MATHEMATICS 2
Summer Semester, 2012/13

Subject Content - Sections, Lectures:

1. The definite integral, properties, evaluation.

2. Applications of definite integrals in geometry, in economics.

3. Systems of linear equations, solution methods, Gauss/Jordan Elimination Method.

4. Algebra of n-tuples (arithmetic vectors), linear space of n-tuples, linear dependence, independence of n-tuples.

5. Matrices, operations with matrices; inverse matrices. Applications of matrices.

6. Determinants, properties and evaluation of determinants. Solution of linear equations systems using determinants.

7. Vector space (linear space), basis and dimension of a linear space. Dot product, the norm of a vector. Orthogonal vectors, applications (optional).

8. Linear mappings, basic notions: matrices corresponding to linear mappings.

9. Euclidean space. Function of more variables, their graphs. Limits and continuity of a function of more variables.

10. Partial derivatives. Extrema of a function of more variables. Applications. Least Squares Method, applications.

11. Differential equations, basic notions. Linear differential equations of the 1st and 2nd order, solution methods.

12. Applications. Differential equations and exponential models.

1. THE DEFINITE INTEGRAL, PROPERTIES, EVALUATION

Motivation. Suppose that we have here the function $F(x)$ such that this function is the rate of change of another function $f(x)$, i.e.

$$\frac{dF}{dx} = f(x) \text{ for any } x \in M,\ \ M \subset R,$$

where we require also $f(x)$ to be a continuous function on the set M.

The question: how to evaluate the total change of the function $f(x)$ for values $x \in \langle a,\, b\rangle$ (supposing that $\langle a,\, b\rangle \subset M$)?

First: recall what does it mean "to evaluate the total change of the function $f(x)$"?

Discuss it in the following simple examples.

Example 1. Suppose the function $f(x) = 30x + 10$ indicates the velocity of some object in km/hour. Compute the length of the distance of the object during the second hour of the motion, i.e. for $x \in \langle 1, 2 \rangle$.

Solution. Exactly the length of such distance for time interval as $x \in \langle 1, 2 \rangle$ is *the total change of the function*, giving the distance as depending on the time.

Compute it: it is necessary to express first the distance $d(x)$ itself, as a function depending on the time x:

as *the velocity is the rate of change of the distance with respect to time*, then the distance at the time x is

$$d(x) = 15x^2 + 10x + c,$$

as the antiderivative, or the indefinite integral of the given function, with c is an arbitrary real constant (this can be checked directly, forming the derivative $d'(x)$ of $d(x) = 15x^2 + 10x + c$: we shall have $d'(x) = f(x)$);

then we are searching for the difference $d(2) - d(1) = 45 + 10 = 55$ km.

Thus: the distance of the object during the 2nd hour of the motion is exactly 55 km. •

Remark (rather obvious one). In our example: the velocity $v(t)$ of the object fails to be a constant - it is changing with time. It follows that the distance of the object for some other time interval, but *of the same length* (e.g., one hour as in our example) will be, in general, different (check it within the 4th hour of the motion, for example).

In other words: bounds of the definite integral are crucial for its value.

Example 2. Think of the function

$$f(x) = \frac{dF}{dx}$$

as *the rate of change* of population of a given region, depending on time x. How to compute the total change - the difference between two values - sizes of population at time $x = b$ and at time $x = a$ with $b > a$?

Solution. Finding the answer consists of two steps:

- *1st step:* determine the function $F(x)$ (beeing just the anti-derivative of the function $f(x)$!)

- *2nd step:* after this, compute the difference $F(b) - F(a)$.

(Try to do it for some specially chosen function $f(x)$.) •

Result of this procedure is denoted as the *definite integral* of the given function within the closed interval $\langle a,\ b\rangle$:

$$\int_a^b f(x)\,dx$$

and it means that

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is an anti-derivative of the function $f(x)$ on the given interval (or, the primitive function to the given function $f(x)$ on the indicated set).

Let us point out that the value of the definite integral does not depend on the choice of the primitive function:

if the function $G(x)$ is a primitive function to the function $f(x)$ over the interval $\langle a,\ b\rangle$ as well, then, based on properties of primitive functions, we have

$$G(x) = F(x) + c$$

and from this it follows

$$G(b) - G(a) = (F(b) + c) - (F(a) + c) = F(b) - F(a).$$

Remark. The sign $\displaystyle\int$ used as a notation for the integral reminds of the letter the capital S, or the symbol for the summation Σ (Greek capital sigma); in fact, later on, we shall show such connection or relation to the summation.

It follows: the value of the definite integral is a *real number*, not a function. The following notation is used as well:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a);$$

this very important formula is called *Newton – Leibniz definition of the definite integral*.

Example 3. The consumption of the gas in some country is increasing, and it is estimated that t years from now, its rate of change will be exponential 5 % per year; its present value is 4 millions of gallons per year. Supposing this trend not changing in the future, how large is the gas consumption in the period of next 3 years?

Solution. Denote Q as the consumption of the gas (in millions of gallons); it depends on time, $Q = Q(t)$, and the exponential function

$$\frac{dQ}{dt} = 4 \cdot e^{0,05t}$$

shows the rate of change of $Q = Q(t)$ (due to exponential growth model), in millions of gallons per year. Hence, we are working again with a function having the property

"to be a rate of change of some other function"

(like velocity of a body is a rate of change of the distance).

