MATHEMATICS 2

LINEAR SPACES

01. Preparatory exercises. Suppose that f, g are two real functions of the variable x defined on the set M, let c is a real number. Suppose both function f, g have some definite property, say p, on the set M. Decide whether function $f+g$ and $c \cdot f$ have the same property p on that set M; in the positive case, provide the reasons; in the negative case, provide the counterexample:
Solve for the property p defined in the following way:

– function is nonnegative,
– function is non-zero,
– $f(x) \geq 0$ for $x \geq 0$,
– function is constant,
– function is linear,
– function is bounded,
– function does not have zero points,
– function is continuous (not continuous),
– function has a finite number of discontinuity points,
– function is even (odd),
– function is periodic,
– function is one-to-one,
– function is increasing (decreasing),
– it holds $f(0) = 0$,
– it holds $f(x) = 0$ for the given M_1, $M_1 \subset M$,
– it holds $f(0) = 1$,
– both functions have the same non-vertical asymptote in ∞,
– both functions have the asymptote in ∞,
– function is concave up (concave down) on M,
– it holds $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = L$.

1. Knowledge of the concept, skills on linear space operations. Decide whether the given set endowed by the usual operations of the sum and the multiple by a real number forms the linear space:

a) E_2 (resp. E_3) – the set of all pairs (triples) of real numbers - vectors, operations in E_2, E_3 defined point-wise;
b) V_n – the set of all n-tuples of real numbers (n-dimensional vectors) for a given n, operations in V_n defined point-wise;
c) V_{n0} – the set of all n-dimensional vectors with the property $x_1 + x_2 + ... + x_n = 0$;
d) V_{n1} – the set of all n-dimensional vectors with the property $x_1 + x_2 + ... + x_n = 1$;
e) $A_1 = \{(x_1, x_2) : x_1, x_2 \in R, x_2 = kx_1, k \in R\}$ for $k \in R$ given;
f) $A_2 = \{(x_1, x_2) : x_1, x_2 \in R, x_2 = 2 - x_1\}$;
g) $A_3 = \{(x_1, x_2, x_3) : x_i \in R, i = 1, 2, 3, x_2 = 0\}$;
h) the set $F(x)$ of all functions $f : R \to R$;
i) the set $F(x)$ of all functions $f : R \to R$, increasing on R;

j) the set $P(x)$ of all polynomial functions with real coefficients;

k) the set $P_n(x)$ of all polynomial functions with real coefficients of the degree maximally n, where n is given;

l) the set $SF(x)$ of all functions $f : R \to R$, continuous on R;

m) the set M_n of all square matrices of the degree n, where n is given;

n) the set SM_n of all square symmetrical matrices of the degree n, where n is given;

o) the set DM_n of all square matrices A of the degree n, n is given, such that $det\, A = 1$;

p) the set RM_n of all square regular matrices A of the degree n, n is given.

2. Knowledge of the concept of the linear combination, skills on linear space operations. Express the vector x in the form of a linear combination of vectors a_1, a_2, ..., a_n and determine coefficients of those linear combinations:

a) $x = (0,\ 3,\ -4)$, $a_1 = (1,\ -1,\ 2)$, $a_2 = (2,\ 1,\ 0)$;
b) $x = (0,\ 3,\ -4)$, $a_1 = (-3,\ 2,\ 1)$, $a_2 = (2,\ 1,\ 0)$;
c) $x = (0,\ 0,\ 0)$, $a_1 = (1,\ 2,\ -1)$, $a_2 = (2,\ 1,\ -1)$, $a_3 = (0,\ 1,\ 1)$;
d) $x = (0,\ 0,\ 0)$, $a_1 = (-1,\ 0,\ 1)$, $a_2 = (0,\ 1,\ 1)$, $a_3 = (1,\ 1,\ 0)$;
e) $x = (6,\ 9,\ 14)$, $a_1 = (-1,\ 0,\ 1)$, $a_2 = (0,\ 1,\ 1)$, $a_3 = (1,\ 1,\ 0)$;
f) $x = (0,\ 0,\ 0,\ 0)$, $a_1 = (1,\ 2,\ -1,\ -2)$, $a_2 = (2,\ 3,\ 0,\ -1)$, $a_3 = (1,\ 2,\ 1,\ 3)$, $a_4 = (1,\ 3,\ -1,\ 0)$;
g) $x = (-1,\ 0,\ 2,\ 3)$, $a_1 = (1,\ -1,\ 0, 2)$, $a_2 = (1,\ 2,\ 0,\ 3)$, $a_3 = (1,\ -4,\ 2,\ 1)$;
h) $x = (4,\ 4,\ -6,\ -18)$, $a_1 = (1,\ 1,\ 0,\ 1)$, $a_2 = (0,\ -1,\ 2,\ 7)$, $a_3 = (2,\ -1,\ 0,\ 1)$;
i) $x = (1,\ -1)$, $a_1 = (1,\ 3)$, $a_2 = (5,\ -1)$, $a_3 = (1,\ 7)$.

