}) = F(\mathtt{t},\mathtt{t}^k\mathtt{x}) \neq 0$, as required. □

fneq0 **Lemma 6.4.** *Let κ be a (formally) real field such that $\tau(\kappa) < +\infty$. Let $F \in \kappa[[\mathtt{x},\mathtt{y}]]$ be a series of order $\geqslant 1$ such that $-F \notin \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$. If $f+\mathtt{z}g \in \mathcal{P}(A)\setminus\{0\}$, then $f \neq 0$.*

Proof. Suppose $f=0$. As $f+\mathtt{z}g \in \mathcal{P}(A)\setminus\{0\}$, we know that $f \in \mathcal{P}(\{F \geqslant 0\})$ and $f^2-Fg^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$. Then $-Fg^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$. If $g(0,0) \neq 0$, then $-F \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$, which is a contradiction. Thus, $g(0,0)=0$ but $g \neq 0$ because $f+\mathtt{z}g \neq 0$. As $-Fg^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$, we have

$$\mathcal{S} := \{\alpha \in \operatorname{Sper}(\kappa[[\mathtt{x},\mathtt{y}]]) :\ F >_\alpha 0\} \subset \{\alpha \in \operatorname{Sper}(\kappa[[\mathtt{x},\mathtt{y}]]) :\ g \in \operatorname{supp}(\alpha)\} =: \mathcal{T}$$

As $-F \notin \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$, there exists $\beta \in \operatorname{Sper}(\kappa[[\mathtt{x},\mathtt{y}]]) :\ F >_\beta 0$. As $F(0,0) \neq 0$, the support of α if not $\mathfrak{m}_2$. By Lemma 5.19 we may assume $\operatorname{ht}(\operatorname{supp}(\beta))=1$. Let α be the ordering of κ induced by β. Let R be the real closure of $(\kappa,<_\alpha)$. Let β_0 be the prime cone of $R[[\mathtt{t}]]$ given by $\mathtt{t}>0$. Let $\eta \in R[[\mathtt{t}]]^2\setminus\{(0,0)\}$ be such that $\eta(0)=(0,0)$ and $\beta=\varphi^{-1}(\beta_0)$ under the homomorphism $\varphi : \kappa[[\mathtt{x},\mathtt{y}]] \to R[[\mathtt{t}]]^2,\ h \mapsto h(\eta)$.

Observe that $F(\eta) >_{\beta_0} 0$. Let $k := \omega(F(\eta)) \geqslant 1$ and write $F(\eta) = a\mathtt{t}^k+\cdots$ for some $a \in \kappa$ such that $a >_\alpha 0$. By Lemma 6.3 we have $G(\mathtt{t},\mathtt{x},\mathtt{y},\mathtt{z}) := g(\eta+\mathtt{t}^{k+1}(\mathtt{x},\mathtt{y},\mathtt{z})) \neq 0$. By the curve selection Lemma [ABR, Thm.VII.4.1] there exists $\theta := (\theta_0,\theta_1,\theta_2,\theta_3) \in \kappa[[\mathtt{t}]]^4$ such that $G(\theta) \neq 0$. If $\theta_0=0$, then $G(0,\theta_1,\theta_2,\theta_3)=g(0,0)=0$, which is a contradiction, so $\theta_0 \neq 0$ and after reparemeterizing we may assume $\theta_0=\lambda\mathtt{t}^q$ for some $q \geqslant 1$ and $\lambda \in \kappa\setminus\{0\}$. Thus,

$$F(\eta(\lambda\mathtt{t}^q)+(\lambda\mathtt{t}^q)^{k+1}(\theta_1,\theta_2,\theta_3)) = a\lambda^k\mathtt{t}^{qk}+\cdots,$$

so $G(\theta)=0$ (because $\mathcal{S} \subset \mathcal{T}$), which is a contradiction. Thus, $f \neq 0$, as required. □

pde

6.2. **Positive definite elements.** Let $F\in\kappa[[\mathtt{x},\mathtt{y}]]$ and $A:=\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F)$. Denote

$$\begin{aligned}
\mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{y}]]):=&\{f\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]):\ f>_\alpha 0\ \forall\alpha\in\operatorname{Sper}(\kappa[[\mathtt{x},\mathtt{y}]]),\ \operatorname{supp}(\alpha)\neq\mathfrak{m}_2\},\\
\mathcal{P}^+(\{F\geqslant 0\}):=&\{f\in\kappa[[\mathtt{x},\mathtt{y}]]:\ f\in\mathcal{P}(\{F\geqslant 0\}),\ f>_\alpha 0\\
&\qquad\forall\alpha\in\operatorname{Sper}(\kappa[[\mathtt{x},\mathtt{y}]]),\ \operatorname{supp}(\alpha)\neq\mathfrak{m}_2,\ F\geqslant_\alpha 0\},\\
\mathcal{P}^{\oplus}(A):=&\{f+\mathtt{z}g\in A:\ f\in\mathcal{P}^+(\{F\geqslant 0\})\text{ and }f^2-Fg^2\in\mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{y}]])\},\\
\mathcal{P}^+(A):=&\{f+\mathtt{z}g\in A:\ f+\mathtt{z}g\in\mathcal{P}(A),\ f+\mathtt{z}g>_\beta 0\ \forall\beta\in\operatorname{Sper}(A),\operatorname{supp}(\beta)\neq\mathfrak{m}_A\},\\
\mathcal{P}(A)=&\{f+\mathtt{z}g\in A:\ f\in\mathcal{P}(\{F\geqslant 0\})\text{ and }f^2-Fg^2\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])\}.
\end{aligned}$$

We have the following chain of inclusions:

$$\mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{y}]])\subset\mathcal{P}^+(\{F\geqslant 0\})\subset\mathcal{P}^{\oplus}(A)\subset\mathcal{P}^+(A)\subset\mathcal{P}(A).$$

We recall here the following approximation result from [Fe8] concerning positivity of the elements of $\kappa[[\mathtt{x},\mathtt{y}]]$.

psdkxy **Corollary 6.5** ([Fe8, Cor.14]). *Let $f\in\mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{y}]])$. There exists $r\geqslant 1$ such that if $g\in\kappa[[\mathtt{x},\mathtt{y}]]$ and $f-g\in\mathfrak{m}_2^r$, then $g\in\mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{y}]])$.*

The following result allows us to construct elements of $\mathcal{P}^{\oplus}(A)$ from elements of $\mathcal{P}(A)$ that are very similar to the original ones (and as close as needed in the $\mathfrak{m}_2$-adic topology).

pd+ **Lemma 6.6** (Construction of positive semidefinite elements,[Fe8, Lem.4.1]). *We have:*

(i) $\mathcal{P}^{\oplus}(A)\subset\mathcal{P}^+(A)$.

(ii) *Let $f+\mathtt{z}g\in\mathcal{P}(A)$ and let $f_i\in\kappa[[\mathtt{x},\mathtt{y}]]$ be such that $\sqrt[r]{(f+\sum_{i=1}^p f_i^2,g)}=\mathfrak{m}_2$ and $\sum_{i=1}^p f_i^2>_\beta 0$ for those $\beta\in\operatorname{Sper}(\kappa[[\mathtt{x},\mathtt{y}]])$ satisfying:* $\operatorname{supp}(\beta)\neq\mathfrak{m}_2$, $F\geqslant_\beta 0$ *and either $f\in\operatorname{supp}(\beta)$ or $f^2-Fg^2\in\operatorname{supp}(\beta)$. Then $f+\sum_{i=1}^p f_i^2+\mathtt{z}g\in\mathcal{P}^{\oplus}(A)$.*

(iii) *Let $f+\mathtt{z}g\in\mathcal{P}(A)$ be such that $f(\mathtt{x},0)\neq 0$, $f^2(\mathtt{x},0)-F(\mathtt{x},0)g^2(\mathtt{x},0)\neq 0$ and $g\neq 0$. For each $M\in\kappa\backslash\{0\}$ there exists a finite set $S\subset\mathbb{N}$ such that $f+M^2\mathtt{y}^{2n}+\mathtt{z}g\in\mathcal{P}^{\oplus}(A)$ for each $n\in\mathbb{N}\backslash S$.*

(iv) *Let $f+\mathtt{z}g\in\mathcal{P}(A)$ be such that $g\neq 0$. Fix $M\in\kappa\backslash\{0\}$ and let h_n be either $M^2(\mathtt{x}^2+\mathtt{y}^2)^n$ or $M^2(\mathtt{x}^{2n}+\mathtt{y}^{2n})$. There exists a finite set $S\subset\mathbb{N}$ such that $f+h_n+\mathtt{z}g\in\mathcal{P}^{\oplus}(A)$ for each $n\in\mathbb{N}\backslash S$.*

(v) *If $f\in\mathcal{P}(\{F\geqslant 0\})$, then $f+\mathtt{z}f=(1+\mathtt{z})f\in\mathcal{P}(A)$.*

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7. Reduction to case when the Pythagoras number is finite

The purpose of this section is to reduce the proof of Theorem 2.3 to the case when Pythagoras numbers are finite. To that end we will use Tougeron's Implicit Function Theorem [T].

7.1. **Tougeron's Implicit Function Theorem.** We recall next for the sake of completeness the statement of celebrated Tougeron's Implicit Function Theorem [T] (see also [Rz1, Ch.V.Prop.1.3]).

tift **Theorem 7.1** ([T]). *Denote $\mathtt{x}:=(\mathtt{x}_1,\ldots,\mathtt{x}_n)$, $\mathtt{y}:=(\mathtt{y}_1,\ldots,\mathtt{y}_q)$ and let $F:=(F_1,\ldots,F_p)\in\kappa[[\mathtt{x},\mathtt{y}]]^p$ be such that $F(0,0)=0$ for some $p\leqslant q$. Let $M:=(\frac{\partial F_i}{\partial\mathtt{y}_j}(\mathtt{x},0))_{1\leqslant i\leqslant q,\ 1\leqslant j\leqslant p}$ be the Jacobian matrix of F and let $\mathfrak{a}\subset\kappa[[\mathtt{x}]]$ be the ideal generated by the p-minors of M and $\mathfrak{a}'\subset\kappa[[\mathtt{x}]]$ be another ideal. Suppose $F_1(\mathtt{x},0),\ldots,F_p(\mathtt{x},0)\in\mathfrak{a}\mathfrak{a}'^2$. Then there exist series $y_1(\mathtt{x}),\ldots,y_q(\mathtt{x})\in\mathfrak{a}\mathfrak{a}'$ such that $F(\mathtt{x},y_1(\mathtt{x}),\ldots,y_q(\mathtt{x}))=0$.*

7.2. **Solution of a system of polynomial equations.** As an application of Tougeron's Implicit Function Theorem we prove the following reduction result strongly inspired by [Rz1, Ch.5.Prop.2.7].

Let $f_1,\dots,f_r,h_1,\dots,h_s\in\mathfrak{m}_2\subset\kappa[[\mathtt{x}]]:=\kappa[[\mathtt{x}_1,\mathtt{x}_2]]$ be a finite family of pairwise relatively prime irreducible series and let $\alpha_i,\beta_j\geqslant 1$ be positive integers. Define $f:=\prod_{i=1}^r f_i^{\alpha_i}$ and $h:=\prod_{j=1}^s h_j^{\beta_j}$ and assume that there exist series $g,F\in\kappa[[\mathtt{x}]]$ such that $h=f^2-Fg^2$. As f and h are relatively prime, also f and Fg are relatively prime. Consider variables $\mathtt{Z}:=(\mathtt{Z}_1,\dots,\mathtt{Z}_r),\mathtt{T}$ and the series

$$\prod_{i=1}^r(f_i+\mathtt{Z}_ih)^{2\alpha_i}-F(g+\mathtt{T}h)^2=f^2-Fg^2+h\Gamma(\mathtt{x},\mathtt{Z},\mathtt{T})=h(1+\Gamma(\mathtt{x},\mathtt{Z},\mathtt{T}))\in\kappa[[\mathtt{x},\mathtt{Z},\mathtt{T}]] \quad (7.1)$$ hprima

for some series $\Gamma\in\kappa[[\mathtt{x},\mathtt{Z},\mathtt{T}]]$. Observe that

$$\begin{aligned}h(1+\Gamma(\mathtt{x},\mathtt{Z},\mathtt{T}))&=\prod_{i=1}^r\Big(\sum_{k_i=0}^{2\alpha_i}\binom{2\alpha_i}{k_i}f_i^{2\alpha_i-k_i}\mathtt{Z}_i^{k_i}h^{k_i}\Big)-Fg^2-2Fgh\mathtt{T}-Fh^2\mathtt{T}^2\\&=h+\sum_{i=1}^r\frac{f^2}{f_i}2\alpha_i\mathtt{Z}_ih-2Fgh\mathtt{T}+h^2\cdot(\text{terms of degree}\geqslant 2\text{ in the variables }\mathtt{Z},\mathtt{T}).\end{aligned}$$

Thus, $\Gamma(\mathtt{x},0,0)=0$ and

$$\Gamma(\mathtt{x},\mathtt{Z},\mathtt{T})=\sum_{i=1}^r\frac{f^2}{f_i}2\alpha_i\mathtt{Z}_i-2Fg\mathtt{T}+h\cdot(\text{terms of degree}\geqslant 2\text{ in the variables }\mathtt{Z},\mathtt{T}).$$

Let $a_1,\dots,a_m\in\mathfrak{m}_2\subset\kappa[[\mathtt{x}]]:=\kappa[[\mathtt{x}_1,\mathtt{x}_2]]$ be a finite family of pairwise relatively prime series and $\gamma_k\geqslant 1$ positive integers. Define $a:=\prod_{k=1}^m a_k^{\gamma_k}$ and consider the (finite) system of polynomial equations:

$$\begin{cases}(1+\mathtt{W}_k)a_k(\mathtt{x}+\mathtt{Y})=a_k^*,\\ \prod_{k=1}^m(1+\mathtt{W}_k)^{\gamma_k}=1,\\ (1+\mathtt{U}_i)(f_i(\mathtt{x}+\mathtt{Y})+\mathtt{z}_ih(\mathtt{x}+\mathtt{Y}))=f_i^*(\mathtt{x}),\\ (1+\mathtt{V}_j)h_j(\mathtt{x}+\mathtt{Y})(1+\Gamma(\mathtt{x}+\mathtt{Y},\mathtt{Z},\mathtt{Y}))^{1/(\beta_js)}=h_j^*(\mathtt{x}),\\ \prod_{i=1}^r(1+\mathtt{U}_i)^{2\alpha_i}=\prod_{j=1}^s(1+\mathtt{V}_j)^{\beta_j}\end{cases} \quad (7.2)$$ system

for some polynomials $a_k^*,f_i^*,h_j^*\in\kappa[\mathtt{x}]$. Denote $a^*:=\prod_{k=1}^m(a_k^*)^{\gamma_k}\in\kappa[\mathtt{x}]$, $f^*:=\prod_{i=1}^r(f_i^*(\mathtt{x}))^{\alpha_i}$ and $h^*:=\prod_{j=1}^s(h_j^*(\mathtt{x}))^{\beta_j}$.

mainred0 **Lemma 7.2.** *Let $(\mathtt{Y},\mathtt{Z},\mathtt{T},\mathtt{U}_i,\mathtt{V}_j,\mathtt{W}_k)=(\psi,\lambda,\mu,u_i,v_j,w_k)$ be a solution of the system of equations (7.2) such that each series belong to $\mathfrak{m}_2^c\subset\kappa[[\mathtt{x}]]$ for some $c\geqslant 2$. Let $\varphi:=(\varphi_1,\varphi_2)$ be the inverse change of coordinates of the change of coordinates $\phi:=(\phi_1,\phi_2):=(\mathtt{x}_1+\psi_1,\mathtt{x}_2+\psi_2)$. Denote $f':=\prod_{i=1}^r(f_i+\lambda_i(\varphi)h)^{\alpha_i}$, $g':=g+\mu(\varphi)h$, $h':=f'^2-Fg'^2=h\cdot(1+\Gamma(\mathtt{x},\lambda(\varphi),\mu(\varphi)))$, $u:=\prod_{i=1}^r(1+u_i)^{\alpha_i}$ and $v:=\prod_{j=1}^s(1+v_j)^{\beta_j}$. Then*

- (i) *$u,v\in\kappa[[\mathtt{x}]]$ are units such that $u(0,0)=1$, $u-1\in\mathfrak{m}_2^c$, $v(0,0)=1$, $v-1\in\mathfrak{m}_2^c$, $u^2=v$*
- (ii) *$\phi_i-\mathtt{x}_i\in\mathfrak{m}_2^c$ for $i=1,2$,*
- (iii) *$a(\phi)=a^*$, $uf'(\phi)=f^*\in\kappa[\mathtt{x}]$, $vh'(\phi)=h^*\in\kappa[\mathtt{x}]$ and*
- (iv) *$f'+\mathtt{z}g'=(f+\mathtt{z}g)W+q(\mathtt{z}^2-F)$ for some unit $W\in\kappa[[\mathtt{x},\mathtt{z}]]$ with $W(0,0,0)=1$ and some series $q\in\kappa[[\mathtt{x},\mathtt{z}]]$.*

Proof. As $\prod_{i=1}^r(1+u_i)^{2\alpha_i}=\prod_{j=1}^s(1+v_j)^{\beta_j}$, we have $u^2=v$. Observe that $\prod_{k=1}^m(1+w_k)^{\gamma_k}=1$. Thus,

$$a^*=\prod_{k=1}^m(a_k^*)^{\gamma_k}=\prod_{k=1}^m(a_k^*)^{\gamma_k}(1+w_k)^{\gamma_k}(a_k(\mathtt{x}+\psi))^{\gamma_k}=\prod_{k=1}^m(a_k(\mathtt{x}+\psi))^{\gamma_k}=a(\mathtt{x}+\psi)=a(\phi).$$

In addition,

$$\begin{aligned}h^* &= \prod_{j=1}^{s}(h_j^*(\mathtt{x}))^{\beta_j} = \prod_{j=1}^{s}((1+v_j)h_j(\mathtt{x}+\psi)(1+\Gamma(\mathtt{x}+\psi,\lambda,\mu))^{1/(\beta_j s)})^{\beta_j}\\ &= \Big(\prod_{j=1}^{s}(1+v_j)^{\beta_j}\prod_{j=1}^{s}(h_j(\mathtt{x}+\psi))^{\beta_j}\Big)(1+\Gamma(\mathtt{x}+\psi,\lambda,\mu))\\ &= vh(\mathtt{x}+\psi)(1+\Gamma(\mathtt{x}+\psi,\lambda,\mu)) = vh'(\phi)\end{aligned}$$

and

$$\begin{aligned}f^* &= \prod_{i=1}^{r}(f_i^*(\mathtt{x}))^{\alpha_i} = \prod_{i=1}^{r}((1+u_i)(f_i(\mathtt{x}+\psi)+\lambda_i h(\mathtt{x}+\psi)))^{\alpha_i}\\ &= \Big(\prod_{i=1}^{r}(1+u_i)^{\alpha_i}\Big)\prod_{i=1}^{r}((f_i(\mathtt{x}+\psi)+\lambda_i h(\mathtt{x}+\psi)))^{\alpha_i}\\ &= u\prod_{i=1}^{r}((f_i(\mathtt{x}+\psi)+\lambda_i h(\mathtt{x}+\psi)))^{\alpha_i} = uf'(\phi).\end{aligned}$$

Observe that

$$\begin{aligned}f'+\mathtt{z}g' = \prod_{i=1}^{r}(f_i+\lambda_i h)^{\alpha_i}+\mathtt{z}(g+\mu h) &= f+\mathtt{z}g+h(\Lambda_1+\mathtt{z}\Lambda_2)\\ &= f+\mathtt{z}g+(f^2-\mathtt{z}^2g^2)(\Lambda_1+\mathtt{z}\Lambda_2)+(\mathtt{z}^2-F)(\Lambda_1+\mathtt{z}\Lambda_2)\end{aligned}$$

for some series $\Lambda_i \in \kappa[[\mathtt{x},\mathtt{y}]]$. Consequently,

$$f'+\mathtt{z}g' = (f+\mathtt{z}g)(1+(f-\mathtt{z}g)(\Lambda_1+\mathtt{z}\Lambda_2))+(\mathtt{z}^2-F)(\Lambda_1+\mathtt{z}\Lambda_2) = (f+\mathtt{z}g)W+(\mathtt{z}^2-F)q,$$

where $W := (1+(f-\mathtt{z}g)(\Lambda_1+\mathtt{z}\Lambda_2)) \in \kappa[[\mathtt{x},\mathtt{z}]]$ is a unit such that $W(0,0,0)=1$ and $q := \Lambda_1+\mathtt{z}\Lambda_2 \in \kappa[[\mathtt{x},\mathtt{z}]]$, as required. □

rpps **Lemma 7.3.** *Let $f_1,\dots,f_r \in \kappa[\mathtt{x},\mathtt{y}]$ be relatively prime series. Then $\mathfrak{a} := (f_1,\dots,f_r)\kappa[[\mathtt{x},\mathtt{y}]]$ contains a power of $\mathfrak{m}_2$.*

Proof. It is enough to prove that $\mathfrak{a}$ has height 2. Suppose that there exists a prime ideal $\mathfrak{p}$ of height 1 that contains $\mathfrak{a}$. As $\kappa[[\mathtt{x},\mathtt{y}]]$ is a UFD, there exists $h \in \kappa[[\mathtt{x},\mathtt{y}]]$ such that $\mathfrak{a} \subset \mathfrak{p} = h\kappa[[\mathtt{x},\mathtt{y}]]$, so h divides $f_1,\dots,f_r$, which is a contradiction because $f_1,\dots,f_r$ are relatively prime. Thus, $\mathfrak{a}$ has height 2, as required. □

The following result guarantees that under some conditions the polynomial system (7.2) has a solution whose components belong to $\mathfrak{m}_2^c$

mainred **Lemma 7.4.** *For each $c \geqslant 2$ there exists $n \geqslant 1$ such that if $a_k^*-a_k, f_i^*-f_i, h_j^*-h_j \in \mathfrak{m}_2^n$ for each k,i,j, then the polynomial system (7.2) has a solution whose components belong to $\mathfrak{m}_2^c$.*

Proof. The proof is conducted in several steps:

STEP 1. *Choice of the suitable p-minors of the Jacobian matrix of the polynomial system* (7.2). Denote $p := (m+1)+(r+s+1)$ and let $\mathfrak{a}$ be the ideal generated by the p-minors of the Jacobian matrix (of the polynomial system (7.2) substituted at $(\mathtt{x},0,0,0,0,0,0)$) of size $p\times(p+r+1)$:

$$M:=\left(\begin{array}{cc|ccc|ccc|ccc|ccc|c}
\frac{\partial a_1}{\partial \mathtt{x}_1} & \frac{\partial a_1}{\partial \mathtt{x}_2} & a_1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\frac{\partial a_m}{\partial \mathtt{x}_1} & \frac{\partial a_m}{\partial \mathtt{x}_2} & 0 & \cdots & a_m & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\
\hline
0 & 0 & \gamma_1 & \cdots & \gamma_m & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\
\hline
\frac{\partial f_1}{\partial \mathtt{x}_1} & \frac{\partial f_1}{\partial \mathtt{x}_2} & 0 & \cdots & 0 & f_1 & \cdots & 0 & 0 & \cdots & 0 & h & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\frac{\partial f_r}{\partial \mathtt{x}_1} & \frac{\partial f_r}{\partial \mathtt{x}_2} & 0 & \cdots & 0 & 0 & \cdots & f_r & 0 & \cdots & 0 & 0 & \cdots & h & 0\\
\hline
\frac{\partial h_1}{\partial \mathtt{x}_1} & \frac{\partial h_1}{\partial \mathtt{x}_2} & 0 & \cdots & 0 & 0 & \cdots & 0 & h_1 & \cdots & 0 & \frac{2\alpha_1}{\beta_1 s}\frac{f^2}{f_1} & \cdots & \frac{2\alpha_r}{\beta_1 s}\frac{f^2}{f_r} & -\frac{2Fg}{\beta_1 s}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\frac{\partial h_s}{\partial \mathtt{x}_1} & \frac{\partial h_s}{\partial \mathtt{x}_2} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & h_s & \frac{2\alpha_1}{\beta_s s}\frac{f^2}{f_1} & \cdots & \frac{2\alpha_r}{\beta_s s}\frac{f^2}{f_r} & -\frac{2Fg}{\beta_s s}\\
\hline
0 & 0 & 0 & \cdots & 0 & 2\alpha_1 & \cdots & 2\alpha_r & -\beta_1 & \cdots & -\beta_s & 0 & \cdots & 0 & 0
\end{array}\right). \tag{7.3}$$

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Let us analyze several p-minors of M:

First minors: Consider the p-minors of M constituted by the columns of M corresponding to the variables $\mathtt{y}_i$, $\mathtt{w}_1,\dots,\mathtt{w}_m$, $\mathtt{u}_1,\dots,\mathtt{u}_r$, $\mathtt{v}_1,\dots,\mathtt{v}_s$, $\mathtt{z}_j$. Namely, the determinants of the matrices:

$$M_{ij}:=\left(\begin{array}{c|ccc|ccc|ccc|c}
\frac{\partial a_1}{\partial \mathtt{x}_i} & a_1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\frac{\partial a_m}{\partial \mathtt{x}_i} & 0 & \cdots & a_m & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\
\hline
0 & \gamma_1 & \cdots & \gamma_m & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\
\hline
\frac{\partial f_1}{\partial \mathtt{x}_i} & 0 & \cdots & 0 & f_1 & \cdots & 0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & h\ (j)\\
\frac{\partial f_r}{\partial \mathtt{x}_i} & 0 & \cdots & 0 & 0 & \cdots & f_r & 0 & \cdots & 0 & 0\\
\hline
\frac{\partial h_1}{\partial \mathtt{x}_i} & 0 & \cdots & 0 & 0 & \cdots & 0 & h_1 & \cdots & 0 & \frac{2\alpha_j}{\beta_1 s}\frac{f^2}{f_j}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\frac{\partial h_s}{\partial \mathtt{x}_i} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & h_s & \frac{2\alpha_j}{\beta_s s}\frac{f^2}{f_j}\\
\hline
0 & 0 & \cdots & 0 & 2\alpha_1 & \cdots & 2\alpha_r & -\beta_1 & \cdots & -\beta_s & 0
\end{array}\right), \tag{7.4}$$

which are equal to the products of the determinants of the square matrices:

$$A_i:=\left(\begin{array}{c|ccc}
\frac{\partial a_1}{\partial \mathtt{x}_i} & a_1 & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots\\
\frac{\partial a_m}{\partial \mathtt{x}_i} & 0 & \cdots & a_m\\
\hline
0 & \gamma_1 & \cdots & \gamma_m,
\end{array}\right), \tag{7.5}$$

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whose determinant is

$$\rho_i:=\det(A_i)=\pm\sum_{k=1}^{m}\gamma_k\frac{\partial a_k}{\partial \mathtt{x}_i}\prod_{\ell\neq k}a_\ell \tag{7.6}$$

rhoi

(for $i=1,2$), and

$$B_j := \left(\begin{array}{ccc|ccc|c} f_1 & \cdots & 0 & 0 & \cdots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & h\ (j)\\ 0 & \cdots & f_r & 0 & \cdots & 0 & 0\\ \hline 0 & \cdots & 0 & h_1 & \cdots & 0 & \frac{2\alpha_j}{\beta_1 s}\frac{f^2}{f_j}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & 0 & 0 & \cdots & h_s & \frac{2\alpha_j}{\beta_s s}\frac{f^2}{f_j}\\ \hline 2\alpha_1 & \cdots & 2\alpha_r & -\beta_1 & \cdots & -\beta_s & 0 \end{array}\right),$$

whose determinant is

$$\eta_j := \det(B_j) = \pm\Big(-h\prod_{k=1}^s h_k + \frac{1}{s}f^2\sum_{k=1}^s\prod_{\ell\neq k}h_\ell\Big)2\alpha_j\prod_{i\neq j}f_i$$

(for $j=1,\dots,r$). A crucial computation is that

$$\det\left(\begin{array}{ccc|c} h_1 & \cdots & 0 & \frac{1}{\beta_1}\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & h_s & \frac{1}{\beta_s}\\ \hline -\beta_1 & \cdots & -\beta_s & 0 \end{array}\right) = \sum_{k=1}^s\prod_{\ell\neq k}h_\ell$$

and the element $\sum_{k=1}^s\prod_{\ell\neq k}h_\ell = 1$ in case $s=1$.

Second minors: Constituted by the columns of M corresponding to the variables $\mathtt{y}_i$, $\mathtt{w}_1,\dots,\mathtt{w}_m$, $\mathtt{u}_1,\dots,\mathtt{u}_r$, $\mathtt{v}_1,\dots,\mathtt{v}_s$, $\mathtt{t}$. Namely, the determinants of the matrices:

$$M_i := \left(\begin{array}{c|ccc|ccc|ccc|c} \frac{\partial a_1}{\partial \mathtt{x}_i} & a_1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \frac{\partial a_m}{\partial \mathtt{x}_i} & 0 & \cdots & a_m & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\ \hline 0 & \gamma_1 & \cdots & \gamma_m & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\ \hline \frac{\partial f_1}{\partial \mathtt{x}_i} & 0 & \cdots & 0 & f_1 & \cdots & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \frac{\partial f_r}{\partial \mathtt{x}_i} & 0 & \cdots & 0 & 0 & \cdots & f_r & 0 & \cdots & 0 & 0\\ \hline \frac{\partial h_1}{\partial \mathtt{x}_i} & 0 & \cdots & 0 & 0 & \cdots & 0 & h_1 & \cdots & 0 & -\frac{2Fg}{\beta_1 s}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \frac{\partial h_s}{\partial \mathtt{x}_i} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & h_s & -\frac{2Fg}{\beta_s s}\\ \hline 0 & 0 & \cdots & 0 & 2\alpha_1 & \cdots & 2\alpha_r & -\beta_1 & \cdots & -\beta_s & 0 \end{array}\right), \tag{7.7}$$

which are equal to the products of the determinants of the square matrices A_i introduced in (7.5) whose determinant is ρ_i (for $i=1,2$, see (7.6)), and

$$B := \left(\begin{array}{ccc|ccc|c} f_1 & \cdots & 0 & 0 & \cdots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0\\ 0 & \cdots & f_r & 0 & \cdots & 0 & 0\\ \hline 0 & \cdots & 0 & h_1 & \cdots & 0 & -\frac{2Fg}{\beta_1 s}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & 0 & 0 & \cdots & h_s & -\frac{2Fg}{\beta_s s}\\ \hline 2\alpha_1 & \cdots & 2\alpha_r & -\beta_1 & \cdots & -\beta_s & 0 \end{array}\right),$$

whose determinant is

$$\xi := \det(B) = \Big(-2\frac{1}{s}Fg\sum_{k=1}^{s}\prod_{\ell\neq k}h_\ell\Big)\prod_{i=1}^{r}f_i.$$

STEP 2. *Computation of the radical ideal of the ideal $\mathfrak{a}$ generated by the p-minors of M.* Consider the ideal $\mathfrak{b}\subset\mathfrak{a}$ generated by the minors chosen in STEP 1, that is,

$$\mathfrak{b} := (\rho_1\eta_1,\dots,\rho_1\eta_r,\rho_2\eta_1,\dots,\rho_2\eta_r,\rho_1\xi,\rho_2\xi).$$

Consider the ideals $\mathfrak{b}_1 := (\rho_1,\rho_2)$ and $\mathfrak{b}_2 := (\eta_1,\dots,\eta_r,\xi)$. Observe that $\mathfrak{b} = \mathfrak{b}_1\cdot\mathfrak{b}_2$ and $\sqrt{\mathfrak{b}} = \sqrt{\mathfrak{b}_1\cdot\mathfrak{b}_2} = \sqrt{\mathfrak{b}_1\cap\mathfrak{b}_2} = \sqrt{\mathfrak{b}_1}\cap\sqrt{\mathfrak{b}_2}$. We claim: $\sqrt{\mathfrak{b}_1} = \mathfrak{m}_2$ *and* $\sqrt{\mathfrak{b}_2} = \mathfrak{m}_2$. Once this is proved we conclude: $\sqrt{\mathfrak{b}} = \mathfrak{m}_2$, so $\sqrt{\mathfrak{a}} = \mathfrak{m}_2$, that is, $\mathfrak{a}$ contains a power of $\mathfrak{m}_2$.

CASE 2.1. We prove first $\sqrt{\mathfrak{b}_1} = \mathfrak{m}_2$.

Define $a^\bullet := a_1\cdots a_m$. We claim: $a^\bullet, \frac{\partial a^\bullet}{\partial \mathtt{x}_1}, \frac{\partial a^\bullet}{\partial \mathtt{x}_2} \in \sqrt{\mathfrak{b}_1}$.

Let $\overline{\kappa}$ be the algebraic closure of κ. By Rückert's Nullstellensatz [Rz1, Prop.IV.2.1] we have to check

$$a^\bullet(\zeta(\mathtt{t})) = 0, \frac{\partial a^\bullet}{\partial \mathtt{x}_1}(\zeta(\mathtt{t})) = 0, \frac{\partial a^\bullet}{\partial \mathtt{x}_2}(\zeta(\mathtt{t})) = 0$$

for each $\zeta(\mathtt{t}) \in \mathcal{Z}(\mathfrak{b}_1)$. Recall that

$$\pm\rho_i = \sum_{j=1}^{m}\alpha_j a_1\cdots\frac{\partial a_j}{\partial \mathtt{x}_i}\cdots a_m.$$

Thus, $\frac{\partial a}{\partial \mathtt{x}_i} = \pm a_1^{\alpha_1-1}\cdots a_m^{\alpha_m-1}\rho_i \in \mathfrak{b}_1$, so $\frac{\partial a}{\partial \mathtt{x}_i}(\zeta(\mathtt{t})) = 0$. This holds for $i = 1,2$, and by [Rz1, Cor.IV.2.5] we deduce $a(\zeta(\mathtt{t})) = 0$, so we may assume for simplicity $a_1(\zeta(\mathtt{t})) = 0$. Consequently, $a^\bullet(\zeta(\mathtt{t})) = 0$. In addition, if we substitute $\mathtt{x} = \zeta(\mathtt{t})$ in a_i, $\frac{\partial a^\bullet}{\partial \mathtt{x}_i}$ and take into account that $a_1(\zeta(\mathtt{t})) = 0$, we have

$$0 = \rho_i(\zeta(\mathtt{t})) = \pm\alpha_1\frac{\partial a_1}{\partial \mathtt{x}_i}(\zeta(\mathtt{t}))a_2(\zeta(\mathtt{t}))\cdots a_m(\zeta(\mathtt{t})) = \pm\alpha_1\frac{\partial a^\bullet}{\partial \mathtt{x}_i}(\zeta(\mathtt{t})),$$

as claimed.

We claim next: $\operatorname{ht}(\mathfrak{b}_1) \geqslant 2$, *that is,* $\mathfrak{b}_1$ *contains a power of* $\mathfrak{m}$.

As $a \in \sqrt{\mathfrak{b}_1}\setminus\{0\}$, we suppose $\operatorname{ht}(\mathfrak{b}_1) = 1$, so there exists a prime ideal $\mathfrak{p} \supset \mathfrak{b}_1$ of height 1. As the ring $\kappa[[\mathtt{x}]]$ is factorial [Rz1, Prop.II.2.1] $\mathfrak{p}$ is principal, say generated by an irreducible series b. Thus,

$$\sqrt{\mathfrak{a}} \subset \mathfrak{p} = b\kappa[[\mathtt{x}]].$$

By the previous claim we deduce that b divides $a^\bullet$ and its derivatives $\frac{\partial a^\bullet}{\partial \mathtt{x}_i}$ for $i = 1,2$. We may assume $b = a_1$ and we also have

$$\frac{\partial a^\bullet}{\partial \mathtt{x}_i} = \frac{\partial a_1}{\partial \mathtt{x}_i}a_2\cdots a_m + a_1\sum_{j=2}^{m}g_2\cdots\frac{\partial a_j}{\partial \mathtt{x}_i}\cdots a_m.$$

Thus, a_1 divides $\frac{\partial a_1}{\partial \mathtt{x}_i}a_2\cdots a_m$, so it divides $\frac{\partial a_1}{\partial \mathtt{x}_i}$ for $i = 1,2$, which is a contradiction. Thus, $\operatorname{ht}(\mathfrak{b}_1) \geqslant 2$, as claimed. ∎

CASE 2.2. We prove next $\sqrt{\mathfrak{b}_2} = \mathfrak{m}_2$. By Lemma 7.3 we have to check that the series $\eta_1,\dots,\eta_r,\xi$ are relatively prime. Recall that the series f_i, f_j are relatively prime and $f = \prod_{i=1}^{r} f_i^{\alpha_i}$ and Fg are relatively prime. Let $b \in \kappa[[\mathtt{x},\mathtt{y}]]$ be a common divisor of $\eta_1,\dots,\eta_r,\xi$. Suppose $b = f_i$ for some i and for simplicity assume $b = f_1$. Then b divides

$$\eta_1 = \pm\Big(-h\prod_{k=1}^{s}h_k + \frac{1}{s}f^2\sum_{k=1}^{s}\prod_{\ell\neq k}h_\ell\Big)\prod_{i\neq 1}f_i,$$

so b divides $-h\prod_{k=1}^s h_k + \frac{1}{s}f^2\sum_{k=1}^s\prod_{\ell\neq k}h_\ell$. As b divides f^2, we deduce that b divides $-h\prod_{k=1}^s h_k$, so b divides both f and $h = f^2 - Fg^2$. Thus, b divides Fg^2, which is a contradiction because f and Fg^2 are relatively prime. Consequently, b is relatively prime to all the series $f_1,\dots,f_r$. As b divides η, we deduce b divides $-2\frac{1}{s}Fg\sum_{k=1}^s\prod_{\ell\neq k}h_\ell$. If b divides $\sum_{k=1}^s\prod_{\ell\neq k}h_\ell$, then $s\geqslant 2$ and b divides $-h\prod_{k=1}^s h_k$, so we may assume $b = h_1$. As b divides $\sum_{k=1}^s\prod_{\ell\neq k}h_\ell$, we deduce that b divides $\prod_{\ell\neq 1}h_\ell$, so we may assume b divides also h_2, which is a contradiction. Consequently, b divides Fg^2. As b divides both Fg^2 and

$$-h\prod_{k=1}^s h_k + \frac{1}{s}f^2\sum_{k=1}^s\prod_{\ell\neq k}h_\ell = f^2\Big(-\prod_{k=1}^s h_k + \frac{1}{s}\sum_{k=1}^s\prod_{\ell\neq k}h_\ell\Big) + Fg^2\prod_{k=1}^s h_k,$$

we deduce b divides both $\zeta := -\prod_{k=1}^s h_k + \frac{1}{s}\sum_{k=1}^s\prod_{\ell\neq k}h_\ell$ and Fg. In addition, we know that b is not an irreducible factor of f nor of h. If $s = 1$, then $\zeta = -h_1 + 1$, so it is a unit, which us a contradiction. Thus, $s\geqslant 2$ and

$$\frac{1}{h_1} + \cdots + \frac{1}{h_s} = s$$

in the quotient field $\operatorname{qf}(\kappa[[\mathtt{x},\mathtt{y}]]/(b)\kappa[[\mathtt{x},\mathtt{y}]])$.

Let $b_1,\dots,b_n\in\kappa[[\mathtt{x},\mathtt{y}]]$ be the irreducible factors of Fg. Let us modify $h_1,\dots,h_s$ to guarantee that ζ and Fg are relatively prime or equivalently

$$\frac{1}{h_1} + \cdots + \frac{1}{h_s} \neq s$$

in the quotient field $\operatorname{qf}(\kappa[[\mathtt{x},\mathtt{y}]]/(b_q)\kappa[[\mathtt{x},\mathtt{y}]])$ for $q = 1,\dots,n$. To that we choose suitable elements $\theta_1,\dots,\theta_s\in\kappa\backslash\{0\}$ such that $\theta_1^{\beta_1}\cdots\theta_s^{\beta_s} = 1$ and substitute h_i by $\frac{h_i}{\theta_i}$. In this way, we have $h = \prod_{i=1}^s h_i^{\beta_i} = \prod_{i=1}^s(\theta_i h_i)^{\beta_i}$. To find the θ_i define

$$\theta_i := \frac{\nu_i^{(\beta_1+\cdots+\beta_s)\beta_1\cdots\beta_{i-1}\beta_{i+1}\cdots\beta_s}}{(\nu_1\cdots\nu_s)^{\beta_1\cdots\beta_s}}$$

and observe that if $\nu_1,\dots,\nu_s\in\kappa\backslash\{0\}$, then $\theta_1^{\beta_1}\cdots\theta_s^{\beta_s} = 1$. Let $\overline{\kappa}$ be the algebraic closure of κ and let $\alpha_q\in\overline{\kappa}[[\mathtt{t}]]^2$ be such that $b_q(\alpha_q) = 0$ for $q = 1,\dots,n$. As b_q is not an irreducible factor of h_i, we deduce $h_i(\alpha_q)\neq 0$ for each pair i,q. Let $\ell_q := \max\{\omega(h_i(\alpha_q)) :\ i = 1,\dots,s\} > 0$ for $q = 1,\dots,n$. Write

$$\frac{1}{h_i(\alpha_q)} := \frac{1}{\mathtt{t}^{\ell_q}}\sum_{e\geqslant 0}c_{ieq}\mathtt{t}^k$$

and observe that $c_{i0q}\neq 0$ for some $q = 1,\dots,n$. Consider the non-zero polynomial

$$H := \prod_{q=1}^n\sum_{i=1}^s c_{i0q}\nu_i^{(\beta_1+\cdots+\beta_s)\beta_1\cdots\beta_{i-1}\beta_{i+1}\cdots\beta_s}$$

and choose $\nu_1,\dots,\nu_s\in\kappa\backslash\{0\}$ such that $H(\nu_1,\dots,\nu_s)\neq 0$. Thus,

$$\omega\Big(\frac{\theta_1}{h_1(\alpha_q)} + \cdots + \frac{\theta_s}{h_s(\alpha_q)}\Big) < 0$$

for $q = 1,\dots,n$ and in particular

$$\frac{\theta_1}{h_1} + \cdots + \frac{\theta_s}{h_s} \neq s$$

in the quotient field $\operatorname{qf}(\kappa[[\mathtt{x},\mathtt{y}]]/(b_q)\kappa[[\mathtt{x},\mathtt{y}]])$ for $q = 1,\dots,n$. Consequently, after this choice we guarantee that $\mathfrak{b}_2$ contains a power of $\mathfrak{m}$. ■

STEP 3. *Application of Tougeron's Implicit Function Theorem.* We consider now the ideal $\mathfrak{g} := \mathfrak{m}_2^c\mathfrak{a}^2\subset\mathfrak{m}$. As $\mathfrak{g}$ contains a power of $\mathfrak{m}$, the canonical homomorphism $\kappa\to\kappa[[\mathtt{x}_1,\mathtt{x}_2]]/\mathfrak{g}$ is finite.

Consequently, there exist monic polynomials $P_1(\mathtt{t}), P_2(\mathtt{t}) \in \kappa[\mathtt{t}]$ such that $P_1(\mathtt{x}_1), P_2(\mathtt{x}_2) \in \mathfrak{g}$. By Rückert's Division Theorem we obtain

$$\begin{aligned} a_k &= a_k^* + Q_{01k}P_1(\mathtt{x}_1) + Q_{02k}P_2(\mathtt{x}_2),\\ f_i &= f_i^* + Q_{11i}P_1(\mathtt{x}_1) + Q_{12i}P_2(\mathtt{x}_2),\\ h_j &= h_j^* + Q_{21j}P_1(\mathtt{x}_1) + Q_{22j}P_2(\mathtt{x}_2), \end{aligned}$$

where each $a_k^*, f_i^*, h_j^* \in \kappa[\mathtt{x}_1,\mathtt{x}_2]$ is a polynomial and each $Q_{e1j}, Q_{e2j} \in \kappa[[\mathtt{x}_1,\mathtt{x}_2]]$. Then $a_k - a_k^*, f_i - f_i^*, h_j - h_j^* \in \mathfrak{g} = \mathfrak{m}_2^c\mathfrak{a}^2$. The Jacobian matrix of the polynomial system (7.2) substituted at $(\mathtt{x},0,0,0,0,0,0)$ is exactly the matrix (7.3) used to define the ideal $\mathfrak{a}$. We apply Tougeron's Implicit Functions Theorem 7.1 with p equations and $q := p+r+1$ variables, the Jacobian ideal $\mathfrak{a}$ and the additional ideal $\mathfrak{a}' := \mathfrak{m}_2^c$ to the polynomial system (7.2) and we find a solution whose components belong to $\mathfrak{m}_2^c$, as required. □

mainred1 **Corollary 7.5.** *Let $a \in \kappa[[\mathtt{x}_1,\mathtt{x}_2]]$ be a non-zero series and let $a = \lambda\prod_{i=1}^m a_i^{\gamma_i}$ be a factorization of a, where the a_i are relatively prime irreducible factors, $\gamma_i \geqslant 1$ is a positive integer and $\lambda \in \kappa\setminus\{0\}$. For each $c \geqslant 2$ there exists $n \geqslant 1$ such that if $a_i^* \in \kappa[[\mathtt{x},\mathtt{y}]]$ and $a_i^* - a_i \in \mathfrak{m}_2^n$ for each $i = 1,\dots,m$ there exist units $u_i \in \kappa[[\mathtt{x},\mathtt{y}]]$ such that $u_i(0,0) = 1$ and $u_i - 1 \in \mathfrak{m}_2^c$ and a change of coordinates $\phi := (\phi_1,\phi_2)$ with $\psi_j = \phi_j - \mathtt{x}_j \in \mathfrak{m}_2^c$ for $j = 1,2$ such that $u_ia_i(\phi) = a_i^*$ and $\prod_{i=1}^m u_i^{\gamma_i} = 1$, so $a(\phi) = \lambda\prod_{i=1}^m (a_i^*)^{\gamma_i}$.*

Proof. Consider the system of polynomial equations:

$$\begin{cases} (1+\mathtt{W}_k)a_k(\mathtt{x}+\mathtt{Y}) = a_k^*,\\ \prod_{k=1}^m (1+\mathtt{W}_k)^{\gamma_k} = 1. \end{cases} \tag{7.8}$$ system2

By Lemma 7.4 for each $c \geqslant 2$ there exists $n \geqslant 1$ such that if $a_i^* \in \kappa[[\mathtt{x},\mathtt{y}]]$ and $a_i^* - a_i \in \mathfrak{m}_2^n$ for each $i = 1,\dots,m$ the previous system of polynomial equations (7.8) has a solution $(w_1,\dots,w_m,\psi_1,\psi_3)$ whose components belong to $\mathfrak{m}_2^c$. The statement follows taking $u_i := 1 + w_i$ and $\phi_j := \mathtt{x}_j + \psi_j$ for $j = 1,2$, as required. □

7.3. **Reduction of the proof of Theorem 2.3 to the case of finite Pythagoras number.** Before reducing the proof of Theorem 2.3 to the case of finite Pythagoras number, we need some preliminary results.

fe **Corollary 7.6.** *Let $a, f, g, F \in \kappa[[\mathtt{x}_1,\mathtt{x}_2]]$ be such that f, gF are relatively prime and denote $h := f^2 - Fg^2$. Suppose $F = \mathtt{x}_1^nF'$ where $n \geqslant 0$ and $F' \in \kappa[[\mathtt{x}_1,\mathtt{x}_2]]$ is square-free and relatively prime to $\mathtt{x}_1$. Then, for each $k \geqslant 2$ there exist (after multiplying $f + \mathtt{z}g$ by a unit of $A := \kappa[[\mathtt{x}_1,\mathtt{x}_2,\mathtt{z}]]/(\mathtt{z}^2 - F)$):*

- *a finitely generated extension of fields $\kappa_0|\mathbb{Q}$,*
- *units $u, v, w_1, w_2 \in \kappa[[\mathtt{x}_1,\mathtt{x}_2]]$ with $u(0,0) = 1$, $v(0,0) = 1$, $w_1(0,0) = 1$, $w_2(0,0) = 1$,*
- *series $\psi_1, \psi_2 \in \mathfrak{m}_2^k$*

such that:

(i) $a(\phi), uf(\phi), vh(\phi), w_1F(\phi), uw_2g(\phi) \in \kappa_0[[\mathtt{x}_1,\mathtt{x}_2]]$,
(ii) $u^2 = v$, $w_1w_2^2 = 1$ *and*
(iii) $u^2h(\phi) = (uf(\phi))^2 - w_1F(\phi)(g(\phi)w_2u)^2 \in \kappa_0[[\mathtt{x}_1,\mathtt{x}_2]]$

where $\phi := (\mathtt{x}_1 + \psi_1, \mathtt{x}_2 + \psi_2)$.

Proof. The proof is conducted in several steps:

Step 1. *Initial preparation.* Let $F'g^2 = \lambda H_1^{\gamma_1}\cdots H_s^{\gamma_s}w$ be a factorization of $F'g^2$ into irreducible series in $\kappa[[\mathtt{x}_1,\mathtt{x}_2]]$, where $\lambda \in \kappa\setminus\{0\}$ and $w \in \kappa[[\mathtt{x}_1,\mathtt{x}_2]]$ is a unit such that $w(0,0) = 1$ and $\gamma_j \geqslant 1$. We may assume that H_j is a Weierstrass polynomial of degree 1 with respecto $\mathtt{x}_1$ for

$j=1,\dots,s'\leqslant s$ and $\omega(H_j(\mathtt{x}_1,0))\geqslant 2$ for $j=s'+1,\dots,s$ (maybe $s'=0$ if $F'g^2$ has no such a factor). Write $H_j=\mathtt{x}_1+\theta_j(\mathtt{x}_2)$ for $j=1,\dots,s'$.

Let $k:=\max\{\omega(\theta_j):\ j=1,\dots,s',\theta_j\neq 0\}+2$, which satisfies $H_j-\mathtt{x}_1=\theta_j\notin\mathfrak{m}^k$ for $j=1,\dots,s'$ if $\theta_j\neq 0$. By Lemma 7.4 (we may assume after multiplying $f+\mathtt{z}g$ by a unit of A) that there exist units $u,v\in\kappa[[\mathtt{x}_1,\mathtt{x}_2]]$ with $u(0,0)=1$, $v(0,0)=1$ and series $\psi_1,\psi_2\in\mathfrak{m}^k$ such that

$$\begin{aligned}a(\phi)&=:a^\bullet\in\kappa[\mathtt{x}_1,\mathtt{x}_2],\\ uf(\phi)&=:f^\bullet\in\kappa[\mathtt{x}_1,\mathtt{x}_2],\\ vh(\phi)&=:h^\bullet\in\kappa[\mathtt{x}_1,\mathtt{x}_2],\\ u^2&=v,\end{aligned}$$

$\phi:=(\phi_1,\phi_2)$ and $\phi_i:=\mathtt{x}_i+\psi_i$ for $i=1,2$. We have

$$h^\bullet=vh(\phi)=u^2h(\phi)=(uf(\phi))^2-F(\phi)(ug(\phi))^2=(f^\bullet)^2-F(\phi)(ug(\phi))^2,$$

so $F(\phi)(ug(\phi))^2\in\kappa[\mathtt{x}_1,\mathtt{x}_2]$. Let κ_0 be the field extension of $\mathbb{Q}$ generated by the coefficients of the polynomials $a^\bullet,f^\bullet,h^\bullet$. Let $F(\phi)(ug(\phi))^2:=\lambda P_1^{\alpha_1}\cdots P_r^{\alpha_r}w'$ be a factorization of $F(\phi)(ug(\phi))^2$ into irreducible factors in $\kappa_0[[\mathtt{x}_1,\mathtt{x}_2]]$, where $\lambda\in\kappa_0\setminus\{0\}$ and $w'\in\kappa_0[[\mathtt{x}_1,\mathtt{x}_2]]$ is a unit an $w'(0,0)=1$. As $\psi_1\in\mathfrak{m}^k$ and $k\geqslant 2$, we deduce that $x_1+\psi_1$ is an irreducible factor of $\kappa[[\mathtt{x}_1,\mathtt{x}_2]]$.

STEP 2. If $n\geqslant 1$, then $\mathtt{x}_1$ is a factor of F, so $\mathtt{x}_1+\psi_1$ is a factor of $F(\phi)(ug(\phi))^2$, so we may assume that $\mathtt{x}_1+\psi_1$ is an irreducible factor of P_1 in $\kappa[[\mathtt{x}_1,\mathtt{x}_2]]$ (if $n\geqslant 1$). We claim: *There exists a unit $u_1\in\kappa[[\mathtt{x},\mathtt{y}]]$ such that $u_1(0,0)=1$ and $\phi_1u_1=(\mathtt{x}_1+\psi_1)u_1=P_1\in\kappa_0[[\mathtt{x}_1,\mathtt{x}_2]]$.*

Let $P_1:=\mu Q_1^{\beta_1}\dots Q_m^{\beta_m}u_1$ be a factorization of P_1 into irreducible series in $\kappa[[\mathtt{x}_1,\mathtt{x}_2]]$ such that each Q_ℓ is a Weierstrass polynomials with respect to either $\mathtt{x}_1$ or $\mathtt{x}_2$, $\mu\in\kappa\setminus\{0\}$ and $u_1\in\kappa[[\mathtt{x}_1,\mathtt{x}_2]]$ is a unit such that $u_1(0,0)=1$. As P_1 is irreducible in $\kappa_0[[\mathtt{x}_1,\mathtt{x}_2]]$ and both κ_0 and κ have characteristic zero, we deduce that each exponent $\beta_\ell=1$. By Weierstrass preparation $\mathtt{x}_1+\psi_1=(\mathtt{x}_1+\zeta_1(\mathtt{x}_2))u_1'$ where $\zeta_1\in\kappa[[\mathtt{x}_2]]$ and $u_1'\in\kappa[[\mathtt{x}_1,\mathtt{x}_2]]$ is a unit such that $u_1'(0,0)=1$. We have $\zeta_1(\mathtt{x}_2)u_1'(0,\mathtt{x}_2)=\psi_1(0,\mathtt{x}_2)\in\mathfrak{m}_2^k$, so $\zeta_1(\mathtt{x}_2)\in\mathfrak{m}_2^k$ because $u_1'(0,0)=1$. We assume $Q_1=\mathtt{x}_1+\zeta_1(\mathtt{x}_2)$. Observe that $Q_1,\dots,Q_s$ are irreducible factors of

$$(\mathtt{x}_1+\zeta_1(\mathtt{x}_2))^nu_1'^nF'(\phi)(ug(\phi))^2=(\mathtt{x}_1+\zeta_1(\mathtt{x}_2))^n\lambda H_1^{\alpha_1}(\phi)\cdots H_s^{\alpha_1}(\phi)w(\phi),$$

that is, each Q_ℓ is associated to $H_j(\phi)$ for some j. Fix $j=1,\dots,s'$ such that $H_j\neq\mathtt{x}_1$. Observe that $H_j(\phi)=\mathtt{x}_1+\psi_1(\mathtt{x}_1,\mathtt{x}_2)+\theta_j(\mathtt{x}_2+\psi_2(\mathtt{x}_1,\mathtt{x}_2))$ and $H_j(\phi)(0,\mathtt{x}_2)=\psi_1(0,\mathtt{x}_2)+\theta_j(\mathtt{x}_2+\psi_2(0,\mathtt{x}_2))$. As $\omega(\psi_1(0,\mathtt{x}_2)),\omega(\psi_2(0,\mathtt{x}_2))\geqslant k\geqslant 2$, we deduce $\omega(H_j(\phi)(0,\mathtt{x}_2))=\omega(\theta_j(\mathtt{x}_2))<k$. Thus, if $H_j\neq\mathtt{x}_1$ and $1\leqslant j\leqslant s'$, we have $H_j(\phi)=(\mathtt{x}_1+\theta_j'(\mathtt{x}_2))u_j'$ where $u_j'\in\kappa[[\mathtt{x}_1,\mathtt{x}_2]]$ is a unit such that $u_j'(0,0)=1$ and $\theta_j'(\mathtt{x}_2)\in\kappa[[\mathtt{x}_2]]$ satisfies $\omega(\theta_j')=\omega(\theta_j)<k$.

Let $\overline{\kappa}$ be the algebraic closure of κ and $G:=G(\overline{\kappa}:\kappa_0)$. Let $\sigma\in G$ and define

$$\Phi_\sigma:\overline{\kappa}[[\mathtt{x}_1,\mathtt{x}_2]]\to\overline{\kappa}[[\mathtt{x}_1,\mathtt{x}_2]],\ \sum_{(\nu_1,\nu_2)}a_{(\nu_1,\nu_2)}\mathtt{x}_1^{\nu_1}\mathtt{x}_2^{\nu_2}\mapsto\sum_{(\nu_1,\nu_2)}\sigma(a_{(\nu_1,\nu_2)})\mathtt{x}_1^{\nu_1}\mathtt{x}_2^{\nu_2},$$

which is a κ_0-isomorphism of rings. As $P_1\in\kappa_0[[\mathtt{x}_1,\mathtt{x}_2]]$, we have

$$\mu(\mathtt{x}_1+\zeta_1(\mathtt{x}_2))Q_2\dots Q_su_1=P_1=\Phi_\sigma(P_1)=\mu(\mathtt{x}_1+\Phi_\sigma(\zeta_1(\mathtt{x}_2)))\Phi_\sigma(Q_2)\dots\Phi_\sigma(Q_s)\Phi_\sigma(u_1).$$

Then, either $\mathtt{x}_1+\zeta_1(\mathtt{x}_2)=\mathtt{x}_1+\Phi_\sigma(\zeta_1(\mathtt{x}_2))$ or $\mathtt{x}_1+\zeta_1(\mathtt{x}_2)$ is associated to $H_j(\phi)$ for some $1\leqslant j\leqslant s'$ such that $H_j\neq\mathtt{x}_1$. As $\omega(\theta_j')<k\leqslant\omega(\zeta_1)=\omega(\Phi_\sigma(\zeta_1))$ for each $1\leqslant j\leqslant s'$ such that $H_j\neq\mathtt{x}_1$, we conclude $\mathtt{x}_1+\zeta_1(\mathtt{x}_2)=\mathtt{x}_1+\Phi_\sigma(\zeta_1(\mathtt{x}_2))$ for each $\sigma\in G$. Thus, $\mathtt{x}_1+\zeta_1(\mathtt{x}_2)\in\kappa_0[[\mathtt{x}_1,\mathtt{x}_2]]$, so $\mathtt{x}_1+\zeta_1(\mathtt{x}_2)$ is associated to P_1 in $\kappa_0[[\mathtt{x}_1,\mathtt{x}_2]]$, that is, there exists a unit $u_1\in\kappa[[\mathtt{x},\mathtt{y}]]$ such that $u_1(0,0)=1$ and $\phi_1u_1=(\mathtt{x}_1+\psi_1)u_1=P_1\in\kappa_0[[\mathtt{x}_1,\mathtt{x}_2]]$.

STEP 3. Let us check: *There exist series $F_1,g_1\in\kappa_0[[\mathtt{x}_1,\mathtt{x}_2]]$ and units $w_1,w_2\in\kappa[[\mathtt{x}_1,\mathtt{x}_2]]$ such that $w_1(0,0)=1$, $w_2(0,0)=1$, $F(\phi)=w_1F_1$, $(gu)(\phi)=w_2g_1$ and $w_1w_2^2=1$.*

Recall that $F = \mathtt{x}_1^n F'$, where $n \geqslant 0$ and $\mathtt{x}_1$ does not divide F'. We distinguish two cases:

CASE 3.1. If $n \geqslant 1$, we write $gu = \mathtt{x}_1^m g'$ where g' is relatively prime with $\mathtt{x}_1$ and $m \geqslant 0$. By the previous claim there exists a unit $u_1 \in \kappa[[\mathtt{x}_1, \mathtt{x}_2]]$ such that $\phi_1 u_1 \in \kappa_0[[\mathtt{x}_1, \mathtt{x}_2]]$, so $F(\phi) = (\phi_1 u_1)^n F'(\phi) u_1^{-n}$ and $(gu)(\phi) = (\phi_1 u_1)^m g'(\phi) u_1^{-m}$. Thus, $F'(\phi)u_1^{-n}(g'(\phi)u_1^{-m})^2 \in \kappa_0[[\mathtt{x}_1, \mathtt{x}_2]]$ is relatively prime with $\phi_1 u_1$. The irreducible factors of $F'(\phi)u_1^{-n}$ in $\kappa[[\mathtt{x}_1, \mathtt{x}_2]]$ have multiplicity one, whereas the irreducible factors of $(g'(\phi)u_1^{-m})^2$ have even multiplicity. ▪

CASE 3.2. If $n = 0$, then $F = F'$. We define $u_1 := 1$ and $g' := g$. The irreducible factors of $F(\phi) = F'(\phi)u_1^{-n}$ in $\kappa[[\mathtt{x}_1, \mathtt{x}_2]]$ have multiplicity one, whereas the irreducible factors of $g(\phi)^2 = (g'(\phi)u_1^{-m})^2$ have even multiplicity. ▪

We are ready to prove: *There exist a series $F_1 \in \kappa_0[[\mathtt{x}_1, \mathtt{x}_2]]$ and a unit $w_1 \in \kappa[[\mathtt{x}_1, \mathtt{x}_2]]$ such that $w_1(0,0) = 1$ and $F(\phi) = w_1 F_1$.*

Let Q be an irreducible factor of $F'(\phi)u_1^{-n}$ in $\kappa[[\mathtt{x}_1, \mathtt{x}_2]]$. Then Q is irreducible factor of $F'(\phi)u_1^{-n}(g'(\phi)u_1^{-m})^2$ of odd multiplicity β. For each $\sigma \in G = G(\overline{\kappa} : \kappa_0)$ we have $\Phi_\sigma(Q)$ is an irreducible factor of $F'(\phi)u_1^{-n}(g'(\phi)u_1^{-m})^2$ of multiplicity β, so $\Phi_\sigma(Q)$ is an irreducible factor of $F'(\phi)u_1^{-n}$ of multiplicity 1. Let $\mathfrak{F} \subset G$ be a finite set such that

$$\{\Phi_\sigma(Q) : \ \sigma \in \mathfrak{F}\} = \{\Phi_\sigma(Q) : \ \sigma \in G\}$$

and $\Phi_\sigma(Q) \neq \Phi_\tau(Q)$ if $\sigma, \tau \in \mathfrak{F}$ and $\sigma \neq \tau$. Then $\prod_{\sigma \in \mathfrak{F}} \Phi_\sigma(Q) \in \kappa_0[[\mathtt{x}_1, \mathtt{x}_2]]$ is an irreducible factor of $F'(\phi)u_1^{-n}$. We proceed like this with all the irreducible factors of $F'(\phi)u_1^{-n}$ and we conclude that there exists a unit $w_1 \in \kappa[[\mathtt{x}_1, \mathtt{x}_2]]$ with $w_1(0,0) = 1$ such that $F'(\phi)u_1^{-n}w_1 \in \kappa_0[[\mathtt{x}_1, \mathtt{x}_2]]$, so $F(\phi)w_1 = (\phi_1 u_1)^n F'(\phi)u_1^{-n}w_1 \in \kappa_0[[\mathtt{x}_1, \mathtt{x}_2]]$.

We prove next: *There exist a series $g_1 \in \kappa_0[[\mathtt{x}_1, \mathtt{x}_2]]$ and a unit $w_2 \in \kappa[[\mathtt{x}_1, \mathtt{x}_2]]$ such that $w_2(0,0) = 1$, $(gu)(\phi) = w_2 g_1$ and $w_1 w_2^2 = 1$.*

Let $w_2 := (\sqrt{w_1})^{-1}$, which satisfies $w_1 w_2^2 = 1$. As $F(\phi)(ug(\phi))^2 \in \kappa_0[[\mathtt{x}_1, \mathtt{x}_2]]$, $F(\phi)w_1 \in \kappa_0[[\mathtt{x}_1, \mathtt{x}_2]]$ and

$$F(\phi)(ug(\phi))^2 = F(\phi)w_1 w_2^2(ug(\phi))^2,$$

we conclude $w_2^2(ug(\phi))^2 \in \kappa_0[[\mathtt{x}_1, \mathtt{x}_2]]$. As the irreducible factors of $(w_2 ug(\phi))^2$ in $\kappa_0[[\mathtt{x}_1, \mathtt{x}_2]]$ have even multiplicity, we conclude $w_2 ug(\phi) \in \kappa_0[[\mathtt{x}_1, \mathtt{x}_2]]$, as required. □

k0 **Lemma 7.7.** *Let $F \in \kappa[[\mathtt{x}, \mathtt{y}]]$ be a non-zero series such that $F(0,0) = 0$ and either F is irreducible or $F := \mathtt{x}^n F'$ where $n \geqslant 0$ and $F' \in \kappa[[\mathtt{x}, \mathtt{y}]]$ is a square-free series relatively prime to $\mathtt{x}$ whose irreducible factors generate real prime ideals. Consider the ring $A := \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/(\mathtt{z}^2 - F)$ and let $f + \mathtt{z}g \in \mathcal{P}(A)$ be such that $\mathtt{x}$ does not divide f if F is reducible and $n \geqslant 2$. Then, for each $k \geqslant 2$ there exist (after multiplying $f + \mathtt{z}g$ by a unit of A):*

- *a change of coordinates $\phi := (\phi_1, \phi_2)$ such that $\phi_1 - \mathtt{x}, \phi_2 - \mathtt{y} \in \mathfrak{m}_2^k$,*
- *a finitely generated subextension $\kappa_0|\mathbb{Q}$ of $\kappa|\mathbb{Q}$ and*
- *units $u, w_1, w_2 \in \kappa_0[[\mathtt{x}, \mathtt{y}]]$*

such that:

(i) $u(0,0) = 1$, $w_1(0,0) = 1$, $w_2(0,0) = 1$, $w_1 w_2^2 = 1$
(ii) $F w_1 \in \kappa_0[[\mathtt{x}, \mathtt{y}]]$ *and*
(iii) $uf(\phi) + \mathtt{z} w_2 ug(\phi) \in \kappa_0[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/(\mathtt{z}^2 - F w_1)$.

Proof. Let $a \in \kappa[[\mathtt{x}, \mathtt{y}]]$ be the greatest common divisor of f and g. Write $f = af_1$ and $g = ag_1$, so that f_1, g_1 are relatively prime. As $f + \mathtt{z}g \in \mathcal{P}(A)$, then $f^2 - Fg^2 \in \mathcal{P}(\kappa[[\mathtt{x}, \mathtt{y}]])$. If F and f_1 are relatively prime, we are done for a moment because f_1 and Fg_1 are relatively prime (and we are under the hypothesis of Corollary 7.6). Otherwise, let b be a common irreducible factor of f_1 and F. We distinguish two cases:

CASE 1. If F is irreducible, then $b = F$. As f_1 and g_1 are relatively prime, F does not divide g_1, so we write $f := aFf_2$ and $g = ag_1$. Thus,

$$f + \mathtt{z}g = aFf_2 + \mathtt{z}ag_1 = \mathtt{z}a(g_1 + \mathtt{z}f_2) - (\mathtt{z}^2 - F)af_2$$

and g_1 and f_2F are relatively prime. ▪

CASE 2. If F is reducible, then $F := \mathtt{x}^nF'$ where $n \geqslant 0$ and $F' \in \kappa[[\mathtt{x},\mathtt{y}]]$ is a square-free series relatively prime to $\mathtt{x}$ whose irreducible factors generate real prime ideals. In this case, b is a real irreducible factor (different from $\mathtt{x}$) and we write $f := abf_2$ and $F := bF_1$. As $f^2 - Fg^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$, also $f_1^2 - Fg_1^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$ because $f^2 - Fg^2 = a^2(f_1^2 - Fg_1^2)$. As b is an irreducible factor of $f_1^2 - Fg_1^2$ that generates a real prime ideal, it divides $f_1^2 - Fg_1^2$ with even multiplicity $\geqslant 2$, so it divides g_1 (because b is a real irreducible factor of F of multiplicity 1), which is a contradiction because f_1 and g_1 are relatively prime. ▪

Thus, we may assume in both cases that we begin with $f' + \mathtt{z}g'$ where $f', Fg' \in \kappa[[\mathtt{x},\mathtt{y}]]$ are relatively prime and we have an additional series $a \in \kappa[[\mathtt{x},\mathtt{y}]]$. In the first case

$$f + \mathtt{z}g = \mathtt{z}a(f' + \mathtt{z}g') - (\mathtt{z}^2 - F)ag',$$

whereas in the second case $f + \mathtt{z}g = a(f' + \mathtt{z}g')$.

Define $h' := f'^2 - Fg'^2$. By Corollary 7.6 there exist a finitely generated subextension $\kappa_0|\mathbb{Q}$ of $\kappa|\mathbb{Q}$, units $u, v, w_1, w_2 \in \kappa[[\mathtt{x}_1,\mathtt{x}_2]]$ with $u(0,0) = 1$, $v(0,0) = 1$, $w_1(0,0) = 1$, $w_2(0,0) = 1$ and series $\psi_1, \psi_2 \in \mathfrak{m}_2^k$ such that $a(\phi), uf'(\phi), vh'(\phi), w_1F(\phi), uw_2g'(\phi) \in \kappa_0[[\mathtt{x}_1,\mathtt{x}_2]]$, $v = u^2$, $w_1w_2^2 = 1$ and

$$u^2h'(\phi) = (uf'(\phi))^2 - w_1F(\phi)(g'(\phi)w_2u)^2,$$

where $\phi := (\mathtt{x} + \psi_1, \mathtt{y} + \psi_2)$. We may assume $\sqrt{w_1}w_2 = 1$. Define $\mathtt{z}' := \mathtt{z}\sqrt{w_1}$ and observe that

$$\begin{aligned} uf'(\phi) + \mathtt{z}ug'(\phi) = uf'(\phi) + \mathtt{z}\sqrt{w_1}(w_2ug'(\phi)) = uf'(\phi) + \mathtt{z}'w_2ug'(\phi) \in \kappa_0[[\mathtt{x},\mathtt{y},\mathtt{z}']],\\ (\mathtt{z}^2 - F(\phi))w_1 = (\mathtt{z}\sqrt{w_1})^2 - F(\phi)w_1 = \mathtt{z}'^2 - F(\phi)w_1 \in \kappa_0[[\mathtt{x},\mathtt{y},\mathtt{z}']],\\ a(\phi) \in \kappa_0[[\mathtt{x},\mathtt{y},\mathtt{z}']], \end{aligned}$$

Consider the κ-homomorphism $\theta : \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]] \to \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$, $(\mathtt{x},\mathtt{y},\mathtt{z}) \to (\phi(\mathtt{x},\mathtt{y}), \mathtt{z}w_2)$. We have

$$\begin{aligned} &w_1\theta(\mathtt{z}^2 - F) = w_1(w_2^2\mathtt{z}^2 - F(\phi)) = \mathtt{z}^2 - F(\phi)w_1 \in \kappa_0[[\mathtt{x},\mathtt{y},\mathtt{z}]]\\ (\text{CASE 1.}):\quad &\sqrt{w_1}u\theta(f + \mathtt{z}g) = \sqrt{w_1}u\theta(\mathtt{z}a(f' + \mathtt{z}g') - (\mathtt{z}^2 - F)ag')\\ &\quad = \sqrt{w_1}w_2\mathtt{z}a(\phi)(uf'(\phi) + \mathtt{z}w_2ug'(\phi)) - \sqrt{w_1}(\mathtt{z}^2w_2^2 - F(\phi)w_1w_2^2)a(\phi)g'(\phi)u\\ &\quad = \mathtt{z}a(\phi)(uf'(\phi) + \mathtt{z}w_2ug'(\phi)) - (\mathtt{z}^2 - F(\phi)w_1)a(\phi)uw_2g'(\phi) \in \kappa_0[[\mathtt{x},\mathtt{y},\mathtt{z}]],\\ (\text{CASE 2.}):\quad &u\theta(f + \mathtt{z}g) = u\theta(a(f' + \mathtt{z}g')) = a(\phi)(uf'(\phi) + \mathtt{z}w_2ug'(\phi)) \in \kappa_0[[\mathtt{x},\mathtt{y},\mathtt{z}]], \end{aligned}$$

as required. □

Let us check that the series of List 2.1 are under the hypothesis of Lemma 7.7:

listhyp *Remarks* 7.8. (i) The series $a\mathtt{x}^2 + b\mathtt{y}^{2k} \in \kappa[[\mathtt{x},\mathtt{y}]]$ where $a \notin -\Sigma\kappa^2$, $b \neq 0$ and $k \geqslant 1$ is either irreducible or the product of two (non-associated) series of order 1 with respect to $\mathtt{x}$, which generate (different) real prime ideals.

(ii) The series $a\mathtt{x}^2 + \mathtt{y}^{2k+1} \in \kappa[[\mathtt{x},\mathtt{y}]]$ where $a \notin -\Sigma\kappa^2$ and $k \geqslant 1$ is irreducible (because it is the product of a times an irreducible Weierstrass polynomial with respect to $\mathtt{x}$).

(iii) The series $a\mathtt{x}^2 \in \kappa[[\mathtt{x},\mathtt{y}]]$ where $a \notin -\Sigma\kappa^2$ is a power of $\mathtt{x}$.

(iv) The series $\mathtt{x}^2\mathtt{y} + (-1)^ka\mathtt{y}^k = \mathtt{y}(\mathtt{x}^2 + (-1)^ka\mathtt{y}^{k-1}) \in \kappa[[\mathtt{x},\mathtt{y}]]$ where $a \notin -\Sigma\kappa^2$ and $k \geqslant 3$ is the product of $\mathtt{y}$ times the series $\mathtt{x}^2 + (-1)^ka\mathtt{y}^{k-1}$, which is either irreducible or the product of two (non-associated) series of order 1 with respect to $\mathtt{x}_1$, which generate (different) real prime ideals.

(v) The series $\mathtt{x}^2\mathtt{y} \in \kappa[[\mathtt{x},\mathtt{y}]]$ is the product of a power of $\mathtt{x}$ times a series of order 1 with respect to $\mathtt{x}$, which generates a real prime ideal.

(vi) The series $\mathtt{x}^3 + a\mathtt{x}\mathtt{y}^2 + b\mathtt{y}^3 \in \kappa[[\mathtt{x},\mathtt{y}]]$ is irreducible (by hypothesis).

(vii) The series $\mathtt{x}^3 + a\mathtt{y}^4 \in \kappa[[\mathtt{x},\mathtt{y}]]$ where $a \notin -\Sigma\kappa^2$ is irreducible (because it is an irreducible Weierstrass polynomial with respect to $\mathtt{x}$).

(viii) The series $\mathtt{x}^3 + \mathtt{x}\mathtt{y}^3 = \mathtt{x}(\mathtt{x}^2 + \mathtt{y}^3) \in \kappa[[\mathtt{x},\mathtt{y}]]$ is the product of $\mathtt{x}$ times the series $\mathtt{x}^2 + \mathtt{y}^3$ is irreducible (because it is an irreducible Weierstrass polynomial with respect to $\mathtt{x}$).

(ix) The series $\mathtt{x}^3 + \mathtt{y}^5 \in \kappa[[\mathtt{x},\mathtt{y}]]$ is irreducible (because it is an irreducible Weierstrass polynomial with respect to $\mathtt{x}$). ∎

The following result is inspired by [Sch3, Cor.4.6] and takes advantage of [ELW].

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Lemma 7.9. *Let $\kappa|\mathbb{Q}$ be a field extension and let $\kappa_0|\mathbb{Q}$ be a subextension. Let $f, g, F \in \kappa_0[[\mathtt{x},\mathtt{y}]]$ be such that $f + \mathtt{z}g \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2 - F))$. Then there exists a subextension $\kappa_1|\kappa_0$ of $\kappa|\kappa_0$ with finite transcendence degree over κ_0 such that $f + \mathtt{z}g \in \mathcal{P}(\kappa_1[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2 - F))$.*

Proof. If κ has finite transcendence degree over κ_0, we take $\kappa_1 := \kappa_0$. Thus, we assume κ has infinite transcendence degree over κ_0. Let $\kappa'|\kappa_0$ be a subextension of $\kappa|\kappa_0$ such that $f + \mathtt{z}g \in \mathcal{P}(\kappa'[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2 - F))$. We have: $f^2 - Fg^2 \in \mathcal{P}(\kappa'[[\mathtt{x},\mathtt{y}]])$ and $f + \mathfrak{p}_\alpha \geqslant 0$ for each $\alpha \in \operatorname{Sper}(\kappa'[[\mathtt{x},\mathtt{y}]])$ such that $F + \mathfrak{p}_\alpha \geqslant 0$, that is, $f \in \mathcal{P}_{\kappa'}(\{F \geqslant 0\})$, where the subindex κ' denotes the field we are working with. We have to find a finitely generated subextension $\kappa_1|\kappa_0$ of $\kappa|\kappa_0$ such that $h := f^2 - Fg^2 \in \mathcal{P}(\kappa_1[[\mathtt{x},\mathtt{y}]])$ and $f \in \mathcal{P}_{\kappa_1}(\{F \geqslant 0\})$.

Let $a := fFh \in \kappa_0[[\mathtt{x},\mathtt{y}]]$ and consider the factorization of $a = \lambda\prod_{k=1}^m a_k^{\gamma_k}$ as the product of its irreducible factors a_k, where the a_k are relatively prime, each $\gamma_k \geqslant 1$ is a positive integer and $\lambda \in \kappa_0\setminus\{0\}$. There exist $\lambda_1, \lambda_2, \lambda_3 \in \kappa_0\setminus\{0\}$ and units $u_1, u_2, u_3 \in \kappa_0[[\mathtt{x},\mathtt{y}]]$ such that $u_i(0,0) = 1$ and $f = \lambda_1 u_1\prod_{k=1}^m a_k^{\gamma_{k1}}$, $F = \lambda_2 u_2\prod_{k=1}^m a_k^{\gamma_{k2}}$ and $h = \lambda_3 u_3\prod_{k=1}^m a_k^{\gamma_{k3}}$ where each γ_{kj} is a non-negative integer and $\gamma_k = \sum_{j=1}^3 \gamma_{kj}$. By Corollary 7.5 there exist polynomials $a_k^* \in \kappa_0[\mathtt{x},\mathtt{y}]$, units $w_k \in \kappa_0[[\mathtt{x},\mathtt{y}]]$ with $w_k(0,0) = 1$ and a change of coordinates $\phi := (\phi_1, \phi_2) \in \kappa_0[\mathtt{x},\mathtt{y}]^2$ with $\phi_1 - \mathtt{x}, \phi_2 - \mathtt{y} \in \mathfrak{m}_2^2$ such that $w_k a_k(\phi) = a_k^*$ and $\prod_{k=1}^m w_k^{\gamma_k} = 1$. Thus, $a(\phi) = \lambda\prod_{k=1}^m (a_k^*)^{\gamma_k}$. Consequently, there exist polynomials $f^*, h^*, F^* \in \kappa_0[\mathtt{x},\mathtt{y}]$ and units $v_1, v_2, v_3 \in \kappa_0[[\mathtt{x},\mathtt{y}]]$ with $v_j(0,0) = 1$ such that $f(\phi) = v_1 f^*$, $h(\phi) = v_2 h^*$ and $F(\phi) = v_3 F^*$. Observe that $h := f^2 - Fg^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$ if and only if $h^* \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$. In addition, $f \in \mathcal{P}_\kappa(\{F \geqslant 0\})$ if and only if $f^* \in \mathcal{P}_\kappa(\{F^* \geqslant 0\})$.

Consider the inclusions $\kappa'[\mathtt{x},\mathtt{y}] \xhookrightarrow{\theta_1} \kappa'[\mathtt{x},\mathtt{y}]_{\mathfrak{m}_2} \xhookrightarrow{\theta_2} \kappa'[[\mathtt{x},\mathtt{y}]]$ and the corresponding spectral maps $\operatorname{Sper}(\theta_2) : \operatorname{Sper}(\kappa'[[\mathtt{x},\mathtt{y}]]) \to \operatorname{Sper}(\kappa'[\mathtt{x},\mathtt{y}]_{\mathfrak{m}_2})$ and $\operatorname{Sper}(\theta_1) : \operatorname{Sper}(\kappa'[\mathtt{x},\mathtt{y}]_{\mathfrak{m}_2}) \to \operatorname{Sper}(\kappa'[\mathtt{x},\mathtt{y}])$. By [DST, Thm.13.3.7] $\operatorname{Sper}(\theta_1)$ is a homeomorphism onto its image, which is $\{\alpha \in \kappa'[\mathtt{x},\mathtt{y}] : \operatorname{supp}(\alpha) \subset \mathfrak{m}_2\}$. By [ABR, Thm.VII.3.2] the map $\operatorname{Sper}(\theta_2)$ is surjective, so the image $\operatorname{Sper}(\theta_2 \circ \theta_1) = \operatorname{Sper}(\theta_2) \circ \operatorname{Sper}(\theta_1)$ is $\{\alpha \in \kappa'[\mathtt{x},\mathtt{y}] : \operatorname{supp}(\alpha) \subset \mathfrak{m}_2\}$. Observe that $h \in \mathcal{P}(\kappa'[[\mathtt{x},\mathtt{y}]])$ if and only if $h \geqslant_\beta 0$ for each $\beta \in \operatorname{Sper}(\kappa'[\mathtt{x},\mathtt{y}])$ such that $\beta \to \alpha$ and $\operatorname{supp}(\alpha) = \mathfrak{m}_2$. In addition, $f^* \in \mathcal{P}_{\kappa'}(\{F^* \geqslant 0\})$ if and only if $f^* \geqslant_\beta 0$ for each $\beta \in \operatorname{Sper}(\kappa'[\mathtt{x},\mathtt{y}])$ such that $\beta \to \alpha$, $\operatorname{supp}(\alpha) = \mathfrak{m}_2$ and $F^* \geqslant_\beta 0$.

Fix $\xi \in \operatorname{Sper}(\kappa') = \{\alpha \in \operatorname{Sper}(\kappa'[[\mathtt{x},\mathtt{y}]]) : \operatorname{supp}(\alpha) = \mathfrak{m}_2\}$ and denote $\operatorname{Sper}(\kappa'[\mathtt{x},\mathtt{y}])_\xi := \{\beta \in \operatorname{Sper}(\kappa'[\mathtt{x},\mathtt{y}]) : \beta \cap \kappa' = \xi\}$. We deduce $h^* \in \mathcal{P}(\kappa'[[\mathtt{x},\mathtt{y}]])$ if and only if $h^* \geqslant_\beta 0$ for each $\beta \in \operatorname{Sper}(\kappa'[\mathtt{x},\mathtt{y}])_\xi$ such that $\operatorname{supp}(\beta) \subset \mathfrak{m}_2$ and each $\xi \in \operatorname{Sper}(\kappa')$. Analogously, $f^* \in \mathcal{P}_{\kappa'}(\{F^* \geqslant 0\})$ if and only if $f^* \geqslant_\beta 0$ for each $\beta \in \operatorname{Sper}(\kappa'[\mathtt{x},\mathtt{y}])_\xi$ such that $\operatorname{supp}(\beta) \subset \mathfrak{m}_2$, $F^* \geqslant_\beta 0$ and this holds for each $\xi \in \operatorname{Sper}(\kappa')$

Let $\mathfrak{R}(\xi)$ be the real closure of $(\kappa', <_\xi)$ and observe that $\operatorname{Sper}(\kappa'[\mathtt{x},\mathtt{y}])_\xi \equiv \operatorname{Sper}(\mathfrak{R}(\xi)[\mathtt{x},\mathtt{y}])$. Under this identification:

(1) $h^* \in \mathcal{P}(\kappa'[[\mathtt{x},\mathtt{y}]])$ if and only if h^* is non-negative on a neighborhood of the origin in $\mathfrak{R}(\xi)^2$, that is, there exists $\varepsilon \in \mathfrak{R}(\xi)$, $\varepsilon > 0$ such that $h^*(x,y) \geqslant 0$ for each $(x,y) \in \mathfrak{R}(\xi)^2$ satisfying $x^2+y^2<\varepsilon^2$ and this holds for each $\xi \in \operatorname{Sper}(\kappa')$.
(2) $f^* \in \mathcal{P}_{\kappa'}(\{F^* \geqslant 0\})$ if and only if there exists an open neighborhood V of the origin in $\mathfrak{R}(\xi)^2$ such that $\{F^* \geqslant 0\} \cap V \subset \{f^* \geqslant 0\} \cap V$, that is, there exists $\varepsilon \in \mathfrak{R}(\xi)$, $\varepsilon > 0$ such that $f^*(x,y) \geqslant 0$ for each $(x,y) \in \mathfrak{R}(\xi)^2$ satisfying $F^*(x,y) \geqslant 0$ and $x^2+y^2<\varepsilon^2$ and this holds for each $\xi \in \operatorname{Sper}(\kappa')$.

Let d be an upper bound for the degrees of f^*, F^*, h^* and let $\{b_k\}_k \subset \kappa_0$ be the collection of the coefficients of f^*, F^*, h^* as polynomials of degree $\leqslant d$. There exists polynomials $P_1, P_2, P_3 \in \mathbb{Q}[\mathtt{x},\mathtt{y},\mathtt{b_k}]_k]$ such that $f^* = P_1(\mathtt{x},\mathtt{y},\{b_k\}_k)$, $F^* = P_2(\mathtt{x},\mathtt{y},\{b_k\}_k)$ and $h^* = P_3(\mathtt{x},\mathtt{y},\{b_k\}_k)$. Recall that each real closed field E that contains κ' defines the element $\xi_E := \kappa' \cap E^2 \in \operatorname{Sper}(\kappa')$ and the real closure $\mathfrak{R}(\xi_E)$ of $(\kappa', <_{\xi_E})$ is contained in E. Consider the statements:

(i)$_{\kappa'}$ For each real closed field R that contains κ' there exists $\varepsilon \in R$, $\varepsilon > 0$ such that $P_3(x,y,\{b_k\}_k) \geqslant 0$ for each $(x,y) \in R^2$ satisfying $x^2+y^2<\varepsilon^2$.
(ii)$_{\kappa'}$ For each real closed field R that contains κ' there exists $\varepsilon \in R$, $\varepsilon > 0$ such that $P_1(x,y,\{b_k\}_k) \geqslant 0$ for each $(x,y) \in R^2$ satisfying $P_2(x,y,\{b_k\}_k) \geqslant 0$ and $x^2+y^2<\varepsilon^2$.

We reformulate properties (1) and (2) above as follows:

(1)' $P_3(\mathtt{x},\mathtt{y},\{b_k\}_k) \in \mathcal{P}_{\kappa'}(\kappa'[[\mathtt{x},\mathtt{y}]])$ if and only if (ii)$_\kappa$ holds.
(2)' $P_1(\mathtt{x},\mathtt{y},\{b_k\}_k) \in \mathcal{P}_{\kappa'}(\{P_2(\mathtt{x},\mathtt{y},\{b_k\}_k) \geqslant 0\})$ if and only if (ii)$_\kappa$ holds.

Consider the additional statements:

(i)' For each real closed field R that contains $\mathbb{Q}(\{b_k\}_k)$ there exists $\varepsilon \in R$, $\varepsilon > 0$ such that $P_3(x,y,\{b_k\}_k) \geqslant 0$ for each $(x,y) \in R^2$ satisfying $x^2+y^2<\varepsilon^2$.
(ii)' For each real closed field R that contains $\mathbb{Q}(\{b_k\}_k)$ there exists $\varepsilon \in R$, $\varepsilon > 0$ such that $P_1(x,y,\{b_k\}_k) \geqslant 0$ for each $(x,y) \in R^2$ satisfying $P_2(x,y,\{b_k\}_k) \geqslant 0$ and $x^2+y^2<\varepsilon^2$.

By Tarski-Seidenberg's Theorem [BCR, Cor.1.4.7] there exist $\psi_1,\ldots,\psi_q \in \mathbb{Q}[\{\mathtt{b_k}\}_k]$ and a finite boolean combination of equalities and strict inequalities involving the polynomials $\psi_1,\ldots,\psi_q$ such that: *Given a tuple $\{c_k\}_k$ and a real closed field R that contains $\mathbb{Q}(\{c_k\}_k)$, statements* (i)' *and* (ii)" *hold if and only if $\mathcal{B}(\{c_k\}_k)$ holds in R.*

Consequently, $P_2(\mathtt{x},\mathtt{y},\{b_k\}_k) \in \mathcal{P}_{\kappa'}(\kappa'[[\mathtt{x},\mathtt{y}]])$ and $P_1(\mathtt{x},\mathtt{y},\{b_k\}_k) \in \mathcal{P}_{\kappa'}(\{P_2(\mathtt{x},\mathtt{y},\{b_k\}_k) \geqslant 0\})$ if and only if $\mathcal{B}(\{b_k\}_k)$ holds in each real closed field R that contains κ.

As $\mathbb{Q}(\{b_k\}_k) \subset \kappa_0(\{b_k\}_k) \subset \kappa$, $P_3(\mathtt{x},\mathtt{y},\{b_k\}_k) \in \mathcal{P}(\kappa'[[\mathtt{x},\mathtt{y}]])$ and

$$P_1(\mathtt{x},\mathtt{y},\{b_k\}_k) \in \mathcal{P}_{\kappa'}(\{P_3(\mathtt{x},\mathtt{y},\{b_k\}_k) \geqslant 0\}),$$

we deduce $\mathcal{B}(\{b_k\}_k)$ holds in each real closed field R that contains κ. Consider the inclusion of fields $\theta : \kappa_0 \hookrightarrow \kappa$ and the spectral map: $\operatorname{Sper}(\theta) : \operatorname{Sper}(\kappa) \to \operatorname{Sper}(\kappa_0)$. We endow the spaces $\operatorname{Sper}(\kappa)$ and $\operatorname{Sper}(\kappa_0)$ with the Harrison topology [ELW], so they become Boolean spaces, that is, they are compact, Hausdorff and totally disconnected. By (i'), (ii'), (i'), (ii") and [ELW, Thm.4.18] $\operatorname{Sper}(\theta)(\operatorname{Sper}(\kappa))$ is a closed subset of

$$C := \{\alpha \in \operatorname{Sper}(\kappa_0) : \ \mathcal{B}(\{b_k\}_k) \text{ holds for } \alpha\}.$$

As C is a finite Boolean combination of Harrison sets $H_{\kappa_0}(b) := \{\alpha \in \operatorname{Sper}(\kappa_0) : b >_\alpha 0\}$ for some $b \in \kappa_0\setminus\{0\}$ (which constitute a subbasis of the Harrison topology of $\operatorname{Sper}(\kappa_0)$), we deduce that C is a clopen subset of $\operatorname{Sper}(\kappa_0)$. By [ELW, Thm.4.18] and [Kn, Prop.3.8] there exists a pure dimensional finitely generated extension $\theta_1 : \kappa_0 \hookrightarrow \kappa_1$ such that the associated spectral map $\operatorname{Sper}(\theta_1) : \operatorname{Sper}(\kappa_1) \to \operatorname{Sper}(\kappa_0)$ satisfies $\operatorname{Sper}(\theta_1)(\operatorname{Sper}(\kappa_1)) = C$. As κ has infinite transcendence degree over κ_0, we may assume $\kappa_1|\kappa_0$ is a finitely generated subextension of $\kappa|\kappa_0$. As $\operatorname{Sper}(\theta_1)(\operatorname{Sper}(\kappa_1)) = C$, the statements (i)$_{\kappa_1}$ and (ii)$_{\kappa_1}$ holds, so $h^* = P_3(\mathtt{x},\mathtt{y},\{a_k\}_k) \in$

$\mathcal{P}(\kappa_1[[\mathtt{x},\mathtt{y}]])$ and $f^* =\in P_1(\mathtt{x},\mathtt{y},\{a_k\}_k) \in \mathcal{P}_{\kappa_1}(\{P_2(\mathtt{x},\mathtt{y},\{b_k\}_k) \geqslant 0\}) = \mathcal{P}_{\kappa_1}(\{F^* \geqslant 0\})$, as required. □

secondextr *Remark* 7.10. If $\kappa_0|\mathbb{Q}$ is finitely generated over $\mathbb{Q}$, say by e elements, and $\kappa_1|\kappa_0$ has finite transcendence degree over κ_0, say d, then $\kappa_1|\mathbb{Q}$ has finite transcendence degree over $\mathbb{Q}$ bounded by $d+e$ and $p(\kappa_1[\mathtt{t}]) \leqslant 2^{\max\{d+e+2,3\}}$, so by Theorem 1.7 $p(\kappa_1[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F)) \leqslant 2^{\max\{d+e+4,5\}}$ is finite. ■

Reduction of Theorem 2.3 *to the case when the Pythagoras number is finite.* Let κ be a (formally) real field and let $F \in \kappa[\mathtt{x},\mathtt{y}]$ be a polynomial of List 2.1. Let $f+\mathtt{z}g \in \mathcal{P}(A)\setminus\{0\}$. We distinguish several cases:

Case 1. Assume first $F \in \{\mathtt{x}^2\mathtt{y}, a\mathtt{x}^2\}$ and $\mathtt{x}$ divides f.

As $f \in \mathcal{P}(\{F \geqslant 0\})$ and $F \in \{\mathtt{x}^2\mathtt{y}, a\mathtt{x}^2\}$ we deduce that $\mathtt{x}^2$ divides f (because otherwise f changes sign when we cross $\mathtt{x}=0$, but this does not happen in $\{F \geqslant 0\}$). Write $f := \mathtt{x}^2 f_1$ where $f_1 \in \kappa[\mathtt{x},\mathtt{y}]$ and let $F_1 \in \{\mathtt{y}, a\}$ be such that $F = \mathtt{x}^2 f_1$. We have $\mathtt{x}^4 f_1^2 - \mathtt{x}^2 F_1 g^2 = f^2 - Fg^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$, so $\mathtt{x}^2 f_1^2 - F_1 g^2 \in \kappa[[\mathtt{x},\mathtt{y}]]$. As $-F_1 \notin \mathcal{P}(\kappa[[\mathtt{y}]])$, we conclude that $g(0,\mathtt{y})=0$, so $\mathtt{x}$ divides g and $g = \mathtt{x}g_1$ for some $g_1 \in \kappa[\mathtt{x},\mathtt{y}]$. Thus, $f+\mathtt{z}g = \mathtt{x}(\mathtt{x}f_1+\mathtt{z}g_1)$.

Subcase 1.1. If $F := \mathtt{x}^2\mathtt{y}$, we consider the parameterization $(\mathtt{s},\mathtt{t}) \mapsto (\mathtt{s},\mathtt{t}^2,\mathtt{s}\mathtt{t})$. Then

$$\mathtt{s}(\mathtt{s}f_1(\mathtt{s},\mathtt{t}^2)+\mathtt{s}\mathtt{t}g(\mathtt{s},\mathtt{t}^2)) \in \mathcal{P}(\kappa[[\mathtt{s},\mathtt{t}]]),$$

so $f_1(\mathtt{s},\mathtt{t}^2)+\mathtt{t}g(\mathtt{s},\mathtt{t}^2) \in \mathcal{P}(\kappa[[\mathtt{s},\mathtt{t}]]) = \Sigma\kappa[[\mathtt{s},\mathtt{t}]]^2$. Thus, there exist series $a_i(\mathtt{s},\mathtt{t}^2), b_i(\mathtt{s},\mathtt{t}^2) \in \kappa[[\mathtt{s},\mathtt{t}^2]]$ such that

$$f_1(\mathtt{s},\mathtt{t}^2)+\mathtt{t}g(\mathtt{s},\mathtt{t}^2) = \sum_{i=1}^p (a_i(\mathtt{s},\mathtt{t}^2)+\mathtt{t}b_i(\mathtt{s},\mathtt{t}^2))^2.$$

Consequently,

$$(f+\mathtt{z}g)(\mathtt{s},\mathtt{t}^2,\mathtt{s}\mathtt{t}) = \mathtt{s}^2\sum_{i=1}^p (a_i(\mathtt{s},\mathtt{t}^2)+\mathtt{t}b_i(\mathtt{s},\mathtt{t}^2))^2 = \sum_{i=1}^p (\mathtt{s}a_i(\mathtt{s},\mathtt{t}^2)+\mathtt{s}\mathtt{t}b_i(\mathtt{s},\mathtt{t}^2))^2$$

and we conclude

$$f(\mathtt{x},\mathtt{y})+\mathtt{z}g(\mathtt{x},\mathtt{y}) = \sum_{i=1}^p (\mathtt{x}a_i(\mathtt{x},\mathtt{y})+\mathtt{z}b_i(\mathtt{x},\mathtt{y}))^2 \in \Sigma A^2.$$

Subcase 1.2. If $F := a\mathtt{x}^2$ where $-a \notin \Sigma\kappa^2$, observe that

$$f+\mathtt{z}g \in \mathcal{P}(\kappa[\sqrt{a}][[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-(\sqrt{a}x)^2)) = \Sigma\kappa[\sqrt{a}][[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-(\sqrt{a}x)^2)^2,$$

where $\sqrt{a}$ is a square root of a. Write $\mathtt{z} = \sqrt{a}\mathtt{x}$ and

$$f+\mathtt{z}g = \mathtt{x}(\mathtt{x}f_1+\sqrt{a}\mathtt{x}g_1) = \mathtt{x}^2(f_1+\sqrt{a}g_1).$$

We deduce that $f_1+\sqrt{a}g_1 \in \mathcal{P}(\kappa[\sqrt{a}][[\mathtt{x},\mathtt{y}]])$, so there exist series $a_i, b_i \in \kappa[[\mathtt{x},\mathtt{y}]]$ such that

$$f_1+\sqrt{a}g_1 = \sum_{i=1}^p (a_i+\sqrt{a}b_i)^2 \rightsquigarrow f_1-\sqrt{a}g_1 = \sum_{i=1}^p (a_i-\sqrt{a}b_i)^2.$$

Consequently,

$$f+\mathtt{z}g = \mathtt{x}^2(f_1\pm\sqrt{a}g_1) = \sum_{i=1}^p (\mathtt{x}a_i\pm\sqrt{a}\mathtt{x}b_i)^2 = \sum_{i=1}^p (\mathtt{x}a_i+\mathtt{z}b_i)^2,$$

so $f+\mathtt{z}g = \sum_{i=1}^p (\mathtt{x}a_i\mathtt{z}b_i)^2 \in \Sigma A^2$. ■

Case 2. Assume next that either $F \neq \mathtt{x}^2\mathtt{y}, a\mathtt{x}^2$ (where $-a \notin \Sigma\kappa^2$) or $F \in \{\mathtt{x}^2\mathtt{y}, a\mathtt{x}^2\}$ and $\mathtt{x}$ does not divide f.

By Corollary 7.7, Remark 7.8 and Lemma 7.9 we may assume (after a change of coordinates of the type $\phi := (\mathtt{x}+\psi_1, \mathtt{y}+\psi_2)$ where $\mathfrak{m}_1^k$ for $k \geqslant 2$ as large as needed) that there exists an extension $\kappa_1|\mathbb{Q}$ with finite transcendence degree over $\mathbb{Q}$ such that $f+\mathtt{z}g \in \kappa_1[[\mathtt{x},\mathtt{y},\mathtt{z}]]$, $F \in \kappa_1[[\mathtt{x},\mathtt{y}]]$ and $f+\mathtt{z}g \in \mathcal{P}(\kappa_1[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F))$. The new $F \in \kappa_1[[\mathtt{x},\mathtt{y}]]$ is a unit $u \in \kappa[[\mathtt{x},\mathtt{y}]]$ with $u(0,0) = 1$ times a series $G \in \kappa[[\mathtt{x},\mathtt{y}]]$ of the List 2.1, that is, $F = uG(\phi)$. Write $u := u_1 + u_2$ where $u_1 \in \kappa[\mathtt{x},\mathtt{y}]$ is a polynomial of degree $\leqslant k-1$ and $u_2 \in (\mathtt{x},\mathtt{y})^k\kappa[\mathtt{x},\mathtt{y}]$. After enlarging κ_1 if necessary (adjoining the coefficients of u_1 to κ_1), we may assume $u_1 \in \kappa_1[\mathtt{x},\mathtt{y}]$ and $u_1^{-1}, u_1^{1/2} \in \kappa_1[[\mathtt{x},\mathtt{y}]]$, keeping the fact that the transcendence degree of κ_1 over $\mathbb{Q}$ is finite. After the change of coordinates $(\mathtt{x},\mathtt{y},\mathtt{z}u_1^{1/2})$, we may assume $F = (1+u_1^{-1}u_2)G(\phi)$. Observe that $F - G \in (\mathtt{x},\mathtt{y})^k\kappa[\mathtt{x},\mathtt{y}]$ for $k \geqslant 2$ large enough:

Subcase 2.1. If G is ℓ-determined for some $\ell \leqslant k$, $\mathtt{z}^2 - F(\mathtt{x},\mathtt{y})$ is contact equivalent to $\mathtt{z}^2 - G(\mathtt{x},\mathtt{y})$ in $\kappa_1[[\mathtt{x},\mathtt{y},\mathtt{z}]]$.

Subcase 2.2. If G is not determined, then $G \in \{\mathtt{x}^2\mathtt{y}, a\mathtt{x}^2\}$ for some $a \in \kappa_1\setminus\{0\}$. If $G = \mathtt{x}^2\mathtt{y}$, we proceed as in the first part of the proof of [Fe8, Lem.3.8] and show that F is right equivalent to G in $\kappa_1[\mathtt{x},\mathtt{y}]$. If $G = a\mathtt{x}^2$ for some $a \in \kappa_1\setminus\{0\}$, we proceed as in the proof of [Fe8, Lem.3.8] and show that F is right equivalent to G in $\kappa_1[\mathtt{x},\mathtt{y}]$. ■

Thus, we may assume: κ_1 is a (formally) real field of finite transcendence degree over $\mathbb{Q}$, $f+\mathtt{z}g \in \kappa_1[[\mathtt{x},\mathtt{y},\mathtt{z}]]$, $F \in \kappa_1[[\mathtt{x},\mathtt{y}]]$ belongs to List 2.1 and $f+\mathtt{z}g \in \mathcal{P}(\kappa_1[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F))$. By Remark 7.10 we know in addition that the Pythagoras number of $\kappa_1[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F)$ is finite and we have finished the reduction of the proof of Theorem 2.3 to the case when the Pythagoras number is finite, as required. □

s8 8. Elephant's improved Theorems

In this section we present the main tools (Elephant's improved Theorems 8.6 and 8.7) to prove in the next one Theorem 2.3.

8.1. **Positive semidefinite quadratic polynomials.** We begin proving a characterization of the sums of squares of a ring $A := \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F)$ where κ is a (formally) real field and $F \in \mathfrak{m}_2 \subset \kappa[[\mathtt{x},\mathtt{y}]]$.

reductionp **Lemma 8.1.** *Let κ be a (formally) real field with $\tau(\kappa) < +\infty$ and denote $A := \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F)$. An element $f+\mathtt{z}g \in A$ belongs to ΣA^2 if and only if there exists $\eta \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$ such that $4\eta(f-F\eta)-g^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$.*

Two key results to prove Lemma 8.1 are the following ones proved in [Fe8].

quadratic **Lemma 8.2** ([Fe8, Lem.5.6])**.** *Let A be a ring of characteristic zero and $P := a_0 + a_1\mathtt{z} + a_2\mathtt{z}^2 \in A[\mathtt{z}]$. Then $P \in \mathcal{P}(A[\mathtt{z}])$ if and only if $a_0, a_2, 4a_0a_2 - a_1^2 \in \mathcal{P}(A)$.*

sosok **Lemma 8.3** ([Fe8, Lem.5.8])**.** *Let κ be a (formally) real field with $\tau(\kappa) < +\infty$ and $\Omega := a_0 + a_1\mathtt{z} + a_2\mathtt{z}^2 \in \kappa[[\mathtt{x},\mathtt{y}]][\mathtt{z}]$ a positive semidefinite quadratic polynomial. Then Ω is a sum of $p := 4\tau(\kappa)$ squares of polynomials of degree $\leqslant 1$ (with respect to $\mathtt{z}$) and coefficients in $\kappa[[\mathtt{x},\mathtt{y}]]$.*

In case $A := \kappa[[\mathtt{x},\mathtt{y}]]$ we can relax the hypothesis of Lemma 8.2 in view of the following result.

pos **Lemma 8.4.** *Let κ be a (formally) real field and let $a,b,c \in \kappa[[\mathtt{x}]]\setminus\{0\}$ such that $a, 4ac-b^2 \in \mathcal{P}(\kappa[[\mathtt{x}]])$. Then $c \in \mathcal{P}(\kappa[[\mathtt{x}]])$.*

Proof. If $a(0) \neq 0$, then a is a unit and $c = \frac{4ac-b^2}{a} + \frac{b^2}{a} \in \mathcal{P}(\kappa[[\mathtt{x}]])$. We assume in the following $a(0) = 0$. To prove that $c \in \mathcal{P}(\kappa[[\mathtt{x}]])$, we have to show by Lemma 5.19 and Remarks 5.20 that for each ordering β of κ and each $\eta \in \mathfrak{R}(\beta)[[\mathtt{t}]]^n$ such that $\eta(0)=0$ it holds $c(\eta)$ is non-negative in $\mathfrak{R}(\beta)[[\mathtt{t}]]$. As $a, 4ac-b^2 \in \mathcal{P}(\kappa[[\mathtt{x}]])$, we deduce

$$\mathcal{S} := \{\alpha \in \operatorname{Sper}(\kappa[[\mathtt{x}]]) :\ c <_\alpha 0\} \subset \{\alpha \in \operatorname{Sper}(\kappa[[\mathtt{x}]]) :\ a \in \operatorname{supp}(\alpha)\} =: \mathcal{T}.$$

If $\mathcal{S} \neq \varnothing$, there exist an ordering α of κ and $\eta \in \mathfrak{R}(\alpha)[[\mathtt{t}]]^n$ such that $c(\eta) < 0$ $\mathfrak{R}(\alpha)[[\mathtt{t}]]$ (for $\mathtt{t} > 0$). Let $2k \geqslant 2$ be the order of the series $c(\eta(\mathtt{t}^2)) = a_{2k}\mathtt{t}^{2k} + \cdots < 0$ and consider the non-zero series $c(\eta(\mathtt{t}^2) + \mathtt{t}^{k+1}\mathtt{x}) \in \mathfrak{R}(\alpha)[[\mathtt{t},\mathtt{x}]]$ (see Lemma 6.3). Observe that also by Lemma 6.3 the series $A(\mathtt{t},\mathtt{x}) := a(\eta(\mathtt{t}^2) + \mathtt{t}^{k+1}\mathtt{x}) \in \mathfrak{R}(\alpha)[[\mathtt{t},\mathtt{x}]]$ is non-zero. By [ABR, Thm.VII.4.1] there exists $\theta := (\theta_0,\theta_1,\dots,\theta_n) \in \mathfrak{R}(\alpha)[[\mathtt{t}]]^{n+1}\setminus\{0\}$ such that $\theta(0)=0$ and $A(\theta) \neq 0$. Observe that $\theta_0 \neq 0$ because if $\theta_0 = 0$, then $A(\theta) = a(\eta(0)) = a(0) = 0$, which is a contradiction. After a reparameterization, we may assume $\theta_0 = \pm\mathtt{t}^q$ for some $q \geqslant 1$. Thus,

$$c(\eta(\mathtt{t}^{2q}) \pm \mathtt{t}^{q(k+1)}(\theta_1,\dots,\theta_n)) = a_{2k}\mathtt{t}^{2k} + \cdots < 0,$$

so $a(\eta(\mathtt{t}^{2q}) \pm \mathtt{t}^{q(k+1)}(\theta_1,\dots,\theta_n)) = 0$, which is a contradicition. Consequently, $\mathcal{S} = \varnothing$ and $c \in \mathcal{P}(\kappa[[\mathtt{x}]])$, as required. □

Proof of Lemma 8.1. Observe first that by Weierstrass division theorem $A \cong \kappa[[\mathtt{x},\mathtt{y}]][\mathtt{z}]/(\mathtt{z}^2 - F)$. Suppose $f + \mathtt{z}g \in \Sigma A^2$. Then there exist $a_i, b_i, \eta \in \kappa[[\mathtt{x},\mathtt{y}]]$ such that

$$f + \mathtt{z}g = \sum_{i=1}^{p} (a_i + \mathtt{z}b_i)^2 - (\mathtt{z}^2 - F)\eta.$$

Thus, $(f - F\eta) + g\mathtt{z} + \eta\mathtt{z}^2 = f + \mathtt{z}g + (\mathtt{z}^2 - F)\eta \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]][\mathtt{z}])$ and by Lemma 8.2 we deduce $\eta, f - F\eta, 4\eta(f - F\eta) - g^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$.

Conversely, suppose $\eta, 4\eta(f - F\eta) - g^2$. By Lemma 8.4 we deduce $f - F\eta \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$. Thus, by Lemma 8.2

$$P := f + \mathtt{z}g + \eta(\mathtt{z}^2 - F) = (f - F\eta) + g\mathtt{z} + \eta\mathtt{z}^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]][\mathtt{z}])$$

and by Lemma 8.3 $P \in \Sigma\kappa[[\mathtt{x},\mathtt{y}]][\mathtt{z}]^2$ (because $\tau(\kappa) < +\infty$), so $f + \mathtt{z}g \in \Sigma A^2$, as required. □

eductionpcor **Corollary 8.5.** *Let κ be a (formally) real field with $\tau(\kappa) < +\infty$ and consider the ring $A := \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2 - F)$. For each element $f + \mathtt{z}g \in A$ there exists $n_0 \geqslant 1$ such that $f + \mathtt{z}g \in \Sigma A^2$ if and only if there exist $n \geqslant n_0$ and series $f_n, g_n \in \kappa[[\mathtt{x},\mathtt{y}]]$ with $f - f_n, g - g_n \in \mathfrak{m}_2^n$ and $\eta_n \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$ such that $4\eta_n(f_n - F\eta) - g_n^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$.*

Proof. The only if part follows from Lemma 8.1. Denote $p := 4\tau(\kappa)$ and let us prove the if part. Consider the polynomial equation

$$f + \mathtt{z}g = \sum_{i=1}^{p} (\mathtt{X}_i + \mathtt{z}\mathtt{Y}_i)^2 - (\mathtt{z}^2 - F)\mathtt{Z} \tag{8.1}$$ smaa

By Strong Artin's approximation (Theorem 5.6) there exists $n_0 \geqslant 1$ such that if the equation (8.1) has an approximated solution module $\mathfrak{m}_2^{n_0}$, then it has also an exact solution in $\kappa[[\mathtt{x},\mathtt{y}]]$.

By hypothesis there exist $n \geqslant n_0$ and series $f_n, g_n \in \kappa[[\mathtt{x},\mathtt{y}]]$ with $f - f_n, g - g_n \in \mathfrak{m}_2^n$ and $\eta_n \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$ such that $4\eta_n(f_n - F\eta) - g_n^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$. By Theorem 1.7 and Lemma 8.1 there exist $a_{in}, b_{in}, q_{in} \in \kappa[[\mathtt{x},\mathtt{y}]]$ such that

$$f_n + \mathtt{z}g_n = \sum_{i=1}^{p} (a_{in} + \mathtt{z}b_{in})^2 - (\mathtt{z}^2 - F)q_{in}.$$

Thus, the equation (8.1) has an approximated solution module $\mathfrak{m}_2^n$ for some $n \geqslant n_0$. By Strong Artin's approximation (§2.1) equation (8.1) has an exact solution in $\kappa[[\mathtt{x},\mathtt{y}]]$, that is, there exist series $a_i, b_i, q \in \kappa[[\mathtt{x},\mathtt{y}]]$ such that

$$f + \mathtt{z}g = \sum_{i=1}^{p} (a_i + \mathtt{z}b_i)^2 - (\mathtt{z}^2 - F)q,$$

so $f+\mathtt{z}g\in\Sigma A^2$, as required. □

8.2. **Elephant's improved Theorems.** The following key results are improvements of Elephant's Theorem [Fe8, Thm.5.5] to approach the remaining (order two) cases of Theorem 2.3 when the Pythagoras number of the ring $A:=\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F)$ is finite.

ert **Theorem 8.6** (Elephant's improved Theorem. Part 1)**.** *Let κ be a (formally) real field such that $\tau(\kappa)<+\infty$. Define $\varphi_q:=(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k},\mathtt{v}\mathtt{x}^k)$ for each $q\geqslant 2$ and let $F\in\kappa[\mathtt{x},\mathtt{y}]\subset\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ be a monic polynomial with respect to $\mathtt{x}$ of degree $2k+1$ (for some $k\geqslant 1$) such that $F(\varphi_q)=\mathtt{x}^{2k+1}(1+\mathtt{x}^2\Gamma_q)$ for some $\Gamma_q\in\kappa[\mathtt{x},\mathtt{u}]$ and $-F(0,\mathtt{y})\notin\Sigma\kappa((\mathtt{y}))^2$. Define $A:=\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F)$ and denote $\mathcal{P}_q(A)$ the set of series $f+\mathtt{z}g\in\mathcal{P}(A)$ such that $f(\mathtt{x},0)\neq 0$ and $\omega(f(\mathtt{x},0))=q\geqslant 2$. Suppose that for each $f+\mathtt{z}g\in\mathcal{P}(A)$ there exist $a_i,b_i,c\in\kappa[[\mathtt{x},\mathtt{u}]]$ such that $\mathtt{x}^{2k}((f+\mathtt{z}g)(\varphi_q))=\sum_{i=1}^p(\mathtt{x}^ka_i+\mathtt{v}\mathtt{x}^kb_i)^2-(\mathtt{z}^2-F)(\varphi_q)c$. Then $\mathcal{P}_q(A)\subset\Sigma A^2$.*

Proof. The proof of the statement is conducted in several steps. For the sake of simplicity once
redest we have finished a step, we reset the local notation involved in such step.

[8.6.i]. Let $f+\mathtt{z}g\in\mathcal{P}_q(A)$ and let us prove $f+\mathtt{z}g\in\Sigma A^2$. If $g=0$, then $f\in\mathcal{P}(A)\backslash\{0\}$. Thus, $f+\mathtt{z}f=f(1+\mathtt{z})\in\mathcal{P}(A)$ and $f\neq 0$. As $\frac{1}{1+\mathtt{z}}$ is a square in A, changing $f+\mathtt{z}g$ by $f+\mathtt{z}f$, we may assume $g\neq 0$. By Corollary 8.5 there exists $n_0\geqslant 1$ that reduces our problem to find $n\geqslant n_0$ and series $f_n,g_n\in\kappa[[\mathtt{x},\mathtt{y}]]$ with $f-f_n,g-g_n\in\mathfrak{m}_2^n$ and $\eta_n\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$ such that
conest $4\eta_n(f_n-\eta F)-g_n^2\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$.

[8.6.ii]. We begin with some preliminary reductions: *For each $n\geqslant 1$ we choose $f_n,g_n\in\kappa[\mathtt{x},\mathtt{y}]\backslash\{0\}$ such that f_n is a Weierstrass polynomial with respect to $\mathtt{x}$ and there exists $r>q$ such that $f_n-\mathtt{y}^{2r}+\mathtt{z}g_n\in\mathcal{P}^{\oplus}(A)$.* We will slightly modify f_n,g_n below.

Write $f(\mathtt{x},0):=c\mathtt{x}^q+\cdots$ for some $c\in\kappa\backslash\{0\}$. Let $\alpha\in\operatorname{Sper}(\kappa)$ and $\mathfrak{R}(\alpha)$ be the real closure of $(\kappa,\leqslant_\alpha)$. Consider the homomorphism $\varphi:\kappa[[\mathtt{x},\mathtt{y}]]\to\mathfrak{R}(\alpha)[[\mathtt{y}]]$, $h\mapsto h(\mathtt{x}^2,0)$. As $\varphi(F)=\mathtt{x}^{4k+2}>0$, we have $f(\mathtt{x}^2,0)=c\mathtt{x}^{2q}+\cdots>0$ (because $f\in\mathcal{P}(\{F\geqslant 0\})$), so $c\geqslant_\alpha 0$. Thus, c is non-negative for each ordering of κ, so $c\in\mathcal{P}(\kappa)=\Sigma\kappa^2$. Hence, we divide $f+\mathtt{z}g$ by c and assume in the following $c=1$.

By Weierstrass preparation theorem there exist a Weierstrass polynomial $P\in\kappa[[\mathtt{y}]][\mathtt{x}]$ of degree $q=\omega(f(\mathtt{x},0))$ and a unit $U\in\kappa[[\mathtt{x},\mathtt{y}]]$ such that $U(0,0)=1$ and $f=PU^2$. We divide $f+\mathtt{z}g$ by U^2 and assume that f is a Weierstrass polynomial with respect to $\mathtt{x}$ of degree q. Observe that $f(\mathtt{x},0)=\mathtt{x}^q$ and $f^2(\mathtt{x},0)-F(\mathtt{x},0)g^2(\mathtt{x},0)=\mathtt{x}^{2q}-\mathtt{x}^{2k+1}g^2(\mathtt{x},0)\geqslant 0$, so $f^2(\mathtt{x},0)-F(\mathtt{x},0)g^2(\mathtt{x},0)\neq 0$. As also $g\neq 0$, if n is large enough, then $f+\mathtt{y}^{2n}+\mathtt{z}g\in\mathcal{P}^{\oplus}(A)$ (use Lemma 6.6(iii)). By Corollaries 6.2 and 6.5 there exists $r>\max\{2n,q\}$ such that for each pair of truncations $f_n',g_n\in\kappa[\mathtt{x},\mathtt{y}]$ of f,g of degree $\leqslant r-1$, which satisfy $f-f_n',g-g_n\in\mathfrak{m}_2^r$, it holds $f_n'+\mathtt{y}^{2n}-\mathtt{y}^{2r}+\mathtt{z}g_n\in\mathcal{P}^{\oplus}(A)$. As f is a Weierstrass polynomial with respect to $\mathtt{x}$, also
onlythis $f_n:=f_n'+\mathtt{y}^{2n}$ is a Weierstrass polynomial with respect to $\mathtt{x}$.

[8.6.iii]. Fix $n\geqslant 1$ and denote $f:=f_n$ and $g:=g_n$ to lighten notations. To find the series $\eta:=\eta_n$ we need some preliminary work.

Write $f^*:=f-\mathtt{y}^{2r}:=\mathtt{x}^q+\sum_{k=0}^{q-1}\mathtt{y}\lambda_k(\mathtt{y})\mathtt{x}^k\in\kappa[[\mathtt{y}]][\mathtt{x}]$. We have

$$f^*(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})=\mathtt{x}^q+\sum_{j=0}^{q-1}\mathtt{u}\mathtt{x}^{q+k}\lambda_j(\mathtt{u}\mathtt{x}^{q+k})\mathtt{x}^j=\mathtt{x}^q\Big(1+\sum_{j=0}^{q-1}\lambda_j(\mathtt{u}\mathtt{x}^{q+k})\mathtt{u}\mathtt{x}^j\Big)\qquad(8.2)$$ f1

$$F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})=\mathtt{x}^{2k+1}(1+a\mathtt{y}^2\mathtt{x}^{2q-1})\qquad(8.3)$$ F1

Define $W := 1 + \mathtt{x}^2\Gamma_q \in \kappa[\mathtt{x},\mathtt{u}]$, which satisfies $W(0,0) = 1$ and $F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) = \mathtt{x}^{2k+1}W$. By hypothesis there exist series $a_i, b_i, c \in \kappa[[\mathtt{x},\mathtt{u}]]$ satisfying

$$\mathtt{x}^{2k}f^*(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) + \mathtt{v}\mathtt{x}^k\mathtt{x}^{2k}g(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) = \sum_{i=1}^{p}(\mathtt{x}^k a_i + \mathtt{v}\mathtt{x}^k b_i)^2 - ((\mathtt{v}\mathtt{x}^k)^2 - \mathtt{x}^{2k+1}W)c.$$

Consequently,

$$\mathtt{x}^{2k}f^*(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) + \mathtt{z}\mathtt{x}^{2k}g(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) = \sum_{i=1}^{p}(\mathtt{x}^k a_i + \mathtt{z}b_i)^2 - (\mathtt{z}^2 - \mathtt{x}^{2k+1}W)c.$$

By Lemma 8.2

$$4c(\mathtt{x}^{2k}f^*(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) - \mathtt{x}^{2k+1}Wc) - (\mathtt{x}^{2k}g(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}))^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]]),$$
$$\mathtt{x}^{2k}f^*(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) - \mathtt{x}^{2k+1}Wc = \sum_{i=1}^{p}(\mathtt{x}^k a_i)^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]]).$$

As $f^* = f - \mathtt{y}^{2r}$, we have $f^*(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) = f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) - (\mathtt{u}\mathtt{x}^{q+k})^{2r}$. As in addition $c \neq 0$ (because otherwise $g = 0$ against the hypothesis), we deduce

$$\xi := 4c(\mathtt{x}^{2k}f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) - \mathtt{x}^{2k+1}Wc) - (\mathtt{x}^{2k}g(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}))^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])\setminus\{0\}, \tag{8.4}$$ $\boxed{\texttt{xi}}$

$$\psi := \mathtt{x}^{2k}f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) - \mathtt{x}^{2k+1}Wc = \mathtt{x}^{2k}(\mathtt{u}\mathtt{x}^{q+k})^{2r} + \sum_{i=1}^{p}(\mathtt{x}^k a_i)^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])\setminus\{0\}. \tag{8.5}$$ $\boxed{\texttt{psi}}$

[8.6.iv]. One would like to construct $\eta \in \kappa[\mathtt{x},\mathtt{y}] \cap \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$ from c substituting $\mathtt{u}$ by $\frac{\mathtt{y}}{\mathtt{x}^{q+k}}$. But at this point this does not work because $c \in \kappa[[\mathtt{x},\mathtt{u}]]$ and the desired substitution is not possible. We need to modify first both c and ξ to have polynomials of $\kappa[\mathtt{x},\mathtt{u}]$ instead of series of $\kappa[[\mathtt{x},\mathtt{u}]]$, but keeping both c and ξ inside $\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$. Once this is done we make the substitution $\mathtt{u} = \frac{\mathtt{y}}{\mathtt{x}^{q+k}}$.

Let $\ell, m \geqslant 1$ be such that $\mathtt{x}^{2\ell}$ divides c but $\mathtt{x}^{2\ell+1}$ does not and $\mathtt{x}^{2m}$ divides ξ but $\mathtt{x}^{2m+1}$ does not. Write $c' := \frac{c}{\mathtt{x}^{2\ell}} \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$ and $\xi' := \frac{\xi}{\mathtt{x}^{2m}} \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$, so $c = c'\mathtt{x}^{2\ell}$ and $\xi = \xi'\mathtt{x}^{2m}$. Observe that $c'(0,\mathtt{u}) \neq 0$ and $\xi'(0,\mathtt{u}) \neq 0$. Thus, $c' + \mathtt{x}^{2n}, \xi' + \mathtt{x}^{2n} \in \mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{u}]])$ for each $n \geqslant 1$. Write $\nu := n_0 + \ell + k$. We have

$$\begin{aligned}\xi^* &:= 4(c + \mathtt{x}^{2\nu+2m})(\mathtt{x}^{2k}f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) + 2\mathtt{x}^{2\nu+2m} - F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})(c + \mathtt{x}^{2\nu+2m})) - (\mathtt{x}^{2k}g(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}))^2\\ &= \xi + 4(\mathtt{x}^{4\nu+4m} + \mathtt{x}^{2\nu+2m}c(2 - F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})) + \mathtt{x}^{2\nu+2m}\psi + \mathtt{x}^{4\nu+4m}(1 - F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})))\\ &= \mathtt{x}^{2m}(\xi' + 4(\mathtt{x}^{4\nu+2m} + \mathtt{x}^{2\nu}c(2 - F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})) + \mathtt{x}^{2\nu}\psi + \mathtt{x}^{4\nu+2m}(1 - F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})))).\end{aligned} \tag{8.6}$$ $\boxed{\texttt{xiast}}$

As $\xi' + \mathtt{x}^{4\nu+2m} \in \mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{u}]])$ and $\psi \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$, also

$$\xi'' := \xi' + 4(\mathtt{x}^{4\nu+2m} + \mathtt{x}^{2\nu}c(2 - F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})) + \mathtt{x}^{2\nu}\psi + \mathtt{x}^{4\nu+2m}(1 - F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}))) \in \mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{u}]]).$$

We have $\xi^* = \mathtt{x}^{2m}\xi''$ and $c' + \mathtt{x}^{2\nu+2m-2\ell} \in \mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{u}]])$. As f is a Weierstrass polynomial, write

$$\begin{aligned}f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) &= \mathtt{x}^q\Big(1 + \sum_{i=1}^{d}\mathtt{x}^i f_i(\mathtt{u})\Big),\\ \mathtt{x}^{2k}g(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) &= \mathtt{x}^s\Big(\sum_{j=0}^{e}\mathtt{x}^j g_j(\mathtt{u})\Big),\\ c'(\mathtt{x},\mathtt{u}) &= \sum_{i\geqslant 0}\mathtt{x}^i c'_i(\mathtt{u})\end{aligned}$$

where $f_i, g_j \in \kappa[\mathtt{u}]$, $c'_i \in \kappa[[\mathtt{u}]]$, $g_0, f_d, g_e \neq 0$ and $c'_0 = c'(0,\mathtt{u}) \neq 0$. Let us collect more information concerning the structure and the properties of c.

H

[8.6.v]. Denote $\varepsilon=1$ if q is odd and $\varepsilon=0$ if q is even. We claim: $\mathtt{x}^{2k}f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})-F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})c=\mathtt{x}^{q+2k+\varepsilon}H$ *where* $H\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$ *and* $c_i'\in\kappa[\mathtt{u}]$ *for* $i=0,\dots,2m-2\ell-2k-q-1$. Recall that $c=c'\mathtt{x}^{2\ell}$.

We have

$$\begin{aligned}\xi&=4c(\mathtt{x}^{2k}f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})-F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})c)-(\mathtt{x}^{2k}g(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}))^2\\&=4\mathtt{x}^{2\ell}\Big(\sum_{i\geqslant0}\mathtt{x}^ic_i'(\mathtt{u})\Big)\Big(\mathtt{x}^{q+2k}\Big(1+\sum_{i=1}^d\mathtt{x}^if_i(\mathtt{u})\Big)-(\mathtt{x}^{2k+1}W)\mathtt{x}^{2\ell}\Big(\sum_{i\geqslant0}\mathtt{x}^ic_i'(\mathtt{u})\Big)\Big)-\mathtt{x}^{2s}\Big(\sum_{j=0}^e\mathtt{x}^jg_j(\mathtt{u})\Big)^2\\&=4\mathtt{x}^{2\ell+2k+q}\Big(\sum_{i\geqslant0}\mathtt{x}^ic_i'(\mathtt{u})\Big)\Big(1+\sum_{i=1}^d\mathtt{x}^if_i(\mathtt{u})-\mathtt{x}^{2\ell+1-q}W\Big(\sum_{i\geqslant0}\mathtt{x}^ic_i'(\mathtt{u})\Big)\Big)-\mathtt{x}^{2s}\Big(\sum_{j=0}^e\mathtt{x}^jg_j(\mathtt{u})\Big)^2.\end{aligned}\tag{8.7}$$

xiexp

Although we have not yet proved it, we will also see:

$$2\ell+1-q\geqslant0\quad\text{and}\quad 2m-2\ell-2k-q\geqslant0.\tag{8.8}$$

bound0

We distinguish two cases:

CASE 1. *q is odd.* As

$$\mathtt{x}^q\Big(1+\sum_{i=1}^d\mathtt{x}^if_i(\mathtt{u})\Big)-\mathtt{x}^{2\ell+1}W\Big(\sum_{i\geqslant0}\mathtt{x}^ic_i'(\mathtt{u})\Big)\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]]),\tag{8.9}$$

qood

its leading form is positive semidefinite, so it cannot have degree q odd. As $c_0'(\mathtt{u})\neq0$ and $W=1+\mathtt{x}^2\Gamma$ (where $q\geqslant2$), we deduce $\mathtt{x}^q-\mathtt{x}^{2\ell+1}c_0'(\mathtt{u})=0$, so $q=2\ell+1$ and $c_0'(\mathtt{u})=1$. Thus, $\mathtt{x}^{2k}f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})-F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})c=\mathtt{x}^{q+2k+1}H$ where $H\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$.

As $\xi\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$, we deduce $2\ell+2k+q+1\leqslant2s$ (see (8.7)). We obtain from (8.7) (using that $q=2\ell+1$, $2\ell+2k+q+1\leqslant2s$ and $c_0'(\mathtt{u})=1$)

$$\begin{aligned}\mathtt{x}^{2m}\xi'=\xi=4\mathtt{x}^{2\ell+2k+q}\Big(\Big(1+\sum_{i\geqslant1}\mathtt{x}^ic_i'(\mathtt{u})\Big)\Big(1+\sum_{i=1}^d\mathtt{x}^if_i(\mathtt{u})-(1+\mathtt{x}^2\Gamma_q)\Big(1+\sum_{i\geqslant1}\mathtt{x}^ic_i'(\mathtt{u})\Big)\Big)\\-\mathtt{x}^{2s-2\ell-2k-q}\Big(\sum_{j=0}^e\mathtt{x}^jg_j(\mathtt{u})\Big)^2\Big)=4\mathtt{x}^{2\ell+2k+q+1}\Big(\Big(1+\sum_{i\geqslant1}\mathtt{x}^ic_i'(\mathtt{u})\Big)\\\cdot\Big(\sum_{i=1}^d\mathtt{x}^{i-1}f_i(\mathtt{u})-\sum_{i\geqslant1}\mathtt{x}^{i-1}c_i'(\mathtt{u})-\mathtt{x}\Gamma_q\Big(1+\sum_{i\geqslant1}\mathtt{x}^ic_i'(\mathtt{u})\Big)\Big)-\mathtt{x}^{2s-2\ell-2k-q-1}\Big(\sum_{j=0}^e\mathtt{x}^jg_j(\mathtt{u})\Big)^2\Big).\end{aligned}$$

Consequently, $2m-2\ell-2k-q-1\geqslant0$ and

$$\begin{aligned}4\Big(1+\sum_{i\geqslant1}\mathtt{x}^ic_i'(\mathtt{u})\Big)\Big(\sum_{i=1}^d\mathtt{x}^{i-1}(f_i(\mathtt{u})-c_i'(\mathtt{u}))-\mathtt{x}\Gamma_q\Big(1+\sum_{i\geqslant1}\mathtt{x}^ic_i'(\mathtt{u})\Big)\Big)\\-\mathtt{x}^{2s-2\ell-q-1}\Big(\sum_{j=0}^e\mathtt{x}^jg_j(\mathtt{u})\Big)^2=\mathtt{x}^{2m-2\ell-q-1}\xi',\end{aligned}$$

or equivalently,

$$\begin{aligned}\sum_{i\geqslant1}\mathtt{x}^{i-1}(f_i-c_i')+\mathtt{x}\Big(\Big(\sum_{i\geqslant1}\mathtt{x}^{i-1}c_i'\Big)\Big(\sum_{i\geqslant1}\mathtt{x}^{i-1}(f_i-c_i')\Big)-\Gamma_q\Big(1+\sum_{i\geqslant1}\mathtt{x}^ic_i'\Big)^2\Big)\\-\mathtt{x}^{2s-2\ell-q-1}\frac14\Big(\sum_{j=0}^e\mathtt{x}^jg_j(\mathtt{u})\Big)^2=\mathtt{x}^{2m-2\ell-q-1}\frac14\xi'\end{aligned}\tag{8.10}$$

solve

where $f_i=0$ for $i\geqslant d+1$. Observe that the coefficient of $\mathtt{x}^i$ on the right hand side of equation (8.10) is zero for $0\leqslant i\leqslant2m-2\ell-q-2$.