Therefore, in the period of the next 3 years, the total consumption of the gas in the country is given as a value of the definite integral

$$Q(3) - Q(0) = \int_0^3 4 \cdot e^{0,05t}\, dt = 4 \cdot \frac{1}{0,05}\left[e^{0,05t}\right]_0^3 = 80 \cdot \left[e^{0,05t}\right]_0^3 =$$

$$= 80 \cdot (e^{0,15} - 1) \doteq 12,95 \text{ mil. of gallons} \quad \bullet$$

The table of definite integrals in specially chosen closed interval:

$$\int_0^1 x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}, \text{ where } n \neq -1$$

$$\int_1^2 \frac{1}{x}\,dx = [\ln x]_1^2 = \ln 2$$

$$\int_a^b k\ \,dx = k\ \cdot [x]_a^b = k \cdot (b-a)$$

$$\int_a^b e^x\,dx = [e^x]_a^b = e^b - e^a$$

$$\int_0^1 e^x\,dx = [e^x]_0^1 = e - 1$$

$$\int_1^e \ln x\,dx = [x \cdot \ln x - x]_1^e = 1$$

$$\int_0^{\pi/2} \cos x \, dx = [\sin x]_0^{\pi/2} = 1$$

$$\int_0^{\pi/2} \sin x \, dx = [-\cos x]_0^{\pi/2} = 1$$

$$\int_0^{1} \frac{1}{1+x^2} \, dx = [arctg\, x]_0^{1} = \frac{\pi}{4}$$

$$\int_0^{\pi/2} \cos^2 x \, dx = \frac{1}{2} \cdot [x + \sin x \cos x]_0^{\pi/2} = \frac{\pi}{4}$$

$$\int_0^{\pi/2} \sin^2 x \, dx = \frac{1}{2} \cdot [x - \sin x \cos x]_0^{\pi/2} = \frac{\pi}{4}$$

Techniques of Integration

Substitution Method for Definite Integral

Suppose the function $u(x)$ is defined on the interval $(a,\ b)$, suppose $u(x)$ maps the interval $(a,\ b)$ onto the interval $(u(a),\ u(b))$, and the function $u(x)$ is a strongly monotonic, continuous with non-zero derivative for any $x \in \langle a,\ b\rangle$. Then the following is true:

$$\int_a^b g(u(x))\,|u'(x)|\,dx = \int_{u(a)}^{u(b)} g(t)\,dt,$$

provided at least one of two integrals does exist.

This theorem is called *Substitution Rule for Definite Integral.*

How to use the Substitution Rule: formally, we apply the substitution

$$u(x) = t, \quad u'(x)\,dx = dt,$$

where $u(x) = t$ is a one-to-one continuous mapping defined on the closed interval $\langle a,\, b\rangle$.

But more, it is important now, that also endpoints of the interval will be transformed as follows:

for $x = a \quad \Longrightarrow \quad$ we shall have the new lower bound $t = u(a)$,

for $x = b \quad \Longrightarrow \quad$ the new upper bound is $t = u(b)$.

In other words:

the whole original interval is transformed or mapped onto the new closed interval $\langle u(a),\, u(b)\rangle$,

and there is no need now to return after the substitution to the original variable (as x in our case).

Example 1. Evaluate the definite integral

$$\int_0^1 8x(x^2+1)^3\,dx.$$

Solution. Apply the substitution

$u = x^2 + 1, \quad du = 2x\,dx;$

new bounds will be

for $x = 0 \quad \Longrightarrow \quad u = 1,$

for $x = 1 \quad \Longrightarrow \quad u = 2$, therefore

$$\int_0^1 8x(x^2+1)^3\,dx = \int_1^2 4u^3\,du = \left[u^4\right]_1^2 = 16 - 1 = 15 \bullet$$

Example 2. Evaluate $\int_0^{\pi/2} \sin^3 x \cdot \cos x \, dx$.

Solution. Put $u = \sin x$; this function fulfils on the interval $\langle 0,\ \pi/2\rangle$ conditions on substitution. It follows

$$du = \cos x \, dx$$

and we transform the bounds in the following way:

the lower bound $x = 0 \quad \Longrightarrow \quad u(0) = \sin 0 = 0,$

the upper bound $x = \pi/2 \quad \Longrightarrow \quad u(1) = \sin \pi/2 = 1,$

therefore

$$\int_0^{\pi/2} \sin^3 x \cdot \cos x \, dx = \int_0^1 u^3 \, du = \left[\frac{u^4}{4}\right]_0^1 = \frac{1}{4} \quad \bullet$$

Example 3. Evaluate $\int_1^e \frac{\ln^2 x}{x}\,dx$.

Solution. Due to the properties of the function in the integral, we put the substitution $u = \ln x$; it could be checked that conditions expressed in the theorem on the substitution are fulfilled on the interval $\langle 1,\, e\rangle$. Thus

$$du = \frac{1}{x}\,dx$$

and bounds are transformed:

the lower bound $x = 1 \quad \Longrightarrow \quad u(1) = \ln 1 = 0,$

the upper bound $x = e \quad \Longrightarrow \quad u(e) = \ln e = 1,$

therefore we have

$$\int_1^e \frac{\ln^2 x}{x}\,dx = \int_0^1 u^2\,du = \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3} \quad \bullet$$

Per Partes Method in the Definite Integral

We have the formula

$$\int_a^b u' \cdot v \ dx = [u \cdot v]_a^b - \int_a^b u \cdot v' \, dx$$

with the only comment: all members are in bounds, i.e., it is very important to realize that the summand $[u \cdot v]_a^b$ is in bounds, therefore, now $[u \cdot v]_a^b$ in the formula is a real number.

