MATHEMATICS 2

Summer Semester, 2012/13

Subject Content - Sections, Lectures:

1. The definite integral, properties, evaluation.
2. Applications of definite integrals in geometry, in economics.
3. Systems of linear equations, solution methods, Gauss/Jordan Elimination Method.
4. Algebra of n-tuples (arithmetic vectors), linear space of n-tuples, linear dependence, independence of n-tuples.
5. Matrices, operations with matrices; inverse matrices. Applications of matrices.
6. Determinants, properties and evaluation of determinants. Solution of linear equations systems using determinants.

7. Vector space (linear space), basis and dimension of a linear space. Dot product, the norm of a vector. Orthogonal vectors, applications (optional).

8. Linear mappings, basic notions: matrices corresponding to linear mappings.

9. Euclidean space. Function of more variables, their graphs. Limits and continuity of a function of more variables.

10. Partial derivatives. Extrema of a function of more variables. Applications. Least Squares Method, applications.

11. Differential equations, basic notions. Linear differential equations of the 1st and 2nd order, solution methods.

12. Applications. Differential equations and exponential models.

SYSTEMS OF LINEAR EQUATIONS

Motivation: two lines problem. Find all common points $P[x,\ y]$ of two lines p_1, p_2 in the plane:

p_1: $2x + 2y = 4$, $\quad p_2$: $3x - 2y = 11$.

Solution of the problem: all ordered couples x, y (as coordinates of the points $P[x,\ y]$, therefore *ordered pairs*), fulfilling both equations. Equations are linear for unknowns x, y:

$$\begin{aligned} 2x+2y &= \ \ 4 \\ 3x-2y &= 11 \end{aligned}$$

Terminology: *a system of 2 linear equations* with two unknowns.

From the geometric interpretation of the solution to two equations in two unknowns we now that we have one of three possible cases:

- either no solution (the lines are parallel),

- one solution (the lines intersect at a single point), or

- infinitely many solutions (the equations are the same line).

Solution method: any pair x, y fulfilling both equations has to fulfil the equation following from the summing up both of them:

$$5x \quad = 15$$

This implies $x = 3$, and then from the first equation $y = -1$. The solution set is a unique ordered couple $x = 3$ and $y = -1$, both lines p_1, p_2 intersect at the point $P[x,\ y] = [3,\ -1]$.

Principle of the method: steps leading to elimination of y, therefore to computation of remaining x.

Systems of linear equations and matrices

Any system of 2 equations is fully determined by the system of coefficients of those equations a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2 (the last 2 are coefficients of the right sides of equations). Write such a system in the form of rectangular scheme called *a matrix* A:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \text{2 rows, 2 columns}$$

We say that A is of the *type* 2×2 (2 rows, 2 columns), it is *a matrix of the system*;

adding the coefficients of the right sides to the matrix A, we get *the augmented matrix of the system* $\bar{A}$ of the type 2×3 (with the priority of the number of rows on the first place there):

$$\bar{A} = \left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right) \qquad \text{2 rows, 3 columns}$$

$$\bar{A} = (A \,|\, b).$$

Matrices, augmented matrices of the system in general case

The system (S) of m linear equations, n unknowns x_1, ... , x_n:

$$
(\mathrm{S})\quad
\begin{array}{rcl}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n &=& b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n &=& b_2 \\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots & & \ldots \\
a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n &=& b_i \\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots & & \ldots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n &=& b_m
\end{array}
$$

There are two matrices related with the system (S):
the matrix of the system A of the type $m \times n$,
and *the augmented matrix of the system* $\bar{A}$ of the type $m \times (n+1)$:

$$
A = \begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
a_{i1} & a_{i2} & \ldots & a_{ij} & \ldots & a_{in} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
a_{m1} & a_{m2} & \ldots & a_{mj} & \ldots & a_{mn}
\end{pmatrix}, \bar{A} = \left(\begin{array}{cccccc|c}
a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} & b_1 \\
a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} & b_2 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
a_{i1} & a_{i2} & \ldots & a_{ij} & \ldots & a_{in} & b_i \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
a_{m1} & a_{m2} & \ldots & a_{mj} & \ldots & a_{mn} & b_m
\end{array}\right)
$$

A solution set to a system (S) is called any ordered n-tuple of numbers x_1, x_2, ... , x_n, for which, after plugging it into any equation of the system (S), this will be satisfied.

Given a system (S) there will be one of three possibilities for solutions to the system:

- exactly one solution (the only one n-tuple $[x_1,\ x_2,\ ...,\ x_n]$),
- infinitely many solutions, i.e. infinitely many n-tuples,
- no solution.

If there is no solution to the system, we call the system *inconsistent*, and if there is at least one solution to the system we call it *consistent*.

Basic notions on matrices

Definition. *A matrix* A is defined as a rectangular form of $m \times n$ entries (esp. real numbers), therefore m rows, n columns of entries. Such a matrix is *of the type* $m \times n$.

The coefficient in the ith row and jth column of A: the entry a_{ij}. Shortly: $A = (a_{ij}), 1 \leq i \leq m, 1 \leq j \leq n$.