Recall that $\Gamma_q \in \kappa[\mathtt{x},\mathtt{u}]$ and $c'_0 = 1$. If one compares coefficients with respect to $\mathtt{x}$ in equation (8.10), one realizes that $c'_i - P_i(f_1,\dots,f_d,g_0,\dots,g_e,c'_1,\dots,c'_{i-1}) = 0$ for some polynomial

$$P_i \in \kappa[\mathtt{u}][\mathtt{x}_1,\dots,\mathtt{x}_d,\mathtt{y}_0,\dots,\mathtt{y}_e,\mathtt{z}_1,\dots,\mathtt{z}_{i-1}]$$

if $i = 1,\dots,2m-2\ell-2k-q-1$. We conclude inductively $c'_i \in \kappa[\mathtt{u}]$ for $i = 0,\dots,2m-2\ell-q-1$. ■

CASE 2. *q is even.* As

$$\mathtt{x}^q\Big(1+\sum_{i=1}^d \mathtt{x}^i f_i(\mathtt{u})\Big) - \mathtt{x}^{2\ell+1}W\Big(\sum_{i\geqslant 0}\mathtt{x}^i c'_i(\mathtt{u})\Big) \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$$

and $c'_0(\mathtt{u}) \neq 0$, we have $q < 2\ell+1$, so $\mathtt{x}^{2k}f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) - F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})c = \mathtt{x}^{q+2k}H$ where $H \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$.

As $\xi \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$ and $\xi = \mathtt{x}^{2m}\xi'$, we deduce $2\ell+2k+q \leqslant 2s$ and $2\ell+2k+q \leqslant 2m$ (see (8.7)). Thus,

$$\Big(\sum_{i\geqslant 0}\mathtt{x}^i c'_i(\mathtt{u})\Big)\Big(1+\sum_{i=1}^d \mathtt{x}^i f_i(\mathtt{u}) - \mathtt{x}^{2\ell+1-q}W\Big(\sum_{i\geqslant 0}\mathtt{x}^i c'_i(\mathtt{u})\Big)\Big) \\ - \mathtt{x}^{2s-2\ell-2k-q}\frac{1}{4}\Big(\sum_{j=0}^e \mathtt{x}^j g_j(\mathtt{u})\Big)^2 = \mathtt{x}^{2m-2\ell-2k-q}\frac{1}{4}\xi'. \qquad (8.11)$$ solve1

Observe that the coefficient of $\mathtt{x}^i$ on the right hand side of equation (8.11) is zero for $0 \leqslant i \leqslant 2m-2\ell-2k-q-1$.

Recall that $W \in \kappa[\mathtt{x},\mathtt{u}]$. If one compares coefficients with respect to $\mathtt{x}$ in equation (8.11), one realizes that $c'_i - Q_i(f_1,\dots,f_d,g_0,\dots,g_e,c'_1,\dots,c'_{i-1}) = 0$ for some polynomial

$$Q_i \in \kappa[\mathtt{u}][\mathtt{x}_1,\dots,\mathtt{x}_d,\mathtt{y}_0,\dots,\mathtt{y}_e,\mathtt{z}_1,\dots,\mathtt{z}_{i-1}]$$

if $i = 0,\dots,2m-2\ell-2k-q-1$. We conclude inductively that $c'_i \in \kappa[\mathtt{u}]$ for $i = 0,\dots,2m-2\ell-2k-q-1$, as claimed. ■

[8.6.vi]. We are ready to modify the series $c,\xi \in \kappa[[\mathtt{x},\mathtt{u}]]$ in order to obtain polynomials $c^\bullet,\xi^\bullet \in \kappa[\mathtt{x},\mathtt{u}] \cap \mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$ (from which we will construct $\eta \in \kappa[\mathtt{x},\mathtt{y}] \cap \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$) that substitute the positive semidefinite series c,ξ already constructed in [8.6.iii].

As $\xi'', c' + \mathtt{x}^{2\nu+2m-2\ell} \in \mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{u}]])$, there exists by Corollary 6.5 $r' \geqslant 1$ such that if $\zeta,\theta \in \kappa[[\mathtt{x},\mathtt{u}]]$ satisfy $\zeta - \xi'', \theta - c' \in \mathfrak{m}_2^{r'}$, then $\zeta, \theta + \mathtt{x}^{2\nu+2m-2\ell} \in \mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{u}]])$. We may assume r' is strictly greater than the maximum of the degrees of the polynomials $\mathtt{x}^i c'_i$ for $i = 0,\dots,2m-2\ell-2k-q-1$.

Let $c^\circ \in \kappa[\mathtt{x},\mathtt{u}]$ be a polynomial such that $\omega(c^\circ - c') \geqslant r' + 2m$. Then

$$c^\circ = \sum_{i=0}^{2m-2\ell-2k-q-1} c'_i\mathtt{x}^i + \mathtt{x}^{2m-2\ell-2k-q}h$$

for some polynomial $h \in \kappa[\mathtt{x},\mathtt{u}]$, so $c^\circ = c' + \mathtt{x}^{2m-2\ell-2k-q}h'$ for some $h' \in \kappa[[\mathtt{x},\mathtt{u}]]$ and $\mathtt{x}^{2\ell}c^\circ = c + \mathtt{x}^{2m-2k-q}h'$. Observe that

$$\omega(c^\circ + \mathtt{x}^{2\nu+2m-2\ell} - (c' + \mathtt{x}^{2\nu+2m-2\ell})) = \omega(c^\circ - c') \geqslant r' + 2m > r'.$$

Thus, $c^\circ + \mathtt{x}^{2\nu+2m-2\ell} \in \kappa[\mathtt{x},\mathtt{u}] \cap \mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{u}]])$ and $c^\bullet := \mathtt{x}^{2\ell}c^\circ + \mathtt{x}^{2\nu+2m} \in \kappa[\mathtt{x},\mathtt{u}] \cap \mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$. Consider the polynomial

$$\xi^\bullet := 4c^\bullet(\mathtt{x}^{2k}f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) + 2\mathtt{x}^{2\nu+2m} - F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})c^\bullet) - (\mathtt{x}^{2k}g(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}))^2. \qquad (8.12)$$ xibull

We claim: *$\xi^\bullet - \xi = \mathtt{x}^{2m}E$ for some $E \in \kappa[[\mathtt{x},\mathtt{u}]]$.*

By (8.4) and [8.6.v] we have:

$$4c(\mathtt{x}^{2k}f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})-F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})c)-(\mathtt{x}^{2k}g(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}))^2=\xi,$$
$$\mathtt{x}^{2k}f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})-F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})c=\mathtt{x}^{q+2k+\varepsilon}H.$$

As $F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})=\mathtt{x}^{2k+1}W$, $\mathtt{x}^{2\ell}c^\circ=c+\mathtt{x}^{2m-2k-q}h$ and $c^\bullet=\mathtt{x}^{2\ell}c^\circ+\mathtt{x}^{2\nu+2m}=c+\mathtt{x}^{2m-2k-q}h'+\mathtt{x}^{2\nu+2m}$, we conclude

$$\begin{aligned}\xi^\bullet&=4c^\bullet(\mathtt{x}^{2k}f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})+2\mathtt{x}^{2\nu+2m}-F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})c^\bullet)-(\mathtt{x}^{2k}g(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}))^2\\
&=4(c+\mathtt{x}^{2m-2k-q}h'+\mathtt{x}^{2\nu+2m})(\mathtt{x}^{2k}f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})+2\mathtt{x}^{2\nu+2m}\\
&\quad-F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})(c+\mathtt{x}^{2m-2k-q}h'+\mathtt{x}^{2\nu+2m}))-(\mathtt{x}^{2k}g(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}))^2\\
&=4c(\mathtt{x}^{2k}f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})-F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})c)-(\mathtt{x}^{2k}g(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}))^2\\
&\quad+4(\mathtt{x}^{2m-2k-q}h'+\mathtt{x}^{2\nu+2m})(\mathtt{x}^{2k}f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})-F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})c)\\
&\quad+4(c+\mathtt{x}^{2m-2k-q}h'+\mathtt{x}^{2\nu+2m})(2\mathtt{x}^{2\nu+2m}-\mathtt{x}^{2k+1}W(\mathtt{x}^{2m-2k-q}h'+\mathtt{x}^{2\nu+2m}))\\
&=\xi+4(\mathtt{x}^{2m-2k-q}h'+\mathtt{x}^{2\nu+2m})\mathtt{x}^{q+2k+\varepsilon}H\\
&\quad+4(c+\mathtt{x}^{2m-2k-q}h'+\mathtt{x}^{2\nu+2m})(2\mathtt{x}^{2\nu+2m}-\mathtt{x}^{2k+1}W(\mathtt{x}^{2m-2k-q}h'+\mathtt{x}^{2\nu+2m})).\end{aligned}$$

Thus, to justify that

$$\begin{aligned}\xi^\bullet-\xi&=4\mathtt{x}^{2m+\varepsilon}(h'+\mathtt{x}^{2\nu+q+2k})H+4\mathtt{x}^{2\nu+2m}(2\mathtt{x}^{2\nu+2m}-\mathtt{x}^{2k+1}W(\mathtt{x}^{2m-2k-q}h'+\mathtt{x}^{2\nu+2m}))\\
&\quad+4(c+\mathtt{x}^{2m-2k-q}h')(2\mathtt{x}^{2\nu+2m}-\mathtt{x}^{2k+1}W\mathtt{x}^{2\nu+2m})+4(c+\mathtt{x}^{2m-2k-q}h')(-\mathtt{x}^{2k+1}W\mathtt{x}^{2m-2k-q}h')\end{aligned}$$

is divisible by $\mathtt{x}^{2m}$, we only have to clarify why the product

$$\begin{aligned}4(c+\mathtt{x}^{2m-2k-q}h')(-\mathtt{x}^{2k+1}W(\mathtt{x}^{2m-2k-q}h'))&=4(\mathtt{x}^{2\ell}c'+\mathtt{x}^{2m-2k-q}h')(-\mathtt{x}^{2k+1}W(\mathtt{x}^{2m-2k-q}h'))\\
&=-4\mathtt{x}^{2m+2\ell+1-q}c'Wh'-4\mathtt{x}^{4m+1-2k-2q}Wh'^2\end{aligned}$$

is divisible by $\mathtt{x}^{2m}$. The first addend on the right hand side is divisible by $\mathtt{x}^{2m}$ because $q\leqslant 2\ell+1$ (see (8.8)), whereas the second addend is divisible by $\mathtt{x}^{2m}$ because, as $2m-2\ell-2k-q\geqslant 0$ and $2\ell+1-q\geqslant 0$ (see (8.8)),

$$4m+1-2k-2q=2m+(2m-2\ell-2k-q)+(2\ell+1-q)\geqslant 2m.$$

We conclude $\xi^\bullet-\xi=\mathtt{x}^{2m}E$ for some $E\in\kappa[[\mathtt{x},\mathtt{u}]]$.

As $\mathtt{x}^{2m}$ divides ξ, also $\mathtt{x}^{2m}$ divides $\xi^\bullet$, so there exists a polynomial $\xi^\circ\in\kappa[\mathtt{x},\mathtt{u}]$ such that $\xi^\bullet=\mathtt{x}^{2m}\xi^\circ$. By (8.6) and (8.12) and using that $c^\bullet=\mathtt{x}^{2\ell}c^\circ+\mathtt{x}^{2\nu+2m}$ we have

$$\begin{aligned}\xi^*&=4(\mathtt{x}^{2\ell}c'+\mathtt{x}^{2\nu+2m})(\mathtt{x}^{2k}f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})+2\mathtt{x}^{2\nu+2m}-F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})(\mathtt{x}^{2\ell}c'+\mathtt{x}^{2\nu+2m}))\\
&\quad-(\mathtt{x}^{2k}g(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}))^2,\\
\xi^\bullet&=4(\mathtt{x}^{2\ell}c^\circ+\mathtt{x}^{2\nu+2m})(\mathtt{x}^{2k}f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})+2\mathtt{x}^{2\nu+2m}-F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})(\mathtt{x}^{2\ell}c^\circ+\mathtt{x}^{2\nu+2m}))\\
&\quad-(\mathtt{x}^{2k}g(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}))^2.\end{aligned}$$

Consequently,

$$\begin{aligned}\xi^\bullet-\xi^*&=4\mathtt{x}^{2\ell}(\mathtt{x}^{2k}f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})+2\mathtt{x}^{2\nu+2m}-F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})\mathtt{x}^{2\nu+2m})(c^\circ-c')\\
&\quad-4\mathtt{x}^{2\nu+2m+2\ell}F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})(c^\circ-c')-4\mathtt{x}^{4\ell}F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k})(c^\circ+c')(c^\circ-c'),\end{aligned}$$

so $\omega(\xi^\bullet-\xi^*)\geqslant\omega(c^\circ-c')$. As $\xi^\bullet-\xi^*=\mathtt{x}^{2m}(\xi^\circ-\xi'')$, we have $\omega(\xi^\circ-\xi'')=\omega(\xi^\bullet-\xi^*)-2m$. As $\omega(c^\circ-c')\geqslant r'+2m$, it holds $\omega(\xi^\circ-\xi'')\geqslant r'$, so $\xi^\circ\in\mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{u}]])$. This means that $\xi^\bullet=\mathtt{x}^{2m}\xi^\circ\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$.

[8.6.vii]. Write $\mathtt{u} := \frac{\mathtt{y}}{\mathtt{x}^q}$ in $\xi^\bullet(\mathtt{x}, \frac{\mathtt{y}}{\mathtt{x}^q}) \in \kappa[\mathtt{x}, \mathtt{u}]$:

$$\xi^\bullet\Big(\mathtt{x}, \frac{\mathtt{y}}{\mathtt{x}^q}\Big) = 4c^\bullet\Big(\mathtt{x}, \frac{\mathtt{y}}{\mathtt{x}^q}\Big)\Big(\mathtt{x}^{2k} f(\mathtt{x},\mathtt{y}) + 2\mathtt{x}^{2\nu+2m} - F(\mathtt{x},\mathtt{y})c^\bullet\Big(\mathtt{x}, \frac{\mathtt{y}}{\mathtt{x}^q}\Big)\Big) - (\mathtt{x}^{2k} g(\mathtt{x},\mathtt{y}))^2,$$

where $c^\bullet \in \kappa[\mathtt{x}, \mathtt{u}]$. After clearing denominators, by Lemma 5.15 we find an integer $\mu \geqslant 1$ and a polynomial $B \in \kappa[\mathtt{x}, \mathtt{y}] \cap \mathcal{P}(\kappa[[\mathtt{x}, \mathtt{y}]])$ such that

$$4B(\mathtt{x}^{2\mu+2k} f + 2\mathtt{x}^{2\nu+2m+2\mu} - FB) - \mathtt{x}^{4\mu+4k} g^2 \in \kappa[\mathtt{x}, \mathtt{y}] \cap \mathcal{P}(\kappa[[\mathtt{x}, \mathtt{y}]]).$$

Thus, there exist series $A_i \in \kappa[[\mathtt{x}, \mathtt{y}]]$ satisfying

$$4B(\mathtt{x}^{2\mu+2k} f + 2\mathtt{x}^{2\nu+2m+2\mu} - FB) - \mathtt{x}^{4\mu+4k} g^2 = \sum_{i=1}^p A_i^2. \tag{8.13}$$ FB

Setting $\mathtt{x} = 0$ in (8.13), we deduce

$$-4F(0,\mathtt{y})B^2(0,\mathtt{y}) = -4a\mathtt{y}^2 B^2(0,\mathtt{y}) = \sum_{i=1}^p A_i^2(0,\mathtt{y}).$$

As $-F(0,\mathtt{y}) \notin -\Sigma\kappa((\mathtt{y}))^2$, we deduce $B(0,\mathtt{y}) = 0$ and $A_i(0,\mathtt{y}) = 0$, so there exist $B', A_i' \in \kappa[[\mathtt{x}, \mathtt{y}]]$ such that $B = \mathtt{x}^2 B'$ (recall that $B \in \mathcal{P}(\kappa[[\mathtt{x}, \mathtt{y}]])$) and $A_i = \mathtt{x} A_i'$. We deduce

$$4\mathtt{x}^4 B'(\mathtt{x}^{2\mu+2k-2} f + 2\mathtt{x}^{2\nu+2m+2\mu-2} - FB') - \mathtt{x}^{4\mu+4k} g^2 = \mathtt{x}^2 \sum_{i=1}^p A_i'^2. \tag{8.14}$$ FB2

Dividing (8.14) by $\mathtt{x}^2$ and setting $\mathtt{x} = 0$ next, we deduce $A_i'(0,\mathtt{y}) = 0$, so there exists $A_i'' \in \kappa[[\mathtt{x}, \mathtt{y}]]$ such that $A_i = \mathtt{x} A_i''$. Consequently,

$$4B'(\mathtt{x}^{2\mu+2k-2} f + 2\mathtt{x}^{2\nu+2m+2\mu-2} - FB') - \mathtt{x}^{4\mu+4k-4} g^2 = \sum_{i=1}^p A_i''^2.$$

Proceeding recursively with μ we find the sought polynomial $\eta \in \kappa[\mathtt{x}, \mathtt{y}] \cap \mathcal{P}(\kappa[[\mathtt{x}, \mathtt{y}]])$ such that

$$4\eta(f + 2\mathtt{x}^{2\nu+2m-2k} - F\eta) - g^2 \in \kappa[\mathtt{x}, \mathtt{y}] \cap \mathcal{P}(\kappa[[\mathtt{x}, \mathtt{y}]]).$$

To finish we define $n := \nu + m - k \geqslant n_0 + \ell$, $f_n := f + 2\mathtt{x}^{2\nu+2m-2k}$, $g_n := g$ and $\eta_n := \eta$, as required. □

The proof of the following result is similar to the one of Theorem 8.6, but the differences between both suggest to include a detailed proof.

ert2 **Theorem 8.7** (Elephant's improved Theorem. Part 2)**.** *Let κ be a (formally) real field such that $\tau(\kappa) < +\infty$ and let $F = a\mathtt{x}^2 + b\mathtt{y}^{2k}$ where $k \geqslant 1$, $a \notin -\Sigma\kappa^2$ and $b \neq 0$. Define $A := \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/(\mathtt{z}^2 - F)$ and $\varphi_q := (\mathtt{x}, \mathtt{u}\mathtt{x}^q, \mathtt{z})$. Denote $\mathcal{P}_q(A)$ the set of series $f + \mathtt{z}g \in \mathcal{P}(A)$ such that $f(\mathtt{x}, 0) \neq 0$ and $q := \omega(f(\mathtt{x}, 0)) \geqslant 1$. Suppose that for each $f + \mathtt{z}g \in \mathcal{P}(A)$ there exist $a_i, b_i, c \in \kappa[[\mathtt{x}, \mathtt{u}]]$ such that $(f + \mathtt{z}g)(\varphi) = \sum_{i=1}^p (a_i + \mathtt{z}b_i)^2 - (\mathtt{z}^2 - F)(\varphi)c$. Then $\mathcal{P}_q(A) \subset \Sigma A^2$.*

Proof. Let $f + \mathtt{z}g \in \mathcal{P}(A)$ be such that $f(\mathtt{x}, 0) \neq 0$ and $q := \omega(f(\mathtt{x}, 0)) \geqslant 1$. Recall that $f \in \mathcal{P}(\{F \geqslant 0\})$ and $f^2 - Fg^2 \in \mathcal{P}(\kappa[[\mathtt{x}, \mathtt{y}]])$. Let us prove first: *$q$ is even.*

Assume q is odd. Let α be an ordering of κ such that $a >_\alpha 0$. Let $\eta := (\mathtt{t}, 0)$ and observe that $F(\eta) = a\mathtt{t}^2 > 0$ independently of the sign of $\mathtt{t}$. On the other hand, $f(\eta) = f(\mathtt{t}, 0) = c\mathtt{t}^q + \cdots$ for some $c \in \kappa\setminus\{0\}$. As q is odd, we can choose the sign of $\mathtt{t}$ in order to have $f(\eta) < 0$, which contradicts the fact that $f \in \mathcal{P}(\{F \geqslant 0\})$. Thus, q is even.

We may assume: $-b \notin \Sigma\kappa^2$.

Otherwise, $-b \in \Sigma\kappa^2$ and

$$(-a)(\mathtt{z}^2 - a\mathtt{x}^2 - b\mathtt{y}^{2k}) = -a\mathtt{z}^2 + (a\mathtt{x})^2 + ab\mathtt{y}^{2k} = (a\mathtt{x})^2 - a\mathtt{z}^2 - a(-b)\mathtt{y}^{2k} = \mathtt{z}'^2 - a\mathtt{x}'^2 - b'\mathtt{y}^{2k},$$

where $\mathtt{z}' := a\mathtt{x}$, $\mathtt{x}' := \mathtt{z}$ and $b' := a(-b)$ satisfies $-b' = (-a)(-b) \notin \Sigma\kappa^2$, because $-a \notin \Sigma\kappa^2$, whereas $-b \in \Sigma\kappa^2$.

Thus, after resetting notations, we assume in addition $-b \notin \Sigma\kappa^2$.

The proof of the statement is now conducted in several steps. For the sake of simplicity once we have finished a step, we reset the local notation involved in such step. redest3

[8.7.i]. Let $f + \mathtt{z}g \in \mathcal{P}(A)$ be such that $\omega(f(\mathtt{x}, 0)) = k \geqslant 2$ and let us prove $f + \mathtt{z}g \in \Sigma A^2$. If $g = 0$, then $f \in \mathcal{P}(A)\backslash\{0\}$. Thus, $f + \mathtt{z}f = f(1 + \mathtt{z}) \in \mathcal{P}(A)$ and $f \neq 0$. As $\frac{1}{1+\mathtt{z}}$ is a square in A, changing $f + \mathtt{z}g$ by $f + \mathtt{z}f$, we may assume $g \neq 0$. By Corollary 8.5 there exists $n_0 \geqslant 1$ that reduces our problem to find $n \geqslant n_0$ and series $f_n, g_n \in \kappa[[\mathtt{x}, \mathtt{y}]]$ with $f - f_n, g - g_n \in \mathfrak{m}_2^n$ and conest3 $\eta_n \in \mathcal{P}(\kappa[[\mathtt{x}, \mathtt{y}]])$ such that $4\eta_n(f_n - \eta F) - g_n^2 \in \mathcal{P}(\kappa[[\mathtt{x}, \mathtt{y}]])$.

[8.7.ii]. We begin with some preliminary reductions: *For each* $n \geqslant 1$ *we choose* $f_n, g_n \in \kappa[\mathtt{x}, \mathtt{y}]\backslash\{0\}$, f_n *is a Weierstrass polynomial with respect to* $\mathtt{x}$ *times* $\lambda \in \kappa\backslash\{0\}$ *and there exists* $r > q$ *such that* $f_n - \mathtt{y}^{2r} + \mathtt{z}g_n \in \mathcal{P}^{\oplus}(A)$. We will slightly modify f_n, g_n below.

As $f(\mathtt{x}, 0) := \lambda\mathtt{x}^q + \cdots$ for some $\lambda \in \kappa\backslash\{0\}$, by Weierstrass preparation theorem there exist a Weierstrass polynomial $P \in \kappa[[\mathtt{y}]][\mathtt{x}]$ of degree $q = \omega(f(\mathtt{x}, 0))$, a unit $U \in \kappa[[\mathtt{x}, \mathtt{y}]]$ and $\lambda \in \kappa\backslash\{0\}$ such that $U(0, 0) = 1$ and $f = \lambda PU^2$. We divide $f + \mathtt{z}g$ by U^2 and assume that f is a Weierstrass polynomial with respect to $\mathtt{x}$ of degree q times $\lambda \in \kappa\backslash\{0\}$. Observe that $f(\mathtt{x}, 0) = \lambda\mathtt{x}^q$ and $f^2(\mathtt{x}, 0) - F(\mathtt{x}, 0)g^2(\mathtt{x}, 0) = \lambda^2\mathtt{x}^{2q} - a\mathtt{x}^2g^2(\mathtt{x}, 0) \geqslant 0$, so $f^2(\mathtt{x}, 0) - F(\mathtt{x}, 0)g^2(\mathtt{x}, 0) \neq 0$ because $a \notin \Sigma\kappa^2$ and in particular it is not a square in κ. As also $g \neq 0$, if n is large enough, then $f + \mathtt{y}^{2n} + \mathtt{z}g \in \mathcal{P}^{\oplus}(A)$ (Lemma 6.6). By Corollaries 6.2 and 6.5 there exists $r > \max\{2n, q\}$ such that for each pair of truncations $f_n', g_n \in \kappa[\mathtt{x}, \mathtt{y}]$ of f, g of degree $\leqslant r - 1$, which satisfy $f - f_n, g - g_n \in \mathfrak{m}_2^r$, it holds $f_n + \mathtt{y}^{2n} - \mathtt{y}^{2r} + \mathtt{z}g_n \in \mathcal{P}^{\oplus}(A)$. As f is a Weierstrass polynomial with respect to $\mathtt{x}$ times $\lambda \in \kappa\backslash\{0\}$, also $f_n := f_n' + \mathtt{y}^{2n}$ is a Weierstrass polynomial with respect onlythis3 to $\mathtt{x}$ times $\lambda \in \kappa\backslash\{0\}$.

[8.7.iii]. Fix $n \geqslant 1$ and denote $f := f_n$ and $g := g_n$ to lighten notations. To find the series $\eta := \eta_n$ we need some preliminary work.

Write $f^* := f - \mathtt{y}^{2r} := \lambda\mathtt{x}^q + \sum_{k=0}^{q-1} \mathtt{y}\lambda_k(\mathtt{y})\mathtt{x}^k \in \kappa[[\mathtt{y}]][\mathtt{x}]$. We have

$$f^*(\mathtt{x}, \mathtt{u}\mathtt{x}^q) = \mathtt{x}^q + \sum_{j=0}^{q-1} \mathtt{u}\mathtt{x}^q\lambda_j(\mathtt{u}\mathtt{x}^q)\mathtt{x}^j = \mathtt{x}^q\Big(1 + \sum_{j=0}^{q-1} \lambda_j(\mathtt{u}\mathtt{x}^q)\mathtt{u}\mathtt{x}^j\Big) \tag{8.15}$$ f13

$$F(\mathtt{x}, \mathtt{u}\mathtt{x}^q) = a\mathtt{x}^2(1 + \tfrac{b}{a}\mathtt{u}^{2k}\mathtt{x}^{2kq-2}) \tag{8.16}$$ F13

Define $W := 1 + \frac{b}{a}\mathtt{u}^{2k}\mathtt{x}^{2kq-2} \in \kappa[\mathtt{x}, \mathtt{u}]$, which satisfies $W(0, 0) = 1$ and $F(\mathtt{x}, \mathtt{u}\mathtt{x}^q) = a\mathtt{x}^2W$. By hypothesis there exist series $a_i, b_i, c \in \kappa[[\mathtt{x}, \mathtt{u}]]$ satisfying

$$f^*(\mathtt{x}, \mathtt{u}\mathtt{x}^q) + \mathtt{z}g(\mathtt{x}, \mathtt{u}\mathtt{x}^q) = \sum_{i=1}^{p}(a_i + \mathtt{z}b_i)^2 - (\mathtt{z}^2 - a\mathtt{x}^2W)c. \tag{8.17}$$

By Lemma 8.2

$$4c(f^*(\mathtt{x}, \mathtt{u}\mathtt{x}^q) - a\mathtt{x}^2Wc) - (g(\mathtt{x}, \mathtt{u}\mathtt{x}^q))^2 \in \mathcal{P}(\kappa[[\mathtt{x}, \mathtt{u}]]),$$

$$f^*(\mathtt{x}, \mathtt{u}\mathtt{x}^q) - a\mathtt{x}^2Wc = \sum_{i=1}^{p}(\mathtt{x}^ka_i)^2 \in \mathcal{P}(\kappa[[\mathtt{x}, \mathtt{u}]]).$$

As $f^* = f - \mathtt{y}^{2r}$, we have $f^*(\mathtt{x},\mathtt{u}\mathtt{x}^q) = f(\mathtt{x},\mathtt{u}\mathtt{x}^q) - (\mathtt{u}\mathtt{x}^q)^{2r}$. As in addition $c \neq 0$ (because otherwise $g = 0$ against the hypothesis), we deduce

$$\xi := 4c(f(\mathtt{x},\mathtt{u}\mathtt{x}^q) - a\mathtt{x}^2Wc) - (g(\mathtt{x},\mathtt{u}\mathtt{x}^q))^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])\backslash\{0\}, \tag{8.18}$$ xi3

$$\psi := f(\mathtt{x},\mathtt{u}\mathtt{x}^q) - a\mathtt{x}^2Wc = (\mathtt{u}\mathtt{x}^q)^{2r} + \sum_{i=1}^{p}(\mathtt{x}^k a_i)^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])\backslash\{0\}. \tag{8.19}$$ psi3

[8.7.iv]. One would like to construct $\eta \in \kappa[\mathtt{x},\mathtt{y}] \cap \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$ from c substituting $\mathtt{u}$ by $\frac{\mathtt{y}}{\mathtt{x}^{q+k}}$. But at this point this does not work because $c \in \kappa[[\mathtt{x},\mathtt{u}]]$ and the desired substitution is not possible. We need to modify first both c and ξ to have polynomials of $\kappa[\mathtt{x},\mathtt{u}]$ instead of series of $\kappa[[\mathtt{x},\mathtt{u}]]$, but keeping both c and ξ inside $\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$. Once this is done we make the substitution $\mathtt{u} = \frac{\mathtt{y}}{\mathtt{x}^{q+k}}$.

Let $\ell, m \geqslant 1$ be such that $\mathtt{x}^{2\ell}$ divides c but $\mathtt{x}^{2\ell+1}$ does not and $\mathtt{x}^{2m}$ divides ξ but $\mathtt{x}^{2m+1}$ does not. Write $c' := \frac{c}{\mathtt{x}^{2\ell}} \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$ and $\xi' := \frac{\xi}{\mathtt{x}^{2m}} \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$, so $c = c'\mathtt{x}^{2\ell}$ and $\xi = \xi'\mathtt{x}^{2m}$. Observe that $c'(0,\mathtt{u}) \neq 0$ and $\xi'(0,\mathtt{u}) \neq 0$. Thus, $c' + \mathtt{x}^{2n}, \xi' + \mathtt{x}^{2n} \in \mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{u}]])$ for each $n \geqslant 1$. Write $\nu := n_0 + \ell$. We have

$$\begin{aligned}\xi^* &:= 4(c + \mathtt{x}^{2\nu+2m})(f(\mathtt{x},\mathtt{u}\mathtt{x}^q) + 2\mathtt{x}^{2\nu+2m} - F(\mathtt{x},\mathtt{u}\mathtt{x}^q)(c + \mathtt{x}^{2\nu+2m})) - (g(\mathtt{x},\mathtt{u}\mathtt{x}^q))^2\\ &= \xi + 4(\mathtt{x}^{4\nu+4m} + \mathtt{x}^{2\nu+2m}c(2 - F(\mathtt{x},\mathtt{u}\mathtt{x}^q)) + \mathtt{x}^{2\nu+2m}\psi + \mathtt{x}^{4\nu+4m}(1 - F(\mathtt{x},\mathtt{u}\mathtt{x}^q)))\\ &= \mathtt{x}^{2m}(\xi' + 4(\mathtt{x}^{4\nu+2m} + \mathtt{x}^{2\nu}c(2 - F(\mathtt{x},\mathtt{u}\mathtt{x}^q)) + \mathtt{x}^{2\nu}\psi + \mathtt{x}^{4\nu+2m}(1 - F(\mathtt{x},\mathtt{u}\mathtt{x}^q)))).\end{aligned} \tag{8.20}$$ xiast3

As $\xi' + \mathtt{x}^{4\nu+2m} \in \mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{u}]])$ and $\psi \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$, also

$$\xi'' := \xi' + 4(\mathtt{x}^{4\nu+2m} + \mathtt{x}^{2\nu}c(2 - F(\mathtt{x},\mathtt{u}\mathtt{x}^q)) + \mathtt{x}^{2\nu}\psi + \mathtt{x}^{4\nu+2m}(1 - F(\mathtt{x},\mathtt{u}\mathtt{x}^q))) \in \mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{u}]]).$$

We have $\xi^* = \mathtt{x}^{2m}\xi''$ and $c' + \mathtt{x}^{2\nu+2m-2\ell} \in \mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{u}]])$. As f is a Weierstrass polynomial, write

$$\begin{aligned}f(\mathtt{x},\mathtt{u}\mathtt{x}^q) &= \mathtt{x}^q\Big(\lambda + \sum_{i=1}^{d}\mathtt{x}^i f_i(\mathtt{u})\Big),\\ g(\mathtt{x},\mathtt{u}\mathtt{x}^q) &= \mathtt{x}^s\Big(\sum_{j=0}^{e}\mathtt{x}^j g_j(\mathtt{u})\Big),\\ c'(\mathtt{x},\mathtt{u}) &= \sum_{i\geqslant 0}\mathtt{x}^i c_i'(\mathtt{u})\end{aligned}$$

where $f_i, g_j \in \kappa[\mathtt{u}]$, $c_i' \in \kappa[[\mathtt{u}]]$, $g_0, f_d, g_e \neq 0$ and $c_0' = c'(0,\mathtt{u}) \neq 0$. Let us collect more information concerning the structure and the properties of c.

H3

[8.7.v]. We claim: $f(\mathtt{x},\mathtt{u}\mathtt{x}^q) - F(\mathtt{x},\mathtt{u}\mathtt{x}^q)c = \mathtt{x}^q H$ *where* $H \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$ *and* $c_i' \in \kappa[\mathtt{u}]$ *for* $i = 0,\dots,2m-2\ell-q-1$. Recall that $c = c'\mathtt{x}^{2\ell}$.

We have

$$\begin{aligned}\xi &= 4c(f(\mathtt{x},\mathtt{u}\mathtt{x}^q) - F(\mathtt{x},\mathtt{u}\mathtt{x}^q)c) - (g(\mathtt{x},\mathtt{u}\mathtt{x}^q))^2\\ &= 4\mathtt{x}^{2\ell}\Big(\sum_{i\geqslant 0}\mathtt{x}^i c_i'(\mathtt{u})\Big)\Big(\mathtt{x}^q\Big(\lambda + \sum_{i=1}^{d}\mathtt{x}^i f_i(\mathtt{u})\Big) - (a\mathtt{x}^2W)\mathtt{x}^{2\ell}\Big(\sum_{i\geqslant 0}\mathtt{x}^i c_i'(\mathtt{u})\Big)\Big) - \mathtt{x}^{2s}\Big(\sum_{j=0}^{e}\mathtt{x}^j g_j(\mathtt{u})\Big)^2\\ &= 4\mathtt{x}^{2\ell+q}\Big(\sum_{i\geqslant 0}\mathtt{x}^i c_i'(\mathtt{u})\Big)\Big(\lambda + \sum_{i=1}^{d}\mathtt{x}^i f_i(\mathtt{u}) - \mathtt{x}^{2\ell+2-q}aW\Big(\sum_{i\geqslant 0}\mathtt{x}^i c_i'(\mathtt{u})\Big)\Big) - \mathtt{x}^{2s}\Big(\sum_{j=0}^{e}\mathtt{x}^j g_j(\mathtt{u})\Big)^2.\end{aligned} \tag{8.21}$$ xiexp3

Although we have not yet proved it, we will also see:

$$2\ell + 2 - q \geqslant 0, \quad 2\ell + q \leqslant 2s \quad \text{and} \quad 2m - 2\ell - q \geqslant 0. \tag{8.22}$$ bound03

Recall that q is even and $\lambda\in\kappa\setminus\{0\}$. As

$$\mathtt{x}^q\Big(\lambda+\sum_{i=1}^d\mathtt{x}^if_i(\mathtt{u})\Big)-\mathtt{x}^{2\ell+2}aW\Big(\sum_{i\geqslant0}\mathtt{x}^ic_i'(\mathtt{u})\Big)\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$$

and $c_0'(\mathtt{u})\neq0$, we have $q\leqslant 2\ell+2$, so $f(\mathtt{x},\mathtt{u}\mathtt{x}^q)-F(\mathtt{x},\mathtt{u}\mathtt{x}^q)c=\mathtt{x}^qH$ where $H\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$.

As $\xi\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$ and $\xi=\mathtt{x}^{2m}\xi'$, we deduce $2\ell+q\leqslant 2s$ and $2\ell+q\leqslant 2m$ (see (8.21)). Thus,

$$\begin{split}\Big(\sum_{i\geqslant0}\mathtt{x}^ic_i'(\mathtt{u})\Big)\Big(\lambda+\sum_{i=1}^d\mathtt{x}^if_i(\mathtt{u})-\mathtt{x}^{2\ell+2-q}aW\Big(\sum_{i\geqslant0}\mathtt{x}^ic_i'(\mathtt{u})\Big)\Big)\\ -\mathtt{x}^{2s-2\ell-q}\frac14\Big(\sum_{j=0}^e\mathtt{x}^jg_j(\mathtt{u})\Big)^2=\mathtt{x}^{2m-2\ell-q}\frac14\xi'.\end{split}\tag{8.23}$$

solve13

Observe that the coefficient of $\mathtt{x}^i$ on the right hand side of equation (8.23) is zero for $0\leqslant i\leqslant 2m-2\ell-q-1$. Recall that $W\in\kappa[\mathtt{x},\mathtt{u}]$. We distinguish several cases:

CASE 1. $2\ell+2-q>0$. If one compares coefficients with respect to $\mathtt{x}$ in equation (8.23), one realizes that $c_i'-P_i(f_1,\dots,f_d,g_0,\dots,g_e,c_1',\dots,c_{i-1}')=0$ for some polynomial

$$P_i\in\kappa[\mathtt{u}][\mathtt{x}_1,\dots,\mathtt{x}_d,\mathtt{y}_0,\dots,\mathtt{y}_e,\mathtt{z}_1,\dots,\mathtt{z}_{i-1}]$$

if $i=0,\dots,2m-2\ell-q-1$. We conclude inductively that $c_i'\in\kappa[\mathtt{u}]$ for $i=0,\dots,2m-2\ell-q-1$. ■

CASE 2. $2\ell+2-q=0$ and $2s-2\ell-q>0$. If $2m-2\ell-q=0$, there is nothing to prove, so let us assume $2m-2\ell-q>0$. As $c_0'\neq0$, we deduce $2\ell+2-q=0$ and $ac_0'=\lambda$, that is, $c_0'=\frac{\lambda}{a}$. Write $W=1+\mathtt{x}^6\Gamma_q$ where $\Gamma_q:=\frac{b}{a}\mathtt{u}^{2k}\mathtt{x}^{2kq-8}\in\kappa[\mathtt{x},\mathtt{u}]$ (recall that $q,k\geqslant2$). We rewrite (8.23) as follows (after simplifying and dividing by $\mathtt{x}$):

$$\begin{split}\Big(\frac{\lambda}{a}+\sum_{i\geqslant1}\mathtt{x}^ic_i'(\mathtt{u})\Big)\Big(\sum_{i=1}^d\mathtt{x}^{i-1}f_i(\mathtt{u})-a\sum_{i\geqslant1}\mathtt{x}^{i-1}c_i'(\mathtt{u})-\mathtt{x}^5\Gamma_q\Big(\lambda+a\sum_{i\geqslant1}\mathtt{x}^ic_i'(\mathtt{u})\Big)\Big)\\ -\mathtt{x}^{2s-2\ell-q-1}\frac14\Big(\sum_{j=0}^e\mathtt{x}^jg_j(\mathtt{u})\Big)^2=\mathtt{x}^{2m-2\ell-q-1}\frac14\xi'.\end{split}\tag{8.24}$$

solve13b

If one compares coefficients with respect to $\mathtt{x}$ in equation (8.24), one realizes that

$$c_i'-Q_i(f_1,\dots,f_d,g_0,\dots,g_e,c_1',\dots,c_{i-1}')=0$$

for some polynomial

$$Q_i\in\kappa[\mathtt{u}][\mathtt{x}_1,\dots,\mathtt{x}_d,\mathtt{y}_0,\dots,\mathtt{y}_e,\mathtt{z}_1,\dots,\mathtt{z}_{i-1}]$$

if $i=0,\dots,2m-2\ell-q-1$. We conclude inductively that $c_i'\in\kappa[\mathtt{u}]$ for $i=0,\dots,2m-2\ell-q-1$. ■

CASE 3. $2\ell+2-q=0$ and $2s-2\ell-q=0$. If $2m-2\ell-q=0$, there is nothing to prove concerning $c_i'\in\kappa[\mathtt{u}]$ for $i=0,\dots,2m-2\ell-q-1$, so let us assume $2m-2\ell-q>0$. We have

$$\Big(\sum_{i\geqslant0}\mathtt{x}^ic_i'(\mathtt{u})\Big)\Big(\lambda+\sum_{i=1}^d\mathtt{x}^if_i(\mathtt{u})-aW\Big(\sum_{i\geqslant0}\mathtt{x}^ic_i'(\mathtt{u})\Big)\Big)-\frac14\Big(\sum_{j=0}^e\mathtt{x}^jg_j(\mathtt{u})\Big)^2=\mathtt{x}^{2m-2\ell-q}\frac14\xi'.\tag{8.25}$$

solve13c

Thus, $c_0'(\lambda-ac_0')-\frac14g_0^2=0$ and $c_0'\in\Sigma\kappa[[\mathtt{u}]]^2$ (because $c'\in\Sigma\kappa[[\mathtt{x},\mathtt{u}]]^2$). Let us modify c' to guarantee $2m-2\ell-q=0$. Denote $f':=\sum_{i=1}^d\mathtt{x}^if_i(\mathtt{u})$ and $g':=\sum_{j=0}^e\mathtt{x}^jg_j(\mathtt{u})$.

If $c_0'(0)=0$, then $\lambda\in\Sigma\kappa^2$ (because $\lambda+f'-aWc'\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$) and $g_0(0)=0$ (because $c_0'(\lambda-ac_0')-\frac14g_0^2=0$). We change c' by $\frac{\lambda}{1+a^2}$ and after this change, we have: $c'\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$, $\lambda+f'-aWc'\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$ and $c'(\lambda+f'-aWc')-\frac14g'^2\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$ are all units, so we have now $2m-2\ell-q=0$.

Assume next $c_0'(0)\neq 0$. Observe that $c'-a\mathtt{u}^2\in\mathcal{P}(\kappa[\mathtt{x},\mathtt{u}])$, $\lambda+f'-aW(c'-a\mathtt{u}^2)=\lambda+f'-aWc'+a^2W\mathtt{u}^2\in\mathcal{P}(\kappa[\mathtt{x},\mathtt{u}])$ and

$$
\begin{aligned}
&(c'-a\mathtt{u}^2)(\lambda+f'-aW(c'-a\mathtt{u}^2))-\frac{1}{4}g'^2\\
&\qquad=\mathtt{x}^{2m-2\ell-q}\frac{1}{4}\xi'+\mathtt{u}^2(a^2-a(\lambda+f'-aW(c'-a\mathtt{u}^2)))\in\mathcal{P}(\kappa[\mathtt{x},\mathtt{u}])
\end{aligned}
$$

because $\lambda+f'-aW(c'-a\mathtt{u}^2)\in\mathfrak{m}_2$ as $\lambda=ac_0'(0)$. We change c' by $c'-a\mathtt{u}^2\in\mathcal{P}(\kappa[\mathtt{x},\mathtt{u}])$ and we have $2m-2\ell-q=0$. ■

[8.7.vi]. We are ready to modify the series $c,\xi\in\kappa[[\mathtt{x},\mathtt{u}]]$ in order to obtain polynomials $c^\bullet,\xi^\bullet\in\kappa[\mathtt{x},\mathtt{u}]\cap\mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$ (from which we will construct $\eta\in\kappa[\mathtt{x},\mathtt{y}]\cap\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$) that substitute the positive semidefinite series c,ξ already constructed in [8.7.iii].

As $\xi'',c'+\mathtt{x}^{2\nu+2m-2\ell}\in\mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{u}]])$, there exists by Lemma 6.5 $r'\geqslant 1$ such that if $\zeta,\theta\in\kappa[[\mathtt{x},\mathtt{u}]]$ satisfy $\zeta-\xi'',\theta-c'\in\mathfrak{m}_2^{r'}$, then $\zeta,\theta+\mathtt{x}^{2\nu+2m-2\ell}\in\mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{u}]])$. We may assume r' is strictly greater than the maximum of the degrees of the polynomials $\mathtt{x}^ic_i'$ for $i=0,\dots,2m-2\ell-q-1$.

Let $c^\circ\in\kappa[\mathtt{x},\mathtt{u}]$ be a polynomial such that $\omega(c^\circ-c')\geqslant r'+2m$. Then $c^\circ=\sum_{i=0}^{2m-2\ell-q-1}c_i'\mathtt{x}^i+\mathtt{x}^{2m-2\ell-q}h$ for some polynomial $h\in\kappa[\mathtt{x},\mathtt{u}]$, so $c^\circ=c'+\mathtt{x}^{2m-2\ell-q}h'$ for some $h'\in\kappa[[\mathtt{x},\mathtt{u}]]$ and $\mathtt{x}^{2\ell}c^\circ=c+\mathtt{x}^{2m-q}h'$. Observe that

$$\omega(c^\circ+\mathtt{x}^{2\nu+2m-2\ell}-(c'+\mathtt{x}^{2\nu+2m-2\ell}))=\omega(c^\circ-c')\geqslant r'+2m>r'.$$

Thus, $c^\circ+\mathtt{x}^{2\nu+2m-2\ell}\in\kappa[\mathtt{x},\mathtt{u}]\cap\mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{u}]])$ and $c^\bullet:=\mathtt{x}^{2\ell}c^\circ+\mathtt{x}^{2\nu+2m}\in\kappa[\mathtt{x},\mathtt{u}]\cap\mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$. Consider the polynomial

$$\xi^\bullet:=4c^\bullet(f(\mathtt{x},\mathtt{u}\mathtt{x}^q)+2\mathtt{x}^{2\nu+2m}-F(\mathtt{x},\mathtt{u}\mathtt{x}^q)c^\bullet)-(g(\mathtt{x},\mathtt{u}\mathtt{x}^q))^2. \tag{8.26}$$ xibull3

We claim: $\xi^\bullet-\xi=\mathtt{x}^{2m}E$ *for some* $E\in\kappa[[\mathtt{x},\mathtt{u}]]$.

By (8.18) and [8.7.v] we have:

$$
\begin{aligned}
&4c(f(\mathtt{x},\mathtt{u}\mathtt{x}^q)-F(\mathtt{x},\mathtt{u}\mathtt{x}^q)c)-(g(\mathtt{x},\mathtt{u}\mathtt{x}^q))^2=\xi,\\
&f(\mathtt{x},\mathtt{u}\mathtt{x}^q)-F(\mathtt{x},\mathtt{u}\mathtt{x}^q)c=\mathtt{x}^qH.
\end{aligned}
$$

As $F(\mathtt{x},\mathtt{u}\mathtt{x}^q)=a\mathtt{x}^2W$, $\mathtt{x}^{2\ell}c^\circ=c+\mathtt{x}^{2m-q}h$ and $c^\bullet=\mathtt{x}^{2\ell}c^\circ+\mathtt{x}^{2\nu+2m}=c+\mathtt{x}^{2m-q}h'+\mathtt{x}^{2\nu+2m}$, we conclude

$$
\begin{aligned}
\xi^\bullet&=4c^\bullet(f(\mathtt{x},\mathtt{u}\mathtt{x}^q)+2\mathtt{x}^{2\nu+2m}-F(\mathtt{x},\mathtt{u}\mathtt{x}^q)c^\bullet)-(g(\mathtt{x},\mathtt{u}\mathtt{x}^q))^2\\
&=4(c+\mathtt{x}^{2m-q}h'+\mathtt{x}^{2\nu+2m})(f(\mathtt{x},\mathtt{u}\mathtt{x}^q)+2\mathtt{x}^{2\nu+2m}-F(\mathtt{x},\mathtt{u}\mathtt{x}^q)(c+\mathtt{x}^{2m-q}h'+\mathtt{x}^{2\nu+2m}))-(g(\mathtt{x},\mathtt{u}\mathtt{x}^q))^2\\
&=4c(f(\mathtt{x},\mathtt{u}\mathtt{x}^q)-F(\mathtt{x},\mathtt{u}\mathtt{x}^q)c)-(g(\mathtt{x},\mathtt{u}\mathtt{x}^q))^2+4(\mathtt{x}^{2m-q}h'+\mathtt{x}^{2\nu+2m})(f(\mathtt{x},\mathtt{u}\mathtt{x}^q)-F(\mathtt{x},\mathtt{u}\mathtt{x}^q)c)\\
&\quad+4(c+\mathtt{x}^{2m-q}h'+\mathtt{x}^{2\nu+2m})(2\mathtt{x}^{2\nu+2m}-a\mathtt{x}^2W(\mathtt{x}^{2m-q}h'+\mathtt{x}^{2\nu+2m}))\\
&=\xi+4(\mathtt{x}^{2m-q}h'+\mathtt{x}^{2\nu+2m})\mathtt{x}^qH+4(c+\mathtt{x}^{2m-q}h'+\mathtt{x}^{2\nu+2m})(2\mathtt{x}^{2\nu+2m}-a\mathtt{x}^2W(\mathtt{x}^{2m-q}h'+\mathtt{x}^{2\nu+2m})).
\end{aligned}
$$

Thus, to justify that

$$
\begin{aligned}
\xi^\bullet-\xi&=4\mathtt{x}^{2m}(h'+\mathtt{x}^{2\nu+q})H+4\mathtt{x}^{2\nu+2m}(2\mathtt{x}^{2\nu+2m}-a\mathtt{x}^2W(\mathtt{x}^{2m-q}h'+\mathtt{x}^{2\nu+2m}))\\
&\qquad+4(c+\mathtt{x}^{2m-q}h')(2\mathtt{x}^{2\nu+2m}-a\mathtt{x}^2W\mathtt{x}^{2\nu+2m})+4(c+\mathtt{x}^{2m-q}h')(-a\mathtt{x}^2W\mathtt{x}^{2m-q}h')
\end{aligned}
$$

is divisible by $\mathtt{x}^{2m}$, we only have to clarify why the product

$$
\begin{aligned}
4(c+\mathtt{x}^{2m-q}h')(-a\mathtt{x}^2W(\mathtt{x}^{2m-q}h'))&=4(\mathtt{x}^{2\ell}c'+\mathtt{x}^{2m-q}h')(-a\mathtt{x}^2W(\mathtt{x}^{2m-q}h'))\\
&=-4\mathtt{x}^{2m+2\ell+2-q}c'aWh'-4\mathtt{x}^{4m+2-2q}aWh'^2
\end{aligned}
$$

is divisible by $\mathtt{x}^{2m}$. The first addend on the right hand side is divisible by $\mathtt{x}^{2m}$ because $q\leqslant 2\ell+2$ (see (8.22)), whereas the second addend is divisible by $\mathtt{x}^{2m}$ because, as $2m-2\ell-q\geqslant 0$ and

$2\ell+2-q\geqslant 0$ (see (8.22)),

$$4m+2-2q=2m+(2m-2\ell-q)+(2\ell+2-q)\geqslant 2m.$$

We conclude $\xi^\bullet-\xi=\mathtt{x}^{2m}E$ for some $E\in\kappa[[\mathtt{x},\mathtt{u}]]$.

As $\mathtt{x}^{2m}$ divides ξ, also $\mathtt{x}^{2m}$ divides $\xi^\bullet$, so there exists a polynomial $\xi^\circ\in\kappa[\mathtt{x},\mathtt{u}]$ such that $\xi^\bullet=\mathtt{x}^{2m}\xi^\circ$. By (8.20) and (8.26) and using that $c^\bullet=\mathtt{x}^{2\ell}c^\circ+\mathtt{x}^{2\nu+2m}$ we have

$$\begin{aligned}\xi^*&=4(\mathtt{x}^{2\ell}c'+\mathtt{x}^{2\nu+2m})(f(\mathtt{x},\mathtt{u}\mathtt{x}^q)+2\mathtt{x}^{2\nu+2m}-F(\mathtt{x},\mathtt{u}\mathtt{x}^q)(\mathtt{x}^{2\ell}c'+\mathtt{x}^{2\nu+2m}))-(g(\mathtt{x},\mathtt{u}\mathtt{x}^q))^2,\\ \xi^\bullet&=4(\mathtt{x}^{2\ell}c^\circ+\mathtt{x}^{2\nu+2m})(f(\mathtt{x},\mathtt{u}\mathtt{x}^q)+2\mathtt{x}^{2\nu+2m}-F(\mathtt{x},\mathtt{u}\mathtt{x}^q)(\mathtt{x}^{2\ell}c^\circ+\mathtt{x}^{2\nu+2m}))-(g(\mathtt{x},\mathtt{u}\mathtt{x}^q))^2.\end{aligned}$$

Consequently,

$$\begin{aligned}\xi^\bullet-\xi^*=4\mathtt{x}^{2\ell}(f(\mathtt{x},\mathtt{u}\mathtt{x}^q)+2\mathtt{x}^{2\nu+2m}-F(\mathtt{x},\mathtt{u}\mathtt{x}^q)\mathtt{x}^{2\nu+2m})(c^\circ-c')\\ -4\mathtt{x}^{2\nu+2m+2\ell}F(\mathtt{x},\mathtt{u}\mathtt{x}^q)(c^\circ-c')-4\mathtt{x}^{4\ell}F(\mathtt{x},\mathtt{u}\mathtt{x}^q)(c^\circ+c')(c^\circ-c'),\end{aligned}$$

so $\omega(\xi^\bullet-\xi^*)\geqslant\omega(c^\circ-c')$. As $\xi^\bullet-\xi^*=\mathtt{x}^{2m}(\xi^\circ-\xi'')$, we have $\omega(\xi^\circ-\xi'')=\omega(\xi^\bullet-\xi^*)-2m$. As $\omega(c^\circ-c')\geqslant r'+2m$, it holds $\omega(\xi^\circ-\xi'')\geqslant r'$, so $\xi^\circ\in\mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{u}]])$. This means that $\xi^\bullet=\mathtt{x}^{2m}\xi^\circ\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{u}]])$.

[8.7.vii]. Write $\mathtt{u}:=\frac{\mathtt{y}}{\mathtt{x}^q}$ in $\xi^\bullet(\mathtt{x},\frac{\mathtt{y}}{\mathtt{x}^q})\in\kappa[\mathtt{x},\mathtt{u}]$:

$$\xi^\bullet\Big(\mathtt{x},\frac{\mathtt{y}}{\mathtt{x}^q}\Big)=4c^\bullet\Big(\mathtt{x},\frac{\mathtt{y}}{\mathtt{x}^q}\Big)\Big(f(\mathtt{x},\mathtt{y})+2\mathtt{x}^{2\nu+2m}-F(\mathtt{x},\mathtt{y})c^\bullet\Big(\mathtt{x},\frac{\mathtt{y}}{\mathtt{x}^q}\Big)\Big)-(g(\mathtt{x},\mathtt{y}))^2,$$

where $c^\bullet\in\kappa[\mathtt{x},\mathtt{u}]$. After clearing denominators, by Lemma 5.15 we find an integer $\mu\geqslant 1$ and a polynomial $B\in\kappa[\mathtt{x},\mathtt{y}]\cap\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$ such that

$$4B(\mathtt{x}^{2\mu}f+2\mathtt{x}^{2\nu+2m+2\mu}-FB)-\mathtt{x}^{4\mu}g^2\in\kappa[\mathtt{x},\mathtt{y}]\cap\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]).$$

Thus, there exist series $A_i\in\kappa[[\mathtt{x},\mathtt{y}]]$ satisfying

$$4B(\mathtt{x}^{2\mu}f+2\mathtt{x}^{2\nu+2m+2\mu}-FB)-\mathtt{x}^{4\mu}g^2=\sum_{i=1}^p A_i^2. \tag{8.27}$$

FB3

Setting $\mathtt{x}=0$ in (8.27), we deduce

$$-4F(0,\mathtt{y})B^2(0,\mathtt{y})=-4b\mathtt{y}^{2k}B^2(0,\mathtt{y})=\sum_{i=1}^p A_i^2(0,\mathtt{y}).$$

As $b\notin-\Sigma\kappa^2$, we deduce $B(0,\mathtt{y})=0$ and $A_i(0,\mathtt{y})=0$, so there exist $B',A_i'\in\kappa[[\mathtt{x},\mathtt{y}]]$ such that $B=\mathtt{x}^2B'$ (recall that $B\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$) and $A_i=\mathtt{x}A_i'$. We deduce

$$4\mathtt{x}^4B'(\mathtt{x}^{2\mu-2}f+2\mathtt{x}^{2\nu+2m+2\mu-2}-FB')-\mathtt{x}^{4\mu}g^2=\mathtt{x}^2\sum_{i=1}^p A_i'^2. \tag{8.28}$$

FB23

Dividing (8.28) by $\mathtt{x}^2$ and setting $\mathtt{x}=0$ next, we deduce $A_i'(0,\mathtt{y})=0$, so there exists $A_i''\in\kappa[[\mathtt{x},\mathtt{y}]]$ such that $A_i=\mathtt{x}A_i''$. Consequently,

$$4B'(\mathtt{x}^{2\mu-2}f+2\mathtt{x}^{2\nu+2m+2\mu-2}-FB')-\mathtt{x}^{4\mu-4}g^2=\sum_{i=1}^p A_i''^2.$$

Proceeding recursively with μ we find the sought polynomial $\eta\in\kappa[\mathtt{x},\mathtt{y}]\cap\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$ such that

$$4\eta(f+2\mathtt{x}^{2\nu+2m}-F\eta)-g^2\in\kappa[\mathtt{x},\mathtt{y}]\cap\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]).$$

To finish, we define $n:=\nu+m-k\geqslant n_0+\ell$, $f_n:=f+2\mathtt{x}^{2\nu+2m-2k}$, $g_n:=g$ and $\eta_n:=\eta$, as required. $\square$

We present a generalization of original Elephant's Theorem [Fe8, Thm.5.5] as a consequence of Theorem 8.6. The original one correspond to the case $k=1$ (that is, $2k+1=3$).

etc **Corollary 8.8** (Elephant's Theorem revisited [Fe8, Thm.5.5]). *Let $F \in \kappa[[\mathtt{x},\mathtt{y}]]$ be a series such that $\omega(F) = \omega(F(\mathtt{x},0)) = 2k+1$ and F is ρ-determined for some $\rho \geqslant 2k+1$. For each $q \geqslant 2$ denote $\mathcal{P}_q(A)$ the set of series $f + \mathtt{z}g \in \mathcal{P}(A)$ such that $f(\mathtt{x},0) \neq 0$ and $\omega(f(\mathtt{x},0)) = q$. Let $A := \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2 - F)$ and fix $f + \mathtt{z}g \in \mathcal{P}(A)\backslash\{0\}$. We obtain:*

(i) *There exist $s \geqslant 1$ and $f_1 + \mathtt{z}g_1 \in \mathcal{P}(A)$ such that $f_1(\mathtt{x},0) \neq 0$ and $f + \mathtt{z}g = \mathtt{y}^{2s}(f_1 + \mathtt{z}g_1)$. In addition, if $\omega(f_1(\mathtt{x},0)) = q \geqslant 2$, then $\omega(g_1) \geqslant 1$.*
(ii) *$\mathcal{P}_q(A) \subset \Sigma A^2$ for each $q \geqslant 2$.*

Proof. By Weierstrass preparation theorem and Tschirnhaus trick there exist $c \in \kappa\backslash\{0\}$, series $a_0,\dots,a_{2k-1} \in \kappa[[\mathtt{y}]]$, a unit $U \in \kappa[[\mathtt{x},\mathtt{y}]]$ with $U(0,0) = 1$ and a Weierstrass polynomial $P := \mathtt{x}^{2k+1} + \sum_{r=0}^{2k-1} a_s(\mathtt{y})\mathtt{y}^{2k+1-r}\mathtt{x}^r$ and $F = cU^2P$. After the change of coordinates $(\mathtt{x},\mathtt{y},\mathtt{z}) \mapsto (c\mathtt{x}, c\mathtt{y}, c^{2k}U\mathtt{z})$, we assume $F = \mathtt{x}^{2k+1} + \sum_{r=0}^{2k-1} a_r(\mathtt{y})\mathtt{y}^{2k+1-r}\mathtt{x}^r$ where $a_r \in \kappa[[\mathtt{y}]]$ for $r = 0,\dots,2k-1$. As F is ρ-determined, we may assume in addition $a_r \in \kappa[\mathtt{y}]$ for $r = 0,\dots,2k-1$.

(i) As $f + \mathtt{z}g \in \mathcal{P}(A)\backslash\{0\}$, we know that $f \in \mathcal{P}(\{F \geqslant 0\})$ and $f^2 - Fg^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$. As $-F$ is not positive semidefinite, if $f = 0$, then $g = 0$, so $f \neq 0$. Assume $f(\mathtt{x},0) = 0$. We claim: $\mathtt{y}^2$ *divides* f.

Otherwise, $f = \mathtt{y}f'$ where $f' \in \kappa[[\mathtt{x},\mathtt{y}]]$ and $f'(\mathtt{x},0) \neq 0$. Thus, there exist a Weierstrass polynomial $Q := \mathtt{x}^q + \sum_{k=0}^{q-1} \mathtt{y}b_k(\mathtt{y})\mathtt{x}^k$ (where $b_k \in \kappa[[\mathtt{y}]]$ for each k), $\mu \in \kappa\backslash\{0\}$ and a unit $V \in \kappa[[\mathtt{x},\mathtt{y}]]$ with $V(0,0) = 1$ such that $f' = Q\mu V^2$.

For each $m \geqslant 1$ consider the homomorphism

$$\varphi : \kappa[[\mathtt{x},\mathtt{y}]] \to \kappa[[\mathtt{t}]],\ h \mapsto h(\mathtt{t}^2, \mathtt{t}^m).$$

If m is odd and large enough,

$$\varphi(F) = \mathtt{t}^{4k+2} + \sum_{r=0}^{2k-1} a_r(\mathtt{t}^m)\mathtt{t}^{(2k+1-r)m+2r} = \mathtt{t}^{4k+2}\Big(1 + \sum_{r=0}^{2k-1} a_r(\mathtt{t}^m)\mathtt{t}^{(2k+1-r)(m-2)}\Big),$$

$$\begin{aligned}\varphi(f) = \varphi(\mathtt{y}f') &= \mathtt{t}^m\Big(\mathtt{t}^{2q} + \sum_{k=0}^{q-1} \mathtt{t}^{m+2k}b_k(\mathtt{t}^m)\Big)\mu V^2(\mathtt{t}^2,\mathtt{t}^m)\\ &= \mu\mathtt{t}^{m+2q}\Big(1 + \sum_{k=0}^{q-1} \mathtt{t}^{m+2k-2q}b_k(\mathtt{t}^m)\Big)V^2(\mathtt{t}^2,\mathtt{t}^m).\end{aligned}$$

Observe that $\varphi(F) \in \Sigma\kappa[[\mathtt{t}]]^2$, whereas $\varphi(f) \notin \Sigma\kappa^2$, as it has odd order. There exists a prime cone $\alpha \in \operatorname{Sper}(\kappa[[\mathtt{t}]])$ such that $\varphi(F) >_\alpha 0$, whereas $\varphi(f) <_\alpha 0$ (consider an ordering of κ and define the sign of $\mathtt{t}$ to get $\mu\mathtt{t} < 0$). This is a contradiction because $f \in \mathcal{P}(\{F \geqslant 0\})$.

We conclude $f'(\mathtt{x},0) = 0$, so $f = \mathtt{y}^2 f''$ where $f'' \in \kappa[[\mathtt{x},\mathtt{y}]]$, as claimed.

Consequently,

$$f^2 - Fg^2 = \mathtt{y}^4 f''^2 - Fg^2 = \mathtt{y}^4 f''^2 - \Big(\mathtt{x}^{2k+1} + \sum_{r=0}^{2k-1} a_r(\mathtt{y})\mathtt{y}^{2k+1-r}\mathtt{x}^r\Big)g^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]).$$

Setting $\mathtt{y} = 0$, we have $-\mathtt{x}^{2k+1}g^2(\mathtt{x},0) \in \mathcal{P}(\kappa[[\mathtt{x}]])$, so $g(\mathtt{x},0) = 0$ and there exists $g' \in \kappa[[\mathtt{x},\mathtt{y}]]$ such that $g = \mathtt{y}g'$. Thus,

$$f^2 - Fg^2 = \mathtt{y}^2(\mathtt{y}^2 f''^2 - Fg'^2) \quad \rightsquigarrow \quad \mathtt{y}^2 f''^2 - Fg'^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]).$$

Setting again $\mathtt{y} = 0$, we have $-\mathtt{x}^{2k+1}g'^2(\mathtt{x},0) \in \mathcal{P}(\kappa[[\mathtt{x}]])$, so $g'(\mathtt{x},0) = 0$ and there exists $g'' \in \kappa[[\mathtt{x},\mathtt{y}]]$ such that $g = \mathtt{y}^2 g''$. This means $f + \mathtt{z}g = \mathtt{y}^2(f'' + \mathtt{z}g'')$ where $f'' + \mathtt{z}g'' \in \mathcal{P}(A)$. Proceeding recursively we find $s \geqslant 1$ and $f_1 + \mathtt{z}g_1 \in \mathcal{P}(A)$ such that $f_1(\mathtt{x},0) \neq 0$ and $f + \mathtt{z}g = \mathtt{y}^{2s}(f_1 + \mathtt{z}g_1)$.

Assume $\omega(f_1(\mathtt{x},0)) = q \geqslant 2$. Substitute $\mathtt{y}=0$ in $f_1^2 - Fg_1^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$ and observe that

$$(f_1^2 - Fg_1^2)(\mathtt{x},0) = f_1^2(\mathtt{x},0) - \mathtt{x}^{2k+1}g_1^2(\mathtt{x},0) \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]).$$

Thus, $\omega(-\mathtt{x}^{2k+1}g_1^2(\mathtt{x},0)) \geqslant 2q$, that is, $\omega(g_1(\mathtt{x},0)) \geqslant q - \frac{2k+1}{2} > 0$, so $\omega(g_1) \geqslant 1$.

(ii) Pick a series $f + \mathtt{z}g \in \mathcal{P}_q(A)$ for some $q \geqslant 2$. Write

$$\begin{aligned} F(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) &= \mathtt{x}^{2k+1} + \sum_{r=0}^{2k-1} a_r(\mathtt{u}\mathtt{x}^{q+k})(\mathtt{u}\mathtt{x}^{q+k})^{2k+1-r}\mathtt{x}^r \\ &= \mathtt{x}^{2k+1}\Big(1 + \mathtt{x}^2\mathtt{u}^{2k+1}\sum_{r=0}^{2k-1} a_r(\mathtt{u}\mathtt{x}^{q+k})\mathtt{x}^{(2k+1-r)(q+k-1)-2}\Big) \end{aligned}$$

and observe that $(2k+1-r)(q+k-1)-2 \geqslant 2$ for $r = 0,\dots,2k-1$, because $q \geqslant 2$ and $k \geqslant 1$. Define $\Gamma_q := \mathtt{u}^{2k+1}\sum_{r=0}^{2k-1} a_r(\mathtt{u}\mathtt{x}^{q+1})\mathtt{x}^{(2k+1-r)q-2} \in \kappa[\mathtt{x},\mathtt{u}]$ and $F = \mathtt{x}^{2k+1}W$, where $W := 1+\mathtt{x}^2\Gamma_q$ satisfies $W(0,0)=1$. Define $A' := \kappa[[\mathtt{x},\mathtt{u},\mathtt{v}]]/(\mathtt{v}^2 - \mathtt{x}W)$ and consider the homomorphism

$$\Phi_q : A \to A',\ \ h(\mathtt{x},\mathtt{y},\mathtt{z}) \mapsto h(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k},\mathtt{v}\mathtt{x}^k).$$

As $f + \mathtt{z}g \in \mathcal{P}(A)$, we have

$$\Phi_q(f+\mathtt{z}g) = f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) + \mathtt{v}\mathtt{x}^k g(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) \in \mathcal{P}(A').$$

As $W(0,0)=1$, there exists by the Implicit Function Theorem $\phi(\mathtt{u},\mathtt{v}) \in \kappa[[\mathtt{u},\mathtt{v}]]$ such that $\mathtt{x} = \phi(\mathtt{u},\mathtt{v})$ is the unique solution of the equation $\mathtt{v}^2 - \mathtt{x}W = 0$ satisfying $\varphi(0,0) = 0$. As $\psi(f+\mathtt{z}g) \in \mathcal{P}(A')$, we get

$$h(\mathtt{u},\mathtt{v}) := f(\phi(\mathtt{u},\mathtt{v}),\mathtt{u}\phi(\mathtt{u},\mathtt{v})^{q+1}) + \mathtt{v}\phi(\mathtt{u},\mathtt{v})g(\phi(\mathtt{u},\mathtt{v}),\mathtt{u}\phi(\mathtt{u},\mathtt{v})^{q+1}) \in \mathcal{P}(\kappa[[\mathtt{u},\mathtt{v}]]) = \Sigma\kappa[[\mathtt{u},\mathtt{v}]]^2,$$

so there exist $h_i \in \kappa[[\mathtt{u},\mathtt{v}]]$ such that $h = \sum_{i=1}^p h_i^2$. Using the relation $\mathtt{v}^2 - \mathtt{x}W$, we find series $A_i, B_i, C \in \kappa[[\mathtt{x},\mathtt{u}]]$ satisfying

$$f(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) + \mathtt{v}\mathtt{x}^k g(\mathtt{x},\mathtt{u}\mathtt{x}^{q+k}) = \sum_{i=1}^p (A_i + \mathtt{v}B_i)^2 - (\mathtt{v}^2 - \mathtt{x}W)C.$$

Thus,

$$\begin{aligned} \Phi_q(\mathtt{x}^{2k}(f+\mathtt{z}g)) &= \sum_{i=1}^p (\mathtt{x}^kA_i + \mathtt{v}\mathtt{x}^kB_i)^2 - ((\mathtt{x}^k\mathtt{v})^2 - \mathtt{x}^{2k+1}W)C \\ &= \sum_{i=1}^p (\mathtt{x}^kA_i + \mathtt{v}\mathtt{y}^kB_i)^2 - \Phi_q((\mathtt{z}^2 - F))C. \end{aligned}$$

We are under the hypotheses of Theorem 8.6(ii), so $\mathcal{P}_q(A) \subset \Sigma A^2$ for each $q \geqslant 2$, as required. □

s9

9. The 2-dimensional case when the Pythagoras number is finite

In this section we prove Theorems 2.2 and 2.3.

9.1. **Reduction to the case of complete rings with finite Pythagoras number.** Let $(A,\mathfrak{m})$ be a local excellent henselian ring subset that its residue field $\kappa := A/\mathfrak{m}$ is (formally) real. Let $\widehat{A}$ be the $\mathfrak{m}$-adic completion of A and consider the inclusion (homomorphism) $A \hookrightarrow \widehat{A}$.

Lemma 9.1. *If $\mathcal{P}(\widehat{A}) = \Sigma A^2$, then $\mathcal{P}(A) = \Sigma A^2$.*

Proof. Let $f \in \mathcal{P}(A)$. By Lemma 5.14(ii) we have $f \in \mathcal{P}(\widehat{A}) = \Sigma A^2$. Thus, there exists $g_1, \dots, g_p \in \widehat{A}$ such that $f = g_1^2 + \cdots + g_p^2$, that is, the polynomial equation $f = \mathtt{X}_1^2 + \cdots + \mathtt{X}_p^2$ has a solution in $\widehat{A}^p$. By Strong Artin's Approximation Theorem there exists $f_1, \dots, f_p \in A$ such that $f = f_1^2 + \cdots + f_p^2 \in \Sigma A^2$, as required. □

Lemma 9.2. *If $\mathcal{P}(A) = \Sigma A^2$ and $p := p(A) < +\infty$, then $\mathcal{P}(\widehat{A}) = \Sigma \widehat{A}^2$.*

Proof. Let $f \in \mathcal{P}(\widehat{A})$ and let $m \geqslant 1$. Let $f_m \in A$ such that $f_m - f \in \mathfrak{m}^{2m+1}\widehat{A}$. By [ABR, Ch.VII.Cor.3.3] each ordering $\alpha \in \operatorname{Sper}(A)$ is the restriction to A of an ordering $\widehat{\alpha} \in \operatorname{Sper}(\widehat{A})$ to A, that is, $A \cap \widehat{\alpha} = \alpha$. Let $x_1, \dots, x_n \in \mathfrak{m}$ be a system of generators of $\mathfrak{m}$ and let us prove: $g_m := f_m + (x_1^2 + \cdots + x_n^2)^m \in \mathcal{P}(\widehat{A})$. Consequently, $g_m \in \widehat{\alpha} \cap A = \alpha$ for each $\alpha \in \operatorname{Sper}(A)$, that is, $g_m \in \mathcal{P}(A)$.

We have $\widehat{A} \cong \kappa[[\mathtt{x}]]/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of $\kappa[[\mathtt{x}]] := \kappa[[\mathtt{x}_1, \dots, \mathtt{x}_n]]$. Let $C := \{\alpha \in \operatorname{Sper}(\kappa[[\mathtt{x}]]) : \mathfrak{a} \subset \operatorname{supp}(\alpha)\}$. We claim: $\mathcal{P}(C) = \mathcal{P}(\widehat{A})$.

Consider the canonical homomorphism $\varphi : \kappa[[\mathtt{x}]] \to \kappa[[\mathtt{x}]]/\mathfrak{a} \cong \widehat{A}$ and consider the spectral map $\operatorname{Sper}(\varphi) : \operatorname{Sper}(\widehat{A}) \cong \operatorname{Sper}(\kappa[[\mathtt{x}]]/\mathfrak{a}) \to \operatorname{Sper}(\kappa[[\mathtt{x}]]),\ \alpha \mapsto \varphi^{-1}(\alpha)$. The image of $\operatorname{Sper}(\varphi)$ is the collection of all prime cones of $\kappa[[\mathtt{x}]]$ such that $\mathfrak{a} \subset \operatorname{supp}(\alpha)$. Consequently, $\mathcal{P}(C) = \mathcal{P}(\widehat{A})$, as claimed.

Suppose that $g_m \notin \mathcal{P}(\widehat{A})$. Then there exists $\beta_0 \in C$ such that $-g_m >_{\beta_0} 0$. Denote $C_1 = C \cap \{\beta \in \operatorname{Sper}(\kappa[[\mathtt{x}]]) : -g_m >_\beta 0\}$ and observe that $-g_m \in \mathcal{P}(C_1)$. By Lemma 5.19 there exists $\gamma \in C_1$ with $\operatorname{ht}(\operatorname{supp}(\gamma)) \geqslant n-1$ such that $-g_m >_\gamma 0$. Let $\gamma \to \gamma_0$ be a specialization of γ such that $\operatorname{supp}(\gamma_0) = \mathfrak{m}_n$. Let $\phi_\gamma : \kappa[[\mathtt{x}]] \to \mathfrak{R}(\gamma_0)[[\mathtt{t}]]$ be such that $h \geqslant_\gamma 0$ if and only if $\phi_\gamma(h) \geqslant 0$ when $\mathtt{t} > 0$. We have $-\phi_\gamma(g_m) > 0$ and $\phi_\gamma(f) > 0$, because $f \in \mathcal{P}(\widehat{A})$. Let $2k := \omega_{\mathtt{t}}(\phi_\gamma(\mathtt{x}_1^2 + \cdots + \mathtt{x}_n^2)) = 2\min\{\omega_{\mathtt{t}}(\phi_\gamma(\mathtt{x}_i)) :\ 1 \leqslant i \leqslant n\}$ and observe that $k \geqslant 1$. If $h \in \mathfrak{m}_n^{2m+1}$, then $\omega_t(\phi_\gamma(h)) \geqslant (2m+1)k$. As $f - g_m \in \mathfrak{m}_n^{2m+1}$, we have $\omega_t(\phi_\gamma(f - g_m)) \geqslant (2m+1)k$. We distinguish two cases:

Case 1. If $\omega_{\mathtt{t}}(\phi_\gamma(f)) \leqslant 2mk$, we have $\phi_\gamma(f) = a_\ell \mathtt{t}^\ell + \cdots$ for some $a_\ell >_{\gamma_0} 0$, so

$$\begin{aligned}\phi_\gamma(g_m) = \phi_\gamma(f_m + (\mathtt{x}_1^2 + \cdots + \mathtt{x}_n^2)^m) &= \phi_\gamma(f) + \phi_\gamma(g_m - f) + \phi_\gamma((\mathtt{x}_1^2 + \cdots + \mathtt{x}_n^2)^m) \\ &= \phi_\gamma((\mathtt{x}_1^2 + \cdots + \mathtt{x}_n^2)^m) + a_\ell \mathtt{t}^\ell + \cdots >_\gamma 0,\end{aligned}$$

which is a contradiction, because $-\phi_\gamma(g_m) > 0$.

Case 2. If $\omega_{\mathtt{t}}(\phi_\gamma(f)) \geqslant 2mk + 1$, we have

$$\phi_\gamma(g_m) = \phi_\gamma(g_m + (\mathtt{x}_1^2 + \cdots + \mathtt{x}_n^2)^m) = \phi_\gamma(f) + \phi_\gamma(g_m - f) + \phi_\gamma((\mathtt{x}_1^2 + \cdots + \mathtt{x}_n^2)^m) >_\gamma 0,$$

because $\omega_{\mathtt{t}}(\phi_\gamma((\mathtt{x}_1^2+\cdots+\mathtt{x}_n^2)^m)) = 2mk$, $\omega_{\mathtt{t}}(\phi_\gamma(f)) \geqslant 2mk+1$ and $\omega_t(\phi_\gamma(g_m - f)) \geqslant (2m+1)k \geqslant 2mk+1$, because $k \geqslant 1$. Again, we have a contradiction, because $-\phi_\gamma(g_m) > 0$.

Consequently, $g_m \in \mathcal{P}(A) = \Sigma A^2 = \Sigma_p A^2$ and $f - g_m \in \mathfrak{m}^m \widehat{A}$ for each $m \geqslant 1$. Thus, the equation $f = \mathtt{X}_1^2 + \cdots + \mathtt{X}_p^2$ has a solution module $\mathfrak{m}^m\widehat{A}$ in $\widehat{A}^p$ for each $m \geqslant 1$. By Strong Artin's Approximation Theorem the previous equation has a solution in $\widehat{A}^p$, that is, $f \in \Sigma\widehat{A}^2$, as required. □

9.2. **Proof of Theorem 2.2.** We recall the proof of Theorem 2.2 (that already appears in [Fe8, Thm.1.5]), which is conducted in several steps. Let $A := \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/(\mathtt{z}^2 - F)$ where κ is a (formally) real field and $F \in \kappa[[\mathtt{x}, \mathtt{y}]]$. Denote the maximal ideal of $\kappa[[\mathtt{x}, \mathtt{y}]]$ with $\mathfrak{m}_2$. We know from Lemma 5.13 that $\omega(F) \leqslant 3$.

ord2 **Lemma 9.3** (Order 2 restrictions)**.** *If $\omega(F) = 2$ and $\mathcal{P}(A) = \Sigma A^2$, then F is right equivalent to one of the following:*

(i) $a\mathtt{x}^2$ *where* $a \notin -\Sigma\kappa^2$.
(ii) $a\mathtt{x}^2 + \mathtt{y}^{2k+1}$ *where* $a \notin -\Sigma\kappa^2$ *and* $k \geqslant 1$.
(iii) $a\mathtt{x}^2 + b\mathtt{y}^{2k}$ *where* $a \notin -\Sigma\kappa^2$, $b \neq 0$ *and* $k \geqslant 1$.

Proof. After a linear change of coordinates, we may assume (by Weierstrass preparation theorem) that there exist $a \in \kappa\backslash\{0\}$, a unit $U \in \kappa[[\mathtt{x},\mathtt{y}]]$ such that $U(0,0) = 1$ and a Weierstrass polynomial $P := \mathtt{x}^2 + 2a_1(\mathtt{y})\mathtt{x} + a_2(\mathtt{y}) \in \kappa[[\mathtt{y}]][\mathtt{x}]$ of degree 2 (with $\omega(a_1) \geqslant 1$ and $\omega(a_2) \geqslant 2$) such that

$$F = a(\mathtt{x}^2 + 2a_1\mathtt{x} + a_2)U^2 = a((\mathtt{x} + a_1)^2 + a_2 - a_1^2)U^2.$$

After the change of coordinates $(\mathtt{x},\mathtt{y},\mathtt{z}) \mapsto (\mathtt{x} - a_1, \mathtt{y}, \mathtt{z}U)$, we may assume

$$F = a\mathtt{x}^2 + \psi(\mathtt{y})$$

where $\omega(\psi) \geqslant 2$. If $\psi = 0$, then $F = a\mathtt{x}^2$. Otherwise, write $\psi = b\mathtt{y}^\ell u^\ell$ where $b \in \kappa\backslash\{0\}$ and $u \in \kappa[[\mathtt{y}]]$ is a unit such that $u(0) = 1$. After the change of coordinates $(\mathtt{x},\mathtt{y},\mathtt{z}) \mapsto (\mathtt{x}, \frac{\mathtt{y}}{u}, \mathtt{z}U)$, we may assume

$$F = a\mathtt{x}^2 + b\mathtt{y}^\ell$$

where $a, b \in \kappa\backslash\{0\}$. If $\ell = 2k+1$ is odd (where $k \geqslant 1$), after the change of coordinates $(\mathtt{x},\mathtt{y},\mathtt{z}) \mapsto (b^{k+1}\mathtt{x}, b\mathtt{y}, b^{k+1}\mathtt{z})$ we can suppose $F = a\mathtt{x}^2 + \mathtt{y}^{2k+1}$. Let us explain now the restrictions concerning the coefficients $a, b \in \kappa\backslash\{0\}$ in the statement:

CASE 1. If $F = a\mathtt{x}^2$ and $\mathcal{P}(A) = \Sigma A^2$, then $a \notin -\Sigma\kappa^2$ by [Sch1, Lem.6.3].

CASE 2. If $F = a\mathtt{x}^2 + b\mathtt{y}^2$ and $\mathcal{P}(A) = \Sigma A^2$, then either a or $b \notin -\Sigma\kappa^2$ by [Sch1, Lem.6.3]. Interchanging $\mathtt{x}$ and $\mathtt{y}$, we may assume $a \notin -\Sigma\kappa^2$.

CASE 3. If $F = a\mathtt{x}^2 + \mathtt{y}^{2k+1}$ (where $k \geqslant 1$) and $a \in -\Sigma\kappa^2$, then $\mathtt{y} \in \mathcal{P}(\{F \geqslant 0\})\backslash\Sigma A^2 \subset \mathcal{P}(A)\backslash\Sigma A^2$.

CASE 4. If $F = a\mathtt{x}^2 + b\mathtt{y}^{2k}$ (where $k \geqslant 2$) and $a \in -\Sigma\kappa^2$, let us find $\varphi \in \mathcal{P}(A)\backslash\Sigma A^2$. We get

$$(b^2+1)^2 - b = b^4 + b^2 + \Big(b - \frac{1}{2}\Big)^2 + \frac{3}{4} \in \Sigma\kappa^2. \tag{9.1}$$ b0

Set $a := -\sum_{i=1}^q a_i^2$ where $a_i \in \kappa\backslash\{0\}$.

Assume first that k is even. Then

$$\begin{aligned}(b^2+1)^2\mathtt{y}^{2k} - a_1^2\mathtt{x}^2 &= ((b^2+1)^2 - b)\mathtt{y}^{2k} + \sum_{k=2}^q a_k^2\mathtt{x}^2 + (a\mathtt{x}^2 + b\mathtt{y}^{2k})\\ &= ((b^2+1)^2 - b)\mathtt{y}^{2k} + \sum_{k=2}^q a_k^2\mathtt{x}^2 + \mathtt{z}^2 \in \mathcal{P}(A) \cap \kappa[[\mathtt{x},\mathtt{y}]] = \mathcal{P}(\{F \geqslant 0\}),\\ (b^2+1)\mathtt{y}^k &\in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]) \subset \mathcal{P}(\{F \geqslant 0\}).\end{aligned}$$

Thus, $\varphi := (b^2+1)\mathtt{y}^k + a_1\mathtt{x} \in \mathcal{P}(\{F \geqslant 0\})\backslash\Sigma A^2 \subset \mathcal{P}(A)\backslash\Sigma A^2$, because it has order 1.

Assume next that k is odd (and recall that $k \geqslant 2$). Then

$$\begin{aligned}(b^2+1)^2\mathtt{y}^{2k+2} - a_1^2\mathtt{x}^2\mathtt{y}^2 &= ((b^2+1)^2 - b)\mathtt{y}^{2k+2} + \sum_{k=2}^q a_k^2\mathtt{x}^2\mathtt{y}^2 + (a\mathtt{x}^2 + b\mathtt{y}^{2k})\mathtt{y}^2\\ &= ((b^2+1)^2 - b)\mathtt{y}^{2k+2} + \sum_{k=2}^q a_k^2\mathtt{x}^2\mathtt{y}^2 + \mathtt{z}^2\mathtt{y}^2 \in \mathcal{P}(A) \cap \kappa[[\mathtt{x},\mathtt{y}]] = \mathcal{P}(\{F \geqslant 0\}),\\ (b^2+1)\mathtt{y}^{k+1} &\in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]) \subset \mathcal{P}(\{F \geqslant 0\}).\end{aligned}$$

Thus, $\varphi := (b^2+1)\mathtt{y}^{k+1} + a_1\mathtt{x}\mathtt{y} \in \mathcal{P}(\{F \geqslant 0\})$. Let us check: $\varphi \notin \Sigma A^2$.

Otherwise, there exist $h_1, \dots, h_p, h \in \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]$ such that

$$\varphi = (b^2 + 1)\mathtt{y}^{k+1} + a_1\mathtt{xy} = \sum_{i=1}^{p} h_i^2 - (\mathtt{z}^2 - a\mathtt{x}^2 - b\mathtt{y}^{2k})h.$$

Comparing initial forms, we find $c \in \Sigma\kappa^2$ such that the quadratic form $\psi := a_1\mathtt{xy} + c\mathtt{z}^2 - ca\mathtt{x}^2$ is a sum of squares of linear forms in the variables $\mathtt{x}, \mathtt{y}, \mathtt{z}$ and coefficients in κ, so $-\frac{a_1^2}{2}(a+c)^2 - 1 = \psi(a_1, -\frac{c^2}{2} - \frac{a^2}{2} - \frac{1}{a_1^2}, 0) \in \Sigma\kappa^2$, which is a contradiction because κ is a (formally) real field. Thus, $\varphi \in \mathcal{P}(\{F \geqslant 0\})\backslash\Sigma A^2 \subset \mathcal{P}(A)\backslash\Sigma A^2$, as required. □

ord3 **Lemma 9.4** (Order 3 restrictions). *If $\omega(F) = 3$ and $\mathcal{P}(A) = \Sigma A^2$, then F is right equivalent to one of the following series:*

(i) $\mathtt{x}^2\mathtt{y}$ *or* $\mathtt{x}^2\mathtt{y} + (-1)^k a\mathtt{y}^k$ *where* $a \notin -\Sigma\kappa^2$ *and* $k \geqslant 3$.
(ii) $\mathtt{x}^3 + a\mathtt{xy}^2 + b\mathtt{y}^3$ *irreducible.*
(iii) $\mathtt{x}^3 + a\mathtt{y}^4$, $\mathtt{x}^3 + \mathtt{xy}^3$ *or* $\mathtt{x}^3 + \mathtt{y}^5$ *where* $a \notin \Sigma\kappa^2$.

First part of the proof of Lemma 9.4. After a linear change of coordinates (Tschirnhaus trick) we may assume that the initial form of F is of type $P := \lambda(\mathtt{x}^3 + a\mathtt{xy}^2 + b\mathtt{y}^3)$ for some $\lambda \in \kappa\backslash\{0\}$. After the change of coordinates $(\mathtt{x}, \mathtt{y}, \mathtt{z}) \mapsto (\lambda\mathtt{x}, \lambda\mathtt{y}, \lambda^2\mathtt{z})$, we may assume $P := \mathtt{x}^3 + a\mathtt{xy}^2 + b\mathtt{y}^3$. If P is reducible, after a linear change of coordinates that only involves the variables $\mathtt{x}, \mathtt{y}$ we can suppose that P is a homogeneous polynomial of one of the following types:

$$\mathtt{x}^3, \mathtt{x}^2\mathtt{y}, \mathtt{y}(\mathtt{x}^2 - a\mathtt{y}^2)$$

where $a \neq 0$. If P is either irreducible or $P = \mathtt{x}(\mathtt{x}^2 + a\mathtt{y}^2)$ with $a \neq 0$, then its discriminant is non-zero and P is 3-determined as we have seen in Example 5.8(iii). Thus, after a change of coordinates we may assume $F = P$.

If $F = \mathtt{y}(\mathtt{x}^2 - a\mathtt{y}^2)$ and $a \in -\Sigma\kappa^2$, then $\mathtt{y} \in \mathcal{P}(\{F \geqslant 0\})\backslash\Sigma A^2 \subset \mathcal{P}(A)\backslash\Sigma A^2$. Thus, $F = \mathtt{x}^2\mathtt{y} - a\mathtt{y}^3$ where $a \notin -\Sigma\kappa^2$.

Assume next $P := \mathtt{x}^2\mathtt{y}$, but $F \neq \mathtt{x}^2\mathtt{y}$. Let $s \geqslant 4$ be the degree of the next non-zero homogeneous form of F and set $F := \mathtt{x}^2\mathtt{y} + a\mathtt{y}^s + b\mathtt{xy}^{s-1} + \mathtt{x}^2\varphi$ where $a, b \in \kappa$ and $\varphi \in \mathfrak{m}_2^{s-2}$. After the change of coordinates $(\mathtt{x}, \mathtt{y}) \mapsto (\mathtt{x} - \frac{1}{2}b\mathtt{y}^{s-2}, \mathtt{y} - \varphi)$ we may assume $F := \mathtt{x}^2\mathtt{y} + a\mathtt{y}^s + \psi$ where $\psi \in \mathfrak{m}^{s+1}$. If $a \neq 0$, after an additional change of coordinates we can suppose $F := \mathtt{x}^2\mathtt{y} + a\mathtt{y}^s$ because $\mathtt{x}^2\mathtt{y} + a\mathtt{y}^s$ is s-determined (Example 5.8(ii)). If $a = 0$, we iterate the previous process until we find $\ell > s$ such that F is right equivalent to $\mathtt{x}^2\mathtt{y} + a'\mathtt{y}^\ell$ for some $a' \neq 0$ or we conclude that F is right equivalent to $\mathtt{x}^2\mathtt{y}$ if such an ℓ does not exist.

If $F := \mathtt{x}^2\mathtt{y} + a\mathtt{y}^{2k+1}$ and $a \in \Sigma\kappa^2$, then $\mathtt{y} \in \mathcal{P}(\{F \geqslant 0\})\backslash\Sigma A^2 \subset \mathcal{P}(A)\backslash\Sigma A^2$. If $F := \mathtt{x}^2\mathtt{y} - a\mathtt{y}^{2k}$ and $a \in \Sigma\kappa^2$, then $\mathtt{x}^2\mathtt{y} = \mathtt{z}^2 + a\mathtt{y}^{2k}$ and $\mathtt{y} \in \mathcal{P}(\{F \geqslant 0\})\backslash\Sigma A^2 \subset \mathcal{P}(A)\backslash\Sigma A^2$. □

Before finishing the proof of Lemma 9.4 we need an intermediate result.

clue0 **Lemma 9.5.** *Let $F \in \kappa[[\mathtt{x}, \mathtt{y}]]$ be a series with inital form $\mathtt{x}^3$. Then*

(i) *There exist a unit $U \in \kappa[[\mathtt{x}, \mathtt{y}]]$ with $U(0,0) = 1$, a unit $W \in \kappa[[\mathtt{y}]]$ with $W(0) = 1$, $b, c \in \kappa$ and $k, \ell \geqslant 0$ such that F is right equivalent to $(\mathtt{x}^3 + b\mathtt{xy}^{k+3} + c\mathtt{y}^{\ell+4}W)U^2$.*
(ii) *If $c \neq 0$ and $k \geqslant \ell + 1$, we may assume $b = 0$ and $W = 1$.*
(iii) *If $\mathcal{P}(A) = \Sigma A^2$, then either $k = 0$ (with $b \neq 0$) or $\ell \leqslant 1$ (with $c \neq 0$) and we may assume $U = 1$.*

Proof. (i) By the Weierstrass preparation theorem and the Tschirnhaus trick there exist a Weierstrass polynomial $P := \mathtt{x}^3 + B(\mathtt{y})x + C(\mathtt{y}) \in \kappa[[\mathtt{y}]][\mathtt{x}]$ and a unit $U_0 \in \kappa[[\mathtt{x}, \mathtt{y}]]$ such that $F = PU_0$. As the initial form of F is $\mathtt{x}^3$, we have in addition $U_0(0,0) = 1$, $\omega(B) \geqslant 3$ and $\omega(C) \geqslant 4$. Let $b, c \in \kappa$, let $k, \ell \geqslant 0$ be integers and $V, W_0 \in \kappa[[\mathtt{y}]]$ units such that $V(0) = 1$, $W_0(0) = 1$ and

$F = (\mathtt{x}^3 + b(V\mathtt{y})^{k+3}\mathtt{x} + cW_0\mathtt{y}^{\ell+4})U_0$. After the change of coordinates $V\mathtt{y} \mapsto \mathtt{y}$ we assume $V = 1$, so $F = (\mathtt{x}^3 + b\mathtt{y}^{k+3}x + cW\mathtt{y}^{\ell+4})U^2$ for units $U \in \kappa[[\mathtt{x}, \mathtt{y}]]$ and $W \in \kappa[[\mathtt{y}]]$ such that $U(0,0) = 1$ and $W(0) = 1$.

(ii) If $c \neq 0$ and $k \geqslant \ell+1$, then $W' := W + \frac{b}{c}\mathtt{x}\mathtt{y}^{k-\ell-1}$ is a unit. After the change of coordinates $W'\mathtt{y} \mapsto \mathtt{y}$, we get $F = (\mathtt{x}^3 + c\mathtt{y}^{\ell+4})U'^2$ for a unit $U' \in \kappa[[\mathtt{x}, \mathtt{y}]]$ with $U'(0,0) = 1$.

(iii) After the change of coordinates $\mathtt{z} \to U\mathtt{z}$ we assume $F = \mathtt{x}^3 + b\mathtt{x}\mathtt{y}^{k+3} + c\mathtt{y}^{\ell+4}W$ with the restrictions described in (i) and (ii). We claim: *If* $k \geqslant 1$ *and* $\ell \geqslant 2$*, there exists* $M \in \Sigma\kappa^2$ *such that* $\varphi = \mathtt{x} + M^2\mathtt{y}^2 \in \mathcal{P}(\{F \geqslant 0\})\backslash\Sigma A^2 \subset \mathcal{P}(A)\backslash\Sigma A^2$.

It is enough to find $M \in \Sigma\kappa^2$ such that: *if* $\alpha \in \operatorname{Sper}(\kappa[[\mathtt{x}, \mathtt{y}]])$ *satisfies* $\varphi <_\alpha 0$*, then* $F <_\alpha 0$.

If $\mathtt{y} \in \operatorname{supp}(\alpha)$, then $\varphi + \operatorname{supp}(\alpha) = \mathtt{x} + \operatorname{supp}(\alpha)$ and $F + \operatorname{supp}(\alpha) = \mathtt{x}^3 + \operatorname{supp}(\alpha)$ and both have the same sign with respect to α.

In order to find the suitable $M \in \Sigma\kappa^2$, let us make first some computations valid for each $M \in \Sigma\kappa^2$. If $\mathtt{y} \notin \operatorname{supp}(\alpha)$ and $\varphi <_\alpha 0$, then $\mathtt{x} <_\alpha -M^2\mathtt{y}^2$, so $-\mathtt{x} >_\alpha M^2\mathtt{y}^2 >_\alpha 0$. As $k \geqslant 1$, we have $((b^2 + 1)\mathtt{y}^4 + b\mathtt{y}^{k+3}) >_\alpha 0$. Thus,

$$-\mathtt{x}b\mathtt{y}^{k+3} >_\alpha -\mathtt{x}(-(b^2 + 1))\mathtt{y}^4) >_\alpha -(b^2 + 1)M^2\mathtt{y}^6,$$

so $\mathtt{x}b\mathtt{y}^{k+3} <_\alpha (b^2 + 1)M^2\mathtt{y}^6$ and $\mathtt{x}^3 <_\alpha -M^6\mathtt{y}^6$. Consequently, as $\ell \geqslant 2$,

$$F = (\mathtt{x}^3 + b\mathtt{x}\mathtt{y}^{k+3} + c\mathtt{y}^{\ell+4}W) <_\alpha (-M^6 + (b^2 + 1)M^2 + (c^2 + 1))\mathtt{y}^6 <_\alpha 0.$$

To guarantee the last inequality for each $\alpha \in \operatorname{Sper}(\kappa[[\mathtt{x}, \mathtt{y}]])$ such that $\varphi <_\alpha 0$, it is enough to find $M \in \Sigma\kappa^2$ such that $M^6 - (b^2 + 1)M^2 - (c^2 + 1) \in \Sigma\kappa^2$. We choose $M := 2(b^2 + 1)(c^2 + 1) \in \Sigma\kappa^2$ and observe that

$$\begin{aligned} &M^6 - (b^2 + 1)M^2 - (c^2 + 1)\\ &\qquad = (c^2 + 1)(16(b^2 + 1)^3(c^2 + 1)^4 - 1)(4(b^2 + 1)^3(c^2 + 1) - 1) \in \Sigma\kappa^2, \end{aligned}$$

as required. □

We are now ready to finish the proof of Lemma 9.4.

Second part of the proof of Lemma 9.4. If $\mathcal{P}(A) = \Sigma A^2$, there exist by Lemma 9.5 a unit $W \in \kappa[[\mathtt{y}]]$ with $W(0) = 1$ and $b, c \in \kappa$ such that F is one among

(i) $\mathtt{x}^3 + b\mathtt{x}\mathtt{y}^3 + c\mathtt{y}^4W$ where $c \neq 0$,
(ii) $\mathtt{x}^3 + b\mathtt{x}\mathtt{y}^3 + c\mathtt{y}^{\ell+5}W$ where $b \neq 0$ and $\ell \geqslant 0$,
(iii) $\mathtt{x}^3 + b\mathtt{x}\mathtt{y}^4 + c\mathtt{y}^5W$ where $c \neq 0$.

We now approach these three cases:

(i) If $F = \mathtt{x}^3 + b\mathtt{x}\mathtt{y}^3 + c\mathtt{y}^4W$ and $c \neq 0$, after the change of coordinates

$$\begin{aligned} \mathtt{x} &\mapsto \mathtt{x} + \frac{b^4}{256c^3}\mathtt{x}^2 - \frac{b^3}{24c^2}\mathtt{x}\mathtt{y} + \frac{b^2}{8c}\mathtt{y}^2,\\ \mathtt{y} &\mapsto \mathtt{y} - \frac{b}{4c}\mathtt{x} \end{aligned}$$

and we may assume $F = \mathtt{x}^3 + c\mathtt{y}^4 + \psi$ where $\psi \in \mathfrak{m}_2^5$. As $\mathtt{x}^3 + c\mathtt{y}^4$ is by Example 5.8(v) 4-determined, we can suppose after an additional change of coordinates that $F = \mathtt{x}^3 + c\mathtt{y}^4$. If $c \in -\Sigma\kappa^2$, we have $\mathtt{x}^3 = \mathtt{z}^2 - c\mathtt{y}^4 \in \Sigma A^2$, so $\mathtt{x} \in \mathcal{P}(\{F \geqslant 0\})\backslash\Sigma A^2 \subset \mathcal{P}(A)\backslash\Sigma A^2$. Thus, $F = \mathtt{x}^3 + c\mathtt{y}^4$ where $c \notin -\Sigma\kappa^2$.

(ii) If $F = \mathtt{x}^3 + b\mathtt{x}\mathtt{y}^3 + c\mathtt{y}^{\ell+5}W$ and $b \neq 0$, after the change of coordinates $(\mathtt{x}, \mathtt{y}, \mathtt{z}) \mapsto (b^2\mathtt{x}, b\mathtt{y}, b^3\mathtt{z})$ we can suppose $b = 1$. As $\mathtt{x}^3 + \mathtt{x}\mathtt{y}^3$ is by Example 5.8(iv) 5-determined, we can suppose after an additional change of coordinates that $F = x^3 + xy^3$ if $\ell \geqslant 1$. Otherwise $\ell = 0$, $c \neq 0$ and $F = x^3 + xy^3 + cy^5W$. After the change of coordinates

$$\begin{aligned} x &\mapsto x - cy^2 - \frac{1}{3}c^3x^2 - c^2xy - \frac{1}{3}c^6x^3 - \frac{5}{3}c^5x^2y - 2c^4xy^2 - \frac{5}{9}c^3y^3, \\ y &\mapsto y + cx - \frac{4}{3}c^2y^2 \end{aligned}$$

there exists a series $\psi \in \mathfrak{m}_2^6$ such that $F = x^3 + xy^3 + \psi$. As $x^3 + xy^3$ is 5-determined, F is right equivalent to $x^3 + xy^3$ and we suppose $F = x^3 + xy^3$.

(iii) If $F = x^3 + bxy^4 + cy^5W$ and $c \neq 0$, after the change of coordinates

$$\begin{aligned} x &\mapsto x - \frac{4b^5}{9375c^4}x^3 + \frac{b^4}{125c^3}x^2y - \frac{4b^3}{75c^2}xy^2 + \frac{2b^2}{15c}y^3, \\ y &\mapsto y - \frac{b}{5c}x, \end{aligned}$$

we assume $F = x^3 + cy^5 + \psi$ where $\psi \in \mathfrak{m}_2^6$. As $x^3 + cy^5$ is by Example 5.8(v) 5-determined, there exists an additional change of coordinates after which $F = x^3 + cy^5$. After the linear change of coordinates $(x, y, z) \mapsto (c^2x, cy, c^3z)$ we get $F = x^3 + y^5$, as required. □

9.2.1. *The non-principal case.* To finish the proof of Theorem 2.2 we explore the case of a ring $A := \kappa[[x, y, z]]/\mathfrak{a}$ of dimension 2 such that $\mathfrak{a}$ is not a principal ideal, but A has the property $\mathcal{P}(A) = \Sigma A^2$. Before we need the following example.

nonprincipal **Theorem 9.6.** *Let $\mathfrak{a}$ be a non-principal ideal of $\kappa[[x, y, z]]$ such that $A := \kappa[[x, y, z]]/\mathfrak{a}$ has dimension 2. Then $\mathcal{P}(A) = \Sigma A^2$ if and only if A is isomorphic to $\kappa[[x, y, z]]/(zx, zy)$.*

Proof. Assume first that $A = \kappa[[x, y, z]]/(zx, zy)$ and let $f \in \mathcal{P}(A)$. If f is a unit, then $f = bU^2$ where $b \in \kappa\setminus\{0\}$ and $U(0, 0, 0) = 1$. If $\mathfrak{m}$ is the maximal ideal of A, then $A/\mathfrak{m} = \kappa$. As $f \in \mathcal{P}(A)$, then $b \in \mathcal{P}(\kappa) = \Sigma\kappa^2$, so $f \in \Sigma A^2$. As a consequence, we assume that f is not a unit. As $zx = 0, zy = 0$, we have $f = f_1(x, y) + f_2(z)$ where $f_1 \in \kappa[[x, y]]$ and $f_2 \in \kappa[[z]]$ satisfy $f_1(0, 0) = 0$ and $f_2(0) = 0$. Observe that $f_1 \in \mathcal{P}(\kappa[[x, y]])$ and $f_2 \in \mathcal{P}(\kappa[[[z]])$ because $\mathfrak{a} := (zx, zy) = (z) \cap (x, y)$, $A/((z)/\mathfrak{a}) \cong \kappa[[x, y]]$ and $A/((x, y)/\mathfrak{a}) \cong \kappa[[z]]$. Thus, there exist $a_i \in \kappa[[x, y]]$ and $b_j \in \kappa[z]$ such that $f = f_1 + f_2 = \sum_i a_i^2 + \sum_j b_j^2 \in \Sigma A^2$.

Assume next $\mathcal{P}(A) = \Sigma A^2$. By Lemma 5.11 we know $\omega(\mathfrak{a}) \leqslant 2$. If $\omega(\mathfrak{a}) = 1$, we may assume $z \in \mathfrak{a}$. But this is impossible because $\dim(A) = 2$ and $\mathfrak{a}$ is not a principal ideal. Thus, $\omega(\mathfrak{a}) = 2$ and by [Sch1, Lem.6.3] $\mathfrak{a}$ is a real radical ideal. Let $\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r \cap \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ be the irredundant primary decomposition of $\mathfrak{a}$ and assume that $\operatorname{ht}(\mathfrak{p}_i) = 1$ for each $i = 1, \ldots, r$ and $\operatorname{ht}(\mathfrak{q}_j) = 2$ for $j = 1, \ldots, s$. Let $\mathfrak{a}_1 := \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ and $\mathfrak{a}_2 := \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$. As each ideal $\mathfrak{p}_i$ is prime and has height 1, there exist $\varphi_i \in \kappa[[x, y, z]]$ irreducible such that $\mathfrak{p}_i = (\varphi_i)$ for $i = 1, \ldots, r$. It holds $\mathfrak{a}_1 = (\varphi)$ where $\varphi := \prod_{i=1}^r \varphi_i$. We claim: $\mathfrak{a} = (\varphi) \cdot \mathfrak{a}_2$.

The inclusion right to left is clear, so let $f \in \mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2$. As $\mathfrak{a}_1 = (\varphi)$, there exists $h \in \kappa[[x, y, z]]$ such that $f = \varphi h$. Let us check: $h \in \mathfrak{a}_2$.

Otherwise, we may assume $h \notin \mathfrak{q}_1$. As $(\prod_{i=1}^r \varphi_i)h = \varphi h \in \mathfrak{a}_2 \subset \mathfrak{q}_1$, we can suppose $\varphi_1 \in \mathfrak{q}_1$, so $\mathfrak{p}_1 \subset \mathfrak{q}_1$, which is a contradiction because the primary decomposition of the real radical ideal $\mathfrak{a}$ is irredundant. Consequently, $h \in \mathfrak{a}_2$ and $f \in (\varphi) \cdot \mathfrak{a}_2$, so $\mathfrak{a} = (\varphi) \cdot \mathfrak{a}_2$.

Thus, $2 = \omega(\mathfrak{a}) = \omega(\varphi) + \omega(\mathfrak{a}_2)$, so $\omega(\varphi) = \omega(\mathfrak{a}_2) = 1$. We may assume $\varphi = z$ and there exists $\psi \in \mathfrak{a}_2$ such that $\omega(\psi) = 1$. We claim: *The initial form of ψ is not a multiple of* z.

Otherwise, we assume that the initial form of ψ is equal to z. By Weierstrass preparation theorem we may write $\psi = z + 2g(x, y)$ for some $g \in \kappa[[x, y]]$ of order $\geqslant 2$. Observe that

$$z(z + 2g) = (z + g)^2 - g^2 \in \mathfrak{a}.$$

As $\omega(g) \geqslant 2$, there exists by Lemma 5.9 $M \in \Sigma\kappa^2$ such that $M^2(\mathtt{x}^2+\mathtt{y}^2)^2 - g^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$. Using the chain of homomorphisms

$$\kappa[[\mathtt{x},\mathtt{y}]] \hookrightarrow \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]] \to A$$

one deduces (using that $(\mathtt{z}+g)^2 - g^2 \in \mathfrak{a}$)

$$\begin{aligned}(M(\mathtt{x}^2+\mathtt{y}^2) - g - \mathtt{z})(M(\mathtt{x}^2+\mathtt{y}^2) + g + \mathtt{z}) &= M^2(\mathtt{x}^2+\mathtt{y}^2)^2 - (g+\mathtt{z})^2 \\ &= M^2(\mathtt{x}^2+\mathtt{y}^2)^2 - g^2 \in \mathcal{P}(A)\end{aligned}$$

and $(M(\mathtt{x}^2+\mathtt{y}^2) - g - \mathtt{z}) + (M(\mathtt{x}^2+\mathtt{y}^2) + g + \mathtt{z}) = 2M(\mathtt{x}^2+\mathtt{y}^2) \in \mathcal{P}(A)$. Thus,

$$M(\mathtt{x}^2+\mathtt{y}^2) - g - \mathtt{z}, M(\mathtt{x}^2+\mathtt{y}^2) + g + \mathtt{z} \in \mathcal{P}(A)\backslash\Sigma A^2$$

because they have order 1 and $\omega(\mathfrak{a}) = 2$. As $\mathcal{P}(A) = \Sigma A^2$, we deduce the initial form of ψ is not a multiple of $\mathtt{z}$.

Thus, we assume $\mathtt{x} \in \mathfrak{a}_2$ and observe that $\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/\mathfrak{a}_2 \cong \kappa[[\mathtt{y},\mathtt{z}]]/\mathfrak{a}_2'$ for some ideal $\mathfrak{a}_2'$ of $\kappa[[\mathtt{y},\mathtt{z}]]$. As $\operatorname{ht}(\mathfrak{a}_2) = 2$, we deduce $\operatorname{ht}(\mathfrak{a}_2') = 1$, so $\mathfrak{a}_2'$ is a principal ideal and there exists $\phi \in \kappa[[\mathtt{y},\mathtt{z}]]$ such that $\mathfrak{a}_2' = (\phi)$. Note that $\mathfrak{a}_2 = (\mathtt{x},\phi)$ and $\mathfrak{a} = (\mathtt{z}\mathtt{x},\mathtt{z}\phi)$. Define $\mathfrak{b} := (\mathtt{z}\phi)$ and $B := \kappa[[\mathtt{y},\mathtt{z}]]/\mathfrak{b}$. We claim: $\mathcal{P}(B) = \Sigma B^2$.

Consider the inclusion of rings $B := \kappa[[\mathtt{y},\mathtt{z}]]/\mathfrak{b} \hookrightarrow A/\mathfrak{a}$. If $f \in \mathcal{P}(B)$, then $f \in \mathcal{P}(A) = \Sigma A^2$, so there exist $a_1,\ldots,a_p,b_1,b_2 \in \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ such that

$$f(\mathtt{y},\mathtt{z}) = a_1^2(\mathtt{x},\mathtt{y},\mathtt{z}) + \cdots + a_p^2(\mathtt{x},\mathtt{y},\mathtt{z}) + \mathtt{z}\mathtt{x}b_1(\mathtt{x},\mathtt{y},\mathtt{z}) + \mathtt{z}\phi(\mathtt{y},\mathtt{z})b_2(\mathtt{x},\mathtt{y},\mathtt{z}).$$

Substituting $\mathtt{x} = 0$ in the previous equality we deduce

$$f(\mathtt{y},\mathtt{z}) = a_1^2(0,\mathtt{y},\mathtt{z}) + \cdots + a_p^2(0,\mathtt{y},\mathtt{z}) + \mathtt{z}\phi(\mathtt{y},\mathtt{z})b_2(0,\mathtt{y},\mathtt{z}),$$

so $f \in \Sigma B^2$. Thus, $\mathcal{P}(B) = \Sigma B^2$.

By Lemma 3.4 we deduce that B is isomorphic to $\kappa[[\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2 - b\mathtt{y}^2)$ for some $b \notin -\Sigma\kappa^2$. This means that $\phi \in \kappa[[\mathtt{y},\mathtt{z}]]$ has order 1 and its initial form is not a multiple of $\mathtt{z}$. After a change of coordinates, we may assume $\phi = \mathtt{y}$. Consequently, $\mathfrak{a} = (\mathtt{z}\mathtt{x},\mathtt{z}\mathtt{y})$, as required. □

9.3. **Reduction to the complete case (general setting).** By [Sch3, Thm.3.9] we have the equivalence between the properties $\mathcal{P}(A) = \Sigma A^2$ and $\mathcal{P}(\widehat{A}) = \Sigma\widehat{A}^2$ if A has dimension 1. Let us prove that $\mathcal{P}(A) = \Sigma A^2$ implies $\mathcal{P}(\widehat{A}) = \Sigma\widehat{A}^2$ if A has dimension 2 and the embedding dimension of A is $\leqslant 3$ (even if $p(A)$ needs not to be bounded).

Proof. We have $\widehat{A} \cong \kappa[[\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3]]/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of $\kappa[[\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3]]$. We may assume $\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3$ correspond to a system of generators of the maximal ideal of A. By Lemma 5.10 there exists after a κ-linear change of coordinates an element of the type $M^2(\mathtt{x}_1^2+\mathtt{x}_2^2) - \mathtt{x}_3^2 \in \mathcal{P}(\widehat{A}) \cap A = \mathcal{P}(A) = \Sigma A^2 \subset \Sigma\widehat{A}^2$ by Lemma 5.14 and [ABR, Ch.VII.Cor.3.3]. Thus, $\omega(\mathfrak{a}) \leqslant 2$. If $\omega(\mathfrak{a}) = 1$, then $\widehat{A} \cong \kappa[[\mathtt{x}_1,\mathtt{x}_2]]/\mathfrak{a}'$. As $2 = \dim(A) = \dim(\widehat{A}) = 2 - \operatorname{ht}(\mathfrak{a}')$, we deduce $\mathfrak{a}' = 0$, so $\widehat{A} \cong \kappa[[\mathtt{x}_1,\mathtt{x}_2]]$ and $\mathcal{P}(\widehat{A}) = \Sigma\widehat{A}^2$. Thus, we may assume $\omega(\mathfrak{a}) = 2$.

Suppose first $\mathfrak{a}$ is a principal ideal. We may assume that $\mathfrak{a}$ is generated by an element of the type $\mathtt{z}^2 - F(\mathtt{x},\mathtt{y})$ where $F \in \kappa[[\mathtt{x},\mathtt{y}]]$. If $\omega(F) \geqslant 4$, we know by Lemma 5.13 that there exists $h \in \mathcal{P}(\widehat{A})\backslash\Sigma\widehat{A}^2$ such that $\omega(h) = 1$. As we have seen above, there exists $h' \in \mathcal{P}(A) = \Sigma A^2 \subset \Sigma\widehat{A}^2$ such that $\omega(h - h') \geqslant 2$, so $\omega(h') = 1$, which is a contradiction, because $\omega(\mathtt{z}^2 - F) = 2$. Consequently, $2 \leqslant \omega(F) \leqslant 3$. If $\omega(F) = 3$ and F is not right equivalent to a series of List 2.1, there exists by [Fe8, Lemm.3.8 & 3.9] there exists $h \in \mathcal{P}(\widehat{A})\backslash\Sigma\widehat{A}^2$ such that $\omega(h) = 1$. As before, we achieve a contradiction. If $\omega(F) = 2$ and F is not right equivalent to a series of List 2.1, there exists by [Fe8, Lem.3.7] either a series of order either 1 and we achieve a contradiction (as we have seen above) or F is right equivalent to $a\mathtt{x}^2 + b\mathtt{y}^{2k}$, where $a \in -\Sigma\kappa^2$, $b \in \kappa\backslash\{0\}$ and $k \geqslant 2$ is odd, there exists a series $h := a_1\mathtt{x}\mathtt{y} + (b^2+1)\mathtt{y}^{k+1} \in \mathcal{P}(\widehat{A})\backslash\Sigma A^2$ of order 2 (for some $a_1 \in \kappa\backslash\{0\}$).

By there exists $h' \in \mathcal{P}(A) = \Sigma A^2 \subset \Sigma\widehat{A}^2$ such that $\omega(h'-h) \geqslant 3$, so the initial form of h' is $a_1\mathtt{xy}$ and (comparing leading forms) there exists $c \in \Sigma\kappa^2$ such that $a_1\mathtt{xy} + c(\mathtt{z}^2 - z\mathtt{x}^2)$ is a sum of squares of linear forms, which is a contradiction (see the proof of [Fe8, Lem.3.7]). Thus, if $\mathcal{P}(A) = \Sigma A^2$ and $\mathfrak{a} = (\mathtt{z}^2 - F)$, then F is right equivalent to a series of 2.1 and $\mathcal{P}(\widehat{A}) = \Sigma\widehat{A}^2$.

Suppose next $\mathfrak{a}$ is not a principal ideal. If (after a change of coordinates) $\mathfrak{a}$ is not an ideal of the type $(\mathtt{zx}, \mathtt{z}\psi)$ where $\psi \in \kappa[[\mathtt{y},\mathtt{z}]]\backslash\{0\}$, there exists by the proof of [Fe8, Thm.3.11] an element $h \in \mathcal{P}(\widehat{A})\backslash\Sigma\widehat{A}^2$ of order 1, which is a contradiction. Let $\mathfrak{b} := (\mathtt{z}\psi)$ and $B := \kappa[[\mathtt{y},\mathtt{z}]]/\mathfrak{b}$. Observe that $\mathfrak{a} \cap \kappa[[y\,z]] = \mathfrak{b}$, so we have an inclusion $B \hookrightarrow \widehat{A}$. We claim: $\mathcal{P}(B) = \Sigma B^2$.

By [Sch3, Lem.3.7] it is enough to check: *If $f \in \mathcal{P}(B)$ is a non-zero divisor, $f \in \Sigma B^2$.*

Let $f \in \mathcal{P}(B)$ be a non-zero divisor. By [Sch3, Lem.3.1] there exists $m \geqslant 1$ such that every element in $f + \mathfrak{m}_2^m$ has the form $u^2 f$ for a unit $u \in B$. By Lemma 5.14 $f \in \mathcal{P}(\widehat{A})$ and as we have seen above there exists $g_m \in \mathcal{P}(A) = \Sigma A^2 \subset \Sigma\widehat{A}^2$ such that $f - g_m \in \mathfrak{m}^m\widehat{A}$. Thus, substituting $\mathtt{x} = 0$, we deduce that $g_m(0,\mathtt{y},\mathtt{z}) \in \Sigma B^2$ and $g_m(0,\mathtt{y},\mathtt{z}) \in f + \mathfrak{m}_2^m$, so there exists a unit $u \in B$ such that $u^2 f = g_m(0,\mathtt{y},\mathtt{z}) \in \Sigma B^2$. Thus, $f \in \Sigma B^2$, as claimed.

By [Fe8, Ex.3.10] we deduce that B is isomorphic to $\kappa[[\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2 - b\mathtt{y}^2)$ for some $b \notin \Sigma\kappa^2$. Consequently, $\phi \in \kappa[[\mathtt{y},\mathtt{z}]]$ has order 1 and its leading form is not a multiple of $\mathtt{z}$. After a change of coordinates, we may assume $\phi = \mathtt{y}$ and $\mathfrak{a} = (\mathtt{zx}, \mathtt{zy})$, so $\mathcal{P}(\widehat{A}) = \Sigma A^2$, as required. □

9.4. **Proof of Theorem 2.3.** In this section we prove Theorem 2.3, when the Pythagoras number of the involved ring is finite, with the help of Elephant's Theorems 8.6 and 8.7.

9.4.1. *Order two cases.* We prove next Theorem 2.3 for the cases (3.i-3.iii) in List 2.1. We begin with the following preliminary result.

ert0 **Lemma 9.7.** *Let κ be a (formally) real field. Let F be equal to either $a\mathtt{y}^2 + \mathtt{x}^{2k+1}$ or $a\mathtt{x}^2 + b\mathtt{y}^{2k}$ where $k \geqslant 1$, $a \notin -\Sigma\kappa^2$ and $b \neq 0$. Define $A := \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2 - F)$ and let $f + \mathtt{z}g \in \mathcal{P}(A)\backslash\{0\}$. Then there exist $s \geqslant 0$ and $f_1, g_1 \in \kappa[[\mathtt{x},\mathtt{y}]]$ such that $f_1(\mathtt{x},0) \neq 0$ and $f + \mathtt{z}g = \mathtt{y}^{2s}(f_1 + \mathtt{z}g_1)$.*

Proof. As $f + \mathtt{z}g \in \mathcal{P}(A)\backslash\{0\}$, we know that $f \in \mathcal{P}(\{F \geqslant 0\})$ and $f^2 - Fg^2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$. By Lemma 6.4 $f \neq 0$. We claim: $\mathtt{y}^2$ *divides* f.

Otherwise, $f = \mathtt{y}f'$ where $f' \in \kappa[[\mathtt{x},\mathtt{y}]]$ and $f'(\mathtt{x},0) \neq 0$. As $f'(\mathtt{x},0) \neq 0$, there exist a Weierstrass polynomial $Q := \mathtt{x}^q + \sum_{k=0}^{q-1} \mathtt{y}b_k(\mathtt{y})\mathtt{x}^k$ (where $b_k \in \kappa[[\mathtt{y}]]$ for each k), $\mu \in \kappa\backslash\{0\}$ and a unit $V \in \kappa[[\mathtt{x},\mathtt{y}]]$ with $V(0,0) = 1$ such that $f' = Q\mu V^2$.

Let α be an ordering of κ such that $a >_\alpha 0$ and let $\mathfrak{R}(\alpha)$ be the real closure of $(\kappa, <_\alpha)$. For each $m \geqslant 1$ consider the homomorphism

$$\varphi : \kappa[[\mathtt{x},\mathtt{y}]] \to \mathfrak{R}(\alpha)[[\mathtt{t}]],\ h \mapsto h(\mathtt{t}^2, \mathtt{t}^m).$$

If m is odd and large enough,

$$\varphi(F) = \begin{cases} a\mathtt{t}^{2m} + \mathtt{t}^{4k+2} = \mathtt{t}^{4k+2}(1 + a\mathtt{t}^{2m-4k-2}) & \text{if } F = a\mathtt{y}^2 + \mathtt{x}^{2k+1}, \\ a\mathtt{t}^4 + b\mathtt{t}^{2km} = a\mathtt{t}^4(1 + \frac{b}{a}\mathtt{t}^{2km-4}) & \text{if } F = a\mathtt{x}^2 + b\mathtt{y}^{2k}, \end{cases}$$

$$\begin{aligned} \varphi(f) = \varphi(\mathtt{y}f') &= \mathtt{t}^m\Big(\mathtt{t}^{2q} + \sum_{k=0}^{q-1} \mathtt{t}^{m+2k} b_k(\mathtt{t}^m)\Big)\mu V^2(\mathtt{t}^2, \mathtt{t}^m) \\ &= \mu\mathtt{t}^{m+2q}\Big(1 + \sum_{k=0}^{q-1} \mathtt{t}^{m+2k-2q} b_k(\mathtt{t}^m)\Big)V^2(\mathtt{t}^2, \mathtt{t}^m). \end{aligned}$$

Define the sign of $\mathtt{t}$ in $\mathfrak{R}(\alpha)[[\mathtt{t}]]$ to get $\mu\mathtt{t} < 0$. Denote β_0 the corresponding prime cone of $\mathfrak{R}(\alpha)[[\mathtt{t}]]$ and $\beta := \varphi^{-1}(\beta_0)$. Observe that $F \in \beta$, whereas $f \notin \beta$. This is a contradiction because $f \in \mathcal{P}(\{F \geqslant 0\})$.

We conclude $f'(\mathtt{x},0)=0$, so $f=\mathtt{y}^2f''$ where $f''\in\kappa[[\mathtt{x},\mathtt{y}]]$, as claimed.

We prove next: $\mathtt{y}^2$ *divides* g. We have:

$$f^2-Fg^2=\mathtt{y}^4f''^2-Fg^2\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]).$$

Setting $\mathtt{y}=0$, we have $-F(\mathtt{x},0)g^2(\mathtt{x},0)\in\mathcal{P}(\kappa[[\mathtt{x}]])=\Sigma\kappa[[\mathtt{x}]]^2$, where

$$-F(\mathtt{x},0)=\begin{cases}-\mathtt{x}^{2k+1} & \text{if } F=a\mathtt{y}^2+\mathtt{x}^{2k+1},\\ -a\mathtt{x}^2 & \text{if } F=a\mathtt{x}^2+b\mathtt{y}^{2k}.\end{cases}$$

Thus, $g(\mathtt{x},0)=0$ and there exists $g'\in\kappa[[\mathtt{x},\mathtt{y}]]$ such that $g=\mathtt{y}g'$. Setting again $\mathtt{y}=0$, we have $-F(\mathtt{x},0)g'^2(\mathtt{x},0)\in\mathcal{P}(\kappa[[\mathtt{x}]])$, so $g'(\mathtt{x},0)=0$ and there exists $g''\in\kappa[[\mathtt{x},\mathtt{y}]]$ such that $g=\mathtt{y}^2g''$. This means $f+\mathtt{z}g=\mathtt{y}^2(f''+\mathtt{z}g'')$ where $f''+\mathtt{z}g''\in\mathcal{P}(A)$.

Proceeding recursively we find $s\geqslant 1$ and $f_1+\mathtt{z}g_1\in\mathcal{P}(A)$ such that $f_1(\mathtt{x},0)\neq 0$ and $f+\mathtt{z}g=\mathtt{y}^{2s}(f_1+\mathtt{z}g_1)$. $\square$

We prove next Theorem 2.3 for (3.ii) of List 2.1 as a consequence of Theorem 8.6.

main1 **Theorem 9.8** (Case (3.ii) of List 2.1)**.** *Let κ be a (formally) real field such that $\tau(\kappa)<+\infty$. Let $A:=\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-a\mathtt{x}^2-\mathtt{y}^{2k+1})$ where $a\notin-\Sigma\kappa^2$ and $k\geqslant 1$. Then $\mathcal{P}(A)=\Sigma A^2$.*

Proof. Let $f+\mathtt{z}g\in\mathcal{P}(A)\setminus\{0\}$. As $-F\notin\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$ (because $a\notin-\Sigma\kappa^2$), we deduce by Lemma 6.4 $f\neq 0$.

By Lemma 9.7 there exist $s\geqslant 1$ and $f_1,g_1\in\kappa[[\mathtt{x},\mathtt{y}]]$ such that $f_1(0,\mathtt{y})\neq 0$ and $f+\mathtt{z}g=\mathtt{x}^{2s}(f_1+\mathtt{z}g_1)$. Denote $q:=\omega(f_1(0,\mathtt{y}))$.

Let us check that we can apply Theorem 8.6 to our current situation. Suppose first $q=1$, that is, the leading form of f_1 is $c\mathtt{x}+d\mathtt{y}$ where $d\in\kappa\setminus\{0\}$.

Let α be an ordering of κ such that $a>_\alpha 0$. Define $\eta:=(\frac{d}{c^2+1}\mathtt{t},\mathtt{t})$ and define the sign of $\mathtt{t}$ in such a way that $d\mathtt{t}<0$. As $\frac{c^2+c+1}{c^2+1}\in\Sigma\kappa^2$, we have

$$f_1(\eta)=\frac{d(c^2+c+1)}{c^2+1}\mathtt{t}+\cdots<0,$$
$$F(\eta)=a\frac{d^2}{(c^2+1)^2}\mathtt{t}^2+\mathtt{t}^{2k+1}>0,$$

so $f_1\notin\mathcal{P}(\{F\geqslant 0\})$, which is a contradiction. Consequently, $q\geqslant 2$.

Consider next $\varphi_q:=(\mathtt{u}\mathtt{y}^{q+k},\mathtt{y},\mathtt{v}\mathtt{y}^k)$ and observe that $F(\varphi_q)=a\mathtt{u}^2\mathtt{y}^{2q+2k}+\mathtt{y}^{2k+1}=\mathtt{y}^{2k+1}(1+a\mathtt{u}^2\mathtt{y}^{2q-1})$ where $2q-1\geqslant 3$, so $F(\varphi_q)=\mathtt{y}^{2k+1}(1+\mathtt{y}^2\Gamma_q)$ for some series $\Gamma_q\in\kappa[[\mathtt{u},\mathtt{y}]]$. In addition, $-F(\mathtt{x},0)=-a\mathtt{x}^2\notin\Sigma\kappa((\mathtt{x}))^2$, because $-a\notin-\Sigma\kappa^2$.

Denote $W:=1+\mathtt{y}^2\Gamma_q\in\kappa[[\mathtt{u},\mathtt{y}]]$, which satisfies $W(0,0)=1$. We have $(\mathtt{z}^2-F)(\varphi)=\mathtt{y}^{2k}(\mathtt{v}^2-\mathtt{y}W)$. Define $A':=\kappa[[\mathtt{u},\mathtt{y},\mathtt{v}]]/(\mathtt{v}^2-\mathtt{y}W)$. As $f_1+\mathtt{z}g_1\in\mathcal{P}(A)$, we have

$$f_1(\mathtt{u}\mathtt{y}^{q+k},\mathtt{y})+\mathtt{v}\mathtt{y}^kg_1(\mathtt{u}\mathtt{y}^{q+k},\mathtt{y})\in\mathcal{P}(A').$$

As $W(0,0)=1$, there exists by the Implicit Function Theorem $\varphi(\mathtt{u},\mathtt{v})\in\kappa[[\mathtt{u},\mathtt{v}]]$ such that $\mathtt{y}=\varphi(\mathtt{u},\mathtt{v})$ is the unique solution of the equation $\mathtt{v}^2-\mathtt{y}W=0$ satisfying $\varphi(0,0)=0$. As $\psi(f_1+\mathtt{z}g_1)\in\mathcal{P}(A')$, we get

$$h(\mathtt{u},\mathtt{v}):=f_1(\mathtt{u}\varphi(\mathtt{u},\mathtt{v})^{q+k},\varphi(\mathtt{u},\mathtt{v}))+\mathtt{v}\varphi(\mathtt{u},\mathtt{v})^kg_1(\mathtt{u}\varphi(\mathtt{u},\mathtt{v})^{q+k},\varphi(\mathtt{u},\mathtt{v}))\in\mathcal{P}(\kappa[[\mathtt{u},\mathtt{v}]])=\Sigma\kappa[[\mathtt{u},\mathtt{v}]]^2,$$

so there exist $h_i\in\kappa[[\mathtt{u},\mathtt{v}]]$ such that $h=\sum_{i=1}^p h_i^2$. Using the relation $\mathtt{v}^2-\mathtt{y}W$, we find series $a_i,b_i,c\in\kappa[[\mathtt{u},\mathtt{y}]]$ satisfying

$$f_1(\mathtt{u}\mathtt{y}^{q+k},\mathtt{y})+\mathtt{v}\mathtt{y}^kg_1(\mathtt{u}\mathtt{y}^{q+k},\mathtt{y})=\sum_{i=1}^p(a_i+\mathtt{v}b_i)^2-(\mathtt{v}^2-\mathtt{y}W)c.$$

Thus,

$$\begin{aligned}\mathtt{y}^{2k}(f_1+\mathtt{z}g_1)(\varphi)=\sum_{i=1}^{p}(\mathtt{y}^k a_i+\mathtt{v}\mathtt{y}^k b_i)^2-((\mathtt{y}^k\mathtt{v})^2-\mathtt{y}^{2k+1}W)c\\ =\sum_{i=1}^{p}(\mathtt{y}^k a_i+\mathtt{v}\mathtt{y}^k b_i)^2-(\mathtt{z}^2-F)(\varphi).\end{aligned}$$

We are under the hypotheses of Theorem 8.6(ii), so $f_1+\mathtt{z}g_1\in\Sigma A^2$ and $f+\mathtt{z}g=\mathtt{x}^{2s}(f_1+\mathtt{z}g_1)\in\Sigma A^2$, as required. □

Our next purpose is to prove Theorem 2.3 for (3.iii) of List 2.1. We will base on Theorem 9.8 and the following result.

limit3 **Lemma 9.9.** *Let κ be a (formally) real field. Let $A:=\kappa[\mathtt{x},\mathtt{y},\mathtt{z}]/(\mathtt{z}^2-a\mathtt{x}^2)$ where $a\notin-\Sigma\kappa^2$ and denote $A_m:=\kappa[\mathtt{x},\mathtt{y},\mathtt{z}]/(\mathtt{z}^2-a\mathtt{x}^2-\mathtt{y}^{2m+1})$ for each $m\geqslant 1$. If $f+\mathtt{z}g\in\mathcal{P}(A)$, then $f+(\mathtt{x}^{2n}+\mathtt{y}^{2n})+\mathtt{z}g\in\mathcal{P}(A_m)$ for each $n\geqslant 1$ and each $m\geqslant 2n$.*

Proof. Let $f+\mathtt{z}g\in\mathcal{P}(A)$ and $n\geqslant 1$. We have

$$\begin{aligned}&f\in\mathcal{P}(\{a\mathtt{x}^2\geqslant 0\}),\\ &f^2-a\mathtt{x}^2g^2\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]).\end{aligned}$$

If $m\geqslant 2n$, we have $2m+1-4n\geqslant 1$ and

$$\begin{aligned}(f+\mathtt{x}^{2n}+\mathtt{y}^{2n})^2-(a\mathtt{x}^2+\mathtt{y}^{2m+1})g^2=(f^2-a\mathtt{x}^2g^2)+2f(\mathtt{x}^{2n}+\mathtt{y}^{2n})\\ +\mathtt{x}^{4n}+2x^{2n}\mathtt{y}^{2n}+\mathtt{y}^{4n}(\sqrt{1-\mathtt{y}^{2m+1-4n}g^2})^2\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]).\end{aligned}$$

Let us check next: $f+\mathtt{x}^{2n}+\mathtt{y}^{2n}\in\mathcal{P}(\{a\mathtt{x}^2+\mathtt{y}^{2m+1}\geqslant 0\})$ for each $m\geqslant 2n$. Once this is proved, we deduce $f+\mathtt{x}^{2n}+\mathtt{y}^{2n}+\mathtt{z}g\in\mathcal{P}(A_m)$ where $A_m:=\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-a\mathtt{x}^2-\mathtt{y}^{2m+1})$.

