Chapter 1

The Effect of Changes in Judicial Rotation Policies

In this chapter we investigate how changes in judicial rotation policies affect the sentencing outcome in the judicial system. We first introduce where this problem arises from and what our goal is in §1.1. Then in §1.2 we concisely describe the dataset we have. In §1.3 we model the judicial system as a three-agent game of defendants, prosecutors and judges and use backward induction to analyze the optimal action of the three agents. Afterwards, we dig into one step of the three-agent game, specifically, the judge shopping step and solve for the optimal judge shopping strategy using dynamic programming in §1.4. Furthermore, we investigate how the sentence imposed on a certain defendant changes with judicial rotation parameters by simulation in §1.5. Finally we discuss our findings and what we plan to do next in §1.6 and conclude in §**??**.

1.1 Introduction

As mentioned in [2], South Carolina, as a state with no binding sentencing guideline, has notably less county-level variation in sentencing than elsewhere, including some states with sentencing guidelines such as Florida, Minnesota, and Washington. Qualitative interviews with 13 South Carolina trial judges to investigate sentencing processes suggest that an important reason for the uniformity is the special judicial rotation policy of South Carolina – Judges in South Carolina do not sit exclusively in one court or county, but instead, they travel among different counties holding court during the year[2]. The rotation not only leads to cross-pollination of ideas and sentencing norms, but more importantly, gives the defendants opportunities to judge shop, i.e., strategically delay the time of going to court to plead in front of lenient judges. Figures 1 and 2 in [2] show that there exists negative correlation between the leniency of judge (evaluated by the judges' incarceration rate and mean prison length imposed repectively) and the number of offenders sentenced, which further implies the existence of the judge shopping behavior.

We aim to build a model for the judge shopping process and gain insights about the factors that influence the system outcomes such as the expected length of sentences imposed on the defendants. Furthermore, we intend to investigate how the changes in judicial rotation policies – e.g., more frequent travel of the judges – affect the uniformity of statewide sentences using simulations built upon the judge shopping model.

1.2 Data

We have access to the dataset of criminal case sentences collected by [3], which covers 17,671 cases representing all offenders who were sentenced by South Carolina Circuit Courts for the fiscal year 2001 and convicted of a felony or serious misdemeanor carrying a maximum sentence of a year or more in prison. This dataset is based on the original sentence data compiled by the South Carolina Sentencing Commission (now disbanded) during the period it was attempting to establish advisory sentencing guidelines. The 46 counties in South Carolina are grouped into 16 judicial circuits and each circuit has a circuit court. We will not go into the details of the dataset here because this research problem is still undergoing and what is covered in this chapter does not require knowing all the data details.

1.3 Three-Agent Game

1.3.1 Setup

The judicial process of sentencing a case involves three agents: the judge, the prosecutor and the defendant. In the U.S., because the judicial system is acutely overloaded, the majority of criminal cases are resolved by plea bargains [5]. A plea bargain is an agreement between the prosecutor and the defendant – In order to get the defendants to plead guilty without a trial, the prosecutors usually offer to reduce or dismiss some charges against the defendant or recommend a relatively short sentence [5]. If an agreement is reached, when the defendant goes to court, he/she will plead guilty in front of a judge; otherwise, the defendant can plead not guilty and the case will be resolved by a trial. In South Carolina, because of the existence of judge rotation, the defendants can delay the time of going to court until a lenient judge comes to town.

The final sentence imposed on a defendant will be determined by the judge he/she chooses, the plea offer given by the prosecutor, and whether he/she accepts the plea offer (i.e., an agreement is reached) or refuses the plea offer and go to trial. To evaluate the expected sentence imposed on a defendant, we have to model the behaviors of the three agents. However, we have to simplify their behaviors because the real interactions among the three agents can be very complicated and hard to predict. Although their interactions can go back and forth, we simplify and flatten the timeline of decisions to be three steps.

Step 1

The defendant chooses the judge/decides when to go to the court. We assume the calendar of judges are scheduled by week. This is the practice in South Carolina. But in reality, sometimes there exist exceptions – e.g., sentences in the same week and same county are given by two or more judges. The reasons for the exceptions are unknown but urgent personal issues of judges can be a possibility. We also assume that the defendant can choose from consecutive weeks, while in reality, the defendant may have to wait for another several weeks to go to court after rejecting a court time proposed.

Step 2

The prosecutor makes the plea offer. We assume the plea offer is represented by the length of incarceration (e.g., 0 if the defendant is not incarcerated), ignoring the cost of non-incarceration punishments such as probation, fines and community service because we do not have those information in the dataset.

Step 3

The defendant decides whether to accept the plea offer or go to trial. In this step we assume the defendants are rational and risk neutral. They compare the expected cost of accepting the plea offer and that of going to trial to choose the option with smaller cost.

In the following section we reason backwards from the third step to the first step to solve for the optimal strategies of the three agents. Relevant assumptions and parameters are introduced in each step.

1.3.2 Backward induction

Step 3

In this step the defendant knows about the plea offer s and the judge he/she is going to face. The harshness level of the judge is denoted by $h \in \mathcal{H}$. Each case is characterized by two quantities – τ, the incarceration sentence if the defendant is convicted, and θ, the probability of conviction at trial – that are independent of the judge who presides over the trial. If going to trial, the defendant has to go through a longer judicial process, which will incur extra costs compared to accepting the plea offer. The extra cost is referred to as the trial cost of the defendant and denoted by c_d ($c_d > 0$).

The cost of accepting the plea offer is s and the cost of going to trial is the sum of expected length of incarceration at trial ($\theta\tau$) and the trial cost c_d. Thus the defendant accepts the offer if and only if

$$s \leq \theta\tau + c_d.$$

Step 2

In this step, the prosecutor needs to propose a plea offer that maximizes his/her utility. When the case is resolved by plea bargain, the utility is defined to be the length of incarceration imposed on the defendant. If the defendant goes to trial, the prosecutor also has to go through a longer judicial process, which will incur extra costs, which we refer to as the trial cost of the prosecutor and denote by c_p ($c_p > 0$). In this case, the utility is defined to be the expected length of incarceration at trial minus the trial cost of the prosecutor.

Since the information on the two sides is actually quite symmetrical and the negotiation of the plea offer can be back and forth in reality, we assume that the prosecutor knows quite well about the case characteristics (θ and τ) and also the defendant's cost functions (i.e., the rationale behind the plea-or-trial decision).

We further assume that judges have a range of plea offers that they can accept, and the lower bound and upper bound of the range are both functions of the harshness level h, denoted by $l(h)$ and $u(h)$, where $l(h) \leq u(h)$ and $l(h), u(h)$ are increasing in h.

The utility of the prosecutor is analyzed as follows.

- If $s > \theta\tau + c_d$, the defendant rejects the plea offer and goes to trial. The prosecutor gets utility $\theta\tau - c_p$.

- If $s < l(h)$ or $s > u(h)$, the judge disallows the plea offer and the case goes to trial. The prosecutor gets utility $\theta\tau - c_p$.

- If $s \leq \theta\tau + c_d$ and $l(h) \leq s \leq u(h)$, the judge allows the plea offer and the defendant accepts it. The prosecutor gets utility s.

Because the prosecutor can foresee the choice of the defendant and knows about the harshness level of the judge, he/she will choose the optimal plea offer $s^* = \max(\min(\theta\tau + c_d, u(h)), \theta\tau - c_p, l(h))$ that brings the maximum utility. The case goes to trial if $u(h) < \theta\tau - c_p$ or $l(h) > \theta\tau + c_d$, i.e., when the judge is too lenient so that prosecutors are not willing to give such lenient plea offers or the judge is too harsh so that defendants are not willing to take such harsh plea offers.

We further assume that the case $u(h) < \theta\tau - c_p$ won't happen (e.g., if we let $u(h) = h\theta\tau$ and assume $h \geq 1$) or even if it happens, the prosecutor has to comply with the judge and could not insist on bringing the case to trial. Then $s^* = \max(\min(\theta\tau + c_d, u(h)), l(h))$. The case goes to trial if $l(h) > \theta\tau + c_d$, i.e., when the judge is too harsh so that defendants are not willing to take such harsh plea offers.

If the case goes to trial ($l(h) > \theta\tau + c_d$), the defendant's cost is $\theta\tau + c_d$. Otherwise ($l(h) \leq \theta\tau + c_d$), the defendant's cost is $s^* = \min(\theta\tau + c_d, u(h))$. Thus the defendant's cost when facing a judge with harness level h is $\min(\theta\tau + c_d, u(h))$ no matter whether he goes to trial or accepts the plea offer.

Step 1

Denote the cost of the defendant by $u_d(\theta, \tau, c_d, h)$. From Step 2 we know that, $u_d(\theta, \tau, c_d, h) = \min(\theta\tau + c_d, u(h))$ no matter whether the defendant goes to trial or accepts a plea offer.

We assume whenever a defendant arrives at the system, he or she can see the schedule of judge assignments for the coming v (mnemonic for visibility) weeks, which is equivalent to assuming that the judge assignment for week $v + t$ will be announced at week t. The defendant either chooses the judge at current week or chooses to delay a week with a delay cost d and make final choice later. We assume that they have to make the choice within r (mnemonic for restrictiveness) weeks from the week they arrive at. We assume the harshness level of the judge assigned to each week is i.i.d. and follows the distribution p_h.

The optimal delaying strategy can be solved by dynamic programming, which we dig into in the next section.

