Cosmological Models

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Lagrangian Formulation of General Relativity: Einstein's equations in vacuum

In this part we derive the Einstein´s equations in vacuum with the help of the Hilber't action. The Einstein field equations in vacuum can be derived by the variational principle, with the so-called Hilbert action:

$$S_H = \int \sqrt{-g} R \, \mathrm{d}^4 x \tag{1}$$

The variational principle state that the equations of motion can be obtained by extremizing the action, ie, when it is fulfilled that:

$$\delta S = 0. \tag{2}$$

Aplying this condition to the Hilbert action (1):

$$\delta S_H = \int \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \mathrm{d}^4 x + \int \sqrt{-g} \delta g^{\mu\nu} R_{\mu\nu} \mathrm{d}^4 x + \int \delta\sqrt{-g} R \mathrm{d}^4 x = \delta S_1 + \delta S_2 + \delta S_3 \tag{3}$$

Let's consider first the variation of the Ricci tensor:

$$\delta S_1 = \int \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \mathrm{d}^4 x \tag{4}$$

But, before, the Rimmann tensor is given by:

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}\,, \tag{5}$$

and with this tensor, we get the Ricci tensor:

$$R_{\sigma\nu} = \partial_\mu \Gamma^{\mu}_{\nu\sigma} + \Gamma^{\mu}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \partial_\nu \Gamma^{\mu}_{\mu\sigma} - \Gamma^{\mu}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{6}$$

taking the variation of the Ricci tensor and considering that the partial derivatives commute with the variation:

$$\delta R_{\sigma\nu} = \partial_\mu \delta\Gamma^{\mu}_{\nu\sigma} - \partial_\nu \delta\Gamma^{\mu}_{\mu\sigma} + \delta\Gamma^{\mu}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} + \Gamma^{\mu}_{\mu\lambda}\delta\Gamma^{\lambda}_{\nu\sigma} - \delta\Gamma^{\mu}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\mu}_{\nu\lambda}\delta\Gamma^{\lambda}_{\mu\sigma} \tag{7}$$

we observe that that the variation of the term $\delta\Gamma^{\lambda}_{\mu\sigma}$ it is the difference of two connections and, therefore, it is a tensor, we calculate its covariant derivatives and make a trick:

$$\nabla_\mu \delta\Gamma^{\mu}_{\nu\sigma} = \partial_\mu \delta\Gamma^{\mu}_{\nu\sigma} + \Gamma^{\mu}_{\mu\lambda}\delta\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\lambda}_{\mu\nu}\delta\Gamma^{\mu}_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma}\delta\Gamma^{\mu}_{\nu\lambda} \tag{8}$$

$$\nabla_\nu \delta\Gamma^{\mu}_{\mu\sigma} = \partial_\nu \delta\Gamma^{\mu}_{\mu\sigma} + \Gamma^{\mu}_{\nu\lambda}\delta\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\lambda}_{\nu\mu}\delta\Gamma^{\mu}_{\lambda\sigma} - \Gamma^{\lambda}_{\nu\sigma}\delta\Gamma^{\mu}_{\mu\lambda} \tag{9}$$

taking the difference of the two previous terms:

$$\nabla_\mu \delta\Gamma^{\mu}_{\nu\sigma} - \nabla_\nu \delta\Gamma^{\mu}_{\mu\sigma} = \partial_\mu \delta\Gamma^{\mu}_{\nu\sigma} - \partial_\nu \delta\Gamma^{\mu}_{\mu\sigma} + \Gamma^{\mu}_{\mu\lambda}\delta\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\lambda}_{\mu\sigma}\delta\Gamma^{\mu}_{\nu\lambda} - \Gamma^{\mu}_{\nu\lambda}\delta\Gamma^{\lambda}_{\mu\sigma} + \Gamma^{\lambda}_{\nu\sigma}\delta\Gamma^{\mu}_{\mu\lambda} \tag{10}$$

Comparing the equations 7 and 10, we see that both are the same , so we have the next identity:

$$\delta R_{\sigma\nu} = \nabla_\mu \delta\Gamma^{\mu}_{\nu\sigma} - \nabla_\nu \delta\Gamma^{\mu}_{\mu\sigma} \tag{11}$$

putting this variation in equation 4

$$\delta S_1 = \int \sqrt{-g} g^{\sigma\nu} [\nabla_\mu \delta\Gamma^\mu_{\nu\sigma} - \nabla_\nu \delta\Gamma^\mu_{\mu\sigma}] \mathrm{d}^4 x \tag{12}$$

changing the indices μ by λ and ν by λ in the first term:

$$\delta S_1 = \int \sqrt{-g} \nabla_\lambda [g^{\sigma\nu} \delta\Gamma^\lambda_{\nu\sigma} - g^{\sigma\lambda} \delta\Gamma^\mu_{\mu\sigma}] \mathrm{d}^4 x \tag{13}$$

The last integral is an integral with respect to the natural volume element of the covariant divergence of a vector; by Stokes's theorem, this is equal to a boundary contribution at infinity, which we can set zero by making the variation vanish at infinity Let's see now how is the variation of the determinate metric root square, for that, we use the next relation:

$$\ln(detM) = Tr(\ln M) \tag{14}$$

where M is square matrix with non vanishing determinant and lnM is defined by $exp(lnM) = M$. Taking the variation of both members of the last relation, we get:

$$\frac{\delta detM}{detM} = Tr(M^{-1}\delta M) \tag{15}$$

taking M like the metric $g_{\mu\nu}$

$$\delta g = g(g^{\mu\nu}\delta g_{\mu\nu}) \tag{16}$$

Let's consider now the next relation

$$g^{\mu\gamma} g_{\mu\alpha} = \delta^\gamma_\alpha \tag{17}$$

taking the variation of this equation:

$$\delta g^{\mu\gamma} g_{\mu\alpha} + g^{\mu\gamma}\delta g_{\mu\alpha} = 0 \Longleftrightarrow \delta g^{\mu\gamma} g_{\mu\alpha} = -g^{\mu\gamma}\delta g_{\mu\alpha} \tag{18}$$

and particularizing for $\gamma \longrightarrow \alpha$

$$\delta g^{\mu\alpha} g_{\mu\alpha} = -g^{\mu\alpha}\delta g_{\mu\alpha} \tag{19}$$

substituting in equation 16, we get:

$$\delta g = -g(g_{\mu\nu}\delta g^{\mu\nu}) \tag{20}$$

On the other hand, varying the root square of the determinant of the metric:

$$\delta\sqrt{-g} = -\frac{\delta g}{2\sqrt{-g}}$$

substituting this result in the equation 38:

$$\delta\sqrt{-g} = \frac{1}{2}\frac{g}{\sqrt{-g}} g_{\mu\nu}\delta g^{\mu\nu} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu} \tag{21}$$

Finally, from the total variation of the Hilbert action, we get:

$$\delta S_H = \int \sqrt{-g}\delta g^{\mu\nu} R_{\mu\nu} \mathrm{d}^4 x + \int \delta\sqrt{-g} R \mathrm{d}^4 x = \int \sqrt{-g}\delta g^{\mu\nu} R_{\mu\nu} \mathrm{d}^4 x + \int -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu} R \mathrm{d}^4 x \tag{22}$$

$$\delta S_H = \int \sqrt{-g}\delta g^{\mu\nu} \left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right] \mathrm{d}^4 x \tag{23}$$

Let's recall that the functional derivative of the action satisfies:

$$\delta S = \int \sum_{i=1}^{n} \left(\frac{\delta S}{\delta \Phi^i} \delta\Phi^i \right) \mathrm{d}^4 x \tag{24}$$

where Φ^i is a complete set of fields being varied (in our case, it is just $g^{\mu\nu}$ and later in the case of k-essence and braiding model we will vary with respect to the field ϕ). So, according to the variational principle, we recover Einstein's equation in vacuum:

$$\frac{1}{\sqrt{-g}} \frac{\delta S_H}{\delta\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \tag{25}$$

Lagrangian Formulation of General Relativity: Einstein's equations with matter

In this part we derive the Einstein's equations in presence of matter, adding a new term: "the action of matter" Now, we are going to obtain the Einstein's equation with an additional term for matter field. What we would really like is to get the non vacuum field equation as well. So lets consider an action of the form:

$$S = \frac{1}{16\pi G} S_H + S_M \tag{26}$$

where S_M is the action for matter and S_H is the Hilbert action. Following the same procedure as above to obtain the Einstein's equation in vacuum, we take the functional derivative:

$$\delta S = \int \left[\sqrt{-g}\delta g^{\mu\nu} \frac{1}{16\pi G} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + \delta S_M \right] \mathrm{d}^4 x \tag{27}$$

and we use the variational principle to obtain:

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi G} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} = 0 \tag{28}$$

Let's use a trick, if we define the energy momentum tensor as:

$$T_{\mu\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \tag{29}$$

and substituting this in the equation 28:

$$\frac{1}{16\pi G} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) - \frac{1}{2} T_{\mu\nu} \tag{30}$$

and therefore:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{31}$$

or equivalently using the Einstein's tensor:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{32}$$

But, why should we think that 29 is really the energy-momentum tensor? Well, in some sense it is only because it is symmetric (due to the fact that $g_{\mu\nu}$ is symmetric), conserved, and (0,2) tensor with dimensions of energy density.

