The physics of guitar string vibrations

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We describe laboratory experiments to study the harmonic content of standing waves in guitar strings. The experimental data were taken by using the magnetic pickup from a guitar and a digital oscilloscope with a Fast Fourier transform capability. The amplitudes of the harmonics in the measured signal depend on the location where the string is plucked, resulting in a different timbre of the sound. The relative amplitudes of transverse standing waves in a string were determined from the experimental data and also predicted from the wave equation with the boundary and initial conditions corresponding to the initial shape of the string.
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I. INTRODUCTION

The research into student understanding and curriculum development in the area of sound waves shows that many students have misconceptions.[1,2] We believe innovative, research-based materials are needed to supplement the curriculum and create an environment where students can actively participate and engage in the learning process through meaningful activities. In this paper, we address the harmonic content of resonant standing waves through the use of the guitar, to facilitate student learning.

Sounds produced by different musical instruments are longitudinal waves propagating in air. In stringed instruments, such as a guitar, violin, or piano, the sound waves in air are excited by transverse resonant standing waves in the strings, while in wind instruments the resonant standing waves are longitudinal waves in air-filled pipes. One can easily distinguish different musical instruments when the same note is played because various instruments have their own specific timbre (or quality of sound). The sound of an instrument playing, for example, the note E_4, consists not only of the fundamental frequency (or first harmonic; $f_1 = 329.63$ Hz for the note E_4), but also of some higher harmonics ($f_n = nf_1$, $n = 1, 2, 3,\ldots$). The amount and relative intensities of the harmonics define the particular tone of a musical instrument, which distinguishes it from others. However, even for the same instrument the tone depends on the way the notes are played by a musician.

To determine the amplitudes of the fundamental and higher harmonics in the oscillating component (for example, a string) of a musical instrument, a Fourier analysis of the string waveform is usually used. A very detailed study of the vibrating string profile has been performed[3] using optical detection of the string vibration. In these experiments, the vibrating string created a vibrating shadow on the photodetector. The Fourier analysis of the signal measured at different initial conditions (how the string was plucked) has shown a good agreement with the calculations of the harmonics (up to the ninth) in the vibrating string from the initial conditions (the initial shape) of the string. An analysis of vibrating string by another optical technique, high speed photography, was recently reported,[4] which also demonstrated good agreement between experiments and calculations.

The role of boundary conditions on the transverse standing waves in a string with one or both ends fixed has been demonstrated,[5] which clearly showed how the positions of nodes and crests of the particular mode of oscillations changed depending on whether the end of the string was free or fixed. Laboratory exercises on oscillation modes in open, closed, and conical pipes (air columns) have been suggested.[6] Placing a movable microphone inside the pipe allowed for determination of the positions of nodes and crests of the standing waves in the pipes of different types and of different sizes.

In this paper, we describe experiments on the harmonic content of resonant standing waves in a guitar string, plucked at different locations along the string. Our method is based on the Fourier analysis of the signal measured by the magnetic guitar pickup placed at a selected location along the string. With such, the students can clearly see that the amplitudes of the fundamental and higher harmonics in the string with both ends fixed strongly depend on the location where the string is plucked. A magnetic pickup connected to an amplifier, while being an indirect method, allows students to perform measurements directly with a guitar, selecting any string or even several strings played in chords. This is unlike the optical arrangement with a single string, combined with an optical system for a direct determination of the mechanical displacement as described in Refs. 3 and 4.

The expected harmonic content of the standing waves in a string plucked at a particular location can be predicted from a Fourier analysis of the initial shape (triangle) of the string pulled at this location. From the Fast Fourier Transform (FFT) of the magnetic pickup signal, the amplitudes of the harmonics in the transverse oscillations of the sting can be determined, and a quantitative comparison of the experimental and calculated results can be performed. This work has been running at our school as an advanced physics lab for undergraduate physics majors, giving them an opportunity to experience wave phenomena, sound waves, standing waves in strings, and the role of harmonics in the timbre of musical instruments. The computational part of the lab requires that the students perform a Fourier analysis of a periodic signal, allowing for quantitative analysis of experimental data in determining the harmonic composition of standing waves in strings.

