Soluciones de los problemas de la primera relación

Notaciones asintóticas

Eficiencia de los Algoritmos

0.i) Demostrar:

$$f \in \Theta(g) \Leftrightarrow \exists c, d \in \Re^{+}, n_0 \in \aleph / n \geq n_0, d \cdot g(n) \leq f(n) \leq c \cdot g(n)$$

$\Rightarrow$

Supongamos que $f \in \theta(g) \Rightarrow \begin{cases} f \in O(g) \Rightarrow \exists n_1 \in \aleph, c > 0 / n \geq n_1,\ f(n) \leq c \cdot g(n) \\ f \in \Omega(g) \Rightarrow \exists n_2 \in \aleph, d > 0 / n \geq n_2,\ f(n) \geq d \cdot g(n) \end{cases}$

Sea n_0 = máximo $\{n_1, n_2\}$ => se da simultáneamente 1 y 2.

Por tanto: $n \geq n_0, \quad d \cdot g(n) \leq f(n) \leq c \cdot g(n)$

$\Leftarrow$ $\quad \exists n_0 \in \aleph, c \in \Re^{+} / n \geq n_0,\ f(n) \leq c \cdot g(n) \Rightarrow f \in O(g)$

$\exists n_0 \in \aleph, d \in \Re^{+} / n \geq n_0,\ f(n) \geq d \cdot g(n) \Rightarrow f \in \Omega(g)$

Luego $f \in \Theta(g)$

0.ii) Demostrar:

$$f(n) \in O(g(n)) \Leftrightarrow g(n) \in \Omega(f(n))$$

$\Rightarrow$	$f(n) \in O(g(n)) \Rightarrow \quad \exists n_0 \in \aleph, c > 0 \,/\, n \geq n_0,\ f(n) \leq c \cdot g(n) \Rightarrow$ $\Rightarrow g(n) \geq \frac{1}{c} \cdot f(n)\ con\ \frac{1}{c} > 0 \Rightarrow g(n) \in \Omega(f(n))$

$\Leftarrow$	$g(n) \in \Omega(f(n)) \Rightarrow \quad \exists n_0 \in \aleph, d > 0 \,/\, n \geq n_0, g(n) \geq d \cdot f(n) \Rightarrow$ $\Rightarrow f(n) \leq \frac{1}{d} \cdot g(n)\ con\ \frac{1}{d} > 0 \Rightarrow f(n) \in O(g(n))$

1.i) Demostrar:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \in \Re^{+} \Rightarrow f(n) \in \Theta(g(n))$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = L \in \Re^{+} \Rightarrow \forall \varepsilon > 0, \exists n_0 \in \aleph \,/\, n \geq n_0, \left| \frac{f(n)}{g(n)} - L \right| < \varepsilon \Rightarrow -\varepsilon < \frac{f(n)}{g(n)} - L < \varepsilon \Rightarrow$$

$$\Rightarrow L - \varepsilon < \frac{f(n)}{g(n)} < L + \varepsilon \Rightarrow (L - \varepsilon) \cdot g(n) < f(n) < (L + \varepsilon) \cdot g(n)$$

$$(g(n) \in \Re^{+}) \quad Sea\ \varepsilon < L,\ d = (L - \varepsilon) > 0,\ c = (L + \varepsilon) > 0$$

$$Entonces\ d \cdot g(n) < f(n) < c \cdot g(n) \Rightarrow (demostración\ ejercicio\ 0.i) \Rightarrow f(n) \in \Theta(g(n))$$

1.ii) Demostrar:

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = 0 \Rightarrow f(n)\in O(g(n)) \ \ pero \ \ f(n)\notin\Theta(g(n))$$

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = 0 \Rightarrow \forall\varepsilon>0,\ \exists n_0 \in\aleph / n\geq n_0,\ \left|\frac{f(n)}{g(n)}\right| < \varepsilon \underset{(f(n),g(n)\in\Re^+)}{\Rightarrow} \frac{f(n)}{g(n)} < \varepsilon \Rightarrow$$

$$\Rightarrow f(n) < \varepsilon\cdot g(n) \Rightarrow f(n)\in O(g(n))$$

$$Supongamos \ \ f(n)\in\Theta(g(n)) \Rightarrow f(n)\in\Omega(g(n)) \Rightarrow \exists c>0, n_0\in\aleph / n\geq n_0,\ f(n)\geq c\cdot g(n$$

$$\Rightarrow \frac{f(n)}{g(n)}\geq c$$

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} \geq \lim_{n\to\infty} c \Rightarrow 0\geq c \quad (absurdo!)$$

$$Luego \ \ f(n)\notin\Theta(g(n))$$

1.iii) Demostrar:

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = +\infty \Rightarrow f(n) \in \Omega(g(n)) \ \ pero \ \ f(n) \notin \Theta(g(n))$$

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = +\infty \Rightarrow \lim_{n\to\infty}\frac{g(n)}{f(n)} = 0 \Rightarrow$$

$$\begin{cases} g(n) \in O(f(n)) \Rightarrow \underset{(ejercicio\ 0.ii)}{\Rightarrow} f(n) \in \Omega(g(n)) \\ g(n) \notin \Omega(f(n)) \Rightarrow f(n) \notin O(g(n)) \underset{(ejercicio\ 0.i)}{\Rightarrow} f(n) \notin \Theta(g(n)) \end{cases}$$

