MATHEMATICS 2
Summer Semester, 2012/13

Subject Content - Sections, Lectures:

1. The definite integral, properties, evaluation.

2. Applications of definite integrals in geometry, in economics.

3. Systems of linear equations, solution methods, Gauss/Jordan Elimination Method.

4. Algebra of n-tuples (arithmetic vectors), linear space of n-tuples, linear dependence, independence of n-tuples.

5. Matrices, operations with matrices; inverse matrices. Applications of matrices.

6. Determinants, properties and evaluation of determinants. Solution of linear equations systems using determinants.

7. Vector space (linear space), basis and dimension of a linear space. Dot product, the norm of a vector. Orthogonal vectors, applications (optional).

8. Linear mappings, basic notions: matrices corresponding to linear mappings.

9. Euclidean space. Function of more variables, their graphs. Limits and continuity of a function of more variables.

10. Partial derivatives. Extrema of a function of more variables. Applications. Least Squares Method, applications.

11. Differential equations, basic notions. Linear differential equations of the 1st and 2nd order, solution methods.

12. Applications. Differential equations and exponential models.

MATRICES, DETERMINANTS

Matrices, Application of Matrices

Motivation: Matrices: a tool useful to express the binary relation between elements of two finite sets.

Example 0. Between 4 people Olga, Ian, Charles and Tom, members of a working group, a relation "x has an influence on y"was studied, and results are expressed in the scheme (where the symbol "$x \longrightarrow y$"means "x has an influence on y") :

$$\begin{array}{ccccc} & & \text{Ian} & & \\ & \swarrow & & \searrow & \\ \text{Tom} & & \longleftarrow & & \text{Olga} \\ & \nwarrow & & \swarrow & \\ & & \text{Charles} & & \end{array}$$

Who is the „strongest member” of the group in the sense: he/she has an influence on possibly the largest subgroup of members of that group? (Then, he/she has a chance to be a leader of this group.)

Example 1. In a plant, from four types of raw material s_1, s_2, s_3, s_4 three products p_1, p_2, p_3 are produced; to produce one unit of any product p_1, p_2, p_3 it is necessary to use number of units of raw material s_1, s_2, s_3, s_4 as indicated in the table:

	p_1	p_2	p_3
s_1	5	6	2
s_2	3	0	1
s_3	0	2	4
s_4	2	3	2

This table: a matrix of the type or the size 4×3 (4 rows, 3 columns);

this matrix represents a relation between 4 types of raw material and 3 products - *consumption matrix*;

entries of the matrix are real numbers, their position in the table - in the matrix is important

Example 2. 4 villages O_1, O_2, O_3, O_4, roads: suppose that from

– O_1 there is a direct road to O_2, O_3, O_4,
– O_2 there is a direct road to O_1 only,
– O_3 there is a direct road to O_1, O_4,
– O_4 there is a direct road to O_1, O_3

(a direct road means: direct connection, not passing through some other village in the list).

Question: to express, in some transparent way, the system of all possible direct connections of those villages. For this purpose, a square matrix $I = (a_{ij})$ of size 4×4 could be used (we have 4 villages in the list):

$a_{ij} = 1 \quad$ for $\quad 1 \leq i, j \leq 4 \quad \Longleftrightarrow$ there is a direct road from O_i to O_j,

otherwise $a_{ij} = 0$ (besides, $a_{ii} = 1$, as it is natural to suppose the existence of the direct road from any O_i to O_i itself.) Therefore, this matrix is of the form

$$I = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \qquad \textit{the incidence matrix}$$

From the matrix I, the "road map"could be reconstructed in a unique way up to the ordering of villages O_1, O_2, O_3, O_4. Such matrices are used also for some other nets, esp. with a large number of elements - entries.

Definition. *A matrix* A is defined as a rectangular array of $m \times n$ entries (esp. real numbers), therefore m rows, n columns of entries. Such a matrix is *of the size or the type* $m \times n$.

Denote the entry in the ith row and jth column of the matrix A as a_{ij}. Then the matrix A could be written in the form

$$A = \begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

with m: the number of rows, n: number of columns in A. The couple m, n denotes *the size - the type of the matrix*: the matrix A is *of size* $m \times n$.

Shortly: $A = (a_{ij}), 1 \leq i \leq m, 1 \leq j \leq n,$ i is the index of the row, j is the index of the column.

For $m = n$: this matrix is said to be *a square matrix of the size* $n \times n$.

Entries a_{ii} in the matrix: *the main diagonal of the matrix* A, or *main diagonal entries* (another notation: "northwest corner - southeast corner for non-square matrix as well).

A symmetrical matrix is any square matrix A with the property

$$a_{ij} = a_{ji} \text{ for all } i,\ j \in \{1,\ 2,\ ...,\ n\}$$

A skew symmetrical matrix is any square matrix A with the property

$$a_{ij} = -a_{ji} \text{ for all } i,\ j \in \{1,\ 2,\ ...,\ n\}$$

Matrices of the sizes $m \times n$ for $m = 1$ or $n = 1$:
a vector (a row vector) for $m = 1$, *a column one* for $n = 1$.