In Sec. II, the amplitudes of the first and higher harmonics of resonant standing waves in a string are calculated as the functions of the location where the string is plucked. In Sec. III, the experimental setup is described in sufficient detail for setting up and running the lab. Section IV represents the experimental results on relative amplitudes of harmonics in the signals picked up by a sensor (magnetic pickup) while playing a string of the guitar. The results are compared with calculated amplitudes of the harmonic, in the standing waves of the guitar string. Finally, we conclude by discussing results and the methods from the point of view of the impact this laboratory work can have on a student's interest in physics, better understanding of concepts of wave phenomena, and mastering their skills in experimenting and analytical work.

II. FOURIER SERIES OF AN INITIAL SHAPE OF A STRING AND HARMONICS OF A STANDING WAVE

When the same note is played on a guitar by plucking a string at different locations, one can hear that the timbre of the instrument is notably different. The sound produced by the string contains the fundamental frequency and higher harmonics. Relative amplitudes of the harmonics depend on the location where the string has been plucked.

The frequencies and amplitudes of the harmonics can be experimentally measured using an FFT analysis of the oscillations measured with a sensor. The expected initial amplitudes of the harmonics when the string is plucked in a particular manner can be calculated as the Fourier coefficients of the infinite harmonic series representing the initial shape of the string. While the initial shape of a guitar string is not exactly triangular due to the bending stiffness of a metal string, we assume that the bending stiffness can be neglected when analyzing the role of the position where the string is plucked.

For a string of length L fixed at both ends ($x=0$ and $x=L$), the transverse displacement $y(x,t)$ depends on both the position x and the time t and satisfies the one-dimensional wave equation[7]

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}, \tag{1}$$

where $v = \sqrt{F_T/\mu}$ is the wave speed, F_T is the tension force in the string, and $\mu = m/L$ is the mass of the string per unit length.

For the case of the string plucked at some position and released at $t=0$, the initial conditions can be described as $y_0(x,0)=f(x)$, where $f(x)$ is the initial shape of the string, and $(\partial y/\partial t)_{t=0} = 0$ for any x. The solution of Eq. (1) with such initial conditions is[7]

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{n\pi vt}{L} = \sum_{n=1}^{\infty} y_n \cos\frac{n\pi vt}{L}, \tag{2}$$

where $y_n = b_n \sin(n\pi x/L)$ and b_n is the amplitude of the nth harmonic. The resonant frequencies of the harmonics are given by the equation for a string fixed at both ends

$$f_n = \frac{nv}{2L}. \tag{3}$$

Such frequencies of the oscillations (harmonics) produced by a string can be varied by changing the force of tension in the string (as when tuning the instrument) or the length L of the string's vibrating section (by pressing a finger against the string at different locations). For each harmonic, the displacement $y_n(x,t)$ due to this harmonic is zero at $x=0$, $x=L$, and at $x=L/n$. The coefficients b_n represent the amplitudes of the harmonics and can be calculated from the initial shape of the string $y_0(x,0)=f(x)$ as

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{4}$$

For the string of length L pulled at the position x_1 from the end of the string a distance $y=A$ (see Fig. 1), $f(x)$ is given by

$$f(x) = \frac{A}{x_1}x \qquad \text{for } 0 \le x \le x_1 \tag{5}$$

and

$$f(x) = \frac{A}{L-x_1}(L-x) \qquad \text{for } x_1 \le x \le L. \tag{6}$$

Hence, the amplitudes of the Fourier components (harmonics) of the initial shape of the string can be calculated as

$$b_n = \frac{2A}{Lx_1}\int_0^{x_1} x\sin\left(\frac{n\pi x}{L}\right)dx + \frac{2A}{L(L-x_1)}\int_{x_1}^{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)dx, \tag{7}$$

with the final result

$$b_n = \frac{2A\sin(n\pi x_1/L)}{\frac{x_1}{L}\left(1-\frac{x_1}{L}\right)\pi^2 n^2}. \tag{8}$$

As a consistency check for Eq. (8), we observe that by symmetry the Fourier amplitude for each harmonic should have the same magnitude when the string is plucked at equal distances on either side of the middle of the string. This means that if we write x_1 in terms of distance from the middle as

$$x_1 = \frac{L}{2} + \alpha\frac{L}{2} = \frac{L}{2}(1+\alpha), \tag{9}$$

where $-1 < \alpha < 1$ is the fraction of the distance from the middle to the end of the string, then b_n should have the same magnitude for either sign of α. Inserting Eq. (9) into Eq. (8), we obtain

$$b_n = \frac{8A\sin\left(\frac{n\pi}{2} + \alpha\frac{n\pi}{2}\right)}{(1-\alpha^2)\pi^2 n^2}. \tag{10}$$

The denominator is clearly unchanged for either sign of α, as is the numerator because the magnitude of the sine function is symmetric about $n\pi/2$ for any n.