1.4 Dynamic Judge Shopping

1.4.1 Problem Formulation

In this section we formulate the decision-making (judge shopping) process of a defendant as a Markov Decision Process[4]. At each time step (week in our problem), the process is in some state x, and the decision maker (defendant in our problem) can choose any action a available in the current state x. The process then either ends or moves to the next time step with a new state x', where the state transition probability $p(x'|x, a)$ is dependent on the current state and the action. The corresponding cost occurs to the decision maker at each time step.

We use Ω to denote the special state of process ending, i.e., when the defendant has made the choice and leaves the system. Other than that, the normal state is defined to be $x = (\mathbf{h}, \epsilon, n)$. $\mathbf{h} = [h_0, ..., h_{v-1}]$ is a v-dimensional vector that represents the schedule of judge assignments and $h_i \in \mathcal{H}$ i.i.d. follows distribution p_h. ϵ is a 2-dimensional vector, where ϵ_0 and ϵ_1 stand for the idiosyncratic shock (observed by the defendant but not by researchers) to the cost of choosing the current week h_0 or delaying one week respectively, and i.i.d follow a Gumbel Min (minimum extreme value type I) distribution with parameter $\mu = -\gamma\beta$ and $\beta = 1$ so that it has mean 0 and a fixed variance that simplifies the notation[1]. n is the number of judges/weeks the defendants can still choose from including the current week. When the defendant arrives at the system, n is equal to r,

[1] γ is Euler's constant. The definition and important results of the Gumbel Min random variable are introduced in Appendix A.

the maximum number of weeks the defendant can stay in the system. Then n decreases by one per week to $n = 1$, the last week the defendant can stay in the system.

The action space (the set of actions available) of the defendant in state x is

$$\mathcal{A}(x) = \begin{cases} \{0, 1\} & n > 1 \\ \{0\} & n = 1 \\ \phi & n \leq 0 \text{ or } x = \Omega. \end{cases}$$

where the action $j = 0$ represents choosing the judge in the current week, $j = 1$ represents delaying a week and ϕ means an empty set.

If the defendant chooses actions $j = 0$, he leaves the system and incurs an immediate cost

$$u_j(x) = u_d(\theta, \tau, c_d, h_0) + \epsilon_j.$$

For notational simplicity, we denote $u_d(\theta, \tau, c_d, h)$ by $u_d(h)$. The state transition probability under action 0 is

$$p(x'|x, a = 0) = \begin{cases} 1 & x' = \Omega \\ 0 & o.w. \end{cases}$$

If the defendant chooses actions $j = 1$, he goes to the next week with a different state and incurs an immediate cost

$$u_j(x) = d + \epsilon_j,$$

where d is delay cost for one week. The decision process ends if the defendant chooses action 0 which otherwise moves to a new state. The state transition probabilities under action 1 are

$$p(x'|x, a = 1) = p(\tilde{x}' = (\mathbf{h}', n')|\tilde{x} = (\mathbf{h}, n), a = 1)g(\epsilon'),$$

where $g(\cdot)$ is the PDF for the GumbelMin variable (Definition A.1.2), $\tilde{x}$ is the part of state that can be observed by outsiders/researchers and

$$p(\tilde{x}' = (\mathbf{h}', n')|\tilde{x} = (\mathbf{h}, n), a = 1) = \begin{cases} p_h(h'_{v-1}) & h'_i = h_{i+1} \; \forall i = 0, ..., v-2, n' = n - 1 \\ 0 & o.w. \end{cases}$$

The state transition probabilities imply the assumptions that the noise ϵ at the next time step is generated independently of the current state and the judge assigned to each week is independent of the judges assigned to other weeks.

1.4.2 Bellman's Equation

Now that we have defined the Markov Decision Process, we write down the corresponding Bellman's equation[1] for deriving the optimal strategy.

Let $v(x)$ be the value (cost) function of the defendant at state x and $A(x)$ be the optimal action to take. Let $v(\tilde{x}) = V(\mathbf{h}, n) = \mathbb{E}_\epsilon[v(x)] = \int v(x)g(\epsilon)d\epsilon$ denote the integrated value function of the defendant at the observable state $\tilde{x} = (\mathbf{h}, n)$. Defining $v(\Omega) = 0$, we have

$$v(x) = \min_{j \in \mathcal{A}(x)} \left\{ u_j(x) + \mathbb{E}[v(x')|x, a = j] \right\}.$$

when $n = 1$, $\forall\, \mathbf{h}, \epsilon$,

$$v(\mathbf{h}, \epsilon, 1) = u_d(h_0) + \epsilon_0, \tag{1.1}$$

$$V(\mathbf{h}, 1) = u_d(h_0), \tag{1.2}$$

$$A(\mathbf{h}, \epsilon, 1) = 0 \tag{1.3}$$

because the defendant has no other choices in this case, i.e., only having one week to choose from.

When $n > 1$,

$$\begin{aligned}
v(x) &= \min_{j \in \mathcal{A}(x)} \left\{ u_j(x) + \mathbb{E}_{x'}[v(x')|x, a = j] \right\} \\
&= \min \left\{ u_d(h_0) + \epsilon_0, d + \epsilon_1 + \iint v(x')p(x' = (\mathbf{h}', n')|x = (\mathbf{h}, n), a = 1)g(\epsilon')d\epsilon d\,\mathbf{h}' \right\} \\
&= \min \left\{ u_d(h_0) + \epsilon_0, d + \epsilon_1 + \int V(\mathbf{h}', n')p(x' = (\mathbf{h}', n')|x = (\mathbf{h}, n), a = 1)d\,\mathbf{h}' \right\} \\
&= \min \left\{ u_d(h_0) + \epsilon_0, d + \epsilon_1 + \mathbb{E}_{k \sim p_h} V([h_1, ..., h_{v-1}, k], n - 1) \right\}.
\end{aligned} \tag{1.4}$$

By the nature of Gumbel Min variables (Corollary A.2.3.2), we know $v(x)$ is the minimum of

$$GumbelMin(\mu - u_d(h_0), 1)$$

and

$$GumbelMin(\mu - d - \mathbb{E}_{k \sim p_h} V([h_1, ..., h_{v-1}, k], n - 1), 1),$$

which follows

$$GumbelMin(\log(\exp(\mu - u_d(h_0)) + \exp(\mu - d - \mathbb{E}_{k \sim p_h} V([h_1, ..., h_{v-1}, k], n - 1))), 1).$$

Thus

$$V(x) = \mathbb{E}_\epsilon(v(x))$$

$$
\begin{aligned}
&= -\log(\exp(\mu - u_d(h_0)) + \exp(\mu - d - \mathbb{E}_{k\sim p_h} V([h_1, ..., h_{v-1}, k], n-1))) - \gamma \\
&= -\log(\exp(-u_d(h_0)) + \exp(-d - \mathbb{E}_{k\sim p_h} V([h_1, ..., h_{v-1}, k], n-1))) - (\mu + \gamma) \\
&= -\log(\exp(-u_d(h_0)) + \exp(-d - \mathbb{E}_{k\sim p_h} V([h_1, ..., h_{v-1}, k], n-1))), \qquad (1.5)
\end{aligned}
$$

and $v(x) = u_d(h_0) + \epsilon_0$ (i.e., the $GumbelMin(\mu - u_d(h_0), 1)$ variable is the smaller one) with probability

$$
\frac{\exp(-u_d(h_0))}{\exp(-u_d(h_0)) + \exp(-d - \mathbb{E}_{k\sim p_h} V([h_1, ..., h_{v-1}, k], n-1))} = \frac{\exp(-u_d(h_0))}{\exp(-V(x))}. \qquad (1.6)
$$

From the analysis above, we know that for any $\mathbf{h}, n$

$$
\mathbb{P}(A(\mathbf{h}, \epsilon, n) = 0) = \frac{\exp(-u_d(h_0))}{\exp(-V(\mathbf{h}, n))}. \qquad (1.7)
$$

With the recursive formula (1.5) and the base case (1.1), we can evaluate the integrated value (cost) function for any state $(\mathbf{h}, n)$ and defendant (θ, τ, c_d, d). With the integrated value function, we can calculate the probability of delaying at each state by (1.7).

1.4.3 Value (Cost) Functions

In this section we try to evaluate the integrated value function $V(\mathbf{h}, n)$.

For $1 \le n \le v$, i.e., when all the choices that are available at the current state can be observed, we have a closed-form solution for $V(\mathbf{h}, n)$.

Lemma 1.4.1. *For $1 \le n \le v$ (when all possible choices are observable), we have*

$$
V(\mathbf{h}, n) = -\log\left(\sum_{i=0}^{n-1} \exp(-id - u_d(h_i))\right).
$$

Proof. This can be shown by mathematical induction.