Cosmological models

In what follow, we take the particular action of K-essence and Braiding to derive the equations of motion for these fields. We use the same method like the one we use to derive the Einstein's equations, e.g., varying the action and then using the variational principle.

K-essence

The purpose of this section is obatin the field equation for this field. Let's consider first the action of k-essence model.

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left[K(X, \phi) + \frac{M_p^2}{2} R \right] \tag{33}$$

where:

$$X = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \tag{34}$$

and

$$M_p^2 = \frac{1}{8\pi G} \tag{35}$$

We vary the action (33) with respect to the inverse metric $g^{\mu\nu}$:

$$\delta S = \int \mathrm{d}^4x \left(\delta\sqrt{-g}\left[K(X,\phi) + \frac{M_p^2}{2}R\right] + \sqrt{-g}\left[\delta K(X,\phi) + \frac{M_p^2}{2}\delta R\right]\right) \tag{36}$$

$$= \int \mathrm{d}^4x \left(\delta\sqrt{-g}\left[K(X,\phi) + \frac{M_p^2}{2}R\right] + \sqrt{-g}\delta K(X,\phi)\right) + \int \mathrm{d}^4x \frac{M_p^2}{2}\delta R \tag{37}$$

$$= \int \mathrm{d}^4x \left(\delta\sqrt{-g}\left[K(X,\phi) + \frac{M_p^2}{2}R\right] + \sqrt{-g}\delta K(X,\phi)\right) + \int \mathrm{d}^4x \frac{M_p^2}{2}\sqrt{-g}\,\delta g^{\mu\nu}R_{\mu\nu} + \int \mathrm{d}^4x \sqrt{-g}\frac{M_p^2}{2}\delta R_{\mu\nu}g^{\mu\nu} \tag{38}$$

The last integral of (38) according to the equation (13) is zero. On the other hand, due to the fact that X depends on $g^{\mu\nu}$, we use the chain rule to vary $K(X,\phi)$:

$$\delta K(X,\phi) = \frac{\delta K}{\delta X}\delta X = K_X\left(-\frac{1}{2}\delta g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right) \tag{39}$$

substituting (39) and (21) in (38), we get:

$$\delta S = \int \mathrm{d}^4x \left(\left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\right)\left[K(X,\phi) + \frac{M_p^2}{2}R\right] + \sqrt{-g}K_X\left(-\frac{1}{2}\delta g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right)\right) + \int \mathrm{d}^4x \frac{M_p^2}{2}\sqrt{-g}\,\delta g^{\mu\nu}R_{\mu\nu} \tag{40}$$

$$= \int \mathrm{d}^4x \sqrt{-g}\delta g^{\mu\nu}\left[-\frac{1}{2}g_{\mu\nu}[K(X,\phi) + \frac{M_p^2}{2}R] - \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi K_X + \frac{M_p^2}{2}R_{\mu\nu}\right] \tag{41}$$

Applying the variational principle and using the equation (24), we get:

$$g_{\mu\nu}\left[K(X,\phi) + \frac{M_p^2}{2}R\right] + \nabla_\mu\phi\nabla_\nu\phi K_X = M_p^2 R_{\mu\nu} \tag{42}$$

and in terms of the Einstein tensor:

$$M_p^2 G_{\mu\nu} = K(X,\phi)g_{\mu\nu} + \nabla_\mu\phi\nabla_\nu\phi K_X \tag{43}$$

from this equation, and using the form of the Einstein's equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{44}$$

we see that the energy momentum for k-essence is:

$$T_{\mu\nu} = K(X,\phi)g_{\mu\nu} + \nabla_\mu\phi\nabla_\nu\phi K_X \tag{45}$$

Now, if we consider the energy-momentum tensor for an arbitrary metric:

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu + P g_{\mu\nu} \tag{46}$$

and if we consider the the k-esscense field like a perfect fluid, we can see that its pressure (P) and its energy density (ρ) are given by:

$$P = K(X,\phi) \tag{47}$$

$$\rho = K_X - K \tag{48}$$

and therefore the corresponding equation of state parameter:

$$\omega = \frac{K}{K_X - K} \tag{49}$$

where we have consider that the velocities have to be normalized:

$$U_\mu = \frac{\nabla_\mu\phi}{|\nabla_\mu\phi|} \tag{50}$$

Braiding

Now for this field we also calculate the field equation. Let's consider now the action for braiding:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[G_3(X,\phi)\Box\phi + \frac{M_p^2}{2}R\right] \tag{51}$$

varying this action action with respect $g^{\mu\nu}$, and taking account that $X = X(g^{\mu\nu})$ and $\Box = \Box(g^{\mu\nu})$:

$$\begin{aligned}\delta S &= \int \mathrm{d}^4x\left(\delta\sqrt{-g}\left[G_3(X,\phi)\Box\phi + \frac{M_p^2}{2}R\right] + \sqrt{-g}\,[\delta G_3(X,\phi)\Box\phi + G_3(X,\phi)\delta\Box\phi\right.\\ &\quad\left. + \frac{M_p^2}{2}\delta R\right]\end{aligned}$$

$$\tag{53}$$

$$\delta S = \int \mathrm{d}^4x\left(\delta\sqrt{-g}\left[G_3(X,\phi)\Box\phi + \frac{M_p^2}{2}R\right] + \sqrt{-g}\left[\delta G_3(X,\phi)\Box\phi + G_3(X,\phi)\delta\Box\phi + \frac{M_p^2}{2}\delta R\right]\right) \tag{54}$$

$$= \int \mathrm{d}^4x\left(\delta\sqrt{-g}\left[G_3(X,\phi)\Box\phi + \frac{M_p^2}{2}R\right] + \sqrt{-g}\,[\delta G_3(X,\phi)\Box\phi + G_3(X,\phi)\delta\Box\phi]\right) + \int \mathrm{d}^4x\sqrt{-g}\frac{M_p^2}{2}\delta R \tag{55}$$

the last term due to the equation (38) is simply:

$$\int \mathrm{d}^4x\frac{M_p^2}{2}\sqrt{-g}\ \delta g^{\mu\nu}R_{\mu\nu} \tag{56}$$

now for the chain rule, we have:

$$\delta G_3(X,\phi) = \frac{\delta G_3(X,\phi)}{\delta X}\delta X \tag{57}$$

$$= G_{3,X}\left(-\frac{1}{2}\delta g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right) \tag{58}$$

putting this in the equation (55) and using (21)

$$\begin{aligned}\delta S &= \int \mathrm{d}^4x\left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\left[G_3(X,\phi)\Box\phi + \frac{M_p^2}{2}R\right] + \sqrt{-g}\left[\left(-\frac{1}{2}\delta g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right)G_{3,X}\Box\phi + G_3(X,\phi)\delta\Box\phi\right]\right)\\ &\quad + \int \mathrm{d}^4x\sqrt{-g}\frac{M_p^2}{2}\delta g^{\mu\nu}R_{\mu\nu}\\ &= \int \mathrm{d}^4x\sqrt{-g}\delta g^{\mu\nu}\left[\left(-\frac{1}{2}g_{\mu\nu}\right)\left(G_3(X,\phi)\Box\phi + \frac{M_p^2}{2}R\right) + G_{3,X}\left(-\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi\right)\Box\phi\right.\\ &\quad\left. + \nabla_\mu\nabla_\nu\phi G_3(X,\phi) + \frac{M_p^2}{2}R_{\mu\nu}\right]\end{aligned} \tag{59}$$

$$\begin{aligned}\delta S &= \int \mathrm{d}^4x\left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\left[G_3(X,\phi)\Box\phi + \frac{M_p^2}{2}R\right] + \sqrt{-g}\left[\left(-\frac{1}{2}\delta g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right)G_{3,X}\Box\phi + G_3(X,\phi)\delta\Box\phi\right]\right)\\ &\quad + \int \mathrm{d}^4x\sqrt{-g}\frac{M_p^2}{2}\delta g^{\mu\nu}R_{\mu\nu}\end{aligned}$$

$$\begin{aligned}\delta S &= \int \mathrm{d}^4x\left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\left[G_3(X,\phi)\Box\phi + \frac{M_p^2}{2}R\right] + \sqrt{-g}\left[\left(-\frac{1}{2}\delta g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right)G_{3,X}\Box\phi\right]\right) + \int \mathrm{d}^4x\sqrt{-g}G_3\delta\Box\phi\\ &\quad + \int \mathrm{d}^4x\sqrt{-g}\frac{M_p^2}{2}\delta g^{\mu\nu}R_{\mu\nu} = (\delta S_1) + (\delta S_2) + (\delta S_3)\end{aligned}$$