Examples.

$$\int_0^1 x \cdot e^{-x}\,dx = \left[-x \cdot e^{-x}\right]_0^1 - \int_0^1 e^{-x}\,dx =$$

$$= -\frac{1}{e} - \left[e^{-x}\right]_0^1 = -\frac{2}{e} + 1$$

(each term in the sum is written in bounds);

$$\int_1^e \ln x\,dx = \left[x \cdot \ln x\right]_1^e - \int_1^e 1\,dx = e - (e-1) = 1$$

$$\int_0^{2\pi} x \cdot \cos x\,dx = \left[x \cdot \sin x\right]_0^{2\pi} - \int_0^{2\pi} \sin x\,dx =$$

$$= \left[\cos x\right]_0^{2\pi} = 1 - 1 = 0 \quad \bullet$$

The Definite Integral and the Area of a Plane Region

First, let us state the following as a fact:

Suppose that $y = f(x)$ is a nonnegative function, defined and continuous on the closed interval $\langle a,\, b\rangle$. Let us denote as R a plane region determined by a part of the graph of the function $f(x)$ for $x \in \langle a,\, b\rangle$, the o_x axis and the vertical lines $x = a$, $x = b$. Then the area $A(R)$ of the region R is determined as the value of the definite integral

$$A(R) = \int_a^b f(x)\, dx \text{ (in area units).}$$

It would be convenient to express the plane region R using the system of inequalities: for coordinates x, y of a point $P[x,\, y] \in R$ one could write for region R the following system of two inequalities:

$$R\colon \quad \begin{array}{c} a \le x \le b, \\ 0 \le y \le f(x) \end{array}$$

The statement could be formulated also for the more general case:

If $y = f(x)$, $y = g(x)$ are two nonnegative functions, both defined and continuous on the closed interval $\langle a,\, b\rangle$ such that, for any $x \in \langle a,\, b\rangle$, the property $g(x) \leq f(x)$ is fulfilled, then the area $A(R)$ of the region R now determined as

$$R\colon \quad \begin{matrix} a \leq x \leq b, \\ g(x) \leq y \leq f(x) \end{matrix}$$

is to be computed as the value of the definite integral

$$A(R) = \int_a^b (f(x) - g(x))\, dx \ \text{ (area units).}$$

Now, the plane region R is determined by parts of the graphs of functions $g(x)$, $f(x)$ for $x \in \langle a,\, b\rangle$, further the o_x axis and the vertical lines $x = a$, $x = b$. Such a region is usually called as *the elementary region of the type* $[x,\, y]$. Its area can be computed as the difference of two areas, any of them is determined by the single function $f(x)$, $g(x)$, resp., as it was written above.

Remark (an important one). The value of a definite integral - the real number - could be interpreted as area of some region even when the problem is formulated not purely as the problem on area computation.

Newton's integral

Motivation:

A question arises: how to interpret the number as a value of the definite integral?

Hint, or connection as a hint: the value of the definite integral (3) in our table:

$$\int_a^b k \ dx = k \ \cdot [x]_a^b = k \cdot (b - a)$$

or: the value of the definite integral of a constant function $f(x) = k$ defined on the interval $I = \langle a,\ b \rangle$ equals numerically to the area *of a rectangle* under the graph of that constant function between the vertical lines $x = a$, $x = b$ and the axis o_x.

This is true in general:

finding the antiderivative within the bounds $\Longleftrightarrow$ evaluation of the plane region

How to prove it - the idea only, without details:

Suppose:

the real function $y = f(x)$ defined on the closed interval $I = \langle a,\, b\rangle$ is nonnegative and continuous on I.

denote as $R(f,\, a,\, b)$ (or shortly R in what follows) a plane region under the graph of that function for $x \in \langle a,\, b\rangle$, determined by the axis o_x and vertical lines $x = a$, $x = b$;

let us search for its area $A(R(f,\, a,\, b))$ or again shortly $A(R)$.

Then, one can prove that the following is true:

$$\int_a^b f(x)\, dx = F(b) - F(a) = A(R) \text{ (area units)}$$

The idea of the proof continues as follows:

Define the real function $A : x \to A(x)$ in the following way:

for the variable $x \in \langle a,\, b\rangle$, the value of $A : x \to A(x)$ is determined as the area under the graph of the function $y = f(x)$ for the interval $\langle a,\, x\rangle$.

Then for this function $A(x)$, in an equivalent expression

$$A(x) = \int_a^x f(t)\, dt$$

the following is true: $A(x)$ is the antiderivative, or the primitive function to the function $f(x)$ on the set $\langle a,\, b\rangle$, i.e., has the property

$$A'(x) = f(x) \text{ for all } x \in \langle a,\, b\rangle.$$

This can be shown following the definition of the derivative of the continuous function as a special limit at an arbitrary point $x_0 \in \langle a,\, b\rangle$.

Remark. The function $A(x)$ is defined correctly. In view of its definition, with the independent variable being the upper bound of the integral, $A(x)$ is called as the *integral as a function of the upper bound*.

Therefore, we shall have

$$\int_a^b f(t)\,dt = A(b) - A(a).$$

But $A(a) = 0$ (no area, or zero area when $x = a$), thus

$$\int_a^b f(t)\,dt = A(b),$$

hence the value of the definite integral on the left side as a value of our function $A(x)$ taken for $x = b$ shows that it is just the area of the plane region described.

Remark. This statement is known to be **the fundamental theorem of calculus**.

