MATHEMATICS 2

MATRICES, DETERMINANTS, APPLICATIONS

01. Preparatory problems. Transform the following matrices onto a reduced echelon form, using GEM:

$$\text{a) } \begin{pmatrix} 3 & -2 & 0 & 1 \\ 2 & -3 & -1 & 4 \\ 0 & 2 & -1 & 6 \end{pmatrix}; \text{ b) } \begin{pmatrix} 0 & 3 & -2 & 0 & 0 \\ 5 & -2 & -3 & 1 & 4 \\ -1 & 0 & 2 & -1 & 6 \end{pmatrix}; \text{ c) } \begin{pmatrix} 5 & 0 & 1 \\ -5 & -1 & 4 \\ -2 & -1 & 4 \\ 2 & -3 & 6 \end{pmatrix}; \text{ d) } \begin{pmatrix} 10 & 20 \\ 20 & -10 \\ -15 & 40 \\ 30 & 0 \end{pmatrix}$$

02. Determine all solutions of the systems of linear equations:

$$\text{a) } \begin{array}{rcl} x_1 - x_2 & = & 2 \\ 3x_1 + 2x_2 - x_3 & = & -2 \\ 4x_1 + 3x_3 & = & 2 \end{array} \quad \text{b) } \begin{array}{rcl} x_1 - x_2 & = & 0 \\ 3x_1 + 2x_2 - x_3 & = & 0 \\ 4x_1 + x_2 - x_3 & = & 0 \end{array} \quad \text{c) } \begin{array}{rcl} 0,3x_1 + 0,2x_2 & = & 5 \\ 0,1x_1 + 0,1x_2 & = & 10 \\ 0,2x_1 + 0,1x_2 & = & 15 \end{array}$$

03. Determine all possible values of real parameters a, b, for which the system has a unique solution, infinitely many solutions, or no solution. Provide all existing solutions, depending on values of the parameters. Check all results:

$$\begin{array}{rcl} ax_1 + bx_2 & = & 1 \\ ax_2 + bx_3 & = & a \\ ax_1 + bx_3 & = & b \end{array}$$

1. Knowledge: the concept of the matrix, skills: matrix operations. Solve the matrix equation $A - 2 \cdot X - 3B = E_n$ for the unknown matrix X, here matrices A, B are given, and E_n is the unit matrix of the corresponding type n:

$$\text{a) } A = \begin{pmatrix} 2 & 3 \\ -1 & 10 \end{pmatrix}, B = \begin{pmatrix} 10 & 30 \\ 10 & 0 \end{pmatrix}; \quad \text{b) } A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix};$$

$$\text{c) } A = \begin{pmatrix} 12 \\ 18 \\ 16 \end{pmatrix}, B = \begin{pmatrix} -12 \\ 24 \\ 36 \end{pmatrix}; \quad \text{d) } A = \begin{pmatrix} 2 & 0 & 30 \\ 20 & 30 & 0 \\ 0 & 30 & 20 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

2. Knowledge: the concept of the matrix, skills: matrix operations. Determine the product of matrices:

$$\text{a) } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}; \text{ b) } \begin{pmatrix} 2 & 3 & 0 \\ 0 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 1 & 5 \end{pmatrix}; \text{ c) } \begin{pmatrix} 2 & 3 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

3. Knowledge: the concept of the matrix, skills: matrix operations. Determine the product of matrices:

$$\text{a) } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}; \text{ b) } \begin{pmatrix} 2 & -3 \\ 4 & -6 \end{pmatrix} \cdot \begin{pmatrix} 9 & -6 \\ 6 & -4 \end{pmatrix}; \text{ c) } \begin{pmatrix} 3 & 4 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$$

4. Knowledge: permutation matrix. A square matrix P of the degree n is called a permutation matrix, if it consists of 0 and 1 only in such a way that, in each row and in each column, there is exactly one 1 only.

a) Determine the total number of all permutation matrices of the degree 3. Provide the list of all such matrices.

b) Provide the product table of all products of the pairs of permutation matrices of the degree 3.

c) Describe the effect of multiplication of a given matrix A by a permutation matrix from the left, or from the right, resp.

Apply the products with the matrix $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$; determine products $A \cdot P$, resp. for $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$ determine the product $P \cdot A$; here P is an arbitrary permutation matrix of the degree 3.

d) For the matrix $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ compute products $A \cdot P$, $P \cdot A$ (P is an arbitrary permutation matrix of the degree 3).

5. Knowledge: the concept of the matrix, skills: matrix operations. Evaluate products of matrices:

a) $\begin{pmatrix} 4 & 3 \\ 7 & 5 \end{pmatrix} \cdot \begin{pmatrix} -28 & 93 \\ 38 & -126 \end{pmatrix} \cdot \begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix}$;

b) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 & 4 \\ 2 & 3 & 0 \\ 2 & 0 & 0 \end{pmatrix}$

6. Knowledge: the concept of the matrix, skills: matrix operations. Determine the powers of matrices:

a) $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}^2$; b) $\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}^3$; c) $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}^4$; d) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n$

7. Knowledge: the concept of the matrix, skills: matrix operations. Determine the powers of matrix A^2, A^3 for the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

8. Knowledge: the concept of the matrix, skills: matrix operations. Decide whether, for any two matrices A, B, the following relation is true:

$(A+B) \cdot (A-B) = A^2 - B^2$. As an illustration, find matrices $(A+B) \cdot (A-B)$, $A^2 - B^2$, where $A = \begin{pmatrix} 2 & 3 \\ -3 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 5 \\ -5 & 1 \end{pmatrix}$.