For systems: the entry a_{ij} of the matrix A with *the row index* i and *the column index* j, $1 \leq i \leq m, 1 \leq j \leq n$, is a coefficient at jth unknown in ith equation of the system.

Entries a_{ii} of the matrix A: *the main diagonal*.

Definition. A square matrix of the $r \times r$ such that

$$a_{ii} = 1, \quad a_{ij} = 0 \text{ pro } \; i \neq j$$

is called **a unit matrix** E_r (of the type $r \times r$).

Definition. Two matrices A, B are said to be equal, $A = B$, if they are of the same type and on the same positions have the same entries. (For matrices of different type, the equality is not defined.)

The column of n unknowns x_1, x_2, ... , x_n forming the solution x, and a column of right sides of equations consisting of m entries b_1, b_2, ... , b_m, denoted as b, are also matrices of types $n \times 1$, resp. $m \times 1$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_i \\ \dots \\ x_n \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_i \\ \dots \\ b_m \end{pmatrix}$$

The augmented matrix will be written also in the form $\bar{A} = (A \,|\, b)$.

Gaussian Elimination Method (GEM)

(S): a system of m linear equations of n unknowns, A the matrix of the system, $\bar{A}$ the augmented matrix of the system.

GEM: a universal method for the solution of any system

Principle: it consists of finite number of steps - operations on rows, ERO, of $\bar{A}$, with the aim to get an equivalent system with the transparent augmented matrix, appropriate for easy finding of solution.

Elementary Row Operations (ERO)

Here are three row operations on the matrix $\bar{A}$, their equivalent equation operations as well as the notation that we will be using to denote each of them. They are called *elementary row operations (ERO)*:

(A) *Interchange rows:* interchanging ith, jth rows R_i, R_j: write

$$R_i \leftrightarrow R_j$$

(B) *Add k times row i to row i, $k \neq 0$:* for ith row write

$$kR_i \to R_i$$

(the sign $\to$ means: instead of the original row R_i we write a new one of the form kR_i)

(C) *Add k times row j to ith row:* to ith row R_i we add k-multiple of the row R_j, write it as

$$R_i + kR_j \to R_i$$

(the ith row of the matrix $\bar{A}$ has been changed).

ERO lead to a system which is equivalent with the original system - we are working with equalities of n real numbers.

Exercise. Solve the system, using ERO:

$$\begin{aligned} 3x + 4y &= 1 \\ x - 2y &= 7 \end{aligned}$$

Solution. We apply ERO on the matrix $\bar{A} = \left(\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right)$

Symbols of ERO will be written close to matrices, the transformation will be denoted by the symbol $\sim$ as a notation for the equivalence of matrices.

$$\left(\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right) R_1 \leftrightarrow R_2 \sim \left(\begin{array}{cc|c} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right) R_2 - 3R_1 \to R_2 \sim$$

$$\sim \left(\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right) 0,1R_2 \to R_2 \sim \left(\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right) R_1 + 2R_2 \to R_1 \sim$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right)$$

It follows the unique solution $x = 3$, $y = -2$ •

Reduced row-echelon form of a matrix

A matrix (any matrix, not just an augmented matrix) is said to be *in reduced row-echelon form* if it satisfies all four of the following conditions.

1. If there are any rows of all zeros then they are at the bottom of the matrix.

2. If a row does not consist of all zeros then its first non-zero entry (i.e. the left most non-zero entry) is a 1. This 1 is called a leading 1 (or a pivot).

3. In any two successive rows, neither of which consists of all zeros, the leading 1 of the lower row is to the right of the leading 1 of the higher row.

4. If a column contains a leading 1 then all the other entries of that column are zero.

A matrix (again any matrix) is said to be *in row-echelon form* if it satisfies items 1 – 3 of the reduced row-echelon form definition.

Notice from these definitions that a matrix that is in reduced row-echelon form is also in row-echelon form while a matrix in row-echelon form may or may not be in reduced row-echelon form.

How to solve the system (S): in finite number of steps, using elementary row operations (ERO) reduce the augmented matrix $\bar{A}$ into the reduced row-echelon form (or, into the row-echelon form).

This sequence of steps fails to be determined in a unique possible way. But is is possible to prove that

- any system of linear equations with matrix in the reduced row-echelon form derived from the matrix $\bar{A}$ is equivalent with the original system (due to ERO) i.e. the solution set are the same.

Description of Gauss - Jordan Elimination Method

Step 1. Locate the leftmost column that does not consist of entirely zeros.

Step 2. Interchange the top row with another row to bring a nonzero entry to the top of the column found in Step 1.

Step 3. Using multiplication of a row by a real number (ERO operation No2), introduce in the first row a pivot 1.

Step 4. Using ERO, multiplication of rows and their additions, form all entries below the pivot 1 as entirely zeros.

Step 5. Now, omit the first row and the first column of the matrix arised as a results of steps 1, 2, 3, and 4, and work with the remaining matrix - a submatrix in the same way, i.e. repeat steps 1 - 4 on that submatrix. Continue in this way until the entire matrix is in row-echelon form.

Step 6. Now, beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows (ERO) above and introduce zeros above the pivot 1'.

As the result of those steps 1 - 6, the last matrix is in reduced row-echelon form.