Operation on matrices - matrix algebra

Definition. If A, B are both matrices of the same size $m \times n$ then we say that $A = B$ provided corresponding entries from each matrix are equal. In other words,

$$A = B \text{ if and only if } a_{ij} = b_{ij}$$

for all i, j, $1 \leq i \leq m$, $1 \leq j \leq n$.

Matrices of different sizes cannot be equal.

Therefore, the equality of two matrices A, B of the same size is defined „element-wise"(entries on the same positions have to be equal). This principle is applied on some other operations defined with matrices.

Next we move on to addition and subtraction of two matrices: these new matrices will be defined simply by adding/subtracting corresponding entries from each matrix.

Definition. *Addition/subtraction of two matrices.* Suppose that $A = (a_{ij})$, $B = (b_{ij})$ are two matrices of the same size $m \times n$ with real numbers as entries. Then their *sum* $C = (c_{ij})$ is a matrix of the same size $m \times n$, denoted as $C = A + B$ such that for the entry c_{ij} we have the sum of corresponding entries, i.e.

$$c_{ij} = a_{ij} + b_{ij}.$$

Similarly, their *difference* $D = (d_{ij})$ is a matrix of the same size $m \times n$, denoted as $D = A - B$ such that for the entry d_{ij} we have again element-wise

$$d_{ij} = a_{ij} - b_{ij}.$$

In view that entries of matrices A, B are real numbers, we see that for the addition of two matrices A, B the commutative law is true:

$$A + B = B + A.$$

(We note that the sum and the difference of matrices of different size is not defined.

For example, let

$$A = \begin{pmatrix} 12 & 6 & 0 & 9 \\ 12 & 3 & 2 & 19 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 9 & 0 & 13 \\ 10 & 6 & 4 & 10 \end{pmatrix}$$

Then for their sum, and difference, resp., we have

$$C = \begin{pmatrix} 12 & 6 & 0 & 9 \\ 12 & 3 & 2 & 19 \end{pmatrix} + \begin{pmatrix} 0 & 9 & 0 & 13 \\ 10 & 6 & 4 & 10 \end{pmatrix} = \begin{pmatrix} 12 & 15 & 0 & 22 \\ 22 & 9 & 6 & 29 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 9 & 0 & 13 \\ 10 & 6 & 4 & 10 \end{pmatrix} - \begin{pmatrix} 12 & 6 & 0 & 9 \\ 12 & 3 & 2 & 19 \end{pmatrix} = \begin{pmatrix} -12 & 3 & 0 & 4 \\ -2 & 3 & 2 & -9 \end{pmatrix}$$

Definition. If A is any matrix of the size $m \times n$ and c is any real number, then *the product (or scalar multiple)*, denoted as $D = c \cdot A$, is a new matrix of the same size - type as A and its entries are found by multiplying the original entries of A by c. In other words, for entries of $D = (d_{ij})$ we have

$$d_{ij} = c \cdot a_{ij}.$$

For $c = -1$, the matrix D is denoted as $-A$, and named as *the reverse matrix*.

The following rules are true (distributive laws for scalar multiples of matrices):

$$c \cdot (A + B) = c \cdot A + c \cdot B, \quad (c + d) \cdot A = c \cdot A + d \cdot A.$$

Definition. *Zero matrix* of the size $m \times n$ is a matrix A with entries entirely equal to zero: for all entries we have $a_{ij} = 0$ for all $1 \leq i \leq m$, $1 \leq j \leq n$.

The sum of the matrix A with its reverse matrix in the zero matrix (of the corresponding size).

If A is a zero matrix of the size $m \times n$, then for any matrix B of the same size $m \times n$ we have $A + B = B + A = B$.

Definition. *The unit matrix E_n of the size $n \times n$* is a square matrix such that

$$a_{ii} = 1 \text{ for all } \quad i = 1, ..., n, \;\; a_{ij} = 0 \text{ for } \quad i \neq j, \;\; 1 \leq i, \; j \leq n$$

$$E_n = \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{pmatrix}$$

Definition. *A matrix transposed to a given matrix.* Suppose $A = (a_{ij})$ is a matrix of the size $m \times n$. A matrix $B = (b_{ij})$ of the size $n \times m$ such that

$$b_{ij} = a_{ji}$$

is said to be *the transposed matrix to the matrix* A, or *a transpose of a given matrix*; denote it as $B = A^T$.

For example, let us find matrices transpose to the following ones:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}:$$

$$A^T = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad B^T = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C^T = \begin{pmatrix} 2 & 3 & -4 \end{pmatrix}.$$

in view of the definition, we have:

$$(A^T)^T = A, \;\; (E_n)^T = E_n, \;\; (A+B)^T = A^T + B^T;$$

moreover, $A^T = A$ for any square symmetrical matrix A.

The product of two matrices

Exercise. Suppose there is a direct bus connection of villages V_1, V_2, V_3, V_4 with some of 3 towns T_1, T_2, T_3 expressed in the first table (with the use of symbols 1, 0 : "there is a bus connection", or "there is no bus connection"). Suppose further that from those towns, you can continue by special trains to one from 4 airports A_1, A_2, A_3, A_4 due to the second table (symbols 1, 0 in the same meaning). Find which village has the bus + train connection with some of airports indicated.