Let's calculate the variation δS_1. Considering the divergence of a vector field:

$$V^{\alpha}{}_{;\alpha} = \frac{1}{\sqrt{-g}}(\sqrt{-g}V^{\alpha})_{,\alpha} \tag{60}$$

$$\delta\Box\phi = \frac{1}{\sqrt{-g}}\left(\delta(\sqrt{-g}\nabla^{\alpha}\phi)\right)_{,\alpha} - \frac{1}{(\sqrt{-g})^2}\delta\sqrt{-g}(\sqrt{-g}\nabla^{\alpha}\phi)_{,\alpha} \tag{61}$$

$$= \frac{1}{\sqrt{-g}}\left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\nabla^{\alpha}\phi + \sqrt{-g}\partial_{\beta}\phi\delta g^{\alpha\beta}\right)_{,\alpha} - \frac{1}{(\sqrt{-g})^2}\left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\right)(\sqrt{-g}\nabla^{\alpha}\phi)_{,\alpha} \tag{62}$$

$$= \frac{1}{\sqrt{-g}}\xi_{,\alpha} + \frac{1}{2}\Box\phi g_{\mu\nu}\delta g^{\mu\nu} \tag{63}$$

therefore δS_2 is:

$$\int \mathrm{d}^4x\sqrt{-g}G_3\delta\Box\phi = \int \mathrm{d}^4x\sqrt{-g}G_3\left(\frac{1}{\sqrt{-g}}\xi_{,\alpha} + \frac{1}{2}\Box\phi g_{\mu\nu}\delta g^{\mu\nu}\right) = \int \mathrm{d}^4x G_3\xi_{,\alpha} + \int \mathrm{d}^4x\frac{\sqrt{-g}}{2}G_3\Box\phi g_{\mu\nu}\delta g^{\mu\nu} \tag{64}$$

$$= \delta S_a + \delta S_b \tag{65}$$

integrating by parts for δS_a

$$\int \mathrm{d}^4x G_3\xi_{,\alpha} = G_3\int \mathrm{d}^4x\xi_{,\alpha} - \int \mathrm{d}^4x\xi G_{3,\alpha} \tag{66}$$

The first term of the right hand side is zero due to the contribution at infinity. So, δS_2 is:

$$\delta S_2 = -\int \mathrm{d}^4x\xi G_{3,\alpha} + \int \mathrm{d}^4x\frac{\sqrt{-g}}{2}G_3\Box\phi g_{\mu\nu}\delta g^{\mu\nu} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{G_3}{2}\Box\phi g_{\mu\nu}\delta g^{\mu\nu} - \frac{\xi}{\sqrt{-g}}G_{3,\alpha}\right] \tag{67}$$

$$= \int \mathrm{d}^4x\sqrt{-g}\left[\frac{G_3}{2}\Box\phi g_{\mu\nu}\delta g^{\mu\nu} - \frac{\xi}{\sqrt{-g}}G_{3,\alpha}\right] \tag{68}$$

$$= \int \mathrm{d}^4x\sqrt{-g}\left[\frac{G_3}{2}\Box\phi g_{\mu\nu}\delta g^{\mu\nu} - \frac{1}{\sqrt{-g}}\left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\nabla^{\alpha}\phi + \sqrt{-g}\partial_{\beta}\phi\delta g^{\alpha\beta}\right)G_{3,\alpha}\right] \tag{69}$$

$$= \int \mathrm{d}^4x\sqrt{-g}\left[\frac{G_3}{2}\Box\phi g_{\mu\nu}\delta g^{\mu\nu} + \frac{G_{3,\alpha}}{2}g_{\mu\nu}\delta g^{\mu\nu}\nabla^{\alpha}\phi - G_{3,\alpha}\partial_{\beta}\phi\delta g^{\alpha\beta}\right] \tag{70}$$

$$= \int \mathrm{d}^4x\sqrt{-g}\left[\frac{G_3}{2}\Box\phi g_{\mu\nu}\delta g^{\mu\nu} + \frac{G_{3,\alpha}}{2}g_{\mu\nu}\delta g^{\mu\nu}\nabla^{\alpha}\phi - G_{3,\mu}\partial_{\nu}\phi\delta g^{\mu\nu}\right] \tag{71}$$

$$= \int \mathrm{d}^4x\sqrt{-g}\delta g^{\mu\nu}\left[\frac{G_3}{2}\Box\phi g_{\mu\nu} + \frac{G_{3,\alpha}}{2}g_{\mu\nu}\nabla^{\alpha}\phi - G_{3,\mu}\partial_{\nu}\phi\right] \tag{72}$$

putting this in equation 60:

$$\delta S = \int \mathrm{d}^4x\sqrt{-g}\delta g^{\mu\nu}\left(-\frac{1}{2}g_{\mu\nu}\left[G_3\Box\phi + \frac{M_p^2}{2}R\right] + \left(-\frac{1}{2}\partial_{\mu}\phi\partial_{\nu}\phi\right)\Box\phi G_{3,X} + \left[\frac{G_3}{2}\Box\phi g_{\mu\nu} + \frac{G_{3,\alpha}}{2}g_{\mu\nu}\nabla^{\alpha}\phi - G_{3,\mu}\partial_{\nu}\phi\right] + \frac{M}{2} \tag{73}$$

we observe from this, that all terms in parenthesis are symmetric except the term that involves $\nabla_{\mu}G_3\nabla_{\nu}\phi$, and due to the fact that $\delta g^{\mu\nu}$ is symmetric, the contraction of both are zero, therefore we consider the symmetric part of this tensor denoted by:

$$\nabla_{(\mu}G_3\nabla_{\nu)}\phi \tag{74}$$

applying the variational principle, we get

$$-\frac{1}{2}g_{\mu\nu}\left[G_3\Box\phi + \frac{M_p^2}{2}R\right] + \left(-\frac{1}{2}\partial_{\mu}\phi\partial_{\nu}\phi\right)\Box\phi G_{3,X} + \left[\frac{G_3}{2}\Box\phi g_{\mu\nu} + \frac{G_{3,\alpha}}{2}g_{\mu\nu}\nabla^{\alpha}\phi - \nabla_{(\mu}G_3\nabla_{\nu)}\phi\right] + \frac{M_p^2}{2}R = 0 \tag{75}$$

rearranging the terms

$$\frac{M_p^2}{2}\left[R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right]-\frac{1}{2}G_{3,X}\partial_\mu\phi\partial_\nu\phi\Box\phi+\frac{1}{2}G_{3,\alpha}g_{\mu\nu}\nabla^\alpha\phi-\nabla_{(\mu}G_3\nabla_{\nu)}\phi=0 \tag{76}$$

in terms of the Einstein's tensor:

$$G_{\mu\nu}=8\pi G[G_{3,X}\Box\phi\nabla_\mu\phi\nabla_\nu\phi+2\nabla_{(\mu}G_3\nabla_{\nu)}\phi-g_{\mu\nu}\nabla^\alpha G_3\nabla_\alpha\phi] \tag{77}$$

therefore, the energy momentum tensor for Braiding field is:

$$T_{\mu\nu}=G_{3,X}\Box\phi\nabla_\mu\phi\nabla_\nu\phi+2\nabla_{(\mu}G_3\nabla_{\nu)}\phi-g_{\mu\nu}\nabla^\alpha G_3\nabla_\alpha\phi \tag{78}$$

Equation of motion for K-essence

The Lagrangian for this field is:

$$\mathcal{L}=K(X,\phi) \tag{79}$$

We calculate the equation of motion given by the Euler-Lagrange equation:

$$\frac{\partial\mathcal{L}}{\partial\phi}-\nabla_\mu\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}=0 \tag{80}$$

by the chain rule, the second term of equation(80):

$$\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}=\frac{\partial K}{\partial X}\frac{\partial X}{\partial\phi_{,\mu}}=K_X\frac{\partial X}{\partial\phi_{,\mu}} \tag{81}$$

on one hand:

$$\frac{\partial X}{\partial\phi_{,\mu}}=\frac{\partial}{\partial\phi_{,\mu}}\left[-\frac{1}{2}g^{\sigma\nu}\phi_{,\sigma}\phi_{,\nu}\right]=-\frac{1}{2}g^{\sigma\nu}\left[\frac{\partial\phi_{,\sigma}}{\partial\phi_{,\mu}}\phi_{,\nu}+\frac{\partial\phi_{,\nu}}{\partial\phi_{,\mu}}\phi_{,\sigma}\right] \tag{82}$$

$$=-\frac{1}{2}g^{\sigma\nu}\left[\delta^\mu_\sigma\phi_{,\nu}+\delta^\mu_\nu\phi_{,\sigma}\right]=-\frac{1}{2}g^{\sigma\nu}\delta^\mu_\sigma\phi_{,\nu}-\frac{1}{2}g^{\sigma\nu}\delta^\mu_\nu\phi_{,\sigma} \tag{83}$$