Exercise. Solve the system, transforming the matrix $\bar{A}$ into the reduced row-echelon form:

$$\begin{aligned} 2x_1 - 2x_2 + x_3 &= 3 \\ 3x_1 + x_2 - x_3 &= 7 \\ x_1 - 3x_2 + 2x_3 &= 0 \end{aligned}$$

Solution. We shall show ERO operations on $\bar{A}$ and in the same time, we shall write the new system arised due to them:

$$\left(\begin{array}{rrr|r} 2 & -2 & 1 & 3 \\ 3 & 1 & -1 & 7 \\ 1 & -3 & 2 & 0 \end{array}\right) R_1 \leftrightarrow R_3 \qquad \begin{aligned} x_1-3x_2+2x_3 &= 0 \\ 3x_1+x_2-x_3 &= 7 \\ 2x_1-2x_2+x_3 &= 3 \end{aligned}$$

$$\left(\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 3 & 1 & -1 & 7 \\ 2 & -2 & 1 & 3 \end{array}\right) R_2 - 3R_1 \rightarrow R_2 \qquad \begin{aligned} x_1-3x_2+2x_3 &= 0 \\ 10x_2-7x_3 &= 7 \\ 2x_1-2x_2+x_3 &= 3 \end{aligned}$$

$$\left(\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 10 & -7 & 7 \\ 2 & -2 & 1 & 3 \end{array}\right) R_3 - 2R_1 \rightarrow R_3 \qquad \begin{aligned} x_1-3x_2+2x_3 &= 0 \\ 10x_2-7x_3 &= 7 \\ 4x_2-3x_3 &= 3 \end{aligned}$$

$$\left(\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 10 & -7 & 7 \\ 0 & 4 & -3 & 3 \end{array}\right) 0,1R_2 \rightarrow R_2 \qquad \begin{aligned} x_1-3x_2+2x_3 &= 0 \\ x_2-7/10x_3 &= 7/10 \\ 4x_2-3x_3 &= 3 \end{aligned}$$

$$\left(\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 1 & -7/10 & 7/10 \\ 0 & 4 & -3 & 3 \end{array}\right) R_3 - 4R_2 \rightarrow R_3 \qquad \begin{aligned} x_1-3x_2+2x_3 &= 0 \\ x_2-7/10x_3 &= 7/10 \\ -1/5x_3 &= 1/5 \end{aligned}$$

$$\left(\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 1 & -7/10 & 7/10 \\ 0 & 0 & -1/5 & 1/5 \end{array}\right) (-5)R_3 \to R_3 \qquad \begin{aligned} x_1 - 3x_2 + 2x_3 &= 0 \\ x_2 - 7/10x_3 &= 7/10 \\ x_3 &= -1 \end{aligned}$$

$$\left(\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 1 & -7/10 & 7/10 \\ 0 & 0 & 1 & -1 \end{array}\right) R_2 + 0,7R_3 \to R_2 \qquad \begin{aligned} x_1 - 3x_2 + 2x_3 &= 0 \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned}$$

$$\left(\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right) R_1 - 2R_3 \to R_1 \qquad \begin{aligned} x_1 - 3x_2 &= 2 \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned}$$

$$\left(\begin{array}{rrr|r} 1 & -3 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right) R_1 + 3R_2 \to R_1 \qquad \begin{aligned} x_1 &= 2 \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned}$$

The last matrix is in the reduced row-echelon form;

it follows that there exists a unique solution of the system, namely
$x_3 = -1$, $x_2 = 0$, $x_1 = 2$. •

How to use linear algebra techniques to solve systems of linear equations:

To solve a system of equations we will first write down the augmented matrix for the system,

We will then use elementary row operations to reduce the augmented matrix to either row-echelon form or to reduced row-echelon form.

Any further work that we'll need to do will depend upon where we stop.

If we go all the way to reduced row-echelon form then in many cases we will not need to do any further work to get the solution. Reducing the augmented matrix to reduced row-echelon form is called **Gauss-Jordan Elimination**.

If we stop at row-echelon form we will have a little more work to do in order to get the solution, but it is generally simple arithmetic. Reducing the augmented matrix to row-echelon form and then stopping is called **Gaussian Elimination Method (GEM)**.

Remark (important!). In view of the purpose - to determine the solution of the system - let us remark that: if, at any step of the elimination, we get some row of the form

$$0 \quad \ldots \quad 0 \;\big|\; c$$

with $c \neq 0$, or a contradictory row, we stop: this means a contradiction. It follows there is no solution of the system, as there is (at least) one contradictory row in the system, therefore the whole system is inconsistent.

Exercise. Find the coefficients a, b, c in the equation of a parabola

$$y = a + bx + cx^2$$

passing through 3 points $A[1,\ -5]$, $B[-2,\ 7]$, $C[3,\ 17]$.

