A tutorial on Heuristic methods

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In this paper we define a heuristic method as a procedure for solving a well-defined mathematical problem by an intuitive approach in which the structure of the problem can be interpreted and exploited intelligently to obtain a reasonable solution. Issues discussed include: (i) the measurement of the quality of a heuristic method, (ii) different types of heuristic procedures, (iii) the interactive role of human beings and (iv) factors that may influence the choice or testing of heuristic methods. A large number of references are included.

1. Introduction

Our objectives in this paper are twofold. First, we wish to provide an introduction to heuristic methods for decision analysts and those managers who are somewhat familiar with the basic techniques of operational research. Our intention is not to give a general recipe for constructing heuristic methods nor do we give an exhaustive comparison of the many existing heuristic procedures and their performances. However our second objective is to identify important issues, related to the use of heuristic methods, on which additional research is required.

There are many possible definitions of a heuristic method of problem solution. Nicholson [42] defines a heuristic method as a procedure "... for solving problems by an intuitive approach in which the structure of the problem can be interpreted and exploited intelligently to obtain a reasonable solution." We shall adopt this definition and in addition we shall assume in this paper that we start with an appropriate well-defined mathematical model of the real world problem under consideration. To be more specific, knowing the values of the decision (or controllable) variables implied by a proposed solution, we can ascertain (i) whether or not the solution is feasible and (ii) the value(s) of the measure(s) of effectiveness of the solution. In other words, we have mathematical representations of the constraints and objective function(s). The procedure will be mathematically well-defined; however, it will not be guaranteed to give a mathematically 'optimal' solution. The word 'intuitive' in the above definition is meant to imply that the practical intuition of the decision maker (who poses the problem) and/or the mathematical intuition of the analyst should probably guide the choice and elaboration of the procedure. We shall use the term 'heuristic solution' to represent the solution obtained by a heuristic method. Finally, at times we shall use the word 'heuristic' as a noun in place of 'heuristic method or procedure.'

Admittedly, in assuming that we have an appropriate mathematical representation of the real world problem, one could argue that we are ignoring the most difficult aspect of an operational research study, namely the development of the mathematical model itself. In this connection White [66] has an interesting discussion of the so-called secondary decisions in an OR study, e.g., what objectives to consider, what constraints to include and what alternatives to test? However, it is important to recognize that, particularly as we address more complicated decision problems, *realistic* formulations are likely to lead to mathematical problems which are very difficult, if not impossible, to solve exactly (this is to a considerable extent due to the combinatorial nature of many practical problems). Thus, we believe that the topic of approximate solution procedures for well-defined mathematical problems is of increasing importance to decision analysts.

There is a vast literature on heuristic methods. We would, however, wish to specifically mention the comprehensive surveys of Klein [25], Müller-Mer-

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bach [36–38], Newell [41], Streim [55], and Weinberg and Zehnder [61]. Most of these surveys attempt to systematize and classify existing heuristic methods. In addition, Streim [55] includes an extensive reference list.

2. Why use a heuristic method?

There are several possible reasons for using a heuristic method of solution. These include:

(a) The mathematical problem is of such a nature that an analytic (closed form) or iterative solution procedure is unknown.

(b) Although an exact analytic or iterative solution procedure may exist, it may be computationally prohibitive to use or perhaps unrealistic in its data requirements. This is particularly true of enumerative methods which, *in theory*, are often applicable where analytic and iterative procedures cannot be found. In this connection Ronald Graham [16] has stated:

> "The discoveries about NP-completeness [1] changed the direction of research on scheduling. Earlier efforts were directed at finding optimum, or exact, solutions to scheduling problems, but now most attention has been turned in the more fruitful direction of determining approximate solutions easily, of finding efficient methods that are guaranteed to give close to optimum results."

(c) The heuristic method, *by design*, may be simpler for the decision maker to understand, hence markedly increasing the chances of implementation. For example, in some inventory control problems, where a mathematically optimal solution can be obtained, the manager prefers to implement a solution of a given form simpler than that of the optimal solution. In the words of Woolsey and Swanson [68, p. 169]: "People would rather live with a problem they cannot solve than accept a solution they cannot understand."

[1] NP-complete, or nondeterministic-polynomial-time-complete problems are a large class of problems having an important characteristic, namely that all algorithms currently known for finding optimal solutions to these problems require a number of computational steps that grows exponentially with the 'size' of the problem (two illustrations of what we mean by 'size' are (i) the number of jobs to be scheduled in a scheduling problem and (ii) the number of cities in a travelling salesman problem). Thus, for large size NP-complete problems it does not appear that any efficient *optimal* solution is possible. A further discussion on problem size and efficiency of algorithms can be found in Lawler [27, pp. 5–8].