We saw earlier that the spatial dependence of the nth harmonic is given by $y_n(x) = b_n\sin(n\pi x/L)$. We can therefore write the amplitude of the nth harmonic as

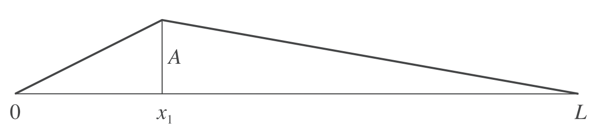

Fig. 1. Initial shape of the string of length L pulled a distance A at the position x_1 from its end.

$$b_n = \frac{y_n(x)}{\sin(n\pi x/L)}. \tag{11}$$

Using Eq. (8), we can write the ratio of the nth to the first Fourier amplitudes as

$$\frac{b_n}{b_1} = \frac{\sin(n\pi x_1/L)}{n^2 \sin(\pi x_1/L)}, \tag{12}$$

where x_1 indicates the location at which the string was plucked.

We can obtain an experimental value of this ratio if we use a magnetic pickup at location x_m and do an FFT to determine the amplitudes of vibration $y_n(x_m)$ and $y_1(x_m)$. From Eq. (11), the experimental ratio of Fourier amplitudes in terms of these measurements is

$$\frac{B_n}{B_1} = \frac{y_n(x_m)\sin(\pi x_m/L)}{y_1(x_m)\sin(n\pi x_m/L)}. \tag{13}$$

Here, we have used the notation B_n/B_1 to indicate measured Fourier amplitude ratios to distinguish from predicted ratios b_n/b_1, based on initial shape of the plucked string.

The idea of this lab is to measure relative amplitudes of the harmonics in the sensor's signals while plucking the string at different locations along the string. We then calculate the amplitudes of harmonics in the standing waves in the strings from the amplitudes of the harmonics in the measured signals to determine the ratio B_n/B_1, which are compared to the theoretically determined amplitudes of harmonics, calculated from the initial shape of the string as given by Eq. (12).

III. EXPERIMENTAL SETUP

The experimental setup is shown in Fig. 2. An electric guitar was used in our experiments, but the active built-in pickups of the electric guitar were not used, to exclude the effect of the frequency response of the built-in amplifiers on the recorded waveform from the pickup coil. Instead, a single coil Neo-D passive pickup[8] was placed above the strings at a selected distance (14.8 cm in this study) from the guitar's bridge using a stand with a clamp. The pickup's signal was amplified by a Studio Linear Amplifier SLA-1 (Applied Research and Technology), and the amplified signal was recorded using an Agilent DSO6012A digital oscilloscope, which has a Fast Fourier Transform (FFT) feature for analysis of measured signals. In addition to the recorded waveform, the oscilloscope displayed the FFT spectrum of the signal, with the frequency (in Hz) along the x-axis and the harmonics amplitude (in dBV) along the y-axis. The external trigger of the oscilloscope was connected to a 6-V battery via a manual switch, so that a single run of the experiment could be initiated by pressing the switch after the string was plucked.

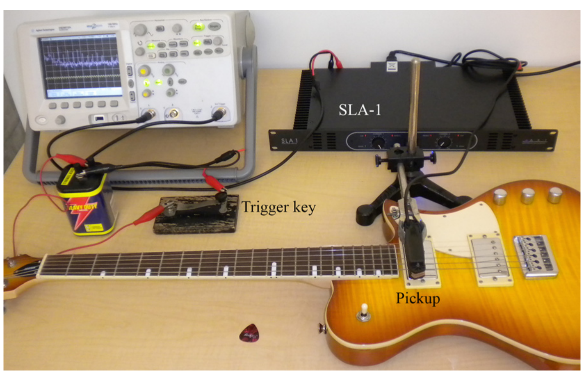

Fig. 2. Experimental setup. The pickup is held by a stand and connected to the input of the linear amplifier SLA-1. The output of SLA-1 is connected to the input of the oscilloscope, which has the FFT Math function selected. The upper trace shows the FFT of the recorded signal (lower trace).