Solution. Let us write conditions for those points A, B, C to be the points of the graph of the parabola in question: it follows that coefficients a, b, c have to be the solution of the following linear equations system:

$$\begin{aligned} a+\ b+\ c &= -5 \\ a-2b+4c &= \ \ \ 7 \\ a+3b+9c &= \ 17 \end{aligned}$$

then we solve it by GEM, reducing its augmented matrix into reduced row-echelon form:

$$
\left(\begin{array}{rrr|r} 1 & 1 & 1 & -5 \\ 1 & -2 & 4 & 7 \\ 1 & 3 & 9 & 17 \end{array}\right) R_2 - R_1 \to R_2,\ R_3 - R_1 \to R_3 \ \sim
$$

$$
\sim \left(\begin{array}{rrr|r} 1 & 1 & 1 & -5 \\ 0 & -3 & 3 & 12 \\ 0 & 2 & 8 & 22 \end{array}\right) \ (-1/3)R_2 \to R_2,\ (1/2)R_3 \to R_3 \ \sim
$$

$$
\sim \left(\begin{array}{rrr|r} 1 & 1 & 1 & -5 \\ 0 & 1 & -1 & -4 \\ 0 & 1 & 4 & 11 \end{array}\right) \ R_3 - 2R_2 \to R_3 \ \sim
$$

$$
\sim \left(\begin{array}{rrr|r} 1 & 1 & 1 & -5 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 5 & 15 \end{array}\right) \ (1/5)R_3 \to R_3 \ \sim
$$

$$
\sim \left(\begin{array}{rrr|r} 1 & 1 & 1 & -5 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right) \ R_1 - R_3 \to R_1,\ R_2 + R_3 \to R_2 \ \sim
$$

$$
\sim \left(\begin{array}{rrr|r} 1 & 1 & 0 & -8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right) \ R_1 - R_2 \to R_1 \ \sim \ \left(\begin{array}{rrr|r} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right)
$$

From the last matrix (in fact, in the reduced row-echelon form) we have

$a = -7, b = -1, c = 3,$

and it follows $y = -7 - x + 3x^2$ •

Remark. This problem is named as *an interpolation problem*,
the quadratic polynomial as the solution is named as *interpolation polynomial*.

Generalization: for $(n + 1)$ given points in the plane $P_i[x_i,\ y_i]$, $i = 0,\ 1, ...,\ n$, find an interpolation polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + ... + a_n$$

of degree n, passing through those given points. In an analogous way, this problem leads to the solution of the set of $n + 1$ equations for the coefficients of the polynomial.

Remark. The unknowns a, b, c can be found also from the row-echelon matrix, omitting the step (4):

$$\begin{aligned} a+b+c &= -5 \\ b-c &= -4 \\ c &= 3 \end{aligned}$$

But we need to put from the 3rd equation $c = 3$ into the 2nd one (this substitution process is called a back substitution):

$b - c = -4, \quad$ therefore $\quad b = -4 + c = -1.$

From the 1st equation it follows then $a = -5 - b - c = -7$.

Existence and the Number of Solutions of a Linear Equations System

Suppose the system (S) consists of m equations with n unknowns, and due to Gauss - Jordan Elimination Method, we have transformed its augmented matrix $\bar{A} = (A|b)$ into the reduced row-echelon form A_{red}; in symbols:

$$\bar{A} = (A|b) \quad \xrightarrow{ERO} \quad A_{red}$$

Denote as r the greatest natural number such that the unit matrix E_r of the size r arises as a submatrix of the matrix A_{red}.

Then, there is exactly one of the three possibilities:

(1) Suppose that $r = n$, and suppose that if there is more than r rows in A_{red}, then all of them are entirely zero rows. Then there exists *the unique solution of the system (S)*, that is, a unique n-tuple $[x_1, \ldots, x_n]$. The system is a consistent one.

(2) Suppose that $r < n$, that is, we have $s = n - r$ columns outside the unit matrix E_r in the reduced matrix A_{red}, corresponding to unknowns $x_{r+1}, \ldots, x_n$; more, suppose again that if there is more than r rows in A_{red}, then all are entirely zero rows. Then *there are infinitely many solutions of the system (S)*: the values of the unknowns $x_{r+1}, \ldots, x_n$ could be taken as arbitrary chosen real numbers (hence, in infinitely many possibilities), and the remaining unknowns $x_1, \ldots, x_r$ are determined by values of those $x_{r+1}, \ldots, x_n$. The system is a consistent one.

(3) Now, suppose that $r \le n$, and in A_{red} the $(r+1)$ row is of the form

$$0 \quad \ldots \quad 0 \,|\, \mathrm{c}$$

Then *there is no solution* of (S), the system is inconsistent.

Homogeneous Systems of Linear Equations

Definition. A system of linear equations (S) with *zero column of coefficients of the right sides of equations* is said to be *a homogeneous system*.

It follows: any homogeneous system has at least one solution, as, for any system, a *trivial* n-tuple consisting entirely of 0: $[0,\ 0, \ldots,\ 0]$, is always a solution (such solution itself is called the trivial solution).

Why it is so: in GEM, a contradictory row cannot be generated.

Therefore, now there is only one of two possibilities:

- a unique solution (therefore, a trivial one),

- infinitely many solutions, with some unknowns freely chosen, they will be variables in the solution set (therefore, also non-trivial solutions).