- *Base Case*:

  When $n = 1$,

  $$
  v(\mathbf{h}, \epsilon, 1) = u_d(h_0) + \epsilon_0
  $$

  with probability 1. Thus

  $$
  V(\mathbf{h}, 1) = \mathbb{E}[v(\mathbf{h}, \epsilon, 1)] = u_d(h_0) = -\log\left(\exp(-u_d(h_0))\right).
  $$

- *Induction step*:

Suppose the claim holds for $1 \leq n \leq v-1$. Then by equations (1.4), (1.5) and (1.6), we have

$$
\begin{aligned}
& V(\mathbf{h}, n+1) \\
&= -\log(\exp(-u_d(h_0)) + \exp(-d - \mathbb{E}_{k\sim p_h} V([h_1, ..., h_{v-1}, k], n))) \\
&= -\log\left(\exp(-u_d(h_0)) + \exp\left(-d - \mathbb{E}_{k\sim p_h}\left[-\log\left(\sum_{i=1}^{n} \exp(-(i-1)d - u_d(h_i))\right)\right]\right)\right) \\
&= -\log\left(\exp(-u_d(h_0)) + \exp(-d)\exp\left(\log\left(\sum_{i=1}^{n} \exp(-(i-1)d - u_d(h_i))\right)\right)\right) \\
&= -\log\left(\exp(-u_d(h_0)) + \exp(-d)\left(\sum_{i=1}^{n} \exp(-(i-1)d - u_d(h_i))\right)\right) \\
&= -\log\left(\exp(-u_d(h_0)) + \sum_{i=1}^{n} \exp(-id - u_d(h_i))\right) \\
&= -\log\left(\sum_{i=0}^{n} \exp(-id - u_d(h_i))\right).
\end{aligned}
$$

$\square$

For $n > v$ (i.e., when the defendants can choose from more judges than he can observe), we look at two specific cases first.

$$
\begin{aligned}
& V(\mathbf{h}, v+1) \\
&= -\log\left[\exp(-u_d(h_0)) + \exp(-d - \mathbb{E}_{k\sim p_h} V([h_1, ..., h_{v-1}, k], v))\right] \\
&= -\log\Bigg[\exp(-u_d(h_0)) \\
&\qquad + \exp\left(-d + \mathbb{E}_{k\sim p_h} \log\left(\sum_{i=1}^{v-1} \exp(-(i-1)d - u_d(h_i)) + \exp(-(v-1)d - u_d(k))\right)\right)\Bigg] \\
&= -\log\left[\exp(-u_d(h_0)) + \exp\left(\mathbb{E}_{k\sim p_h} \log\left(\sum_{i=1}^{v-1} \exp(-id - u_d(h_i)) + \exp(-vd - u_d(k))\right)\right)\right].
\end{aligned}
$$

$$
\begin{aligned}
& V(h, v+2) \\
&= -\log\left[\exp(-u_d(h_0)) + \exp(-d - \mathbb{E}_{k\sim p_h} V([h_1, ..., h_{v-1}, k], v+1))\right] \\
&= -\log\Bigg[\exp(-u_d(h_0)) + \exp\Bigg(-d + \mathbb{E}_{k\sim p_h} \log\Bigg[\exp(-u_d(h_1)) + \exp\Bigg(\mathbb{E}_{k_1\sim p_h} \\
&\qquad \log\left(\sum_{i=2}^{v-1} \exp(-(i-1)d - u_d(h_i)) + \exp(-(v-1)d - u_d(k)) + \exp(-vd - u_d(k_1))\right)\Bigg)\Bigg]\Bigg)\Bigg]
\end{aligned}
$$

$$= -\log\Bigg[\exp(-u_d(h_0)) + \exp\Bigg(\mathbb{E}_{k\sim p_h}\log\Bigg[\exp(-d-u_d(h_1)) + \exp\Bigg(\mathbb{E}_{k_1\sim p_h}$$
$$\log\Bigg(\sum_{i=2}^{v-1}\exp(-id-u_d(h_i)) + \exp(-vd-u_d(k)) + \exp(-(v+1)d-u_d(k_1))\Bigg)\Bigg)\Bigg]\Bigg)\Bigg].$$

From the two cases above, we can observe the pattern for general $v < n \le r$. We will not write down the closed-form formula for it because there is not a clean and clear way to write it down. But we can evaluate the value function for any $(\mathbf{h}, n)$ using the recursive formula

$$\exp(-V(\mathbf{h}, n)) = \exp(-u_d(h_0)) + \exp(-d - \mathbb{E}_{k\sim p_h} V([h_1, ..., h_{v-1}, k], n-1)). \tag{1.8}$$

1.4.4 Monotonicity and Convexity of the Value Function

Monotonicity and convexity results can be shown for the integrated value function $V(\mathbf{h}, n)$.

Monotonicity

In the following we show the integrated value function $V(\mathbf{h}, n)$ is non-increasing in n, the number of available weeks, for any given observable judge assignment $\mathbf{h}$. This makes intuitive sense because the integrated value function is the minimum expected cost the defendant can achieve at the state $(\mathbf{h}, n)$. Greater n means that the defendant has more choices, which will not result in greater cost when facing the same judge assignment.

Lemma 1.4.2. *$V(\mathbf{h}, n)$ is non-increasing in n.*

Proof. We prove the statement by induction.

- *Base Case*:

  For $n = 1$, $V(\mathbf{h}, n) = \mathbb{E}_\epsilon(u_d(h_0) + \epsilon_0) = u_d(h_0)$. For $n = 2$, there are two cases: either $v = 1$, then we have

  $$V(\mathbf{h}, 2) = \mathbb{E}_\epsilon \min\{u_d(h_0) + \epsilon_0, d + \epsilon_1 + \mathbb{E}_{k\sim p_h}[u_d(k)]\} \le \mathbb{E}_\epsilon[u_d(h_0) + \epsilon_0] = u_d(h_0) = V(\mathbf{h}, 1)$$

  or if $v > 1$, we have

  $$V(\mathbf{h}, 2) = \mathbb{E}_\epsilon \min\{u_d(h_0) + \epsilon_0, d + \epsilon_1 + u_d(h_1)\} \le \mathbb{E}_\epsilon[u_d(h_0) + \epsilon_0] = u_d(h_0) = V(\mathbf{h}, 1).$$

  In both cases we have $V(\mathbf{h}, 2) \le V(\mathbf{h}, 1)$.

- *Induction step*:

Assuming $V(\mathbf{h}, n) \leq V(\mathbf{h}, n-1)$, we can show $V(\mathbf{h}, n+1) \leq V(\mathbf{h}, n)$.

$$\begin{aligned} V(\mathbf{h}, n+1) &= \mathbb{E}_\epsilon \min\{u_d(h_0) + \epsilon_0, d + \epsilon_1 + \mathbb{E}_{k\sim p_h}[V([h_1, h_2, \cdots, h_{v-1}, k], n)]\} \\ &\leq \mathbb{E}_\epsilon \min\{u_d(h_0) + \epsilon_0, d + \epsilon_1 + \mathbb{E}_{k\sim p_h}[V([h_1, h_2, \cdots, h_{v-1}, k], n-1)]\} = V(\mathbf{h}, n), \end{aligned}$$

where in the second line, we used our induction assumption that

$$V([h_1, h_2, \cdots, h_{v-1}, k], n) \leq V([h_1, h_2, \cdots, h_{v-1}, k], n-1).$$

□

Since $V(\mathbf{h}, n)$ is non-increasing in n and is positive, the limit exists

$$\lim_{n\to\infty} V(\mathbf{h}, n) = V^*(\mathbf{h}).$$

We can take $n \to \infty$ on both sides of the recursion

$$\exp(-V(\mathbf{h}, n)) = \exp(-u_d(h_0)) + \exp(-d - \mathbb{E}_{k\sim p_h} V([h_1, ..., h_{v-1}, k], n-1)),$$

and get

$$\exp(-V^*(\mathbf{h})) = \exp(-u_d(h_0)) + \exp(-d - \mathbb{E}_{k\sim p_h} V^*([h_1, ..., h_{v-1}, k])), \tag{1.9}$$

which can possibly be used to solve for $V^*(\mathbf{h})$ recursively. For example when $v = 1$, we will have

$$-V^*(h_0) = \log\left[e^{-u_d(h_0)} + e^{-d-\mathbb{E}_{k\sim p_h} V^*(k)}\right]. \tag{1.10}$$

It gives that

$$-V^*(h_0) = \log\left[e^{-u_d(h_0)} + e^{-d+A}\right], \tag{1.11}$$

where A is a constant that satisfies the fixed point equation

$$A = \mathbb{E}_{h_0\sim p_h}\left[\log\left[e^{-u_d(h_0)} + e^{-d+A}\right]\right]. \tag{1.12}$$

From (1.7) we know that the probability of a defendant choosing the current judge at the state $(\mathbf{h}, n)$ is

$$\mathbb{P}(A(\mathbf{h}, \epsilon, n) = 0) = \frac{\exp(-u_d(h_0))}{\exp(-V(\mathbf{h}, n))},$$

and the probability at the state $(\mathbf{h}, n-1)$ is

$$\mathbb{P}(A(\mathbf{h}, \epsilon, n-1) = 0) = \frac{\exp(-u_d(h_0))}{\exp(-V(\mathbf{h}, n-1))}.$$

The monotonicity the value function implies that as n decreases, i.e., as it approaches the deadline of making the final choice, when facing the same judge assignment $\mathbf{h}$, the probability of the defendant choosing the current judge increases (to be more rigorous, is non-decreasing). This shows that our model captures the intuition that the defendant has a higher tendency to rush into a choice when it becomes more urgent.

Convexity

From Lemma 1.4.2, we know $V(\mathbf{h}, n)$ is non-increasing in n. Then its expectation over $\mathbf{h}$, $\mathbb{E}_{\mathbf{h}}[V(\mathbf{h}, n)]$ is also non-increasing in n. In the following we show the convexity of $\mathbb{E}_{\mathbf{h}}[V(\mathbf{h}, n)]$, which means it decreases more slowly as n increases. This implies that on average, increasing the number of possible choices (n) has diminishing marginal returns. Note that it does not hold in general for any $\mathbf{h}$. Numerical experiments show that $V(\mathbf{h}, n) - V(\mathbf{h}, n+1)$ can be greater or less than $V(\mathbf{h}, n+1) - V(\mathbf{h}, n+2)$ depending on the value of $\mathbf{h}$.