Remarks. Let us mention the following properties of the notion *area of the plane region* which have to be reflected by its definite integral representation (and in fact, one can check them directly):

(1) for the constant function $f(x) = k$, $k > 0$ a real constant, the area of the rectangle is $A(R(f,\, a,\, b)) = k \cdot (b - a)$;

(2) the area has the property *of the additivity with respect to the interval*, i.e. for real numbers a, b, c such that $a < c < b$, the following is true:

$$A(R(f,\, a,\, b)) = A(R(f,\, a,\, c)) + A(R(f,\, c,\, b));$$

(3) the area is *monotonic with respect the functions graphs:* if two functions $y = f(x)$, $y = g(x)$ defined and continuous on $\langle a,\, b\rangle$ have the property

$$f(x) \leq g(x) \quad \text{for all} \quad x \in \langle a,\, b\rangle,$$

then it follows that

$$A(R(f,\, a,\, b)) \leq A(R(g,\, a,\, b))$$

(area under the smaller function is smaller)

Examples

Example 1. Find the area of the region R enclosed by $y = x^2$, $y = \sqrt{x}$.

Solution. First of all, just what do we mean by *area enclosed by*: the region R must have one of the two curves on every boundary of the region.

To determine what R is, graph the two functions: their graphs have two points of intersection $[0,\ 0]$, $[1,\ 1]$; these will be the limits of integration. More, for x in the closed interval $I = \langle 0,\ 1\rangle$ we have $x^2 \le \sqrt{x}$. In other words, R can be described as the set of all points $[x,\ y]$ in the plane with properties

$$R\colon \quad \begin{array}{c} 0 \le x \le 1, \\ x^2 \le y \le \sqrt{x} \end{array}$$

So, the integral that we need to compute to find the area $A(R)$ is, in area units,

$$A(R) = \int_a^b (f(x) - g(x))\,dx = \int_0^1 (\sqrt{x} - x^2)\,dx =$$

$$= \left[\frac{2}{3} \cdot x^{3/2} - \frac{1}{3} \cdot x^3\right]_0^1 = \frac{1}{3} \bullet$$

Remarks.

(1) First, in almost all of these problems, graphs are almost required. Often the bounding region, which will give the limits of integration, is difficult to determine without graphs.

(2) Also, one has to decide which of the given functions is the *upper function* and with is the *lower function*: this can be done using graphs of those functions as well.

Example 2. Determine the area of the region R enclosed by $y = 2x^2 + 10$ and $y = 4x + 16$.

Solution. In this case the intersection points could be found by setting the two equations equal:

$2x^2 + 10 = 4x + 16$ or $2x^2 - 4x - 6 = 0$, therefore $2(x+1)(x-3) = 0$; so the two curves will intersect at $x = -1$ and $x = 3$. Then two intersection points are $[-1,\ 12]$ and $[3,\ 28]$.

The region in question can be written as: $R\colon \begin{array}{c} -1 \le x \le 3, \\ 2x^2 + 10 \le y \le 4x + 16 \end{array}$

Then the area $A(R)$ is:

$$A(R) = \int_{-1}^{3} ((4x + 16) - (2x^2 + 10))\, dx = \int_{-1}^{3} (-2x^2 + 4x + 6)\, dx =$$

$$= \left[-\frac{2}{3} \cdot x^3 + 2x^2 + 6x \right]_{-1}^{3} = \frac{64}{3} \text{ (area units)}\bullet$$

Example 3. Determine the area of the region bounded by $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$ and $x = 5$.

Solution. Two functions used in this problem are identical to the functions from the first problem. The difference is that we have extended the bounded region out from the intersection points.

Draw the picture of this region: between $x = -2$, $x = 5$, the line $y = 4x + 16$ intersects the parabola graph twice, at $x - 1$ and then at $x = 3$.

Therefore, as a subset of the plane, the region is the set union of three elementary regions R_1, R_2, R_3, expressed in the following inequality series:

$$R_1: \quad \begin{array}{c} -2 \leq x \leq -1, \\ 4x + 16 \leq y \leq 2x^2 + 10 \end{array}$$

$$R_2: \quad \begin{array}{c} -1 \leq x \leq 3, \\ 2x^2 + 10 \leq y \leq 4x + 16 \end{array} \qquad \text{and} \quad R = R_1 \cup R_2 \cup R_3$$

$$R_3: \quad \begin{array}{c} 3 \leq x \leq 5, \\ 4x + 16 \leq y \leq 2x^2 + 10 \end{array}$$

$$A(R) = A(R_1) + A(R_2) + A(R_3), \text{ with}$$

$$A(R_1) = \int_{-2}^{-1} ((2x^2 + 10) - (4x + 16))\, dx = \int_{-2}^{-1} (2x^2 - 4x - 6)\, dx =$$

$$= \left[\frac{2}{3} \cdot x^3 - 2x^2 - 6x\right]_{-2}^{-1} = \frac{14}{3},$$

$$A(R_2) = \int_{-1}^{3} ((4x + 16) - (2x^2 + 10))\, dx = \int_{-1}^{3} (-2x^2 + 4x + 6)\, dx =$$

$$= \left[-\frac{2}{3} \cdot x^3 + 2x^2 + 6x\right]_{-1}^{3} = \frac{64}{3},$$

$$A(R_3) = \int_{3}^{5} ((2x^2 + 10) - (4x + 16))\, dx = \int_{3}^{5} (2x^2 - 4x - 6)\, dx =$$

$$= \left[\frac{2}{3} \cdot x^3 - 2x^2 - 6x\right]_{3}^{5} = \frac{64}{3};$$

Example 4. Find the area of the region bounded by the parabola $x = y^2 + 2$ and the line $y = x - 8$.

Solution. Sketch the graphs of those functions; we see that this region fails to be an elementary one of the type $[x,\ y]$. So, let us made first the general observations.