1.iv) Demostrar:

$$\Theta(f^2(n)) = \Theta(f(n))^2$$

$$\Theta(f(n))^2 = \Theta(f(n)) \cdot \Theta(f(n)) = \{t : \aleph \rightarrow \Re \;/\; \exists g, h \in \Theta(f(n)), \exists n \in \aleph \;/\; \forall n \geq n_0, t(n) = g$$

a)

$$\Theta(f^2(n)) < \Theta(f(n))^2$$

$$Sea\ t(n) \in \Theta(f^2(n)) \Rightarrow \exists n_0 \in \aleph, c, d \in \Re^+ / n \geq n_0 \Rightarrow d \cdot f^2(n) \leq t(n) \leq c \cdot f^2(n) \underset{c,d,t(n)>0}{\Rightarrow}$$

$$\Rightarrow \sqrt{d} \cdot f(n) \leq \sqrt{t(n)} \leq \sqrt{c} \cdot f(n) \Rightarrow \sqrt{t(n)} \in \Theta(f(n))$$

$$Como\ t(n) = \sqrt{t(n)} \cdot \sqrt{t(n)} \Rightarrow t(n) \in \Theta(f(n))$$

1.iv) Demostrar:

$$\Theta(f^2(n)) = \Theta(f(n))^2$$

b)

$$\Theta(f(n))^2 < \Theta(f^2(n))$$

$$Sea\ t(n) \in \Theta(f(n))^2 \Rightarrow \exists r(n), s(n) \in \Theta f(n)) \,/\, t(n) = r(n)\cdot s(n)$$

$$Si\ r(n) \in \Theta(f(n)) \Rightarrow \exists n_0^r \in \aleph, \begin{matrix} c_r > 0 \\ d_r > 0 \end{matrix} / \ n \geq n_0^r, c_r \cdot f(n) \leq r(n) \leq d_r \cdot f(n)$$

$$Si\ s(n) \in \Theta(f(n)) \Rightarrow \exists n_0^s \in \aleph, \begin{matrix} c_s > 0 \\ d_s > 0 \end{matrix} / \ n \geq n_0^s, c_s \cdot f(n) \leq s(n) \leq d_s \cdot f(n)$$

$$Sea\ n_0 = \max(n_0^s, n_0^r), entonces:$$

$$c_r \cdot c_s \cdot f(n) \cdot f(n) \leq r(n) \cdot s(n) \leq d_r \cdot d_s \cdot f(n) \cdot f(n) \underset{\substack{c = c_r \cdot c_s \\ d = d_r \cdot d_s}}{\Rightarrow} c \cdot f^2(n) \leq t(n) \leq d \cdot f^2(n) \Rightarrow$$

$$\Rightarrow t(n) \in \Theta(f^2(n))$$

$$Demostradas\ ambas\ inclusiones,\ se\ tiene\ \Theta(f(n))^2 = \Theta(f^2(n))$$

2.a) Demostrar: $\forall k > 0, k \cdot f \in O(f)$

$$k \cdot f \leq k \cdot f, \forall n \geq 1 \quad \Rightarrow \quad k \cdot f \in O(f)$$

2.b) Demostrar: $f \in O(g), h \in O(g) \Rightarrow (f+h) \in O(g)$

$$f \in O(g) \Rightarrow \exists n_1 \in \aleph, c_1 > 0 \ / \ \forall n \geq n_1,\ f(n) \leq c_1 \cdot g(n)$$

$$h \in O(g) \Rightarrow \exists n_2 \in \aleph, c_2 > 0 \ / \ \forall n \geq n_2,\ h(n) \leq c_2 \cdot g(n)$$

$$Sea\ n_0 = \max\{n_1, n_2\}$$

$$\forall n \geq n_0,\ f(n) + h(n) \leq c_1 \cdot g(n) + c_2 \cdot g(n) = (c_1 + c_2) \cdot g(n) \underset{d=(c_1+c_2)>0}{=} d \cdot g(n)$$

$$Luego\ f + h \in O(g)$$

2.b') Demostrar: $f \in O(g) \Rightarrow (f+g) \in O(g)$

$$g \underset{(2.a)}{\in} O(g), \; ya \; que \; g = 1 \cdot g$$

$$Luego \quad \left. \begin{array}{l} f \in O(g) \\ g \in O(g) \end{array} \right\} \Rightarrow (f+g) \in O(g)$$

2.c) Demostrar: $f \in O(g),\ g \in O(h) \Rightarrow f \in O(h)$

$$f \in O(g) \Rightarrow \exists n_1 \in \aleph, c_1 > 0 \ /\ \forall n \geq n_1,\ f(n) \leq c_1 \cdot g(n)$$

$$g \in O(h) \Rightarrow \exists n_2 \in \aleph, c_2 > 0 \ /\ \forall n \geq n_2,\ g(n) \leq c_2 \cdot h(n)$$

$$Sea\ n_0 = \max\{n_1, n_2\}$$

$$\forall n \geq n_0,\ f(n) \leq c_1 \cdot g(n) \leq (c_1 \cdot c_2) \cdot g(n) \underset{d=(c_1 \cdot c_2)>0}{=} d \cdot h(n)$$

$$Luego\ f \in O(h)$$

2.d) Demostrar: $n^r \in O(n^5)$ si $0 \le r \le 5$

$$\lim_{n \to \infty} \frac{n^5}{n^r} = \lim_{n \to \infty} n^{5-r} \underset{0 \le r \le 5}{=} \begin{cases} +\infty & 0 \le r < 5 \Rightarrow n^r \in O(n^5) \\ 1 & r = 5 \Rightarrow n^r \in O(n^5) \end{cases}$$