	T_1	T_2	T_3
V_1	1	1	0
V_2	0	0	1
V_3	1	0	1
V_4	0	0	1

	A_1	A_2	A_3	A_4
T_1	1	1	0	1
T_2	0	0	1	1
T_3	1	0	0	0

Solution. Suppose the tables are incidence matrices A, B. Our aim is to find the relation between 4 villages and 4 airports - use again the incidence matrix C with entries c_{ik} as 1 or 0 (of the size 4×4);

When there is a direct connection between the village V_i and the airport A_k, i.e. when $c_{ik} = 1$ for $1 \leq i \leq 4, 1 \leq k \leq 4$:

it is necessary, but also sufficient, to find at least one town, say T_j, such that there is a direct bus connection from the village V_i to T_j, and this town T_j has a direct connection with the kth airport A_k.

This means $c_{ik} = 1$ exactly when there exists the index j, $1 \leq j \leq 3$ such that $a_{ij} = 1$ and in the same time $b_{jk} = 1$.

On the other hand, when $c_{ik} = 0$ is true: there is no j such that we have such incidence of symbols 1 on the same positions - i.e., searching for couples in the ith row of the first matrix A and in the same time in the kth column of the matrix B, we obtain for any $1 \leq j \leq 3$ a couple from the following set of combinations only: $\{00,\ 10,\ 01\}$.

In this way, we combine all *rows of the first matrix* A (in total: 4 rows) with all *columns of the second matrix* B (in total: 4 columns). Finally, we get 16 entries of the matrix in question $C = (c_{ik})$: we have

	T_1	T_2	T_3
V_1	1	1	0
V_2	0	0	1
V_3	1	0	1
V_4	0	0	1

	A_1	A_2	A_3	A_4
T_1	1	1	0	1
T_2	0	0	1	1
T_3	1	0	0	0

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Definition. Suppose $A = (a_{ij})$ is a matrix of the size $m \times n$, $B = (b_{jk})$ is a matrix of the size $n \times r$. Then, *the product of two matrices* A, B is a matrix $C = (c_{ik})$ of the size $m \times r$ with its entry c_{ik} as

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{ij}b_{jk} + \ldots + a_{in}b_{nk}$$

How to find the entry c_{ik} of the matrix $C = A \cdot B$ combining entries of the ith *row* of the matrix A with entries of the kth *column* of the matrix B:

$$\begin{pmatrix} a_{11} & \ldots & a_{1j} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \boxed{a_{i1}} & \ldots & \boxed{a_{ij}} & \ldots & \boxed{a_{in}} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & \ldots & a_{mj} & \ldots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \ldots & \boxed{b_{1k}} & \ldots & b_{1r} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ b_{j1} & \ldots & \boxed{b_{jk}} & \ldots & b_{jr} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ b_{n1} & \ldots & \boxed{b_{nk}} & \ldots & b_{nr} \end{pmatrix} =$$

$$= \begin{pmatrix} c_{11} & \ldots & c_{1k} & \ldots & c_{1r} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{i1} & \ldots & \boxed{c_{ik}} & \ldots & c_{ir} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{m1} & \ldots & c_{mk} & \ldots & c_{mr} \end{pmatrix}$$

Using the Greek capital Σ as the symbol for the sum as e.g.

$$s_1 + s_2 + ... + s_n = \sum_{j=1}^{n} s_j,$$

then, for c_{ik}, it is possible to write

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + ... + a_{ij}b_{jk} + ... + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk}.$$

Remark. Be careful: our definition implies that the product of two matrices is defined for two matrices A, B of such sizes only that *the number of columns of the matrix A is the same as the number of rows of the matrix B* !!!

Exercise. Compute products $A \cdot B$ and $B \cdot A$ for given matrices A, B:

$$A \cdot B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

We see that in general, *the product of two matrices fails to be a commutative operation*, therefore this operation with matrices differs from the usual multiplication of two real numbers: due to our example, in general we have

$$A \cdot B \neq B \cdot A$$

Therefore, the product of two matrices depends on when the matrix A is multiplied by the matrix B *from the right (side)* or *from the left (side)*!!!

Exercise - matrix multiplication. A metal foundry produces bronze alloy of 2 types, A_1, A_2; quantities of copper (Cu), tin (Sn) and zinc (Zn) which are necessary for to produce one unit of the alloy of any of those types are given in the first table. Copper, tin and zinc are available from two suppliers S_1, S_2 and the second table shows prices rated by suppliers (in dollars per unit of quantity):

	Cu	Sn	Zn
A_1	4800	600	300
A_2	6000	1400	700

	S_1	S_2
Cu	0,75	0,70
Sn	6,50	6,70
Zn	0,40	0,50

Question: how to decide when choosing a supplier to get the most appropriate price(s)?