Lemma 1.4.3. *Let $\bar{V}(n) = \mathbb{E}_{\mathbf{h}} V(\mathbf{h}, n)$. We have $\bar{V}(n) - \bar{V}(n+1) \leq \bar{V}(n-1) - \bar{V}(n)$.*

Proof.

$$\begin{aligned}
&V(\mathbf{h}, n) - V(\mathbf{h}, n+1) \\
=&\mathbb{E}_\epsilon \min\{u_d(h_0) + \epsilon_0, d + \epsilon_1 + \mathbb{E}_{k\sim p_h}[V([h_1, h_2, \cdots, h_{v-1}, k], n-1)]\} \\
&- \mathbb{E}_\epsilon \min\{u_d(h_0) + \epsilon_0, d + \epsilon_1 + \mathbb{E}_{k\sim p_h}[V([h_1, h_2, \cdots, h_{v-1}, k], n)]\} \\
=&\mathbb{E}_\epsilon \Big[\min\{u_d(h_0) + \epsilon_0, d + \epsilon_1 + \mathbb{E}_{k\sim p_h}[V([h_1, h_2, \cdots, h_{v-1}, k], n-1)]\} \\
&- \min\{u_d(h_0) + \epsilon_0, d + \epsilon_1 + \mathbb{E}_{k\sim p_h}[V([h_1, h_2, \cdots, h_{v-1}, k], n)]\} \Big] \\
\leq&\mathbb{E}_\epsilon \Big[\mathbb{E}_{k\sim p_h}[V([h_1, h_2, \cdots, h_{v-1}, k], n-1)] - \mathbb{E}_{k\sim p_h}[V([h_1, h_2, \cdots, h_{v-1}, k], n)] \Big] \\
=&\mathbb{E}_{k\sim p_h}[V([h_1, h_2, \cdots, h_{v-1}, k], n-1)] - \mathbb{E}_{k\sim p_h}[V([h_1, h_2, \cdots, h_{v-1}, k], n)].
\end{aligned}$$

Thus

$$\begin{aligned}
\bar{V}(n) - \bar{V}(n+1) &= \mathbb{E}_{\mathbf{h}}[V(\mathbf{h}, n) - V(\mathbf{h}, n+1)] \\
&\leq \mathbb{E}_{\mathbf{h}}[\mathbb{E}_{k\sim p_h}[V([h_1, h_2, \cdots, h_{v-1}, k], n-1)] - \mathbb{E}_{k\sim p_h}[V([h_1, h_2, \cdots, h_{v-1}, k], n)]] \\
&= \bar{V}(n-1) - \bar{V}(n).
\end{aligned}$$

□

1.4.5 Probability of Choosing the Judge in a Certain Week

With the integrated value function, the defendant can make the optimal delaying decision based on a simple threshold strategy, which is to choose the judge in the current week if

$$u_d(h_0) + \epsilon_0 \leq d + \epsilon_1 + \mathbb{E}_{k\sim p_h} V([h_1, ..., h_{v-1}, k], n-1)$$

and delay a week otherwise at the state $(\mathbf{h}, n)$ $(n > 1)$.

Based on the strategy, we can calculate the probability that a defendant (characterized by (θ, τ, c_d)) chooses the i^{th} judge among the choices he has, when he can observe the coming v weeks and has to make a choice within r weeks. We denote the probability by $p_{i,v,r}$. Although the parameters (θ, τ, c_d) are not included in the notation of the probability for the sake of simplicity, we should keep in mind that the probability differs from defendant to defendant.

Lemma 1.4.4. *Let $\tilde{\mathbf{h}}$ be the judge assignment of the $r+v-1$ weeks since the defendant arrives at the system. With $\tilde{\mathbf{h}}_{i:j}$ being the subarray $[\tilde{h}_i, ..., \tilde{h}_j]$, $\mathbf{h} = \tilde{\mathbf{h}}_{0:(r-1)} \in \mathcal{R}^r$ represents the r judges the defendant can possibly choose. The probability that the defendant ends up choosing judge h_i among $\mathbf{h}$ is*

$$p_{i,v,r}(\mathbf{h}) = \frac{\exp(-id - u_d(h_i))}{\exp(-V(\mathbf{h}_{0:(v-1)}, r))} \times$$
$$\exp\left(-\sum_{j=0}^{\min(i-1,r-v-1)} \mathbb{E}_{k\sim p_h}[V([\tilde{h}_{j+1}, ..., \tilde{h}_{j+v-1}, k], r-j-1) - V(\tilde{\mathbf{h}}_{(j+1):(j+v)}, r-j-1)]\right).$$

Proof. Since the defendant can observe the judge assignment for the coming v weeks and he has to make the choice within r weeks since entering the system, then he will observe at most $r+v-1$ weeks of judge assignments, i.e., $\tilde{h}$, when being in the system. Thus $p_{i,v,r}$, the probability that the defendant ends up choosing judge h_i $(0 \leq i \leq r-1)$ can be seen as a function of $\tilde{\mathbf{h}}$.

For $0 \leq i \leq r-1$, by (1.5) and (1.7), we have[2]

$$\begin{aligned}
&p_{i,v,r}(\tilde{\mathbf{h}}) \\
&= \prod_{j=0}^{i-1} P(A(\tilde{\mathbf{h}}_{j:(j+v-1)}, \epsilon, r-j) = 1) P(A(\tilde{\mathbf{h}}_{i:(i+v-1)}, \epsilon, r-i) = 0) \\
&= \prod_{j=0}^{i-1} \frac{\exp(-d - \mathbb{E}_{k\sim p_h} V([\tilde{h}_{j+1}, ..., \tilde{h}_{j+v-1}, k], r-j-1))}{\exp(-V(\tilde{\mathbf{h}}_{j:(j+v-1)}, r-j))} \frac{\exp(-u_d(\tilde{h}_i))}{\exp(-V(\tilde{\mathbf{h}}_{i:(i+v-1)}, r-i))}
\end{aligned}$$

[2] We use the conventions $\prod_{j=0}^{-1} b_j = 1$ and $\sum_{i=0}^{-1} b_j = 0$, where b_j represents any sequence.

$$
\begin{aligned}
&= \frac{\exp(-id - u_d(\tilde{h}_i))}{\exp(-V(\tilde{\mathbf{h}}_{0:(v-1)}, r))} \prod_{j=0}^{i-1} \frac{\exp(-\mathbb{E}_{k\sim p_h} V([\tilde{h}_{j+1}, ..., \tilde{h}_{j+v-1}, k], r-j-1))}{\exp(-V(\tilde{\mathbf{h}}_{(j+1):(j+v)}, r-j-1))} \\
&= \frac{\exp(-id - u_d(h_i))}{\exp(-V(\mathbf{h}_{0:(v-1)}, r))} \times \\
&\quad \exp\left(-\sum_{j=0}^{i-1} \mathbb{E}_{k\sim p_h} [V([\tilde{h}_{j+1}, ..., \tilde{h}_{j+v-1}, k], r-j-1) - V(\tilde{\mathbf{h}}_{(j+1):(j+v)}, r-j-1)] \right) \\
&= \frac{\exp(-id - u_d(h_i))}{\exp(-V(\mathbf{h}_{0:(v-1)}, r))} \times \\
&\quad \exp\left(-\sum_{j=0}^{\min(i-1, r-v-1)} \mathbb{E}_{k\sim p_h} [V([\tilde{h}_{j+1}, ..., \tilde{h}_{j+v-1}, k], r-j-1) - V(\tilde{\mathbf{h}}_{(j+1):(j+v)}, r-j-1)] \right).
\end{aligned}
$$

where the ϵ's in the first line are i.i.d. copies instead of the same variable, and the last equation comes from Lemma 1.4.1. Note that $p_{i,v,r}(\tilde{\mathbf{h}})$ does not depend on $\tilde{\mathbf{h}}_{r:(r+v-2)}$. Thus we can write $p_{i,v,r}$ as a function of the sub-array of $\tilde{\mathbf{h}}$, which is $\mathbf{h}$.

$$
\begin{aligned}
&p_{i,v,r}(\mathbf{h}) = \mathbb{E}_{\tilde{\mathbf{h}}_{r:(r+v-2)}}[p_{i,v,r}(\tilde{\mathbf{h}})] \\
&= \frac{\exp(-id - u_d(h_i))}{\exp(-V(\mathbf{h}_{0:(v-1)}, r))} \times \\
&\quad \exp\left(-\sum_{j=0}^{\min(i-1, r-v-1)} \mathbb{E}_{k\sim p_h} [V([\tilde{h}_{j+1}, ..., \tilde{h}_{j+v-1}, k], r-j-1) - V(\tilde{\mathbf{h}}_{(j+1):(j+v)}, r-j-1)] \right).
\end{aligned}
$$

□

As shown in the proof above, we can write $p_{i,v,r}$ as a function of any vector $\mathbf{h}$ with a dimension greater than r and we have $p_{i,v,r}(\mathbf{h}) = p_{i,v,r}(\mathbf{h}_{0:(r-1)})$. Also, for $\mathbf{h} \in \mathcal{R}^{r+1}, i \geq 1$, we have

$$
p_{i,v,r+1}(\mathbf{h}) = P(A(\mathbf{h}_{0:(v-1)}, \epsilon, r+1) = 1) p_{i-1,v,r}(\mathbf{h}_{1:r}). \tag{1.13}
$$

Corollary 1.4.4.1. *When $v = r$, the probability that the defendant ends up choosing judge h_i among $\mathbf{h} \in \mathcal{R}^r$, the r judges that he can possibly choose, is*

$$
p_{i,v,r}(\mathbf{h}) = \frac{\exp(-id - u_d(h_i))}{\exp(-V(\mathbf{h}, r))} = \frac{\exp(-id - u_d(h_i))}{\sum_{k=0}^{r-1} \exp(-kd - u_d(h_k))} = \frac{\exp(-id - u_d(h_i))}{\sum_{k=0}^{v-1} \exp(-kd - u_d(h_k))}.
$$

Proof. By Lemma 1.4.4, when $r - v = 0$,

$$
p_{i,v,r}(\mathbf{h}) = \frac{\exp(-id - u_d(h_i))}{\exp(-V(\mathbf{h}, r))}.
$$

By Lemma 1.4.1, we have

$$\exp(-V(\mathbf{h}, r)) = \sum_{k=0}^{r-1} \exp(-kd - u_d(h_k)).$$

□

1.5 Sentence Imposed on A Certain Defendant

In §1.3 and §1.4 we have introduced the setup of the judge shopping game in the judicial system and analyzed the optimal action of the three agents involved. Now we try to analyze how the system outcome, specifically, the sentence imposed on a certain defendant, changes with these parameters.