Let us check: $f(0,\mathtt{y})\in\Sigma\kappa[[\mathtt{y}]]^2$. Fix $\gamma\in\operatorname{Sper}(\kappa[[\mathtt{y}]])$ and consider the prime ideal $\mathfrak{p}:=(\mathtt{x})\kappa[[\mathtt{x},\mathtt{y}]]$ of $\kappa[[\mathtt{x},\mathtt{y}]]$. Consider the surjective homomorphism $\varphi:\kappa[[\mathtt{x},\mathtt{y}]]\to\kappa[[\mathtt{y}]],\ h(\mathtt{x},\mathtt{y})\mapsto h(0,y)$ and the prime cone $\gamma_0:=\varphi^{-1}(\gamma)$. Observe that $a\mathtt{x}^2\in\mathfrak{p}=\operatorname{supp}(\gamma_0)$, so $f\geqslant_{\gamma_0}0$. Thus, $f(0,y)=\varphi(f)\geqslant_\gamma 0$. Consequently, $f(0,\mathtt{y})\in\mathcal{P}(\kappa[[\mathtt{y}]])=\Sigma\kappa[[\mathtt{y}]]^2$.

Let $\beta\in\operatorname{Sper}(\kappa[[\mathtt{x},\mathtt{y}]])$ be such that $a\mathtt{x}^2+\mathtt{y}^{2m+1}\geqslant_\beta 0$. We distinguish two cases:

Case 1. $a\mathtt{x}^2\geqslant_\beta 0$. Then $f\geqslant_\beta 0$, so $f+\mathtt{x}^{2n}+\mathtt{y}^{2n}\geqslant_\beta 0$. ■

Case 2. $a\mathtt{x}^2<_\beta 0$. Then $-a\mathtt{x}^2>_\beta 0$, so $\mathtt{y}^{2m+1}\geqslant_\beta -a\mathtt{x}^2>_\beta 0$ and $\mathtt{y}>_\beta 0$. Let $\alpha\in\operatorname{Sper}(\kappa)$ be such that $\beta\to\alpha$ and let $\mathfrak{R}(\alpha)$ be the real closure of $(\kappa,\leqslant_\alpha)$. As $\mathtt{y}>_\beta 0$, we have $\mathtt{y}\notin\operatorname{supp}(\beta)$, so $\operatorname{supp}(\beta)\neq\mathfrak{m}_2$ is a prime ideal of height $\leqslant 1$. By Lemma 5.19 we may assume $\operatorname{ht}(\operatorname{supp}(\beta))=1$.

We have $-a>_\alpha 0$. By Lemma 5.21 there exists a homomorphism $\phi:\kappa[[\mathtt{x},\mathtt{y}]]\to\mathfrak{R}(\alpha)[[\mathtt{y}]]$ such that $\phi(\mathtt{y})=\mathtt{y}^q$ for some $q\geqslant 1$ and $h\geqslant_\beta 0$ if and only if $\phi(h)=h(\phi(\mathtt{x}),\mathtt{y}^q)\geqslant 0$. In addition, $h\in\operatorname{supp}(\beta)$ if and only if $h(\phi(\mathtt{x}),\mathtt{y}^q)=0$.

As $a\mathtt{x}^2+\mathtt{y}^{2m+1}\geqslant_\beta 0$, then $a\phi(\mathtt{x})^2+\mathtt{y}^{(2m+1)q}\geqslant 0$. As $-a>_\alpha 0$, we deduce that $2\omega(\phi(\mathtt{x}))=\omega(\phi(\mathtt{x})^2)\geqslant\omega(\mathtt{y}^{(2m+1)q})=(2m+1)q$, so $\omega(\phi(\mathtt{x}))\geqslant(m+\frac{1}{2})q>mq\geqslant 2nq$.

Write $f:=\mathtt{x}f_1(\mathtt{x},\mathtt{y})+f(0,\mathtt{y})$ and observe that

$$\begin{aligned}\phi(f+\mathtt{x}^{2n}+\mathtt{y}^{2n})=\phi(\mathtt{x})f_1(\phi(\mathtt{x}),\mathtt{y}^q)+f(0,\mathtt{y}^q)+\phi(\mathtt{x})^{2n}+\mathtt{y}^{2nq}\\ =\phi(\mathtt{x})(f_1(\phi(\mathtt{x}),\mathtt{y}^q)+\phi(\mathtt{x})^{2n-1})+f(0,\mathtt{y}^q)+\mathtt{y}^{2nq}.\end{aligned}$$

As $\omega(\phi(\mathtt{x}))>2nq=\omega(\mathtt{y}^{2nq})$, we deduce that $\phi(f+\mathtt{x}^{2n}+\mathtt{y}^{2n})\geqslant_\beta 0$ because $\omega(f(0,\mathtt{y}^q)+\mathtt{y}^{2nq})\leqslant 2nq$ and $f(0,\mathtt{y}^q)+\mathtt{y}^{2nq}\in\Sigma\kappa[[\mathtt{y}]]^2$.

We conclude $f+\mathtt{x}^{2n}+\mathtt{y}^{2n}\in\mathcal{P}(\{\{a\mathtt{x}^2+\mathtt{y}^{2m+1}\geqslant 0\}\})$ for each $m\geqslant 2n$, as required. □

main2 **Theorem 9.10** (Case (3.iii) of List 2.1)**.** *Let κ be a (formally) real field such that $\tau(\kappa)<+\infty$. Let $A:=\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-a\mathtt{x}^2)$ where $a\notin-\Sigma\kappa^2$ and $k\geqslant 1$. Then $\mathcal{P}(A)=\Sigma A^2$.*

Proof. For each $k\geqslant 1$ denote $A_k:=\kappa[\mathtt{x},\mathtt{y},\mathtt{z}]/(\mathtt{z}^2-a\mathtt{x}^2-\mathtt{y}^{2k+1})$ for each $k\geqslant 1$. Let $f+\mathtt{z}g\in\mathcal{P}(A)$ and let $k\geqslant 1$. By Lemma 9.9 $f+(\mathtt{x}^{2k}+\mathtt{y}^{2k})+\mathtt{z}g\in\mathcal{P}(A_{2k})$. By Theorem 9.8 $f+(\mathtt{x}^{2k}+\mathtt{y}^{2k})+\mathtt{z}g\in\Sigma A_{2k}^2$. As this holds for each $k\geqslant 1$ and $p(A)<+\infty$ (Theorem 1.7), we deduce by Theorem 5.6 that $f+\mathtt{z}g\in\Sigma A^2$, as required. □

We finally prove Theorem 2.3 for the remaining case (3.i) of List 2.1. We will take advantage of Theorems 8.7 and 9.8.

main3 **Theorem 9.11** (Case (3.i) of List 2.1)**.** *Let κ be a (formally) real field such that $\tau(\kappa)<+\infty$. Let $A:=\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-a\mathtt{x}^2-b\mathtt{y}^{2k})$ where $a\notin-\Sigma\kappa^2$, $b\neq 0$ and $k\geqslant 1$. Then $\mathcal{P}(A)=\Sigma A^2$.*

Proof. Let $f+\mathtt{z}g\in\mathcal{P}(A)\setminus\{0\}$. As $-F\notin\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$ (because $a\notin-\Sigma\kappa^2$), we deduce by Lemma 6.4 $f\neq 0$.

By Lemma 9.7 there exist $s\geqslant 1$ and $f_1,g_1\in\kappa[[\mathtt{x},\mathtt{y}]]$ such that $f_1(\mathtt{x},0)\neq 0$ and $f+\mathtt{z}g=\mathtt{y}^{2s}(f_1+\mathtt{z}g_1)$. Denote $q:=\omega(f_1(\mathtt{x},0))$ and consider $\varphi_q:=(\mathtt{x},\mathtt{u}\mathtt{x}^q,\mathtt{v})$. We have $(\mathtt{z}^2-F)(\varphi_q)=(\mathtt{z}^2-F)(\mathtt{x},\mathtt{u}\mathtt{x}^q,\mathtt{v})=\mathtt{y}^{2k}(\mathtt{v}^2-a\mathtt{x}^2-b\mathtt{u}^{2k}\mathtt{x}^{2kq})$. Define $A':=\kappa[[\mathtt{x},\mathtt{u},\mathtt{v}]]/(\mathtt{v}^2-a\mathtt{x}^2(1+b\mathtt{u}^{2k}\mathtt{x}^{2kq-2}))$. As $f_1+\mathtt{z}g_1\in\mathcal{P}(A)$, we have $f_1(\mathtt{x},\mathtt{u}\mathtt{x}^q)+\mathtt{v}g_1(\mathtt{x},\mathtt{u}\mathtt{x}^q)\in\mathcal{P}(A')$. Define $W:=1+b\mathtt{u}^{2k}\mathtt{x}^{2kq-2}$. As $W(0,0)=1$, there exists $V\in\kappa[[\mathtt{x},\mathtt{u}]]$ such that $V(0,0)=1$ and $V^2=W$. Consider the isomorphism $\psi_1:\kappa[[\mathtt{x},\mathtt{u},\mathtt{v}]]\to\kappa[[\mathtt{s},\mathtt{u},\mathtt{v}]]$, $(\mathtt{x}V,\mathtt{u},\mathtt{v})\mapsto(\mathtt{s},\mathtt{u},\mathtt{v})$. The previous isomorphism induces an isomorphism $\overline{\psi}_1$ between A' and $A'':=\kappa[[\mathtt{s},\mathtt{y},\mathtt{v}]]/(\mathtt{v}^2-a\mathtt{s}^2)$. Thus, $\overline{\psi}_1(f_1(\mathtt{x},\mathtt{u}\mathtt{x}^q)+\mathtt{v}g_1(\mathtt{x},\mathtt{u}\mathtt{x}^q))\in\mathcal{P}(A'')=\Sigma A''^2$ (Theorem 9.10) and as $\overline{\psi}_1$ is an isomorphism, we find series $a_i,b_i,c\in\kappa[[\mathtt{x},\mathtt{u}]]$ satisfying

$$f_1(\mathtt{x},\mathtt{u}\mathtt{x}^q)+\mathtt{v}g_1(\mathtt{x},\mathtt{u}\mathtt{x}^q)=\sum_{i=1}^p(a_i+\mathtt{v}b_i)^2-(\mathtt{v}^2-a\mathtt{x}^2W)c.$$

Consequently,

$$(f_1+\mathtt{z}g_1)(\varphi_q)=\sum_{i=1}^p(a_i+\mathtt{v}b_i)^2-(\mathtt{z}^2-F)(\varphi_q).$$

By Theorem 8.6(ii) we deduce that $f_1+\mathtt{z}g_1\in\Sigma A^2$, so $f+\mathtt{z}g=\mathtt{y}^{2s}(f_1+\mathtt{z}g_1)\in\Sigma A^2$, as required. □

9.4.2. *Order three cases.* We are ready to revisit Theorem 2.3 for the rings $A=\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F)$ when $F\in\kappa[[\mathtt{x},\mathtt{y}]]$ is one of the series (3.iv-3.ix) of List 2.1. For the sake of completeness we borrow the following result from [Fe8, Thm.5.9] in order to provide a complete proof of Theorem 2.3. Recall that by Remarks 5.5 (ii.2) a series $b:=\sum_{k\geqslant\ell}b_k\mathtt{t}^k\in\kappa[[\mathtt{t}]]$ with $b_\ell\neq 0$ does not belong to $\Sigma\kappa((\mathtt{t}))^2$ if and only if either ℓ is odd or ℓ is even and $b_\ell\notin\Sigma\kappa^2$.

order3 **Theorem 9.12** (Order three: Cases (3.iv)-(3.ix) of List 2.1)**.** *Let κ be a (formally) real field such that $\tau(\kappa)<+\infty$. Denote $A:=\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F)$ where $F\in\kappa[[\mathtt{x},\mathtt{y}]]$ is one of the series* (3.iv) *to* (3.ix) *of List* 2.1*. Then $\mathcal{P}(A)=\Sigma A^2$.*

Proof. The general strategy to prove this result (except for case (v)) is to show that (maybe after a suitable change of coordinates when needed) there exists no positive semidefinite element $f+\mathtt{z}g\in\mathcal{P}(A)$ such that $f(\mathtt{x},0)$ has order 1, whereas $F(\mathtt{x},0)$ has order 3 and F satisfies the remaining hypotheses in the statement of Corollary 8.8 (for $k=1$). Once this is done, Corollary 8.8 (for $k=1$) applies and we conclude $\mathcal{P}(A)=\Sigma A^2$.

We use Lemma 5.14 freely along the proof. By Lemma 6.1 we do not have to care about positive semidefinite units of A.

(3.iv) This case is somehow special because with the current coordinates $F(\mathtt{x},0)=0$, so a suitable change of coordinates is needed to apply Corollary 8.8 (for $k=1$). We prove first: *if* $f+\mathtt{z}g\in\mathcal{P}(A)$*, then* $\omega(f)\geqslant 2$.

Suppose that $f+\mathtt{z}g\in\mathcal{P}(A)$ and $\omega(f)=1$. Observe that $f\in\mathcal{P}(\{F\geqslant 0\})$. If $\omega(f(\mathtt{x},0))=1$, there exist $\zeta\in\kappa[[\mathtt{y}]]$ of order $\geqslant 1$, $b\in\kappa\backslash\{0\}$ and a unit $U\in\kappa[[\mathtt{x},\mathtt{y}]]$ such that $U(0,0)=1$ and $f=(\mathtt{x}-\zeta(\mathtt{y}))bU^2$. Consider the homomorphism

$$\varphi:\kappa[[\mathtt{x},\mathtt{y}]]\to\kappa[[\mathtt{t}]],\ h\mapsto h(\zeta(\mathtt{t}^3)-b\mathtt{t}^2,\mathtt{t}^3).$$

We have $\varphi(F)=(\zeta(\mathtt{t}^3)-b\mathtt{t}^2)^2\mathtt{t}^3+(-1)^ka\mathtt{t}^{3k}$ and $\varphi(f)=-b^2\mathtt{t}^2\varphi(U)^2\in-\Sigma\kappa[[\mathtt{t}]]^2$. As $b\neq 0$ and $k\geqslant 3$, we deduce $\omega(\varphi(F))=7$ and $\varphi(F)=b^2\mathtt{t}^7+\cdots$. Fix an ordering $\alpha\in\operatorname{Sper}(\kappa)$ and let $\beta_0\in\operatorname{Sper}(\kappa[[\mathtt{t}]])$ be a prime cone that extends α such that $\mathtt{t}>_\beta 0$. Thus, $\varphi(F)>_{\beta_0}0$, whereas $\varphi(f)=-b^2\mathtt{t}^2\varphi(U)^2<_{\beta_0}0$, which is a contradiction because $f\in\mathcal{P}(\{F\geqslant 0\})$. Consequently, $\omega(f(\mathtt{x},0))\geqslant 2$.

Assume next $\omega(f(\mathtt{x},0))\geqslant 2$ and $\omega(f(0,\mathtt{y}))=1$. There exist $\xi:=\sum_{\ell\geqslant 2}\xi_\ell\mathtt{x}^\ell\in\kappa[[\mathtt{x}]]$ of order $\geqslant 2$, $c\in\kappa\backslash\{0\}$ and a unit $V\in\kappa[[\mathtt{x},\mathtt{y}]]$ such that $V(0,0)=1$ and $f=(\mathtt{y}-\xi(\mathtt{x}))cV^2$. Let $\alpha\in\operatorname{Sper}(\kappa)$ and $M:=\xi_2^2+1$. Consider the homomorphism

$$\phi:\kappa[[\mathtt{x},\mathtt{y}]]\to\kappa[[\mathtt{t}]],\ h\mapsto h(\mathtt{t},M\mathtt{t}^2).$$

We have $\phi(F)=M\mathtt{t}^4-a(-1)^k\mathtt{t}^{2k}>0$ because $k\geqslant 3$, so $\phi(f)=c(M\mathtt{t}^2-\xi(\mathtt{t}))\phi(V)^2>0$. As $M>\xi_2$, we deduce $c>_\alpha 0$. As this holds for each $\alpha\in\operatorname{Sper}(\kappa)$, we conclude $c\in\Sigma\kappa^2\backslash\{0\}$.

Thus, we may assume $f=\mathtt{y}-\xi(\mathtt{x})$. As the coefficient $a\notin-\Sigma\kappa^2$, there exists $\alpha\in\operatorname{Sper}(\kappa)$ such that $a>_\alpha 0$. Consider the homomorphism

$$\psi:\kappa[[\mathtt{x},\mathtt{y}]]\to\kappa[[\mathtt{t}]],\ h\mapsto h(\mathtt{t}^k,-\mathtt{t}^2).$$

We have $\psi(F)=-\mathtt{t}^{2k+2}+a\mathtt{t}^{2k}>0$ and $\psi(f)=-\mathtt{t}^2-\xi(\mathtt{t}^k)<0$ because $k\geqslant 3$. Thus, there exists a prime cone $\beta_1\in\operatorname{Sper}(\kappa[[\mathtt{t}]])$ extending α such that $\psi(F)>_{\beta_1}0$, whereas $\psi(f)<_{\beta_1}0$, which is a contradiction because $f\in\mathcal{P}(\{F\geqslant 0\})$.

Consequently, $\omega(f)\geqslant 2$. By Example 5.8(ii) we know that F is k-determined. After a linear change of coordinates $\mathtt{y}\mapsto\lambda\mathtt{x}+\mathtt{y}$ (for some $\lambda\in\kappa$) the series F becomes $F_\lambda(\mathtt{x},\mathtt{y})=\mathtt{x}^2(\lambda\mathtt{x}+\mathtt{y})+(-1)^ka(\lambda\mathtt{x}+\mathtt{y})^k$ and f becomes $f_\lambda(\mathtt{x},\mathtt{y})=f(\mathtt{x},\lambda\mathtt{x}+\mathtt{y})$. It holds $F_\lambda(\mathtt{x},0)=\lambda\mathtt{x}^3+(-1)^ka\lambda^k\mathtt{x}^k$, $F(0,\mathtt{y})=(-1)^ka\mathtt{y}^k$ and $f_\lambda(\mathtt{x},0)=f(\mathtt{x},\lambda\mathtt{x})$. We choose $\lambda\in\kappa$ such that $\lambda(1-a\lambda^2)\neq 0$ and $\omega(f(\mathtt{x},\lambda\mathtt{x}))=\omega(f)\geqslant 2$ (in both cases $\omega(F_\lambda(\mathtt{x},0))=3$, $\omega(F_\lambda(0,\mathtt{y}))=k$, $f_\lambda(\mathtt{x},0)\neq 0$ and $\omega(f_\lambda(\mathtt{x},0))=\omega(f_\lambda)\geqslant 2$). Observe that

$$-F(0,\mathtt{y})=(-1)^{k+1}a\mathtt{y}^k=\begin{cases}a\mathtt{y}^k & \text{if } k \text{ is odd},\\ -a\mathtt{y}^k & \text{if } k \text{ is even},\end{cases}$$

so $-F(0,\mathtt{y})\notin\Sigma\kappa((\mathtt{y}))^2$ (because $a\notin-\Sigma\kappa^2$).

(3.v) For this case we use a 'limit argument'. Let $f+\mathtt{z}g\in\mathcal{P}(A)$. We may assume $g\neq 0$ (Lemma 6.6(iv)). Then by (6.1)

$$f\in\mathcal{P}(\{\mathtt{x}^2\mathtt{y}\geqslant 0\})\subset\mathcal{P}(\{\mathtt{y}\geqslant 0\}),$$
$$f^2-\mathtt{x}^2\mathtt{y}g^2\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]).$$

By Lemma 6.6 if n is large enough, then $f+(\mathtt{x}^2+\mathtt{y}^2)^n+\mathtt{z}g\in\mathcal{P}^{\oplus}(A)$, that is, by §6.2

$$f+(\mathtt{x}^2+\mathtt{y}^2)^n\in\mathcal{P}^+(\{\mathtt{x}^2\mathtt{y}\geqslant 0\})\subset\mathcal{P}(\{\mathtt{y}\geqslant 0\}),$$
$$(f+(\mathtt{x}^2+\mathtt{y}^2)^n)^2-\mathtt{x}^2\mathtt{y}g^2\in\mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{y}]]).$$

By Corollary 6.5 we deduce that if $r\geqslant 1$ is large enough, then

$$(f+(\mathtt{x}^2+\mathtt{y}^2)^n)-(\mathtt{x}^2\mathtt{y}+\mathtt{y}^{2r})g^2\in\mathcal{P}^+(\kappa[[\mathtt{x},\mathtt{y}]]).$$

Let us check: *if* $r\geqslant 2n+2$ *is large enough, then* $f+(\mathtt{x}^2+\mathtt{y}^2)^n\in\mathcal{P}^+(\{\mathtt{x}^2\mathtt{y}+\mathtt{y}^{2r}\geqslant 0\})$.

As $f+(\mathtt{x}^2+\mathtt{y}^2)^n \in \mathcal{P}^+(\{\mathtt{x}^2\mathtt{y} \geqslant 0\})$, the series $f+(\mathtt{x}^2+\mathtt{y}^2)^n$ is positive for each $\gamma \in \operatorname{Sper}(\kappa[[\mathtt{x},\mathtt{y}]])$ such that $\operatorname{supp}(\gamma) = (\mathtt{x})$. This means that there exist $1 \leqslant s \leqslant n$, $c \in \Sigma\kappa^2\backslash\{0\}$ and a unit $u \in \kappa[[\mathtt{y}]]$ such that $u(0) = 1$ and $f(0,\mathtt{y}) + \mathtt{y}^{2n} = \mathtt{y}^{2s}cu^2$. Let $\beta \in \operatorname{Sper}(\kappa[[\mathtt{x},\mathtt{y}]])$ be such that $\mathtt{y}(\mathtt{x}^2 + \mathtt{y}^{2r-1}) \geqslant_\beta 0$. If $\mathtt{y} \geqslant_\beta 0$, then $\mathtt{x}^2\mathtt{y} \geqslant_\beta 0$, so $f \geqslant_\beta 0$ and $f + (\mathtt{x}^2 + \mathtt{y}^2)^n \geqslant_\beta 0$. If $\mathtt{y} <_\beta 0$, then $\mathtt{x}^2 + \mathtt{y}^{2r-1} \leqslant_\beta 0$ and $\mathtt{y} \notin \operatorname{supp}(\beta)$. Thus, as $2r-1 \geqslant 4n+3$,

$$\mathtt{x}^2 \leqslant_\beta -\mathtt{y}^{2r-1} \leqslant_\beta \mathtt{y}^{4n+2} \leqslant_\beta \mathtt{y}^{4s+2}.$$

As $\mathtt{y} <_\beta 0$, then $|\mathtt{x}|_\beta <_\beta -\mathtt{y}^{2s+1}$ where

$$|h|_\beta := \begin{cases} h & \text{if } h \geqslant_\beta 0, \\ -h & \text{if } h <_\beta 0 \end{cases}$$

for each $h \in \kappa[[\mathtt{x},\mathtt{y}]]$. Write

$$f + (\mathtt{x}^2+\mathtt{y}^2)^n = f(0,\mathtt{y}) + \mathtt{y}^{2n} + \mathtt{x}\xi = \mathtt{y}^{2s}cu^2 + \mathtt{x}\xi$$

where $\xi \in \kappa[[\mathtt{x},\mathtt{y}]]$. We have

$$f + (\mathtt{x}^2+\mathtt{y}^2)^n = \mathtt{y}^{2s}cu^2 + \mathtt{x}\xi \geqslant_\beta \mathtt{y}^{2s}cu^2 - |\mathtt{x}|_\beta|\xi|_\beta \geqslant \mathtt{y}^{2s}cu^2 + \mathtt{y}^{2s+1}(1+\xi^2(0,0)) >_\beta 0.$$

Thus, $f + (\mathtt{x}^2+\mathtt{y}^2)^n \in \mathcal{P}^+(\{\mathtt{x}^2\mathtt{y} + \mathtt{y}^{2r} \geqslant 0\})$, so $f + (\mathtt{x}^2+\mathtt{y}^2)^n + \mathtt{z}g \in \mathcal{P}^{\oplus}(A_r)$ where $A_r := \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2 - \mathtt{x}^2\mathtt{y} - \mathtt{y}^{2r})$. As $\mathcal{P}(A_r) = \Sigma A_r^2$ and $p := p(A_r) < +\infty$, there exist $a_{in}, b_{in}, q_n \in \kappa[[\mathtt{x},\mathtt{y}]]$ such that:

$$f + (\mathtt{x}^{2n}+\mathtt{y}^{2n}) + \mathtt{z}g = (a_{1n}+\mathtt{z}b_{1n})^2 + \cdots + (a_{pn}+\mathtt{z}b_{pn})^2 - (\mathtt{z}^2 - \mathtt{x}^2\mathtt{y} - \mathtt{y}^{2r})q_n.$$

Consequently,

$$f + \mathtt{z}g = (a_{1n}+\mathtt{z}b_{1n})^2 + \cdots + (a_{pn}+\mathtt{z}b_{pn})^2 - (\mathtt{z}^2 - \mathtt{x}^2\mathtt{y})q_n \mod \mathfrak{m}_2^{2n}.$$

By Strong Artin's approximation there exist $a_i, b_i, q \in \kappa[[\mathtt{x},\mathtt{y}]]$ such that

$$f + \mathtt{z}g = (a_1+\mathtt{z}b_1)^2 + \cdots + (a_p+\mathtt{z}b_p)^2 - (\mathtt{z}^2 - \mathtt{x}^2\mathtt{y})q,$$

that is, $f + \mathtt{z}g \in \Sigma A^2$.

(3.vi) In this case $F(\mathtt{x},0) = \mathtt{x}^3$, $-F(0,\mathtt{y}) = -b\mathtt{y}^3 \notin \Sigma\kappa((\mathtt{y}))^2$ and F is 3-determined (Example 5.8(iii)). We have to prove: *if $f + \mathtt{z}g \in \mathcal{P}(A)$ and $f(\mathtt{x},0) \neq 0$, then $\omega(f(\mathtt{x},0)) \geqslant 2$.*

Assume $\omega(f(\mathtt{x},0)) = 1$. There exist $\zeta \in \kappa[[\mathtt{y}]]$ of order $\geqslant 1$, $c \in \kappa\backslash\{0\}$ and a unit $U \in \kappa[[\mathtt{x},\mathtt{y}]]$ such that $U(0,0) = 1$ and $f = (\mathtt{x} - \zeta(\mathtt{y}))cU^2$. Consider the homomorphism

$$\varphi : \kappa[[\mathtt{x},\mathtt{y}]] \to \kappa[[\mathtt{t}]],\ h \mapsto h(\zeta(\mathtt{t}) - c\mathtt{t}^2, \mathtt{t}).$$

We have $\varphi(F) = (\zeta(\mathtt{t}) - c\mathtt{t}^2)^3 + a\mathtt{t}^2(\zeta(\mathtt{t}) - c\mathtt{t}^2) + b\mathtt{t}^3$ and $\varphi(f) = -c^2\mathtt{t}^2U^2(\zeta(\mathtt{t}) - c\mathtt{t}^2, \mathtt{t})$. As $b \neq 0$, we deduce

$$\omega((\zeta(\mathtt{t}) - c\mathtt{t}^2)^3 + a\mathtt{t}^2(\zeta(\mathtt{t}) - c\mathtt{t}^2) + b\mathtt{t}^3) = \begin{cases} \omega(F(d,1)\mathtt{t}^3 + \cdots) = 3 & \text{if } \omega(\zeta) = 1, \\ \omega(b\mathtt{t}^3 + \cdots) = 3 & \text{if } \omega(\zeta) \geqslant 2, \end{cases}$$

for some $d \in \kappa\backslash\{0\}$. We only have to explain the first row. As $\zeta \in \kappa[[\mathtt{t}]]$ and $\omega(\zeta) = 1$, then $\zeta = d\mathtt{t} + \cdots$, so

$$\varphi(F) = (\zeta(\mathtt{t}) - c\mathtt{t}^2)^3 + a\mathtt{t}^2(\zeta(\mathtt{t}) - c\mathtt{t}^2) + b\mathtt{t}^3 = (d^3 + ad + b)\mathtt{t}^3 + \cdots = F(d,1)\mathtt{t}^3 + \cdots.$$

As $F(\mathtt{x},1) \in \kappa[\mathtt{x}]$ is an irreducible polynomial, $F(d,1) \neq 0$ and $\varphi(F)$ has order 3.

Fix an ordering α of κ and let $\beta \in \operatorname{Sper}(\kappa[[\mathtt{t}]])$ be a prime cone that extends α such that $F(d,1)\mathtt{t} > 0$ in the first case and $b\mathtt{t} > 0$ in the second case. Thus, $\varphi(F) >_\beta 0$, whereas $\varphi(f) = -c^2\mathtt{t}^2U^2(\zeta(\mathtt{t}) - c\mathtt{t}^2, \mathtt{t}) <_\beta 0$. This is a contradiction because $f \in \mathcal{P}(\{F \geqslant 0\})$. Consequently, $\omega(f(\mathtt{x},0)) \geqslant 2$.

(3.vii) In this case $F(\mathtt{x},0) = \mathtt{x}^3$, $-F(0,\mathtt{y}) = -a\mathtt{y}^4 \notin \Sigma\kappa((\mathtt{y}))^2$ (because $a \notin -\Sigma\kappa^2$) and F is 4-determined (Example 5.8(v)). We have to prove: *if $f + \mathtt{z}g \in \mathcal{P}(A)$ and $f(\mathtt{x},0) \neq 0$, then $\omega(f(\mathtt{x},0)) \geqslant 2$.*

Assume $\omega(f(\mathtt{x},0))=1$. There exists $\zeta\in\kappa[[\mathtt{y}]]$ of order $\geqslant 1$, $b\in\kappa\backslash\{0\}$ and a unit $U\in\kappa[[\mathtt{x},\mathtt{y}]]$ such that $U(0,0)=0$ and $f=(\mathtt{x}-\zeta(\mathtt{y}))bU^2$. As $a\notin-\Sigma\kappa^2$, there exists an ordering $\alpha\in\operatorname{Sper}(\kappa)$ such that $a>_\alpha 0$. Consider the homomorphism

$$\varphi:\kappa[[\mathtt{x},\mathtt{y}]]\to\kappa[[\mathtt{t}]],\ h\mapsto h(\zeta(\mathtt{t})-b\mathtt{t}^2,\mathtt{t}).$$

We have $\varphi(F)=(\zeta(\mathtt{t})-b\mathtt{t}^2)^3+a\mathtt{t}^4$ and $\varphi(f)=-b^2\mathtt{t}^2\varphi(U)^2\in-\Sigma\kappa[[\mathtt{t}]]^2$. In addition,

$$\omega((\zeta(\mathtt{t})-b\mathtt{t}^2)^3+a\mathtt{t}^4)=\begin{cases}3 & \text{if } \omega(\zeta)=1,\\ 4 & \text{if } \omega(\zeta)\geqslant 2.\end{cases}$$

Let $\beta\in\operatorname{Sper}(\kappa[[\mathtt{t}]])$ be a prime cone that extends α and satisfies $(\zeta(\mathtt{t})-b\mathtt{t}^2)^3+a\mathtt{t}^4>_\beta 0$. Observe that $\varphi(f)=-b^2\mathtt{t}^2\varphi(U)^2<_\beta 0$. This is a contradiction because $f\in\mathcal{P}(\{F\geqslant 0\})$. Consequently, $\omega(f(\mathtt{x},0))\geqslant 2$.

(3.viii) We prove first: *if $f+\mathtt{z}g\in\mathcal{P}(A)$, then $\omega(f(\mathtt{x},0))\geqslant 2$.*

Suppose that $f+\mathtt{z}g\in\mathcal{P}(A)$ and $\omega(f(\mathtt{x},0))=1$. There exist $\zeta\in\kappa[[\mathtt{y}]]$ of order $\geqslant 1$, $a\in\kappa\backslash\{0\}$ and a unit $U\in\kappa[[\mathtt{x},\mathtt{y}]]$ such that $U(0,0)=1$ and $f=(\mathtt{x}+\zeta(\mathtt{y}))aU^2$. Let $\alpha\in\operatorname{Sper}(\kappa)$ and consider the homomorphism

$$\varphi:\kappa[[\mathtt{x},\mathtt{y}]]\to\kappa[[\mathtt{t}]],\ h\mapsto h(\mathtt{t}^2,0).$$

We have $\varphi(F)=\mathtt{t}^6>0$, so $\varphi(f)=a\mathtt{t}^2\phi(U)^2>0$. This means that $a>_\alpha 0$ and as this holds for each $\alpha\in\operatorname{Sper}(\kappa)$, we conclude $a\in\Sigma\kappa^2\backslash\{0\}$. Thus, we may assume $f=\mathtt{x}+\zeta(\mathtt{y})$. Consider the homomorphism

$$\varphi:\kappa[[\mathtt{x},\mathtt{y}]]\to\kappa[[\mathtt{t}]],\ h\mapsto h(0,\mathtt{t}).$$

We have $\varphi(F)=0$ and $\varphi(f)=\zeta(\mathtt{t})$. As $f\in\mathcal{P}(\{F\geqslant 0\})$ and $\varphi(F)=0$, we deduce $\zeta(\mathtt{t})\in\mathcal{P}(\kappa[[\mathtt{t}]])=\Sigma\kappa[[\mathtt{t}]]^2$. In particular, $\omega(\zeta(\mathtt{t}))\geqslant 2$ and we choose $M:=\zeta_2^2+1$ where $\zeta(\mathtt{t}):=\sum_{k\geqslant 2}\zeta_k\mathtt{t}^k$. Consider the homomorphism

$$\varphi:\kappa[[\mathtt{x},\mathtt{y}]]\to\kappa[[\mathtt{t}]],\ h\mapsto h(-M\mathtt{t}^2,-\mathtt{t}).$$

Choose $\beta\in\operatorname{Sper}(\kappa[[\mathtt{t}]])$ such that $\mathtt{t}>_\beta 0$. Then $\varphi(F)=-M^3\mathtt{t}^6+M\mathtt{t}^5>_\beta 0$ and $\varphi(f)=-M\mathtt{t}^2+\zeta(\mathtt{t})<_\beta 0$ because $M>_\beta\zeta_2$. This is a contradiction because $f\in\mathcal{P}(\{F\geqslant 0\})$. Consequently, $\omega(f(\mathtt{x},0))\geqslant 2$.

By Example 5.8(iv) F is 5-determined. After the change of coordinates $\mathtt{x}\mapsto\mathtt{x}+\mathtt{y}^2$ the series F becomes $F(\mathtt{x},\mathtt{y})=(\mathtt{x}+\mathtt{y}^2)^3+(\mathtt{x}+\mathtt{y}^2)\mathtt{y}^3$ and satisfies $F(\mathtt{x},0)=\mathtt{x}^3$ and $-F(0,\mathtt{y})=-\mathtt{y}^5-\mathtt{y}^6\notin\Sigma\kappa((\mathtt{y}))^2$. As we have already proved, there exists no element $f+\mathtt{z}g\in\mathcal{P}(A)$ such that $\omega(f(\mathtt{x},0))=1$ (and this enough in our situation, because the performed change of coordinates was $\mathtt{x}\mapsto\mathtt{x}+\mathtt{y}^2$).

(3.ix) In this case $F(\mathtt{x},0)=\mathtt{x}^3$, $-F(0,\mathtt{y})=-\mathtt{y}^5\notin\Sigma\kappa((\mathtt{t}))^2$ and F is 5-determined (Example 5.8(v)). We have to prove: *if $f+\mathtt{z}g\in\mathcal{P}(A)$ and $f(\mathtt{x},0)\neq 0$, then $\omega(f(\mathtt{x},0))\geqslant 2$.*

Assume $\omega(f(\mathtt{x},0))=1$. There exists $\zeta\in\kappa[[\mathtt{y}]]$ of order $\geqslant 1$, $b\in\kappa\backslash\{0\}$ and a unit $U\in\kappa[[\mathtt{x},\mathtt{y}]]$ such that $U(0,0)=1$ and $f=(\mathtt{x}-\zeta(\mathtt{y}))bU^2$. Consider the homomorphism

$$\varphi:\kappa[[\mathtt{x},\mathtt{y}]]\to\kappa[[\mathtt{t}]],\ h\mapsto h(\zeta(\mathtt{t})-b\mathtt{t}^2,\mathtt{t}).$$

We have $\varphi(F)=(\zeta(\mathtt{t})-b\mathtt{t}^2)^3+\mathtt{t}^5$ and $\varphi(f)=-b^2\mathtt{t}^2\varphi(U)^2\in-\Sigma\kappa[[\mathtt{t}]]^2$. In addition,

$$\omega((\zeta(\mathtt{t})-b\mathtt{t}^2)^3+\mathtt{t}^5)=\begin{cases}3 & \text{if } \omega(\zeta)=1,\\ 5 & \text{if } \omega(\zeta)\geqslant 2.\end{cases}$$

Fix an ordering $\alpha\in\operatorname{Sper}(\kappa)$ and let $\beta\in\operatorname{Sper}(\kappa[[\mathtt{t}]])$ be a prime cone that extends α such that $(\zeta(\mathtt{t})-b\mathtt{t}^2)^3+\mathtt{t}^5>0$. Thus, $\varphi(F)>_\beta 0$, whereas $\varphi(f)=-b^2\mathtt{t}^2\varphi(U)^2<_\beta 0$. This is a contradiction because $f\in\mathcal{P}(\{F\geqslant 0\})$. Consequently, $\omega(f(\mathtt{x},0))\geqslant 2$, as required. $\square$

Part 3. The 1-dimensional case

s10

10. 1-dimensional rings with the property PSD=SOS

In this section we prove the equivalence between the statements of Main Theorem 3.1 and Theorem 3.2 and the ‘if part’ of Theorem 3.2. As usual we denote the algebraic closure of a field κ with $\overline{\kappa}$. We begin simplifying the generators of some ideals of $\kappa[[x,y,z]]$ in order to lighten the presentation along this section:

deg4bcd

Remarks 10.1. (i) Let $\mathfrak{a}=(y^2-zx, a_0x^2+a_1xy+a_2xz+yz, a_0xy+a_1xz+a_2yz+z^2)$ be an ideal of $\kappa[[x,y,z]]$. After the change of coordinates $(x,y,z)\mapsto(x, y-\frac{a_2}{3}x, z-\frac{2}{3}a_2y+\frac{1}{9}a_2^2x)$, we obtain

$$\begin{aligned}\mathfrak{a}=(y^2-xz,\,&(a_0-\tfrac{1}{3}a_1a_2+\tfrac{2}{27}a_2^3)x^2+(a_1-\tfrac{1}{3}a_2^2)xy+yz-\tfrac{2}{3}a_2(y^2-xz),\\ &(a_0-\tfrac{1}{3}a_1a_2+\tfrac{2}{27}a_2^3)xy+(a_1-\tfrac{1}{3}a_2^2)xz+z^2-\tfrac{2}{9}a_2^2(y^2-xz)\\ &-\tfrac{1}{3}a_2((a_0-\tfrac{1}{3}a_1a_2+\tfrac{2}{27}a_2^3)x^2+(a_1-\tfrac{1}{3}a_2^2)xy+yz)).\end{aligned}$$

Thus, $\mathfrak{a}=(y^2-xz, qx^2+pxy+yz, qxy+pxz+z^2)$ where $p:=a_1-\frac{1}{3}a_2^2$ and $q:=a_0-\frac{1}{3}a_1a_2+\frac{2}{27}a_2^3$. Observe that $P(t-\frac{a_2}{3})=t^3+pt^2+q$.

(ii) Let $\mathfrak{a}:=(y^2-zx, z^2+a_3yz+a_2xz+a_1xy+a_0x^2)$ be an ideal of $\kappa[[x,y,z]]$. After the change of coordinates $(x,y,z)\mapsto(x, y-\frac{a_3}{4}x, z+\frac{a_3}{16}(a_3x-8y))$, there exist

$$b_0:=-\frac{a_2}{2}+\frac{3}{16}a_3^2,\ c_0:=-\frac{a_1}{4}+\frac{1}{8}a_2a_3-\frac{1}{32}a_3^3,\ d_0:=a_0-\frac{1}{4}a_1a_3+\frac{1}{16}a_2a_3^2-\frac{3}{256}a_3^4$$

in κ such that $\mathfrak{a}=(y^2-zx, z^2-2b_0xz-4c_0xy+d_0x^2)$. Observe that if $P:=t^4+a_3t^3+a_2t^2+a_1t+a_0$, then $P_0:=P(t-\frac{a_3}{4})=t^4-2b_0t^2-4c_0t+d_0$. If $c_0=0$, denote $b:=b_0$ and $d:=d_0$. If $c_0\neq 0$, after the additional change of coordinate $(x,y,z)\mapsto(x, c_0y, c_0^2z)$, we have $\mathfrak{a}=(y^2-zx, z^2-2bxz-4c^2xy+dx^2)$ where $b:=\frac{b_0}{c_0^2}$, $c:=\frac{1}{c_0}$ and $d:=\frac{d_0}{c_0^4}$. In addition, $Q:=\frac{1}{c_0^4}P_0(c_0t)=t^4-2bt^2-4c^2t+d$. ■

We prove next that the statements of Main Theorem 3.1 and Theorem 3.2 are equivalent:

equivm

10.1. Equivalence between the statements of Main Theorem 3.1 and Theorem 3.2.

We will use freely that as $\overline{\kappa}[[x_1,\dots,x_n]]$ is a $\kappa[[x_1,\dots,x_n]]$-module, we have

$$(\kappa[[x_1,\dots,x_n]]/\mathfrak{a})\otimes_{\kappa[[x_1,\dots,x_n]]}\overline{\kappa}[[x_1,\dots,x_n]]\cong\overline{\kappa}[[x_1,\dots,x_n]]/\mathfrak{a}\overline{\kappa}[[x_1,\dots,x_n]].$$

for each ideal $\mathfrak{a}\subset\kappa[[x_1,\dots,x_n]]$ (see [AM, Ch.2.Ex.2, pag.31]).

Recall that by [ABR, Thm.VII.3.2] A is a real (reduced) ring if and only if the completion $\widehat{A}$ is a real (reduced) ring. As $\widehat{A}\cong\kappa[[x,y,z]]/\mathfrak{a}$ (for some (formally) real field κ), we deduce that $\widehat{A}$ is real (reduced) if and only if $\mathfrak{a}$ is a real ideal. Consider the following type ideals:

(i) $\mathfrak{a}:=(y,z)$.
(ii) $\mathfrak{a}:=(y^2-ax^2,z)$ for some $a\notin-\Sigma\kappa^2$.
(iii) $\mathfrak{a}:=(y^2-ax^2,xz,yz)$ for some $a\notin-\Sigma\kappa^2$.
(iv) $\mathfrak{a}:=(y^2-xz, yz+pyx+qx^2, qxy+pxz+z^2)$ where the polynomial $P:=t^3+pt+q\in\kappa[t]$ is irreducible.

It holds that the previous ideals are real, so the quotient rings $\kappa[[x,y,z]]/\mathfrak{a}$ are real (reduced). In addition, $(\kappa[[x,y,z]]/\mathfrak{a})\otimes_{\kappa[[x,y,z]]}\overline{\kappa}[[x,y,z]]\cong\overline{\kappa}[[x,y,z]]/\mathfrak{a}\overline{\kappa}[[x,y,z]]$. In case (i) $\widehat{A}\cong\kappa[[x]]$ and $\widehat{A}\otimes_{\kappa[[x]]}\overline{\kappa}[[x]]\cong\overline{\kappa}[[x]]$. In case (ii) $\widehat{A}\cong\kappa[[x,y]]/(y^2-ax^2)$ and $\widehat{A}\otimes_{\kappa[[x,y]]}\overline{\kappa}[[x,y]]\cong\overline{\kappa}[[w_1,w_2]]/(w_1w_2)$ because $y^2-ax^2=(y-\sqrt{a}x)(y+\sqrt{a}x)$. In case (iii) $\widehat{A}\otimes_{\kappa[[x,y,z]]}\overline{\kappa}[[x,y,z]]\cong\overline{\kappa}[[w_1,w_2,w_3]]/(w_1w_2,w_1w_3,w_2w_2)$ because $y^2-ax^2=(y-\sqrt{a}x)(y+\sqrt{a}x)$, $xz+\sqrt{a}yz=(x+$

$\sqrt{a}\mathtt{y})\mathtt{z}$ and $\mathtt{xz}-\sqrt{a}\mathtt{yz}=(\mathtt{x}-\sqrt{a}\mathtt{y})\mathtt{z}$. In case (iv) let us prove $\widehat{A}\otimes_{\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]}\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]\cong\overline{\kappa}[[\mathtt{w}_1,\mathtt{w}_2,\mathtt{w}_3]]/(\mathtt{w}_1\mathtt{w}_2,\mathtt{w}_1\mathtt{w}_3,\mathtt{w}_2\mathtt{w}_3)$.

Let $\xi_1,\xi_2,\xi_3\in\overline{\kappa}$ be the roots of P (which are pairwise different because $P\in\kappa[\mathtt{t}]$ is irreducible). Observe that $(\mathtt{w}_1\mathtt{w}_2,\mathtt{w}_1\mathtt{w}_3,\mathtt{w}_2\mathtt{w}_3)$ is the zero ideal corresponding to the set constituted by the projective points $[1:0:0],[0:1:0],[0:0:1]\in\overline{\kappa}\mathbb{P}^2$. We make a change of coordinates to transform the previous points onto the independent projective points $[1:\xi_1:\xi_1^2],[1:\xi_2:\xi_2^2],[1:\xi_3:\xi_3^2]$ and we obtain the ideal (L_1L_2,L_1L_3,L_2L_3) where

$$\begin{aligned}L_1&:=\xi_1\xi_2\mathtt{x}-\xi_1\mathtt{y}-\xi_2\mathtt{y}+\mathtt{z},\\ L_2&:=\xi_2\xi_3\mathtt{x}-\xi_2\mathtt{y}-\xi_3\mathtt{y}+\mathtt{z},\\ L_3&:=\xi_1\xi_3\mathtt{x}-\xi_1\mathtt{y}-\xi_3\mathtt{y}+\mathtt{z}.\end{aligned}$$

Write $\delta:=(\xi_1-\xi_2)(\xi_1-\xi_3)(\xi_2-\xi_3)\neq0$, which is a square root of the discriminant $\Delta(P)$ of the irreducible P. The reader can check that

$$\begin{aligned}\delta(\mathtt{y}^2-\mathtt{xz})&=(\xi_3-\xi_1)L_1L_2+(\xi_2-\xi_3)L_1L_3+(\xi_1-\xi_2)L_2L_3,\\ \delta(\mathtt{yz}+p\mathtt{yx}+q\mathtt{x}^2)&=(\xi_3^2-\xi_1^2)L_1L_2+(\xi_2^2-\xi_3^2)L_1L_3+(\xi_1^2-\xi_2^2)L_2L_3,\\ \delta(q\mathtt{xy}+p\mathtt{xz}+\mathtt{z}^2)&=-\xi_2(\xi_3^2-\xi_1^2)L_1L_2-\xi_1(\xi_2^2-\xi_3^2)L_1L_3-\xi_3(\xi_1^2-\xi_2^2)L_2L_3\end{aligned}$$

and that the matrix

$$\begin{pmatrix}\xi_3-\xi_1&\xi_2-\xi_3&\xi_1-\xi_2\\ \xi_3^2-\xi_1^2&\xi_2^2-\xi_3^2&\xi_1^2-\xi_2^2\\ -\xi_2(\xi_3^2-\xi_1^2)&-\xi_1(\xi_2^2-\xi_3^2)&-\xi_3(\xi_1^2-\xi_2^2)\end{pmatrix}$$

has determinant $\delta^2\neq0$. Consequently, $(L_1L_2,L_1L_3,L_2L_3)=(\mathtt{y}^2-\mathtt{xz},\mathtt{yz}+p\mathtt{yx}+q\mathtt{x}^2,q\mathtt{xy}+p\mathtt{xz}+\mathtt{z}^2)$ and

$$\widehat{A}\otimes_{\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]}\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]=\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(L_1L_2,L_1L_3,L_2L_3)\cong\overline{\kappa}[[\mathtt{w}_1,\mathtt{w}_2,\mathtt{w}_3]]/(\mathtt{w}_1\mathtt{w}_2,\mathtt{w}_1\mathtt{w}_3,\mathtt{w}_2\mathtt{w}_3).$$

Suppose conversely that A is a real (reduced) ring of embedding dimension $n\leqslant3$ and $\widehat{A}\otimes_{\kappa[[\mathtt{x}_1,\dots,\mathtt{x}_n]]}\overline{\kappa}[[\mathtt{x}_1,\dots,\mathtt{x}_n]]\cong\overline{\kappa}[[\mathtt{w}_1,\dots,\mathtt{w}_n]]/(\mathtt{w}_i\mathtt{w}_j:\ i\neq j)$. We distinguish several cases:

CASE 1. If $n=1$, then $\widehat{A}\otimes_{\kappa[[\mathtt{x}]]}\overline{\kappa}[[\mathtt{x}]]\cong\overline{\kappa}[\mathtt{w}_1]$. As A is a real (reduced) ring, also $\widehat{A}$ is a real reduced ring, so $\widehat{A}\cong\kappa[\mathtt{x}]$ (because it is a 1-dimensional regular formal ring whose residue field is the (formally) real field κ). ∎

CASE 2. If $n=2$, then $\widehat{A}\otimes_{\kappa[[\mathtt{x},\mathtt{y}]]}\overline{\kappa}[[\mathtt{x},\mathtt{y}]]\cong\overline{\kappa}[\mathtt{w}_1,\mathtt{w}_2]/(\mathtt{w}_1\mathtt{w}_2)$. Consequently, $\widehat{A}=\kappa[[\mathtt{x},\mathtt{y}]]/(F)$ where $F\in\kappa[[\mathtt{x},\mathtt{y}]]$ and there exist series $\varphi_1,\varphi_2\in\overline{\kappa}[[\mathtt{w}_1,\mathtt{w}_2]]$ such that $\mathtt{w}_1\mathtt{w}_2=F(\varphi_1,\varphi_2)$ and the Jacobian matrix of (φ_1,φ_2) is a unit of $\overline{\kappa}[[\mathtt{w}_1,\mathtt{w}_2]]$. Thus F has order 2 and its leading form is a quadratic form with coefficients in κ of maximal rank 2. By [Fe8, Thm.2.5] we may assume that $F\in\kappa[[\mathtt{x},\mathtt{y}]]$ equals $\mathtt{x}^2-a\mathtt{y}^2$ for some $a\in\kappa\backslash\{0\}$. As $\widehat{A}$ is a real ideal, we deduce $a\notin-\Sigma\kappa^2$. ∎

CASE 3. If $n=3$, then $\widehat{A}\otimes_{\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]}\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]\cong\overline{\kappa}[\mathtt{w}_1,\mathtt{w}_2,\mathtt{w}_3]/(\mathtt{w}_1\mathtt{w}_2,\mathtt{w}_2\mathtt{w}_3,\mathtt{w}_2\mathtt{w}_3)$. Write $\widehat{A}=\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/\mathfrak{a}$ where $\mathfrak{a}$ is a real ideal of $\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ (because $\widehat{A}$ is a real (reduced) ring, as A is a real (reduced) ring). As $\widehat{A}\otimes_{\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]}\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]=\overline{\kappa}[\mathtt{w}_1,\mathtt{w}_2,\mathtt{w}_3]/\mathfrak{a}\overline{\kappa}[\mathtt{w}_1,\mathtt{w}_2,\mathtt{w}_3]$, there exist series $\varphi_1,\varphi_2,\varphi_3\in\overline{\kappa}[[\mathtt{w}_1,\mathtt{w}_2,\mathtt{w}_3]]$ such that the Jacobian matrix of $\varphi:=(\varphi_1,\varphi_2,\varphi_3)$ is a unit of $\overline{\kappa}[[\mathtt{w}_1,\mathtt{w}_2,\mathtt{w}_3]]$ and series $H_1,H_2,H_3\in\mathfrak{a}\overline{\kappa}[\mathtt{x},\mathtt{y},\mathtt{z}]$ such that $H_1(\varphi)=\mathtt{w}_1\mathtt{w}_2$, $H_2(\varphi)=\mathtt{w}_1\mathtt{w}_2$ and $H_3(\varphi)=\mathtt{w}_2\mathtt{w}_3$. Consider the ideal $\mathfrak{b}:=\{H(\varphi):\ H\in\mathfrak{a}\overline{\kappa}[\mathtt{x},\mathtt{y},\mathtt{z}]\}$. As $\widehat{A}\otimes_{\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]}\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]\cong\overline{\kappa}[\mathtt{w}_1,\mathtt{w}_2,\mathtt{w}_3]/(\mathtt{w}_1\mathtt{w}_2,\mathtt{w}_2\mathtt{w}_3,\mathtt{w}_2\mathtt{w}_3)=\overline{\kappa}[\mathtt{w}_1,\mathtt{w}_2,\mathtt{w}_3]/\mathfrak{b}$, we have $\mathfrak{b}=(\mathtt{w}_1\mathtt{w}_2,\mathtt{w}_2\mathtt{w}_3,\mathtt{w}_2\mathtt{w}_3)$.

Let $\psi_1,\psi_2,\psi_3\in\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ be series such that the Jacobian matrix of $\psi:=(\psi_1,\psi_2,\psi_3)$ is a unit of $\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ and $\varphi_1(\psi)=\mathtt{x}$, $\varphi_2(\psi)=\mathtt{y}$, $\varphi_3(\psi)=\mathtt{z}$, $\psi_1(\varphi)=\mathtt{w}_1$, $\psi_2(\varphi)=\mathtt{w}_2$ and $\psi_3(\varphi)=\mathtt{w}_3$. In particular, $\mathfrak{a}\overline{\kappa}[\mathtt{x},\mathtt{y},\mathtt{z}]=(H_1,H_2,H_3)\overline{\kappa}[\mathtt{x},\mathtt{y},\mathtt{z}]$ and $H_1=\psi_1\psi_2$, $H_2=\psi_1\psi_3$ and $H_3=\psi_2\psi_3$. As the determinant of the Jacobian matrix of $\psi:=(\psi_1,\psi_2,\psi_3)$ is a unit of $\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]$, the three systems of equations $\{\psi_1=0,\psi_2=0\}$, $\{\psi_1=0,\psi_3=0\}$ and $\{\psi_2=0,\psi_3=0\}$ have Jacobian matrices with 2-minors that are units of $\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]$. By the Implicit Function Theorem the three systems of equations $\{\psi_1=0,\psi_2=0\}$, $\{\psi_1=0,\psi_3=0\}$ and $\{\psi_2=0,\psi_3=0\}$ have unique solutions in either $\overline{\kappa}[[\mathtt{x}]]$ or $\overline{\kappa}[[\mathtt{y}]]$ or $\overline{\kappa}[[\mathtt{z}]]$ (depending on the non-zero 2-minors of the Jacobian matriz of ψ at the origin).

By the Implicit Function Theorem we may assume that there exist unique series $\lambda,\mu\in(\mathtt{x})\overline{\kappa}[[\mathtt{x}]]$ such that $\psi_1(\mathtt{x},\lambda,\mu)=0$ and $\psi_2(\mathtt{x},\lambda,\mu)=0$. Thus, $H_i(\mathtt{x},\lambda,\mu)=0$ for $i=1,2,3$, so $H(\mathtt{x},\lambda,\mu)=0$ for each $H\in\mathfrak{a}$. Write $\lambda:=\sum_{k\geqslant1}\lambda_k\mathtt{x}^k$ and $\mu:=\sum_{k\geqslant1}\mu_k\mathtt{x}^k$. Let σ be an element of the Galois group $G(\overline{\kappa}:\kappa)$ and denote $\lambda^\sigma:=\sum_{k\geqslant1}\sigma(\lambda_k)\mathtt{x}^k$ and $\mu^\sigma:=\sum_{k\geqslant1}\sigma(\mu_k)\mathtt{x}^k$. If $H\in\mathfrak{a}\subset\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$, then $H(\mathtt{x},\lambda^\sigma,\mu^\sigma)=0$ for each $\sigma\in G(\overline{\kappa}:\kappa)$. Thus, $H_i(\mathtt{x},\lambda^\sigma,\mu^\sigma)=0$ for $i=1,2,3$, so either $[\psi_1(\mathtt{x},\lambda^\sigma,\mu^\sigma)=0$ and $\psi_2(\mathtt{x},\lambda^\sigma,\mu^\sigma)=0]$, or $[\psi_1(\mathtt{x},\lambda^\sigma,\mu^\sigma)=0$ and $\psi_3(\mathtt{x},\lambda^\sigma,\mu^\sigma)=0]$, or $[\psi_2(\mathtt{x},\lambda^\sigma,\mu^\sigma)=0$ and $\psi_3(\mathtt{x},\lambda^\sigma,\mu^\sigma)=0]$. This means that there exist $\sigma_0:=\mathrm{id},\sigma_1,\sigma_2\in G(\overline{\kappa}:\kappa)$ such that $\sigma(\kappa(\lambda_k,\mu_k:\ k\geqslant1))=\sigma_{j_\sigma}(\kappa(\lambda_k,\mu_k:\ k\geqslant1))$ for some $j_\sigma=0,1,2$ for each $\sigma\in G(\overline{\kappa}:\kappa)$. Consequently, the subextension $E:=\kappa(\lambda_k,\mu_k:\ k\geqslant1)|\kappa$ of $\overline{\kappa}|\kappa$ has degree $m\leqslant3$ over κ. Let $\rho\in E$ be such that $E=\kappa[\rho]$ and let $P\in\kappa[\mathtt{t}]$ be the irreducible polynomial of θ over κ, which has degree m. We distinguish the three possible situations:

SUBCASE 3.1. $m=3$. Write $\lambda=\mathtt{x}\eta_0+\mathtt{x}\eta_1\rho+\mathtt{x}\eta_2\rho^2$ (where $\eta_\ell\in\kappa[[\mathtt{x}]]$) and $\mu:=\mathtt{x}\xi_0+\mathtt{x}\xi_1\rho+\mathtt{x}\xi_2\rho^2$ (where $\xi_\ell\in\kappa[[\mathtt{x}]]$). Let $\rho_2,\rho_3\in\overline{\kappa}$ be the remaining roots of P different from $\rho_1:=\rho$. As we have seen above the solutions of the three systems of equations $\{\psi_1=0,\psi_2=0\}$, $\{\psi_1=0,\psi_3=0\}$ and $\{\psi_2=0,\psi_3=0\}$ are respectively (maybe after reordering them)

$$(\mathtt{x},\mathtt{x}\eta_0+\mathtt{x}\eta_1\rho_1+\mathtt{x}\eta_2\rho_1^2,\mathtt{x}\xi_0+\mathtt{x}\xi_1\rho_1+\mathtt{x}\xi_2\rho_1^2),$$
$$(\mathtt{x},\mathtt{x}\eta_0+\mathtt{x}\eta_1\rho_2+\mathtt{x}\eta_2\rho_2^2,\mathtt{x}\xi_0+\mathtt{x}\xi_1\rho_2+\mathtt{x}\xi_2\rho_2^2),$$
$$(\mathtt{x},\mathtt{x}\eta_0+\mathtt{x}\eta_1\rho_3+\mathtt{x}\eta_2\rho_3^2,\mathtt{x}\xi_0+\mathtt{x}\xi_1\rho_3+\mathtt{x}\xi_2\rho_3^2).$$

Thus, the vectors

$$(1,\eta_0(0)+\eta_1(0)\rho_1+\eta_2(0)\rho_1^2,\xi_0(0)+\xi_1(0)\rho_1+\xi_2(0)\rho_1^2),$$
$$(1,\eta_0(0)+\eta_1(0)\rho_2+\eta_2(0)\rho_2^2,\xi_0(0)+\xi_1(0)\rho_2+\xi_2(0)\rho_2^2),$$
$$(1,\eta_0(0)+\eta_1(0)\rho_3+\eta_2(0)\rho_3^2,\xi_0(0)+\xi_1(0)\rho_3+\xi_2(0)\rho_3^2)$$

are κ-linearly independent, because the Jacobian matrix of ψ at the origin is invertible. As the vectors $(1,\rho_1,\rho_1^2)$, $(1,\rho_2,\rho_2^2)$ and $(1,\rho_3,\rho_3^2)$ are κ-linearly independent (because the roots ρ_i are different), the matrices

$$M_0:=\begin{pmatrix}1&0&0\\ \eta_0(0)&\eta_1(0)&\eta_2(0)\\ \xi_0(0)&\xi_1(0)&\xi_2(0)\end{pmatrix}\quad\text{and}\quad M:=\begin{pmatrix}1&0&0\\ \eta_0&\eta_1&\eta_2\\ \xi_0&\xi_1&\xi_2\end{pmatrix}$$

are invertible. As

$$\begin{pmatrix}1&1&1\\ \eta_0+\eta_1\rho_1+\eta_2\rho_1^2&\mathtt{x}\eta_0+\mathtt{x}\eta_1\rho_2+\mathtt{x}\eta_2\rho_2^2&\eta_0+\eta_1\rho_3+\eta_2\rho_3^2\\ \xi_0+\xi_1\rho_1+\xi_2\rho_1^2&\xi_0+\xi_1\rho_2+\xi_2\rho_2^2&\xi_0+\xi_1\rho_3+\xi_2\rho_3^2\end{pmatrix}=M\begin{pmatrix}1&1&1\\ \rho_1&\rho_2&\rho_3\\ \rho_1^2&\rho_2^2&\rho_3^2\end{pmatrix},$$

after a change of coordinates (given by the inverse matrix of M), we may assume $\lambda=\rho_1\mathtt{x}$ and $\mu=\rho_1^2\mathtt{x}$. This means that the solutions of the three systems of equations $\{\psi_1=0,\psi_2=0\}$, $\{\psi_1=0,\psi_3=0\}$ and $\{\psi_2=0,\psi_3=0\}$ are $(\mathtt{x},\rho_1\mathtt{x},\rho_1^2\mathtt{x})$, $(\mathtt{x},\rho_2\mathtt{x},\rho_2^2\mathtt{x})$ and $(\mathtt{x},\rho_3\mathtt{x},\rho_3^2\mathtt{x})$. If $P:=a_0+a_1\mathtt{t}+a_2\mathtt{t}^2+\mathtt{t}^3$, then the quadratic forms

$$\mathtt{y}^2-\mathtt{z}\mathtt{x},a_0\mathtt{x}^2+a_1\mathtt{x}\mathtt{y}+a_2\mathtt{x}\mathtt{z}+a_3\mathtt{y}\mathtt{z},a_0\mathtt{x}\mathtt{y}+a_1\mathtt{x}\mathtt{z}+a_2\mathtt{y}\mathtt{z}+\mathtt{z}^2\in\mathfrak{a}\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]\cap\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]].$$

As $\mathfrak{a}\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ is generated by three $\overline{\kappa}$-linearly independent quadratic forms, and the latter ones are $\overline{\kappa}$-linearly independent, we deduce that $\mathfrak{a}\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ is generated by the previous there quadratic forms. By [ABR, Prop.VII.1.7(b),(c), Prop.VII.1.13] we have $\mathfrak{a}\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]\cap\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]=\mathfrak{a}$, so

$$\mathfrak{a}=(\mathtt{y}^2-\mathtt{z}\mathtt{x},a_0\mathtt{x}^2+a_1\mathtt{x}\mathtt{y}+a_2\mathtt{x}\mathtt{z}+\mathtt{y}\mathtt{z},a_0\mathtt{x}\mathtt{y}+a_1\mathtt{x}\mathtt{z}+a_2\mathtt{y}\mathtt{z}+\mathtt{z}^2)\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]].$$

By Remark 10.1(i) we may assume after a change of coordinates $\mathfrak{a} = (\mathtt{y}^2 - \mathtt{xz}, q\mathtt{x}^2 + p\mathtt{xy} + \mathtt{yz}, q\mathtt{xy} + p\mathtt{xz} + \mathtt{z}^2)\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ where $P(\mathtt{t}) = \mathtt{t}^3 + p\mathtt{t}^2 + q$ is an irreducible polynomial.

Subcase 3.2. $m = 2$. We may assume $\rho = \sqrt{a}$ for some $a \in \kappa\setminus\kappa^2$ and we write $\lambda = \mathtt{x}\eta_0 + \mathtt{x}\eta_1\sqrt{a}$ (where $\eta_\ell \in \kappa[[\mathtt{x}]]$) and $\mu := \mathtt{x}\xi_0 + \mathtt{x}\xi_1\sqrt{a}$ (where $\xi_\ell \in \kappa[[\mathtt{x}]]$). After the change of coordinates $(\mathtt{x},\mathtt{y},\mathtt{z}) \mapsto (\mathtt{x},\mathtt{y} - \eta_0(\mathtt{x}), \mathtt{z} - \xi_0(\mathtt{x}))$, we may assume $\eta_0 = 0$ and $\xi_0 = 0$. The vectors $(1, \eta_1(0)\sqrt{a}, \xi_1(0)\sqrt{a})$ and $(1, -\eta_1(0)\sqrt{a}, -\xi_1(0)\sqrt{a})$ are κ-linearly independent, because the Jacobian matrix of ψ at the origin is invertible. Thus, after reordering the variables we may assume $\eta_1(0) \neq 0$. Thus, after the change of coordinates $(\mathtt{x},\mathtt{y},\mathtt{z}) \mapsto (\mathtt{x}, \frac{\mathtt{y}}{\eta_1}, \mathtt{z} - \frac{\mathtt{y}\xi_1}{\eta_1})$, we may assume $\eta_1 = 1$ and $\xi_1 = 0$, so $\lambda = \mathtt{x}\sqrt{a}$ and $\mu = 0$. Consequently, $(\mathtt{x}, \mathtt{x}\sqrt{a}, 0)$ is the solution of the system $\{\psi_1 = 0, \psi_2 = 0\}$ and $(\mathtt{x}, -\mathtt{x}\sqrt{a}, 0)$ is the solution of the system $\{\psi_1 = 0, \psi_3 = 0\}$. The solution of the system $\{\psi_2 = 0, \psi_3 = 0\}$ is of the form $(\mathtt{x}\gamma_0, \mathtt{x}\gamma_1, \mathtt{x}\gamma_2)$ where each $\gamma_i \in \kappa[[\mathtt{x}]]$. The vectors $(1, \sqrt{a}, 0), (1, -\sqrt{a}, 0), (\gamma_0(0), \gamma_1(0), \gamma_2(0))$ are κ-linearly independent, because the Jacobian matrix of ψ at the origin is invertible. Consequently, $\gamma_2(0) \neq 0$ and after the change $(\mathtt{x},\mathtt{y},\mathtt{z}) \mapsto (\mathtt{x} - \frac{\gamma_0}{\gamma_2}\mathtt{z}, \mathtt{y} - \frac{\gamma_1}{\gamma_2}\mathtt{z}, \frac{1}{\gamma_2}\mathtt{z})$, we may assume in addition $\gamma_0 = 0, \gamma_1 = 0, \gamma_2 = 1$, so the solution of $\{\psi_2 = 0, \psi_3 = 0\}$ is $\{\mathtt{x} = 0, \mathtt{y} = 0\}$. Then the quadratic forms

$$\mathtt{y}^2 - a\mathtt{x}^2, \mathtt{xz}, \mathtt{yz} \in \mathfrak{a}\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]] \cap \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]].$$

As $\mathfrak{a}\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ is generated by three $\overline{\kappa}$-linearly independent quadratic forms, and the latter ones are $\overline{\kappa}$-linearly independent, we deduce that $\mathfrak{a}\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ is generated by $\mathtt{y}^2 - a\mathtt{x}^2, \mathtt{xz}, \mathtt{yz}$. By [ABR, Prop.VII.1.7(b),(c), Prop.VII.1.13] we have $\mathfrak{a}\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]] \cap \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]] = \mathfrak{a}$, so $\mathfrak{a} = (\mathtt{y}^2 - a\mathtt{x}^2, \mathtt{xz}, \mathtt{yz})\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$.

Subcase 3.3. $m = 1$. In this case we may assume that the solutions of the systems $\{\psi_1 = 0, \psi_2 = 0\}$, $\{\psi_1 = 0, \psi_3 = 0\}$ and $\{\psi_2 = 0, \psi_3 = 0\}$ are of the form $(\gamma_{i0}\mathtt{x}, \gamma_{i1}\mathtt{x}, \gamma_{i2}\mathtt{x})$ where $i = 1,2,3$ and each $\gamma_{ij} \in \kappa[[\mathtt{x}]]$. As the Jacobian matrix of ψ at the origin is invertible we deduce that the matrices

$$M_0 := \begin{pmatrix} \gamma_{00}(0) & \gamma_{01}(0) & \gamma_{02}(0) \\ \gamma_{10}(0) & \gamma_{11}(0) & \gamma_{12}(0) \\ \gamma_{20}(0) & \gamma_{21}(0) & \gamma_{22}(0) \end{pmatrix} \quad \text{and} \quad M := \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} \\ \gamma_{10} & \gamma_{11} & \gamma_{12} \\ \gamma_{20} & \gamma_{21} & \gamma_{22} \end{pmatrix}$$

are invertible. After the change of coordinates $(\mathtt{x},\mathtt{y},\mathtt{z}) \mapsto M^{-1}(\mathtt{x},\mathtt{y},\mathtt{z})$ we may assume $\gamma_{ii} = 1$ and $\gamma_{ij} = 0$ if $i \neq j$. Then the quadratic forms

$$\mathtt{xy}, \mathtt{xz}, \mathtt{yz} \in \mathfrak{a}\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]] \cap \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]].$$

As $\mathfrak{a}\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ is generated by three $\overline{\kappa}$-linearly independent quadratic forms, and the latter ones are $\overline{\kappa}$-linearly independent, we deduce that $\mathfrak{a}\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ is generated by the previous there quadratic forms. By [ABR, Prop.VII.1.7(b),(c), Prop.VII.1.13] we have $\mathfrak{a}\overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]] \cap \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]] = \mathfrak{a}$, so $\mathfrak{a} = (\mathtt{xy}, \mathtt{xz}, \mathtt{yz})\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$. After the change of coordinates $(\mathtt{x},\mathtt{y},\mathtt{z}) \mapsto (\mathtt{x} - \mathtt{y}, \mathtt{x} + \mathtt{y}, \mathtt{z})$ we have $\mathfrak{a} = (\mathtt{x}^2 - \mathtt{y}^2, (\mathtt{x} - \mathtt{y})\mathtt{z}, (\mathtt{x} + \mathtt{y})\mathtt{z}) = (\mathtt{x}^2 - \mathtt{y}^2, \mathtt{xz}, \mathtt{yz})\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$, as required. □

10.2. **One dimensional rings with the property psd=sos.** We prove next the 'if part' of Theorem 3.2.

rep **Theorem 10.2.** *Let A be one of the following formal rings:*

(i) $\kappa[[\mathtt{x}]]$.
(ii) $\kappa[\mathtt{x},\mathtt{y}]]/(\mathtt{y}^2 - a\mathtt{x}^2)$ *for some* $a \notin -\Sigma\kappa^2$.
(iii) $\kappa[\mathtt{x},\mathtt{y},\mathtt{z}]/\mathfrak{a}$ *where* $\mathfrak{a} = (\mathtt{y}^2 - \mathtt{xz}, \mathtt{yz} + p\mathtt{yx} + q\mathtt{x}^2, q\mathtt{xy} + p\mathtt{xz} + \mathtt{z}^2)$ *and the polynomial* $P := \mathtt{t}^3 + p\mathtt{t} + q \in \kappa[\mathtt{t}]$ *is irreducible.*
(iv) $\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{y}^2 - a\mathtt{x}^2, \mathtt{xz}, \mathtt{yz})$ *for some* $a \notin -\Sigma\kappa^2$.
(v) $\kappa[\mathtt{x},\mathtt{y},\mathtt{z}]/(\mathtt{z}^2 - \mathtt{xy}, \mathtt{y}^2 - 2b\mathtt{yx} - 4c^2\mathtt{zx} + d\mathtt{x}^2)$ *where* $P := \mathtt{t}^4 - 2b\mathtt{t}^2 - 4c^2\mathtt{t} + d \in \kappa[\mathtt{t}]$ *is a chimeric polynomial.*

Then $\mathcal{P}(A) = \Sigma A^2$.

Proof. (i) $A = \kappa[[\mathtt{x}]]$. The property $\mathcal{P}(A) = \Sigma A^2$ is straightforward.

(ii) $A = \kappa[[\mathtt{x}, \mathtt{y}]]/(\mathtt{y}^2 - a\mathtt{x}^2)$ (where $-a \notin \Sigma\kappa^2$).

Assume first $\mathtt{y}^2 - a\mathtt{x}^2$ is irreducible. By Remark 5.5 (ii) we may assume $f(0,0) = 0$. Observe that $\omega(f) \geqslant 2$, because $f(\mathtt{t}, \pm\sqrt{a}\mathtt{t}) \in \mathcal{P}(\kappa[\pm\sqrt{a}][[\mathtt{t}]])$. Write $f(\mathtt{t}, \sqrt{a}\mathtt{t}) = \mathtt{t}^{2k}(b_0 + b_1\sqrt{a})(u_0(t) + \sqrt{a}\mathtt{t}u_1(\mathtt{t}))^2$, where $b_0, b_1 \in \kappa$, $u_0, u_1 \in \kappa[[\mathtt{t}]]$ and $u_0(0) = 1$. Observe that $b_0 + b_1\sqrt{a} \in \mathcal{P}(\kappa[\sqrt{a}])$, so there exist $c_{0i}, c_{1i} \in \kappa$ such that $b_0 + b_1\sqrt{a} = \sum_{i=1}^{p}(c_{0i} + \sqrt{a}c_{1i})^2$. Thus,

$$\begin{aligned} f(\mathtt{t}, \sqrt{a}\mathtt{t}) &= \mathtt{t}^{2k}(b_0 + b_1\sqrt{a})(u_0(\mathtt{t}) + \sqrt{a}\mathtt{t}u_1(\mathtt{t})) \\ &= \mathtt{t}^{2k-2}\sum_{i=1}^{p}(c_{0i}\mathtt{t} + \sqrt{a}\mathtt{t}c_{1i})^2(u_0(\mathtt{t}) + \sqrt{a}\mathtt{t}u_1(\mathtt{t}))^2. \end{aligned}$$

Consequently, $f(\mathtt{x}, \mathtt{y}) = \mathtt{x}^{2k-2}\sum_{i=1}^{p}(c_{0i}\mathtt{x} + \mathtt{y}c_{1i})^2(u_0(\mathtt{x}) + \mathtt{y}u_1(\mathtt{x}))^2$.

Suppose next $\mathtt{y}^2 - a\mathtt{x}^2$ is reducible. After a change of coordinates $A = \kappa[[\mathtt{x}, \mathtt{y}]]/(\mathtt{x}\mathtt{y})$. By Remark 5.5 (ii) we may assume $f(0,0) = 0$. We have $\omega(f) \geqslant 2$ and we write $f = f_1(\mathtt{x}) + f_2(\mathtt{y})$ for some series $f_1, f_2 \in \mathcal{P}(\kappa[[\mathtt{t}]]) = \Sigma\kappa[[\mathtt{t}]]^2$, so $f \in \Sigma A^2$.

(iii) $A = \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/(\mathtt{y}^2 - \mathtt{x}\mathtt{z}, \mathtt{y}\mathtt{z} + p\mathtt{y}\mathtt{x} + q\mathtt{x}^2, q\mathtt{x}\mathtt{y} + p\mathtt{x}\mathtt{z} + \mathtt{z}^2)$, where $P := \mathtt{t}^3 + p\mathtt{t} + q \in \kappa[\mathtt{t}]$ is the irreducible polynomial over κ of some $\rho \in \overline{\kappa}$, whereas $Q := \mathtt{t}^3 + 2p\mathtt{t}^2 + p^2\mathtt{t} - q^2$ is the irreducible polynomial of ρ^2 over κ. Observe that

$$\begin{aligned} \mathtt{z}^3 + 2p\mathtt{x}\mathtt{z}^2 + p^2\mathtt{x}^2\mathtt{z} - q^2\mathtt{x}^3 &= \mathtt{z}(q\mathtt{x}\mathtt{y} + p\mathtt{x}\mathtt{z} + \mathtt{z}^2) \\ &\quad - (\mathtt{y}\mathtt{z} + p\mathtt{y}\mathtt{x} + q\mathtt{x}^2)(q\mathtt{x} - p\mathtt{y}) - (\mathtt{y}^2 - \mathtt{x}\mathtt{z})(p^2\mathtt{x} + p\mathtt{z}) \end{aligned}$$

By Remark 5.5 (ii) we may assume $f(0,0,0) = 0$. We have $\omega(f) \geqslant 2$, because $f(\mathtt{t}, \eta\mathtt{t}, \eta^2\mathtt{t}) \in \mathcal{P}(\kappa[\rho][[\mathtt{t}]])$ and the irreducible polynomial of η over κ has degree 3. Write $f(\mathtt{t}, \rho\mathtt{t}, \rho^2\mathtt{t}) = \mathtt{t}^{2k}(b_0 + b_1\rho + b_2\rho^2)(u_0(\mathtt{t}) + \rho\mathtt{t}u_1(\mathtt{t}) + \rho^2\mathtt{t}u_2(\mathtt{t}))^2$, where $b_0, b_1, b_2 \in \kappa$, $u_0, u_1, u_2 \in \kappa[[\mathtt{t}]]$ and $u_0(0) = 1$. Observe that $b_0 + b_1\rho + b_2\rho^2 \in \mathcal{P}(\kappa[\sqrt{a}])$, so there exist $c_{0i}, c_{1i}, c_{2i} \in \kappa$ such that $b_0 + b_1\rho + b_2\rho^2 = \sum_{i=1}^{p}(c_{0i} + \rho c_{1i} + \rho^2 c_{2i})^2$. Thus,

$$\begin{aligned} f(\mathtt{t}, \rho\mathtt{t}, \rho^2\mathtt{t}) &= \mathtt{t}^{2k}(b_0 + b_1\rho + b_2\rho^2)(u_0(\mathtt{t}) + \rho\mathtt{t}u_1(\mathtt{t}) + \rho^2\mathtt{t}u_2(\mathtt{t})) \\ &= \mathtt{t}^{2k-2}\sum_{i=1}^{p}(c_{0i}\mathtt{t} + \rho\mathtt{t}c_{1i} + \rho^2\mathtt{t}c_{2i})^2(u_0(\mathtt{t}) + \rho\mathtt{t}u_1(\mathtt{t}) + \rho^2\mathtt{t}u_2(\mathtt{t}))^2. \end{aligned}$$

Consequently, $f(\mathtt{x}, \mathtt{y}, \mathtt{z}) = \mathtt{x}^{2k-2}\sum_{i=1}^{p}(c_{0i}\mathtt{x} + \mathtt{y}c_{1i} + \mathtt{z}c_{2i})^2(u_0(\mathtt{x}) + \mathtt{y}u_1(\mathtt{x}) + \mathtt{z}u_2(\mathtt{x}))^2$.

(iv) $A = \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/(\mathtt{x}^2 - a\mathtt{y}^2, \mathtt{x}\mathtt{z}, \mathtt{y}\mathtt{z})$ for some $a \notin -\Sigma\kappa^2$. By Remark 5.5 (ii) we may assume $f(0,0,0) = 0$. We have $\omega(f) \geqslant 2$ and write $f = f_1(\mathtt{x}, \mathtt{y}) + f_2(\mathtt{z})$ for some series $f_1 \in \mathcal{P}(\kappa[[\mathtt{x}, \mathtt{y}]]/(\mathtt{x}^2 - a\mathtt{y}^2)) = \Sigma\kappa[[\mathtt{x}, \mathtt{y}]]/(\mathtt{x}^2 - a\mathtt{y}^2)^2$ and $f_2 \in \mathcal{P}(\kappa[[\mathtt{z}]]) = \Sigma\kappa[[\mathtt{z}]]^2$, so $f \in \Sigma A^2$.

(v) $A = \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/(\mathtt{y}^2 - \mathtt{x}\mathtt{z}, \mathtt{z}^2 - 2b\mathtt{x}\mathtt{z} - 4c^2\mathtt{y}\mathtt{x} + d\mathtt{x}^2)$, where $P := \mathtt{t}^4 - 2b\mathtt{t}^2 - 4c^2\mathtt{t} + d \in \kappa[\mathtt{t}]$ is a chimeric polynomial over κ. Observe that

$$\begin{aligned} &\mathtt{z}^4 - 4b\mathtt{z}^3\mathtt{x} + 4b^2\mathtt{z}^2\mathtt{x}^2 + 2d\mathtt{z}^2\mathtt{x}^2 - 4bd\mathtt{z}\mathtt{x}^3 - 16c^4\mathtt{z}\mathtt{x}^3 + d^2\mathtt{x}^4 \\ &\qquad = (\mathtt{z}^2 - 2b\mathtt{x}\mathtt{z} - 4c^2\mathtt{y}\mathtt{x} + d\mathtt{x}^2)(\mathtt{z}^2 - 2b\mathtt{x}\mathtt{z} + 4c^2\mathtt{y}\mathtt{x} + d\mathtt{x}^2) + 16c^4\mathtt{x}^2(\mathtt{y}^2 - \mathtt{x}\mathtt{z}), \quad (10.1) \quad \boxed{\texttt{eqr1}} \\ &\mathtt{y}^4 - 2b\mathtt{y}^2\mathtt{x}^2 - 4c^2\mathtt{y}\mathtt{x}^3 + d\mathtt{x}^4 = (\mathtt{z}^2 - 2b\mathtt{x}\mathtt{z} - 4c^2\mathtt{y}\mathtt{x} + d\mathtt{x}^2)\mathtt{x}^2 + (\mathtt{y}^2 - \mathtt{x}\mathtt{z})(-2b\mathtt{x}^2 + \mathtt{x}\mathtt{z} + \mathtt{y}^2). \quad (10.2) \quad \boxed{\texttt{eqr2}} \end{aligned}$$

Consider the (formally) real field $L := \kappa[\mathtt{t}]/(P)$ and the (well-defined) homomorphism $\varphi : A \to L[[\mathtt{x}]], f \mapsto f(\mathtt{x}, \mathtt{t}\mathtt{x}, \mathtt{t}^2\mathtt{x})$. By Remark 5.5 (ii) we may assume $f(0,0,0) = 0$ and $f \in \mathcal{P}(A)$. By Lemma 5.14 $\varphi(f) \in \mathcal{P}(L[[\mathtt{x}]])$. Using equations (10.1) and (10.2) we may assume

$$f := \mathtt{x}f_0(\mathtt{x}) + \mathtt{z}f_1(\mathtt{x}) + \mathtt{z}^2 f_2(\mathtt{x}) + \mathtt{z}^3 f_3(\mathtt{x}) + \mathtt{y}f_4(\mathtt{x}) + \mathtt{y}\mathtt{z}f_5(\mathtt{x}) + \mathtt{y}\mathtt{z}^2 f_6(\mathtt{x}) + \mathtt{y}\mathtt{z}^3 f_7(\mathtt{x})$$

where each $f_j \in \kappa[[x]]$. Then

$$\varphi(f) = xf_0(x) + t^2xf_1(x) + t^4x^2f_2(x) + t^6x^3f_3(x) + txf_4(x) + t^3x^2f_5(x) + t^5x^3f_6(x) + t^7x^4f_7(x).$$

As $\varphi(f) \in \mathcal{P}(L[[x]])$, we deduce x divides f_0, f_1, f_4, which means in particular that $\omega(f) \geqslant 2$ and we may write

$$f(x, tx, t^2x) = x^{2k}(b_0 + b_1t + b_2t^2 + b_3t^3 + xu_0(x) + txu_1(x) + t^2xu_2(x) + t^3xu_3(x)),$$

where $b_0, b_1, b_2, b_3 \in \kappa$, $u_0, u_1, u_2, u_3 \in \kappa[[x]]$, $b_0 + b_1t + b_2t^2 + b_3t^3 \neq 0$ and $k \geqslant 1$. Observe that $b_0 + b_1t + b_2t^2 + b_3t^3 \in \mathcal{P}(L) = \Sigma L^2$. As P is a chimeric polynomial, there exist $c_{0i}, c_{1i}, c_{2i} \in \kappa$ and $\mu \in \Sigma\kappa^2$ such that $b_0 + b_1t + b_2t^2 + b_3t^3 + \mu P = \sum_{i=1}^p (c_{0i} + c_{1i}t + c_{2i}t^2)^2$. Assume $c_{01} + c_{11}t + c_{21}t^2 \neq 0$. Consider the equation in $L[[x, Y]]$:

$$H(x, Y) := b_0 + b_1t + b_2t^2 + b_3t^3 + \mu P + xu_0(x) + txu_1(x) + t^2xu_2(x) + t^3xu_3(x) - (c_{01} + c_{11}t + c_{21}t^2 + Y)^2 - \sum_{i=2}^p (c_{0i} + c_{1i}t + c_{2i}t^2)^2.$$

Observe that $H(0, 0) = 0$ and

$$\frac{\partial H}{\partial Y}(0, 0) = -2(c_{01} + c_{11}t + c_{21}t^2 + Y)|_{(0,0)} = -2(c_{01} + c_{11}t + c_{21}t^2) \neq 0.$$

By the Implicit Function Theorem there exists $g(x) = \sum_{\ell\geqslant 1}(d_{0\ell} + d_{1\ell}t + d_{2\ell}t^2 + d_{3\ell}t^3)x^\ell \in xL[[x]]$ such that

$$H(x, g(x)) = b_0 + b_1t + b_2t^2 + b_3t^3 + \mu P + xu_0(x) + txu_1(x) + t^2xu_2(x) + t^3xu_3(x) - (c_{01} + tc_{11} + t^2c_{21} + g(x))^2 - \sum_{i=2}^p (c_{0i} + tc_{1i} + t^2c_{2i})^2 = 0.$$

Consequently,

$$f(x, tx, t^2x) = x^{2k-2}\Big((c_{01}x + c_{11}tx + c_{21}t^2x + \sum_{\ell\geqslant 1}(d_{0\ell}x^2 + d_{1\ell}tx^2 + d_{2\ell}t^2x^2 + d_{3\ell}t^3x^2)x^{\ell-1})^2 + \sum_{i=2}^p (c_{0i}x + c_{1i}tx + c_{2i}t^2x)^2\Big),$$

Thus,

$$f(x, y, z) = x^{2k-2}\Big((c_{01}x + c_{11}y + c_{21}z + \sum_{\ell\geqslant 1}(d_{0\ell}x^2 + d_{1\ell}yx + d_{2\ell}zx + d_{3\ell}yz)x^{\ell-1})^2 + \sum_{i=2}^p (c_{0i}x + c_{1i}y + c_{2i}z)^2\Big)$$

modulo $\mathfrak{a}$, that is, $f \in \mathcal{P}(A)$, as required. $\square$

We show next that if $A := \kappa[[x, y, z]]/\mathfrak{a}$ where $\mathfrak{a} := (y^2 - zx, z^2 - 2bxz - 4c^2xy + dx^2)$ where $P := t^4 - 2bt^2 - 4c^2t + d \in \kappa[t]$ is an irreducible polynomial such that $L := \kappa[t]/(P)$ is a (formally) real field, the elements $f \in \mathcal{P}(A)\backslash\Sigma A^2$ (if any) have order $\leqslant 2$. The proof is quite similar to the one of Theorem 10.2(v), but we include the precise details below for the sake of completeness.

geq3 **Lemma 10.3.** *Let* $\mathfrak{a} := (y^2 - zx, z^2 - 2bxz - 4c^2xy + dx^2)$ *where* $P := t^4 - 2bt^2 - 4c^2t + d \in \kappa[t]$ *is an irreducible polynomial such that* $L := \kappa[t]/(P)$ *is a (formally) real field. Denote* $A := \kappa[[x, y, z]]/\mathfrak{a}$ *and let* $f \in \mathcal{P}(A)$ *be such that* $\omega(f) \geqslant 3$. *Then* $f \in \Sigma A^2$.

Proof. Consider the (well-defined) homomorphism $\varphi : A \to L[[\mathtt{x}]], f \mapsto f(\mathtt{x}, \mathtt{t}\mathtt{x}, \mathtt{t}^2\mathtt{x})$. By Remark 5.5 (ii) we may assume $f(0,0,0) = 0$ and $f \in \mathcal{P}(A)$. Proceeding as we have done in the proof of Theorem 10.2(v) we may write

$$f(\mathtt{x}, \mathtt{t}\mathtt{x}, \mathtt{t}^2\mathtt{x}) = \mathtt{x}^{2k}(b_0 + b_1\mathtt{t} + b_2\mathtt{t}^2 + b_3\mathtt{t}^3 + \mathtt{x}u_0(\mathtt{x}) + \mathtt{t}\mathtt{x}u_1(\mathtt{x}) + \mathtt{t}^2\mathtt{x}u_2(\mathtt{x}) + \mathtt{t}^3\mathtt{x}u_3(\mathtt{x})),$$

where $b_0, b_1, b_2, b_3 \in \kappa$, $u_0, u_1, u_2, u_3 \in \kappa[[\mathtt{x}]]$, $b_0 + b_1\mathtt{t} + b_2\mathtt{t}^2 + b_3\mathtt{t}^3 \neq 0$ and $k \geqslant 2$ (because $\omega(f) \geqslant 4$ and $f(\mathtt{x}, \mathtt{t}\mathtt{x}, \mathtt{t}^2\mathtt{x}) \in \mathcal{P}(L[[\mathtt{x}]])$). Observe that $b_0 + b_1\mathtt{t} + b_2\mathtt{t}^2 + b_3\mathtt{t}^3 \in \mathcal{P}(L) = \Sigma L^2$. Thus, there exist $c_{0i}, c_{1i}, c_{2i}, c_{3i}, d_0, d_1, d_2 \in \kappa$ and $\mu \in \Sigma\kappa^2$ such that $b_0 + b_1\mathtt{t} + b_2\mathtt{t}^2 + b_3\mathtt{t}^3 + (d_0 + d_1\mathtt{t} + d_2\mathtt{t}^2)P = \sum_{i=1}^p (c_{0i} + c_{1i}\mathtt{t} + c_{2i}\mathtt{t}^2 + c_{3i}\mathtt{t}^3)^2$. Assume $c_{01} + c_{11}\mathtt{t} + c_{21}\mathtt{t}^2 + c_{31}\mathtt{t}^3 \neq 0$. Consider the equation in $L[[\mathtt{x}, \mathtt{Y}]]$:

$$\begin{aligned} H(\mathtt{x}, \mathtt{Y}) := {}& b_0 + b_1\mathtt{t} + b_2\mathtt{t}^2 + b_3\mathtt{t}^3 + (d_0 + d_1\mathtt{t} + d_2\mathtt{t}^2)P + \mathtt{x}u_0(\mathtt{x}) + \mathtt{t}\mathtt{x}u_1(\mathtt{x}) + \mathtt{t}^2\mathtt{x}u_2(\mathtt{x}) \\ & + \mathtt{t}^3\mathtt{x}u_3(\mathtt{x}) - (c_{01} + c_{11}\mathtt{t} + c_{21}\mathtt{t}^2 + c_{31}\mathtt{t}^3 + \mathtt{Y})^2 - \sum_{i=2}^p (c_{0i} + c_{1i}\mathtt{t} + c_{2i}\mathtt{t}^2 + c_{3i}\mathtt{t}^3)^2. \end{aligned}$$

Observe that $H(0,0) = 0$ and

$$\frac{\partial H}{\partial \mathtt{Y}}(0,0) = -2(c_{01} + c_{11}\mathtt{t} + c_{21}\mathtt{t}^2 + c_{31}\mathtt{t}^3 + \mathtt{Y})|_{(0,0)} = -2(c_{01} + c_{11}\mathtt{t} + c_{21}\mathtt{t}^2 + c_{31}\mathtt{t}^3) \neq 0.$$

By the Implicit Function Theorem there exists $g(\mathtt{x}) = \sum_{\ell \geqslant 1}(d_{0\ell} + d_{1\ell}\mathtt{t} + d_{2\ell}\mathtt{t}^2 + d_{3\ell}\mathtt{t}^3)\mathtt{x}^\ell \in \mathtt{x}L[[\mathtt{x}]]$ such that

$$\begin{aligned} H(\mathtt{x}, g(\mathtt{x})) = {}& b_0 + b_1\mathtt{t} + b_2\mathtt{t}^2 + b_3\mathtt{t}^3 + (d_0 + d_1\mathtt{t} + d_2\mathtt{t}^2)P + \mathtt{x}u_0(\mathtt{x}) + \mathtt{t}\mathtt{x}u_1(\mathtt{x}) + \mathtt{t}^2\mathtt{x}u_2(\mathtt{x}) \\ & + \mathtt{t}^3\mathtt{x}u_3(\mathtt{x}) - (c_{01} + c_{11}\mathtt{t} + c_{21}\mathtt{t}^2 + c_{31}\mathtt{t}^3 + g(\mathtt{x}))^2 - \sum_{i=2}^p (c_{0i} + c_{1i}\mathtt{t} + c_{2i}\mathtt{t}^2 + c_{3i}\mathtt{t}^3)^2 = 0. \end{aligned}$$

Consequently,

$$\begin{aligned} f(\mathtt{x}, \mathtt{t}\mathtt{x}, \mathtt{t}^2\mathtt{x}) = {}& \mathtt{x}^{2k-4}\Big((c_{01}\mathtt{x}^2 + c_{11}\mathtt{t}\mathtt{x}^2 + c_{21}\mathtt{t}^2\mathtt{x}^2 + c_{31}\mathtt{t}^3\mathtt{x}^2 \\ & + \sum_{\ell \geqslant 1}(d_{0\ell}\mathtt{x}^2 + d_{1\ell}\mathtt{t}\mathtt{x}^2 + d_{2\ell}\mathtt{t}^2\mathtt{x}^2 + d_{3\ell}\mathtt{t}^3\mathtt{x}^2)\mathtt{x}^\ell)^2 + \sum_{i=2}^p (c_{0i}\mathtt{x}^2 + c_{1i}\mathtt{t}\mathtt{x}^2 + c_{2i}\mathtt{t}^2\mathtt{x}^2 + c_{3i}\mathtt{t}^3\mathtt{x}^2)^2\Big), \end{aligned}$$

Thus,

$$\begin{aligned} f(\mathtt{x}, \mathtt{y}, \mathtt{z}) = \mathtt{x}^{2k-4}\Big(&(c_{01}\mathtt{x}^2 + c_{11}\mathtt{y}\mathtt{x} + c_{21}\mathtt{z}\mathtt{x} + c_{31}\mathtt{z}\mathtt{y} + \sum_{\ell \geqslant 1}(d_{0\ell}\mathtt{x}^2 + d_{1\ell}\mathtt{y}\mathtt{x} + d_{2\ell}\mathtt{z}\mathtt{x} + d_{3\ell}\mathtt{y}\mathtt{z})\mathtt{x}^\ell)^2 \\ & + \sum_{i=2}^p (c_{0i}\mathtt{x}^2 + c_{1i}\mathtt{y}\mathtt{x} + c_{2i}\mathtt{z}\mathtt{x} + c_{3i}\mathtt{z}\mathtt{y})^2\Big) \end{aligned}$$

modulo $\mathfrak{a}$, that is, $f \in \mathcal{P}(A)$, as required. □

10.2.1. *Higher embedding dimension.* In higher embedding dimension we have the following examples:

lilb *Example* 10.4 (Linearly independent linear branches). Consider the 1-dimensional real ideal $\mathfrak{a}_n$ of $\kappa[[\mathtt{x}_1, \ldots, \mathtt{x}_n]]$ generated by the monomials $H_{ij} := \mathtt{x}_i\mathtt{x}_j$ for $i \neq j$ and the real quotient ring $C_n := \kappa[[\mathtt{x}_0, \ldots, \mathtt{x}_n]]/\mathfrak{a}_n$. The ring C_n has multiplicity n, embedding dimension n and it is not complete intersections if $n \geqslant 3$. We claim: $\mathcal{P}(C_n) = \Sigma C_n^2$ *for each* $n \geqslant 1$. ■

Proof. Let $f \in \mathcal{P}(C_n)$. By Remark 5.5(ii) we may assume $f(0, \ldots, 0) = 0$. As $\mathtt{x}_i\mathtt{x}_j \in \mathfrak{a}_n$, then

$$f = f(\mathtt{x}_1, 0, \ldots, 0) + f(0, \mathtt{x}_2, 0, \ldots, 0) + \cdots + f(0, \ldots, 0, \mathtt{x}_n) \mod \mathfrak{a}_n.$$

As $f \in \mathcal{P}(C_n)$, it holds that $f_k(0, \ldots, 0, \mathtt{x}_k, 0, \ldots, 0) \in \mathcal{P}(\kappa[[\mathtt{x}_k]]) = \Sigma\kappa[[\mathtt{x}_k]]^2$ for each $k = 1, \ldots, n$, so $f \in \Sigma C_n^2$, as required. □

s11 11. Minimal systems of generators of zero ideals

Before proving the 'only if' part of Theorem 3.2 in the following sections we prove here Lemma 3.4 and Theorem 3.5, but we need before some preliminary results.

11.1. **General case.** We prove (after some initial preparation) that if $A := \kappa[[\mathtt{x}_1, \dots, \mathtt{x}_n]]/\mathfrak{a}$ is a 1-dimensional formal ring of embedding dimension n such that $\mathcal{P}(A) = \Sigma A^2$ then $\mathfrak{a}$ contains $n-1$ elements of order 2 such that their (quadratic) leading forms are κ-linearly independent. Given $\alpha \in \operatorname{Sper}(\kappa)$, define

$$|\theta|_\alpha := \begin{cases} \theta & \text{if } \theta \geqslant_\alpha 0, \\ -\theta & \text{if } \theta <_\alpha 0 \end{cases} \tag{11.1}$$ absval

for each $\theta \in \mathfrak{R}(\alpha)$.

boundroot **Lemma 11.1.** *Let κ be a (formally) real field, $f := \sum_{k=0}^d a_k \mathtt{t}^k \in \kappa[[\mathtt{t}]]$ be a non-zero polynomial and define $M := d + \sum_{k=0}^{d-1} (\frac{a_k}{a_d})^2$. If $\alpha \in \operatorname{Sper}\kappa$, $\mathfrak{R}(\alpha)$ is the real closure of $(\kappa, <_\alpha)$ and $\theta \in \mathfrak{R}(\alpha)$ is a non-zero root of f, then $\frac{1}{M} <_\alpha |\theta|_\alpha <_\alpha M$.*

Proof. After dividing f by a_d, we may assume $a_d = 1$. Assume $|\theta|_\alpha \geqslant 1$ and observe

$$|\theta|_\alpha^d = \Big| - \sum_{k=0}^{d-1} a_k \theta^k \Big|_\alpha \quad \rightsquigarrow \quad |\theta|_\alpha \leqslant \sum_{k=0}^{d-1} |a_k|_\alpha |\theta|_\alpha^{k-d} \leqslant \sum_{k=0}^{d-1} |a_k|_\alpha \leqslant d + \sum_{k=0}^{d-1} a_k^2 = M,$$

because $|a_k|_\alpha \leqslant 1 + a_k^2$ for each $k = 0, \dots, d-1$. As $\theta \neq 0$, we deduce $\frac{1}{\theta}$ is a root of $g := \mathtt{t}^d f(\frac{1}{\mathtt{t}})$. Thus, $|\frac{1}{\theta}|_\alpha <_\alpha M$, so $\frac{1}{M} <_\alpha |\theta|_\alpha$, as required. □

Given a real reduced formal ring A of dimension 1, we find next an ample family of quadratic forms that are positive semidefinite elements on A.

p2v **Lemma 11.2.** *Let κ be a (formally) real field and let $A := \kappa[[\mathtt{x}]]/\mathfrak{a}$ be a reduced formal ring of dimension 1. After a linear change of coordinates there exist $M \in \kappa \backslash \{0\}$ such that $M^2 \mathtt{x}_i^2 - \mathtt{x}_j^2 \in \mathcal{P}(A)$ for each pair $1 \leqslant i, j \leqslant n$.*

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal prime ideals associated to A and define $A_i := A/\mathfrak{p}_i$ for $i = 1, \dots, r$. Let K be the total ring of fractions of A. We have $K = K_1 \times \cdots \times K_r$, where $K_i := \operatorname{qf}(A_i)$, and the integral closure A' of A in K is $A'_1 \times \cdots \times A'_r$, where A'_i is the integral closure of A_i in K_i. The integral closure A'_i of A_i is isomorphic to a power series ring $\kappa_i[[\mathtt{t}]]$ where κ_i is a finite real extension of κ, see [AR]. Consequently, $K_i = \kappa_i((\mathtt{t}))$ for $i = 1, \dots, r$. Consider the embedding

$$\theta : A \hookrightarrow A_1 \times \cdots \times A_r \subset A' = A'_1 \times \cdots \times A'_r \subset K = K_1 \times \cdots \times K_r.$$

given by $a \mapsto (a + \mathfrak{p}_1, \dots, a + \mathfrak{p}_r)$. Denote $\theta_i : A_i \hookrightarrow A'_i$ the inclusion.

Let $\pi_i : K \to K_i$ denote the the i-th projection and let ω_i the function order in the discrete valuation ring $\kappa_i[[\mathtt{t}]]$. We extend ω_i to $K_i = \kappa_i((\mathtt{t}))$ in the usual way and denote it again with ω_i. Let $\rho_i \in \kappa_i$ be such that $\kappa_i = \kappa[\rho_i]$ for $i = 1, \dots, r$. Let $P_i := \mathtt{t}^{d_i} + \sum_{k=0}^{d_i - 1} c_{ik} \mathtt{t}^k \in \kappa[\mathtt{t}]$ be the irreducible polynomial of ρ_i. Let $E_i|\kappa$ be the Galois closure of $\kappa_i|\kappa$ and let $G(E_i : \kappa)$ be its Galois group.

After a linear change of coordinates (with coefficients in κ), we may assume that $\omega_i(\theta_i(\mathtt{x}_{j_1})) = \omega_i(\theta_i(\mathtt{x}_{j_2}))$ for each $i = 1, \dots, r$ and $1 \leqslant j_1, j_2 \leqslant n$. Write

$$\omega_i(\theta_i(\mathtt{x}_j)) = \sum_{k \geqslant \ell_i} b_{ijk} \mathtt{t}^k \in \kappa[\rho_i][[\mathtt{t}]],$$

where $b_{ij\ell_i} \neq 0$.

It is enough to prove $f_M := M^2\mathtt{x}_1^2 - \mathtt{x}_2^2 \in \mathcal{P}(A)$ for some $M \in \kappa\backslash\{0\}$. By Lemma 5.19 and Remark 5.20(ii) we have to show: $f_M \geqslant_\beta 0$ *for each* $\beta \in \operatorname{Sper}(A)$ *such that* $\operatorname{ht}(\operatorname{supp}(\beta)) = 1$.

We fix such a $\beta \in \operatorname{Sper}(A)$. As $\dim(A) = 1$, then $\operatorname{supp}(\beta) = \mathfrak{p}_i$ for some $i = 1, \ldots, r$. Let $\alpha := \beta \cap \kappa$ and observe that $\beta \to \alpha$. Let $\mathfrak{R}(\alpha)$ be the real closure of $(\kappa, <_\alpha)$ and let $\zeta \in \mathfrak{R}(\alpha)$ be a root of P_i. By Lemma 11.1 $|\zeta|_\alpha \leqslant_\alpha d_i + \sum_{k=0}^{d_i-1} b_{ik}^2 =: N_i \in \Sigma\kappa^2$. There exists $\sigma \in G(E_i : \kappa)$ such that $\sigma(\rho_i) = \zeta$. Define $\theta_i^\sigma(\mathtt{x}_j) := \sum_{k \geqslant \ell_i} \sigma(b_{ijk})\mathtt{t}^k$ and consider the homomorphism

$$\theta_i^\sigma : A_i \to \kappa[\zeta][[\mathtt{t}]],\ g \to g(\theta_i^\sigma(\mathtt{x}_1), \ldots, \theta_i^\sigma(\mathtt{x}_n)).$$

Given $g \in A$ it holds $g \geqslant_\beta 0$ if and only if $\theta_i^\sigma(g) = g(\theta_i^\sigma(\mathtt{x}_1), \ldots, \theta_i^\sigma(\mathtt{x}_n)) \geqslant_\alpha 0$ in the unique ordering of $\mathfrak{R}(\alpha)[[\mathtt{t}]]$ that makes $\mathtt{t} > 0$. Write $\frac{\sigma(b_{i2\ell_i})}{\sigma(b_{i1\ell_i})} := \sum_{k=0}^{d_i-1} \lambda_{ik}\zeta^k$ where $\lambda_{ik} \in \kappa$ for each pair i, k. There exists a unit $U_{12i} \in \kappa[\zeta][[\mathtt{t}]]$ such that $U_{12i}(0) = 1$ and (use Cauchy-Schwarz's inequality)

$$\begin{aligned}\frac{\theta_i^\sigma(\mathtt{x}_2)^2}{\theta_i^\sigma(\mathtt{x}_1)^2} = \frac{\sigma(b_{i2\ell_i})^2}{\sigma(b_{i1\ell_i})^2}U_{12i}^2 <_\alpha 1 + \frac{\sigma(b_{i2\ell_i})^2}{\sigma(b_{i1\ell_i})^2} = 1 + \Big(\sum_{k=0}^{d_i-1} \lambda_{ik}\zeta^k\Big)^2 \\ \leqslant_\alpha 1 + \Big(\sum_{k=0}^{d_i-1} \lambda_{ij}^2\Big)\Big(\sum_{k=0}^{d_i-1} \zeta^{2k}\Big) \leqslant_\alpha 1 + \Big(\sum_{k=0}^{d_i-1} \lambda_{ij}^2\Big)\Big(\sum_{k=0}^{d_i-1} N_i^{2k}\Big) =: M_i \in \Sigma\kappa^2\end{aligned}$$

Define $M := M_1 + \cdots + M_r \in \Sigma\kappa^2$ and observe that $\frac{\theta_i^\sigma(\mathtt{x}_2)^2}{\theta_i^\sigma(\mathtt{x}_1)^2} \leqslant_\alpha M^2$ for each $i = 1, \ldots, r$, each $\sigma \in G(E_i : \kappa)$ such that $\kappa[\sigma(\rho_i)]$ is a (formally) real field and each $\alpha \in \operatorname{Sper}(\kappa[\sigma(\rho_i)])$. Thus, $M^2\mathtt{x}_1^2 - \mathtt{x}_2^2 \in \mathcal{P}(A)$, as required. □

p2v2 **Lemma 11.3** (Special generators of order two). *Let* $A := \kappa[[\mathtt{x}_1, \ldots, \mathtt{x}_n]]/\mathfrak{a}$ *be a real formal ring such that* $\mathcal{P}(A) = \Sigma A^2$ *and* $M^2\mathtt{x}_n^2 - \sum_{i=1}^{n-1} \mathtt{x}_i^2 \in \Sigma A^2$ *for some* $M \in \kappa\backslash\{0\}$. *Then there exists*

$$(\mathtt{x}_1 - g_1(\mathtt{x}_2, \ldots, \mathtt{x}_n))^2 + \sum_{k=2}^{n-1} \lambda_k U_k^2(\mathtt{x}_k - g_k(\mathtt{x}_{k+1}, \ldots, \mathtt{x}_n))^2 - \mu_n \mathtt{x}_n^2 U_n^2 \in \mathfrak{a},$$

where $g_k \in \kappa[[\mathtt{x}_{k+1}, \ldots, \mathtt{x}_n]]$, $U_k \in \kappa[[\mathtt{x}_2, \ldots, \mathtt{x}_n]]$ *is a unit with* $U_k(0, \ldots, 0) = 1$, $\lambda_k \in \Sigma\kappa^2\backslash\{0\}$ *and* $\mu_n \in \kappa\backslash(-\Sigma\kappa^2)$.

Proof. As $M^2\mathtt{x}_n^2 - \sum_{i=1}^{n-1} \mathtt{x}_i^2 \in \Sigma A^2$, there exists $f_i \in \mathfrak{m}_n \subset \kappa[[\mathtt{x}_1, \ldots, \mathtt{x}_n]]$ such that

$$f := M^2\mathtt{x}_n^2 - \sum_{i=1}^{n-1} \mathtt{x}_i^2 - \sum_{i=1}^{p} f_i^2 \in \mathfrak{a}. \qquad (11.2)$$ expf0

The series $-f(\mathtt{x}_1, 0, \ldots, 0) = \mathtt{x}_1^2 + \sum_{i=1}^p f_i(\mathtt{x}_1, 0, \ldots, 0)^2$ has order 2. By Weierstrass preparation theorem there exist $\sigma_1 \in \Sigma\kappa^2\backslash\{0\}$, a unit $V_1 \in \kappa[[\mathtt{x}_1, \ldots, \mathtt{x}_n]]$ such that $V_1(0, \ldots, 0) = 1$ and a Weierstrass polynomial $P_1 := \mathtt{x}_1^2 - 2g_1(\mathtt{x}_2, \ldots, \mathtt{x}_n)\mathtt{x}_1 + h_1(\mathtt{x}_2, \ldots, \mathtt{x}_n)$ such that $-f = P_1V_1^2\sigma_1$ and $h_1, g_1 \in \kappa[[\mathtt{x}_2, \ldots, \mathtt{x}_n]]$. Write

$$P_1 = (\mathtt{x}_1 - g_1(\mathtt{x}_2, \ldots, \mathtt{x}_n))^2 + h_1(\mathtt{x}_2, \ldots, \mathtt{x}_n) - (g_1(\mathtt{x}_2, \ldots, \mathtt{x}_n))^2.$$

We deduce (using the definition of f and substituting $\mathtt{x}_1 = g_1(\mathtt{x}_2, \ldots, \mathtt{x}_n)$ and $\mathtt{x}_n = 0$ in (11.2)) that

$$\begin{aligned}&h_1(\mathtt{x}_2, \ldots, \mathtt{x}_{n-1}, 0) - (g_1(\mathtt{x}_2, \ldots, \mathtt{x}_{n-1}, 0))^2 \\ &\quad= \frac{\sigma_1}{V_1(g_1(\mathtt{x}_2, \ldots, \mathtt{x}_n), \mathtt{x}_2, \ldots, \mathtt{x}_{n-1}, 0)^2\sigma_1^2}\Big((g_1(\mathtt{x}_2, \ldots, \mathtt{x}_{n-1}, 0))^2 + \sum_{i=2}^{n-1} \mathtt{x}_i^2 \\ &\qquad + \sum_{i=1}^{p} f_i(g_1(\mathtt{x}_2, \ldots, \mathtt{x}_{n-1}, 0), \mathtt{x}_2, \ldots, \mathtt{x}_{n-1}, 0)^2\Big).\end{aligned}$$

By Weierstrass preparation theorem there exist $\sigma_2 \in \Sigma\kappa^2\backslash\{0\}$, a unit $V_2 \in \kappa[[\mathtt{x}_2,\dots,\mathtt{x}_n]]$ such that $V_2(0,\dots,0)=1$ and a Weierstrass polynomial $P_2 := \mathtt{x}_2^2 - 2g_2(\mathtt{x}_3,\dots,\mathtt{x}_n)\mathtt{x}_2 + h_2(\mathtt{x}_3,\dots,\mathtt{x}_n)$ such that $h_1(\mathtt{x}_2,\dots,\mathtt{x}_n) - (g_1(\mathtt{x}_2,\dots,\mathtt{x}_n))^2 = P_2V_2^2\sigma_2$ and $h_2, g_2 \in \kappa[[\mathtt{x}_3,\dots,\mathtt{x}_n]]$. Thus,

$$P_1 = (\mathtt{x}_1 - g_1(\mathtt{x}_2,\dots,\mathtt{x}_n))^2 + V_2^2\sigma_2((\mathtt{x}_2 - g_2(\mathtt{x}_3,\dots,\mathtt{x}_n))^2 + h_2(\mathtt{x}_3,\dots,\mathtt{x}_n) - g_2(\mathtt{x}_3,\dots,\mathtt{x}_n)^2).$$

Proceeding inductively we find series $g_k, h_k \in \kappa[[\mathtt{x}_{k+1},\dots,\mathtt{x}_n]]$, units $V_k \in \kappa[[\mathtt{x}_k,\dots,\mathtt{x}_n]]$ with $V_k(0,\dots,0)=1$ and $\sigma_k \in \Sigma\kappa^2\backslash\{0\}$ such that

$$P_1 = (\mathtt{x}_1 - g_1(\mathtt{x}_2,\dots,\mathtt{x}_n))^2 + \sum_{k=2}^{n-1}\Big(\prod_{j=2}^{k}\sigma_jV_j^2\Big)(\mathtt{x}_k - g_k(\mathtt{x}_{k+1},\dots,\mathtt{x}_n))^2 + \Big(\prod_{j=2}^{n-1}\sigma_jV_j^2\Big)(h_n(\mathtt{x}_n) - g_n(\mathtt{x}_n)^2).$$

Denote $\lambda_k := \prod_{j=2}^k \sigma_j$ and $U_k := \prod_{j=2}^k V_j \in \kappa[[\mathtt{x}_2,\dots,\mathtt{x}_n]]$, which satisfies $U_k(0,\dots,0)=1$. If $h_n(\mathtt{x}_n)^2 - g_n(\mathtt{x}_n) = 0$, we use that $\mathfrak{a}$ is radical [Sch2, Lem.6.3] to deduce $\mathtt{x}_k - g_k(\mathtt{x}_{k+1},\dots,\mathtt{x}_n) \in \mathfrak{a}$ for $k=1,\dots,n-1$, which contradicts the fact that $\omega(\mathfrak{a})=2$. Thus, $h_n(\mathtt{x}_n) - g_n(\mathtt{x}_n)^2 \neq 0$ and we find $k \geqslant 2$, $\eta \in \kappa\backslash\{0\}$ and a unit $V_n \in \kappa[[\mathtt{x}_n]]$ such that $V_n(0)=1$ and $h_n(\mathtt{x}_n) - g_n(\mathtt{x}_n)^2 = -\eta\mathtt{x}_n^kV_n^2$. Let us check: $k=2$.

If k is odd, then $\eta\mathtt{x}_n \in \mathcal{P}(A)\backslash\Sigma A^2$, which is a contradiction. Thus, $k=2\ell$ and let us check: $\ell = 1$.

Assume $\ell \geqslant 2$ and pick $\beta \in \operatorname{Sper}(A)$. We have

$$\begin{aligned}(\mathtt{x}_1 - g_1(\mathtt{x}_2,\dots,\mathtt{x}_n))^2 &\leqslant_\beta (\mathtt{x}_1 - g_1(\mathtt{x}_2,\dots,\mathtt{x}_n))^2 \\ &\quad + \sum_{k=2}^{n-1}\lambda_kU_k^2(\mathtt{x}_k - g_k(\mathtt{x}_{k+1},\dots,\mathtt{x}_n))^2 = \eta V_n^2\mathtt{x}_n^{2\ell}\Big(\prod_{j=2}^{n-1}\sigma_jV_j^2\Big) \\ &\leqslant_\beta (\eta^2+1)^2\mathtt{x}_n^{2\ell}V_n^2\Big(\prod_{j=2}^{n-1}(\sigma_k^2+1)^2V_k^2\Big) \leqslant_\beta (\eta^2+1)^2\mathtt{x}_n^4V_n^2\Big(\prod_{j=2}^{n-1}(\sigma_k^2+1)^2V_k^2\Big),\end{aligned}$$

so $(\eta^2+1)\mathtt{x}_n^2V_n(\prod_{j=2}^{n-1}(\sigma_k^2+1)V_k) - (\mathtt{x}_1 - g_1(\mathtt{x}_2,\dots,\mathtt{x}_n)) \in \mathcal{P}(A)\backslash\Sigma A^2$, which is a contradiction. Consequently, $\ell = 1$ and if we write $\mu_n := \eta\prod_{j=2}^{n-1}\sigma_j$ and $U_j := \prod_{j=2}^n V_j$, we have

$$(\mathtt{x}_1 - g_1(\mathtt{x}_2,\dots,\mathtt{x}_n))^2 + \sum_{k=2}^{n-1}\lambda_kU_k^2(\mathtt{x}_k - g_k(\mathtt{x}_{k+1},\dots,\mathtt{x}_n))^2 - \mu_n\mathtt{x}_n^2U_n^2 \in \mathfrak{a}.$$

If $\mu_n \in -\Sigma\kappa^2$, we use that $\mathfrak{a}$ is radical [Sch2, Lem.6.3] to deduce $\mathtt{x}_k - g_k(\mathtt{x}_{k+1},\dots,\mathtt{x}_n), \mathtt{x}_n \in \mathfrak{a}$ for $k=1,\dots,n-1$, which contradicts the fact that $\omega(\mathfrak{a})=2$. Consequently, $\mu_n \in \kappa\backslash-\Sigma\kappa^2$, as required. □

ngen2 **Corollary 11.4** (Number of generators of order two)**.** *Let $A := \kappa[[\mathtt{x}_1,\dots,\mathtt{x}_n]]/\mathfrak{a}$ be a real formal ring of dimension 1 and embedding dimension n such that $\mathcal{P}(A) = \Sigma A^2$. Then there exist $n-1$ elements of order 2 in $\mathfrak{a}$ whose leading forms (of order 2) are κ-linearly independent.*

Proof. By Lemma 11.2 there exists (after a change of coordinates) $M \in \kappa\backslash\{0\}$ such that $M^2\mathtt{x}_i^2 - \mathtt{x}_j^2 \in \mathcal{P}(A)$ for each pair $1 \leqslant i,j \leqslant n$ with $i \neq j$. Then $(n-1)M^2\mathtt{x}_i^2 - \sum_{j\neq i}\mathtt{x}_i^2 \in \mathcal{P}(A)\backslash\Sigma A^2$. By Lemma 11.3 there exist for each $i=2,\dots,n$

$$f_i := (\mathtt{x}_1 - g_{2i}(\mathtt{x}_2,\dots,\mathtt{x}_n))^2 + \sum_{j\neq i}\lambda_jU_j^2(\mathtt{x}_j - g_{ji}(\mathtt{x}_{j+1},\dots,\mathtt{x}_n))^2 - \mu_i\mathtt{x}_i^2U_i^2 \in \mathfrak{a}$$

where $\lambda_j \in \Sigma\kappa^2\backslash\{0\}$, $\mu_i \in \kappa\backslash-\Sigma\kappa^2$, $U_j \in \kappa[[\mathtt{x}_2,\dots,\mathtt{x}_n]]$ are units such that $U_j(0,\dots,0)=1$.

Suppose there exist $\theta_i \in \kappa\setminus\{0\}$ such that $\sum_{i=2}^n \theta_i f_i = 0$. As $\mathcal{P}(A) = \Sigma A^2$, we have $\dim_r(A) = \dim(A) = 1$ and there exists a specialization $\beta \to \alpha$ such that $\operatorname{supp}(\alpha) = \mathfrak{m}$ and $\alpha \in \operatorname{Sper}(\kappa)$. Observe that $A/\operatorname{supp}(\beta)$ is a real ring of dimension 1. Let $\mathcal{F}_0 := \{i = 2, \dots, n : \ \lambda_i = 0\}$, $\mathcal{F}_1 := \{i = 2, \dots, n : \ \lambda_i >_\alpha 0\}$ and $\mathcal{F}_{-1} := \{i = 2, \dots, n : \ \lambda_i <_\alpha 0\}$. Suppose $\mathcal{F}_0 \neq \{i = 2, \dots, n\}$. If $\mathcal{F}_1 = \varnothing$, then $\sum_{i\in\mathcal{F}_{-1}} \theta_i f_i = 0$. Making $\mathtt{x}_2 = 0, \dots, \mathtt{x}_n = 0$, we have $\sum_{i\in\mathcal{F}_{-1}} \lambda_i \mathtt{x}_1^2 = 0$, which is a contradiction, because $\lambda_i <_\alpha 0$ for each $i \in \mathcal{F}_{-1}$. Here we have used that we have chosen $n-1$ elements of order 2.

Consequently, $\mathcal{F}_1 \neq \varnothing$. Analogously, $\mathcal{F}_{-1} \neq \varnothing$. Reordering the variables $\mathtt{x}_2, \dots, \mathtt{x}_n$, we may assume $\mathcal{F}_1 = \{\ell, \dots, r\}$ for some $2 \leqslant \ell < r < n$ and $\mathcal{F}_{-1} = \{r+1, \dots, n\}$. Write $\eta_i := -\theta_i$ for each $i \in \mathcal{F}_{-1}$. We have $\sum_{i\in\mathcal{F}_1} \theta_i f_i - \sum_{i\in\mathcal{F}_{-1}} \eta_i f_i = 0$, so $\sum_{i=\ell}^r \theta_i f_i = \sum_{i=r+1}^n \eta_i f_i$. Denote q_i the leading form of f_i for $i = 2, \dots, r$. We have

$$\sum_{i=r+1}^n \eta_i q_i = \sum_{k=1}^r \rho_k(\mathtt{x}_k - \ell_k(\mathtt{x}_{k+1}, \dots, \mathtt{x}_n))^2 + p_1(\mathtt{x}_{r+1}, \dots, n),$$

where $\rho_k \in \kappa\setminus\{0\}$, $\rho_k >_\alpha 0$, $\ell_k \in \kappa[\mathtt{x}_{k+1}, \dots, \mathtt{x}_n]$ is a linear form and $p_1 \in \kappa[\mathtt{x}_{k+1}, \dots, \mathtt{x}_n]$ is a quadratic form, which is not positive semidefinite because $A/\operatorname{supp}(\beta)$ is a real ring of dimension 1. In addition,

$$\sum_{i=\ell}^r \theta_i q_i = \sum_{k=1}^{\ell} \sigma_k(\mathtt{x}_k - m_k(\mathtt{x}_{k+1}, \dots, \mathtt{x}_n))^2 + \sum_{k=r+1}^n \sigma_k(\mathtt{x}_k - m_k(\mathtt{x}_{\ell+1}, \dots, \mathtt{x}_r, \mathtt{x}_{k+1}, \dots, \mathtt{x}_n))^2 + p_2(\mathtt{x}_{\ell+1}, \dots, \mathtt{x}_r)$$

where $\sigma_k \in \kappa\setminus\{0\}$, $\sigma_k >_\alpha 0$, the linear forms $m_k \in \kappa[\mathtt{x}_{k+1}, \dots, \mathtt{x}_n]$ for $k = 1, \dots, \ell$ and $m_k \in \kappa[\mathtt{x}_{\ell+1}, \dots, \mathtt{x}_r, \mathtt{x}_{k+1}, \dots, \mathtt{x}_n]$ is a linear form and $p_2 \in \kappa[\mathtt{x}_{\ell+1}, \dots, \mathtt{x}_r]$ is a quadratic form, which is not positive semidefinite because $A/\operatorname{supp}(\beta)$ is a real ring of dimension 1. As $\sum_{i=\ell}^r \theta_i q_i = \sum_{i=r+1}^n \eta_i q_i$, we deduce after making $\mathtt{x}_k = \ell_k(\mathtt{x}_{k+1}, \dots, \mathtt{x}_n)$ for $k = 1, \dots, \ell$ that

$$p_1(\mathtt{x}_{r+1}, \dots, n) = \sum_{k=1}^{\ell} \sigma_k(\ell_k(\mathtt{x}_{k+1}, \dots, \mathtt{x}_n) - m_k(\mathtt{x}_{k+1}, \dots, \mathtt{x}_n))^2 + \sum_{k=r+1}^n \sigma_k(\mathtt{x}_k - m_k(\mathtt{x}_{\ell+1}, \dots, \mathtt{x}_r, \mathtt{x}_{k+1}, \dots, \mathtt{x}_n))^2 + p_2(\mathtt{x}_{\ell+1}, \dots, \mathtt{x}_r).$$

If we make $\mathtt{x}_{\ell+1} = 0, \dots, \mathtt{x}_r = 0$, we deduce

$$p_1(\mathtt{x}_{r+1}, \dots, n) = \sum_{k=1}^{\ell} \sigma_k(\ell_k(\mathtt{x}_{k+1}, \dots, \mathtt{x}_\ell, 0, \dots, 0, \mathtt{x}_{r+1}, \cdots, \mathtt{x}_n) - m_k(\mathtt{x}_{k+1}, \dots, \mathtt{x}_\ell, 0, \dots, 0, \mathtt{x}_{r+1}, \cdots, \mathtt{x}_n))^2 + \sum_{k=r+1}^n \sigma_k(\mathtt{x}_k - m_k(0, \dots, 0, \mathtt{x}_{k+1}, \dots, \mathtt{x}_n))^2,$$

so p_1 is a positive semidefinite quadratic form, which is a contradiction. Consequently, $\mathcal{F}_0 = \{i = 2, \dots, n\}$ and the leading forms (of order 2) of the $n-1$ generators f_i are κ-linearly independent, as required. □

Remark 11.5. We do not know if the previous bound is sharp (even if $n = 3$, because we conjecture there exists no chimeric polynomial over a (formally) real field), but in any case if we consider the quadratic forms $F_k := \sum_{j\neq k} \mathtt{x}_j^2 - n\mathtt{x}_k$, it holds $F_1 + \cdots + F_n = 0$. ■

We are ready to prove Lemma 3.4.

Proof of Lemma 3.4. If $n=1$, then $A\cong\kappa[[\mathtt{t}]]$ (because $\dim(A)=1$). Thus, we assume $n=2$. By Lemma 11.3 we may assume after a linear change of coordinates that $\mathtt{x}^2-a\mathtt{y}^2\in\mathfrak{a}$ for some $a\notin-\Sigma A^2$. As $\dim(A)=1$, A has embedding dimension 2 and $\omega(\mathtt{x}^2-a\mathtt{y}^2)=2$, we deduce $\mathfrak{a}=(\mathtt{x}^2-a\mathtt{y}^2)\kappa[[\mathtt{x},\mathtt{y}]]$ for some $a\notin-\Sigma A^2$, as required. □

11.2. **Minimal systems of generators.** Our next purpose is to prove Theorem 3.5 after some preliminary preparation.

generators0 **Lemma 11.6.** *Let* $\widehat{A}=\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/\mathfrak{a}$ *be a formal ring of embedding dimension* 3 *such that* $\mathcal{P}(A)=\Sigma A^2$. *After a change of coordinates* $\mathtt{z}^2-\mathtt{x}\mathtt{y},F+\mathtt{z}G\in\mathfrak{a}$ *for some series* $F,G\in\kappa[\mathtt{x},\mathtt{y}]$ *such that* $\omega(F)=2$.

Proof. By Lemma 11.2 we may assume after a linear change of coordinates that there exist $M\in\kappa\backslash\{0\}$ such that $M^2\mathtt{z}^2-\mathtt{x}^2,M^2\mathtt{z}^2-\mathtt{y}^2\in\mathcal{P}(A)$. Thus, $2M^2\mathtt{z}^2-\mathtt{x}^2-\mathtt{y}^2\in\mathcal{P}(A)=\Sigma A^2$. By Lemma 11.3 there exist

$$(\mathtt{x}+g_1(\mathtt{y},\mathtt{z}))^2+\lambda_2U_2(\mathtt{y},\mathtt{z})^2(\mathtt{y}+g_2(\mathtt{z}))^2-\mu\mathtt{z}^2\lambda_2U_2(\mathtt{y},\mathtt{z})^2U_3(\mathtt{z})^2\in\mathfrak{a}\qquad(11.3)$$ eqdeg2

for some series $g_1,h_1\in\kappa[[\mathtt{y},\mathtt{z}]]$, $g_2\in\kappa[[\mathtt{z}]]$, $h_2\in\kappa[[\mathtt{y}]]$, units $U_2,V_2\in\kappa[[\mathtt{y},\mathtt{z}]]$, $U_3\in\kappa[[\mathtt{z}]]$, $V_2\in\kappa[[\mathtt{y}]]$ such that $U_2(0,0)=1$, $V_2(0,0)=1$, $U_3(0)=1$, $V_3(0)=1$, sums of squares $\lambda_2,\eta_3\in\Sigma\kappa^2\backslash\{0\}$ and $\mu,\rho\in\kappa\backslash-\Sigma\kappa^2$.

We already know from Lemma 11.4 that $\mathfrak{a}$ contains two generators of order 2 with κ-linearly independent leading forms (one of them the one in (11.3)), however we need to work a little bit more to understand the shape of the other generator of order 2. After a change of coordinates of the type $\mathtt{y}:=\mathtt{y}'+\mu\mathtt{z}$, we may assume that equation (11.3) is regular with respect to $\mathtt{z}$. Observe that this change does not affect the series $M^2\mathtt{x}^2-\mathtt{z}^2\in\mathcal{P}(A)$ and $M^2\mathtt{z}^2-\mathtt{x}^2\in\mathcal{P}(A)$. In addition, there exist a unit $U'\in\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ such that $U'(0,0,0)=1$ and a Weierstrass polynomial $Q\in\kappa[[\mathtt{x},\mathtt{y}]][\mathtt{z}]$ of degree 2 such that

$$a^2\mathtt{z}^2W^2V^2-a(\mathtt{x}+g_1(\mathtt{y},\mathtt{z}))^2-a\lambda((\mathtt{y}+g_1'(\mathtt{z}))V)^2=a^2QU'.$$

Observe that $-aQ(\mathtt{x},\mathtt{y},0)\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]])$ and its leading form is a positive definite quadratic form (of rank 2). Let $Q'\in\mathfrak{a}$ be a series of order 2. By Weierstrass division theorem we may assume $Q'=\mathtt{z}G+F$, where $F,G\in\kappa[[\mathtt{x},\mathtt{y}]]$. Suppose all the series of order 2 of $\mathfrak{a}$ of the form $\mathtt{z}G+F$ have $\omega(F)\geqslant 3$. As $M^2\mathtt{x}^2-\mathtt{z}^2\in\mathcal{P}(A)=\Sigma A^2$, there exist $\theta_1,\theta_2\in\kappa$ such that the leading form of $M^2\mathtt{x}^2-\mathtt{z}^2+\theta_1Q+\theta_2(\mathtt{z}G+F)$ is a positive semidefinite quadratic form. In particular, $\theta_1-1\in\Sigma\kappa^2$, so $\theta_1\in\Sigma\kappa^2\backslash\{0\}$. In addition, if we make $\mathtt{z}=0$, the leading form of $M^2\mathtt{x}^2-\frac{\theta_1}{a}(-aQ(\mathtt{x},\mathtt{y},0))$ is a positive semidefinite quadratic form. But this is a contradiction because $-aQ(\mathtt{x},\mathtt{y},0)$ is positive definite quadratic form (of rank 2) and $-a\notin\Sigma\kappa^2$. Consequently, there exist $F,G\in\kappa[[\mathtt{x},\mathtt{y}]]$ such that $\omega(F)=2$ and $F+\mathtt{z}G\in\mathfrak{a}$.