Exercise. Find all solutions of the homogeneous system:

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & & & - & 6x_4 & = & 0 \\ 2x_1 & + & x_2 & + & 3x_3 & & & = & 0 \\ 7x_1 & + & 8x_2 & + & 6x_3 & - & 18x_4 & = & 0 \end{array}$$

Solution. We shall provide GEM written in the table:

$$\begin{array}{rrrr|r} x_1 & x_2 & x_3 & x_4 & \\ \hline 1 & 2 & 0 & -6 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ 7 & 8 & 6 & -18 & 0 \\ \hline 1 & 2 & 0 & -6 & 0 \\ 0 & -3 & 3 & 12 & 0 \\ 0 & -6 & 6 & 24 & 0 \end{array} \qquad \begin{array}{rrrr|r} x_1 & x_2 & x_3 & x_4 & \\ \hline 1 & 2 & 0 & -6 & 0 \\ 0 & 1 & -1 & -4 & 0 \\ 0 & 1 & -1 & -4 & 0 \\ \hline 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & -1 & -4 & 0 \\ & & & & \end{array}$$

There are infinitely many solutions; 2 unknowns could be freely chosen, let take those as x_3, x_4. Then $x_2 = x_3 + 4x_4$, $x_1 = -2x_3 - 2x_4$. Therefore, solutions are 4-tuples of the form $[-2x_3 - 2x_4,\ x_3 + 4x_4,\ x_3,\ x_4]$.

The trivial solution $[0,\ 0,\ 0,\ 0]$, a special case, corresponds to $x_3 = x_4 = 0$.

Two other 4-tuples are special ones: $x_3 = 1$, $x_4 = 0$: we get $(-2,\ 1,\ 1,\ 0)$,

$x_3 = 0$, $x_4 = 1$: we get $(-2,\ 4,\ 0,\ 1)$ •

Remark. For any system of linear equations, a homogeneous or non-homogeneous one, the solution set could be given in a specific, universally applicable form. As a showcase, let us write the solution set for the non-homogeneous system with its augmented matrix in the reduced row-echelon form:

$$\begin{aligned} x_1 \quad -2x_3 + 5x_4 &= 7 \\ x_2 + \ x_3 - 3x_4 &= 5 \end{aligned}$$

we see that there are infinitely many solution of that system; a 4-tuple as a solution could be given in the form

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 + 2x_3 - 5x_4 \\ 5 - x_3 + 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -5 \\ 3 \\ 0 \\ 1 \end{pmatrix} x_4,$$

$x_3,\ x_4 \in R$ arbitrarily chosen. This solution has the following structure:

the vector x as the solution of the non-homogeneous is a sum of

– the vector $x^\star = \begin{pmatrix} 7 \\ 5 \\ 0 \\ 0 \end{pmatrix}$, which is the solution of the non-homogeneous system (in general, it arises due to the specially chosen unknowns $x_3 = x_4 = 0$, therefore, it is called as *the particular solution of the non-homogeneous system*;

– the vector given as the *linear combination* $x_{hom} = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -5 \\ 3 \\ 0 \\ 1 \end{pmatrix} x_4$, showing the complete set of the corresponding homogeneous system derived from our non-homogeneous system (putting zeros on the right sides of equations in any row); in view of its form, it is denoted as *the general solution of the adjacent homogeneous system*.

This is true in general as well, and in fact it follows from GEM: for any system, the structure of its solution set is as follows (let us call what follows as "the structure theorem"):

– the vector x *as the solution of the non-homogeneous system is the sum of two vectors,*

first of them being an arbitrary taken vector $x^{\star}$ *as a particular solution of the non-homogeneous system,*

the second one is a vector x_{hom} *being the general solution of the adjacent homogeneous system.*

Remark. The previous formulation includes also the case when there exists the unique solution of the non-homogeneous system: in that case, the reduced matrix of the system has to be a square matrix, therefore $n = m = r$ is true, and the homogeneous system adjacent to the non-homogeneous one has the unique solution as well, therefore this has to be the trivial one. Therefore, in the structure theorem, the vector x_{hom} is the zero vector.

Applications of Systems of Linear Equations - Problems

In next problems, formulate the corresponding system of linear equations, discuss the condition(s) on the existence of solution, find all possible solutions (using GEM Method, or using Cramer's Rule), and finally check results:

1. *Interpolation Problem.* Find the equation of parabola $y = ax^2 + bx + c$ that passes through the points P, Q, R:

(a) $P\,[-1,\ 6]$, $Q\,[1,\ 4]$, $R\,[2,\ 9]$;

(b) $P\,[-1,\ -4]$, $Q\,[-2,\ 6]$, $R\,[-3,\ 22]$;

(c) $P\,[1,\ -1]$, $Q\,[-1,\ 9]$, $R\,[2,\ -3]$.

2. *Interpolation Problem.* Find the equation of a cubic parabola of the form $y = ax^3 + bx^2 + cx + d$ that passes through the points P, Q, R, S:

(a) $P\,[-2,\ 0]$, $Q\,[1,\ 6]$, $R\,[-1,\ 6]$, $S[3,\ 30]$;

(b) $P\,[1,\ 6]$, $Q\,[-1,\ 10]$, $R\,[3,\ 26]$, $S\,[-2,\ 6]$;

(c) $P\,[-1,\ 0]$, $Q\,[1,\ 4]$, $R\,[2,\ 3]$, $S\,[3,\ 16]$.