(d) For a well-defined problem that can be solved optimally a heuristic method can be used for learning purposes, e.g., to develop an intuitive feeling as to what variables are important. (This closely parallels one of the primary reasons for using simulation methods in operational research.)

(e) A heuristic may be used as part of an iterative procedure that guarantees the finding of an optimal solution. Two distinct possibilities exist:

(i) to easily obtain an initial feasible solution, e.g., the so-called Northwest Corner Rule (see e.g., Wagner [60]) for obtaining an initial solution to the transportation problem. The Rule often does not give a particularly good solution, but this is unimportant in this case in that the subsequent computerized optimization routine finds the optimal solution quickly from *any* initial feasible solution.

(ii) to make a decision at an intermediate step of an exact solution procedure, e.g., the rule for selecting the variable to be entered into the basis in the Simplex Method is heuristic in that it does not necessarily minimize the number of steps needed to reach the optimal solution.

(f) In implicit enumeration approaches to problem solving a good starting solution can give a bound that drastically reduces the computational effort. Heuristics can be used to give such 'good' starting solutions.

With all these possible reasons for using heuristic methods the reader may begin to wonder why anyone would ever bother trying to obtain an optimal solution to a problem. We are not advocating the abandonment of optimization methods *where their use (including implementation considerations) is appropriate*. Moreover, the use of heuristic methods is not an easy way out. As we shall see, it is not simple to properly develop and evaluate a heuristic procedure.

3. Measuring the quality of a heuristic

Our definition of a heuristic method included the words "for solving problems (well defined mathematically) by . . . to obtain a reasonable solution." In this section we interpret the meaning of the word 'reasonable.' (Wheeling [64, p. 372] provided an early discussion of this issue.)

In our view a good heuristic should possess the following four properties:

(1) Realistic computational effort to obtain the solution.

(2) The solution should be close to the optimum on the average, i.e., we want good performance on the average.

(3) The chance of a very poor solution (i.e., far from the optimum) should be low.

(4) The heuristic should be as simple as possible for the user to understand, preferably explainable in intuitive terms, particularly if it is to be used manually. Carefully prepared documentation should help in this regard.

If a heuristic method involves repetitive solution tries with only the best solution being retained, then in Properties 2 and 3 we are only referring to this 'best' solution (and not all of the solutions from which it is selected).

An important factor in deciding on the relative importance of Properties 2 and 3 is the number of times that the problem, on which the heuristic is being employed, will be solved. As the frequency of solution increases, one would expect more weight to be given to Property 2 relative to Property 3.

Property 4, which is much more difficult to quantify than the other three properties, has been largely overlooked in the literature, yet it may well be the most important of the four desired characteristics. An illustration of the application of this philosophy in the development of decision systems for inventory management and production planning is provided by Peterson and Silver [45].

Here we concentrate on the measurement of quality primarily in terms of Properties 2 and 3. We shall discuss several different measures, but before doing that, the following point is worth emphasizing. Rather than precisely determining values of the measures of quality of a particular heuristic, it may be more worthwhile to find under what conditions (i.e., values of the uncontrollable variables) the heuristic does particularly poorly and hence should not be used. As an example, a heuristic for ascertaining the best number of service channels in a problem context, where service is provided to randomly arriving customers, may perform very well as long as the utilization of the channels is not too high. A second example is afforded by Blackburn and Millen [3] who provide guidelines as to the choice among several heuristic solution methods for the dynamic lot size problem (selection of timing and sizes of replenishments of a single item facing a deterministic, but time-varying demand pattern).

3.1. Comparison with the optimum solution

One would like to be able to compare the heuristic solution with the best possible solution over a large number of problem instances. Usually this is not possible, in that, as discussed earlier, a major reason for using a heuristic procedure in the first place is that it may be impossible or prohibitive from a computational standpoint to obtain the optimal solution. One may have to resort to simulation to estimate the value of the best solution (in a case where the objective function is only indirectly expressed in terms of the controllable and uncontrollable variables). Also, it may be necessary to concentrate on small scale problems (of smaller size than at least some of the instances of interest) to reduce the computational effort to a reasonable level.

Even when the optimum solution can be found there remains the question of what problem instances to use for testing purposes, i.e., what set of values to assign to the uncontrollable variables. This is clearly a question of experimental design, a subject on which there is a large literature (see e.g., [7,18 and 35]. Typically there are too many uncontrollable factors to permit a complete factorial design of experiments. Insight (partly from an understanding of the related theory), as well as an investigation of the results of preliminary experiments can suggest which variables are likely to be important. Often, a dimensional analysis (see e.g., Naddor [40]) can reduce the number of distinct parameters that need be considered.