If we make the change of coordinates $\mathtt{x}+g_1(\mathtt{y}'+\mu\mathtt{z},\mathtt{z})\mapsto\mathtt{x}$, $\mathtt{y}'+\mu\mathtt{z}+g_2(\mathtt{z})\mapsto\mathtt{y}$, $\mathtt{z}\mapsto\mathtt{z}$, we find $a(\mathtt{x}^2+\lambda\mathtt{y}^2)-\mathtt{z}^2,F+\mathtt{z}G\in\mathfrak{a}$ where $-a\notin\Sigma\kappa^2$, $\lambda\in\Sigma\kappa^2\backslash\{0\}$ and $\omega(F)=2$. The last condition hold because if we make $\mathtt{z}=0$ in the change $\mathtt{x}+g_1(\mathtt{y}'+\mu\mathtt{z},\mathtt{z})\mapsto\mathtt{x}$, $\mathtt{y}'+\mu\mathtt{z}+g_2(\mathtt{z})\mapsto\mathtt{y}$, $\mathtt{z}\mapsto\mathtt{z}$, we obtain $\mathtt{x}+g_1(\mathtt{y}',0)\mapsto\mathtt{x}$ and $\mathtt{y}'\mapsto\mathtt{y}$.

Let us check: *We may assume (after a change of coordinates) that* $\mathtt{z}^2-\mathtt{x}\mathtt{y}\in\mathfrak{a}$. We distinguish several cases:

Case 1. $\omega(G)=1$ *and the leading form of* $F+\mathtt{z}G$ *has rank* 3. Assume first the leading form of F has rank 2. After a linear change of coordinates that involves only the variables $\mathtt{x},\mathtt{y}$ we may assume $F=\mathtt{x}^2+\mu\mathtt{y}^2+F_3$ and $G=b\mathtt{x}+c\mathtt{y}+G_2$ where $\mu,b,c\in\kappa\backslash\{0\}$, $\mu\neq 0$, b or c are non-zero, $F_3\in\mathfrak{m}_2^3$ and $G_2\in\mathfrak{m}_2^2$. We distinguish two situations:

Subcase 1.1. $b\neq 0$. The leading form of $F+\mathtt{z}G$ is $H:=\mathtt{x}^2+\mu\mathtt{y}^2+\mathtt{z}(b\mathtt{x}+c\mathtt{y})$. After the change of coordinates

$$\mathtt{x}\mapsto\mathtt{x}-\tfrac{b\mathtt{z}}{2},\ \mathtt{z}\mapsto\mathtt{z}+\tfrac{2c}{b^2}\mathtt{y}$$

we have $H = \mathtt{x}^2 - \frac{b^2}{4}\mathtt{z}^2 + (\frac{c^2}{b^2} + \mu)\mathtt{y}^2$. After the change of coordinates

$$\mathtt{x} + \tfrac{b}{2}\mathtt{z} \mapsto (\tfrac{c^2}{b^2} + \mu)\mathtt{x},\ \mathtt{x} - \tfrac{b}{2}\mathtt{z} \mapsto \mathtt{y},\ \mathtt{y} \mapsto \mathtt{z},$$

we have $H = (\frac{c^2}{b^2} + \mu)(\mathtt{xy} - \mathtt{z}^2)$. By Theorem 5.7 $F + \mathtt{z}G$ is right equivalent to $(\frac{c^2}{b^2} + \mu)(\mathtt{z}^2 - \mathtt{xy})$.

Subcase 1.2. $b = 0, c \neq 0$. The leading form of $F + \mathtt{z}G$ is $H := \mathtt{x}^2 + \mu\mathtt{y}^2 + c\mathtt{zy}$. After the change of coordinates $\mathtt{y} \mapsto \mathtt{y} - \frac{c}{2\mu}\mathtt{z}$ we have $H = \mu(\mathtt{y}^2 - (\frac{c}{2\mu}\mathtt{z})^2) + \mathtt{x}^2$. After the change of coordinates

$$\mu(\mathtt{y} + \frac{c}{2\mu}\mathtt{z}) \mapsto \mathtt{x},\ \mathtt{y} - \frac{c}{2\mu}\mathtt{z} \mapsto \mathtt{y},\ \mathtt{x} \mapsto \mathtt{z},$$

we have $H = \mathtt{xy} - \mathtt{z}^2$. By Theorem 5.7 $F + \mathtt{z}G$ is right equivalent to $\mathtt{z}^2 - \mathtt{xy}$.

If the leading form of F has rank 1, we may assume (after a linear change of coordinates) that it is $\mathtt{x}^2$ and the leading form of $F + \mathtt{z}G$ is $H := \mathtt{x}^2 + \mathtt{z}(b\mathtt{x} + c\mathtt{y})$ where $b, c \in \kappa$ and $c \neq 0$. After the change $\mathtt{x} \mapsto \mathtt{z},\ b\mathtt{x} + c\mathtt{y} \mapsto \mathtt{y},\ \mathtt{z} \mapsto \mathtt{x}$, we deduce $H = \mathtt{z}^2 - \mathtt{xy}$. By Theorem 5.7 $F + \mathtt{z}G$ is right equivalent to $\mathtt{z}^2 - \mathtt{xy}$. ∎

Case 2. $\omega(G) = 1$ *and the leading form of* $F + \mathtt{z}G$ *has rank* 2.This means that the leading form H of F has rank 1 and it is proportional to G^2, that is, $H = G(G + \mathtt{z})$ (maybe after rescaling $\mathtt{z}$). After interchanging $\mathtt{x}, \mathtt{y}$ and rescaling the variables if necessary, we may assume that the leading form of G is either $\mathtt{x}$ or $\mathtt{x} + \mathtt{y}$. We distinguish both situations:

Subcase 2.1. The leading form of G is $\mathtt{x}$ and the leading form of $F + \mathtt{z}G$ is $\mathtt{x}^2 + \mathtt{xz}$. We have

$$a(\mathtt{x}^2 + \lambda\mathtt{y}^2) - \mathtt{z}^2 - a(F + \mathtt{z}G) \in \mathfrak{a}$$

and its leading form is $a\lambda\mathtt{y}^2 - \mathtt{z}^2 - a\mathtt{xz}$, which is right equivalent to $a\lambda(\mathtt{z}^2 - \mathtt{xy})$ (use for instance the change of coordinates $\mathtt{y} \mapsto \mathtt{z},\ \mathtt{z} - a\mathtt{x} \mapsto \mathtt{x},\ \mathtt{z} \mapsto \mathtt{y}$). By Theorem 5.7 $a(\mathtt{x}^2 + \lambda\mathtt{y}^2) - \mathtt{z}^2 - a(F + \mathtt{z}G)$ is right equivalent to $\mathtt{z}^2 - \mathtt{xy}$.

Subcase 2.2. The leading form of G is $\mathtt{x} + \mathtt{y}$ and the leading form of $F + \mathtt{z}G$ is $(\mathtt{x} + \mathtt{y})^2 + (\mathtt{x} + \mathtt{y})\mathtt{z}$. The leading form of $a(\mathtt{x}^2 + \lambda\mathtt{y}^2) - \mathtt{z}^2 - a(F + \mathtt{z}G) \in \mathfrak{a}$ is

$$H := -2a\mathtt{xy} + a(\lambda - 1)\mathtt{y}^2 - \mathtt{z}^2 - a(\mathtt{x} + \mathtt{y})\mathtt{z} = -2a\mathtt{xy} + a(\lambda - 1)\mathtt{y}^2 - (\mathtt{z} + \tfrac{a}{2}(\mathtt{x} + \mathtt{y}))^2 + \tfrac{a^2}{4}(\mathtt{x}^2 + 2\mathtt{xy} + \mathtt{y}^2)$$

After the change of coordinates $\mathtt{z} + \frac{a}{2}(\mathtt{x} + \mathtt{y}) \to \mathtt{z}$ we have

$$H = \tfrac{a^2}{4}\mathtt{x}^2 + (\tfrac{1}{2}a^2 - 2a)\mathtt{xy} + (\tfrac{1}{4}a^2 - a + a\lambda)\mathtt{y}^2 - \mathtt{z}^2 = (\tfrac{a}{2}\mathtt{x} + (\tfrac{1}{2}a - 2)\mathtt{y})^2 - \mathtt{z}^2 + \mu\mathtt{y}^2$$

where $\mu = a(\lambda + 1) - 4$. If $\mu \neq 0$, after the change of coordinates

$$\mu(\tfrac{a}{2}\mathtt{x} + (\tfrac{1}{2}a - 2)\mathtt{y} - \mathtt{z}) \mapsto \mathtt{x},\ \mu(\tfrac{a}{2}\mathtt{x} + (\tfrac{1}{2}a - 2)\mathtt{y} + \mathtt{z}) \mapsto \mathtt{y},\ \mathtt{y} \mapsto \mathtt{z},$$

we have $H = \mu(\mathtt{xy} - \mathtt{z}^2)$, so by Theorem 5.7 $a(\mathtt{x}^2 + \lambda\mathtt{y}^2) - \mathtt{z}^2 - a(F + \mathtt{z}G) \in \mathfrak{a}$ is right equivalent to $\mu(\mathtt{xy} - \mathtt{z}^2)$.

Suppose next $\mu = 0$, so $a(\lambda + 1) = 4$ and $a \neq 4$ (because $\lambda \neq 0$). After the change of coordinates $\mathtt{x} \mapsto \mathtt{x} + \mathtt{y}$, we have

$$a\mathtt{x}^2 + 2a\mathtt{xy} + a(\lambda + 1)\mathtt{y}^2 - \mathtt{z}^2 = a((\mathtt{x} + \mathtt{y})^2 + \lambda\mathtt{y}^2) - \mathtt{z}^2 \in \mathfrak{a}.$$

Thus, $a\mathtt{x}^2 + 2a\mathtt{xy} + 4\mathtt{y}^2 - \mathtt{z}^2 \in \mathfrak{a}$. Observe that

$$a\mathtt{x}^2 + 2a\mathtt{xy} + 4\mathtt{y}^2 - \mathtt{z}^2 = (2\mathtt{y} + \tfrac{a}{2}\mathtt{x})^2 - \mathtt{z}^2 + (a - \tfrac{a^2}{4})\mathtt{y}^2 = (2\mathtt{y} + \tfrac{a}{2}\mathtt{x} - \mathtt{z})(2\mathtt{y} + \tfrac{a}{2}\mathtt{x} + \mathtt{z}) + \mu\mathtt{y}^2$$

where $\mu = \frac{a}{4}(4 - a) \neq 0$. After the change of coordinates

$$2\mathtt{y} + \tfrac{a}{2}\mathtt{x} - \mathtt{z} \mapsto -\mu\mathtt{x},\ 2\mathtt{y} + \tfrac{a}{2}\mathtt{x} + \mathtt{z} \mapsto \mathtt{y},\ \mathtt{y} \mapsto \mathtt{z}$$

we have $\mu(-\mathtt{xy} + \mathtt{z}^2) \in \mathfrak{a}$. ∎

Case 3. $\omega(G) \geqslant 2$. We distinguish several situations:

SUBCASE 3.1. $F := \mathtt{x}\mathtt{y} + F'$ *where* $F' \in \mathfrak{m}_2^3$. Consider the element of $\mathfrak{a}$:

$$\begin{aligned}a(1+\lambda)(F+\mathtt{z}G)+a(\mathtt{x}^2+\lambda\mathtt{y}^2)-\mathtt{z}^2 &= a(1+\lambda)(\mathtt{x}\mathtt{y}+F'+\mathtt{z}G)+a(\mathtt{x}^2+\lambda\mathtt{y}^2)-\mathtt{z}^2\\ &= a(\mathtt{x}^2+\mathtt{x}\mathtt{y}(1+\lambda)+\lambda\mathtt{y}^2)+a(1+\lambda)F'+a^2(1+\lambda)^2\frac{G^2}{4}-\Big(\mathtt{z}-a(1+\lambda)\frac{G}{2}\Big)^2\\ &= a\Big(\Big(\mathtt{x}+\frac{(1+\lambda)}{2}\mathtt{y}\Big)^2-\Big(\frac{(1-\lambda)}{2}\mathtt{y}\Big)^2\Big)+\Big(\frac{a(1+\lambda)}{2}G\Big)^2-\Big(\mathtt{z}-a(1+\lambda)\frac{G}{2}\Big)^2+a(1+\lambda)F'\end{aligned}$$

After the change of coordinates

$$a(\mathtt{x}+\mathtt{y})\mapsto\mathtt{x},\ \mathtt{x}+\lambda\mathtt{y}\mapsto\mathtt{y},\ \mathtt{z}-a(1+\lambda)G\mapsto\mathtt{z},$$

we obtain the series $\mathtt{x}\mathtt{y}-\mathtt{z}^2+h\in\mathfrak{a}$ for some $h\in\mathfrak{m}_2^3$. By Theorem 5.7 $\mathtt{x}\mathtt{y}-\mathtt{z}^2+h$ is right equivalent to $\mathtt{z}^2-\mathtt{x}\mathtt{y}$.

SUBCASE 3.2. *Either* $\omega(F(\mathtt{x},0))=2$ *or* $\omega(F(0,\mathtt{y}))=2$. After interchanging the variables $\mathtt{x}$ and $\mathtt{y}$, we may assume the leading form of F is $\mathtt{x}^2+b\mathtt{x}\mathtt{y}+c\mathtt{y}^2$ for some $b,c\in\kappa$. Thus, the leading form of $a(\mathtt{x}^2+\lambda\mathtt{y}^2)-\mathtt{z}^2-(a-m^2)(F+\mathtt{z}G)\in\mathfrak{a}$ is

$$H := m^2\mathtt{x}^2+(a-m^2)b\mathtt{x}\mathtt{y}+(a\lambda-ac+cm^2)\mathtt{y}^2-\mathtt{z}^2$$

for each $m\in\kappa$. We choose $m\in\kappa$ such that the quadratic form

$$m^2\mathtt{x}^2+(a-m^2)b\mathtt{x}\mathtt{y}+(a\lambda-ac+cm^2)\mathtt{y}^2$$

has rank 2. We have

$$H=\Big(m\mathtt{x}+\frac{(a-m^2)b}{2m}\mathtt{y}\Big)^2-\mathtt{z}^2+\mu\mathtt{y}^2$$

where $\mu := a\lambda-ac+cm^2-\frac{(a-m^2)^2b^2}{4m^2}\in\kappa\backslash\{0\}$. After the change of coordinates

$$m\mathtt{x}+\frac{(a-m^2)b}{2m}\mathtt{y}-\mathtt{z}\mapsto\mu\mathtt{x},\ m\mathtt{x}+\frac{(a-m^2)b}{2m}\mathtt{y}+\mathtt{z}\mapsto\mathtt{y},\ \mathtt{y}\mapsto\mathtt{z}$$

we have $H=\mu(\mathtt{x}\mathtt{y}-\mathtt{z}^2)$. By Theorem 5.7 $\mathtt{x}\mathtt{y}-\mathtt{z}^2+h$ is right equivalent to $\mathtt{z}^2-\mathtt{x}\mathtt{y}$.

We have proved that after a change of coordinates $\mathtt{z}^2-\mathtt{x}\mathtt{y}\in\mathfrak{a}$ and we have additional a series $F+\mathtt{z}G$, where $\omega(F)=2$. □

By Lemma 11.6 the proof of Theorem 3.5 is reduced to prove the following.

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Theorem 11.7. *Let* $A := \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/\mathfrak{a}$ *be a* 1*-dimensional formal ring such that* $\mathtt{z}^2-\mathtt{x}\mathtt{y}\in\mathfrak{a}$*, there exists* $F+\mathtt{z}G\in\mathfrak{a}$ *with* $\omega(F)=2$*,* $\omega(G)\geqslant 1$ *and* $\omega(\mathfrak{a})=2$*. Then either:*

(i) $\mathfrak{a}=(\mathtt{z}^2-\mathtt{x}\mathtt{y},F+\mathtt{z}G)$*, or*

(ii) $\mathfrak{a}=(\mathtt{z}^2-\mathtt{x}\mathtt{y},(\mathtt{z}+F_1)G_2-F_2G_1,(\mathtt{z}-F_1)G_1-F_3G_2)$ *for some series* $F_i,G_j\in\kappa[[\mathtt{x},\mathtt{y}]]$ *such that* $\omega(F_2)=\omega(F_3)=1$*,* $\mathtt{x}\mathtt{y}=F_1^2+F_2F_3$ *and either* $\omega(G_1)=1$ *or* $\omega(G_2)=1$*.*

In both cases $\mathfrak{a}\cap\kappa[[\mathtt{x},\mathtt{y}]]=H\kappa[[\mathtt{x},\mathtt{y}]]$ *is a principal ideal. In addition, in case* (i)

$$H=\begin{cases}F & \text{if } F \text{ divides } G,\\ (F_*^2-\mathtt{x}\mathtt{y}G_*^2)D & \text{if } D=\gcd(F,G),\ \omega(D)=1,\ F=DF_* \text{ and } G=DG_*,\end{cases}$$

whereas in case (ii)

$$H=\begin{cases}2F_1G_1Q_2-F_2G_1+F_3G_1Q_2^2 & \text{if } \gcd(G_1,G_2)=G_1 \text{ and } G_2=G_1Q_2,\\ 2F_1Q_1G_2-F_2Q_1^2G_2+F_3G_2 & \text{if } \gcd(G_1,G_2)=G_1 \text{ and } G_2=Q_1G_2,\\ 2F_1G_1G_2-F_2G_1^2+F_3G_2^2 & \text{if } G_1,G_2 \text{ are relatively prime,}\end{cases}$$

where H *divides the series* $P := 2F_1G_1G_2-F_2G_1^2+F_3G_2^2\in\mathfrak{a}\cap\kappa[[\mathtt{x},\mathtt{y}]]$ *of order* 3 *and* P *divides* $F^2-\mathtt{x}\mathtt{y}G^2$ *as well.*

Proof. Recall that $\operatorname{depth}(A)$ is the minimum number of elements of a *regular sequence of* $\mathfrak{a}$, that is, $f_1, \dots, f_r \in \mathfrak{m}$ such that f_1 is not a zero divisor of $\mathfrak{a}$ and f_k is not a zero divisor of $A/(f_1, \dots, f_{k-1})A$ for $k = 2, \dots, r$ (see [JP, Def.6.5.1]). In addition, $\operatorname{pd}(A)$ is the minimum n such that the exact sequence

$$0 \longrightarrow M_n \xrightarrow{\alpha_n} \cdots \xrightarrow{\alpha_1} M_0 \xrightarrow{\alpha_0} A \longrightarrow 0$$

is a free resolution of A, where each M_k is a finitely generated free $\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$-module and $M_n \neq 0$. By [JP, Thm.6.5.20] (Auslander-Buchsbaum Formula) we have $\operatorname{depth}(A) + \operatorname{pd}(A) = \operatorname{depth}(\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]) = 3$. As $\mathfrak{a}$ is a radical ideal of height 2, then $\operatorname{pd}(A) \geqslant \operatorname{ht}(\mathfrak{a}) = 2$ and by [JP, Ex.6.5.6] we have $\operatorname{depth}(A) = 1$. Thus, $\operatorname{pd}(A) = 2 = \operatorname{ht}(\mathfrak{a})$, so $\operatorname{ht}(\mathfrak{a})$ is a *perfect ideal* [E, p.122]. By [E, Thm.1, p.122] we deduce that $\mathfrak{a}$ is *determinantal*. This means (by [E, Thm.1, p.122] and [JP, Thm.6.5.26] (Hilbert-Burch)) that there exists an exact sequence

$$0 \longrightarrow M_2 \xrightarrow{\alpha_2} M_1 \xrightarrow{\alpha_1} \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]] \to A \longrightarrow 0$$

where M_2 and M_1 are $\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$-modules such that $\operatorname{rank}(M_2) = \operatorname{rank}(M_1) - 1$ and $\mathfrak{a}$ is generated by the minors of order $\operatorname{rank}(M_2)$ of the matrix of α_2 times a non-zero divisor $a \in \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$. In fact, as $\operatorname{ht}(\mathfrak{a}) = 2$, one deduces by Krull's principal ideal theorem that a is a unit [E, p.122]. Consequently, $\mathfrak{a}$ is generated by the minors of order $\operatorname{rank}(M_2)$ of the matrix of α_2 and we may assume that $\mathtt{z}^2 - \mathtt{x}\mathtt{y}$ is one of those minors. As $\mathtt{z}^2 - \mathtt{x}\mathtt{y} \in \mathfrak{a}$, it is the determinant of a minor of order $\operatorname{rank}(M_2)$ of the matrix of α_2. We claim: *we may assume* $\operatorname{rank}(M_2) \leqslant 2$.

Suppose $\operatorname{rank}(M_2) > 2$. As $\omega(\mathtt{z}^2 - \mathtt{x}\mathtt{y}) = 2$ and $\operatorname{rank}(M_2) > 2$, there exist an entry of the matrix of α_2, which is a unit, and we may assume after a change of coordinates in M_2 and M_1 that the matriz of α_2 has the form:

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & M \end{array}\right).$$

Thus, one has new exact sequence

$$0 \longrightarrow M_2' \xrightarrow{\alpha_2} M_1' \xrightarrow{\alpha_1} \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]] \to A \longrightarrow 0$$

where M_2' and M_1' are $\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$-modules such that $\operatorname{rank}(M_2') = \operatorname{rank}(M_2) - 1$, $\operatorname{rank}(M_1') = \operatorname{rank}(M_1) - 1$, $\operatorname{rank}(M_2') = \operatorname{rank}(M_1') - 1$ and $\mathfrak{a}$ is generated by the minors of order $\operatorname{rank}(M_2')$ of the matrix of α_2 times a non-zero divisor $a \in \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$.

Proceeding recursively we reduce the rank of M_2 until it is $\leqslant 2$. Thus, the matrix of α_2 has one of the two following forms:

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \mathtt{z}^2 - \mathtt{x}\mathtt{y} \\ 0 & H_1 + \mathtt{z}H_2 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|c} \mathtt{z} + F_1 & F_2 \\ \hline F_3 & \mathtt{z} - F_1 \\ G_1 & G_2 \end{array}\right),$$

where $F_i, G_i, H_i \in \kappa[[\mathtt{x},\mathtt{y}]]$ and $(\mathtt{z} + F_1)(\mathtt{z} - F_1) - F_2F_3 = \mathtt{z}^2 - \mathtt{x}\mathtt{y}$. As $\mathtt{x}\mathtt{y} = F_1^2 + F_2F_3$, we deduce $\omega(F_2F_3) = 2$, so $\omega(F_2) = \omega(F_3) = 1$. In the first case $\mathfrak{a} = (\mathtt{z}^2 - \mathtt{x}\mathtt{y}, H_1 + \mathtt{z}H_2)$, whereas in the second case $\mathfrak{a} = (\mathtt{z}^2 - \mathtt{x}\mathtt{y}, (\mathtt{z} + F_1)G_2 - F_2G_1, (\mathtt{z} - F_1)G_1 - F_3G_2)$, which proves (ii) except for the fact that either $\omega(G_1) = 1$ or $\omega(G_2) = 1$, which we prove below.

If $\mathfrak{a} = (\mathtt{z}^2 - \mathtt{x}\mathtt{y}, H_1 + \mathtt{z}H_2)$, then $F + \mathtt{z}G = (H_1 + \mathtt{z}H_2)(A + \mathtt{z}B) - (\mathtt{z}^2 - \mathtt{x}\mathtt{y})H_2B$ where $A, B \in \kappa[[\mathtt{x},\mathtt{y}]]$, so $F = H_1A + \mathtt{x}\mathtt{y}H_2B$ and $G = H_1B + H_2A$. As $\omega(\mathfrak{a}) = 2$, we have $\omega(H_1 + \mathtt{z}H_2) \geqslant 2$. Then $\omega(H_2) \geqslant 1$, so $\omega((\mathtt{z}^2 - \mathtt{x}\mathtt{y})H_2B) \geqslant 3$. Thus, $\omega(H_1 + \mathtt{z}H_2) + \omega(A + \mathtt{z}B) = \omega(F + \mathtt{z}G) = 2$, so $\omega(H_1 + \mathtt{z}H_2) = 2$ (because $\omega(\mathfrak{a}) = 2$) and $\omega(A + \mathtt{z}B) = 0$. Consequently, we may assume $H_1 + \mathtt{z}H_2 = F + \mathtt{z}G$, so $\mathfrak{a} = (\mathtt{z}^2 - \mathtt{x}\mathtt{y}, F + \mathtt{z}G)$, which proves (i).

We prove next the remaining part of the statement. As A has dimension 1 and $\mathtt{z}^2 - \mathtt{x}\mathtt{y} \in \mathfrak{a}$, we deduce $\mathfrak{a} \cap \kappa[[\mathtt{x},\mathtt{y}]] \neq 0$ and has height 1. As $\kappa[[\mathtt{x},\mathtt{y}]]$ has dimension 2, this means that $\mathfrak{a} \cap \kappa[[\mathtt{x},\mathtt{y}]]$ is a principal ideal generated by $H \in \kappa[[\mathtt{x},\mathtt{y}]]$, which divides

$$F^2 - \mathtt{x}\mathtt{y}G^2 = (F + \mathtt{z}G)(F - \mathtt{z}G) + (\mathtt{z}^2 - \mathtt{x}\mathtt{y})G^2 \in \mathfrak{a} \cap \kappa[[\mathtt{x},\mathtt{y}]]. \tag{11.4}$$

FG

Observe that $\omega(F^2-\mathtt{xy}G^2)=4$, because $\omega(F)=2$ and the leading form of F^2 is a square, whereas the one of $\mathtt{xy}G^2$ is not.

Assume first that we are under the hypothesis of (i). Observe that

$$H=(\mathtt{z}^2-\mathtt{xy})\sum_{k\geqslant 0}Q_k\mathtt{z}^k+(Q_{01}+\mathtt{z}Q_{11})(F+\mathtt{z}G),$$

where $Q_k,Q_{j1}\in\kappa[[\mathtt{x},\mathtt{y}]]$, so

$$\begin{aligned}H&=-\mathtt{xy}Q_0+Q_{01}F,\\ 0&=-\mathtt{xy}Q_1+Q_{01}G+FQ_{11},\\ 0&=Q_0-\mathtt{xy}Q_2+Q_{11}G,\\ 0&=Q_k-\mathtt{xy}Q_{k+2},\ k\geqslant 1.\end{aligned}$$

Proceeding recursively we deduce that $Q_k=\mathtt{xy}^\ell Q_{k+2\ell}$ for each $\ell\geqslant 1$ and $k\geqslant 1$. Consequently, $Q_k=0$ for $k\geqslant 1$. Thus,

$$\begin{aligned}H&=-\mathtt{xy}Q_0+Q_{01}F,\\ 0&=Q_{01}G+FQ_{11},\\ 0&=Q_0+Q_{11}G.\end{aligned}$$

If F divides G, then $G=FG_*$ for some $G_*\in\kappa[[\mathtt{x},\mathtt{y}]]$, so $F+\mathtt{z}G=F(1+\mathtt{z}G^*)$ and $\mathfrak{a}=(\mathtt{z}^2-\mathtt{xy},F)$, that is, we may assume $G=0$. Thus, if $G=0$, we have $Q_{11}=0$, so $Q_0=0$. Consequently, $H=Q_{01}F$ and as $F\in\mathfrak{a}\cap\kappa[[\mathtt{x},\mathtt{y}]]$, we may take $H=F$. Suppose next F does not divide G. Let $D=\gcd(F,G)\in\kappa[[\mathtt{x},\mathtt{y}]]$, which has order $\leqslant 1$, and write $F=F_*D$ and $G=G_*D$, where $F_*,G_*\in\kappa[[\mathtt{x},\mathtt{y}]]$. Thus, $0=Q_{01}G+FQ_{11}=(Q_{01}G_*+F_*Q_{11})D$, so $Q_{01}G_*+F_*Q_{11}=0$. As $\gcd(F_*,G_*)=1$, we have $Q_{01}=F_*Q^*$ and $Q_{11}=-G_*Q^*$ where $Q^*\in\kappa[[\mathtt{x},\mathtt{y}]]$. Thus, $Q_0=G_*^2DQ^*$ and $H=D(F_*^2-\mathtt{xy}G_*^2)Q^*$, so we may assume $H=D(F_*^2-\mathtt{xy}G_*^2)$, because H generates $\mathfrak{a}\cap\kappa[[\mathtt{x},\mathtt{y}]]$ and $D(F_*^2-\mathtt{xy}G_*^2)=(F_*-\mathtt{z}G^*)(F+\mathtt{z}G)+(\mathtt{z}^2-\mathtt{xy})G^*G\in\mathfrak{a}\cap\kappa[[\mathtt{x},\mathtt{y}]]$.

Assume next that we are under the hypothesis of (ii). As $\omega(\mathfrak{a})=2$, we deduce $\omega(G_i)\geqslant 1$, whereas $\omega(F_2)=\omega(F_3)=1$ and $\mathtt{xy}=F_1^2+F_2F_3$ imply $\omega(F_1)\geqslant 1$. There exist series $A_i,B_i\in\kappa[[\mathtt{x},\mathtt{y}]]$ such that

$$\begin{aligned}F+\mathtt{z}G=&((\mathtt{z}+F_1)G_2-F_2G_1)(A_1+\mathtt{z}B_1)\\ &+((\mathtt{z}-F_1)G_1-F_3G_2))(A_2+\mathtt{z}B_2)-(\mathtt{z}^2-\mathtt{xy})(G_2B_1+G_1B_2).\end{aligned}\tag{11.5}$$

$\boxed{\texttt{expFg}}$

Consequently,

$$F=(F_1G_2-F_2G_1)A_1-(F_1G_1+F_3G_2)A_2+\mathtt{xy}(G_2B_1+G_1B_2),\tag{11.6}$$

$\boxed{\texttt{expF}}$

$$G=(F_1G_2-F_2G_1)B_1-(F_1G_1+F_3G_2)B_2+A_1G_2+A_2G_1.\tag{11.7}$$

$\boxed{\texttt{expG}}$

As $\omega(F)=2$, either $\omega(A_1)=0$ and $\omega(F_1G_2-F_2G_1)=2$ or $\omega(A_2)=0$ and $\omega(F_1G_1+F_3G_2)=2$. Observe that either $\omega(G_1)=1$ or $\omega(G_2)=1$ (which is the remaining part of (ii)). The series

$$2F_1G_1G_2-F_2G_1^2+F_3G_2^2=((\mathtt{z}+F_1)G_2-F_2G_1)G_1-((\mathtt{z}-F_1)G_1-F_3G_2))G_2\in\mathfrak{a}\cap\kappa[[\mathtt{x},\mathtt{y}]]$$

has order $\geqslant 3$. In addition,

$$\begin{aligned}(F_1G_2-F_2G_1)^2-\mathtt{xy}G_2^2&=(F_1G_2-F_2G_1+\mathtt{z}G_2)(F_1G_2-F_2G_1-\mathtt{z}G_2)+(\mathtt{z}^2-\mathtt{xy})G_2^2\in\mathfrak{a},\\ (F_1G_1+F_3G_2)^2-\mathtt{xy}G_1^2&=(F_1G_1+F_3G_2+\mathtt{z}G_1)(F_1G_1+F_3G_2-\mathtt{z}G_1)+(\mathtt{z}^2-\mathtt{xy})G_1^2\in\mathfrak{a}.\end{aligned}$$

If $\omega(F_1G_2-F_2G_1)=2$, then $\omega((F_1G_2-F_2G_1)^2-\mathtt{xy}G_2^2)=4$, because $\omega(F_1G_2-F_2G_1)=2$ and the leading form of $(F_1G_2-F_2G_1)^2$ is a square, whereas the one of $\mathtt{xy}G_2^2$ is not. Analogously, if $\omega(F_1G_1+F_3G_2)=2$, then $\omega((F_1G_1+F_3G_2)^2-\mathtt{xy}G_1^2)=4$. As $\mathtt{xy}=F_1^2+F_2F_3$, we deduce

$$\begin{aligned}(F_1G_2-F_2G_1)^2-\mathtt{xy}G_2^2&=-F_2(2F_1G_1G_2-F_2G_1^2+F_3G_2^2),\\ (F_1G_1+F_3G_2)^2-\mathtt{xy}G_1^2&=F_3(2F_1G_1G_2-F_2G_1^2+F_3G_2^2).\end{aligned}$$

As either $\omega((F_1G_2-F_2G_1)^2-\mathtt{xy}G_2^2)=4$ or $\omega((F_1G_1+F_3G_2)^2-\mathtt{xy}G_1^2)=4$ and $\omega(F_2),\omega(F_3)=1$, we conclude

$$\omega(2F_1G_1G_2-F_2G_1^2+F_3G_2^2)=3.$$

Using equations (11.6), (11.7), $\mathtt{xy}=F_1^2+F_2F_3$ and after simplifying the final expression:

$$\begin{aligned}F^2-\mathtt{xy}G^2=&-(2F_1G_1G_2-F_2G_1^2+F_3G_2^2)(-B_1^2F_1^2F_2-B_1^2F_2^2F_3-2B_1B_2F_1^3\\&-2B_1B_2F_1F_2F_3+B_2^2F_1^2F_3+B_2^2F_2F_3^2-2A_1B_2F_1^2-2A_1B_2F_2F_3\\&+2A_2B_1F_1^2+2A_2B_1F_2F_3+A_1^2F_2+2A_1A_2F_1-A_2^2F_3),\end{aligned}$$

so $F^2-\mathtt{xy}G^2$ is a multiple of $2F_1G_1G_2-F_2G_1^2+F_3G_2^2$.

In addition, as H generates $\mathfrak{a}\cap\kappa[[\mathtt{x},\mathtt{y}]]$, there exist $Q_k\in\kappa[[\mathtt{x},\mathtt{y}]]$ for each $k\geqslant 0$ and $Q_{01},Q_{11},Q_{02},Q_{12}\in\kappa[[\mathtt{x},\mathtt{y}]]$ such that

$$H=(\mathtt{z}^2-\mathtt{xy})\sum_{k\geqslant 0}Q_k\mathtt{z}^k+(Q_{01}+\mathtt{z}Q_{11})((\mathtt{z}-F_1)G_1-F_3G_2)+(Q_{02}+\mathtt{z}Q_{12})((\mathtt{z}+F_1)G_2-F_2G_1).$$

We have

$$\begin{aligned}H&=-\mathtt{xy}Q_0-Q_{01}(F_1G_1+F_3G_2)+Q_{02}(F_1G_2-F_2G_1),\\0&=-Q_1\mathtt{xy}-Q_{11}(F_1G_1+F_3G_2)+Q_{01}G_1+Q_{12}(F_1G_2-F_2G_1)+Q_{02}G_2,\\0&=Q_0-\mathtt{xy}Q_2+Q_{11}G_1+Q_{12}G_2,\\0&=Q_k-\mathtt{xy}Q_{k+2},\ k\geqslant 1.\end{aligned}$$

Proceeding recursively we deduce that $Q_k=\mathtt{xy}^\ell Q_{k+2\ell}$ for each $\ell\geqslant 1$ and $k\geqslant 1$. Consequently, $Q_k=0$ for $k\geqslant 1$. Thus,

$$\begin{aligned}H&=-\mathtt{xy}Q_0-Q_{01}(F_1G_1+F_3G_2)+Q_{02}(F_1G_2-F_2G_1),\\0&=-Q_{11}(F_1G_1+F_3G_2)+Q_{01}G_1+Q_{12}(F_1G_2-F_2G_1)+Q_{02}G_2,\\0&=Q_0+Q_{11}G_1+Q_{12}G_2,\end{aligned}$$

In particular, $\omega(H)\geqslant 2$.

Suppose first $\gcd(G_1,G_2)=1$. As

$$(-Q_{11}F_3+Q_{12}F_1+Q_{02})G_2=(Q_{11}F_1+Q_{12}F_2-Q_{01})G_1,$$

we deduce G_1 divides $-Q_{11}F_3+Q_{12}F_1+Q_{02}$ and G_2 divides $Q_{11}F_1+Q_{12}F_2-Q_{01}$, so $\omega(Q_{01})\geqslant 1$ and $\omega(Q_{02})\geqslant 1$. In addition, as $Q_0=-Q_{11}G_1-Q_{12}G_2$, we have $\omega(Q_0)\geqslant 1$. As $H=-\mathtt{xy}Q_0-Q_{01}(F_1G_1+F_3G_2)+Q_{02}(F_1G_2-F_2G_1)$, we deduce $\omega(H)\geqslant 3$. As $P\in\mathfrak{a}\cap\kappa[[\mathtt{x},\mathtt{y}]]=H\kappa[[\mathtt{x},\mathtt{y}]]$ and $\omega(P)=3$, we may assume $H=P$.

Suppose next $\omega(G_1)=1$ and G_1,G_2 are not relatively prime. Then G_1 divides G_2. Write $G_2=G_1Q_2$ where $Q_2\in\kappa[[\mathtt{x},\mathtt{y}]]$ and observe that

$$(-2F_1Q_2-F_3Q_2^2+F_2)G_1=((\mathtt{z}-F_1)G_1-F_3G_1Q_2)Q_2-((\mathtt{z}+F_1)G_1Q_2-F_2G_1)\in\mathfrak{a}\cap\kappa[[\mathtt{x},\mathtt{y}]].$$

We claim: $\omega(-2F_1Q_2-F_3Q_2^2+F_2)=1$

If $\omega(Q_2)\geqslant 1$, we have $\omega(-2F_1Q_2-F_3Q_2^2+F_2)=1$, because $\omega(F_2)=1$. Suppose $\omega(Q_2)=0$ and $\omega(-2F_1Q_2-F_3Q_2^2+F_2)\geqslant 2$. Let F_{i1} be the homogeneous form of F_i of degree 1 (for $i=1,2,3$) and $q=Q_2(0)$. We have $F_{21}=2F_{11}q+F_{31}q^2$. As $\mathtt{xy}=F_1^2+F_2F_3$, we have

$$\mathtt{xy}=F_{11}^2+F_{21}F_{31}=F_{11}^2+(2F_{11}q+F_{31}q^2)F_{31}=(F_{11}+qF_{31})^2,$$

which is a contradiction, because a square of a linear form has rank 1 (as a quadratic form), whereas $\mathtt{xy}$ has rank 2 (as a quadratic form). Consequently, $\omega(-2F_1Q_2-F_3Q_2^2+F_2)=1$.

As $\mathfrak{a}\cap\kappa[[\mathtt{x},\mathtt{y}]]=H\kappa[[\mathtt{x},\mathtt{y}]]$, $\omega(H)\geqslant 2$ and $\omega((-2F_1Q_2-F_3Q_2^2+F_2)G_1)=2$, we may assume $H=(-2F_1Q_2-F_3Q_2^2+F_2)G_1$.

If $\omega(G_2) = 1$ and G_1, G_2 are not relatively prime, the discussion is similar and we omit the precise details. $\square$

s12 12. Non-complete intersections: expected cases

In this section we approach the "only if" part of the proof of Theorem 3.2 when the completion of A has embedding dimension 3 and it is a non-complete intersection. Namely, we prove the following result.

nci **Theorem 12.1** (Non-complete intersections)**.** *Consider an ideal* $\mathfrak{a} := (\mathtt{z}^2 - \mathtt{xy}, (\mathtt{z} + F_1)G_2 - F_2G_1, (\mathtt{z} - F_1)G_1 - F_3G_2)$ *of* $\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ *for some series* $F_1, F_2, F_3, G_1, G_2 \in \kappa[[\mathtt{x},\mathtt{y}]]$ *such that* $\omega(F_2) = \omega(F_3) = 1$ *and* $\mathtt{xy} = F_1^2 + F_2F_3$*. Assume that the complete ring* $A := \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/\mathfrak{a}$ *has the property* $\mathcal{P}(A) = \Sigma A^2$*. Then after a change of coordinates either:*

(i) $\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{y}^2 - a\mathtt{x}^2, \mathtt{xz}, \mathtt{yz})$ *for some* $a \notin -\Sigma\kappa^2$ *or*
(ii) $\mathfrak{a} = (\mathtt{y}^2 - \mathtt{xz}, \mathtt{yz} + p\mathtt{yx} + q\mathtt{x}^2, q\mathtt{xy} + p\mathtt{xz} + \mathtt{z}^2)$ *and the polynomial* $P := \mathtt{t}^3 + p\mathtt{t} + q \in \kappa[\mathtt{t}]$ *is irreducible.*

Remark 12.2. In case (i) the generators satisfy the relation $\mathtt{z}(\mathtt{y}^2 - a\mathtt{x}^2) + a\mathtt{x}(\mathtt{xz}) - \mathtt{y}(\mathtt{yz}) = 0$, whereas in case (ii) the generators satisfy the relation

$$q\mathtt{x}(\mathtt{y}^2 - \mathtt{xz}) + \mathtt{z}(\mathtt{yz} + p\mathtt{yx} + q\mathtt{x}^2) - \mathtt{y}(q\mathtt{xy} + p\mathtt{xz} + \mathtt{z}^2) = 0.$$

Proof of Theorem 12.1. By Lemma 11.7 the series $P := 2F_1G_1G_2 - F_2G_1^2 + F_3G_2^2 \in \mathfrak{a} \cap \kappa[[\mathtt{x},\mathtt{y}]]$ has order 3. Thus, either G_1 or G_2 has order 1, so we may assume $\omega(G_1) = 1$ and in particular it is an irreducible series. As $\mathcal{P}(A) = \Sigma A^2$, the ideal $\mathfrak{a}$ is in addition by [Sch2, Lem.6.3] real radical.

General procedure. Along this proof we need to find in several situations elements $f \in \mathcal{P}(A) \backslash \Sigma A^2$. To that end, we first choose ideal $\mathfrak{a}_1, \dots, \mathfrak{a}_s$ of A such that $\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_s = (0)$. By Lemma 5.16 an element $f \in \mathcal{P}(A)$ if and only if $f \in \mathcal{P}(A/\mathfrak{a}_j)$ for each $j = 1, \dots, s$. In all the cases we deal with below we find $f \in A$ such that $f + \mathfrak{a}_j \in \Sigma(A/\mathfrak{a}_j)^2$ for $j = 1, \dots, s$ (so $f \in \mathcal{P}(A)$), whereas $f \notin \Sigma A^2$, because f has a representative in $\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ of order 1, while $\omega(\mathfrak{a}) = 2$ (Lemma 5.12). We will use freely this procedure along the proof without providing again further details.

The proof is now conducted in several steps:

Step 1. *G_1 is a divisor of P.* We claim: *After a change of coordinates* $\mathfrak{a} = (\mathtt{z}^2 - a\mathtt{x}^2, \mathtt{y}) \cap (\mathtt{z}^2 - b_1\mathtt{x}^2, \mathtt{x}, \mathtt{z}) = (\mathtt{z}^2 - a\mathtt{x}^2, \mathtt{xy}, \mathtt{xz})$ *where* $a \notin -\Sigma\kappa^2$ (which corresponds to statement (i)).

As G_1 is a divisor of P, we have G_1 divides $F_3G_2^2$, so G_1 divides either G_2 or F_3. We distinguish both situations:

Case 1.1. *G_1 divides G_2.* Let us prove: *After a change of coordinates* $\mathfrak{a} = (\mathtt{z}^2 - b_1\mathtt{x}^2, \mathtt{y}) \cap (\mathtt{z}^2 - b_1\mathtt{x}^2, \mathtt{x}, \mathtt{z}) = (\mathtt{z}^2 - b_1\mathtt{x}^2, \mathtt{xy}, \mathtt{xz})$ *where* $b_1 \notin -\Sigma\kappa^2$.

Write $G_2 = G_1Q_2$ for some $Q_2 \in \kappa[[\mathtt{x},\mathtt{y}]]$. As $P := 2F_1G_1G_2 - F_2G_1^2 + F_3G_2^2 = G_1^2(2F_1Q_2 - F_2 + F_3Q_2^2)$ has order 3 and G_1 has order 1, we deduce that $Q := 2F_1Q_2 - F_2 + F_3Q_2^2$ has order 1. We have

$$\begin{aligned}\mathfrak{a} := (\mathtt{z}^2 - \mathtt{xy}, G_1) \cap (\mathtt{z}^2 - \mathtt{xy}, (\mathtt{z} + F_1)Q_2 - F_2, (\mathtt{z} - F_1) - F_3Q_2))&\\ = (\mathtt{z}^2 - \mathtt{xy}, G_1) \cap (\mathtt{z}^2 - \mathtt{xy}, \mathtt{z} - F_3Q_2 - F_1, 2F_1Q_2 - F_2 + F_3Q_2^2).&\end{aligned}$$

As $\mathtt{xy} = F_1^2 + F_2F_3$, we have $\mathtt{z}^2 - \mathtt{xy} - (\mathtt{z}^2 - (F_3Q_2 - F_1)^2) = (-F_2 + 2F_1Q_2 + F_3Q_2^2)F_3$ and consequently

$$\begin{aligned}\mathfrak{a} := (\mathtt{z}^2 - \mathtt{xy}, G_1) \cap (\mathtt{z}^2 - \mathtt{xy}, (\mathtt{z} + F_1)Q_2 - F_2, (\mathtt{z} - F_1) - F_3Q_2))\\ = (\mathtt{z}^2 - \mathtt{xy}, G_1) \cap (-F_3Q_2 - F_1 + \mathtt{z}, 2F_1Q_2 - F_2 + F_3Q_2^2).\end{aligned}$$

Denote $a\mathtt{x} + b\mathtt{y}$ the leading form of Q, where $a, b \in \kappa$ and non both of them are zero. By Weierstrass preparation theorem we assume (interchanging $\mathtt{x}$ and $\mathtt{y}$ if necessary) $G_1 = \mathtt{y} - h_1(\mathtt{x})$ where $h_1 \in \mathtt{x}\kappa[[\mathtt{x}]]$.

Subcase 1.1.1. If $h_1 = 0$, then $(\mathtt{z}^2 - \mathtt{xy}, G_1) = (\mathtt{z}^2, \mathtt{y})$. As $\mathfrak{a}$ is a radical ideal,

$$\mathfrak{a} = (\mathtt{y}, \mathtt{z}) \cap (-F_3Q_2 - F_1 + \mathtt{z}, Q).$$

As $\omega(Q) = 1$, after a change of coordinates of the type $\mathtt{x} \mapsto \mathtt{x} + \lambda\mathtt{y}$ for some $\lambda \in \kappa$ and applying Weierstrass' preparation theorem, we may assume $Q = \mathtt{y} - h_2$ where $h_2 \in \mathtt{x}\kappa[[\mathtt{x}]]$. If $\omega(h_2) \geqslant 2$, then there exists $M \in \kappa\backslash\{0\}$ such that $M^2\mathtt{x}^2 - h_2(\mathtt{x}) \in \mathcal{P}(\kappa[[\mathtt{x}]])$. Observe that $f := (\mathtt{y} - h_2) + M^2\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$. We deduce $\omega(h_2) = 1$ and after a change of coordinates, we may assume $\mathfrak{a} = (\mathtt{y}, \mathtt{z}) \cap (\mathtt{z} - h_3, \mathtt{x})$, where $h_3 := (F_3Q_2 + F_1)(0, \mathtt{y}) \in \kappa[[\mathtt{y}]]$. If $\omega(h_3) \geqslant 2$, there exists $M^2 \in \kappa\backslash\{0\}$ such that $M^2\mathtt{y}^2 - h_3(\mathtt{y}) \in \mathcal{P}(\kappa[[\mathtt{y}]])$. Thus, $\mathtt{z} - h_3 + M\mathtt{y}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$, so $\omega(h_3) = 1$. After the change $h_3 \mapsto \mathtt{y}$, we have

$$\mathfrak{a} = (\mathtt{y}, \mathtt{z}) \cap (\mathtt{z} - \mathtt{y}, \mathtt{x}) = (\mathtt{z} - \mathtt{y}, \mathtt{y}) \cap (\mathtt{z} - \mathtt{y}, \mathtt{x}) = (\mathtt{z} - \mathtt{y}, \mathtt{xy}),$$

which contradicts the fact that $\omega(\mathfrak{a}) = 2$. Consequently, $h_1 \neq 0$.

Subcase 1.1.2. Assume now $\mathfrak{a} = (\mathtt{z}^2 - \mathtt{x}h_1, \mathtt{y} - h_1) \cap (-F_3Q_2 - F_1 + \mathtt{z}, Q)$, where $h_1 \neq 0$ and recall that $Q = a\mathtt{x} + b\mathtt{y} + \cdots$ for some $a, b \in \kappa$ not both zero. Write $h_1 := \sum_{k \geqslant \ell} b_k\mathtt{x}^k$ where $b_\ell \neq 0$. We claim: $\ell = 1$.

Suppose first $\ell \geqslant 2$ is even (and let us obtain a contradiction). Assume first $a = 0$, so $b \neq 0$. Let $M \in \Sigma\kappa^2$ be such that $M\mathtt{x}^2 + Q(\mathtt{x}, h_1(\mathtt{x})) \in \mathcal{P}(\kappa[[\mathtt{x}]])$. Then $Q + M\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$, so $a \neq 0$. As ℓ is even, then $b_\ell\mathtt{x} = \frac{\mathtt{z}^2}{h_1/b_\ell} \in \mathcal{P}(\kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/(\mathtt{z}^2 - \mathtt{x}h_1, \mathtt{y} - h_1))$. Let $M \in \Sigma\kappa^2$ be such that $M\mathtt{x}^2 + ab_\ell(Q - a\mathtt{x})(\mathtt{x}, h_1) \in \mathcal{P}(\kappa[[\mathtt{x}]])$. Then

$$ab_\ell Q + M\mathtt{x}^2 + \mathtt{y}^2 = a^2b_\ell\mathtt{x} + (M\mathtt{x}^2 + ab_\ell(Q - a\mathtt{x})) + \mathtt{y}^2 \in \mathcal{P}(A)\backslash\Sigma A^2.$$

Assume now $\ell \geqslant 3$ is odd, so $\mathtt{x}h_1$ has even order $\ell + 1 \geqslant 4$. By Lemma 5.13 there exists $M \in \Sigma\kappa^2$ such that $M\mathtt{x}^2 + \mathtt{z} \in \mathcal{P}(\kappa[[\mathtt{x}, \mathtt{y}]]/(\mathtt{z}^2 - \mathtt{x}h_1)$. Assume first $a = 0$, so $b \neq 0$. Thus, $\omega(Q(\mathtt{x}, h_1, \pm\sqrt{h_1\mathtt{x}})) \geqslant 2$ and there exists $M \in \kappa\backslash\{0\}$ such that $(M^2\mathtt{x}^2 + Q)(\mathtt{x}, h_1, \pm\sqrt{h_1\mathtt{x}}) \in \Sigma\kappa[\sqrt{b_\ell}][[\mathtt{x}]]^2$, so $M\mathtt{x}^2 + Q \in \mathcal{P}(A)\backslash\Sigma A^2$.

Consequently, $a \neq 0$ and we assume $a = 1$. Write $(-F_3Q_2 - F_1)(\mathtt{x}, 0) = \lambda\mathtt{x} + \cdots$, so $Q' := -F_3Q_2 - F_1 + \mathtt{z} - \lambda Q \in (-F_3Q_2 - F_1 + \mathtt{z}, Q)$ has order $\geqslant 2$ with respect to $\mathtt{x}$. Thus, $\omega(Q'(\mathtt{x}, h_1, \pm\sqrt{h_1\mathtt{x}})) \geqslant 2$ and there exists $M \in \Sigma\kappa^2$ such that $(M\mathtt{x}^2 + Q')(\mathtt{x}, h_1, \pm\sqrt{h_1\mathtt{x}}) \in \Sigma\kappa[\sqrt{b_\ell}][[\mathtt{x}]]^2$, so $M\mathtt{x}^2 + Q' \in \mathcal{P}(A)\backslash\Sigma A^2$.

We conclude $\ell = 1$, as claimed.

As $\ell = 1$, we deduce $\mathtt{z}^2 - \mathtt{x}h_1 = \mathtt{z}^2 - \sum_{k \geqslant 1} b_k\mathtt{x}^{k+1} \in \mathfrak{a}$ has leading form $\mathtt{z}^2 - b_1\mathtt{x}^2$, where $b_1 \neq 0$. If $b_1 \in -\Sigma\kappa^2\backslash\{0\}$, then $\mathtt{z}, \mathtt{x}, \mathtt{y}$ belong to the real radical of $(\mathtt{z}^2 - \mathtt{xy}, G_1)$, so $\mathfrak{a} = (-F_3Q_2 - F_1 + \mathtt{z}, F_3Q_2^2 + 2F_1Q_2 - F_2)$, which is a contradiction, because $\omega(\mathfrak{a}) = 2$. Consequently, $b_1 \notin -\Sigma\kappa^2$.

Write $h_1 = b_1\mathtt{x}u_1$ where $u_1 \in \kappa[[\mathtt{x}]]$ is a unit such that $u_1(0) = 1$. After the change of coordinates $u_1\mathtt{y} \mapsto \mathtt{y}$ and $\mathtt{z}\sqrt{u_1} \mapsto \mathtt{z}$, which keeps the equation $\mathtt{z}^2 - \mathtt{xy}$ invariant, we may assume $\mathfrak{a} = (\mathtt{z}^2 - b_1\mathtt{x}^2, \mathtt{y} - b_1\mathtt{x}) \cap (\mathtt{z} - h, Q)$ where $h, Q \in \kappa[[\mathtt{x}]]$ and $\omega(Q) = 1$. Interchanging $\mathtt{x}$ and $\mathtt{y}$ if necessary, Q is regular of order 1 with respect to $\mathtt{y}$ and using Weierstrass' preparation Theorem we may assume $Q = \mathtt{y} - \zeta_2(\mathtt{x})$ for some $\zeta_2 \in \mathtt{x}\kappa[[\mathtt{x}]]$. Observe that if we have interchanged the variables $\mathtt{x}$ and $\mathtt{y}$, the equation $\mathtt{z}^2-\mathtt{xy}$ keeps invariant and the ideal $(\mathtt{z}^2-b_1\mathtt{x}^2,\mathtt{y}-b_1\mathtt{x})$ keeps the same form (after some manipulations and changing b_1 by $\frac{1}{b_1}$). As

$$(\mathtt{z}-h(\mathtt{x},\zeta_2(\mathtt{x})),\mathtt{y}-\zeta_2(\mathtt{x}))=(\mathtt{z}-h,Q)\subset(\mathtt{z}^2-\mathtt{xy}),$$

we deduce $h(\mathtt{x},\zeta_2(\mathtt{x}))^2=\mathtt{x}\zeta_2(\mathtt{x})$. This means that ζ_2 has odd order and we write $\zeta_2=\mathtt{x}^{2k+1}\lambda^2u_2^2$ for some $\lambda\in\kappa\backslash\{0\}$ and a unit $u_2\in\kappa[[\mathtt{x}]]$ with $u_2(0)=1$, so we may assume $h(\mathtt{x},\zeta_2(\mathtt{x}))=\mathtt{x}^{k+1}\lambda u_2$, that is, $\mathfrak{a}=(\mathtt{z}^2-b_1\mathtt{x}^2,\mathtt{y}-b_1\mathtt{x})\cap(\mathtt{z}-\mathtt{x}^{k+1}\lambda u_2,\mathtt{y}-\mathtt{x}^{2k+1}\lambda u_2^2)$. After the change of coordinates, $\mathtt{y}\mapsto\mathtt{y}+b_1\mathtt{x}$,

$$\mathfrak{a}=(\mathtt{z}^2-b_1\mathtt{x}^2,\mathtt{y})\cap(\mathtt{z}-\mathtt{x}^{k+1}\lambda u_2,\mathtt{y}+b_1\mathtt{x}-\mathtt{x}^{2k+1}\lambda^2u_2^2).$$

Assume first $k=0$ and $b_1=\lambda^2\neq0$. Thus,

$$\mathfrak{a}=(\mathtt{z}-\lambda\mathtt{x},\mathtt{y})\cap(\mathtt{z}+\lambda\mathtt{x},\mathtt{y})\cap(\mathtt{z}-\lambda\mathtt{x}u_2,\mathtt{y}-\mathtt{x}^2g_2(\mathtt{x}))$$

for some $g_2\in\kappa[[\mathtt{x}]]\backslash\{0\}$ (recall that $\omega(\mathfrak{a})=2$). Thus, there exists $M\in\kappa\backslash\{0\}$ such that $\mathtt{x}^2g_2(\mathtt{x})+M^2\mathtt{x}^2\in\mathcal{P}(\kappa[[\mathtt{x}]])$. Then $\mathtt{y}+M^2\mathtt{x}^2\in\mathcal{P}(A)\backslash\Sigma A^2$.

Suppose next either $k\geqslant1$ or $k=0$ and $b_1-\lambda^2\neq0$. By the Implicit Function Theorem there exist series $g_1,g_3\in\kappa[[\mathtt{y}]]$ such that

$$\mathfrak{a}=(\mathtt{z}^2-b_1\mathtt{x}^2,\mathtt{y})\cap(\mathtt{x}-g_1,\mathtt{z}-g_3)=((\mathtt{z}-g_3(\mathtt{y}))^2-b_1(\mathtt{x}-g_1(\mathtt{y}))^2,\mathtt{y})\cap(\mathtt{x}-g_1(\mathtt{y}),\mathtt{z}-g_3(\mathtt{y}))$$

and after the change $\mathtt{x}-g_1(\mathtt{y})\mapsto\mathtt{x},\mathtt{y}\mapsto\mathtt{y},\mathtt{z}-g_3\mapsto\mathtt{z}$, we have

$$\mathfrak{a}=(\mathtt{z}^2-b_1\mathtt{x}^2,\mathtt{y})\cap(\mathtt{z}^2-b_1\mathtt{x}^2,\mathtt{x},\mathtt{z})=(\mathtt{z}^2-b_1\mathtt{x}^2,\mathtt{xy},\mathtt{xz})$$

where $b_1\notin-\Sigma\kappa^2$, as claimed. ■

CASE 1.2. *G_1 and G_2 are relatively prime, but G_1 divides F_3.* Let us prove: *After a change of coordinates $\mathfrak{a}=(\mathtt{z}^2-\lambda\mathtt{x}^2,\mathtt{y})\cap(\mathtt{z}^2-\lambda\mathtt{x}^2,\mathtt{x},\mathtt{z})=(\mathtt{z}^2-\lambda\mathtt{x}^2,\mathtt{xy},\mathtt{xz})$ where $\lambda\notin-\Sigma\kappa^2$.*

As $\omega(F_3)=1$, there exists $\mu\in\kappa\backslash\{0\}$ such that $F_3=\mu G_1$. Thus, $(\mathtt{z}-F_1)G_1-F_3G_2=G_1(\mathtt{z}-F_1-\mu G_2)$. Consequently,

$$\begin{aligned}\mathfrak{a}&=(\mathtt{z}^2-\mathtt{xy},(\mathtt{z}+F_1)G_2-F_2G_1,(\mathtt{z}-F_1-\mu G_2)G_1)\\&=(\mathtt{z}^2-\mathtt{xy},(\mathtt{z}+F_1)G_2-F_2G_1,(\mathtt{z}-F_1-\mu G_2))\cap(\mathtt{z}^2-\mathtt{xy},(\mathtt{z}+F_1)G_2,G_1)\\&=(\mathtt{z}^2-\mathtt{xy},(\mathtt{z}+F_1)G_2-F_2G_1,(\mathtt{z}-F_1-\mu G_2))\cap(\mathtt{z}^2-\mathtt{xy},(\mathtt{z}+F_1),G_1)\cap(\mathtt{z}^2-\mathtt{xy},G_1,G_2).\end{aligned}$$

As G_1,G_2 are relatively prime, the ideal (G_1,G_2) contains a power of $\mathfrak{m}_2$, so $(\mathtt{z}^2-\mathtt{xy},G_1,G_2)$ contains a power of $\mathfrak{m}_3$. As $\mathfrak{a}$ is a real radical ideal, we conclude

$$\mathfrak{a}=(\mathtt{z}^2-\mathtt{xy},(\mathtt{z}+F_1)G_2-F_2G_1,(\mathtt{z}-F_1-\mu G_2))\cap(\mathtt{z}^2-\mathtt{xy},\mathtt{z}+F_1,G_1).$$

As $\mathtt{xy}=F_1^2+F_2F_3$, we have $F_1^2-\mathtt{xy}=-F_2F_3=-\mu G_1F_2$, so $-\mathtt{xy}=-F_1^2-\mu G_1F_2$ and

$$(F_1+\mu G_2)^2-\mathtt{xy}=(F_1+\mu G_2)^2-F_1^2-\mu G_1F_2=\mu((2F_1+\mu G_2)G_2-F_2G_1).$$

Then, $\mathfrak{a}=\mathfrak{a}_1\cap\mathfrak{a}_2$, where

$$\begin{aligned}\mathfrak{a}_1:=&(\mathtt{z}^2-\mathtt{xy},(\mathtt{z}+F_1)G_2-F_2G_1,\mathtt{z}-(F_1+\mu G_2))\\=&((F_1+\mu G_2)^2-\mathtt{xy},(2F_1+\mu G_2)G_2-F_2G_1,\mathtt{z}-(F_1+\mu G_2))\\=&((F_1+\mu G_2)^2-\mathtt{xy},\mathtt{z}-(F_1+\mu G_2)),\\\mathfrak{a}_2:=&(\mathtt{z}^2-\mathtt{xy},\mathtt{z}+F_1,G_1)=(F_1^2-\mathtt{xy},\mathtt{z}+F_1,G_1)=(\mathtt{z}+F_1,G_1).\end{aligned}$$

Observe that $\omega((F_1+\mu G_2)^2-\mathtt{xy})=2$. After a change of coordinates, we may assume $(F_1+\mu G_2)^2-\mathtt{xy}$ is regular of order 2 with respect to $\mathtt{y}$, and using Weierstrass preparation theorem, we may assume, taking into account that $G_1\in\kappa[[\mathtt{x},y]]$ has order 1, that

$$\mathfrak{a}=(\mathtt{y}^2+a_1\mathtt{y}+a_0,\mathtt{z}-a_3(\mathtt{x},\mathtt{y}))\cap(\mathtt{y},\mathtt{z}).$$

where $a_1\in\mathtt{x}\kappa[[\mathtt{x}]]$, $a_0\in\mathtt{x}^2\kappa[[\mathtt{x}]]$, $a_3\in\mathfrak{m}_2$ and $a_3(\mathtt{x},0)\neq0$ (because otherwise $z-a_3(\mathtt{x},\mathtt{y})\in(\mathtt{y},\mathtt{z})$, which contradicts the fact that $\omega(\mathfrak{a})=2$). In addition $\omega(a_3(\mathtt{x},0))=1$, because otherwise there exists $M \in \kappa$ such that $-a_3(\mathtt{x},0) + M^2\mathtt{x}^2 \in \mathcal{P}(\kappa[[\mathtt{x}]])$. Thus, $\mathtt{z} - a_3(\mathtt{x},\mathtt{y}) + M^2\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$. Observe that $\mathtt{y}$ divides $a_3 - a_3(\mathtt{x},0)$. After the change of coordinates $a_3(\mathtt{x},0) \mapsto \mathtt{x}$, $\mathtt{z} - (a_3 - a_3(\mathtt{x},0)) \mapsto \mathtt{z}$, we may assume

$$\mathfrak{a} = (\mathtt{y}^2 + a_1\mathtt{y} + a_0, \mathtt{z} - \mathtt{x}) \cap (\mathtt{z} - \mathtt{y}a_4, \mathtt{y}) = (\mathtt{y}^2 + a_1\mathtt{y} + a_0, \mathtt{z} - \mathtt{x}) \cap (\mathtt{y},\mathtt{z}),$$

for some series $a_4 \in \kappa[[\mathtt{x},\mathtt{y}]]$. After the change of coordinates $\mathtt{x} \mapsto \mathtt{x} + \mathtt{z}$, we deduce

$$\mathfrak{a} = (\mathtt{y}^2 + a_1(\mathtt{x}+\mathtt{z})\mathtt{y} + a_0(\mathtt{x}+\mathtt{z}), \mathtt{x}) \cap (\mathtt{y},\mathtt{z}) = (\mathtt{y}^2 + a_1(\mathtt{z})\mathtt{y} + a_0(\mathtt{z}), \mathtt{x}) \cap (\mathtt{y},\mathtt{z}).$$

Write $\mathtt{y}^2 + a_1(\mathtt{z})\mathtt{y} + a_0(\mathtt{z}) = (\mathtt{y} + \frac{a_1(\mathtt{z})}{2})^2 + a_0(\mathtt{z}) - \frac{a_1(\mathtt{z})^2}{4}$, where $\frac{a_1(\mathtt{z})}{2} \in \mathtt{z}\kappa[[\mathtt{z}]]$ and $b := a_0(\mathtt{z}) - \frac{a_1(\mathtt{z})^2}{4} \in \mathtt{z}^2\kappa[[\mathtt{z}]]$. As $\mathfrak{a}$ is real radical and $\omega(\mathfrak{a}) = 2$, we deduce $b \neq 0$, so we write $b = -\mathtt{z}^\ell\lambda u^\ell$, where $\ell \geqslant 2$, $\lambda \in \kappa\backslash\{0\}$, $u \in \kappa[[\mathtt{x}]]$ is a unit such that $u(0) = 1$. After the change of coordinates $\mathtt{y} + \frac{a_1(\mathtt{z})}{2} \mapsto \mathtt{y}$, $\mathtt{z}u \mapsto \mathtt{z}$, we have

$$\mathfrak{a} = (\mathtt{y}^2 - \lambda\mathtt{z}^\ell, \mathtt{x}) \cap \Big(\mathtt{z}, \mathtt{y} - \frac{a_1^*(\mathtt{z})}{2}\Big) = (\mathtt{y}^2 - \lambda\mathtt{z}^\ell, \mathtt{x}) \cap (\mathtt{y},\mathtt{z}) = (\mathtt{y}^2 - \lambda\mathtt{z}^\ell, \mathtt{x}\mathtt{y}, \mathtt{x}\mathtt{z})$$

for some series $a_1^* \in \mathtt{z}\kappa[[\mathtt{z}]]$. If $\ell \geqslant 3$ is odd, then $\lambda\mathtt{z} \in \mathcal{P}(A)\backslash\Sigma A^2$, so $\ell \geqslant 2$ is even. If $\ell \geqslant 4$, by Lemma 5.13 there exists $M \in \kappa$ such that $M\mathtt{x}^2 + \mathtt{z} \in \mathcal{P}(A)\backslash\Sigma A^2$. Consequently, $\ell = 2$ and as $\mathfrak{a} = (\mathtt{y}^2 - \lambda\mathtt{z}^2, \mathtt{x}\mathtt{y}, \mathtt{x}\mathtt{z})$ is a real radical ideal, $\lambda \notin -\Sigma\kappa^2$, as claimed. ∎

Intermediate Preparation. In the following we assume that G_1 does not divide P, so $H = P$ and $\mathfrak{a} \cap \kappa[[\mathtt{x},\mathtt{y}]] = P\kappa[[\mathtt{x},\mathtt{y}]]$ (Lemma 11.7). After a change of coordinates of the type $\mathtt{x} \mapsto \mathtt{x} + \lambda\mathtt{y}$ (where $\lambda \in \kappa$) we may assume by Weierstrass preparation that P is a Weierstrass polynomial of degree 3 with respect to $\mathtt{y}$ and $\omega(P) = 3$ (as a series of $\kappa[[\mathtt{x},\mathtt{y}]]$). The quotient $\kappa[[\mathtt{x},\mathtt{y}]]/(P\kappa[[\mathtt{x},\mathtt{y}]])$ is isomorphic to $A_0 := \kappa[[\mathtt{x}]][\mathtt{y}]/(P\kappa[[\mathtt{x}]][\mathtt{y}])$. In addition, as $\mathtt{z}^2 - (\mathtt{x}+\lambda\mathtt{y})\mathtt{y} \in \mathfrak{a}$, we deduce A is isomorphic to $\kappa[[\mathtt{x},\mathtt{y}]][\mathtt{z}]/(\mathfrak{a} \cap \kappa[[\mathtt{x},\mathtt{y}]][\mathtt{z}])$. As $\mathfrak{a} \cap \kappa[[\mathtt{x},\mathtt{y}]] = P\kappa[[\mathtt{x},\mathtt{y}]]$, we have $A_0 \hookrightarrow A$. As $G_1 \in \mathfrak{a}$ has order 1, it is relatively prime with P, so there exists a monomorphism of A into the total ring of fractions B of A_0, that maps $\mathtt{z}$ to $\frac{F_1G_1+F_3G_2}{G_1}$, because $(\mathtt{z} - F_1)G_1 - F_3G_2 \in \mathfrak{a}$.

Step 2. *Assume P is irreducible in $\kappa[[\mathtt{x}]][\mathtt{y}]$.* Then, A_0 is an integral domain and $K := B$ is its field of fractions. As $A_0 \hookrightarrow A \overset{\theta}{\hookrightarrow} K$, we deduce A is an integral domain, so $\mathfrak{a}$ is a prime ideal and K is the field of fractions of A. As $A_0 \cong \kappa[[\mathtt{x}]][\mathtt{y}]/(P\kappa[[\mathtt{x}]][\mathtt{y}])$ and $P \in \kappa[[\mathtt{x}]][\mathtt{y}]$ is a monic polynomial, $A_0 = \kappa[[\mathtt{x}]][\zeta]$ for some root $\zeta \in \overline{\kappa}[[\mathtt{x}^*]]$ of P of order $\geqslant 1$ (Lemma A.4). In addition, $K = \kappa((\mathtt{x}))[\zeta]$. By Lemma A.1 there exists a finite real extension $\kappa[\rho]|\kappa$ and a minimal $n \geqslant 1$ such that $\zeta = g_0(\mathtt{x}^{1/n})$ for some $g_0 \in \kappa[\rho][[\mathtt{t}]]$ of order $\geqslant n$ and $K = \kappa((\mathtt{x}))[g_0(\mathtt{x}^{1/n})]$. The extension of fields $K|\kappa((\mathtt{x}))$ has degree 3, because P has order 3. By Corollary A.2 either:

- $n = 3$ and $\zeta = g(\tau\mathtt{x}^{1/3})$ for some $g \in \kappa[[\mathtt{t}]]$ of order $\geqslant 3$ and $\tau \in \overline{\kappa}\backslash\{0\}$ such that $a := \tau^3 \in \kappa\backslash\{0\}$, or
- $n = 1$, $\omega(g_0) \geqslant 1$ and $\kappa[\rho]|\kappa$ is an algebraic extension of degree 3.

We analyze both cases:

Case 2.1. *$\kappa[\rho] = \kappa$ and $K = \kappa((\mathtt{x}))[g(\tau\mathtt{x}^{1/3})]$ has degree 3 over $\kappa((\mathtt{x}))$.* We claim: $\mathcal{P}(A) \neq \Sigma A^2$.

Write $\theta(\mathtt{x}) = a\mathtt{t}^3$ and $\theta(\mathtt{y}) = g(\mathtt{t})$ (use Remark A.3), where $\omega(g(\mathtt{t})) \geqslant 3$. In addition, $\theta(\mathtt{z}) = \frac{(F_1G_1+F_3G_2)(a\mathtt{t}^3,g(\mathtt{t}))}{G_1(a\mathtt{t}^3,g(\mathtt{t}))} \in \kappa((\mathtt{t}))$ and $\theta(\mathtt{z})^2 = (\theta(\mathtt{x}) + \lambda\theta(\mathtt{y}))\theta(\mathtt{y})$, so $\theta(\mathtt{z}) \in \kappa[[\mathtt{t}]]$ (because $\kappa[[\mathtt{t}]]$ is integrally closed in $\kappa((\mathtt{t}))$) and $\omega(\theta(\mathtt{z})) \geqslant 3$. Write $\theta(\mathtt{y}) = \sum_{k\geqslant 3} b_k\mathtt{t}^k$ and $\theta(\mathtt{z}) = \sum_{k\geqslant 3} c_k\mathtt{t}^k$. After a change of coordinates, we may assume $\theta(\mathtt{x}) = \mathtt{t}^3$, $\theta(\mathtt{y}) = b_4\mathtt{t}^4 + b_5\mathtt{t}^5 + \sum_{k\geqslant 6} b_k\mathtt{t}^k$ and $\theta(\mathtt{z}) = c_4\mathtt{t}^4 + c_5\mathtt{t}^5 + \sum_{k\geqslant 6} c_k\mathtt{t}^k$. If $b_4, b_5 = 0$, then $\mathtt{y} + (b_6^2 + 1)\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$. Analogously, if $c_4, c_5 = 0$, then $\mathtt{z} + (b_6^2 + 1)\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$. In addition, if $b_4 \neq 0$, then $b_4\mathtt{y} \in \mathcal{P}(A)\backslash\Sigma A^2$, whereas if $c_4 \neq 0$, then $c_4\mathtt{z} \in \mathcal{P}(A)\backslash\Sigma A^2$. Suppose next that both b_4, c_4 are zero and both b_5, c_5 are non-zero. Thus, $c_5\mathtt{y} - b_5\mathtt{z} + ((b_6 - c_6)^2 + 1)\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$. Consequently, in all cases $\mathcal{P}(A) \neq \Sigma A^2$. ∎

CASE 2.2. *$\kappa[\rho]|\kappa$ is an extension of degree 3, $g_0 \in \kappa[\rho][[\mathtt{x}]]$ and $K = \kappa((\mathtt{x}))[g_0]$ has degree 3.* Let us prove: *$\mathfrak{a} = (\mathtt{y}^2 - \mathtt{xz}, \mathtt{yz} + p\mathtt{yx} + q\mathtt{x}^2, q\mathtt{xy} + p\mathtt{xz} + \mathtt{z}^2)$ and the polynomial $P := \mathtt{t}^3 + p\mathtt{t} + q \in \kappa[\mathtt{t}]$ is irreducible*, so we are in case (ii).

Write $\theta(\mathtt{y}) = \sum_{k\geqslant 1} b_k\mathtt{x}^k$ and $\theta(\mathtt{z}) = \sum_{k\geqslant 1} c_k\mathtt{x}^k$ (recall that $\theta(\mathtt{x}) = \mathtt{x}$). Observe that both $b_1, c_1 \notin \kappa$ because otherwise either $\mathtt{y} - b_1\mathtt{x} + (b_2^2+1)\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$ (if $b_1 \in \kappa$) or $\mathtt{z} - c_1\mathtt{x} + (b_2^2+1)\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$ (if $c_1 \in \kappa$). We may assume $b_1 = \rho$, so there exist series $\zeta_i, \xi_i, \chi_i \in \kappa[[\mathtt{x}]]$ such that

$$\begin{aligned}\theta(\mathtt{y}) &= \zeta_1\mathtt{x}^2 + \rho\mathtt{x}(1+\xi_i\mathtt{x}) + \chi_1\rho^2\mathtt{x}^2,\\ \theta(\mathtt{z}) &= \zeta_2\mathtt{x} + \rho\xi_2\mathtt{x} + \chi_2\rho^2\mathtt{x}.\end{aligned}$$

As $\theta(\mathtt{x}) = \mathtt{x}$, after the change of coordinates $\mathtt{y} \mapsto \mathtt{y} - \zeta_1\mathtt{x}^2$, $\mathtt{z} \mapsto \mathtt{z} - \zeta_2\mathtt{x}$, we may assume $\zeta_i = 0$, that is,

$$\begin{aligned}\theta(\mathtt{y}) &= \rho\mathtt{x} + \rho\mathtt{x}^2\xi_i + \chi_1\rho^2\mathtt{x}^2,\\ \theta(\mathtt{z}) &= \rho\mathtt{x}\xi_2 + \chi_2\rho^2\mathtt{x}.\end{aligned}$$

After the additional change of coordinates $\frac{\mathtt{y}}{1+\xi_i\mathtt{x}} \mapsto \mathtt{y}$, we may assume $\xi_1 = 0$, that is,

$$\begin{aligned}\theta(\mathtt{y}) &= \rho\mathtt{x} + \chi_1\rho^2\mathtt{x}^2,\\ \theta(\mathtt{z}) &= \rho\mathtt{x}\xi_2 + \chi_2\rho^2\mathtt{x}.\end{aligned}$$

Finally, the ulterior change of coordinates $\mathtt{z} \mapsto \mathtt{z} - \mathtt{y}\xi_2$, allow us to assume $\xi_2 = 0$, that is,

$$\begin{aligned}\theta(\mathtt{y}) &= \rho\mathtt{x} + \chi_1\rho^2\mathtt{x}^2,\\ \theta(\mathtt{z}) &= \chi_2\rho^2\mathtt{x}.\end{aligned}$$

Observe that $\chi_2(0) \neq 0$, because (as we have seen above) otherwise $\mathcal{P}(A) \neq \Sigma A^2$. Thus, after an additional change of coordinates, we may assume $\theta(\mathtt{x}) = \mathtt{x}$, $\theta(\mathtt{z}) = \rho^2\mathtt{x}$ and $\theta(\mathtt{y}) = \rho\mathtt{x}$. In particular, $K = \kappa((\mathtt{x}))[\rho] = \kappa[\rho]((\mathtt{x}))$. Let P be the irreducible polynomial of ρ over κ, which is a polynomial of degree 3. Using Tschirnhaus trick we may assume $P := \mathtt{t}^3 + p\mathtt{t} + q \in \kappa[\mathtt{t}]$. Consequently,

$$\mathfrak{a} = (\mathtt{y}^2 - \mathtt{xz}, \mathtt{yz} + p\mathtt{yx} + q\mathtt{x}^2, q\mathtt{xy} + p\mathtt{xz} + \mathtt{z}^2),$$

because it is the kernel of the κ-homomorphism $\theta^* : \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]] \to \kappa[\rho]((\mathtt{x}))$, $f \mapsto f(\mathtt{x}, \rho\mathtt{x}, \rho^2\mathtt{x})$. Only the inclusion $\ker(\theta^*) \subset \mathfrak{a}$ requires some comment.

Observe that $\mathtt{y}^3 + p\mathtt{yx}^2 + q\mathtt{x}^3 = (\mathtt{y}^2 - \mathtt{xz})\mathtt{y} + \mathtt{x}(\mathtt{yz} + p\mathtt{yx} + q\mathtt{x}^2) \in \mathfrak{a}$. Let $f \in \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]$ and by Weierstrass division theorem there exist series $g_j \in \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]$, $a_i, b_i \in \kappa[[\mathtt{x}]]$ and $\lambda \in \kappa$ such that

$$\begin{aligned}f = g_1(q\mathtt{xy} + p\mathtt{xz} + \mathtt{z}^2) &+ g_1(\mathtt{y}^3 + p\mathtt{yx}^2 + q\mathtt{x}^3) + \mathtt{z}(a_2(\mathtt{x})\mathtt{y}^2 + a_1(\mathtt{x})\mathtt{y} + \mathtt{x}a_0(\mathtt{x}) + \lambda)\\ &+ b_2(\mathtt{x})\mathtt{y}^2 + b_1(\mathtt{x})\mathtt{y} + b_0(\mathtt{x}) = g(q\mathtt{xy} + p\mathtt{xz} + \mathtt{z}^2) + g_1(\mathtt{y}^3 + p\mathtt{yx}^2 + q\mathtt{x}^3)\\ &+ (\mathtt{yz} + p\mathtt{yx} + q\mathtt{x}^2)(a_2(\mathtt{x})\mathtt{y} + a_1(\mathtt{x})) + (\mathtt{xz} - \mathtt{y}^2)a_0(\mathtt{x}) + \lambda\mathtt{z}\\ &+ (b_2(\mathtt{x}) - p\mathtt{x}a_2(\mathtt{x}) + a_0(\mathtt{x}))\mathtt{y}^2 + (b_1(\mathtt{x}) - q\mathtt{x}^2a_2(\mathtt{x}) - p\mathtt{x}a_1(\mathtt{x}))\mathtt{y} + b_0(\mathtt{x}) - q\mathtt{x}^2a_1(\mathtt{x}).\end{aligned}$$

Thus, there exists series $c_i \in \kappa[[\mathtt{x}]]$ and $\lambda \in \kappa$ such that f is congruent with

$$c_2(\mathtt{x})\mathtt{y}^2 + c_1(\mathtt{x})\mathtt{y} + c_0(\mathtt{x}) + \lambda\mathtt{z}$$

modulo $\mathfrak{a}$. If $f \in \ker(\theta^*)$, we have

$$0 = c_2(\mathtt{x})\rho^2\mathtt{x}^2 + c_1(\mathtt{x})\rho\mathtt{x} + c_0(\mathtt{x}) + \lambda\rho^2\mathtt{x} = (c_2(\mathtt{x})\mathtt{x}^2 + \lambda\mathtt{x})\rho^2 + c_1(\mathtt{x})\rho\mathtt{x} + c_0(\mathtt{x}).$$

As $\{1, \rho, \rho^2\}$ are κ-linearly independent, we deduce $c_2 = 0, c_1 = 0, c_0 = 0, \lambda = 0$, so $f \in \mathfrak{a}$, as required. ∎

STEP 3. *Assume P is reducible in $\kappa[[\mathtt{x}]][\mathtt{y}]$, it is the product of an irreducible Weierstrass polynomial of degree* 1 *and an irreducible Weierstrass polynomial of degree* 2 *(and G_1 does not* *divide P).* As G_1 has order 1, it is not a zero divisor of $\kappa[[\mathtt{x},\mathtt{y}]]/(P)$. Denote the factors of P with $L := \mathtt{y} - h(\mathtt{x})$ and $Q := \mathtt{y}^2 + h_1(\mathtt{x})\mathtt{x}\mathtt{y} + h_2(\mathtt{x})\mathtt{x}^2$. Recall that $\deg_{\mathtt{y}}(L) = 1 = \omega(L)$ and $\deg_{\mathtt{y}}(Q) = 2 = \omega(Q)$.

We have $\mathfrak{a} = (\mathtt{z}^2 - \mathtt{x}\mathtt{y}, (\mathtt{z} + F_1)G_2 - F_2G_1, (\mathtt{z} - F_1)G_1 - F_3G_2, LQ) = \mathfrak{a}_1 \cap \mathfrak{a}_2$ where

$$\mathfrak{a}_1 := (\mathtt{z}^2 - \mathtt{x}\mathtt{y}, (\mathtt{z} + F_1)G_2 - F_2G_1, (\mathtt{z} - F_1)G_1 - F_3G_2, L),$$
$$\mathfrak{a}_2 := (\mathtt{z}^2 - \mathtt{x}\mathtt{y}, (\mathtt{z} + F_1)G_2 - F_2G_1, (\mathtt{z} - F_1)G_1 - F_3G_2, Q).$$

As $L := \mathtt{y}-h(\mathtt{x}) \in \mathfrak{a}_1$ does not divide G_1 (recall that $\omega(G_1) = 1$ and $P = LQ$ and G_1 are relatively prime) and $G_1\mathtt{z}-(F_1G_1+F_3G_2), \mathtt{z}^2-\mathtt{x}\mathtt{y} \in \mathfrak{a}_1$, there exists $h'(\mathtt{x}) \in \kappa[[\mathtt{x}]]$ such that $\mathtt{z}-h'(\mathtt{x}) \in \mathfrak{a}_1$. Then either $\mathfrak{a}_1 = (\mathtt{y} - h(\mathtt{x}), \mathtt{z} - h'(\mathtt{x}))$ or there exists a non-zero power series $h''(\mathtt{x}) \in \kappa[[\mathtt{x}]]$ (if $(\mathtt{z}+F_1)G_2-F_2G_1 \notin (\mathtt{y}-h(\mathtt{x}), \mathtt{z}-h'(\mathtt{x})))$ such that $\mathfrak{a}_1 = (\mathtt{y}-h(\mathtt{x}), \mathtt{z}-h'(\mathtt{x}), h''(\mathtt{x}))$, which means that $\mathfrak{a}_1$ contains a power of $\mathfrak{m}_3$. In the first case $\mathfrak{a}_1$ is a non-maximal prime ideal and $\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2$, whereas in the second case $\mathfrak{a} = \mathfrak{a}_2$, because $\mathfrak{a}$ is a real radical ideal. As $Q\kappa[[\mathtt{x},\mathtt{y}]] \subset \mathfrak{a}_2 \cap \kappa[[\mathtt{x}]]$ and $\mathfrak{a} \cap \kappa[[\mathtt{x}]] = P\kappa[[\mathtt{x}]]$, we deduce $\mathfrak{a} \neq \mathfrak{a}_2$, so $\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2$.

The ring $A_1 := \kappa[[\mathtt{x}]][\mathtt{y}]/(Q\kappa[[\mathtt{x}]][\mathtt{y}])$ is an integral domain and $K_1 = \kappa((\mathtt{x}))[\mathtt{y}]/(Q\kappa((\mathtt{x}))[\mathtt{y}])$ is its field of fractions. As $Q \in \kappa[[\mathtt{x}]][\mathtt{y}]$ is a monic polynomial, $A_1 = \kappa[[\mathtt{x}]][\zeta]$ for some root $\zeta \in \overline{\kappa}[[\mathtt{x}^*]]$ of Q of order $\geqslant 1$ (Lemma A.4). In addition, $K_1 = \kappa((\mathtt{x}))[\zeta]$. By Lemma A.1 there exists a finite real extension $\kappa[\rho]|\kappa$ and a minimal $n \geqslant 1$ such that $\zeta = g_0(\mathtt{x}^{1/n})$ for some $g_0 \in \kappa[\rho][[\mathtt{x}]]$ of order $\geqslant n$ and $K = \kappa((\mathtt{x}))[g_0(\mathtt{x}^{1/n})]$. The extension of fields $K|\kappa((\mathtt{x}))$ has degree 2, because Q has degree 2. By Corollary A.2 either:

- $n = 2$ and $\zeta = g(\tau\mathtt{x}^{1/2})$ for some $g \in \kappa[[\mathtt{x}]]$ of order $\geqslant 1$ and $\tau \in \overline{\kappa}\backslash\{0\}$ such that $a := \tau^2 \in \kappa\backslash\{0\}$, or
- $n = 1$, $\omega(g_0) \geqslant 1$ and $\kappa[\rho]|\kappa$ is an algebraic extension of degree 2, so we assume $\rho := \sqrt{a}$ for some $a \in \kappa\backslash(\kappa^2 \cup -\Sigma\kappa^2)$.

We analyze both cases:

CASE 3.1. *$\kappa[\rho] = \kappa$ and $K = \kappa((\mathtt{x}))[g(\tau\mathtt{x}^{1/2})]$ has degree 2 over $\kappa((\mathtt{x}))$.* We claim: $\mathcal{P}(A) \neq \Sigma A^2$.

Write $\theta(\mathtt{x}) = a\mathtt{t}^2$ and $\theta(\mathtt{y}) = g(\mathtt{t})$ (use Remark A.3), where $\omega(g(\mathtt{t})) \geqslant 2$. In addition, $\theta(\mathtt{z}) = \frac{(F_1G_1+F_3G_2)(a\mathtt{t}^2,g(\mathtt{t}))}{G_1(a\mathtt{t}^2,g(\mathtt{t}))} \in \kappa((\mathtt{t}))$ (recall that G_1 and Q are relatively prime) and $\theta(\mathtt{z})^2 = (\theta(\mathtt{x})+\lambda\theta(\mathtt{y}))\theta(\mathtt{y})$, so $\theta(\mathtt{z}) \in \kappa[[\mathtt{t}]]$ (because $\kappa[[\mathtt{t}]]$ is integrally closed in $\kappa((\mathtt{t}))$) and $\omega(\theta(\mathtt{z})) \geqslant 2$. Write $\theta(\mathtt{y}) = \sum_{k\geqslant 2} b_k\mathtt{t}^k$ and $\theta(\mathtt{z}) = \sum_{k\geqslant 2} c_k\mathtt{t}^k$, which holds because $\deg_{\mathtt{y}}(Q) = 2 = \omega(Q)$). After a change of coordinates, we may assume $b_2 = c_2 = 0$.

Let $\mu_1, \mu_2 \in \kappa$ (not both 0) be such that $\mu_1h(\mathtt{x}) + \mu_2h'(\mathtt{x})$ has order $\geqslant 2$. If $b_3, c_3 = 0$, there exists $M \in \kappa$ such that $\mu_1(\sum_{k\geqslant 4} b_k\mathtt{t}^k - h(a\mathtt{t}^2)) + \mu_2(\sum_{k\geqslant 4} c_k\mathtt{t}^k - h'(a\mathtt{t}^2)) + M^2a^2\mathtt{t}^4 \in \mathcal{P}(\kappa[[\mathtt{t}]])$. Observe that $\mu_1(\mathtt{y}-h(\mathtt{x}))+\mu_2(\mathtt{z}-h'(\mathtt{x}))+M^2\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$. Thus, either b_3 or c_3 are non-zero and after a change of coordinates, we may assume $\theta(\mathtt{x}) = \mathtt{t}^2$, $\theta(\mathtt{y}) = \mathtt{t}^3$, $\theta(\mathtt{z}) = 0$ (because the semigroup of $\mathbb{N}$ generated by $2, 3$ contains all natural numbers $\geqslant 2$). Recall that $\omega(\mathfrak{a}) = 2$, so $\mathtt{z} \notin \mathfrak{a}$. Thus, $h' \neq 0$ and $\lambda(\mathtt{z} - h'(\mathtt{x})) + M^2\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$ for suitable $\lambda, M \in \kappa$. More precisely, we take $\lambda := -\frac{\partial h'}{\partial \mathtt{x}}(0)$ and $M = 1$ if $\frac{\partial h'}{\partial \mathtt{x}}(0) \neq 0$, whereas we choose $\lambda = 1$ and $M = (\frac{\partial^2 h'}{\partial \mathtt{x}^2}(0))^2 + 1$ if $\frac{\partial h'}{\partial \mathtt{x}}(0) = 0$. ■

CASE 3.2. *$\kappa[\rho]|\kappa$ has degree 2 and $\rho := \sqrt{a}$ for some $a \in \kappa\backslash(\kappa^2 \cup -\Sigma\kappa^2)$.* Let us prove: *After a change of coordinates, we may assume $\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{y}^2 - a\mathtt{x}^2, \mathtt{x}\mathtt{z}, \mathtt{y}\mathtt{z})$ for some $a \notin -\Sigma\kappa^2$*, which corresponds to case (i).

After the change of coordinates $\mathtt{y}-h(\mathtt{x}) \mapsto \mathtt{y}, \mathtt{z}-h(\mathtt{x}) \mapsto \mathtt{z}$, we may assume $\mathfrak{a}_1 = (\mathtt{y},\mathtt{z})$. Write $\theta(\mathtt{y}) = \sum_{k\geqslant 1} b_k\mathtt{x}^k$ and $\theta(\mathtt{z}) = \sum_{k\geqslant 1} c_k\mathtt{x}^k$ (and recall $\theta(\mathtt{x}) = \mathtt{x}$). Observe first that both b_1, c_1 are non-zero. Otherwise, $\mathtt{y}+\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma(^2A)$ if $b_1 \neq 0$ and $\mathtt{y}+\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma(^2A)$ if $c_1 \neq 0$. Suppose that both $b_1, c_1 \in \kappa$ and consider $c_1\mathtt{y} - b_1\mathtt{z} + ((c_1b_2 - c_2b_1)^2 + 1)\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$. Interchanging

$\mathtt{y}$ and $\mathtt{z}$ if necessary, we may assume $b_1 \in \kappa[\rho]\backslash\kappa$. Write $b_1 = a_0 + a_1\sqrt{a}$ for some $a_i \in \kappa$ (with $a_1 \neq 0$) and $a \in \kappa\backslash(\kappa^2 \cup -\Sigma\kappa^2)$, so there exist series $\zeta_i, \xi_i \in \kappa[[\mathtt{x}]]$ such that

$$\theta(\mathtt{y}) = \zeta_1\mathtt{x}^2 + \rho\mathtt{x}(1 + \xi_1 x),$$
$$\theta(\mathtt{z}) = \zeta_2\mathtt{x} + \rho\xi_2\mathtt{x}.$$

After the change of coordinates $\mathtt{y} \mapsto \frac{\mathtt{y}-\zeta_1\mathtt{x}^2}{1+\xi_1 x}, \mathtt{z} \mapsto \mathtt{z} - \zeta_2$, we may assume $\theta(\mathtt{y}) = \rho\mathtt{x}$ and $\theta(\mathtt{z}) = \rho\xi_2\mathtt{x}$ and after the additional change of coordinates $\mathtt{z} \mapsto \mathtt{z}-\mathtt{y}\xi_2$, we assume in fact $\theta(\mathtt{z}) = 0$. After the previous changes of coordinates we have $\mathfrak{a}_2 = (\mathtt{y}^2 - a\mathtt{x}^2, \mathtt{z})$ and $\mathfrak{a}_1 = (\mathtt{y} - h_2(\mathtt{x}), \mathtt{z} - h_3(\mathtt{x}))$ for some series $h_2, h_3 \in \kappa[\mathtt{x}]$. As $\omega(\mathfrak{a}) = 2$, we deduce $h_3 \neq 0$ and $\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2$. Write $h_3 = \sum_{k\geqslant 1} d_k\mathtt{x}^k$. If $d_1 = 0$, then $\mathtt{z}-h_3(\mathtt{x})+(d_1^2+1)\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$, so $d_1 \neq 0$. By Weiertrass preparation theorem there exists $b \in \kappa[[\mathtt{z}]]\backslash\kappa$ such that $\mathtt{x} - b(\mathtt{z}) \in \mathfrak{a}_1$, so also $\mathtt{y} - h_2(b(\mathtt{z})) \in \mathfrak{a}_1$. After the change of coordinates $\mathtt{x} \mapsto \mathtt{x} + b(\mathtt{z}), \mathtt{y} \mapsto \mathtt{y} + h_2(b(\mathtt{z}))$, we have $\mathfrak{a}_2 = (\mathtt{y}^2 - a\mathtt{x}^2, \mathtt{z})$ and $\mathfrak{a}_1 = (\mathtt{x}, \mathtt{y})$, so $\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2 = (\mathtt{y}^2 - a\mathtt{x}^2, \mathtt{xz}, \mathtt{yz})$ for some $a \notin -\Sigma A^2$. ■

STEP 4. *Assume the series P is reducible in $\kappa[[\mathtt{x}, \mathtt{y}]]$, it is the product of three irreducible Weierstrass polynomials of degree 1 with respect to $\mathtt{y}$ (and G_1 is not a factor of P).* We claim: *After a change of coordinates, $\mathfrak{a} = (\mathtt{x}^2 - \mathtt{y}^2, \mathtt{xz}, \mathtt{yz})$, that is, we are in case (i) of the statement.*

Write the factors of P as $\mathtt{y} - h_1, \mathtt{y} - h_2, \mathtt{y} - h_3$ where $h_i \in \mathtt{x}\kappa[[\mathtt{x}]]$. We have $\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2 \cap \mathfrak{a}_3$ where

$$\mathfrak{a}_i = (\mathtt{z}^2 - \mathtt{xy}, G_1\mathtt{z} - (F_1G_1 + F_3G_2), (\mathtt{z} + F_1)G_2 - F_2G_1, \mathtt{y} - h_i).$$

As G_1 is a non zero divisor of $\kappa[[\mathtt{x}, \mathtt{y}]]/(P)$ and $G_1\mathtt{z} - (F_1G_1 + F_3G_2), \mathtt{z}^2 - \mathtt{xy} \in \mathfrak{a}$, there exist series $h_i'(\mathtt{x}) \in \mathtt{x}\kappa[[\mathtt{x}]]$ such that $(\mathtt{y}-h_i(\mathtt{x}), \mathtt{z}-h_i'(\mathtt{x})) \subset \mathfrak{a}_i$. If $(\mathtt{y}-h_i(\mathtt{x}), \mathtt{z}-h_i'(\mathtt{x})) \subsetneq \mathfrak{a}_i$, there exists a power of the maximal ideal $\mathfrak{m}_3$ contained in $\mathfrak{a}_i$. As $\mathfrak{a}$ is a radical ideal, we deduce $\mathfrak{a} = \bigcap_{j\neq i} \mathfrak{a}_j$. But this contradicts the fact that $\mathfrak{a} \cap \kappa[[\mathtt{x}, \mathtt{y}]] = P\kappa[[\mathtt{x}, \mathtt{y}]]$, because $\bigcap_{j\neq i} \mathfrak{a}_j \cap \kappa[[\mathtt{x}, \mathtt{y}]]$ contains elements of order 2, whereas $\omega(P) = 3$. We conclude $\mathfrak{a} = \mathfrak{a}_1\cap\mathfrak{a}_2\cap\mathfrak{a}_3$ and $\mathfrak{a}_i = (\mathtt{y}-h_i(\mathtt{x}), \mathtt{z}-h_i'(\mathtt{x}))$ for $i = 1, 2, 3$.

Interchanging $\mathtt{y}$ and $\mathtt{z}$, we may assume $\omega(h_1 - h_2) \leqslant \omega(h_1' - h_2')$. Thus, there exists $d_3 \in \kappa[[\mathtt{x}]]$ such that $h_1' - h_2' = d_3(h_1 - h_2)$. Consider the equation $g := \mathtt{z} - h_2' - d_3(\mathtt{y} - h_2) \in \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]$ and observe that $g_3(\mathtt{x}, h_i(\mathtt{x}), h_i'(\mathtt{x})) = h_i' - h_2' - d_3(h_i - h_2)$ for $i = 1, 2$. Thus, $g_3 \in \mathfrak{a}_1 \cap \mathfrak{a}_2$ and after the change of coordinates $\mathtt{x} \mapsto \mathtt{x}, \mathtt{y} \mapsto \mathtt{y}, g_3 \mapsto \mathtt{z}$, we deduce $\mathtt{z} \in \mathfrak{a}_1 \cap \mathfrak{a}_2$ and we have $\mathfrak{a}_1\cap\mathfrak{a}_2 = (\mathtt{y}-h_1, \mathtt{z})\cap(\mathtt{y}-h_2, \mathtt{z}) = ((\mathtt{y}-h_1)(\mathtt{y}-h_2), \mathtt{z})$, so we may assume $\mathfrak{a} = \mathfrak{a}_1\cap\mathfrak{a}_2\cap\mathfrak{a}_3 = (\mathtt{y}-h_1, \mathtt{z})\cap(\mathtt{y}-h_2, \mathtt{z})\cap(\mathtt{y}-h_3, \mathtt{z}-h_3')$ for some $h_3' \neq 0$ (because otherwise $\mathtt{z} \in \mathfrak{a}$, whereas $\omega(\mathfrak{a}) = 2$). In fact, $\omega(h_3') = 1$, because otherwise there exists $M \in \kappa$ such that $\mathtt{z} + M^2\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$. Let $d_2 \in \kappa[[\mathtt{x}]]$ be such that $h_1 - h_3 = d_2(-h_3')$. Consider the equation $g_2 := \mathtt{y} - h_3 - d_2(\mathtt{z} - h_3')$ and denote $h_1' = 0$. Observe that $g_2(\mathtt{x}, h_i(\mathtt{x}), h_i'(\mathtt{x})) = h_i - h_3 - d_2(h_i' - h_3')$ for $i = 1, 3$, so $g_2 \in \mathfrak{a}_1\cap\mathfrak{a}_3$. After the change of coordinates $\mathtt{x} \mapsto \mathtt{x}, g_2 \mapsto \mathtt{y}, \mathtt{z} \mapsto \mathtt{z}$, we may assume $\mathfrak{a}_1 = (\mathtt{y}, \mathtt{z})$, $\mathfrak{a}_2 = (\mathtt{y}-h_2, \mathtt{z})$ and $\mathfrak{a}_3 = (\mathtt{y}, \mathtt{z} - h_3')$ for some series $h_2, h_3' \in \mathtt{x}\kappa[[\mathtt{x}]]$. Observe that $h_2 \neq 0$ because otherwise $\mathtt{y} \in \mathfrak{a}$, whereas $\omega(\mathfrak{a}) = 2$. In fact, $\omega(h_2) = 1$, because otherwise there exists $M \in \kappa$ such that $\mathtt{y} + M^2\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$. As $\omega(h_2) = 1$ and $\omega(h_3') = 1$, there exists by Weierstrass preparation theorem units $u_2, u_3' \in \kappa[[\mathtt{x}, \mathtt{y}]]$ and Weierstrass polynomials $\mathtt{x} - b_2(\mathtt{y})$, $\mathtt{x} - b_3'$ where $b_2 \in \mathtt{y}\kappa[[\mathtt{y}]]$, $b_3' \in \mathtt{z}\kappa[[\mathtt{z}]]$, $\mathtt{y} - h_2 = (\mathtt{x} - b_2(\mathtt{y}))u_2$ and $\mathtt{z} - h_3' = (\mathtt{x} - b_3'(\mathtt{z}))u_3'$. Thus, $\mathfrak{a}_2 = (\mathtt{x} - b_2(\mathtt{y}), \mathtt{z})$ and $\mathfrak{a}_3 = (\mathtt{x} - b_3'(\mathtt{z}), \mathtt{y})$ and after the change of coordinates $\mathtt{x} \mapsto \mathtt{x} + b_2(\mathtt{y}) + b_3'(\mathtt{z}), \mathtt{y} \mapsto \mathtt{y}, \mathtt{z} \mapsto \mathtt{z}$, we have $\mathfrak{a}_1 = (\mathtt{x}, \mathtt{y})$, $\mathfrak{a}_2 = (\mathtt{x}, \mathtt{z})$ and $\mathfrak{a}_3 = (\mathtt{x}, \mathtt{y})$, so $\mathfrak{a} = (\mathtt{xy}, \mathtt{yz}, \mathtt{xz})$. Consequently, after the change of coordinates $\mathtt{x} := \mathtt{u} - \mathtt{v}$, $\mathtt{y} := \mathtt{u} + \mathtt{v}$, we conclude

$$\mathfrak{a} = (\mathtt{u}^2 - \mathtt{v}^2, (\mathtt{u} + \mathtt{v})\mathtt{z}, (\mathtt{u} - \mathtt{v})\mathtt{z}) = (\mathtt{u}^2 - \mathtt{v}^2, 2\mathtt{uz}, 2\mathtt{vz}) = (\mathtt{u}^2 - \mathtt{v}^2, \mathtt{uz}, \mathtt{vz}),$$

as required. □

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13. Complete intersections: unexpected case

13.1. **Proof of the "only if" part of Theorem 3.2 for complete intersections.** Let us determine under what conditions $A = \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/\mathfrak{a}$ where $\mathfrak{a} = (\mathtt{z}^2 - \mathtt{x}\mathtt{y}, F + \mathtt{z}G)$ satisfies $\mathcal{P}(A) = \Sigma A^2$. Let use prove: *If $\mathcal{P}(A) = \Sigma A^2$, then, after a change of coordinates, $\mathfrak{a} = (\mathtt{y}^2 - \mathtt{x}\mathtt{z}, \mathtt{z}^2 - 2b\mathtt{y}^2 + 4c^2\mathtt{y}\mathtt{x} + d\mathtt{x}^2)$, where the polynomial $P := \mathtt{t}^4 - 2b\mathtt{t}^2 - 4c^2\mathtt{t} + d \in \kappa[\mathtt{t}]$ is chimeric.* To ease the proof of this result we conduct it into several steps.

13.1.1. *First reduction.* We analyze first what happens when the leading form of F is $\mathtt{x}\mathtt{y}$.

Lemma 13.1. *If the leading form of F is (a multiple of) $\mathtt{x}\mathtt{y}$, then $\mathcal{P}(A) \neq \Sigma A^2$.*

Proof. We distinguish several cases:

Case 1. Assume first $\omega(g) \geqslant 2$. Write $F = \mathtt{x}\mathtt{y} + F_3 + F_4$, where $F_3 := a\mathtt{x}^3 + b\mathtt{x}^2\mathtt{y} + c\mathtt{x}\mathtt{y}^2 + d\mathtt{y}^3 \in \kappa[[\mathtt{x},\mathtt{y}]]$ is an homogeneous polynomial of degree 3 and $F_4 \in \mathfrak{m}_2^4$. We have

$$(\mathtt{x}\mathtt{y} + F_3 + F_4 + \mathtt{z}g)(1 - (b\mathtt{x} + c\mathtt{y})) = \mathtt{x}\mathtt{y} + a\mathtt{x}^3 + d\mathtt{y}^3 + F_4' + \mathtt{z}g' \in \mathfrak{a}$$

for some $F_4', g' \in \kappa[[\mathtt{x},\mathtt{y}]]$, where $\omega(F_4') \geqslant 4$ and $\omega(g') \geqslant 2$, so we may assume from the beginning $b, c = 0$. Consider the following element of $\mathfrak{a}$:

$$(\mathtt{x}\mathtt{y} + F_3 + F_4 + \mathtt{z}g) + (\mathtt{z}^2 - \mathtt{x}\mathtt{y}) = \Big(\mathtt{z} + \frac{g}{2}\Big)^2 + F_3 + F_4 - \frac{g^2}{4}.$$

By Lemma 5.13 we deduce either a or d are non-zero. Suppose $d = 0$, then

$$(\mathtt{x}\mathtt{y} + a\mathtt{x}^3 + F_4 + \mathtt{z}g)(1 + a\mathtt{y}) = \mathtt{x}\mathtt{y} + a\mathtt{x}^3 + a\mathtt{x}\mathtt{y}^2 + F_4' + \mathtt{z}g' \in \mathfrak{a}$$

for some $F_4', g' \in \kappa[[\mathtt{x},\mathtt{y}]]$, where $\omega(F_4') \geqslant 4$ and $\omega(g') \geqslant 2$. Then

$$(\mathtt{x}\mathtt{y} + a\mathtt{x}^3 + a\mathtt{x}\mathtt{y}^2 + F_4' + \mathtt{z}g') + (\mathtt{z}^2 - \mathtt{x}\mathtt{y}) = \Big(\mathtt{z} + \frac{g'}{2}\Big)^2 + a\mathtt{x}^3 + a\mathtt{x}\mathtt{y}^2 + F_4' - \frac{g'^2}{4}.$$

As the series $\mathtt{z}'^2 + a\mathtt{x}^3 + a\mathtt{x}\mathtt{y}^2$ is 3-determined, after a change of coordinates, we may assume $\mathtt{z}^2 + a\mathtt{x}(\mathtt{x}^2 + \mathtt{y}^2) \in \mathfrak{a}$, so $-a\mathtt{x} \in \mathcal{P}(A) \setminus \Sigma A^2$. Thus, we may assume $d \neq 0$ and interchanging the roles of $\mathtt{x},\mathtt{y}$ we may assume that also $a \neq 0$. We write $F_3 = a(\mathtt{x}^3 + d'\mathtt{y}^3)$ and after the change $\mathtt{y} \mapsto -d'\mathtt{y}$, we may write $F_3 = a(\mathtt{x}^3 - d'^4\mathtt{y}^3)$. Consider

$$\begin{aligned}&(\mathtt{x}\mathtt{y} + a(\mathtt{x}^3 - d'^4\mathtt{y}^3) + F_4 + \mathtt{z}g)(1 - a(\mathtt{x} - d'^4\mathtt{y}))\\&\quad= \mathtt{x}\mathtt{y} + a(\mathtt{x}^3 - \mathtt{x}^2\mathtt{y} + d'^4\mathtt{x}\mathtt{y}^2 - d'^4\mathtt{y}^3) + F_4' + \mathtt{z}g' = \mathtt{x}\mathtt{y} + a(\mathtt{x} - \mathtt{y})(\mathtt{x}^2 + d'^4\mathtt{y}^2) + F_4' + \mathtt{z}g' \in \mathfrak{a}\end{aligned}$$

for some $F_4', g' \in \kappa[[\mathtt{x},\mathtt{y}]]$, where $\omega(F_4') \geqslant 4$ and $\omega(g') \geqslant 2$. Then

$$(\mathtt{x}\mathtt{y} + a(\mathtt{x} - \mathtt{y})(\mathtt{x}^2 + d'^4\mathtt{y}^2) + F_4' + \mathtt{z}g') + (\mathtt{z}^2 - \mathtt{x}\mathtt{y}) = \Big(\mathtt{z} + \frac{g'}{2}\Big)^2 + a(\mathtt{x} - \mathtt{y})(\mathtt{x}^2 + d'^4\mathtt{y}^2) + F_4' - \frac{g'^2}{4}.$$

As the series $\mathtt{z}'^2 + a(\mathtt{x} - \mathtt{y})(\mathtt{x}^2 + d'^4\mathtt{y}^2)$ is 3-determined, after a change of coordinates, we may assume $\mathtt{z}^2 + a(\mathtt{x} - \mathtt{y})(\mathtt{x}^2 + d'^4\mathtt{y}^2) \in \mathfrak{a}$, so $-a(\mathtt{x} - \mathtt{y}) \in \mathcal{P}(A) \setminus \Sigma A^2$. ■

To lighten notations, we reset all notations at this point.

Case 2. Assume next $\omega(g) = 1$ and write $g = a\mathtt{x} + b\mathtt{y} + g_2$ where $a, b \in \kappa$ are not both zero and $g_2 \in \mathfrak{m}_2^2$. Write $F = \mathtt{x}\mathtt{y} + F_3$, where $F_3 \in \mathfrak{m}_2^3$. We distinguish two subcases:

Subcase 2.1. Suppose first $b = 0$ and $a \neq 0$ (the case $a = 0, b \neq 0$ is symmetric). Then

$$P := (\mathtt{x}\mathtt{y} + F_3)^2 - \mathtt{x}\mathtt{y}(a\mathtt{x} + g_2)^2 = \mathtt{x}^2\mathtt{y}(\mathtt{y} - a^2\mathtt{x}) + 2\mathtt{x}\mathtt{y}F_3 + F_3^2 - 2a\mathtt{x}^2\mathtt{y}g_2 - \mathtt{x}\mathtt{y}g_2^2 \in \mathfrak{a}$$

As the initial form of P is $\mathtt{x}^2\mathtt{y}(\mathtt{y} - a^2\mathtt{x})$, it is reducible and has two irreducible factors P_1 and P_2 whose leading forms are $\mathtt{y}$ and $\mathtt{y} - a^2\mathtt{x}$. We divide P by P_1P_2 and the quotient P_3 has leading form $\mathtt{x}^2$. We claim: $P_1P_2 := \mathtt{y}(\mathtt{y} - a^2\mathtt{x}) + \cdots \in \mathcal{P}(A)$.

It is enough to check that $P_1P_2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]/(P))$. By Corollary 5.16 it is enough to check that $P_1P_2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]/(P_i))$ for $i = 1, 2, 3$. For $i = 1, 2$ the result is clear. If P_3 is reducible in $\overline{\kappa}[[\mathtt{x},\mathtt{y}]]$, we may write $P_3 = P_4P_5$, where $P_k = \mathtt{x} - g_k(\mathtt{y})$ and $g_k \in (\mathtt{y}^2)\overline{\kappa}[[\mathtt{y}]]$. Thus, $P_1P_2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]/(P))$. Suppose next P_3 is irreducible in $\overline{\kappa}[[\mathtt{x},\mathtt{y}]]$ and $\kappa[[\mathtt{x},\mathtt{y}]]/(P_3)$ is a (formally) real field (because otherwise $P_1P_2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]/(P_3)) = \kappa[[\mathtt{x},\mathtt{y}]]/(P_3))$. By Corollary 5.22 there exist $\alpha_1, \alpha_2 \in \kappa[[\mathtt{t}]]$ with $\omega(\alpha_2) < \omega(\alpha_1)$ (because the initial form of P_3 is $\mathtt{x}^2$) such that $P_1P_2 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]/(P))$ if and only if $(P_1P_2)(\alpha_1,\alpha_2) \in \mathcal{P}(\kappa[[\mathtt{t}]])$, which is true, because $\omega(\alpha_2) < \omega(\alpha_1)$.

If $P_1P_2 \in \Sigma A^2$, there exist $\lambda, \mu \in \kappa$ such that the quadratic form $q := \mathtt{y}(\mathtt{y} - a^2\mathtt{x}) + \lambda(\mathtt{z}^2 - \mathtt{x}\mathtt{y}) + \mu(\mathtt{x}\mathtt{y} + a\mathtt{z}\mathtt{x})$ is positive semidefinite. Observe that

$$q(\mathtt{x}, 1, 0) = 1 - (a^2 + \lambda - \mu)\mathtt{x}.$$

If $a^2 + \lambda - \mu \neq 0$, we substitute $\mathtt{x} := \frac{2}{(a^2+\lambda-\mu)}$, so $q(\frac{2}{(a^2+\lambda-\mu)}, 1, 0) = -1 \in -\Sigma\kappa^2$. Thus, $a^2 + \lambda - \mu = 0$ and $\mu = a^2 + \lambda$, that is,

$$q = \mathtt{y}(\mathtt{y} - a^2\mathtt{x}) + \lambda(\mathtt{z}^2 - \mathtt{x}\mathtt{y}) + (a^2 + \lambda)(\mathtt{x}\mathtt{y} + a\mathtt{z}\mathtt{x}) = \mathtt{y}^2 + \lambda\mathtt{z}^2 + a(a^2 + \lambda)\mathtt{z}\mathtt{x}$$

and $q(\mathtt{x}, 0, 1) = \lambda + a(a^2 + \lambda)\mathtt{x}$ and $q(0, 0, 1) = \lambda \in \Sigma\kappa^2$ (so in particular, $a^2 + \lambda \neq 0$). We substitute $\mathtt{x} := \frac{-\lambda^2-1}{a(a^2+\lambda)}$ and we have $q(\frac{-\lambda^2-1}{a(a^2+\lambda)}, 0, 1) = \lambda - \lambda^2 - 1 \in -\Sigma\kappa^2\backslash\{0\}$, which is a contradiction, so $P_1P_2 \in \mathcal{P}(A)\backslash\Sigma A^2$.

Subcase 2.2. Assume next $a, b \neq 0$. Then

$$P := (\mathtt{x}\mathtt{y} + F_3)^2 - \mathtt{x}\mathtt{y}(a\mathtt{x} + b\mathtt{y} + g_2)^2 \in \mathfrak{a}$$

and its leading form is $-\mathtt{x}\mathtt{y}(a^2\mathtt{x}^2 + (2ab - 1)\mathtt{x}\mathtt{y} + b^2\mathtt{y}^2)$. Thus, P is reducible and has two irreducible factors P_1 and P_2 whose leading forms are $\mathtt{x}$ and $\mathtt{y}$. We divide P by $-P_1P_2$ and the quotient P_3 has leading form $a^2\mathtt{x}^2 + (2ab - 1)\mathtt{x}\mathtt{y} + b^2\mathtt{y}^2$. If P_3 is reducible in $\overline{\kappa}[[\mathtt{x},\mathtt{y}]]$, we may write $P_3 = P_4P_5$, where $P_k = \mathtt{x} - g_k(\mathtt{y})$, $g_k \in (\mathtt{y})\overline{\kappa}[[\mathtt{y}]]$ and $\omega(g_k) = 1$. Suppose next P_3 is irreducible in $\overline{\kappa}[[\mathtt{x},\mathtt{y}]]$ and $\operatorname{qf}(\kappa[[\mathtt{x},\mathtt{y}]]/(P_3))$ is a (formally) real field (because otherwise $\mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]/(P_3)) = \kappa[[\mathtt{x},\mathtt{y}]]/(P_3))$. By Corollary 5.22 there exist $c \in \kappa$ and $\alpha_1, \alpha_2 \in \kappa[\sqrt{c}][[\mathtt{t}]]$ with $\omega(\alpha_2) = \omega(\alpha_1)$ (because the initial form of P_3 is $a^2\mathtt{x}^2 + (2ab - 1)\mathtt{x}\mathtt{y} + b^2\mathtt{y}^2$) such that $h \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]/(P_3))$ if and only if $h(\alpha_1,\alpha_2) \in \mathcal{P}(\kappa[\sqrt{c}][[\mathtt{t}]])$.

For each $M \in \kappa\backslash\{0\}$ define $f_M := M^2\mathtt{x}^2 - P_1P_2$ and observe that $f_M \in \mathcal{P}(A)$ if $f_M \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]/(P))$. By Corollary 5.16 $f_M \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]/(P))$ if and only if $f_M \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]/(P_i))$ for each $i = 1, 2, 3$. It is clear that $f_M \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]/(P_i))$ for $i = 1, 2$. If $\operatorname{qf}(\kappa[[\mathtt{x},\mathtt{y}]]/(P_3))$ is not a (formally) real field, then each $f_M \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]/(P))$. Suppose next $P_3 = P_4P_5$ is reducible in $\overline{\kappa}[[\mathtt{x},\mathtt{y}]]$ and $\operatorname{qf}(\kappa[[\mathtt{x},\mathtt{y}]]/(P_3))$ is a (formally) real field. We have $\operatorname{qf}(\kappa[[\mathtt{x},\mathtt{y}]]/(P_3)) \cong \kappa((\mathtt{y}))[g_3(\mathtt{y})]$ and $g_3 \in \mathfrak{R}(\alpha \cap \kappa)[[\mathtt{y}]]$ for each $\alpha \in \operatorname{Sper}(\kappa[[\mathtt{x},\mathtt{y}]]/(P_3))$. Consequently, $f_M \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]/(P_3))$ if and only if $f_M(g_3(\mathtt{y}),\mathtt{y}) \in \mathcal{P}(\mathfrak{R}(\alpha \cap \kappa)[[\mathtt{y}]])$ for each $\alpha \in \operatorname{Sper}(\kappa[[\mathtt{x},\mathtt{y}]]/(P_3))$. As $\omega(g_3) = 1$, there exists by Lemma 11.1 $M \in \kappa\backslash\{0\}$ such that $f_M(g_3(\mathtt{y}),\mathtt{y}) \in \mathcal{P}(\mathfrak{R}(\alpha \cap \kappa)[[\mathtt{y}]])$ for each $\alpha \in \operatorname{Sper}(\kappa[[\mathtt{x},\mathtt{y}]]/(P_3))$, so $f_M \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]/(P))$. It is enough to apply Lemma 11.1 to the independent coefficient of $\frac{\alpha_2}{\alpha_1}$, which belongs to $\mathfrak{R}(\alpha \cap \kappa)$ for each $\alpha \in \operatorname{Sper}(\kappa[[\mathtt{x},\mathtt{y}]]/(P_3))$.

If P_3 is irreducible in $\overline{\kappa}[[\mathtt{x},\mathtt{y}]]$ and $\operatorname{qf}(\kappa[[\mathtt{x},\mathtt{y}]]/(P_3))$ is a (formally) real field, then $f_M \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]/(P_3))$ if $f_M(\alpha_1,\alpha_2) \in \mathcal{P}(\kappa[[\mathtt{t}]])$. As $\omega(\alpha_1) = \omega(\alpha_2)$, we find $M \in \kappa\backslash\{0\}$ such that $f_M \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]/(P))$. Namely, if ρ is the independent coefficient of $\frac{\alpha_2}{\alpha_1}$, it is enough to take $M := \rho^2 + 1$.

Let $M \in \kappa$ be such that $f_M \in \mathcal{P}(A)$ and suppose that $f_M \in \Sigma A^2$. Then, there exist $\lambda, \mu \in \kappa$ such that the quadratic form $q := M^2\mathtt{x}^2 - \mathtt{x}\mathtt{y} + \lambda(\mathtt{z}^2 - \mathtt{x}\mathtt{y}) + \mu(\mathtt{x}\mathtt{y} + a\mathtt{z}\mathtt{x} + b\mathtt{z}\mathtt{y})$ is positive semidefinite. Observe that

$$q(1, \mathtt{y}, 0) = M^2 - (1 + \lambda - \mu)\mathtt{y}.$$

If $1 + \lambda - \mu \neq 0$, we substitute $\mathtt{y} := \frac{2M^2}{(1+\lambda-\mu)}$, so $q(1, \frac{2M^2}{(1+\lambda-\mu)}, 0) = -M^2 \in -\Sigma\kappa^2\backslash\{0\}$. Thus, $1 + \lambda - \mu = 0$ and $\mu = 1 + \lambda$, that is,

$$q = M^2\mathtt{x}^2 - \mathtt{x}\mathtt{y} + \lambda(\mathtt{z}^2 - \mathtt{x}\mathtt{y}) + (1 + \lambda)(\mathtt{x}\mathtt{y} + a\mathtt{z}\mathtt{x} + b\mathtt{z}\mathtt{y}) = M^2\mathtt{x}^2 + \lambda\mathtt{z}^2 + (1 + \lambda)(a\mathtt{z}\mathtt{x} + b\mathtt{z}\mathtt{y}).$$

Consequently, $q(0,\mathtt{y},1) = \lambda + b(1+\lambda)\mathtt{y}$ and $q(0,0,1) = \lambda \in \Sigma\kappa^2$ (so in particular, $1+\lambda \neq 0$). We substitute $\mathtt{y} := \frac{-\lambda^2-1}{b(1+\lambda)}$ and obtain $q(0, \frac{-\lambda^2-1}{b(1+\lambda)}, 1) = \lambda - \lambda^2 - 1 \in -\Sigma\kappa^2\setminus\{0\}$, which is a contradiction, so $f_M \in \mathcal{P}(A)\setminus\Sigma A^2$. □

13.1.2. *Second reduction.* In the following we assume that the leading form of F is not a multiple of $\mathtt{xy}$. Thus, either $F(0,\mathtt{y})$ is (a non-zero multiple of) $\mathtt{y}^2$ or $F(\mathtt{x},0)$ is (a non-zero multiple of) $\mathtt{x}^2$, so interchanging $\mathtt{x}$ and $\mathtt{y}$, we assume in the following $F(0,\mathtt{y}) = \mathtt{y}^2$. We prove next that if there exists a reducible element of $\mathfrak{a}$ of order 2, then $\mathcal{P}(A) \neq \Sigma A^2$. Observe that this includes the case when $\gcd(F,G)$ has order 1.

Lemma 13.2. *If $F(0,\mathtt{y}) = \mathtt{y}^2$ and there exists a reducible element of $\mathfrak{a}$ of order 2, then $\mathcal{P}(A) \neq \Sigma A^2$.*

Proof. Suppose there exists a reducible element $H \in \mathfrak{a}$ of order 2. As $\mathtt{z}^2 - \mathtt{xy}$ is irreducible and $\mathfrak{a} = (\mathtt{z}^2 - \mathtt{xy}, F + \mathtt{z}G)$, we may assume there exists $H_{ij} \in \kappa[[\mathtt{x},\mathtt{y}]]$ such that $(H_{11} + \mathtt{z}H_{12})(H_{21} + \mathtt{z}H_{22}) = (\mathtt{z}^2 - \mathtt{xy})H_{12}H_{22} + F + \mathtt{z}G$. Thus, $F = H_{11}H_{21} + \mathtt{xy}H_{12}H_{22}$. As $F(0,\mathtt{y})$ is (a multiple of) $\mathtt{y}^2$, we deduce $H_{11}(\mathtt{y},0)$ and $H_{21}(\mathtt{y},0)$ are series of order 1. Thus, after multiplying by suitable units of $\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ we may change (by Weierstrass preparation Theorem) $H_{i1} + \mathtt{z}H_{12}$ by $\mathtt{y} - h_i(\mathtt{x},\mathtt{z})$ where $h_i \in \kappa[[\mathtt{x},\mathtt{z}]]$ for $i = 1,2$. Observe that

$$\begin{aligned}\mathfrak{a} &= (\mathtt{z}^2 - \mathtt{xy}, \mathtt{y} - h_1(\mathtt{x},\mathtt{z})) \cap (\mathtt{z}^2 - \mathtt{xy}, \mathtt{y} - h_2(\mathtt{x},\mathtt{z}))\\ &= (\mathtt{z}^2 - \mathtt{x}h_1(\mathtt{x},\mathtt{z}), \mathtt{y} - h_1(\mathtt{x},\mathtt{z})) \cap (\mathtt{z}^2 - \mathtt{x}h_2(\mathtt{x},\mathtt{z}), \mathtt{y} - h_2(\mathtt{x},\mathtt{z}))\end{aligned}$$

By Corollary 5.16 an element $q \in \mathcal{P}(A)$ if and only $q(\mathtt{x}, h_i(\mathtt{x},\mathtt{z}), \mathtt{z}) \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{z}]]/(\mathtt{z}^2 - \mathtt{x}h_i(\mathtt{x},\mathtt{z})))$ for $i = 1,2$.

Suppose first $\omega(h_1(\mathtt{x},\mathtt{z}) - h_2(\mathtt{x},\mathtt{z})) \geqslant 2$. There exists by Lemma 5.9 $M \in \kappa$ such that $h_1(\mathtt{x},\mathtt{z}) - h_2(\mathtt{x},\mathtt{z}) + M^2(\mathtt{x}^2 + \mathtt{z}^2) \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{z}]])$. We have $f(\mathtt{x}, h_i(\mathtt{x},\mathtt{z}), \mathtt{z}) = h_i(\mathtt{x},\mathtt{z}) - h_2(\mathtt{x},\mathtt{z}) + M^2(\mathtt{x}^2 + \mathtt{z}^2) \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{z}]]/(\mathtt{z}^2 - \mathtt{x}h_i(\mathtt{x},\mathtt{z})))$ for $i = 1,2$. Thus, $f := \mathtt{y} - h_2(\mathtt{x},\mathtt{z}) + M^2(\mathtt{x}^2 + \mathtt{z}^2) \in \mathcal{P}(A)\setminus\Sigma A^2$ (because f has order 1 and $\omega(\mathfrak{a}) \geqslant 2$). So we assume in the following $\omega(h_1(\mathtt{x},\mathtt{z}) - h_2(\mathtt{x},\mathtt{z})) = 1$. We distinguish several cases:

Case 1. $\omega(h_2) \geqslant 2$. As $\omega(h_1 - h_2) = 1$, we deduce $\omega(h_1) = 1$. We distinguish two subcases:

Subcase 1.1. Assume next $\omega(h_1(\mathtt{x},0)) = 1$. As $\mathfrak{a}$ is a real ideal, its minimal associated primes are real prime ideals [BCR, Lem.4.1.5], so $(\mathtt{z}^2 - \mathtt{x}h_1(\mathtt{x},\mathtt{z}))\kappa[[\mathtt{x},\mathtt{z}]]$ is a real prime ideal. By Weierstrass preparation theorem we may substitute $\mathtt{z}^2 - \mathtt{x}h_2(\mathtt{x},\mathtt{z})$ by a Weierstrass polynomial $P_2 := \mathtt{z}^2 - \mathtt{x}^2a_1(\mathtt{x})\mathtt{z} + a_0(\mathtt{x})\mathtt{x}^3 \in \kappa[[\mathtt{x}]][\mathtt{z}]$. By Corollaries 5.16 and 5.22 there exist $c \in \kappa$ (which may be a square) and series $\alpha_{1i}, \alpha_{3i} \in \kappa[\sqrt{c}][[\mathtt{t}]]$ (where i is either 1 or 2 and $\alpha_{1i} = \theta_i^i\mathtt{t}^i$ for some $\theta_i \in \kappa\setminus\{0\}$) with $\omega(\alpha_{3i}) \geqslant \frac{3}{2}\omega(\alpha_{1i}) > \omega(\alpha_{1i})$ (because $P_2(\theta_i^i\mathtt{t}^i, \alpha_{3i}) = 0$) such that $Q \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{z}]]/(P_2)$ if and only if $Q(\theta_i^i\mathtt{t}^i, \alpha_{3i}) \in \mathcal{P}(\kappa[\sqrt{c}][[\mathtt{t}]])$ for each i. The symbol $+\cdots$ means 'plus terms of higher order'. We distinguish two situations:

Subcase 1.1.1. The leading form of $\mathtt{z}^2 - \mathtt{x}h_1(\mathtt{x},\mathtt{z})$ is the square of a linear form. We claim that $f := \mathtt{z}^2 - \mathtt{x}h_1(\mathtt{x},\mathtt{z}) \in \mathcal{P}(A)$.

We have to check that $f \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{z}]]/(\mathtt{z}^2 - \mathtt{x}h_2(\mathtt{x},\mathtt{z}))$. As $\omega(h_1(\mathtt{x},0)) = 1$, we have

$$f(\theta_i^i\mathtt{t}^i, \alpha_{3i}) = \theta_i^{2i}\mathtt{t}^{2i} + \cdots \in \mathcal{P}(\kappa[[\mathtt{t}]]),$$

so $f \in \mathcal{P}(A)$.

If $f \in \Sigma A^2$, there exists $p_1, p_2 \in \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ such that

$$g := f + p_1(\mathtt{z}^2 - \mathtt{xy}) + p_2(\mathtt{y} - h_1(\mathtt{x},\mathtt{z}))(\mathtt{y} - h_2(\mathtt{x},\mathtt{z})) \in \Sigma\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]^2.$$

If we substitute $\mathtt{y} = h_1(\mathtt{x},\mathtt{z})$, we have $(1 + p_1)(\mathtt{z}^2 - \mathtt{x}h_1(\mathtt{x},\mathtt{z})) \in \Sigma\kappa[[\mathtt{x},\mathtt{z}]]^2$. As $\mathtt{z}^2 - \mathtt{x}h_1(\mathtt{x},\mathtt{z})$ generates a real ideal, we deduce $1 + p_1 = (\mathtt{z}^2 - \mathtt{x}h_1(\mathtt{x},\mathtt{z}))Q$, where $Q \in \Sigma\kappa[[\mathtt{x},\mathtt{z}]]^2$. Thus,

$p_1 = -1 + (\mathtt{z}^2 - \mathtt{x}h_1(\mathtt{x},\mathtt{z}))Q(\mathtt{x},\mathtt{z})$. If we substitute $\mathtt{y} = h_2(\mathtt{x},\mathtt{z})$, we have

$$\begin{aligned} &Q(\mathtt{x},\mathtt{z})(\mathtt{z}^2 - \mathtt{x}h_1(\mathtt{x},\mathtt{z}))(\mathtt{z}^2 - \mathtt{x}h_2(\mathtt{x},\mathtt{z})) - h_1(\mathtt{x},\mathtt{z})\mathtt{x} + h_2(\mathtt{x},\mathtt{z})\mathtt{x} \\ &\qquad = \mathtt{z}^2 - \mathtt{x}h_1(\mathtt{x},\mathtt{z}) + (-1 + (\mathtt{z}^2 - \mathtt{x}h_1(\mathtt{x},\mathtt{z}))Q(\mathtt{x},\mathtt{z}))(\mathtt{z}^2 - \mathtt{x}h_2(\mathtt{x},\mathtt{z})) \in \Sigma\kappa[[\mathtt{x},\mathtt{z}]]^2 \end{aligned}$$

The leading form of the previous sum of squares coincides with the leading form of $-\mathtt{x}h_1(\mathtt{x},\mathtt{z})$, so the leading form of $-\mathtt{x}h_1(\mathtt{x},\mathtt{z})$ is of the type $\mathtt{x}^2\sigma$ for some $\sigma \in \Sigma\kappa^2\backslash\{0\}$, which is a contradiction, because the leading form $\mathtt{z}^2 + \sigma\mathtt{x}^2$ of $\mathtt{z}^2 - \mathtt{x}h_1(\mathtt{x},\mathtt{z})$ should be the square of a linear form.