$$=-\frac{1}{2}g^{\mu\nu}\phi_{,\nu}-\frac{1}{2}g^{\sigma\mu}\phi_{,\sigma}=-\frac{1}{2}g^{\mu\nu}\phi_{,\nu}-\frac{1}{2}g^{\mu\nu}\phi_{,\nu}=-g^{\mu\nu}\phi_{,\nu} \tag{84}$$

therefore:

$$\frac{\partial X}{\partial\phi_{,\mu}}=-g^{\mu\nu}\phi_{,\nu} \tag{85}$$

so, according to Euler-Lagrange equation:

$$\nabla_\mu\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}=\nabla_\mu\left[K_X\frac{\partial X}{\partial\phi_{,\mu}}\right]=\nabla_\mu K_X\frac{\partial X}{\partial\phi_{,\mu}}+K_X\nabla_\mu\frac{\partial X}{\partial\phi_{,\mu}} \tag{86}$$

since K depends on X and ϕ:

$$\nabla_\mu K_X=\frac{\partial K_X}{\partial X}\nabla_\mu X+\frac{\partial K_X}{\partial\phi}\nabla_\mu\phi \tag{87}$$

for this, let's see that:

$$\nabla_\mu X=\nabla_\mu\left[-\frac{1}{2}g^{\lambda\nu}\phi_{,\lambda}\phi_{,\nu}\right]=-\frac{1}{2}g^{\lambda\nu}\nabla_\mu\phi_{,\lambda}\phi_{,\nu}-\frac{1}{2}g^{\lambda\nu}\phi_{,\lambda}\nabla_\mu\phi_{,\nu}=-\frac{1}{2}g^{\lambda\nu}\phi_{;\lambda\mu}\phi_{,\nu}-\frac{1}{2}g^{\lambda\nu}\phi_{,\lambda}\phi_{;\nu\mu} \tag{88}$$

making the change $\nu\longleftrightarrow\lambda$ in the last term:

$$\nabla_\mu X=-\frac{1}{2}g^{\lambda\nu}\phi_{;\lambda\mu}\phi_{,\nu}-\frac{1}{2}g^{\lambda\nu}\phi_{,\nu}\phi_{;\lambda\mu}=-g^{\lambda\nu}\phi_{;\lambda\mu}\phi_{,\nu} \tag{89}$$

substituting this in the equation (90):

$$\nabla_\mu K_X=K_{XX}[-g^{\lambda\nu}\phi_{;\lambda\mu}\phi_{,\nu}]+K_{X\phi}\phi_{,\mu}=-g^{\lambda\nu}K_{XX}\phi_{;\lambda\mu}\phi_{,\nu}+K_{X\phi}\phi_{,\mu} \tag{90}$$

substituting this and equation (85) in equation (86):

$$\nabla_\mu \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} = \left[-g^{\lambda\nu} K_{XX} \phi_{;\lambda\mu} \phi_{,\nu} + K_{X\phi} \phi_{,\mu}\right] \left[-g^{\mu\sigma} \phi_{,\sigma}\right] + K_X \nabla_\mu [-g^{\mu\nu} \phi_{,\nu}] \tag{91}$$

$$= - \left[-g^{\lambda\nu} K_{XX} \phi_{;\lambda\mu} \phi_{,\nu} + K_{X\phi} \phi_{,\mu}\right] g^{\mu\sigma} \phi_{,\sigma} - K_X g^{\mu\nu} \phi_{;\nu\mu} \tag{92}$$

substituting this in equation (80), we have:

$$K_\phi + \left[-g^{\lambda\nu} K_{XX} \phi_{;\lambda\mu} \phi_{,\nu} + K_{X\phi} \phi_{,\mu}\right] g^{\mu\sigma} \phi_{,\sigma} + K_X g^{\mu\nu} \phi_{;\nu\mu} = 0 \tag{93}$$

in other form:

$$K_{XX} \phi_{;\lambda\mu} \phi_{,\nu} \phi_{,\sigma} g^{\lambda\nu} g^{\mu\sigma} - K_{X\phi} \phi_{,\sigma} \phi_{,\mu} g^{\mu\sigma} - K_X \phi_{;\nu\mu} g^{\mu\nu} - K_\phi = 0 \tag{94}$$

we consider a spatially-flat Friedman-Robertson-Walker (FRW) metric

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2 \tag{95}$$

where the Christoffel symbols what we need are given by:

$$\Gamma^0{}_{00} = 0; \Gamma^0{}_{11} = a\dot{a}; \Gamma^0{}_{22} = a\dot{a}r^2; \Gamma^0{}_{33} = a\dot{a}r^2 \sin^2\theta \tag{96}$$

First, according to the FRW metric, we calculate the covariant derivatives of equation 94

$$K_{XX}[\partial_\mu \phi_\lambda - \Gamma^\rho{}_{\mu\lambda} \phi_\rho] \phi_{,\nu} \phi_{,\sigma} g^{\lambda\nu} g^{\mu\sigma} - K_{X\phi} \phi_{,\sigma} \phi_{,\mu} g^{\mu\sigma} - K_X [\partial_\mu \phi_{,\nu} - \Gamma^\rho{}_{\mu\nu} \phi_\rho] g^{\mu\nu} - K_\phi = 0 \tag{97}$$

$$K_{XX} \partial_\mu \phi_\lambda \phi_{,\nu} \phi_{,\sigma} g^{\lambda\nu} g^{\mu\sigma} - K_{XX} \Gamma^\rho{}_{\mu\lambda} \phi_{,\rho} \phi_{,\nu} \phi_{,\sigma} g^{\lambda\nu} g^{\mu\sigma} - K_{X\phi} \phi_{,\sigma} \phi_{,\mu} g^{\mu\sigma} - K_X \partial_\mu \phi_{,\nu} g^{\mu\nu} + K_X \Gamma^\rho{}_{\mu\nu} \phi_\rho g^{\mu\nu} - K_\phi = 0 \tag{98}$$

considering the cosmological case ($\phi = \phi(t)$), all derivatives are respect to time. In the firs term $\mu = \lambda = \nu = \sigma = 0$. In the second term $\rho = \nu = \sigma = 0$ and therefore $\lambda = \mu = 0$. In the third term $\mu = \sigma = 0$. In the fourth term $\mu\nu = 0$. In the five term $\rho = 0$

$$K_{XX} \ddot{\phi} \dot{\phi} \dot{\phi} g^{00} g^{00} - K_{XX} \Gamma^0{}_{00} \dot{\phi} \dot{\phi} \dot{\phi} g^{00} g^{00} - K_{X\phi} \dot{\phi}^2 g^{00} - K_X \ddot{\phi} g^{00} + K_X \Gamma^0{}_{\mu\nu} \dot{\phi} g^{\mu\nu} - K_\phi = 0 \tag{99}$$

simplifying the terms

$$K_{XX} \ddot{\phi} \dot{\phi}^2 + K_{X\phi} \dot{\phi}^2 + K_X \ddot{\phi} + K_X [\Gamma^0{}_{11} g^{11} + \Gamma^0{}_{22} g^{22} + \Gamma^0{}_{33} g^{33}] \dot{\phi} - K_\phi = 0 \tag{100}$$

$$K_{XX} \ddot{\phi} \dot{\phi}^2 + K_{X\phi} \dot{\phi}^2 + K_X \ddot{\phi} + K_X [\frac{1}{a^2} a\dot{a} + \frac{1}{a^2 r^2} a\dot{a} r^2 + \frac{1}{a^2 r^2 \sin^2\theta} a\dot{a} r^2 \sin^2\theta] \dot{\phi} - K_\phi = 0 \tag{101}$$

therefore, the equation of motion according to FRW metric for k-essence model is:

$$K_{XX} \ddot{\phi} \dot{\phi}^2 + K_{X\phi} \dot{\phi}^2 + K_X \ddot{\phi} + 3HK_X \dot{\phi} - K_\phi = 0 \tag{102}$$

A particular case: Purely kinetic k essence

Let's consider the field equation for the k-essence model (equation 102), and take the particular case for the Lagrangian:

$$K(X, \phi) = F(X)V(\phi) \tag{103}$$

dividing by $K(X, \phi) = K(X)V(\phi)$

$$F_{XX} \ddot{\phi} \dot{\phi}^2 + F_X \frac{V_\phi}{V} \dot{\phi}^2 + F_X \ddot{\phi} + 3HF_X \dot{\phi} - F \frac{V_\phi}{V} = 0 \tag{104}$$

according to the cosmological case, the scalar field's kinetic density X is:

$$X = \frac{1}{2} \dot{\phi}^2 \longrightarrow \dot{\phi}^2 = 2X \tag{105}$$

so

$$2XF_{XX} \ddot{\phi} + 2XF_X \frac{V_\phi}{V} + F_X \ddot{\phi} + 3HF_X \dot{\phi} - F \frac{V_\phi}{V} = 0 \tag{106}$$

rearranging terms, we have the following equation of motion:

$$(F_X + 2XF_{XX}) \ddot{\phi} + 3HF_X \dot{\phi} + (2XF_X - F) \frac{V_\phi}{V} = 0 \tag{107}$$