Remark (an important one). In general, it is not possible to express any plane region as that of the type $[x,\ y]$ (this shows even the very elementary region being the triangle $\Delta\ ABC$ with vertices $A[0,\ 0]$, $B[1,\ 1]$, $C[2,\ 0]$).

Instead of a partition of that region into (finite numbers of) plane regions of the type $[x,\ y]$, it is possible to continue using some other approach.

Supposing the special conditions true for graphs of bounded functions, it is convenient to treat some regions "with respect to y", to introduce plane regions of the type $[y,\, x]$:

let $x = \varphi(y)$, $x = \psi(y)$ be two nonnegative real functions of the variable y, both continuous for $y \in \langle c,\, d\rangle$ and such that, for all y, $y \in \langle c,\, d\rangle$, it holds

$$\psi(y) \leq \varphi(y)$$

The plane region R consisting of all points $[x,\, y]$ with coordinates as

$$R\colon \quad \begin{array}{c} c \leq y \leq d, \\ \psi(y) \leq x \leq \varphi(y) \end{array}$$

is said to *the elementary region of the type* $[y,\, x]$ (the principle: all with respect to y first).

Analogously, the area of the region now is the value of the definite integral

$$A(R) = \int_c^d (\varphi(y) - \psi(y))\, dy \text{ (area units)}$$

Now, let us solve the problem in our example. Let us find the points of intersection: put

$$y + 8 = y^2 + 2, \quad y^2 - y - 6 = 0, \quad (y - 3)(y + 2) = 0,$$

thus $y = 3$ or $y = -2$. So, points of intersection are $[11,\ 3]$ and $[6,\ -2]$. Our region of the type $[y,\ x]$ is:

$$R\colon \quad \begin{array}{c} -2 \le y \le 3, \\ y^2 + 2 \le x \le y + 8 \end{array}$$

The area (in area units) is

$$A(R) = \int_{-2}^{3} ((y + 8) - (y^2 + 2))\, dy = \int_{-2}^{3} (y + 6 - y^2)\, dy =$$

$$= \left[\frac{y^2}{2} + 6y - \frac{y^3}{3}\right]_{-2}^{3} = \frac{125}{6} \bullet$$

Remark. *The even and the odd functions.* Recall that:

An even function is any function $f(x)$ defined on the domain Df symmetrical with respect to the number 0 on the real line which satisfies

$$f(x) = f(-x) \text{ for any } x, -x \in Df.$$

Examples of typical even functions defined for any real number x are $f(x) = x^2$, $f(x) = x^4$, $f(x) = |x|$ or $f(x) = \cos x$.

An *odd function* is any function $f(x)$ defined on the domain Df symmetrical with respect to the number 0 on the real line such that

$$f(x) = -f(-x) \text{ for any } x, -x \in Df.$$

Typical odd functions defined on the set of all real numbers are $f(x) = x^3$, $f(x) = x^5$ or $f(x) = \sin x$.

Integrate them on $I = \langle -a,\, a\rangle$, and in view that the definite integral means the area of the region, one gets:

for an even function $f(x)$: $\displaystyle\int_{-a}^{a} f(x)\,dx = 2 \cdot \int_{0}^{a} f(x)\,dx,$

for an odd function $f(x)$: $\displaystyle\int_{-a}^{a} f(x)\,dx = 0.$

Results are true due to the symmetry of the region.

More, in case of odd function, the value of the definite integral means the area of the region *together with its sign, positive or negative*. For example:

$$\int_{-1}^{1} (10x^4 + x^2 - 1)\, dx = 2 \cdot \int_0^a f(x)\, dx =$$

$$= 2 \cdot \left[\frac{10x^5}{5} + \frac{x^3}{3} - x \right]_0^1 = 2(2 + \frac{1}{3} - 1) = \frac{8}{3},$$

as we have an even function $f(x)$ as the integrand; but

$$\int_{-10}^{10} (9x^3 - x + 2\sin x)\, dx = 0,$$

as the integrand $f(x) = 9x^3 - x + 2\sin x$ is the odd function now.

Remark. Here you can find the list of the most important properties of the definite integral.

(1) $\displaystyle\int_a^a f(x)\,dx = 0$ (due to definition)

(2) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (due to definition)

(3) $\displaystyle\int_a^b (c_1 f_1(x) + c_2 f_2(x))\,dx = c_1 \int_a^b f_1(x)\,dx + c_2 \int_a^b f_2(x)\,dx$

for arbitrary $c_1,\, c_2 \in R$

(4) $\displaystyle\int_a^b f(x)\,dx \geq 0$, if $f(x) \geq 0$ on $\langle a,\, b\rangle$

(5) $\displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$, if $f(x) \leq g(x)$ on $\langle a,\, b\rangle$

(6) $m \cdot (b-a) \leq \int_a^b f(x)\,dx \leq M \cdot (b-a)$,

with m as the minimum, and M as the maximum of $f(x)$ on $\langle a,\, b\rangle$ (the area estimation)

(7) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$,

if $f(x)$ is continuous on intervals $\langle a,\, b\rangle$, $\langle b,\, c\rangle$

Example 5. *The area of the plane region in a different context - Oil production problem.* Using data from the first 3 years of the production as well as the geological studies, the management of an oil company estimates that oil will be pumped from a producing field at a rate given as

$$R(t) = \frac{100}{t+10} + 10, \text{ where } 0 \le t \le 15$$

and $R(t)$ is the rate of production in thousands of barrels per 1 year, t years after the pumping had started. Find the total amounts of the oil pumped over the interval $\langle 5,\ 10\rangle$. Interpret this result as the area of an appropriate plane region. Estimate also the amount of oil pumped for $t \in \langle 10,\ 15\rangle$ (which is the time interval of the same length).

