MATHEMATICS 2

METHODS FOR SOLVING LINEAR EQUATIONS SYSTEMS

01. Preparatory problems. Determine coordinates of all points $P[x,\, y]$ which are common to the 2 lines l_1, l_2 in the plane:

a) p_1: $2x+2y=-1$, p_2: $3x-2y=11$; b) p_1: $x-3y=1$, p_2: $2x+y=9$;
c) p_1: $x+3y=2$, p_2: $2x+6y=4$; d) p_1: $x-3y=2$, p_2: $2x-6y=14$;
e) p_1: $\dfrac{x}{2}+\dfrac{y}{5}=1$: p_2: $y=1+2x$.

02. Preparatory problems. Determine all possible solutions of the system of linear equations, check all solutions:

a) $\begin{aligned} x_1+2x_2 &= 3 \\ 3x_1+4x_2 &= 11 \end{aligned}$ b) $\begin{aligned} 3x_1-6x_2 &= 9 \\ x_1-2x_2 &= 3 \end{aligned}$ c) $\begin{aligned} 2x_1+3x_2 &= 0 \\ 3x_1-x_2 &= 0 \end{aligned}$

1. Knowledge, skills: Gaussian Elimination Method (GEM). Applying GEM, find all possible solutions of the system of linear equations, check all solutions:

a) $\begin{aligned} x_1+x_2 &= 0 \\ x_2+x_3 &= 1 \\ x_1+x_3 &= 2 \end{aligned}$ b) $\begin{aligned} 2x_1+x_2 &= 0 \\ 2x_2+x_3 &= 0 \\ 2x_1 +x_3 &= 0 \end{aligned}$ c) $\begin{aligned} 3x_1+2x_2-x_3 &= 8 \\ -x_1+3x_2+2x_3 &= 3 \\ 2x_1-x_2+4x_3 &= -4 \end{aligned}$

2. Knowledge, skills: Gaussian Elimination Method (GEM). Applying GEM, find all possible solutions of the system of linear equations, check all solutions:

a) $\begin{aligned} x_1+2x_2+3x_3 &= 2 \\ 3x_1+3x_2+5x_3 &= 2 \\ 3x_1+x_2+5x_3 &= 2 \end{aligned}$ b) $\begin{aligned} 40x_1+20x_2+30x_3 &= 40 \\ 20x_1+10x_2+20x_3 &= 20 \\ 40x_1+10x_2+10x_3 &= 40 \end{aligned}$ c) $\begin{aligned} x_1+x_2-3x_3 &= -2 \\ 3x_1+2x_2-2x_3 &= 5 \\ 4x_1-3x_2+2x_3 &= -1 \end{aligned}$

d) $\begin{aligned} 2x_1+3x_2+2x_3 &= 3 \\ 4x_1+3x_2+5x_3 &= 4 \\ 2x_1 +3x_3 &= 2 \end{aligned}$ e) $\begin{aligned} 2x_1+3x_2+4x_3 &= -1 \\ x_1+3x_2+2x_3 &= 1 \\ 3x_1+2x_2+2x_3 &= 4 \end{aligned}$

3. Knowledge, skills: Gaussian Elimination Method (GEM). Applying GEM, find all possible solutions of the system of linear equations, check all solutions:

a) $\begin{aligned} 3x_1+5x_2+6x_3 &= 1 \\ 4x_1+3x_2+2x_3 &= 5 \\ 3x_1+5x_2+x_3 &= 1 \end{aligned}$ b) $\begin{aligned} -4x_1+2x_2+5x_3 &= 4 \\ 3x_1+6x_2+3x_3 &= 0 \\ 3x_1-2x_2+3x_3 &= 0 \end{aligned}$ c) $\begin{aligned} x_1-2x_2+2x_3 &= -9 \\ 3x_1+5x_2+4x_3 &= 10 \\ 5x_1+12x_2+6x_3 &= 29 \end{aligned}$