Solution. Suppose tables are taken as matrices M, N. Entries in the product $M \cdot N$ of those matrices show relation between (quantities of) alloys A_1 and A_2 (rows) and prices (per unit) rated by suppliers S_1, S_2 (columns):

$$C = M \cdot N = \begin{pmatrix} 4800 & 6004 & 300 \\ 6000 & 1400 & 700 \end{pmatrix} \cdot \begin{pmatrix} 0,75 & 0,70 \\ 6,50 & 6,70 \\ 0,40 & 0,50 \end{pmatrix} = \begin{pmatrix} 7620 & 7530 \\ 13880 & 13930 \end{pmatrix}$$

It follows that foundry has to pay to the first supplier S_1 in total

$$7\,620 + 13\,880 = 21\,700 \text{ dollars}$$

(the first column sum), to the second supplier S_2

$$7\,530 + 13\,930 = 21\,460 \text{ dollars}$$

(the second column sum); hence, this matrix multiplication provides a tool how to decide which supplier has to be chosen. •

Further remarks on matrix multiplication

(1) Let us recall again that we have seen that multiplication of matrices is, in general, non-commutative: for $A \cdot B, \quad B \cdot A$ we have

$$A \cdot B \neq B \cdot A$$

(2) Multiplication of matrices is *associative*: for matrices A, B, C of appropriate sizes, we have the associative law

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

More, two *distributive laws* are true as well in the sense:

$$(A + B) \cdot C = A \cdot C + B \cdot C, \quad C \cdot (A + B) = C \cdot A + C \cdot B$$

(3) For the unit matrix E_n of the size $n \times n$ and for an arbitrary square matrix A of the size $n \times n$ we have

$$A \cdot E_n = E_n \cdot A = A$$

(4) For any square matrix A of the size $n \times n$, its mth power is defined due to induction as follows:

$$\text{we put} \quad A^1 = A, \qquad A^m = A^{m-1} \cdot A;$$

it is convenient to put also $A^0 = E_n$.

(5) A product of the matrix A of the size $m \times n$ with a column matrix - vector B of the size $n \times 1$ is a column vector as well, its size is $m \times 1$; similarly, for a row matrix - vector D of the size $1 \times m$ this product $D \cdot A$ is a row vector of the size $1 \times n$.

(6) A product of the matrix of the size $1 \times n$ (row vector) with a matrix of the size $n \times 1$ (column vector) is interpreted (mostly) as *a real number*, therefore not a matrix (it depends on the context). This operation is called as *the scalar product* of two vectors, or better or more frequently as *a dot product*. E.g.

$$\begin{pmatrix} -1 & 1 & 6 \end{pmatrix} \cdot \begin{pmatrix} 15 \\ 10 \\ 10 \end{pmatrix} = -15 + 10 + 60 = 55.$$

(7) In contrary to multiplication of two real numbers, both different from zero, *the product of two non-zero matrices could be a zero matrix*, as the following example shows:

$$A \cdot B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Inverses of matrices

Definition. Let A be a square matrix of degree n. If there is a matrix X such that

$$A \cdot X = X \cdot A = E_n,$$

then this matrix X is said to be *the inverse matrix* to the matrix A and it is denoted as A^{-1}.

From the definition of the inverse matrix A^{-1} it follows:

(1) A^{-1} is a square matrix of degree n, and it fulfills

$$A \cdot A^{-1} = A^{-1} \cdot A = E_n$$

(2) The inverse matrix to a given matrix (provided it exists) is determined uniquely.

(3) It holds $(A^{-1})^{-1} = A$ (the inverse matrix to a matrix A^{-1} is the original matrix A).

(4) For the unit matrix, it follows from the definition that $E_n^{-1} = E_n$.

(5) It can be proved that the inverse matrix to the product $A \cdot B$ of two matrices is the product of their inverse matrices *in the reverse order*, i.e.

$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1},$$

provided both inverse matrices A^{-1}, B^{-1} do exist.

How to compute the inverse matrix to a given matrix: the method for finding the inverse matrix A^{-1} to a given matrix A is based on ERO operations applied on rows of A. We shall present this method on a matrix of degree 3 (generalization for matrices of degree n is a technical way and formally we need to enlarge number of entries in rows and columns only). Denote

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix};$$

for the product it holds $A \cdot A^{-1} = E_3$. Denote columns of the inverse matrix A^{-1} as x, y, z (starting from the left): so, in a shorter notation, we have $A^{-1} = (x \ y \ z)$. Further, denote columns of the unit matrix E_3 as

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

then, the equality $A \cdot A^{-1} = E_3$ implies that for finding the inverse matrix A^{-1}, we have to solve 3 matrix equations, or equivalently 3 systems of linear equations with the same matrix A:

$$A \cdot x = e_1, \quad A \cdot y = e_2, \quad A \cdot z = e_3$$

Solution: apply GEM with aim to reduce the matrix A on the unit matrix E_3 (if possible), it means *simultaneously for 3 right sides of those equations*. It means, on the right side we shall have the unit matrix E_3. Therefore: write E_3 next to A; then provide ERO operations on A, and in the same time, *provide the same ERO operations* on E_3:

$$(A \mid E) = \left(\begin{array}{ccc|ccc} a_1 & b_1 & c_1 & 1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 1 \end{array}\right)$$

Due to this algorithm, we shall have instead the original matrix A the unit matrix E_3,

and on the place where E_3 has been written originally, we shall get the inverse matrix A^{-1}.