9. Knowledge: the concept of the matrix, skills: matrix operations. To the given matrix A, find all possible matrices B such that the following is fulfilled: $A{\cdot}B = B{\cdot}A$. Solve for matrices A:

$$\text{a)}\ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix};\quad \text{b)}\ \begin{pmatrix} 7 & -3 \\ 5 & -2 \end{pmatrix};\quad \text{c)}\ \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

10. Knowledge: the concept of the matrix inverse to the given one, skills: matrix operations. Evaluate the matrix inverse to the given one, check the solution:

$$\text{a)}\ \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix};\quad \text{b)}\ \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix};\quad \text{c)}\ \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

$$\text{d)}\ \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix};\quad \text{e)}\ \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix};\quad \text{f)}\ \begin{pmatrix} 5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix};$$

$$\text{g)}\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix};\quad \text{h)}\ \begin{pmatrix} 2 & 3 & -10 \\ -1 & -8 & 15 \\ 3 & -2 & -5 \end{pmatrix};\quad \text{i)}\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

11. Knowledge: the concept of the matrix inverse to the given one, skills: matrix operations. Determine the matrix A^{-1}, and the matrix which is inverse to the matrix A^2 (i.e. $(A^2)^{-1}$); solve for the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

12. Knowledge: the concept of the matrix inverse to the given one, matrix equations, skills: matrix operations. Find the matrix X fulfilling the property:

$$\text{a)}\ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot X = \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix};\quad \text{b)}\ \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix} \cdot X \cdot \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 14 & 16 \\ 9 & 10 \end{pmatrix};$$

$$\text{c)}\ X \cdot \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -5 & 6 \end{pmatrix};\quad \text{d)}\ \begin{pmatrix} 1 & 2 & -3 \\ 3 & 2 & -4 \\ 2 & -1 & 0 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & -3 & 0 \\ 10 & 2 & 7 \\ 10 & 7 & 8 \end{pmatrix}$$

13. Knowledge: matrix equations, skills: matrix operations. Find solution of the system of linear equations, using the inverse matrix A^{-1}:

$$\text{a)}\ \begin{pmatrix} 5 & -8 \\ 5 & 4 \end{pmatrix} \cdot x = \begin{pmatrix} 0 \\ 6 \end{pmatrix};\quad \text{b)}\ \begin{pmatrix} 3 & 8 \\ -7 & 5 \end{pmatrix} \cdot x = \begin{pmatrix} -20 \\ -48 \end{pmatrix};$$

$$\text{c)}\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot x = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix};\quad \text{d)}\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot x = \begin{pmatrix} 10 \\ 20 \\ -30 \end{pmatrix}$$

Solutions:

1. a), b), c), d): in all cases $X = \frac{1}{2}(A - 3B - E_n)$ • **2.** a) $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$; b) $\begin{pmatrix} 3 & 4 \\ 3 & 5 \end{pmatrix}$; c) $\begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ • **3.** a) $\begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\gamma \end{pmatrix}$; b) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; c) $\begin{pmatrix} 8 & 11 \end{pmatrix}$ • **4.** c), d) multiplication of the matrix A by some of permutation matrices from the right (from the left) leads to the change in the columns order (rows order) of the matrix A into the new order of columns (rows), and this new order is determined by column (row) indices of entries 1, in the sense counting them by columns (rows) from the left to the right • **5.** a) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$; b) $\begin{pmatrix} 0 & 3 & 8 \\ 0 & 6 & 4 \\ 0 & -6 & -4 \end{pmatrix}$ • **6.** a) $\begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix}$; b) $\begin{pmatrix} -35 & -30 \\ 45 & 10 \end{pmatrix}$; c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; d) $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ • **7.** $A^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $A^3 = E_3$ • **8.** In general, it is not true: $(A+B)(A-B) = \begin{pmatrix} -61 & -10 \\ -32 & -17 \end{pmatrix}$, $A^2 - B^2 = \begin{pmatrix} -29 & 6 \\ -6 & -33 \end{pmatrix}$ • **9.** a) $\begin{pmatrix} a & 2b \\ 3b & a+3b \end{pmatrix}$ ($a, b \in R$ lib.); b) $\begin{pmatrix} a & 3b \\ -5b & a+9b \end{pmatrix}$ ($a, b \in R$ lib.); c) $\begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$ ($a, b, c \in R$ lib.) •

10. a) $\begin{pmatrix} 0,4 & 0,2 \\ -0,3 & 0,1 \end{pmatrix}$; b) $\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}$; c) $\frac{1}{ad - bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ (it exists in case $ad - bc \neq 0$ only); d) does not exist; e) $1/9 \cdot \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$; f) $\begin{pmatrix} 1/5 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$; g) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$; h) does not exist; i) $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ • **11.** $A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $(A^2)^{-1} = A$ • **12.** a) $\begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix}$; b) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$; c) $\begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}$; d) $\begin{pmatrix} 6 & 4 & 5 \\ 2 & 1 & 2 \\ 3 & 3 & 3 \end{pmatrix}$ • **13.** a) $\begin{pmatrix} 0,8 \\ 0,5 \end{pmatrix}$; b) $\begin{pmatrix} 4 \\ -4 \end{pmatrix}$; c) $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$; • d) $\begin{pmatrix} 10 \\ 20 \\ -30 \end{pmatrix}$ •