d) $\begin{aligned} x_1-2x_2 &= -3 \\ 2x_1-x_2 &= 0 \\ 4x_1-5x_2 &= -6 \end{aligned}$ e) $\begin{aligned} x_1+2x_2+3x_3 &= 0 \\ 4x_1+7x_2+5x_3 &= 0 \\ x_1+6x_2+10x_3 &= 0 \\ x_1+x_2-4x_3 &= 0 \end{aligned}$ f) $\begin{aligned} -x_1 +x_3+x_4 &= 3 \\ -x_2 +x_4 &= 3 \\ -x_1 &= 1 \\ -x_1+2x_2+4x_3+x_4 &= 4 \end{aligned}$

4. Knowledge, skills: Gaussian Elimination Method (GEM). Applying GEM, find all possible solutions of the system of linear equations, check all solutions:

a) $\begin{aligned} 3x_1 - 2x_2 + 5x_3 - 6x_4 &= 0 \\ 7x_1 + x_2 - 3x_3 - 4x_4 &= 1 \\ 6x_1 + 5x_2 - 13x_3 + 3x_4 &= 1 \\ 2x_1 - 13x_2 + 40x_3 - 16x_4 &= 13 \end{aligned}$
b) $\begin{aligned} 7x_1 + 2x_2 + 2x_3 &= 5 \\ 3x_1 + 6x_2 + 4x_3 &= -2 \\ 5x_1 + 2x_2 + 4x_3 &= 2 \end{aligned}$
c) $\begin{aligned} -2x_1 + x_2 &= -1 \\ x_1 + 2x_2 &= 8 \\ -6x_1 + 3x_2 &= 3 \end{aligned}$

d) $\begin{aligned} 5x_1 + 12x_2 + 9x_3 + 25x_4 &= 15 \\ 15x_1 + 34x_2 + 25x_3 + 64x_4 &= 40 \\ 20x_1 + 46x_2 + 34x_3 + 89x_4 &= 70 \\ 10x_1 + 23x_2 + 17x_3 + 44x_4 &= 25 \end{aligned}$
e) $\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 3x_1 - 5x_2 - 2x_3 &= -3 \\ 7x_1 - 3x_2 + x_3 &= 16 \end{aligned}$
f) $\begin{aligned} x + y &= a \\ y + z &= b \\ z + x &= c \end{aligned}$

g) $\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 12 \\ x_1 + x_2 - x_3 + x_4 &= 13 \\ x_1 - x_2 + x_3 + x_4 &= 5 \\ -x_1 + x_2 + x_3 + x_4 &= 8 \end{aligned}$
h) $\begin{aligned} x + y + z &= 10 \\ y + z + u &= 20 \\ z + u + x &= 40 \\ u + x + y &= 80 \end{aligned}$
i) $\begin{aligned} x_1 + 3x_2 - 6x_3 - 6x_4 &= 7 \\ 2x_1 + x_2 - 4x_3 - 2x_4 &= 15 \\ 4x_1 - x_2 - 5x_3 + 5x_4 &= 30 \\ 5x_1 + 10x_2 - 20x_3 - 22x_4 &= 38 \end{aligned}$

5. Knowledge, skills: Cramer's Rule. Solve the systems of linear equations using Cramer's Rule, check all solutions (the unknowns x_i and the coefficients of the matrix of the system are written in the form of a table):

a)

x_1	x_2	x_3	
3	2	−1	8
−1	3	2	3
2	−1	4	−4

b)

x_1	x_2	x_3	
2	−3	1	0
1	2	−1	3
2	1	1	12

c)

x_1	x_2	x_3	x_4	
3	1	−1	2	0
1	2	1	−1	0
2	−1	2	1	0
1	3	1	3	0

d)

x_1	x_2	x_3	
3	−2	1	11
−1	1	−3	7
9	−7	11	11

e)

x_1	x_2	x_3	x_4	
1	2	−1	−2	−2
2	1	1	1	8
1	−1	−1	1	1
1	2	2	−1	4

f)

x_1	x_2	x_3	x_4	
2	−3	6	−1	1
1	2	−1	0	0
1	3	−1	−1	−2
9	−1	15	−5	1

6. Knowledge, skills: Cramer's Rule. Solve the systems of linear equations in 2 a, b, c, d; 3 a, b, f. Check all solutions.

