MATHEMATICS 2

DEFINITE INTEGRALS, PROPERTIES, EVALUATION

01. Preparatory problems. Evaluate indefinite integrals:

a) $\int (x^2 - x + 1)^2 \, dx$ b) $\int (x^2 + 2)\sqrt{1 - x} \, dx$ c) $\int \frac{x + 3}{(2x + 1)(1 - x)} \, dx$

d) $\int \frac{8}{x(x^2 - 4)} \, dx$ e) $\int \left(\frac{x + 1}{2 - x}\right)^2 \, dx$ f) $\int \frac{x^2 + 2}{x^2 + 4x + 3} \, dx$

g) $\int \frac{x + 3}{x^2 + 3} \, dx$ h) $\int \frac{1}{3x^2 + 8} \, dx$ i) $\int \frac{3x + 2}{x^2 + x + 1} \, dx$

j) $\int (e^x + e^{-x})^2 \, dx$ k) $\int (1 - 3x)e^{-x} \, dx$ l) $\int (x - 3)^2 e^{-5x} \, dx$

m) $\int (\ln x + \ln^2 x) \, dx$ n) $\int \cos^2 x \, dx$ o) $\int \sin^3 x \, dx$

p) $\int x^2 \ln(x + 1) \, dx$ r) $\int e^x \cos x \, dx$ s) $\int e^{-2x} \sin 3x \, dx$

02. Preparatory problems. Discuss and verify results:

$$\int_0^1 x^n \, dx = \left[\frac{x^{n+1}}{n + 1}\right]_0^1 = \frac{1}{n + 1} \text{ (for } n \neq -1)$$

$$\int_a^b k \;\, dx = k \;\cdot [x]_a^b = k \cdot (b - a)$$

$$\int_1^2 \frac{1}{x} \, dx = [\ln x]_1^2 = \ln 2$$

$$\int_a^b e^x \, dx = [e^x]_a^b = e^b - e^a$$

$$\int_0^{\pi/2} \cos x \, dx = [\sin x]_0^{\pi/2} = 1, \quad \int_0^{\pi/2} \sin x \, dx = [-\cos x]_0^{\pi/2} = 1$$

$$\int_0^{\pi} \cos x \, dx = [\sin x]_0^{\pi} = 0, \quad \int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi} = 0$$

$$\int_0^{\pi/2} \cos^2 x \, dx = \frac{1}{2} \cdot [x + \sin x \cos x]_0^{\pi/2} = \frac{\pi}{4}$$

$$\int_0^{\pi/2} \sin^2 x \, dx = \frac{1}{2} \cdot [x - \sin x \cos x]_0^{\pi/2} = \frac{\pi}{4} \bullet$$

1. Knowledge + skills. Evaluate the definite integrals:

a) $\int_0^1 (x^2 - 3x + 1) \, dx$ b) $\int_{-1}^1 (6x^5 + 3) \, dx$ c) $\int_0^2 \sqrt[3]{x} \, dx$

d) $\int_0^8 \sqrt{8 - x} \, dx$ e) $\int_0^1 \frac{1}{x} \, dx$ f) $\int_1^2 \frac{8}{x^2} \, dx$

g) $\int_2^4 \frac{2x}{x^2 + 4x + 3} \, dx$ h) $\int_0^1 \frac{x + 1}{x^2 + 1} \, dx$ i) $\int_0^1 (e^x + e^{-x})^2 \, dx$

Solution. a) $-\frac{1}{6}$; b 6; c) $\frac{3}{2} \cdot \sqrt[3]{2}$; d) $\frac{32}{3} \cdot \sqrt{2}$; e) undefined; f) 4; g) $\ln \frac{1029}{625}$; h) $\frac{\ln 2}{2} + \frac{\pi}{4}$;

i) $\frac{e^2}{2} + 2 - \frac{e^{-2}}{2}$ $\bullet$

2. Concepts knowledge + skills: evaluate the definite integrals:

a) $\displaystyle\int_0^1 x^2\sqrt{x}\,dx$; b) $\displaystyle\int_{-1}^1 x(x^2-1)^2\,dx$; c) $\displaystyle\int_1^2 \left(x-\frac{1}{x}\right)^2\,dx$;