$Luego,\ para$ $0 \le r \le 5,\ n^r \in O(n^5)$

2.f) Demostrar: $\log_b n \in O(n^k), \forall b > 1, k > 0$

$$\lim_{n\to\infty} \frac{\log_b n}{n^k} \underset{L'hôpital}{=} \lim_{n\to\infty} \frac{\frac{1}{n}\cdot \log_b e}{n^k} = \lim_{n\to\infty} \frac{\log_b e}{n^{k+1}} = 0$$

Luego,

$$\log_b n \in O(n^k), \quad b > 1, k > 0$$

2.g) Demostrar: $Max(n^3, 10n^2) \in O(n^3)$

$$Sea\ n_0 = 11, c = 1 \qquad n \geq n_0 = 11, \max(n^3, 10n^2) = n^3 \leq 1 \cdot n^3$$

$$\Rightarrow \max(n^3, 10n^2) \in O(n^3)$$

2.h) Demostrar: $\sum_{i=1}^{n} i^k \in \Theta(n^{k+1}), \forall k \in \aleph$

$$\sum_{i=1}^{n} i^k = 1^k + 2^k + \cdots + n^k \leq n \cdot n^k = n^{k+1}$$

$$Sean \left.\begin{matrix} n_0 = 1 \\ c = 1 \end{matrix}\right\} \forall n \geq n_0, \sum_{i=1}^{n} i^n \leq c \cdot n^{k+1}$$

$$Luego \sum_{i=1}^{n} i^k \in O(n^{k+1})$$

2.h) Demostrar: $\sum_{i=1}^{n} i^k \in \Theta(n^{k+1}), \forall k \in \aleph$

Por inducción sobre n

$n = 1 \quad \sum_{i=1}^{n} i^k = 1^k = 1 = 1^{k+1} = n^{k+1} \Rightarrow$ *se cumple*

Supongamos que es cierto para $n \Rightarrow \sum_{i=1}^{n} i^k \in \Omega(n^{k+1}) \Rightarrow$

$\Rightarrow \exists n_0 \in \aleph, c > 0 \,/\, \forall n \geq n_0, \sum_{i=1}^{n} i^k \geq c \cdot n^{k+1}$

¿$n+1$?

2.h) Demostrar: $\sum_{i=1}^{n} i^k \in \Theta(n^{k+1}), \forall k \in \aleph$

Por inducción sobre n

$$n = 1 \quad \sum_{i=1}^{n} i^k = 1^k = 1 = 1^{k+1} = n^{k+1} \Rightarrow \textit{se cumple}$$

Supongamos que es cierto para $n \Rightarrow \sum_{i=1}^{n} i^k \in \Omega(n^{k+1}) \Rightarrow$

$$\Rightarrow \exists n_0 \in \aleph, c > 0 \,/\, \forall n \geq n_0, \sum_{i=1}^{n} i^k \geq c \cdot n^{k+1}$$

¿$n+1$?

$$\sum_{i=1}^{n+1} i^k = \sum_{i=1}^{n} i^k + (n+1)^k \underset{hip.}{\geq} c \cdot n^{k+1} + (n+1)^k \geq c \cdot n^{k+1}. \quad \textit{Con lo que} \quad \sum_{i=1}^{n+1} i^k \in \Omega(n^{k+1})$$

2.h) Demostrar: $\sum_{i=1}^{n} i^k \in \Theta(n^{k+1}), \forall k \in \aleph$

$$\left.\begin{aligned} \lim_{n\to\infty} \frac{(n+1)^{k+1}}{n^{k+1}} = 1 \Rightarrow (n+1)^{k+1} \in \Theta(n^{k+1}) \\ \lim_{n\to\infty} \frac{n^{k+1}}{(n+1)^{k+1}} = 1 \Rightarrow n^{k+1} \in \Theta((n+1)^{k+1}) \end{aligned}\right\} \Rightarrow \Theta(n^{k+1}) = \Theta((n+1)^{k+1})$$

Y en particular $\Omega(n^{k+1}) = \Omega((n+1)^{k+1})$

Luego $\sum_{i=1}^{n+1} i^k \in \Omega(n^{k+1}) = \Omega((n+1)^{k+1}) \Rightarrow$ *Demostrado*

Luego $\sum_{i=1}^{n+1} i^k \in \Theta(n^{k+1})$

$$2.\text{i})\ \log_a n \in \Theta(\log_b n), \forall a,b > 1$$

$$\log_a n = \frac{\log_b n}{\log_b a} = \frac{1}{\log_b a} \cdot \log_b n$$

$$Sea\ c = \frac{1}{\log_b a} > 0 \quad (a > 1)$$

$$Entonces \log_a n = c \cdot \log_b n \quad \Rightarrow$$

$$c \cdot \log_b n \leq \log_a n \leq c \cdot \log_b n \Rightarrow \log_a n \in \Theta(\log_b n)$$