Ideally it would be desirable to develop the probability distribution of the size of the cost penalty associated with using the heuristic solution instead of the optimal solution. In order to do this one would first have to decide on appropriate probability distributions for the uncontrollable variables. In theory, one could then develop the desired distribution. However, initial efforts by several researchers have indicated that the analysis becomes extremely complicated, even under the questionable simplifying assumption of independence of the uncontrollable variables. Thus, from a pragmatic standpoint the major line of attack of a probabilistic analysis would appear to be the development of a histogram of cost penalties by the empirical generation of values of the uncontrollable variables according to their specified probability distributions, i.e., a numerical determination of the desired probability distribution. Conditional distributions are also of interest, e.g., the distribution of errors given that a particular parameter takes on a specific value. This kind of conditional distribution helps decide under which circumstances it is attractive to use the heuristic under consideration.

3.2. Problem relaxation–bounding

When the optimum solution can not be found (often the case when a heuristic is used), an alternative approach is to relax the problem so that a solution can be evaluated that is at least as good as the optimum solution, hence providing a bound on the value of the optimum solution, i.e., the value of the optimum solution cannot be better than the bound. The relaxed solution itself needs not to be obtained; all one needs is its value or, even less, a bound on its value. Then we check how close the heuristic solution is to this bound. This is really only a one-way test. We know that the value of the optimal solution must lie between the value of the heuristic solution and the bound. Thus, if the value of the heuristic is very close to the bound, it must be very close to the value of the optimum solution. On the other hand, a large gap between the value of the heuristic solution and the bound may be caused by the heuristic being poor, the bound being too loose or both. Note that the heuristic method itself gives a bound of the opposite type, i.e., the value of the optimum solution cannot be worse than the value of the heuristic solution.

The most common way of relaxing a problem is to ignore one or more constraints. We present two illustrations:

(i) In an integer linear programming problem, a bound can be obtained by ignoring the integer constraints and solving the much simpler, continuous variable problem. (Peterson and Silver [45, pp. 536–537] provide a related example of bounding the solution of a coordinated control inventory problem).

(ii) In a travelling salesman problem the difficulty of solution is caused by the constraint that every city must be visited precisely once *in a single tour*. Removing the single tour constraint leads directly to a bound on the original problem.

With problem relaxation we still must face the issue of what examples to use for test purposes.

3.3. Extreme value statistical methods

Often a heuristic method is used that involves generating many solutions and choosing the best of these. Under such circumstances the value of each generated solution can be considered as a random variable. Then we are interested in the extreme of many of these (not necessarily independent) random variables. The theory of extreme value statistics can be used to develop estimates of the value of the optimal solution of the problem at hand. Interesting references on this subject are Clough [6], Dannenbring [10] and Golden [15].

3.4. Other comparisons

For discussion purposes here we shall assume that we are dealing with a problem where we wish to *maximize* some objective function. Thus any bound obtained in Section 3.2 would be an *upper* bound. Where such a bound can not be easily obtained or is suspected to be poor, one can resort to comparisons of the heuristic solution with other types of solutions, where the latter produce *lower* bounds on the value of the optimum solution. Possibilities include:

(i) Comparison with an enumerative method, requiring much more computational effort, that is terminated after a large amount of computation, but likely without having found the optimal solution. An example would be a branch-and-bound procedure (see Wagner [60] for a description of such procedures) stopped after a certain large number of branches were generated. Such a 'truncated' branch-and-bound procedure would also give us an upper bound (in a maximization problem) on the value of the optimal solution.

(ii) Comparison with performance of the decision maker, either during an earlier time frame or directly in parallel – there are compelling arguments for this type of comparison. Identification of significant improvement over existing procedures is probably much more important for encouraging implementation than any proof of optimality, or nearness to optimality. An interesting reference is Bowman's article [4].

(iii) Comparison with other heuristic procedures – where other heuristic solution methods have been proposed and/or used, one certainly can compare 'our' heuristic against the others. The danger is that the other heuristics may be particularly bad so that, even though 'our' heuristic's performance was judged the best, it may still be a poor solution method.

(iv) Comparison with a 'random' decision rule – an extreme type of heuristic is where one makes a decision completely at random. As an illustration, some researchers (see e.g., Conway [8]), in looking at heuristics for sequencing jobs at work centers in a job shop context, use for a baseline comparison the decision rule of choosing the next job to process at a work center by a random selection among the jobs waiting at the center. In general, a 'random' rule should produce a rather poor bound on the value of the optimum solution, hence can be useful for quickly rejecting a poor heuristic method.

