MATHEMATICS 2
Summer Semester, 2016/17

Subject Content - Sections, Lectures:

1. The definite integral, properties, evaluation.

2. Applications of definite integrals in geometry, in economics.

3. Systems of linear equations, solution methods, Gauss/Jordan Elimination Method.

4. Algebra of n-tuples (arithmetic vectors), linear space of n-tuples, linear dependence, independence of n-tuples.

5. Matrices, operations with matrices; inverse matrices. Applications of matrices.

6. Determinants, properties and evaluation of determinants. Solution of linear equations systems using determinants.

7. Vector space (linear space), basis and dimension of a linear space. Dot product, the norm of a vector. Orthogonal vectors, applications (optional).

8. Linear mappings, basic notions: matrices corresponding to linear mappings.

9. Euclidean space. Function of more variables, their graphs. Limits and continuity of a function of more variables.

10. Partial derivatives. Extrema of a function of more variables. Applications. Least Squares Method, applications.

11. Differential equations, basic notions. Linear differential equations of the 1st and 2nd order, solution methods.

12. Applications. Differential equations and exponential models.

VECTOR SPACES (LINEAR SPACES)

Motivation: Linear equations system, GEM Method: using ERO, we transform the augmented matrix of a system to a matrix in the reduced row form, at any step of the method, working in fact with *linear combinations* of rows.

Now, this idea of *linear combinations of elements of some set* will be crucial for us. Therefore, we would like to study sets - better, *structures* where it is possible to define the notion of such linear combinations of its elements.

The notion of linear combinations occurred also when solving the solution set of a homogeneous system of m linear equations with n unknowns $A \cdot x = 0$, where A is a matrix of that system. We know that the structure of the solution set is as follows:

- we need to find some basic, special set of solutions (as a special case, such a set could consists of a trivial solution only as well), denoted as *a base of solutions*,
- then, any *linear combination of those base solutions* is a solution of the system as well,
- but more: any other solution of the system is a linear combination of those base solution (or, there are no other solution outside linear combinations of base solution set).

Therefore: structures with *linear combinations of their elements*, properties, applications.

1. Vector Space $V_n(\mathbb{R})$

Definition. Let V be a set on which addition and scalar multiplication are defined:

- if α and β are objects in V and c is a scalar - a real number, $c \in \mathbb{R}$, then $\alpha + \beta$ and $c \cdot \alpha$ are elements in V.

Let the following axioms are true for all $\alpha,\ \beta,\ \gamma \in V$ and all $c \in \mathbb{R}$:

(1) $\alpha + \beta = \beta + \alpha$ (commutative law);
(2) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ (associativity);
(3) in V, there is a neutral (zero) element denoted as $0 \in V$ w.r. to addition, i.e. $\alpha + 0 = \alpha$;
(4) for every $\alpha \in V$ there is an element $\delta \in V$ as negative to α with property $\alpha + \delta = 0$, denoted as $\delta = -\alpha$;
(5) $(c + d) \cdot \alpha = c \cdot \alpha + d \cdot \alpha$ (distributivity w.r. to c, $d \in \mathbb{R}$)
(6) $c \cdot (\alpha + \beta) = c \cdot \alpha + c \cdot \beta$ (distributivity w.r. to α, $\beta \in V$)
(7) $(cd) \cdot \alpha = c \cdot (d \cdot \alpha)$;
(8) $1 \cdot \alpha = \alpha$.

Then V is called *a vector space* (or *a linear space*), and the objects of V are called *vectors*.

Remarks.

A real vector space: more than a set, this is *a structure* - a set endowed by some special operations. Using those operations it is possible to form new elements of the vector space.

In view of the definition, we are restricted here to use real numbers as scalars, $c \in \mathbb{R}$, therefore we generally call the vector space *a real vector space*.

Axioms: these are simply the rules under which we operate.

So, addition and multiple by scalars c, $c \in \mathbb{R}$, could be defined on a given set V in different ways. But: if all axioms are valid, any such system forms the real vector space.

Examples on real vector spaces:

Example 1. If n is any positive integer then the set $V_n(\mathbb{R})$ of all n-tuples of real numbers with the standard addition element-wise and scalar multiplication element-wise is a real vector space: one can verify all axioms are true.