Equation of motion for Braiding

Let's consider now the Lagrangian for this field given by:

$$\mathcal{L} = G_3(X,\phi)\Box\phi \tag{108}$$

where:

$$\Box\phi = g^{\mu\nu}\phi_{;\nu\mu} \tag{109}$$

and

$$X = -\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} \tag{110}$$

The Euler Lagrange equations for this case read:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \nabla_\mu\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}} + \nabla_\mu\nabla_\nu\frac{\partial\mathcal{L}}{\partial\phi_{;\mu\nu}} = 0 \tag{111}$$

Let's calculate the field equation for this field. First, let's calculate the second term of 111:

$$\nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\right) = \nabla_\mu\left(\frac{\partial G_3}{\partial\phi_{,\mu}}\Box\phi\right) = \nabla_\mu\left(\frac{\partial G_3}{\partial X}\frac{\partial X}{\partial\phi_{,\mu}}\Box\phi\right) \tag{112}$$

$$= \nabla_\mu G_{3,X}\frac{\partial X}{\partial\phi_{,\mu}}\Box\phi + G_{3,X}\nabla_\mu\frac{\partial X}{\partial\phi_{,\mu}}\Box\phi + G_{3,X}\frac{\partial X}{\partial\phi_{,\mu}}\nabla_\mu\Box\phi \tag{113}$$

using the equations 85 and 90

$$\nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\right) = \left(-G_{3,XX}g^{\lambda\sigma}\phi_{;\lambda\mu}\phi_{,\sigma} + G_{3,X\phi}\phi_{,\mu}\right)(-g^{\mu\nu}\phi_{,\nu})\Box\phi + G_{3,X}\nabla_\mu(-g^{\mu\nu}\phi_{,\nu})\Box\phi + G_{3,X}(-g^{\mu\nu}\phi_{,\nu})\nabla_\mu\Box\phi \tag{114}$$

$$= -\left(-G_{3,XX}g^{\lambda\sigma}\phi_{;\lambda\mu}\phi_{,\sigma} + G_{3,X\phi}\phi_{,\mu}\right)g^{\mu\nu}\phi_{,\nu}\Box\phi - G_{3,X}g^{\mu\nu}\phi_{;\nu\mu}\Box\phi - G_{3,X}g^{\mu\nu}\phi_{,\nu}\nabla_\mu\Box\phi \tag{115}$$

$$= -\left(-G_{3,XX}g^{\lambda\sigma}\phi_{;\lambda\mu}\phi_{,\sigma} + G_{3,X\phi}\phi_{,\mu}\right)g^{\mu\nu}\phi_{,\nu}\Box\phi - G_{3,X}(\Box\phi)^2 - G_{3,X}g^{\mu\nu}\phi_{,\nu}\nabla_\mu\Box\phi \tag{116}$$

let's calculate the third term of 111:

$$\nabla_\mu\nabla_\nu\left(\frac{\partial\mathcal{L}}{\partial\phi_{;\mu\nu}}\right) = \nabla_\mu\nabla_\nu G_3\frac{\partial}{\partial\phi_{;\mu\nu}}(g^{\mu\nu}\phi_{;\mu\nu}) = g^{\mu\nu}\nabla_\mu\nabla_\nu G_3 = \Box G_3 \tag{117}$$

therefore the equation of motion given by the Euler Lagrange equation is:

$$G_{3,\phi}\Box\phi + \left(-G_{3,XX}g^{\lambda\sigma}\phi_{;\lambda\mu}\phi_{,\sigma} + G_{3,X\phi}\phi_{,\mu}\right)g^{\mu\nu}\phi_{,\nu}\Box\phi + G_{3,X}(\Box\phi)^2 + G_{3,X}g^{\mu\nu}\phi_{,\nu}\nabla_\mu\Box\phi + \Box G_3 = 0 \tag{118}$$

Let's calculate the terms $\nabla_\mu\Box\phi$ and $\Box G_3$. Using the formula for the divergence for a vector field V^α:

$$V^\alpha{}_{;\alpha} = \frac{1}{\sqrt{-g}}(\sqrt{-g}V^\alpha)_{,\alpha} \tag{119}$$

$$\nabla_\mu\Box\phi = \nabla_\mu(g^{\lambda\sigma}\nabla_\lambda\nabla_\sigma\phi) = \nabla_\mu(\nabla_\sigma\nabla^\sigma\phi) = \nabla_\mu\left[\frac{1}{\sqrt{-g}}(\sqrt{-g}g^{\lambda\sigma}\phi_{,\sigma})_{,\lambda}\right] = \partial_\mu\left[\frac{1}{\sqrt{-g}}(\sqrt{-g}g^{\lambda\sigma}\phi_{,\sigma})_{,\lambda}\right] \tag{120}$$

$$\Box G_3 = \nabla_\mu(\nabla^\mu G_3) = \frac{1}{\sqrt{-g}}(\sqrt{-g}\nabla^\mu G_3)_{,\mu} = \frac{1}{\sqrt{-g}}[\sqrt{-g}g^{\alpha\mu}\nabla_\alpha G_3]_{,\mu} = \frac{1}{\sqrt{-g}}\left[\sqrt{-g}g^{\alpha\mu}(-G_{3,X}g^{\lambda\sigma}\phi_{;\lambda\alpha}\phi_{,\sigma} + G_{3,\phi}\phi_{,\alpha})\right]_{,\mu} \tag{121}$$

therefore, the equation of motion for the metric $g_{\alpha\beta}$ is:

$$G_{3,\phi}\Box\phi + \left(-G_{3,XX}g^{\lambda\sigma}\phi_{;\lambda\mu}\phi_{,\sigma} + G_{3,X\phi}\phi_{,\mu}\right)g^{\mu\nu}\phi_{,\nu}\Box\phi + G_{3,X}(\Box\phi)^2 + G_{3,X}g^{\mu\nu}\phi_{,\nu}\partial_\mu\left[\frac{1}{\sqrt{-g}}\left(\sqrt{-g}g^{\lambda\sigma}\phi_{,\sigma}\right)_{,\lambda}\right] + \frac{1}{\sqrt{-g}}\left[\sqrt{-g}g^{\alpha} \right. \tag{122}$$

Now, if we consider the spatially-flat FRW metric (95). Let's calculate term by term of equation 122. The first term:

$$\Box\phi = \nabla_\mu\nabla^\mu\phi = \frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}\nabla^\mu\phi] = \frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}g^{\mu\nu}\nabla_\nu\phi] = \frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}g^{\mu0}\dot{\phi}] \tag{123}$$

$$= -\frac{1}{\sqrt{-g}}\partial_t[\sqrt{-g}\dot{\phi}] = -\frac{1}{a^3r^2\sin\theta}\partial_t(a^3r^2\sin\theta\dot{\phi}) = -\frac{1}{a^3}\partial_t(a^3\dot{\phi}) = -\frac{1}{a^3}(3a^2\dot{a} + a^3\ddot{\phi}) \tag{124}$$

$$\Rightarrow \Box\phi = -3H\dot{\phi} - \ddot{\phi} \tag{125}$$

therefore, the first term is:

$$G_{3,\phi}\Box\phi = -3G_{3,\phi}H\dot{\phi} - G_{3,\phi}\ddot{\phi} \tag{126}$$

The second term:

$$\left(-G_{3,XX}g^{\lambda\sigma}\phi_{;\lambda\mu}\phi_{,\sigma} + G_{3,X\phi}\phi_{,\mu}\right)g^{\mu\nu}\phi_{,\nu}\Box\phi = \left(-G_{3,XX}g^{\lambda\sigma}\phi_{;\lambda\mu}\phi_{,\sigma}g^{\mu\nu}\phi_{,\nu} + G_{3,X\phi}\phi_{,\mu}g^{\mu\nu}\phi_{,\nu}\right)\Box\phi \tag{127}$$

$$= \left(-G_{3,XX}g^{\lambda\sigma}\phi_{;\lambda\mu}\phi_{,\sigma}g^{\mu\nu}\phi_{,\nu} + G_{3,X\phi}\phi_{,\mu}g^{\mu\nu}\phi_{,\nu}\right)\Box\phi \tag{128}$$

$$= \left(-G_{3,XX}g^{\lambda\sigma}[\partial_\mu\partial_\lambda\phi - \Gamma^\rho{}_{\lambda\mu}\phi_{,\rho}]\phi_{,\sigma}g^{\mu\nu}\phi_{,\nu} + G_{3,X\phi}\phi_{,\mu}g^{\mu\nu}\phi_{,\nu}\right)\Box\phi \tag{129}$$

$$= \left(-G_{3,XX}g^{\lambda\sigma}\partial_\mu\partial_\lambda\phi\phi_{,\sigma}g^{\mu\nu}\phi_{,\nu} + G_{3,XX}g^{\lambda\sigma}\Gamma^\rho{}_{\lambda\mu}\phi_{,\rho}\phi_{,\sigma}g^{\mu\nu}\phi_{,\nu} + G_{3,X\phi}\phi_{,\mu}g^{\mu\nu}\phi_{,\nu}\right)\Box\phi \tag{130}$$