d) $\displaystyle\int_1^2 \left(\frac{2}{x^3}-\frac{3}{x^4}\right)\,dx$; e) $\displaystyle\int_{-\pi/2}^{\pi/2}(x-\cos x)\,dx$; f) $\displaystyle\int_{-1}^1 (e^x-e^{-x})^2\,dx$;

g) $\displaystyle\int_0^2 \frac{x}{x^2-9}\,dx$; h) $\displaystyle\int_{-1}^1 \frac{2x}{4-x^2}\,dx$; i) $\displaystyle\int_1^2 \frac{3+2\sqrt{x}}{x^2}\,dx$;

j) $\displaystyle\int_0^1 \frac{e^x}{1+e^x}\,dx$; k) $\displaystyle\int_1^e \frac{1}{x(1+\ln x)}\,dx$; l) $\displaystyle\int_1^{-1} \frac{1}{x+x^2}\,dx$;

m) $\displaystyle\int_{-3}^3 \frac{3x+1}{9+x^2}\,dx$; n) $\displaystyle\int_0^1 \frac{1}{x^2+3x+2}\,dx$; o) $\displaystyle\int_0^3 \frac{2x+1}{x-4}\,dx$.

Solution. a) $2/7$; b) 0; c) $5/6$; d) $-1/8$; e) -2; f) $e^2-e^{-2}-4$; g) $1/2\cdot\ln(5/9)$; h) 0; i) $\dfrac{5+8\sqrt{2}}{3}$; j) $\ln\dfrac{1+e}{2}$; k) $\ln 2$; l) the function fails to be defined in the interval given; m) $\dfrac{\pi}{6}$; n) $\ln(4/3)$; o) $6-18\ln 2$ •

3. Concepts knowledge + skills. Evaluate the the definite integrals, using the substitution method, verify the solution:

a) $\displaystyle\int_0^1 8x(x^2+1)^3\,dx = 15$

b) $\displaystyle\int_0^1 \sqrt{4x+9}\,dx = 2/9\cdot(5\sqrt{5}-8)$

c) $\displaystyle\int_0^{\pi/2} \sin x\cdot\cos^2 x\,dx = \frac{1}{3}$

4. Concept knowledge, skills in application of the method. Evaluate the the definite integrals, using per partes method, check the solution:

a) $\displaystyle\int_0^1 x\cdot e^{-x}\,dx$

b) $\displaystyle\int_0^{\pi} x\cdot\sin x\,dx$

c) $\displaystyle\int_1^e \ln x\,dx$

Solution. a) $-\dfrac{2}{e}+1$; b) π; c) 1 •

5. Concept knowledge, skills in application of the method. Evaluate the definite integrals, interpret them as area $A(R)$ of the appropriate plane region R, in the special case as the area of the subgraph region:

a) $\displaystyle\int_0^1 xe^x\,dx$; b) $\displaystyle\int_0^\pi x\sin x\,dx$; c) $\displaystyle\int_0^\pi x^2\sin x\,dx$; d) $\displaystyle\int_1^e x\ln x\,dx$;

e) $\displaystyle\int_0^2 \frac{x}{x^2-9}\,dx$; f) $\displaystyle\int_3^4 \frac{1}{x^2-3x+2}\,dx$; g) $\displaystyle\int_{-1}^1 \frac{x^2+1}{4-x^2}\,dx$; h) $\displaystyle\int_1^e \frac{1}{x(1+\ln x)}\,dx$;

i) $\displaystyle\int_0^1 x\sqrt{1-x^2}\,dx$; j) $\displaystyle\int_1^3 \frac{1}{x+x^2}\,dx$; k) $\displaystyle\int_0^1 \frac{x+3}{9+x^2}\,dx$; l) $\displaystyle\int_0^1 \frac{1}{2x^2+3}\,dx\cdot$

m) $\displaystyle\int_{-1}^2 |x|\,dx$; n) $\displaystyle\int_0^2 x\cdot|x-1|\,dx$; o) $\displaystyle\int_0^3 min\{1,\,x^2\}\,dx\cdot$