3.5. Worst case behaviour

The thrust of much recent research on large-scale, deterministic, combinatorial-type problems has been to identify the worst error possible under the use of a particular heuristic solution procedure (see [10,11]). It is comforting to know that a heuristic's cost penalty will never be worse than a particular value. However, this information is almost certainly not enough to choose among competing heuristic procedures. As mentioned earlier, a decision maker is likely more interested in average performance and/or the *probability* that the cost penalty is larger than a certain size. A particular heuristic may have a very poor worst case behaviour caused by rare (essentially pathological) values of the uncontrollable variables, yet the same heuristic may perform excellently under almost all other conditions. An example is Quicksort, the sorting routine frequently used in software related to operational research (see Nievergelt et al. [43]). On the other hand, worst case analyses, as a side benefit, may indicate under what conditions the heuristic does poorly, hence signaling when not to use the method.

4. Types of heuristic methods

We now turn to a categorization of heuristic methods. It should be emphasized that the categories are not meant to be mutually exclusive. Indeed, it often makes sense to blend two or more of the types in the solution of a particular class of problems. In addition, one should not overlook the possibility of using two distinct heuristic methods in parallel to solve the same problem, choosing the better of the two solutions, i.e., do not necessarily bank on good performance of a single method. Finally, where possible, the developer of a heuristic method is well advised to first have a sound appreciation of the theoretical work that has been done on the particular problem under consideration; often such theory will suggest specific lines of attack that are likely to be fruitful in developing an effective heuristic procedure. As an aside, since heuristics for complex problems are often developed from exact solutions methods for simpler problems (see e.g., de Werra [63]), this is a good, perhaps the most important, reason for teaching students exact procedures for simple, academic problems.

4.1. Decomposition methods

Here the problem under consideration is broken into smaller parts that are solved separately, but taking account, at least in a crude way, of possible interactions among the parts. Decomposition is prevalant in the traditional separation of problems into the functional areas of an organization. A second common type of decomposition is the separation of system design from system operation. Commonly the operational effects are ignored in the design phase and then the operational rules are developed *based on the design having been chosen.* Another good illustration of decomposition is the hierarchical planning work of Hax and Meal [17] where medium-range aggregate planning decisions are separated from short-range scheduling questions, but with appropriate coupling of the two problem areas. An important virtue of such decomposition is that it is consistent with how most organizations actually function.

4.2. Inductive methods

The idea here is to generalize from smaller (or somewhat simpler) versions of the same problem. For example, in a problem involving the location of several facilities (such as plants and warehouses) to satisfy customer demand, the solution may be relatively easy to obtain for the case of but a few facilities. Properties of the solutions for such simpler cases may be profitably used to develop a heuristic for the more general case of several facilities (see Bilde and Vidal [2]). In addition, sometimes the situation where a particular parameter becomes very large is particularly easy to analyze, again providing insight for the more difficult case of an intermediate (not too large or not too small) value of the parameter. An example is in the general area of renewal processes where the aggregate effect of a large number of processes has a Poisson behaviour. (Silver [54] has made use of this property in the development of a procedure for coordinated control of inventoried items under probabilistic demand.)

4.3. Feature extraction (or reduction) methods

The general approach here is to first obtain the optimal solutions to several numerical cases under consideration. Common features of these solutions are extracted and are assumed to hold in general. Examples might include that a particular variable always takes on the value 0 or that certain control variables are highly correlated or that a particular constraint never appears to be binding. A more extreme version of reduction is to assume (by analysis of simpler versions of the problem or simply by 'gut' feel) that good solutions must satisfy certain properties, where these properties substantially simplify the analysis. Then once the solution is obtained one verifies that, indeed, the properties are met (see White [65]). A good example of this approach is the use of the assumption that in any reasonable inventory control strategy the frequency of stockout occasions will be quite low. Another example is provided by Hitchings and Cottam [19] in the context of a facilities layout problem. They recognize that, if certain facilities are judiciously fixed in location or certain pairs are constrained to be adjacent to one another, then this reduces the dimensionality of the problem, thus permitting each potential solution to be evaluated much more quickly.

4.4. Methods involving model manipulation

The idea here is to change the nature of the model in some way so as to facilitate the solution and then use the solution of the revised model as representative of the solution of the original mathematical problem. In a more general sense, this is what one does when using a mathematical model of a real world problem and then interpreting the solution of the model as being the solution for the original problem. Examples of model manipulation include:

(i) modification of the objective function, e.g., linearization of a non-linear function.

(ii) relaxation of certain constraints, some of which may be flexible in any event (e.g., a budget constraint need not necessarily be rigid).