3. Knowledge of the concept of the linear combination, skills on linear space operations. Find the expression of the vector $f(x) = 4x^2 + 2x - 14$ in the form of the linear combination of vectors $f_1(x)$, $f_2(x)$, $f_3(x)$ for all real numbers x and determine coefficients of those linear combinations:

a) $f_1(x) = x + 1$, $f_2(x) = x^2 + x - 3$, $f_3(x) = 1 - x^2$;
b) $f_1(x) = x$, $f_2(x) = 1 + x$, $f_3(x) = (1 + x)^2$.

4. Knowledge of the concept of the linear independent, linearly dependent vectors, skills on linear space operations. Decide whether the given set of vectors is linearly dependent, or linearly independent:

a) $a_1 = (1,\ 2,\ 3)$, $a_2 = (3,\ 6,\ 7)$;
b) $a_1 = (4,\ -2,\ 6)$, $a_2 = (6,\ -3,\ 9)$;
c) $a_1 = (2,\ -3,\ 1)$, $a_2 = (3,\ -1,\ 5)$, $a_3 = (1,\ -1,\ 4)$;
d) $a_1 = (5,\ 4,\ 3)$, $a_2 = (3,\ 3,\ 2)$, $a_3 = (8,\ 1,\ 3)$;
e) $a_1 = (0,\ 1,\ 1)$, $a_2 = (1,\ 0,\ 1)$, $a_3 = (1,\ 1,\ 0)$;
f) $a_1 = (4,\ -5,\ 2,\ 6)$, $a_2 = (2,\ -2,\ 1,\ 3)$, $a_3 = (6,\ -3,\ 3,\ 9)$, $a_4 = (4,\ -1,\ 5,\ 6)$;
g) $a_1 = (-2,\ -3,\ -5)$, $a_2 = (1,\ 0,\ -2)$, $a_3 = (-1,\ -7,\ -9)$, $a_4 = (-1,\ 0,\ 1)$;
h) $a_1 = (-2,\ -5,\ -7,\ -3)$, $a_2 = (1,\ -1,\ 0,\ 0)$, $a_3 = (-1,\ 0,\ -2,\ -3)$.

5. Knowledge of the concept of the linear independent, linearly dependent vectors, skills on linear space operations. Decide whether the given sets of vectors in the linear space $P_2(x)$ are linearly dependent, or linearly independent, provide reasons:

a) $1 + x,\ 1 - x,\ 2 + x - x^2$;   b) $1 - x,\ x - x^2,\ x^2 - 1$.

6. Knowledge of the concept of the linear independent, linearly dependent vectors, skills on linear space operations. Decide whether the given sets of vectors in the linear space $P_3(x)$ are linearly dependent, or linearly independent, provide reasons:

a) $1,\ 1+x,\ 1+x^2,\ 1+x^3$; b) $1-x,\ x-x^2,\ x^2-x^3,\ x^3-1$;
c) $x,\ x+x^2,\ x^2+x^3,\ x^3+1$; d) $1+x,\ 1-x,\ x^3+2x,\ x^3-2x^2$.