3. *Equation of a circle.* Find the equation of the circle $(x-x_0)^2+(y-y_0)^2 = r^2$ (i.e., find coordinates of the centre $[x_0,\ y_0]$ and radius r of that circle) that passes through the points A, B, C:

(a) $A\,[1,\ 2]$, $B\,[1,\ -2]$, $C\,[0,\ -1]$;

(b) $A\,[1,\ 5]$, $B\,[-4,\ 0]$, $C\,[4,\ -4]$;

(c) $A\,[-1,\ 5]$, $B\,[-2,\ -2]$, $C\,[5,\ 5]$.

4. *Equation of a circle.* Find the equation of the circle in the form $ax^2 + ay^2 + bx + cy + d = 0$ that passes through the given points A, B, C:

(a) $A\,[1,\ 1]$, $B\,[2,\ 1]$, $C\,[3,\ 2]$;

(b) $A\,[4,\ 3]$, $B\,[1,\ 2]$, $C\,[2,\ 0]$.

5. Find a cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ such that $p(1) = 5$, the derivative $p'(1) = 5$, $p(2) = 17$, and the derivative $p'(2) = 21$.

6. *A game.* Find three numbers whose sum is 34, when the sum of of the first and the second is 7, and the sum of the second and the third is 22.

7. *Charges in Zoo.* A zoo charges 6 euros for adults, 3 euros for students, and 0,5 euro for children. One morning 79 people enter and pay a total of 207 euro. Determine the possible numbers of adults, students and children.

(*Solution:* Let A, S, C denote the numbers of adults, students, children. Possible solutions are:

$A = 5k$, $S = 67 - 11k$, $C = 12 + 6k$, where $k = 0,\ 1, \ldots,\ 6$.)

8. *Flows in Networks - Traffic Flow.* Set up the system of linear equations that describes the traffic flow (here data x_i means: numbers of vehicles per minute into/out of a node as a symbol for a crossroads in the diagram).

- Determine the possible flows,

- determine the maximum for the flow x_3,

- find flows x_1, x_2, x_4, if the flow x_3 is supposed to be regulated as 50.

Remark: all flows are considered as nonnegative ones, and the necessary condition for any node is

the total flow into the node + the total flow out of the node has to be zero.

$$\begin{array}{ccccc} 200 & & x_1 & & 100 \\ \longrightarrow & \oplus & \longrightarrow & \oplus & \longrightarrow \\ & x_2 \downarrow & & \uparrow x_3 & \\ \longrightarrow & \oplus & \longrightarrow & \oplus & \longrightarrow \\ 150 & & x_4 & & 250 \end{array}$$

9. *Flows in Networks - Traffic Flow.* Set up the system of linear equations that describes the traffic flow;

- determine the flows x_1, x_2, x_3, if $x_4 = 100$,

- determine the maximum and the minimum values for the flow x_4, if all flows are constrained to be nonnegative.

$$
\begin{array}{ccccc}
 & 800 \downarrow & & \uparrow 400 & \\
400 & & x_1 & & 600 \\
\longrightarrow & \oplus & \longrightarrow & \oplus & \longrightarrow \\
 & x_4 \downarrow & & \uparrow x_2 & \\
\longleftarrow & \oplus & \longleftarrow & \oplus & \longleftarrow \\
400 & & x_3 & & 200 \\
 & 200 \downarrow & & \uparrow 200 &
\end{array}
$$

10. *Flows in Networks - Traffic Flow.* Set up the system of linear equations that describes the traffic flow;

- determine the flows x_1, x_2, x_3, if $x_4 = 100$,

- determine the maximum and the minimum values for the flow x_4, if all flows are constrained to be nonnegative.

$$
\begin{array}{ccccc}
 & 500\downarrow & & \uparrow 400 & \\
700 & & x_1 & & 600 \\
\longrightarrow & \oplus & \longrightarrow & \oplus & \longrightarrow \\
 & x_4\downarrow & & \uparrow x_2 & \\
\longleftarrow & \oplus & \longleftarrow & \oplus & \longleftarrow \\
400 & & x_3 & & 600 \\
 & 600\downarrow & & \uparrow 200 &
\end{array}
$$

11. *Production Scheduling.* A small producer produces two types of guitars, A and B. Costs per one guitar of any type, necessary for labour and materials (in dollars) are shown in the table:

type of guitar	A	B
labour cost	30	40
material cost	20	30

As the total investment for labour and materials, 3000 dollars per week are projected, and the producer assumes three production variants how to use up this sum, due to the table:

possibility No.:	1.	2.	3.
- labour cost	1800	1750	1720
- material cost	1200	1250	1280

Compute how many guitars should be produced respecting any of those three restrictions, assuming also the full using of the investment indicated.