This method is named as *the Jordan's method for finding the inverse matrix*. In general, for a matrix A of degree n in a symbolic notation

$$(A \mid E_n) \quad \longrightarrow \quad (E_n \mid A^{-1}).$$

Problem of the existence of the inverse matrix: the method indicated above means that in this way, we work on finding the reduced row-echelon form matrix to the given (square)matrix A. Thus, the existence of the inverse matrix is equivalent to the existence of that reduced row-echelon form as a unit matrix of the corresponding degree n.

Remark. In general, the system of m linear equations with n unknowns, with its matrix A and a column of right sides of equations b (therefore, with the augmented matrix of the size $m \times (n+1)$) could be expressed as a matrix multiplication

$$A \cdot x = b,$$

where x is *the column vector* consisting of unknowns x_1, x_2, ..., x_n. Such an equation represents, in fact, an equation where *a matrix is unknown* - therefore, a matrix equation (we shall deal with those equation in next chapters). Suppose that $m = n$ (square system), and suppose there exists the inverse matrix A^{-1} to the matrix A.

Then, the solution of this system is given as

$$x = A^{-1} \cdot b$$

and this provides a new method of how to solve such a system (under special assumptions expressed above). The method could be named as *solution method using the inverse matrix*.

Matrix equations

An equation of the type $A \cdot X = B$ for given matrices A, B and the matrix X as unknown is called a matrix equation.
How to solve it: suppose there exists the inverse matrix A^{-1} to the matrix A. Then multiply the equation by A^{-1} *from the left* and we get

$$A \cdot X = B, \quad A^{-1} \cdot A \cdot X = A^{-1} \cdot B, \quad X = A^{-1} \cdot B$$

In the similar way, supposing the inverse matrix A^{-1} does exist, we solve the matrix equation $X \cdot A = B$ by multiplying the equation by A^{-1} *from the right*:

$$X = B \cdot A^{-1}$$

Those steps could be combined, e.g. for the matrix equation $A \cdot X \cdot B = C$, assuming there exist inverse matrices A^{-1}, B^{-1}, we get:

$$X = A^{-1} \cdot C \cdot B^{-1}$$

Determinants

Determinant of a square matrix

In what follows, we shall suppose to work with square matrices only. The idea: we adjoin to those matrices - in a unique way - *a real number*, and this number will be called as *the determinant of the matrix*; it will be defined in an inductive way.

1. For a matrix A of the size 2×2, where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, we define determinant

$$det\ A = a_{11}a_{22} - a_{21}a_{12}$$

This determinant is called as determinant of the 2nd order (or of the 2nd degree), and we shall write

$$det\ A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Examples: $\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - 4 \cdot 1 = 0,$ $\begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} = 1 \cdot 1 = 1,$ $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

2. Determinant of a matrix A of size 3×3 will be the number defined using three determinants of the 2nd degrees as follows:

$$det\ A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix}$$

This computation of determinants of the 3rd degree is named as *Sarrus' Rule*; it can be expressed as a sum of 6 products, any product as of 3 entries of the matrix; the first product is determined as a product of main diagonal entries. Further, three of them are taken with $+$ sign, and three with $-$ sign:

$$det\ A = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{21}a_{12}a_{33} - a_{11}a_{32}a_{23}$$

3. Determinant of degree n with $n \geq 4$ is determined as the number by *expanding the matrix by a row, or expanding by a column*, following induction principle w.r. to the degree n. Notation: for a square matrix A of size $n \times n$, let us denote its determinant again as

$$det\ A = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}$$

Now, we will compute $det\ A$: let us choose indices of a row i and a column j, resp. for $1 \leq i \leq n$, $1 \leq j \leq n$, arbitrary, but in what follows as fixed ones.

Now, remove the ith row and the jth column in the matrix A. In this way, we get from the original matrix A a new square matrix of the size $(n-1) \times (n-1)$; its determinant is called *the minor*, or *the sub-determinant of the degree* $n-1$, and is denoted as M_{ij}.

The (algebraic) co-factor A_{ij} *corresponding to the entry* a_{ij} of the matrix A is called the determinant M_{ij} (the minor) together with its sign $+1$ or -1 due to the formula

$$A_{ij} = (-1)^{i+j} \cdot M_{ij}$$

Remark. We can summarize this idea in the following "sign matrix" that will tell us if we should leave the minor alone (i.e. tack on a $+$) or change its sign (i.e. tack on a $-$) when writing down the co-factor: to the positions of entries given by couples (ij) we have

$$A = \begin{pmatrix} + & - & + & \ldots \\ - & + & - & \ldots \\ + & - & + & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

Now, suppose that i is the index of an arbitrary chosen, but in what follows a fixed row of the matrix A. Then the determinant $det\ A$ is the number

$$det\ A = a_{i1}A_{i1} + a_{i2}A_{i2} + ... + a_{in}A_{in} = \sum_{j=1}^{n} a_{ij}A_{ij}$$

This method of finding the determinant of degree n is called *expanding determinant along some row*, in our notation by the ith row (or in general, *the co-factor expansion*). This method contains *the recurrence principle*:

the determinant of degree n is determined by determinants of lower degrees $(n-1)$, and so on, and we stop at determinants of degrees 3 or 2, then we use Sarrus' Rule or direct computation.