7. Knowledge of the concept of the linear independent, linearly dependent vectors, skills on linear space operations. Suppose x, y, z are 3 vectors in the linear space L which are linearly independent. Decide whether the following vectors are linearly dependent, or linearly independent, provide reasons:

a) $x+y,\ y+z,\ z+x$; b) $x+2y,\ y+3z,\ z+4x$;
c) $x,\ x+y,\ x+y+z$; d) $x-y,\ y-z,\ z-x$;
e) $x+y,\ x-y,\ x+y+z$.

8. Knowledge of the concept of the linear independent, linearly dependent vectors, skills on linear space operations. Suppose x, y, z, u are 4 vectors in the linear space L such they are linearly independent. Decide whether the following vectors are linearly dependent, or linearly independent, provide reasons:

a) $x+y,\ y+z,\ z+u,\ u+x$;
b) $x+y+z,\ y+z+u,\ z+u+x,\ u+x+y$;
c) $x+y+z+u,\ x-y+z+u,\ x+y+z-u,\ x-y+z-u$.

9. Knowledge of the concept of the linear independent, linearly dependent vectors, skills on linear space operations. Find the rank of the set of vectors (the maximal number of linearly independent vectors in this set):

a) $a_1=(1,\ 1,\ 0)$, $a_2=(3,\ 6,\ 7)$;
b) $a_1=(4,\ -2,\ 0)$, $a_2=(1,\ 1,\ -1)$, $a_3=(5,\ -1,\ -1)$;
c) $a_1=(0,\ 1,\ 1)$, $a_2=(1,\ 0,\ 1)$, $a_3=(1,\ 1,\ 0)$;
d) $a_1=(2,\ 0,\ 0)$, $a_2=(0,\ -3,\ 0)$, $a_3=(0,\ 0,\ 4)$;
e) $a_1=(10,\ 1,\ 1)$, $a_2=(0,\ 10,\ 1)$, $a_3=(1,\ 0,\ 10)$;
f) $a_1=(3,\ 0,\ 1)$, $a_2=(1,\ 2,\ 1)$, $a_3=(1,\ 1,\ 1)$;
g) $a_1=(2,\ 0,\ -1)$, $a_2=(-4,\ 0,\ 2)$, $a_3=(10,\ 0,\ -1)$;
h) $a_1=(2,\ 1,\ -1,\ 0)$, $a_2=(3,\ 2,\ 1,\ 2)$, $a_3=(2,\ 0,\ 1,\ 1)$, $a_4=(3,\ 0,\ 2,\ 1)$.

10. Knowledge of the concept of the matrix rank. Determine the rank $r(A)$ of the given matrix A:

a) $\begin{pmatrix} 2 & -3 \\ 4 & -6 \end{pmatrix}$ b) $\begin{pmatrix} 2 & -3 \\ 4 & -8 \end{pmatrix}$ c) $\begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ d) $\begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \end{pmatrix}$

e) $\begin{pmatrix} 2 & 3 & 0 \\ 2 & 3 & 4 \\ 2 & 0 & 0 \end{pmatrix}$ f) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ g) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 3 \\ 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \end{pmatrix}$ h) $\begin{pmatrix} 1 & 1 & 4 & 3 \\ 2 & 1 & 11 & 13 \\ 2 & 3 & 5 & -1 \\ 1 & -2 & 13 & 24 \end{pmatrix}$

11. Knowledge: Frobenius Theorem. Apply Theorem of Frobenius and decide on the existence or non-existence of a solution of the given system of linear equations. Determine all existing solutions. The systems are written in the form of the table:

a)
$$\begin{array}{rrrr|r} x_1 & x_2 & x_3 & x_4 & \\ \hline 3 & 1 & -1 & 2 & 6 \\ 2 & 0 & 1 & -1 & 1 \\ -5 & 1 & 3 & -4 & -12 \\ 1 & -5 & 3 & -3 & 3 \end{array}$$

b)
$$\begin{array}{rrrr|r} x_1 & x_2 & x_3 & x_4 & \\ \hline 1 & 1 & 1 & 1 & 0 \\ 2 & 3 & -1 & 4 & 1 \\ -2 & 0 & 3 & 2 & 2 \\ 4 & 1 & -1 & -2 & 3 \end{array}$$

c)
$$\begin{array}{rrrr|r} x_1 & x_2 & x_3 & x_4 & \\ \hline 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}$$

d)
$$\begin{array}{rrrrr|r} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline 1 & 3 & -3 & -6 & 2 & 2 \\ 1 & 2 & 2 & -3 & 4 & 1 \\ 2 & 5 & 2 & -3 & 3 & 9 \end{array}$$

e)
$$\begin{array}{rrrr|r} x_1 & x_2 & x_3 & x_4 & \\ \hline -1 & 1 & 3 & -1 & 4 \\ 2 & 0 & 4 & -1 & 8 \\ 4 & -2 & -2 & 4 & 0 \\ 3 & -1 & 1 & 0 & 4 \end{array}$$