In the first term $\nu = \mu = \sigma = \lambda = 0$. In the second term $\rho = \nu = \mu = \sigma = \lambda = 0$. In the third term $\mu = \nu = 0$

$$= \left(-G_{3,XX}g^{00}\ddot{\phi}\dot{\phi}g^{00}\dot{\phi} + G_{3,XX}g^{00}\Gamma^0{}_{00}\dot{\phi}\dot{\phi}g^{00}\dot{\phi} + G_{3,X\phi}\dot{\phi}g^{00}\dot{\phi}\right)\Box\phi \tag{131}$$

$$= \left(-G_{3,XX}\ddot{\phi}\dot{\phi}^2 - G_{3,X\phi}\dot{\phi}^2\right)\Box\phi = \left(-G_{3,XX}\ddot{\phi}\dot{\phi}^2 - G_{3,X\phi}\dot{\phi}^2\right)(-3H\dot{\phi} - \ddot{\phi}) \tag{132}$$

$$= 3HG_{3,XX}\ddot{\phi}\dot{\phi}^3 + G_{3,XX}\ddot{\phi}^2\dot{\phi}^2 + 3HG_{3,X\phi}\dot{\phi}^3 + G_{3,X\phi}\dot{\phi}^2\ddot{\phi} \tag{133}$$

where $\Gamma^0{}_{00} = 0$. The third term:

$$G_{3,X}\Box\phi^2 = G_{3,X}(-3H\dot{\phi} - \ddot{\phi})^2 = G_{3,X}(9H^2\dot{\phi}^2 + 6H\dot{\phi}\ddot{\phi} + \ddot{\phi}^2) \tag{134}$$

The fourth term:

$$G_{3,X}g^{\mu\nu}\phi_{,\nu}\nabla_\mu\Box\phi = G_{3,X}g^{\mu\nu}\phi_{,\nu}\partial_\mu\left[\frac{1}{\sqrt{-g}}(\sqrt{-g}g^{\lambda\sigma}\phi_{,\sigma})_{,\lambda}\right] = G_{3,X}\dot{\phi}\partial_t\left[\frac{1}{a^3r^2\sin\theta}\partial_t(a^3r^2\sin\theta\dot{\phi})\right] \tag{135}$$

$$= G_{3,X}\dot{\phi}\partial_t\left[\frac{1}{a^3}(3a^2\dot{a}\dot{\phi} + a^3\ddot{\phi})\right] = G_{3,X}\dot{\phi}\partial_t(3H\dot{\phi} + \ddot{\phi}) = G_{3,X}\dot{\phi}(3\dot{H}\dot{\phi} + 3H\ddot{\phi} + \partial_t\ddot{\phi}) \tag{136}$$

The fifth term:

$$\Box G_3 = \frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\alpha\mu}(-G_{3,X}g^{\lambda\sigma}\phi_{;\lambda\alpha}\phi_{,\sigma} + G_{3,\phi}\phi_{,\alpha})\right] = \frac{1}{\sqrt{-g}}\partial_\mu\left[-\sqrt{-g}g^{\alpha\mu}G_{3,X}g^{\lambda\sigma}\phi_{;\lambda\alpha}\phi_{,\sigma} + \sqrt{-g}g^{\alpha\mu}G_{3,\phi}\phi_{,\alpha}\right] \tag{137}$$

$$= \frac{1}{\sqrt{-g}}\partial_\mu\left[-\sqrt{-g}g^{\alpha\mu}G_{3,X}g^{\lambda\sigma}(\partial_\alpha\partial_\lambda\phi - \Gamma^\rho{}_{\lambda\alpha}\partial_\rho\phi)\phi_{,\sigma} + \sqrt{-g}g^{\alpha\mu}G_{3,\phi}\phi_{,\alpha}\right] \tag{138}$$

$$= \frac{1}{\sqrt{-g}}\partial_\mu\left[-\sqrt{-g}g^{\alpha\mu}G_{3,X}g^{\lambda\sigma}\partial_\alpha\partial_\lambda\phi\phi_{,\sigma} + \sqrt{-g}g^{\alpha\mu}G_{3,X}g^{\lambda\sigma}\Gamma^\rho{}_{\lambda\alpha}\partial_\rho\phi\phi_{,\sigma} + \sqrt{-g}g^{\alpha\mu}G_{3,\phi}\phi_{,\alpha}\right] \tag{139}$$

In the first term $\sigma = \lambda = \alpha = 0$. I the second term $\sigma = \rho = \lambda = 0$. In the third term $\alpha = 0$

$$= \frac{1}{\sqrt{-g}}\partial_\mu\left[-\sqrt{-g}g^{0\mu}G_{3,X}g^{00}\ddot{\phi}\dot{\phi} + \sqrt{-g}g^{\alpha\mu}G_{3,X}g^{00}\Gamma^0{}_{0\alpha}\dot{\phi}\dot{\phi} + \sqrt{-g}g^{0\mu}G_{3,\phi}\dot{\phi}\right] \tag{140}$$

$$= \frac{1}{\sqrt{-g}}\partial_t\left[-\sqrt{-g}G_{3,X}\ddot{\phi}\dot{\phi} - \sqrt{-g}G_{3,\phi}\dot{\phi}\right] = -\frac{1}{\sqrt{-g}}\partial_t\left[\sqrt{-g}(G_{3,X}\ddot{\phi}\dot{\phi} + G_{3,\phi}\dot{\phi})\right] \tag{141}$$

where $\Gamma^0{}_{0\alpha} = 0\forall\alpha$

$$\Box\phi = -\frac{1}{a^3r^2\sin\theta}\partial_t\left[a^3r^2\sin\theta(G_{3,X}\ddot{\phi}\dot{\phi} + G_{3,\phi}\dot{\phi})\right] = -\frac{1}{a^3}\partial_t\left[a^3(G_{3,X}\ddot{\phi}\dot{\phi} + G_{3,\phi}\dot{\phi})\right] \tag{142}$$

$$= -\frac{1}{a^3}\left[3a^2\dot{a}(G_{3,X}\ddot{\phi}\dot{\phi} + G_{3,\phi}\dot{\phi}) + a^3(\partial_tG_{3,X}\ddot{\phi}\dot{\phi} + G_{3,X}\partial_t\ddot{\phi}\dot{\phi} + G_{3,X}\ddot{\phi}^2 + \partial_tG_{3,\phi}\dot{\phi} + G_{3,\phi}\ddot{\phi})\right] \tag{143}$$

$$= -\frac{1}{a^3}\left[3a^2\dot{a}(G_{3,X}\ddot{\phi}\dot{\phi} + G_{3,\phi}\dot{\phi}) + a^3([G_{3,XX}\dot{\phi}\ddot{\phi} + G_{3,X\phi}\dot{\phi}]\ddot{\phi}\dot{\phi} + G_{3,X}\partial_t\ddot{\phi}\dot{\phi} + G_{3,X}\ddot{\phi}^2 + [G_{3,\phi X}\dot{\phi}\ddot{\phi} + G_{3,\phi\phi}\dot{\phi}]\dot{\phi} + G_{3,\phi}\ddot{\phi})\right] \tag{144}$$

$$\Box G_3 = -3H(G_{3,X}\ddot{\phi}\dot{\phi}+G_{3,\phi}\dot{\phi})-(G_{3,XX}\dot{\phi}^2\ddot{\phi}^2+G_{3,X\phi}\dot{\phi}^2\ddot{\phi}+G_{3,X}\partial_t\ddot{\phi}\dot{\phi}+G_{3,X}\ddot{\phi}^2+G_{3,\phi X}\dot{\phi}^2\ddot{\phi}+G_{3,\phi\phi}\dot{\phi}^2+G_{3,\phi}\ddot{\phi}) \tag{145}$$