1.5.1 Derivation of the Expectation and Variance of the Sentence Length

From Step 2 in the backward induction (§ 1.3.2), we know that a defendant characterized by (θ, τ, c_d), if choosing a judge with harshness level h, will obtain an expected sentence of $\theta\tau$ if going to trial ($l(h) > \theta\tau + c_d$) and a sentence of $\min(\theta\tau + c_d, u(h))$ if accepting the plea offer ($l(h) \leq \theta\tau + c_d$). Denoting the expected sentence by $e(\theta, \tau, c_d, h)$, we have

$$e(\theta, \tau, c_d, h) = \begin{cases} \theta\tau & l(h) > \theta\tau + c_d \\ \min(\theta\tau + c_d, u(h)) = u_d(\theta, \tau, c_d, h) & l(h) \leq \theta\tau + c_d. \end{cases}$$

Since $l(h)$ is increasing in h, for a certain defendant (θ, τ, c_d), there exists a corresponding harshness threshold h^* such that the case goes to trial when facing judges harsher than the threshold.

For a certain defendant (θ, τ, c_d) entering the system and facing a random judge assignment, the expectation of the sentence imposed on him is

$$M(v, r) = \mathbb{E}_{\mathbf{h}} \left[\sum_{i=0}^{r-1} e(h_i, \theta, \tau, c_d) p_{i,v,r}(\mathbf{h}) \right], \tag{1.14}$$

where the term inside the expectation, $\sum_{i=0}^{r-1} e(h_i, \theta, \tau, c_d) p_{i,v,r}(\mathbf{h})$, is the expected sentence imposed on a defendant (θ, τ, c_d) facing the judge assignment $\mathbf{h}$. If we consider a special case that all the defendants in the judicial system are of the same type (same (θ, τ, c_d) in our model), the quantity $M(v, r)$ can be seen as the expectation of the length of sentences imposed on all the defendants entering the system.

The variance of the sentence imposed on him is

$$\sigma^2(v, r) = \mathbb{E}_{\mathbf{h}} \left[\sum_{i=0}^{r-1} e^2(h_i, \theta, \tau, c_d) p_{i,v,r}(\mathbf{h}) \right] - M(v, r)^2. \tag{1.15}$$

Similarly, this quantity can also be seen as the variance of the length of sentences imposed on all the defendants entering the system in the special case that all the defendants in the system have the same (θ, τ, c_d).

1.5.2 Simulation Setup

Before evaluating the expectation and variance of the length of sentences by simulations, we need to decide the form of $l(h)$ and $u(h)$. We choose $u(h) = h\theta\tau$ and $l(h) = (h - \delta)\theta\tau$, which assume that the maximum and minimum plea offer acceptable to the judge with harshness level h increases linearly with the case severity (characterized by the expected sentence at trial $\theta\tau$) and the range $(u(h) - l(h) = \delta\theta\tau)$ – which is independent of h – also becomes larger for more severe cases. Since it does not make sense for a judge to require the plea offer to be even smaller than the expected sentence at trial $(\theta\tau)$, we restrict h to be greater than 1.

When we only consider one type of defendant (θ, τ, c_d), no matter how many different judges we have, they can be classified into three types.

- $h < \frac{c_d}{\theta\tau} + 1$

  In this case we have $\theta\tau + c_d > u(h) > l(h)$. The defendant accepts the plea offer $u(h) = h\theta\tau$.

- $\frac{c_d}{\theta\tau} + 1 \leq h \leq \frac{c_d}{\theta\tau} + 1 + \epsilon$

  In this case we have $l(h) \leq \theta\tau + c_d \leq u(h)$. The defendant accepts the plea offer $\theta\tau + c_d$.

- $h > \frac{c_d}{\theta\tau} + 1 + \epsilon$

  In this case we have $l(h) > \theta\tau + c_d$. The defendant rejects the plea offer and goes to trial.

In our simulations, we consider a certain defendant with $\theta\tau = 1$ and $c_d = 2$.[3] Then we choose $\epsilon = 0.5$ and $h \in \mathcal{H} = \{1.5, 3, 4.5\}$ so that we cover the judges of all the three types in the simulation and the minimum acceptable plea offer of the most lenient judge is no less than $\theta\tau$. We start with a uniform distribution of $\mathbf{h}$ over the set $\mathcal{H}$.

Our main goal is to investigate how the expectation and variance of the sentence imposed on this defendant change with v and r. Aside from that, we also look at how the delay cost d influences the effects of v and r.

1.5.3 Observations from Simulation

Expectation of the sentence

When the restrictiveness r is 10 weeks, as v increases from 1 to 10 (we only consider $v \leq r$ since the cases $v \geq r$ are equivalent to $v = r$), as shown in Figure 1.1, the expectation of the sentence

[3] If $\frac{c_d}{\theta\tau}$ is too small, the judge shopping phenomenon can be negligible because the sentences given by judges of the three types are all very similar.

decreases with v and converges to a certain value when v is large enough. When the delay costs d increases, the expectation converges faster but its amplitude increases.

These phenomena can be explained intuitively. For a certain r, as v increases, the defendants have more information when making the decisions at each week, which naturally leads to better decisions, i.e., shorter sentences, for the defendants. As the delay cost increases, delaying too many weeks becomes almost impossible and thus the effective number of choices the defendants have actually decreases. Then the amplitude of expectation should be larger because that corresponds to worse choices. Also, the expectation should converge faster because more visibility won't help any more when the extra observable choices are impossible to be chosen.

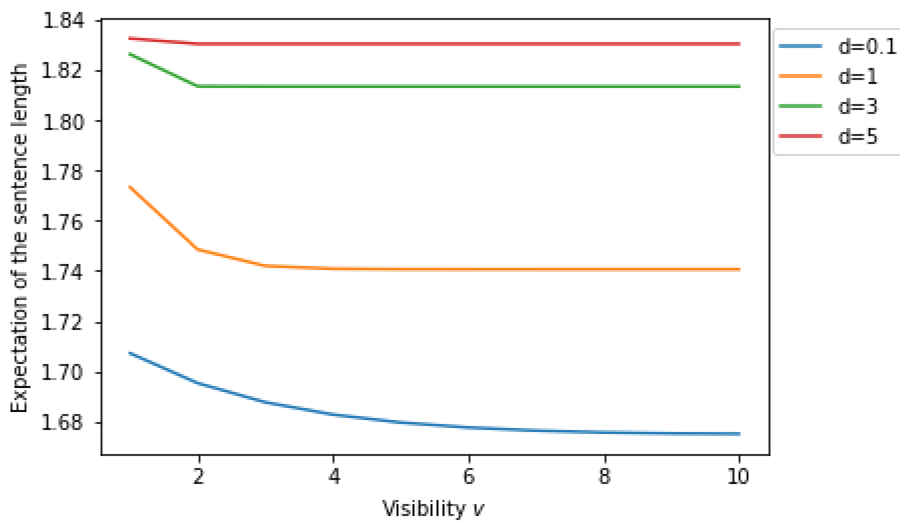

Figure 1.1: The expectation of the sentence length $M(v, r)$ versus the visibility v when $r = 10$ under different delay costs.

Similar observations exist for the effect of r. Let the visibility r be 1 week, as r increases from 1 to 10, as shown in Figure 1.2, the expectation of the sentence decreases with r and converges to a certain value when r is large enough. When the delay costs d increases, the expectation converges faster but its amplitude increases. When $v = r = 1$, the expectation is the same for different delay costs because in that case no judge shopping can exist and every defendant has only one choice.

Explanations for these phenomena are also similar to that in the case of v. For a certain v, as r increases, the defendants have more possible choices, which also naturally leads to better decisions, i.e., shorter sentences, as more information (greater v) will do. As for the delay cost, the amplitude of expectation becomes larger as the delay cost increases, for the same reason as in the case of v that the number of "meaningful" choices the defendants have decreases as d increases. Also, the expectation converges faster when d is greater because more choices will not help any more when the extra available choices are impossible to be chosen.