2.j) $\sum_{i=1}^{n} i^{-1} \in \Theta(\log_2 n)$

$$\sum_{i=1}^{n} i^{-1} \in O(\log_2 n)$$

Por inducción:

$$n = 2 \quad \sum_{i=1}^{n} i^{-1} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2} \leq 2 \cdot 1 = 2 \cdot \log_2 2$$

$$Sup.\ para\ n \Rightarrow \exists n_0 \in \aleph, c > 0 \ / \ \forall n \geq n_0, \sum_{i=1}^{n} i^{-1} \leq c \cdot \log n$$

$¿n+1?$

$$\sum_{i=1}^{n+1} i^{-1} = \sum_{i=1}^{n} i^{-1} + \frac{1}{n+1} \underset{hip.}{\leq} c \cdot \log n + \frac{1}{n+1} \leq c \cdot \log n + \log n = (c+1) \cdot \log n$$

$$Luego \sum_{i=1}^{n} i^{-1} \in O(\log n)$$

2.j) $\sum_{i=1}^{n} i^{-1} \in \Theta(\log_2 n)$

$$\sum_{i=1}^{n} i^{-1} \in \Omega(\log_2 n)$$

Por inducción:

$$n = 2 \quad \sum_{i=1}^{n} i^{-1} = \frac{3}{2} \geq 1 = \log_2 2$$

$$Sup.\ para\ n \Rightarrow \exists n_0 \in \aleph, d > 0 \ / \ \forall n \geq n_0, \sum_{i=1}^{n} i^{-1} \geq d \cdot \log n$$

¿$n+1$?

$$\sum_{i=1}^{n+1} i^{-1} = \sum_{i=1}^{n} i^{-1} + \frac{1}{n+1} \underset{hip.}{\geq} d \cdot \log n + \frac{1}{n+1} \geq d \cdot \log n$$

$$Luego \sum_{i=1}^{n} i^{-1} \in \Omega(\log n)$$

2.j) $\sum_{i=1}^{n} i^{-1} \in \Theta(\log_2 n)$

$$\lim_{n\to\infty} \frac{\log n}{\log n+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty} \frac{n+1}{n} = 1 \Rightarrow \Theta(\log n) = \Theta(\log(n+1))$$

$$Luego \sum_{i=1}^{n+1} i^{-1} \in \Theta(\log(n+1)) \Rightarrow \sum_{i=1}^{n+1} i^{-1} \in \Theta(\log n)$$

$$2.k)\quad f \in O(g) \Leftrightarrow \frac{1}{f} \in \Omega\left(\frac{1}{g}\right)$$

$\Rightarrow$	$f \in O(g) \Rightarrow \exists n_0 \in \aleph, c > 0 \,/\, \forall n \geq n_0,\ f(n) \leq c \cdot g(n) \Rightarrow \frac{1}{f(n)} \geq \frac{1}{c} \cdot \frac{1}{g(n)} \Rightarrow$ $\Rightarrow \frac{1}{f(n)} \geq d \cdot \frac{1}{g(n)} \Rightarrow \frac{1}{f} \in \Omega\left(\frac{1}{g}\right)$
$\Leftarrow$	Seguir el sentido contrario

2.l) $f(n) = c \cdot g(n),\, c > 0 \Rightarrow \Theta(f) = \Theta(g)$

$$f(n) = c \cdot g(n),\, c > 0 \Rightarrow c \cdot g(n) \leq f(n) \leq c \cdot g(n) \Rightarrow f(n) \in \Theta(g(n))$$

$$f(n) = c \cdot g(n),\, c > 0 \Rightarrow g(n) = \frac{1}{c} \cdot f(n) \Rightarrow \frac{1}{c} \cdot f(n) \leq g(n) \leq \frac{1}{c} \cdot f(n) \Rightarrow g(n) \in \Theta(f(n))$$

$$Luego\ \Theta(f(n)) = \Theta(g(n))$$

3.i) Demostrar: $f(n) \in O(n^a),\ g(n) \in O(n^b) \Rightarrow f(n) \cdot g(n) \in O(n^{a+b})$

$$f(n) \cdot g(n) \underset{\substack{\textit{regla del} \\ \textit{producto}}}{\in} O(n^a \cdot n^b) = O(n^{a+b})$$

3.i) Demostrar: $f(n) \in O(n^a),\ g(n) \in O(n^b) \Rightarrow f(n) \cdot g(n) \in O(n^{a+b})$

$$f(n) \cdot g(n) \underset{\substack{\textit{regla del}\\ \textit{producto}}}{\in} O(n^a \cdot n^b) = O(n^{a+b})$$

3.ii) Demostrar: $f(n) \in O(n^a),\ g(n) \in O(n^b) \Rightarrow f(n) + g(n) \in O(n^{\max\{a,b\}})$

$$f(n) + g(n) \underset{\substack{\textit{regla de}\\ \textit{la suma}}}{\in} O(\max(n^a, n^b) = O(n^{\max\{a,b\}})$$

4. Encontrar el menor entero k tal que $f(n) \in O(n^k)$

i) $f(n) = 13n^2 + 4n - 73$

$$\lim_{n\to\infty} \frac{13n^2 + 4n - 73}{n^k} = \begin{cases} +\infty & k < 2 \Rightarrow f(n) \notin O(n^k) \\ 13 & k = 2 \Rightarrow f(n) \in O(n^k) \\ 0 & k > 2 \Rightarrow f(n) \in O(n^k) \end{cases}$$