IV. RESULTS AND DISCUSSION

As a first step, the frequencies f_n of the harmonics of the strings of different lengths (pressed on different frets) were measured, using the FFT on the oscilloscope, to confirm that the measured frequencies were well described by Eq. (3).

Next, the open string #1 (the thinnest one) was plucked at a particular location, and the voltage from the magnetic pickup over one time sweep of the oscilloscope was used to produce the FFT signal. An example of the FFT spectra is shown in Fig. 3 as an oscilloscope screen shot. The horizontal axis is frequency and the vertical axis is signal intensity β_V, measured in decibel-volts (dBV). The intensity of the signal is proportional to the square of the voltage, so we have

$$\beta_V = 20 \log \frac{V}{V_0}. \tag{14}$$

If the difference in two amplitudes in dBV is taken from the FFT, then the ratio of the voltages at the oscilloscope input can be calculated from

$$\beta_{V_n} - \beta_{V_1} = 20 \log \frac{V_n}{V_1}. \tag{15}$$

This equation tells us that a 20 dBV difference in amplitudes of two harmonics would correspond to the ratio $V_2/V_1 = 10$ of the two amplitudes.

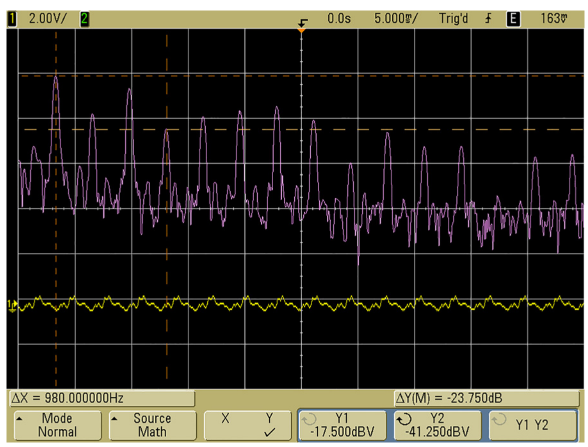

Fig. 3. The top trace on the screen displays the FFT of the guitar pickup signal on a logarithmic scale. The scales: 500 Hz/div (horizontal) and 20 dBV/div (vertical). Notice that the 13th harmonics is not seen, because it has a node at the selected magnetic pickup position $x_m = 14.8$ cm. This missing harmonic can be understood from Eq. (11). With $n = 13$, $x_m = 14.8$ cm, and $L = 64.3$ cm, we have $nx_m/L = 3.00$, so $\sin(n\pi x_m/L) = 0$, and the pickup signal of the 13th harmonic should be zero.

The background in the FFT spectrum changes slightly with frequency, and this fact could lead to errors when taking the difference of two peak heights located at different frequencies. However, in the range of the harmonics up to the seventh (analyzed in this work) the change was less than the noise level as seen in Fig. 3, and was ignored. If analysis of the eighth and higher harmonics were necessary, then a background level analysis would be required. The cause of the background is unknown, but we have noticed that the change in the background was larger in earlier experiments with an acoustic guitar compared to the current experiments with an electric guitar. We speculate that the guitar body response to the vibrations at different frequencies could be one of the causes.

The amplitudes β_n (in dBV) of the fundamental and several higher harmonics of the signal were measured from the FFT and the measurements were repeated several times for each pluck location of the string. The ratios V_n/V_1 of the harmonic amplitudes (for $n = 2$–7) of the measured waveforms were calculated for each harmonic as averages from multiple measurements using Eq. (15). We can relate this ratio of voltages for the harmonics to the ratio of each harmonic's amplitude as discussed below. The relative amplitudes of harmonics in the measured signal depend not only on the amplitudes of harmonics of standing waves in a string, but also on other factors such as the position of the sensor along the string, frequency dependences of the sensor and the amplifier, the decay times for different harmonics in the string, and the background.