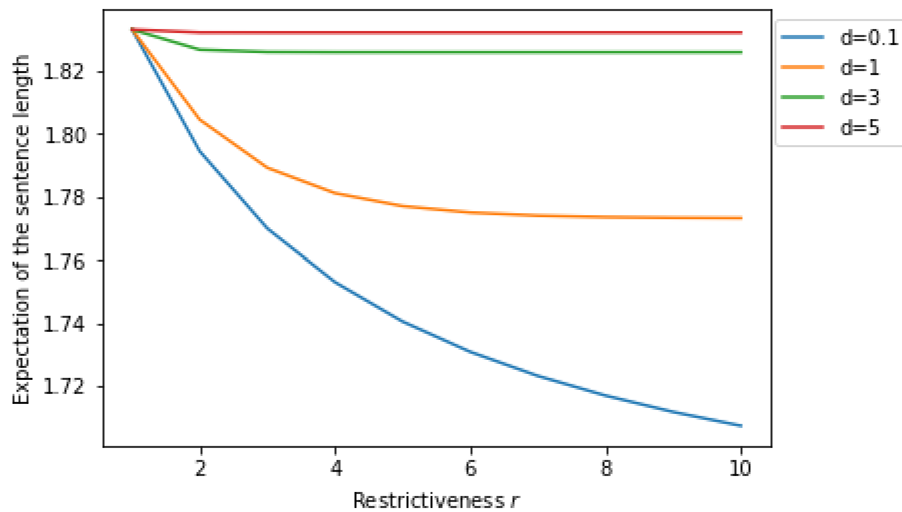

Figure 1.2: The expectation of the sentence length $M(v, r)$ versus the restrictiveness r when $v = 1$ under different delay costs.

To have a more comprehensive exhibition of how the expectation of sentence length decreases with v and r, we look at $v = 1, 2, \cdots, 10$ and for each v calculate the expectation of sentence length with $r = v, \cdots, 10$. The 10 expectation versus restrictiveness curves are shown in the same plot (Figure 1.3).

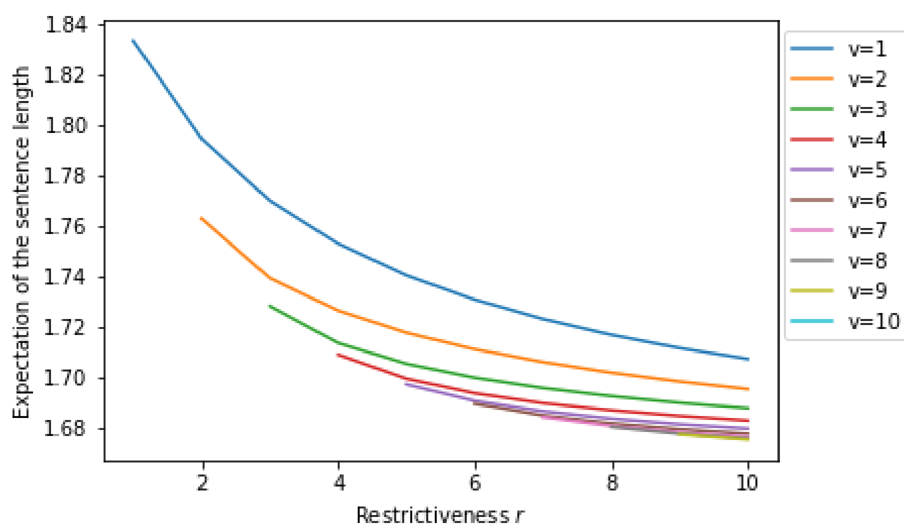

Figure 1.3: The expectation of the sentence length $M(v, r)$ versus the restrictiveness r for different visibility $v = 1, \cdots, 10$ ($d = 0.1$).

Variance of the sentence

Under the same set of parameters, we have the same observations about the variance as for the expectation.

When the restrictiveness r is 10 weeks, as v increases from 1 to 10, as shown in Figure 1.4, the variance of the sentence length decreases with v and converges to a certain value when v is large enough. When the delay costs d increases, the variance converges faster but its amplitude increases.

Similarly, when the visibility r is 1 week, as r increases from 1 to 10, as shown in Figure 1.5, the variance of the sentence length decreases with r and converges to a certain value when r is large enough. When the delay costs d increases, the variance converges faster but its amplitude increases.

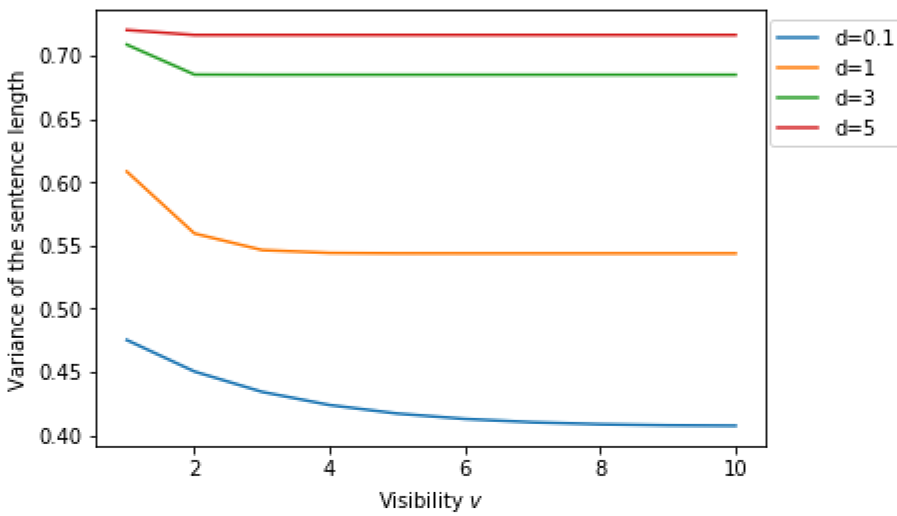

Figure 1.4: The variance of the sentence length $M(v, r)$ versus the visibility v when $r = 10$ under different delay costs.

Figure 1.6 shows how the variance changes with both v and r (the delay cost d is set to 0.1 by default).

The reason for these phenomena is similar to that in the case of expectation. Because we choose the uniform distribution of the judge harshness level in our simulation, when $v = r = 1$, i.e., the defendant has no freedom of judge shopping and thus has equal probability of choosing a judge from each type, then the distribution of sentence should be an uniform distribution over the three sentences given by the three types of judges. When the defendant can make better choices (as r or v increases especially when d is small), the uniform distribution is shifted towards a distribution with more weights on more lenient judges, which results in the decrease of variance.

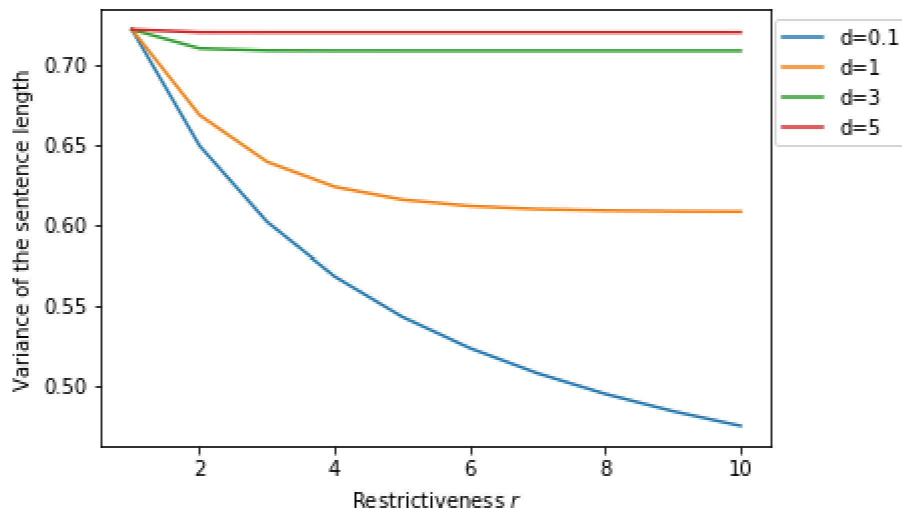

Figure 1.5: The variance of the sentence length $M(v, r)$ versus the restrictiveness r when $v = 1$ under different delay costs.

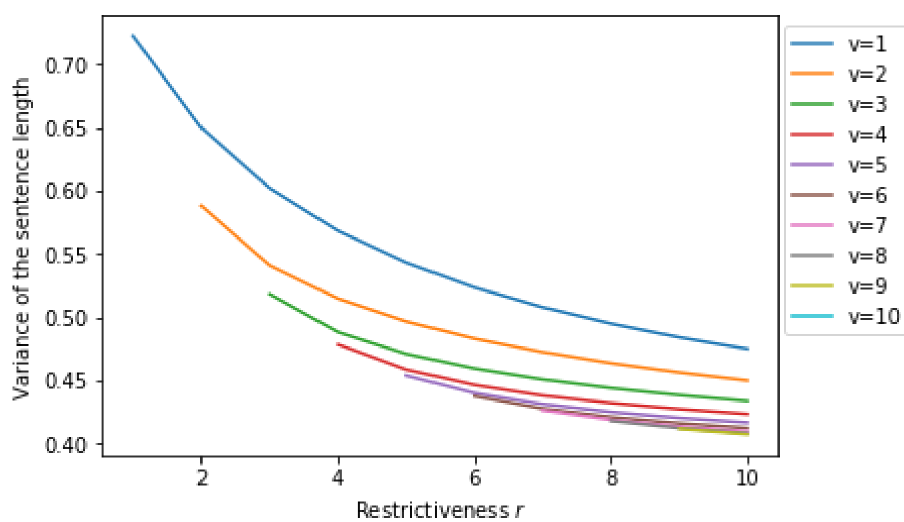

Figure 1.6: The variance of the sentence length $M(v, r)$ versus the restrictiveness r for different visibility $v = 1, \cdots, 10$ ($d = 0.1$).