12. *Concert - the scheduling problem.* In a concert hall, there are 10 000 seats in total. Tickets are charged as 4 dollars (cheaper seats) or 8 dollars (more expensive seats). How many cheaper and how many expensive tickets should be sold for any of 3 concerts in the hall, if managers will expect revenues as shown in the table?

concert	1st	2nd	3rd
number of tickets	10 000	10 000	10 000
revenue expected	56 000	60 000	68 000

13. *Diet Problem - Scheduling.* In a hospital, a special diet has to be prepared for the patient, combining three basic foods K, L, M in a way that the diet has to contain exactly 200 units of calcium (Ca), 80 units of iron (Fe), and 120 units of vitamin A (vit. A). The number of units of those components in one unit of food K, L, M is written in the table:

	K	L	M
Ca	30	10	20
Fe	10	10	20
vit. A	10	30	20

How many units of any of K, L, M should be used for the exact composition of the diet?

14. *Diet Problem - Scheduling, continuation.* Repeat the diet problem 13 assuming that

a) the diet is to include exactly 400 units of calcium, 160 units of iron, and 240 units of vitamin A, or

b) food M is no longer available,or

c) the vitamin A requirement is deleted.

14. *Diet Problem - Scheduling Problem.* In a new hospital, a special diet has to be prepared for the patient, combining four basic foods K, L, M, N in a way that the diet has to contain exactly 110 units of calcium (Ca), 120 units of iron (Fe), 130 units of copper (Cu) and 140 units of vitamin A (vit. A). The number of units of those components in one unit of food K, L, M, N is written in the table:

	K	L	M	N
Ca	10	20	30	40
Fe	20	30	40	10
Cu	30	40	10	20
vit. A	40	10	20	30

How many units of any of four K, L, M, N should be used for the exact composition of the diet?

15. *Production Scheduling problem.* A furniture factory produces three types of chairs A, B, C; each piece of a chair of any type requires three operations in cutting, assembling and finishing department. Each operation requires the number of hours as indicated in the table. The workers in the factory can provide 700 hours of cutting, 660 hours in assembling, and 230 hours in finishing departments each work week. How many chairs of any of those three type should be produced so that all available labour-hours are fully used?

	A	B	C
cutting dept.	1 hr.	2 hr.	3 hr.
assembling dept.	1,2 hr.	1,8 hr.	2,4 hr.
finishing dept.	0,4 hr.	0,6 hr.	1 hr.

16. *Production Scheduling.* A small manufacturing plant makes three types of inflatable boats: one-person, two-person, and four-person models. Each boat requires the services of three departments as listed in the table. The cutting, assembly and packaging departments have available a maximum of 380, 330 and 120 labour-hours per week, respectively. How many boats of any of those three type should be produced so that all available labour-hours are fully used?

	1-person boat	2person boat	4-person boat
cutting dept.	0,5 hr.	1 hr.	1,5 hr.
assembly dept.	0,6 hr.	0,9 hr.	1,2 hr.
packaging dept.	0,2 hr.	0,3 hr.	0,5 hr.

17. *Production Scheduling - continuation.* Solve the "boat problem 16ässuming

a) the cutting, assembly and packaging departments have available a maximum of 350, 330 and 115 labour-hours per week;

b) the packaging department is no longer used;

c) the four-person boat is no longer produced.

18. *Scheduling Problem.* A garment industry manufactures three shirt styles $M1$, $M2$, $M3$. Each style shirt requires the services of three departments as listed in the table. The cutting, sewing and packaging departments have available a maximum of 1160, 1560 and 480 labour-hours per week. How many of each style shirt must be produced each week for the plant to operate at full capacity?

	style $M1$	style $M2$	style $M3$	time available
cutting dept.	0,2 hr.	0,4 hr.	0,3 hr.	1160 hr.
sewing dept.	0,3 hr.	0,5 hr.	0,4 hr.	1560 hr.
packaging dept.	0,1 hr.	0,2 hr.	0,1 hr.	480 hr.

19. *Production Scheduling Problem.* A metal foundry produces bronze alloy of 3 types A, B, C. Each alloy requires the services of any of two departments, smelting department and finishing department, with their maximal week capacities as 350 labour hours, and 150 hours, respectively. Each operation requires the number of hours as indicated in the table.

	alloy A	alloy B	alloy C	max. of capacity
smelting dept.	30 hr.	10 hr.	10 hr.	350 hr.
finishing dept.	10 hr.	10 hr.	30 hr.	150 hr.

How many of alloys A, B, C should be produced each week to operate on full capacity? (Remark. Number of alloys has to be a nonnegative integer.)

20. *Systems of linear equations.* A reservoir is filled of water running from three possible taps; if all three taps are opened, the reservoir will be filled up within 24 minutes; in case when the first and the second taps are opened, the reservoir will be filled up within 36 minutes; finally, if the first and the third taps are opened, the reservoir will be filled up within 48 minutes. Compute the time in minutes which is necessary for the reservoir will be filled up by the first tap only, or the second one, or the third one, respectively.

21. *Systems of linear equations.* (Another) reservoir is filled of water running from (another) three taps; from the first and the second one together, it will be filled up within 45 minutes; from the first and the third one together it will be filled up within 50 minutes; when the second tap is only used, then from this, the reservoir will be filled up 2 hours earlier than from the third tap only. Find the time in minutes which is necessary for the reservoir will be filled up by the first tap only, or the second one, or the third one, respectively.