For quantitative comparison of the measured sensor signal harmonics, given by B_n/B_1, to the calculated (expected) harmonics of the standing waves, given by b_n/b_1, we have taken into account the factors mentioned above as follows. First, a magnetic pickup is always placed very close to the string (a few mm) so the position of the sensor can easily be taken into account while calculating the harmonic amplitudes of the standing waves in the string from the FFT analysis of the measured signal. At the distance x_m from the bridge, the string displacement y_n due to the n^{th} harmonic with amplitude B_n is given by Eq. (2): $y_n = B_n \sin(n\pi x_n/L)$. With this, the relative amplitudes of measured harmonics depend on the relative displacements of the string at the pickup location x_m due to the particular harmonics of the standing wave as shown in Eq. (13).

Second, the signal voltage (emf) is induced in the magnetic pickup due to motion of a metal string near the pickup's coil. According to Faraday's Law, the induced emf is proportional to the time rate of change of the magnetic flux Φ. As a result, we can expect that the contribution of the string displacement y_n due to the nth harmonic of the standing wave in the string to the signal voltage V_n will be proportional to a frequency of the harmonic $f_n = nf_I$. Hence, we can assume that $V_n/V_1 = n(y_n/y_1)$. Combining this factor with Eq. (13), the relative amplitudes of harmonics B_n/B_1 of the standing wave in the string can be calculated from the relative amplitudes of the magnetic pickup signal V_n/V_1 as

$$\frac{B_n}{B_1} = \frac{V_n}{V_1} \frac{\sin(\pi x_m/L)}{n \sin(n\pi x_m/L)}. \quad (16)$$

This equation can be used for quantitative comparison of the experimental data on the relative amplitudes B_n/B_1 of the harmonics of the standing waves to the relative Fourier components b_n/b_1 of the initial shape of the string. The relative amplitudes of harmonics of the string displacement B_n/B_1 have been calculated from the relative amplitudes of the harmonics in the measured signal V_n/V_1, and are presented in Table I.

Table I. Measured average relative amplitudes of harmonics of the sensor signal recorded when open string #1 (the thinnest one) was plucked at different locations x_1. The string length $L = 64.3$ cm and the sensor was a distance 14.8 cm from the bridge.

Fret	x_1 (cm)	x_1/L	B_2/B_1	B_3/B_1	B_4/B_1	B_5/B_1	B_6/B_1	B_7/B_1
1	3.7	0.058	0.544	0.367	0.279	0.155	0.104	0.074
2	7.1	0.110	0.462	0.232	0.163	0.072	0.034	0.009
3	10.4	0.162	0.453	0.220	0.115	0.015	0.018	0.024
4	13.4	0.208	0.392	0.159	0.060	0.024	0.027	0.020
5	16.3	0.253	0.408	0.123	0.032	0.052	0.031	0.009
6	19.0	0.295	0.364	0.052	0.071	0.044	0.013	0.020
7	21.6	0.336	0.297	0.023	0.087	0.027	0.015	0.020
8	24.0	0.373	0.236	0.070	0.101	0.010	0.029	0.014
9	26.3	0.409	0.172	0.094	0.082	0.027	0.024	0.010
10	28.5	0.443	0.110	0.115	0.065	0.034	0.016	0.015
11	30.5	0.464	0.061	0.120	0.045	0.038	0.012	0.018
12	32.5	0.505	0.019	0.128	0.026	0.032	0.015	0.014

Using Eq. (12), the Fourier components were calculated for different locations x_1 of where the string was plucked. The calculated relative amplitudes b_n/b_1 are shown in Fig. 4, together with the experimentally measured amplitudes for harmonics 2–7. The correlation between the measured amplitudes of the standing waves and the Fourier components of the initial shape of the string is very good.

As an extension of this experiment, the decay rates of different harmonics can be studied by triggering the oscilloscope later in time relative to when the string is plucked. Our preliminary experiments indicate that the higher harmonics decay faster compared to the fundamental. This results in the string sound timbre being more "crisp" initially, to a more "flat" (or pure) sound (containing mostly the first harmonic), regardless of where the string is plucked. For the analysis of the decay rates of different harmonics, a recently published paper[9] with analysis of the motion of a single Fourier mode as an example of a transient, free decay of coupled linear oscillators looks very promising.