Solution. As the function $R(t)$ is interpreted as *the rate of change* of the total production with respect to time t in years, we deduce that the total amount of the oil pumped in the given time interval $\langle 5,\ 10\rangle$ is just the definite integral. Take variables as

$$5 \le t \le 10,$$
$$0 \le y \le \frac{100}{t+10} + 10$$

and the total amount of oil is then

$$\int_5^{10} \left(\frac{100}{t+10} + 10 \right) dt = [100 \cdot \ln(10+t)]_5^{10} + 10 \cdot 5 =$$
$$= 100 \cdot \ln \frac{20}{15} + 50 = 100 \cdot \ln \frac{4}{3} + 50;$$

this is approximately 78,786 thousands of barrels.

In view of the computation, this number could be represented as the area of the region R determined by the graph of the function $R(t) = \dfrac{100}{t+10} + 10$ for $0 \le t \le 25$ (see inequalities series).

Concerning the estimation of the production during the next 5-years period, in evaluation it means the change of limits in the definite integral only. As the function

$$R(t) = \frac{100}{t+10} + 10$$

is decreasing for $t \in \langle 10,\ 15 \rangle$, one can expect the value of the integral

$$\int_{10}^{15} \left(\frac{100}{t+10} + 10 \right)\, dt$$

or the amount of oil for $t \in \langle 10,\ 15 \rangle$ less than 78,786 thousands of barrels. More, it follows obvious that in case we are interesting for the total amount of the oil pumped over the whole 15 years period, i.e. $t \in \langle 0,\ 15 \rangle$, it is of no sense to take this value as multiplying 3 times the definite integral

$$\int_{5}^{10} \left(\frac{100}{t+10} + 10 \right)\, dt \bullet$$

Example 6. *"The useful life"problem.* Suppose that the total accumulated costs $C(t)$ and revenues $R(t)$ (both in thousands of dollars), resp., for a coin-operated photocopying machine satisfy conditions (expressing their rates of changes)

$$C'(t) = 3, \quad R'(t) = 8 \cdot e^{-0{,}5t}$$

where t is the time in years. Find the useful life of the machine to the nearest year and the total profit accumulated during the useful life period of the machine. Draw the corresponding plane region.

Solution. The useful life of the machine t_0 is determined by the condition

the rate of change of revenues of the machine > the rate of change of its costs;

therefore, solving the (exponential) equation first

$$C'(t) = 3 = 8 \cdot e^{-0,5t} = R'(t)$$

we get $t_0 = 2 \cdot \ln \dfrac{8}{3} \doteq 1,96$ years; the nearest integer is then 2, and it follows

the total profit accumulated during the period of 2 years is given as the definite integral

$$\int_0^2 (R'(t) - C'(t))\, dt = \int_0^2 (8e^{-0,5t} - 3)\, dt =$$

$$= -16 \cdot \left[e^{-0,5t}\right]_0^2 - 6 = 10 - \frac{16}{e} \doteq 4,11 \text{ thousands of dollars } \bullet$$

Application: Net Excess Profit

Example 7. Suppose that x years from now, the first investment plan will generate the profit at the rate $R_1(x) = 50 + x^2$ thousands of dollars per year, and the second investment plan generates the profit at the rate $R_2(x) = 200 + 5x$ thousands of dollars per year. Compute how many years x_0 will be the second plan more profitable than the first one, and compute how much *excess profit* one can earn investing in the second plan instead of the first one during the period $\langle 0,\ x_0 \rangle$.

Solution. First, both functions $R_1(x)$, $R_2(x)$ are rate of change of the total profit followed from the investment. We see that

$$R_1(x) = R_2(x) \text{ for } x_0 = 15 \text{ years;}$$

it is true that $R_1(x) \leq R_2(x)$ for $0 \leq x \leq 15$ (after 15 years period, the converse is true: $R_1(x) \geq R_2(x)$).

During that 15 years period, investment to the 2nd plan will generate the difference in profit, compared with the investment to the 1st plan; this difference (in thousands of dollars), in view of $R_1(x)$, $R_2(x)$), is given as the value of the following definite integral:

$$\int_0^{15} (R_1(x) - R_2(x))\,dx = \int_0^{15} (150 + 5x - x^2)\,dx = \frac{3375}{2} = 1\,687,50 \bullet$$

In general: *Net Excess Profit of Investment for the period of N years* is:

$$\int_0^{N} (R_1(x) - R_2(x))\,dx$$

Application: Consumer Surplus, Producer Surplus

Suppose the market with exactly one product.

Let $p = D(x)$ is a demand function, giving the different prices p that consumers are prepared to pay for various quantities of that good; in general, it is a decreasing function, and we see that it has a property *to be a rate of change of some other function*. So, when $x = x_0$ (units of that product), then they will pay the price p_0. Therefore, the total amount of money spent on x_0 goods is the product $x_0 \cdot p_0$, and this amount could be represented as an area of a rectangle.

But the demand function shows that for quantities up to x_0 consumers would actually be willing to pay the higher price given by the demand curve. It follows that paying the fixed price p_0 will represent the benefit for consumers, which is called *Consumer Surplus, CS* and which can be computed using the definite integral (and interpreted as an area of some plane region):

$$CS = \int_0^{x_0} D(x)\,dx - x_0 \cdot p_0$$

The same situation could be analyzed from the point of view of a supplier - producer: suppose that $p = S(x)$ is a supply function, in general an increasing one and of the type to be a rate of change of some other function as well. Again we are working with different prices at which now producers are prepared to supply various quantities of a good. When an amount x_0 is sold for the price p_0, then the total amount of money received is $x_0 \cdot p_0$.