(iii) change nature of probability distributions, e.g., the assumption of normality instead of a more complex distribution.

(iv) aggregation of variables, the idea being to reduce the number of decision variables (however, eventually the value of the aggregate variable has to be apportioned among the original disaggregated variables). An interesting reference on the cost implications of aggregation is the work of Geoffrion [14].

4.5. Constructive methods

The methods presented here contrast with local improvement methods, the latter to be discussed in the next section. The basic idea of a constructive method is to literally build up to (i.e., construct) a single feasible solution, often in a deterministic, sequential fashion. An example is the so-called nearest-neighbour rule for solving the travelling salesman problem (see Rosenkrantz et al. [48]). To establish a single circuit that passes exactly once through each city one starts with a particular city (say i) and first goes to the nearest city (call it j) to i. From j one next goes to city k, the nearest city to j that has not yet been visited, etc. This is an example of a so-called 'greedy' algorithm, greedy in the sense that it does the best it can in each single step. For certain classes of problems (not the case for the travelling salesman problem) a greedy heuristic leads to very good solutions (Cornuejols et al. [9], Jenkyns [21] and Lawler [27]). However, the nearest-neighbour rule generally provides a poor solution to the travelling salesman problem; it suffers from its myopic viewpoint of only considering the very next node. This is in contrast with heuristic methods that use so-called 'look-ahead' rules (see Müller–Merbach [37]); an example for the travelling salesman problem is given by Müller–Merbach [36, Section 3.3] where the look-ahead rule is based on bound calculations (see Section 3.2 of the current paper).

Another example of a constructive method is afforded by the problem of locating a number of work centers (n) at a number (m) of potential sites in a manufacturing plant so as to keep material handling costs as low as possible. One possible constructive heuristic (see Lee and Moore [28]) is to first locate a particular center (call it i_1) at a particular location (say j_1). Now with i_1 at j_1 we choose a second center, i_2, and place it at another location, j_2, then a third center, i_3 at j_3, etc. No complete solution is available until the last remaining center is located. To generate a new solution, the *whole* procedure is repeated.

Another type of constructive method relates to problems having many decision variables subject to a relatively small number of aggregate constraints. An illustration is the problem of selecting the timing and sizes of replenishments of several items that are produced on the same capacity-limited machine. A constructive approach (see van Nunen and Wessels [44]) involves first selecting the lot sizes of each individual item separately, ignoring the capacity constraints (a set of problems each relatively easy to solve). Then, if the implied aggregate production hours exceed any of the capacities, the individual item solutions are adjusted until a feasible solution is achieved.

4.6. Local improvement methods

In contrast with the constructive procedures of the previous section, here we start with a *feasible* solution and improve upon it iteratively. The work center location problem can be solved in this fashion (see e.g. Hitchings and Cottam [19]). The initial feasible solution may be the existing layout in the manufacturing plant or it may be the solution of a constructive procedure. One method of local improvement is to attempt to switch pairs of centers; when an attractive switch is found, it is made and the process is continued until no further improvement can be achieved by any single switch. A relevant question is whether or not to continue searching when the number of possible switches remaining is extremely large; in other words, what is an appropriate stopping rule? Similarly, if repeated trials are made with a constructive method, one would like to have a reasonable rule for when to stop generating solutions. Randolph et al. [47] have examined the use of Bayesian (subjective probability) approaches to constructing such rules. Even when an exact enumeration method (for example, branch and bound) is used, one is often faced with the same type of stopping problem in that considerable further computation may lead to an insignificant improvement in the solution. Heuristic reasoning is likely to be useful in the development of stopping rules.

Further details on local improvement methods and neighbourhood search can be found in Müller–Merbach [38] and Wheeling [64]. Lin and Kernighan [30] use a local improvement approach in their method for solving the travelling salesman problem.

5. Interactive role of humans during the execution of a heuristic method

A properly designed interactive system affords the following possible advantages in the execution of a heuristic procedure:

(i) Capability of interactively guiding the search for a good solution, particularly making use of the experience of the decision maker (see e.g., Segal and Weinberger [52] and Tobin [56]). The human has the ability to suggest new solutions, as old ones are evaluated, with the drudgery of the detailed computations being left to the computer. Interactive procedures are frequently used in engineering design, an example being the AIDES (Adaptive Initial Design Synthesizer) program (Rudd et al. [50]).

(ii) Ability for the analyst to learn about the system, particularly how the mathematics itself behaves, with the intention of possibly modifying the model or heuristic procedure based on the observed behavior.

(iii) Facility for the decision maker himself to learn about the system and/or model and to gain confidence in the use of the model. This is emphasized by Little [31] as related to decision making in the marketing area.