E.g., take $n = 2$: addition and multiples of 2-component vectors is defined as

$$\text{for } \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{we have} \quad \vec{x} + \vec{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix},$$

$$\text{for } \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, c \in \mathbb{R} \quad \text{we have} \quad c \cdot \vec{x} = \begin{pmatrix} cx_1 \\ cx_2 \end{pmatrix}.$$

Zero element (or neutral element) in $V_2(\mathbb{R})$: vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

This linear space $V_2(\mathbb{R})$, or $V_n(\mathbb{R})$ in general: *linear space of arithmetical vectors*.

We shall use row vectors or column vectors, but preferably column ones (row vectors: for saving place only in writing them).

Preferably, for elements of the vector space, we shall write $\alpha,\ \beta, \ldots$ instead of $\vec{\alpha},\ \vec{\beta}, \ldots$.

Recall that equality of vectors in $V_n(\mathbb{R})$ is defined in the usual way also in element-wise way: for $\alpha = (a_1, \ldots, a_n)$, $\beta = (b_1,\ \ldots,\ b_n)$ we have

$$\alpha = \beta \text{ if and only if when } a_i = b_i \text{ for all } i = 1, \ldots, n;$$

zero vector: $0 = (0,\ \ldots,\ 0)$,

the negative vector to the vector α is the vector $(-a_1,\ \ldots,\ -a_n)$ denoted as $-\alpha$ (also called as the reverse vector).

In view how to apply operations in an arbitrary vector space, we define:

Definition. Let V be a vector space, let elements α_1, α_2, ..., $\alpha_k \in L$, c_1, c_2, ..., $c_k \in \mathbb{R}$. The vector

$$x = c_1\alpha_1 + c_2\alpha_2 + ... + c_k\alpha_k$$

is said to be *the linear combination* of vectors α_1, α_2, ..., α_k. Real numbers - scalars c_1, c_2, ..., c_k are coefficients of such linear combination.

Remark. **1.** Due to the definition, coefficients of a linear combination could be taken as arbitrary numbers: non-zero ones, or zero ones. Therefore, from this definition it follows:

zero vector of any V – vector 0 is a linear combination of any set of vectors!!! It is sufficient to take all coefficients as zeros: $c_1 = c_2 = ... = c_k = 0$.

Remark. A question whenever a given vector x is a linear combination of a given set of vectors α_1, α_2, ..., α_k states: the corresponding system of linear equations has to be solved!!! Let us provide some example, in which we shall work in the linear space $V_3(\mathbb{R})$ as an illustration.

Exercise. Determine whether the vector $x = (0,\ -3,\ 4) \in V_3(\mathbb{R})$ could be expressed as a linear combination of vectors α_1, α_2 of this vector space, with
a) $\alpha_1 = (1,\ -1,\ 2)$, $\alpha_2 = (2,\ 1,\ 0)$, or
b) $\alpha_1 = (-3,\ 2,\ 1)$, $\alpha_2 = (2,\ 1,\ 0)$.

Solution. In both cases, we are searching for real numbers c_1, c_2 such that

$$c_1\alpha_1 + c_2\alpha_2 = x.$$

a) It has to be $c_1\alpha_1 + c_2\alpha_2 = c_1(1,\ -1,\ 2) + c_2(2,\ 1,\ 0) = (0,\ -3,\ 4)$.
The condition vectors to be equal means to formulate 3 linear equations for their components or entries, that is for 2 unknowns c_1, c_2:

$$\begin{aligned} c_1 + 2c_2 &= 0 \\ -c_1 + c_2 &= -3 \\ 2c_1 \qquad &= 4 \end{aligned}$$

There is the unique solution of that system, namely $c_1 = 2$, $c_2 = -1$; thus, the vector x is the following linear combination

$$x = 2\alpha_1 - \alpha_2.$$

b) Now, for the pair of vectors α_1, α_2, our expression of x as

$$x = c_1\alpha_1 + c_2\alpha_2$$

means to solve the linear for 3 components of given vectors:

$$\begin{aligned} -3c_1 + 2c_2 &= 0 \\ 2c_1 + c_2 &= -3 \\ c_1 &= 4 \end{aligned}$$