Similarly, the supply function shows that for quantities up to x_0 producers would actually be willing to accept the lower price given by the supply curve. It follows that accepting the fixed price p_0 will provide the benefit for producers, which is called *Producer Surplus, PS* and which can be computed using the definite integral as well (and interpreted as an area of some plane region):

$$PS = x_0 \cdot p_0 - \int_0^{x_0} S(x)\, dx$$

Remark: the amount x_0 will result from the equilibrium on the market: in general

$$D(x_0) = S(x_0) \text{ when equilibrium occurs}$$

with the price p_0 derived from equilibrium for demand and supply on a market.

Example 8. Solve the Consumer Surplus, Producer Surplus problem for the case that $D(x) = \frac{x^2}{25} - \frac{7}{5}x + 10$ dollars per unit, $S(x) = \frac{2}{25}x^2 + 2$ dollars per unit, supposing the condition of equilibrium on the market.

Solution. The solution of the condition of equilibrium $S(x) = D(x)$ means to solve a quadratic equation, where the positive root(s) are acceptable only, with respect to the practical meaning: this is exactly one root $x_0 = 5$ (units). The corresponding equilibrium price is $p_0 = 4$ dollars, and we have:

$$CS = \int_0^5 \left(\frac{x^2}{25} - \frac{7}{5}x + 10 \right) dx - 5 \cdot 4 = 30 - \frac{95}{6} \doteq 14,167 \text{ dollars}$$

when 5 units have been bought,

$$PS = 20 - \int_0^5 \left(\frac{2}{25}x^2 + 2 \right) dx = \frac{20}{3} \text{ dollars}$$

when 5 units were supplied. •

Average Value of the Function

For the function $f(x)$ defined and continuous on the closed interval $\langle a, b\rangle$ we can find at least one real number $\alpha \in (a, b)$ with the property

$$f(\alpha) = \frac{1}{b-a} \cdot \int_a^b f(x)\,dx.$$

The number on the right side $AV = \dfrac{1}{b-a} \cdot \displaystyle\int_a^b f(x)\,dx$ is called *the Average Value (AV) (or Mean Value)* of the function $f(x)$ on $\langle a, b\rangle$. In other words,

$$\int_a^b f(x)\,dx = (b-a) \cdot f(\alpha).$$

The equality means: the area of the plane region determined by the graph of the function $f(x)$ (on the left side of the equality) is equal to the area of the rectangle with its side as the closed interval $\langle a, b\rangle$ and the size of the second side is just a value $f(\alpha)$ on the open interval (a, b) (on the right side).

Therefore, it is quite natural to call this value as an average value of the given function. It is important that, in case of continuous function $f(x)$ we have:

$$AV = f(\alpha) \text{ for some } \alpha \in (a,\, b).$$

Example 1. Find the average value AV of the function
a) $f(x) = 1 - x^2$ on the interval $\langle -1,\ 1\rangle$;
b) $g(x) = \sin x$ on the interval $\langle 0,\ \pi\rangle$.

Solution.

$$\text{a) } AV = \frac{1}{2}\int_{-1}^{1}(1-x^2)\,dx = \int_0^1 (1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_0^1 = \frac{2}{3}.$$

Therefore, the area of the parabolic segment in the plane could be expressed as the area of the rectangle with one side as the interval $\langle -1,\ 1\rangle$ of the length 2 and the size of the other side will be exactly $\frac{2}{3}$.

Let us find points of the interval $\alpha \in (-1,\ 1)$ at which AV is taken on as a value of the function $f(x) = 1 - x^2$: we have to solve the equation

$$AV = \frac{2}{3} = 1 - x^2 \iff x_{1,2} = \pm\frac{\sqrt{3}}{3} \text{ (two points).}$$

$$\text{b) } AV = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{2}{\pi} \quad \bullet$$

Example 2. The velocity of an object $v(x)$ in meter per minute varies during the first 20 minutes of its motion as follows:

– from the start of the motion ($x = 0$) to 4th minute it was $v(x) = 0,5x$ m/min,
– from the 4th minute to the 10th minute it was constant $v(x) = 2$ m/min, and then
– from 10th to 20th minute it was $v(x) = 0,8x - 6$ m/min.

Compute the average value AV of the object velocity within 20 minutes. Determine the time when the velocity was exactly the average value.

Solution. For the average value AV of velocity $v(x)$ at the time x, $x \in \langle 0,\ 20\rangle$ we have

$$AV = \frac{1}{20}\int_0^{20} v(x)\,dx = \frac{1}{20}\left(\int_0^4 0,5x\,dx + \int_4^{10} 2\,dx + \int_{10}^{20} (0,8x - 6)\,dx\right) =$$

$$= \frac{1}{20}\left(\left[\frac{0,5x^2}{2}\right]_0^4 + 12 + \left[\frac{0,8x^2}{2} - 6x\right]_{10}^{20}\right) = \frac{76}{20} = 3,8 \text{ m/min.}$$

As the function $v(x)$ is continuous on the interval $\langle 0,\ 20\rangle$, we deduce there exists at least one time moment at which the velocity of the object was exactly the same as AV.

From the definition of the function (sketch its graph) it follows it occurs exactly once for $x \in \langle 10,\ 20\rangle$; for to find it, one has to solve the equation

$$3,8 = 0,8x - 6; \text{ it follows that } x = 12,125 \text{ min.} \bullet$$

Inventory Problem

Let us solve a practical inventory problem, based on the the application of the notion of mean value.