We conclude this section with three remarks which, in fact, are relevant to methods obtaining exact solutions, as well as heuristic procedures:

(i) In many important problems it is difficult to obtain, a priori, from the decision maker an explicit statement of the relevant objective(s) and constraints. In an interactive mode these can be elicited by computer questioning as they are needed during the development of the solution. This is the spirit of the work on multiobjective decision problems by Roy and Bertier [49] and Zionts and Wallenius [69], as well as the paper by Little [31]. (See also Keeney and Raiffa [24].)

(ii) It is likely that an important factor in the effectiveness of an interactive system is the ability to graphically portray solutions, e.g., the display of a map in a vehicle routing system (Krolak et al. [26]).

(iii) The increasing availability of microcomputers should contribute to a substantial growth in the use of interactive methods.

6. Factors that may be important in the choice or testing of a heuristic method

In this section we attempt to identify some of the factors that may be significant in the choice of a particular heuristic method or in its evaluation. The factors are not meant to be independent; in particular, the first three are certainly interrelated. An important research area is the development of more explicit guidelines for designing and testing heuristics.

(a) Strategic (system design) versus tactical (system operation) problem–strategic decisions (e.g., capital investment in plant facilities) are one-time major decisions for which a rather elaborate analysis is justified, in contrast with tactical decisions (e.g., quality control, production scheduling, etc.) that are more minor and repetitive in nature. In addition, in tactical decisions there are opportunities for adapting to compensate for errors made in earlier decisions.

(b) Frequency of the decision – is it a one-off or a repeated decision problem? As mentioned earlier, a criterion of good *average* performance makes more sense with repetitive decisions.

(c) Amount of computational effort permitted – this factor obviously relates to the importance and frequency of the decision. More generally, the selection of a heuristic method is dependent upon the allowed solution time and/or monetary budget.

(d) Analytic qualifications of the decision maker – as discussed earlier it is desirable that the heuristic method be understandable to the decision maker. Thus, the latter's capacity to understand analytic reasoning should influence the choice of heuristic method, particularly if the method is to be used manually.

(e) Number of decision (controllable) variables – the number of such variables affects the need for a heuristic in the first place and should influence the choice of approach.

(f) Number of uncontrollable variables – this factor is particularly important in terms of testing heuristic methods. The larger the number of uncontrollable variables, the less likely that a probabilistic analysis is possible.

(g) Size of the problem – as an example, consider the travelling salesman problem. One heuristic method may be best for a case with 30 cities, whereas another heuristic may be better for a different case having 100 cities.

(h) Discrete versus continuous variables – most of the general literature on heuristic methods has been concerned with combinatorial problems involving integer variables. Perhaps different approaches are more appropriate when variables are continuous.

(i) Deterministic versus probabilistic variables – most of the literature has been concerned with large-scale deterministic models.

Again, perhaps it makes sense to devise very different heuristic procedures for probabilistic problems that also usually involve a more limited number of variables. Howard [20] and Naddor [39], among others, have looked at the effects of using just the first few moments of probability distributions, instead of the entire distributions, in dealing with specific decision problems.

7. Summary

In this paper we have discussed heuristic methods of solving well-defined mathematical problems. This has included the measurement of performance of heuristics, the different types of heuristic procedures and a listing of a number of factors that are likely to be relevant in the selection and testing of a heuristic method for any particular problem. Hopefully, this paper will have achieved its two objectives, namely (i) introducing decision analysts to the basic concepts of heuristic methods and (ii) stimulating analysts to undertake research on a number of important aspects of heuristic procedures. Related to the latter point, a challenging research topic is, given a particular problem and a maximum allowed computing time (or complexity), to find a solution method that gives a best possible solution within these conditions.

References

[1] E. Bartee, Problem solving with ordinal measurement, Manag. Sci. 17 (1971) 622–633.
[2] O. Bilde and R.V.V. Vidal, On the connections between locations and scheduling problems, in: M. Johnson and S. Ashour (Eds.), Simulation Councils Proc. Ser. 3 (2) (1973).
[3] J. Blackburn and R. Millen, Selecting a lot-sizing technique for a single-level assembly process, Working paper, School of Management, Boston University, Boston, MA (1978).
[4] E.H. Bowman, Consistency and optimality in managerial decision making, Manag. Sci. 9 (1963) 310–321.
[5] R.E. Burkard, Heuristische Verfahren zur Lösung quadratischer Zuordnungsprobleme, Z. Operations Res. 19 (1975) 183–193.
[6] D.J. Clough, An asymptotic extreme value sampling theory for estimation of a global maximum, CORS J. 7 (1969) 102–115.
[7] W.G. Cochran and G.M. Cox, Experimental Designs, 2nd Edition (Wiley, New York, 1957).
[8] R.W. Conway, Priority dispatching and work-in-process inventory in a job shop, J. Indust. Engrg. XVI (1965) 123–130.
[9] G. Cornuejols, M.L. Fisher and G.L. Nemhauser, Location of bank accounts to optimize float: an analytic study of exact and approximate algorithms, Manag. Sci. 23 (1977) 789–810.
[10] D. Dannenbring, Procedures for estimating optimal