$Luego\ k = 2$

4. Encontrar el menor entero k tal que $f(n) \in O(n^k)$

ii) $f(n) = \dfrac{1}{(n+1)}$

$$\lim_{n\to\infty} \frac{n^k}{1/(n+1)} = \lim_{n\to\infty} \frac{n^k}{(n+1)^{-1}} = \begin{cases} +\infty & k > -1 \Rightarrow f(n) \in O(n^k) \\ 1 & k = -1 \Rightarrow f(n) \in O(n^k) \\ 0 & k < -1 \Rightarrow f(n) \notin O(n^k) \end{cases}$$

$Luego\ k = -1$

4. Encontrar el menor entero k tal que $f(n) \in O(n^k)$

iv) $f(n) = (n-1)^3$

$$\lim_{n\to\infty} \frac{n^k}{(n-1)^3} = \begin{cases} +\infty & k > 3 \Rightarrow f(n) \in O(n^k) \\ 1 & k = 3 \Rightarrow f(n) \in O(n^k) \\ 0 & k < 3 \Rightarrow f(n) \notin O(n^k) \end{cases}$$

$Luego\ k = 3$

4. Encontrar el menor entero k tal que $f(n) \in O(n^k)$

v) $f(n) = \dfrac{(n^3 + 2n - 1)}{(n+1)}$

$$\lim_{n\to\infty} \frac{(n^3+2n-1)/(n+1)}{n^k} = \lim_{n\to\infty} \frac{(n^3+2n-1)}{(n+1)\cdot n^k} = \begin{cases} +\infty & k < 2 \Rightarrow f(n) \notin O(n^k) \\ 1 & k = 2 \Rightarrow f(n) \in O(n^k) \\ 0 & k > 2 \Rightarrow f(n) \in O(n^k) \end{cases}$$

Luego $k = 2$

4. Encontrar el menor entero *k* tal que $f(n) \in O(n^k)$

vi) $f(n) = \sqrt{n^2 - 1}$

$$n^2 - 1 \le n^2 + n^2 = 2 \cdot n^2, \forall n \quad \Rightarrow \quad \sqrt{n^2 - 1} \le \sqrt{2} \cdot n, \forall n \quad \Rightarrow \quad \sqrt{n^2 - 1} \in O(n) = O(n^1)$$

$$Supongamos\ \sqrt{n^2 - 1} \in O(n^0) = O(1) \Rightarrow \exists n_0 \in \aleph, c > 0 \,/\, \forall n \ge n_0, \sqrt{n^2 - 1} \le c \cdot 1 = c$$

$$Sea\ n_1 = c + n_0 > n_0 \Rightarrow \sqrt{n_1^2 - 1} \le c \Rightarrow \sqrt{c^2 + n_0^2 + 2cn_0 - 1} \le c \Rightarrow c^2 + n_0^2 + 2cn_0 - 1 \le c \Rightarrow$$

$$\Rightarrow n_0^2 + 2cn_0 - 1 \le 0 \Rightarrow n_0^2 + 2cn_0 - 1 \le 1$$

$$Pero\ el\ mínimo\ valor\ \ para\ n_0 = 1 \Rightarrow 1 + 2c \le 1 \Rightarrow 2c \le 0 \Rightarrow c \le 0 \quad (absurdo!)$$

$$Luego\ \sqrt{n^2 + 1} \notin O(n^0) = O(1)$$

4. Encontrar el menor entero k tal que $f(n) \in O(n^k)$

$$\left.\begin{array}{l} Supongamos\ \exists k < 0\ tal\ que\ \sqrt{n^2 - 1} \in O(n^k) \\ \qquad \lim_{n \to \infty} \dfrac{n^k}{n^0} = \lim_{n \to \infty} \dfrac{1}{n^{-k}} = 0 \Rightarrow n^k \in O(n^0) \end{array}\right\} \sqrt{n^2 - 1} \in O(n^0) \quad (absurdo!)$$

$$Esto\ implica\ que\ \neg\exists k < 0 \,/\, \sqrt{n^2 - 1} \in O(n^k)$$

$$Luego\ k = 1$$

5. Demostrar por inducción que existe c>0 tal que $\sum_{k=1}^{n} k^2 \geq c \cdot n^3$

Caso n=1: $\sum_{k=1}^{n} k^2 = 1 \geq c \cdot 1^3 \Leftrightarrow 0 < c \leq 1$

¿n+1?

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 \underset{hip.}{\geq} c \cdot n^3 + (n+1)^2$$

$$c \cdot n^3 + (n+1)^2 \geq c \cdot (n+1)^3 \Leftrightarrow c \cdot n^3 + n^2 + 2n + 1 \geq c \cdot (n^3 + 3n^2 + 3n + 1) \Leftrightarrow$$

$$\Leftrightarrow n^2 + 2n + 1 \geq c \cdot (3n^2 + 3n + 1) \Leftrightarrow \begin{Bmatrix} 1 \geq 3c \\ 2 \geq 3c \\ 1 \geq c \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} c \leq 1/3 \\ c \leq 2/3 \\ c \leq 1 \end{Bmatrix} \Leftrightarrow c \leq 1/3$$

$$Luego\ sea\ c = 1/3 \quad \Rightarrow \quad \sum_{k=1}^{n} k^2 \geq c \cdot n^3$$