12. Knowledge: Frobenius Theorem. Apply Theorem of Frobenius and decide on the existence or non-existence of a solution of the given system of linear equations. Determine all existing solutions. The systems are written in the form of the table:

a)
$$\begin{array}{rrr|r} x_1 & x_2 & x_3 & \\ \hline 1 & 2 & -2 & 0 \\ 2 & -4 & 3 & 0 \\ 3 & -2 & 1 & 0 \end{array}$$

b)
$$\begin{array}{rrr|r} x_1 & x_2 & x_3 & \\ \hline 3 & -4 & 2 & 0 \\ 1 & -2 & 3 & 0 \\ -2 & 2 & 1 & 0 \end{array}$$

c)
$$\begin{array}{rrrr|r} x_1 & x_2 & x_3 & x_4 & \\ \hline 5 & 2 & 1 & -6 & 0 \\ -2 & 1 & -5 & 7 & 0 \\ -5 & -2 & 6 & -1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}$$

d)
$$\begin{array}{rrr|r} x_1 & x_2 & x_3 & \\ \hline 2 & -1 & -2 & 0 \\ 2 & -1 & 3 & 0 \\ 4 & -2 & 1 & 0 \\ 6 & 3 & 4 & 0 \end{array}$$

14. Knowledge: the rank of the matrix. Determine the rank $r(A)$ of the given matrix A, depending on values on the real parameter p:

a) $\begin{pmatrix} p & 4 \\ 1 & p \end{pmatrix}$ b) $\begin{pmatrix} p^2 & 3 & 2 \\ 0 & 1 & 4 \\ p & -1 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & p & 1 \\ 1 & 1 & p \end{pmatrix}$ d) $\begin{pmatrix} p & 1 & -1 \\ -1 & p & 1 \\ 1 & -1 & p \end{pmatrix}$

e) $\begin{pmatrix} -p & 1 & 1 & 1 \\ 1 & -p & 1 & 1 \\ 1 & 1 & -p & 1 \\ 1 & 1 & 1 & -p \end{pmatrix}$

15. Knowledge of the concept, skills to determine: the basis of the linear space. Decide whether the following sets od vectors form the basis of the linear space V_3; in the positive case, determine the coordinates of the vector $x = (1, 2, 3)$ in the basis given:

a) $a_1 = (2, 0, 0)$, $a_2 = (0, 3, 0)$, $a_3 = (0, 0, 4)$;
b) $a_1 = (1, 1, 1)$, $a_2 = (0, 1, 1)$, $a_3 = (0, 0, 1)$;
c) $a_1 = (3, 0, 1)$, $a_2 = (1, 2, 1)$, $a_3 = (1, 1, 1)$;
d) $a_1 = (1, 1, 2)$, $a_2 = (3, 2, 4)$, $a_3 = (-2, -1, -2)$.

16. Knowledge of the concept, skills to determine: the basis of the linear space. Decide whether the following sets od vectors form the basis of the linear space V_4, provide reasons:

a) $a_1 = (1, 5, 4, 3)$, $a_2 = (1, 2, 1, 4)$, $a_3 = (-1, -3, -2, -1)$, $a_4 = (2, 1, 3, 2)$;
b) $a_1 = (1, 1, 0, -1)$, $a_2 = (2, 0, 1, 2)$, $a_3 = (-1, 2, 2, 1)$, $a_4 = (3, 1, 1, 3)$;
c) $a_1 = (1, 0, 2, 3)$, $a_2 = (-1, 1, 0, 0)$, $a_3 = (2, 5, 7, 3)$;
d) $a_1 = (2, 1, -1, 0)$, $a_2 = (3, 2, 1, 2)$, $a_3 = (2, 0, 1, 1)$, $a_4 = (3, 0, 2, 1)$; in the positive case, determine the coordinates of the vector $x = (5, -1, 5, 3)$ in the basis given.