solution values for large combinatorial problems, Manag. Sci. 23 (1977) 1273–1283.
[11] M.L. Fisher, Worst-cast analysis of heuristics, Working paper 78-02-06, Department of Decision Sciences, Wharton School, University of Pennsylvania, Philadelphia, PA (1978).
[12] G. Gallus, Heuristische Verfahren zur Lösung allgemeiner ganzzahliger linearen Optimierungsprobleme (ein Überblick), Z. Operations Res. 20 (1976) 89–104.
[13] M. Garey, R. Graham and D. Johnson, Performance guarantees for scheduling algorithms, Operations Res. 26 (1978) 3–21.
[14] A. Geoffrion, A priori error bounds for procurement commodity aggregation in logistics planning models, Naval Res. Logist. Quart. To appear
[15] B. Golden, A statistical approach to the TSP, Networks 7 (1977) 209–225.
[16] R. Graham, The combinatorial mathematics of scheduling, Scientific American 238 (1978) 124–132.
[17] A.C. Hax and H.C. Meal, Hierachial integration of production planning and scheduling, in: M.A. Geisler (Ed.), Studies in Management Sciences, Vol. I, Logistics (North-Holland, Amsterdam, 1975).
[18] C.R. Hicks, Fundamental Concepts in the Design of Experiments, 2nd Edition (Holt, Rinehart and Winston, New York, 1973).
[19] G.G. Hitchings and M. Cottam, An efficent heuristic procedure for solving the layout design problem, OMEGA 4 (1976) 205–214.
[20] R.A. Howard, Proximal decision analysis, Manag. Sci. 17 (1971) 507–541.
[21] T.A. Jenkyns, The efficacy of the greedy algorithm, Proc. 7th S.E. Conference on Combinatorics, Graph Theory and Computing (1976) 341–350.
[22] D.S. Johnson, Approximate algorithms for combinatorial problems, J. Comput. System Sciences 9 (1974) 256–278.
[23] R.M. Karp, The probabilistic analysis of some combinatorial search algorithms, in: J.F. Traub (Ed.), Algorithms and Complexity: New Directions and Recent Results (Academic Press, New York, 1976) 1–19.
[24] R.L. Keeney and H. Raiffa, Decisions with Multiple Objectives: Preferences and Value Tradeoffs (Wiley, New York, 1976) Chapter 3.
[25] H.K. Klein, Heuristische Entscheidungsmodelle. Neue Techniken des Programmierens und Entscheidens für das Management (Wiesbaden, 1971).
[26] P. Krolak, W. Felts and J. Nelson, A man–machine approach toward solving the generalized truck-dispatching problem, Transportation Sci. 6 (1972) 149–170.
[27] E. Lawler, Combinatorial Optimization (Holt, Rinehart and Winston, New York, 1976).
[28] R.C. Lee and J.M. Moore, CORELAP – computerized relationship layout planning, J. Indust. Engrg. 18 (1967) 195–200.
[29] S. Lin, Heuristic programming as an aid to network design, Networks 5 (1975) 33–44.
[30] S. Lin and B.W. Kernighan, An effective heuristic algorithm for the travelling-salesman problem, Operations Res. 21 (1973) 498–516.
[31] J.D.C. Little, Models and managers: the concept of a decision calculus, Manag. Sci. 16 (1970) 466–485.
[32] F.W. Matthäus, Heuristische Lösungs Verfahren für Lieferplansprobleme, Z. Operations Res. 19 (1975) 163–181.
[33] R.C. Meier, W.T. Newell and H.L. Pazer, Simulation in Business and Economics (Prentice-Hall, Englewood Cliffs, NJ 1969).
[34] G.C. Michael, A review of heuristic programming, Decision Sciences 3 (1972) 74–100.
[35] D.C. Montgomery, Design and Analysis of Experiments (Wiley, New York, 1976).
[36] H. Müller–Merbach, Heuristic methods: structures, applications, computational experience, in: R. Cottle and J. Krarup (Eds.), Optimization Methods for Resource Allocation (English Universities Press, 1974) 401–416.
[37] H. Müller–Merbach, Modelling techniques and heuristics for combinatorial problems, in: B. Roy (Ed.), Combinatorial Programming: Methods and Applications (Reidel, Dordrecht, 1975) 3–27.
[38] H. Müller–Merbach, Morphologie heuristischer Verfahren, Z. Operations Res. 20 (1976) 69–87.
[39] E. Naddor, Optimal and heuristic decisions in single– and multi-item inventory systems, Manag. Sci. 21 (1975) 1234–1249.
[40] E. Naddor, Dimensions in operations research, Operations Res. 10 (1962) 508–514.
[41] A. Newell, Heuristic programming: ill-structured problems, in: J.S. Aronofsky (Ed.), Progress in Operations Research, Vol. III (John Wiley, New York, 1969).
[42] T. Nicholson, Optimization in Industry, Vol. 1 Optimization Techniques (Longman Press, London, 1971) Chapter 10.
[43] J. Nievergelt, J.C. Farrar and E.M. Reingold, Computer Approaches to Mathematical Problems (Prentice-Hall, Englewood Cliffs, NJ, 1974) Chapter 2.
[44] J.A.E.E. van Nunen and J. Wessels, Multi-item lot size determination and scheduling under capacity constraints, European J. Operational Res. 2 (1978) 36–41.
[45] R. Peterson and E.A. Silver, Decision Systems for Inventory Management and Production Planning (Wiley, New York, 1979).
[46] G. Polya, How to Solve It (Doubleday, 1957).
[47] P. Randolph, G. Swinson and C.E. Ellingsen, Stopping rules for sequencing problems, Operations Res. 21 (1973) 1309–1315.
[48] D.J. Rosenkrantz, R.E. Stearns and P.M. Lewis, Approximate algorithms for the traveling salesperson problem, Proc. 15th Annual IEEE Symposium of Switching and Automata Theory (1974) 33–42.
[49] B. Roy and P. Bertier, La méthode electre II, une application au média planning (the electre II method, an application to media planning), in: M. Ross (Ed.), Operational Research '72 (North-Holland Elsevier, Amsterdam 1973) 291–302.
[50] D.F. Rudd, G.J. Powers and J.J. Siirola, Process Synthesis (Prentice-Hall, Englewood Cliffs, NJ, 1973).
[51] S. Sahni, General techniques for combinatorial approximation, Operations Res. 25 (1977) 920–936.
[52] M. Segal and D. Weinberger, Turfing, Operations Res. 25 (1977) 367–386.
[53] N. Shelley and G. Bryan, Human Judgements and Optimality (Wiley, New York, 1964).