7. Knowledge, skills. Solve homogeneous systems applying the appropriate method (GEM or Cramer's Rule), check solutions (the unknowns x_i and the coefficients of the matrix of the system are written in the form of a table):

a)

x_1	x_2	x_3	
3	5	2	0
1	−2	4	0
3	1	0	0

b)

x_1	x_2	x_3	
5	3	2	0
4	3	1	0
3	3	0	0

c)

x_1	x_2	x_3	
1	−4	2	0
4	2	−5	0

d)

x_1	x_2	x_3	
3	1	−1	0
1	−3	1	0
−1	1	3	0

e)

x_1	x_2	x_3	
1	1	1	0
2	2	2	0
1	2	3	0
3	4	5	0

f)

x_1	x_2	x_3	
1	2	3	0
3	1	2	0
2	3	1	0

g)

x_1	x_2	x_3	
1	2	3	0
3	1	2	0

h)

x_1	x_2	
11	17	0
37	−53	0
−29	61	0

i)

x_1	x_2	x_3	x_4	
4	−1	2	6	0
5	10	−1	−2	0
7	3	3	3	0
3	2	1	5	0

j)

x_1	x_2	x_3	x_4	x_5	x_6	
0	0	1	0	−1	1	0
−1	1	0	1	0	0	0
0	−1	1	−1	0	−1	0
0	1	0	−1	−1	0	0
−1	0	1	0	0	1	0

8. Knowledge, skills. Solve homogeneous systems applying the appropriate method (GEM or Cramer's Rule), check solutions:

a) $\begin{aligned} 2x_1+3x_2 &= 0 \\ 4x_1+6x_2 &= 0 \end{aligned}$

b) $\begin{aligned} 3x_1+4x_2-7x_3 &= 0 \\ 5x_1+2x_2-6x_3 &= 0 \end{aligned}$

c) $\begin{aligned} 5x_1+2x_2-18x_3 &= 0 \\ 2x_1+x_2-8x_3 &= 0 \end{aligned}$

d) $\begin{aligned} 4x_1-6x_2+5x_3 &= 0 \\ 6x_1-9x_2+10x_3 &= 0 \end{aligned}$

e) $\begin{aligned} x_1+2x_2-3x_3+x_4 &= 0 \\ -x_1+x_2-x_3+x_4 &= 0 \\ 2x_1+3x_2+4x_3-x_4 &= 0 \\ -2x_1+x_2+x_3-3x_4 &= 0 \end{aligned}$

f) $\begin{aligned} x_1+4x_2-3x_3 &= 0 \\ x_1-3x_2-x_3 &= 0 \\ 2x_1+x_2-4x_3 &= 0 \end{aligned}$

g) $\begin{aligned} x_1-3x_2-26x_3+22x_4 &= 0 \\ x_1-8x_3+7x_4 &= 0 \\ x_1+x_2-2x_3+2x_4 &= 0 \\ 4x_1+5x_2-2x_3+3x_4 &= 0 \end{aligned}$

h) $\begin{aligned} 8x_1-5x_2-6x_3+3x_4 &= 0 \\ 4x_1-x_2-3x_3+2x_4 &= 0 \\ 12x_1-7x_2-9x_3+5x_4 &= 0 \end{aligned}$

i) $\begin{aligned} 2x_1+x_2-x_3 &= 0 \\ -x_1-x_2 &= 0 \\ -x_1+x_2 &= 0 \\ x_1-x_3 &= 0 \end{aligned}$