SUBCASE 1.1.2. The leading form of $\mathtt{z}^2 - \mathtt{x}h_1(\mathtt{x},\mathtt{z})$ is not the square of a linear form. Write the leading form of h_1 as $\ell_1 := c\mathtt{x} + 2d\mathtt{z}$. Then $\mathtt{z}^2 - \mathtt{x}(c\mathtt{x} + 2d\mathtt{z}) = (\mathtt{z} - d\mathtt{x})^2 - (c + 4d^2)\mathtt{x}^2$ and $c + 4d^2 \neq 0$. As $(\mathtt{z}^2 - \mathtt{x}h_1(\mathtt{x},\mathtt{z}))\kappa[[\mathtt{x},\mathtt{z}]]$ is a real ideal, we deduce (using finite determinacy) that the quadratic form $\mathtt{z}^2 - \mathtt{x}(c\mathtt{x} + 2d\mathtt{z})$, which has rank 2 is not positive semidefinite. Let $\rho \in \kappa$ be such that the leading forms of $h_1(\mathtt{x},0) + \rho\mathtt{z}$ and h_1 are linearly independent. We claim: $f := (h_1(\mathtt{x},\mathtt{z}) - \mathtt{y})(h_1(\mathtt{x},0) + \rho\mathtt{z}) \in \mathcal{P}(A)$.

We only have to check that

$$f_1(\mathtt{x},\mathtt{z}) := f(\mathtt{x}, h_2(\mathtt{x},\mathtt{z}),\mathtt{z}) = (h_1(\mathtt{x},\mathtt{z}) - h_2(\mathtt{x},\mathtt{z}))(h_1(\mathtt{x},0) + \rho\mathtt{z}) \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{z}]]/(\mathtt{z}^2 - \mathtt{x}h_2(\mathtt{x},\mathtt{z}))).$$

By Weierstrass preparation theorem we may substitute $\mathtt{z}^2 - \mathtt{x}h_2(\mathtt{x},\mathtt{z})$ by a Weierstrass polynomial $P_2 := \mathtt{z}^2 - \mathtt{x}^2 a_1(\mathtt{x})\mathtt{z} + a_0(\mathtt{x})\mathtt{x}^3 \in \kappa[[\mathtt{x}]][\mathtt{z}]$. By Corollaries 5.16 and 5.22 there exist $c \in \kappa$ (which me be a square) and series $\alpha_{1i}, \alpha_{3i} \in \kappa[[\mathtt{t}]]$ (where i is either 1 or 2 and $\alpha_{1i} = \theta_i^i\mathtt{t}^i$ for some $\theta_i \in \kappa\backslash\{0\}$) with $\omega(\alpha_{3i}) \geqslant \frac{3}{2}\omega(\alpha_{1i}) > \omega(\alpha_{1i}) = i$ (because $P_2(\alpha_{1i},\alpha_{3i}) = 0$) such that $f_1 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{z}]]/(P_2)$ if and only if $f_1(\alpha_{1i},\alpha_{3i}) \in \mathcal{P}(\kappa[[\mathtt{t}]])$ for each i. Observe that $f_1(\theta_i^i\mathtt{t}^i,\alpha_{3i}) = h_1(\theta_i^i\mathtt{t}^i, 0)^2 + \cdots$, where $+\cdots$ means 'plus terms of higher order'. Thus, $f_1 \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{z}]]/(P_2)$ and $f \in \mathcal{P}(A)$.

Suppose $f \in \Sigma A^2$ and denote ℓ_1 the leading form of h_1 and ℓ_2 the leading form of $h_1(\mathtt{x},0) + \rho\mathtt{z}$ (which are linearly independent). Then there exist $\lambda,\mu \in \kappa$ such that the quadratic form

$$q = (\ell_1 - \mathtt{y})\ell_2 + \lambda(\mathtt{z}^2 - \mathtt{x}\mathtt{y}) + \mu(\mathtt{y} - \ell_1)\mathtt{y} = (\ell_1 - \mathtt{y})(\ell_2 - \mu\mathtt{y}) + \lambda(\mathtt{z}^2 - \mathtt{x}\mathtt{y})$$

is positive semidefinite. Substituting $\mathtt{y}$ by ℓ_1, we deduce that $\lambda(\mathtt{z}_2 - \mathtt{x}\ell_1)$ is positive semidefinite. As the quadratic form $\mathtt{z}_2 - \mathtt{x}\ell_1$ is not positive semidefinite, we deduce $\lambda = 0$ and $q = (\ell_1 - \mathtt{y})(\ell_2 - \mu\mathtt{y})$ is positive semidefinite. Thus, $(\ell_1 - \mathtt{y})$ divides $\ell_2 - \mu\mathtt{y}$ and there exists $a \in \kappa\backslash\{0\}$ such that $\ell_2 - \mu\mathtt{y} = a(\ell_1 - \mathtt{y})$. Substituting $\mathtt{y} = 0$, we deduce $\ell_2 = a\ell_1$, which is a contradiction, because ℓ_1 and ℓ_2 are not proportional.

SUBCASE 1.2. Assume now $\omega(h_1(\mathtt{x},0)) \geqslant 2$. As $\omega(h_1) = 1$, there exists a unit $u \in \kappa[[\mathtt{x},\mathtt{z}]]$ such that $u(0,0) = 1$ and $h_1 := (a\mathtt{z} + h_1'(\mathtt{x}))u(\mathtt{x},\mathtt{z})$ where $h_1' \in \mathfrak{m}_1^2$ and $a \in \kappa\backslash\{0\}$. We have $\mathtt{z}^2 - \mathtt{x}h_1 = \mathtt{z}^2 - \mathtt{x}(a\mathtt{z} + h_1'(\mathtt{x}))u(\mathtt{x},\mathtt{z})$. As its leading form is $\mathtt{z}(\mathtt{z} - a\mathtt{x})$, there exist series $g_1, g_2 \in \mathfrak{m}_1^2\kappa[[\mathtt{x}]]$ and a unit $v \in \kappa[[\mathtt{x},\mathtt{z}]]$ such that $\mathtt{z}^2 - \mathtt{x}h_1 = (\mathtt{z} - g_1)(\mathtt{z} - a\mathtt{x} - g_2)v$. As $\mathtt{z}^2 - \mathtt{x}\mathtt{y} \in \mathfrak{a}$, we have

$$\begin{aligned} \mathfrak{a} &= (\mathtt{y} - (a\mathtt{z} + h_1'(\mathtt{x}))u(\mathtt{x},\mathtt{z}), (\mathtt{z} - g_1)(\mathtt{z} - a\mathtt{x} - g_2)) \cap (\mathtt{y} - h_2(\mathtt{x},\mathtt{z}), \mathtt{z}^2 - \mathtt{x}h_2(\mathtt{x},\mathtt{z})) \\ &= (\mathtt{y} - a^2\mathtt{x}u_1^2(\mathtt{x}), \mathtt{z} - a\mathtt{x}u_1(\mathtt{x})) \cap \Big(\mathtt{y} - \frac{g_1^2}{\mathtt{x}}, \mathtt{z} - g_1\Big) \cap (\mathtt{y} - h_2(\mathtt{x},\mathtt{z}), \mathtt{z}^2 - \mathtt{x}h_2(\mathtt{x},\mathtt{z})) \end{aligned}$$

for some unit $u_1 \in \kappa[[\mathtt{x}]]$ such that $u_1(0) = 1$ (recall that $\mathtt{z}^2 - \mathtt{x}\mathtt{y} \in \mathfrak{a}$). We claim: $f := a(\mathtt{y} - a^2\mathtt{x}u_1^2(\mathtt{x}))(\mathtt{z} - a\mathtt{x}u_1(\mathtt{x})) \in \mathcal{P}(A)$.

By Lemma 5.16 we have to check that $0 = f(\mathtt{x}, a^2\mathtt{x}u_1^2(\mathtt{x}), a\mathtt{x}u_1(\mathtt{x})) \in \mathcal{P}(\kappa[[\mathtt{x}]])$ (which is clearly true), $f(\mathtt{x}, \frac{g_1^2}{\mathtt{x}}, g_1) = a(a^2\mathtt{x}u_1^2(\mathtt{x}) - \frac{g_1^2}{\mathtt{x}})(a\mathtt{x}u_1(\mathtt{x}) - g_1) \in \mathcal{P}(\kappa[[\mathtt{x}]])$ (which is true because $\omega(g_1) \geqslant 2$) and $f(\mathtt{x}, h_2(\mathtt{x},\mathtt{z}),\mathtt{z}) \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{z}]]/(\mathtt{z}^2 - \mathtt{x}h_2(\mathtt{x},\mathtt{z})))$. To check the latter we have to prove

$$f(\theta_i^i\mathtt{t}^i, h_2(\theta_i^i\mathtt{t}^i,\alpha_{3i}),\alpha_{3i}) = a(a^2\theta_i^i\mathtt{t}^iu_1^2(\theta_i^i\mathtt{t}^i) - h_2(\theta_i^i\mathtt{t}^i,\alpha_{3i})(a\theta_i^i\mathtt{t}^iu_1(\theta_i^i\mathtt{t}^i) - \alpha_{3i}) \in \mathcal{P}(\kappa[[\mathtt{t}]]),$$

which is true because $f(\theta_i^i\mathtt{t}^i, h_2(\theta_i^i\mathtt{t}^i,\alpha_{3i}),\alpha_{3i}) = a^4\theta_i^{2i}\mathtt{t}^{2i} + \cdots$. We have use used that $\omega(h_2) \geqslant 2$ and $\omega(\alpha_{3i}) > i$.

Suppose $f \in \Sigma A^2$. Then, there exist $\lambda, \mu \in \kappa$ such that the quadratic form
$$q = a(\mathtt{y} - a^2\mathtt{x})(\mathtt{z} - a\mathtt{x}) + \lambda(\mathtt{z}^2 - \mathtt{x}\mathtt{y}) + \mu(\mathtt{y} - a\mathtt{z})\mathtt{y}$$
is positive semidefinite. If we substitute $\mathtt{x} = 0$ and $\mathtt{y} = 0$ in q, we deduce $\lambda \in \Sigma\kappa^2\backslash\{0\}$, whereas if we substitute $\mathtt{x} = 0$ and $\mathtt{z} = 0$ in q, we deduce $\mu \in \Sigma\kappa^2\backslash\{0\}$. We substitute next $\mathtt{y} = a^2\mathtt{x}$ in q and obtain
$$q(\mathtt{x}, a^2\mathtt{x}, \mathtt{z}) = (a\mathtt{x} - \mathtt{z})(a^3\mu\mathtt{x} - a\lambda\mathtt{x} - \lambda\mathtt{z}),$$
so $a\mathtt{x} - z$ divides $a^3\mu\mathtt{x} - a\lambda\mathtt{x} - \lambda\mathtt{z}$ (because q is positive semidefinite). Substituting $\mathtt{z} = a\mathtt{x}$ in $a^3\mu\mathtt{x} - a\lambda\mathtt{x} - \lambda\mathtt{z}$, we deduce $a\mathtt{x}(a^2\mu - 2\lambda)$, so $\lambda = \frac{a^2\mu}{2}$. If we substitute $\lambda = \frac{a^2\mu}{2}$ and $\mathtt{z} = a\mathtt{x}$ in q, we obtain
$$q(\mathtt{x}, \mathtt{y}, a\mathtt{x}) = \mu\frac{(a^2\mathtt{x} - 2\mathtt{y})(a^2\mathtt{x} - \mathtt{y})}{2},$$
which is not positive semidefinite, because
$$q\Big(\mathtt{x}, \frac{3}{4}a^2\mathtt{x}, a\mathtt{x}\Big) = \frac{-a^4\mu\mathtt{x}^2}{16}$$
and $\mu \in \Sigma\kappa^2\backslash\{0\}$. ∎

The case $\omega(h_1) \geqslant 2$ is analogous to the previous one, so we assume in the following $\omega(h_1) = 1, \omega(h_2) = 1, \omega(h_1 - h_2) = 1$. The leading form q_i of $\mathtt{z}^2 - \mathtt{x}h_i(\mathtt{x}, \mathtt{z})$ is a quadratic form. Denote $\mathit{lf}(\cdot)$ the operator to compute the leading form of a series. We analyze next what happens when $\mathit{lf}(h_2 - h_1)$ divides or not one of the quadratic forms q_i. Interchanging q_1 and q_2 if necessary, we may assume $i = 2$. We distinguish two cases:

Case 2. $\mathit{lf}(h_2 - h_1)$ divides q_2, that is, $q_2 = \mathit{lf}(h_2 - h_1)\ell_1$ where $\ell_1 \in \kappa[\mathtt{x}, \mathtt{y}]$ is a linear form. Thus, $q_1 = \mathtt{z}^2 - \mathtt{x}\mathit{lf}(h_1) = (\mathtt{z}^2 - \mathtt{x}\mathit{lf}(h_2)) + \mathtt{x}\mathit{lf}(h_2 - h_1) = (\ell_1 + \mathtt{x})\mathit{lf}(h_2 - h_1)$. Consequently, $\mathit{lf}(h_2 - h_1)$ also divides q_1.

If ℓ_1 is proportional to $\mathit{lf}(h_1 - h_2)$, then $\ell_1 + \mathtt{x}$ is not proportional to $\mathit{lf}(h_1 - h_2)$. Otherwise, $\mathit{lf}(h_2 - h_1)$ is proportional to $\mathtt{x}$, but $\mathtt{x}$ does not divide q_1 nor q_2. Thus, we may assume ℓ_1 is not proportional to $\mathit{lf}(h_1 - h_2)$ (interchanging if necessary q_1 and q_2). As q_2 is the product of two non-proportional linear forms, $\mathtt{z}^2 - \mathtt{x}h_2(\mathtt{x}, \mathtt{z}) = f_1f_2$, where $f_1, f_2 \in \kappa[[\mathtt{x}, \mathtt{z}]]$ are series of order 1 (with respect to $\mathtt{z}$) whose respective leading forms are $\mathit{lf}(h_2 - h_1)$ and ℓ_1. As $f_i(0, \mathtt{z})$ have order 1, by Weierstrass preparation Theorem there exists $g_i \in (\mathtt{x})\kappa[[\mathtt{x}]]$ such that f_i and $\mathtt{z} - g_i(\mathtt{x})$ generate the same ideal in $\kappa[\mathtt{x}, \mathtt{z}]$ for $i = 1, 2$. As the initial form of f_1 and f_2 are not proportional, the series $g_1 - g_2 \in \kappa[[\mathtt{x}]]$ has order $\omega(g_1 - g_2) = 1$. As the initial form of f_1 and $h_1 - h_2$ coincide, $\omega((h_1 - h_2)(\mathtt{x}, g_1)) \geqslant 2$. We have:
$$\mathfrak{a} = (\mathtt{y} - h_1, \mathtt{z}^2 - \mathtt{x}h_1) \cap (\mathtt{y} - h_2, \mathtt{z} - g_1) \cap (\mathtt{y} - h_2, \mathtt{z} - g_2).$$
We claim: $f := (\mathtt{y} - h_1)(\mathtt{z} - g_2) + m^2(\mathtt{z} - g_2)^2 \in \mathcal{P}(A)$ *for each* $m \in \kappa\backslash\{0\}$.

By Corollary 5.16 it is enough to check that $f(\mathtt{x}, h_1, \mathtt{z}) = m^2(\mathtt{z} - g_2)^2 \in \mathcal{P}(\kappa[[\mathtt{x}, \mathtt{z}]]/(\mathtt{z}^2 - \mathtt{x}h_1))$ (which is true), $f(\mathtt{x}, h_2, g_1) = (h_2 - h_1)(\mathtt{x}, g_1)(g_1 - g_2) + m^2(g_1 - g_2)^2 \in \mathcal{P}(\kappa[[\mathtt{x}]])$ (which is true, because $\omega((h_1 - h_2)(\mathtt{x}, g_1)) \geqslant 2$) and $0 = f(\mathtt{x}, h_2, g_2) \in \mathcal{P}(\kappa[[\mathtt{x}, \mathtt{z}]]/(f_2))$ (which is true). Consequently, $f \in \mathcal{P}(A)$ for each $m \in \kappa\backslash\{0\}$.

Suppose $f \in \Sigma A^2$. Then there exist $e_1, e_2 \in \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]$ such that
$$(\mathtt{y} - h_1)(\mathtt{z} - g_2) + m^2(\mathtt{z} - g_2)^2 + e_1(\mathtt{z}^2 - \mathtt{x}\mathtt{y}) + e_2(\mathtt{y} - h_1)(\mathtt{y} - h_2) \in \Sigma\kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]^2.$$
We substitute $\mathtt{y} = h_2$ and we have
$$\begin{aligned}(h_2 - h_1)(\mathtt{z} - g_2) &+ m^2(\mathtt{z} - g_2)^2 + e_1(\mathtt{x}, h_2, \mathtt{z})(\mathtt{z}^2 - \mathtt{x}h_2)\\ &= (h_2 - h_1)(\mathtt{z} - g_2) + m^2(\mathtt{z} - g_2)^2 + e_1(\mathtt{x}, h_2, \mathtt{z})(\mathtt{z} - g_1)(\mathtt{z} - g_2)w,\end{aligned}$$

for some unit $w \in \kappa[\mathtt{x},\mathtt{y}]$ such that $w(0,0)=1$. As $lf(h_1-h_2)$ is the leading form of f_1 and ℓ_1 is the leading form of f_2, the quadratic form

$$Q := lf(h_1-h_2)\ell_1 + m^2\ell_1^2 + \lambda lf(h_1-h_2)\ell_1 = \ell_1((1+\lambda)lf(h_1-h_2)+m^2\ell_1)$$

is positive semidefinite for $\lambda := e_1(0,0,0)$, so $\lambda=-1$. We substitute $\mathtt{y}=h_1$ and we obtain $m^2(\mathtt{z}-g_2)^2+e_1(\mathtt{x},h_1,\mathtt{z})(\mathtt{z}^2-\mathtt{x}h_1)$, whose leading form is

$$m^2\ell_1^2-(\mathtt{z}^2-\mathtt{x}lf(h_1)).$$

We make $\mathtt{x}=0$ and obtain $m^2(\ell_1^2(0,\mathtt{z}))-\mathtt{z}^2$. If $\ell_1(0,1)=0$, we take any $m\in\kappa\backslash\{0\}$ and Q is not positive semidefinite. If $\beta:=\ell_1(0,1)\neq 0$, we take $m:=\frac{1}{1+\beta^2}$ and substitute $\mathtt{z}=1$. Thus,

$$\frac{\beta^2}{(1+\beta^2)^2}-1=\frac{-1-\beta^2-\beta^4}{(1+\beta^2)^2}\in -\Sigma\kappa^2\backslash\{0\},$$

which contradicts the fact that Q is positive semidefinite. Thus, $f\notin\Sigma A^2$. ∎

Case 3. $lf(h_2-h_1)$ does not divide q_2. Let $\ell_2\in\kappa[\mathtt{x},\mathtt{z}]$ be a linear form κ-linearly independent to $lf(h_2-h_1)$ such that ℓ_2 does not divide q_2. As $q_2(1,\mathtt{z})$ is a monic polynomial of degree 2, it has two roots $\rho_j\in\overline{\kappa}$ (which may be equal). As $lf(h_2-h_1)(1,\rho_j)\neq 0$ for $j=1,2$ and it is algebraic over κ, there exists by Lemma 11.1 $m\in\kappa\backslash\{0\}$ such that $lf(h_2-h_1)^2(1,\rho_j)>_\alpha m^2$ for each $\alpha\in\operatorname{Sper}(\kappa)$ and $j=1,2$. As $lf(h_2-h_1)(1,\rho_j)\ell_2(1,\rho_j)\neq 0$ for $j=1,2$ and it is algebraic over κ, there exists by Lemma 11.1 $M\in\kappa\backslash\{0\}$ such that $|lf(h_2-h_1)(1,\rho_j)\ell_2(1,\rho_j)|_\alpha <_\alpha M^2$ for each $\alpha\in\operatorname{Sper}(\kappa)$ and $j=1,2$. We claim: $f:=(\mathtt{y}-h_1)^2-\frac{m^2}{M^2}\ell_2(\mathtt{y}-h_1)\in\mathcal{P}(A)$.

By Lemma 5.16 it is enough to check that $0=f(\mathtt{x},h_1,\mathtt{z})\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{z}]]/(\mathtt{z}^2-\mathtt{x}h_1))$ (which is true) and $f_2:=(h_2-h_1)^2-\frac{m^2}{M^2}\ell_2(\mathtt{x},h_2)(h_2-h_1)=f(\mathtt{x},h_2,\mathtt{z})\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{z}]]/(\mathtt{z}^2-\mathtt{x}h_2))$. By Weierstrass preparation theorem there exists a Weierstrass polynomial $P_2:=\mathtt{z}^2+\mathtt{x}a_1w_1(\mathtt{x})\mathtt{z}+\mathtt{x}^2a_0w_0(\mathtt{x})$ where $a_0,a_1\in\kappa$ and $w_1,w_2\in\kappa[[\mathtt{x}]]$ are units such that $w_i(0)=1$. By Corollaries 5.16 and 5.22 there exist $c\in\kappa$ (which may be a square) and series $\alpha_{1i},\alpha_{3i}\in\kappa[\sqrt{c}][[\mathtt{t}]]$ (where i is either 1 or 2 and $\alpha_{1i}=\theta_i^i\mathtt{t}^i$ for some $\theta_i\in\kappa\backslash\{0\}$) with $\omega(\alpha_{3i})=\omega(\alpha_{1i})=i$ (because $P_2(\alpha_{1i},\alpha_{3i})=0$) such that $f_2\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{z}]]/(P_2)$ if and only if $f_2(\alpha_{1i},\alpha_{3i})\in\mathcal{P}(\kappa[\sqrt{c}][[\mathtt{t}]])$ for each i. We have

$$\begin{aligned}f_2(\theta_i^i\mathtt{t}^i,\alpha_{3i})&=(h_2-h_1)^2(\theta_i^i\mathtt{t}^i,\alpha_{3i})-\frac{m^2}{M^2}\ell_2(\theta_i^i\mathtt{t}^i,\alpha_{3i})(h_2-h_1)(\theta_i^i\mathtt{t}^i,\alpha_{3i})\\&=\theta_i^{2i}\mathtt{t}^{2i}\Big(lf(h_2-h_1)(1,\rho_j)-\frac{m^2}{M^2}\ell_2(1,\rho_j)lf(h_2-h_1)(1,\rho_j)\Big)+\cdots\end{aligned}$$

for $j=1,2$, so $f_2\in\mathcal{P}(\kappa[[\mathtt{t}]])$. Thus, $f\in\mathcal{P}(A)$.

Assume $f\in\Sigma A^2$. Then there exist $e_1,e_2\in\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ such that $f+e_1(\mathtt{z}^2-\mathtt{x}\mathtt{y})+e_2(\mathtt{y}-h_1)(\mathtt{y}-h_2)\in\Sigma\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]^2$. Making $\mathtt{y}=h_1$, we deduce that $e_1(\mathtt{x},h_1,\mathtt{z})(\mathtt{z}^2-\mathtt{x}h_1)\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{z}]])$. If $e_1(0,0,0)\neq 0$, then $e_1(0,0,0)(\mathtt{z}^2-\mathtt{x}h_1)\in\mathcal{P}(\kappa[[\mathtt{x},\mathtt{z}]])=\Sigma\kappa[[\mathtt{x},\mathtt{z}]]^2$. By Lemma 5.18 we deduce that $(\mathtt{z}^2-\mathtt{x}h_1)\kappa[[\mathtt{x},\mathtt{z}]]$ is an intersection of minimal prime ideals, which are real, because $\mathfrak{a}$ is a real ideal, so all its associated minimal prime ideals are real. As $\omega((\mathtt{z}^2-\mathtt{x}h_1)\kappa[[\mathtt{x},\mathtt{z}]])=2$, it contains no element of order 1, so $\mathtt{z}^2-\mathtt{x}h_1$ cannot be a sum of squares, because it generates a real ideal.

Consequently, $e_1(0,0,0)=0$ and there exists $\mu\in\kappa$ such that quadratic form

$$q:=(\mathtt{y}-lf(h_1))^2-\frac{m^2}{M^2}\ell_2(\mathtt{y}-lf(h_1))+\mu(\mathtt{y}-lf(h_1))(\mathtt{y}-lf(h_2))$$

is positive semidefinite. Thus,

$$q':=(\mathtt{y}-lf(h_1))-\frac{m^2}{M^2}\ell_2+\mu(\mathtt{y}-lf(h_2))$$

is a multiple of $(\mathtt{y} - lf(h_1))$. As $\ell_2, lf(h_1 - h_2) \in \kappa[[\mathtt{x},\mathtt{z}]]$, if we substitute $\mathtt{y} = lf(h_1)$ in q', we have

$$0 = -\frac{m^2}{M^2}\ell_2 + \mu(lf(h_1 - h_2)) \quad \rightsquigarrow \quad \ell_2 = \frac{M^2}{m^2}\mu(lf(h_1 - h_2)),$$

which is a contradiction, because $lf(h_1 - h_2)$ and ℓ_2 are κ-linearly independent. Thus, $f \notin \Sigma A^2$, as required. $\square$

13.1.3. *Third reduction.* In the following we assume $F(0,\mathtt{y}) = \mathtt{y}^2$ and there exists no reducible element of order 2 in $\mathfrak{a}$. Let us analyze what happens if F divides G and F is irreducible in $\kappa[[\mathtt{x},\mathtt{y}]]$. Before that we need a preliminary result.

nfrf **Lemma 13.3.** *Let κ be a (formally) real field and let $P \in \kappa[\mathtt{t}]$ be an irreducible polynomial. Suppose that the field $\kappa[\mathtt{t}]/(P)$ is not formally real. Then $P \in \Sigma\kappa[\mathtt{t}]^2$.*

Proof. Let $\alpha \in \operatorname{Sper}(\kappa(\mathtt{t}))$ and let $\mathfrak{R}(\alpha)$ the real closure of $(\kappa(\mathtt{t}), \leqslant_\alpha)$. Let $R := \mathfrak{R}(\beta)$ be the real closure of $(\kappa, \leqslant_\beta)$, where $\beta := \alpha \cap \kappa$, and consider the inclusion $R \hookrightarrow \mathfrak{R}(\alpha)$. Consider the inclusion $R[\mathtt{t}] \hookrightarrow R(\mathtt{t}) \hookrightarrow \mathfrak{R}(\alpha)$. Assume that $P \in -\alpha$, so $\sqrt{-P} \in \mathfrak{R}(\alpha)$. Consider the ring homomorphism $\varphi : R[\mathtt{t}][\mathtt{x},\mathtt{y}]/(\mathtt{x}^2 + P(\mathtt{t}), \mathtt{xy} - 1) \to \mathfrak{R}(\alpha)$ that is the identity on $R[\mathtt{t}]$ and maps $\mathtt{x}$ onto $\sqrt{-P}$ and $\mathtt{y}$ onto $1/\sqrt{-P}$. By Artin-Lang's homomorphism theorem [BCR, Thm.4.1.2] there exists a homomorphism $\theta : R[\mathtt{t}][\mathtt{x}]/(\mathtt{x}^2 + P(\mathtt{t})) \to R$, so there exists $(t_0, x_0, y_0) \in R^3$ such that $P(t_0) + x_0^2 = 0$ and $x_0y_0 - 1 = 0$. Thus, $P(t_0) = -x_0^2 < 0$. As P is monic, there exists $M \in R$ such that $P(M) > 0$. By Bolzano's theorem [BCR, Prop.1.2.4] there exists $z_0 \in R$ such that $P(z_0) = 0$. By [BCR, Prop.1.3.7] we deduce that $\kappa[\mathtt{t}]/(P)$ admits an ordering that extends $\leqslant_\beta$, which contradicts the fact that $\kappa[\mathtt{t}]/(P)$ is not formally real. Consequently, $P \in \alpha$ for $\alpha \in \operatorname{Sper}(\kappa(\mathtt{t}))$, so $P \in \Sigma\kappa(\mathtt{t})^2$ and by [C] P is a sum of squares in $\kappa(\mathtt{t})$, as required. $\square$

Lemma 13.4. *Suppose $F(0,\mathtt{y}) = \mathtt{y}^2$, F divides G, F is irreducible in $\kappa[[\mathtt{x},\mathtt{y}]]$ and $\mathcal{P}(A) = \Sigma A^2$. Then, after a change of coordinates $\mathfrak{a} = (\mathtt{z}^2 - \mathtt{xy}, \mathtt{y}^2 - 2b\mathtt{yx} + d\mathtt{x}^2)$, where $\mathtt{t}^4 - 2b\mathtt{t}^2 + d \in \kappa[\mathtt{t}]$ is an irreducible polynomial such that the field $\kappa[\mathtt{t}]/(P)$ is (formally) real.*

Proof. As F divides G, there exists $Q \in \kappa[[\mathtt{x},\mathtt{y}]]$ such that $F = GQ$, so $F + \mathtt{z}G = F(1 + \mathtt{z}Q)$ and $\mathfrak{a} = (\mathtt{z}^2 - \mathtt{xy}, F)$. By Weierstrass preparation theorem we may assume $F = \mathtt{y}^2 + a(\mathtt{x})\mathtt{x} + b(\mathtt{x})\mathtt{x}^2$ is a Weierstrass polynomial where $a, b \in \kappa[[\mathtt{x}]]$. We distinguish two cases:

Case 1. Assume first that the roots of F belong to $\kappa[[\mathtt{x}^{1/2}]]$ (see Corollary A.2 and Remark A.3). As F has order 2, the roots of F are (by Lemma A.4) of the form $\zeta := \mathtt{x}c_1(\mathtt{x}) \pm \mathtt{x}\mathtt{x}^{1/2}c_2(\mathtt{x})$ where $c_1, c_2 \in \kappa[[\mathtt{x}]]$ and $c_2 \neq 0$, because F is irreducible. As $\mathfrak{a} \cap \kappa[[\mathtt{x},\mathtt{y}]] = (F)\kappa[[\mathtt{x},\mathtt{y}]]$, we have $\kappa[[\mathtt{x},\mathtt{y}]]/(F)\kappa[[\mathtt{x},\mathtt{y}]] \hookrightarrow A$. Thus, to prove that $\mathcal{P}(A) \neq \Sigma A^2$ it is enough to find $h \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]/(F))$ of order 1. By Corollary 5.22 $h \in \mathcal{P}(\kappa[[\mathtt{x},\mathtt{y}]]/(F))$ if and only if $h(t^2, \mathtt{t}^2c_1(\mathtt{t}^2) + \mathtt{t}^3c_2(\mathtt{t}^2)) \geqslant 0$ in $\kappa[[\mathtt{t}]]$. Observe that $h := \mathtt{x}$ satisfies the required conditions. ▪

Case 2. Assume next there exists $c \in \kappa\backslash(\kappa^2 \cup -\Sigma\kappa^2)$ such that the roots of F belong to $\kappa[\sqrt{c}][[\mathtt{x}]]$, that is, the roots of F are of the form $\mathtt{x}c_1(\mathtt{x}) \pm \sqrt{c}\mathtt{x}c_2(\mathtt{x})$ where $c_1, c_2 \in \kappa[[\mathtt{x}]]$. Thus, $F := (\mathtt{y} - \mathtt{x}c_1(\mathtt{x}))^2 - c\mathtt{x}^2c_2(\mathtt{x})^2$. If $\omega(c_2) \geqslant 1$, we deduce by Lemma 5.13 that $\mathcal{P}(A) \neq \Sigma A^2$.

Assume in the following $\omega(c_2) = 0$, so $\omega(c_1(\mathtt{x}) \pm \sqrt{c}c_2(\mathtt{x})) = 0$ and $c_2(0) \neq 0$. Choose the root $\theta := \mathtt{x}c_1(\mathtt{x}) + \sqrt{c}\mathtt{x}c_2(\mathtt{x})$ and define $\tau := \mathtt{x}\sqrt{c_1(\mathtt{x}) + \sqrt{c}c_2(\mathtt{x})}$, which satisfy $\mathtt{x}\theta = \tau^2$. Denote $\zeta := c_1(0) + \sqrt{c}c_2(0) \in \kappa[\sqrt{c}]$ and $\eta := \sqrt{c_1(0) + \sqrt{c}c_2(0)}$, where $\eta^2 = \zeta$. We have $\kappa[\sqrt{c}] = \kappa[\eta^2] \subset \kappa[\eta]$. Observe that $c_1(\mathtt{x}) + \sqrt{c}c_2(\mathtt{x}) = d_0\mathtt{x} + d_1\eta^2$ and $\sqrt{c_1(\mathtt{x}) + \sqrt{c}c_2(\mathtt{x})} = \sqrt{d_0\mathtt{x} + d_1\eta^2}$ where $d_i \in \kappa[[\mathtt{x}]]$ and $d_1(0) = 1$. As $d_1 + \mathtt{x}\frac{d_0}{\eta^2} \in \kappa[\eta^2][[\mathtt{x}]]$ and $d_1(0) = 1$, there exists $e_i \in \kappa[[\mathtt{x}]]$ such that $e_1(0) = 0$ and $\sqrt{\frac{d_0}{\eta^2}\mathtt{x} + d_1} = e_1 + \mathtt{x}e_3\eta^2$. We have $\sqrt{c_1(\mathtt{x}) + \sqrt{c}c_2(\mathtt{x})} = \eta e_1 + \eta^3\mathtt{x}e_3$. Consider the change of coordinates φ given by

$$(\mathtt{x}, \mathtt{w}_1, \mathtt{w}_2) \mapsto (\mathtt{x}, \mathtt{x}^2d_0 + \mathtt{w}_1d_1, \mathtt{w}_2e_1 + \mathtt{w}_1\mathtt{w}_2e_3)$$

and observe that

$$\varphi(\mathtt{x},\eta^2\mathtt{x},\pm\eta\mathtt{x})=(\mathtt{x},(d_0\mathtt{x}+d_1\eta^2)\mathtt{x},\pm(e_1\eta+e_3\mathtt{x}\eta^3)\mathtt{x})$$

Thus, $\varphi^{-1}(\mathtt{x},(d_0\mathtt{x}+d_1\eta^2)\mathtt{x},\pm(e_1\eta+e_3\mathtt{x}\eta^3)\mathtt{x})=(\mathtt{x},\eta^2\mathtt{x},\pm\eta\mathtt{x})$. As $\mathfrak{a}$ is generated by two series of order 2, we have $\mathfrak{a}=(\mathtt{z}^2-\mathtt{xy},\mathtt{y}^2-2b\mathtt{yx}+d\mathtt{x}^2)$ where $\mathtt{t}^2-2b\mathtt{t}+d\in\kappa[\mathtt{t}]$ is the irreducible polynomial of η^2 over κ. At this point, we distinguish two subcases:

Subcase 2.1. $\kappa[\eta]=\kappa[\eta^2]$. Thus, $\eta=a_0+a_1\eta^2$ where $a_0,a_1\in\kappa$. Consider the change of coordinates $(\mathtt{x},\mathtt{y},\mathtt{z})\mapsto(\mathtt{x},\mathtt{y},\mathtt{z}-a_0-a_1\mathtt{y})$, which transforms

$$(\mathtt{x},\eta^2\mathtt{x},\eta\mathtt{x})\mapsto(\mathtt{x},\eta^2\mathtt{x},0)\quad\text{and}\quad(\mathtt{x},\eta^2\mathtt{x},-\eta\mathtt{x})\mapsto(\mathtt{x},\eta^2\mathtt{x},-2\eta\mathtt{x}).$$

Thus, $\mathtt{z}(\mathtt{z}+2a_0\mathtt{x}+2a_1\mathtt{y})\in\mathfrak{a}$, which is a contradiction, because $\mathfrak{a}$ contains no reducible elements of order 2.

Subcase 2.2. $\kappa[\eta]\subsetneq\kappa[\eta^2]$. The irreducible polynomial of η over κ (which has degree four) is $P=\mathtt{t}^4-2b\mathtt{t}^2+d$. Observe that

$$P\Big(\frac{\mathtt{z}}{\mathtt{x}}\Big)\mathtt{x}^4=\mathtt{z}^4-2b\mathtt{z}^2\mathtt{x}^2+d\mathtt{x}^4=(\mathtt{z}^2-\mathtt{xy})(\mathtt{z}^2-2b\mathtt{x}^2+\mathtt{xy})+\mathtt{x}^2(\mathtt{y}^2-2b\mathtt{yx}+d\mathtt{x}^2)\in\mathfrak{a}.$$

Suppose the field $\kappa[\mathtt{t}]/(P)$ is not (formally) real. By Lemma 13.3 $P\in\Sigma\kappa[\mathtt{t}]^2$, so $P(\frac{\mathtt{z}}{\mathtt{x}})\mathtt{x}^4=\mathtt{z}^4-2b\mathtt{z}^2\mathtt{x}^2+d\mathtt{x}^4\in\mathfrak{a}\cap\Sigma\kappa[\mathtt{x},\mathtt{y}]^2$. Thus, there exists $\lambda_{i1},\lambda_{i0}\in\kappa$ such that $\mathtt{z}^4-2b\mathtt{z}^2\mathtt{x}^2+d\mathtt{x}^4=\sum_{i=1}^p(\mathtt{z}^2-\lambda_{i1}\mathtt{xz}+\lambda_{i0}\mathtt{x}^2)^2$. As $\mathfrak{a}$ is a real ideal, we deduce $\mathtt{z}^2-\lambda_{i1}\mathtt{xz}+\lambda_{i0}\mathtt{x}^2\in\mathfrak{a}$ for $i=1,\dots,p$. As $\mathfrak{a}$ is a radical ideal and $\omega(\mathfrak{a})\geqslant 2$, we deduce that either λ_{i1} or λ_{i0} is not zero for $i=1,\dots,p$. There exists $\mu_{i1},\mu_{i2}\in\kappa$ such that

$$\mathtt{z}^2-\lambda_{i1}\mathtt{xz}+\lambda_{i0}\mathtt{x}^2=\mu_{i1}(\mathtt{z}^2-\mathtt{xy})+\mu_{i2}(\mathtt{y}^2-2b\mathtt{yx}+d\mathtt{x}^2).$$

Making $\mathtt{x}=0$, we deduce $\mu_{i1}=1$ and $\mu_{i2}=0$, so $\mathtt{z}^2-\lambda_{i1}\mathtt{xz}+\lambda_{i0}\mathtt{x}^2=\mathtt{z}^2-\mathtt{xy}$, which is a contradiction. Consequently, the field $\kappa[\mathtt{t}]/(P)$ is (formally) real, as required. □

13.1.4. *Forth reduction.* In the following we assume $F(0,\mathtt{y})=\mathtt{y}^2$, $\mathfrak{a}$ contains no reducible elements of order 2 (so in particular $\gcd(F,G)$ is not an irreducible factor of order 1) and F does not divide G (so in particular $G\neq 0$). As $\omega(F)=2$, we assume in the following that F,G are relatively prime. By Theorem 11.7 we have that $\mathfrak{a}\cap\kappa[[\mathtt{x},\mathtt{y}]]=(F^2-\mathtt{xy}G^2)\kappa[[\mathtt{x},\mathtt{y}]]$ and $\omega(F^2-\mathtt{xy}G^2)=4$. As $\mathfrak{a}$ is a radical ideal, also $\mathfrak{a}\cap\kappa[[\mathtt{x},\mathtt{y}]]$ is a radical ideal. This means in particular that $F^2-\mathtt{xy}G^2$ has no multiple factors. As $\mathfrak{a}\cap\kappa[[\mathtt{x},\mathtt{y}]]=(F^2-\mathtt{xy}G^2)\kappa[[\mathtt{x},\mathtt{y}]]$ and $\mathtt{z}^2-\mathtt{xy}\in\mathfrak{a}$, the extension of rings $A_0:=\kappa[[\mathtt{x},\mathtt{y}]]/(F^2-\mathtt{xy}G^2)\hookrightarrow A=\kappa[[\mathtt{x},\mathtt{y}]][\mathtt{z}]/\mathfrak{a}$ is integral. Let as check: *G is not a zero divisor of A_0.*

Suppose there exist non-zero $H,Q\in\kappa[[\mathtt{x},\mathtt{y}]]$ such that $GH=(F^2-\mathtt{xy}G^2)Q$. As G and $F^2-\mathtt{xy}G^2$ are relatively primer (because F and G are relatively prime), we deduce that $F^2-\mathtt{xy}G^2$ divides H, so $H\in\mathfrak{a}\cap\kappa[[\mathtt{x},\mathtt{y}]]$. Thus, G is not a zero divisor of A_0.

As $F+\mathtt{z}G\in\mathfrak{a}$, we deduce $\mathtt{z}=\frac{F}{G}\in B_0$, which is the total ring of fractions of A_0. As $A=A_0[\mathtt{z}]=A_0[\frac{F}{G}]$, we conclude $A_0\hookrightarrow A\hookrightarrow B_0$, that is, A is a subring of B_0 that contains A_0. Thus, if $\rho\in\overline{\kappa}$ and $\alpha_1,\alpha_2\in\kappa[\rho][[\mathtt{t}^{1/p}]]$ are series such that $(F^2-\mathtt{xy}G^2)(\alpha_1,\alpha_2)=0$ and $G(\alpha_1,\alpha_2)\neq 0$, we have $\alpha_3:=\frac{F(\alpha_1,\alpha_2)}{G(\alpha_1,\alpha_2)}\in\overline{\kappa}((\mathtt{t}^{1/p}))$ satisfies $\alpha_3^2=\alpha_1\alpha_2$, so $2\omega(\alpha_3)=\omega(\alpha_1)+\omega(\alpha_2)\geqslant 0$ and $\alpha_3\in\kappa[\rho]((\mathtt{t}^{1/p}))$.

We distinguish next what happens with all possible factorizations of $F^2-\mathtt{xy}G^2$.

Lemma 13.5 (Four irreducible factors)**.** *If $F^2-\mathtt{xy}G^2$ has four (different) irreducible factors of order 1, then $\mathcal{P}(A)\neq\Sigma A^2$.*

Proof. Suppose $\mathcal{P}(A)=\Sigma A^2$. The irreducible factors of $F^2-\mathtt{xy}G^2$ are of the type $P_k:=\mathtt{y}-\mathtt{x}a_k(\mathtt{x})$ where $a_k\in\kappa[[\mathtt{x}]]$ for $k=1,2,3,4$. There exist series $\beta_i,\gamma_i\in\kappa[[\mathtt{t}]]$ such that $\mathfrak{a}=\mathfrak{p}_1\cap\mathfrak{p}_2\cap\mathfrak{p}_3\cap\mathfrak{p}_4$ where $\mathfrak{p}_i$ is the kernel of the substitution homomorphism $\Gamma_i:\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]\to\kappa[[\mathtt{t}]]$, $h\mapsto h(\zeta_i)$, where $\zeta_i:=(\mathtt{t},\beta_i\mathtt{t},\gamma_i\mathtt{t})$. For each $i=1,2,3,4$ the vector $v_i:=(1,\beta_i(0),\gamma_i(0))$

is non-zero and we consider the projective points $p_i := [v_i] \in \kappa\mathbb{P}^2$. Pick two of them p_i, p_j and suppose that they are distinct. After a linear change of coordinates, we may assume $v_i := (1,0,0)$ and $v_j := (0,1,0)$. After reparameterization, we may assume $\alpha_i = 1$, that is, $\zeta_i := (\mathtt{t}, \mathtt{t}^2\beta_i^\bullet, \mathtt{t}^2\gamma_i^\bullet)$ where $\beta_i^\bullet, \gamma_i^\bullet \in \kappa[[\mathtt{t}]]$ and after the change of coordinates $(\mathtt{x},\mathtt{y},\mathtt{z}) \mapsto (\mathtt{x}, \mathtt{y} - \mathtt{x}^2\beta_i^\bullet(\mathtt{x}), \mathtt{z} - \mathtt{x}^2\gamma_i^\bullet(\mathtt{x}))$, we may assume $\zeta_i = (\mathtt{t}, 0, 0)$. Next we reparameterize ζ_j to have $\zeta_j := (\mathtt{t}^2\alpha_j^\bullet, \mathtt{t}, \mathtt{t}^2\gamma_j^\bullet)$ where $\alpha_j^\bullet, \gamma_j^\bullet \in \kappa[[\mathtt{t}]]$. After the change of coordinates $(\mathtt{x},\mathtt{y},\mathtt{z}) \mapsto (\mathtt{x} - \mathtt{y}^2\alpha_j^\bullet(\mathtt{y}), \mathtt{y}, \mathtt{z} - \mathtt{y}^2\gamma_i^\bullet(\mathtt{y}))$ that keeps ζ_i invariant, we may assume also $\zeta_j = (0, \mathtt{t}, 0)$. Thus, we may assume $\mathtt{z} \in \mathfrak{p}_i \cap \mathfrak{p}_j$, so $\mathfrak{p}_i \cap \mathfrak{p}_j$ contains a series of order 1 if $p_i \neq p_j$. We distinguish several cases:

Case 1. $p_1 \neq p_2$ and $p_3 \neq p_4$. Then there exists series $g_{12} \in \mathfrak{p}_1 \cap \mathfrak{p}_2$ and $g_{34} \in \mathfrak{p}_3 \cap \mathfrak{p}_4$ of order 1. The series $g_{12}g_{34} \in \mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3 \cap \mathfrak{p}_4$, which is a contradiction, because $\mathfrak{a}$ contains no reducible elements of order 2. ▪

Case 2. $p_1 = p_2 = p_3 \neq p_4$. After a linear change of coordinates, we may assume $p_1 := [1:0:0]$ and $p_4 := [0:1:0]$, so we may assume:

$$\zeta_1 := (\mathtt{t}, \beta_1^*\mathtt{t}^2, \gamma_1^*\mathtt{t}^2),\ \zeta_2 := (\mathtt{t}, \beta_2^*\mathtt{t}^2, \gamma_2^*\mathtt{t}^2),\ \zeta_3 := (\mathtt{t}, \beta_3^*\mathtt{t}^2, \gamma_3^*\mathtt{t}^2),\ \zeta_4 := (\alpha_4^*\mathtt{t}^2, \mathtt{t}, \gamma_4^*\mathtt{t}^2)$$

where $\alpha_i^*, \beta_i^*, \gamma_i^* \in \kappa[[\mathtt{t}]]$. By Corollary 5.16 there exists $M := \sum_{i=1}^4 (1 + \gamma_i(0)^2) \in \kappa \backslash \{0\}$ such that $\mathtt{z} + M^2\mathtt{x}^2 + \mathtt{y}^2) \in \mathcal{P}(A) \backslash \Sigma A^2$. ▪

Case 3. All the points p_1, p_2, p_3, p_4 coincide. We may assume $p_1 := [1:0:0]$, so we may assume:

$$\zeta_1 := (\mathtt{t}, \beta_1^*\mathtt{t}^2, \gamma_1^*\mathtt{t}^2),\ \zeta_2 := (\mathtt{t}, \beta_2^*\mathtt{t}^2, \gamma_2^*\mathtt{t}^2),\ \zeta_3 := (\mathtt{t}, \beta_3^*\mathtt{t}^2, \gamma_3^*\mathtt{t}^2),\ \zeta_4 := (\mathtt{t}, \beta_4^*\mathtt{t}^2, \gamma_4^*\mathtt{t}^2)$$

where $\alpha_i^*, \beta_i^*, \gamma_i^* \in \kappa[[\mathtt{t}]]$. By Corollary 5.16 there exists $M := \sum_{i=1}^4 (1 + \gamma_i(0)^2) \in \kappa \backslash \{0\}$ such that $\mathtt{z} + M^2\mathtt{x}^2 \in \mathcal{P}(A) \backslash \Sigma A^2$. ▪

We conclude $\mathcal{P}(A) \neq \Sigma A^2$, as required. □

Lemma 13.6 (Three irreducible factors). *If $F^2 - \mathtt{xy}G^2$ has an irreducible factor P_1 of order 2 and two (different) irreducible factors P_2, P_3 of order 1, then $\mathcal{P}(A) \neq \Sigma A^2$.*

Proof. Suppose $\mathcal{P}(A) = \Sigma A^2$. By Weierstrass preparation theorem we may assume $P_1 := \mathtt{y}^2 + a_1(\mathtt{x})\mathtt{xy} + a_0(\mathtt{x})\mathtt{x}^2$, $P_2 := \mathtt{y} - a_2(\mathtt{x})\mathtt{x}$ and $P_3 := \mathtt{y} - a_3(\mathtt{x})\mathtt{x}$ where $a_i \in \kappa[[\mathtt{x}]]$. Let $\eta \in \overline{\kappa}[[\mathtt{x}^*]]$ be a root of P_i, which has $\omega(\eta) \geqslant 1$ (Lemma A.4). The roots of P_2, P_3 belong to $\kappa[[\mathtt{x}]]$ and have order $\geqslant 1$. The roots of P_1 belong to either $\kappa[\sqrt{a}][[\mathtt{x}]]$ for some $a \in \kappa \backslash \kappa^2$ or to $\kappa[[\mathtt{x}^{1/2}]]$. We distinguish several cases:

Case 1. Assume first that the roots of P_1 belong to $\kappa[[\mathtt{x}^{1/2}]]$ (see Corollary A.2 and Remark A.3). As P_1 has order 2, there exist (by Lemma A.4) series $b_i, c_i \in \kappa[[\mathtt{t}]]$ and $\beta_i, \gamma_i \in \kappa[[\mathtt{x}]]$ such that the minimal prime ideals $\mathfrak{p}_i$ associated to $\mathfrak{a}$ are the kernels of the homomorphisms $\Gamma_i : \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]] \to \kappa[[\mathtt{t}]]$, $h \mapsto h(\zeta_i)$, where

$$\zeta_1 := (\mathtt{t}^2, b_1\mathtt{t}^2, c_1\mathtt{t}^2),\ \zeta_2 := (\mathtt{t}, \beta_2\mathtt{t}, \gamma_2\mathtt{t}),\ \zeta_3 := (\mathtt{t}, \beta_3\mathtt{t}, \gamma_3\mathtt{t}).$$

Consider the vectors

$$v_1 := (1, b_1(0), c_1(0)), v_2 := (1, \beta_2(0), \gamma_2(0)), v_3 := (1, \beta_3(0), \gamma_3(0))$$

and the projective points $p_i := [v_i] \in \kappa\mathbb{P}^2$ for $i = 1,2,3$. We distinguish two subcases:

Subcase 1.1 The points p_1, p_2, p_3 are projectively κ-independent. We choose a line that contains p_2, p_3 (but does not contain p_1). After a change of coordinates, we may assume (from the beginning) that such line is $\mathtt{z} = 0$, so we may assume $\gamma_i(0) = 0$ for $i = 2,3$ and

$$\zeta_1 := (\mathtt{t}^2, b_1\mathtt{t}^2, c_1\mathtt{t}^2),\ \zeta_2 := (\mathtt{t}, \beta_2\mathtt{t}, \gamma_2^*\mathtt{t}^2),\ \zeta_3 := (\mathtt{t}, \beta_3\mathtt{t}, \gamma_3^*\mathtt{t}^2)$$

where $\gamma_i^* \in \kappa[[\mathtt{x}]]$ and $c_1(0) \neq 0$ (because p_1 does not belong to the line $\mathtt{z} = 0$). By Corollary 5.16 there exist $M := 3 + c_1(0)^2 + \gamma_2^*(0)^2 + \gamma_3^*(0)^2 \in \kappa \backslash \{0\}$ such that $\mathtt{z} + M^2\mathtt{x}^2 \in \mathcal{P}(A) \backslash \Sigma A^2$.

SUBCASE 1.2 The points p_1, p_2, p_3 are projectively κ-dependent, that is, they belong to the same projective line. After a change of coordinates, we may assume (from the beginning) that they belong to the line $\mathtt{z} = 0$, that is, $c_1(0) = 0$, $\gamma_i(0) = 0$ for $i = 2, 3$ and

$$\zeta_1 := (\mathtt{t}^2, b_1\mathtt{t}^2, c_1^*\mathtt{t}^3),\ \zeta_2 := (\mathtt{t}, \beta_2\mathtt{t}, \gamma_2^*\mathtt{t}^2),\ \zeta_3 := (\mathtt{t}, \beta_3\mathtt{t}, \gamma_3^*\mathtt{t}^2)$$

where $c_1^* \in \kappa[[\mathtt{t}]]$ and $\gamma_i^* \in \kappa[[\mathtt{x}]]$. If $c_i^*(0) = 0$, there exists $M := 3 + (c_i^*)'^2(0) + (\gamma_2^*(0))^2 + (\gamma_3^*(0))^2 \in \kappa$ such that $\mathtt{z} + M^2\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$.

Suppose next $c_i^*(0) \neq 0$. After a change of coordinates that involves only the variable $\mathtt{x}$ and $\mathtt{y}$, we may assume

$$\zeta_1 := (\mathtt{t}^2, b_1^*\mathtt{t}^4, c_1^*\mathtt{t}^3),\ \zeta_2 := (\mathtt{t}, \beta_2\mathtt{t}, \gamma_2^*\mathtt{t}^2),\ \zeta_3 := (\mathtt{t}, \beta_3\mathtt{t}, \gamma_3^*\mathtt{t}^2)$$

where $b_1^* \in \kappa[[\mathtt{x}]]$. After a change of coordinates, we may assume (taking into account that the semigroup of $\mathbb{N}$ generated by $2, 3$ contains all the positive integers greater than 2)

$$\zeta_1 := (\mathtt{t}^2, 0, c_1^*\mathtt{t}^3),\ \zeta_2 := (\mathtt{t}, \beta_2\mathtt{t}, \gamma_2^*\mathtt{t}^2),\ \zeta_3 := (\mathtt{t}, \beta_3\mathtt{t}, \gamma_3^*\mathtt{t}^2).$$

If $\beta_2(0), \beta_3(0) = 0$, there exists $M := 2+\beta_2(0)^2+\beta_3(0)^2 \in \kappa\backslash\{0\}$ such that $\mathtt{y}+M^2\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$, so we may assume (maybe after interchanging the indices 2 and 3) that $\beta_2(0) \neq 0$. After the new change of coordinates $(\mathtt{x}, \mathtt{y}, \mathtt{z}) \mapsto (\mathtt{x} - \frac{1}{\beta_2(\mathtt{x})}\mathtt{y}, \mathtt{y}, \mathtt{z} - \frac{\gamma_2^*(\mathtt{x})}{\beta_2(\mathtt{x})}\mathtt{y}^2)$ and using the fact that $\omega(\gamma_2^*\mathtt{t}^2) \geqslant 2$, we may assume

$$\zeta_1 := (\mathtt{t}^2, 0, c_1^*\mathtt{t}^3),\ \zeta_2 := (0, \mathtt{t}, 0),\ \zeta_3 := (\alpha_3\mathtt{t}, \beta_3\mathtt{t}, \gamma_3^*\mathtt{t}^2)$$

for some $\alpha_3 \in \kappa[[\mathtt{t}]]$. If $\alpha_3(0) = 0$, then $\beta_3(0) \neq 0$ and there exists $M := \frac{1+\alpha_3'(0)^2}{\beta_3(0)} \in \kappa\backslash\{0\}$ such that $\mathtt{x} + M^2\mathtt{y}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$, so we assume $\alpha_3(0) \neq 0$. After a reparameterization of ζ_3, we may assume $\alpha_3 = 1$, that is, $\zeta_3 := (\mathtt{t}, \beta_3\mathtt{t}, \gamma_3^*\mathtt{t}^2)$. After the new change of coordinates $(\mathtt{x}, \mathtt{y}, \mathtt{z}) \mapsto (\mathtt{x}, \mathtt{y}, \mathtt{z} - \gamma_3^*(\mathtt{x})\mathtt{x}^2)$ and using the fact that $\omega(\gamma_3^*(\mathtt{x})\mathtt{x}^2) \geqslant 2$, we may assume

$$\zeta_1 := (\mathtt{t}^2, 0, c_1^*\mathtt{t}^3),\ \zeta_2 := (0, \mathtt{t}, 0),\ \zeta_3 := (\alpha_3\mathtt{t}, \beta_3\mathtt{t}, 0).$$

Thus, $\mathtt{yz} \in \mathfrak{a}$, which is a contradiction, because $\mathfrak{a}$ contains no reducible elements of order 2. ■

CASE 2. Assume next that the roots of P_1 belong to $\kappa[\sqrt{a}][[\mathtt{x}]]$ for some $a \in \kappa\backslash\kappa^2$. There exist series $b_i, c_i, \alpha_i, \beta_i, \gamma_i \in \kappa[[\mathtt{x}]]$ such that the minimal prime ideals $\mathfrak{p}_i$ associated to $\mathfrak{a}$ are the kernels of the homomorphisms $\Gamma_i : \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]] \to \kappa[[\mathtt{t}]],\ h \mapsto h(\zeta_i)$, where

$$\zeta_1 := (\mathtt{t}, (b_1 + b_2\sqrt{a})\mathtt{t}, (c_1 + c_2\sqrt{a})\mathtt{t}),\ \zeta_3 := (\mathtt{t}, \beta_3\mathtt{t}, \gamma_3\mathtt{t}),\ \zeta_4 := (\mathtt{t}, \beta_4\mathtt{t}, \gamma_4\mathtt{t}).$$

Consider the vectors

$$v_1 := (1, (b_1(0) + b_2(0)\sqrt{a}), (c_1(0) + c_2(0)\sqrt{a})),\ v_3 := (1, \beta_3(0), \gamma_3(0)),\ v_4 := (1, \beta_4(0), \gamma_4(0))$$

and the projective points $p_i := [v_i] \in \kappa\mathbb{P}^2$ for $i = 1, 3, 4$.

SUBCASE 2.1. If p_1, p_3, p_4 are κ-dependent, that is, they belong to the same projective line, we may assume that such line is $\mathtt{z} = 0$, so

$$\zeta_1 := (\mathtt{t}, (b_1 + b_2\sqrt{a})\mathtt{t}, (c_1^* + c_2^*\sqrt{a})\mathtt{t}^2),\ \zeta_3 := (\mathtt{t}, \beta_3\mathtt{t}, \gamma_3^*\mathtt{t}^2),\ \zeta_4 := (\mathtt{t}, \beta_4\mathtt{t}, \gamma_4^*\mathtt{t}^2)$$

where $c_i^*, \gamma_i^* \in \kappa[[\mathtt{x}]]$. Thus, there exists $M := 3 + 4c_1^*(0)^2 + 4c_2^*(0)^2(a^2 + 1) + \gamma_3^*(0)^2 + \gamma_4^*(0)^2$ such that $\mathtt{z} + M^2\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$ for some $M \in \Sigma\kappa^2$.

SUBCASE 2.2. Assume p_1, p_3, p_4 are κ-independent. After a change of coordinates, we may assume

$$\zeta_1 := (\mathtt{t}, b_2\sqrt{a}\mathtt{t}, c_2\sqrt{a}\mathtt{t}),\ \zeta_3 := (\mathtt{t}, \beta_3\mathtt{t}, \gamma_3\mathtt{t}),\ \zeta_4 := (\mathtt{t}, \beta_4\mathtt{t}, \gamma_4\mathtt{t}).$$

We may assume $\omega(b_2) \leqslant \omega(c_2)$, so after the new change of coordinates $(\mathtt{x}, \mathtt{y}, \mathtt{z}) \mapsto (\mathtt{x}, \mathtt{y}, \mathtt{z} - \frac{c_2(\mathtt{x})}{b_2(\mathtt{x})}\mathtt{y})$, we have

$$\zeta_1 := (\mathtt{t}, b_2\sqrt{a}\mathtt{t}, 0),\ \zeta_3 := (\mathtt{t}, \beta_3\mathtt{t}, \gamma_3\mathtt{t}),\ \zeta_4 := (\mathtt{t}, \beta_4\mathtt{t}, \gamma_4\mathtt{t}).$$

If $\gamma_3(0) = 0$ and $\gamma_4(0) = 0$, there exists $M := 2 + \gamma_3'(0)^2 + \gamma_4'(0)^2 \in \kappa\backslash\{0\}$ such that $\mathtt{z} + M^2\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$. So we may assume $\gamma_3(0) \neq 0$. We reparameterize ζ_3 to have $(\alpha_3\mathtt{t}, \beta_3^*\mathtt{t}, \mathtt{t})$ where

$\alpha_3, \beta_3^* \in \kappa[[\mathtt{t}]]$. After the change of coordinates $(\mathtt{x},\mathtt{y},\mathtt{z}) \mapsto (\mathtt{x}-\alpha_3(\mathtt{z})\mathtt{z}, \mathtt{y}-\beta_3^*(\mathtt{z})\mathtt{z}, \mathtt{z})$, we may assume

$$\zeta_1 := (\mathtt{t}, b_2\sqrt{a}\mathtt{t}, 0),\ \zeta_3 := (0,0,\mathtt{t}), \zeta_4 := (\alpha_4\mathtt{t}, \beta_4\mathtt{t}, \gamma_4\mathtt{t})$$

for some $\alpha_4 \in \kappa[[\mathtt{t}]]$. If $\beta_4 = 0$, then $\mathtt{yz} \in \mathfrak{a}$, which is a contradiction, because $\mathfrak{a}$ contains no reducible elements of order 2. If $\alpha_4 = 0$, then $\mathtt{xz} \in \mathfrak{a}$, which is again a contradiction. Thus, we may assume $\alpha_4 \neq 0, \beta_4 \neq 0$ and $\omega(\alpha_4) \leqslant \omega(\beta_4)$, so there exists a series $\delta_4 \in \kappa[[\mathtt{t}]]$ such that $\beta_4 = \alpha_4\delta_4$. Thus, $(\mathtt{y}-\delta_4(\mathtt{x})\mathtt{x})\mathtt{z} \in \mathfrak{a}$, which is again a contradiction, because $\mathfrak{a}$ contains no reducible elements of order 2. ▪

We conclude $\mathcal{P}(A) \neq \Sigma A^2$, as required. □

Lemma 13.7 (Two irreducible factors of order 2)**.** *If $F^2 - \mathtt{xy}G^2$ has two irreducible factors P_1, P_2 of order 2, then $\mathcal{P}(A) \neq \Sigma A^2$.*

Proof. Suppose $\mathcal{P}(A) = \Sigma A^2$. By Weierstrass preparation theorem we may assume $P_i := \mathtt{y}^2 + a_{i1}(\mathtt{x})\mathtt{xy} + a_{i0}(\mathtt{x})\mathtt{x}^2$ where $a_{ij} \in \kappa[[\mathtt{x}]]$ for $j = 0,1$ and $i = 1,2$. Let $\eta \in \overline{\kappa}[[\mathtt{x}^*]]$ be a root of P_i, which has $\omega(\eta) \geqslant 1$ (Lemma A.4). The roots of P_i belong to either $\kappa[\sqrt{a}][[\mathtt{x}]]$ for some $a \in \kappa\backslash\kappa^2$ or to $\kappa[[\mathtt{x}^{1/2}]]$. We distinguish several cases:

Case 1. Assume first that the roots of P_1, P_2 belong to $\kappa[[\mathtt{x}^{1/2}]]$ (see Corollary A.2 and Remark A.3). By Corollary 5.22 and Remark 5.23 we may assume, after reparameterization, that there exist $\varepsilon_2 \in \{-1,1\}$ and series $b_i, c_i \in \kappa[[\mathtt{t}]]$ such that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2$, where $\mathfrak{p}_i$ is the kernel of the homomorphism $\Gamma_i : \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]] \to \kappa, h \mapsto h(\zeta_i)$ and

$$\zeta_1 := (\mathtt{t}^2, b_1\mathtt{t}^2, c_1\mathtt{t}^2),\ \zeta_2 := (\varepsilon_2\mathtt{t}^2, b_2\mathtt{t}^2, c_2\mathtt{t}^2).$$

Consider the vectors $v_1 := (1, b_1(0), c_1(0))$ and $v_2 := (\varepsilon_2, b_2(0), c_2(0))$ and the projective points $p_i := [v_i] \in \kappa\mathbb{P}^2$ for $i = 1,2$. We distinguish two subcases:

Subcase 1.1. If the points p_1 and p_2 are equal, we may assume after the change of coordinates $(\mathtt{x},\mathtt{y},\mathtt{z}) \mapsto (\mathtt{x}, \mathtt{y}-b_1(0)\mathtt{x}, \mathtt{z}-c_1(0)\mathtt{x})$

$$\zeta_1 := (\mathtt{t}^2, b_1^*\mathtt{t}^3, c_1^*\mathtt{t}^3),\ \zeta_2 := (\varepsilon_2\mathtt{t}^2, b_2^*\mathtt{t}^3, c_2^*\mathtt{t}^3).$$

for some $b_i^*, c_i^* \in \kappa[[\mathtt{t}]]$. If $c_1^*(0) = 0$ and $c_2^*(0) = 0$, there exists $M := 2+((c_1^*)'(0))^2+((c_2^*)'(0))^2 \in \kappa\backslash\{0\}$ such that $\mathtt{z} + M^2\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$. So we suppose $c_1^*(0) \neq 0$ and after a change of coordinates, we may assume (taking into account that the semigroup of $\mathbb{N}$ generated by $2,3$ contains all the positive integers greater than 2):

$$\zeta_1 := (\mathtt{t}^2, 0, c_1^*\mathtt{t}^3),\ \zeta_2 := (a\mathtt{t}^2, b_2^*\mathtt{t}^3, c_2^*\mathtt{t}^3).$$

Observe that $b_2^*(0) \neq 0$, because otherwise there exists $M := 1 + (b_2^*)'(0)^2 \in \kappa\backslash\{0\}$ such that $\mathtt{y} + M^2\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$. After a change of coordinates, we may assume (taking into account that the semigroup of $\mathbb{N}$ generated by $2,3$ contains all the positive integers greater than 2 and the even ones $2m \geqslant 4$ correspond to powers $(\mathtt{t}^2)^k$ for $k \geqslant 2$)

$$\zeta_1 := (\mathtt{t}^2, 0, c_1^*\mathtt{t}^3), \zeta_2 := (a\mathtt{t}^2, b_2^*\mathtt{t}^3, 0).$$

Thus, $\mathtt{yz} \in \mathfrak{a}$, which is a contradiction, because $\mathfrak{a}$ contains no reducible elements of order 2.

Subcase 1.2. If the points p_1 and p_2 are different, we may assume after a change of coordinates that $p_1 = [1:0:0]$ and $p_2 = [0:1:0]$, so

$$\zeta_1 := (\mathtt{t}^2, b_1^*\mathtt{t}^3, c_1^*\mathtt{t}^3),\ \zeta_2 := (a_2^*\mathtt{t}^3, \mathtt{t}^2, c_2^*\mathtt{t}^3).$$

for some $a_i^*, b_i^*, c_i^* \in \kappa[[\mathtt{t}]]$. If $c_1^*(0) = 0$ and $c_2^*(0) = 0$, there exists $M := 2+(c_1^*)'(0)^2+(c_2^*)'(0)^2 \in \kappa\backslash\{0\}$ such that $\mathtt{z} + M^2(\mathtt{x}^2+\mathtt{y}^2) \in \mathcal{P}(A)\backslash\Sigma A^2$. So let us suppose $c_1^*(0) \neq 0$ and after a change of coordinates, we may assume (taking into account that the semigroup of $\mathbb{N}$ generated by $2,3$ contains all the positive integers greater than 2)

$$\zeta_1 := (\mathtt{t}^2, 0, c_1^*\mathtt{t}^3),\ \zeta_2 := (a_2^*\mathtt{t}^3, \mathtt{t}^2, c_2^*\mathtt{t}^3).$$

If $a_2^*(0) = 0$, there exists $M := 1 + (a_2^*)'(0)^2 \in \kappa\backslash\{0\}$ such that $\mathtt{x} + M^2\mathtt{y}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$, so we assume $a_2^*(0) \neq 0$. After a change of coordinates (taking into account that the semigroup of $\mathbb{N}$ generated by $2, 3$ contains all the positive integers greater than 2), we have

$$\zeta_1 := (\mathtt{t}^2, 0, c_1^\bullet \mathtt{t}^2),\ \zeta_2 := (a_2^*\mathtt{t}^3, \mathtt{t}^2, 0)$$

for some $c_1^\bullet \in \kappa[[\mathtt{t}]]$. Thus, $\mathtt{yz} \in \mathfrak{a}$, which is a contradiction, because $\mathfrak{a}$ contains no reducible elements of order 2. ∎

Case 2. Assume next that the roots of P_1 belong to $\kappa[[\mathtt{x}^{1/2}]]$ (see Corollary A.2 and Remark A.3), whereas the roots of P_2 belong to $\kappa[\sqrt{a}][[\mathtt{x}]]$ for some $a \in \kappa\backslash\kappa^2$. As P_1, P_2 have order 2, there exist series $b_1, c_1 \in \kappa[[\mathtt{t}]]$ and $\alpha_2, \beta_2, \gamma_2, \delta_2 \in \kappa[[\mathtt{x}]]$ such that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2$, where $\mathfrak{p}_i$ is the kernel of the homomorphism $\Gamma_i : \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]] \to \kappa, h \mapsto h(\zeta_i)$ and

$$\zeta_1 := (\mathtt{t}^2, b_1\mathtt{t}^2, c_1\mathtt{t}^2),\ \zeta_2 := (\mathtt{t}, (\alpha_2 + \sqrt{a}\beta_2)\mathtt{t}, (\gamma_2 + \sqrt{a}\delta_2)\mathtt{t})).$$

After the change of coordinates $(\mathtt{x}, \mathtt{y}, \mathtt{z}) \mapsto (\mathtt{x}, \mathtt{y} - \alpha_2(\mathtt{x})\mathtt{x}, \mathtt{z} - \gamma_2(\mathtt{x})\mathtt{x})$, we may assume

$$\zeta_1 := (\mathtt{t}^2, b_1\mathtt{t}^2, c_1\mathtt{t}^2),\ \zeta_2 := (\mathtt{t}, \sqrt{a}\beta_2\mathtt{t}, \sqrt{a}\delta_2\mathtt{t}).$$

We may assume $\omega(\beta_2) \leqslant \omega(\delta_2)$ and, after the change of coordinates $(\mathtt{x}, \mathtt{y}, \mathtt{z}) \mapsto (\mathtt{x}, \mathtt{y}, \mathtt{z} - \frac{\delta_2(\mathtt{x})}{\beta_2(\mathtt{x})}\mathtt{y})$, we have

$$\zeta_1 := (\mathtt{t}^2, b_1\mathtt{t}^2, c_1\mathtt{t}^2),\ \zeta_2 := (\mathtt{t}, \sqrt{a}\beta_2\mathtt{t}, 0).$$

If $c_1(0) \neq 0$, then $c_1(0)\mathtt{z} \in \mathcal{P}(A)\backslash\Sigma A^2$, so $c_1(0) = 0$ and we write

$$\zeta_1 := (\mathtt{t}^2, b_1\mathtt{t}^2, c_1^*\mathtt{t}^3),\ \zeta_2 := (\mathtt{t}, \sqrt{a}\beta_2\mathtt{t}, 0)$$

where $c_1^* \in \kappa[[\mathtt{t}]]$. If $c_1^*(0) = 0$, there exists $M := 1 + (c_1^*)'(0)^2 \in \kappa\backslash\{0\}$ such that $\mathtt{z} + M^2\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$, so $c_1^*(0) \neq 0$. After a change of coordinates (taking into account that the semigroup of $\mathbb{N}$ generated by $2, 3$ contains all the positive integers greater than 2)

$$\zeta_1 := (\mathtt{t}^2, 0, c_1^*\mathtt{t}^3),\ \zeta_2 := (\mathtt{x}, \alpha_2^*\mathtt{x} + \sqrt{a}\beta_2\mathtt{x}, 0)$$

for some $\alpha_2^* \in \kappa[[\mathtt{x}]]$. Thus, $\mathtt{yz} \in \mathfrak{a}$, which is a contradiction, because $\mathfrak{a}$ contains no reducible elements of order 2. ∎

Case 3. Assume next that the roots of P_1 belong to $\kappa[\sqrt{a}][[\mathtt{x}]]$, whereas the roots of P_2 belong to $\kappa[\sqrt{b}][[\mathtt{x}]]$ for some $a, b \in \kappa\backslash\kappa^2$. As $P_1(\mathtt{y}, 0)$, $P_2(\mathtt{y}, 0)$ have order 2, there exist series $\alpha_i, \beta_i, \gamma_i, \delta_i \in \kappa[[\mathtt{t}]]$ such that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_3$, where $\mathfrak{p}_i$ is the kernel of the homomorphism $\Gamma_i : \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]] \to \kappa, h \mapsto h(\zeta_i)$ and

$$\zeta_1 := (\mathtt{t}, (\alpha_1 + \sqrt{a}\beta_1)\mathtt{t}, (\gamma_1 + \sqrt{a}\delta_1)\mathtt{t}),\ \zeta_3 := (\mathtt{t}, (\alpha_2 + \sqrt{b}\beta_2)\mathtt{t}, (\gamma_2 + \sqrt{b}\delta_2)\mathtt{t}),$$

After a change of coordinates, we may assume

$$\zeta_1 := (\mathtt{t}, \sqrt{a}\beta_1\mathtt{t}, 0),\ \zeta_3 := (\mathtt{t}, (\alpha_2 + \sqrt{b}\beta_2)\mathtt{t}, (\gamma_2 + \sqrt{b}\delta_2)\mathtt{t}).$$

If $\gamma_2(0) = 0$ and $\delta_2(0) = 0$, there exists $M := 1 + 4(\gamma_2'^2(0) + \delta_2'^2(0)(b^2 + 1)) \in \kappa\backslash\{0\}$ such that $\mathtt{z} + M^2\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$. Suppose first $\delta_2(0) \neq 0$, after the change of coordinates $(\mathtt{x}, \mathtt{y}, \mathtt{z}) \mapsto (\mathtt{x}, \mathtt{y} - \frac{\beta_2(\mathtt{x})}{\delta_2(\mathtt{x})}\mathtt{z} + \frac{\gamma_2(\mathtt{x})}{\delta_2(\mathtt{x})}\mathtt{x}, \mathtt{z})$, we may assume

$$\zeta_1 := (\mathtt{t}, (\alpha_1^* + \sqrt{a}\beta_1)\mathtt{t}, 0),\ \zeta_3 := (\mathtt{t}, 0, (\gamma_2 + \sqrt{b}\delta_2)\mathtt{t})$$

for some $\alpha_1^* \in \kappa[[\mathtt{t}]]$. Thus, $\mathtt{yz} \in \mathfrak{a}$, which is a contradiction, because $\mathfrak{a}$ contains no reducible elements of order 2. We deduce $\delta_2(0) = 0$, so $\gamma_2(0) \neq 0$ (because we have seen that both cannot be zero). After the change of coordinates $(\mathtt{x}, \mathtt{y}, \mathtt{z}) \mapsto (\mathtt{x} - \frac{1}{\gamma_2(\mathtt{x})}\mathtt{z}, \mathtt{y} - \frac{\alpha_2(\mathtt{x})}{\gamma_2(\mathtt{x})}, \mathtt{z})$ (and using the fact that $\delta_2(0) = 0$), we may assume

$$\zeta_1 := (\mathtt{t}, \sqrt{a}\beta_1\mathtt{t}, 0),\ \zeta_3 := (-\sqrt{b}\frac{\delta_2^*}{\gamma_2}\mathtt{t}^2, \sqrt{b}\beta_2\mathtt{t}, (\gamma_2 + \sqrt{b}\delta_2^*\mathtt{t})\mathtt{t})$$

for some series $\delta_2^* \in \kappa[[\mathtt{t}]]$. Observe that both δ_2^* and β_2 cannot be zero. If $\omega(\beta_2\mathtt{t}) \leqslant \omega(-\frac{\delta_2^*}{\gamma_2}\mathtt{t}^2)$, there exists $\rho_2 \in \kappa[[\mathtt{t}]]$ such that $-\frac{\delta_2^*}{\gamma_2}\mathtt{t}^2 = \beta_2\mathtt{t}\rho_2$, so $\mathtt{z}(\mathtt{x} - \rho_2\mathtt{y}) \in \mathcal{P}(A)\backslash\Sigma A^2$, which is a contradiction, because $\mathfrak{a}$ because $\mathfrak{a}$ contains no reducible elements of order 2. If $\omega(\beta_2\mathtt{t}) > \omega(-\frac{\delta_2^*}{\gamma_2}\mathtt{t}^2)$, there exists $\rho_2^* \in \kappa[[\mathtt{t}]]$ such that $-\frac{\delta_2^*}{\gamma_2}\mathtt{t}^2\rho_2^* = \beta_2\mathtt{t}$, so $\mathtt{z}(\rho_2'\mathtt{x} - \mathtt{y}) \in \mathcal{P}(A)\backslash\Sigma A^2$, which is a contradiction, because $\mathfrak{a}$ contains no reducible elements of order 2. ▪

We conclude $\mathcal{P}(A) \neq \Sigma A^2$, as required. □

Lemma 13.8 (Two irreducible factors of different orders)**.** *If $F^2 - \mathtt{xy}G^2$ has and irreducible factor P_1 of order 3 and an irreducible factor P_2 of order 1, then $\mathcal{P}(A) \neq \Sigma A^2$.*

Proof. Suppose $\mathcal{P}(A) = \Sigma A^2$. By Weierstrass preparation theorem we may assume $P_1 := \mathtt{y}^3 + a_2(\mathtt{x})\mathtt{x}\mathtt{y}^2 + a_1(\mathtt{x})\mathtt{x}^2\mathtt{y} + a_0(\mathtt{x})\mathtt{x}^3$ and $P_2 := \mathtt{y} - a_3(\mathtt{x})\mathtt{x}$ where $a_i \in \kappa[[\mathtt{x}]]$. Let $\eta \in \overline{\kappa}[[\mathtt{x}^*]]$ be a root of P_i, which has $\omega(\eta) \geqslant 1$ (Lemma A.4). The root of P_2 belongs to $\kappa[[\mathtt{x}]]$, whereas the roots of P_1 belong to either $\kappa[\rho][[\mathtt{x}]]$ for some $\rho \in \overline{\kappa}$ such that $[\kappa[\rho] : \kappa] = 3$ or to $\kappa[[\mathtt{x}^{1/3}]]$. We distinguish several cases:

Case 1. The roots of P_1 belongs to $\kappa[[\mathtt{t}^{1/3}]]$ (see Corollary A.2 and Remark A.3). Thus, there exist series $b, c \in \kappa[[\mathtt{t}]]$ and $\beta, \gamma \in \kappa[[\mathtt{x}]]$ such that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2$, where $\mathfrak{p}_i$ is the kernel of the homomorphism $\Gamma_i : \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]] \to \kappa, h \mapsto h(\zeta_i)$ and

$$\zeta_1 := (\mathtt{t}^3, b\mathtt{t}^3, c\mathtt{t}^3),\ \zeta_2 := (\mathtt{t}, \beta\mathtt{t}, \gamma\mathtt{t}).$$

After a change of coordinates, we may assume

$$\zeta_1 := (\mathtt{t}^3, b^*\mathtt{t}^4, c^*\mathtt{t}^5),\ \zeta_2 := (\mathtt{t}, \beta\mathtt{t}, \gamma\mathtt{t})$$

where $b^*, c^* \in \kappa[[\mathtt{t}]]$. If $\gamma(0) = 0$ and $c^*(0) = 0$, there exists $M := 2 + (c^*)'(0)^2 + \gamma'(0)^2 \in \kappa\backslash\{0\}$ such that $\mathtt{z} + M^2\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$. Thus, either $\gamma(0) \neq 0$ or $c^*(0) \neq 0$. We distinguish two subcases:

Subcase 1.1. $\gamma(0) = 0$ and $c^*(0) \neq 0$. After the change of coordinates $(\mathtt{x}, \mathtt{y}, \mathtt{z}) \mapsto (\mathtt{x}, \mathtt{y}, \mathtt{z} - \gamma(\mathtt{x})\mathtt{x})$, we may assume

$$\zeta_1 := (\mathtt{t}^3, b^*\mathtt{t}^4, c^*\mathtt{t}^5),\ \zeta_2 := (\mathtt{t}, \beta\mathtt{t}, 0).$$

Suppose $b^*(0) \neq 0$ and let $H \in \mathfrak{a}$ be an element of order 2. The leading form of H has the form $Q := \lambda_1\mathtt{x}^2 + \lambda_2\mathtt{y}^2 + \lambda_3\mathtt{z}^2 + \lambda_4\mathtt{xy} + \lambda_5\mathtt{xz} + \lambda_6\mathtt{yz}$. As $H(\zeta_i) = 0$ for $i = 1, 2$, we deduce $\lambda_1 = 0$ and $\lambda_4 = 0$. If $\beta(0) \neq 0$, we deduce $\lambda_2 = 0$, so $Q = (\lambda_3\mathtt{z} + \lambda_5\mathtt{x} + \lambda_6\mathtt{z})\mathtt{z}$, which is reducible. Thus, the leading forms of all the elements of $\mathfrak{a}$ of order 2 are reducible quadratic forms, which is a contradiction (because before all the previous changes of coordinates $\mathtt{z}^2 - \mathtt{xy} \in \mathfrak{a}$). Consequently, $\beta(0) = 0$. After the change of coordinates $(\mathtt{x}, \mathtt{y}, \mathtt{z}) \mapsto (\mathtt{x}, \mathtt{y} - \beta(\mathtt{x})\mathtt{x}, \mathtt{z})$, we may assume

$$\zeta_1 := (\mathtt{t}^3, b^*\mathtt{t}^4, c^*\mathtt{t}^5),\ \zeta_2 := (\mathtt{t}, 0, 0).$$

Consequently, $b^*(0)\mathtt{y} \in \mathcal{P}(A)\backslash\Sigma A^2$. Thus, $b^*(0) = 0$ and after a change of coordinates, we may assume

$$\zeta_1 := (\mathtt{t}^3, b^\bullet\mathtt{t}^7, c^*\mathtt{t}^5),\ \zeta_2 := (\mathtt{t}, \beta\mathtt{t}, 0)$$

where $b^\bullet \in \kappa[[\mathtt{t}]]$. If $\beta(0) = 0$, there exists $M := 1 + \beta(0)^2 \in \kappa\backslash\{0\}$ such that $\mathtt{y} + M^2\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$. Thus, $\beta(0) \neq 0$ and after a change of coordinates and a reparameterization, we may assume

$$\zeta_1 := (\mathtt{t}^3, c^*\mathtt{t}^5, b^\bullet\mathtt{t}^7),\ \zeta_2 := (0, 0, \mathtt{t}) \qquad (13.1)$$ 357

Observe that $c^*(0)\mathtt{xy} \in \mathcal{P}(A)$. Let $h \in \mathfrak{a}$ be a series of order 2. Its leading form is $Q := \lambda_1\mathtt{x}^2 + \lambda_2\mathtt{xy} + \lambda_3\mathtt{xz} + \lambda_4\mathtt{y}^2 + \lambda_5\mathtt{yz} + \lambda_6\mathtt{z}^2$. As $h(0, 0, \mathtt{t}) = 0$, we have $\lambda_6 = 0$. As $h(\mathtt{t}^3, c^*\mathtt{t}^5, b^\bullet\mathtt{t}^7) = 0$, we deduce $\lambda_1 = 0$, $\lambda_2 = 0$, so $q = \lambda_3\mathtt{xz} + \lambda_4\mathtt{y}^2 + \lambda_5\mathtt{yz}$. If $c^*(0)\mathtt{xy} \in \Sigma A^2$ there exists $h \in \mathfrak{a}$ of order 2 such that $\mathtt{xy} + h \in \Sigma A^2$, so it leading form $\mathtt{xy} + q$ is a sum of squares of linear forms, that is,

$$Q := \mathtt{xy} + \lambda_3\mathtt{xz} + \lambda_4\mathtt{y}^2 + \lambda_5\mathtt{yz}$$

is a sum of squares of linear forms for some $\lambda_3, \lambda_4, \lambda_5 \in \kappa$, which is a contradiction because $Q(-1 - \lambda_4, 1, 0) = -1$.

SUBCASE 1.2. $\gamma(0) \neq 0$. After a change of coordinates and a reparameterization of ζ_1, we may assume

$$\zeta_1 := (\mathtt{t}^3, b^*\mathtt{t}^4, c^*\mathtt{t}^5),\ \zeta_2 := (0,0,\mathtt{t}).$$

If $b^*(0) \neq 0$, we have $b^*(0)\mathtt{y} \in \mathcal{P}(A)\backslash\Sigma A^2$, so $b^*(0) = 0$ and

$$\zeta_1 := (\mathtt{t}^3, b^\bullet\mathtt{t}^5, c^*\mathtt{t}^5),\ \zeta_2 := (0,0,\mathtt{t})$$

for some $b^\bullet \in \kappa[[\mathtt{t}]]$. If $b^\bullet(0) = 0$, there exists $M := 1+(b^\bullet)'(0)^2$ such that $\mathtt{y}+M^2\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$, so $b^\bullet(0) \neq 0$. After a change of coordinates (taking into account that the semigroup of $\mathbb{N}$ generated by $3, 5$ contains all the positive integers greater than 5 except for 7), we may assume

$$\zeta_1 := (\mathtt{t}^3, b^\bullet\mathtt{t}^5, c^\bullet\mathtt{t}^7),\ \zeta_2 := (0,0,\mathtt{t})$$

where $c^\bullet \in \kappa[[\mathtt{t}]]$. But this corresponds to (13.1) and we have already proved $\mathcal{P}(A)\backslash\Sigma A^2$. ▪

CASE 2. A root of P_1 belongs to $\kappa[\rho][[\mathtt{x}]]$ where $\rho \in \overline{\kappa}$ satisfies $[\kappa[\rho] : \kappa] = 3$. Thus, there exist series $a_i, b_i \in \kappa[[\mathtt{t}]]$ and $\beta, \gamma \in \kappa[[\mathtt{t}]]$ such that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2$, where $\mathfrak{p}_i$ is the kernel of the homomorphism $\Gamma_i : \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]] \to \kappa, h \mapsto h(\zeta_i)$ and

$$\zeta_1 := (\mathtt{t}, (a_0 + a_1\rho + a_2\rho^2)\mathtt{t}, (b_0 + b_1\rho + b_2\rho^2)\mathtt{t}),\ \zeta_4 := (\mathtt{t}, \beta\mathtt{t}, \gamma\mathtt{t})$$

As $\mathtt{z}^2 - \mathtt{xy} \in \mathfrak{a}$ we have $a_0 + a_1\rho_i + a_2\rho_i^2 = (b_0 + b_1\rho_i + b_2\rho_i^2)^2$. If $(b_0 + b_1\rho_i + b_2\rho_i^2)(0) \in \kappa$, also $(a_0+a_1\rho_i+a_2\rho_i^2)(0) \in \kappa$. After a change of coordinates, we may assume $(b_0+b_1\rho_i+b_2\rho_i^2)(0) = 0$, $(a_0 + a_1\rho_i + a_2\rho_i^2)(0) = 0$ and

$$\zeta_1 := (\mathtt{t}, \xi_1\mathtt{t}^2, \xi_2\mathtt{t}^2),\ \zeta_4 := (\mathtt{t}, \beta\mathtt{t}, \gamma\mathtt{t})$$

where $\xi_1 \in \kappa[\rho_i][[\mathtt{t}]]$. After a new change of coordinates that only involves second and third variables, we may assume

$$\zeta := (\mathtt{t}, \xi_1\mathtt{t}^2, \xi_2\mathtt{t}^2),\ \zeta_4 := (\mathtt{t}, \beta\mathtt{t}, \gamma^*\mathtt{t}^2)$$

for some $\gamma^* \in \kappa[[\mathtt{t}]]$. Thus, there exists $M := 1 + \xi_2(0)^2 + \gamma^*(0)^2 \in \kappa\backslash\{0\}$ such that $\mathtt{z} + M^2\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$. Consequently, $(b_0 + b_1\rho + b_2\rho^2)(0) \notin \kappa$ is a generator of the extension $\kappa[\rho]|\kappa$, so we may assume $(b_0 + b_1\rho + b_2\rho^2)(0) = \rho$. Thus,

$$\zeta_1 := (\mathtt{t}, (\rho + (b_0 + b_1\rho + b_2\rho^2)\mathtt{t})^2\mathtt{t}, (\rho + (b_0 + b_1\rho + b_2\rho^2)\mathtt{t})\mathtt{t}),\ \zeta_4 := (\mathtt{t}, \beta\mathtt{t}, \gamma\mathtt{t})$$

Consider the change of coordinates

$$\psi : (\mathtt{x},\mathtt{y},\mathtt{z}) \mapsto (\mathtt{x}, \mathtt{y} + 2\mathtt{z}(b_0\mathtt{x} + b_1\mathtt{z} + b_2\mathtt{y}) + (b_0\mathtt{x} + b_1\mathtt{z} + b_2\mathtt{y})^2\mathtt{x}, \mathtt{z} + (b_0\mathtt{x} + b_1\mathtt{z} + b_2\mathtt{y})\mathtt{x}),$$

which maps $\mathtt{z}^2 - \mathtt{xy}$ onto $\mathtt{z}^2 - \mathtt{xy}$ and $(\mathtt{t}, \rho^2\mathtt{t}, \rho\mathtt{t})$ onto ζ_1, so we may assume

$$\zeta_1 := (\mathtt{t}, \rho^2\mathtt{t}, \rho\mathtt{t}),\ \zeta_4 := (\mathtt{t}, \beta\mathtt{t}, \gamma\mathtt{t})$$

and $\mathtt{z}^2 - \mathtt{xy}, F + \mathtt{z}G \in \mathfrak{a}$ where $\omega(F) = 2$ and $F(0,\mathtt{y}) = \mathtt{y}^2 + \cdots$. Denote $v_1 := (1, \rho^2, \rho)$ and $v_4 := (1, \beta(0), \gamma(0)) \in \kappa^3\backslash\{(0,0,0)\}$, where $\beta(0) = \gamma(0)^2$. Let $P := \mathtt{t}^3 + \lambda_2\mathtt{t}^2 + \lambda_1\mathtt{t} + \lambda_0 \in \kappa[\mathtt{t}]$ be the irreducible polynomial of ρ over κ and consider the quadratic form $Q_3 := \mathtt{yz} + \lambda_2\mathtt{yx} + \lambda_1\mathtt{zx} + \lambda_0\mathtt{x}^2$. We have $Q_3(\zeta_1) = \mathtt{t}^3P(\rho) = 0$. If $Q_3(v_4) = 0$, then

$$0 = Q_3(v_4) = Q_3(1, (\gamma(0))^2, \gamma(0)) = (\gamma(0))^3 + \lambda_2(\gamma(0))^2 + \lambda_1\gamma(0) + \lambda_0 = P(\gamma(0)),$$

which is a contradiction because $\gamma(0) \in \kappa$ and the polynomial $\mathtt{t}^3 + \lambda_2\mathtt{t}^2 + \lambda_1\mathtt{t} + \lambda_0 \in \kappa[\mathtt{t}]$ is irreducible.

Observe that $f := Q_3(v_4)Q_3 \in \mathcal{P}(A)$, because $f(\zeta_1) = 0$ and $f(\zeta_4) = Q_3(v_4)^2\mathtt{t}^2 + \cdots$.

Suppose $f \in \Sigma A^2$. Then there exist series $f_1, f_2 \in \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ such that $f + f_1(\mathtt{z}^2 - \mathtt{xy}) + f_2(F + \mathtt{z}G) \in \Sigma\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]^2$. Consequently, if $Q_1 := f_1$ and Q_2 is the leading form of f_2, there exist $\lambda_1, \lambda_2 \in \kappa$ such that

$$Q_3(v_4)Q_3 + \lambda_1Q_1 + \lambda_2Q_2 = \sum_{\ell=1}^{p}(c_\ell\mathtt{x} + d_\ell\mathtt{y} + e_\ell\mathtt{z})^2$$

for some $c_\ell, d_\ell, e_\ell \in \kappa$. Let $\alpha_0 \in \operatorname{Sper}(\kappa)$ and let $\mathfrak{R}(\alpha) \subset \overline{\kappa}$ be its real closure. As $P := \mathtt{t}^3+\lambda_2\mathtt{t}^2+\lambda_1\mathtt{t}+\lambda_0$ has degree 3, we may assume $\rho \in \mathfrak{R}(\alpha)$. As $(Q_3(v_4)Q_3+\lambda_1Q_1+\lambda_2Q_2)(v_1) = 0$, we deduce $\sum_{i=1}^p (c_\ell 1+d_\ell\rho^2+e_\ell\rho)^2 = 0$, that is, $c_\ell 1+d_\ell\rho^2+e_\ell\rho = 0$ for $\ell = 1,\dots,p$. As $1, \rho, \rho^2$ are κ-linearly independent, we conclude $c_\ell, d_\ell, e_\ell = 0$ for $\ell = 1,\dots,p$, so $Q_3(v_4)Q_3+\lambda_1Q_1+\lambda_2Q_2 = 0$, which is a contradiction, because

$$(Q_3(v_4)Q_3 + \lambda_1Q_1 + \lambda_2Q_2)(v_4) = Q_3(v_4)^2 \neq 0.$$

Consequently, $f \in \mathcal{P}(A)\backslash\Sigma A^2$. ▪

We conclude $\mathcal{P}(A) \neq \Sigma A^2$, as required. □

Lemma 13.9 (One irreducible factor)**.** *Suppose $F^2 - \mathtt{xy}G^2$ is irreducible and $\mathcal{P}(A) = \Sigma A^2$. Then we may assume $\mathfrak{a} = (\mathtt{z}^2 - \mathtt{xy}, \mathtt{y}^2 - 2b\mathtt{xy} - 4c^2\mathtt{zx} + d\mathtt{x}^2)$ where $P := \mathtt{t}^4 - 2b\mathtt{t}^2 - 4c^2\mathtt{t} + d$ is an irreducible polynomial of $\kappa[\mathtt{t}]$ such that the field $\kappa[\mathtt{t}]/(P)$ is (formally) real.*

Proof. Suppose $\mathcal{P}(A) = \Sigma A^2$. By Weierstrass preparation theorem we may assume $P := F^2 - \mathtt{xy}G^2 := \mathtt{y}^4 + a_3(\mathtt{x})\mathtt{xy}^3 + a_2(\mathtt{x})\mathtt{x}^2\mathtt{y}^2 + a_1(\mathtt{x})\mathtt{x}^3\mathtt{y} + a_0(\mathtt{x})\mathtt{x}^4$ where $a_i \in \kappa[[\mathtt{x}]]$. Let $\eta \in \overline{\kappa}[[\mathtt{x}^*]]$ be a root of P, which has $\omega(\eta) \geqslant 1$ (Lemma A.4). We will see that the roots of P belong to either $\kappa[\rho][[\mathtt{x}]]$ for some $\rho \in \overline{\kappa}$ such that $[\kappa[\rho] : \kappa] = 4$ or to $\kappa[\theta][\mathtt{x}^{1/2}]$ where $[\kappa[\theta] : \kappa]$ divides 4 or to $\kappa[[\mathtt{x}^{1/4}]]$. We distinguish several cases:

Case 1. The roots of P belongs to $\kappa[[\mathtt{x}^{1/4}]]$ (see Corollary A.2 and Remark A.3). There exists series $b, c \in \kappa[[\mathtt{t}]]$ such that $\mathfrak{a}$ is the kernel of the homomorphism $\Gamma : \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]] \to \kappa, h \mapsto h(\zeta)$ where $\zeta := (\mathtt{t}^4, b\mathtt{t}^4, c\mathtt{t}^4)$. Thus, $\mathtt{x} \in \mathcal{P}(A)\backslash\Sigma A^2$, which is a contradiction. ▪

Case 2. The roots of P belong to $\overline{\kappa}[[\mathtt{x}^{1/2}]]\backslash\overline{\kappa}[[\mathtt{x}]]$. Define $E_0 := \kappa(a_k : k \geqslant 0)$. By Lemma A.1 $E_0|\kappa$ is a finite extension. Let $E_1|\kappa$ be the Galois closure of $E_0|\kappa$, which is a finite Galois extension. The roots of P belong to $E_1[[\mathtt{x}^{1/2}]] = \kappa[[\mathtt{x}]][\rho, \mathtt{x}^{1/2}]$ for some primitive element of $E_1|\kappa$. Let $h := \sum_{k\geqslant 0} a_k\mathtt{x}^k \in E_1[[\mathtt{x}]]$ be such that $\xi = h(\mathtt{x}^{1/2}) = \sum_{k\geqslant 0} a_k\mathtt{x}^{k/2}$ is a root of P. Observe that there exists a smallest odd integer $k_0 \geqslant 1$ such that $a_{k_0} \neq 0$ and $\xi^* := \sum_{k\geqslant 0} a_k(-1)^k\mathtt{x}^{k/2}$ is another root of P. For each $\sigma \in G(E_1 : \kappa)$ the automorphism

$$\widetilde{\sigma} : E_1[[\mathtt{x}^{1/2}]] \to E_1[[\mathtt{x}^{1/2}]],\ \sum_{k\geqslant 0} c_k\mathtt{x}^{k/2} \mapsto \sum_{k\geqslant 0} \sigma(c_k)\mathtt{x}^{k/2}$$

provides a root $\zeta_\sigma := \sum_{k\geqslant 0} \sigma(a_k)\mathtt{x}^{k/2}$.

Define $E_0 := \kappa(a_k : k \geqslant 0)$ and consider the involution

$$\tau : E_1[[\mathtt{x}^{1/2}]] \to E_1[[\mathtt{x}^{1/2}]],\ \sum_{k\geqslant 0} c_k\mathtt{x}^{k/2} \mapsto \sum_{k\geqslant 0} c_k(-1)^k\mathtt{x}^{k/2}.$$

Observe that $\tau \circ \widetilde{\sigma} = \widetilde{\sigma} \circ \tau$ for each $\sigma \in G(E_1 : \kappa)$. If $\sigma|_{E_0} = \mathrm{id}_{E_0}$ for each $\sigma \in G(E_1 : \kappa)$, then $h \in \kappa[[\mathtt{x}]]$ and $(\mathtt{y} - h(\mathtt{x}^{1/2}))(\mathtt{y} - h(-\mathtt{x}^{1/2})) \in \kappa[[\mathtt{x}]][\mathtt{y}]$ has degree 2 and divides the irreducible polynomial P of degree 4, which is a contradiction. Suppose next that for each $\sigma \in G(E_1 : \kappa)$ either $\widetilde{\sigma} = \mathrm{id}$ (that is, $\sigma|_{E_0} = \mathrm{id}_{E_0}$) or $\widetilde{\sigma} = \tau$ (that is, $\sigma(a_k) = (-1)^k a_k$ for each $k \geqslant 1$). Thus, $a_k \in \kappa$ for each k even and $a_k a_\ell \in \kappa$ for each pair (k, ℓ) such that $k + \ell$ is even. For each k odd such that $a_k \neq 0$, the irreducible polynomial of a_k over κ is $\mathtt{t}^2 - a_k^2$. Thus, $a_k = b_k a_{k_0}$ where $b_k \in \kappa$ for each odd integer $k \geqslant 1$. Consequently, $h = \sum_{\ell\geqslant 0} a_{2\ell}\mathtt{x}^{2\ell} + a_{k_0}\sum_{\ell\geqslant 0} b_{2\ell+1}\mathtt{x}^{2\ell+1}$ and

$$(\mathtt{y} - h(\mathtt{x}^{1/2}))(\mathtt{y} - h(-\mathtt{x}^{1/2})) = \mathtt{y}^2 - 2\sum_{\ell\geqslant 0} a_{2\ell}\mathtt{x}^\ell\mathtt{y} + \Big(\sum_{\ell\geqslant 0} a_{2\ell}\mathtt{x}^\ell\Big)^2 - a_{k_0}^2\mathtt{x}\Big(\sum_{\ell\geqslant 0} b_{2\ell+1}\mathtt{x}^\ell\Big) \in \kappa[[\mathtt{x}]][\mathtt{y}]$$

has degree 2 and divides the irreducible polynomial P of degree 4, which is a contradiction. Consequently, there exists $\sigma_0 \in G(E_1 : \kappa)$ such that $\widetilde{\sigma_0} \neq \mathrm{id}$ and $\widetilde{\sigma_0} \neq \tau$. As P has four roots in $E_1[[\mathtt{x}^{1/2}]]$, we deduce that these roots are $\zeta_1 := \zeta$, $\zeta_2 := \tau(\zeta)$, $\zeta_3 := \widetilde{\sigma_0}(\zeta)$ and $\zeta_4 := \tau(\widetilde{\sigma_0}(\zeta))$. Define $E := \kappa((\mathtt{x}))[\zeta_1, \zeta_2, \zeta_3, \zeta_4]$, which is the splitting field of P over $\kappa((\mathtt{x}))$. We claim: *For* *each $k \geqslant 0$ the irreducible polynomial of a_k over is κ has degree a divisor of 2 if k is even and a divisor of 4 is k is odd.*

As the roots of P are ζ, $\tau(\zeta)$, $\widetilde{\sigma_0}(\zeta)$ and $\tau(\widetilde{\sigma_0}(\zeta))$ and for each $\sigma \in G(E_1 : \kappa)$ the image of ζ under $\widetilde{\sigma}$ is a root of P, we deduce that the images of a_k under the elements of $G(E_1 : \kappa)$ are a_k, $(-1)^k a_k$, $\sigma_0(a_k)$ and $(-1)^k\sigma_0(a_k)$. Thus, if k is even, the roots of the irreducible polynomial of a_k over κ are a_k and $\sigma_0(a_k)$ (if they are different) and a_k if they coincide. If k is odd, the possible roots of the irreducible polynomial Q of a_k over κ are a_k, $(-1)^k a_k$, $\sigma_0(a_k)$ and $(-1)^k\sigma_0(a_k)$. Thus, Q may have degrees 1, 2 or 4.

We claim: $[E_0 : \kappa] = 2$ *and* $E_0 = \kappa(a_{k_0})$ *for some integer* $\kappa_0 \geqslant 0$*, or* $[E_0 : \kappa] = 4$ *and either* $E_0 = \kappa(a_{k_0})$ *for some odd integer* $\kappa_0 \geqslant 0$*, or* $E_0 = \kappa(a_{k_0}, a_{k_1})$ *for a pair of integers* $k_0, k_1 \geqslant 1$.

Let $k_0 \geqslant 1$ be an integer such that $[\kappa(a_{k_0}) : \kappa]$ is maximal. Suppose first $[\kappa(a_{k_0}) : \kappa] = 4$. Let us check: $a_k \in \kappa(a_{k_0})$ for each $k \geqslant 1$. Otherwise, there exists $a_{k_1} \notin \kappa(a_{k_0})$ and the irreducible polynomial of a_{k_1} over $\kappa(a_{k_0})$ has degree $2 \leqslant d \leqslant 4$. Denote $\eta_1 := a_{k_0}, \eta_2, \eta_3, \eta_4$ the roots of the irreducible polynomial of a_{k_0} over κ and $\chi_1 := a_{k_1}, \chi_2, \ldots, \chi_d$ the roots of the irreducible polynomial of a_{k_1} over $\kappa(a_{k_0})$. For each pair of indices $i = 1, 2, 3, 4$ and $j = 1, \ldots, d$ there exists and automorphism $\varphi_{ij} : E_1 \to E_1$ such that $\varphi_{ij}(\eta_1) = \eta_i$ and $\varphi(\chi_1) = \chi_j$, which provides $4d$ possibilities and we only have 4, because P has exactly four roots. Consequently, $E_0 = \kappa(a_{k_0})$ and $[E_0 : \kappa] = 4$.

Suppose next $[\kappa(a_{k_0}) : \kappa] = 2$. If $a_k \in \kappa(a_{k_0})$ for each $k \geqslant 1$, then $E_0 = \kappa(a_{k_0})$ and $[E_0 : \kappa] = 2$. Otherwise there exists a_{k_1} such that $[\kappa(a_{k_1}) : \kappa] = 2$ and $a_{k_1} \notin \kappa(a_{k_0}$. Thus, $[\kappa(a_{k_0}, a_{k_1}) : \kappa] = 4$. Let us check: $a_k \in \kappa(a_{k_0}, a_{k_1})$ for each $k \geqslant 1$. Otherwise, there exists $a_{k_2} \notin \kappa(a_{k_0}, a_{k_1})$ and the irreducible polynomial of a_{k_2} over $\kappa(a_{k_0}, a_{k_1})$ has degree 2. Denote $\eta_{i1} := a_{k_i}, \eta_{i2}$ the roots of the irreducible polynomial of a_{k_i} over κ. For each pair of indices $i = 1, 2, 3$ and $j = 1, 2$ there exists and automorphism $\varphi_{ij} : E_1 \to E_1$ such that $\varphi_{ij}(\eta_{i1}) = \eta_{ij}$, which provides 8 possibilities and we only have 4, because P has exactly four roots. Consequently, $E_0 = \kappa(a_{k_0}, a_{k_1})$ and $[E_0 : \kappa] = 4$.

We distinguish three subcases:

Subcase 2.1. $[E_0 : \kappa] = 2$. In this case $E_0 = \kappa[\theta]$, where $\theta \in \overline{\kappa}\backslash\kappa$ and $\theta^2 \in \kappa$. The roots of P are $\zeta_{\varepsilon_1,\varepsilon_2} := \sum_{k\geqslant 0}(c_k + \varepsilon_1\rho d_k)(\varepsilon_2)^k \mathtt{x}^{k/2}$ where $c_k, d_k \in \kappa$ and $(\varepsilon_1, \varepsilon_2) \in \{-1, 1\}^2$. Suppose $\rho^2 \in -\Sigma\kappa^2$. Then

$$\begin{aligned}P = \Big(\Big(\mathtt{y} - \sum_{\ell\geqslant 0} c_{2\ell}\mathtt{x}^\ell\Big)^2 + \rho^2\Big(\sum_{\ell\geqslant 0} d_{2\ell}\mathtt{x}^\ell\Big)^2 - \mathtt{x}\Big(\sum_{\ell\geqslant 0} c_{2\ell+1}\mathtt{x}^\ell\mathtt{x}^{1/2}\Big)^2 - \mathtt{x}\rho^2\Big(\sum_{\ell\geqslant 0} d_{2\ell+1}\mathtt{x}^\ell\Big)^2\Big)^2 \\ - 4\rho^2\Big(\Big(\mathtt{y} - \sum_{\ell\geqslant 0} c_{2\ell}\mathtt{x}^\ell\Big)\Big(\sum_{\ell\geqslant 0} d_{2\ell}\mathtt{x}^\ell\Big) - \mathtt{x}\Big(\sum_{\ell\geqslant 0} c_{2\ell+1}\mathtt{x}^\ell\mathtt{x}^{1/2}\Big)\Big(\sum_{\ell\geqslant 0} d_{2\ell+1}\mathtt{x}^\ell\Big)\Big)^2,\end{aligned}$$

so P is a sum of squares in $\kappa[[\mathtt{x}]][\mathtt{y}]$, which is a contradiction, because P generates the real prime ideal $\mathfrak{a} = \mathfrak{p} \cap \kappa[[\mathtt{x}, \mathtt{y}]]$. Thus, $\rho^2 \notin -\Sigma\kappa^2$. We conclude that $\mathfrak{a}$ is the kernel of the homomorphism $\Gamma : \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]] \to \kappa[[\mathtt{t}]],\ h \mapsto h(\zeta)$ where $\zeta := (\mathtt{t}^2, (a_1^* + \rho a_2^*)\mathtt{t}^2, (b_1^* + \rho b_2^*)\mathtt{t}^2)$ and $a_i^*, b_i^* \in \kappa[[\mathtt{t}]]$. Thus, $\mathtt{x} \in \mathcal{P}(A)\backslash\Sigma A^2$.

Subcase 2.2. $[E_0 : \kappa] = 4$. Suppose first $E_0 = \kappa(a_{k_0}, a_{k_1})$ for a pair of integers $k_0, k_1 \geqslant 1$. As $[\kappa(a_{k_i}) : \kappa] = 2$, there exists $\rho_i \in \overline{\kappa}\backslash\kappa$ such that $\rho_i^2 \in \kappa$ and $\kappa(a_{k_0}, a_{k_1}) = \kappa(\rho_0, \rho_1) = \kappa(\rho) = \kappa(\theta)$ where $\rho := \rho_0 + \rho_1$ and $\theta := \rho_0 - \rho_1$. The roots of the irreducible polynomial of ρ over κ are $\pm\rho$ and $\pm\theta$. Suppose next $E_0 = \kappa(a_{k_0})$ for some odd integer $\kappa_0 \geqslant 0$. The roots of the irreducible polynomial of $\rho := a_{k_0}$ over κ are $\pm\rho$ and $\pm\theta$ where $\theta := \sigma_0(\rho)$. Thus, we can unify both situations. Observe that $\mathtt{t}^4 - (\rho^2 + \theta^2)\mathtt{t}^2 + \rho^2\theta^2 \in \kappa[\mathtt{t}]$ is the irreducible polynomial of $\rho, \theta, -\rho, -\theta$ over κ and $\widetilde{\sigma_0} = \tau$. In addition, $[\kappa[\rho^2] : \kappa] = 2$. Write $a_k := a_{k0} + a_{k1}\rho + a_{k2}\rho^2 + a_{k3}\rho^3$ where $a_{kj} \in \kappa$. Thus,

$$(-1)^k(a_{k0} + a_{k1}\rho + a_{k2}\rho^2 + a_{k3}\rho^3) = (-1)^k a_k = \sigma_0^2(a_k) = a_{k0} - a_{k1}\rho + a_{k2}\rho^2 - a_{k3}\rho^3$$

Thus, if k is even, $a_{k1}=0$, $a_{k3}=0$. If k is odd, $a_{k0}=0$, $a_{k2}=0$, that is,
$$a_k=\begin{cases}a_{k0}+a_{k2}\rho^2 & \text{if } k \text{ is even,}\\ (a_{k1}+a_{k3}\rho^2)\rho & \text{if } k \text{ is odd.}\end{cases}$$
Consequently, as $\omega(\xi)\geqslant 1$, a root of P is
$$\xi=\sum_{\ell\geqslant 1}(a_{2\ell,0}+a_{2\ell,2}\rho^2)\mathtt{x}^\ell+(\rho\mathtt{x}^{1/2})\sum_{\ell\geqslant 1}(a_{2\ell+1,0}+a_{2\ell+1,2}\rho^2)\mathtt{x}^\ell.$$
Write $\mathtt{s}:=\rho\mathtt{x}^{1/2}$ and $\delta:=\rho^2$, so $\mathtt{x}=\frac{\mathtt{s}^2}{\delta}$ and
$$\xi\Big(\frac{\mathtt{s}^2}{\delta}\Big):=\sum_{\ell\geqslant 1}(a_{2\ell,0}+a_{2\ell,2}\delta)\frac{\mathtt{s}^{2\ell}}{\delta^\ell}+\mathtt{s}\sum_{\ell\geqslant 1}(a_{2\ell+1,0}+a_{2\ell+1,2}\delta)\frac{\mathtt{s}^{2\ell}}{\delta^\ell}.$$
The ideal $\mathfrak{a}$ is the kernel of a homomorphism $\Gamma:\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]\to\kappa[[\mathtt{t}]]$, $h\mapsto h(\zeta)$ where $\zeta:=(\frac{\mathtt{s}^2}{\delta},\xi(\frac{\mathtt{s}^2}{\delta}),\chi(\frac{\mathtt{s}^2}{\delta}))$ where $\chi\in\kappa[[\mathtt{x}]][\xi]$ and $\xi=\chi^2$. As $\chi^2=\xi\mathtt{x}$ and $\omega(\xi)\geqslant 1$, we deduce $\omega(\chi)\geqslant 1$ and we may write
$$\chi\Big(\frac{\mathtt{s}^2}{\delta}\Big):=\sum_{\ell\geqslant 1}(b_{2\ell,0}+b_{2\ell,2}\delta)\frac{\mathtt{s}^{2\ell}}{\delta^\ell}+\mathtt{s}\sum_{\ell\geqslant 1}(b_{2\ell+1,0}+b_{2\ell+1,2}\delta)\frac{\mathtt{s}^{2\ell}}{\delta^\ell}.$$
where $b_{ij}\in\kappa$ and $\xi(\frac{\mathtt{s}^2}{\delta})=(\chi(\frac{\mathtt{s}^2}{\delta}))^2$. After the change of coordinates $(\mathtt{x},\mathtt{y},\mathtt{z})\mapsto(\mathtt{x},\mathtt{y}-\sum_{\ell\geqslant 1}a_{2\ell,0}\mathtt{x}^\ell,\mathtt{z}-\sum_{\ell\geqslant 1}b_{2\ell,0}\mathtt{x}^\ell)$, we may assume
$$\xi\Big(\frac{\mathtt{s}^2}{\delta}\Big)=\sum_{\ell\geqslant 1}a_{2\ell,2}\delta\frac{\mathtt{s}^{2\ell}}{\delta^\ell}+\mathtt{s}\sum_{\ell\geqslant 1}(a_{2\ell+1,0}+a_{2\ell+1,2}\delta)\frac{\mathtt{s}^{2\ell}}{\delta^\ell},$$
$$\chi\Big(\frac{\mathtt{s}^2}{\delta}\Big)=\sum_{\ell\geqslant 1}b_{2\ell,2}\delta\frac{\mathtt{s}^{2\ell}}{\delta^\ell}+\mathtt{s}\sum_{\ell\geqslant 1}(b_{2\ell+1,0}+b_{2\ell+1,2}\delta)\frac{\mathtt{s}^{2\ell}}{\delta^\ell}.$$
If either $a_{2,2}$ or $b_{2,2}$ are non zero, then either $a_{2,2}\mathtt{y}\in\mathcal{P}(A)\backslash\Sigma A^2$ or $b_{2,2}\mathtt{z}\in\mathcal{P}(A)\backslash\Sigma A^2$, so we may assume $a_{2,2}=0$ and $b_{2,2}=0$. Thus,
$$\xi\Big(\frac{\mathtt{s}^2}{\delta}\Big)=\mathtt{s}^4\sum_{\ell\geqslant 2}a_{2\ell,2}\delta\frac{\mathtt{s}^{2\ell-4}}{\delta^\ell}+\mathtt{s}^3\sum_{\ell\geqslant 1}(a_{2\ell+1,0}+a_{2\ell+1,2}\delta)\frac{\mathtt{s}^{2\ell-2}}{\delta^\ell},$$
$$\chi\Big(\frac{\mathtt{s}^2}{\delta}\Big)=\mathtt{s}^4\sum_{\ell\geqslant 2}b_{2\ell,2}\delta\frac{\mathtt{s}^{2\ell-4}}{\delta^\ell}+\mathtt{s}^3\sum_{\ell\geqslant 1}(b_{2\ell+1,0}+b_{2\ell+1,2}\delta)\frac{\mathtt{s}^{2\ell-2}}{\delta^\ell}.$$
If $a_{3,0}+a_{3,2}\delta$ and $b_{3,0}+b_{3,2}\delta$ are κ-proportional, we may assume that $a_{3,0}+a_{3,2}\delta=\mu(b_{3,0}+b_{3,2}\delta)$ for some $\mu\in\kappa$ and after the change of coordinates $(\mathtt{x},\mathtt{y},\mathtt{z})\mapsto(\mathtt{x},\mathtt{y}-\mu\mathtt{z},\mathtt{z})$, we may assume $a_{3,0}+a_{3,2}\delta=0$, so there exists $M\in\kappa\backslash\{0\}$ such that $\mathtt{y}+\frac{M^2}{\delta^2}\mathtt{x}^2\in\mathcal{P}(A)\backslash\Sigma A^2$. Thus, we assume $a_{3,0}+a_{3,2}\delta$ and $b_{3,0}+b_{3,2}\delta$ are κ-linearly independent.

Let $H\in\mathfrak{a}$ be any series of order 2 and let $q:=\lambda_1\mathtt{x}^2+\lambda_2\mathtt{x}\mathtt{y}+\lambda_3\mathtt{x}\mathtt{z}+\lambda_4\mathtt{y}^2+\lambda_5\mathtt{y}\mathtt{z}+\lambda_6\mathtt{z}^2$ where $\lambda_j\in\kappa$ be the leading form of H. As $H(\zeta)=0$, we deduce $\lambda_1=0$ and $(\lambda_2(a_{3,0}+a_{3,2}\delta)+\lambda_3(b_{3,0}+b_{3,2}\delta))\mathtt{s}^5=0$. As $a_{3,0}+a_{3,2}\delta$ and $b_{3,0}+b_{3,2}\delta$ are κ-linearly independent, we deduce $\lambda_2=0$ and $\lambda_3=0$, so $q=\lambda_4\mathtt{y}^2+\lambda_5\mathtt{y}\mathtt{z}+\lambda_6\mathtt{z}^2$, which is a quadratic form of rank 2 (for any $H\in\mathfrak{a}$). This is a contradiction because $\mathfrak{a}$ contains a quadratic form of rank 3 (that is, $\mathtt{z}^2-\mathtt{x}\mathtt{y}$ before the changes of coordinates we have performed along the proof). ∎

Case 3. The roots of P belong to $\kappa[\eta][[\mathtt{x}]]$ for some $\eta\in\overline{\kappa}$ such that $\kappa[\eta]$ is real and $[\kappa[\eta]:\kappa]=4$. Then $\mathfrak{a}$ is the kernel of the homomorphism $\Gamma:\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]\to\kappa[[\mathtt{t}]]$, $h\mapsto h(\zeta)$ where
$$\zeta:=(\mathtt{t},(a_0+a_1\eta+a_2\eta^2+a_3\eta^3)^2\mathtt{t},(a_0+a_1\eta+a_2\eta^2+a_3\eta^3)\mathtt{t})$$

for some series $a_j \in \kappa[[\mathtt{x}]]$ (recall that $\mathtt{z}^2 - \mathtt{xy} \in \mathfrak{a}$). The element $a_0(0) + a_1(0)\eta + a_2(0)\eta^2 + a_3(0)\eta^3 \in \kappa[\eta]$ has either degree 1, 2 or 4. If $a_0(0) + a_1(0)\eta + a_2(0)\eta^2 + a_3(0)\eta^3$ has degree 1 or 2, after a change or coordinates we may assume

$$\zeta := (\mathtt{t}, (b_0 + b_1\eta + b_2\eta^2 + b_3\eta^3)\mathtt{t}, (c_0 + c_1\eta + c_2\eta^2 + c_3\eta^3)\mathtt{t}^2)$$

for some $c_j, b_j \in \kappa[[\mathtt{x}]]$, so $\mathtt{z} + M^2\mathtt{x}^2 \in \mathcal{P}(A)\backslash\Sigma A^2$ for some $M \in \kappa\backslash\{0\}$. Thus, $a_0(0) + a_1(0)\eta + a_2(0)\eta^2 + a_3(0)$ has degree 4 for each $i = 1, 2, 3, 4$ and we may assume $a_0(0) + a_1(0)\eta + a_2(0)\eta^2 + a_3(0) = \eta$ for each $i = 1, 2, 3, 4$. We have the branches

$$\zeta := (\mathtt{t}, (\eta^2 + (b_0 + b_1\eta + b_2\eta^2 + b_3\eta^3)\mathtt{t})\mathtt{t}, (\eta + (c_0 + c_1\eta + c_2\eta^2 + c_3\eta^3)\mathtt{t})\mathtt{t})$$

where $b_j, c_j \in \kappa[[\mathtt{x}]]$. We claim: *After a change of coordinates,* $\zeta = (\mathtt{t}, \eta^2\mathtt{t}, \eta\mathtt{t})$.

Consider the change of coordinates φ given by

$$(\mathtt{x}, \mathtt{w}_1, \mathtt{w}_2) \mapsto$$
$$(\mathtt{x}, \mathtt{w}_1 + \mathtt{x}^2 b_0(\mathtt{x}) + \mathtt{x}\mathtt{w}_2 b_1(\mathtt{x}) + \mathtt{x}\mathtt{w}_1 b_2(\mathtt{x}) + \mathtt{w}_1\mathtt{w}_2 b_3(\mathtt{x}), \mathtt{w}_2 + \mathtt{x}^2 c_0(\mathtt{x}) + \mathtt{x}\mathtt{w}_2 c_1(\mathtt{x}) + \mathtt{x}\mathtt{w}_1 c_2(\mathtt{x}) + \mathtt{w}_1\mathtt{w}_2 c_3(\mathtt{x}))$$

and observe that $\varphi(\mathtt{t}, \eta^2\mathtt{t}, \eta\mathtt{t}) = \zeta$. Thus, $\varphi^{-1}(\zeta) = (\mathtt{t}, \eta^2\mathtt{t}, \eta\mathtt{t})$, as claimed.

Let $P := \mathtt{t}^4 + \lambda_3\mathtt{t}^3 + \lambda_2\mathtt{t}^2 + \lambda_1\mathtt{t} + \lambda_0 \in \kappa[\mathtt{t}]$ be the irreducible polynomial of η over κ. Observe that $\mathtt{z}^4 + \lambda_3\mathtt{z}^3\mathtt{x} + \lambda_2\mathtt{z}^2\mathtt{x}^2 + \lambda_1\mathtt{z}\mathtt{x}^3 + \lambda_0\mathtt{x}^4 \in \mathfrak{a}$. As $\mathtt{z}^2 - \mathtt{yx} \in \mathfrak{a}$, we have

$$\mathtt{y}^2\mathtt{x}^2 + \lambda_3\mathtt{zyxx} + \lambda_2\mathtt{yxx}^2 + \lambda_1\mathtt{zx}^3 + \lambda_0\mathtt{x}^4 \in \mathfrak{a},$$

so also $\mathtt{y}^2 + \lambda_3\mathtt{zy} + \lambda_2\mathtt{yx} + \lambda_1\mathtt{zx} + \lambda_0\mathtt{x}^2 \in \mathfrak{a}$ (because $\mathfrak{a}$ is a prime ideal and $\mathtt{x} \notin \mathfrak{a}$). As $\mathfrak{a}$ is generated by two series of order 2, we conclude

$$\mathfrak{a} = (\mathtt{z}^2 - \mathtt{xy}, \mathtt{y}^2 + \lambda_3\mathtt{zy} + \lambda_2\mathtt{yx} + \lambda_1\mathtt{zx} + \lambda_0\mathtt{x}^2).$$

By Remark 10.1(ii) we may assume

$$\mathfrak{a} = (\mathtt{z}^2 - \mathtt{xy}, \mathtt{y}^2 - 2b\mathtt{yx} - 4c^2\mathtt{zx} + d\mathtt{x}^2).$$

where $P := \mathtt{t}^4 - 2b\mathtt{t}^2 - 4c^2\mathtt{t} + d \in \kappa[\mathtt{t}]$ is an irreducible polynomial. Observe that $Q := P(\frac{\mathtt{z}}{\mathtt{x}})\mathtt{x}^4 = \mathtt{z}^4 - 2b\mathtt{z}^2\mathtt{x}^2 - 4c^2\mathtt{zx}^3 + d\mathtt{x}^4 \in \mathfrak{a}$ and $\mathfrak{a} \cap \kappa[[\mathtt{x}, \mathtt{z}]] = Q\kappa[[\mathtt{x}, \mathtt{z}]]$, which is a real prime ideal, because $\mathfrak{a}$ is a real prime ideal different form $\mathfrak{m}_2$ (recall that it does not contain series of order 1). Suppose the field $\kappa[\mathtt{t}]/(P)$ is not (formally) real. By Lemma 13.3 $P \in \Sigma\kappa[\mathtt{t}]^2$, so $Q := P(\frac{\mathtt{z}}{\mathtt{x}}) \in \mathfrak{a} \cap \Sigma\kappa[\mathtt{x}, \mathtt{z}]^2$, which is a contradiction, because Q generates a real prime ideal. Consequently, the field $\kappa[\mathtt{t}]/(P)$ is (formally) real. ▪

We conclude $\mathcal{P}(A) \neq \Sigma A^2$, as required. □

13.1.5. *Final step.* Until this moment we have proved that if $A := \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/\mathfrak{a}$, where $\mathfrak{a} = (\mathtt{z}^2 - \mathtt{xy}, F + \mathtt{z}G)$, satisfies $\mathcal{P}(A) = \Sigma A^2$, then there exists a change of coordinates such that $\mathfrak{a} = (\mathtt{z}^2 - \mathtt{xy}, \mathtt{y}^2 - 2b\mathtt{yx} - 4c^2\mathtt{zx} + d\mathtt{x}^2)$ where $P := \mathtt{t}^4 - 2b\mathtt{t}^2 - 4c^2\mathtt{t} + d \in \kappa[\mathtt{t}]$ is an irreducible polynomial such that the field $\kappa[\mathtt{t}]/(P)$ is (formally) real. We finish this section by proving that P is in addition a chimeric polynomial (over κ), that is, for each element $Q := a_0 + a_1\mathtt{t} + a_2\mathtt{t}^2 + a_3\mathtt{t}^3 \in \mathcal{P}(\kappa[\mathtt{t}]/(P))$ there exists $\mu \in \kappa$ such that $P + \mu Q \in \mathcal{P}(\kappa[\mathtt{t}])$. In fact, $\mu \in \Sigma\kappa^2\backslash\{0\}$.

pschi **Lemma 13.10** (Chimeric irreducible factor)**.** *Let* $\mathfrak{a} := (\mathtt{z}^2 - \mathtt{xy}, \mathtt{y}^2 - 2b\mathtt{yx} - 4c^2\mathtt{zx} + d\mathtt{x}^2)$ *be such that* $P := \mathtt{t}^4 - 2b\mathtt{t}^2 - 4c^2\mathtt{t} + d \in \kappa[\mathtt{t}]$ *is an irreducible polynomial such that the field* $\kappa[\mathtt{t}]/(P)$ *is (formally) real and denote* $A := \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/\mathfrak{a}$*. If* $\mathcal{P}(A) = \Sigma A^2$*, the polynomial* $P := \mathtt{t}^4 - 2b\mathtt{t}^2 - 4c^2\mathtt{t} + d \in \kappa[\mathtt{t}]$ *is chimeric.*

Proof. Suppose the polynomial P is not chimeric. By Lemma 14.3 below there exists a polynomial $H := a_0 + a_1\mathtt{t} + a_2\mathtt{t}^2 + \mathtt{t}^3 \in \kappa[\mathtt{t}]$ such that the remainder $Q := b_0 + b_1\mathtt{t} + b_2\mathtt{t}^2 + b_3\mathtt{t}^3 \in \kappa[\mathtt{t}]$ of

H^2 satisfies Q that $Q+\mu P \notin \mathcal{P}(\kappa[\mathtt{t}]) = \Sigma\kappa[\mathtt{t}]^2$ for each $\mu \in \kappa$. Write $H^2 = Q+(c_0+c_1\mathtt{t}+c_2\mathtt{t}^2)P$. Define $h := a_0\mathtt{x}^2 + a_1\mathtt{zx} + a_2\mathtt{z}^2 + a_3\mathtt{yz}$ and $q := b_0\mathtt{x}^2 + b_1\mathtt{zx} + b_2\mathtt{z}^2 + b_3\mathtt{yz}$ and observe that

$$\begin{aligned}\mathtt{x}^6 q = b_0\mathtt{x}^8 + b_1\mathtt{zx}^7 + b_2\mathtt{z}^2\mathtt{x}^6 + b_3(\mathtt{yx})(\mathtt{zx})\mathtt{x}^4 &= Q\Big(\frac{\mathtt{z}}{\mathtt{x}}\Big)\mathtt{x}^4\mathtt{x}^4 - b_3(\mathtt{z}^2-\mathtt{xy})\mathtt{zxx}^4\\ &= \Big(H\Big(\frac{\mathtt{z}}{\mathtt{x}}\Big)\mathtt{x}^4\Big)^2 + (c_0\mathtt{x}^2 + c_1\mathtt{zx} + c_2\mathtt{z}^2)\mathtt{x}^2 P\Big(\frac{\mathtt{z}}{\mathtt{x}}\Big)\mathtt{x}^4 - b_3(\mathtt{z}^2-\mathtt{xy})\mathtt{zxx}^4\end{aligned}$$

In addition,

$$\begin{aligned}P\Big(\frac{\mathtt{z}}{\mathtt{x}}\Big)\mathtt{x}^4 &= \mathtt{z}^4 - 2b\mathtt{z}^2\mathtt{x}^2 - 4c^2\mathtt{zx}^3 + d\mathtt{x}^4\\ &= \mathtt{x}^2(\mathtt{y}^2 - 2b\mathtt{yx} - 4c^2\mathtt{zx} + d\mathtt{x}^2) + (\mathtt{z}^2-\mathtt{xy})(\mathtt{z}^2+\mathtt{xy}-2b\mathtt{x}^2).\end{aligned}$$

Consequently, $q \in \mathcal{P}(A) = \Sigma A^2$. As q is a quadratic form,

$$q = \sum_{j=1}^{p}(d_{0j}\mathtt{x} + d_{1j}\mathtt{y} + d_{2j}\mathtt{z})^2 - \lambda(\mathtt{z}^2-\mathtt{xy}) - \mu(\mathtt{y}^2 - 2b\mathtt{yx} - 4c^2\mathtt{zx} + d\mathtt{x}^2)$$

for some $d_{ij},\lambda,\mu \in \kappa$. Substitute $\mathtt{x}=1, \mathtt{y}=\mathtt{t}^2, \mathtt{z}=\mathtt{t}$ in the previous expression to obtain

$$Q = \sum_{j=1}^{p}(d_{0j} + d_{1j}\mathtt{t}^2 + d_{2j}\mathtt{t})^2 - \mu P,$$

so $Q+\mu P \in \Sigma\kappa[\mathtt{t}]^2$, which is a contradiction. Consequently, P is chimeric, as required. □

s14 14. Chimeric polynomials over a (formally) real field

In this section we study some main properties of chimeric polynomials and we present large families of (formally) real fields over which there exist no chimeric polynomial.

14.1. **Generalities about chimeric polynomials.** Let κ be a (formally) real field with algebraic closure $\overline{\kappa}$ and let $P \in \kappa[\mathtt{t}]$ be an irreducible polynomial of degree 4 such that $L := \kappa[\mathtt{t}]/(P)$ is (formally) real. We say that P is *chimeric (over κ)* if for each element $Q := a_0 + a_1\mathtt{t} + a_2\mathtt{t}^2 + a_3\mathtt{t}^3 \in \mathcal{P}(\kappa[\mathtt{t}]/(P))$ there exists $\mu \in \kappa$ such that $P + \mu Q \in \mathcal{P}(\kappa[\mathtt{t}])$. Observe that $\mu \in \Sigma\kappa^2\setminus\{0\}$.

equivd **Lemma 14.1.** *Let $P := \mathtt{t}^4 + \lambda_3\mathtt{t}^3 + \lambda_2\mathtt{t}^2 + \lambda_1\mathtt{t} + \lambda_0 \in \kappa[\mathtt{t}]$ be an irreducible polynomial such that $L := \kappa[\mathtt{t}]/(P)$ is (formally) real. Consider the ideal $\mathfrak{a} = (\mathtt{z}^2 - \mathtt{yx}, \mathtt{y}^2 + \lambda_3\mathtt{zy} + \lambda_2\mathtt{z}^2 + \lambda_1\mathtt{zx} + \lambda_0\mathtt{x}^2)$ of $\kappa[\mathtt{x},\mathtt{y},\mathtt{z}]$ and the quotient ring $A := \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/\mathfrak{a}$. Then P is chimeric over κ if and only if each quadratic form positive semidefinite on A is a sum of squares of elements of A.*

In addition, if P is chimeric, $P(\eta) \in \Sigma\kappa^2$ for each $\eta \in \kappa$.