Remarks. (1) This method of finding $det\ A$ is correct; using mathematical induction, one can prove that the value of the determinant is a unique one and does not depend on choice of the row index.

(2) The value of $det\ A$ could be determined by expanding the determinant along some column (independently on the choice of that column). E.g., expand along the jth column:

$$det\ A = a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj} = \sum_{i=1}^{n} a_{ij}A_{ij}$$

(3) For expansion, it is convenient (and recommended) to choose a row or a column consisting of mostly zeros.

Exercise 1. Find $det\ A$ expanding along: a) 1st row; b) 3rd column; c) 4th row; the matrix A is

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & -1 & 2 & -1 \\ 1 & 1 & -3 & -1 \\ 3 & 0 & 0 & 5 \end{pmatrix}$$

Solution. a) Expanding along 1st row: we have there three entries as zero, therefore

$$det\ A = 0 + 2 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 1 & 2 & -1 \\ 1 & -3 & -1 \\ 3 & 0 & 5 \end{vmatrix} + 0 + 0 = -2 \cdot \begin{vmatrix} 1 & 2 & -1 \\ 1 & -3 & -1 \\ 3 & 0 & 5 \end{vmatrix} = 80$$

b) Expanding along 3rd column:

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & -1 & 2 & -1 \\ 1 & 1 & -3 & -1 \\ 3 & 0 & 0 & 5 \end{pmatrix}$$

we shall not write summands to zero entries a_{ij} (we see that $a_{13} = a_{43} = 0$):

$$det\, A = 2\cdot(-1)^{2+3}\cdot \begin{vmatrix} 0 & 2 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & 5 \end{vmatrix} - 3\cdot(-1)^{3+3}\cdot \begin{vmatrix} 0 & 2 & 0 \\ 1 & -1 & -1 \\ 3 & 0 & 5 \end{vmatrix} = 32+48 = 80$$

c) Expanding along the 4th row:

$$det\, A = 3\cdot(-1)^{4+1}\cdot \begin{vmatrix} 2 & 0 & 0 \\ -1 & 2 & -1 \\ 1 & -3 & -1 \end{vmatrix} + 5\cdot(-1)^{4+4}\cdot \begin{vmatrix} 0 & 2 & 0 \\ 1 & -1 & 2 \\ 1 & 1 & -3 \end{vmatrix} = 30+50 = 80 \quad \bullet$$

Properties of Determinants

In what follows, we provide results on properties of determinants; proofs could be found in textbooks on linear algebra. We provide ideas of proofs on determinants of degree 3 only, and, in general case as well, these are based mainly on expansion of a determinant along a row/column.

Property (P1). Suppose that A is a square matrix, and A^T is a matrix transpose to A. Then it holds: $det\ A = det\ A^T$. It follows that all properties formulated for rows of a determinant are valid also for columns, and vice versa. (Nevertheless, often we shall formulate both cases, for rows and column as well.)

Property (P2). For a given determinant D, create a new determinant determinant D_1 in such a way that all entries in a given row will be multiplied by a real number k. Then $D_1 = k \cdot D$.

As an illustration, take D_1 as

$$D_1 = \begin{vmatrix} 30 & 60 & 90 \\ 3 & 8 & 5 \\ 12 & 0 & 18 \end{vmatrix} = 30 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 3 & 8 & 5 \\ 12 & 0 & 18 \end{vmatrix} = 30 \cdot 6 \cdot 2 \cdot \begin{vmatrix} 1 & 1 & 3 \\ 3 & 4 & 5 \\ 2 & 0 & 3 \end{vmatrix}$$

(Sarrus' rule follows; advantage: we will work with smaller numbers).

Property (P3). If some row (or some column) in a determinant D consists entirely of zeros (zero row, or zero column in D), then for the value of a determinant we have $D = 0$.

(It follows from an expansion along this zero row (zero column).)

Property (P4). Suppose that all entries of one row (one column), say ith one, of a determinant D are the sum of two summands. Then the value of a determinant D is also a sum of two summands, namely determinants D_1, D_2: first of them D_1 has in its ith row first members of the sum, and the second one D_2 contains in its ith row those second summands.

Property (P5). Suppose that a determinant D^* has arisen from a determinant D in result of a mutual change of two rows (columns). Then $D^* = -D$.

Property (P6) (corollary of (P5)). If there are two identical rows (columns) in a determinant D, then $D = 0$.

Property (P7) (corollary of (P6), (P2)). If a row (column in a determinant D is a k-multiple of some other row (column), then it follows $D = 0$. (In other words: if at least two rows (columns) in a determinant are proportional, then $D = 0$.)

Property (P8) (corollary of (P4), (P7)). Value of a determinant will not change when a multiple of a row (column) will be added to some other row (column).

Conclusion: for a large number n, evaluate D in the following way:

(1) first, apply operations on D, using properties **(P1) – (P8)**,

(2) then expand D along some row (or some column).

Let us mention that operations on D could lead to a new determinant $det\ A$ of an upper triangle matrix (i.e., all sub-diagonal entries are zero); the determinant of an upper triangle matrix is simply the product of all diagonal entries:

$$det\ A = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$$

as a special case, we get that a determinant of a diagonal matrix (its nonzero entries are in the main diagonal only) is equal to the product of those diagonal entries. (More, the determinant of any unit matrix E_n has the value 1.)