6. Sean f(n) y g(n) asintóticamente no negativas. Demostrar la veracidad o falsedad de:

6.a) $Max(f(n), g(n)) \in O(f(n) + g(n))$

$$Max(f(n), g(n)) \leq f(n) + g(n),\ \forall n \underset{\substack{n_0=1 \\ c=1}}{\Rightarrow} Max(f(n), g(n)) \in O(f(n) + g(n))$$

6.b) $Max(f(n), g(n)) \in \Omega(f(n) + g(n))$

$$Max(f(n), g(n)) \geq \frac{1}{2}(f(n) + g(n)),\ \forall n \underset{\substack{n_0=1 \\ c=1/2}}{\Rightarrow} Max(f(n), g(n)) \in \Omega(f(n) + g(n))$$

7. Expresar en notación ***O*** el orden de un algoritmo cuyo T(n) fuese f(n) si:

7.a) $f(n) = \log(n!)$

$$T(n) = \log(n!) = \log(n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 1) = \log n + \log(n-1) + \cdots + \log(1) \leq n \cdot \log(n)$$

$$Luego\ T(n) = \log(n!) \in O(n \log n)$$

7. Expresar en notación ***O*** el orden de un algoritmo cuyo T(n) fuese f(n) si:

7.b) $T(n) = n!$

$n! \in O(n!)$ (como cota menor)

También:

$$n! = n(n-1)(n-2)\cdot\cdots\cdot 1 \leq n \cdot n \cdot n \cdot\cdots\cdot n = n^n \Rightarrow n! \in O(n^n)$$

O también (por 7.a) $n! = 2^{\log_2(n!)} \leq 2^{n\cdot\log_2 n}$

En general: $n! \in O(a^{n\cdot\log_a n}) \quad \forall a > 1$

Que es lo mismo que la expresión anterior, ya que $a^{n\cdot\log_a n} =$

$$a^{n\cdot\log_a n} = a^{\log_a n^n} = n^n$$

8. Dadas las siguientes funciones de n, indicar para cada par (i,j) si

$$f_i(n) \in O(f_j(n))$$

$$f_i(n) \in \Omega(f_j(n))$$

$$f_1(n) = n^2$$

$$f_2(n) = n^2 + 1000n$$

$$f_3(n) = \begin{cases} n & n \ impar \\ n^3 & n \ par \end{cases}$$

$$f_4(n) = \begin{cases} n & n \leq 100 \\ n^3 & n > 100 \end{cases}$$

(1, 2) (2, 1)	$\lim_{n \to \infty} \frac{n^2}{n^2 + 1000n} = 1 \Rightarrow \Theta(n^2) = \Theta(n^2 + 1000n)$

(3, 4)	$Sea\ n_0 = 101,\ c = 1$ $\forall n \geq n_0 \Rightarrow \begin{cases} n\ impar & f_3(n) = n \leq n^3 = f_4(n) \Rightarrow f_3(n) \in O(f_4(n)) \\ n\ par & f_3(n) = n^3 = n^3 = f_4(n) \Rightarrow f_3(n) \in O(f_4(n)) \end{cases} \Rightarrow$ $\Rightarrow f_3(n) \in O(f_4(n))$ $¿\, f_3(n) \in \Omega(f_4(n))?$ $Sup.\ f_3(n) \in \Omega(f_4(n)) \Rightarrow \exists n_0 \in \aleph,\ c > 0 \,/\, \forall n \geq n_0,\ f_3(n) \geq c \cdot f_4(n)$ $\left.\begin{array}{r} n_1 > n_0 \\ Sea\ \ n_1\ \ impar \\ n_1 > 1/\sqrt{c} \end{array}\right\} \Rightarrow f_3(n_1) \geq c \cdot f_4(n_1) \Rightarrow n_1 \geq c \cdot n_1^3$ $Pero\ n_1 > \frac{1}{\sqrt{c}} \Rightarrow n_1^2 > \frac{1}{c} \Rightarrow c > \frac{1}{n_1^2} \Rightarrow c \cdot n^3 > \frac{n_1^3}{n_1^2} = n\ (abs.)$ $Luego\ \ f_3(n) \notin \Omega(f_4(n))$

(4, 3)	$f_3(n) \in O(f_4(n)) \Rightarrow f_4(n) \in \Omega(f_3(n))$ $f_3(n) \notin \Omega(f_4(n)) \Rightarrow f_4(n) \notin O(f_3(n))$

(1, 3)	$Sup.\ f_1(n) \in O(f_3(n)) \Rightarrow \exists n_0 \in \aleph, c > 0 \ / \ \forall n \geq n_0,\ f_1(n) \leq c \cdot f_3(n)$ $\left.\begin{array}{r} n_1 \geq n_0 \\ Sea\ n_1\ impar \\ n_1 > c \end{array}\right\} \Rightarrow f_1(n_1) \leq c \cdot f_3(n_1) \Rightarrow n_1^2 \leq c \cdot n_1. \quad n_1 > c \Rightarrow n_1^2 > c \cdot n_1\ (abs.)$ $Luego\ \ f_1(n) \notin O(f_3(n))$ $Sup.\ f_1(n) \in \Omega(f_3(n)) \Rightarrow \exists n_0 \in \aleph, c > 0 \ / \ \forall n \geq n_0,\ f_1(n) \geq c \cdot f_3(n)$ $\left.\begin{array}{r} n_1 \geq n_0 \\ Sea\ \ n_1\ \ par \\ n_1 > 1/c \end{array}\right\} \Rightarrow f_1(n_1) \geq c \cdot f_3(n_1) \Rightarrow n_1^2 \geq c \cdot n_1^3$ $Pero\ n_1 > 1/c \Rightarrow c > 1/n_1 \Rightarrow c \cdot n_1^3 > n_1^2 \quad (absurdo)$ $Luego\ \ f_1(n) \notin \Omega(f_3(n))$