Proof. Observe that $L := \kappa[\mathtt{t}]/(P)$ is a (formally) real field. The number of orderings of L extending an ordering of $\leqslant_\alpha$ of κ coincides with the number of roots of P in the real closure $\mathfrak{R}(\alpha)$ of $(\kappa,\leqslant_\alpha)$, it is bounded by 4 and is congruent with 4 modulo 2 (see [BCR, Prop.1.3.7]). Fix an ordering α of κ such that P has at least one root in $\eta \in \mathfrak{R}(\alpha)$.

By Lemma 5.19 and Remark 5.20(ii) a quadratic form $q = a_0\mathtt{x}^2 + a_1\mathtt{xz} + a_2\mathtt{xy} + a_3\mathtt{yz} \in \mathcal{P}(A)$ (observe that $\mathtt{z}^2 - \mathtt{xy}, \mathtt{z}^2 + \lambda_3\mathtt{zy} + \lambda_2\mathtt{z}^2 + \lambda_1\mathtt{zx} + \lambda_0\mathtt{x}^2 \in \mathfrak{a}$) if and only if $q(\phi) \in \mathcal{P}(\mathfrak{R}(\alpha)[[\mathtt{t}]])$ for each $\phi := (\phi_1,\phi_2,\phi_3) \in \mathfrak{R}(\alpha)[[\mathtt{t}]]$ such that $H(\phi) = 0$ for each $H \in \mathfrak{a}$. It is enough to check the latter property with the generators of $\mathfrak{a}$. Observe that if $\phi_1 = 0$, then $\phi_3 = 0$ and consequently $\phi_2 = 0$, so $\phi_1 \neq 0$ and we may assume $\phi_1 = \pm\mathtt{t}^q$ for some $q \geqslant 1$. Observe that $\phi_2\phi_1 = \phi_3^2$ and

$$\phi_2^2 + \lambda_3\phi_3\phi_2 + \lambda_2\phi_3^2 + \lambda_1\phi_3\phi_1 + \lambda_0\phi_1^2.$$

Thus, $2\omega(\phi_3) = \omega(\phi_1) + \omega(\phi_2)$, so we may assume $\omega(\phi_1) \leqslant \min\{\omega(\phi_2),\omega(\phi_3)\}$. As $\lambda_0 \neq 0$ (because P is irreducible), we deduce that either $\omega(\phi_1) = \omega(\phi_2)$ or $\omega(\phi_1) = \omega(\phi_3)$, so we conclude $\omega(\phi_1) = \omega(\phi_2) = \omega(\phi_3)$. Write $\phi_2 = \mu_2\mathtt{t}^q u_2$ and $\phi_3 = \mu_3\mathtt{t}^q u_3$ where $\mu_2,\mu_3 \in \mathfrak{R}(\alpha)\setminus\{0\}$

and $u_i \in \mathfrak{R}(\alpha)[[\mathtt{t}]]$ is a unit with $u_i(0) = 1$. As both $\mathtt{z}^2 - \mathtt{yx}$ and $\mathtt{y}^2 + \lambda_3\mathtt{zy} + \lambda_2\mathtt{z}^2 + \lambda_1\mathtt{zx} + \lambda_0\mathtt{x}^2$ are homogeneous polynomials, we may assume $q = 1$, that $\mathtt{t} >_\alpha 0$ and $\phi_1 = \mathtt{t}$, so $\phi_2 = \mu_2\mathtt{t}u_2$ and $\phi_3 = \mu_3\mathtt{t}u_3$ where $\mu_2 = \mu_3^2$ and $u_2 = u_3^2$, that is, $\phi = (\mathtt{t}, \mu_3^2\mathtt{t}u_3^2, \mu_3\mathtt{t}u_3)$. In addition,

$$u_3^4\mu_3^4 + \lambda_3 u_3^3\mu_3^3 + \lambda_2 u_3^2\mu_3^2 + \lambda_1 u_3\mu_3 + \lambda_0 = 0,$$

so $u_3 = 1$ and $\mu_3 = \eta$ is a root of P, that is, $\phi = (\mathtt{t}, \eta^2\mathtt{t}, \eta\mathtt{t})$. Thus, $q(\phi) \in \mathcal{P}(\mathfrak{R}(\alpha)[[\mathtt{t}]])$ if and only if the polynomial $Q := a_0 + a_1\mathtt{t} + a_2\mathtt{t}^2 + a_3\mathtt{t}^3$ satisfies $Q(\eta) \geqslant_\alpha 0$. Consequently, $q \in \mathcal{P}(A)$ if and only if $Q \in \mathcal{P}(\kappa[\mathtt{t}]/(P))$.

Suppose first that P is a chimeric polynomial over κ. As P is chimeric over κ, there exists $\mu \in \kappa$ such that $Q + \mu P \in \mathcal{P}(\kappa[\mathtt{t}]) = \Sigma\kappa[\mathtt{t}]^2$. As $Q + \mu P$ has degree 4, there exist polynomials $b_{0j} + b_{1j}\mathtt{t} + b_{2j}\mathtt{t}^2 \in \kappa[\mathtt{t}]$ such that $Q + \mu P = \sum_{j=1}^p (b_{0j} + b_{1j}\mathtt{t} + b_{2j}\mathtt{t}^2)^2$. Consequently,

$$q(\mathtt{x}, \mathtt{y}, \mathtt{z}) - \sum_{j=1}^p (b_{0j}\mathtt{x} + b_{1j}\mathtt{z} + b_{2j}\mathtt{y})^2 \in \mathfrak{a},$$

that is, $q \in \Sigma A^2$.

Conversely, suppose that each quadratic form positive semidefinite on A is a sum of squares of elements of A. Let $Q := a_0 + a_1\mathtt{t} + a_2\mathtt{t}^2 + a_3\mathtt{t}^3 \in \mathcal{P}(\kappa[\mathtt{t}]/(P))$ and consider the quadratic form $q := a_0\mathtt{x}^2 + a_1\mathtt{zx} + a_2\mathtt{yx} + a_3\mathtt{yz}$. As we have seen above, $Q \in \mathcal{P}(\kappa[\mathtt{t}]/(P))$ if and only if $q \in \mathcal{P}(A)$. Thus, $q - \sum_{j=1}^p (b_{0j}\mathtt{x} + b_{1j}\mathtt{z} + b_{2j}\mathtt{y})^2 \in \mathfrak{a}$ for some $b_{ij} \in \kappa$, so there exists $H \in \kappa[\mathtt{t}]$ such that $Q + HP = \sum_{j=1}^p (b_{0j} + b_{1j}\mathtt{t} + b_{2j}\mathtt{t}^2)^2$ is a sum of squares of degree $\leqslant 2$. As Q has degree $\leqslant 3$, we deduce $H = \mu \in \kappa$, so P is a chimeric polynomial over κ, as required.

Suppose next P is chimeric over κ. Let $M \in \kappa$ be such that $M^2 - (\frac{1}{\eta_i^2} + \eta_i^2) >_\beta 0$ in $\mathfrak{R}(\beta)$ for each ordering β of κ and $i = 1, 2, 3, 4$ (whenever $\eta_i \in \mathfrak{R}(\beta)$, see Lemma 11.1). Thus, $q(\mathtt{t}, \eta_i^2\mathtt{t}, \eta_i\mathtt{t}) = \mathtt{t}^2\eta_i^2(M - (\frac{1}{\eta_i^2} + \eta_i^2) >_\beta 0)$ in $\mathfrak{R}(\beta)$ for each ordering β of κ and $i = 1, 2, 3, 4$ (whenever $\eta_i \in \mathfrak{R}(\beta)$), so $q \in \mathcal{P}(A)$ (as we have seen above). As P is chimeric, $q \in \Sigma A^2$. Consequently, there exist $\mu_1, \mu_2 \in \kappa$ such that the quadratic form

$$M^2\mathtt{z}^2 - \mathtt{x}^2 - \mathtt{y}^2 + \mu_1(\mathtt{z}^2 - \mathtt{xy}) + \mu_2(\mathtt{y}^2 + \lambda_3\mathtt{zy} + \lambda_2\mathtt{z}^2 + \lambda_1\mathtt{zx} + \lambda_0\mathtt{x}^2)$$

is a sum of squares. In particular $\mu_2 - 1, \mu_2\lambda_0 - 1 \in \Sigma\kappa^2$, so $\lambda_0 \in \Sigma\kappa^2\backslash\{0\}$

In addition, as P is chimeric, $P_\eta = P(\mathtt{t} + \eta)$ is also chimeric for each $\eta \in \kappa$. Thus, $P(\eta) = P_\eta(0) \in \Sigma\kappa^2\backslash\{0\}$ for each $\eta \in \kappa$, as required. □

Remark 14.2. If $\kappa = R$ is a real closed field, there exists no chimeric polynomial over R. Otherwise, there exists a chimeric polynomial P over R. By Lemma 14.1 $P(x) = 0$ for each $x \in R$ and, as R is real closed, we deduce that $P \in \Sigma R[\mathtt{t}]^2$, so $R[\mathtt{t}]/(P)$ is not a (formally) real field, which is a contradiction. ■

The following result reveals the obstruction for a real irreducible polynomial to be in fact a chimeric polynomial.

obstruct **Lemma 14.3** (Obstruction). *Let $P := \mathtt{t}^4 + \lambda_3\mathtt{t}^3 + \lambda_2\mathtt{t}^2 + \lambda_1\mathtt{t} + \lambda_0 \in \kappa[\mathtt{t}]$ be an irreducible polynomial such that $\kappa[\mathtt{t}]/(P)$ is a (formally) real field. Then P is chimeric over κ if and only if for each polynomial $Q := \mathtt{t}^3 + \lambda_2\mathtt{t}^2 + \lambda_1\mathtt{t} + \lambda_0 \in \kappa[\mathtt{t}]$ there exists $\mu \in \Sigma\kappa^2\backslash\{0\}$ such that the remainder R of Q^2 divided by P satisfies $R + \mu P \in \Sigma\kappa[\mathtt{t}]^2$.*

Proof. The only if part follows from the definition of chimeric polynomial, so let us prove the converse. Let $H := a_3\mathtt{t}^3 + a_2\mathtt{t}^2 + a_1\mathtt{t} + a_0 \in \kappa[\mathtt{t}]$ be such that $H \in \mathcal{P}(\kappa[\mathtt{t}]/(P))$. As $\kappa[\mathtt{t}]/(P)$ is a (formally) real field, $\mathcal{P}(\kappa[\mathtt{t}]/(P)) = \Sigma(\kappa[\mathtt{t}]/(P))^2$. Thus, there exist polynomials $b_{3i}\mathtt{t}^3 + b_{2i}\mathtt{t}^2 + b_{1i}\mathtt{t} + b_{0i}, c_2\mathtt{t}^2 + c_1\mathtt{t} + c_0 \in \kappa[\mathtt{t}]$ such that $H = \sum_{i=1}^p (b_{3i}\mathtt{t}^3 + b_{2i}\mathtt{t}^2 + b_{1i}\mathtt{t} + b_{0i})^2 + (c_2\mathtt{t}^2 + c_1\mathtt{t} + c_0)P$. We want to find $\mu \in \kappa$ such that $H + \mu P \in \Sigma\kappa[\mathtt{t}]^2$. If each $b_{3i} = 0$, then $c_2 = 0$, $c_1 = 0$ and $\mu := c_0 = \sum_{i=1}^p b_{2i}^2 \in \Sigma\kappa^2$, so we may assume that some $b_{3i} \neq 0$ and in fact reindexing the indices, we may assume that $b_{3i} \neq 0$ for $i = 1, \dots, s$ and $b_{3i} = 0$ for $i = s+1, \dots, q$. Let R_i be the remainder of $(b_{3i}\mathtt{t}^3 + b_{2i}\mathtt{t}^2 + b_{1i}\mathtt{t} + b_{0i})^2$ divided by P for $i = 1, \dots, s$. Observe that

$$H - \sum_{i=1}^{s} R_i - \sum_{i=s+1}^{p} (b_{2i}\mathtt{t}^2 + b_{1i}\mathtt{t} + b_{0i})^2 \in (P)$$

and $R_i \in \mathcal{P}(\kappa[\mathtt{t}]/(P))$, so it is enough to find $\mu_i \in \kappa$ such that $R_i + \mu_i P \in \Sigma\kappa[\mathtt{t}]^2$. But this follows from the hypothesis that for each polynomial $Q := \mathtt{t}^3 + \lambda_2\mathtt{t}^2 + \lambda_1\mathtt{t} + \lambda_0 \in \kappa[\mathtt{t}]$ there exists $\mu \in \Sigma\kappa^2$ such that the remainder R of Q^2 divided by P satisfies $R + \mu P \in \Sigma\kappa[\mathtt{t}]^2$. □

Remark 14.4. Consequently, to prove that an irreducible polynomial $P \in \kappa[\mathtt{t}]$ such that $\kappa[\mathtt{t}]/(P)$ is a (formally) real field is not chimeric it is enough to find a polynomial $Q := \mathtt{t}^3+\lambda_2\mathtt{t}^2+\lambda_1\mathtt{t}+\lambda_0 \in \kappa[\mathtt{t}]$ such that the remainder R of Q^2 divided by P satisfies $R + \mu P \notin \Sigma\kappa[\mathtt{t}]^2$ for each $\mu \in \Sigma\kappa^2$. However, from the practice point of view the complexity of general computations is far from manageable. ▪

14.2. **Real roots of a chimeric polynomial.** We will use Sturm's sequences [BCR, Def.1.2.8] of polynomials of degree 4 with coefficients in a (formally) real field and Sturm's Theorem [BCR, Cor.1.2.10] to determine the number their number of roots in the real closed field $\mathfrak{R}(\alpha)$ for $\alpha \in \operatorname{Sper}(\kappa)$. We will also take advantage of the following result that follows [BPR, Prop.4.5].

deg4c **Proposition 14.5.** *Let $P \in \kappa[\mathtt{t}]$ be a polynomial of degree 4 with non zero discriminant $\Delta(P)$ and leading coefficient λ_4. Let $\alpha \in \operatorname{Sper}(\kappa)$ be such that $\lambda_4 >_\alpha 0$ and denote r_α the number of roots of P in $\mathfrak{R}(\alpha)$. Then:*

(i) $\Delta(P) >_\alpha 0$ *if and only if* $r_\alpha \equiv 0 \mod 4$.
(ii) $\Delta(P) <_\alpha 0$ *if and only if* $r_\alpha \equiv 2 \mod 4$.

Remark 14.6. Given a polynomial $P \in \kappa[\mathtt{t}]$ of degree 4, denote $\mathfrak{S}_k$ the set of $\alpha \in \operatorname{Sper}(\kappa)$ such that P has exactly k roots in $\mathfrak{R}(\alpha)$. If P is irreducible, we have

$$\mathfrak{S}_0 \cup \mathfrak{S}_4 = \{\alpha \in \operatorname{Sper}(\kappa) : \ \Delta(P) >_\alpha 0\},$$
$$\mathfrak{S}_2 = \{\alpha \in \operatorname{Sper}(\kappa) : \ \Delta(P) <_\alpha 0\}$$

and $\operatorname{Sper}(\kappa) = \mathfrak{S}_0 \sqcup \mathfrak{S}_2 \sqcup \mathfrak{S}_4$. ▪

ajustem **Lemma 14.7.** *Let $P \in \kappa[\mathtt{t}]$ be an irreducible polynomial and let $m \in \kappa$. Then there exists $m_1 \in \kappa\backslash\{0\}$ such that $(\mathtt{t}^2 - m)(\mathtt{t}^2 - (m + m_1^4)) \in \mathcal{P}(\kappa[\mathtt{t}]/(P))$.*

Proof. Consider the polynomials $H := P(\sqrt{t})P(-\sqrt{t}) \in \kappa[\mathtt{t}]$ and $Q := H(m + \mathtt{t}^2)$. Let $m_1 \in \kappa\backslash\{0\}$ be such that $m_1^2 <_\beta \theta$ for each root $\theta \in \mathfrak{R}(\beta)$ of Q and each $\beta \in \operatorname{Sper}(\kappa)$ (use Lemma 11.1). Suppose that there exists a root η of P such that $m <_\beta \eta^2 <_\beta m + m_1^4$ for some $\beta \in \operatorname{Sper}(\kappa)$. Then $\eta^2 - m >_\beta 0$ and $\sqrt{\eta^2 - m} \in \mathfrak{R}(\beta)$ satisfies $H(\sqrt{\eta^2 - m}) = 0$, so $m_1^2 <_\beta \sqrt{\eta^2 - m}$ and $m_1^4 < \eta^2 - m$, that is, $m + m_1^4 <_\beta \eta^2$, which is a contradiction. Consequently, there exists no root η of P such that $m <_\beta \eta^2 <_\beta m + m_1^4$ for some $\beta \in \operatorname{Sper}(\kappa)$. Thus, $(\eta_i - m)(\eta_i - (m + m_1^4)) >_\beta 0$ for each $\beta \in \operatorname{Sper}(\kappa)$ and $i = 1, 2, 3, 4$, so $(\mathtt{t} - m)(\mathtt{t} - (m + m_1^4)) \in \mathcal{P}(\kappa[\mathtt{t}]/(P))$, as required. □

After some preparation, we prove next that chimeric polynomials over a (formally) real field κ have either no root or four roots in $\mathfrak{R}(\alpha)$ for each $\alpha \in \operatorname{Sper}(\kappa)$. If $P \in \kappa[\mathtt{t}]$ is a chimeric polynomial, after a translation of the variable, we may assume $P := \mathtt{t}^4 - 2b_0\mathtt{t}^2 - 4c_0\mathtt{t} + d_0$ where $d_0 \in \Sigma\kappa^2\backslash\{0\}$. If $c_0 = 0$, write $b := b_0$, $c := 0$ and $d := d_0$. Otherwise, $P_1 := \frac{P(c_0\mathtt{t})}{c_0^4} = \mathtt{t}^4 - 2b\mathtt{t}^2 - 4c^2\mathtt{t} + d$ where $b := \frac{b_0}{c_0^2}$, $c := \frac{1}{c_0}$ and $d := \frac{d_0}{c_0^4} \in \Sigma\kappa^2\backslash\{0\}$.

crit42 **Lemma 14.8** (Number of real roots of a chimeric polynomial)**.** *Let κ be a (formally) real field and let $P := \mathtt{t}^4 - 2b\mathtt{t}^2 - 4c^2\mathtt{t} + d \in \kappa[\mathtt{t}]$ (where $b, c \in \kappa$ and $d \in \Sigma\kappa^2$) be a chimeric polynomial. Then $\Delta(P) \in \Sigma\kappa^2\backslash\{0\}$, $\mathfrak{S}_4 \neq \varnothing$ and $\operatorname{Sper}(\kappa) = \mathfrak{S}_0 \sqcup \mathfrak{S}_4$.*

Proof. The derivate of P is $P' := 4(\mathtt{t}^3 - b\mathtt{t} - c^2)$. We compute the remainder of $(P'/4)^2$ modulo P and obtain the polynomial $Q := 2c^2\mathtt{t}^3 + (b^2-d)\mathtt{t}^2 + 2bc^2\mathtt{t} + c^4 \in \mathcal{P}(\kappa[\mathtt{t}]/(P))$. We look for $\mu \in \kappa$ such that $Q + \mu P \in \mathcal{P}(\kappa[\mathtt{t}]) = \Sigma\kappa[\mathtt{t}]^2$. As Q has degree $\leqslant 3$, we deduce $\mu \in \Sigma\kappa^2 \backslash \{0\}$.

The discriminant of $H := \mu P + Q$ is

$$\Delta(H) = 16(b^4d + 2b^3c^4 - 2b^2d^2 - 18bc^4d - 27c^8 + d^3)(4\mu^3 - 4b\mu^2 + (b^2-d)\mu - c^4)^2.$$

The discriminant of P is $\Delta(P) = 256(b^4d + 2b^3c^4 - 2b^2d^2 - 18bc^4d - 27c^8 + d^3)$ and it is non-zero because P is irreducible. If $H \in \mathcal{P}(\kappa[\mathtt{t}])$, then by Lemma 14.5 $\Delta(H) >_\alpha 0$ for each $\alpha \in \operatorname{Sper}(\kappa)$. Otherwise, there exists an ordering $\beta \in \operatorname{Sper}(\kappa)$ such that $\Delta(H) <_\beta 0$ and for such ordering $P + \mu Q$ has two different roots in $\mathfrak{R}(\beta)$ and two different (complex) roots in $\mathfrak{R}(\beta)[\sqrt{-1}] \backslash \mathfrak{R}(\beta)$, which contradicts the fact that $P + \mu Q \in \mathcal{P}(\kappa[\mathtt{t}])$. We distinguish two cases:

Case 1. If $\Delta(P) \in \Sigma\kappa^2 \backslash \{0\}$, by Lemma 14.5 the polynomial P has no roots in $\mathfrak{R}(\alpha)$ or has four roots in $\mathfrak{R}(\alpha)$ for each $\alpha \in \operatorname{Sper}(\kappa)$, that is, $\operatorname{Sper}(\kappa) = \mathfrak{S}_0 \sqcup \mathfrak{S}_4$. As $\kappa[\mathtt{t}]/(P)$ is a (formally) real field, $\mathfrak{S}_4 \neq \varnothing$. ■

Case 2. If $\Delta(P) \in \kappa \backslash \Sigma\kappa^2$, then $4\mu^3 - 4b\mu^2 + (b^2-d)\mu - c^4 = 0$, so

$$d = \frac{-c^4 + b^2\mu - 4b\mu^2 + 4\mu^3}{\mu}.$$

Let $\rho \in \overline{\kappa}$ be such that $\mu = \rho^2$. Then

$$\rho^2 P = (\mathtt{t}^2\rho + 2\mathtt{t}\rho^2 + 2\rho^3 - b\rho - c^2)(\mathtt{t}^2\rho - 2\mathtt{t}\rho^2 + 2\rho^3 - b\rho + c^2)$$

and $\rho \in \overline{\kappa} \backslash \kappa$ because P is irreducible. The roots of P are

$$\zeta_1 := \rho + \frac{\sqrt{-\rho^4 + b\rho^2 + c^2\rho}}{\rho}, \quad \zeta_2 := \rho - \frac{\sqrt{-\rho^4 + b\rho^2 + c^2\rho}}{\rho},$$
$$\zeta_3 := -\rho + \frac{\sqrt{-\rho^4 + b\rho^2 - c^2\rho}}{\rho}, \quad \zeta_4 := -\rho - \frac{\sqrt{-\rho^4 + b\rho^2 - c^2\rho}}{\rho}.$$

Pick $\alpha \in \mathfrak{S}_2$. As $\mu \in \Sigma\kappa^2 \backslash \{0\}$, it holds $\rho \in \mathfrak{R}(\alpha)$. We may assume $\rho >_\alpha 0$. If $-\rho^4 + b\rho^2 - c^2\rho >_\alpha 0$, then

$$-\rho^4 + b\rho^2 + c^2\rho = (-\rho^4 + b\rho^2 - c^2\rho) + 2c^2\rho >_\alpha 0,$$

so $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathfrak{R}(\alpha)$ and $\alpha \in \mathfrak{S}_4$, which is a contradiction. Thus, $-\rho^4 + b\rho^2 - c^2\rho <_\alpha 0$ and $-\rho^4 + b\rho^2 + c^2\rho >_\alpha 0$, so $-\rho^3 + b\rho - c^2 <_\alpha 0$ and $-\rho^3 + b\rho + c^2 >_\alpha 0$. As $P_2 := \mathtt{t}^2\rho - 2\mathtt{t}\rho^2 + 2\rho^3 - b\rho + c^2$ has no roots in $\mathfrak{R}(\alpha)$ and it is a monic polynomial, then $P_2(\rho) >_\alpha 0$ and $P_2(-\rho) >_\alpha 0$. As ρ is between ζ_2 and ζ_1 in $\mathfrak{R}(\alpha)$ and $P_1 := \mathtt{t}^2\rho + 2\mathtt{t}\rho^2 + 2\rho^3 - b\rho - c^2$ is a monic polynomial we have $P_1(\rho) <_\alpha 0$. In addition, $P_1(-\rho) = 5\rho^3 - b\rho - c^2$.

By Lemma 14.1 we know that

$$P(0) = \frac{(-2\rho^3 + b\rho + c^2)(-2\rho^3 + b\rho - c^2)}{\rho^2} >_\alpha 0.$$

As $-\rho^3 + b\rho - c^2 <_\alpha 0$, also $-2\rho^3 + b\rho - c^2 = -\rho^3 + (-\rho^3 + b\rho - c^2) <_\alpha 0$, so also $-2\rho^3 + b\rho + c^2 <_\alpha 0$. Thus, $5\rho^3 - b\rho - c^2 >_\alpha 0$. We deduce $P(\rho) = P_1(\rho)P_2(\rho) >_\alpha 0$ and $P(-\rho) = P_1(-\rho)P_2(-\rho) <_\alpha 0$.

By Lemma 14.7 there exists $m_1 \in \kappa \backslash \{0\}$ such that $Q := (\mathtt{t}^2 - \mu)(\mathtt{t}^2 - (\mu + m_1^4)) \in \mathcal{P}(\kappa[\mathtt{t}]/(P))$. As P is a chimeric polynomial, there exists $\lambda \in \kappa$ such that $Q + \lambda P \in \Sigma\kappa[\mathtt{t}]^2$. We have $0 \leqslant_\alpha (Q + \lambda P)(\rho) = \lambda P(\rho)$ and $0 \leqslant_\alpha (Q + \lambda P)(-\rho) = \lambda P(-\rho)$. As $P(\rho) >_\alpha 0$, we deduce $\lambda \geqslant_\alpha 0$, whereas as $P(-\rho) <_\alpha 0$, we have $\lambda \leqslant_\alpha 0$, so $\lambda = 0$. Consequently, $Q \in \Sigma\kappa[\mathtt{t}]^2$, which is a contradiction, because Q has four simple roots in $\mathfrak{R}(\alpha)$. Thus, P is not chimeric over κ.

This means that there exists no chimeric polynomial satisfying the conditions of Case 2., as required. □

density0 *Remarks* 14.9. (i) Let $P\in\kappa[\mathtt{t}]$ be a chimeric polynomial. If $\alpha\in\operatorname{Sper}(\kappa)$ satisfies that P has four root in $\mathfrak{R}(\alpha)$, say $\zeta_1<_\alpha\zeta_2<_\alpha\zeta_3<_\alpha\zeta_4$, then by Lemma 14.1 $\kappa\cap((\zeta_1,\zeta_2)_\alpha\cup(\zeta_3,\zeta_4)_\alpha)=\varnothing$.

(ii) Let κ be a (formally) real field. If κ is dense in $\mathfrak{R}(\alpha)$ for each $\alpha\in\operatorname{Sper}(\kappa)$, there exists by (i) no chimeric polynomial over κ. This happens for instance if all the orderings of κ are Archimedean, but there are other examples: real closed fields, pseudo real closed fields [Pr1], pseudo classically closed fields [Pop] and RC_π-fields [Er]. See [ADF] for further details. ▪

nein2 **Corollary 14.10.** *Let κ be a (formally) real field and let $P:=\mathtt{t}^4+\lambda_3\mathtt{t}^3+\lambda_2\mathtt{t}^2+\lambda_1\mathtt{t}+\lambda_0$ be an irreducible polynomial. Let β be an ordering of $\kappa[\mathtt{t}]/(P)$ and let $\alpha:=\beta\cap\mathbb{Q}(\lambda_0,\lambda_1,\lambda_2,\lambda_3)$. Suppose $(\mathbb{Q}(\lambda_0,\lambda_1,\lambda_2,\lambda_3),<_\alpha)$ is Archimedean. Then P is not a chimeric polynomial over κ.*

Proof. The polynomial $P\in\mathbb{Q}(\lambda_0,\lambda_1,\lambda_2,\lambda_3)[\mathtt{t}]$ is irreducible. Observe that $\mathfrak{R}(\alpha)\subset\mathbb{R}$. Consider the (formally) real field

$$\mathbb{Q}(\lambda_0,\lambda_1,\lambda_2,\lambda_3)[\mathtt{t}]/(P)\hookrightarrow\kappa[\mathtt{t}]/(P).$$

Define $\gamma:=\beta\cap\mathbb{Q}(\lambda_0,\lambda_1,\lambda_2,\lambda_3)[\mathtt{t}]/(P)$. As $(\mathbb{Q}(\lambda_0,\lambda_1,\lambda_2,\lambda_3),<_\alpha)$ is Archimedean, we deduce

$$(\mathbb{Q}(\lambda_0,\lambda_1,\lambda_2,\lambda_3)[\mathtt{t}]/(P),<_\gamma)\hookrightarrow\mathfrak{R}(\gamma)\subset\mathbb{R}.$$

Thus, P has one root in $\mathfrak{R}(\alpha)\subset\mathbb{R}$. If P is a chimeric polynomial over κ, then P has by Lemma 14.8 four roots four roots in $\mathfrak{R}(\alpha)\subset\mathbb{R}$. As $\mathbb{Q}$ is dense in $\mathbb{R}$, it is also dense in $\mathfrak{R}(\alpha)$. By Remark 14.9(i) P is not chimeric over κ, as required. □

14.3. **Real fields that admit no chimeric polynomials.** To determine whether there exists or not a (formally) real field that admits chimeric polynomials seems a difficult problem:

(i) To prove that a (formally) real field κ admits no chimeric polynomials means to find for each irreducible polynomial $P\in\kappa[\mathtt{t}]$ of degree 4 a polynomial $Q\in\kappa[\mathtt{t}]$ of degree 3 such that its remainder R modulo P satisfies $R+\mu P\notin\mathcal{P}(\kappa[\mathtt{t}])$ for each $\mu\in\kappa$ (Lemma 14.3). Many computations, when there is no general argument! (contrast Lemmas 14.11 and 14.12 with the proof of Theorem 14.14 in Appendix D).

(ii) To show that a (formally) real field κ admits a chimeric polynomial means to find an irreducible polynomial $P\in\kappa[\mathtt{t}]$ of degree 4 and to prove that for each polynomial $Q\in\kappa[\mathtt{t}]$ of degree 3 there exists $\mu\in\kappa$ such that the remainder R of Q^2 modulo P satisfies $R+\mu P\notin\mathcal{P}(\kappa[\mathtt{t}])$. Surely many computations!

We analyze next two situations, which provides many additional (formally) real fields that admit no chimeric polynomials, by means of Corollaries 14.25 and 14.26.

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14.3.1. *Real fields with enough Archimedean orderings have no chimeric polynomials.* If κ is a (formally) real field, we denote $\mathfrak{D}_\kappa$ the set of all $\alpha\in\operatorname{Sper}(\kappa)$ such that κ is dense in $\mathfrak{R}(\alpha)$. Recall that if κ is a pseudo real closed field, then $\mathfrak{D}_\kappa=\operatorname{Sper}(\kappa)$ (see [Pr1, Prop.1.4]). We show that there is no chimeric polynomial over a (formally) real field κ such that $\mathfrak{D}_\kappa$ is dense in $\operatorname{Sper}(\kappa)$.

density1 **Lemma 14.11.** *Let κ be a (formally) real field such that $\mathfrak{D}_\kappa$ is dense in* $\operatorname{Sper}(\kappa)$. *Then there exists by no chimeric polynomial over κ.*

Proof. Suppose there exists a chimeric polynomial P over κ. We may assume $P:=\mathtt{t}^4-2b\mathtt{t}^2-4c^2\mathtt{t}+d\in\kappa[\mathtt{t}]$. By Lemma 14.8 there exists $\beta\in\operatorname{Sper}(\kappa)$ such that P has four roots in $\mathfrak{R}(\beta)$.

Then, a Sturm's sequence for P is:

$$\begin{aligned}
F_0 &:= P := \mathtt{t}^4 - 2b\mathtt{t}^2 - 4c^2\mathtt{t} + d,\\
F_1 &:= P' := 4(\mathtt{t}^3 - b\mathtt{t}^2 - c^2),\\
F_2 &:= b\mathtt{t}^2 + 3c^2\mathtt{t} - d,\\
F_3 &:= 4(b^3 - bd - 9c^4)\mathtt{t} + 4c^2(b^2 + 3d),\\
F_4 &:= \Delta(P)
\end{aligned}$$

and the leading coefficients of $[F_0, F_1, F_2, F_3, F_4]$, which are $[1, 4, b, 4(b^3 - bd - 9c^4), \Delta(P)]$, are all positive with respect to β, that is, $b >_\beta 0, b^3 - bd - 9c^4 >_\beta 0, \Delta(P) >_\beta 0$, or equivalently, $\beta \in \{b > 0, b^3 - bd - 9c^4 > 0, \Delta(P) > 0\}$. As $\mathfrak{D}_\kappa$ is dense in $\operatorname{Sper}(\kappa)$, there exists $\alpha \in \mathfrak{D}_\kappa$ such that $\alpha \in \{b > 0, b^3 - bd - 9c^4 > 0, \Delta(P) > 0\}$. Consequently, all the terms of the sequence $[1, 4, b, 4(b^3 - bd - 9c^4), \Delta(P)]$ are positive with respect to α. Thus, P has four roots in $\mathfrak{R}(\alpha)$, say $\zeta_1 <_\alpha \zeta_2 <_\alpha \zeta_3 <_\alpha \zeta_4$. As κ is dense in $\mathfrak{R}(\alpha)$, we find $k \in \kappa \cap (\zeta_1, \zeta_2)_\alpha$. Thus, $P(k) <_\alpha 0$, which is a contradiction by Lemma 14.1. We deduce that there exists by no chimeric polynomial over κ, as required. □

nein4 **Lemma 14.12.** *Let $\mathfrak{p}$ be a real prime ideal of $\mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_n]$. Consider the (formally) real field $\kappa := \operatorname{qf}(\mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_n]/\mathfrak{p})$. Then $\mathfrak{D}_\kappa$ is dense in $\operatorname{Sper}(\kappa)$ and, in particular, there exists no chimeric polynomial over κ.*

Proof. By Noether's normalization we may assume that $\mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_d] \hookrightarrow \mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_n]/\mathfrak{p}$ is an integral extension. Let $p_1, \dots, p_r \in \mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_d]$ and consider the open set $\mathcal{U} := \{\alpha \in \operatorname{Sper}(\kappa) : p_1 >_\alpha 0, \dots, p_r >_\alpha 0\}$. Consider the (formally) real field $\kappa_1 := \kappa[\sqrt{p_1}, \dots, \sqrt{p_r}]$ and consider the homomorphism

$$\varphi : \mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_n, \mathtt{y}_1, \dots, \mathtt{y}_r] \to (\mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_n]/\mathfrak{p})[\sqrt{p_1}, \dots, \sqrt{p_r}] \subset \kappa_1$$

that maps $\mathtt{x}_i$ onto $\mathtt{x}_i + \mathfrak{p}_i$ for $i = 1, \dots, n$ and $\mathtt{y}_j$ onto $\sqrt{p_j}$ for $j = 1, \dots, r$. The kernel $\mathfrak{q} := \ker(\varphi)$ is a real (proper) prime ideal of $\mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_n][\mathtt{t}]$ [BCR, Thm.4.3.7] that contains $\mathfrak{p}$ and $\mathtt{y}_1^2 - p_1, \dots, \mathtt{y}_r^2 - p_r$. Observe that $\mathfrak{q} \cap \mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_n] = \mathfrak{p}$ (because κ is the quotient field of $\mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_n]/\mathfrak{p}$). Observe that $\mathfrak{q} \cap \mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_n] = \mathfrak{p}$ (because κ is the quotient field of $\mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_n]/\mathfrak{p}$). We have the diagram

$$\mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_d] \hookrightarrow \mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_n]/\mathfrak{p} \hookrightarrow \mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_n, \mathtt{y}_1, \dots, \mathtt{y}_r]/\mathfrak{q} \hookrightarrow \kappa[\sqrt{p_1}, \dots, \sqrt{p_r}] = \kappa_1$$

and $\mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_d] \hookrightarrow \mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_n, \mathtt{y}_1, \dots, \mathtt{y}_r]/\mathfrak{q}$ is an integral extension, because $\mathtt{y}_1^2 - p_1, \dots, \mathtt{y}_r^2 - p_r \in \mathfrak{q}$. In addition, $\kappa[\sqrt{p_1}, \dots, \sqrt{p_r}]$ is the quotient field of $\mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_n, \mathtt{y}_1, \dots, \mathtt{y}_r]/\mathfrak{q}$. By the primitive element theorem there exist $\lambda_{d+2}, \cdots, \lambda_n, \mu_1, \dots, \mu_r \in \mathbb{Z}$ such that $u := \mathtt{x}_{d+1} + \sum_{k=d+2}^n \lambda_k \mathtt{x}_k + \sum_{j=1}^r \mu_j \mathtt{y}_j$ (modulo $\mathfrak{q}$) is a primitive element of $\operatorname{qf}(\mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_n, \mathtt{t}]/\mathfrak{q})$ and let $f \in \mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_d][\mathtt{u}]$ be (up to a non-zero element of $\mathbb{Q}(\mathtt{x}_1, \dots, \mathtt{x}_d)$) the irreducible polynomial of u over $\mathbb{Q}(\mathtt{x}_1, \dots, \mathtt{x}_d)$. Thus,

$$\begin{aligned}
\mathbb{Q}(\mathtt{x}_1, \dots, \mathtt{x}_d)[\mathtt{u}]/(f) &= \operatorname{qf}(\mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_d, \mathtt{t}]/(f))\\
&= \operatorname{qf}(\mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_n, \mathtt{y}_1, \dots, \mathtt{y}_r]/\mathfrak{q}) = \kappa[\sqrt{p_1}, \dots, \sqrt{p_r}] = \kappa_1
\end{aligned}$$

and $\kappa = \operatorname{qf}(\mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_n]/\mathfrak{p}) \hookrightarrow \kappa_1$. Observe that $\mathtt{x}_k = \frac{a_k(\mathtt{x}_1, \dots, \mathtt{x}_d, \mathtt{u})}{c(\mathtt{x}_1, \dots, \mathtt{x}_d)}$ and $\mathtt{y}_j = \frac{b_j(\mathtt{x}_1, \dots, \mathtt{x}_d, \mathtt{u})}{c(\mathtt{x}_1, \dots, \mathtt{x}_d)}$ for some polynomials $a_k, b_j \in \mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_d][\mathtt{u}]$ (of degree $\leqslant \deg_{\mathtt{u}}(f) - 1$ with respect to $\mathtt{u}$) and $c \in \mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_d] \setminus \{0\}$ where $k = d + 1, \dots, n$ and $j = 1, \dots, r$.

Suppose f does not change sign in $\mathbb{R}^{d+1}$ and assume (changing f by $-f$) that $f \geqslant 0$. Here we use that $\mathbb{Q}$ admits a unique immersion in $\mathbb{R}$ as (formally) real fields. By [St, Thm.4] there exist an integer $m \geqslant 0$, $r_i, s_i \in \mathbb{Q}$, $r_i, s_i > 0$ and $g_i, h_i \in \mathbb{Q}[\mathtt{x}_1, \dots, \mathtt{x}_d, \mathtt{t}] \setminus \{0\}$ such that

$$f^{2m+1} + \Big(\sum_{i=1}^p r_i g_i^2\Big) f = \sum_{i=1}^p s_i h_i^2.$$

As $\kappa_1=\mathbb{Q}(\mathtt{x}_1,\dots,\mathtt{x}_d)[\mathtt{u}]/(f)$ is a (formally) real field, $h_i=0$ modulo (f), so f divides h_i for $i=1,\dots,p$. Let $k\geqslant 1$ be the largest power of f that divides $h_1,\dots,h_p$. Write $h_i:=f^kh_i'$ for some $h_i'\in\mathbb{Q}[\mathtt{x}_1,\dots,\mathtt{x}_d,\mathtt{u}]$ and

$$f^{2m}+\Big(\sum_{i=1}^p r_ig_i^2\Big)=f^{2k-1}\sum_{i=1}^p s_ih_i'^2,$$

so f divides $\sum_{i=1}^p r_ig_i^2$. We may assume f does not divide h_1'. As $\kappa[\mathtt{t}]/(P)=\mathbb{Q}(\mathtt{x}_1,\dots,\mathtt{x}_d)[\mathtt{u}]/(f)$ is a (formally) real field, $g_i=0$ modulo (f), so f divides g_i for $i=1,\dots,p$. Let $\ell\geqslant 1$ be the largest power of f that divides $g_1,\dots,g_p$. Write $g_i:=f^\ell g_i'$ for some $g_i'\in\mathbb{Q}[\mathtt{x}_1,\dots,\mathtt{x}_d,\mathtt{u}]$ and

$$f^{2m-1}+f^{2\ell-1}\Big(\sum_{i=1}^p r_ig_i'^2\Big)=f^{2k-2}\sum_{i=1}^p s_ih_i'^2.$$

We may assume f does not divide g_1'. If $2m-1\leqslant\min\{2\ell-1,2k-2\}$, then

$$1+f^{2\ell-2m}\Big(\sum_{i=1}^p r_ig_i'^2\Big)=f^{2k-2m-1}\sum_{i=1}^p s_ih_i'^2,$$

which contradicts the fact that $\mathbb{Q}(\mathtt{x}_1,\dots,\mathtt{x}_d)[\mathtt{u}]/(f)$ is a (formally) real field. Thus, $2m-1>\min\{2\ell-1,2k-2\}$. If $2\ell-1\geqslant 2k-2$, then

$$f^{2m+1-2k}+f^{2\ell+1-2k}\Big(\sum_{i=1}^p r_ig_i'^2\Big)=\sum_{i=1}^p s_ih_i'^2.$$

which contradicts the fact that $\mathbb{Q}(\mathtt{x}_1,\dots,\mathtt{x}_d)[\mathtt{u}]/(f)$ is a (formally) real field, because f does not divide h_1'. Thus, $2\ell-1<2k-2$ and

$$f^{2m-2\ell}+\Big(\sum_{i=1}^p r_ig_i'^2\Big)=f^{2k-2\ell-1}\sum_{i=1}^p s_ih_i'^2.$$

which contradicts the fact that $\mathbb{Q}(\mathtt{x}_1,\dots,\mathtt{x}_d)[\mathtt{u}]/(f)$ is a (formally) real field, because f does not divide g_1'.

We conclude f change sign in $\mathbb{R}^{d+1}$.

Write $f:=f_0+\mathtt{u}f_1+\cdots+\mathtt{u}^ef_e$ where $f_j\in\mathbb{Q}[\mathtt{x}_1,\dots,\mathtt{x}_d]$ and $e\geqslant 1$. By [BCR, Thm.4.5.1] the zero ideal of $X=\{(x',u)\in\mathbb{R}^{d+1}:\ f(x',u)=0\}$ is the ideal generated by f. If $\frac{\partial f}{\partial\mathtt{t}}(x',u)=0$ for each $(x',u)\in\{f=0\}$, then f divides $\frac{\partial f}{\partial\mathtt{u}}$ in $\mathbb{R}[\mathtt{x}_1,\dots,\mathtt{x}_d][\mathtt{u}]$, so $\frac{\partial f}{\partial\mathtt{u}}=fg$ for some $g\in\mathbb{R}[\mathtt{x}_1,\dots,\mathtt{x}_d,\mathtt{u}]$, which contradicts the fact that $\deg_{\mathtt{u}}(\frac{\partial f}{\partial\mathtt{u}})\leqslant\deg_{\mathtt{u}}(f)-1$. Thus, there exists a point $(x_0',u_0)\in\mathbb{R}^{d+1}$ such that $f(x_0',u_0)=0$ and $\frac{\partial f}{\partial\mathtt{u}}(x_0',u_0)\neq 0$. Consider the function $u\mapsto f(x_0',u)$, which is strictly monotone on an open interval containing u_0, so f changes sign at (x_0',u). By [EGH, Lem.2.3] there exists a dense transcendence basis $\mathcal{B}$ of $\mathbb{R}$ over $\mathbb{Q}$. By the Implicit Function Theorem [BCR, Cor.2.9.8] there exist open neighborhood $U\subset\mathbb{R}^d$ of x_0 and $V\subset\mathbb{R}$ of t_0 and a Nash function $\varphi:U\to\mathbb{R}$ such that $(U\times V)\cap\{f=0\}=\operatorname{graph}(f)$. Pick a point $x:=(x_1,\dots,x_d)\in\mathcal{B}^d\cap V$ such that $c(x_1,\dots,x_d)\neq 0$, $f_e(x_1,\dots,x_d)\neq 0$ and let $u:=\varphi(x)$. As $x\in\mathcal{B}^d$ and u is a root of $f(x,\mathtt{u})$, we deduce that $\mathbb{Q}[\mathtt{x}_1,\dots,\mathtt{x}_d,\mathtt{u}]/(f)\cong\mathbb{Q}[x,u]\subset\mathbb{R}$. Thus, there exist (injective) homomorphisms

$$\mathbb{Q}\hookrightarrow\kappa\hookrightarrow\kappa_1=\mathbb{Q}(\mathtt{x}_1,\dots,\mathtt{x}_d)[\mathtt{t}]/(f)\hookrightarrow\mathbb{R}.$$

and let $\alpha\in\operatorname{Sper}(\kappa)$ be the ordering induced by the previous homomorphism. Observe that $\mathfrak{R}(\alpha)\subset\mathbb{R}$, so the ordering of $\mathfrak{R}(\alpha)$ is Archimedean and κ is dense in $\mathfrak{R}(\alpha)$, so $\alpha\in\mathfrak{D}_\kappa$. Observe in addition that $p_1>_\alpha 0,\dots,p_r>_\alpha 0$ (because $p_j=(\sqrt{p_j})^2>_\alpha 0$ in $\kappa_1\hookrightarrow\mathbb{R}$ for $j=1,\dots,r$), that is, $\alpha\in\mathcal{U}$. Consequently, $\alpha\in\mathcal{U}\cap\mathfrak{D}_\kappa\neq\varnothing$, so $\mathfrak{D}_\kappa$ is dense in $\operatorname{Sper}(\kappa)$, as required. $\square$

Remark 14.13. The previous proof works *verbatim* if we substitute $\mathbb{Q}$ by a countable (formally) real field that admits a unique ordering and such ordering is in addition Archimedean. ▪

14.3.2. *Fields of iterated Laurent power series that admit no chimeric polynomials.* We state next that many fields of iterated Laurent power series do not admit chimeric polynomials. This provides further examples of (formally) real fields κ with no chimeric polynomials for which $\mathfrak{D}_\kappa$ is not dense in $\operatorname{Sper}(\kappa)$. As the proof of the following result is quite technical and involve many computations (that make it cumbersome), we postpone it until Appendix D.

nonarch **Theorem 14.14.** *Let κ be a (formally) real field. We have:*

(i) *If $P \in \kappa[\mathtt{t}]$ is a chimeric polynomial over κ, then $P \in \kappa((\mathtt{x}))[\mathtt{t}]$ is a chimeric polynomial over $\kappa((\mathtt{x}))$.*
(ii) *Suppose that all the orderings of κ are Archimedean. Then there exists no chimeric polynomial over $\kappa((\mathtt{x}_1))\cdots((\mathtt{x}_n))$ for each $n \geqslant 1$.*

bapa

14.4. **Elementary equivalent fields and chimeric polynomials.** We prove next (under mild conditions) that if a (formally) real field κ admits no chimeric polynomials (over κ), then any field κ_1 elementarily equivalent to κ admits no chimeric polynomials (over κ_1).

Definition 14.15. We say that two fields κ_1 and κ_2 are *elementarily equivalent* if they satisfy the same first order formulas.

14.4.1. *Preliminaries on the τ-invariant and Pythagoras numbers.* Fix a (formally) real field κ. For each $d \geqslant 1$ denote

$$\tau_{2d}(\kappa) := \sup\{s(F) : \ [F:\kappa] \leqslant 2d, \text{ non (formally) real}\}$$

where $s(F)$ is the level of F. We have $\tau_{2e}(\kappa) \leqslant \tau_{2d}(\kappa)$ if $e \leqslant d$ and $\tau(\kappa) = \sup\{\tau_{2d}(\kappa) : \ d \geqslant 1\}$.

For each $d \geqslant 1$ denote $\kappa_d[\mathtt{t}]$ the κ-linear space of polynomial of $\kappa[\mathtt{t}]$ of degree $\leqslant d$. Denote

$$p_{2d}(\kappa[\mathtt{t}]) = \inf\{p \geqslant 1 : \ \Sigma\kappa[\mathtt{t}]^2 \cap \kappa_{2d}[\mathtt{t}] = \Sigma_p\kappa[\mathtt{t}]^2 \cap \kappa_{2d}[\mathtt{t}]\}$$

We have $p_{2e}(\kappa[\mathtt{t}]) \leqslant p_{2d}(\kappa[\mathtt{t}])$ if $e \leqslant d$ and $p(\kappa[\mathtt{t}]) = \sup\{p_{2d}(\kappa[\mathtt{t}]) : \ d \geqslant 1\}$.

Remark 14.16. A natural question is if the condition $p(\kappa[\mathtt{t}]) < +\infty$ is stronger than $p_{2d}(\kappa[\mathtt{t}]) < +\infty$ for each $d \geqslant 1$.

(i) If $p(\kappa) = +\infty$, we choose elements $a_p \in \Sigma\kappa^2 \setminus \Sigma_{p-1}\kappa^2$ for each $p \geqslant 1$ and consider the polynomials $F_{2d,p} := \mathtt{t}^{2d} + a_p$ for each $d \geqslant 1$ and $p \geqslant 1$. Observe that $F_{2d,p} \in (\Sigma\kappa[\mathtt{t}]^2 \cap \kappa_{2d}[\mathtt{t}]) \setminus \Sigma_{p-1}\kappa[\mathtt{t}]^2$. Consequently, $p_{2d}(\kappa[\mathtt{t}]) = +\infty$ for each $d \geqslant 1$ and $p(\kappa[\mathtt{t}]) = +\infty$.

(ii) We do not know if there exists a field κ with $p(\kappa) < +\infty$, $p_{2d}(\kappa[\mathtt{t}]) < +\infty$ for each $d \geqslant 1$ and $p(\kappa[\mathtt{t}]) = +\infty$. Equivalently, we do not know if there exists a field κ with $p(\kappa) < +\infty$, $\tau_{2d}(\kappa) < +\infty$ for each $d \geqslant 1$ and $\tau(\kappa) = +\infty$. ∎

By Cassels [C] a polynomial $f \in \kappa_{2d}[\mathtt{t}] \cap \Sigma_p\kappa(\mathtt{t})^2$ if and only if $f \in \kappa_{2d}[\mathtt{t}] \cap \Sigma_p\kappa[\mathtt{t}]^2$. Following the proof of [Pf2, Satz 2] and by [L, Ch.IX.Cor.2.3] one deduces

$$1 + \tau_{2d}(\kappa) \leqslant p_{2d}(\kappa[\mathtt{t}]) \leqslant 2\tau_{2d}(\kappa). \tag{14.1}$$ tpd

In any case, we provide a detailed proof of the previous inequalities in Appendix B for the sake of completeness.

14.4.2. *Several first order language formulas.* Denote $\mathtt{w} := (\mathtt{w}_d, \ldots, \mathtt{w}_1, \mathtt{w}_0)$ and consider the polynomial $G_{d,\mathtt{w}} := \mathtt{w}_d\mathtt{t}^d + \cdots + \mathtt{w}_1\mathtt{t} + \mathtt{w}_0 \in \mathbb{Z}[\mathtt{w}, \mathtt{t}]$. Let us provide several first order language formulas to express some of the properties we need:

Formula 0. *Finite Pythagoras numbers and (formally) real fields.* The formula ζ_p that express that a field κ has finite Pythagoras number p is

$$\forall \mathtt{x}_1, \ldots, \mathtt{x}_{p+1} \exists \mathtt{y}_1, \ldots, \mathtt{y}_p \Big(\sum_{j=1}^{p+1} \mathtt{x}_j^2 = \sum_{i=1}^{p} \mathtt{y}_i^2 \Big).$$

This property allows to represent recursively (in finitely many steps) each element of $\Sigma\kappa^2$ as an element of $\Sigma_p\kappa^2$. If κ is a field with $p(\kappa)=p$, the formula τ_p that express that κ is (formally) real is

$$\neg\Big(\exists \mathtt{x}_1,\dots,\mathtt{x}_p\Big(-1=\sum_{i=1}^p \mathtt{x}_i\Big)\Big).$$

frf **Lemma 14.17.** *Let κ be a field of Pythagoras number p and let κ_1 be a field elementarily equivalent to κ. Then κ is formally real if and only if κ_1 is (formally) real.*

Proof. As κ_1 is elementarily equivalent to κ, we deduce $p(\kappa_1)=p(\kappa)=p$ (because κ_1 satisfies the formula ζ_p as κ satisfies formula ζ_p). In addition, $\kappa\models\tau_p$ if and only if $\kappa_1\models\tau_p$. Consequently, κ is formally real if and only if κ_1 is (formally) real, as required. □

Formula 1. *Irreducibility of $G_{d,a}$ for $a:=(a_d,\dots,a_0)\in\kappa^{d+1}$.* Let $\varphi_{d,\ell}$ be the formula of first order in the language of rings that express that $G_{d,a}$ is not the product of a polynomial of degree ℓ and a polynomial of degree $d-\ell$:

$$\neg\exists \mathtt{y}:=(\mathtt{y}_\ell,\dots,\mathtt{y}_0),\mathtt{z}:=(\mathtt{z}_{d-\ell},\dots,\mathtt{z}_0)$$
$$\forall \mathtt{t}\Big(\bigvee_{i=0}^{\ell}\mathtt{y}_i\neq 0\wedge\bigvee_{i=0}^{d-\ell}\mathtt{z}_i\neq 0\wedge(G_{d,\mathtt{w}}(\mathtt{w},\mathtt{t})=G_{\ell,\mathtt{y}}(\mathtt{t})\cdot G_{d-\ell,\mathtt{z}}(\mathtt{t})\Big)$$

where the formula $G_{d,\mathtt{w}}(\mathtt{t})=G_{\ell,\mathtt{y}}(\mathtt{t})\cdot G_{d-\ell,\mathtt{z}}(\mathtt{t})$ can be written as the conjunction of finitely many equalities that arise when making equal the coefficient of $\mathtt{t}^j$ in the left hand side with the coefficient of $\mathtt{t}^j$ in the right hand side. The formula that express the irreducibility of $G_{d,a}$ in $\kappa[\mathtt{t}]$ is $\phi_d:=\bigwedge_{\ell=1}^{\lfloor d/2\rfloor}\varphi_{d,\ell}$. Thus,

$$\{a\in\kappa^{d+1}:\ G_{d,a}\text{ is irreducible in }\kappa[\mathtt{t}]\}=\{a\in\kappa^{n+1}:\ \kappa\models\phi_d(a)\}.$$

Formula 2. *-1 is a sum of 2^m squares in $\kappa[\mathtt{t}]/(G_{2d,p})$ for $a:=(a_{2d},\dots,a_0)\in\kappa^{2d+1}$.* Let $\psi_{2d,m}$ be the formula of first order in the language of rings that express the existence of tuples $q_j:=(q_{j,2d-1},\dots,q_{j,0})\in\kappa^{2d}$ for $j=1,\dots,2^m$ and $s:=(s_{2d-2},\dots,s_0)\in\kappa^{2d-1}$ such that

$$-1=\sum_{j=1}^{2^m}G_{2d-1,q_j}^2+G_{2d,\mathtt{w}}G_{2d-2,s},$$

that is,

$$\phi_{2d,m}(\mathtt{w}):=\exists\mathtt{y}_1,\dots,\mathtt{y}_{2^m},\mathtt{s}\Big(-1=\sum_{j=1}^{2^m}G_{2d-1,\mathtt{y}_j}^2+G_{2d,\mathtt{w}}G_{2d-2,\mathtt{s}}\Big).$$

Then,

$$\{a:=(a_{2d},\dots,a_0)\in\kappa^{2d+1}:\ L_{2d,a}:=\kappa[t]/(G_{2d,a})\text{ is a non (formally) real field}$$
$$\text{and }-1\in\Sigma_{2^m}L_{2d,a}^2\}=\{a:=(a_n,\dots,a_0)\in\kappa^{2d+1}:\ \kappa\models\phi_{2d}(a)\wedge\psi_{2d,m}(a)\}.$$

Remark 14.18. The field κ satisfies $\tau_{2d}(\kappa)\leqslant 2^m$ if for each $a:=(a_{2d},\dots,a_0)\in\kappa^{2d+1}$ such that $G_{2d,a}$ is irreducible and $L_{2d,a}:=\kappa[t]/(G_{2d,a})$ is a non (formally) real field, the formula $\psi_{2d,m}(a)$ holds. ▪

t2d **Lemma 14.19.** *Let κ and κ_1 be elementarily equivalent fields. If $\tau_{2d}(\kappa)\leqslant 2^m$, then $\tau_{2d}(\kappa_1)\leqslant 2^m$.*

Proof. Suppose there exists $a\in\kappa_1^{2d+1}$ such that $G_{2d,a}$ is irreducible (so $\phi_{2d}(a)$ holds) and $L_{1,a}:=\kappa_1[t]/(G_{2d,a})$ is a non (formally) real field, but the formula $\psi_{2d,m}(a)$ does not hold. As $L_{1,a}$ is a non (formally) real field, there exists $\ell\geqslant 1$ such that $\psi_{2d,\ell}(a)$ holds. Thus, in $\kappa_1\models\exists\mathtt{w}(\phi_{2d}(\mathtt{w})\wedge\psi_{2d,\ell}(\mathtt{w})\wedge\neg\psi_{2d,m}(\mathtt{w}))$. As κ_1 is elementarily equivalent to κ, we have $\kappa\models\exists\mathtt{w}(\phi_{2d}(\mathtt{w})\wedge$

$\psi_{2d,\ell}(\mathtt{w}) \wedge \neg\psi_{2d,m}(\mathtt{w}))$. Consequently, there exists $b \in \kappa^{2d+1}$ such that $\phi_{2d}(b) \wedge \psi_{2d,\ell}(b) \wedge \neg\psi_{2d,m}(b)$, that is, $G_{2d,b} \in \kappa[\mathtt{t}]$ is irreducible, the field $L_b = \kappa[t]/(G_{2d,b})$ is not (formally) real, but -1 is not a sum of 2^m squares in $L_{p'}$, which is a contradiction, because $\tau_d(\kappa) \leqslant 2^m$. □

As a straightforward consequence of the previous lemma, we have the following result.

Corollary 14.20. *Let κ and κ_1 be elementarily equivalent fields. We have:*

(i) *If $\tau_{2d}(\kappa) < +\infty$ for all $d \geqslant 1$, then $\tau_{2d}(\kappa_1) = \tau_{2d}(\kappa) < +\infty$ for all $d \geqslant 1$.*
(ii) *If $\tau(\kappa) < +\infty$, then $\tau(\kappa_1) = \tau(\kappa) < +\infty$.*

FORMULA 3. *$\tau_d(\kappa) \leqslant 2^m$ and the field $\kappa[t]/(G_{d,p})$ is (formally) real.* If we assume $\tau_d(\kappa) \leqslant 2^m$, we have the following equality

$$\{p \in \kappa^{d+1} : \ \kappa[t]/(G_{d,p}) \text{ is (formally) real}\} = \{p \in \kappa^{d+1} : \kappa \models \phi_d(p) \vee \neg\psi_{2d,m}(p)\}$$

FORMULA 4. *Every polynomial $G_{2d,a} \in \Sigma\kappa^2[\mathtt{t}] \cap \kappa_{2d}[\mathtt{t}]$ is a sum of p squares in $\kappa[\mathtt{t}]$.* Consider the sentence $\chi_{2d,p}$

$$\forall \mathtt{w}\Big(\exists \mathtt{z}_1,\ldots,\mathtt{z}_{p+1}\forall \mathtt{t}\Big(G_{2d,\mathtt{w}}(\mathtt{t}) = \sum_{i=1}^{p+1} G_{d,\mathtt{z}_i}^2\Big) \to \Big(\exists \mathtt{y}_1,\ldots,\mathtt{y}_p\forall \mathtt{t}\Big(G_{2d,\mathtt{w}}(\mathtt{t}) = \sum_{j=1}^{p} G_{d,\mathtt{y}_j}^2\Big)\Big)$$

Suppose the field $\kappa \models \chi_{2d,p}$ and denote $\Sigma_0\kappa[\mathtt{t}]^2 = \{0\}$. Let $a \in \kappa^{2d+1}$ be such that $G_{2d,a} \in \Sigma\kappa[\mathtt{t}]^2$. Then there exists $q \geqslant p+1$ such that $G_{2d,a} \in \Sigma_q\kappa[\mathtt{t}]^2$. Then there exists $a', a'' \in \kappa^{2d+1}$ such that $G_{2d,a} = G_{2d,a'} + G_{2d,a''}$, $G_{2d,a'} \in \Sigma_{p+1}\kappa[\mathtt{t}]^2 = \Sigma_p\kappa[\mathtt{t}]^2$ (because $\kappa \models \chi_{2d,p}$) and $G_{2d,a''} \in \Sigma_{q-p}\kappa[\mathtt{t}]^2$, so $G_{2d,a} \in \Sigma_{q-1}\kappa[\mathtt{t}]^2$ and proceeding recursively $G_{2d,a} \in \Sigma_p\kappa[\mathtt{t}]^2$. Thus, $p_{2d}(\kappa[\mathtt{t}]) \leqslant p$. Conversely, if $p_{2d}(\kappa[\mathtt{t}]) \leqslant p$, then $\kappa \models \chi_{2d,p}$.

p2d **Lemma 14.21.** *Let κ be a field such that $p_{2d}(\kappa[\mathtt{t}]) < +\infty$ for some $d \geqslant 1$. If κ_1 is elementarily equivalent to κ, then $p_{2d}(\kappa_1[\mathtt{t}]) = p_{2d}(\kappa[\mathtt{t}])$.*

Proof. Write $p := p_{2d}(\kappa[\mathtt{t}])$. Then $\kappa \models \chi_{2d,p}$, so $\kappa_1 \models \chi_{2d,p}$ (because κ_1 is elementarily equivalent to κ). Thus, $p_{2d}(\kappa_1[\mathtt{t}]) \leqslant p$. If $p_{2d}(\kappa_1[\mathtt{t}]) \leqslant p-1$, then $\kappa_1 \models \chi_{2d,p-1}$, so $\kappa \models \chi_{2d,p-1}$ and $p = p_{2d}(\kappa[\mathtt{t}]) \leqslant p-1$, which is a contradiction. Consequently, $p_{2d}(\kappa_1[\mathtt{t}]) = p$, as required. □

As a straightforward consequence of the previous lemma, we have the following result.

Corollary 14.22. *Let κ and κ_1 be elementarily equivalent fields. We have:*

(i) *If $p_{2d}(\kappa[\mathtt{t}]) < +\infty$ for all $d \geqslant 1$, then $p_{2d}(\kappa_1[\mathtt{t}]) < +\infty$ for all $d \geqslant 1$.*
(ii) *If $p(\kappa[\mathtt{t}]) < +\infty$, then $p(\kappa_1[\mathtt{t}]) < +\infty$.*

FORMULA 5. *The polynomial $G_{4,a}$ is chimeric for $a := (0, -2b, -4c^2, d) \in \kappa^3$.* Recall that the polynomial $G_{4,a} := \mathtt{t}^4 - 2b\mathtt{t}^2 - 4c^2\mathtt{t} + d \in \kappa[\mathtt{t}]$ is chimeric if:

(i) It is irreducible (that is, it satisfies formula $\phi_4(a)$).
(ii) $L_a := \kappa[\mathtt{t}]/(G_{4,a})$ is a (formally) real field.
(iii) For each $Q \in \mathcal{P}(L_a) = \Sigma L_a^2$ there exists $\mu \in \kappa$ such that $Q + \mu G_{4,a} \in \Sigma\kappa[\mathtt{t}]^2$.

If $\tau_4(\kappa) = 2^m < +\infty$, the condition (ii) is expressed by $\neg\psi_m(a)$. Let us express the condition (iii) if $p_6 := p_6(\kappa[t]) < +\infty$. Denote also $p_4 := p_4(\kappa[t]) \leqslant p_6$. As L_a is a (formally) real field, $\mathcal{P}(L_a) = \Sigma L_a^2$. The representatives of the elements of L_a have degree $\leqslant 3$, so a sum of squares in ΣL_a^2 can be represented by a sum of squares of polynomials of degree 3, that is an element of

$\Sigma\kappa[\mathtt{t}]^2 \cap \kappa_6[\mathtt{t}] = \Sigma_{p_6}\kappa[\mathtt{t}]^2 \cap \kappa_6[\mathtt{t}]$. Consequently, if $\mathtt{v} := (0, -2\mathtt{v}_2, -4\mathtt{v}_1^2, \mathtt{v}_0)$, condition (iii) can be written using the following formula $\xi_{p_6}(\mathtt{v})$:

$$\forall \mathtt{w}\Big(\exists \mathtt{z}_1, \dots \mathtt{z}_{p_6}\exists \mathtt{u}\forall \mathtt{t}\Big(G_{3,\mathtt{w}} + G_{2,\mathtt{u}}G_{4,\mathtt{v}} = \sum_{j=1}^{p_6} G_{3,\mathtt{z}_j}^2\Big)\Big)$$
$$\to \Big(\exists \mathtt{y}_1, \dots \mathtt{y}_{p_4}\exists \mathtt{x}\forall \mathtt{t}\Big(G_{3,\mathtt{w}} + \mathtt{x}G_{4,\mathtt{v}} = \sum_{j=1}^{p_4} G_{2,\mathtt{y}_j}^2\Big)\Big)$$

We have the following result:

kk1chi **Lemma 14.23.** *Let κ be a (formally) field such that $\tau_4(\kappa) = 2^m < +\infty$ and $p_6 := p_6(\kappa[\mathtt{t}]) < +\infty$ and let κ_1 be a field elementarily equivalent to κ. Then there exists no chimeric polynomial over κ if and only if there exists no chimeric polynomial over κ_1.*

Proof. As $p_6 < +\infty$, we have $p(\kappa) \leqslant p_6 < +\infty$, so by Lemma 14.17 also κ_1 has $p(\kappa_1) = p(\kappa) < +\infty$ and it is (formally) real. By Lemmas 14.19 and 14.21 we have $\tau_4(\kappa_1) < +\infty$ and $p_6(\kappa_1[\mathtt{t}]) < +\infty$. Consider the sentence χ_{m,p_6} given by

$$\neg\exists \mathtt{v}(\phi_4(\mathtt{v}) \wedge \neg\psi_{4,m}(\mathtt{v}) \wedge \xi_{p_6}(\mathtt{v})),$$

where $\mathtt{v} := (0, -2\mathtt{v}_2, -4\mathtt{v}_1^2, \mathtt{v}_0)$. The previous sentence express that there exists no chimeric polynomials over a (formally) real field F with $\tau_4(F) \leqslant 2^m$ and $p_6(F[\mathtt{t}]) \leqslant p_6$. As κ_1 be a field elementarily equivalent to κ, we have $\kappa \models \chi_{m,p_6}$ if and only if $\kappa_1 \models \chi_{m,p_6}$. Consequently, there exists no chimeric polynomial over κ if and only if there exists no chimeric polynomial over κ_1, as required. □

Remark 14.24. We deduce from formula (14.1) that $\tau_6(\kappa) < +\infty$ implies $p_6 \leqslant \tau_6(\kappa) < +\infty$ (and also $\tau_4(\kappa) \leqslant \tau_6(\kappa) < +\infty$ and $p_4 \leqslant p_6 < +\infty$). Thus, in the hypothesis of Lemma 14.23 it is enough to ask either $\tau_6(\kappa) < +\infty$ or (equivalently) $p_6(\kappa[\mathtt{t}]) < +\infty$. ▪

14.4.3. *Ultraproducts of fields that admit no chimeric polynomials.* We recall next how ultraproducts of fields are constructed. We follow [CK, Ch.4] and we propose the reader this book for further reference. Let I be a non-empty set and let D be an ultrafilter over I. For each $i \in I$ let κ_i be a field. Consider the cartesian product $\prod_{i\in I}\kappa_i$. We say that two elements $x := (x_i)_{i\in I}, y := (y_i)_{i\in I} \in \prod_{i\in I}\kappa_i$ of the cartesian product are equivalent $x =_D y$ if and only if $\{i \in I : \ x_i = y_i\} \in D$. The previous relation is an equivalence relation and we consider the quotient set $\prod_D \kappa_i$, which is called the *ultraproduct of the fields κ_i modulo D*. If all the field $\kappa_i = \kappa$ are the same field, we call $\prod_D \kappa$ the *ultrapower of κ modulo D*. We endow $\prod_{i\in I}\kappa_i$ with the estructure of product ring (coordinate by coordinate) and consider the set

$$\mathfrak{m} := \{x := (x_i)_{i\in I} \in \prod_{i\in I}\kappa_i : \ x_i = 0\} \in D.$$

It holds: *$\mathfrak{m}_D$ is an ideal of $\prod_{i\in I}\kappa_i$ and in fact it is a maximal ideal* [CK, Ex.4.1.30]. The map

$$\theta : \Big(\prod_{i\in I}\kappa_i\Big)/\mathfrak{m} \to \prod_D \kappa_i$$

is well-defined and bijective and endows $\prod_D \kappa_i$ with a structure of field. This type of construction was already used by Artin to prove the existence of the algebraic closure of a field.

The Theorem of Łoś [CK, Thm.4.1.9] states that for any sentence φ of the language of first order of rings $\prod_D \kappa_i \models \varphi$ if and only if $\{i \in I : \ \kappa_i \models \varphi\} \in D$. In particular, a field κ and any ultrapower of κ modulo an ultrafilter D are elementarily equivalent [CK, Cor.4.1.10]. By [CK, Cor.4.1.13] the natural (diagonal) embedding $\kappa \hookrightarrow \prod_D \kappa$ provides an elementary embedding.

We have the following two results concerning the non existence of chimeric polynomials over ultraproducts and ultrapowers:

ultra1 **Corollary 14.25.** *Let $\{\kappa_i\}_{i\in I}$ be a collection of (formally) real fields and let $m \geqslant 1$ such that $\tau_6(\kappa_i) = 2^m$ for each $i \in I$. Suppose there exists no chimeric polynomials over κ_i for each $i \in I$ and let D be an ultrafilter over I. Then there exist no chimeric polynomials over $\prod_D \kappa_i$.*

ultra2 **Corollary 14.26.** *Let κ be a (formally) real field such that $\tau_6(\kappa) < +\infty$ and there exists no chimeric polynomials over κ and let I be a non empty set. For each ultrafilter D over I there exists no chimeric polynomials over the ultrapower $\prod_D \kappa$.*

Remark 14.27. By [CK, Prop.4.3.7] we can construct from a (formally) real field κ with $\tau_6(\kappa) < +\infty$ and no chimeric polynomials over κ new (formally) real fields κ_α of cardinality $\alpha > \operatorname{Card}(\kappa)$ with $\tau_6(\kappa_\alpha) = \tau_6(\kappa) < +\infty$ and no chimeric polynomials over κ_α. ▪

Part 4. Higher embedding dimension and global-local results

s15 15. Examples in higher embedding dimension

We begin recalling some aspects about rational singularities and observing that all the singularities on List 2.1 are either rational singularities or limits of rational singularities.

15.1. **Rational singularities.** As Pignatelli pointed out (when a part of these results where presented in a seminar in Trento in 2022), the complete rings in List 2.1(3) that have isolated singularities correspond to 2-dimensional *rational (hypersurface) singularities* [A, Li] of multiplicity 2 (also known as *Du Val singularities*). He also observed that if $\overline{\kappa}$ is the algebraic closure of the (formally) real field κ, for each rational singularity $B_G := \overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2 - G(\mathtt{x},\mathtt{y}))$ of multiplicity 2 (and embedding dimension 3) over $\overline{\kappa}$ there exists $A_F := \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2 - F(\mathtt{x},\mathtt{y}))$ in List 2.1(3) such that $A_F \otimes_{\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]} \overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]] \cong B_G$. In addition, the complete rings in List 2.1(3) that do not have isolated singularities correspond to "limit" singularities of families of rational singularities of multiplicity 2 (see [vS, pp.6, 121]).

Recall that by [A, Cor.6] the *embedding dimension* $\operatorname{emb\,dim}(A)$ of a rational singularity A equals its *multiplicity* $\operatorname{mult}(A)$ plus 1. In addition, rational singularities $A := \kappa[[\mathtt{x}_1,\ldots,\mathtt{x}_n]]/\mathfrak{a}$ are isolated singularities [A], [Li, Thm.4.1], Cohen-Macaulay [W, (2.6)] and the ideal $\mathfrak{a}$ is generated by $\binom{n-1}{2}$ formal series of order 2 whose leading (quadratic) forms are κ-linearly independent [W, Rmk.2.2.1]. In §16 we propose some obstructions that must satisfy local henselian excellent rings A of dimension 2 of higher embedding dimension such that $\mathcal{P}(A) = \Sigma A^2$. Our best result in general embedding dimension n is Corollary 16.1 where we show that if $A := \kappa[[\mathtt{x}_1,\ldots,\mathtt{x}_n]]/\mathfrak{a}$ is a complete ring of embedding dimension n that satisfies $\mathcal{P}(A) = \Sigma A^2$, there exist, after a change of coordinates, sums of squares $\sigma_k \in \Sigma\kappa^2\setminus\{0\}$ and a series $F \in \kappa[[\mathtt{x},\mathtt{y}]]$ belonging to List 2.1 such that $\mathtt{x}_3^2 - F(\mathtt{x}_1,\mathtt{x}_2) + \sum_{k=4}^n \sigma_k \mathtt{x}_k^2 \in \mathfrak{a}$. In Subsections §15.2 and §15.3 we present several families of examples in higher embedding dimension of 2-dimensional complete rings A with the property $\mathcal{P}(A) = \Sigma A^2$:

(i) one of the families corresponds to Veronese cones (over rational normal curves), which are rational singularities (Examples 15.2),
(ii) another family corresponds to generalizations (for higher embedding dimension) of Whitney's umbrella (which are not isolated singularities, Examples 15.3), so they are not rational singularities,
(iii) some 2-dimensional rings of the type $\kappa[[\mathtt{x}_1,\ldots,\mathtt{x}_n]]/\mathfrak{a}$, where $\mathfrak{a}$ is generated by square-free monomials of degree 2 (Theorem 15.17).

As we comment in §15.2, the irreducible singularities in items (i) and (ii) are Cohen-Macaulay, but not Goresntein, when the embedding dimension is $\geqslant 4$. However, there are many reducible 2-dimensional singularities with the property psd=sos when the embedding is $\geqslant 4$ that are not Cohen-Macaulay (Lemma 15.10 and Remark 15.11).

Question 15.1. Natural questions (whose answer is unknown for us) are the following: *Let $A := \kappa[[\mathtt{x}_1, \dots, \mathtt{x}_n]]/\mathfrak{a}$ be a complete integral domain of multiplicity m and embedding dimension n with the property $\mathcal{P}(A) = \Sigma A^2$:*

(i) *Is A a Cohen-Macaulay ring?*
(ii) *Does the inequality $m \leqslant n-1$ hold?*
(iii) *Is A generated by $\binom{n-1}{2}$ formal series of order 2 whose leading (quadratic) forms are κ-linearly independent?*
(iv) *If A has an isolated singularity, does A correspond to a rational singularity?*
(v) *If A has a rational singularity with (formally) real residue field κ, does there exists a complete ring A' with residue field κ such that $\mathcal{P}(A') = \Sigma A'^2$ and $A \otimes_{\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]} \overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]] \cong A' \otimes_{\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]} \overline{\kappa}[[\mathtt{x},\mathtt{y},\mathtt{z}]]$?*
(vi) *If A does not have an isolated singularity, does A correspond to a "limit" of a family of rational singularities (of the same multiplicity)?*

irredn

15.2. **Irreducible examples in higher embedding dimensions.** In this section we prove that for each embedding dimension and each (formally) real field κ there exist local excellent henselian integral domains of dimension 2 such that every positive semidefinite element is a sum of squares. We present two families, one of rational singularities and another one of non-rational singularities.

vrs *Example* 15.2 (Veronese's rational singularities). Consider the ring κ-homomorphism

$$\varphi_n : \kappa[[\mathtt{x}_0, \dots, \mathtt{x}_n]] \to \kappa[[\mathtt{s},\mathtt{t}]],\ f \mapsto f(\mathtt{s}^n, \mathtt{s}^{n-1}\mathtt{t}, \dots, \mathtt{s}\mathtt{t}^{n-1}, \mathtt{t}^n),$$

which is an integral inclusion. Define $\mathfrak{p}_n := \ker(\varphi_n)$, which is a real prime ideal of $\kappa[[\mathtt{x}_0, \dots, \mathtt{x}_n]]$ such that $A_n := \kappa[[\mathtt{x}_0, \dots, \mathtt{x}_n]]/\mathfrak{p}_n$ is a 2-dimensional real (reduced) ring. The real prime ideal $\mathfrak{p}_n$ is generated by the $\binom{n}{2}$ homogeneous polynomials

$$F_{ij} := \mathtt{x}_i\mathtt{x}_j - \mathtt{x}_{i-1}\mathtt{x}_{j+1}, \quad 1 \leqslant i \leqslant j \leqslant n-1.$$

One can check that A_n has multiplicity n (using Van der Monde determinant one shows that $n+1$ different points of the zero set of $\mathfrak{p}_n$ in $\overline{\kappa}^n$ are $\overline{\kappa}$-linearly independent), embedding dimension $n+1$ (because $\omega(\mathfrak{p}_n) = 2$) and it is a (determinantal) rational singularity [W, §2-3]. We prove below $\mathcal{P}(A_n) = \Sigma A_n^2$ (Theorem 15.6) when $\tau(\kappa) < +\infty$. These real (reduced) rings A_n are not complete intersections if $n \geqslant 3$, but they are at least normal, hence Cohen-Macaulay (but not Gorenstein [W, (2.5)]). In particular, $A_2 := \kappa[[\mathtt{x}_0,\mathtt{x}_1,\mathtt{x}_2]]/(\mathtt{x}_1^2 - \mathtt{x}_0\mathtt{x}_2)$ appears in List 2.1(3.i). If $i = j$, then $\mathtt{x}_i^2 - \mathtt{x}_{i-1}\mathtt{x}_{i+1} \in \mathfrak{a}$ is right equivalent to the element List 2.1(3.i) (Corollary 16.1(iii)), whereas if $i \neq j$, then $\mathtt{x}_i\mathtt{x}_j - \mathtt{x}_{i-1}\mathtt{x}_{j+1}$ is right equivalent to $\mathtt{u}^2 + \mathtt{v}^2 - (\mathtt{x}^2 + \mathtt{y}^2)$ (Corollary 16.1(iii)). If $n \geqslant 3$, we have

$$\begin{cases} \sum_{i=1}^{\frac{n-1}{2}} \left(\mathtt{x}_i - \frac{1}{2}\mathtt{x}_0\right)^2 + \sum_{i=\frac{n+1}{2}}^{n-1} \left(\mathtt{x}_i - \frac{1}{2}\mathtt{x}_n\right)^2 - \frac{1}{4}\left(\frac{n-1}{2}\right)(\mathtt{x}_0^2 + \mathtt{x}_n^2) \in \mathfrak{q}_n & \text{if } n \text{ is odd,} \\ \sum_{k=1}^{\frac{n-2}{2}} \left(\mathtt{x}_{2k} - \frac{1}{2}\mathtt{x}_0 - \frac{1}{2}\mathtt{x}_n\right)^2 + \sum_{k=1}^{\frac{n}{2}} \mathtt{x}_{2k-1}^2 - \frac{1}{4}\left(\frac{n-2}{2}\right)\left(\left(\mathtt{x}_0 + \frac{n+2}{n-2}\mathtt{x}_n^2\right)^2 - \frac{8n}{(n-2)^2}\mathtt{x}_n^2\right) \in \mathfrak{q}_n & \\ & \text{if } n \geqslant 4 \text{ is even,} \end{cases}$$

which are right equivalent to

$$\begin{cases} \sum_{i=1}^{n-1} \mathtt{y}_i^2 - \frac{n-1}{2}(\mathtt{y}_0^2 + \mathtt{y}_n^2) & \text{if } n \text{ is odd,} \\ \sum_{i=1}^{n-1} \mathtt{y}_i^2 - \frac{n-2}{2}(\mathtt{y}_0^2 - \frac{n}{2}\mathtt{y}_n^2) & \text{if } n \text{ is even} \end{cases}$$

(Corollary 16.1(ii)). ∎

wnrs *Example* 15.3 (Whitney's non-rational singularities). Consider the κ-ring homomorphism

$$\psi_n : \kappa[[\mathtt{x}_0, \dots, \mathtt{x}_n]] \to \kappa[[\mathtt{s},\mathtt{t}]],\ f \mapsto f(\mathtt{s}, \mathtt{s}\mathtt{t}, \dots, \mathtt{s}\mathtt{t}^{n-1}, \mathtt{t}^n),$$

which is finite, injective and $(s)\kappa[[s,t]] \subset \operatorname{im}(\psi_n)$, because $\mathtt{s}^i\mathtt{t}^j = \mathtt{s}^{i-1}(\mathtt{s}\mathtt{t}^r)\mathtt{t}^{nq} = \psi_n(\mathtt{x}_0^{i-1}\mathtt{x}_r\mathtt{x}_n^q)$ where $j = nq + r$ and $0 \leqslant r < n$. Define $\mathfrak{q}_n := \ker(\psi_n)$, which is a real prime ideal of

$\kappa[[\mathtt{x}_0,\dots,\mathtt{x}_n]]$ such that $B_n:=\kappa[[\mathtt{x}_0,\dots,\mathtt{x}_n]]/\mathfrak{q}_n$ is a 2-dimensional real (reduced) ring. The real prime ideal $\mathfrak{q}_n$ is generated by the $\binom{n}{2}$ homogeneous polynomials

$$G_{ij}:=\mathtt{x}_i\mathtt{x}_j-\mathtt{x}_0\mathtt{x}_\ell\mathtt{x}_n^q:\quad 1\leqslant i\leqslant j\leqslant n-1,\ i+j=qn+\ell,\ 0\leqslant\ell\leqslant n-1.$$

One can check that B_n has multiplicity n (using Van der Monde determinant one shows that $n+1$ different points of the zero set of $\mathfrak{p}_n$ in $\overline{\kappa}^n$ are $\overline{\kappa}$-linearly independent), embedding dimension $n+1$ (because $\omega(\mathfrak{p}_n)=2$) and it is not a rational singularity, because it is non-isolated ($\operatorname{Sing}(B_n)=\{\mathtt{x}_0=0\}=\{\mathtt{x}_0=0,\dots,\mathtt{x}_{n-1}=0\}$), but it is a determinantal singularity. These real rings B_n are not complete intersections if $n\geqslant 3$, they are neither normal ($\mathtt{x}_1/\mathtt{x}_0$ is integral over B_n) nor Gorenstein (by Stanley's Criterion, [E, 21.14], [St]). On the positive, they are Cohen-Macaulay: $\operatorname{depth}(B_n)\leqslant\dim(B_n)=2$ and $\{\mathtt{x}_0,\mathtt{x}_n\}$ is a regular sequence. The first ring $B_2:=\kappa[[\mathtt{x}_0,\mathtt{x}_1,\mathtt{x}_2]]/(\mathtt{x}_1^2-\mathtt{x}_0^2\mathtt{x}_2)$ appears in List 2.1(v) and it is known as Whitney's umbrella singularity. We prove below: $\mathcal{P}(B_n)=\Sigma B_n^2$ *for each* $n\geqslant 2$ (Theorem 15.9). If $i=j$ and $2i=qn$, then $q=1$, n is even and $\mathtt{x}_i^2-\mathtt{x}_0^2\mathtt{x}_n\in\mathfrak{a}$ is right equivalent to the element List 2.1(3.v), whereas if $2i<n$, then $\mathtt{x}_i^2-\mathtt{x}_0\mathtt{x}_{2i}\in\mathfrak{a}$ is right equivalent to the element List 2.1(3.i) (Corollary 16.1(iii)). In addition, if $i+j<n$, then $\mathtt{x}_i\mathtt{x}_j-\mathtt{x}_0\mathtt{x}_{i+j}$ is right equivalent to $\mathtt{u}^2+\mathtt{v}^2-(\mathtt{x}^2+\mathtt{y}^2)$ (Corollary 16.1(iii)). If $n\geqslant 3$, we have

$$\begin{cases}\sum_{i=1}^{\frac{n-1}{2}}\Big(\mathtt{x}_i-\frac{1}{2}\mathtt{x}_0\mathtt{x}_n\Big)^2+\sum_{i=\frac{n+1}{2}}^{n-1}\Big(\mathtt{x}_i-\frac{1}{2}\mathtt{x}_0\Big)^2-\frac{1}{4}\Big(\frac{n-1}{2}\Big)\mathtt{x}_0^2(1+\mathtt{x}_n^2)\in\mathfrak{q}_n & \text{if } n \text{ is odd,}\\ \sum_{k=1}^{\frac{n-2}{2}}\Big(\mathtt{x}_{2k}-\frac{1}{2}\mathtt{x}_0-\frac{1}{2}\mathtt{x}_0\mathtt{x}_n\Big)^2+\sum_{k=1}^{\frac{n}{2}}\mathtt{x}_{2k-1}^2-\frac{1}{4}\Big(\frac{n-2}{2}\Big)\mathtt{x}_0^2(1-\mathtt{x}_n)^2-\mathtt{x}_0^2\mathtt{x}_n\in\mathfrak{q}_n & \text{if } n \text{ is even,}\end{cases}$$

which are right equivalent to

$$\begin{cases}\sum_{i=1}^{n-1}\mathtt{y}_i^2-\frac{n-1}{2}\mathtt{y}_0^2 & \text{if } n \text{ is odd,}\\ \sum_{i=1}^{n-1}\mathtt{y}_i^2-\frac{n-2}{2}\mathtt{y}_0^2 & \text{if } n \text{ is even}\end{cases}$$

(Corollary 16.1(ii)). ■

Before proving $\mathcal{P}(A_n)=\Sigma A_n^2$ when $\tau(\kappa)<+\infty$ we need some preliminary results.

expcan **Lemma 15.4.** *For each $f\in\kappa[[\mathtt{x}_0,\dots,\mathtt{x}_n]]$ there exist series $f_0,f_1,\dots,f_{n-1}\in\kappa[[\mathtt{x}_n]]$ and $g_i\in\kappa[[\mathtt{x}_0,\mathtt{x}_n]]$ such that $f=f_0(\mathtt{x}_n)+\sum_{i=1}^{n-1}f_i(\mathtt{x}_n)\mathtt{x}_i+\mathtt{x}_0(g_0(\mathtt{x}_0,\mathtt{x}_n)+\sum_{i=1}^{n}\mathtt{x}_ig_i(\mathtt{x}_0,\mathtt{x}_n))\mod\mathfrak{p}_n$.*

Proof. For each $\nu:=(\nu_1,\dots,\nu_{n-1})$ consider the homogeneous polynomial

$$G_\nu:=\mathtt{x}_1^{\nu_1}\cdots\mathtt{x}_n^{\nu_{n-1}}-\mathtt{x}_0^{d-1-k}\mathtt{x}_n^k\mathtt{x}_i$$

where $d:=\nu_1+\cdots+\mu_{n-1}$, $0\leqslant i<n$ and $\sum_{j=1}^{n-1}j\nu_j=nk+i$. As

$$G_\nu\circ\varphi_n=\prod_{j=1}^{n-1}(\mathtt{s}^{n-j}\mathtt{t}^j)^{\nu_j}-\mathtt{s}^{n(d-1-k)}\mathtt{t}^{nk}\mathtt{s}^{n-i}\mathtt{t}^i=\mathtt{s}^{nd-nk-i}\mathtt{t}^{nk+i}-\mathtt{s}^{n(d-k)-i}\mathtt{t}^{nk+i}=0,$$

it holds $G_\nu\in\mathfrak{p}_n$. For each $\nu=(0,\dots,0,\overset{(i)}{n},0,\dots,0)$ (where $i=1,\dots,n-1$) we deduce $\mathtt{x}_i^n-\mathtt{x}_n^i\mathtt{x}_0^{n-i}\in\mathfrak{p}_n$. Thus, we divide $f\in\kappa[[\mathtt{x}_0,\dots,\mathtt{x}_n]]$ successively by the polynomials $\mathtt{x}_i^n-\mathtt{x}_n^i\mathtt{x}_0^{n-i}$ to obtain

$$f=\sum_{0\leqslant\nu_1,\dots,\nu_{n-1}<n}a_\nu(\mathtt{x}_0,\mathtt{x}_n)\mathtt{x}_1^{\nu_1}\cdots\mathtt{x}_{n-1}^{\nu_{n-1}}\mod\mathfrak{p}_n.$$

As $G_\nu\in\mathfrak{p}_n$ for each $\nu:=(\nu_1,\dots,\nu_{n-1})$, there exist $b_0,b_1,\dots,b_{n-1}\in\kappa[[\mathtt{x}_0,\mathtt{x}_n]]$ such that $f=b_0(\mathtt{x}_0,\mathtt{x}_n)+\sum_{i=1}^{n-1}b_i(\mathtt{x}_0,\mathtt{x}_n)\mathtt{x}_i\mod\mathfrak{p}_n$. Write $b_i(\mathtt{x}_0,\mathtt{x}_n):=f_i(\mathtt{x}_n)+\mathtt{x}_0g_i(\mathtt{x}_0,\mathtt{x}_n)$ where $f_i\in\kappa[[\mathtt{x}_n]]$ and $g_i\in\kappa[[\mathtt{x}_0,\mathtt{x}_n]]$, so there exists $g_k\in\kappa[[\mathtt{x}_0,\mathtt{x}_n]]$ such that

$$f=f_0(\mathtt{x}_n)+\sum_{i=1}^{n-1}f_i(\mathtt{x}_n)\mathtt{x}_i+\mathtt{x}_0(g_0(\mathtt{x}_0,\mathtt{x}_n)+\mathtt{x}_1g_1(\mathtt{x}_0,\mathtt{x}_n)+\cdots+\mathtt{x}_ng_n(\mathtt{x}_0,\mathtt{x}_n))\mod\mathfrak{p}_n,$$

as required. □

We need also the following polynomial density result.

appcrn **Lemma 15.5** (Polynomial density)**.** *Let $\mathfrak{P}_n := \kappa[\mathtt{x}_0,\dots,\mathtt{x}_n] \cap \mathfrak{p}_n$ and consider the polynomial ring $P_n := \kappa[\mathtt{x}_0,\dots,\mathtt{x}_n]/\mathfrak{P}_n$. For each $f \in \mathcal{P}(A_n)$ and each $k \geqslant 1$ there exists a polynomial $f_k \in \mathcal{P}(P_n)$ such that $\omega(f - f_k) > k$.*

Proof. Denote $\mathtt{x} := (\mathtt{x}_0,\dots,\mathtt{x}_n)$, $\|\mathtt{x}\|^2 := \mathtt{x}_0^2 + \cdots + \mathtt{x}_n^2$ and the maximal ideal of $\kappa[[\mathtt{x}]]$ with $\mathfrak{m}_n$. Let $f \in \mathcal{P}(A_n)$ and $f + \|\mathtt{x}\|^{2k} \in \mathcal{P}(A_n)$. Let $g_k \in \kappa[\mathtt{x}]$ be such that $\omega(f + \|\mathtt{x}\|^{2k} - g_k) \geqslant 2k+1$. We claim: $g_k \in \mathcal{P}(A_n)$.

If $f(0) \neq 0$, then $f(0) \in \mathcal{P}(\kappa) = \Sigma\kappa^2$ (Remark 5.5(ii)) and $g_k \in \mathcal{P}(A_n)$, because $g_k(0) = f(0)$. Suppose $f(0) = 0$ and $g_k \notin \mathcal{P}(A_n)$. This means that $\mathcal{U} := \{\beta \in \operatorname{Sper}(A_n) :\ g_k <_\beta 0\} \neq \varnothing$. Pick $\beta \in \mathcal{U}$ and let $\alpha = \beta \cap \kappa$. As $n \geqslant 2$, there exists $\eta : A_n \to \mathfrak{R}(\alpha)[[\mathtt{t}]]$ such that the two orderings η_+ and η_- induced by η in A_n belong to $\mathcal{U}$ (see Lemma 5.21). Thus, $g_k(\eta) < 0$ both if $\mathtt{t} < 0$ or $\mathtt{t} > 0$. We have

$$g_k(\eta) = f(\eta) + \|\eta\|^{2k} + h(\eta)$$

for some $h \in \mathfrak{m}_n^{2k+1}$. Observe that $\omega(h(\eta)) \geqslant (2k+1)\omega(\|\eta\|)$. Thus, if $\omega(f(\eta)) \geqslant 2k\omega(\|\eta\|)$, then $g_k(\eta) > 0$ (because $f(\eta) \geqslant 0$ and $\|\eta\|^{2k} > 0$). If $\omega(f(\eta)) < 2k\omega(\|\eta\|)$, then $\omega(g_k(\eta)) = \omega(f(\eta)) < 2k\omega(\|\eta\|) < \omega(h(\eta))$. Consequently, $g_k(\eta) > 0$, which is a contradiction. Consequently, $\mathcal{U} = \varnothing$ and $g_k \in \mathcal{P}(A_n)$.

Consider the homomorphisms

$$\Phi_n^\epsilon : P_n \to \kappa[\mathtt{s},\mathtt{t}],\ Q \mapsto Q(\epsilon\mathtt{s}^n, \epsilon\mathtt{s}^{n-1}\mathtt{t},\dots,\epsilon\mathtt{s}\mathtt{t}^{n-1},\epsilon\mathtt{t}^n), \tag{15.1}$$ phin+

where $\epsilon = \pm$. We claim: $\operatorname{Sper}(P_n) = \operatorname{im}(\Phi_n^-) \cup \operatorname{im}(\Phi_n^+)$ and if n is odd $\operatorname{Sper}(P_n) = \operatorname{im}(\Phi_n^-) = \operatorname{im}(\Phi_n^+)$.

Let $\alpha \in \operatorname{Sper}(P_n)$ and consider the homomorphism $\pi_\alpha : P_n \to P_n/\operatorname{supp}(\alpha) \hookrightarrow \mathfrak{R}(\alpha)$. Define $p_i := \pi_\alpha(\mathtt{x}_i)$ for each $i = 0,\dots,n$ and observe that $\pi_\alpha(f) = f(p_0,\dots,p_n)$. We distinguish two cases:

Case 1. If n is odd, $s := \sqrt[n]{p_0} \in \mathfrak{R}(\alpha)$ and $t := \sqrt[n]{p_0} \in \mathfrak{R}(\alpha)$. As $\mathfrak{P}_n \subset \ker(\pi_\alpha)$, we deduce $(p_0,\dots,p_n) = (s^n, s^{n-1}t,\dots,st^{n-1},t^n) = (-(-s)^n, -(-s)^{n-1}(-t),\dots,-(-s)(-t)^{n-1},-(-t)^n)$. ■

Case 2. If n is even, $\mathtt{x}_0\mathtt{x}_n - \mathtt{x}_{n/2}^2 \in \mathfrak{P}_n$ and $p_0p_n - p_{n/2}^2 = 0$, so either $p_0p_n = 0$ or p_0 and p_n have the same sign in $\mathfrak{R}(\alpha)$. If $p_0p_n = 0$, then $p_1 = 0,\dots,p_{n-1} = 0$, because $\mathfrak{P}_n \subset \ker(\pi_\alpha)$. If either $p_0 < 0$ or $p_n < 0$, define $(s,t) := (\sqrt[n]{-p_0}, \sqrt[n]{-p_n}) \in \mathfrak{R}(\alpha)^2$. As $\mathfrak{P}_n \subset \ker(\pi_\alpha)$, we deduce $(p_0,\dots,p_n) = (-s^n, -s^{n-1}t,\dots,-st^{n-1},-t^n)$. If both $p_0 \geqslant 0$ and $p_n \geqslant 0$, define $(s,t) := (\sqrt[n]{p_0}, \sqrt[n]{p_n}) \in \mathfrak{R}(\alpha)^2$. As $\mathfrak{P}_n \subset \ker(\pi_\alpha)$, we deduce $(p_0,\dots,p_n) = (s^n, s^{n-1}t,\dots,st^{n-1},t^n)$.

Consider the homomorphism $\varpi : \kappa[\mathtt{s},\mathtt{t}] \to \mathfrak{R}(\alpha),\ g \mapsto g(s,t)$ and $\beta := \varpi^{-1}(\mathfrak{R}(\alpha)^2)$. It holds $\operatorname{Sper}(\Phi_n^\epsilon)(\beta) = \alpha$ for some $\epsilon = \pm$. In fact, if n is odd, $\operatorname{Sper}(P_n) = \operatorname{im}(\Phi_n^-) = \operatorname{im}(\Phi_n^+)$.

Let us check: *There exists $M \in \kappa$ such that $g_k + M^2\|\mathtt{x}\|^{2k} \in \mathcal{P}(P_n)$.*

If $\{\beta \in \operatorname{Sper}(P_n) :\ g_k <_\beta 0\} = \varnothing$, we take $M = 0$. Otherwise, as $\operatorname{Sper}(P_n) = \operatorname{im}(\Phi_n^-) \cup \operatorname{im}(\Phi_n^+)$, it is enough to find $M \in \kappa$ such that $\Phi_n^\epsilon(g_k) + M^2\Phi_n^\epsilon(\|\mathtt{x}\|^{2k}) \in \mathcal{P}(\kappa[\mathtt{s},\mathtt{t}])$ for $\epsilon = \pm$. Let $\mathcal{V}^\epsilon := \{\beta \in \operatorname{Sper}(\kappa[\mathtt{s},\mathtt{t}]) :\ \Phi_n^\epsilon(g_k) <_\beta 0\}$ and let $\beta \in \mathcal{V}^\epsilon$. We claim: *There exists $\varepsilon_\beta \in \kappa\backslash\{0\}$ such that $\varepsilon_\beta^4 <_\beta \mathtt{s}^2 + \mathtt{t}^2$.*

Otherwise, $\mathtt{s}^2 + \mathtt{t}^2 <_\beta \varepsilon^4$ for each $\varepsilon \in \kappa\backslash\{0\}$, so $|\mathtt{s}|_\beta <_\beta \varepsilon^2$ and $|\mathtt{t}|_\beta <_\beta \varepsilon^2$ for each $\varepsilon \in \kappa\backslash\{0\}$. Denote $\mathfrak{n}_2 := (\mathtt{s},\mathtt{t})\kappa[\mathtt{s},\mathtt{t}]$ and let us check: $\mathfrak{n}_2$ *is a β-convex ideal*, that is, *if $Q_1, Q_2 \in \beta$ and $Q_1 + Q_2 \in \mathfrak{n}_2$, then $Q_1, Q_2 \in \mathfrak{n}_2$.*

If $Q_1(0,0) \neq 0$, then $Q_2(0,0) = -Q_1(0,0)$, so $Q_1(0,0)$ and $Q_2(0,0)$ are non-zero and have opposite signs in $\mathfrak{R}(\beta)$ (we have used that $\operatorname{supp}(\beta) \cap \kappa = \{0\}$). Denote $g_i := Q_i - Q_i(0,0) \in$

$(\mathtt{s},\mathtt{t})\kappa[\mathtt{s},\mathtt{t}]$ and write $g_i=\sum_{1\leqslant|\nu|\leqslant d}a_\nu\mathtt{s}^{\nu_1}\mathtt{t}^{\nu_2}$ where $a_\nu\in\kappa$ and $d\geqslant 0$. We have

$$|g_i|_\beta\leqslant_\beta\sum_{1\leqslant|\nu|\leqslant d}|a_\nu|_\beta|\mathtt{s}|_\beta^{\nu_1}|\mathtt{t}|_\beta^{\nu_2}<_\beta\sum_{1\leqslant|\nu|\leqslant d}|a_\nu|_\beta\varepsilon^2<_\beta\Big(\sum_{1\leqslant|\nu|\leqslant d}(a_\nu^2+1)\Big)^2\varepsilon^2$$

for each $\varepsilon\in\kappa\backslash\{0\}$. Thus, $|g_i|_\beta<\varepsilon^2$ for each $\varepsilon\in\kappa\backslash\{0\}$. In particular, as $Q_i(0,0)\neq 0$, we have $|g_i|_\beta<\frac{Q_i(0,0)^2}{(1+Q_i(0,0)^2)^2}<_\beta|Q_i(0,0)|_\beta$. As $Q_i=g_i+Q_i(0,0)\in\beta$, we deduce $Q_i(0,0)>_\beta 0$, against the fact that $Q_1(0,0)$ and $Q_2(0,0)$ have opposite signs in $\mathfrak{R}(\beta)$. Consequently, $Q_i(0,0)=0$ for $i=1,2$ and $Q_i\in\mathfrak{n}_2$.

By [BCR, Prop.4.3.8] there exists a specialization $\beta\to\alpha$ such that $\operatorname{supp}(\alpha)=\mathfrak{n}_2$. By [ABR, Thm.VII.3.2] there exists a specialization $\widehat{\beta}\to\widehat{\alpha}$ of A_n lying over $\beta\to\alpha$, that is, $\widehat{\alpha}\cap P_n=\alpha$ and $\widehat{\beta}\cap P_n=\beta$. As $\Phi_n^\epsilon(g_k)\in\mathcal{P}(\kappa[[\mathtt{s},\mathtt{t}]])$, we deduce $\Phi_n^\epsilon(g_k)>_\beta 0$, which is a contradiction. Consequently, the claim holds.

As $\varepsilon_\beta^4<_\beta\mathtt{s}^2+\mathtt{t}^2$, we obtain $1<_\beta\frac{\mathtt{s}^2+\mathtt{t}^2}{\varepsilon_\beta^4}$. Fix $d\geqslant 1$ and observe that

$$(\mathtt{s}^2+\mathtt{t}^2)^{dn}=\Big(\sum_{\ell=0}^d\binom{n}{\ell}(\mathtt{s}^\ell\mathtt{t}^{(n-\ell)})^2\Big)^d=\Phi_n^+\Big(\Big(\sum_{\ell=0}^n\binom{n}{\ell}\mathtt{x}_\ell^2\Big)^d\Big)$$

If $\mathtt{s}^k\mathtt{t}^\ell$ is a monomial and $k+\ell\leqslant dn$, then

$$\mathtt{s}^{2k}\mathtt{t}^{2\ell}\leqslant_\beta(\mathtt{s}^2+\mathtt{t}^2)^{k+\ell}<_\beta\frac{(\mathtt{s}^2+\mathtt{t}^2)^{dn}}{\varepsilon_\beta^{4(dn-k-\ell)}}.$$

In addition, $a^2+1\pm a>_\beta$ for each $a\in\kappa$. Thus, if $Q\in\kappa[\mathtt{s},\mathtt{t}]$ is a polynomial of degree $\leqslant d$, there exists $N_\beta\in\Sigma\kappa^2$ such that $Q^2<_\beta N_\beta^2(\mathtt{s}^2+\mathtt{t}^2)^{2dn}$. In particular, for $\Phi_n^\epsilon(g_k)$ we find $M_\beta^\epsilon\in\Sigma\kappa^2$ such that $(\Phi_n^\epsilon(g_k))^2\leqslant_\beta(M_\beta^\epsilon)^2(\mathtt{s}^2+\mathtt{t}^2)^{2dn}$ where $d:=\max\{\deg(\Phi_n^\epsilon(g_k)),k\}$. We deduce:

$$\Phi_n^\epsilon\Big(g_k+(M_\beta^\epsilon)^2\Big(\sum_{\ell=0}^n\binom{n}{\ell}\mathtt{x}_\ell^2\Big)^d\Big)=\Phi_n^\epsilon(g_k)+(M_\beta^\epsilon)^2(\mathtt{s}^2+\mathtt{t}^2)^{dn}>_\beta 0$$

for each $\beta\in\mathcal{V}^\epsilon$. Define

$$\mathcal{V}_\beta^\epsilon:=\Big\{\gamma\in\operatorname{Sper}(\kappa[\mathtt{s},\mathtt{t}]):\ \Phi_n^\epsilon\Big(g_k+(M_\beta^\epsilon)^2\Big(\sum_{\ell=0}^n\binom{n}{\ell}\mathtt{x}_\ell^2\Big)^d\Big)>_\gamma 0\Big\},$$

which is an open subset of $\operatorname{Sper}(\kappa[\mathtt{s},\mathtt{t}])$ that contains β. As $\mathcal{V}^\epsilon$ is by [BCR, Cor.7.1.13] quasi-compact, there exist $\beta_1^\epsilon,\dots,\beta_s^\epsilon\in\mathcal{V}^\epsilon$ such that $\mathcal{V}^\epsilon\subset\bigcup_{i=1}^s\mathcal{V}_{\beta_i^\epsilon}^\epsilon$. Define

$$f_k:=g_k+\sum_{i=1}^s((M_{\beta_i^-}^-)^2+(M_{\beta_i^+}^+)^2)\Big(\sum_{\ell=0}^n\binom{n}{\ell}\mathtt{x}_\ell^2\Big)^d$$

and observe that

$$\Phi_n^\epsilon(f_k)=\Phi_n^\epsilon(g_k)+\sum_{k=1}^s((M_{\beta_i^-}^-)^2+(M_{\beta_i^+}^+)^2)(\mathtt{s}^2+\mathtt{t}^2)^{dn}\in\mathcal{P}(\kappa[[\mathtt{s},\mathtt{t}]])$$

for $\epsilon=\pm$, so $f_k\in\mathcal{P}(P_n)$ and $\omega(f-f_k)\geqslant 2d-1\geqslant k$, as required. □

We prove next that $\mathcal{P}(A_n)=\Sigma A_n^2$ for each $n\geqslant 2$.

ultred **Theorem 15.6.** *If $\tau(\kappa)<+\infty$, it holds $\mathcal{P}(A_n)=\Sigma A_n^2$ for each $n\geqslant 2$.*

Proof. Let $p := p(\kappa[[\mathtt{x}_0,\mathtt{x}_1]])$, which is by (2.1) finite because $\tau(\kappa) < +\infty$. By Lemma 15.5 and M. Artin's Strong Approximation Theorem, it is enough to prove $\mathcal{P}(P_n) \subset \Sigma_p A_n^2$. Consider the homomorphism

$$\Gamma_n : P_n \to \kappa[\mathtt{x}_0,\mathtt{x}_1]_{\mathtt{x}_0},\ Q \mapsto Q\Big(\mathtt{x}_0,\mathtt{x}_1,\frac{\mathtt{x}_1^2}{\mathtt{x}_0},\dots,\frac{\mathtt{x}_1^k}{\mathtt{x}_0^{k-1}},\dots,\frac{\mathtt{x}_1^n}{\mathtt{x}_0^{n-1}}\Big).$$

Let $f \in P_n$ and

$$g = \Gamma_n(f) = f\Big(\mathtt{x}_0,\mathtt{x}_1,\frac{\mathtt{x}_1^2}{\mathtt{x}_0},\dots,\frac{\mathtt{x}_1^k}{\mathtt{x}_0^{k-1}},\dots,\frac{\mathtt{x}_1^n}{\mathtt{x}_0^{n-1}}\Big) = \frac{Q(\mathtt{x}_0,\mathtt{x}_1)}{\mathtt{x}_0^{2r}},$$

where $r \geqslant 0$ and $Q \in \kappa[\mathtt{x}_0,\mathtt{x}_1]$. By Lemma 5.14 $Q \in \mathcal{P}(\kappa[\mathtt{x}_0,\mathtt{x}_1])$ and by [Sch3, Cor.4.6] $Q \in \Sigma\kappa[[\mathtt{x}_0,\mathtt{x}_1]]^2 = \Sigma_p\kappa[[\mathtt{x}_0,\mathtt{x}_1]]^2$. Thus, $\mathtt{x}_0^{2r}g = P = A_1^2+\cdots+A_p^2$ for some $A_i \in \kappa[[\mathtt{x}_0,\mathtt{x}_1]]$, so

$$\mathtt{x}_0^{2r}f = A_1^2+\cdots+A_p^2 \mod \mathfrak{p}_n. \tag{15.2}$$ ast69

By Lemma 15.4, there exist series $a_{i0},a_{i1},\dots,a_{i,n-1} \in \kappa[[\mathtt{x}_n]]$ and $b_i \in \kappa[[\mathtt{x}_0,\dots,\mathtt{x}_n]]$ such that

$$a = a_{i0}(\mathtt{x}_n) + \sum_{j=1}^{n-1} a_{ij}(\mathtt{x}_n)\mathtt{x}_j + \mathtt{x}_0 b_i \mod \mathfrak{p}_n.$$

Consequently,

$$\mathtt{x}_0^{2r}f = \sum_{i=1}^{p}\Big(a_{i0}(\mathtt{x}_n) + \sum_{j=1}^{n-1} a_{ij}(\mathtt{x}_n)\mathtt{x}_j + \mathtt{x}_0 b_i\Big)^2 \mod \mathfrak{p}_n.$$

Applying Φ_n^+ to the previous expresion, we deduce

$$s^{2rn}\Phi_n^+(f) = \sum_{i=1}^{p}\Big(a_{i0}(\mathtt{t}^n) + \sum_{j=1}^{n-1} a_{ij}(\mathtt{t}^n)\mathtt{s}^{n-j}\mathtt{t}^j + \mathtt{s}^n(\Phi_n^+(b_i))\Big)^2$$

and comparing orders respect to s

$$\omega_{\mathtt{s}}\Big(a_{i0}(\mathtt{t}^n) + \sum_{j=1}^{n-1} a_{ij}(\mathtt{t}^n)\mathtt{s}^{n-j}\mathtt{t}^j + \mathtt{s}^n(\Phi_n^+(b))\Big) \geqslant rn.$$

Thus, we deduce that $a_{ij}(\mathtt{t}^n) = 0$ for $j = 0,\dots n-1$. Consequently, $a_{ij} = 0$ for $j = 0,\dots n-1$. Therefore, $\mathtt{x}_0^{2r}f = \mathtt{x}_0^2(b_1^2+\cdots+b_p^2) \mod \mathfrak{p}_n$. As $\mathtt{x}_0 \notin \mathfrak{p}_n$ and $\mathfrak{p}_n$ is a prime ideal, we conclude $\mathtt{x}_0^{2r-2}f = (b_1^2+\cdots+b_p^2) \mod \mathfrak{p}_n$. We begin again the argument from (15.2) and, proceeding inductively, we deduce $f \in \Sigma A_n^2$, as required. □

Before proving $\mathcal{P}(B_n) = \Sigma B_n^2$ for each $n \geqslant 2$, we need some preliminary work.

igualo **Lemma 15.7.** *Let $a_1,\dots,a_p \in \kappa[[\mathtt{x}_1,\dots,\mathtt{x}_m]]$ be such that $a_1 \neq 0$. Then there exist $b_1,\dots,b_p \in \kappa[[\mathtt{x}_1,\dots,\mathtt{x}_m]]$ such that $\omega(b_i) = \min\{\omega(a_j) :\ j = 1,\dots,p\}$ for $i = 1,\dots,p$ and $a_1^2+\cdots+a_p^2 = b_1^2+\cdots+b_p^2$.*

Proof. We may assume $\omega(a_i) \leqslant \omega(a_{i+1})$ for $i = 1,\dots,p-1$. We proceed by induction on p. For the case $p = 1$ it is enough to take $b_1 = a_1$. Assume we have found $b_{1,k},\dots,b_{k,k} \in \kappa[[\mathtt{x}_1,\dots,\mathtt{x}_m]]$ such that $\omega(b_i) = \omega(a_1)$ and $a_1^2+\cdots+a_k^2 = b_{1,k}^2+\cdots+b_{k,k}^2$. If $\omega(a_{k+1}) = \omega(a_1)$, we define $b_{j,k+1} := b_{j,k+1}$ for $j = 1,\dots,k$ and $b_{k+1,k+1} := a_{k+1}$. Otherwise, $\omega(b_{k,k+1}) = \omega(a_1) < \omega(a_{k+1})$. We have

$$(\tfrac{3}{5}b_{k,k} - \tfrac{4}{5}a_{k+1})^2 + (\tfrac{4}{5}b_{k,k} + \tfrac{3}{5}a_{k+1})^2 = b_{k,k}^2 + a_{k+1}^2,$$

so if we define $b_{j,k+1} := b_{j,k}$ for $j = 1,\dots,k-1$,

$$b_{k,k+1} := \tfrac{3}{5}b_{k,k} - \tfrac{4}{5}a_{k+1} \quad\text{and}\quad b_{k+1,k+1} := \tfrac{4}{5}b_{k,k} + \tfrac{3}{5}a_{k+1},$$

we have $\omega(b_{j,k+1}) = \omega(b_{j,k}) = \omega(a_1)$ for $j = 1, \dots, k-1$, $\omega(b_{k,k+1}) = \min\{\omega(b_{k,k}), \omega(a_{k+1})\} = \omega(a_1)$, $\omega(b_{k+1,k+1}) = \min\{\omega(b_{k,k}), \omega(a_{k+1})\} = \omega(a_1)$ and

$$\begin{aligned} a_1^2 + \cdots + a_{k+1}^2 &= (b_{1,k}^2 + \cdots + b_{k-1,k}^2) + (b_{k,k}^2 + a_{k+1}^2) \\ &= (b_{1,k+1}^2 + \cdots + b_{k-1,k+1}^2) + (b_{k,k+1}^2 + b_{k+1,k+1}^2). \end{aligned}$$

To finish, it is enough to define $b_j := b_{p,j}$ for $j = 1, \dots, p$, as required. □

The following result is strongly inspired in [L, Lem.XI.1.2].

matrixsos **Lemma 15.8** (Matrices associated to sums of squares). *Let $u_i \in \kappa[[\mathtt{x}_1, \dots, \mathtt{x}_m]]$ be units. Then there exists an $2^k \times 2^k$ invertible square matrix M with coefficients in $\kappa[[\mathtt{x}_1, \dots, \mathtt{x}_m]]$ such that its first row is $(u_1, \dots, u_{2^k})$ and $MM^t = M^tM = (u_1^2 + \cdots + u_{2^k}^2)I_{2^k}$.*

Proof. We proceed by induction on k. If $k = 0$, it is enough to take $M := (u_1)$. Suppose the result true for $k-1$ and let us check that it is also true for k. Split the set $u_1, \dots, u_{2^k}$ into two equal parts, and re-label the elements as $v_1, \dots, v_{2^{k-1}}$ and $w_1, \dots, w_{2^{k-1}}$. Write $v := \sum_{i=1}^{2^{k-1}} v_i^2$ and $w := \sum_{i=1}^{2^{k-1}} w_i^2$, so $u_1^2 + \cdots + u_{2^k}^2) = v + w$. By inductive hypothesis there exist $2^{k-1} \times 2^{k-1}$ invertible square matrix V, W with coefficients in $\kappa[[\mathtt{x}_1, \dots, \mathtt{x}_m]]$ such that the first row of V is $(v_1, \dots, v_{2^{k-1}})$, the first row of W is $(w_1, \dots, w_{2^{k-1}})$, $VV^t = V^tV = vI_{2^{k-1}}$ and $WW^t = W^tW = wI_{2^{k-1}}$. Consider the $2^k \times 2^k$ matrix

$$M := \begin{pmatrix} V & W \\ -v^{-1}V^tW^tV & V^t \end{pmatrix}$$

with coefficients in $\kappa[[\mathtt{x}_1, \dots, \mathtt{x}_m]]$, whose first row is

$$(v_1, \dots, v_{2^{k-1}}, w_1, \dots, w_{2^{k-1}}) = (u_1, \dots, u_{2^k}).$$

A straightforward computation shows $MM^t = M^tM = (u_1^2 + \cdots + u_{2^k}^2)I_{2^k}$. As $\det(M)^2 = \det(MM^t) = u$, we conclude that M is invertible, as required. □

We prove next that $\mathcal{P}(B_n) = \Sigma B_n^2$ for each $n \geqslant 2$.

ultred2 **Theorem 15.9.** *For each $n \geqslant 2$ it holds $\mathcal{P}(B_n) = \Sigma B_n^2$.*

Proof. Consider the surjective (well-defined) homomorphism

$$\theta : B_n \to \kappa[[\mathtt{x}_n]],\ g \mapsto g(0, \dots, 0, \mathtt{x}_n),$$

whose kernel is the ideal $(\mathtt{x}_0, \dots, \mathtt{x}_{n-1})$ (which contains $\mathfrak{q}_n$). By Lemma 5.14 $\theta(\mathcal{P}(B_n)) \subset \mathcal{P}(\kappa[[\mathtt{x}_n]])$.

Let $f \in \mathcal{P}(B_n)$ and assume by Remark 5.5(ii) $f(0, \dots, 0) = 0$. As $\psi_n(f) \in \mathcal{P}(\kappa[[\mathtt{s}, \mathtt{t}]]) = \Sigma\kappa[[\mathtt{s}, \mathtt{t}]]^2$ (Lemma 5.14), there exist $a_{ik} \in \kappa[[\mathtt{s}, \mathtt{t}^n]]$ and $b_{ik} \in \kappa[[\mathtt{t}^n]]$ such that

$$\psi_n(f) = \sum_{i=1}^m \Big(\sum_{k=0}^{n-1} (a_{ik}(\mathtt{s}, \mathtt{t}^n)\mathtt{s}\mathtt{t}^k + b_{ik}(\mathtt{t}^n)\mathtt{t}^k) \Big)^2 \tag{15.3}$$ sos1

We distinguish two cases:

Case 1. If $\mathtt{s}$ divides $\psi_n(f)$, then $b_{ik}(\mathtt{t}^n) = 0$ for each pair i, k. Thus,

$$\psi_n(f) = \sum_{i=1}^m \Big(\sum_{k=0}^{n-1} (a_{ik}(\mathtt{s}, \mathtt{t}^n)\mathtt{s}\mathtt{t}^k) \Big)^2$$

As $(\mathtt{s})\kappa[[\mathtt{s}, \mathtt{t}]] \subset \operatorname{im}(\psi_n)$, we deduce $a_{ik}(\mathtt{s}, \mathtt{t}^n)\mathtt{s}\mathtt{t}^k \in \operatorname{im}(\psi_n)$, so $f \in \mathcal{P}(B_n)$. ▪

CASE 2. If $\mathtt{s}$ does not divides $\psi_n(f)$, we write

$$\psi_n(f(0,\dots,0,\mathtt{x}_n))=f(0,\dots,0,\mathtt{t}^n)=\psi_n(f)(0,\mathtt{t})=\sum_{i=1}^m\Big(\sum_{k=0}^{n-1}b_{ik}(\mathtt{t}^n)\mathtt{t}^k\Big)^2.$$

As $f(0,\dots,0,\mathtt{x}_n)\in\mathcal{P}(\kappa[[\mathtt{x}_n]])$, there exist $c\in\Sigma\kappa^2\setminus\{0\}$, $q\geqslant 1$ and a unit $v\in\kappa[[\mathtt{x}_n]]$ such that $v(0)=1$ and $f(0,\dots,0,\mathtt{x}_n)=\mathtt{x}_n^{2q}cv(\mathtt{x}_n)$. Thus,

$$\sum_{i=1}^m\Big(\sum_{k=0}^{n-1}b_{ik}(\mathtt{t}^n)\mathtt{t}^k\Big)^2=\mathtt{t}^{2qn}cv(\mathtt{t}^n).$$

In particular, $\omega(b_{ik}(\mathtt{t}^n))\geqslant nq$ for each i,k and there exists some index $i=1,\dots,m$ such that $\omega(b_{i0}(\mathtt{t}^n))=nq$. Multiplying f by $\frac{c}{c^2v(\mathtt{t}^n)^2}$ we may assume that $cv(\mathtt{t}^n)^2=1$. We may assume $m=2^r$. By Lemma 15.7 we may assume $\omega(b_{i0}(\mathtt{t}^n))=nq$ for each $i=1,\dots,m$. We have

$$1=\sum_{i=1}^m\Big(\sum_{k=0}^{n-1}\frac{b_{ik}(\mathtt{t}^n)}{(\mathtt{t}^n)^q}\mathtt{t}^k\Big)^2\quad\rightsquigarrow\quad(\mathtt{t}^n)^q=\sum_{i=1}^m\Big(\sum_{k=0}^{n-1}b_{ik}(\mathtt{t}^n)\mathtt{t}^k\Big)\Big(\sum_{k=0}^{n-1}\frac{b_{ik}(\mathtt{t}^n)}{(\mathtt{t}^n)^q}\mathtt{t}^k\Big)\tag{15.4}$$ matrixunit

and $\omega(\frac{b_{i0}(\mathtt{t}^n)}{(\mathtt{t}^n)^q})=0$, so $\frac{b_{i0}(\mathtt{t}^n)}{(\mathtt{t}^n)^q}$ is a unit of $\kappa[[\mathtt{t}]]$ for each $i=1,\dots,m$. By Lemma 15.8 we find an invertible square matrix M of size $m\times m$ and coefficients in $\kappa[[\mathtt{t}]]$ such that its first row is

$$\Big(\sum_{k=0}^{n-1}\frac{b_{1k}(\mathtt{t}^n)}{(\mathtt{t}^n)^q}\mathtt{t}^k,\dots,\sum_{k=0}^{n-1}\frac{b_{mk}(\mathtt{t}^n)}{(\mathtt{t}^n)^q}\mathtt{t}^k\Big)$$

and $MM^t=M^tM=\mathrm{id}_m$. Write

$$w:=\Big(\sum_{k=0}^{n-1}(a_{1k}(\mathtt{s},\mathtt{t}^n)\mathtt{s}\mathtt{t}^k+b_{1k}(\mathtt{t}^n)\mathtt{t}^k),\dots,\sum_{k=0}^{n-1}(a_{mk}(\mathtt{s},\mathtt{t}^n)\mathtt{s}\mathtt{t}^k+b_{ik}(\mathtt{t}^n)\mathtt{t}^k)\Big)$$

We have

$$\psi_n(f)=\sum_{i=1}^m\Big(\sum_{k=0}^{n-1}(a_k(\mathtt{s},\mathtt{t}^n)\mathtt{s}\mathtt{t}^k+b_{ik}(\mathtt{t}^n)\mathtt{t}^k)\Big)^2=wM^tMw^t$$

and the first addend of the new representation of $\psi_n(f)=A_1^2+\cdots+A_m^2$ is by (15.4)

$$A_1:=\sum_{k=0}^{n-1}a'_{1k}(\mathtt{s},\mathtt{t}^n)\mathtt{s}\mathtt{t}^k+\sum_{i=1}^m\Big(\sum_{k=0}^{n-1}b_{ik}(\mathtt{t}^n)\mathtt{t}^k\Big)\Big(\sum_{k=0}^{n-1}\frac{b_{ik}(\mathtt{t}^n)}{(\mathtt{t}^n)^q}\mathtt{t}^k\Big)=\sum_{k=0}^{n-1}a'_{1k}(\mathtt{s},\mathtt{t}^n)\mathtt{s}\mathtt{t}^k+(\mathtt{t}^n)^q$$

for some series $a'_{1k}(\mathtt{s},\mathtt{t}^n)\in\kappa[[\mathtt{s},\mathtt{t}^n]]$. Observe that

$$(\mathtt{t}^n)^{2q}=\psi_n(f)(0,\mathtt{t})=A_1(0,\mathtt{t})^2+\cdots+A_m(0,\mathtt{t})^2=(\mathtt{t}^n)^{2q}+A_2(0,\mathtt{t})^2+\cdots+A_m(0,\mathtt{t})^2,$$

so $A_k(0,\mathtt{t})=0$ for $k=2,\dots,m$. Thus, $\mathtt{s}$ divides A_k for $k=2,\dots,m$ and, as $(\mathtt{s})\kappa[[\mathtt{s},\mathtt{t}]]\subset\mathrm{im}(\psi_n)$, we have $A_k\in\mathrm{im}(\psi_n)$ for $k=2,\dots,m$ and $\sum_{k=0}^{n-1}a'_{1k}(\mathtt{s},\mathtt{t}^n)\mathtt{s}\mathtt{t}^k\in\mathrm{im}(\psi_n)$. In addition, $(\mathtt{t}^n)^q\in\mathrm{im}(\psi_n)$, so $f\in\Sigma B_n^2$.

We conclude $\mathcal{P}(B_n)=\Sigma B_n^2$, as required. □

redn

15.3. **Reducible examples in higher embedding dimensions.** We begin with some expected construction that provides examples of ring that have the property psd=sos, but are not integral domains:

nsv **Lemma 15.10.** *Let $\mathfrak{a}\subset\kappa[[\mathtt{x}]]:=\kappa[[\mathtt{x}_1,\dots,\mathtt{x}_n]]$ and $\mathfrak{b}\subset\kappa[\mathtt{y}]:=\kappa[[\mathtt{y}_1,\dots,\mathtt{y}_m]]$ be ideals such that the rings $A:=\kappa[[\mathtt{x}]]/\mathfrak{a}$ and $B:=\kappa[[\mathtt{y}]]/\mathfrak{b}$ satisfy $\mathcal{P}(A)=\Sigma A^2$ and $\mathcal{P}(B)=\Sigma B^2$. Consider the ring $C:=\kappa[[\mathtt{x},\mathtt{y}]]/\mathfrak{c}$ where $\mathfrak{c}:=(\mathfrak{a}\kappa[[\mathtt{x},\mathtt{y}]]+(\mathtt{y}_1,\dots,\mathtt{y}_m)\kappa[[\mathtt{x},\mathtt{y}]])\cap((\mathtt{x}_1,\dots,\mathtt{x}_n)\kappa[[\mathtt{x},\mathtt{y}]]+\mathfrak{b}\kappa[[\mathtt{x},\mathtt{y}]])$. Then $\mathcal{P}(C)=\Sigma C^2$.*

Proof. Pick $f \in \mathcal{P}(C)$ and assume, by Remark 5.5(ii), that $f \in \mathfrak{m}_C$. As $\mathtt{x}_i\mathtt{y}_j \in \mathfrak{c}$ for each $i = 1, \dots, n$ and $j = 1, \dots, m$, we may assume $f = f_1 + f_2$, where $f_1 \in \kappa[[\mathtt{x}]]$ and $f_2 \in \kappa[[\mathtt{y}]]$. Let $\alpha \in \operatorname{Sper}(A)$ and let $\mathfrak{p}$ be a prime ideal of $\kappa[[\mathtt{x}]]$ that contains $\mathfrak{a}$ such that $\operatorname{supp}(\alpha) = \mathfrak{p}/\mathfrak{a}$. Consider the ideal $\mathfrak{P} := \mathfrak{p} + (\mathtt{y}_1, \dots, \mathtt{y}_m)$, which is a prime ideal of $\kappa[[\mathtt{x}, \mathtt{y}]]$ that contains $\mathfrak{c}$ and satisfies $C/(\mathfrak{P}/\mathfrak{c}) \cong \kappa[[\mathtt{x}, \mathtt{y}]]/\mathfrak{P} \cong \kappa[[\mathtt{x}]]/\mathfrak{p} \cong A/\operatorname{supp}(\alpha)$. Thus, α define a prime cone α' of C with support $\mathfrak{P}/\mathfrak{c}$. As $f_2 \in \mathfrak{P}/\mathfrak{c}$ and $f \in \mathcal{P}(C)$, we deduce that $f_1 \in \alpha$. Consequently, $f_1 \in \mathcal{P}(A) = \Sigma A^2$. Analogously, one shows that $f_2 \in \mathcal{P}(B) = \Sigma B^2$. Consequently, $f_1 = \sigma_1 + g_1$ where $\sigma_1 \in \Sigma\kappa[[\mathtt{x}]]^2$ and $g_1 \in \mathfrak{a}$ and $f_2 = \sigma_2 + g_2$ where $\sigma_2 \in \Sigma\kappa[[\mathtt{y}]]^2$ and $g_2 \in \mathfrak{a}_2$. As

$$\begin{aligned}\mathfrak{c} := (\mathfrak{a}\kappa[[\mathtt{x}, \mathtt{y}]] + (\mathtt{y}_1, \dots, \mathtt{y}_m)\kappa[[\mathtt{x}, \mathtt{y}]]) \cap ((\mathtt{x}_1, \dots, \mathtt{x}_n)\kappa[[\mathtt{x}, \mathtt{y}]] + \mathfrak{b}\kappa[[\mathtt{x}, \mathtt{y}]])\\ = \mathfrak{a}\kappa[[\mathtt{x}, \mathtt{y}]] + \mathfrak{b}\kappa[[\mathtt{x}, \mathtt{y}]] + (\mathtt{x}_i\mathtt{y}_j :\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m)\kappa[[\mathtt{x}, \mathtt{y}]]\end{aligned}$$

we deduce that $f_1 + f_2 = \sigma_1 + \sigma_2 + g_1 + g_2$, where $g_1 + g_2 \in \mathfrak{c}$. Consequently, $f \in \Sigma C^2$, as required. □

nsvr *Remark* 15.11. Under the hypothesis of the previous result, if $\dim(A) = 2$, $\mathtt{x}_1$ is not a zero divisor of A and $\mathtt{y}_1$ is not a zero divisor of B, then C is not Cohen-Macaulay. Observe that $n \geqslant 2$ and $m \geqslant 2$. As $\mathcal{P}(A) = \Sigma A^2$ and $\mathcal{P}(B) = \Sigma B^2$, the rings A and B are by [Sch2, Lem.6.3] real (reduced), so both rings are reduced. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal prime ideals associated to $\mathfrak{a}$ and let $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ be the minimal prime ideals associated to $\mathfrak{b}$. Define $\mathfrak{P}_i := \mathfrak{p}_i\kappa[[\mathtt{x}, \mathtt{y}]] + (\mathtt{y}_1, \dots, \mathtt{y}_m)\kappa[[\mathtt{x}, \mathtt{y}]]$ for $i = 1, \dots, r$ and $\mathfrak{Q}_j := (\mathtt{x}_1, \dots, \mathtt{x}_n)\kappa[[\mathtt{x}, \mathtt{y}]] + \mathfrak{q}_j$ for $j = 1, \dots, s$. Thus, $\mathfrak{c} = \bigcap_{i=1}^r \mathfrak{P}_i \cap \bigcap_{j=1}^s \mathfrak{Q}_j$ and the minimal prime ideals associated to $\mathfrak{c}$ are $\mathfrak{P}_1, \dots, \mathfrak{P}_r, \mathfrak{Q}_1, \dots, \mathfrak{Q}_s$. The element $\mathtt{x}_1 - \mathtt{y}_1$ is not a unit in C and let us check: $\mathtt{x}_1 - \mathtt{y}_1$ *is not a zero divisor of* C.

We have to check: $\mathtt{x}_1 - \mathtt{y}_1 \notin \bigcup_{i=1}^r \mathfrak{P}_i \cup \bigcup_{j=1}^s \mathfrak{Q}_j$. If $\mathtt{x}_1 - \mathtt{y}_1 \in \mathfrak{P}_i$, then $\mathtt{x}_1 \in \mathfrak{p}_i$ (because $\mathtt{y}_1 \in \mathfrak{P}_i$ and $\mathfrak{P}_i \cap \kappa[[\mathtt{x}]] = \mathfrak{p}_i$), which is a contradiction, because $\mathtt{x}_1$ is not a zero divisor of A. Analogously, if $\mathtt{x}_1 - \mathtt{y}_1 \in \mathfrak{Q}_j$, then $\mathtt{y}_1 \in \mathfrak{q}_j$ (because $\mathtt{x}_1 \in \mathfrak{Q}_j$ and $\mathfrak{Q}_j \cap \kappa[[\mathtt{y}]] = \mathfrak{q}_j$), which is a contradiction, because $\mathtt{y}_1$ is not a zero divisor of A. Thus, $\mathtt{x}_1 - \mathtt{y}_1$ is not a zero divisor of C.