Sensitivity to the distribution of judge harshness

The above observations about the expectation and variance of the sentence length suggest that in our setting, i.e., in a judicial system with only one type of defendants and an uniform distribution of judge harshness level, "easier" judge shopping – whether caused by more visibility of judge assignment or more available choices – lowers the expectation and variance of sentences imposed on the defendants in the system.

However, these observations are sensitive to the distribution of judge harshness level.

If we put more density (see (1.16) for the distribution of h) on the most harsh judge ($h = 4.5$) who happens to give the shortest sentence in our setting because the defendant will choose to go to trial when facing this judge, as shown in Figure 1.7, "easier" judge shopping leads to higher expectation of the sentence even though Lemma 1.4.2 suggests that it leads to lower cost of the defendant.[4] This is because the worst option (the most harsh judge) happens to achieve the shortest sentence and is a very common choice (with probability 0.8 by construction) when no judge shopping exists. Thus deviation from the default status under no judge shopping leads to increase in the expected sentence length.

$$P(h) = \begin{cases} 0.1 & h = 1.5 \\ 0.1 & h = 3 \\ 0.8 & h = 4.5, \end{cases} \tag{1.16}$$

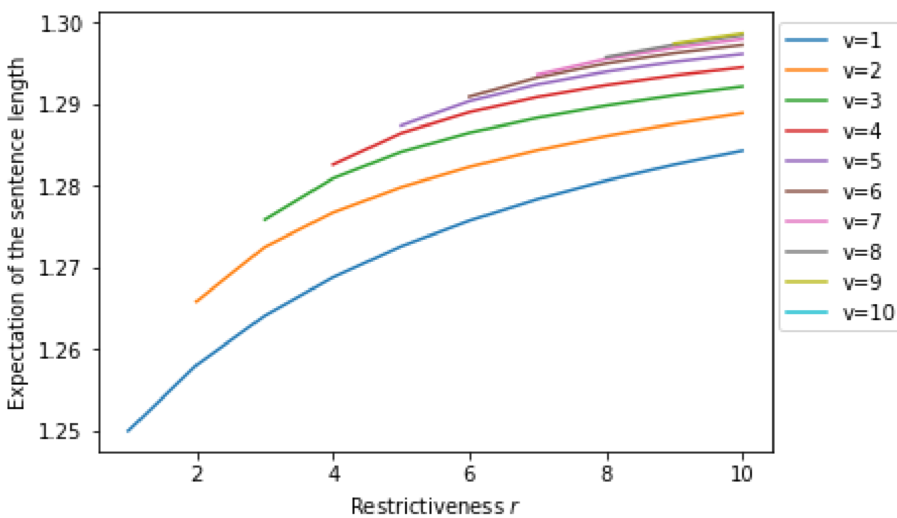

Figure 1.7: The expectation of the sentence length $M(v, r)$ versus the restrictiveness r for different visibility $v = 1, \cdots, 10$ under the distribution of judge harshness as defined in (1.16), $d = 0.1$.

If we put more density (see (1.17) for the distribution of h) on the judge who gives the longest sentence, which is the judge with harness level $h = 3$ in our setting, as opposed to Figure 1.6, Figure 1.8 shows that the variance of the sentence length increases with r and v. The reason is explained in the following paragraph.

[4]Lemma 1.4.2 only shows that the cost is non-increasing as r increases. But the monotonicity w.r.t. v comes naturally from the optimality of dynamic programming – when we can observe $\mathbf{h} \in \mathcal{H}^v$ it is sub-optimal to only use $\mathbf{h}_{0:v-2}$ to make decisions, which is equivalent to the case with visibility $v - 1$.

Since in this setting most judges have the harshness level $h = 3$, when $v = r = 1$, i.e., the defendant has no freedom of judge shopping and thus has equal probability of choosing a judge from each type, the distribution of sentence should be a discrete distribution over a set of three sentences where most (0.8) of the density is given to the longest sentence. Although this distribution has a high expectation, its variance is relatively low. However, when the defendant can make better choices (as r or v increases), this distribution of sentence shifts towards a distribution with more weight on the lenient judge ($h = 1.5$), which results in a more uniform distribution compared to the no-shopping case and thus leads to the increase of variance.

$$P(h) = \begin{cases} 0.1 & h = 1.5 \\ 0.8 & h = 3 \\ 0.1 & h = 4.5, \end{cases} \tag{1.17}$$

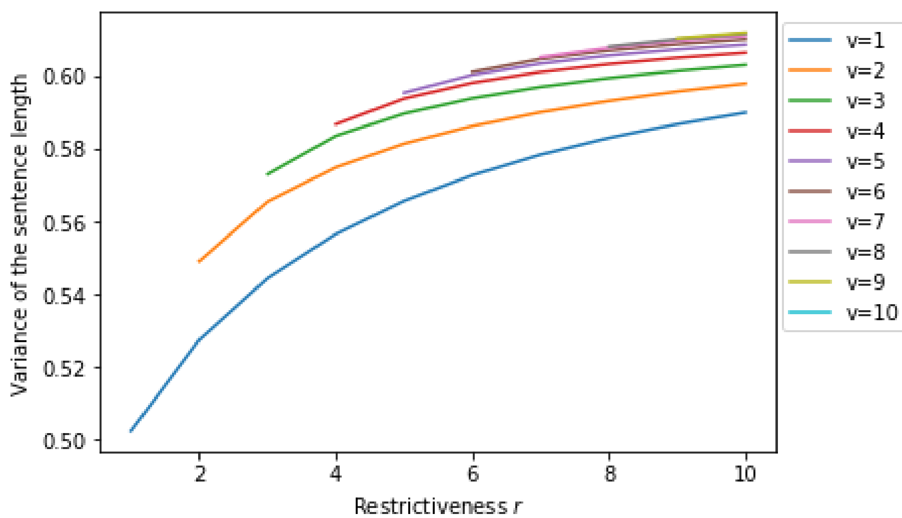

Figure 1.8: The variance of the sentence length $M(v, r)$ versus the restrictiveness r for different visibility $v = 1, \cdots, 10$ under the distribution of judge harshness as defined in (1.17), $d = 0.1$.

1.6 Next Steps

The remaining steps are mainly simulations.

1.6.1 Simulation 1

We want to look at the relationship between the number of defendants sentenced by a certain judge and the mean sentence length given by the judge.

Description

Since a defendant has to make a choice within r weeks and can only observe the coming v weeks, the arriving rate faced by a certain judge is decided by the judge schedule within r weeks before it and v weeks after it. Consider a judge assignment $\mathbf{h} = [h_0, ..., h_{r-1}, ..., h_{r+v-2}, ..., h_{2r-2}]$ and look at the judge h_{r-1}. Note that $[h_{r+v-1}, ..., h_{2r-2}]$ doesn't actually have influence. We include it only for simplicity of notification.

Let $(\theta_{i,j}, \tau_{i,j}), i = 0, 1, ...N_j$ be the defendants coming to the system at week $j, j = 0, ..., r-1$, where N_j follows a Poisson distribution with rate λ and $\theta_{i,j}\tau_{i,j}$ follows another distribution. Let $I_{i,j}$ be the indicator random variable for whether the defendant $(\theta_{i,j}, \tau_{i,j})$ chooses judge h_{r-1}. Thus $P(I_{i,j} = 1) = p_{r-1-j,v,r}(\mathbf{h}_{j:(j+r-1)}, \theta_{i,j}, \tau_{i,j})$. Note that $\theta_{i,j}, \tau_{i,j}$ are not listed in the arguments of the function $p_{i,v,r}$ in Lemma 1.4.4 but they do affect the function.

Then the number of defendants who choose judge h_{r-1} is

$$N = \sum_{j=0}^{r-1} \sum_{i=1}^{N_j} I_{i,j}.$$

The average sentence of defendants who choose judge h_{r-1} is

$$S = \frac{\sum_{j=0}^{r-1} \sum_{i=1}^{N_j} e(h_{r-1}, \theta_{i,j}, \tau_{i,j}, c_d) I_{i,j}}{\sum_{j=0}^{r-1} \sum_{i=1}^{N_j} I_{i,j}}.$$

The expected arriving rate is

$$\mathbb{E}[N] = \sum_{j=0}^{r-1} \lambda \mathbb{E}_{\theta,\tau,\mathbf{h}\,/h_{r-1}}[p_{r-1-j,v,r}(\mathbf{h}_{j:(j+r-1)}, \theta, \tau)]$$

and expected average sentence is

$$\mathbb{E}_{\theta_{i,j},\tau_{i,j},\mathbf{h}\,/h_{r-1}}[S],$$

where $\mathbf{h}\,/h_{r-1}$ represents all the elements in h except for h_{r-1}.

We want to calculate/simulate the two quantities for different value of h_{r-1} and expect to see that $\mathbb{E}[N]$ decreases with h_{r-1} and $\mathbb{E}[S]$ increases with h_{r-1}. When we look at the relationship between $\mathbb{E}[S]$ and $\mathbb{E}[N]$, we expect to see negative correlation.