Now, we sum all the calculate terms of the equation of motion for the FRW metric is given by the equation 122:

$$-3G_{3,\phi}H\dot{\phi}-G_{3,\phi}\ddot{\phi}+(3HG_{3,XX}\ddot{\phi}\dot{\phi}^3+G_{3,XX}\ddot{\phi}^2\dot{\phi}^2+3HG_{3,X\phi}\dot{\phi}^3+G_{3,X\phi}\dot{\phi}^2\ddot{\phi})+G_{3,X}(9H^2\dot{\phi}^2+6H\dot{\phi}\ddot{\phi}+\ddot{\phi}^2) \tag{146}$$

$$+G_{3,X}\dot{\phi}(3\dot{H}\dot{\phi}+3H\ddot{\phi}+\partial_t\ddot{\phi})-3H(G_{3,X}\ddot{\phi}\dot{\phi}+G_{3,\phi}\dot{\phi})-(G_{3,XX}\dot{\phi}^2\ddot{\phi}^2+G_{3,X\phi}\dot{\phi}^2\ddot{\phi}+G_{3,X}\partial_t\ddot{\phi}\dot{\phi}+G_{3,X}\ddot{\phi}^2+G_{3,\phi X}\dot{\phi}^2\ddot{\phi}+G_{3,\phi\phi}\dot{\phi}^2+G_{3,} \tag{147}$$

rearranging the terms

$$\ddot{\phi}[-G_{3,\phi}+3HG_{3,XX}\dot{\phi}^3+G_{3,X\phi}\dot{\phi}^2+6H\dot{\phi}G_{3,X}+3H\dot{\phi}G_{3,X}-3HG_{3,X}\dot{\phi}-G_{3,X\phi}\dot{\phi}^2-G_{3,\phi}-G_{3,\phi X}\dot{\phi}^2]\ddot{\phi} \tag{148}$$

$$-3G_{3,\phi}H\dot{\phi}+3HG_{3,X\phi}\dot{\phi}^3+9H^2G_{3,X}\dot{\phi}^2+3\dot{H}\dot{\phi}^2G_{3,X}-3HG_{3,\phi}\dot{\phi}-G_{3,\phi\phi}\dot{\phi}^2=0 \tag{149}$$

therefore the equation of motion for braiding is:

$$\ddot{\phi}[6H\dot{\phi}G_{3,X}+3HG_{3,XX}\dot{\phi}^3-2G_{3,\phi}-G_{3,\phi X}\dot{\phi}^2] \tag{150}$$

$$+9H^2G_{3,X}\dot{\phi}^2-6G_{3,\phi}H\dot{\phi}+3HG_{3,X\phi}\dot{\phi}^3+3\dot{H}\dot{\phi}^2G_{3,X}-G_{3,\phi\phi}\dot{\phi}^2=0 \tag{151}$$

Energy-momentum tensor for the cosmological case

k Essence

Let's recall that the energy-momentum tensor for k-essence model is:

$$T_{\mu\nu} = g_{\mu\nu}K(x,\phi) + \phi_{,\mu}\phi_{,\nu}K_X \tag{152}$$

and the FRW metric is

$$g_{\mu\nu} = diag(-1, a^2, a^2r^2, a^2r^2\sin^2\theta) \tag{153}$$

The explicit form of the energy momentum tensor for k-essence field in the cosmological case is symply:

$$T_{\mu\nu} = \begin{pmatrix} -K(X,\phi)+\dot{\phi}^2K_X & 0 & 0 & 0 \\ 0 & a^2K(X,\phi) & 0 & 0 \\ 0 & 0 & a^2K(X,\phi) & 0 \\ 0 & 0 & 0 & a^2r^2\sin^2\theta K(X,\phi) \end{pmatrix} \tag{154}$$

Braiding

The energy-momentum tensor for Braiding model is:

$$T_{\mu\nu} = G_{3,X}\Box\phi\phi_{,\mu}\phi_{,\nu} + 2\nabla_{(\mu}G_3\nabla_{\nu)}\phi - g_{\mu\nu}\nabla^\alpha G_3\nabla_\alpha\phi \tag{155}$$

We are also going to find the explicit form of the energy-momentum tensor. For this, let's recall that the D'Alembertian operator is:

$$\Box\phi = -3H\dot{\phi} - \ddot{\phi} \tag{156}$$

The symmetric tensor:

$$\nabla_{(\mu}G_3\nabla_{\nu)}\phi = \frac{1}{2}\left(\nabla_\mu G_3\nabla_\nu\phi + \nabla_\mu\phi\nabla_\nu G_3\right) \tag{157}$$

The covariant derivative of G_3 is:

$$\nabla_\mu G_3 = -G_{3,X}g^{\lambda\sigma}\phi_{;\lambda\mu}\phi_{,\sigma} + G_{3,\phi}\phi_{,\mu} \tag{158}$$

Whit, this, we can find that the energy-momentum tensor for the Braiding field is:

$$T_{00} = -G_{3,X}(3H\dot{\phi}+\ddot{\phi}) + (G_{3,X}\ddot{\phi}\dot{\phi} + G_{3,\phi}\dot{\phi})\dot{\phi} \tag{159}$$

$$T_{11} = a^2(G_{3,X}\ddot{\phi}\dot{\phi} + G_{3,\phi}\dot{\phi})\dot{\phi} \tag{160}$$

$$T_{22} = a^2r^2(G_{3,X}\ddot{\phi}\dot{\phi} + G_{3,\phi}\dot{\phi})\dot{\phi} \tag{161}$$

$$T_{33} = a^2r^2\sin^2\theta(G_{3,X}\ddot{\phi}\dot{\phi} + G_{3,\phi}\dot{\phi})\dot{\phi} \tag{162}$$

Pressure and energy density for Braiding

From the explicit form of the energy-momentum tensor for the perfect fluid:

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & a^2 P & 0 & 0 \\ 0 & 0 & a^2 r^2 P & 0 \\ 0 & 0 & 0 & a^2 r^2 \sin^2\theta P \end{pmatrix} \tag{163}$$

we compare with the energy-momentun tensor for Brading to find that the energy density and the pressure for this field are given by:

$$\rho = -G_{3,X}(3H\dot{\phi} + \ddot{\phi}) + (G_{3,X}\ddot{\phi}\dot{\phi} + G_{3,\phi}\dot{\phi})\dot{\phi} \tag{164}$$

$$P = (G_{3,X}\ddot{\phi}\dot{\phi} + G_{3,\phi}\dot{\phi})\dot{\phi} \tag{165}$$

Therefore, the parameter of the equation of state is:

$$\omega = \frac{P}{\rho} = \frac{(G_{3,X}\ddot{\phi}\dot{\phi} + G_{3,\phi}\dot{\phi})\dot{\phi}}{-G_{3,X}(3H\dot{\phi} + \ddot{\phi}) + (G_{3,X}\ddot{\phi}\dot{\phi} + G_{3,\phi}\dot{\phi})\dot{\phi}} \tag{166}$$

$$\omega = \frac{1}{1 - \frac{G_{3,X}(3H\dot{\phi}+\ddot{\phi})}{(G_{3,X}\ddot{\phi}\dot{\phi}+G_{3,\phi}\dot{\phi})\dot{\phi}}} \tag{167}$$

Numerical Calculations

The first thing what we are going to do is solve numerically the case of k essence model.

Purely kinetic k essence

Solving the Friedman equation and the equation of motion for this field we get the following set of graphics:

Analytical solution for X=X(a)

In the plot 1, we present the analytical solution for $X(a)$, this solution is given by Scherer, where he consider X near X_0 (the minimum of the Lagrangian.)

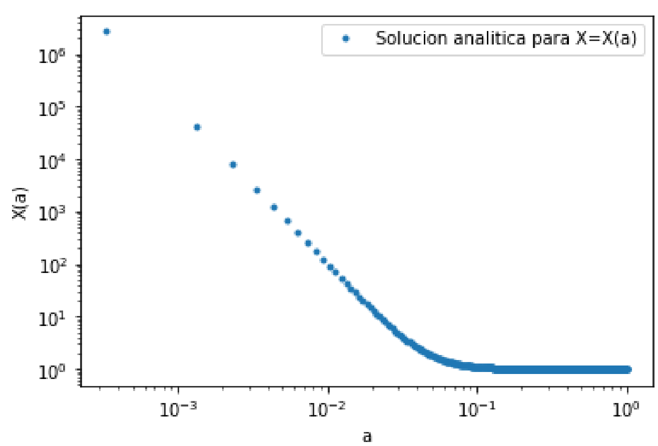

Figure 1: Analytical solution for X=X(a)

Numerical and Analytical solution for $\dot{\phi} = \dot{\phi}(a)$

The analytical solution for $\dot{\phi}$ in terms of the the scale factor is proportional to the analytical solution of $X(a)$, (due to the cosmological case), and the numerical solution is given by the system of coupled differential equations in terms of a and $\dot{\phi}$, which result from the Friedman equation and the equation of motion for the k essence field. In the plot 2 we present the comparison between the analytical solution and the numerical solution for $X(a)$. We see that both graphics have the same form.

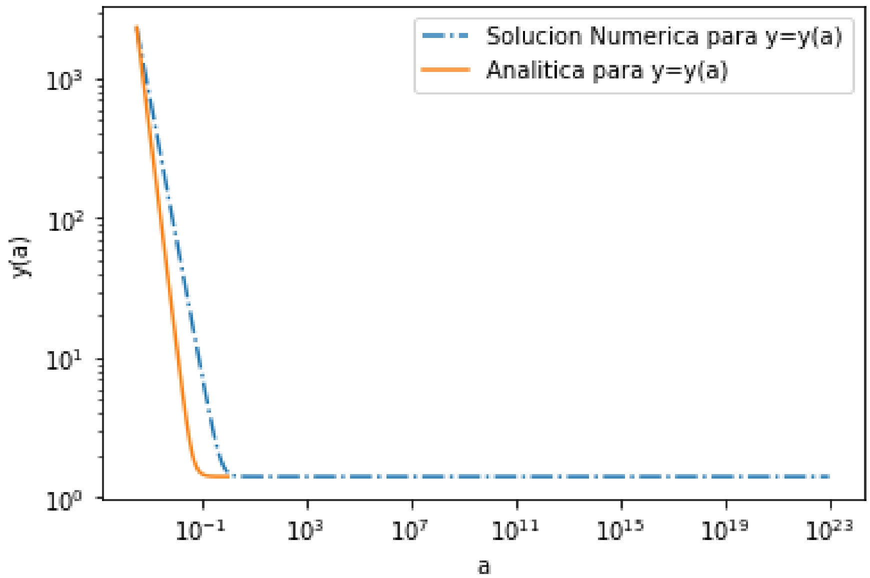

Figure 2: $\dot{\phi} = \dot{\phi}(a)$

Numerical and Analytical solution for $\dot{\phi} = \dot{\phi}(t)$

We present in the plot 3 the theoretical and the numerical solution for $\dot{\phi}$. We see this two graphics are very similar, at least in its form.