There is no solution of this system; the vector x fails to be the linear combination of those given vectors α_1, α_2. ●

Examples of real vector spaces of special meaning: one can prove that the following sets with the binary operation of addition and with multiple of an element by a real number form a real vector space:

(a) all matrices of the size $m \times n$, $m, n \geq 1$, with addition and a multiple of a matrix in the sense of matrices operations defined; remark that zero "vectorïs the zero matrix of the size $m \times n$;

(b) all square matrices of 2nd degree, with addition and a multiple of a matrix in the sense of matrices operations defined; again, we have a zero element in this space as the square zero matrix of 2nd degree;

(c) the set $F(x)$ of all real functions of one variable defined on $\mathbb{R}$ (addition and multiple point-wise); it follows that the zero element in this space is the constant zero function;

(d) the set $F_{\langle a,\, b\rangle}(x)$ of all real functions of one variable defined on the closed interval $\langle a,\, b\rangle$ (addition and multiple in the same way, i.e. as derived from the previous example); it follows that the zero element in this space is the constant zero function defined now on $\langle a,\, b\rangle$ only;

(e) the set $F_0(x)$ of all real functions of one variable defined on $\mathbb{R}$ (addition and multiple point-wise), with the property $f(0) = 0$ for any $f \in F_0(x)$;

(f) the set $C(x)$ of all real functions of one variable defined on $\mathbb{R}$ (addition and multiple point-wise), *continuous* on $\mathbb{R}$ (the constant zero function is continuous);

(g) the set $P(x)$ of all polynomial functions of one variable, addition and multiple as usual operations;

(h) for a given $n \in N$, all polynomial functions of the form

$$P_n(x) = \{a_0 + a_1 x + a_2 x^2 + ... + a_n x^n, a_i \in \mathbb{R}\},$$

addition and multiple as usual operations;

(i) the solution set of a homogeneous equations system $A \cdot x = 0$ for a given matrix A of size $m \times n$ (with zero element as the zero n-tuple - this is, in fact, always a solution of that linear system).

How to check axioms are true: the key step is to verify that

- the sum of two elements and
- the multiple of any element by a real number are elements of the system as well.

The following sets with operation + and multiple by a real number fail to form a linear space:

(a) the set of all regular square matrices of the second degree, with addition operation and multiples by a real number in the usual way;

(b) the set $F_1(x)$ of all real functions of one variable defined on $\mathbb{R}$ (addition and multiple point-wise), with the property $f(0) = 1$ for any $f \in F_1(x)$;

(c) all polynomial functions $P(x)$ with non-negative coefficients, addition and multiple derived from the vector space of all polynomials.

Remark. Let $V(\ R)$ is an arbitrary real vector space, and let 0 is the zero element of this space. Then the subset $\{0\}$ is a vector space itself: sums and real multiples of 0 produce 0 only, and all axioms are satisfied. Then $\{0\}$ is called the *zero vector space.*

2. Subspaces of a vector space

In our examples of special, important vector spaces: some vector spaces were subsets of others (see examples of real functions with different properties), but more: they were also substructures:

Definition. Suppose that V is a vector space and W is a subset of V. If, under the addition and scalar multiplication that is defined on V, W is also a vector space, then we call W a *subspace of* V.

As the axioms valid in V are valid also in the subset, therefore also in W, the question when W is a subspace of the given space V needs verifying W is closed w.r. the addition and multiples by real numbers: we require that

- the sum of any two elements $\alpha, \beta \in W$ is back in W,

- the scalar multiple $c \cdot \alpha$ of any element $\alpha \in W$ will be back in W.

We also need to verify that the zero vector is in W, and that each element $\alpha \in W$ has a negative that is also in W. So, we have the theorem:

Theorem. Suppose that $W \neq \emptyset$ is subset of the vector space V. Then W will be a subspace of the space V if the following two conditions are true:

(a) If $\alpha,\ \beta \in W$, then $\alpha + \beta \in W$ as well (i.e. W is closed under addition);

(b) If $\alpha \in W$ and c is any scalar, then $c \cdot \alpha \in W$ as well (i.e. W is closed under scalar multiplication).