Exercise. Evaluate the determinant $D = \begin{vmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & -1 & 2 \\ 0 & 2 & 3 & 0 \\ 3 & 3 & -1 & 1 \end{vmatrix}$

Solution. Apply ERO operations first: to the 2nd row, (-2)-multiple of the 1st row will be added, then (-3)-multiple of the 1st row will be added to the 4th row:

$$D = \begin{vmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & -1 & 2 \\ 0 & 2 & 3 & 0 \\ 3 & 3 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -5 & -4 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & -7 & -8 \end{vmatrix}$$

and now expand along the 1st column:

$$\begin{vmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -5 & -4 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & -7 & -8 \end{vmatrix} = \begin{vmatrix} 1 & -5 & -4 \\ 2 & 3 & 0 \\ 3 & -7 & -8 \end{vmatrix}$$

To the 2nd row, (-2)-multiple of the 1st row will be added; to the 3rd row, (-3)-multiple of the 1st row is added:

$$\begin{vmatrix} 1 & -5 & -4 \\ 2 & 3 & 0 \\ 3 & -7 & -8 \end{vmatrix} = \begin{vmatrix} 1 & -5 & -4 \\ 0 & 13 & 8 \\ 0 & 8 & 4 \end{vmatrix} = \begin{vmatrix} 13 & 8 \\ 8 & 4 \end{vmatrix} = 4 \cdot \begin{vmatrix} 13 & 2 \\ 8 & 1 \end{vmatrix} = -12 \bullet$$

Remark. A practical use of determinants: it can be proved that the area A of a triangle $\Delta\, P_1P_2P_3$ with vertices $P_i[x_i,\ y_i]$, $i = 1,\ 2,\ 3$, is given as the absolute value of a special determinant of the degree 3, namely

$$A(\Delta\, P_1P_2P_3) = \frac{1}{2} \cdot \left\| \begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix} \right\|$$

For three points $P_i[x_i,\ y_i]$, $i = 1,\ 2,\ 3$, in the plane, the condition

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

is a necessary and sufficient one for all those three points are points of the same line, say p.

More, the equation of a line p in the plane, determined by two different points $A[x_1,\ y_1]$, $B[x_2,\ y_2]$, is given by the following equation, using again a special determinant of the degree 3:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

All proofs could be studied in linear algebra textbooks (or, in plane geometry, the use of vector operations).

How to find the inverse matrix using determinants

The inverse matrix to a given matrix $A = (a_{ij})$, $1 \leq i, j \leq n$ with its non-zero determinant $det\ A$ could be computed by the use of determinants, in the following way: it holds

$$A^{-1} = \frac{1}{det\ A} \cdot \begin{pmatrix} A_{11} & A_{21} & \dots & A_{i1} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{i2} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots & \dots & \\ A_{1j} & A_{2j} & \dots & A_{ij} & \dots & A_{nj} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{in} & \dots & A_{nn} \end{pmatrix}$$

Entries of the inverse matrix are co-factors of the matrix A written in such a way that in jth row and ith column of the inverse matrix, we have the co-factors A_{ij} of an entry a_{ij} from ith row and jth column of the original matrix A.
In other words, the inverse matrix is constructed in two steps:
- write the matrix with entries as co-factors of the given matrix A,
- form the transpose of the matrix of co-factors (such a matrix is call *the adjacent matrix to the given matrix* A, notation: $adj\ A$),
- multiply the previous matrix by the number $\dfrac{1}{det\ A}$.

Matrices having the inverses are called also *invertible matrices*. Let us remark that the inverse matrix A^{-1} does exist to the square matrix A fulfilling the condition $det\ A \neq 0$; a matrix with this property is called *a regular matrix*. It follows invertible matrices are just regular matrices only. Matrices with $det\ A = 0$) are *singular* ones.

Exercise. Find the inverse matrix to the given matrix $A = \begin{pmatrix} 2 & 0 \\ -1 & -2 \end{pmatrix}$.

Solution. The matrix A is a regular one: one can easily check that $det\, A = -4 \neq 0$. Co-factors are:

$$A_{11} = -2,\ A_{12} = 1,\ A_{21} = 0,\ A_{22} = 2,$$

therefore the inverse matrix A^{-1} is

$$A^{-1} = -\frac{1}{4} \cdot \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} = -\frac{1}{4} \cdot \begin{pmatrix} -2 & 0 \\ 1 & 2 \end{pmatrix}.$$

Verify that we have the inverse:

$$A \cdot A^{-1} = -\frac{1}{4} \cdot \begin{pmatrix} 2 & 0 \\ -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} -2 & 0 \\ 1 & 2 \end{pmatrix} = -\frac{1}{4} \cdot \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$A^{-1} \cdot A = -\frac{1}{4} \cdot \begin{pmatrix} -2 & 0 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ -1 & -2 \end{pmatrix} = -\frac{1}{4} \cdot \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ •

Exercise. Find the inverse matrix to the given matrix $B = \begin{pmatrix} 2 & 3 & 1 \\ 7 & 9 & 5 \\ 3 & 4 & 3 \end{pmatrix}$.

