MATHEMATICS 2:

How to find the solution of the system of linear equations (following in a proper way)

Problem. Find all solution of the homogeneous system of linear equation, express them in the vector form, and check all possible solutions:

$$\begin{aligned} x_1+2x_2 \qquad - 6x_4 &= 0 \\ 2x_1+ x_2+3x_3 \qquad &= 0 \\ 7x_1+8x_2+6x_3-18x_4 &= 0 \end{aligned}$$

Solution. We use Gaussian Elimination Method, having $m = 3$ equations and $n = 4$ unknowns. That is, we shall apply elementary row operations (ERO) on the augmented matrix of the system $\bar{A} = (A\,|b)$ with the entirely zeros column b, with the aim to transform the matrix $\bar{A}$ into the reduced form. Operations will be applied on rows of the table, the matrix $\bar{A}$ is written there in the form of the table. We shall not provide notations for single ERO operations:

x_1	x_2	x_3	x_4	
1	2	0	-6	0
2	1	3	0	0
7	8	6	-18	0
1	2	0	-6	0
0	-3	3	12	0
0	-6	6	24	0
1	2	0	-6	0
0	1	-1	-4	0
0	1	-1	-4	0
1	0	2	2	0
0	1	-1	-4	0

The matrix in the last table is in the reduced form, the number r of non-zero rows of the reduced matrix is $r = 2$, so this form suggests that we shall have infinitely many solutions. Now, we shall express solutions.

It follows that $n - r = 2$ unknowns of total 4 ones could be chosen arbitrary. Such a pair of free unknowns could be chosen in different ways, in total, in $\binom{4}{2} = 6$ ways (expressed as a number of possible combinations). Let us decide that x_3, x_4 will be free unknowns. Then for x_2, x_1 we shall have

$x_2 = x_3 + 4x_4$ (follows from the 2nd row of the table),

$x_1 = -2x_3 - 2x_4$ (follows from the 1st row of the table).

Therefore, any solution of the system is of the form $x = [-2x_3 - 2x_4,\, x_3 + 4x_4,\, x_3,\, x_4]$.

Write these solutions in the column vector form, transform into the sum of vectors:

$$x = \begin{pmatrix} -2x_3 - 2x_4 \\ x_3 + 4x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \cdot x_3 + \begin{pmatrix} -2 \\ 4 \\ 0 \\ 1 \end{pmatrix} \cdot x_4$$

We see that in case we have specially chosen $x_3 = x_4 = 0$, then we get the trivial solution $[0, 0, 0, 0]$ (we know that any homogeneous system has at least the trivial vector as the solution, so called trivial solution).

Further special cases: when

$x_3 = 1$, $x_4 = 0$: we get $(-2, 1, 1, 0)$,

$x_3 = 0$, $x_4 = 1$: we get $(-2, 4, 0, 1)$.

In view of the structure of the solution x, it is possible to formulate that any solution of the given homogeneous system

$$x = \begin{pmatrix} -2x_3 - 2x_4 \\ x_3 + 4x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \cdot x_3 + \begin{pmatrix} -2 \\ 4 \\ 0 \\ 1 \end{pmatrix} \cdot x_4$$

is the so called linear combination of only two special solutions

$$X = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad Y = \begin{pmatrix} -2 \\ 4 \\ 0 \\ 1 \end{pmatrix}$$

where x_3, x_4 are coefficients of this linear combinations. Therefore, those two special solutions X, Y form **the basis for all possible solutions**, and in fact any solution of the system is uniquely determined exactly by that basis of solutions. The solution x (therefore the whole set of them) is called the **general solution** of the homogeneous system of linear equations.

In conclusion, when we need to check whether the vector x is the solution of the system, it is sufficient to check for this property only for those basis vectors. Let us provide this in the matrix form, i.e. for checking whether $A \cdot x = 0$ is true, where A is the matrix of the coefficients of the system (the matrix of the system), we verify that $A \cdot X = 0$, $A \cdot Y = 0$ (it follows from matrix algebra rules, and could be easily seen):

$$\begin{pmatrix} 1 & 2 & 0 & -6 \\ 2 & 1 & 3 & 0 \\ 7 & 8 & 6 & -18 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 2 & 0 & -6 \\ 2 & 1 & 3 & 0 \\ 7 & 8 & 6 & -18 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 & -6 \\ 2 & 1 & 3 & 0 \\ 7 & 8 & 6 & -18 \end{pmatrix} \cdot Y = \begin{pmatrix} 1 & 2 & 0 & -6 \\ 2 & 1 & 3 & 0 \\ 7 & 8 & 6 & -18 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Remark. For any system of linear equations (non-homogeneous, homogeneous) the set of all solutions could be described in general in the special form - based on the previous problem just solved. Show this on a system to be supposed also with its augmented matrix in the reduced form as follows:

$$\begin{aligned} x_1 \quad -2x_3+5x_4 &= 7 \\ x_2+\ x_3-3x_4 &= 5 \end{aligned}$$

This system has infinitely many solutions. The general solution is

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7+2x_3-5x_4 \\ 5-x_3+3x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -5 \\ 3 \\ 0 \\ 1 \end{pmatrix} x_4, \quad x_3,\ x_4 \in R \text{ arbitrary.}$$

This solution has the following structure: vector x as the general solution of the non-homogeneous system is the sum

– of the vector $x^\star = \begin{pmatrix} 7 \\ 5 \\ 0 \\ 0 \end{pmatrix}$, the solution of the non-homogeneous system (given by the choice $x_3 = x_4 = 0$, therefore this is called as *a particular solution of the non-homogeneous system*;

– of the vector given as the linear combination $x_{hom} = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -5 \\ 3 \\ 0 \\ 1 \end{pmatrix} x_4$, which represents all possible solutions of the homogeneous system corresponding to the given non-homogeneous system of linear equations (right sides column changed by entirely zeros column): the vector x_{hom} is exactly *the general solution of the homogeneous system of linear equations.*

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