Where the definition of addition and scalar multiplication on W are the same as on V.

Proof can be done verifying all axioms for vector spaces.

Fact - due to the definition of subspace: Every vector space V has at least two subspaces: namely, V itself (as the largest one), and $W = \{0\}$ (the zero space, as the smallest possible one).

Example. Let us determine if the set P_n of all polynomials of degree at most n is a subspace of the vector space $F(x)$ of all real functions defined on $\mathbb{R}$.

Solution. As we know, a polynomial is said to have degree n if its largest exponent is n. Suppose we have two polynomials of degrees at most n:

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n, \;\; q_n(x) = b_0 + ba_1 x + b_2 x^2 + ... + b_n x^n,$$

$a_i,\ b_i \in \mathbb{R}$ and let $c \in \mathbb{R}$ be any scalar. Then,

$$p_n(x) + q_n(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + ... + (a_n + b_n)x^n,$$

$$c \cdot p_n(x) = ca_0 + ca_1 x + ca_2 x^2 + ... + ca_n x^n.$$

In both cases, the degree of the new polynomial is not greater than n. (Of course in the case of scalar multiplication it will remain degree n, but with the sum, the degree of the sum is at most n, which is sufficient for us.)

Therefore, P_n is closed under addition and scalar multiplication, and so it will be a subspace of of the vector space $F(x)$ of all real functions defined on $\mathbb{R}$. •

3. Span of a set in the linear space

Motivation. Suppose that $M \neq \emptyset$ is a subset in a linear space V, not necessarily a subspace of V. Question: is there a smallest subspace of V containing the set M? And, in positive case, how to find - to construct it?

We know that, in case $M \neq \emptyset$, in the subspace with elements from M we shall necessarily have also sums and multiples of elements from M,
- then sum and multiples of those sums and multiples, ... , and so on,
but it seems that, on the other hand, to create all linear combinations is sufficient as well; it follows as a result we shall obtain the following:

Theorem. Suppose that $M,\ M \neq \emptyset$ is a subset in a linear space V. Then the smallest subspace of V containing the set M is the subspace consisting of all possible linear combinations

$$\{c_1\alpha_1 + c_2\alpha_2 + \ldots + c_k\alpha_k, c_i \in \mathbb{R}, \alpha_i \in M\}$$

of elements from the set M.

Proof. We have to show that this set - the set of all possible linear combinations of elements from M - is closed w.r. to sums and scalar multiples. The principle of the proof is based on two observations: for elements expressed as linear combinations of some elements in M of a vector space V we have

– the sum of two linear combinations is a linear combination as well,

– a scalar multiple of a linear combination is a linear combination as well.

Moreover, as we know, the zero vector is a linear combination of any set of elements of M. Next step of the proof is then verification of axioms of vector space, and one can show that this space of all linear combinations of elements from M is, in fact, the smallest vector space containing the "generating"set M.
•

The linear space constructed in this way: *the span* of the set M, its notation: $[M]$.

In case when the set M consists of finite number of vectors from the vector space V, i.e.

$$M = \{x_1, \, x_2, \ldots x_k\}$$

we shall write $[M] = span\{x_1, \, x_2, \ldots x_k\}$. This vector space consists of all linear combinations of vector indicated in the set. Those vectors

$$x_1, \, x_2, \ldots x_k$$

are denoted also as *generators of the vector space* $[M] = [\{x_1, \, x_2, \ldots x_k\}]$, or, this vector space is generated by that set of vectors (the case of infinitely many vectors is not excluded as well).

Repeat again that the span $[M]$ of the set M in the vector space V

(a) is a subspace of V, and

(b) is the smallest subspace of V that contains all of the vectors $x_1, \, x_2, \ldots x_k$.