9. Knowledge, skills. Solve systems applying the appropriate method, check solutions (the unknowns x_i and the coefficients of the matrix of the system are written in the form of a table):

a)

x_1	x_2	x_3	
3	−2	5	0
1	−7	11	0

b)

x	y	z	
2	−1	3	0
1	1	−3	−1
5	−1	3	7

c)

x_1	x_2	x_3	x_4	
1	3	2	2	3
2	4	1	0	12
1	3	2	1	4
3	2	4	6	−1

d)

x_1	x_2	x_3	x_4	
11	−3	7	−2	0
1	4	−4	2	0
9	−9	2	10	0
−3	4	4	−5	0

e)

x_1	x_2	x_3	x_4	
3	2	−1	1	0
1	1	−1	5	0
2	1	3	−1	0

f)

x_1	x_2	x_3	
0,4	−0,3	0,1	2
0,2	0,1	−0,3	−0,4
0,3	−0,4	0,2	2

g)

x_1	x_2	x_3	x_4	x_5	
1	2	3	4	5	0
1	1	1	1	1	0
5	3	5	3	5	0
2	5	5	5	2	0

h)

x_1	x_2	x_3	
1	2	-1	2
3	-1	2	7
1	0	-1	-2
2	1	1	7

i)

x_1	x_2	x_3	
7	14	-21	7
1	2	-3	1
5	10	15	5
3	6	-9	3

j)

x_1	x_2	x_3	x_4	
1	2	3	4	7
2	4	6	3	4
4	3	2	1	3
3	6	4	2	6

k)

x_1	x_2	x_3	x_4	
1	1	4	3	7
2	1	11	13	18
2	3	5	-1	10
-1	2	-13	-24	-19

l)

x_1	x_2	x_3	x_4	
2	-1	0	0	0
-1	2	-1	0	0
0	-1	2	-1	0
0	0	-1	2	5

m)

x_1	x_2	x_3	x_4	
1	1	1	1	0
2	3	-1	4	1
-2	0	3	2	2
4	1	-1	-2	3

n)

x_1	x_2	x_3	x_4	
3	4	2	5	3
7	3	1	2	2
-4	3	-3	2	-5
6	10	0	9	1

10. Knowledge, skills. Solve systems applying the appropriate method, check solutions:

a)
$$\begin{aligned} 15x_1-3x_2+12x_3 &= 0\\ 7x_1-x_2+4x_3 &= 0\\ 2x_1-5x_2+20x_3 &= 0 \end{aligned}$$

b)
$$\begin{aligned} x_1-x_2-3x_4 &= -1\\ 7x_1-2x_2-2x_3-10x_4 &= -5\\ 7x_1-x_2+x_3-9x_4 &= -7\\ 2x_1-2x_3-4x_4 &= -6\\ 6x_1-x_2+2x_3-7x_4 &= -4 \end{aligned}$$

c)
$$\begin{aligned} 3x_1+3x_2-4x_3+4x_4 &= 0\\ 2x_1+x_2-2x_3+x_4 &= 0\\ x_1-x_2-2x_4 &= 0\\ 6x_1+6x_2-8x_3+8x_4 &= 0 \end{aligned}$$

d)
$$\begin{aligned} x_1+x_2+x_3+x_4 &= 1\\ 2x_1+2x_2+2x_3 &= 0\\ x_1+x_2+5x_3-x_4+6x_5 &= 1\\ x_1+x_2-3x_3+x_4-6x_5 &= -1 \end{aligned}$$

e)
$$\begin{aligned} 27x_1-19x_2+22x_3-35x_4 &= 6\\ 20x_1-13x_2+14x_3-13x_4 &= -23\\ 8x_1-2x_2+6x_3-10x_4 &= 10\\ 9x_1-4x_2+7x_3-8x_4 &= -3\\ 18x_1-9x_2+12x_3-17x_4 &= 3 \end{aligned}$$

f)
$$\begin{aligned} x_1+2x_2+3x_3+4x_4+5x_5 &= 1\\ -2x_1+3x_2+4x_3+5x_4+6x_5 &= 2\\ -3x_1+4x_2+5x_3+6x_4+7x_5 &= 3 \end{aligned}$$