(3, 1)	$f_1(n) \notin O(f_3(n)) \Rightarrow f_3(n) \notin \Omega(f_1(n))$ $f_1(n) \notin \Omega(f_3(n)) \Rightarrow f_3(n) \notin O(f_1(n))$

(2, 3)	$\left.\begin{array}{r} Sup.\ f_2(n) \in O(f_3(n)) \\ Sabemos\ \ f_1(n) \in O(f_2(n)) \end{array}\right\} \Rightarrow f_1(n) \in O(f_3(n)) \quad (falso)$ $Luego\ \ f_2(n) \notin O(f_3(n))$ $\left.\begin{array}{r} Sup.\ f_2(n) \in \Omega(f_3(n)) \\ Sabemos\ \ f_1(n) \in \Omega(f_2(n)) \end{array}\right\} \Rightarrow f_1(n) \in \Omega(f_3(n)) \quad (falso)$ $Luego\ \ f_2(n) \notin \Omega(f_3(n))$

(3, 2)	$f_2(n) \notin O(f_3(n)) \Rightarrow f_3(n) \notin \Omega(f_2(n))$ $f_2(n) \notin \Omega(f_3(n)) \Rightarrow f_3(n) \notin O(f_2(n))$

(1, 4)	$Sea \left.\begin{array}{r} n_0 = 101 \\ c = 1 \end{array}\right\} \underset{n \geq n_0}{\Rightarrow} \begin{array}{l} f_1(n) = n^2 \\ f_4(n) = n^3 \end{array}$ $\lim_{n \to \infty} \dfrac{n^2}{n^3} = 0 \Rightarrow \begin{array}{l} n^2 \in O(n^3) \\ n^2 \notin \Omega(n^3) \end{array} \Rightarrow \begin{array}{l} f_1(n) \in O(f_4(n)) \\ f_1(n) \notin \Omega(f_4(n)) \end{array}$

(4, 1)	$f_1(n) \in O(f_4(n)) \Rightarrow f_4(n) \in \Omega(f_1(n))$ $f_1(n) \notin \Omega(f_4(n)) \Rightarrow f_4(n) \notin O(f_1(n))$

(2, 4)	$\left.\begin{array}{l} f_2(n) \in O(f_1(n)) \\ f_1(n) \in O(f_4(n)) \end{array}\right\} \Rightarrow Luego\ \ f_2(n) \in O(f_4(n))$ $\left.\begin{array}{r} Sup.\ \ f_2(n) \in \Omega(f_4(n)) \\ Sabemos\ \ f_1(n) \in \Omega(f_2(n)) \end{array}\right\} \Rightarrow f_1(n) \in \Omega(f_4(n)) \quad (absurdo)$ $Luego\ \ f_2 \notin \Omega(f_4(n))$

(4, 2)	$f_2(n) \in O(f_4(n)) \Rightarrow f_4(n) \in \Omega(f_2(n))$ $f_2(n) \notin \Omega(f_4(n)) \Rightarrow f_4(n) \notin O(f_2(n))$

9. Decir cuáles de las siguientes afirmaciones son verdaderas y demostrarlo:

9.a) $2^{n+1} \in O(2^n)$ *Cierto*

$$2^{n+1} = 2 \cdot 2^n, \quad \forall n \quad \underset{\substack{n_0=1 \\ c=2}}{\Rightarrow} \quad 2^{n+1} \in O(2^n)$$

9.b) $(n+1)! \in O(n!)$ *Falso*

$$\lim_{n \to \infty} \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{n!}{(n+1) \cdot n!} = \lim_{n \to \infty} \frac{1}{(n+1)} = 0 \Rightarrow$$

$$n! \notin \Omega((n+1)!) \quad \Rightarrow \quad (n+1)! \notin O(n!)$$

$$9.c)\ \forall f : \aleph \to \Re^{+},\ f(n) \in O(n) \Rightarrow f^{2}(n) \in O(n^{2})$$ *Cierto*

$$f(n) \in O(n) \Rightarrow \exists n_0 \in \aleph, c > 0 \,/\, \forall n \geq n_0,\ f(n) \leq c \cdot n \Rightarrow$$

$$\Rightarrow f^{2}(n) \leq c^{2} \cdot n^{2} \Rightarrow f^{2}(n) \in O(n^{2}$$

9.d) $\forall f : \aleph \to \Re^+,\ f(n) \in O(n) \Rightarrow 2^{f(n)} \in O(2^n)$ Falso

$$Sea\ \ f(n) = 2n$$

$$2^{f(n)} = 2^{2n} = 4^n$$

$$\lim_{n\to\infty} \frac{2^n}{4^n} = \lim_{n\to\infty} \left(\frac{2}{4}\right)^n = 0$$

$$Luego\ 2^n \notin \Omega(4^n) \Rightarrow 4^n \notin O(2^n) \Rightarrow 2^{f(n)} \notin O(2^n)$$