Example. Suppose that during one year, expenditures of goods to an inventory are regular, each as 12 000 kg of some goods, and the next expenditure will come exactly when inventory is empty. Goods are distributed to sale chains *uniformly* at the rate 300 kg per week. Compute the mean value of goods stored in the inventory within one expenditure interval (full state - empty state).

Solution. We see that in this way the inventory will be totally empty within 40 weeks; for the variable t as time (in weeks) we get *the linear* function $S(t)$ showing the actual size of goods stored; we have

$$S(t) = 12\,000 - 300t \text{ kg},$$

therefore, the average value for the period of 40 weeks is

$$AV = \frac{1}{40}\int_0^{40} (12\,000 - 300t)\, dt = \frac{15}{2}\left[40t - \frac{t^2}{2}\right]_0^{40} = 6\,000 \text{ kg}.$$

It means the following: one can deduce that, with expenditures functioning as shown in the example during one year, the amount of goods stored at any moment within this year can be treated exactly as the mean value evaluated. •

Remark. One can easily formulate the general case of that problem, together with its solution, when working with a linear function $S(t)$. On the other hand, it might happen, and it corresponds to the practical situations, that the function $S(t)$ showing the actual size of goods stored could be *non-linear* as well, and the result just obtained fails to be true; in such cases, it is necessary to evaluate the corresponding definite integral.

Riemann’s Integral

Motivation - the geometric derivation of the Fundamental Theorem of Calculus. Repeat the question on finding the area $A(R)$ of the plane region R given as the set of all points $P[x,\ y]$ of the plane with properties

$$R\colon\quad \begin{array}{l} a \le x \le b \\ 0 \le y \le f(x), \end{array}$$

where $f(x)$ is a continuous nonnegative real function defined on the closed interval $\langle a,\ b\rangle$.

As an approximation of the area, instead of the exact $A(R)$ one can take as the area of the rectangle, one side being the interval $\langle a,\ b\rangle$ of the length $b - a$, and the other side as a vertical section in the region given by the value of the function at $x = b$, that is

$$A(R) \doteq f(b) \cdot (b - a).$$

In general, a more precise approximation is given by the division of the interval into 2 parts of the equal lengths with the aid of the center x_1 of the interval $\langle a,\, b\rangle$, i.e. put $x_1 = a + \dfrac{b-a}{2}$ and then compute the area as

$$A(R) \doteq f(x_1)\cdot\frac{b-a}{2} + f(b)\cdot\frac{b-a}{2}.$$

This can be generalized: due to the same principle, divide the interval in general into n equal subintervals of the same length $\Delta\, x = \dfrac{b-a}{n}$, using $(n+1)$ points

$$x_0 = a < x_1 < \ldots < x_n = b,$$

forming the partition denoted as D.

Then take values of the function $f(x_i)$ at $x_i = a + i\cdot\dfrac{b-a}{n}$, $i = 1, 2, \ldots, n$.

Then the approximation of the area of the region could be taken as the sum of n rectangles, the base of each rectangle is the corresponding subinterval:

$$A(R) \doteq (f(x_1) + f(x_2) + \ldots + f(x_n))\cdot\Delta\, x$$

It follows that this sum, called as *the integral sum corresponding to the function* $f(x)$ *and to the partition* D, is an approximation to the corresponding definite integral

$$\int_a^b f(x)\,dx.$$

The idea was "the sum of the areas of rectangles approaches the actual area under the curve", and this approximation will be better and better, when the number of rectangles will increase without bounds. We shall not provide all necessary steps with full details, but let us only summarize that in fact three steps are crucial.

Step 1. Form the partition D of the interval $I = \langle a,\, b \rangle$ of n subinterval of the same length: as it was said above, using $n + 1$ points

$$x_0 = a < x_1 < \dots < x_n = b$$

we divide the entire interval on n equal parts, their length is $\dfrac{b-a}{n} = \Delta\, x$.

Step 2. Using the vertical lines at these partition points, divide the region R on n partial regions R_i. The area of the region R will be the sum of partial regions: this property is known as *the additivity of area*). In each partial interval $\langle x_{i-1},\, x_i \rangle$, $i = 0, 1, \dots, n$, take one point α_i, $x_{i-1} \leq \alpha_i \leq x_i$ arbitrary. The area of any partial region $A(R_i)$ could be approximated as the area of the rectangle with one side as $\Delta\, x$ and the other side as $f(\alpha_i)$:

$$A(R_i) \doteq f(\alpha_i) \cdot \Delta\, x.$$

Then the total area $A(R)$ equals approximately to the integral sum

$$A(R) = \sum_{i=1}^{n} A(R_i) \doteq \sum_{i=1}^{n} f(\alpha_i) \cdot \Delta\, x$$

Step 3. This approximation will be more and more precise provided the number n is greater and greater, thus smaller number $\Delta\, x$. This can be done using even the whole sequence of partitions D_i of the interval I, not only the unique partition D taken in the beginning, with lengths of partial subintervals converging to zero. Then, such a construction leads to the applying of the limit of the sequence of integral sums for n increasing, $n \to \infty$:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(\alpha_i) \cdot \Delta\, x.$$

If this limit does exist and it is a finite number, then it defines *the definite integral in the sense of Riemann* of the function $f(x)$ on the interval $\langle a,\ b\rangle$, and we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(\alpha_i) \cdot \Delta\, x = \int_a^b f(x)\, dx.$$