Define $D := C/(\mathtt{x}_1 - \mathtt{y}_1)C$ and observe that $D \cong \kappa[[\mathtt{x}, \mathtt{y}]]/\mathfrak{d}$, where $\mathfrak{d} := \mathfrak{c} + (\mathtt{x}_1 - \mathtt{y}_1)\kappa[[\mathtt{x}, \mathtt{y}]]$, and it is isomorphic to $\kappa[\mathtt{x}, \mathtt{y}']/\mathfrak{d}'$, where $\mathtt{y}' := (\mathtt{y}_2, \dots, \mathtt{y}_m)$,

$$\mathfrak{d}' := \mathfrak{a}\kappa[[\mathtt{x}, \mathtt{y}']] + \mathfrak{b}'\kappa[[\mathtt{x}, \mathtt{y}']] + (\mathtt{x}_1^2, \mathtt{x}_1\mathtt{x}_2, \dots, \mathtt{x}_1\mathtt{x}_n, \mathtt{x}_i\mathtt{y}_j :\ 1 \leqslant i \leqslant n,\ 2 \leqslant j \leqslant m)\kappa[[\mathtt{x}, \mathtt{y}]]$$

and $\mathfrak{b}' := \{g(\mathtt{x}_1, \mathtt{y}') :\ g \in \mathfrak{b}\}$. The maximal ideal of D is $(\mathtt{x}, \mathtt{y}')/\mathfrak{d}'$ and $\mathtt{x}_1 \notin \mathfrak{d}'$. Observe that $\mathtt{x}_1 h \in \mathfrak{d}'$ for each $h \in (\mathtt{x}, \mathtt{y}')$, so all the elements of D are either units or zero divisors of D. Thus, $\operatorname{depth}(D) = 0$ and by [JP, Cor.6.5.5(i)] we have $\operatorname{depth}(C) = \operatorname{depth}(D) + 1 = 1$. We claim: $\dim(C) \geqslant 2$. Consequently, $\operatorname{depth}(C) < \dim(C)$ and C is not a Cohen-Macaulay ring.

To prove the later inequality, we may assume $\mathfrak{p}_1 \subsetneq \mathfrak{p} \subsetneq (\mathtt{x})\kappa[[\mathtt{x}]]$ is a chain of primes ideals in A of length 2. Then $\mathfrak{P}_1 \subsetneq \mathfrak{P} \subsetneq (\mathtt{x}, \mathtt{y})\kappa[[\mathtt{x}, \mathtt{y}]]$ where $\mathfrak{P} = \mathfrak{p}\kappa[[\mathtt{x}, \mathtt{y}]] + (\mathtt{y}_1, \dots, \mathtt{y}_m)\kappa[[\mathtt{x}, \mathtt{y}]]$. Thus, $\dim(C) \geqslant 2$, as required. ■

planei **Lemma 15.12.** *Denote* $\kappa[[\mathtt{x}]] := \kappa[[\mathtt{x}_1, \dots, \mathtt{x}_n]]$ *and let* $A := \kappa[[\mathtt{x}]]/\mathfrak{a}_1$ *be such that* $\mathcal{P}(A) = \Sigma A^2$. *Assume that* $\mathtt{x}_{n-1} \in \mathfrak{a}_1$ *and let* $\mathfrak{a}_2 := \{\mathtt{x}_1, \dots, \mathtt{x}_{n-2}\}\kappa[[\mathtt{x}]]$ *denote* $\mathfrak{b} := \mathfrak{a}_1 \cap \mathfrak{a}_2$. *Then* $B := \kappa[[\mathtt{x}]]/\mathfrak{b}$ *satisfies* $\mathcal{P}(B) = \Sigma B^2$.

Proof. Consider the surjective homomorphisms $\pi_1 : B \to A = B/(\mathfrak{a}_1/\mathfrak{b})$ and the projection $\pi_2 : B \to \kappa[[\mathtt{x}_{n-1}, \mathtt{x}_n]] = B/(\mathfrak{a}_2/\mathfrak{b})$. Let $f \in \mathcal{P}(B)$. By Lemma 5.16 we have $f \in \mathcal{P}(A) = \Sigma A^2$ and $f \in \mathcal{P}(\kappa[[\mathtt{x}_{n-1}, \mathtt{x}_n]]) = \Sigma\kappa[[\mathtt{x}_{n-1}, \mathtt{x}_n]]^2$. Write $f = \sigma_i + a_i$ where $\sigma_1 \in \Sigma\kappa[[\mathtt{x}_1, \dots, x_{n-2}, \mathtt{x}_n]]^2$, $\sigma_2 \in \kappa[[\mathtt{x}_{n-1}, \mathtt{x}_n]]$ and $a_i \in \mathfrak{a}_i$ for $i = 1, 2$. Observe that $\tau := \sigma_1(0, \dots, 0, \mathtt{x}_n) = \sigma_2(0, \dots, 0, \mathtt{x}_n) \in \kappa[[\mathtt{x}_n]]$. If this common series is identically zero, $\sigma_1 \in \mathfrak{b}$ (because $\mathtt{x}_1, \dots, \mathtt{x}_{n-2} \in \mathfrak{a}_2$) and $\sigma_2 \in \mathfrak{a}_1$ (because $\mathtt{x}_{n-1} \in \mathfrak{a}_1$). Thus, $\sigma = \sigma_1 + \sigma_2 \in \Sigma B^2$ and $f = \sigma$ in B, because $\sigma = \sigma_1 + \sigma_2 = \sigma_1 + a_1 = f$ modulo $\mathfrak{a}_1$ and $\sigma = \sigma_1 + \sigma_2 = \sigma_2 + a_2 = f$ modulo $\mathfrak{a}_2$.

Suppose in the following that $\tau \neq 0$. By Lemma 15.7 we may assume (using that $\mathtt{x}_{n-1} \in \mathfrak{a}_1$)

$$\sigma_1 = \sum_{j=1}^{2^m} (\mathtt{x}_n^\ell u_j + \mathtt{x}_1 h_{j1} + \cdots + \mathtt{x}_{n-2} h_{j,n-2})^2$$

where $\ell, m \geqslant 1$, $h_{jk} \in \kappa[[\mathtt{x}]]$ and $u_j \in \kappa[[\mathtt{x}_n]]$ is a unit such that $u_j(0) \neq 0$. We have $\tau = \mathtt{x}_n^{2\ell}(u_1^2 + \cdots + u_{2^m}^2)$. By Lemma 15.8 there exists a $2^m \times 2^m$ matrix M and coefficients in $\kappa[[\mathtt{x}_n]]$ such that its first row is $(u_1, \ldots, u_{2^m})$ and $MM^t = M^tM = (u_1^2 + \cdots + u_{2^m}^2)I_{2^m}$. Write

$$w := (\mathtt{x}_n^\ell u_1 \mathtt{x}_1 h_{1,1} + \cdots + \mathtt{x}_{n-2} h_{1,n-2}, \ldots, \mathtt{x}_n^\ell u_{2^m} \mathtt{x}_1 h_{2^m,1} + \cdots + \mathtt{x}_{n-2} h_{2^m,n-2})$$

and consider $\sigma_1(u_1^2+\cdots+u_{2^m}^2) = ww^t(u_1^2+\cdots+u_{2^m}^2) = wM^tMw^t$. Observe that the first addend of the sum of squares wM^tMw^t is the square of $h := \sum_{j=1}^{2m}(\mathtt{x}_n^\ell u_j^2 \mathtt{x}_1 h_{1,1} + \cdots + \mathtt{x}_{n-2} h_{1,n-2} u_j)$. As

$$(u_1^2 + \cdots + u_{2^m}^2)\sigma_1(0, \ldots, 0, \mathtt{x}_n) = (u_1^2 + \cdots + u_{2^m}^2)\tau = \mathtt{x}^{2\ell}(u_1^2 + \cdots + u_{2^m}^2)^2$$

and $h^2(0, \ldots, 0, \mathtt{x}_n) = \mathtt{x}^{2\ell}(u_1^2 + \cdots + u_{2^m}^2)^2 = (u_1^2 + \cdots + u_{2^m}^2)\sigma_1(0, \ldots, 0, \mathtt{x}_n)$. Thus, $(u_1^2 + \cdots + u_{2^m}^2)\sigma_1 - h^2 \in \Sigma((\mathtt{x}_1, \ldots, \mathtt{x}_{n-2})\kappa[[\mathtt{x}]])^2$. We may write $(u_1^2+\cdots+u_{2^m}^2) = \lambda u^2$ where $\lambda \in \Sigma\kappa^2\backslash\{0\}$, $u \in \kappa[[\mathtt{x}_n]]$ and $u(0) = 1$. Thus, we may assume

$$\sigma_1 = \eta(\mathtt{x}_n^\ell U + \mathtt{x}_1 h_{11} + \cdots + \mathtt{x}_{n-2} h_{1,n-2})^2 + \sum_{j=2}^{p} (\mathtt{x}_1 h_{j1} + \cdots + \mathtt{x}_{n-2} h_{j,n-2})^2$$

where $h_{jk} \in \kappa[[\mathtt{x}]]$, $\eta \in \Sigma\kappa^2\backslash\{0\}$ and $U \in \kappa[[\mathtt{x}_n]]$ is a unit such that $\eta U(0) = 1$.

By Lemma 15.7 we may assume (using that $\mathtt{x}_1, \ldots, \mathtt{x}_{n-2} \in \mathfrak{a}_2$)

$$\sigma_2 = \sum_{j=1}^{2^r} (\mathtt{x}_n^{\ell'} v_j + \mathtt{x}_{n-1} g_{j,n-1})^2$$

where $\ell', r \geqslant 1$, $g_{j,n-1} \in \kappa[[\mathtt{x}]]$ and $v_j \in \kappa[[\mathtt{x}_n]]$ is a unit such that $v_j(0) \neq 0$. We have $\tau = \mathtt{x}_n^{2\ell'}(v_1^2+\cdots+v_{2^m}^2)$, so $\ell' = \ell$. By Lemma 15.8 there exists a $2^r \times 2^r$ matrix N and coefficients in $\kappa[[\mathtt{x}_n]]$ such that its first row is $(v_1, \ldots, v_{2^r})$ and $NN^t = N^tN = (v_1^2 + \cdots + v_{2^r}^2)I_{2^r}$. Write

$$w' := (\mathtt{x}_n^\ell v_1 + \mathtt{x}_{n-1} g_{1,n-1}, \ldots, \mathtt{x}_n^\ell v_{2^r} + \mathtt{x}_{n-1} g_{2^r,n-1})$$

and consider $\sigma_2(v_1^2 + \cdots + v_{2^r}^2) = w'w'^t(v_1^2 + \cdots + v_{2^r}^2) = w'N^tNw'^t$. Observe that the first addend of the sum of squares $w'N^tNw'^t$ is the square of $g := \sum_{j=1}^{2^r}(\mathtt{x}_n^\ell v_j^2 + \mathtt{x}_{n-1} g_{1,n-1} v_j)$. As

$$(v_1^2 + \cdots + v_{2^r}^2)\sigma_2(0, \ldots, 0, \mathtt{x}_n) = (v_1^2 + \cdots + v_{2^r}^2)\tau = \mathtt{x}^{2\ell}(v_1^2 + \cdots + v_{2^r}^2)^2$$

and $g^2(0, \ldots, 0, \mathtt{x}_n) = \mathtt{x}^{2\ell}(v_1^2 + \cdots + v_{2^r}^2)^2 = (v_1^2 + \cdots + v_{2^r}^2)\sigma_2(0, \ldots, 0, \mathtt{x}_n)$. Thus, $(v_1^2 + \cdots + v_{2^r}^2)\sigma_2 - g^2 \in \Sigma((\mathtt{x}_{n-1})\kappa[[\mathtt{x}]])^2$. We may write $(v_1^2 + \cdots + v_{2^r}^2) = \lambda' v^2$ where $\lambda' \in \Sigma\kappa^2\backslash\{0\}$, $v \in \kappa[[\mathtt{x}_n]]$ and $v(0) = 1$. Thus, we may assume

$$\sigma_2 = \eta'(\mathtt{x}_n^\ell V + \mathtt{x}_{n-1} g_{j,n-1})^2 + \sum_{j=1}^{2^r} (\mathtt{x}_{n-1} g_{j,n-1})^2$$

where $g_{j,n-1} \in \kappa[[\mathtt{x}]]$, $\eta' \in \Sigma\kappa^2\backslash\{0\}$ and $V \in \kappa[[\mathtt{x}_n]]$ is a unit such that $\eta' V(0) = 1$. As $\sigma_1(0, \ldots, 0, \mathtt{x}_n) = \sigma_2(0, \ldots, 0, \mathtt{x}_n)$, we deduce $\eta' = \eta$ and $V^2 = U^2$.

If $r < m$ define $g_{j,n-1} = 0$ for $2^r + 1 \leqslant j \leqslant 2^m$, whereas if $m < r$ define $h_{j,k} = 0$ for $2^r + 1 \leqslant j \leqslant 2^m$ and $k = 1, \ldots, n-2$. Thus, we may assume $r = m$. Define

$$\sigma := \eta(\mathtt{x}_n^\ell u + \mathtt{x}_1 h_{11} + \cdots + \mathtt{x}_{n-2} h_{1,n-2} + \mathtt{x}_{n-1} g_{1,n-1})^2 + \sum_{j=1}^{2^m} (\mathtt{x}_1 h_{j1} + \cdots + \mathtt{x}_{n-2} h_{j,n-2} + \mathtt{x}_{n-1} g_{j,n-1})^2$$

and let us check that $f-\sigma\in\mathfrak{b}$. As $\mathtt{x}_{n-1}\in\mathfrak{a}_1$, we have $f-\sigma=f-\sigma_1+\sigma_1-\sigma=a_1+\sigma_1-\sigma\in\mathfrak{a}_1$. As $\mathtt{x}_1,\dots,\mathtt{x}_{n-2}\in\mathfrak{a}_2$, we have $f-\sigma=f-\sigma_2+\sigma_2-\sigma=a_2+\sigma_2-\sigma\in\mathfrak{a}_2$, so $f-\sigma\in\mathfrak{a}_1\cap\mathfrak{a}_2=\mathfrak{b}$. Thus, $f=\sigma\in\Sigma B^2$, as required. $\square$

Definitions 15.13. A finite family I of pairs $(i,j)\in\{1,\dots,n\}^2$ such that $i<j$ is *suitable* if there exists a permutation of the indices $1,\dots,n$ after which

$$I=\{(i_k,j_k):\ k=2,\dots,d,\ 1\leqslant i_k\leqslant i_{k+1}\leqslant n,\ 1\leqslant j_k<j_{k+1}\leqslant n,\ i_k<j_k\}$$

for some $1\leqslant d\leqslant n$. This is a *standard presentation of* I. If I is standardly presented, then

$$I_\ell=\{(i_k,k):\ k=2,\dots,\ell,\ 1\leqslant i_k\leqslant i_{k+1}\leqslant n,\ 1\leqslant j_k<j_{k+1}\leqslant n,\ i_k<j_k\}.$$

is a suitable family standardly presented for each $\ell=2,\dots,d$.

stp *Example* 15.14. Consider the real prime ideal

$$\mathfrak{p}_{ij}:=(\mathtt{x}_1,\dots,\mathtt{x}_{i-1},\mathtt{x}_{i+1},\dots,\mathtt{x}_{j-1},\mathtt{x}_{j+1},\dots,\mathtt{x}_n)$$

for each $1\leqslant i<j\leqslant n$ and let $I\subset\{1,\dots,n\}^2$ be a suitable family standardly presented. Consider the intersection $\mathfrak{a}_\ell:=\bigcap_{(i,j)\in I_\ell}\mathfrak{p}_{i,j}$ for each $\ell=2,\dots,n$. Then $A_\ell:=\kappa[[\mathtt{x}]]/\mathfrak{a}_\ell$ has the property $\mathcal{P}(A_\ell)=\Sigma A_\ell^2$ for each $k=2,\dots,n$. We will take advantage of Lemmas 15.10 and 15.12. We proceed by induction on ℓ. If $\ell=2$, then $I_2=\{(1,2)\}$, $\mathfrak{a}_2=\mathfrak{p}_{12}$ and $A_2:=\kappa[[\mathtt{x}]]/\mathfrak{a}_2\cong\kappa[[\mathtt{x}_1,\mathtt{x}_2]]$, which satisfies $\mathcal{P}(A_2)=\Sigma A_2^2$. By induction hypothesis assume the result true for $\ell<n-1$, that is, $\mathcal{P}(A_\ell)=\Sigma A_\ell^2$, and let us check that it is also true for $\ell+1$. Observe that $I_{\ell+1}=I_\ell\cup\{(i_{\ell+1},j_{\ell+1})\}$. If $j_\ell<i_{\ell+1}$, then $\mathcal{P}(A_{\ell+1})=\Sigma A_{\ell+1}^2$ by Lemma 15.10. Otherwise, $i_{\ell+1}\leqslant j_\ell$. We have $\mathtt{x}_{j_{\ell+1}}\in\mathfrak{a}_{\ell+1}$ and $\mathfrak{p}_{i_{\ell+1},j_{\ell+1}}$ contains all the variables except for $\mathtt{x}_{i_{\ell+1}}$ and $\mathtt{x}_{j_{\ell+1}}$. By Lemma 15.12 we deduce that $A_{\ell+1}=\kappa[[\mathtt{x}]]/(\mathfrak{a}_\ell\cap\mathfrak{p}_{i_{\ell+1},j_{\ell+1}})$ satisfies $\mathcal{P}(A_{\ell+1})=\Sigma A_{\ell+1}^2$. Consequently, $\mathcal{P}(A_n)=\Sigma A_n^2$, as required. ∎

ounterpsdsos *Example* 15.15. Consider the real prime ideal

$$\mathfrak{p}_{ij}:=(\mathtt{x}_1,\dots,\mathtt{x}_{i-1},\mathtt{x}_{i+1},\dots,\mathtt{x}_{j-1},\mathtt{x}_{j+1},\dots,\mathtt{x}_n)$$

for each $1\leqslant i<j\leqslant n$ and $n\geqslant 4$. Let

$$I:=\{(i_k,j_k):\ k=2,\dots,n,\ \ 1\leqslant i_k\leqslant i_{k+1}\leqslant n,\ 1\leqslant j_k<j_{k+1}\leqslant n,\ i_k<j_k\}$$

be a suitable family standardly presented and let (i_k,m) with $1\leqslant i_k<m<j_k\leqslant n$ and $I'\subset\{1,\dots,n\}^2$ be such that the elements of I' have coordinates (i,j) with $i<j$ and I' is disjoint to $I\cup\{(i_k,m)\}$. Define $J:=I\sqcup\{(i_k,m)\}\sqcup I'$ and $\mathfrak{a}:=\bigcap_{(i,j)\in J}\mathfrak{p}_{i,j}$ and $A:=\kappa[[\mathtt{x}]]/\mathfrak{a}$. We claim: $\mathcal{P}(A)\neq\Sigma A^2$.

The quotient ideals $\mathfrak{q}_{i,j}:=\mathfrak{p}_{i,j}/\mathfrak{a}$ are prime ideals of A. Thus we have surjective homomorphisms $A\to A/\mathfrak{q}_{i,j}=\kappa[[\mathtt{x}]]/\mathfrak{p}_{i,j}$. Define

$$f_\varepsilon:=\sum_{i=1}^n\mathtt{x}_i^2-2\sum_{(i,j)\in I}\mathtt{x}_i\mathtt{x}_j+\varepsilon 2\mathtt{x}_{i_k}\mathtt{x}_m.$$

where $\varepsilon=\pm1$. Observe that $f_\varepsilon=(\mathtt{x}_i-\mathtt{x}_j)^2$ modulo $\mathfrak{p}_{i,j}$ for $(i,j)\in I$, $f_\varepsilon=(\mathtt{x}_{i_k}+\varepsilon\mathtt{x}_m)^2$ modulo $\mathfrak{p}_{i_k,m}$ and $f=\mathtt{x}_i^2+\mathtt{x}_j^2$ modulo $\mathfrak{p}_{i,j}$ if $(i,j)\in I'$, so $f_\varepsilon\in\mathcal{P}(A/\mathfrak{q}_{i,j})$ for each $(i,j)\in J$. Let $\alpha\in\operatorname{Sper}(A)$ and write $\operatorname{supp}(\alpha)=\mathfrak{p}/\mathfrak{a}$ for some ideal $\mathfrak{p}$ of $\kappa[[\mathtt{x}]]$ that contains $\mathfrak{a}$. By [AM, Prop.1.11] we have that $\operatorname{supp}(\alpha)$ contains $\mathfrak{p}_{i,j}$ for some $(i,j)\in J$, so α provides also a prime cone of $A/\mathfrak{q}_{i,j}=\kappa[[\mathtt{x}]]/\mathfrak{p}_{i,j}$. As $f\in\mathcal{P}(A/\mathfrak{q}_{i,j})$ for each $(i,j)\in J$, we deduce $f\in\mathcal{P}(A)$. Let us check: $f\notin\Sigma A^2$.

Suppose that $f\in\Sigma A^2$ and let $\lambda_{i,j}\in\kappa$ be such that

$$F_\varepsilon:=f_\varepsilon+\sum_{(i,j)\notin J}\lambda_{i,j}\mathtt{x}_i\mathtt{x}_j$$

is a sum of squares of linear forms in the variables $\mathtt{x}_1,\dots,\mathtt{x}_n$ with coefficients in κ. Define $v_{i,j} := (0,\dots,0,\overset{(i)}{1},0,\dots,0,\overset{(j)}{1},0,\dots,0)$ and observe that $F(v_{i,j}) = 0$ for $(i,j) \in I$. In addition, the vectors $\{v_{i,j}\}_{(i,j)\in I}$ are linearly independent (because the matrix $(v_{i,j})_{(i,j)\in I}$ has rank $n-1$). Let $\beta \in \operatorname{Sper}(\kappa)$ and let $R := \mathfrak{R}(\beta)$. The symmetric matrix of F_ε has rank $\leqslant 1$ and it is a sum of squares of linear forms with coefficients in R, so F is the square of a linear form. Thus,

$$F_\varepsilon = (a_1\mathtt{x}_1 + \cdots + a_n\mathtt{x}_n)^2$$

for some $a_i \in R$. Thus, $a_i = \pm 1$ for $i = 1,\dots,n$ and we may assume $a_1 = 1$. As $a_ia_j = -1$ for $(i,j) \in I$, the sign of $a_{i_k}a_m$ is determined by the set I because $1 \leqslant i_k < m \leqslant n$. Thus either F_{+1} or F_{-1} is not the square of a linear form. Consequently, either $f_{+1} \notin \Sigma A^2$ or $f_{-1} \notin \Sigma A^2$. We conclude $\mathcal{P}(A) \neq \Sigma A^2$, as required. ∎

Let us provide a more subtle family of non-integral domains that have the property psd=sos. We will take advantage once more of Lemma 15.8. Denote

$$\mathfrak{p}_{i,j} := \{\mathtt{x}_1,\dots,\mathtt{x}_{i-1},\mathtt{x}_{i+1},\dots,\mathtt{x}_{j-1},\mathtt{x}_{j+1},\dots,\mathtt{x}_n\}\kappa[[\mathtt{x}]]$$

if $1 \leqslant i < j \leqslant n$ and

$$\mathfrak{q}_i := \{\mathtt{x}_1,\dots,\mathtt{x}_{k-1},\mathtt{x}_{k+1},\dots,\mathtt{x}_n\}\kappa[[\mathtt{x}]]$$

if $1 \leqslant k \leqslant n$.

pijqk *Remarks* 15.16. (i) Let $\mathfrak{p}_{i,j}$ for some $1 \leqslant i < j \leqslant n$ and $\mathfrak{q}_k$ for some $1 \leqslant k \leqslant n$. Then $\mathfrak{p}_{i,j} \not\subset \mathfrak{q}_k$ if and only if $k \notin \{i,j\}$.

(ii) Let $\mathfrak{a}$ be an ideal of $\kappa[[\mathtt{x}_1,\dots,\mathtt{x}_n]]$ of height $\geqslant n-2$ and generated by square-free monomials of degree 2 and let $\mathfrak{p}$ be a minimal prime ideal associated to $\mathfrak{a}$. As $\mathfrak{a} \subset \mathfrak{m}_n$ and $\mathfrak{a} \neq \mathfrak{m}_n$, we have $\mathfrak{p} \neq \mathfrak{m}_n$. The prime ideal $\mathfrak{p}$ is generated by monomials of degree 1, because the generators of $\mathfrak{a}$ are of the type $\mathtt{x}_i\mathtt{x}_j$ with $1 \leqslant i < j \leqslant n$. In particular $\mathfrak{a}$ is a radical ideal. As $\dim(A) = n - \operatorname{ht}(\mathfrak{a}) \leqslant 2$, we deduce that $\operatorname{ht}(\mathfrak{a}) \geqslant n-2$, so $\operatorname{ht}(\mathfrak{p}) \geqslant n-2$. Thus, either $\mathfrak{p} = \mathfrak{p}_{ij}$ for some $1 \leqslant i < j \leqslant n$ or $\mathfrak{p} = \mathfrak{q}_i$ for some $i = 1,\dots,n$.

(iii) Under the hypothesis of (ii) let $\{\mathfrak{p}_{i,j}\}_{(i,j)\in I}$ (where $I \subset \{1,\dots,n\}^2$) be the collection of the minimal prime ideals associated to $\mathfrak{a}$ of height $n-2$ and let $\{\mathfrak{p}_k\}_{k\in K}$ (where $K \subset \{1,\dots,n\}$) be the collection of the minimal prime ideals associated to $\mathfrak{a}$ of height $n-1$. After reordering the indices $\{1,\dots,n\}$ we may assume by (i) (and suing the fact that $\omega(\mathfrak{a}) = 2$) that there exists $0 \leqslant d \leqslant n$ ($d \neq 1$) such that $I \subset \{1,\dots,d\}^2$ and $K = \{d+1,\dots,n\}$. ∎

redpsdsos **Theorem 15.17.** *Let κ be a (formally) real field. Let $\mathfrak{a}$ be an ideal of $\kappa[[\mathtt{x}_1,\dots,\mathtt{x}_n]]$ of height $\geqslant n-2$ and generated by square-free monomials of degree 2 and denote $A := \kappa[[\mathtt{x}_1,\dots,\mathtt{x}_n]]/\mathfrak{a}$. Let $\{\mathfrak{p}_{i,j}\}_{(i,j)\in I}$ (where $I \subset \{1,\dots,d\}^2$) be the collection of the prime ideals of A of height $n-2$ and let $\{\mathfrak{p}_\ell\}_{d+1\leqslant\ell\leqslant n}$ (for some $0 \leqslant d \leqslant n$ with $d \neq 1$) be the collection of the prime ideals of A of height $n-1$ (see Remarks 15.16). Then $\mathcal{P}(A) = \Sigma A^2$ if and only if I is either empty or a suitable family.*

Proof. We distinguish several cases:

Case 1. $\operatorname{ht}(\mathfrak{a}) = n-1$, that is, $I = \varnothing$. We have $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \mathfrak{q}_n$, so $\mathfrak{a} = (\mathtt{x}_i\mathtt{x}_j :\ 1 \leqslant i < j \leqslant n)$. By Example 10.4 we deduce $\mathcal{P}(A) = \Sigma A^2$. ∎

Case 2. $\operatorname{ht}(\mathfrak{a}) = n-2$ and $d = n$, that is, each minimal prime ideal associated to $\mathfrak{a}$ has height $n-2$. Assume $\mathcal{P}(A) = \Sigma A^2$. Let us check: *$I \subset \{1,\dots,n\}^2$ is a suitable family.*

Let us reorder the indices of the variables $\mathtt{x}_1,\dots,\mathtt{x}_n$ in order to prove that I is a suitable family (standardly presented). We may assume $(1,2) \in I$. We order the elements of I lexicographically and suppose the first k elements constitute a suitable family I_k (standardly presented). Consider the element $(i_{k+1},\ell) \in I\backslash I_k$ (observe that $\ell \leqslant m$ for each $(i_{k+1},m) \in I$). By Example 15.15 it holds $\ell \geqslant j_k + 1$ (because otherwise $\mathcal{P}(A) \neq \Sigma A^2$), so after reordering the indices $\{k+1,\dots,n\}$, we may assume $(i_{k+1},\ell)=(i_{k+1},\max\{i_{k+1},j_k\}+1)$. Thus, after finitely many steps, we deduce that if $\mathcal{P}(A)=\Sigma A^2$, then I is a suitable family (that we have standardly presented). As $\omega(\mathfrak{a})=2$, we have $I=\{(1,2),\dots,(i_k,j_k),\dots,(i_n,j_n)\}$ where $i_k<j_k$, $i_k\leqslant i_{k+1}$ and $j_k<j_{k+1}$.

We have proved in Example 15.14 that if I is a suitable family (standardly presented), then $\mathcal{P}(A)=\Sigma A^2$. ■

CASE 3. $\operatorname{ht}(\mathfrak{a})=n-2$ and $2\leqslant d<n$. We have

$$\mathfrak{a}=\bigcap_{(i,j)\in I}\mathfrak{p}_{i,j}\cap\bigcap_{d\leqslant\ell\leqslant n}\mathfrak{q}_\ell$$

If I is not a suitable family, there exists by Example 15.15

$$f\in\kappa[[\mathtt{x}_1,\dots,\mathtt{x}_d]]\cap(\mathcal{P}(\kappa[[\mathtt{x}]]/\mathfrak{a}')\backslash\Sigma(\kappa[[\mathtt{x}]]/\mathfrak{a}')^2)$$

where $\mathfrak{a}':=\bigcap_{(i,j)\in I}\mathfrak{p}_{i,j}$. Observe that $f=0$ modulo $\mathfrak{a}'':=\bigcap_{d\leqslant\ell\leqslant n}\mathfrak{q}_\ell$. Thus, $f\in\mathcal{P}(A)\backslash\Sigma A^2$, so I is a suitable family. By Examples 15.14 and 10.4 we have $\mathcal{P}(\kappa[[\mathtt{x}]]/\mathfrak{a}')=\Sigma(\kappa[[\mathtt{x}]]/\mathfrak{a}')^2$ and $\mathcal{P}(\kappa[[\mathtt{x}]]/\mathfrak{a}'')=\Sigma(\kappa[[\mathtt{x}]]/\mathfrak{a}'')^2$. By Lemma 15.10 we deduce $\mathcal{P}(A)=\Sigma A^2$, as required. □

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16. OBSTRUCTIONS IN HIGHER EMBEDDING DIMENSIONS

We provide next some obstructions in higher embedding dimension for the 2-dimensional case.

obstruction **Corollary 16.1.** *Let $A:=\kappa[[\mathtt{x}_1,\dots,\mathtt{x}_n]]/\mathfrak{a}$ be a formal ring of dimension 2. After a change of coordinates:*

(i) *There exists $M\in\Sigma\kappa^2$ such that $M^2(\mathtt{x}_1^2+\mathtt{x}_2^2)-\mathtt{x}_k^2\in\mathcal{P}(A)$ for $k=3,\dots,n$.*

Suppose in the following $\mathcal{P}(A)=\Sigma A^2$ and A has embedding dimension n. After a change of coordinates:

(ii) *There exist $\sigma_k\in\Sigma\kappa^2\backslash\{0\}$ and $F\in\kappa[[\mathtt{x},\mathtt{y}]]$ belonging to List 2.1(3) such that $\mathtt{x}_3^2-F(\mathtt{x}_1,\mathtt{x}_2)+\sum_{k=4}^n\sigma_k\mathtt{x}_k^2\in\mathfrak{a}$.*
(iii) *If $q-G\in\mathfrak{a}$ satisfies: (1) q is a positive semidefinite quadratic form of rank $\leqslant n-2$ that does not depend on variables $\mathtt{x}_i,\mathtt{x}_j$, and (2) $G\in\kappa[[\mathtt{x}_i,\mathtt{x}_j]]$, then G is right equivalent to a series belonging to* List 2.1(3) *times $\sigma\in\Sigma\kappa^2\backslash\{0\}$.*

Proof. (i) Using local parametrization (adapted to the general cases of a formal ring over an arbitrary ((formally) real) field κ), one prove that there exist polynomials

$$P_k:=\mathtt{x}_k^{r_k}+\sum_{j=0}^{r_k-1}a_{kj}(\mathtt{x}_1,\mathtt{x}_2)\mathtt{x}_k^j\in\mathfrak{a}\cap\kappa[[\mathtt{x}_1,\mathtt{x}_2)]][\mathtt{x}_k]$$

such that $\omega(a_{kj})\geqslant r_k-j$ for $0\leqslant j\leqslant r_k-1$. By Lemma 5.10 there exists $N\in\Sigma\kappa^2$ such that the quadratic form $(N^2r_k^2+1)^2(\mathtt{x}_1^2+\mathtt{x}_2^2)-\mathtt{x}_k^2\in\mathcal{P}(A)$ for $k=3,\dots,n$. The statement holds if we take $M:=N^2(r_3^2+\cdots+r_n^2)+1$.

(ii) By statement (i) $(n-2)M^2(\mathtt{x}_1^2+\mathtt{x}_2^2)-\sum_{k=3}^n\mathtt{x}_k^2\in\mathcal{P}(A)=\Sigma A^2$. Thus, there exist $f\in\mathfrak{a}$ and $f_i\in\kappa[[\mathtt{x}_1,\dots,\mathtt{x}_n]]$ such that

$$f:=(n-2)M^2(\mathtt{x}_1^2+\mathtt{x}_2^2)-\sum_{k=3}^n\mathtt{x}_k^2-\sum_{i=1}^p f_i\in\mathfrak{a}. \qquad (16.1)$$ expf01

If we make $\mathtt{x}_1=0,\mathtt{x}_2=0$, we deduce

$$-f(0,0,\mathtt{x}_3,\dots,\mathtt{x}_n)=\sum_{k=3}^n\mathtt{x}_k^2+\sum_{i=1}^p f_i(0,0,\mathtt{x}_3,\dots,\mathtt{x}_n).$$

The series $-f(0,0,\mathtt{x}_3,0,\ldots,0)=\mathtt{x}_3^2+\sum_{i=1}^p f_i(0,0,\mathtt{x}_3,0,\ldots,0)^2$ has order 2. By Weierstrass preparation theorem there exist $\tau_3\in\Sigma\kappa^2\setminus\{0\}$, a unit $V_3\in\kappa[[\mathtt{x}_1,\ldots,\mathtt{x}_n]]$ such that $V_3(0,\ldots,0)=1$ and a Weierstrass polynomial

$$P_3:=\mathtt{x}_3^2-2g_3(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_4,\ldots,\mathtt{x}_n)\mathtt{x}_3+h_3(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_4,\ldots,\mathtt{x}_n)$$

such that $-f=P_3V_3^2\tau_3$ and $g_3,h_3\in\kappa[[\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_4,\ldots,\mathtt{x}_n]]$. Write

$$P_3=(\mathtt{x}_3-g_3(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_4,\ldots,\mathtt{x}_n))^2+h_3(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_4,\ldots,\mathtt{x}_n)-(g_3(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_4,\ldots,\mathtt{x}_n))^2.$$

We deduce (using the definition of f and substituting $\mathtt{x}_3=g_3(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_4,\ldots,\mathtt{x}_n)$ and $\mathtt{x}_1=0,\mathtt{x}_2=0$ in (16.1)) that

$$\begin{aligned}h_3(0,0,\mathtt{x}_4,\ldots,\mathtt{x}_n)&-(g_3(0,0,\mathtt{x}_4,\ldots,\mathtt{x}_n))^2\\&=\frac{\tau_3}{V_3(0,0,g_3(0,0,\mathtt{x}_4,\ldots,\mathtt{x}_n),x_4,\ldots,\mathtt{x}_n)^2\tau_3^2}\Big((g_3(0,0,\mathtt{x}_4,\ldots,\mathtt{x}_n))^2\\&\qquad+\sum_{i=4}^n\mathtt{x}_i^2+\sum_{i=1}^p f_i(0,0,g_3(0,0,\mathtt{x}_4,\ldots,\mathtt{x}_n),x_4,\ldots,\mathtt{x}_n)^2\Big).\end{aligned}$$

By Weierstrass preparation theorem there exist $\tau_4\in\Sigma\kappa^2\setminus\{0\}$, a unit $V_4\in\kappa[[\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_4,\ldots,\mathtt{x}_n]]$ such that $V_2(0,\ldots,0)=1$ and a Weierstrass polynomial $P_4:=\mathtt{x}_4^2-2g_4(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_5,\ldots,\mathtt{x}_n)\mathtt{x}_4+h_4(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_5,\ldots,\mathtt{x}_n)$ such that $h_4(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_5,\ldots,\mathtt{x}_n)-(g_4(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_5,\ldots,\mathtt{x}_n))^2=P_4V_4^2\tau_4$ and $h_4,g_4\in\kappa[[\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_5,\ldots,\mathtt{x}_n]]$. Thus,

$$\begin{aligned}P_3=(\mathtt{x}_3-g_3(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_4,\ldots,\mathtt{x}_n))^2&+V_4^2\tau_4((\mathtt{x}_4-g_4(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_5,\ldots,\mathtt{x}_n))^2\\&+h_4(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_5,\ldots,\mathtt{x}_n)-g_4(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_5,\ldots,\mathtt{x}_n)^2).\end{aligned}$$

Proceeding inductively we find $g_k,h_k\in\kappa[[\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_{k+1},\ldots,\mathtt{x}_n]]$, units $V_k\in\kappa[[\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_k,\ldots,\mathtt{x}_n]]$ with $V_k(0,\ldots,0)=1$ and $\tau_k\in\Sigma\kappa^2\setminus\{0\}$ such that

$$\begin{aligned}P_3=(\mathtt{x}_3-g_3(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_4,\ldots,\mathtt{x}_n))^2&+\sum_{k=4}^n\Big(\prod_{j=4}^k\tau_jV_j^2\Big)(\mathtt{x}_k-g_k(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_{k+1},\ldots,\mathtt{x}_n))^2\\&+\Big(\prod_{j=4}^n\tau_jV_j^2\Big)(h_n(\mathtt{x}_1,\mathtt{x}_2)-g_n(\mathtt{x}_1,\mathtt{x}_2)^2).\end{aligned}$$

Write $\sigma_k:=\prod_{j=2}^k\tau_j$ and $U_k:=\prod_{j=2}^k V_j\in\kappa[[\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_4,\ldots,\mathtt{x}_n]]$, which satisfies $U_k(0,\ldots,0)=1$. We have

$$\begin{aligned}P_3=(\mathtt{x}_3-g_3(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_4,\ldots,\mathtt{x}_n))^2&+\sum_{k=4}^n\sigma_kU_k^2(\mathtt{x}_k-g_k(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_{k+1},\ldots,\mathtt{x}_n))^2\\&+\sigma_nU_n^2(h_n(\mathtt{x}_1,\mathtt{x}_2)-g_n(\mathtt{x}_1,\mathtt{x}_2)^2).\end{aligned}$$

After a change of coordinates, we may assume

$$P_3:=\mathtt{x}_3^2+\sum_{k=4}^n\sigma_k\mathtt{x}_k^2-H(\mathtt{x}_1,\mathtt{x}_2)\in\mathfrak{a},$$

for some $H\in\mathfrak{m}_2\subset\kappa[[\mathtt{x}_1,\mathtt{x}_2]]$. Now, we follow the proof of [Fe8, Thm.1.5] in its Section 3 (List of candidates). There it is proved that if H is not right equivalent to a series in List 2.1(3) there exists a series $g\in\mathcal{P}(A)$ of order 1. As $\omega(\mathfrak{a})=2$, we conclude that $\mathcal{P}(A)\neq\Sigma A^2$, which is a contradiction. Consequently, after an additional change of coordinates

$$P_3:=\mathtt{x}_3^2-F(\mathtt{x}_1,\mathtt{x}_2)+\sum_{k=4}^n\sigma_k\mathtt{x}_k^2\in\mathfrak{a}$$

where $F\in\kappa[[\mathtt{x},\mathtt{y}]]$ belongs to List 2.1(3).

(iii) After a change of coordinates (and proceeding similarly to (ii)) we may assume

$$q - G = \mathtt{x}_3^2 + \sum_{k=4}^{r} \lambda_k \mathtt{x}_k^2 - G'(\mathtt{x}_1, \mathtt{x}_2)$$

for some $\lambda_k \in \Sigma\kappa^2\backslash\{0\}$, $4 \leqslant r \leqslant n$ and $G' \in \mathfrak{m}_2 \subset \kappa[[\mathtt{x}_1, \mathtt{x}_2]]$. Again, we follow the proof of [Fe8, Thm.1.5] in its Section 3 (List of candidates). There it is proved that if G' is not right equivalent to a series in List 2.1(3) there exists a series $g \in \mathcal{P}(A)$ of order 1. As $\omega(\mathfrak{a}) = 2$, we conclude that $\mathcal{P}(A) \neq \Sigma A^2$, which is a contradiction. Consequently, G' and thus G are right equivalent to a series belonging to List 2.1(3) times $\sigma \in \Sigma\kappa^2\backslash\{0\}$, as required. □

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17. Global-local property

In this section we prove the following global-local property.

glp **Theorem 17.1** (Global-local property)**.** *Let A be a ring that contains a (formally) real field κ_0. Let $\mathfrak{n}$ be a maximal ideal of A such that $A/\mathfrak{n} = \kappa_0$. Assume that $A_\mathfrak{n}$ is a local excellent ring and let $\mathtt{x}_1, \dots, \mathtt{x}_n \in A$ be a system of generators of $\mathfrak{m} := \mathfrak{n}A_\mathfrak{n}$. Assume that the family of sets $\mathcal{U}_a := \{\beta \in \operatorname{Sper}(A) : x_1^2 + \cdots + x_n^2 <_\beta a^2\}$ where $a \in \kappa_0\backslash\{0\}$ constitutes a basis of neighborhoods in A of the set of prime cones with support $\mathfrak{n}$. Assume that $\mathcal{P}(A) = \Sigma A^2$ and $p := p(\widehat{A_\mathfrak{n}}) < +\infty$. Then:*

(i) $\mathcal{P}(\widehat{A_\mathfrak{n}}) = \Sigma\widehat{A_\mathfrak{n}}^2$.
(ii) *Both $\widehat{A_\mathfrak{n}}$ and $A_\mathfrak{n}$ are real reduced rings.*
(iii) $\dim_r(A_\mathfrak{n}) \leqslant \dim(A_\mathfrak{n}) = \dim(\widehat{A_\mathfrak{n}}) = \dim_r(\widehat{A_\mathfrak{n}}) \leqslant 2$.

Before proving Theorem 17.1 we need some preliminary results.

approx12 **Lemma 17.2.** *Let $\mathfrak{a} \subset \kappa[[\mathtt{x}]]$ be a real ideal and let $A := \kappa[[\mathtt{x}]]/\mathfrak{a}$. For each $f \in \mathcal{P}(A)$ and each $m \geqslant 1$ there exists $f_m \in \kappa[\mathtt{x}] \cap \mathcal{P}(A)$ such that $f - f_m \in \mathfrak{m}_n^m$.*

Proof. Write $f = \sum_\nu a_\nu \mathtt{x}^\nu$ and $g_m := \sum_{\nu,\ |\nu| \leqslant 2m} a_\nu \mathtt{x}^\nu$. Consider the constructible set $C := \{\alpha \in \operatorname{Sper}(A) : \mathfrak{a} \subset \operatorname{supp}(\alpha)\}$. We claim: $\mathcal{P}(C) = \mathcal{P}(A)$.

Consider the canonical homomorphism $\varphi : A \to A/\mathfrak{a}$ and consider the spectral map $\operatorname{Sper}(\varphi) : \operatorname{Sper}(A/\mathfrak{a}) \to \operatorname{Sper}(A),\ \alpha \mapsto \varphi^{-1}(\alpha)$. The image of $\operatorname{Sper}(\varphi)$ is the collection of all prime cones of A such that $\mathfrak{a} \subset \operatorname{supp}(\alpha)$. Consequently, $\mathcal{P}(C) = \mathcal{P}(A)$, as claimed.

By Lemma 5.19 $h \in \mathcal{P}(C)$ if and only if $f \geqslant_\alpha 0$ for each $\alpha \in C$ such that $\operatorname{ht}(\operatorname{supp}(\alpha)) \geqslant n-1$. Denote C_1 the set of the $\alpha \in C$ such that $\operatorname{ht}(\operatorname{supp}(\alpha)) \geqslant n-1$. Let $\alpha \to \alpha_0$ be a specialization of α such that $\operatorname{supp}(\alpha_0) = \mathfrak{m}_n$. By Lemma 5.21 we have $\phi_\alpha : \kappa[[\mathtt{x}]] \to \mathfrak{R}(\alpha_0)[[\mathtt{t}]]$ such that $h \geqslant_\alpha 0$ if and only if $\phi_\alpha(h) \geqslant 0$ when $\mathtt{t} > 0$.

As $f \in \mathcal{P}(C)$, for each $\alpha \in C_1$ we have $\phi_\alpha(f) \geqslant 0$. Let $2k_\alpha = \omega_\mathtt{t}(\phi_\alpha(\mathtt{x}_1^2 + \cdots + \mathtt{x}_n^2)) = 2\min\{\omega_\mathtt{t}(\phi_\alpha(\mathtt{x}_i)) : 1 \leqslant i \leqslant n\}$ and observe that $k \geqslant 1$. If $h \in \mathfrak{m}_n^{2m+1}$, then $\omega_t(\phi_\alpha(h)) \geqslant (2m+1)k$. As $f - g_m \in \mathfrak{m}_n^{2m+1}$, we have $\omega_t(\phi_\alpha(f - g_m)) \geqslant (2m+1)k$. Define $f_m := g_m + (\mathtt{x}_1^2 + \cdots + \mathtt{x}_n^2)^m \in \kappa[\mathtt{x}]$. We claim: $\phi_\alpha(f_m) \geqslant 0$ *if* $\mathtt{t} > 0$. We distinguish two case:

Case 1. If $\omega_\mathtt{t}(\psi_\alpha(f)) \leqslant 2mk$, we have $\psi_\alpha(f) = a_\ell \mathtt{t}^\ell + \cdots$ for some $a_\ell >_{\alpha_0} 0$, so

$$\begin{aligned}\psi_\alpha(f_m) = \psi_\alpha(g_m + (\mathtt{x}_1^2 + \cdots + \mathtt{x}_n^2)^m) &= \psi_\alpha(f) + \psi_\alpha(g_m - f) + \psi_\alpha((\mathtt{x}_1^2 + \cdots + \mathtt{x}_n^2)^m)\\ &= \psi_\alpha((\mathtt{x}_1^2 + \cdots + \mathtt{x}_n^2)^m) + a_\ell \mathtt{t}^\ell + \cdots \geqslant_\alpha 0.\end{aligned}$$

Case 2. If $\omega_\mathtt{t}(\psi_\alpha(f)) \geqslant 2mk + 1$, we have

$$\psi_\alpha(f_m) = \psi_\alpha(g_m + (\mathtt{x}_1^2 + \cdots + \mathtt{x}_n^2)^m) = \psi_\alpha(f) + \psi_\alpha(g_m - f) + \psi_\alpha((\mathtt{x}_1^2 + \cdots + \mathtt{x}_n^2)^m) \geqslant_\alpha 0,$$

because $\omega_{\mathtt{t}}(\psi_\alpha((\mathtt{x}_1^2+\cdots+\mathtt{x}_n^2)^m)) = 2mk$, $\omega_{\mathtt{t}}(\psi_\alpha(f)) \geqslant 2mk+1$ and $\omega_t(\phi_\alpha(g_m-f)) \geqslant (2m+1)k \geqslant 2mk+1$, because $k \geqslant 1$.

As this happens for each $\alpha \in C_1$, we conclude $f \in \mathcal{P}(C)$, so $f \in \mathcal{P}(A) \cap \kappa[\mathtt{x}]$, as required. □

approx13 **Corollary 17.3.** *Let A be a ring and let $\mathfrak{p}$ be a prime ideal of A such that $A_{\mathfrak{p}}$ is a local excellent ring. Let $f \in \mathcal{P}(\widehat{A_{\mathfrak{p}}})$. For each $m \geqslant 1$ there exists $f_m \in \mathcal{P}(A_{\mathfrak{p}}) \cap A$ and $a_m \in A \backslash \mathfrak{p}$ such that $f - \frac{f_m}{a_m^2} \in (\mathfrak{p}\widehat{A_{\mathfrak{p}}})^m$ for each $m \geqslant 1$.*

Proof. The residue field of $\widehat{A_{\mathfrak{p}}}$ is $\kappa := A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = (A\backslash\mathfrak{p})^{-1}(A/\mathfrak{p}) = \operatorname{qf}(A/\mathfrak{p})$. Then $\widehat{A_{\mathfrak{p}}} \cong \kappa[[\mathtt{y}_1,\dots,\mathtt{y}_\ell]]/\mathfrak{b}$ for some $\ell \geqslant 1$ and some ideal $\mathfrak{b}$ of $\kappa[[\mathtt{y}_1,\dots,\mathtt{y}_\ell]]$. In addition, the maximal ideal of $\widehat{A_{\mathfrak{p}}}$ is $\mathfrak{p}A_{\mathfrak{p}}\widehat{A_{\mathfrak{p}}}$. Let $h_1,\dots,h_\ell \in A$ be a system of generators of $\mathfrak{p}A_{\mathfrak{p}}$.

For each $m \geqslant 1$ there exists by Lemma 17.2 a polynomial $g_m \in \kappa[\mathtt{y}_1,\dots,\mathtt{y}_m] \cap \mathcal{P}(\widehat{A_{\mathfrak{p}}})$ such that $g_m - f \in (\mathfrak{p}\widehat{A_{\mathfrak{p}}})^m$. Let $a_m \in A\backslash\mathfrak{p}$ be such that $a_m^2 g_m \in A[\mathtt{y}_1,\dots,\mathtt{y}_m]$. Define $f_m := a_m^2 g_m(h_1,\dots,h_\ell) \in A \cap \mathcal{P}(A_{\mathfrak{p}})$, which satisfies $f - \frac{f_m}{a_m^2} \in (\mathfrak{p}\widehat{A_{\mathfrak{p}}})^m$, as required. □

Remark 17.4. If $\mathfrak{p} = \mathfrak{n}$ is a maximal ideal of A such that $\kappa := A/\mathfrak{n}$ is isomorphic to a subfield of A, we may assume that $a_m = 1$ for each $m \geqslant 1$. ∎

We are ready to prove Theorem 17.1.

Proof of Theorem 17.1. (i) Let $f \in \mathcal{P}(\widehat{A_{\mathfrak{n}}})$. By Corollary 17.3 for each $m \geqslant 1$ there exists $f_m \in A \cap \mathcal{P}(\widehat{A_{\mathfrak{n}}})$ such that $f - f_m \in \mathfrak{n}^m\widehat{A_{\mathfrak{n}}}$.

Let $\mathcal{V} := \{\beta \in \operatorname{Sper}(A) :\ f_m <_\beta 0\}$. If $\mathcal{V} = \varnothing$, we have $f \in \mathcal{P}(A)$. Otherwise, we pick $\beta \in \mathcal{V}$. Let $\beta \to \beta_0$ be a maximal specialization of β. As $\operatorname{Sper}_{\max}(A)$ is a Hausdorff space, either β_0 has $\mathfrak{n}$ as support or there exists $a_\beta \in \kappa\backslash\{0\}$ such that both $\beta, \beta_0 \in \{\gamma \in \operatorname{Sper}(A) :\ a_\beta^2 <_\beta x_1^2+\cdots+x_n^2\}$.

Suppose next $\operatorname{supp}(\beta_0) = \mathfrak{n}$. By [ABR, Thm.VII.3.2] there exists a specialization $\widehat{\beta} \to \widehat{\beta}_0$ of $\widehat{A_{\mathfrak{n}}}$ lying over $\beta \to \beta_0$, that is, $\widehat{\beta}_0 \cap A = \beta_0$ and $\widehat{\beta} \cap A = \beta$. As $f_m \in \mathcal{P}(\widehat{A_{\mathfrak{n}}})$, we deduce that $f_m \in \widehat{\beta} \cap A = \beta$, which is a contradiction. Consequently, $\operatorname{supp}(\beta_0) \neq \mathfrak{n}$.

As $\mathcal{V}$ is by [BCR, Cor.7.1.13] quasi-compact, there exist $\beta_1,\dots,\beta_s \in \mathcal{V}$ such that

$$\mathcal{V} \subset \bigcup_{j=1}^{s} \{\gamma \in \operatorname{Sper}(A) :\ a_{\beta_j}^2 <_\gamma x_1^2+\cdots+x_n^2\}.$$

Let $a \in A\backslash\mathfrak{n}$ be such

$$\frac{1}{a} = 1 + \sum_{j=1}^{s} \frac{1}{a_{\beta_j}^2}$$

and observe that $a <_\eta a_{\beta_j}^2$ for each $j = 1,\dots,s$ and each $\eta \in \operatorname{Sper}(A)$. Thus, $\mathcal{V} \subset \{\beta \in \operatorname{Sper}(A) :\ a^2 <_\beta x_1^2+\cdots+x_n^2\}$. We claim: *There exists $M \in \kappa\backslash\{0\}$ such that $F_m := f_m + M^2(x_1^2+\cdots+x_n^2)^m \in \mathcal{P}(A) = \Sigma A^2$.*

We have proved that

$$1 <_\beta \frac{x_1^2+\cdots+x_n^2}{a^2}$$

for each $\beta \in \mathcal{V}$. Fix $\beta \in \mathcal{V}$. If $x^\nu := x_1^{\nu_1}\cdots x_n^{\nu_n}$ satisfies $|\nu| \leqslant m$, then

$$|x^{2\nu}|_\beta \leqslant_\beta (x_1^2+\cdots+x_n^2)^{|\nu|} <_\beta \frac{(x_1+\cdots+x_n^2)^{2m}}{a^{2(2m-|\nu|)}} \rightsquigarrow |x^\nu|_\beta <_\beta \frac{(x_1+\cdots+x_n^2)^{2m}}{a^{2(2m-|\nu|)}}.$$

In addition, $|b|_\beta < b^2+1$ for each $b \in \kappa$. Thus, for $f_m \in \kappa[x]$, there exists $N_\beta \in \Sigma\kappa^2\backslash\{0\}$ such that $|f_m|_\beta <_\beta N_\beta^2 + (x_1^2+\cdots+x_n^2)^m$. Thus, $f_m + N_\beta^2 + (x_1^2+\cdots+x_n^2)^m >_\beta 0$ and $\mathcal{V} \subset \bigcup_{\beta\in\mathcal{V}} \mathcal{U}_\beta$, where $\mathcal{U}_\beta = \{\gamma \in \operatorname{Sper}(A) :\ f_m + N_\beta^2 + (x_1^2+\cdots+x_n^2)^m >_\gamma 0\}$. As $\mathcal{V}$ is by

[BCR, Cor.7.1.13] quasi-compact, there exist $\beta_1,\dots,\beta_s\in\mathcal{V}$ such that $\mathcal{V}\subset\bigcup_{j=1}^s\mathcal{U}_{\beta_j}$. We have $F_m:=f_m+\sum_{j=1}^s N_{\beta_j}^2+(x_1^2+\cdots+x_n^2)^m\in\mathcal{P}(A)$.

Observe that $f-F_m\in\mathfrak{n}^m\widehat{A_\mathfrak{n}}$ for each $m\geqslant 1$. We have $F_m\in\Sigma A^2\subset\Sigma\widehat{A_\mathfrak{n}}^2=\Sigma_p\widehat{A_\mathfrak{n}}^2$ for each $m\geqslant 1$. By Strong Artin's Approximation we deduce that $f\in\Sigma_p\widehat{A_\mathfrak{n}}^2$. Consequently, $\mathcal{P}(\widehat{A_\mathfrak{n}})=\Sigma\widehat{A_\mathfrak{n}}^2$.

(ii) By [Sch2, Lem.6.3] we deduce $\widehat{A_\mathfrak{n}}$ is a real reduced ring. As $A_\mathfrak{n}\hookrightarrow\widehat{A_\mathfrak{n}}$, we deduce $\widehat{A_\mathfrak{n}}$ is a real reduced ring too.

(iii) By Theorem 1.4 and §1.3.1 we have $\dim(\widehat{A_\mathfrak{n}})=\dim_r(\widehat{A_\mathfrak{n}})\leqslant 2$. By [ABR, Thm.VII.7.2] we have $\dim_r(A_\mathfrak{n})\leqslant\dim_r(\widehat{A_\mathfrak{n}})$, whereas by [AM, Cor.11.19] we have $\dim(\widehat{A_\mathfrak{n}})=\dim(A_\mathfrak{n})$. We conclude

$$\dim_r(A_\mathfrak{n})\leqslant\dim(A_\mathfrak{n})=\dim(\widehat{A_\mathfrak{n}})=\dim_r(\widehat{A_\mathfrak{n}})\leqslant 2,$$

as required. $\square$

gle *Examples* 17.5. (i) Let $A:=\kappa[\mathtt{x}]/\mathfrak{a}$ be a polynomial ring and let $\mathfrak{n}:=\{\mathtt{x}\}A$, which satisfies $A/\mathfrak{n}=\kappa$. We claim: *The family of sets $\mathcal{U}_a:=\{\beta\in\operatorname{Sper}(A):\ \mathtt{x}_1^2+\cdots+\mathtt{x}_n^2<_\beta a^2\}$ where $a\in\kappa_0\setminus\{0\}$ constitutes a basis of neighborhoods in A of the set of prime cones with support $\mathfrak{n}$.*

Let $\beta\in\operatorname{Sper}_{\max}(A)$ and suppose that $\mathtt{x}_1^2+\cdots+\mathtt{x}_n^2<_\beta a^2$ for each $a\in\kappa\setminus\{0\}$. Let us check: $\mathfrak{n}$ is a β-convex ideal, that is, if $Q_1,Q_2\in\beta$ and $Q_1+Q_2\in\mathfrak{n}$, then $Q_1,Q_2\in\mathfrak{n}$.

If $Q_1(0)\neq 0$, then $Q_2(0)=-Q_1(0)$, so $Q_1(0)$ and $Q_2(0)$ are non-zero and have opposite signs in $\mathfrak{R}(\beta)$ (we have used that $\operatorname{supp}(\beta)\cap\kappa=\{0\}$). Denote $g_i:=Q_i-Q_i(0)\in(\mathtt{s},\mathtt{t})\kappa[\mathtt{s},\mathtt{t}]$ and write $g_i=\sum_{1\leqslant|\nu|\leqslant d}a_\nu\mathtt{x}^\nu$ where $a_\nu\in\kappa$ and $d\geqslant 0$. We have

$$|g_i|_\beta\leqslant_\beta\sum_{1\leqslant|\nu|\leqslant d}|a_\nu|_\beta|\mathtt{x}|_\beta^\nu<_\beta\sum_{1\leqslant|\nu|\leqslant d}|a_\nu|_\beta a^2<_\beta\Big(\sum_{1\leqslant|\nu|\leqslant d}(a_\nu^2+1)\Big)^2a^2$$

for each $a\in\kappa\setminus\{0\}$. Thus, $|g_i|_\beta<a^2$ for each $a\in\kappa\setminus\{0\}$. In particular, as $Q_i(0)\neq 0$, we have $|g_i|_\beta<\frac{Q_i(0)^2}{(1+Q_i(0)^2)^2}<_\beta|Q_i(0)|_\beta$. As $Q_i=g_i+Q_i(0)\in\beta$, we deduce $Q_i(0)>_\beta 0$, against the fact that $Q_1(0)$ and $Q_2(0)$ have opposite signs in $\mathfrak{R}(\beta)$. Consequently, $Q_i(0)=0$ for $i=1,2$ and $Q_i\in\mathfrak{n}$.

As $\beta\in\operatorname{Sper}_{\max}(A)$, we deduce by [BCR, Prop.4.3.8] $\operatorname{supp}(\beta)=\mathfrak{n}$. Consequently, $\{\mathcal{U}_a\}_{a\in\kappa\setminus\{0\}}$ is a basis of neighborhoods in A of the set of prime cones with support $\mathfrak{n}$

(ii) Let $(A,\mathfrak{m})$ be a local excellent henselian ring such that $\kappa:=A/\mathfrak{m}$ is a subfield of A. By [ABR, Prop.II.2.4] each non-refinable specialization chain finishes on a prime cone α whose support is $\mathfrak{m}$. Thus, $\operatorname{Sper}(A/\mathfrak{m})\subset\mathcal{U}_a$ for each $a\in\kappa\setminus\{0\}$.

(iii) Let $A:=\mathcal{O}(X)=\mathcal{O}(\Omega)/\mathcal{J}(X)$ be the ring of global analytic on a global analytic set $X\subset\Omega$, where $\Omega\subset\mathbb{R}^n$ is an open set, and assume $0\in X$. Denote $\mathfrak{m}_0$ the maximal ideal of A associated to 0 and observe that $A/\mathfrak{m}_x=\mathbb{R}$. We claim: *The family of sets $\mathcal{U}_a:=\{\beta\in\operatorname{Sper}(A):\ \mathtt{x}_1^2+\cdots+\mathtt{x}_n^2<_\beta a^2\}$ where $a\in\mathbb{R}\setminus\{0\}$ constitutes a basis of neighborhoods in A of the unique prime cone whose support $\mathfrak{n}$.*

By [ABR, Prop.VIII.4.4] the localization $A_{\mathfrak{m}_x}$ is a local excellent ring. Let $\beta_0\in\operatorname{Sper}_{\max}(A)$ such that $\beta\in\mathcal{U}_a$ for each $a\in K\setminus\{0\}$. By [ABR, VIII.Prop.5.4] we have that $\mathcal{C}_a=\{\beta\in\operatorname{Sper}(A):\ \mathtt{x}_1^2+\cdots+\mathtt{x}_n^2\leqslant_\beta a^2\}$ is contained in the set of all generizations of the prime cones with supports maximal ideals associated to points of X, because $\mathcal{C}_a\cap X$ is a compact set. In particular, as β_0 is a maximal prime cone, we deduce $\operatorname{supp}(\beta_0)=\mathfrak{m}_x$ for some $x\in X$. As $\beta_0\in\bigcap_{a\in\mathbb{R}\setminus\{0\}}\mathcal{U}_a\subset\bigcap_{a\in\mathbb{R}\setminus\{0\}}\mathcal{C}_a=\{\mathfrak{m}_0\}$, we deduce $\operatorname{supp}(\beta_0)=\mathfrak{m}_0$, as required. ▪

In [Sch1, Cor.4.6] Scheiderer proved the following.

schlocal **Corollary 17.6.** *Let κ be a field, and let $f \in \kappa[\mathtt{x},\mathtt{y}]$. Then f is a sum of squares in the power series ring $\kappa[[\mathtt{x},\mathtt{y}]]$ if, and only if, for every real closure R of κ, the polynomial f has non-negative values in a neighborhood of the origin in R^2.*

As a consequence of Theorem 17.1 and Example 17.5(i) we have the following improvement, which shows that we can choose the same ratio for all the neighborhoods of the origin in R^n.

impschlocal **Corollary 17.7.** *Let κ be a field, and let $f \in \kappa[\mathtt{x}]$. Then $f \in \mathcal{P}(\kappa[[\mathtt{x}]])$ if and only if there exists $\varepsilon \in \kappa\backslash\{0\}$ such that for every real closure R of κ, the polynomial f has non-negative values in the neighborhood $\{\|\mathtt{x}\|^2 \leqslant \varepsilon^2\}$ of the origin in R^n.*

Proof. Suppose first that $f \in \mathcal{P}(\kappa[[\mathtt{x}]])$. As we are in the hypothesis of Theorem 17.1 as a consequence of Example 17.5(i), we deduce from the proof of Theorem 17.1 that there exists $\varepsilon \in \kappa\backslash\{0\}$ such that $\{\alpha \in \operatorname{Sper}(\kappa[\mathtt{x}]):\ f <_\alpha 0\} \subset \{\alpha \in \operatorname{Sper}(\kappa[\mathtt{x}]):\ \varepsilon^2 <_\alpha \mathtt{x}_1^2 + \cdots + \mathtt{x}_n^2\}$. Thus,

$$\{\alpha \in \operatorname{Sper}(\kappa[\mathtt{x}]):\ \mathtt{x}_1^2 + \cdots + \mathtt{x}_n^2 \leqslant_\alpha \varepsilon^2\} \subset \{\alpha \in \operatorname{Sper}(\kappa[\mathtt{x}]):\ f \geqslant_\alpha 0\}.$$

Conversely, suppose that for every real closure R of κ, the polynomial f has non-negative values in the neighborhood $\{\|\mathtt{x}\|^2 \leqslant \varepsilon^2\}$ of the origin in R^n. Proceeding as in the proof of [Sch1, Cor.4.6] one shows that $f \in \mathcal{P}(\kappa[[\mathtt{x}]])$, as required. □

17.1. **Global analytic functions.** In [Fe7] we consider the case $A := \mathcal{O}(X) = \mathcal{O}(\Omega)/\mathcal{J}(X)$ is the ring of global analytic functions on a global analytic subset X of an open subset $\Omega \subset \mathbb{R}^n$. Denote $\mathcal{O}_\Omega$ the sheaf of rings of analytic germs on Ω and consider the sheaf of (quotient) rings $\mathcal{O}_X := \mathcal{O}_\Omega/(\mathcal{J}(X)\mathcal{O}_\Omega)$. By [ABR, Prop.VIII.4.4] the localization $\mathcal{O}(X)_{\mathfrak{m}_x}$ is a local excellent ring, the homomorphism $\mathcal{O}(X)_{\mathfrak{m}_x} \to \mathcal{O}_{X,x}$ is faithfully flat and it extends to an isomorphism between their respective completions. Let $\operatorname{Sing}(X)$ be the set of points of X such that $\mathcal{O}(X)_{\mathfrak{m}_x}$ is not a regular local ring. As $X\backslash\operatorname{Sing}(X) \neq \varnothing$, we pick a point $x \in X\backslash\operatorname{Sing}(X)$. Then $\mathcal{O}(X)_{\mathfrak{m}_x}$ is a real reduced ring and by Theorem 17.1 we have $\dim(\mathcal{O}(X)_{\mathfrak{m}_x}) = \dim(\mathcal{O}(X)_{\mathfrak{m}_x}) \leqslant 2$. Thus,

$$\dim(X) = \max\{\dim(\mathcal{O}(X)_{\mathfrak{m}_x}):\ x \in X\backslash\operatorname{Sing}(X)\} \leqslant 2,$$

so $\dim(\mathcal{O}(X)_{\mathfrak{m}_x}) \leqslant 2$ for each $x \in X$. Thus, $\dim(\widehat{\mathcal{O}(X)_{\mathfrak{m}_x}}) \leqslant 2$ and as it is a local henselian excellent ring with residue field $\mathbb{R}$ has $\tau(\mathbb{R}) = 1$, we deduce by Theorem 1.7 and Example 17.5(iii) that $p(\widehat{\mathcal{O}(X)_{\mathfrak{m}_x}}) < +\infty$ for each $x \in X$. By Theorem 17.1 we have $\mathcal{P}(\widehat{\mathcal{O}(X)_{\mathfrak{m}_x}}) = \Sigma\widehat{\mathcal{O}(X)_{\mathfrak{m}_x}}^2$, the rings $\mathcal{O}(X)_{\mathfrak{m}_x}$ and $\widehat{\mathcal{O}(X)_{\mathfrak{m}_x}} = \widehat{\mathcal{O}_{X,x}}$ are real reduced and $\dim(\widehat{\mathcal{O}(X)_{\mathfrak{m}_x}}) = \dim(\mathcal{O}(X)_{\mathfrak{m}_x}) = \dim_r(\mathcal{O}(X)_{\mathfrak{m}_x}) = \dim_r(\widehat{\mathcal{O}(X)_{\mathfrak{m}_x}}) \leqslant 2$ for each $x \in X$. As $\widehat{\mathcal{O}_{X,x}}$ is real reduced, we have $\mathcal{J}(X)\mathcal{O}_{\Omega,x}$ is a real radical ideal for each $x \in X$. Consequently, $\mathcal{O}_X$ is a coherent sheaf of ideals. As $\mathcal{O}_{X,x}$ is a local excellent henselian ring and $\mathcal{P}(\widehat{\mathcal{O}_{X,x}}) = \Sigma\widehat{\mathcal{O}_{X,x}}^2$, we deduce $\mathcal{P}(\mathcal{O}_{X,x}) = \Sigma\mathcal{O}_{X,x}^2$ for each $x \in X$. In addition, by the curve selection lemma and Lemma 5.19 we have

$$\mathcal{P}(\mathcal{O}_{X,x}) = \{f_x \in \mathcal{O}_{X,x}:\ f \geqslant 0 \text{ on a small neighborhood } V \subset X \text{ of } x\}.$$

In [Fe7, Thm.1.6] we proved a kind of converse of Theorem 17.1. Namely,

global1 **Theorem 17.8.** *Let $X \subset \mathbb{R}^n$ be a global analytic subset of an open subset $\Omega \subset \mathbb{R}^n$. Suppose that the local embedding dimension of X is bounded by 3. Then $\mathcal{P}(\mathcal{O}(X)) = \Sigma\mathcal{O}(X)^2$ if and only if $\mathcal{P}(\mathcal{O}_{X,x}) = \Sigma\mathcal{O}_{X,x}^2$ for each $x \in X$ and $(X,\mathcal{O}_X)$ is a (real) coherent analytic set. If such is the case $p(\mathcal{O}(X)) \leqslant 6$.*

17.2. **Irreducible components.** Let κ be a (formally) real field, denote $\mathtt{x} := (\mathtt{x}_1,\ldots,\mathtt{x}_n)$ and let $A := \kappa[[\mathtt{x}]]/\mathfrak{a}$ be such that $\omega(\mathfrak{a}) = 2$ and $\mathcal{P}(A) = \Sigma A^2$, so $\dim(A) = d \leqslant 2$. Let us apply Theorem 17.1 to A. Let $\mathfrak{p}$ be a prime ideal of A. We distinguish several cases:

Case 1. If $\operatorname{ht}(\mathfrak{p}) = d$, then $\mathfrak{p} = \mathfrak{m}$ and $A_\mathfrak{m} = A$, which has the property $\mathcal{P}(A) = \Sigma A^2$ by hypothesis.

CASE 2. If $\mathrm{ht}(\mathfrak{p}) = 0$, then $\mathfrak{p}$ is a minimal (real) prime ideal of A and let $\mathfrak{q}$ be the ideal of $\kappa[[\mathtt{x}]]$ that contains $\mathfrak{a}$ and satisfies $\mathfrak{p} = \mathfrak{q}/\mathfrak{a}$. We have $A_\mathfrak{p} = \kappa[[\mathtt{x}]]_\mathfrak{q}/\mathfrak{q}\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]_\mathfrak{q} = (\kappa[[\mathtt{x}]]/\mathfrak{q})_{(0)} = \mathrm{qf}(\kappa[[\mathtt{x}]]/\mathfrak{q})$, which is a (formally) real field and has (by default) the property $\mathcal{P}(A_\mathfrak{p}) = \Sigma A_\mathfrak{p}^2$.

CASE 3. If $d = 2$ and $\mathrm{ht}(\mathfrak{p}) = 1$, we will have further information. Let $\mathfrak{q}$ be a prime ideal of $\kappa[[\mathtt{x}]]$ such that $\mathfrak{p} = \frac{\mathfrak{q}}{\mathfrak{a}}$. As $A/\mathfrak{p} \cong \kappa[[\mathtt{x}]]/\mathfrak{q}$ has dimension 1, we have $\mathrm{ht}(\mathfrak{q}) = n-1$ and $A_\mathfrak{p} = \kappa[[\mathtt{x}]]_\mathfrak{q}/\mathfrak{a}\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]_\mathfrak{q}$ is a ring of dimension 1. Observe that $\widehat{A_\mathfrak{p}}$ is a complete ring whose residue field is $L := A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \cong \kappa[\mathtt{x}]_\mathfrak{q}/(\mathfrak{q}\kappa[\mathtt{x}]_\mathfrak{q}) = (\kappa[[\mathtt{x}]]/\mathfrak{q})_{(0)} = \mathrm{qf}(\kappa[[\mathtt{x}]]/\mathfrak{q})$. By Rückert's parameterization there exists, after a change of coordinates, an irreducible Weierstrass polynomial $P \in \kappa[[\mathtt{x}_1]][\mathtt{x}_2]$ of discriminant $\Delta \in \mathtt{x}_1\kappa[[\mathtt{x}_1]]\setminus\{0\}$ and polynomials $Q_3,\ldots,Q_n \in \kappa[[\mathtt{x}_1]][\mathtt{x}_2]$ such that $\Delta^q\mathfrak{q} \subset \{P, \Delta\mathtt{x}_3 - Q_3, \ldots, \Delta\mathtt{x}_n - Q_n\} \subset \mathfrak{q}$. We may also assume that all the minimal prime ideals $\mathfrak{p}_i$ associated to $\mathfrak{a}$ of height $n-2$ satisfy $\mathfrak{p}_i \cap \kappa[[\mathtt{x}_1,\mathtt{x}_2]] = \{0\}$. Observe that $\{P, \Delta\mathtt{x}_3 - Q_3, \ldots, \Delta\mathtt{x}_n - Q_n\}$ is a system of $n-2$ generators of $\mathfrak{q}\kappa[\mathtt{x}]_\mathfrak{q}$. Thus,

$$\widehat{A_\mathfrak{p}} = L[[\mathtt{y}_1,\ldots,\mathtt{y}_{n-1}]]/\mathfrak{b}$$

for the ideal $\mathfrak{b}$ of $L[[\mathtt{y}_1,\ldots,\mathtt{y}_{n-1}]]$. We have $L = \mathrm{qf}(\kappa[[\mathtt{x}]]/\mathfrak{q}) = \kappa((\mathtt{x}_1))[\mathtt{x}_2]/(P)$, where $P \in \kappa((\mathtt{x}_1))[\mathtt{x}_2]$ is an irreducible monic polynomial, and it is a (formally) real field, because $\mathfrak{q}$ is a real prime ideal of $\kappa[[\mathtt{x}]]$. Thus, $L = \kappa((\mathtt{x}_1))[\zeta]$ for some $\zeta \in \overline{\kappa}[[\mathtt{x}_1^{1/m}]]$ for some $m \geqslant 1$ (recall that P is an irreducible Weierstrass polynomial). By Lemma A.1 there exists a finite algebraic extension $F|\kappa$ such that $\zeta \in F((\mathtt{x}_1))[[\mathtt{x}_1^{1/m}]] = F((\mathtt{x}_1))[\mathtt{s}]/(\mathtt{s}^m - \mathtt{x}_1) \cong F((\mathtt{s}))$ and F is (formally) real. In case $\kappa = R$ is a real closed field, then $F = \kappa = R$ and $L = R((\mathtt{s}))$. Let us check: $\mathcal{P}(\widehat{A_\mathfrak{p}}) = \Sigma\widehat{A_\mathfrak{p}}^2$.

ssos **Theorem 17.9.** *Under the previous hypothesis,* $\mathcal{P}(\widehat{A_\mathfrak{p}}) = \Sigma\widehat{A_\mathfrak{p}}^2$.

Proof. Let $f \in \mathcal{P}(\widehat{A_\mathfrak{p}})$ and let $m \geqslant 1$. By Corollary 17.3 there exists $f_m \in \mathcal{P}(A_\mathfrak{p}) \cap A$ and $a_m \in A\setminus\mathfrak{p}$ such that $f - \frac{f_m}{a_m^2} \in (\mathfrak{p}\widehat{A_\mathfrak{p}})^m$. Define $\mathcal{V}_m := \{\alpha \in \mathrm{Sper}(A) :\ \Delta^2\mathtt{x}_2^2 f_m < 0\}$. Pick $\beta \in \mathcal{V}_m$. As $f_m \in \mathcal{P}(A_\mathfrak{p})$ and $\Delta \in \mathtt{x}_1\kappa[[\mathtt{x}_1]]\setminus\{0\}$, we deduce that $\mathrm{supp}(\beta) \not\subset \mathfrak{p}$ (because $\beta \in \mathcal{V}_m$) and $\mathtt{x}_1\mathtt{x}_2 \notin \mathrm{supp}(\beta)$.

If $\mathrm{ht}(\mathrm{supp}(\beta)) = n$, then $\mathrm{supp}(\beta) = \mathfrak{m}_n$ and $\Delta^2\mathtt{x}_2^2 f_m \geqslant_\beta 0$, so $\beta \notin \mathcal{V}$.

Let us check: *There exists* $k_\beta \geqslant 1$ *such that* $(\mathtt{x}_1\mathtt{x}_2)^{2k_\beta} <_\beta P^2 + (\Delta\mathtt{x}_3 - Q_3)^2 + \cdots + (\Delta\mathtt{x}_n - Q_n)^2$.

If $\mathrm{ht}(\mathrm{supp}(\beta)) = n-1$, then there exists $\phi : A \to \mathfrak{R}(\beta)[[\mathtt{t}]]$ such that $h \geqslant_\beta 0$ if and only if $\phi(h) \geqslant 0$ when $\mathtt{t} > 0$. As $\mathrm{supp}(\beta) \not\subset \mathfrak{p}$ and $\mathtt{x}_1\mathtt{x}_2 \notin \mathrm{supp}(\beta)$, we have $\phi(P^2 + (\Delta\mathtt{x}_3 - Q_3)^2 + \cdots + (\Delta\mathtt{x}_n - Q_n)^2) \neq 0$ and $\phi(\mathtt{x}_1\mathtt{x}_2) \neq 0$. Consequently, there exists $k_\beta \geqslant 1$ such that $2k_\beta\omega_\mathtt{t}(\phi(\mathtt{x}_1\mathtt{x}_2)) > \omega_t(\phi(P^2 + (\Delta\mathtt{x}_3 - Q_3)^2 + \cdots + (\Delta\mathtt{x}_n - Q_n)^2))$. Consequently, $\phi(P^2 + (\Delta\mathtt{x}_3 - Q_3)^2 + \cdots + (\Delta\mathtt{x}_n - Q_n)^2 - (\mathtt{x}_1\mathtt{x}_2)^{2k_\beta}) > 0$ if $\mathtt{t} > 0$, so $P^2 + (\Delta\mathtt{x}_3 - Q_3)^2 + \cdots + (\Delta\mathtt{x}_n - Q_n)^2 - (\mathtt{x}_1\mathtt{x}_2)^{2k_\beta} \geqslant_\beta 0$.

Suppose next $\mathrm{ht}(\mathrm{supp}(\beta)) = n-2$ and $P^2 + (\Delta\mathtt{x}_3 - Q_3)^2 + \cdots + (\Delta\mathtt{x}_n - Q_n)^2 \leqslant_\beta (\mathtt{x}_1\mathtt{x}_2)^{2k}$ for each $k \geqslant 1$. Consequently, $P^2 \leqslant_\beta (\mathtt{x}_1\mathtt{x}_2)^{2k}$, $(\Delta\mathtt{x}_3 - Q_3)^2 <\leqslant_\beta (\mathtt{x}_1\mathtt{x}_2)^{2k}$, …, $(\Delta\mathtt{x}_n - Q_n)^2 <\leqslant_\beta (\mathtt{x}_1\mathtt{x}_2)^{2k}$ for each $k \geqslant 1$. Suppose that $\mathfrak{p}$ is not β-convex. Then there exists $h_1, h_2 \in \beta$ such that $h_1 + h_2 \in \mathfrak{p}$, but $h_1, h_2 \notin \mathfrak{p}$.

We multiplying h_1, h_2 by a large power of Δ in order to find $g_1, g_2 \in \mathfrak{p}$ such that $r_1 := h_1 - g_1, r_2 := h_2 - g_2 \in \kappa[[\mathtt{x}_1]][\mathtt{x}_2]$ and have degree $< \deg(P)$. As $\mathrm{ht}(\mathrm{supp}(\beta)) = n-2$ and $\dim(A) = 2$, we deduce $\mathrm{supp}(\beta)$ is a minimal prime ideal associated to $\mathfrak{a}$ of height $n-2$. Thus, after the change of coordinates we have done above, the minimal prime ideal $\mathrm{supp}(\beta)$ of height $n-2$ satisfies $\mathrm{supp}(\beta) \cap \kappa[[\mathtt{x}_1,\mathtt{x}_2]] = \{0\}$. Consequently, $h_i \notin \mathrm{supp}(\beta)$ for $i = 1,2$. As $P^2 \leqslant_\beta (\mathtt{x}_1\mathtt{x}_2)^{2k}$, $(\Delta\mathtt{x}_3 - Q_3)^2 <\leqslant_\beta (\mathtt{x}_1\mathtt{x}_2)^{2k}$, …, $(\Delta\mathtt{x}_n - Q_n)^2 <\leqslant_\beta (\mathtt{x}_1\mathtt{x}_2)^{2k}$ for each $k \geqslant 1$ and $|\mathtt{x}_i|_\beta < 1$ for $i = 1,2$, we deduce that $h_i - r_i \geqslant_\beta 0$ for $i = 1,2$, because otherwise

$$0 \leqslant_\beta h_i = h_i - g_i + g_i \leqslant_\beta h_i - g_i + |g_i|_\beta \leqslant_\beta \frac{1}{2}(h_i - g_i) <_\beta 0,$$

which is a contradiction. We have $(h_i-g_i)\in\beta\setminus\operatorname{supp}(\beta)$ and $(h_1-g_1)+(h_2-g_2)\in\mathfrak{p}\cap\kappa[[\mathtt{x}_1]][\mathtt{x}_2]=\{0\}$, so $h_2-g_2=-(h_1-g_1)$ and both are strictly positive with respect to $<_\beta$, which is a contradiction. Consequently, $\mathfrak{p}$ is β-convex. By [BCR, Prop.4.3.8] there exists a specialization $\beta\to\alpha$ such that $\operatorname{supp}(\alpha)=\mathfrak{p}$. We conclude $\operatorname{supp}(\beta)\subset\mathfrak{p}$, which is a contradiction.

We deduce there exists $k_\beta\geqslant 1$ such that $(\mathtt{x}_1\mathtt{x}_2)^{2k_\beta}<_\beta P^2+(\Delta\mathtt{x}_3-Q_3)^2+\cdots+(\Delta\mathtt{x}_n-Q_n)^2$. As $\mathcal{V}$ is by [BCR, Cor.7.1.13] quasi-compact, there exist $\beta_1,\dots,\beta_s\in\mathcal{V}$ such that

$$\begin{aligned}\mathcal{V}\subset\bigcup_{j=1}^s\{\gamma\in\operatorname{Sper}(A):\ (\mathtt{x}_1\mathtt{x}_2)^{2k_{\beta_j}}<_\beta P^2+(\Delta\mathtt{x}_3-Q_3)^2+\cdots+(\Delta\mathtt{x}_n-Q_n)^2\}\\ \subset\{\gamma\in\operatorname{Sper}(A):\ (\mathtt{x}_1\mathtt{x}_2)^{2k}<_\beta P^2+(\Delta\mathtt{x}_3-Q_3)^2+\cdots+(\Delta\mathtt{x}_n-Q_n)^2\},\end{aligned}$$

where $k:=\max\{k_{\beta_j}:\ j=1,\dots,s\}$. We claim: *There exists $M\in\kappa_1:=\kappa((\mathtt{x}_1))[\mathtt{x}_2]/(P)\setminus\{0\}$ such that $F_m:=f_m+M^2(q_2^2+\cdots+q_n^2)^m\in\mathcal{P}(A)=\Sigma A^2$, where $q_2:=P$, $q_\ell:=\Delta\mathtt{x}_\ell-Q_\ell$ for $\ell=3,\dots,n$.*

We have proved that

$$1<_\beta\frac{q_2^2+\cdots+q_n^2}{(\mathtt{x}_1\mathtt{x}_2)^{2k}}$$

for each $\beta\in\mathcal{V}$. Fix $\beta\in\mathcal{V}$. If $q^\nu:=q_2^{\nu_2}\cdots q_n^{\nu_n}$ satisfies $|\nu|\leqslant m$, then

$$|q^{2\nu}|_\beta\leqslant_\beta(q_2^2+\cdots+q_n^2)^{|\nu|}<_\beta\frac{(q_2+\cdots+q_n^2)^{2m}}{(\mathtt{x}_1\mathtt{x}_2)^{2k(2m-|\nu|)}}\ \rightsquigarrow\ |q^\nu|_\beta<_\beta\frac{(q_2+\cdots+q_n^2)^{2m}}{(\mathtt{x}_1\mathtt{x}_2)^{2k(2m-|\nu|)}}.$$

In addition, $|b|_\beta<b^2+1$ for each $b\in\kappa$. Thus, for $f_m\in\kappa_1[q]$, there exists $N_\beta\in\Sigma\kappa^2\setminus\{0\}$ such that $|f_m|_\beta<_\beta N_\beta^2+(q_2^2+\cdots+q_n^2)^m$. Thus, $f_m+N_\beta^2+(q_2^2+\cdots+q_n^2)^m>_\beta 0$ and $\mathcal{V}\subset\bigcup_{\beta\in\mathcal{V}}\mathcal{U}_\beta$, where $\mathcal{U}_\beta=\{\gamma\in\operatorname{Sper}(A):\ f_m+N_\beta^2+(q_2^2+\cdots+q_n^2)^m>_\gamma 0\}$. As $\mathcal{V}$ is by [BCR, Cor.7.1.13] quasi-compact, there exist $\beta_1,\dots,\beta_s\in\mathcal{V}$ such that $\mathcal{V}\subset\bigcup_{j=1}^s\mathcal{U}_{\beta_j}$. Let $N_1\in\kappa[[\mathtt{x}_1]][\mathtt{x}_2]/(P)\setminus\{0\}$ and $p\geqslant 0$ such that $\sum_{j=1}^s N_{\beta_j}^2:=\frac{N_1}{\mathtt{x}_1^{2p}}$

We have $F_m:=\mathtt{x}_1^{2p}f_m+\frac{N_1}{(}q_2^2+\cdots+q_n^2)^m\in\mathcal{P}(A)$. Observe that $f-\frac{F_m}{a^2\mathtt{x}_1^{2p}}\in\mathfrak{n}^m\widehat{A_\mathfrak{p}}$ for each $m\geqslant 1$. We have $F_m\in\Sigma A^2\subset\Sigma\widehat{A_\mathfrak{p}}^2=\Sigma_p\widehat{A_\mathfrak{p}}^2$ for each $m\geqslant 1$. By Strong Artin's Approximation we deduce that $f\in\Sigma_p\widehat{A_\mathfrak{p}}^2$. Consequently, $\mathcal{P}(\widehat{A_\mathfrak{p}})=\Sigma\widehat{A_\mathfrak{p}}^2$, as required. □

Let us see some consequence of Theorem 17.9.

Example 17.10 (Embedding dimension 3). Let $A:=R[[\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3]]/(g)$ be a non-isolated singularity, where R is a real closed field, such that $\mathcal{P}(A)=\Sigma A^2$. Let $\mathfrak{q}\in\operatorname{Sper}(A)$ be such that $A_\mathfrak{q}$ is a non-regular local ring. Then $A_\mathfrak{q}$ has embedding dimension 2, its residue field is $R((\mathtt{s}))$ and $\mathcal{P}(\widehat{A_\mathfrak{q}})=\Sigma\widehat{A_\mathfrak{q}}^2$. Thus, $\widehat{A_\mathfrak{q}}\cong R((\mathtt{s}))[\mathtt{x},\mathtt{y}]/(\mathtt{x}^2-b\mathtt{y}^2)$ where $b\in R((\mathtt{s}))\setminus(-R((\mathtt{s}))^2)$, so $b=\mathtt{s}^\varepsilon a^2$ where $a\in R((\mathtt{s}))\setminus\{0\}$ and $\varepsilon\in\{0,1\}$. After a change of coordinates we may assume $a=1$, so $\widehat{A_\mathfrak{q}}$ is either $R((\mathtt{s}))[[\mathtt{x},\mathtt{y}]]/(\mathtt{y}^2-\mathtt{x}^2)$ or $R((\mathtt{s}))[[\mathtt{x},\mathtt{y}]]/(\mathtt{y}^2-\mathtt{s}\mathtt{x}^2)$. Recall that there are exactly two non-isolated singularities in List 2.1 when $\kappa=R$ is a real closed field: $A=R[[\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3]]/(\mathtt{x}_3^2-\mathtt{x}_1^2)$ or $A=R[[\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3]]/(\mathtt{x}_3^2-\mathtt{x}_2\mathtt{x}_1^2)$. ▪

Example 17.11 (Embedding dimension 4). Let $A:=R[[\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3,\mathtt{x}_4]]/\mathfrak{a}$ be a non-isolated singularity of dimension 2, where R is a real closed field, such that $\mathcal{P}(A)=\Sigma A^2$. Let $\mathfrak{q}\in\operatorname{Sper}(A)$ be such that $A_\mathfrak{q}$ is a non-regular local ring. Then $A_\mathfrak{q}$ has embedding dimension 3, its residue field is $R((\mathtt{s}))$ and $\mathcal{P}(\widehat{A_\mathfrak{q}})=\Sigma\widehat{A_\mathfrak{q}}^2$. Thus, $\widehat{A_\mathfrak{q}}\cong R((\mathtt{s}))[\mathtt{x},\mathtt{y},\mathtt{z}]/\mathfrak{b}$ where $\mathfrak{b}$ belongs to the following list:

(i) $\mathfrak{b}:=(\mathtt{y}^2-b\mathtt{x}^2,\mathtt{x}\mathtt{z},\mathtt{y}\mathtt{z})$ for some $b\notin-\Sigma R((\mathtt{s}))^2$.
(ii) $\mathfrak{b}:=(\mathtt{y}^2-\mathtt{x}\mathtt{z},\mathtt{y}\mathtt{z}+p\mathtt{y}\mathtt{x}+q\mathtt{x}^2,q\mathtt{x}\mathtt{y}+p\mathtt{x}\mathtt{z}+\mathtt{z}^2)$ where $P:=\mathtt{t}^3+p\mathtt{t}+q\in R((\mathtt{s}))[\mathtt{t}]$ is irreducible.

We analyze first case (i). As the elements $b \in R((\mathtt{s}))\backslash(-R((\mathtt{s}))^2)$ can be written as $b = \mathtt{s}^\varepsilon a^2$ where $a \in R((\mathtt{s}))\backslash\{0\}$ and $\varepsilon \in \{0,1\}$, after a change of coordinates we may assume $a = 1$, so $\widehat{A_\mathfrak{q}}$ is either $R((\mathtt{s}))[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{y}^2-\mathtt{x}^2,\mathtt{xz},\mathtt{yz}) = R((\mathtt{s}))[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{y}^2-\mathtt{x}^2,(\mathtt{x}+\mathtt{y})\mathtt{z},(\mathtt{x}-\mathtt{y})\mathtt{z}) \cong R((s))[[\mathtt{u},\mathtt{v},\mathtt{z}]]/(\mathtt{uv},\mathtt{uz},\mathtt{vz})$ or $R((\mathtt{s}))[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{y}^2-\mathtt{s}\mathtt{x}^2,\mathtt{xz},\mathtt{yz})$. By Lemma 15.12 and Example 15.14 the 2-dimensional singularities $A = R[[\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3,\mathtt{x}_4]]/(\mathtt{x}_1\mathtt{x}_2,\mathtt{x}_1\mathtt{x}_3,\mathtt{x}_2\mathtt{x}_4)$ and $A = R[[\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3,\mathtt{x}_4]]/(\mathtt{x}_2^2-\mathtt{x}_4\mathtt{x}_1^2,\mathtt{x}_1\mathtt{x}_3,\mathtt{x}_2\mathtt{x}_3)$ have the property $\mathcal{P}(A) = \Sigma A^2$.

Concerning case (ii), we recall from Example 15.3 the complete ring

$$A = B_4 := \kappa[[\mathtt{x}_0,\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3]]/(\mathtt{x}_1^2-\mathtt{x}_0\mathtt{x}_2,\mathtt{x}_1\mathtt{x}_2-\mathtt{x}_0^2\mathtt{x}_3,\mathtt{x}_2^2-\mathtt{x}_0\mathtt{x}_1\mathtt{x}_3),$$

which corresponds to the irreducible polynomial $P := \mathtt{t}^3-\mathtt{x}_3 \in R((\mathtt{x}_3))[\mathtt{t}]$. We know from Lemma 5.13 that $A = C_4 := \kappa[[\mathtt{x}_0,\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3]]/(\mathtt{x}_1^2-\mathtt{x}_0\mathtt{x}_2,\mathtt{x}_1\mathtt{x}_2-\mathtt{x}_0^2\mathtt{x}_3^2,\mathtt{x}_2^2-\mathtt{x}_0\mathtt{x}_1\mathtt{x}_3^2)$, which corresponds to the irreducible polynomial $P := \mathtt{t}^3-\mathtt{x}_3^2 \in R((\mathtt{x}_3))[\mathtt{t}]$, does not have the property $\mathcal{P}(C_4) = \Sigma C_4^2$ (due to the relation $\mathtt{x}_2^2-\mathtt{x}_0\mathtt{x}_1\mathtt{x}_3^2$). The same happens whenever the irreducible polynomial $P = \mathtt{t}^3+p\mathtt{t}+q \in R[[\mathtt{x}_3]][\mathtt{t}]$ satisfies $\omega_{\mathtt{x}_3}(p) \geqslant 1$ and $\omega_{\mathtt{x}_3}(q) \geqslant 2$. Thus, further restrictions concerning the irreducible polynomial $P := \mathtt{t}^3+p\mathtt{t}+q \in R((\mathtt{s}))[\mathtt{t}]$ should be found to guarantee that $\mathcal{P}(A) = \Sigma A^2$. ∎

In the forthcoming article [Fe9] we take advantage of Theorem 17.9 and the previous examples to prove the following.

global2 **Theorem 17.12.** *Let $X \subset \mathbb{R}^n$ be a global analytic surface and suppose that for each $x \in \operatorname{Sing}(X)$ the germ X_x has an isolated singularity or all the irreducible components of the analytic set germ X_x are regular. If $\mathcal{P}(\mathcal{O}(X_x)) = \Sigma_2\mathcal{O}(X_x)^2$ for each $x \in \operatorname{Sing}(X)$, then $\mathcal{P}(\mathcal{O}(X)) = \Sigma_3\mathcal{O}(X)^2$.*

psps18

18. Further applications

18.1. **Principal saturated preorderings of low order.** In [Sch6, Sch7] Scheiderer studied the problem of determining when a *finitely generated preordering* T of an excellent henselian local ring $(A,\mathfrak{m})$ of dimension 2 is *saturated*. Recall that T is generated by $h_1,\dots,h_r \in A$, if $T := \{\sum_{\nu\in\{0,1\}^r}\sigma_\nu h_1^{\nu_1}\cdots h_r^{\nu_r} :\ \sigma_i \in \Sigma A^2\}$. If we denote $X(T) := \{\alpha \in \operatorname{Sper}(A) :\ f \geqslant_\alpha 0\ \forall f \in T\}$, the *saturation of* T is $\operatorname{Sat}(T) := \{f \in A :\ f \geqslant_\alpha 0\ \forall \alpha \in X(T)\}$. As one can expect, T is *saturated* if $T = \operatorname{Sat}(T)$. The preordering $\widehat{T}$ generated by T in the completion $\widehat{A}$ is

$$\widehat{T} := \Big\{\sum_{\nu\in\{0,1\}^r}\sigma_\nu h_1^{\nu_1}\cdots h_r^{\nu_r} :\ \sigma_i \in \Sigma\widehat{A}^2\Big\}.$$

In [Sch6, Cor.3.25] Scheiderer established a criterion to decide when the saturation of $\widehat{T}$ implies the saturation of T.

scht **Corollary 18.1.** *Let A be an excellent henselian local ring with $p(A) < +\infty$ and let T be a finitely generated preordering in A. Then T is saturated in A if and only if $\widehat{T}$ is saturated in $\widehat{A}$.*

The previous result reduces the quoted problem to the study of the saturation of finitely generated preorderings on complete rings with finite Pythagoras number. He focused on the characterization of the principal saturated preorderings $T := \operatorname{PO}(F) := \{s_1+s_2F :\ s_1,s_2 \in \Sigma A^2\}$ of low order in a 2-dimensional excellent henselian regular local ring $(A,\mathfrak{m})$ whose residue field $\kappa := A/\mathfrak{m}$ is (formally) real and has $\tau(\kappa) < +\infty$. If $F \in A\backslash\mathfrak{m}^2$, then T is always saturated [Sch4, Lem.3.1], so we concentrate on elements $F \in \mathfrak{m}^2$. A general strategy in [Sch6, Sch7] to prove that a principal preordering $T := \operatorname{PO}(F)$ of $\kappa[[\mathtt{x},\mathtt{y}]]$ is saturated is to consider the extended ring $B := \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F)$ and to observe that $\operatorname{Sat}(T) = \mathcal{P}(\{F \geqslant 0\}) \subset \mathcal{P}(B)$. If $f \in \operatorname{Sat}(T)$ and $\mathcal{P}(B) = \Sigma B^2$, there exist $a_i,b_i,q \in \kappa[[\mathtt{x},\mathtt{y}]]$ for $i = 1,\dots,p$ such that

$$f = \sum_{i=1}^p(a_i+b_i\mathtt{z})^2-(\mathtt{z}^2-F)q \quad\rightsquigarrow\quad f = \sum_{i=1}^p a_i^2+F\sum_{i=1}^p b_i^2 \in T.$$

The previous implication is a key to prove Corollary 18.2 below, which is a relevant application of Theorem 2.3. We refer the reader to a weaker version of the following result proved in [Fe8, §6].

ord23p **Corollary 18.2.** *Let $(A,\mathfrak{m})$ be an excellent henselian regular local ring of dimension 2 such that its residue field $\kappa := A/\mathfrak{m}$ is (formally) real and has $\tau(\kappa) < +\infty$. Let $F \in \mathfrak{m}^2 \backslash \mathfrak{m}^4$ and let T be the principal preordering of A generated by F. Then T is saturated in A if and only if there exists $c \in \kappa\backslash 0$ such that c^2F is 'right equivalent' in $\widehat{A} \cong \kappa[[\mathtt{x},\mathtt{y}]]$ to one of the series in List 2.1.*

Sketch of proof. By [Sch2, Prop.3.9] and (2.1) we have $p(A) = p(\widehat{A}) = p(\kappa[[\mathtt{x},\mathtt{y}]]) \leqslant 2\tau(\kappa) < +\infty$. The result now follows straightforwardly from Corollary 18.1, the proof of Theorem 2.2 (which corresponds to the one of [Fe8, Thm.1.5]) and Theorem 2.3. A key fact to prove the 'only if' implication is that if the difference $\mathcal{P}(A)\backslash\Sigma A^2 \neq \varnothing$, then $\mathcal{P}(\{F \geqslant 0\})\backslash\Sigma A^2 \neq \varnothing$ (see the proof of [Fe8, Thm.1.5]). As a guideline the reader can follow closely [Sch7, §6.5]. □

The definition of 'right equivalence of series' is recalled in §5.1.4. In [Sch7, Rem.6.8] Scheiderer explains that going beyond series of order three seems difficult because it is not know whether the saturation of $\mathrm{PO}(F)$ depends only on the pair $(\widehat{A},F)$. In [Sch5, Thm.6.3 & Thm.6.6] Scheiderer extended the analogous results to Corollary 18.2 when κ is a real closed field to the setting of excellent regular local rings of dimension 2, that is, he erases the 'henselian condition'. The strategy proposed there uses strongly the fact that the residue field is real closed. It seems to us that the extension of Corollary 18.2 to the setting of an excellent regular local ring of dimension 2 when the residue field κ is (formally) real and has $\tau(\kappa) < +\infty$ needs a new strategy.

tsos

18.2. **Transference of sums of squares.** In [Sch8] Scheiderer (see also [CS]) gave a negative answer (if $n \geqslant 2$) to the following question raised by Sturmfels:

Question 18.3. Let $f \in \mathbb{Q}[\mathtt{x}] := \mathbb{Q}[\mathtt{x}_1,\dots,\mathtt{x}_n]$ be a polynomial, which is a sum of squares in $\mathbb{R}[\mathtt{x}] := \mathbb{R}[\mathtt{x}_1,\dots,\mathtt{x}_n]$. Is f necessarily a sum of squares in $\mathbb{Q}[\mathtt{x}_1,\dots,\mathtt{x}_n]$?.

The previous question can be formulated as:

Question 18.4. Does the equality $\Sigma\mathbb{R}[\mathtt{x}]^2 \cap \mathbb{Q}[\mathtt{x}] = \Sigma\mathbb{Q}[\mathtt{x}]^2$ hold?

18.2.1. *Sturmfel's question.* We can formulate a similar question for rings $A := \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2 - F(\mathtt{x},\mathtt{y}))$ where κ admits a unique ordering, R is the real closure of κ (endowed with its unique ordering) and $F \in \kappa[[\mathtt{x},\mathtt{y}]]$. We know how to answer Sturmfel's question when: *$\tau(\kappa) < +\infty$ and $F \in \kappa[[\mathtt{x},\mathtt{y}]]$ is a series such that $\omega(F) = \omega(F(\mathtt{x},0)) = 2k+1$ and F is ρ-determined for some $\rho \geqslant 2k+1$.* We refer the reader to List 2.1 to check which of them have the property $\mathcal{P}(A) = \Sigma A^2$ (an important restriction to belong to List 2.1 is that $k = 1$, that is, $2k+1 = 3$).

Let $B := R[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2 - F)$. By Lemma 5.14, an adapted version of Lemma 5.21 and [ABR, Prop.VII.5.1] one deduces that $\mathcal{P}(A) = \mathcal{P}(B) \cap A$. Let us prove: $A \cap \Sigma B^2 = \Sigma A^2$.

Let $f + \mathtt{z}g \in A \cap \Sigma B^2 \subset A \cap \mathcal{P}(B) = \mathcal{P}(A)$. We claim: $f + \mathtt{z}g \in \Sigma A^2$.

By Corollary 8.8 there exists $s \geqslant 0$ and $f_1 + \mathtt{z}g_1 \in \mathcal{P}(A)$ such that $f_1(\mathtt{x},0) \neq 0$ and $f + \mathtt{z}g = \mathtt{y}^{2s}(f_1 + \mathtt{z}g_1)$. As $f + \mathtt{z}g \in \Sigma B^2$, there exist $a_i, b_i, q \in R[[\mathtt{x},\mathtt{y}]]$ such that

$$\mathtt{y}^{2s}(f_1 + \mathtt{z}g_1) = \sum_{i=1}^{p}(a_i + \mathtt{z}b_i)^2 - (\mathtt{z}^2 - F)q, \tag{18.1}$$ 514

$$\mathtt{y}^{2s}f_1 = \sum_{i=1}^{p} a_i^2 + F\sum_{i=1}^{p} b_i^2, \tag{18.2}$$ 515

$$\mathtt{y}^{2s}g_1 = 2\sum_{i=1}^{p} a_ib_i. \tag{18.3}$$

Observe that $q = \sum_{i=1}^p b_i^2$. If $s \geqslant 1$, we set $\mathtt{y} = 0$ in (18.2) and deduce

$$0 = \sum_{i=1}^p a_i^2(\mathtt{x}, 0) + \mathtt{x}^{2k+1} \sum_{i=1}^p b_i^2(\mathtt{x}, 0),$$

so $a_i(\mathtt{x}, 0) = 0$ and $b_i(\mathtt{x}, 0) = 0$ for each i. There exist $a_i', b_i', q' \in R[[\mathtt{x}, \mathtt{y}]]$ satisfying $a_i = \mathtt{y}a_i'$, $b_i = \mathtt{y}b_i'$ and $q = \sum_{i=1}^p b_i^2 = \mathtt{y}^2 q'$. This means that we can divide (18.1) by $\mathtt{y}^2$. Proceeding recursively we conclude $f_1 + \mathtt{z}g_1 \in A \cap \Sigma B^2$. Thus, either $f_1 + \mathtt{z}g_1$ is a unit in A and $f_1 + \mathtt{z}g_1 \in \Sigma A^2$ (by Remark 5.5(ii)) or there exist series $c_i, d_i \in R[[\mathtt{x}]]$ such that $f_1(\mathtt{x}, 0) = \sum_{i=1}^p c_i^2 + \mathtt{x}^{2k+1} \sum_{i=1}^p d_i^2$ (set $\mathtt{y} = 0, \mathtt{z} = 0$ in a representation of $f_1 + \mathtt{z}g_1$ as a sum of squares in B). Consequently, $\omega(f_1(\mathtt{x}, 0)) = q \geqslant 2$ and by Theorem 8.8(ii) $f_1 + \mathtt{z}g_1 \in \Sigma A^2$, so $f + \mathtt{z}g \in \Sigma A^2$. Thus, the answer to the corresponding Sturmfels question for this family of rings is positive! (see also [Fe8, Rem.5.10(iii)] for a weaker version) even if most of them satisfy $\mathcal{P}(A) \neq \Sigma A^2$.

Remarks 18.5. (i) The condition $\tau(\kappa) < +\infty$ should be avoided using the results in Section 7. However the results seem cumbersome and we avoid entering into further details. ■

(ii) We do not know a similar result when $\omega(F)$ is even, because we do not know an analogous result to Corollary 8.8 in such situation.

18.2.2. *Extension of coefficients and descent.* Let κ be a (formally) real field that admits a unique ordering and let R be a real closed field that contains κ (endowed with its unique ordering) as an ordered subfield. Let $\mathfrak{a}$ be an ideal of $\kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]$ and define $A := \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/\mathfrak{a}$ and $B := R[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/(\mathfrak{a}R[[\mathtt{x}, \mathtt{y}, \mathtt{z}]])$. Observe that $B = A \otimes_{\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]} R[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]$. Recall that $\tau(R) = 1$ and $p(B) < +\infty$ by Theorem 1.7.

transfd **Theorem 18.6.** *If $\mathcal{P}(B) = \Sigma B^2$, then $\mathcal{P}(A) = \Sigma A^2$.*

Proof. Suppose first $\dim(B) = 1$ and denote $C := R[\sqrt{-1}]$. As $\mathcal{P}(B) = \Sigma B^2$, we know by Theorem 1.8 that $B \cong R[[\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3]]/(\mathtt{x}_1\mathtt{x}_2, \mathtt{x}_1\mathtt{x}_3, \mathtt{x}_2\mathtt{x}_3)$, so $B \otimes_R C \cong C[[\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3]]/(\mathtt{x}_1\mathtt{x}_2, \mathtt{x}_1\mathtt{x}_3, \mathtt{x}_2\mathtt{x}_3)$. By §10.1 (the fact that C is not the algebraic closure of κ is not relevant in the proof) we deduce that after a κ-change of coordinates $\mathfrak{a}$ is in one of the following situations:

(i) $\mathfrak{a} := (\mathtt{y}, \mathtt{z})$.
(ii) $\mathfrak{a} := (\mathtt{y}^2 - a\mathtt{x}^2, \mathtt{z})$ for some $a \notin -\Sigma\kappa^2$.
(iii) $\mathfrak{a} := (\mathtt{y}^2 - a\mathtt{x}^2, \mathtt{xz}, \mathtt{yz})$ for some $a \notin -\Sigma\kappa^2$.
(iv) $\mathfrak{a} := (\mathtt{y}^2 - \mathtt{xz}, \mathtt{yz} + p\mathtt{yx} + q\mathtt{x}^2, q\mathtt{xy} + p\mathtt{xz} + \mathtt{z}^2)$ where the polynomial $P := \mathtt{t}^3 + p\mathtt{t} + q \in \kappa[\mathtt{t}]$ is irreducible.

Consequently, by Theorem 3.2 that $\mathcal{P}(A) = \Sigma A^2$ and A has dimension 1.

Suppose next $\dim(B) = 2$. As $\mathcal{P}(B) = \Sigma B^2$, we know by [Fe8, Lem.3.2] that $\omega(\mathfrak{a}R[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]) \leqslant 2$, so $\omega(\mathfrak{a}) \leqslant 2$. If $\omega(\mathfrak{a}) = 1$, then $A \cong \kappa[[\mathtt{x}, \mathtt{y}]]/\mathfrak{a}'$ for some ideal $\mathfrak{a}'$ of $\kappa[[\mathtt{x}, \mathtt{y}]]$ and $B \cong \kappa[[\mathtt{x}, \mathtt{y}]]/\mathfrak{a}'\kappa[[\mathtt{x}, \mathtt{y}]]$. As B has dimension 2 and $\mathcal{P}(B) = \Sigma B^2$, we deduce by Theorem 1.6 that $\operatorname{ht}(\mathfrak{a}'\kappa[[\mathtt{x}, \mathtt{y}]]) = 0$, so $\mathfrak{a}'\kappa[[\mathtt{x}, \mathtt{y}]] = (0)$ and $\mathfrak{a}' = (0)$. Thus, $A \cong \kappa[[\mathtt{x}, \mathtt{y}]]$ and $\mathcal{P}(A) = \Sigma A^2$. Suppose next $\omega(\mathfrak{a}) = 2$. Suppose first there exists a R-change of coordinates φ such that $\varphi^*(\mathfrak{a})R[[\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3]] = (\mathtt{x}_3\mathtt{x}_1, \mathtt{x}_3\mathtt{x}_2) = (\mathtt{x}_3) \cap (\mathtt{x}_1, \mathtt{x}_2)$. Consider the homomorphism

$$\varphi : \kappa[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]/\mathfrak{a} \to R[[\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3]]/\varphi^*(\mathfrak{a})R[[\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3]],\ f + \mathfrak{a} \mapsto f(\varphi) + \varphi^*(\mathfrak{a})R[[\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3]],$$

which is well-defined, because $\mathfrak{a} \subset \mathfrak{a}R[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]$.

The ideal $\mathfrak{a}$ is not principal, because otherwise $\mathfrak{a}R[[\mathtt{x}, \mathtt{y}, \mathtt{z}]]$ is also a principal ideal. Let $f_1, \ldots, f_r \in \mathfrak{a}$ be a system of generators of $\mathfrak{a}$. Write $\mathtt{zx} = f_1(\varphi)g_{11} + \cdots + f_r(\varphi)g_{1r}$ and $\mathtt{zy} = f_1(\varphi)g_{21} + \cdots + f_r(\varphi)g_{2r}$ where $g_{ij} \in R[[\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3]]$. As $\mathtt{x}_3\mathtt{x}_1$ and $\mathtt{x}_3\mathtt{x}_2$ are independent quadratic forms and $\omega(\mathfrak{a}) = 2$, we may assume that $\omega(f_1) = \omega(f_2) = 2$ and their initial forms are R-linearly independent. As $\mathtt{x}_3\mathtt{x}_1$ and $\mathtt{x}_3\mathtt{x}_1$ generate $\varphi^*(\mathfrak{a})R[[\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3]]$, we deduce that $\mathtt{x}_3$ divides both f_1 and f_2. After a κ-linear change of coordinates, we may assume by Weierstrass Preparation

Theorem that f_1, f_2 are Weierstrass polynomials with respect to $\mathtt{z}$. Using the resultant of f_1, f_2 with respect to $\mathtt{z}$, we deduce f_1 and f_2 share an irreducible factor h, which has order 1, because f_1 and f_2 have independent leading forms of order 2. After a κ-change of coordinates, we may assume $h = \mathtt{z}$, so $f_1 = \mathtt{z}g_1$ and $f_2 = \mathtt{z}g_2$ where $g_1, g_2 \in \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ have order 1. As the leading forms of f_1, f_2 are κ-linearly independent, the leading forms of g_1, g_2 are κ-linearly independent. After a κ-change of coordinates, we may assume $g_1 = \mathtt{x}$ and $g_2 = \mathtt{y}$. This means that

$$\mathfrak{a}R[[\mathtt{x},\mathtt{y},\mathtt{z}]] = (\mathtt{xz},\mathtt{yz})R[[\mathtt{x},\mathtt{y},\mathtt{z}]]. \tag{18.4}$$ 123

This means that $\mathtt{z}$ divides in $R[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ each element of $\mathfrak{a}$, so it divides each element of $\mathfrak{a}$ in $\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$. Pick $f \in \mathfrak{a}$ and write $f = \mathtt{z}g$ where $g \in \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$. Using (18.4) we find $q_1, q_2 \in R[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ such that $g = \mathtt{x}q_1 + \mathtt{y}q_2$, so $g(0,0,\mathtt{z}) = 0$, that is, there exist $q_1', q_2' \in \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ such that $g = \mathtt{x}q_1' + \mathtt{y}q_2'$. Consequently, $\mathfrak{a} = (\mathtt{xz},\mathtt{yz})$ and $A = \kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/\mathfrak{a}$ satisfies $\mathcal{P}(A) = \Sigma A^2$.

We assume en the following that $\mathfrak{a}R[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ is a principal ideal. Let $H \in \mathfrak{a}$ of order 2 and observe that H generates $\mathfrak{a}R[[\mathtt{x},\mathtt{y},\mathtt{z}]]$. After a κ-linear change of coordinates, we may assume by Weierstrass Preparation Theorem we may assume H is a Weierstrass polynomial with respect to $\mathtt{z}$. Using the unicity of the quotient and remainder in Weierstrass Division Theorem we deduce $\mathfrak{a}$ is generated by H. After a κ-change of coordinates, we may assume $H = \mathtt{z}^2 - F(\mathtt{x},\mathtt{y})$ where $F \in \kappa[[\mathtt{x},\mathtt{y}]]$ is a series of order $\geqslant 2$. As H generates $\mathfrak{a}R[[\mathtt{x},\mathtt{y},\mathtt{z}]]$ and $\mathcal{P}(B) = \Sigma B^2$, we deduce by Lemma 5.13 that $2 \leqslant \omega(F) \leqslant 3$. We distinguish both cases:

Case 1. Suppose first $\omega(F)$ has order 2. Then after a κ-change of coordinates we may assume either $F = a\mathtt{x}^2$ where $a \in \kappa\setminus\{0\}$ or $F = a\mathtt{x}^2 + b\mathtt{y}^\ell$ where $a, b \in \kappa\setminus\{0\}$ and $\ell \geqslant 2$. By Theorem 1.6 we deduce (as $\mathcal{P}(B) = \Sigma B^2$) that $a > 0$ in both cases, so $\mathcal{P}(A) = \Sigma A^2$ (interchanging $\mathtt{z}$ and $\mathtt{x}$, if necessary, one realizes that the sign of b is irrelevant) by Theorem 2.3.

Case 2. Suppose next $\omega(F)$ has order 3. After a κ-change of coordinates, we may assume (see the proof of [Fe8, Lem.3.8]) that the leading form P of F has one of the following forms:

Subcase 2.1. An irreducible polynomial $\mathtt{x}^3 + a\mathtt{x}\mathtt{y}^2 + b\mathtt{y}^3$. In this case we may assume by Example 5.8 that $F = \mathtt{x}^3 + a\mathtt{x}\mathtt{y}^2 + b\mathtt{y}^3$, so $\mathcal{P}(A) = \Sigma A^2$ by Theorem 2.3.

Subcase 2.2. $\mathtt{x}(\mathtt{x}^2 - a\mathtt{y}^2)$ where $a \neq 0$. In this case we may assume by Example 5.8 that $\mathtt{x}(\mathtt{x}^2 - a\mathtt{y}^2)$. As $\mathcal{P}(B) = \Sigma B^2$, we deduce by Theorem 1.6 that $a > 0$ and by Theorem 2.3 we have $\mathcal{P}(A) = \Sigma A^2$.

Subcase 2.3. $\mathtt{x}^2\mathtt{y}$. Then either F is right equivalent to $\mathtt{x}^2\mathtt{y}$ and $\mathcal{P}(A) = \Sigma A^2$ by Theorem 2.3, or F is right equivalent to $\mathtt{x}^2\mathtt{y} + a\mathtt{y}^\ell$ for some $a \in \kappa\setminus\{0\}$ and $\ell \geqslant 3$ (see the proof of [Fe8, Lem.3.8]). As $\mathcal{P}(B) = \Sigma B^2$, we deduce by Theorem 1.6 $a(-1)^\ell < 0$. By Theorem 2.3 we conclude $\mathcal{P}(A) = \Sigma A^2$.

Subcase 2.4. $\mathtt{x}^3$. By Lemma [Fe8, Lem.3.9](i) we may assume after a κ-change of coordinates that $F := \mathtt{x}^3 + b\mathtt{x}\mathtt{y}^{k+3} + c\mathtt{y}^{\ell+4}W$ where $b, c \in \kappa$, $k, \ell \geqslant 0$ and $W \in \kappa[[\mathtt{y}]]$ is a unit such that $W(0) = 1$. In addition, if $c \neq 0$ and $\kappa \geqslant \ell + 1$, we may assume by Lemma [Fe8, Lem.3.9](ii) that $b = 0$ and $W = 1$. As $\mathcal{P}(B) = \Sigma B^2$, we deduce by Lemma [Fe8, Lem.3.9](iii) that either $k = 0$ (and $b \neq 0$) or $\ell \leqslant 1$ (and $c \neq 0$). Thus, we have the following possibilities:

- (2.4.i) $\ell = 0$ (and $c \neq 0$), that is, $F := \mathtt{x}^3 + b\mathtt{x}\mathtt{y}^3 + c\mathtt{y}^4W$ with $c \neq 0$. By the proof of [Fe8, Lem.3.8] we deduce that F is right equivalent to $\mathtt{x}^3 + c\mathtt{y}^4$ for some $c \in \kappa\setminus\{0\}$. As $\mathcal{P}(B) = \Sigma B^2$, we have by Theorem 1.6 that $c > 0$. By Theorem 2.3 we conclude $\mathcal{P}(A) = \Sigma A^2$.
- (2.4.ii) $k = 0$ ($b \neq 0$) and $\ell \geqslant 1$, that is, $F := \mathtt{x}^3 + b\mathtt{x}\mathtt{y}^3 + c\mathtt{y}^{5+m}W$ with $b \neq 0$ and $m \geqslant 0$. By the proof of [Fe8, Lem.3.8] we deduce that F is right equivalent to $\mathtt{x}^3 + \mathtt{x}\mathtt{y}^3$ and by Theorem 2.3 we have $\mathcal{P}(A) = \Sigma A^2$.
- (2.4.iii) $\ell = 1$ (and $c \neq 0$), that is, $F := \mathtt{x}^3 + b\mathtt{x}\mathtt{y}^3 + c\mathtt{y}^5W$ with $c \neq 0$. By the proof of [Fe8, Lem.3.8] we deduce that F is right equivalent to $\mathtt{x}^3 + \mathtt{y}^5$ and by Theorem 2.3 we have $\mathcal{P}(A) = \Sigma A^2$.

We have proved that $\mathcal{P}(A)=\Sigma A^2$ and $\dim(A)=2$ in all cases, as required. $\square$

Remarks 18.7. The converse of Theorem 18.6 is not true in general.

(i) In the 1-dimensional case we conjecture that there exists no chimeric polynomials over a (formally) real field κ (Conjecture 3.7), so we expect no 1-dimensional ring $A:=\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/\mathfrak{a}$ such that $\mathcal{P}(A)=\Sigma A^2$, whereas $\mathcal{P}(B)\neq\Sigma B^2$ for $B:=R[[\mathtt{x},\mathtt{y},\mathtt{z}]]/\mathfrak{a}R[[\mathtt{x},\mathtt{y},\mathtt{z}]]$.

(ii) Let κ be a (formally) real field that admits a unique ordering and has an irreducible polynomial $P\in\kappa[\mathtt{t}]$ of degree 3 and only one root in R (take for instance $\kappa=\mathbb{Q}$, $P:=\mathtt{t}^3-2$ and R any real closed field). Let $F:=\mathtt{y}^3P(\frac{\mathtt{x}}{\mathtt{y}})\in\kappa[[\mathtt{x},\mathtt{y}]]$ and define $A:=\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F)$ and $B:=R[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F)$. By Theorem 1.6 (or 2.3) we deduce that $\mathcal{P}(A)=\Sigma A^2$. Observe that F is reducible in $\mathbb{R}[[\mathtt{x},\mathtt{y}]]$ and it is the product of a linear form of $\mathbb{R}[[\mathtt{x},\mathtt{y}]]$ and an irreducible quadratic form of $\mathbb{R}[[\mathtt{x},\mathtt{y}]]$. By Theorem 1.6 we deduce that $\mathcal{P}(B)\neq\Sigma B^2$. ■

Part 5. Appendices

A Appendix A. Rings and fields of Puiseux series

Let κ be a field of characteristic zero and $\overline{\kappa}$ its algebraic closure. Let $\overline{\kappa}[[\mathtt{t}^*]]=\bigcup_{p\geqslant1}\overline{\kappa}[[\mathtt{t}^{1/p}]]$ be the ring of Puiseux series with coefficients in $\overline{\kappa}$ and $\overline{\kappa}((\mathtt{t}^*))$ its field of fractions, which is an algebraically closed field. For the sake of completeness we include the following two results:

finite **Lemma A.1.** *Let κ be a field of characteristic zero and let $\overline{\kappa}$ be its algebraic closure. If $f:=\sum_{k\geqslant0}a_k\mathtt{t}^k\in\overline{\kappa}[[\mathtt{t}]]$ is algebraic over $\kappa((\mathtt{t}))$, then there exists a finite Galois extension $F|\kappa$ such that $f\in F[[\mathtt{t}]]$. In addition, the roots of the irreducible polynomial of f over $\kappa((\mathtt{t}))$ belong to $F[[\mathtt{t}]]$.*

Proof. Consider the algebraic extension $E:=\kappa(a_k:\ k\geqslant0)|\kappa$. Let $F|\kappa$ be the Galois closure of $E|\kappa$. Let $\gamma:F\to F$ be a κ-automorphism. Denote $G(F:\kappa)$ the Galois group $F|\kappa$. Then

$$\Gamma:F[[\mathtt{t}]]\to F[[\mathtt{t}]],\ \sum_{k\geqslant0}b_k\mathtt{t}^k\to\sum_{k\geqslant0}\sigma(b_k)\mathtt{t}^k$$

is a $\kappa[[\mathtt{t}]]$-automorphism that extends to a $\kappa((\mathtt{t}))$-automorphism of $F((\mathtt{t}))$. Observe that $\Gamma(f)$ is a root of the irreducible polynomial P of f over $\kappa((\mathtt{t}))$ for each κ-automorphism γ of F. As P has finitely many roots, there exist κ-automorphisms $\gamma_1,\ldots,\gamma_r\in G(F:\kappa)$ of F such that $\{\Gamma_1(f),\ldots,\Gamma_r(f)\}=\{\Gamma(f):\ \gamma\in G(F:\kappa)\}$. We may assume that $\Gamma_i(f)\neq\Gamma_j(f)$ is $i\neq j$.

Consider $L:=\kappa(\gamma_1(a_k),\ldots,\gamma_r(a_k):\ k\geqslant0)\subset F$. Pick $x\in L$ and let $\lambda_{ik}\in\kappa$ for $i=1,\ldots,r$ such that $x=\sum_{i=1}^r\sum_{k=0}^\ell\lambda_{ik}\gamma_i(a_k)$. Let $\gamma\in G(F:\kappa)$ and consider the compositions $\gamma\circ\gamma_i$ for $i=1,\ldots,r$. Let $i_1,\ldots,i_r\in\{1,\ldots,r\}$ be such that $(\Gamma\circ\Gamma_j)(f)=\Gamma_{i_j}(f)$ for $j=1,\ldots,r$. As $\Gamma_i(f)\neq\Gamma_j(f)$ is $i\neq j$, we deduce $i_{j_1}\neq i_{j_2}$ if $j_1\neq j_2$. Thus,

$$\gamma(x)=\sum_{j=1}^r\sum_{k=0}^\ell\lambda_{jk}(\gamma\circ\gamma_j)(a_k)=\sum_{i=1}^r\sum_{k=0}^\ell\lambda_{jk}\gamma_{i_j}(a_k)\in L.$$

This means that each $\gamma\in G(F:\kappa)$ restricts to an automorphism $\gamma|_L\in G(L:\kappa)$. If $x\in L\backslash\kappa\subset F\backslash\kappa$, there exists $\gamma\in G(F:\kappa)$ such that $\gamma|_L(x)\gamma(x)\neq x$. Thus, $L|\kappa$ is a Galois extension. As it is contained in F and $L|\kappa$ is a Galois extension, we conclude $F=L$.

If $x\in F$, there exist $y_1,\ldots,y_r\in E$ such that $x=\sum_{i=1}^r\gamma_i(y_i)$. If $\gamma\in G(L:\kappa)$, there exists a permutation $\sigma\in\mathfrak{S}_r$ such that $\gamma(x)=\sum_{i=1}^r\gamma_{\sigma(i)}(y_i)$. Thus, the irreducible polynomial of x over κ has degree $\leqslant r!$.

Let $x_0\in F$ be such that $[\kappa(x_0):\kappa]\leqslant r!$ is maximal. If $y\in F\backslash\kappa(x_0)$, we choose $\lambda\in\kappa$ such that $\kappa(x_0,y)=\kappa(x_0+\lambda y)$. As $[\kappa(x_0):\kappa]$ is maximal, but $\kappa(x_0)\subsetneq\kappa(x_0+\lambda y)=\kappa(x_0,y)$, we achieve a contradiction. Consequently, $F=\kappa(x_0)|\kappa$ is a finite extension and $f\in F[[\mathtt{t}]]$.

If $\zeta \in F((\mathtt{t}))$, there exist $g \in F[[\mathtt{t}]]$ and $m \geqslant 0$ such that $\zeta = \frac{g}{\mathtt{t}^m}$. If ρ is a primitive element of the extension $F|\kappa$, then $g = \sum_{\ell=0}^{n-1} g_\ell \rho^\ell$. If $\theta : F((\mathtt{t})) \to F((\mathtt{t}))$ is a $\kappa((\mathtt{t}))$-isomorphism, then $\theta(\zeta) = \sum_{\ell=0}^{n-1} \frac{g_\ell}{\mathtt{t}^m}\theta(\rho)^\ell$, so θ is determined by its restriction to F, which provides an element $\theta|_F \in G(F:\kappa)$. Thus, the roots of the irreducible polynomial of f over $\kappa((\mathtt{t}))$ belong to $F[[\mathtt{t}]]$. □

safef **Corollary A.2.** *Let κ be a field of characteristic zero and let $\overline{\kappa}$ be its algebraic closure. Let $f := \sum_{k\geqslant 0} b_k \mathtt{t}^{k/n} \in \overline{\kappa}[[\mathtt{t}^{1/n}]]$ and assume that there exists an integer $k_0 \geqslant 1$ relatively prime with n such that $b_k \neq 0$. Suppose that the irreducible polynomial P of f over $\kappa((\mathtt{t}))$ has degree n. Then there exist $\alpha \in \overline{\kappa}$ and $a_k \in \kappa$ such that $\alpha^n \in \kappa$ and $f = \sum_{k\geqslant 0} a_k \alpha^k \mathtt{t}^{k/n}$.*

Proof. Let $F|\kappa$ be the Galois closure of $\kappa(b_k : k \geqslant 0)$ (recall that $f := \sum_{k\geqslant 0} b_k \mathtt{t}^{k/n}$). By Lemma A.1 $F|\kappa$ is a finite Galois extension, so there exists $\theta \in F$ such that $F = \kappa(\theta)$ and we assume that the degree $[F:\kappa] = m$. Let $\gamma : F((\mathtt{t}^{1/n})) \to F((\mathtt{t}^{1/n}))$ be a $\kappa((\mathtt{t}))$-homomorphism and denote $\omega := e^{2\pi \mathtt{i}/n}$. As $\gamma(\mathtt{t}) = \mathtt{t}$, there exists an integer $r = 0, \dots, n-1$ such that $\gamma(\mathtt{t}^{1/n}) = \omega^r \mathtt{t}^{1/n}$. Write

$$f := \sum_{k=0}^{n-1}\sum_{\ell=0}^{m-1} f_{k\ell}(\mathtt{t})\theta^\ell \mathtt{t}^{k/n}$$

where $f_{k\ell} \in \kappa[[\mathtt{t}]]$. Then $\gamma(f) = \sum_{k=0}^{n-1}\sum_{\ell=0}^{m-1} f_{k\ell}(\mathtt{t})\gamma(\theta)^\ell \omega^{rk}\mathtt{t}^{k/n}$. As γ is a κ-automorphism, then $\gamma(\theta)$ is a root of the irreducible polynomial of θ over κ, so $\gamma|_F : F \to F$ is a κ-automorphism of F, that is, an element of the Galois group $G(F:K)$. In fact, all the assignations of the previous type provide $\kappa((\mathtt{t}))$-homomorphisms of $F((\mathtt{t}^{1/n}))$.

As k_0 is relatively prime with n and $b_{k_0} \neq 0$, the images of f under the $\kappa((\mathtt{t}))$-homomorphisms

$$\gamma_r : F((\mathtt{t}^{1/n})) \to F((\mathtt{t}^{1/n})),\ \frac{g(\mathtt{t})}{\mathtt{t}^{p/n}} \mapsto \frac{g(\omega^r \mathtt{t})}{(\omega^r \mathtt{t})^{p/n}}$$

are all different and are roots of the polynomial P. As P has degree n, these are all the roots of P in the algebraically closed field $\overline{\kappa}((\mathtt{t}^*))$ of Puiseux series with coefficients in $\overline{\kappa}$. Thus, if $\sigma : F \to F$ is a κ-automorphism, we consider the $\kappa((\mathtt{t}))$-homomorphism $\Gamma_\sigma : F((\mathtt{t}^{1/n})) \to F((\mathtt{t}^{1/n}))$ whose restriction to F is σ and maps $\mathtt{t}^{1/n}$ to $\mathtt{t}^{1/n}$. Then $\Gamma_\sigma(f)$ is a root of P, so there exists $r_\sigma = 0, \dots, n-1$ such that $\Gamma_\sigma(f) = \gamma_{r_\sigma}(f)$. We have

$$\sum_{k\geqslant 0} \sigma(b_k)\mathtt{t}^{k/n} = \Gamma_\sigma(f) = \gamma_{r_\sigma}(f) = \sum_{k\geqslant 0} b_k \omega^{r_\sigma k}\mathtt{t}^{k/n},$$

so $\sigma(b_k) = b_k\omega^{r_\sigma k}$ for each $k \geqslant 0$. Thus, $\sigma(b_k^n) = b_k^n$ for each $\sigma \in G(F:\kappa)$, that is, $b_k^n \in \kappa$ for each $k \geqslant 0$. In addition, $b_k \in \kappa$ if k is a multiple of n because $\omega^{r_\sigma k} = 1$.