f(n) es un contraejemplo

10. Sea x un número real, $0<x<1$. Ordenar las tasas de crecimiento de las siguientes funciones:

$$n \cdot \log(n), \quad n^8, \quad (1+x)^n, \quad (n^2+8n+\log^3(n))^4, \quad \frac{n^2}{\log(n)}$$

$$n^2 + 8n + \log^3(n) \in O(n^2),\ ya\ que$$

$$\lim_{n\to\infty} \frac{n^2}{\log^3(n)} = \lim_{n\to\infty} \frac{2n}{3 \cdot \frac{1}{n} \cdot \log^2(n)} = \lim_{n\to\infty} \frac{2n^2}{3 \cdot \log^2(n)} = \lim_{n\to\infty} \frac{2n^2}{3 \cdot \log^2(n)} = \lim_{n\to\infty} \frac{4n^2}{6 \cdot \log^2(n)} =$$

$$= \lim_{n\to\infty} \frac{8n^2}{6} = +\infty \Rightarrow \log^3(n) \in O(n^2)$$

$$Si\ \ f(n) \in O(n^2) \Rightarrow \exists n_0 \in \aleph, c > 0 \ /\ \forall n \geq n_0,\ f(n) \leq c \cdot n^2 \Rightarrow f(n)^4 \leq c^4 \cdot n^8 \Rightarrow$$

$$\Rightarrow f(n)^4 \in O(n^8) \Rightarrow (n^2 + 8n + \log^3(n))^4 \in O(n^8)$$

$$Además \quad (n^2 + 8n + \log^3(n))^4 \geq n^8 \underset{\substack{n_0 = 1 \\ c = 1}}{\Rightarrow} n^8 \in O((n^2 + 8n + \log^3(n))^4)$$

$$Luego \quad O(n^8) = O((n^2 + 8n + \log^3(n))^4)$$

$$\lim_{n\to\infty}\frac{(1+x)^n}{n^8}=\lim_{n\to\infty}\frac{(1+x)^n\cdot\log_{1+x}e}{8n^7}=\cdots=\lim_{n\to\infty}\frac{(1+x)^n\cdot\log_{1+x}^8 e}{8!}=+\infty\Rightarrow$$

$$\Rightarrow\begin{cases} n^8\in O((1+x)^n) \\ n^8\notin\Omega((1+x)^n)\Rightarrow(1+x)^n\notin O(n^8)\end{cases}$$

$$Luego\quad O(n^8)\lhd O((1+x)^n)$$

$$O((n^2+8n+\log^3(n))^4)\lhd((1+x)^n)$$

$$\lim_{n\to\infty} \frac{n^8}{\frac{n^2}{\log(n)}} = \lim_{n\to\infty} n^6 \cdot \log(n) = \infty \Rightarrow \begin{cases} \dfrac{n^2}{\log(n)} \in O(n^8) \\ n^8 \notin O\left(\dfrac{n^2}{\log(n)}\right) \end{cases}$$

$$Luego\ O\left(\frac{n^2}{\log(n)}\right) \triangleleft O(n^8)$$

$$\lim_{n\to\infty}\frac{\frac{n^2}{\log(n)}}{n^{1+x}} = \lim_{n\to\infty}\frac{n^2}{n^{1+x}\cdot\log(n)} = \lim_{n\to\infty}\frac{2n}{(1+x)\cdot n^x\cdot\log(n)+n^x} =$$

$$\lim_{n\to\infty}\frac{2n}{n^x\cdot\left[(1+x)\cdot\log(n)+1\right]} = \lim_{n\to\infty}\frac{2n^{1-x}}{(1+x)\cdot\log(n)+1} = \lim_{n\to\infty}\frac{2\cdot(1-x)\cdot n^x}{(1+x)\cdot\frac{1}{n}} =$$

$$\lim_{n\to\infty} 2\cdot\frac{(1-x)}{(1+x)}\cdot n^{1+x} = +\infty \Rightarrow \begin{cases} n^{1+x} \in O\left(\frac{n^2}{\log(n)}\right) \\ \frac{n^2}{\log(n)} \notin O(n^{1+x}) \end{cases}$$

$$Luego\ O(n^{1+x}) \lhd O\left(\frac{n^2}{\log(n)}\right)$$

$$\lim_{n\to\infty}\frac{n\cdot\log(n)}{n^{1+x}}=\lim_{n\to\infty}\frac{1+\log(n)}{(1+x)\cdot n^{x}}=\lim_{n\to\infty}\frac{n^{-1}}{x\cdot(1+x)\cdot n^{x-1}}=\lim_{n\to\infty}\frac{1}{x\cdot(1+x)\cdot n^{x}}=0\Rightarrow$$

$$\Rightarrow\begin{cases} n\cdot\log(n)\in O(n^{1+x}) \\ n^{1+x}\notin O(n\cdot\log(n)) \end{cases}$$

$$Luego\ O(n\cdot\log(n))\lhd O(n^{1+x})$$

11. Demostrar que $\log(n) \in O(\sqrt{n})$ pero $\sqrt{n} \notin O(\log(n))$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\log(n)} = \lim_{n\to\infty} \frac{\frac{1}{2}}{\sqrt{n}\cdot\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} = \infty \Rightarrow \begin{cases} \log(n) \in O(\sqrt{n}) \\ \log(n) \notin \Omega(\sqrt{n}) \Rightarrow \sqrt{n} \notin O(\log(n)) \end{cases}$$