Solution. a) 1; b) π; c) π^2-4; d) $1/4(e^2+1)$; e) $1/5(\ln 5-\ln 9)$; f) $2\ln 2-\ln 3$; g) 0,7454; h) $\ln 2$; i) $1/3$; j) 0,4052; k) 0,3744; l) 0,2400; m) $5/2$; n) 1; o) $10/3$ •

6. Knowledge and skills: area of the plane region. Using the definite integral, evaluate the area $A(R)$ of the plane region R determined by the graph of the parabola $y=-x^2+4x-3$ and the axis o_x. Sketch the region in question.

Solution. $\dfrac{4}{3}$ area units •

7. Knowledge and skills: area of the plane region. Using the definite integral, evaluate the area $A(R)$ of the plane region R determined by the graphs of parabolas $y=x^2$, $y=\sqrt{x}$. Sketch the region in question.

Solution. $\dfrac{1}{3}$ area units •

8. Knowledge and skills: area of the plane region. Using the definite integral, evaluate the area $A(R)$ of the plane region R determined by the graphs of functions $y=\ln x$, $y=\ln^2 x$. Sketch the region in question.

Solution. $3-e$ area units •

9. Knowledge, skills: functions and their graphs, curves and their graphs, area of the plane region. Sketch graphs of functions or curves in the coordinate system, and verify that graphs determine the bounded plane region. Next, using the definite integral, evaluate the area $A(R)$ of the plane corresponding region R in question:

a) $y=2x$, $y=6-x$, $y=0$; b) $y=2x^2$, $y=6-x^2$, $x=0$;
c) $y=x$, $y=3x$, $x=10$; d) $y=x^2$, $y=3x^2$, $y=10$;
e) $x+y=4$, $xy=1$; f) $y=4-x^2$, $y=x^2$;
g) $y^2=x$, $x=5$; h) $y=x^2$, $y^2=x$;
i) $y=x^2$, $y=x^3$; j) $y=\sqrt{2x}$, $2y=x$;
k) $y=e^x$, $y=e^{-x}$, $y=2$; l) $y=e^{x-1}$, $y=e^{-2x}$, $y=4$;
m) $y=e^x$, $y=e^{2x}$, $x=6$; n) $y=2^{-x}$, $y=5$, $x=0$;
o) $y=e^x$, $x=5$, $y=0$, $x=0$; p) $y=x$, $y=2x$, $x^2+y^2=4$.

10. Knowledge and skills: area of the plane region. Using the definite integral, evaluate the area $A(R)$ of the plane region R determined by the graphs of functions:

a) $y=\cos x$ for $0\le x\le \pi/2$;

b) $y=\sin x$, $y=\cos x$ and the axis o_x pro $0\le x\le\pi/2$;

c) $y = \sin x$, $y = \cos x$ and the axis o_y in the 1st quadrant.

Solution. a) 1 ; b) $2 - \sqrt{2}$; c) $\sqrt{2} - 1$ area units •

11. Knowledge and skills: area of the plane region. Using the definite integral, evaluate the area $A(R)$ of the plane region R determined by the graphs of functions:

a) $y = \dfrac{1}{4 - x^2}$, $y = 1$; b) $y = \dfrac{4}{(x-1)^2}$, $y = 4$, $y = 16$
c) $y = \sqrt{8 - x}$, $y = \sqrt{2x - 1}$, $y = 0$; d) $y = 4 - x^2$, $y = 3x^2 - 6x$;
e) $y = \dfrac{2}{x}$, $y = \dfrac{x}{2}$, $y = 2$; f) $y = e^{-x}$, $y = e^{2x}$, $x = 3$.