[54] E. Silver, A control system for coordinated inventory replenishment, Internat. J. Prod. Res. 12 (1974) 647–671.
[55] H. Streim, Heuristische Lösungsverfahren: Versuch einer Begriffsklärung, Z. Operations Res. 19 (1975) 143–162.
[56] N.R. Tobin, Timesharing, interactive models and operational research, Operational Res. Quart. 27 (1976) 531–545.
[57] F. Tonge, The use of heuristic programming in management science, Manag. Sci. 7 (1961) 231–237.
[58] J.F. Traub, (Ed.), Algorithms and Complexity (Academic Press, New York, 1976).
[59] G.L. Urban, Building models for decision makers, Interfaces 4 (1974) 1–11.
[60] H.M. Wagner, Principles of Operations Research, 2nd Edition (Prentice-Hall, Englewood Cliffs, NJ, 1975).
[61] F. Weinberg and C.A. Zehnder (ed.), Heuristiche Planungsmethoden (Springer, Berlin, 1969).
[62] P. Weiner, Heuristics, Networks 5 (1975) 101–103.
[63] D. de Werra, Construction of school timetables by flow methods, INFOR 9 (1971) 12–22.
[64] R.F. Wheeling, Heuristic search: structured problems, in: J.S. Aronofsky (Ed.), Progress in Operations Research, Vol. III (Wiley, New York, 1969).
[65] D.J. White, Problems involving infrequent but significant contingencies, Operational Res. Quart. 20 (1969) 45–57.
[66] D.J. White, Decision Methodology (Wiley, New York, 1975).
[67] E. Wilson, Elementary scientific method, in: R.G. Brown (Ed.), Source Book in Production Management (Dryden Press, Hinsdale, IL, 1970) Chapter 14.
[68] R.E.D. Woolsey and H.S. Swanson, Operations Research for Immediate Applications (Harper and Row, New York, 1975).
[69] S. Zionts and J. Wallenius, An iterative programming method for solving the multiple criteria problem, Manag. Sci. 22 (1976) 652–663.