Solution. Again $det\, B = 54 + 28 + 45 - (27 + 40 + 63) = -3 \neq 0$. Using the same notation for co-factors, now co-factors A_{ij} are:

$$A_{11} = \begin{vmatrix} 9 & 5 \\ 4 & 3 \end{vmatrix} = 7,\ \ A_{21} = -\begin{vmatrix} 3 & 1 \\ 4 & 3 \end{vmatrix} = -5,\ \ A_{31} = \begin{vmatrix} 3 & 1 \\ 9 & 5 \end{vmatrix} = 6,$$

$$A_{12} = -\begin{vmatrix} 7 & 5 \\ 3 & 3 \end{vmatrix} = -6,\ \ A_{22} = \begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix} = 3,\ \ A_{32} = -\begin{vmatrix} 2 & 1 \\ 7 & 5 \end{vmatrix} = -3,$$

$$A_{13} = \begin{vmatrix} 7 & 9 \\ 3 & 4 \end{vmatrix} = 1,\ \ A_{23} = -\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 1,\ \ A_{33} = \begin{vmatrix} 2 & 3 \\ 7 & 9 \end{vmatrix} = -3;$$

$$B^{-1} = -\frac{1}{3} \cdot \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \frac{1}{3} \cdot \begin{pmatrix} -7 & 5 & -6 \\ 6 & -3 & 3 \\ -1 & -1 & 3 \end{pmatrix}.$$

Let us verify the property $B \cdot B^{-1} = B^{-1} \cdot B = E_3$:

$$B \cdot B^{-1} = \begin{pmatrix} 2 & 3 & 1 \\ 7 & 9 & 5 \\ 3 & 4 & 3 \end{pmatrix} \cdot \frac{1}{3} \cdot \begin{pmatrix} -7 & 5 & -6 \\ 6 & -3 & 3 \\ -1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$B^{-1} \cdot B = \frac{1}{3} \cdot \begin{pmatrix} -7 & 5 & -6 \\ 6 & -3 & 3 \\ -1 & -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 & 1 \\ 7 & 9 & 5 \\ 3 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \bullet$$

For the inverse matrix, the following is true:

(1) For arbitrary square matrices A, B of the same size $n \times n$, for the determinant of the product we have

$$det\,(A \cdot B) = det\, A \cdot det\, B$$

From this, the following relation follows for determinants of a matrix, and its inverse matrix:

$$det\, A \cdot det\, A^{-1} = det\, E_n = 1$$

(2) If matrices A, B are regular ones, of the same size $n \times n$, then also the matrix $A \cdot B$ is a regular one and it holds

$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$$

(3) If $det\, A \neq 0$ and a matrix A^T is a transpose to the matrix A, then we have

$$(A^{-1})^T = (A^T)^{-1}$$

(4) The inverse matrix to an upper triangular matrix is an upper triangular matrix as well.

Cramer's Rule (general case)

Theorem (Cramer's Rule). Suppose $A \cdot x = b$ is a system of n linear equations, suppose that D is a determinant of the matrix A of that system: $D = det\,A$. If $D \neq 0$, then the system has a unique solution, the vector $x = (x_1,\ x_2, \ldots,\ x_n)$, such that for its entries we have

$$x_i = \frac{D_i}{D},\ \ i = 1,\ \ldots,\ n;$$

here D_i is the determinant arisen from D by replacing the ith column of D with the column b of right sides, $i = 1, \ldots, n$.

Proof. Suppose that $D = det\,A \neq 0$. First we shall prove that if there is a solution of that system, then this solution i determined in a unique way. For that purpose, suppose that two vectors x, y are solutions of the system, i.e. it holds

$$A \cdot x = b \ \text{ and in the same way } \ A \cdot y = b$$

From those matrix equations it follows that

$x = A^{-1} \cdot b, \quad y = A^{-1} \cdot b, \quad$ therefore $x = y$.

Now, let us search the ith entry of the solution vector; in the product of matrices, let us write the inverse matrix in a symbolic way as a matrix of co-factors divided by $det\ A$:

$$x = A^{-1} \cdot b = \frac{1}{det\ A} \cdot (A_{ji}) \cdot b$$

Thus, due to the rule for the multiplication of matrices, the entry x_i as the product of ith row of the inverse matrix A^{-1} with the column b, is

$$x_i = \frac{1}{det\ A} \cdot (A_{1i} \cdot b_1 + A_{2i} \cdot b_2 + ... + A_{ni} \cdot b_n)$$

The term $(A_{1i} \cdot b_1 + A_{2i} \cdot b_2 + ... + A_{ni} \cdot b_n)$ represents exactly such a number which is the value of the determinant D_i, in case D_i is computed by expansion along the ith columns, $i = 1, ..., n$. It follows

$$x_i = \frac{D_i}{D} \bullet$$

Remark. From our considerations it follows: using the method of solution of a system consisting of n linear equations by Cramer's rule requires to evaluate in total $(n + 1)$ determinants of the degree n. Usually, we apply first basic properties of determinants, including expansion along some row/column.