Examples. Find the spans $[M]$ of the following sets of vectors:
a) $M = \{(3,\ 4),\ (1,\ 2),\ (0,\ 1)\}$ in V_2;
b) $M = \{(3,\ 4,\ 1), (2,\ 0,\ 1)\}$ in V_3.
Solution. a) In view of previous considerations: span $[M]$ consists of all linear combinations - vectors $y \in V_2$ of the form
$y = c_1 \cdot (3,\ 4) + c_2 \cdot 1,\ 2) + c_3 \cdot (0,\ 1)$, where $c_1,\ c_2,\ c_3$ are arbitrary real numbers, coefficients of these linear combinations. If all $c_1,\ c_2,\ c_3 = 0$, then we see that the vector $(0,\ 0) \in [M]$.

How to get the overview of the span $[M]$? For example, put the question: what about reducing the number 3 of those vector to 2 only? Or, in other words: would be possible to generate the vector $(0,\ 1)$ from the remaining two as their linear combination? Solve:
we ask on the existence of coefficients $d_1,\ d_2$ with the property
$d_1 \cdot (3,\ 4) + d_2 \cdot 1,\ 2) = (0,\ 1),$
which is equivalent for the search of the solutions of the system of 2 linear equations

$$\begin{aligned} 3d_1 + \ d_2 &= 0 \\ 4d_1 + 2d_2 &= 1 \end{aligned}$$

Apply Cramer's rule for the system with the square matrix A: for its determinant we have

$$det\ A = \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = 6 - 4 = 2 \neq 0,$$

therefore such system has a unique solution $d_1 = -1/2$, $d_2 = 3/2$; conclusion: $[M] = [\{(3,\ 4),\ (1,\ 2),\ (0,\ 1)\}] = [\{(3,\ 4),\ (1,\ 2)\}]$, or the vector $(0,\ 1)$ is redundant here.

But return to the previous system of linear equations: take an arbitrary vector $\alpha = (a,\ b) \in V_2$; could be true that $\alpha \in [M]$? Answer will be given by the existence/nonexistence of solutions of the system of linear equations with the right side $(a,\ b)$ as a column:

$$\begin{aligned} 3d_1 + \ d_2 &= a \\ 4d_1 + 2d_2 &= b \end{aligned}$$

But the same argumentation is valid: $det\ A = \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} \neq 0$,, hence we have even that $[M] = V_2$! We formulate: the set M generates the whole linear space V_2. (Question: which property of the given vectors lead to the result? Try to repeat the search for the pair $\beta = (3,\ 4)$, $\gamma = (6,\ 8)$ - the vector $(0,\ 1)$ fails to be the linear combination of $\beta,\ \gamma$, thus the span $[\beta,\ \gamma] \subset V_2$.

4. Linearly dependent/independent vectors

Question: Suppose that we have k vectors α_1, α_2, ..., $\alpha_k \in V$ with property

$$[\alpha_1, \alpha_2, ..., \alpha_k] = V$$

(that is, generators of V). Is there possible to find some smaller subset of that set such that, also vectors in such smaller set still will be generators of V?

Which vectors could be left: vectors of linear combinations of some others vectors. But which vectors then remain, how to find them?

Definition. Let α_1, α_2, ..., $\alpha_n \in V$. Those vectors are said to be *linearly dependent*, if there are real numbers c_1, c_2, ..., c_n, at least one of them being different from zero, such that

$$c_1 \cdot \alpha_1 + c_2 \cdot \alpha_2 + ... + c_n \cdot \alpha_n = 0.$$

Remark. It is evident that the condition - the equality

$$c_1 \cdot \alpha_1 + c_2 \cdot \alpha_2 + ... + c_n \cdot \alpha_n = 0$$

is surely true with all coefficient as zeros: $c_i = 0$. But this is a trivial property, and there no sense to involve such a trivial "definition";

so, the definition in fact expresses the non-trivial case:

- for those vectors, it is possible to find real numbers in such a way that the zero vector on the right side is a *non-trivial* linear combination, i.e., there is at least one real number $c_i \neq 0$ such that

$$c_1 \cdot \alpha_1 + c_2 \cdot \alpha_2 + ... + c_n \cdot \alpha_n = 0.$$

For example: vectors $\alpha_1 = (1,\ 1)$, $\alpha_2 = (1,\ 2)$, $\alpha_3 = (2,\ -3) \in V_2(\ R)$ are linearly dependent:

$$7\alpha_1 - 5\alpha_2 - \alpha_3 = 0$$

Conversely, if the equation in question is fulfilled *exclusively* for all coefficients as zeros, such a set of vectors is called as linearly independent one:

Definition. Vectors α_1, α_2, ..., $\alpha_n \in V$ are said to be *linearly independent* provided

$$c_1 \cdot \alpha_1 + c_2 \cdot \alpha_2 + ... + c_n \cdot \alpha_n = 0$$

implies $c_i = 0$ for all $i = 1, ..., n$.