Solutions:

1. d) f) i) o) p) not; all others yes • **2.** a) $x = -a_1 + a_2$; b) x fails to be the linear combination (the corresponding system of linear equations has no solution); c) zero coefficients only; d) infinitely many solutions: $x = ta_1 - ta_2 + ta_3$, $t \in R$ arbitrary; e) x fails to be the linear combination (the corresponding system of linear equations has no solution); f) infinitely many solutions: $x = (-2a_1 + a_2 - a_3 + a_2)t$, $t \in R$ arbitrary; g) x fails to be the linear combination (the corresponding system of linear equations has no solution); h) $x = 2a_1 - 3a_2 + a_3$; i) infinitely many solutions; $x = (2-9t)a_1 + ta_2 + (4t-1)a_3$, $t \in R$ arbitrary • **3.** a) $f = -2f_1 + 4f_2$; b) $f = 12f_1 - 18f_2 + 4f_3$ • **4.** a) lin. independent; b) lin. dependent; c) lin. independent; d) lin. dependent; e) lin. independent; f) lin. dependent; g) lin. dependent; h) lin. independent • **5.** a) lin. independent; b) lin. dependent • **6.** a) lin. independent, b) lin. independent, c) lin. dependent, d) lin. independent • **7.** a) lin. independent; b) lin. independent; c) lin. independent; d) lin. dependent; e) lin. independent • **8.** a) lin. dependent; b) lin. independent; c) lin. dependent • **9.** a) 2; b) 2; c) 3; d) 3; e) 3; f) 3; g) 1; h) 4 • **10.** a) 1; b) 2; c) 3; d) 2; e) 3; f) 3; g) 4; h) 2 • **11.** a) $h(A) = h(\bar{A}) = 4$, therefore a unique solution: $x = (1, -1, 2, 3)$; b) $h(A) = 3$, $h(\bar{A}) = 4$, therefore no solution; c) $h(A) = h(\bar{A}) = 4$, therefore a unique solution only: $x = (1, 2, 3, 4)$; d) $h(A) = h(\bar{A}) = 3$, $x = (41 - 10{,}5u + 10v,\ -11 + 6{,}5u - 3v,\ 2 - u + v,\ 0{,}5u,\ v)$, $u, v \in R$ arbitrary; e) $h(A) = h(\bar{A}) = 3$, $x = (4-2t,\ 8-5t,\ t,\ 0)$, $t \in R$ arbitrary • **12.** a) $h(A) = 2$, $(2t, 7t, 8t)$, $t \in R$ arbitrary; b) $h(A) = 2$, $(8t, 7t, 2t)$, $t \in R$ arbitrary; c) $h(A) = 4$, a unique solution only: $(0, 0, 0, 0)$; d) $h(A) = 3$, a unique solution only: $(0, 0, 0)$ • **14.** a) for $p \neq \pm 2$ is $h(A) = 2$; for $p = \pm 2$ is $h(A) = 1$; b) for $p \neq 0$, $p \neq -2$ is $h(A) = 3$; for $p = 0$ or pro $p = -2$ is $h(A) = 2$; c) for $p \neq 1$ is $h(A) = 3$; for $p = 1$ is $h(A) = 1$; d) for $p \neq 0$ is $h(A) = 3$; for $p = 0$ is $h(A) = 2$; e) for $p \neq 3$, $p \neq -1$ is $h(A) = 4$; for $p = 3$ is $h(A) = 3$; for $p = -1$ is $h(A) = 1$ • **15.** a) forms the basis, $x = 1/2a_1 + 2/3a_2 + 3/4a_3$; b) forms the basis, $x = a_1 + a_2 + a_3$; c) forms the basis, $x = a_1 - 2a_2 + 6a_3$; d) the set of vectors fails to be a basis V_3 • **16.** a) not; b) yes; c) not; d) yes; $x = -a_1 + 2a_3 + a_4$ •