(Hint. For the time t_i, $i = 1, 2, 3$ corresponding to the use of the only tap, define *the spring-discharge* x_i as a number of liters of water per minute running from a tap: then we shall have that $x_i \cdot t_i = 1$, and filling the reservoir from two or three taps simultaneously simply means their spring-discharges are added - spring-discharges form ädditive quantities".)

22. *Systems of linear equations with a parameter.* Find all possible values of a real parameter p, for which the system has a unique solution, infinitely many solutions, or no solution, and evaluate all existing solutions, depending on corresponding values of the parameter p:

a) $$\begin{aligned} px_1 + 4x_2 &= 1 \\ x_1 + px_2 &= p \end{aligned}$$

b) $$\begin{aligned} p^2 x_1 + 3x_2 + 2x_3 &= 0 \\ x_2 + 4x_3 &= 8 \\ px_1 - x_2 + x_3 &= 0 \end{aligned}$$

c) $$\begin{aligned} x_1 + x_2 + px_3 &= 0 \\ (p+1)x_1 + x_3 &= 1 \\ (p+1)x_2 + x_3 &= 1 \end{aligned}$$

d) $$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + px_2 + x_3 &= p \\ x_1 + x_2 + px_3 &= p^2 \end{aligned}$$

e) $$\begin{aligned} x_1 + px_2 + x_3 &= 1 \\ x_1 + x_2 + x_3 &= 1 \\ x_1 + x_2 + p^2 x_3 &= 1 \end{aligned}$$

f) $$\begin{aligned} px_1 + x_3 &= 0 \\ px_2 + x_3 &= 0 \\ x_1 + x_2 + (p-1)x_3 &= 0 \end{aligned}$$

23. *Systems of linear equations with a parameter.* Find all possible values of a real parameter p, for which the system has a unique solution, infinitely many solutions, or no solution, and evaluate all existing solutions, depending on corresponding values of the parameter p:

a) $$\begin{aligned} px_1+4x_2 &= 0 \\ x_1+px_2 &= 0 \end{aligned}$$

b) $$\begin{aligned} px_1+x_2 &= p \\ px_1-x_2 &= 0 \end{aligned}$$

c) $$\begin{aligned} px_1+x_2 &= 0 \\ px_1-x_2 &= 0 \end{aligned}$$

d) $$\begin{aligned} 5x-3y &= qx \\ -3x+5y &= qy \end{aligned}$$

24. *Systems of linear equations with a parameter.* Find all possible values of a real parameter p, for which the system has a unique solution, infinitely many solutions, or no solution, and evaluate all existing solutions, depending on corresponding values of the parameter p:

a) $$\begin{aligned} x_1+x_2+x_3&=1\\ x_1+px_2+x_3&=1\\ px_1+x_2+x_3&=1 \end{aligned}$$

b) $$\begin{aligned} x_1+x_2+x_3&=0\\ x_1+px_2+x_3&=0\\ px_1+x_2+x_3&=0 \end{aligned}$$

c) $$\begin{aligned} 5x_1+(p+1)x_2+10x_3&=0\\ x_1-x_2+4x_3&=0\\ (p+2)x_1+4x_2+2x_3&=0 \end{aligned}$$

d) $$\begin{aligned} px_1+x_2-x_3&=0\\ -x_1+px_2+x_3&=0\\ x_1-x_2+px_3&=0 \end{aligned}$$

e) $$\begin{aligned} x_1+px_2+x_3&=1\\ (2p-1)x_1+(p-1)x_3&=0\\ (p-1)x_1+(p-1)x_3&=0 \end{aligned}$$

f) $$\begin{aligned} x_1+x_2+x_3&=2\\ 2x_1+3x_2+4x_3&=3\\ 3x_1+2x_2+px_3&=6 \end{aligned}$$

25. *Systems of linear equations with a parameter.* Find all possible values of a real parameter p, for which the system has a unique solution, infinitely many solutions, or no solution, and evaluate all existing solutions, depending on corresponding values of the parameter p:

a) $$\begin{array}{rcrcl} (2p-3)x_1 + & & (p-2)x_3 & = & 0 \\ (p-2)x_1 + & & (p-2)x_3 & = & 0 \\ x_1 + & (p-1)x_2 + & x_3 & = & 1 \end{array}$$

b) $$\begin{array}{rrrrl} -ax_1 + & x_2 + & x_3 + & x_4 & = 2 \\ x_1 - & ax_2 + & x_3 + & x_4 & = 2 \\ x_1 + & x_2 - & ax_3 + & x_4 & = 2 \\ x_1 + & x_2 + & x_3 - & ax_4 & = 2 \end{array}$$

26. *Systems of linear equations with a parameter.* Find all possible values of a real parameter q, for which the system has a nonzero solution; evaluate all existing solutions, depending on corresponding values of the parameter q:

a) $$\begin{array}{rrl} & 3y & = qx \\ 2x - & y & = qy \end{array}$$

b) $$\begin{array}{rrrl} x - & y + & z & = qx \\ & 2y - & z & = qy \\ & & 3z & = qz \end{array}$$

c) $$\begin{array}{rrrl} 4x + & y + & 5z & = 0 \\ & -3y + & z & = 0 \\ 3x - & 3y + & qz & = 0 \end{array}$$