Exercise. Two vectors $\alpha_1 = (1, 1)$, $\alpha_2 = (1, 2) \in V_2(R)$ are linearly independent: the condition for coefficients c_1, c_2 for their linear combination to be zero

$$c_1 \cdot \alpha_1 + c_2 \cdot \alpha_2 = 0 \quad \text{or} \quad c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

leads to a homogeneous equations system with $D = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \neq 0$,

thus, there is only a trivial solution $c_1 = c_2 = 0$, i.e. vectors α_1, α_2 are linearly independent. •

In general: two vectors $\alpha_1 = (a,\, b),\, \alpha_2 = (c,\, d) \in V_2(\,R)$ are linearly independent if and only if $D = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc \neq 0.$

Special case: unit vectors $(1,\, 0,\, 0),\, (0,\, 1,\, 0),\, (0,\, 0,\, 1) \in V_3(\,R)$ are linearly independent.

Vectors $(1,\ 1,\ 0),\ (1,\ 0,\ 1),\ (0,\ 1,\ 1)\ \in\ V_3(\ R)$ are linearly independent as well: let us search for numbers c_1, c_2, c_3 with properties

$$c_1 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We get the linear equations system:

$$\begin{aligned} c_1 + c_2 \qquad &= 0 \\ c_1 \qquad + c_3 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

Its determinant: $D = -2 \neq 0$, thus, there is only a trivial solution

$$c_1 = c_2 = c_3 = 0,$$

and vectors are linearly independent.

Remark - an important one.

How to decide the problem on linear dependence, or linear independence of vectors in $V_n\,R$: we have to show the existence of non-trivial solution, or the trivial solution only, of a homogeneous linear equations system with unknowns as coefficients of a linear combination of the vectors in question.

Suppose we have now exactly n vectors $\alpha_1, \alpha_2, ..., \alpha_n \in V_n(\,R)$ - so, the special case: n vectors in $V_n(\,R)$, and suppose that D is the determinant consisting of those vectors as its columns. Apply conditions for solutions of the square homogeneous system (Cramer's Rule) with D as its determinant:

$$\boxed{\begin{array}{l} \alpha_1, \alpha_2, ..., \alpha_n \in V_n(\,R) \text{ lin. independent } \leftrightarrow \text{ trivial solution only} \leftrightarrow D \neq 0 \\[2em] \alpha_1, \alpha_2, ..., \alpha_n \in V_n(\,R) \text{ lin. dependent } \leftrightarrow \text{ non-trivial solution} \leftrightarrow D = 0 \end{array}}$$

Criterion:

Theorem. Vectors α_1, α_2, ..., α_n in V are linearly dependent if and only if at least one of them is a linear combination of the remaining ones.

Proof. Suppose that vectors α_1, α_2, ..., α_n are linearly dependent. Then in the equation

$$c_1 \cdot \alpha_1 + c_2 \cdot \alpha_2 + ... + c_n \cdot \alpha_n = 0$$

we have at least one coefficient, say c_j, with $c_j \neq 0$. Then express it as

$$\alpha_j = -\frac{1}{c_j}(c_1 \cdot \alpha_1 + ... + c_{j-1} \cdot \alpha_{j-1} + c_{j+1} \cdot \alpha_{j+1} + ... + c_n \cdot \alpha_n)$$

thus the vector α_j is the linear combination of those vectors remaining vectors.