Challenges and issues

For different value of $\mathbf{h}, \theta, \tau$, we need to recalculate $p_{r-1-j,v,r}(\mathbf{h}_{j:(j+r-1)}, \theta, \tau)$, which includes calculating the value function V. The computation cost can be very high especially for large v and r. Besides, if we consider continuous density for $\theta\tau$ and h, the two quantities can only be approximated by simulations instead of being calculated based on formula, which further increase the computation cost.

Besides, we may want to look at how the relationship changes with v, r, d, c_d. Thus I think we do want to compute it as efficiently as possible.

Another challenge is to take into consideration the capacity constraint, i.e., the maximum number of cases that can be sentenced by a judge per week. Because there are no closed-form formula for $\mathbb{E}[N]$ as a function of h_{r-1}, it is hard to impose the capacity constraint by shifting the area of the $\mathbb{E}[N]$ vs. h curve. Since now we are tackling the problem with simulations, we can consider simulating the decision of each defendant and incorporating the capacity constraint naturally in the decision process – in which case we need to decide what will happen when a defendant doesn't get the judge he wants.

1.6.2 Simulation 2

We want to simulate the South Carolina judicial system with 46 counties grouped into 16 judicial circuits to look into how judge rotation affects the county-level variance of sentences. After that, we can look into how the county-level variance of sentence changes as the amount of judge rotation changes.

Description

We'll start with one county and should be able to simulate the mean and variance of sentences in the county given a certain judge assignment. This part should be somehow to similar to Simulation 1 described above and will have similar computational challenges.

After that, we need to build the judge rotation between different counties/circuits. Suppose we have in total n judges and each judge has his home county/circuit. Given the amount of judge rotation, say the percentage of time a judge stays in home county/circuit, we need to be able to generate the judge assignments in all counties. After generating the judge assignments, we can deal with each county and evaluate the county-level variance of sentence – before which we need to refer to the literature to see how that is quantified. Then we can investigate the relationship between the amount of rotation and the county-level variances.

We should start with simple model (say fewer counties and circuits) and arbitrary parameters to get a feel of how the system works. Then we will try to estimate parameters from the South Carolina data and also literature – this part is full of question marks.

Challenges and issues

There can be many details we need to make choices on when building the simulation system. Computational cost and real parameter estimation/choosing are also big challenges.

Appendix A

Gumbel (Min) variable

A.1 Definition

In the following we define Gumbel variable and GumbelMin variable.

Definition A.1.1. *Let X follows Gumbel distribution $Gumbel(\mu, \beta)$. The cumulative distribution function of X is*

$$F(x; \mu, \beta) = e^{-e^{-(x-\mu)/\beta}},$$

and the probability distribution function is

$$f(x; \mu, \beta) = \frac{1}{\beta} e^{-e^{-(x-\mu)/\beta}} e^{-(x-\mu)/\beta}$$

Definition A.1.2. *A random variable X follows $GumbelMin(\mu, \beta)$ if $-X$ follows $Gumbel(\mu, \beta)$ as defined in Definition A.1.1.*

A.2 Properties

Then we show some properties of Gumbel and GumbelMin random variables.

A.2.1 Mean and Variance

A $Gumbel(\mu, \beta)$ variable X has mean $\mathbb{E}(X) = \mu + \gamma\beta$, where $\gamma \approx 0.5772$ is the Euler constant, and standard deviation $\sigma = \beta\pi/\sqrt{6}$ [6].

A.2.2 Constant Addition

Lemma A.2.1. *For $X \sim Gumbel(\mu, \beta)$, we have $U = X + \alpha$ follows $Gumbel(\mu + \alpha, \beta)$.*

Proof.

$$P(u \leq t) = P(x \leq t - \alpha) = F(t - \alpha; \mu, \beta) = e^{-e^{-(t-\alpha-\mu)/\beta}}.$$

□

Corollary A.2.1.1. *If X follows $GumbelMin(\mu, \beta)$. Then $X + \alpha$ follows $GumbelMin(\mu - \alpha, \beta)$.*

Proof. Since $-X$ follows *Gumbel*(μ, β), by Lemma A.2.1, $-X - \alpha$ follows *Gumbel*$(\mu - \alpha, \beta)$. Thus $X + \alpha$ follows $GumbelMin(\mu - \alpha, \beta)$ by definition. □

A.2.3 Maximum of IID Gumbel Variables

Lemma A.2.2. *Consider $U_i \overset{i.i.d.}{\sim}$ Gumbel$(\mu_i, \beta), 1 \leq i \leq J$. We have*

$$\max\{U_1, ..., U_J\} \sim \mathit{Gumbel}\left(\beta \log\left(\sum_{i=1}^{J} e^{\mu_i/\beta}\right), \beta\right).$$

Proof.

$$\begin{aligned} P(\max\{U_1, ..., U_J\} \leq t) &= \prod_{i=1}^{J} P(U_i \leq t) = \prod_{i=1}^{J} e^{-e^{-(t-\mu_i)/\beta}} = e^{-\sum_{i=1}^{J} e^{-(t-\mu_i)/\beta}} \\ &= e^{-e^{\log(e^{-t/\beta}\sum_{i=1}^{J} e^{\mu_i/\beta})}} = e^{-e^{-t/\beta + \log(\sum_{i=1}^{J} e^{\mu_i/\beta})}} \end{aligned}$$

□

Lemma A.2.3. *Consider $U_i \overset{i.i.d.}{\sim}$ Gumbel$(\mu_i, \beta), i = 1, 2$. We have*

$$P(U_1 > U_2) = \frac{\exp(\mu_1/\beta)}{\exp(\mu_1/\beta) + \exp(\mu_2/\beta)}.$$

Proof.

$$\begin{aligned} P(U_1 > U_2) &= \int f(u_2; \mu_2, \beta) du_2 \int_{u_2}^{\infty} f(u_1; \mu_1, \beta) du_1 \\ &= \int \frac{1}{\beta} e^{-e^{-(u_2-\mu_2)/\beta}} e^{-(u_2-\mu_2)/\beta} (1 - e^{-e^{-(u_2-\mu_1)/\beta}}) du_2 \\ &= 1 - \int \frac{1}{\beta} e^{-e^{-u_2/\beta}(e^{\mu_2/\beta} + e^{\mu_1/\beta})} e^{-(u_2-\mu_2)/\beta} du_2 \\ &= 1 - \int \frac{1}{\beta} e^{-e^{-u_2/\beta}(e^{\mu_2/\beta} + e^{\mu_1/\beta})} e^{-u_2/\beta} (e^{\mu_2/\beta} + e^{\mu_1/\beta}) \frac{e^{\mu_2/\beta}}{e^{\mu_2/\beta} + e^{\mu_1/\beta}} du_2 \\ &= 1 - \frac{e^{\mu_2/\beta}}{e^{\mu_2/\beta} + e^{\mu_1/\beta}} \\ &= \frac{e^{\mu_1/\beta}}{e^{\mu_2/\beta} + e^{\mu_1/\beta}}. \end{aligned}$$

□

By Lemma A.2.2 and A.2.3, we have the following corollary.

Corollary A.2.3.1. *Consider* $U_i \overset{i.i.d.}{\sim} Gumbel(\mu_i, \beta), i = 1, ..., J$. *We have*

$$P(U_1 > \max\{U_2, ..., U_j\}) = \frac{\exp(\mu_1/\beta)}{\sum_{i=1}^{J} \exp(\mu_i/\beta)}.$$

Proof.

$$P(U_1 > \max\{U_2, ..., U_j\}) = \frac{\exp(\mu_1/\beta)}{\exp(\mu_1/\beta) + \exp(\beta \log(\sum_{i=2}^{J} e^{\mu_i/\beta})/\beta)} = \frac{\exp(\mu_1/\beta)}{\sum_{i=1}^{J} \exp(\mu_i/\beta)}.$$

□

A.2.4 Minimum of IID GumbelMin Variables

Corollary A.2.3.2. *Consider* $U_i \overset{i.i.d.}{\sim} GumbelMin(\mu_i, \beta), i = 1, ..., J$. *We have*

$$P(U_1 < \min\{U_2, ..., U_j\}) = \frac{\exp(\mu_1/\beta)}{\sum_{i=1}^{J} \exp(\mu_i/\beta)},$$

and

$$\min\{U_1, ..., U_J\} \sim GumbelMin\left(\beta \log\left(\sum_{i=1}^{J} e^{\mu_i/\beta}\right), \beta\right),$$

with mean $-\beta \log(\sum_{i=1}^{J} e^{\mu_i/\beta}) - \gamma\beta$ *and standard deviation* $\sigma = \beta\pi/\sqrt{6}$.

Proof. By Corollary A.2.3.1, we have

$$\begin{aligned} P(U_1 < \min\{U_2, ..., U_j\}) &= P(-U_1 > -\min\{U_2, ..., U_j\}) \\ &= P(-U_1 > \max\{-U_2, ..., -U_j\}) \\ &= \frac{\exp(\mu_1/\beta)}{\sum_{i=1}^{J} \exp(\mu_i/\beta)}. \end{aligned}$$

By Lemma A.2.2,

$$\min\{U_1, ..., U_J\} = -\max\{-U_1, ..., -U_J\} \sim GumbelMin\left(\beta \log\left(\sum_{i=1}^{J} e^{\mu_i/\beta}\right), \beta\right),$$

which has mean $-\left(\left(\beta \log\left(\sum_{i=1}^{J} e^{\mu_i/\beta}\right) + \gamma\beta\right)\right)$ and standard deviation $\sigma = \beta\pi/\sqrt{6}$ [6]. □

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