Let $p, q \in \mathbb{Z}$ be such that $1 = pn + qk_0$ and observe that $b_{k_0}^{-qn} \in \kappa$. Consider the irreducible polynomial $P(\mathtt{t}b_{k_0}^{-qn}, \mathtt{y}) \in \kappa((\mathtt{t}))[\mathtt{y}]$. Observe that its roots are

$$\sum_{k\geqslant 0} b_k b_{k_0}^{-kq}\omega^{rk}\mathtt{t}^{k/n}$$

for $r = 0, \dots, n-1$. We apply the previous argument for f to $g := \sum_{k\geqslant 0} b_k b_{k_0}^{-kq}\mathtt{t}^{k/n}$. Consequently, for each $\sigma \in G(F:\kappa)$ there exists $r_\sigma = 0, \dots, n-1$ such that

$$\Gamma_\sigma(g) = \sum_{k\geqslant 0} b_k b_{k_0}^{-kq}\omega^{r_\sigma k}\mathtt{t}^{k/n}$$

As $b_{k_0} b_{k_0}^{-k_0 q} = (b_{k_0}^n)^p \in \kappa$, we have $b_{k_0} b_{k_0}^{-k_0 q}\omega^{r_\sigma k_0} = \sigma(b_{k_0} b_{k_0}^{-k_0 q}) = b_{k_0} b_{k_0}^{-k_0 q}$, so $r_\sigma k_0 = sn$ for some integer s. As k_0 and n are relatively prime, we deduce $r_\sigma = 0$ for each $\sigma \in G(F:\kappa)$. Thus, $\sigma(b_k b_{k_0}^{-kq}) = b_k b_{k_0}^{-kq}$ for each $k \geqslant 0$ and each $\sigma \in G(F:\kappa)$, that is, $a_k := b_k b_{k_0}^{-kq} \in \kappa$ for each $k \geqslant 0$ and $b_k = a_k(b_{k_0}^q)^k$. If we take $\alpha := b_{k_0}^q$, we have $f = \sum_{k\geqslant 0} a_k\alpha^k\mathtt{t}^{k/n}$ and $\alpha^n = b_{k_0}^{qn} \in \kappa$, as required. □

safefr *Remark* A.3. Under the previous hypothesis, we may assume after the change of coordinates $\mathtt{t} \mapsto \mathtt{t}/\alpha^n$ (which has coefficients in κ) that $f \in \kappa[[\mathtt{t}^{1/n}]]$. ∎

We recall the following result concerning the roots in $\overline{\kappa}((\mathtt{t}^*))$ of Weierstrass polynomials of $\kappa[[\mathtt{t}]][\mathtt{x}]$.

dow **Lemma A.4.** *Let $P \in \kappa[[\mathtt{t}]][\mathtt{x}]$ be a Weierstrass polynomial such that* $\deg(P) = \omega(P)$. *Let $\zeta \in \overline{\kappa}((\mathtt{t}^*))$. Then $\zeta \in \overline{\kappa}[[\mathtt{t}^*]]$ and $\omega(\zeta) \geqslant 1$.*

Proof. Write $P := \mathtt{x}^d + \sum_{k=0}^{d-1} a_k(\mathtt{t})\mathtt{x}^k$, where $d \geqslant 1$ and $\omega(a_k) \geqslant d-k$ for $k = 0, \dots, d-1$, and substitute ζ in P to have

$$0 = P(\mathtt{t}, \zeta) = \zeta^d + \sum_{k=0}^{d-1} a_k(\mathtt{t})\zeta^k \rightsquigarrow \zeta^d = -\sum_{k=0}^{d-1} a_k(\mathtt{t})\zeta^k.$$

Consequently,

$$\begin{aligned} d\omega(\zeta) &= \omega\Big(\sum_{k=0}^{d-1} a_k(\mathtt{t})\zeta^k\Big) \\ &\geqslant \min\{\omega(a_k) + k\omega(\zeta) :\ k = 0, \dots, d-1\} \geqslant \min\{d - k + k\omega(\zeta) :\ k = 0, \dots, d-1\}. \end{aligned}$$

Let $k_0 \in \{0, \dots, d-1\}$ be such that $d\omega(\zeta) \geqslant d - k_0 + k_0\omega(\zeta)$, so $(d-k_0)\omega(\zeta) \geqslant d - k_0$. As $d - k_0 > 0$, we have $\omega(\zeta) \geqslant 1$, so $\zeta \in \overline{\kappa}[[\mathtt{t}^*]]$, as required. □

B Appendix B. Proof of a formula about Pythagoras numbers

The purpose of this appendix is to prove formula (14.1), namely,

$$1 + \tau_{2d}(\kappa) \leqslant p_{2d}(\kappa[\mathtt{t}]) \leqslant 2\tau_{2d}(\kappa),$$

which relates the τ-invariant of κ and the Pythagoras number of $\kappa[\mathtt{t}]$.

Proof of formula (14.1). We will use freely along this proof that the product of two sums of 2^k squares in a field κ is a sum of 2^k squares in κ.

Suppose $\tau_{2d}(\kappa) = 2^m$. We prove first: $p(\kappa) \leqslant 2^{m+1}$.

Pick $a \in \Sigma\kappa^2 \setminus \{0\}$ and consider the polynomial $f := \mathtt{t}^2 + a$, which is irreducible, because κ is (formally) real and so f has no roots in κ. Thus, the field $L := \kappa[\mathtt{t}]/(f)$ is a non (formally) real field and $-1 = \sum_{k=1}^{2^m}(a_k + \mathtt{t}b_k)^2 - (t^2 + a)c$, where $a_k, b_k, c \in \kappa$. Thus, $-1 = \sum_{k=1}^{2^m} a_k^2 - ac$ and $\sum_{k=1}^{2^m} b_k^2 = c$, so $a(\sum_{k=1}^{2^m} b_k^2) = 1 + \sum_{k=1}^{2^m} a_k^2$ and a is a sum of 2^{m+1} squares in κ.

Let $f \in \Sigma\kappa[\mathtt{t}]^2 \cap \kappa_{2d}[\mathtt{t}]$. The leading coefficient of f is a sum of squares in κ, so it is a sum of 2^{m+1} squares in κ. Thus, we may assume f is a monic polynomial and in fact that it is square free. Write $f = \sum_{i=1}^s g_i^2$ where $g_i \in \kappa[\mathtt{t}]$. Let f_1 be a monic irreducible factor of f (which has degree $2e$ smaller than or equal to $2d$). In $L_1 = \kappa[\mathtt{t}]/(f_1)$ we have $\sum_{i=1}^s g_i^2 = 0$ modulo (f_1). If L_1 is a (formally) real field, then f_1 divides g_i for $i = 1, \dots, s$ and the square-free polynomial f has a multiple factor. Consequently, L_1 is a non (formally) real field and there exist polynomials $h_j \in \kappa[\mathtt{t}]$ of degree smaller than or equal to $2e-1$, a monic polynomial $h \in \kappa[\mathtt{t}]$ of degree smaller than or equal to $2e-2$ and $\lambda \in \kappa\setminus\{0\}$ (recall that κ is (formally) real) such that $-1 = \sum_{j=1}^{2^m} h_j^2 - f_1 h\lambda$, so $f_1 h\lambda = 1 + \sum_{j=1}^{2^m} h_j^2$. As $f_1 h$ is monic, $\lambda \in \Sigma_{2^{m+1}}\kappa^2$, so $f_1 h \in \Sigma_{2^{m+1}}\kappa[\mathtt{t}]^2$. We distinguish two cases:

Case 1. $e = 1$. Then $h = 1$ and $f \in \Sigma_{2^{m+1}}\kappa[\mathtt{t}]^2$.

Case 2. $e \geqslant 2$. The monic irreducible factor p_k of h have degree smaller than or equal to $2e-2$ and the field $\kappa[\mathtt{t}]/(p_k)$ is non (formally) real, because $0 = 1 + \sum_{j=1}^{2^m} h_j^2$ module (p_k). By induction hypothesis each $p_k \in \Sigma_{2^{m+1}}\kappa[\mathtt{t}]^2$. Thus, $h \in \Sigma_{2^{m+1}}\kappa(\mathtt{t})^2$, so $f_1 \in \Sigma_{2^{m+1}}\kappa(\mathtt{t})^2$. Now, by [C] we have $f_1 \in \Sigma_{2^{m+1}}\kappa[\mathtt{t}]^2$.

We deduce that $f \in \Sigma_{2^{m+1}}\kappa(\mathtt{t})^2$ and again by [C] we conclude $f \in \Sigma_{2^{m+1}}\kappa[\mathtt{t}]^2 \cap \kappa_{2d}[\mathtt{t}]$. Consequently, $p_{2d}(\kappa[\mathtt{t}]) \leqslant 2^{m+1} = 2\tau_{2d}(\kappa)$.

Suppose next $p_{2d}(\kappa[\mathtt{t}]) = p$ and let $L|\kappa$ be a non (formally) real extension of κ of degree $\leqslant 2d$. Then there exists a monic polynomial $f_1 \in \kappa[\mathtt{t}]$ such that $L \cong \kappa[\mathtt{t}]/(f_1)$. Proceeding similarly as we have done above, we deduce $f_1 \in \Sigma\kappa[\mathtt{t}]^2 \cap \kappa_{2d}[\mathtt{t}] = \Sigma_p\kappa[\mathtt{t}]^2 \cap \kappa_{2d}[\mathtt{t}]$. Thus, there exist polynomials $g_j \in \kappa[\mathtt{t}]$ such that $\deg(g_j) \leqslant d$, $g_{2\ell} \neq 0$ and $f = \sum_{j=1}^p g_j^2$. Thus, $-1 = \sum_{j=1}^{p-1} \frac{g_j^2}{g_{2\ell}^2}$ in L, so L has level $\leqslant p-1$. Consequently, $\tau_{2d}(\kappa) \leqslant p-1$, as required. □

$\boxed{\mathtt{C}}$

Appendix C. Additional obstruction to be a chimeric polynomial

We present next an additional curious obstruction that satisfy chimeric polynomials.

Theorem C.1. *Let κ be a (formally) real field and let $P := \mathtt{t}^4 - 2b\mathtt{t}^2 - 4c^2\mathtt{t} + d \in \kappa[\mathtt{t}]$ be a chimeric over κ. Then $b^6 >_\alpha d^3 >_\alpha b^5$ for each $\alpha \in \mathfrak{S}_4$ such that $b >_\alpha 1$ and $b^6 >_\alpha d^3 >_\alpha b^7$ for each $\alpha \in \mathfrak{S}_4$ such that $b <_\alpha 1$.*

Proof. By Lemma 14.8 it holds $\mathfrak{S}_4 \neq \varnothing$. Let $\alpha \in \mathfrak{S}_4$. We claim: $P(\pm\sqrt{b}) <_\alpha 0$.

We have $P(\sqrt{b}) = -b^2 - 4c^2\sqrt{b} + d = -(b^2-d) - 4c^2\sqrt{b}$. As P has four roots in $\mathfrak{R}(\alpha)$, we have $b >_\alpha 0$ and $b(b^2-d) - 9c^4 = b^3 - bd - 9c^4 >_\alpha 0$, so $b^2 - d >_\alpha 0$ and $P(\sqrt{b}) = -(b^2-d) - 4c^2\sqrt{b} <_\alpha 0$. By Lemma 14.7 there exists $m_1 \in \kappa\backslash\{0\}$ such that $H := (\mathtt{t}^2 - b)(\mathtt{t}^2 - (b + m_1^4)) \in \mathcal{P}(\kappa[\mathtt{t}]/(P))$. As P is chimeric, there exists $\mu \in \kappa$ such that $H + \mu P \in \mathcal{P}(\kappa[\mathtt{t}])$. As $b >_\alpha 0$, then H has four different roots in $\mathfrak{R}(\alpha)$, so $\mu \neq 0$. We have $0 \leqslant_\alpha (H + \mu P)(\pm\sqrt{b}) = \mu P(\pm\sqrt{b})$, so $P(\sqrt{b})$ has the same sign as $P(-\sqrt{b})$ in $\mathfrak{R}(\alpha)$, that is, $P(-\sqrt{b}) <_\alpha 0$.

A Sturm's sequence for P is:

$$\begin{aligned}
F_0 &:= P := \mathtt{t}^4 - 2b\mathtt{t}^2 - 4c^2\mathtt{t} + d,\\
F_1 &:= P' := 4(\mathtt{t}^3 - b\mathtt{t}^2 - c^2),\\
F_2 &:= b\mathtt{t}^2 + 3c^2\mathtt{t} - d,\\
F_3 &:= 4(b^3 - bd - 9c^4)\mathtt{t} + 4c^2(b^2 + 3d),\\
F_4 &:= \Delta(P).
\end{aligned}$$

We have

$$[F_0(0), F_1(0), F_2(0), F_3(0), F_4(0)] = [d, -c^2, -d, 4c^2(b^2+3d), \Delta(P)],$$

so the number of change signs is 2 for each $\alpha \in \mathfrak{S}_4$ (because by Lemma 14.5 $\Delta(P) >_\alpha 0$). On the other hand, the number of change signs of the sequence of the leading coefficients $(1, 4, b, 4(b^3 - bd - 9c^4), \Delta(P))$ of the Sturm's sequence of P is 0 for each $\alpha \in \mathfrak{S}_4$. This means that P has two positive roots in $\mathfrak{R}(\alpha)$ for each $\alpha \in \mathfrak{S}_4$. In addition, as P is chimeric, we have $P(\frac{d}{d+4c^2}) >_\alpha 0$ for each $\alpha \in \operatorname{Sper}(\kappa)$ and

$$\begin{aligned}
&\Big[F_0\Big(\frac{d}{d+4c^2}\Big), F_1\Big(\frac{d}{d+4c^2}\Big), F_2\Big(\frac{d}{d+4c^2}\Big), F_3\Big(\frac{d}{d+4c^2}\Big), F_4\Big(\frac{d}{d+4c^2}\Big)\Big]\\
&\quad= \Big[P\Big(\frac{d}{d+4c^2}\Big), \frac{-4(64c^8 + 48c^6d + 16bc^4d + 12c^4d^2 + 8bc^2d^2 + c^2d^3 + bd^3 - d^3)}{4c^2+d)^3},\\
&\qquad\frac{d(-4c^4 - 5c^2d + bd - d^2)}{(4c^2+d)^2}, \frac{4(4b^2c^4 + b^2c^2d + 3c^4d + b^3d + 3c^2d^2 - bd^2)}{b^2(4c^2+d)}, \Delta(P)\Big].
\end{aligned}$$

As $b >_\alpha 0$ and $b >_\alpha 0$ and $b(b^2-d) - 9c^4 = b^3 - bd - 9c^4 >_\alpha 0$, we have $b^3d - bd^2 = db(b^2-d) >_\alpha 0$, so $F_3\Big(\frac{d}{d+4c^2}\Big) >_\alpha 0$. In addition, if $b >_\alpha 1$, we have $F_1\Big(\frac{d}{d+4c^2}\Big) <_\alpha 0$. Thus, the number of change signs of the previous Sturm's sequence at $\frac{4d}{3b^2}$ for α is exactly 2 (because $F_0(\frac{d}{d+4c^2}) >_\alpha 0$ and $\Delta(P) >_\alpha 0$). Thus, $\frac{d}{4c^2+d}$ is a lower bound for the two positive roots of P for each $\alpha \in \mathfrak{S}_4$.

If $\alpha \in \mathfrak{S}_2$, then $\Delta(P) <_\alpha 0$ and the number of changes of signs in

$$[F_0(0), F_1(0), F_2(0), F_3(0), F_4(0)] = [d, -c^2, -d, 4c^2(b^2+3d), \Delta(P)],$$

is 1. On the other hand, the number of change signs of the sequence of the leading coefficients $[1, 4, b, 4(b^3-bd-9c^4), \Delta(P)]$ of the Sturm's sequence of P is at most 1, because it is smaller that or equal to the number of changes signs in $[F_0(0), F_1(0), F_2(0), F_3(0), F_4(0)]$. As $\Delta(P) <_\alpha 0$, we deduce that such number of changes of signs is also 1, so P has no positive roots in $\mathfrak{R}(\alpha)$ for each $\alpha \in \mathfrak{S}_2$.

Consider the polynomial $Q := (\mathtt{t}^3 - 2b\mathtt{t}^2 - 4c^2) + 4c^2 + d$. Pick $\alpha \in \mathfrak{S}_4$. Let $\eta_1, \eta_2, \eta_3, \eta_4 \in \mathfrak{R}(\alpha)$ be the roots of P. As $P(\eta_i) = \eta_i(\eta_i^3 - 2b\eta_i^2 + 4c^2) + d = 0$, we have

$$Q(\eta_i) = \eta_i^3 - 2b\eta_i^2 - 4c^2 + 4c^2 + d = \frac{-d}{\eta_i} + 4c^2 + d.$$

If $\eta_i <_\alpha 0$, then $Q(\eta_i) = \frac{-d}{\eta_i} + 4c^2 + d >_\alpha 0$, whereas if $\eta_i >_\alpha 0$, then $Q(\eta_i) = \frac{4c^2+d}{\eta_i}(\eta_i - \frac{d}{4c^2+d}) >_\alpha 0$. Let $\alpha \in \mathfrak{S}_2$, then the roots $\eta_i \in \mathfrak{R}(\alpha)$ are negative, so $Q(\eta_i) = \frac{-d}{\eta_i} + 4c^2 + d >_\alpha 0$.

Consequently, $Q \in \mathcal{P}(\kappa[\mathtt{t}]/(P))$ and as P is chimeric, there exists $\mu \in \Sigma\kappa^2$ such that $Q + \mu P \in \Sigma\kappa[\mathtt{t}]^2$. Thus,

$$(Q + \mu P)(\sqrt{b}) = -b\sqrt{b} + d + \mu P(\sqrt{b}) >_\beta 0.$$

in $\mathfrak{R}(\beta)$ for each $\beta \in \operatorname{Sper}(\kappa)$.

If $\alpha \in \mathfrak{S}_4$ and $b_\alpha > 1$, then $\mu P(\sqrt{b}) <_\alpha 0$, so $-b\sqrt{b} + d >_\alpha 0$ and $d^2 >_\alpha b^3$. As P is chimeric,

$$0 <_\alpha P\Big(\frac{b^2}{d}\Big) = \frac{b^8}{d^4} - 2\frac{b^5}{d^2} - \frac{4c^2b^2}{d} + d.$$

Thus, either $\frac{b^8}{d^4} - \frac{b^5}{d^2} >_\alpha 0$ or $-\frac{b^5}{d^2} + d >_\alpha 0$. Consequently, either $b^3 >_\alpha d^2$ or $d^3 >_\alpha b^5$. As we have already proved that $d^2 >_\alpha b^3$ and $b^2 - d >_\alpha 0$, we conclude $b^6 >_\alpha d^3 >_\alpha b^5$.

If $\alpha \in \mathfrak{S}_4$ and $b_\alpha < 1$, consider the chimeric polynomial

$$\frac{P(b\mathtt{t})}{b^4} = \mathtt{t}^4 - 2\frac{1}{b}\mathtt{t}^2 - 4\frac{c^2}{b^3}\mathtt{t} + \frac{d}{b^4}.$$

As $b >_\alpha 0$, we can apply the same kind of arguments as above, to conclude (because $\frac{1}{b} >_\alpha 1$) that $(\frac{d}{b^4})^3 >_\alpha (\frac{1}{b})^5$, so $b^6 >_\alpha d^3 >_\alpha b^7$ (because $b^2 - d >_\alpha 0$), as required. □

D

Appendix D. Fields of Laurent series admitting no chimeric polynomials

In this appendix we prove Theorem 14.14.

D.1. **Preliminary results.** Before proving Theorem 14.14 we develop some preliminary work to lighten its proof.

hh0 **Lemma D.1.** *Let $H := A_0 + A_1\mathtt{t} + A_2\mathtt{t}^2 + A_3\mathtt{t}^3 + A_4\mathtt{t}^4 \in \kappa[[\mathtt{x}]][\mathtt{t}]$ be a polynomial such that $H_0 := H(0, \mathtt{t})$ has degree 4, the discriminant $\Delta(H_0) \neq 0$ and $H_0 \in \mathcal{P}(\kappa[\mathtt{t}])$. Then $H \in \mathcal{P}(\kappa[[\mathtt{x}]][\mathtt{t}])$ and $\{H = 0\} \cap \mathfrak{R}(\alpha) = \varnothing$ for each $\alpha \in \operatorname{Sper}(\kappa[[\mathtt{x}]])$.*

Proof. If $H \notin \mathcal{P}(\kappa[[\mathtt{x}]][\mathtt{t}])$, there exists a homomorphism $\varphi : \kappa[[\mathtt{x}]][\mathtt{t}] \to E$, where E is a real closed field, such that $\varphi(H) < 0$. Let $\mathfrak{p} := \ker(\varphi) \cap \kappa[[\mathtt{x}]]$, which is (a prime ideal) either equal to (0) or $(\mathtt{x})$. If $\mathfrak{p} = (\mathtt{x})$, then

$$\kappa[\mathtt{t}] \cong \kappa[[\mathtt{x}]][\mathtt{t}]/((\mathtt{x})\kappa[[\mathtt{x}]][\mathtt{t}]) \mapsto \kappa[[\mathtt{x}]][\mathtt{t}]/\ker(\varphi) \hookrightarrow E.$$

As $H_0 \in \mathcal{P}(\kappa[\mathtt{t}])$, we deduce $\varphi(H_0) \geqslant 0$, which is a contradiction. Consequently, $\mathfrak{p} = (0)$.

Consider the prime cone β induced by φ in $\kappa[[\mathtt{x}]]$ whose support is (0) and observe that $\kappa[[\mathtt{x}]] \hookrightarrow \kappa((\mathtt{x})) \hookrightarrow \mathfrak{R}(\beta)$. Denote $\beta_0 := \beta \cap \kappa \in \operatorname{Sper}(\kappa)$ and $\mathfrak{R}(\beta_0)$ its real closure. By [ABR, Ex.II.3.13] the real closure of $(\kappa((\mathtt{x})), \leqslant_\beta)$ is $\mathfrak{R}(\beta_0)((\mathtt{x}^*))$. As $H_0 \in \mathcal{P}(\kappa[\mathtt{t}]) = \Sigma\kappa[\mathtt{t}]^2$, its leading coefficient is a (non-zero) sum of squares of κ. Thus, the leading coefficient of H is a (non-zero) sum of squares of $\kappa[[\mathtt{x}]]$ that is in addition a unit, so we may assume H is a monic polynomial. As $\Delta(H_0) \in \kappa\setminus\{0\}$, we deduce $\Delta(H) \in \kappa[[\mathtt{x}]]\setminus\{0\}$ is a unit of $\kappa[[\mathtt{x}]]$ (recall that H is a monic polynomial).

As $H_0 \in \mathcal{P}(\kappa[\mathtt{t}])\setminus\{0\} = \Sigma\kappa[\mathtt{t}]^2\setminus\{0\}$ and $\Delta(H_0) \neq 0$, we have that H_0 has no roots in $\mathfrak{R}(\gamma)$ for each $\gamma \in \operatorname{Sper}(\kappa)$. Thus, $\Delta(H) \in \Sigma\kappa[[\mathtt{x}]]^2$ is a unit. Consequently, by Lemma 14.5 H has either 0 or 4 simple roots in $\mathfrak{R}(\beta)$. As $H \in \mathfrak{R}(\beta)[\mathtt{t}]$ change sign in E, which is a real closed field that contains the real closed field $\mathfrak{R}(\beta)$, it also changes sign in $\mathfrak{R}(\beta)$. Thus, H has a root in $\mathfrak{R}(\beta)$ of multiplicity 1, so it has 4 roots of multiplicity one in $\mathfrak{R}(\beta)$.

Let $G_0 := H \in \kappa[[\mathtt{x}]][\mathtt{t}]$, $G_1 := \frac{\partial H}{\partial \mathtt{t}} \in \kappa[[\mathtt{x}]][\mathtt{t}]$, $G_2 := -\operatorname{remainder}(G_0, G_1, \mathtt{t}) \in \kappa[[\mathtt{x}]][\mathtt{t}]$, $G_3 := -\operatorname{remainder}(G_1, G_2, \mathtt{t}) \in \kappa[[\mathtt{x}]][\mathtt{t}]$ and $H_4 := \Delta(H)$ be a Sturm's sequence for H in $\kappa[[\mathtt{x}]][\mathtt{t}]$. Observe that G_1 is a monic polynomial times 4. As H has four different roots in $\mathfrak{R}(\beta)$, we deduce that the leading coefficients B_2 of G_2 and B_3 of G_3 are positive in $\mathfrak{R}(\beta)$.

As H is a monic polynomial, the roots of H belongs to the ring of Puiseux series $\mathfrak{R}(\beta_0)[[\mathtt{x}^*]]$. If $\zeta \in \mathfrak{R}(\beta_0)[[\mathtt{x}^*]]$ is such a root of H, then $H(\mathtt{x}, \zeta) = 0$ and making $\mathtt{x} = 0$, we deduce $0 = H(0, \zeta(0)) = H_0(\zeta(0))$, so H_0 has a root $\zeta(0) \in \mathfrak{R}(\beta_0)$. But this is a contradiction because $H_0 \in \mathcal{P}(\kappa[\mathtt{t}])$ has not root in $\mathfrak{R}(\beta_0)$.

Consequently, $H \in \mathcal{P}(\kappa[[\mathtt{x}]][\mathtt{t}])$. As H has no multiple roots, then it is a square-free polynomial, so $\{H = 0\} \cap \mathfrak{R}(\alpha) = \varnothing$ for each $\alpha \in \operatorname{Sper}(\kappa[[\mathtt{x}]])$, because $H \in \mathcal{P}(\kappa[[\mathtt{x}]][\mathtt{t}])$. $\square$

spm **Lemma D.2.** *Let $P \in \kappa[\mathtt{t}]$ be a chimeric polynomial and let $H_0 \in \kappa[\mathtt{t}]$ be a polynomial of degree 4 such that $H_0 \in \mathcal{P}(\kappa[\mathtt{t}])$. Then there exist infinitely many $\theta \in \kappa$ such that $H := H_0 + \theta P \in \mathcal{P}(\kappa[\mathtt{t}])$, $\{H = 0\} \cap \mathfrak{R}(\alpha) = \varnothing$ for each $\alpha \in \operatorname{Sper}(\kappa)$ and $\Delta(H) \neq 0$.*

Proof. The proof is conducted in several steps:

STEP 0. INITIAL PREPARATION. Observe first that if $H_1 \in \mathcal{P}(\kappa[\mathtt{t}])$ and $\Delta(H_1) \neq 0$, then $\{H_1 = 0\} \cap \mathfrak{R}(\alpha) = \varnothing$ for each $\alpha \in \operatorname{Sper}(\kappa)$ (because H_1 is a square-free polynomial).

The polynomial $\Delta(H_0 + \mathtt{s}P) \in \kappa[\mathtt{s}]$ has degree 6 and its leading coefficient is $\Delta(P) \neq 0$ (because P is irreducible). Thus, if $\theta \in \kappa$ is not a root of $\Delta(H_0 + \mathtt{s}P)$ and $H_0 + \theta P$ has degree 4, then $\Delta(H_0 + \theta P) \neq 0$. Consequently, the key part is to find $\theta \in \kappa$ such that $H_0 + \theta P \in \mathcal{P}(\kappa[\mathtt{t}])$ and θ is not a root of $\Delta(H_0 + \mathtt{s}P) \in \kappa[\mathtt{s}]$.

If $\Delta(H_0) \neq 0$, it is enough to take $\theta = 0$ and $H := H_0$. Thus, we assume $\Delta(H_0) = 0$. This means that H_0 has a multiple factor. As H_0 has degree 4 and $H_0 \in \mathcal{P}(\kappa[\mathtt{t}])$, we deduce we have the following cases:

(i) H_0 has a double root in κ and an irreducible factor of degree 2 in $\kappa[\mathtt{t}]$.
(ii) H_0 has two double roots in κ (which may coincide).
(iii) H_0 is the square of an irreducible factor G_0 of degree 2.

STEP 1. CASES (i) AND (ii) As P is a chimeric polynomial, $P(a) \in \Sigma\kappa^2\setminus\{0\}$ for each $a \in \kappa$ (Lemma 14.1). In particular, this happens for the roots of H_0 in κ (if any). As $P \notin \mathcal{P}(\kappa[\mathtt{t}])$ and P is irreducible in $\kappa[\mathtt{t}]$, we deduce that P and H_0 share no root in $\overline{\kappa}$. In case (i) the irreducible factor of degree 2 cannot have roots in $\mathfrak{R}(\alpha)$ for each $\alpha \in \operatorname{Sper}(\kappa)$, because it divides H_0 with multiplicity 1. Thus, in cases (i) and (ii) the set $\{P \leqslant_\alpha 0\} \subset \{H_0 >_\alpha 0\}$, because $\{H_0 = 0\} \subset \{P >_\alpha 0\}$ for each $\alpha \in \operatorname{Sper}(\kappa)$.

Let P' be the derivative of P (with respecto to $\mathtt{t}$). Let $\zeta_1, \zeta_2, \zeta_3 \in \overline{\kappa}$ be the roots of P' and let $\xi_1, \xi_2, \xi_3, \xi_4 \in \overline{\kappa}$ be the roots of P. Let $\alpha \in \operatorname{Sper}(\kappa)$ and observe that for each $x \in$

$\{P \leqslant_\alpha 0\} \subset \mathfrak{R}(\alpha)$ it holds $|P(x)|_\alpha \leqslant \max\{|P(\zeta_1)|_\alpha, |P(\zeta_2)|_\alpha, |P(\zeta_3)|_\alpha\}$ (use that P is a monic polynomial of even degree and Rolle's theorem). As $P(\zeta_1), P(\zeta_2), P(\zeta_3) \in \overline{\kappa}$, there exists by Lemma 11.1 $M \in \Sigma\kappa^2$ such that $\frac{1}{M} <_\alpha |P(\zeta_i)| <_\alpha M$ for $i = 1, 2, 3$. Thus, $|P(x)| <_\alpha M$ for each $x \in \{P \leqslant_\alpha 0\} \subset \mathfrak{R}(\alpha)$ and each $\alpha \in \operatorname{Sper}(\kappa)$.

Suppose $\{P \leqslant_\alpha 0\} \subset \{H_0 >_\alpha 0\}$ for each $\alpha \in \operatorname{Sper}(\kappa)$. Let $\eta_1, \eta_2, \eta_3 \in \overline{\kappa}$ be the roots of H_0'. As $H_0(\eta_1), H_0(\eta_2), H_0(\eta_3)$ are algebraic over κ, there exists by Lemma 11.1 $N \in \Sigma\kappa^2$ such that $\frac{1}{N} <_\alpha |H_0(\eta_i)|_\alpha <_\alpha N$ for $i = 1, 2, 3$ and $\frac{1}{N} <_\alpha |H_0(\xi_j)|_\alpha <_\alpha N$ for $j = 1, 2, 3, 4$. Observe that for each $x \in \{P \leqslant_\alpha 0\} \subset \mathfrak{R}(\alpha)$ it holds

$$H_0(x) \geqslant_\alpha \min_\alpha\{|H_0(\eta_i)|_\alpha, |H_0(\xi_j)|_\alpha,\ i = 1, 2, 3,\ j = 1, 2, 3, 4\} \geqslant_\alpha \frac{1}{N}$$

There exists $k_0 \geqslant 1$ such that if $k \geqslant k_0$, then $\frac{1}{(M+k)(N+k)}$ is not a root of the polynomial $\Delta(H_0 + \mathtt{s}P) \in \kappa[\mathtt{s}]$. Thus, $H_0 + \frac{1}{(M+k)(N+k)}P$ satisfies for $k \geqslant k_0$ the required properties.

STEP 2. CASE (iii) If $\{P \leqslant_\alpha 0\} \subset \{H_0 := G_0^2 >_\alpha 0\}$, the same proof done in STEP 1 works, so let us assume $\{P \leqslant_\alpha 0\} \not\subset \{H_0 := G_0^2 >_\alpha 0\}$ for some $\alpha \in \operatorname{Sper}(\kappa)$. Let $\eta, \eta' \in \overline{\kappa}$ be the roots of G_0. Let $\alpha \in \operatorname{Sper}(\kappa)$ be such that $\{P \leqslant_\alpha 0\} \not\subset \{G_0^2 >_\alpha 0\}$. Thus, $\eta, \eta' \in \mathfrak{R}(\alpha)$ and we may assume $P(\eta) <_\alpha 0$ and $G_0(\eta) = 0$. We distinguish several cases:

CASE (iii.1) *Suppose first* $P(\eta')_\alpha > 0$. By Lemma 14.7 there exists $G_0^* \in \kappa[\mathtt{t}]$ of degree 2 such that $G_0G_0^* \in \mathcal{P}(\kappa[\mathtt{t}]/(P))$ and G_0, G_0^* are not proportional (that is, they roots are different). As P is a chimeric polynomial over κ, there exists $\lambda \in \Sigma\kappa^2$ such that $G_0G_0^* + \lambda P \in \mathcal{P}(\kappa[\mathtt{t}])$. Observe that $\lambda \neq 0$ because G_0, G_0^* are not proportional. We substitute η, η' in $G_0G_0^* + \lambda P$ and obtain $\lambda P(\eta) >_\alpha 0$ and $\lambda P(\eta') >_\alpha 0$. Thus, $\lambda^2 P(\eta)P(\eta') <_\alpha 0$, which is a contradiction because $P(\eta) >_\alpha 0$ whereas $P(\eta') >_\alpha 0$. We deduce that if $P(\eta) <_\alpha 0$, then also $P(\eta') <_\alpha 0$.

CASE (iii.2) $P(\eta') <_\alpha 0$ *and suppose there exists another ordering* $\beta \in \operatorname{Sper}(\kappa)$ *such that* $\eta, \eta' \in \mathfrak{R}(\beta)$, $P(\eta) >_\beta 0$ *and* $P(\eta') >_\beta 0$. By Lemma 14.7 there exists $G_0^* \in \kappa[\mathtt{t}]$ of degree 2 such that $G_0G_0^* \in \mathcal{P}(\kappa[\mathtt{t}]/(P))$. As P is a chimeric polynomial over κ, there exists $\lambda \in \Sigma\kappa^2$ such that $G_0G_0^* + \lambda P \in \mathcal{P}(\kappa[\mathtt{t}])$. Observe that $\lambda \neq 0$ because G_0, G_0^* are not proportional. We substitute η in $G_0G_0^* + \lambda P$ and obtain $\lambda P(\eta) >_\alpha 0$ and $\lambda P(\eta) >_\beta 0$, whereas $P(\eta) <_\alpha 0$ and $P(\eta) >_\beta 0$. This contradicts the fact that $\lambda \in \Sigma\kappa^2\backslash\{0\}$. Consequently, $P(\eta) <_\beta 0$ and $P(\eta') <_\beta 0$ for each $\beta \in \operatorname{Sper}(\kappa)$.

CASE (iii.3) $P(\eta) <_\beta 0$ *and* $P(\eta') <_\beta 0$ *for each* $\beta \in \operatorname{Sper}(\kappa)$. Write $H_0 = \sigma G_0^2$ for some $\sigma \in \kappa\backslash\{0\}$ in such a way that we can assume $G_0 \in \kappa[\mathtt{t}]$ is a monic polynomial of degree 2. As $H_0 \in \mathcal{P}(\kappa[\mathtt{t}])$, we deduce $\sigma \in \Sigma\kappa^2\backslash\{0\}$, so we may assume $H_0 = G_0^2$. Consider the polynomial $G_0^2 - \frac{1}{4}P$, which has leading coefficient $\frac{3}{4}$ and let $\chi_1, \chi_2, \chi_3, \chi_4 \in \overline{\kappa}$ be the roots of $G_0^2 - \frac{1}{4}P$. Observe that $\{G_0^2 - \frac{1}{4}P \leqslant_\alpha 0\} = [\chi_1, \chi_2]_\alpha \cup [\chi_3, \chi_4]_\alpha$ for each $\alpha \in \operatorname{Sper}(\kappa)$ after reordering the indices of the roots χ_i to have $\chi_1 <_\alpha \chi_2 <_\alpha \chi_3 <_\alpha \chi_4$ in case all the $\chi_i \in \mathfrak{R}(\alpha)$). It may happen that one of the two previous intervals is empty (if some of the roots χ_i does not belong to $\mathfrak{R}(\alpha)$).

Fix $\alpha \in \operatorname{Sper}(\kappa)$. As $P(\eta) <_\alpha 0$ and $P(\eta') <_\alpha 0$, we have $(G_0^2 - \frac{1}{4}P)(\eta) > 0$ and $(G_0^2 - \frac{1}{4}P)(\eta') > 0$. As $\{G_0^2 > 0\} = \mathfrak{R}(\alpha)\backslash\{\eta, \eta'\}$, we deduce G_0^2 is strictly positive on $\{G_0^2 - \frac{1}{4}P \leqslant_\alpha 0\}$. Let $\rho \in \overline{\kappa}$ be the root of the derivative G_0' and observe that the roots of the derivative $(G_0^2)' = 2G_0G_0'$ are η, η', ρ. As $G_0^2 - \frac{1}{4}P$ is a polynomial of degree 4 and leading coefficient $\frac{3}{4}$, for each $x \in \{G_0^2 - \frac{1}{4}P \leqslant_\alpha 0\}$ it holds

$$G_0^2(x) >_\alpha \min_\alpha(\{|G_0^2(\chi_1)|_\alpha, |G_0^2(\chi_2)|_\alpha, |G_0^2(\chi_3)|_\alpha, |G_0^2(\chi_4)|_\alpha, |G_0^2(\rho)|_\alpha\}\backslash\{0\}).$$

As $G_0^2(\chi_1), G_0^2(\chi_2), G_0^2(\chi_3), G_0^2(\chi_4), G_0^2(\rho) \in \overline{\kappa}$ are algebraic over κ, there exists by Lemma 11.1 $N \in \Sigma\kappa^2\backslash\{0\}$ such that

$$\min_\alpha(\{|G_0^2(\chi_1)|_\alpha, |G_0^2(\chi_2)|_\alpha, |G_0^2(\chi_3)|_\alpha, |G_0^2(\chi_4)|_\alpha, |G_0^2(\rho)|_\alpha\}\backslash\{0\}) >_\alpha \frac{1}{N}$$

for each $\alpha \in \operatorname{Sper}(\kappa)$. Thus, $G_0^2(x) >_\alpha \frac{1}{N}$ for each $x \in \{G_0^2 - \frac{1}{4}P \leqslant_\alpha 0\}$ and each $\alpha \in \operatorname{Sper}(\kappa)$.

Let $\zeta_1, \zeta_2, \zeta_3$ be the roots of P'. As P is a polynomial of degree 4 and leading coefficient 1, we have

$$|P(x)|_\alpha < \max_\alpha\{|P(\chi_1)|_\alpha, |P(\chi_2)|_\alpha, |P(\chi_3)|_\alpha, |P(\chi_4)|_\alpha, |P(\zeta_1)|_\alpha, |P(\zeta_2)|_\alpha, |P(\zeta_3)|_\alpha\}$$

for each $x \in \{G_0^2 - \frac{1}{4}P \leqslant_\alpha 0\}$ and each $\alpha \in \operatorname{Sper}(\kappa)$. As

$$P(\chi_1), P(\chi_2), P(\chi_3), P(\chi_4), P(\zeta_1), P(\zeta_2), P(\zeta_3) \in \overline{\kappa}$$

are algebraic over κ there exists by Lemma 11.1 $M \in \Sigma\kappa^2 \setminus \{0\}$ such that

$$\max_\alpha\{|P(\chi_1)|_\alpha, |P(\chi_2)|_\alpha, |P(\chi_3)|_\alpha, |P(\chi_4)|_\alpha, |P(\zeta_1)|_\alpha, |P(\zeta_2)|_\alpha, |P(\zeta_3)|_\alpha\} <_\alpha M$$

for each $\alpha \in \operatorname{Sper}(\kappa)$. Thus, $|P(x)|_\alpha < M$ for each $x \in \{G_0^2 - \frac{1}{4}P \leqslant_\alpha 0\}$ and each $\alpha \in \operatorname{Sper}(\kappa)$. There exists $k_0 \geqslant 1$ such that if $k \geqslant k_0$ then $\Delta(G^2 - \frac{1}{4(N+k)(M+k)}P) \neq 0$ (see Step 0). We claim: $H_k := G^2 - \frac{1}{4(N+k)(M+k)}P \in \mathcal{P}(\kappa[\mathtt{t}])$ for each $k \geqslant k_0$.

Suppose there exists $\beta \in \operatorname{Sper}(\kappa[\mathtt{t}])$ such that $H_k <_\beta 0$ and let $\alpha := \beta|_\kappa \in \operatorname{Sper}(\kappa)$. If $\operatorname{supp}(\beta) \neq (0)$, then $\kappa[\mathtt{t}]/\operatorname{supp}(\beta)$ is an algebraic extension of κ (because $\kappa[\mathtt{t}]$ is a PID). By the unicity of real closure we have $\kappa \hookrightarrow \kappa[\mathtt{t}]/\operatorname{supp}(\beta) \hookrightarrow \mathfrak{R}(\alpha)$ and there exists $x_0 \in \mathfrak{R}(\alpha)$ such that $H_k(x_0) <_\alpha 0$. If $\operatorname{supp}(\beta) = (0)$, then $\kappa \hookrightarrow \kappa[\mathtt{t}] \hookrightarrow \kappa(\mathtt{t}) \hookrightarrow \mathfrak{R}(\beta)$. As $\mathfrak{R}(\alpha) \subset \mathfrak{R}(\beta)$, we have

$$\kappa \hookrightarrow \kappa[\mathtt{t}] \hookrightarrow \mathfrak{R}(\alpha)[\mathtt{t}] \hookrightarrow \mathfrak{R}(\beta)$$

By Artin-Lang's Theorem [BCR, Thm.4.1.2] there exists an homomorphism $\mathfrak{R}(\alpha)[\mathtt{t}] \to \mathfrak{R}(\alpha)$ that maps H_k to a negative element of $\mathfrak{R}(\alpha)$. Thus, there exists $x_0 \in \mathfrak{R}(\alpha)$ such that $H(x_0) <_\alpha 0$. Consequently,

$$\begin{aligned} H_k(x_0) = G^2(x_0) - \frac{1}{4(N+k)(M+k)}P(x_0) <_\alpha 0 \\ \rightsquigarrow \quad G^2(x_0) <_\alpha \frac{1}{4(N+k)(M+k)}P(x_0) < \frac{1}{4}P(x_0). \end{aligned}$$

As $x_0 \in \{G_0^2 - \frac{1}{4}P \leqslant_\alpha 0\}$, we have $G^2(x_0) > \frac{1}{N}$ and $|P(x_0)| < M$, so

$$\begin{aligned} H(x_0) = G^2(x_0) - \frac{1}{4(N+k)(M+k)}P(x_0) &\geqslant G^2(x_0) - \frac{1}{4(N+k)(M+k)}|P(x_0)| \\ &> \frac{1}{N} - \frac{M}{4(N+k)(M+k)} > \frac{3}{4N} > 0, \end{aligned}$$

which is a contradiction. Consequently, $H_k \in \mathcal{P}(\kappa[\mathtt{t}])$ for each $k \geqslant k_0$, as required. □

D.1.1. *Archimedean fields.* Let κ be a (formally) real field and let P be a positive cone (for instance, $P = \Sigma\kappa^2$). Consider the ring $I(P,\kappa) := \{x \in \kappa : \ \exists n \in \mathbb{N}, -n <_P x <_P n\}$. By [Schl, §8.Thm.1] all the orderings of κ compatible with P are Archimedean if and only if $I(P,\kappa) = \kappa$. Let us recall a rational density property that follows from the Archimedean property.

density **Lemma D.3** (Density). *Let κ be a (formally) real field, let P be a positive cone and suppose that $I(P,\kappa) = \kappa$. Then:*

(i) *If $F := \bigcap_{\alpha \in \operatorname{Sper}(\kappa),\ P \subset \alpha} \mathfrak{R}(\alpha)$ and $Q := \bigcap_{\beta \in \operatorname{Sper}(F),\ P \subset \beta} \beta$, then $I(Q,F) = F$.*
(ii) *If $x, y \in F$ satisfies $0 <_Q x <_Q y$, there exist $\beta_i \in \operatorname{Sper}(F)$ that contains Q and $q_i \in \mathbb{Q}$ for $i = 1,2$ such that $x <_Q q_1 <_{\beta_1} y$ and $x <_{\beta_2} < q_2 <_Q y$.*

Proof. (i) The real closure $\mathfrak{R}(\alpha)$ is a subfield of the algebraic closure $\overline{\kappa}$ of κ for each $\alpha \in \operatorname{Sper}(\kappa)$. Consequently, $F := \bigcap_{\alpha \in \operatorname{Sper}(\kappa),\ P \subset \alpha} \mathfrak{R}(\alpha)$ is a subfield of $\overline{\kappa}$ (so it is an algebraic extension of κ) and it is (formally) real, because $F \subset \mathfrak{R}(\alpha)$ for each $\alpha \in \operatorname{Sper}(\kappa)$ that contains P. In addition, as $I(P,\kappa) = \kappa$, all the orderings of κ compatible with P are Archimedean [Schl, §8.Thm.1].

Thus, $\mathfrak{R}(\alpha)$ is a subfield of the real numbers $\mathbb{R}$ for each α compatible with P. Let us check: $I(Q,F)=F$, where $Q:=\bigcap_{\beta\in\operatorname{Sper}(F),\ P\subset\beta}\beta$.

Let $\beta\in\operatorname{Sper}(F)$ be such that $Q\subset\beta$. Then $P\subset\alpha:=\beta\cap\kappa$, so α is compatible with P. Let $\mathfrak{R}'(\beta)$ be the real closure of $(F,\leqslant_\beta)$. Then $\mathfrak{R}'(\beta)\subset\overline{F}=\overline{\kappa}$ is a real closed field that contains $(\kappa,\leqslant_\alpha)$ as an ordered subfield. Thus, $\mathfrak{R}(\alpha)$ is an ordered subfield of $\mathfrak{R}'(\beta)$. As $\mathfrak{R}'(\beta)[\sqrt{-1}]=\overline{F}=\overline{\kappa}=\mathfrak{R}(\alpha)[\sqrt{-1}]$, we conclude that $\mathfrak{R}(\alpha)=\mathfrak{R}'(\beta)$. As $P\subset\alpha$, we know that α is an Archimedean ordering. Consequently, $(\kappa,\leqslant_\alpha)$ is an ordered subfield of $\mathbb{R}$, so $\mathfrak{R}(\alpha)=\mathfrak{R}'(\beta)$ is an ordered subfield of $\mathbb{R}$. We conclude that β is an Archimedean field of F, so $I(Q,F)=F$.

(ii) We have $0<_Q y-x$. As $I(Q,F)=F$, there exists by [Schl, §6.7] an integer $k\geqslant 1$ such that $0<_Q\frac{1}{2^k}<_Q y-x$, so $1<_Q(y-x)2^k$ and $x2^k+1<_Q y2^k$. As $I(Q,F)=F$, there exists $n_0\in\mathbb{N}$ such that $0<_Q x2^k<_Q n_0$. Let m be the minimum of the non-emptyset $S:=\{n\in\mathbb{N}:\ 0<_Q x2^k<_Q n\}$. We have $x2^k<_Q m$ and there exists $\beta_1\in\operatorname{Sper}(\kappa)$ such that $Q\subset\beta_1$ and $m-1<_{\beta_1}x2^k$. Thus, $x2^k<_Q m<_{\beta_1}x2^k+1<_Q y2^k$, so $x<_Q q_1:=\frac{m}{2^k}<_{\beta_1}y$.

If we consider next $0<_Q\frac{1}{y}<_Q\frac{1}{x}$, there exists (as we have already proven) another ordering $\beta_2\in\operatorname{Sper}(\kappa)$ such that $Q\subset\beta_2$ and $q_3\in\mathbb{Q}$ satisfying $\frac{1}{y}<_Q q_3<_{\beta_2}\frac{1}{x}$, so $x<_{\beta_2}q_2:=\frac{1}{q_3}<_Q y$, as required. □

Suppose next that κ is a (formally) real field that is dense in $\mathfrak{R}(\alpha)$ for each $\alpha\in\operatorname{Sper}(\kappa)$. Let P be a positive cone of κ and define $F:=\bigcap_{\alpha\in\operatorname{Sper}(\kappa),\ P\subset\alpha}\mathfrak{R}(\alpha)$ and $Q:=\bigcap_{\beta\in\operatorname{Sper}(F),\ P\subset\beta}\beta$. Let $x,y\in F$ be such that $0<_\beta x<_\beta y$ for each $\beta\in\operatorname{Sper}(F)$ satisfying $P\subset\beta$. A natural question to generalize Lemma D.3 (and consequently Theorem 14.14 below) to this situation is the following:

questdensity *Question* D.4. Do there exist $\beta_i\in\operatorname{Sper}(F)$ that contains Q and $q_i\in\kappa$ for $i=1,2$ such that $x<_Q q_1<_{\beta_1}y$ and $x<_{\beta_2}<q_2<_Q y$?

Of course, the previous question has a trivial positive answer if $\operatorname{Sper}(\kappa)$ is a singleton and κ is dense in its unique real closure.

D.2. **Proof of Theorem 14.14.** We are ready to prove Theorem 14.14. We proceed in several steps and divide the proof into several lemmas to make it clearer. We begin proving Theorem 14.14(i).

Lemma D.5. *Let $P\in\kappa[\mathtt{t}]$ be a chimeric polynomial over κ. Then P is a chimeric polynomial over $\kappa((\mathtt{x}))$.*

Proof. Let $P\in\kappa[\mathtt{t}]$ be a chimeric polynomial over κ. Let us check: *P is a chimeric polynomial over $\kappa((\mathtt{x}))$.*

Observe first that P is irreducible over $\kappa((\mathtt{x}))$. Otherwise, there exist monic polynomials $P_1,P_2\in\kappa((\mathtt{x}))[\mathtt{t}]$ of degrees $\geqslant 1$ such that $P=P_1P_2$. Observe that $\deg_{\mathtt{t}}(P_1)+\deg_{\mathtt{t}}(P_2)=4$. Let $k_i\geqslant 0$ (be a minimal integer) such that $Q_i:=\mathtt{x}^{k_i}P_i\in\kappa[[\mathtt{x}]][\mathtt{t}]$ but $Q_i(0,\mathtt{t})\neq 0$. Then $\mathtt{x}^{k_1+k_2}P=Q_1Q_2$ where $k_1+k_2\geqslant 0$. If we substitute $\mathtt{x}=0$, we obtain either $P(\mathtt{t})=P(0,\mathtt{t})=Q_1(0,\mathtt{t})Q_2(0,\mathtt{t})$ if $k_1+k_2=0$ and $0=Q_1(0,\mathtt{t})Q_2(0,\mathtt{t})$ if $k_1+k_2\geqslant 1$. As $Q_i(0,\mathtt{t})\neq 0$ for $i=1,2$, we deduce $k_1+k_2=0$, so $k_1=0$ and $k_2=0$. Thus, $P_i=Q_i\in\kappa[[\mathtt{x}]][\mathtt{t}]$ for $i=1,2$ and $P=P_1(0,\mathtt{t})P_2(0,\mathtt{t})$. As P is irreducible in $\kappa[\mathtt{t}]$, we may assume $P_1(0,\mathtt{t})\in\kappa\backslash\{0\}$, so $\deg_{\mathtt{t}}(P_2(0,\mathtt{t}))=4$. Thus, $4\leqslant\deg_{\mathtt{t}}(P_2)\leqslant 4$, so $\deg_{\mathtt{t}}(P_1)=0$. As $P_1(0,\mathtt{t})\neq 0$ and has degree 0 with respect to $\mathtt{t}$, it is a unit in $\kappa[[\mathtt{x}]]$. Thus, P is irreducible in $\kappa((\mathtt{x}))[\mathtt{t}]$.

Observe that $\kappa((\mathtt{x}))[\mathtt{t}]/(P)$ is isomorphic to $(\kappa[\mathtt{t}]/(P))((\mathtt{x}))$. As the field $\kappa[\mathtt{t}]/(P)$ is (formally) real, also $(\kappa[\mathtt{t}]/(P))((\mathtt{x}))\cong\kappa((\mathtt{x}))[\mathtt{t}]/(P)$ is a (formally) real field. Let $Q\in\kappa((\mathtt{x}))[\mathtt{t}]$ a polynomial of degree $\leqslant 3$ such that $Q\in\mathcal{P}((\kappa[\mathtt{t}]/(P))((\mathtt{x})))$. Multiplying by an even power of

$\mathtt{x}$ we may assume that $Q \in \kappa[[\mathtt{x}]][\mathtt{t}]$ and either $Q(0,\mathtt{t}) \neq 0$ or $\frac{Q}{\mathtt{x}}(0,\mathtt{t}) \neq 0$. As P is chimeric over κ, there exists $\alpha \in \operatorname{Sper}(\kappa)$ such that P has four roots $\zeta_i \in \mathfrak{R}(\alpha)$. As $\deg_{\mathtt{t}}(Q) \leqslant 3$, we have $Q(\mathtt{x},\zeta_i) \neq 0$ and either $Q(0,\zeta_i) \neq 0$ or $\frac{Q}{\mathtt{x}}(0,\zeta_i) \neq 0$ for $i = 1,2,3,4$, so

$$0 < Q(\mathtt{x},\zeta_i) \begin{cases} Q(0,\zeta_i) + \cdots & \text{if } Q(0,\mathtt{t}) \neq 0 \\ \mathtt{x}(\frac{Q}{\mathtt{x}}(0,\zeta_i)) + \cdots & \text{if } \frac{Q}{\mathtt{x}}(0,\zeta_i) \neq 0 \end{cases}$$

for $i = 1,2,3,4$ and either $\mathtt{x} > 0$ or $\mathtt{x} < 0$, so $Q(0,\mathtt{t}) \neq 0$. Consequently, $Q(0,\mathtt{t}) \in \mathcal{P}(\kappa[\mathtt{t}]/(P))$. As P is a chimeric polynomial, there exists $\mu \in \kappa\backslash\{0\}$ such that $Q(0,\mathtt{t}) + \mu P \in \mathcal{P}(\kappa[\mathtt{t}]) = \Sigma\kappa[\mathtt{t}]^2$. By Lemmas D.1 and D.2 there exists $\mu' \in \kappa\backslash\{0\}$ such that $Q + \mu P \in \mathcal{P}(\kappa[[\mathtt{x}]][\mathtt{t}])$. Consequently, P is chimeric polynomial over $\kappa((\mathtt{x}))$, as required. □

We prove next Theorem 14.14(ii) by means of an induction procedure. The following result will help with the inductive process.

dser **Lemma D.6.** *Let $a \in \kappa((\mathtt{x}_1))\cdots((\mathtt{x}_n))\backslash\{0\}$ be a non-zero iterated Laurent power series. Then there exist units $u_k \in \kappa((\mathtt{x}_1))\cdots((\mathtt{x}_{k-1}))[[\mathtt{x}_k]]$ such that $u_k(\mathtt{x}_1,\ldots,\mathtt{x}_{k-1},0) = 1$, integers $m_k \in \mathbb{Z}$, $\varepsilon_k \in \{0,1\}$ and $a_0 \in \kappa\backslash\{0\}$ such that $a = a_0\prod_{k=1}^n \mathtt{x}_k^{2m_k+\varepsilon_k}\prod_{k=1}^n u_k^2$.*

Proof. Write $a = \mathtt{x}_n^{2m_n+\varepsilon_n}a_{n-1}u_n^2$ where $m_n \in \mathbb{Z}$, $\varepsilon_n \in \{0,1\}$, $u_n \in \kappa((\mathtt{x}_1))\cdots((\mathtt{x}_{n-1}))[[\mathtt{x}_n]]$ is a unit such that $u_n(\mathtt{x}_1,\ldots,\mathtt{x}_{n-1},0) = 1$ and $a_{n-1} \in \kappa((\mathtt{x}_1))\cdots((\mathtt{x}_{n-1}))\backslash\{0\}$. By induction hypothesis there exist units $u_k \in \kappa((\mathtt{x}_1))\cdots((\mathtt{x}_{k-1}))[[\mathtt{x}_k]]$ such that $u_k(\mathtt{x}_1,\ldots,\mathtt{x}_{k-1},0) = 1$, integers $m_k \in \mathbb{Z}$, $\varepsilon_k \in \{0,1\}$ and $a_0 \in \kappa\backslash\{0\}$ such that $a_{n-1} = a_0\prod_{k=1}^{n-1}\mathtt{x}_k^{2m_k+\varepsilon_k}\prod_{k=1}^{n-1}u_k^2$, so $a = a_0\prod_{k=1}^n \mathtt{x}_k^{2m_k+\varepsilon_k}\prod_{k=1}^n u_k^2$, as required. □

D.2.1. *Statement.* We have to prove: *Let κ be a (formally) real fields such that all its orderings are Archimedean. Then there exists no chimeric polynomial over $\kappa((\mathtt{x}_1))\cdots((\mathtt{x}_n))$ for each $n \geqslant 1$.*

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D.2.2. *Initial preparation.* As all the orderings of κ are Archimedean, there exists by Lemma 14.11 no chimeric polynomial over κ. We assume by induction hypothesis that there exists no chimeric polynomial over $\kappa((\mathtt{x}_1))\cdots((\mathtt{x}_{n-1}))$ and let us check: *There exists no chimeric polynomial over $\kappa((\mathtt{x}_1))\cdots((\mathtt{x}_n))$.*

Denote $\kappa_0 := \kappa((\mathtt{x}_1))\cdots((\mathtt{x}_{n-1}))$ and $\mathtt{x} := \mathtt{x}_n$ and suppose that there exists a chimeric polynomial $P \in \kappa_0((\mathtt{x}))[\mathtt{t}]$. Multiplying P by a power of $\mathtt{x}^4$ we may assume $P \in \kappa_0[[\mathtt{x}]][\mathtt{t}]$ and after a change of coordinates $\mathtt{t} \mapsto \frac{\mathtt{t}}{\mathtt{x}^\ell}$ for some $\ell \geqslant 0$, we may assume $P \in \kappa_0[[\mathtt{x}]][\mathtt{t}]$ and it is monic. Consequently, the four roots of P belong to the ring of Puiseux series $\overline{\kappa_0}[[\mathtt{x}^*]]$. This means that there exists (a smallest) $p \geqslant 1$ such that the roots of P belong to the ring $\overline{\kappa_0}[[\mathtt{x}^{1/p}]]$. As $P \in \kappa_0[[\mathtt{x}]][\mathtt{t}]$ and has degree 4, some immediate restrictions appear concerning coefficients and exponents. Let $\eta \in \overline{\kappa_0}$ be a pth root of unity. As the set of roots of P is invariant under the action of the $\overline{\kappa_0}$-automorphism $\psi_k : \overline{\kappa_0}[[\mathtt{x}^{1/p}]] \to \overline{\kappa_0}[[\mathtt{x}^{1/p}]],\ \mathtt{t} \mapsto \eta^k\mathtt{t}$ for $k = 0,\ldots,p-1$ and there exists $\alpha \in \operatorname{Sper}(\kappa_0((\mathtt{x})))$ such that the four roots of P in $\overline{\kappa_0}[[\mathtt{x}^{1/p}]]$ belong to $\mathfrak{R}(\alpha)$ we deduce $p \leqslant 2$. Thus, all the roots of P belong to $\overline{\kappa_0}[[\mathtt{x}^{1/2}]]$. Let $\zeta = \sum_{k\geqslant 0} a_k\mathtt{t}^{k/2}$ be a root of P and define $F := \kappa_0(a_k :\ k \geqslant 0)$. By Lemma A.1 we know that $F|\kappa_0$ is a finite extension of fields and we denote $E|\kappa_0$ the Galois closure of $F|\kappa_0$ in $\overline{\kappa_0}|\kappa_0$. The Galois group $G(E:\kappa_0) \times \mathbb{Z}_2$ acts on the set of roots of P as follows: *given the root ζ of P, we have*

$$\zeta = \sum_{k\geqslant 0} a_k\mathtt{t}^{k/2} \mapsto \sum_{k\geqslant 0}\sigma(a_k)(-1)^{k\ell}\mathtt{t}^{k/2},$$

where $\sigma \in G(E:\kappa_0)$ and $\ell = 0,1$. As P has exactly four roots in $\overline{\kappa_0}[[\mathtt{x}^*]]$, either $p = 2$ and $F|\kappa_0$ has degree 2 or $p = 1$ and $F|\kappa_0$ has degree 4. We distinguish both cases in Lemmas D.7 and D.8. A main tool will be the Sturm's sequence of a polynomial of the form $H_0 := Q + \mathtt{m}P$ where $Q \in \kappa_0((\mathtt{x}))[\mathtt{t}]$ is a polynomial of degree $\leqslant 3$ such that $Q \in \mathcal{P}(\kappa_0((\mathtt{x}))[\mathtt{t}]/(P))$.

As P is a chimeric polynomial, there exists $\mu_0 \in \kappa_0((\mathtt{x}))$ such that $H_0(\mu_0) = Q + \mu_0 P \in \mathcal{P}(\kappa_0((\mathtt{x}))[\mathtt{t}])$. By Lemma D.2 there exists infinitely many $\mu \in \kappa_0((\mathtt{x}))$ such that $H_0(\mu) = Q + \mu_0 P \in \mathcal{P}(\kappa_0((\mathtt{x}))[\mathtt{t}])$. A Sturm's sequence for H_0 with respect to $\mathtt{t}$ is: $H_0 := \mathtt{x}Q_0 + \mathtt{m}P$, $H_1 := \mathtt{x}Q_0' + \mathtt{m}P'$, $H_2 := -\text{remainder}(H_0, H_1, \mathtt{t})$, $H_3 := -\text{remainder}(H_1, H_2, \mathtt{t})$, $H_4 := \Delta(\mathtt{x}Q_0 + \mathtt{m}P, \mathtt{t})$. A Sturm's for $H_0(\mu)$ with respect to $\mathtt{t}$ is $[H_0(\mu), H_1(\mu), H_2(\mu), H_3(\mu), H_4(\mu)]$ if $\deg_t(H_2(\mu)) = 2$, $\deg_t(H_3(\mu)) = 1$ and $H_4(\mu) = \Delta(\mathtt{x}Q_0 + \mu P, \mathtt{t}) \neq 0$. As the leading coefficients of H_2, H_3, H_4 are polynomials with respecto to $\mathtt{m}$, we may assume that they remain non-zero when substituting $\mathtt{m}$ by some $\mu \in \kappa_0((\mathtt{x}))$ such that $H_0(\mu) \in \mathcal{P}(\kappa_0((\mathtt{x}))[\mathtt{t}])$, so its Sturm's sequence with respect to $\mathtt{t}$ is $[H_0(\mu), H_1(\mu), H_2(\mu), H_3(\mu), H_4(\mu)]$. We only have to care about the leading coefficients $[B_0(\mu), B_1(\mu), B_2(\mu), B_3(\mu), H_4(\mu)] = [\mu, 4\mu, B_2(\mu), B_3(\mu), H_4(\mu)]$ of the previous Sturm's sequence. As $H_4(\mu) \neq 0$ (so $H_0(\mu)$ has no multiple roots in $\overline{\kappa}$), to prove that $H_0(\mu)$ is not a sum of squares we have to guarantee that it has roots in $\mathfrak{R}(\alpha)$ for some $\alpha \in \operatorname{Sper}(\kappa)$. If $H_4(\mu) <_\alpha 0$ for some $\alpha \in \operatorname{Sper}(\kappa)$, then by Proposition 14.5 $H_0(\mu)$ has two roots in $\mathfrak{R}(\alpha)$, so $H_0(\mu) \notin \mathcal{P}(\kappa[\mathtt{t}])$. If $B_2(\mu) >_\alpha 0$ and $B_3(\mu) >_\alpha 0$ for some $\alpha \in \operatorname{Sper}(\kappa)$, then by Proposition 14.5 $H_0(\mu)$ has either two roots (if $H_4(\mu) <_\alpha 0$) or four roots (if $H_4(\mu) >_\alpha 0$) in $\mathfrak{R}(\alpha)$, so $H_0(\mu) \notin \mathcal{P}(\kappa[\mathtt{t}])$. Our purpose in the next two result is to prove that for each $\mu \in \Sigma\kappa^2$ either both $B_2(\mu) >_\alpha 0$ and $B_3(\mu) >_\alpha 0$ or $H_4(\mu) <_\alpha 0$ for some $\alpha \in \operatorname{Sper}(\kappa)$. Observe that $\mathfrak{R}(\beta) = \mathfrak{R}(\alpha)((\mathtt{x}^*))$ where $\alpha := \beta \cap \kappa_0 \in \operatorname{Sper}(\kappa_0)$ and the sign of $\mathtt{x}$ is determined by $\beta \in \operatorname{Sper}(\kappa_0((\mathtt{x})))$.

case2 **Lemma D.7.** *If $p = 2$ and $F|\kappa_0$ has degree 2, then P is not a chimeric polynomial over κ.*

Proof. After a translation of the variable we may assume $\zeta = c_1\sqrt{a} + c_2\mathtt{x}^{1/2} + c_3\sqrt{a}\mathtt{x}^{1/2}$ where $a \in \kappa\backslash\kappa^2$ and $c_i \in \kappa_0[[\mathtt{x}]]$. Substituting $\mathtt{t}$ by $c_i\mathtt{t}$ where c_i is one of the series of lowest order between c_1, c_2, c_3 and factoring out c_i^4 from $P(\mathtt{x}, c_i\mathtt{t})$ we may assume that one of the series c_i has order 0 and in fact that $c_i(0) = 1$ for some $i = 1, 2, 3$. Observe that $\kappa_0((\mathtt{x}))[\zeta] \subset \kappa_0((\mathtt{x}))[\sqrt{a}, \mathtt{x}^{1/2}]$ and as both fields have degree 4 over $\kappa_0((\mathtt{x}))$, we deduce $L = \kappa_0((\mathtt{x}))[\zeta] = \kappa_0((\mathtt{x}))[\sqrt{a}, \mathtt{x}^{1/2}]$, so it is a Galois extension and both $a = (\sqrt{a})^2$ and $\mathtt{x} = (\mathtt{x}^{1/2})^2$ are positive in all the orderings of L. Using the isomorphism $\kappa_0[[\mathtt{x}]][\mathtt{t}] \to \kappa_0[[\mathtt{x}]][\mathtt{t}]$, $f(\mathtt{x}, \mathtt{t}) \mapsto f(a\mathtt{x}, \mathtt{t})$ we may assume that $\omega(c_2) \leqslant \omega(c_3)$. As $[\kappa_0((\mathtt{x}))[\zeta] : \kappa_0((\mathtt{x}))] = 4$, we deduce that $c_2 \neq 0$. We will find a polynomial $Q_0 \in \kappa_0[[\mathtt{x}]][\mathtt{t}] \cap \mathcal{P}(L)$ of degree $\leqslant 3$ such that $H_0(\mu) := Q_0\mathtt{x}^p + \mu P \notin \mathcal{P}(\kappa[\mathtt{t}])$ for each $\mu \in \kappa_0((\mathtt{x}))$. If such $H_0 \in \mathcal{P}(\kappa_0((\mathtt{x}))[\mathtt{t}])$, we may assume by Lemma D.2 that a corresponding Sturm's sequence for H_0 is: $H_0 := \mathtt{x}Q_0 + \mu P$, $H_1 := \mathtt{x}Q_0' + \mu P'$, $H_2 := -\text{remainder}(H_0, H_1, \mathtt{t})$, $H_3 := -\text{remainder}(H_1, H_2, \mathtt{t})$, $H_4 := \Delta(\mathtt{x}Q_0 + \mu P, \mathtt{t})$ (and $\deg_t(H_2) = 2$, $\deg_t(H_3) = 1$ and $\deg_t(H_4) = 0$) (see §D.2.2). As $\Delta(\mathtt{x}Q_0 + \mu P, \mathtt{t}) \neq 0$ the four roots of $\mathtt{x}Q_0 + \mu P$ in the algebraically closed field $\overline{\kappa_0((\mathtt{x}))}$ are all simple, so $\mathtt{x}Q_0 + \mu P \in \mathcal{P}(\kappa_0((\mathtt{x}))[\mathtt{t}])$ implies that the four roots of $\mathtt{x}Q_0 + \mu P$ in $\overline{\kappa_0((\mathtt{x}))} = \overline{\kappa_0}((\mathtt{x}^*))$ does not belong to $\mathfrak{R}(\beta)$ for each $\beta \in \operatorname{Sper}(\kappa_0((\mathtt{x})))$. Observe that $\mathfrak{R}(\beta) = \mathfrak{R}(\alpha)((\mathtt{x}^*))$ where $\alpha := \beta \cap \kappa_0 \in \operatorname{Sper}(\kappa_0)$ and the sign of $\mathtt{x}$ is determined by β. By Proposition 14.5 we deduce $\Delta(\mathtt{x}Q_0 + \mu P, \mathtt{t}) >_\beta 0$ for each $\beta \in \operatorname{Sper}(\kappa_0((\mathtt{x})))$, or equivalently, $\Delta(\mathtt{x}Q_0 + \mu P, \mathtt{t}) \in \Sigma\kappa_0((\mathtt{x}))^2\backslash\{0\}$. We will consider the sequence $[B_0, B_1, B_2, B_3, B_4]$ of the leading coefficients of the Sturm's sequence $[H_0, H_1, H_2, H_3, H_4]$ in order to achieve a contradiction. As $B_0 = \mu$ and $B_1 = 4\mu$, we only have to care about the signs of $[B_2, B_3, B_4]$ We distinguish several cases:

Case 1. $c_1(0) = 1$. We write $\zeta = \sqrt{a} + c_{21}\mathtt{x}^p\mathtt{x}^{1/2} + c_{31}\mathtt{x}^p\mathtt{x}^m\sqrt{a}\mathtt{x}^{1/2}$ where $c_{i1} \in \kappa_0[[\mathtt{x}]]$, $c_{21}(0) \neq 0$, $p, m \geqslant 0$ and either $c_{31}(0) \neq 0$ or $c_{31} = 0$. We show: *There exists $Q_0 := (\mathtt{t}^2 - a)(b_1\mathtt{t} + b_0) + d_1\mathtt{x}^p\mathtt{t} + d_0\mathtt{x}^p \in \kappa_0[\mathtt{t}] \cap \mathcal{P}(L)$ such that $H_0(\mu) := Q_0\mathtt{x}^{p+1} + \mu P \notin \mathcal{P}(\kappa[\mathtt{t}])$ for each $\mu \in \kappa_0((\mathtt{x}))$.*

To assure that $Q_0 \in \mathcal{P}(L)$, it is enough that $P(\pm\sqrt{a}) = \pm d_1\sqrt{a} + d_0 >_\alpha 0$ for each $\alpha \in \operatorname{Sper}(\kappa_0)$ such that $a >_\alpha 0$. Suppose we have chosen $b_0, b_1, d_0, d_1 \in \kappa_0$ such that $Q_0 \in \mathcal{P}(L)$ and let us find additional restrictions to guarantee that $H_0(\mu) := Q_0\mathtt{x}^{p+1} + \mu P \notin \mathcal{P}(\kappa[\mathtt{t}])$ for each $\mu \in \kappa_0((\mathtt{x}))$.

we suppose that $\mu \in \kappa_0[[\mathtt{x}]]$ satisfies $\mu(0) \neq 0$. In this case

$$B_2 = 16a^3\mu^2 + \mathtt{x}B_{21}, \quad B_3 = 1024\mu^4 a^5 \mathtt{x}(4a^2c_3^2\mu + 4ac_2^2\mu - aa_2 - a_0 + \mathtt{x}B_{31})$$

where $B_{21}, B_{31} \in \kappa[\mathtt{t}]$. In order to have that $H_0 \in \mathcal{P}(\kappa[t])$ one needs by Proposition 14.5 and Sturm's sequence that $H_4 \in \Sigma({}^2\kappa_0((\mathtt{x})))\setminus\{0\}$ and $B_2B_3 <_\alpha 0$ for each $\alpha \in \mathrm{Sper}(\kappa((\mathtt{x})))$. As $B_2 \in \Sigma({}^2\kappa_0((\mathtt{x})))$, we deduce that $-B_3 \in \Sigma({}^2\kappa_0((\mathtt{x})))$, so in particular it has even order. Thus,

$$\mu =$$

Case 2. $c_1(0) = 0$, $c_2(0) = 1$ and $\omega(c_3) \leqslant \omega(c_1)$. As $[\kappa_0((\mathtt{x}))[\zeta] : \kappa_0((\mathtt{x}))] = 4$, we deduce that $c_3 \neq 0$. We write $\zeta = \sqrt{a}c_{11}\mathtt{x}^p\mathtt{x}^m + \mathtt{x}^{1/2} + \sqrt{a}\mathtt{x}^{1/2}c_{31}\mathtt{x}^p$ where $c_{i1} \in \kappa_0[[\mathtt{x}]]$, $c_{31}(0) \neq 0$, $p, m \geqslant 0$ and either $c_{11}(0) \neq 0$ or $c_{11} = 0$ and $m >> 0$. We show: *There exists*

$$Q_0 := \begin{cases} a_0 + a_1\mathtt{t} - a_2\mathtt{t}^2 + a_3\mathtt{t}^3 \in & \textit{if } p = 0, \\ a_0\mathtt{x} + a_1\mathtt{x}\mathtt{t} + (-a_0 + \mathtt{x}^p a_{21})\mathtt{t}^2 + (-a_1 + \mathtt{x}^p a_{31})\mathtt{t}^3 & \textit{if } p \geqslant 1 \end{cases}$$

such that $a_j \in \kappa$, $Q_0 \in \mathcal{P}(L)$ *and* $H_0(\mu) := Q_0\mathtt{x}^p + \mu P \notin \mathcal{P}(\kappa[\mathtt{t}])$ *for each* $\mu \in \kappa_0((\mathtt{x}))$.

Case 3. $c_1(0) = 0$, $c_2(0) = 1$ and $\omega(c_1) < \omega(c_3)$. As $[\kappa_0((\mathtt{x}))[\zeta] : \kappa_0((\mathtt{x}))] = 4$, we deduce that $c_1 \neq 0$. We write $\zeta = \sqrt{a}c_{11}\mathtt{x}^p + \mathtt{x}^{1/2} + \sqrt{a}\mathtt{x}^{1/2}c_{31}\mathtt{x}^p\mathtt{x}^m$ where $c_{i1} \in \kappa_0[[\mathtt{x}]]$, $c_{31}(0) \neq 0$, $p, m \geqslant 0$ and either $c_{31}(0) \neq 0$ or $c_{31} = 0$ and $m >> 0$. We show: *There exists* $Q_0 := a_0\mathtt{x}^{p+2} + a_1\mathtt{x}\mathtt{t} - a1\mathtt{t}^3$ *such that* $a_j \in \kappa$, $Q_0 \in \mathcal{P}(L)$ *and* $H_0(\mu) := Q_0\mathtt{x}^p + \mu P \notin \mathcal{P}(\kappa[\mathtt{t}])$ *for each* $\mu \in \kappa_0((\mathtt{x}))$.

□

To deal with the proof of the following lemma we suggest the reader to have in mind the polynomial

$$\begin{aligned} P &:= \mathtt{t}^4 - 6\mathtt{x}^2\mathtt{t}^2 - 4\mathtt{t}^2 + 9\mathtt{x}^4 - 12\mathtt{x}^2 + 4 \\ &= (\mathtt{t} - \sqrt{2} - \sqrt{3}\mathtt{t})(\mathtt{t} - \sqrt{2} + \sqrt{3}\mathtt{t})(\mathtt{t} + \sqrt{2} - \sqrt{3}\mathtt{t})(\mathtt{t} + \sqrt{2} + \sqrt{3}\mathtt{t}), \end{aligned}$$

which was our inspiration to prove this result.

case1 **Lemma D.8.** *If* $p = 1$ *and* $F|\kappa_0$ *has degree* 4, *then* P *is not a chimeric polynomial over* κ.

Proof. Let θ be a primitive element of F over κ_0. Write $\zeta = \sum_{j=0}^3 \theta^j b_j$ where $b_j \in \kappa_0[[\mathtt{x}]]$. After the translation of the variable $\mathtt{t} \mapsto \mathtt{t} - b_0$ in $\kappa_0[[\mathtt{x}]][\mathtt{t}]$, we may assume $b_0 = 0$. This means that the leading coefficient τ_1 of the series $\zeta = \sum_{j=1}^3 \theta^j b_j$ has degree either 2 or 4. After a change of variable in $\kappa_0[[\mathtt{x}]][\mathtt{t}]$ that maps $\mathtt{x}$ to $\frac{\mathtt{x}}{\mathtt{t}^m}$ for some $m \geqslant 0$, we may assume $\zeta(0) = \tau_1$. We study both cases concerning τ_1:

Case 1. τ_1 has degree 4 over κ_0. Then, $P(0, \mathtt{t})$ is an irreducible polynomial of degree 4 and one of its roots is τ_1. Let $\tau_2, \tau_3, \tau_4 \in \overline{\kappa_0}$ be the remaining roots of $P(0, \mathtt{t})$. Observe that the roots $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ of P have initials forms $\tau_1, \tau_2, \tau_3, \tau_4$. Let $Q \in \kappa_0[\mathtt{t}]$ be a polynomial of degree $\leqslant 3$ such that $Q(\tau_i) >_\alpha 0$ for each $i = 1, 2, 3, 4$ and each $\alpha \in \mathrm{Sper}(\kappa_0)$. As the orderings of $\kappa_0((\mathtt{x}))$ are determined by and ordering of κ_0 and the sign of the variable, we deduce $Q(\zeta_i) = Q(\tau_i) + \cdots >_\beta 0$ for each $\beta \in \mathrm{Sper}(\kappa_0((\mathtt{x})))$. If $Q \in \mathcal{P}(\kappa_0[\mathtt{t}])$, we take $\mu = 0$. Otherwise, as P is a chimeric polynomial over $\kappa_0((\mathtt{x}))$, there exists $\mu \in \Sigma\kappa_0((\mathtt{x}))^2$ such that $Q + \mu P \in \mathcal{P}(\kappa_0((\mathtt{x}))[\mathtt{t}])$. As $Q \in \kappa_0[\mathtt{t}]$ does not belong to $\mathcal{P}(\kappa_0[\mathtt{t}])$, the order of μ with respect to $\mathtt{x}$ is $\leqslant 0$ (because otherwise making $\mathtt{x} = 0$, we have $Q + \mu(0)P(0, \mathtt{t}) = Q \notin \mathcal{P}(\kappa_0[\mathtt{t}])$, which is a contradicition). If the order of μ with respect to $\mathtt{x}$ is < 0, we multiply $Q + \mu P$ by the smallest power $\mathtt{x}^{2s}$ such that $\lambda := \mathtt{x}^{2s}\mu \in \kappa_0[[\mathtt{x}]]$, where $s \geqslant 0$ is an integer. If we make $\mathtt{x} = 0$ in $\mathtt{x}^{2s}Q + \mathtt{x}^{2s}\mu P$, we obtain $\lambda(0)P(0, \mathtt{t}) \in \mathcal{P}(\kappa_0[\mathtt{t}])$, which is a contradiction, because $P(0, \mathtt{t})$ has four different real roots in $\mathfrak{R}(\alpha)$ for some $\alpha \in \mathrm{Sper}(\kappa_0)$. Thus, $\omega(\mu) = 0$ and after making $\mathtt{x} = 0$, we deduce $Q + \mu P(0, \mathtt{t}) \in \mathcal{P}(\kappa_0[\mathtt{t}])$. As this happens for each $Q \in \mathcal{P}(\kappa_0[\mathtt{t}]/(P(0, \mathtt{t})))$, we deduce that $P(0, \mathtt{t}) \in \kappa_0[\mathtt{t}]$ is a chimeric polynomial, which is a contradiction. ▪

Case 2. τ_1 has degree 2 over κ_0. Write $\tau_1 := c_1 + c_2\sqrt{a}$ where $a, c_i \in \kappa_0$ and $a \in \kappa_0\setminus\kappa_0^2$. By Lemma D.6 we may assume, after a homothety of the variable $\mathtt{t}$, that $a = a_0\prod_{k=1}^n \mathtt{x}_k^{\varepsilon_k}$ where $\epsilon_k \in \{0, 1\}$ and $a_0 \in \kappa\setminus\{0\}$. Define $\varepsilon_0 := 1$ and let $m := \max\{k = 0, \ldots, n : \ \varepsilon_k = 1\}$. If $m = 0$, then $a = a_0 \in \kappa\backslash\kappa^2$. If $1 \leqslant m \leqslant n$, then $a = a_0(\prod_{k=1}^{m-1} \mathtt{x}_k^{\varepsilon_k})\mathtt{x}_m$ and after the change of coordinates $a_0(\prod_{k=1}^{m-1} \mathtt{x}_k^{\varepsilon_k})\mathtt{x}_m$ to $\mathtt{x}_m$ we may assume $a = \mathtt{x}_m$.

After a translation of the variable and a homothety we may assume $\zeta = \sqrt{a} + \mathtt{x}\zeta^*$ where $\zeta^* \in F[[\mathtt{x}]]$. As $F = \kappa_0[\theta]|\kappa_0$ has degree 4, we have $\kappa_0[\theta]|\kappa_0[\sqrt{a}]$ has degree 2, so we may assume $\theta = \sqrt{d_0 + \sqrt{a}d_1}$ where $d_0, d_1 \in \kappa_0$ and $d_0 \neq 0$. Thus, we may assume

$$\zeta_1 := \zeta = \sqrt{a} + \sqrt{d_0 + \sqrt{a}d_1}\mathtt{x}^{\ell+1}\lambda_1 + \sqrt{a}\sqrt{d_0 + \sqrt{a}d_1}\mathtt{x}^{\ell+1}\lambda_2$$

where $\lambda_1, \lambda_2 \in \kappa_0[[\mathtt{t}]]$, the values $\lambda_1(0), \lambda_2(0) \in \kappa_0$ are not simultaneously zero and $\ell \geqslant 0$ is an integer. Consequently, the remaining roots of P are of the form:

$$\begin{aligned}
\zeta_2 &:= \sqrt{a} - \sqrt{d_0 + \sqrt{a}d_1}\mathtt{x}^{\ell+1}\lambda_1 - \sqrt{a}\sqrt{d_0 + \sqrt{a}d_1}\mathtt{x}^{\ell+1}\lambda_2,\\
\zeta_3 &:= -\sqrt{a} + \sqrt{d_0 - \sqrt{a}d_1}\mathtt{x}^{\ell+1}\lambda_1 - \sqrt{a}\sqrt{d_0 - \sqrt{a}d_1}\mathtt{x}^{\ell+1}\lambda_2,\\
\zeta_4 &:= -\sqrt{a} - \sqrt{d_0 - \sqrt{a}d_1}\mathtt{x}^{\ell+1}\lambda_1 + \sqrt{a}\sqrt{d_0 - \sqrt{a}d_1}\mathtt{x}^{\ell+1}\lambda_2.
\end{aligned}$$

Consider the polynomial

$$Q := (\mathtt{t} - \sqrt{a}(1 + \mathtt{x}^{\ell+1}b_1) + \mathtt{x}^{\ell+1}b_0)(\mathtt{t} + \sqrt{a}(1 + \mathtt{x}^{\ell+1}b_1) + \mathtt{x}^{\ell+1}b_0)(\mathtt{t} + b_2)\mathtt{x}^{\ell+1} \in \kappa_0((\mathtt{x}))[t]. \quad \text{(D.1)}$$ defq

We distinguish two situations:

SUBCASE 2.1 $a = a_0 \in \kappa\backslash\kappa^2$. Under this hypothesis, we choose $b_0, b_1 \in \kappa_0$ such that:

- $q_1 := \sqrt{a}b_1 + \sqrt{d_0 + \sqrt{a}d_1}\lambda_1(0) + \sqrt{a}\sqrt{d_0 + \sqrt{a}d_1}\lambda_2(0) + b_0 >_\alpha 0$,
- $q_2 := \sqrt{a}b_1 - \sqrt{d_0 + \sqrt{a}d_1}\lambda_1(0) - \sqrt{a}\sqrt{d_0 + \sqrt{a}d_1}\lambda_2(0) + b_0 >_\alpha 0$,
- $q_3 := -\sqrt{a}b_1 + \sqrt{d_0 - \sqrt{a}d_1}\lambda_1(0) - \sqrt{a}\sqrt{d_0 - \sqrt{a}d_1}\lambda_2(0) + b_0 >_\alpha 0$,
- $q_4 := -\sqrt{a}b_1 - \sqrt{d_0 - \sqrt{a}d_1}\lambda_1(0) + \sqrt{a}\sqrt{d_0 - \sqrt{a}d_1}\lambda_2(0) + b_0 >_\alpha 0$

in $\mathfrak{R}(\alpha)$ for each $\alpha \in \operatorname{Sper}(\kappa_0)$. In addition, we will choose $b_2 \in \kappa_0 \cap (0, \sqrt{a})_{\kappa_0(\sqrt{a})}$ for each $\alpha \in \operatorname{Sper}(F)$ and close enough to $\sqrt{a}$ in $\mathfrak{R}(\beta)$ for some $\beta \in \operatorname{Sper}(F)$ such that $a >_\beta 0$, $d_0 \pm \sqrt{a}d_1 >_\beta 0$. Under these assumptions $Q(\xi_i) >_\gamma 0$ for each $\gamma \in \operatorname{Sper}(F((\mathtt{x})))$ and each $i = 1, 2, 3, 4$. Consequently, $Q \in \mathcal{P}(L)$, where $L := \kappa_0((\mathtt{x}))[t]/(P)$. We claim: *$P + \mu Q \notin \mathcal{P}(\kappa_0((\mathtt{x}))[\mathtt{t}])$ for each $\mu \in \Sigma\kappa_0((\mathtt{x}))^2$ if $a - b_2^2 >_\beta 0$ for each $\beta \in \operatorname{Sper}(\kappa_0)$ such that $\sqrt{a}, \sqrt{d_0 + \sqrt{a}d_1}, \sqrt{d_0 - \sqrt{a}d_1} \in \mathfrak{R}(\beta)$ and b_2^2 is close enough to a with respect to one of such orderings.*

We take advantage of Sturm's sequence to prove our claim. We compute a Sturm's sequence for $P + \mu Q$ where $\mu \in \kappa_0((\mathtt{x}))$. Namely, $H_0 := P + \mu Q$, $H_1 := P' + \mu Q'$, $H_2 := -\text{remainder}(H_0, H_1, \mathtt{t})$, $H_3 := -\text{remainder}(H_1, H_2, \mathtt{t})$, $H_4 := \Delta(P + \mu Q, \mathtt{t})$. Assume first $\mu \in \Sigma\kappa_0[[\mathtt{x}]]^2$. The leading coefficients of $[H_0, H_1, H_2, H_3, H_4]$ are, after clearing squares from the denominators and making $\mathtt{x} = 0$, of the form

$$\begin{aligned}
&[1, 4, a, a(16a^2d_0\lambda_2^2(0) + 32a^2d_1\lambda_1(0)\lambda_2(0) + 8ab_1b_2\mu(0) + 16ad_0\lambda_1^2(0) + b_2^2\mu^2(0) - 8ab_0\mu(0)\\
&+ a\mu^2(0)), H_{40} + H_{41}\mu(0) + H_{42}\mu^2(0) - 16^2a^3(b_1b_2 + b_0)(a - b_2^2)\mu^3(0) + 16a^2(a - b_2^2)^2\mu^4(0)]
\end{aligned}$$

where $H_{4j} \in \mathbb{Q}[a, b_0, b_1, b_2, d_0, d_1, \lambda_1(0), \lambda_2(0)]$. To prove our claim it is enough to show that for each $\mu(0) \in \Sigma\kappa_0^2$ there exists $\beta \in \operatorname{Sper}(\kappa_0)$ such that $\sqrt{a}, \sqrt{d_0 + \sqrt{a}d_1}, \sqrt{d_0 - \sqrt{a}d_1} \in \mathfrak{R}(\beta)$, $a - b_2^2$ is close to 0 in $\mathfrak{R}(\beta)$ and either

$$\begin{aligned}
E_1 :=& a(16a^2d_0\lambda_2^2(0) + 32a^2d_1\lambda_1(0)\lambda_2(0) + 8ab_1b_2\mu(0) + 16ad_0\lambda_1^2(0)\\
&+ b_2^2\mu^2(0) - 8ab_0\mu(0) + a\mu^2(0)) >_\alpha 0, \text{or}\\
E_2 :=& H_{40} + H_{41}\mu(0) + H_{42}\mu^2(0) - 16^2a^3(b_1b_2 + b_0)(a - b_2^2)\mu^3(0) + 16a^2(a - b_2^2)^2\mu^4(0) <_\alpha 0.
\end{aligned}$$

If such is the case, we have that $P + \mu Q$ has either four or two different roots in $\mathfrak{R}(\beta)((\mathtt{t}^*))$ if a and b_2^2 are close enough with respect to β.

We divide E_2 by E_1 with respect to $\mu(0)$ and write $E_2 = E_1T_1 + T_2$ where
$$T_1, T_2 \in \mathbb{Q}[a, b_0, b_1, b_2, d_0, d_1, \lambda_1(0), \lambda_2(0)][\mu(0)],$$
$\deg_\mu(T_1) = 2$ and $\deg_\mu(T_2) \leqslant 1$. Write
$$T_1 = \frac{1}{(b_2^2+a)^3}(4096b_2^{14}(b_2\lambda_2(0) - \lambda_1(0))^2(d_0 - b_2d_1) + (a - b_2^2)S_1),$$
$$T_2 = \frac{1}{(b_2^2+a)^3}(-32768b_2^{14}(b_2\lambda_2(0) - \lambda_1(0))^4(d_0 - b_2d_1)^2 + (a - b_2^2)S_2)$$
where $S_1, S_2 \in \mathbb{Q}[a, b_0, b_1, b_2, d_0, d_1, \lambda_1(0), \lambda_2(0)][\mu(0)]$. The roots of the polynomial
$$a(16a^2d_0\lambda_2^2(0) + 32a^2d_1\lambda_1(0)\lambda_2(0) + 8ab_1b_2\mathtt{t} + 16ad_0\lambda_1^2(0) + b_2^2\mathtt{t}^2 - 8ab_0\mathtt{t} + a\mathtt{t}^2)$$
are bounded by
$$3 + (16a^2d_0\lambda_2^2(0) + 32a^2d_1\lambda_1(0)\lambda_2(0))^2 + (8ab_1b_2 - 8ab_0)^2 + (a + b_2^2)^2 \leqslant_\alpha N := 4$$
$$+ (16a^2d_0\lambda_2^2(0) + 32a^2d_1\lambda_1(0)\lambda_2(0))^2 + (8ab_1)^2(a^2+1) + (8ab_0)^2 + (8ab_18ab_0)^2(a^2+1) + (2a)^2$$
if $b_2^2 <_\alpha a <_\alpha a^2 + 1$. Let $M \in \Sigma\kappa_0^2$ be such that $|S_i|_\alpha < M$ for each $\alpha \in \operatorname{Sper}(\kappa_0)$ whenever $\mu(0) \in \kappa_0$ is smaller with respect to α than $N \in \Sigma\kappa_0^2$. Recall that $a \notin \kappa_0^2$, so $\sqrt{a}\lambda_2(0) - \lambda_1(0) \neq 0$ and $d_0 - \sqrt{a}d_1 \neq 0$. Let $n \geqslant 1$ be such that
$$4096b_2^{14}(b_2\lambda_2(0) - \lambda_1(0))^2(d_0 - b_2d_1) >_\alpha \frac{1}{n},$$
$$32768b_2^{14}(b_2\lambda_2(0) - \lambda_1(0))^4(d_0 - b_2d_1)^2 >_\alpha \frac{1}{n}$$
whenever $b_2 \in \kappa_0$ and $\alpha \in \operatorname{Sper}(\kappa_0)$ satisfy $a >_\alpha 0$, $d_0 - \sqrt{a}d_1 >_\alpha 0$, $b_2 <_\alpha \sqrt{a}$, $d_0 - b_2d_1 > \frac{1}{2}(d_0 - \sqrt{a}d_1)$ and
$$|\sqrt{a}\lambda_2(0) - \lambda_1(0)|_\alpha >_\alpha |b_2\lambda_2(0) - \lambda_1(0)|_\alpha >_\alpha \tfrac{1}{2}|\sqrt{a}\lambda_2(0) - \lambda_1(0)|_\alpha.$$

Let $\mathscr{P}$ be the intersection of the cones $\beta \in \operatorname{Sper}(\kappa_0)$ such that $\sqrt{a}, \sqrt{d_0 + \sqrt{a}d_1}, \sqrt{d_0 - \sqrt{a}d_1} \in \mathfrak{R}(\beta)$. Recall that all the orderings of κ_0 are Archimedean, so there exists $m \geqslant 1$ such that $nM < 2^m$. Consider the field $\kappa_0[\sqrt{a}] = \{k_0 + \sqrt{a}k_1 : k_0, k_1 \in \kappa\}$ and observe that all the orderings of $\kappa_0[\sqrt{a}]$ that extend an ordering of κ_0 is also Archimedean.

By [Schl, Thm.1] and Lemma D.3 there exist $b_2 \in \mathbb{Q}$ and α compatible with $\mathscr{P}$ such that $\sqrt{a} - \frac{1}{2^m} <_\alpha b_2 <_{\mathscr{P}} \sqrt{a}$. For this choice of $b_2 \in \mathbb{Q}$, one has $T_1 >_\alpha 0$ and $T_2 <_\alpha 0$. Thus, if $E_1 <_\alpha 0$, then $E_2 = E_1T_1 + T_2 <_\alpha 0$, so $P + \mu Q$ has exactly two different roots in $\mathfrak{R}(\alpha)((\mathtt{x}^*))$ for each $\mu \in \Sigma\kappa_0[[\mathtt{t}]]^2$. If $E_1 >_\alpha 0$ and $E_2 >_\alpha 0$, then $P + \mu Q$ has exactly four different roots in $\mathfrak{R}(\alpha)((\mathtt{x}^*))$ for each $\mu \in \Sigma\kappa_0[[\mathtt{t}]]^2$, whereas if $E_1 >_\alpha 0$ and $E_2 <_\alpha 0$, the polynomial $P + \mu Q$ has again exactly two different roots in $\mathfrak{R}(\alpha)((\mathtt{x}^*))$ for each $\mu \in \Sigma\kappa_0[[\mathtt{t}]]^2$.

Suppose next that $\mu \in \Sigma\kappa_0((\mathtt{x}))^2 \setminus \Sigma\kappa_0[[\mathtt{x}]]^2$. This is equivalent to show that there exists no $\mu \in \Sigma\kappa_0[[\mathtt{x}]]^2$ such that $Q + \mu P \in \mathcal{P}(\kappa_0[[\mathtt{x}]][\mathtt{t}])$. Write $\mu = \mathtt{x}^{2k}\lambda$ where $k \geqslant 1$, $\lambda \in \Sigma\kappa_0[[\mathtt{x}]]^2$ and $\lambda(0) \neq 0$. We compute the corresponding Sturm's sequence: $G_0 := Q + \mu P$, $G_1 := Q' + \mu P'$, $G_2 := -\text{remainder}(G_0, G_1, \mathtt{t})$, $G_3 := -\text{remainder}(G_1, G_2, \mathtt{t})$, $G_4 := \Delta(Q + \mu P, \mathtt{t})$. The leading coefficients $[A_0, A_1, A_2, A_3, A_4]$ of $[G_0, G_1, G_2, G_3, G_4]$ are, after clearing squares from the denominators,

$$A_0 = \mu,$$
$$A_1 = 4\mu,$$
$$A_2 = \begin{cases} \mathtt{x}^{2k}(a\lambda(0) + \mathtt{x}H_{21}) & \text{if } 2k < \ell+1, \\ \frac{16(a-b_2^2)+(4b_2^2\lambda(0)-1)^2+2}{16\lambda(0)}\mathtt{x}^{2k}(1+\mathtt{x}H_{22}) & \text{if } 2k = \ell+1, \\ \frac{3\mathtt{x}^{2\ell+2}}{16\lambda(0)\mathtt{x}^{2k}}(1+\mathtt{x}H_{23}) & \text{if } 2k > \ell+1, \end{cases}$$
$$A_3 = \begin{cases} \frac{(b_2^2+a)\mathtt{x}^{2k}}{a}(1+\mathtt{x}H_{31}) & \text{if } 2k < \ell+1, \\ \frac{32\lambda(0)((8a\lambda(0)^2+(4b_2\lambda(0)-\frac{3}{2})^2+\frac{3}{4})(a-b_2^2)+4b_2^2(2b_2\lambda(0)-1)^2)}{(16a\lambda(0)^2-8b_2\lambda(0)+3)^2}(1+\mathtt{x}H_{32}) & \text{if } 2k = \ell+1, \\ \frac{32\lambda(0)(b_2^2+3a)\mathtt{x}^{2k}}{9}(1+\mathtt{x}H_{33}) & \text{if } 2k > \ell+1, \end{cases}$$
$$A_4 = \begin{cases} 16a^2\lambda(0)^2(a-b_2^2)^2\mathtt{x}^{4k+4\ell+4}(1+\mathtt{x}H_{41}) & \text{if } 2k < \ell+1, \\ 4a(4(a-b_2^2)\lambda(0)^2+(2b_2\lambda(0)-1)^2)(a-b_2^2)^2\mathtt{x}^{6\ell+6}(1+\mathtt{x}H_{42}) & \text{if } 2k = \ell+1, \\ 4a(a-b_2^2)^2\mathtt{x}^{6\ell+6}(1+\mathtt{x}H_{43}) & \text{if } 2k > \ell+1, \end{cases}$$

where $H_{ij} \in \kappa_0[[\mathtt{x}]]$. Consequently, the signs (with respect to α) of $[A_0, A_1, A_2, A_3, A_4]$ are $[+,+,+,+,+]$ for each $\alpha \in \operatorname{Sper}(\kappa_0)$ such that $a >_\alpha 0$ and each $b_2 \in \kappa_0$ satisfying $a - b_2^2 >_\alpha 0$. This means that $Q + \mu P$ has exactly four different roots in $\mathfrak{R}(\alpha)$ for each $\mu \in \Sigma\kappa_0[[\mathtt{x}]]^2$. We deduce $Q + \mu P \notin \mathcal{P}(\kappa_0((\mathtt{x}))[\mathtt{t}])$ for each $\mu \in \Sigma\kappa_0[[\mathtt{x}]]^2$.

We conclude $Q + \mu P \notin \mathcal{P}(\kappa_0((\mathtt{x}))[\mathtt{t}])$ for each $\mu \in \Sigma\kappa_0((\mathtt{x}))^2$.

Subcase 2.2 $a = \mathtt{x}_m$ with $m \leqslant n-1$. Under this hypothesis we choose $b_2 >_\alpha \sqrt{a}$ in $\mathfrak{R}(\alpha)$ for each $\alpha \in \operatorname{Sper}(\kappa_0)$. In addition, we choose $b_1 <_\alpha \sqrt{a}$ in $\mathfrak{R}(\alpha)$ for each $\alpha \in \operatorname{Sper}(\kappa_0)$ such that:

- $q_1 := -\sqrt{a}b_1 + \sqrt{d_0 + \sqrt{a}d_1}\lambda_1(0) + \sqrt{a}\sqrt{d_0+\sqrt{a}d_1}\lambda_2(0) + b_0 >_\alpha 0$,
- $q_2 := -\sqrt{a}b_1 - \sqrt{d_0 + \sqrt{a}d_1}\lambda_1(0) - \sqrt{a}\sqrt{d_0+\sqrt{a}d_1}\lambda_2(0) + b_0 >_\alpha 0$,
- $q_3 := -\sqrt{a}b_1 - \sqrt{d_0 - \sqrt{a}d_1}\lambda_1(0) + \sqrt{a}\sqrt{d_0-\sqrt{a}d_1}\lambda_2(0) - b_0 >_\alpha 0$,
- $q_4 := -\sqrt{a}b_1 + \sqrt{d_0 - \sqrt{a}d_1}\lambda_1(0) - \sqrt{a}\sqrt{d_0-\sqrt{a}d_1}\lambda_2(0) - b_0 >_\alpha 0$

in $\mathfrak{R}(\alpha)$ for each $\alpha \in \operatorname{Sper}(\kappa_0)$. We choose $b_1 \in \kappa_0$ such that $\omega_{\mathtt{x}_k}(b_1(\mathtt{x}_1,\dots,\mathtt{x}_k,0,\dots,0)) \geqslant 0$ for $k = 1,\dots,n-1$. We assume $\omega_{\mathtt{x}_k}(b_0(\mathtt{x}_1,\dots,\mathtt{x}_k,0,\dots,0)) \geqslant \omega_{\mathtt{x}_k}(b_1(\mathtt{x}_1,\dots,\mathtt{x}_k,0,\dots,0))$ for $k = 1,\dots,n-1$ and $\omega_{\mathtt{x}_m}(b_0(\mathtt{x}_1,\dots,\mathtt{x}_m,0,\dots,0)) \geqslant 1$.

As $d_0 - \sqrt{a}d_1 >_\alpha 0$, $d_0 + \sqrt{a}d_1 >_\alpha 0$ and $a >_\alpha 0$ for each $\alpha \in \operatorname{Sper}(\kappa_0)$ such that $\xi_1,\xi_2,\xi_3,\xi_4 \in \mathfrak{R}(\alpha)$, we deduce that $d_0^2 - \mathtt{x}_m d_1^2 >_\alpha 0$ for such orderings. We may assume from the beginning (after a change of coordinates in the variable $\mathtt{x}_n$) that $\omega_{\mathtt{x}_k}(d_i(\mathtt{x}_1,\dots,\mathtt{x}_k,0,\dots,0)) \geqslant 0$ and

$$\omega_{\mathtt{x}_k}(\lambda_j(\mathtt{x}_1,\dots,\mathtt{x}_k,0,\dots,0)(0)) \geqslant 0$$

for $k = 1,\dots,n-1$, $i = 1,2$, $j = 1,2$ and in fact that $0 \leqslant \omega_{\mathtt{x}_k}(d_0(\mathtt{x}_1,\dots,\mathtt{x}_k,0,\dots,0)) \leqslant 1$ for $k = 1,\dots,n-1$. Consequently, there exists $0 \leqslant k_0 \leqslant n-1$ such that $\omega_{\mathtt{x}_k}(d_0(\mathtt{x}_1,\dots,\mathtt{x}_k,0,\dots,0)) \leqslant \omega_{\mathtt{x}_k}(d_1(\mathtt{x}_1,\dots,\mathtt{x}_k,0,\dots,0))$ for $k = k_0+1,\dots,n-1$ and

$$\omega_{\mathtt{x}_{k_0}}(d_0(\mathtt{x}_1,\dots,\mathtt{x}_{k_0},0,\dots,0)) < \omega_{\mathtt{x}_{k_0}}(d_1(\mathtt{x}_1,\dots,\mathtt{x}_{k_0},0,\dots,0)).$$

If $\omega_{\mathtt{x}_m}(d_0(\mathtt{x}_1,\dots,\mathtt{x}_k,0,\dots,0)) = 1$, we write $d_0 = \mathtt{x}_m d_0' = a d_0'$ and $d_1 = \mathtt{x}_m d_1'$ where $d_0', d_1' \in \kappa((\mathtt{x}_1))\cdots((\mathtt{x}_{n-1}))$ satisfies $0 \leqslant \omega_{\mathtt{x}_k}(d_0'(\mathtt{x}_1,\dots,\mathtt{x}_k,0,\dots,0)) \leqslant 1$, $\omega_{\mathtt{x}_m}(d_0'(\mathtt{x}_1,\dots,\mathtt{x}_m,0,\dots,0)) = 0$ and $\omega_{\mathtt{x}_k}(d_0'(\mathtt{x}_1,\dots,\mathtt{x}_k,0,\dots,0)) \leqslant \omega_{\mathtt{x}_k}(d_1'(\mathtt{x}_1,\dots,\mathtt{x}_k,0,\dots,0))$ for $k = k_0+1,\dots,n-1$ and

$$\omega_{\mathtt{x}_{k_0}}(d_0(\mathtt{x}_1,\dots,\mathtt{x}_{k_0},0,\dots,0)) < \omega_{\mathtt{x}_{k_0}}(d_1(\mathtt{x}_1,\dots,\mathtt{x}_{k_0},0,\dots,0)).$$

Interchanging the roles of λ_1 and λ_2 and substituting d_j by d_j' we may assume that

$$\omega_{\mathtt{x}_m}(d_0(\mathtt{x}_1,\dots,\mathtt{x}_m,0,\dots,0)) = 0.$$

As each value $q_p >_\alpha 0$ for each $p = 1,2,3,4$, there exists $0 \leqslant k_1 \leqslant n-1$ such that

$$\omega_{\mathtt{x}_k}(\lambda_1(\mathtt{x}_1,\dots,\mathtt{x}_k,0,\dots,0)(0)) \geqslant \omega_{\mathtt{x}_k}(\lambda_1(\mathtt{x}_1,\dots,\mathtt{x}_k,0,\dots,0)(0))$$

for $k = k_0+1,\dots,n-1$ and

$$\omega_{\mathtt{x}_{k_1}}(\lambda_1(\mathtt{x}_1,\dots,\mathtt{x}_{k_1},0,\dots,0)(0)) \geqslant \omega_{\mathtt{x}_{k_1}}(\lambda_1(\mathtt{x}_1,\dots,\mathtt{x}_{k_1},0,\dots,0)(0)).$$

In addition, if $k_1 \leqslant m$, then

$$\omega_{\mathtt{x}_m}(\lambda_1(\mathtt{x}_1,\dots,\mathtt{x}_m,0,\dots,0)(0)) \geqslant \omega_{\mathtt{x}_m}(\lambda_1(\mathtt{x}_1,\dots,\mathtt{x}_m,0,\dots,0)(0)).$$

Thus, we may write $\lambda_1(0) = \mathtt{x}_m\lambda_1'(0)$ and after the change of coordinates $\frac{\mathtt{x}_{k_1}}{\mathtt{x}_m} \mapsto \mathtt{x}_{k_1}$ if $m < k_1$, we may assume

$$\omega_{\mathtt{x}_k}(\lambda_1'(\mathtt{x}_1,\dots,\mathtt{x}_k,0,\dots,0)(0)) \geqslant 0$$

for each $k = 1,\dots,n-1$.

Under these assumptions $Q(\xi_i) >_\gamma 0$ for each $\gamma \in \operatorname{Sper}(F((\mathtt{x})))$ and each $i = 1,2,3,4$. Consequently, $Q \in \mathcal{P}(L)$, where $L := \kappa_0((\mathtt{x}))[t]/(P)$. We claim: *$P + \mu Q \notin \mathcal{P}(\kappa_0((\mathtt{x}))[\mathtt{t}])$ for each $\mu \in \Sigma\kappa_0((\mathtt{x}))^2$.*

We take advantage of Sturm's sequence to prove our claim. We compute a Sturm's sequence for $P+\mu Q$ where $\mu \in \kappa_0((\mathtt{x}))$. Namely, $H_0 := P+\mu Q$, $H_1 := P'+\mu Q'$, $H_2 := -\text{remainder}(H_0,H_1,\mathtt{t})$, $H_3 := -\text{remainder}(H_1,H_2,\mathtt{t})$, $H_4 := \Delta(P+\mu Q,\mathtt{t})$. Assume first $\mu \in \Sigma\kappa_0[[\mathrm{x}]]^2$. The leading coefficients of $[H_0,H_1,H_2,H_3,H_4]$ are, after clearing squares from the denominators and making $\mathtt{x} = 0$, of the form

$$\begin{aligned}[1,4,a,a((b_2^2+a)\mu(0)^2 + 8(b_1b_2 - b0)\mu(0) + 16ad_0\lambda_1'(0)^2 + 32ad_1\lambda_1'(0)\lambda_2(0) + 16d_0\lambda_2^2(0)),\\ a^5(H_{40} + H_{41}\mu(0) + H_{42}\mu^2(0) - 16^2a(b_1b_2+b_0)(a-b_2^2)\mu^3(0) + 16a(a-b_2^2)^2\mu^4(0))]\end{aligned}$$

where $H_{4j} \in \mathbb{Q}[a,b_0,b_1,b_2,d_0,d_1,\lambda_1(0),\lambda_2(0)]$. If $\omega_{\mathtt{x}_k}(\mu(\mathtt{x}_1,\dots,\mathtt{x}_k,0,\dots,0)(0)) < 0$ for some $m \leqslant k \leqslant n-1$ and $\omega_{\mathtt{x}_{k'}}(\mu(\mathtt{x}_1,\dots,\mathtt{x}_{k'},0,\dots,0)(0)) \geqslant 0$ for $k \leqslant k' \leqslant n-1$, then the sign of

$$H_{40} + H_{41}\mu(0) + H_{42}\mu^2(0) - 16^2a(b_1b_2+b_0)(a-b_2^2)\mu^3(0) + 16a(a-b_2^2)^2\mu^4(0)$$

coincides with the sign of $16a(a-b_2^2)^2\mu^4(0)$, which is positive. Analogously, the sign of

$$(b_2^2+a)\mu(0)^2 + 8(b_1b_2 - b0)\mu(0) + 16ad_0\lambda_1'(0)^2 + 32ad_1\lambda_1'(0)\lambda_2(0) + 16d_0$$

coincides with the sign of $(b_2^2+a)\mu(0)^2$, which is positive. Consequently, $P+\mu Q$ has four roots in $\mathfrak{R}(\alpha)$ for each $\alpha \in \operatorname{Sper}(\kappa_0)$ such that $a >_\alpha 0$. Thus, we assume that

$$\omega_{\mathtt{x}_k}(\mu(\mathtt{x}_1,\dots,\mathtt{x}_k,0,\dots,0)(0)) \geqslant 0$$

for each $m \leqslant k \leqslant n-1$. We substitute $\mathtt{x}_{n-1} = 0,\dots,\mathtt{x}_m = 0$ (after clearing $a = \mathtt{x}_m$) and obtain the sequence

$$[1,4,1,b_2^2\mu(0)^2 + 8b_1b_2\mu(0) + 16d_0\lambda_2(0)^2 - 8b_0\mu(0), -1024b_0^2b_2^2\mu(0)^2]$$

if $\mu(\mathtt{x}_1,\dots,\mathtt{x}_{m-1},0,\dots,0) \neq 0$, which means that $P+\mu Q$ has two roots in $\mathfrak{R}(\alpha)$ for each $\alpha \in \operatorname{Sper}(\kappa_0)$ such that $a >_\alpha 0$. Otherwise, if $\mu(\mathtt{x}_1,\dots,\mathtt{x}_{m-1},0,\dots,0) = 0$, we obtain the sequence

$$[1,4,1,16(ad_0\lambda_1(0)^2 + 2ad_1\lambda_1(0)\lambda_2 + d_0\lambda_2(0)^2), 4096(a\lambda_1(0)^2 - \lambda_2(0)^2)^2(d_0^2 - ad_1^2)].$$

As

$$ad_0(ad_0\lambda_1(0)^2 + 2ad_1\lambda_1(0)\lambda_2 + d_0\lambda_2(0)^2) = (ad_0\lambda_1(0) + ad_1\lambda_2(0))^2 + a\lambda_2^2(0)(d_0^2 - ad_1^2)$$

is positive, we deduce $P+\mu Q$ has four roots in $\mathfrak{R}(\alpha)$ for each $\alpha \in \operatorname{Sper}(\kappa_0)$ such that $a >_\alpha 0$, $d_0^2 - ad_1^2 >_\alpha 0$ and $d_0 >_\alpha 0$.

Suppose next that $\mu \in \Sigma\kappa_0((\mathtt{x}))^2 \setminus \Sigma\kappa_0[[\mathtt{x}]]^2$. This is equivalent to show that there exists no $\mu \in \Sigma\kappa_0[[\mathtt{x}]]^2$ such that $Q + \mu P \in \mathcal{P}(\kappa_0[[\mathtt{x}]][\mathtt{t}])$. Write $\mu = \mathtt{x}^{2k}\lambda$ where $k \geqslant 1$, $\lambda \in \Sigma\kappa_0[[\mathtt{x}]]^2$ and $\lambda(0) \neq 0$. We compute the corresponding Sturm's sequence: $G_0 := Q + \mu P$, $G_1 :=$

$Q' + \mu P'$, $G_2 := -\text{remainder}(G_0, G_1, \mathtt{t})$, $G_3 := -\text{remainder}(G_1, G_2, \mathtt{t})$, $G_4 := \Delta(Q + \mu P, \mathtt{t})$. We distinguish several situations:

(a) Suppose first $\ell+1 = 2k$. The leading coefficients $[A_0, A_1, A_2, A_3, A_4]$ of $[G_0, G_1, G_2, G_3, G_4]$ are, after clearing squares from the denominators,

$$[\lambda, 4\lambda, a(16\lambda^2 - 8b_2\lambda + 3a), a(8(b_2^2 + a)\lambda^2 - (4b_2^3 + 12ab_2)\lambda + ab_2^2 + 3a^2), a^6(4\lambda^2 - 4b_2\lambda + a)].$$

Let $\alpha \in \operatorname{Sper}(\kappa_0)$ be such that $a >_\alpha 0$. If $4\lambda^2 - 4b_2\lambda + a <_\alpha 0$, then $Q + \mu P$ has two roots in $\mathfrak{R}(\alpha)$. Let us check that if $4\lambda^2 - 4b_2\lambda + a >_\alpha 0$, then both $16\lambda^2 - 8b_2\lambda + 3a >_\alpha 0$ and $8(b_2^2 + a)\lambda^2 - (4b_2^3 + 12ab_2)\lambda + ab_2^2 + 3a^2 >_\alpha 0$. The roots of $4\lambda^2 - 4b_2\lambda + a$ are

$$\eta_1 := \frac{b_2}{2} - \frac{\sqrt{b_2^2 - a}}{2}, \ \eta_2 := \frac{b_2}{2} + \frac{\sqrt{b_2^2 - a}}{2}.$$

Let us show that the roots of $16\lambda^2 - 8b_2\lambda + 3a >_\alpha 0$ and $8(b_2^2 + a)\lambda^2 - (4b_2^3 + 12ab_2)\lambda + ab_2^2 + 3a^2 >_\alpha 0$ belong to the interval $[\eta_1, \eta_2]_\alpha$. Those roots are respectively:

$$\tau_1 := \frac{b_2}{4} - \frac{\sqrt{b_2^2 - 3a}}{4}, \ \tau_2 := \frac{b_2}{4} + \frac{\sqrt{b_2^2 - 3a}}{4},$$
$$\theta_1 := \frac{b_2(b_2^2 + 3a)}{4(b_2^2 + a)} - \frac{\sqrt{(b_2^2 - a)(b_2^2 + 2a)(b_2^2 + 3a)}}{4(b_2^2 + a)},$$
$$\theta_2 := \theta_2 := \frac{b_2(b_2^2 + 3a)}{4(b_2^2 + a)} + \frac{\sqrt{(b_2^2 - a)(b_2^2 + 2a)(b_2^2 + 3a)}}{4(b_2^2 + a)}.$$

As $a = \mathtt{x}_m$, it holds that $\tau_i, \theta_i <_\alpha \eta_2$ for $i = 1, 2$. It holds also $\tau_1 <_\alpha \tau_2$ and $\theta_1 <_\alpha \theta_2$. In addition, we have

$$\tau_1 - \eta_1 = \frac{1}{8b_2}a + \cdots >_\alpha 0 \quad \text{and} \quad \theta_1 - \eta_1 = \frac{1}{8b_2^3}a^2 + \cdots >_\alpha 0.$$

Consequently, $\tau_i, \theta_i \in [\eta_1, \eta_2]_\alpha$ for $i = 1, 2$.

Thus, if $4\lambda^2 - 4b_2\lambda + a >_\alpha 0$, then both $16\lambda^2 - 8b_2\lambda + 3a >_\alpha 0$ and $8(b_2^2 + a)\lambda^2 - (4b_2^3 + 12ab_2)\lambda + ab_2^2 + 3a^2 >_\alpha 0$. Consequently, $Q + \mu P$ has for roots in $\mathfrak{R}(\alpha)$, which is a contradiction.

(b) Suppose next $\ell + 1 < 2k$. The leading coefficients of $[G_0, G_1, G_2, G_3, G_4]$ are, after clearing squares from the denominators,

$$[\lambda, 4\lambda, a\lambda, \frac{b_2^2 + a}{a}, 16a^2\lambda^2(-b_2^2 + a)^2].$$

Thus, $Q + \mu P$ has for roots in $\mathfrak{R}(\alpha)$ for each $\alpha \in \operatorname{Sper}(\kappa_0)$ such that $a >_\alpha 0$, which is a contradiction.

(c) Suppose next $\ell + 1 > 2k$. The leading coefficients of $[G_0, G_1, G_2, G_3, G_4]$ are, after clearing squares from the denominators,

$$[\lambda, 4\lambda, \frac{3}{16\lambda}, \frac{32\lambda(b_2^2 + 3a)}{9}, 4a(-b_2^2 + a)^2].$$

Thus, $Q + \mu P$ has for roots in $\mathfrak{R}(\alpha)$ for each $\alpha \in \operatorname{Sper}(\kappa_0)$ such that $a >_\alpha 0$, which is a contradiction, as required. □

Remark D.9. (i) In the proof of Theorem 14.14(ii) we have only used the hypothesis: *All the orderings of κ are Archimedean*:

- to show that there exists no chimeric polynomials over κ (used to prove Case 1 of Lemma D.8).
- to take advantage of Lemma D.3 to deal with Subcase 2.1 in Lemma D.8.

For the remaining cases we do not need such hypothesis. To prove CASE 1 of Lemma D.8 a sufficient (milder) condition is that $\mathfrak{D}_\kappa$ is dense in $\operatorname{Sper}(\kappa)$ (see Lemma 14.11), whereas to approach SUBCASE 2.1 of Lemma D.8 it would be enough to consider those (formally) real fields κ for which Question D.4 has a positive answer. This includes the mentioned (formally) real fields for which all orderings are Archimedean or those (formally) real fields κ that admit a unique ordering α and κ is dense in $\mathfrak{R}(\alpha)$.

(ii) We guess that Theorem 14.14(ii) can be extended to generalized field of power series $\kappa((\mathtt{t}^G))$ where G is a linearly ordered abelian group, adapting the previous proof. However, the details seem further cumbersome and exceed the purposes of this article. ∎

E

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REFERENCES

[ABR] C. Andradas, L. Bröcker, J.M. Ruiz: Constructible sets in real geometry. *Ergebnisse der Mathematik und ihrer Grenzgebiete* (3) **33**. Springer-Verlag, Berlin, (1996).

[AR] C. Andradas, J.M. Ruiz: On local uniformization of orderings. *Recent advances in real algebraic geometry and quadratic forms* (Berkeley, CA, 1990/1991; San Francisco, CA, 1991), 19–46. *Contemp. Math.*, **155** American Mathematical Society, Providence, RI, (1994).

[ADF] S. Anscombe, Ph. Dittmann, A. Fehm: Denseness results in the theory of algebraic fields. *Ann. Pure Appl. Logic* **172** (2021), no.8, Paper No. 102973, 20 pp.

[Ar] E. Artin: Über die Zerlegung definiter Funktionen in Quadrate. *Abh. Math. Sem. Univ. Hamburg* **5** (1927), no. 1, 100–115.

[AS1] E. Artin, O. Schreier: Algebraische Konstruktion reeller Körper. *Abh. Math. Sem. Univ. Hamburg* **5** (1927), no. 1, 85–99.

[AS2] E. Artin, O. Schreier: Eine Kennzeichnung der reell abgeschlossenen Körper. *Abh. Math. Sem. Univ. Hamburg* **5** (1927), no. 1, 225–231.

[A] M. Artin: On isolated rational singularities of surfaces. *Amer. J. Math.* **88** (1966), 129–136.

[AGV] M. Artin, A. Grothendieck, J. L. Verdier: Théorie des topos et cohomologie étale des schémas. Tome 3. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4). Avec la collaboration de P. Deligne et B. Saint-Donat. *Lecture Notes in Math.*, **305** Springer-Verlag, Berlin-New York, (1973).

[AM] M.F. Atiyah, I.G. MacDonald: Introduction to commutative algebra. *Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont.* (1969).

[BPR] S. Basu, R. Pollack, M.-F. Roy: Algorithms in real algebraic geometry.*Algorithms Comput. Math.*, **10** Springer-Verlag, Berlin, (2006).

[BGV] K.-J. Becher, D. Grimm, J. Van Geel: Sums of squares in algebraic function fields over a complete discretely valued field. *Pacific J. Math.* **267** (2014), no. 2, 257–276.

[BCR] J. Bochnak, M. Coste, M.-F. Roy: Real algebraic geometry. *Ergeb. Math.* **36**, Springer-Verlag, Berlin (1998).

[CS] J. Capco, C. Scheiderer: Two remarks on sums of squares with rational coefficients.(English summary)Algebra, logic and number theory, 25–36. *Banach Center Publ.*, **121** Polish Academy of Sciences, Institute of Mathematics, Warsaw, (2020).

[C] J. W. S. Cassels: On the representation of rational functions as sums of squares. *Acta Arith.* **9** (1964), 79–82.

[CK] C. C. Chang, H.J. Keisler: Model theory. *Stud. Logic Found. Math.*, **73** North-Holland Publishing Co., Amsterdam, (1990).

[CDLR] M.D. Choi, Z.D. Dai, T.Y.Lam, B. Reznick: The Pythagoras number of some affine algebras and local algebras, *J. Reine Angew. Math.* **336** (1982), 45–82.

[CLRR] M.D. Choi, T.Y.Lam, B. Reznick, A. Rosenberg: Sums of squares in some integral domains, *J. Algebra* **65** (1980), no. 1, 234–256.

[CJ] J.-L. Colliot-Thélène, U. Jannsen: Sommes de carrés dans les corps de fonctions. (French. English summary) [Sums of squares in function fields] *C. R. Acad. Sci. Paris Sér. I Math.* **312** (1991), no. 11, 759–762.

[DST] M. Dickmann, N. Schwartz, M. Tressl: Spectral spaces. *New Mathematical Monographs*, **35**. Cambridge University Press, Cambridge, (2019).

[D] D.W. Dubois: A nullstellensatz for ordered fields. *Ark. Mat.* **8** (1969), 111–114.

[E] D. Eisenbud: Some directions of recent progress in commutative algebra. *Algebraic geometry* (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), pp. 111–128. *Amer. Math. Soc.*, Providence, R.I., (1975).

[ELW] R. Elman, T.Y. Lam, A.R. Wadsworth: Orderings under field extensions. *J. Reine Angew. Math.* **306** (1979), 7–27.

[EGH] P. Erdös, L. Gillman, M. Henriksen: An isomorphism theorem for real-closed fields. *Ann. of Math.* (2) **61** (1955), 542–554.

[Er] Y. Ershov: Relative regular closeness and π-valuations, *Algebra Log.* **31** (6) (1992) 342–360.

[Fe1] J.F. Fernando: On the Pythagoras numbers of real analytic rings. *J. of Algebra*, **243** (2001), 321–338.

[Fe2] J.F. Fernando: Positive semidefinite germs in real analytic surfaces, *Math. Ann.* **322** (2002), no. 1, 49–67.

[Fe3] J.F. Fernando: Sums of squares in real analytic rings, *Trans. Amer. Math. Soc.* **354** (2002), no. 5, 1909–1919.

[Fe4] J.F. Fernando: Analytic germs of minimal Pythagoras number. *Math. Z.* **244** (2003), no. 4, 725–752.

[Fe5] J.F. Fernando: Erratum: Analytic germs of minimal Pythagoras number. *Math. Z.* **250** (2005), no. 4, 967–969.

[Fe6] J.F. Fernando: Sums of squares in excellent henselian local rings. *Advances in Mathematics Research.* **7**. Edited by Gabriel Oyibo. Nova Science Publishers, Inc., Hauppauge, NY, (2007). Pages: 75–100, ISBN: 1-59454-458-1.

[Fe7] J.F. Fernando: On the positive extension property and Hilbert's 17th problem for real analytic sets. *J. Reine Angew. Math.* **618** (2008), 1–49.

[Fe8] J.F. Fernando: Representation of positive semidefinite elements as sum of squares in 2-dimensional local rings. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* RACSAM **116** (2022), no. 2, Paper No. 59, 65 pp.

[Fe9] J.F. Fernando: Sums of two squares in global real analytic sets. *Work in progress* (2025).

[FR1] J.F. Fernando, J.M. Ruiz: Positive semidefinite germs on the cone, *Pacific J. Math.* **205** (2002), no. 1, 109–118.

[FR2] J.F. Fernando, J.M. Ruiz: On the Pythagoras numbers of real analytic set germs. *Bull. Soc. Math. France* **133** (2005), no. 3, 349–362.

[FRS1] J.F. Fernando, J.M. Ruiz, C. Scheiderer: Sums of squares in real rings. *Trans. Amer. Math. Soc.* **356** (2004), no. 7, 2663–2684.

[FRS2] J.F. Fernando, J.M. Ruiz, C. Scheiderer: Sums of squares of linear forms. *Math. Res. Lett.* **13** (2006), no. 6, 945–954.

[Fo] A. Fornasiero: Embedding Henselian fields into power series. *J. Algebra* **304** (2006), no. 1, 112–156.

[H1] D. Hilbert: Über die Darstellung definiter Formen als Summe von Formenquadraten. *Math. Ann.* **32** (1888), no. 3, 342–350.

[H2] D. Hilbert: Über die vollen Invariantensysteme. *Math. Ann.* **42** (1893), no. 3, 313–373.

[H3] D. Hilbert: Hermann Minkowski.*Math. Ann.* **68** (1910), no. 4, 445–471.

[H4] D. Hilbert: Hilbert's invariant theory papers. Translated from the German by Michael Ackerman. With comments by Robert Hermann. *Lie Groups: History, Frontiers and Applications*, **VIII** Math Sci Press, Brookline, MA, (1978).

[Hu] Y. Hu: The Pythagoras number and the u-invariant of Laurent series fields in several variables. *J. Algebra* **426** (2015), 243–258.

[J1] U. Jannsen: Principe de Hasse cohomologique. (French) [Cohomological Hasse principle] Séminaire de Théorie des Nombres, Paris, 1989-90, 121–140, Progr. Math., **102**, Birkhäuser Boston, Boston, MA, (1992).

[J2] U. Jannsen: Hasse principles for higher-dimensional fields. *Ann. of Math.* (2) **183** (2016), no. 1, 1–71.

[JP] T. de Jong, G. Pfister: Local analytic geometry. Basic theory and applications. *Advanced Lectures in Mathematics*. Friedr. Vieweg & Sohn, Braunschweig (2000).

[K] K. Kato: A Hasse principle for two-dimensional global fields. With an appendix by Jean-Louis Colliot-Thélène. *J. Reine Angew. Math.* **366** (1986), 142–183.

[Kn] M. Knebusch: Generic splitting of quadratic forms. I. *Proc. London Math. Soc.* (3) **33** (1976), no.1, 65–93.

[L] T. Y. Lam: Introduction to quadratic forms over fields. *Graduate Studies in Mathematics*, **67**. American Mathematical Society, Providence, RI (2005).

[Li] J. Lipman: Rational singularities, with applications to algebraic surfaces and unique factorization. *Inst. Hautes Études Sci. Publ. Math.* (1969), **36**, 195–279.

[Pf1] A. Pfister: Zur Darstellung definiter Funktionen als Summe von Quadraten. *Invent. Math.* **4** (1967), 229–237.

[Pf2] A. Pfister: Quadratic forms with applications to algebraic geometry and topology. *London Mathematical Society Lecture Note Series*, **217**. Cambridge University Press, Cambridge, (1995).

[Pf3] A. Pfister: On the Milnor conjectures: history, influence, applications. *Jahresber. Deutsch. Math.-Verein.* **102** (2000), no. 1, 15–41.

[PP] G. Pfister, D. Popescu: Die strenge Approximationeigenschaft lokaler Ringe. *Invent. Math.* **30** (1975), 145–174.

[Po] D. Popescu: General Néron desingularization and approximation. *Nagoya Math. J.* **104** (1986), 85–115.

[Pop] F. Pop: Classically projective groups and pseudo classically closed fields, in: F.-V. Kuhlmann, S. Kuhlmann, M. Marshall (Eds.), Valuation Theory and Its Applications, in: *Fields Inst. Commun.*, vol. II, American Mathematical Society, (2003), pp. 251–283.

[Pl] D. Plaumann: Sums of squares on reducible real curves. *Math. Z.* **265** (2010), no. 4, 777–797.

[P] Y. Pourchet: Sur la représentation en somme de carrés des polynômes à une indéterminée sur un corps de nombres algébriques. *Acta Arith.* **19** (1971), 89–104.

[Pr1] A. Prestel: Pseudo real closed fields. *Set theory and model theory* (Bonn, 1979), pp. 127–156 *Lecture Notes in Math.*, **872** Springer-Verlag, Berlin-New York, (1981).

[Pr2] A. Prestel: Lectures on formally real fields. *Lecture Notes in Math.*, **1093** Springer-Verlag, Berlin, (1984).

[R] J.-J. Risler: Une caractérisation des idéaux des variétés algébriques réelles. *C. R. Acad. Sci. Paris* Sér. A-B **271** (1970), A1171–A1173.

[Rt1] C. Rotthaus: On the approximation property of excellent rings. *Invent. Math.* **88** (1987), no. 1, 39–63.

[Rt2] C. Rotthaus: Rings with approximation property. *Math. Ann.* **287** (1990), no. 3, 455–466.

[Rz1] J.M. Ruiz: The basic theory of series. *Advanced Lectures in Mathematics*. Friedr. Vieweg & Sohn, Braunschweig, (1993).

[Rz2] J.M. Ruiz: Sums of two squares in analytic rings, *Math. Z.* **230** (1999), no. 2, 317–328.

[Sch1] C. Scheiderer: Real and étale cohomology. *Lecture Notes in Math.*, **1588** Springer-Verlag, Berlin, (1994).

[Sch2] C. Scheiderer: Sums of squares of regular functions on real algebraic varieties, *Trans. Amer. Math. Soc.* **352** (1999) no. 3, 1039–1069.

[Sch3] C. Scheiderer: On sums of squares in local rings. *J. reine angew. Math.* **540** (2001), 205–227.

[Sch4] C. Scheiderer: Sums of squares on real algebraic surfaces. *Manuscr. Math.* **119** (2006), 395–410.

[Sch5] C. Scheiderer: Sums of squares on real algebraic curves. *Math. Z.* **245** (2003), no. 4, 725–760.

[Sch6] C. Scheiderer: Weighted sums of squares in local rings and their completions, I. *Math. Z.* **266** (2010), no. 1, 1–19.

[Sch7] C. Scheiderer: Weighted sums of squares in local rings and their completions, II. *Math. Z.* **266** (2010), no. 1, 21–42.

[Sch8] C. Scheiderer: Sums of squares of polynomials with rational coefficients. *J. Eur. Math. Soc.* (JEMS) **18** (2016), no. 7, 1495–1513.

[Schl] A. Schleiermacher: On Archimedean fields. *J. Geom.* **92** (2009), no.1–2, 143–173.

[S] J.-P. Serre: Galois cohomology. Springer-Verlag, Berlin, (1997).

[St] G. Stengle: A nullstellensatz and a positivstellensatz in semialgebraic geometry. *Math. Ann.* **207** (1974), 87–97.

[vS] D. van Straten: Doktorarbeit: Weakly normal Surface Singularities and their improvements, (1987). `https://download.uni-mainz.de/mathematik/Algebraische Geometrie/DvS-Publikationen/DvS_1987_weakly normal Surface Singularities and their improvements.pdf`

[T] J.-C. Tougeron: Idéaux de fonctions différentiables. *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Band **71**. Springer-Verlag, Berlin-New York, (1972).

[V1] V. Voevodsky: Reduced power operations in motivic cohomology. *Publ. Math. Inst. Hautes Études Sci.* **98** (2003), 1–57.

[V2] V. Voevodsky: Motivic cohomology with $\mathbb{Z}/2$-coefficients. *Publ. Math. Inst. Hautes Études Sci.* **98** (2003), 59–104.

[W] J.M. Wahl: Equations defining rational singularities. *Ann. Sci. École Norm. Sup.* (4) **10** (1977), no.2, 231–263.

[ZS] O. Zariski, P. Samuel: Commutative algebra. Vol. II. *Graduate Texts in Mathematics*, **29**. Springer-Verlag, New York-Heidelberg, (1975).

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