The statistical errors expected in our measurements depend on the measurements from the digital oscilloscope and the location of the magnetic pickup x_m, and their corresponding errors. The details for the calculations are shown in the Appendix.

V. CONCLUSION

As a result of this laboratory exercise, students have the opportunity to understand that harmonic composition of standing waves in the strings of musical instruments is strongly dependent on the way the strings are played. Generally, prior to the lab, students assume that playing a specific note on any instrument produces oscillations only at this frequency. Observing visually that other harmonics are also present and comparing this observation with how the note sounds when the string is plucked at different locations allows students to make connections between the sound waves produced and complex

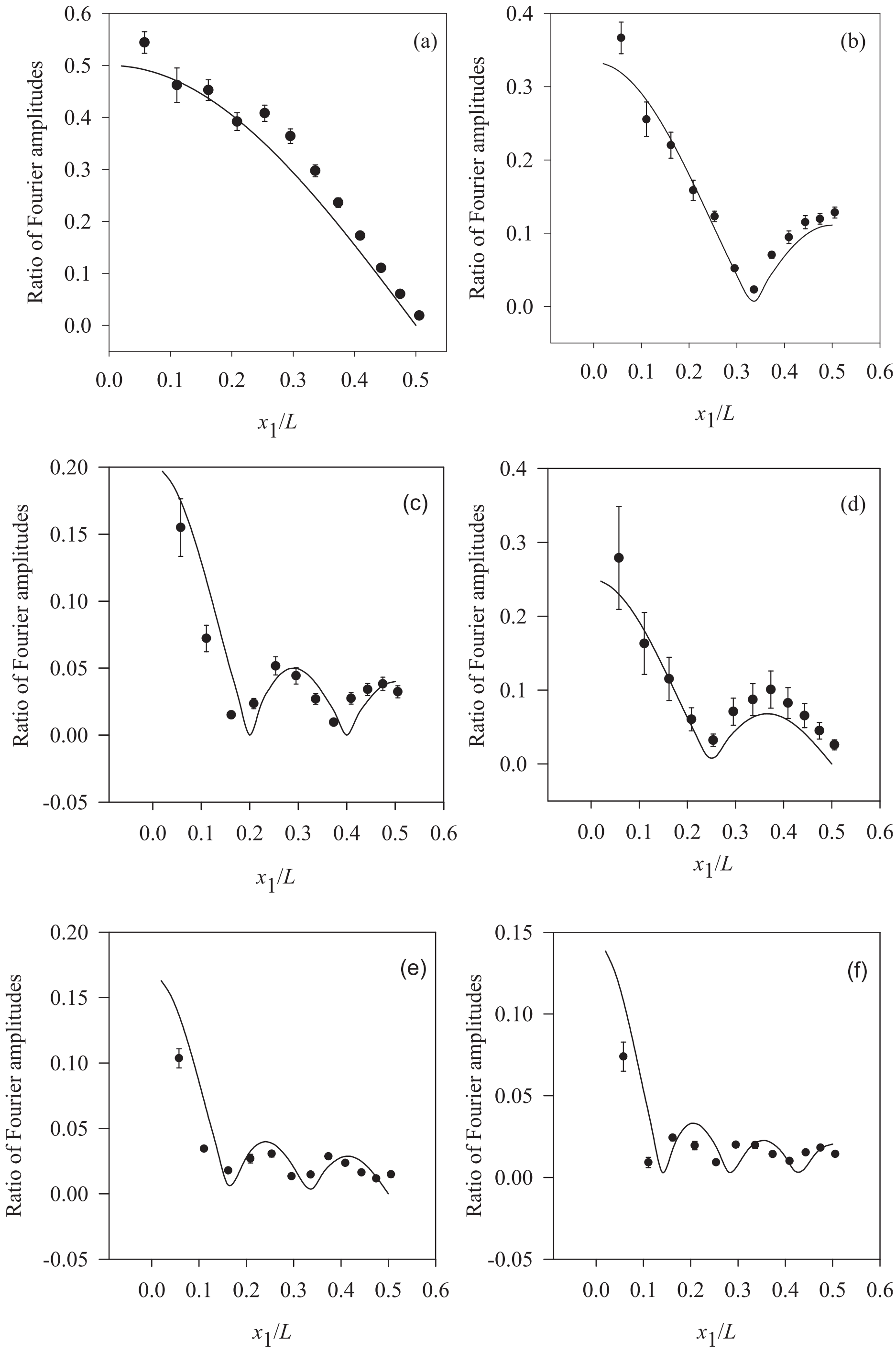

Fig. 4. Relative amplitudes of harmonics (a) $n=2$ through (f) $n=7$ of standing waves in a string calculated from Eq. (1) and from the experimentally measured signals as a function of the pluck location x_1/L. The points are determined experimentally as B_n/B_1 from Eqs. (16) and (A9) while the theory (solid) is calculated as b_n/b_1 from Eq. (12).