11. Knowledge, skills. In the system of linear equations with parameters, determine all possible values of a real parameter p, for which the system has a unique solution, infinitely many solutions, or no solution. Provide all existing solutions, depending on values of the parameter. Check all results:

a) $\begin{aligned} px_1+4x_2 &= 1 \\ x_1+px_2 &= p \end{aligned}$ b) $\begin{aligned} p^2x_1+3x_2+2x_3 &= 0 \\ x_2+4x_3 &= 8 \\ px_1-x_2+x_3 &= 0 \end{aligned}$ c) $\begin{aligned} x_1+x_2+px_3 &= 0 \\ (p+1)x_1+x_3 &= 1 \\ (p+1)x_2+x_3 &= 1 \end{aligned}$

d) $\begin{aligned} x_1+x_2+x_3 &= 1 \\ x_1+px_2+x_3 &= p \\ x_1+x_2+px_3 &= p^2 \end{aligned}$ e) $\begin{aligned} x_1+px_2+x_3 &= 1 \\ x_1+x_2+x_3 &= 1 \\ x_1+x_2+p^2x_3 &= 1 \end{aligned}$ f) $\begin{aligned} px_1+x_3 &= 0 \\ px_2+x_3 &= 0 \\ x_1+x_2+(p-1)x_3 &= 0 \end{aligned}$

Solutions:

1. a) $(1/2,\ -1/2,\ 3/2)$; b) $(0,\ 0,\ 0)$; c) $(1,\ 2,\ -1)$ • **2.** a) $(-1,\ 0,\ 1)$; b) $(1,\ 0,\ 0)$; c) $(1,\ 3,\ 2)$; d) no solution; e) $(2,\ 1,\ -2)$ • **3.** a) $(2,\ -1,\ 0)$; b) $(-4/9,\ 0,\ 4/9)$; c) $(1-18a,\ 2a+3,\ 11a-2)$, a arbitrary real number; d) $(1,\ 2)$; e) $(0,\ 0,\ 0)$; f) $(-1,\ -4,\ 3,\ -1)$ • **4.** a) $(1,\ 1,\ 1,\ 1)$; b) $(1,\ -1/2,\ -1/2)$; c) no solution; d) no solution; e) $(3,\ 2,\ 1)$; f) $(0,5(a-b+c),\ 0,5(a+b-c),\ 0,5(-a+b+c))$; g) $(11/2,\ 7,\ 3,\ 7/2)$; h) $(30,\ 10,\ -30,\ 40)$; i) $(10,\ 5,\ 2,\ 1)$ • **5.** a) $(1,\ 2,\ -1)$; b) $(2,\ 3,\ 5)$; c) $(0,\ 0,\ 0,\ 0)$; d) no solution; e) $(1,\ 2,\ 1,\ 3)$; f) no solution • **7.** a) $(0,\ 0,\ 0)$; b) $(-a,\ a,\ a)$, a arbitrary real number; c) $(16a,\ 13a,\ 18a)$, a arbitrary real number; d) the trivial solution only; e) $(1,\ -2,\ 1)t$, $t \in R$; f) $(0,\ 0,\ 0)$; g) $(-1,\ -7,\ 5)t$, $t \in R$ • h) the trivial solution only; i) the trivial solution only; j) $(a,\ c,\ a-b,\ a-c,\ a,\ b)$, $a,\ b,\ c \in R$ • **8.** a) $(3a,\ -2a)$, a arbitrary real number; b) $(10a,\ 17a,\ 14a)$, a arbitrary real number; c) $(2a,\ 4a,\ a)$, $a \in R$ lib.; d) $(3a,\ 2a,\ 0)$, $a \in R$ lib.; e) the trivial solution only; f) $(13a,\ 2a,\ 7a)$, $a \in R$ lib.; g) $(8a-7b,\ -6a+5b,\ a,\ b)$, $a,\ b \in R$ lib.; h) $(3a,\ 0,\ 4a,\ 0)$, $a \in R$ lib.; i) the trivial solution only • **9.** a) $(13a,\ -28a,\ -19a)$, $a \in R$ lib.; b) $(1,\ -2+3t,\ t)$, $t \in R$ lib.; c) $(3,\ 2,\ -2,\ -1)$; d) the trivial solution only; e) $(10a,\ -16a,\ -a,\ a)$, $a \in R$ lib.; f) $(4,\ 0,\ 4)$; g) $(-a,\ 2a,\ 0,\ -2a,\ a)$, $a \in R$ lib.; h) $(1,\ 2,\ 3)$; i) $(1-2a,\ a,\ 0)$ j) $(0,\ 1,\ -1,\ 2)$; k) $(11-7a-10b,\ -4+3a+7b,\ a,\ b)$, $a,\ b \in R$ lib.; l) $(1,\ 2,\ 3,\ 4)$; m) no solution; n) no solution • **10.** a) $(0,\ 4a,\ a)$, $a \in R$ lib.; b) $(1,\ -4,\ 0,\ 2)$; c) $(a,\ a,\ a,\ a)$, $a \in R$ lib.; d) $(3a,\ 3b+1,\ -1-3a-3b,\ 1,\ 2a+2b+1)$, $a,\ b \in R$; e) $(1,\ 2,\ -4,\ -3)$; f) $(0,\ 2+a+2b,\ -1-2a-3b,\ a,\ b)$, $a,\ b \in R$ lib. • **11.** a) when $det\,A = p^2-4 \neq 0$, then there is a unique solution $\left(\dfrac{-3p}{p^2-4},\ \dfrac{p^2-1}{p^2-4}\right)$; in case $p = \pm 2$ no solution; b) if $det\,A = 5p(p+2) \neq 0$, then there is a unique solution $\left(\dfrac{-8}{p(p+2)},\ \dfrac{8(p-2)}{5(p+2)},\ \dfrac{8(p+3)}{5(p+2)}\right)$; for $p = 0$, $p = -2$ no solution; c) if $det\,A = (p+2)(p^2-1) \neq 0$, then there is a unique solution $\left(\dfrac{p}{(p-1)(p+2)},\ \dfrac{p}{(p-1)(p+2)},\ \dfrac{-2}{(p-1)(p+2)}\right)$; for $p = 1$ infinitely many solutions of the form $(1-x_2,\ x_2,\ 1)$, $x_2 \in R$ arbitrary; for $p = 1$ and for $p = -2$ no solution; d) when $det\,A = (p-1)^2 \neq 0$, then there is a unique solution $(-p-1,\ 1,\ p+1)$; for $p = 1$ infinitely many solutions of the form $(1-x_2-x_3,\ x_2,\ x_3)$, $x_2,\ x_3 \in R$ arbitrary; e) if $det\,A = (p-1)(p^2-1) \neq 0$, then there is a unique solution $(1,\ 0,\ 0)$; for $p = 1$ infinitely many solutions of the form $(1-x_2-x_3,\ x_2,\ x_3)$, $x_2,\ x_3 \in R$ arbitrary; for $p = -1$ infinitely many solutions of the form $(1-x_3,\ 0,\ x_3)$, $x_3 \in R$ arbitrary; f) if $det\,A = p(p-2)(p+1) \neq 0$, then there is a trivial solution only; pro $p = 0$ infinitely many solutions of the form $(x_1,\ -x_1,\ 0)$, $x_1 \in R$ arbitrary; for $p = 2$ infinitely many solutions of the form $(x_2,\ x_2,\ -2x_2)$, $x_2 \in R$ arbitrary; for $p = -1$ infinitely many solutions of the form $(x_1,\ x_1,\ x_1)$, $x_1 \in R$ arbitrary •