12. Knowledge and skills: area of the plane region. Using the definite integral, evaluate the area $A(R)$ of the plane region R; sketch the corresponding region, and estimate the evaluation as well first:

a) $y = e^x$, $y = e^{-x}$, $x = \ln 2$; b) the circle $x^2 + y^2 = 8$ and the parabola $y^2 = 2x$;
c) $y = x^2$, $y = \dfrac{2}{1 + x^2}$; d) $y = \dfrac{8}{x^2}$, $y = x$, $y = 8x$;
e) $y = \sqrt{x}$, $y = \dfrac{7}{3}\sqrt{x}$, $y = x - 2$

Solution. a) 1/2; b) $2\pi + 4/3$, $6\pi - 4/3$ (the region not given in a unique way, two possible interpretation); c) $\pi - 2/3$; d) 5/2; e) 49/6 •

13. Average Value of the function. Evaluate the Average Value AV of the given function within the interval $\langle a,\, b\rangle$ given, sketch graphically:

a) $f(x) = 1 + x$, $\langle 0,\, 1\rangle$, resp. $\langle -1,\, 1\rangle$; b) $f(x) = x^3 - x$, $\langle -1,\, 1\rangle$;
c) $f(x) = x(2 - x)$, $\langle 0,\, 2\rangle$; d) $f(x) = e^x$, $\langle 0,\, 1\rangle$;
e) $f(x) = \sin x$, $\langle 0,\, \pi\rangle$; f) $f(x) = \cos x$, $\langle 0,\, \dfrac{\pi}{2}\rangle$;
g) $f(x) = x^3 + x^2$, $\langle -1,\, 1\rangle$; h) $f(x) = \ln x$, $\langle 1,\, e\rangle$.

Solution. a) 3/2; 1; b) 0; c) 2/3; d) $e - 1$; e) $\dfrac{2}{\pi}$; f) $\dfrac{2}{\pi}$; g) 1/3; h) $\dfrac{1}{e-1}$ •

14. Average Value of the function. The object is moving at the velocity $v(t) = \sqrt{1 + t}$ m/s. Evaluate the Average Value AV of its velocity within first 15 seconds of its motion.

Solution. 2,8 m/s •

15. Knowledge and skills: Average Value. The object is moving at the velocity $v(t)$ in m/sec. given as $v(t) = 10 - \sqrt{1 + 2t}$. Evaluate the Average Value AV of its velocity within the first 40 seconds of its motion. Sketch the graph of the velocity function and indicate AV.

Solution.

16. Knowledge and skills: Average Value. The object is moving at the velocity $v(t)$ given as $f(t) = \sqrt{1 + 4t}$ m/sec. during the first 6 seconds of its motion, and then at the velocity $g(t) = 8 - 0,5t$ m/sec until the end of the 16th seconds. Evaluate the Average Value AV of its velocity within the 16 seconds of its motion. Sketch the graph of the velocity function and indicate AV.

Solution.

17. Knowledge and skills: Average Value. The object is moving at the velocity $v(t)$ in km/h given as $v(t) = \sqrt{1+3t}$ for $0 \le t \le 5$, and $v(t) = 4(t-6)^2$ for $5 \le t \le 6$. Evaluate the Average Value AV of its velocity within the first 6 hours of its motion. Sketch the graph of the velocity function and indicate AV.

Solution.

18. Knowledge and skills: Average Value. Evaluate the Average Value AV of the given function within the interval $\langle a,\, b\rangle$ given, sketch graphically:

a) $f(x) = 1 - |x|,\ \ \langle -2,\, 2\rangle$; b) $f(x) = |x^2 - x|,\ \ \langle 0,\, 2\rangle$;
c) $f(x) = |x^2 - 2x|,\ \ \langle 0,\, 3\rangle$; d) $f(x) = |1 - x^2|,\ \ \langle -1,\, 1\rangle$;
e) $f(x) = |1 - x^2|,\ \ \langle -1,\, 4\rangle$; f) $f(x) = x \cdot \sqrt{5 - x},\ \ \langle 0,\, 5\rangle$;
g) $f(x) = x \cdot \sqrt{9 - x^2},\ \ \langle 0,\, 3\rangle$; h) $f(x) = |\ln x|,\ \ \langle 1/e,\, e\rangle$.

Solution.