mechanical oscillations of the string. Solving the wave equation with boundary conditions according to the initial shape of the string allows students to clearly see how the wave theory is related to the sounding of the musical instruments.

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APPENDIX: ERROR CALCULATIONS

We obtain measurements of the ratio of amplitudes as given by Eq. (16) and designate the dbV differences between the nth and the first harmonic as δ_n so that

$$\delta_n = \beta_{V_n} - \beta_{V_1} = 20\log\frac{V_n}{V_1} \tag{A1}$$

and

$$\frac{V_n}{V_1} = 10^{0.05\delta_n}. \tag{A2}$$

To simplify notation, we designate the ratio of interest as $R_n = B_n/B_1$, which depends on the measured quantities δ_n and x_m so that $R_n = R_n(\delta_n, x_m)$. The squared error in R_n is thus given by

$$\sigma_{R_n}^2 = \left(\frac{\partial R_n}{\partial \delta_n}\right)^2 \sigma_{\delta_n}^2 + \left(\frac{\partial R_n}{\partial x_m}\right)^2 \sigma_{x_m}^2. \tag{A3}$$

From Eqs. (16) and (A2) we have

$$R_n = \frac{B_n}{B_1} = 10^{0.05\delta_n}\frac{\sin(\pi x_m/L)}{n\sin(n\pi x_m/L)}, \tag{A4}$$

$$\frac{\partial R_n}{\partial \delta_n} = \frac{\sin(\pi x_m/L)}{n\sin(n\pi x_m/L)}\frac{\partial}{\partial \delta_n}\left(10^{0.05\delta_n}\right), \tag{A5}$$

and

$$\frac{\partial}{\partial \delta_n}\left(10^{0.05\delta_n}\right) = 0.05\ln 10\left(10^{0.05\delta_n}\right). \tag{A6}$$

So from Eq. (A5) we get

$$\frac{\partial R_n}{\partial \delta_n} = 0.05\ln 10\left(10^{0.05\delta_n}\right)\frac{\sin(\pi x_m/L)}{n\sin(n\pi x_m/L)}. \tag{A7}$$

The last term of Eq. (A3) contains $\partial R_n/\partial x_m$, which gives

$$\frac{\partial R_n}{\partial x_m} = \frac{\pi V_n}{LV_1}\left[\left(\frac{\cos(\pi x_m/L)}{n\sin(n\pi x_m/L)}\right) - \left(\frac{\sin(\pi x_m/L)\cos(n\pi x_m/L)}{\sin^2(n\pi x_m/L)}\right)\right]. \tag{A8}$$

The error squared is then given by Eq. (A3) with these terms inserted and noting that $R_n = B_n/B_1$, giving

$$\sigma_{B_n/B_1}^2 = (0.05\ln 10)^2\left(10^{0.05\delta_n}\right)^2\left[\frac{\sin(\pi x_m/L)}{n\sin(n\pi x_m/L)}\right]^2\sigma_{\delta_n}^2 + \left(\frac{\pi}{L}10^{0.05\delta_n}\right)^2\left[\frac{\cos(\pi x_m/L)}{n\sin(n\pi x_m/L)} - \frac{\sin(\pi x_m/L)\cos(n\pi x_m/L)}{\sin^2(n\pi x_m/L)}\right]^2\sigma_{x_m}^2. \tag{A9}$$

The quantity σ_{x_m} was estimated as 3 mm, while the quantity δ_n was measured 5 times for each fret and the mean value and error in the mean were used for calculations. If several identical measurements occurred and the error in the mean was less than $\pm$ half the least count (0.625), then the larger value for σ_{δ_n} was used ($\sigma_{\delta_n} = 0.625/2$).

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