Conversely, suppose that some of vectors α_1, α_2, ..., α_n is the linear combination of those remaining ones; let α_k with $1 \leq k \leq n$ is just this vector. Then the linear combination condition could be rewritten as the condition for the linear dependence of vector with at least one coefficient as non-zero one, namely on condition on linear dependence of all vectors, with one non-zero coefficient c_k (as $c_k = 1$). •

Remark. It follows:

(1) Any group of vectors containing zero vector is a set of linearly dependent vectors.
(2) Two vectors are linearly dependent if and only if one of them is a multiple of the second one.

If vectors α_1, α_2, ..., α_n are generators of the linear space V, i.e.

$$[\alpha_1, \alpha_2, ..., \alpha_n] = L,$$

then it is possible to find a largest group in the set of those vectors with the property they are generators of V as well.

Definition. A linear space is said to be *a space of finite dimension* if there exists at least one finite set of vectors $A = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ in this space such that A is the set of generators of V, or equivalently

$$[A] = [\alpha_1, \alpha_2, ..., \alpha_n] = V.$$

5. Basis of a linear space

Definition. The set B, $B \subset V$, of vectors in the vector space V is said to be *a basis of the vector space*, if B has any of the following two properties:

(1) B is a set of linearly independent vectors;

(2) $[B] = L$, or V is the span of B (or, equivalently, B is the set of generators of V).

Examples. Vectors $e_1 = (1,\ 0,\ 0)$, $e_2 = (0,\ 1,\ 0)$, $e_3 = (0,\ 0,\ 1)$ form a basis of the vector space $V_3(\ R)$: this set
- consists of linearly independent vectors,
- any vector $x \in V_3(\ R)$ is a linear combination of those vectors:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

moreover, coefficients in this linear combination are coordinates of the vector in question. This basis: *the standard* or usual, obvious, canonical, unit basis of that space $V_3(\ R)$ (analogously for the space $V_n(\ R)$). •

It holds:

Theorem. Let $B = \{\beta_1, \beta_2, ..., \beta_k\}$ be a basis of the vector space V, let $x \in L$ be a linear combination of elements in the basis as

$$x = c_1 \cdot \beta_1 + c_2 \cdot \beta_2 + ... + c_k \cdot \beta_k.$$

Then coefficients c_i, $i = 1, ..., k$, are determined in a unique way.

Proof. Suppose that there is possible to express the vector x in the form of linear combination as $x = \sum_{j=1}^{k} c_j \cdot \beta_j$ and, in the same time, using coefficients d_j as $x = \sum_{j=1}^{k} d_j \cdot \beta_j$. Then it follows

$$\sum_{j=1}^{k} (c_j - d_j) \cdot \beta_j = 0;$$

but in view of linearly independent elements of the basis we get

$$c_j - d_j = 0 \text{ for all } j = 1, 2, ..., k$$

and our theorem is proved. •

Remark. Let $B = \{\beta_1, \beta_2, ..., \beta_k\}$ be a basis of the vector space V, suppose that for $x \in V$ we have $x = \sum_{j=1}^{k} c_j \cdot \beta_j$.

Coefficients c_j: *coordinates of the vector x with respect to the basis B*;

the choice of the basis B, and also the order of elements of the basis, determine the coordinates of the vector!!!

E.g., $x = (-2, 1) \in V_2(R)$ has coordinates $x_1 = -2$, $x_2 = 1$ with respect the standard, unit basis $\{(1, 0), (0, 1)\}$;

but when we choose the basis of $V_2(R)$ as $B = \{(3, -1), (-1, 0)\}$, the same vector x has now coordinates due to the system of equations:

$$x = \begin{pmatrix} -2 \\ 1 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Solution: coordinates of x with respect to the basis B: $c_1 = c_2 = -1$; we shall write it as $x = (-1, -1)_B$ (the subscript B shows the basis, and we shall omit it only in case of the standard - unit basis).

Definition. *The dimension of the vector space* V is called the number of elements in the basis of the space. Notation: $dim\ V$.

It can be proved that, in case the vector space has a basis with finite number of elements, then all bases are finite as well, and they have the same number of basis elements.

Vector space with finite numbers of basis elements: vector spaces of finite dimension: $dim\ L = n$,

otherwise: $dim\ L = \infty$.

Example at the end: the vector space $V_n(\mathbb{R})$ is a space of finite dimension n (as its standard basis consists of n unit vectors).