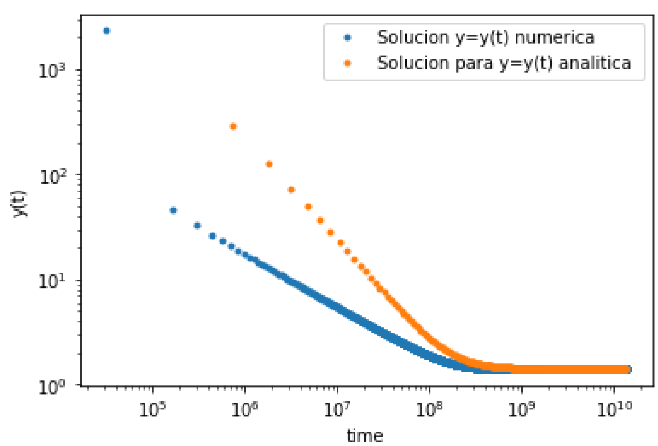

Figure 3: $\dot{\phi} = \dot{\phi}(t)$

Analytical solution for a=a(t)

The following plot (figure 4) is the analytical solution for $a(t)$, this solution goes from the equality time until nowadays.

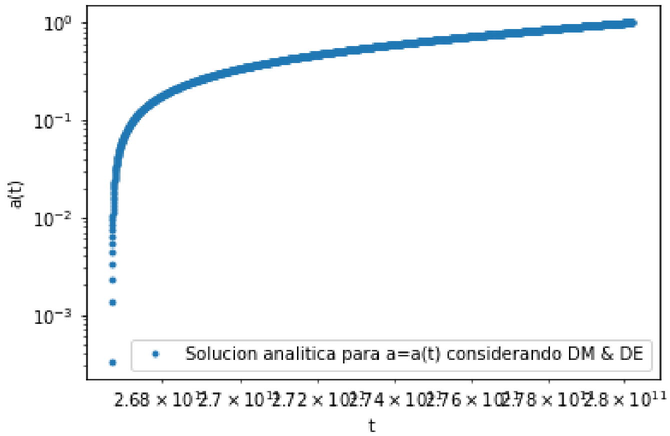

Figure 4: Analytical solution fora(t)

Analytical solution for a=a(t) by parts

We present in the figure 5 two graphics, considering DM and DE separated for estimate, this graphics should be consist with the figure 4.

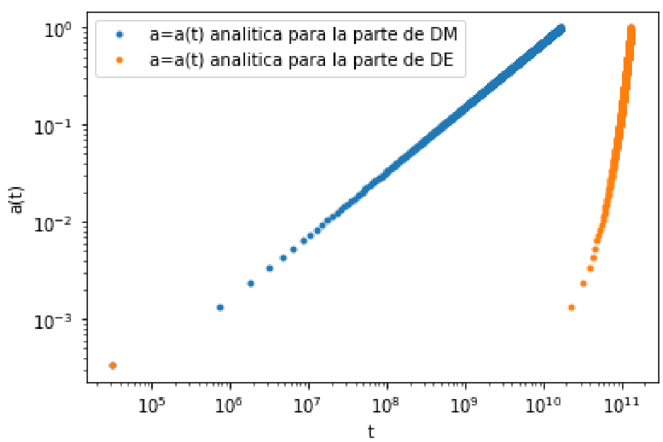

Figure 5: Analytical solution for a(t)

Linear model for Braiding. $G(\phi) = \mu\phi$

En lo que sigue consideramos el lagrangiano de Braiding de una forma lineal del campo, es decir, tomamos el caso:

$$G_3(X, \phi) = G(\phi) = \kappa\phi \tag{168}$$

las ecuaciones diferenciales a resolver las obtenemos de la ecuacion de Friedmann y de la ecuacion de movimiento obtenida para Braiding:

$$\dot{a} = \sqrt{\frac{\kappa}{3M_p^2}} ya \tag{169}$$

$$\dot{y} = -3\sqrt{\frac{\kappa}{3M_p^2}} y^2 \tag{170}$$

donde

$$\dot{\phi} = y \tag{171}$$

Solving numerically this system of deferential equations we get the nexto results (from figure 6 - 11):

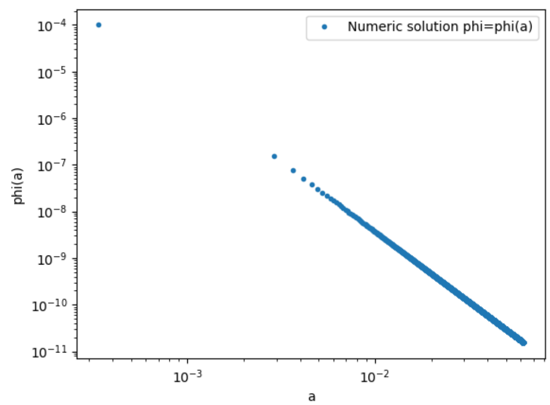

Figure 6: $\dot{\phi}(a)$

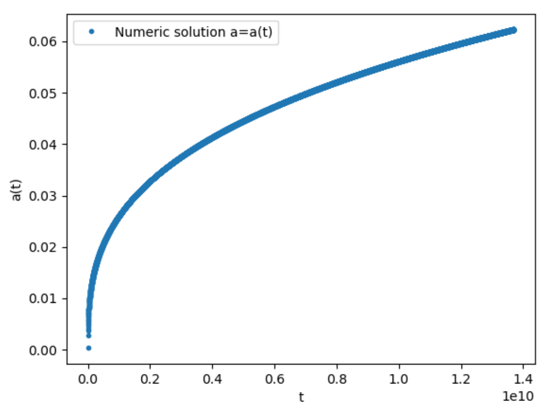

Figure 7: a(t)

Quadratic model for Braiding : $G(\phi) = \mu\phi^2$

Now we

$$G_3(X,\phi) = G(\phi) = \kappa\phi^2 \tag{172}$$

The system of differential equations to resolve are obtained with the Friedman equation and with the equation of motion for this field:

$$\dot{a} = \sqrt{\frac{2\kappa\phi}{3M_p^2}}ya \tag{173}$$

$$\dot{y} = -3\sqrt{\frac{2\kappa\phi}{3M_p^2}}y^2 + \frac{y^2}{2\phi} \tag{174}$$

$$\dot{\phi} = y \tag{175}$$

Solving this system of differential equations we get the next results(from figure 12 to figure 17)

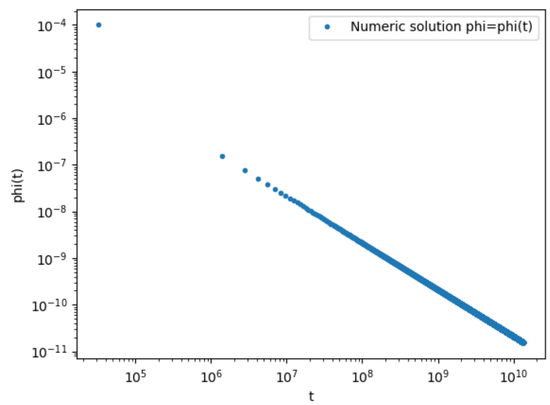

Figure 8: $\dot{\phi}(t)$

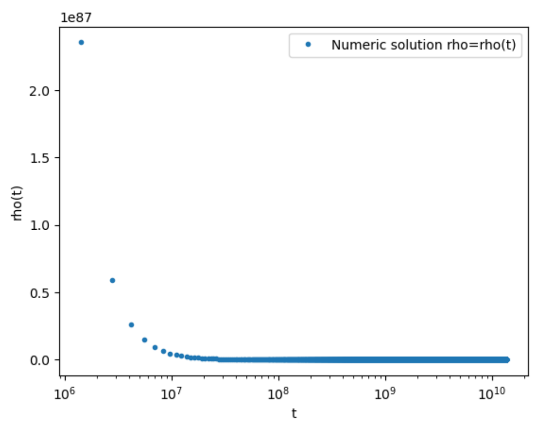

Figure 9: $\rho(t)$

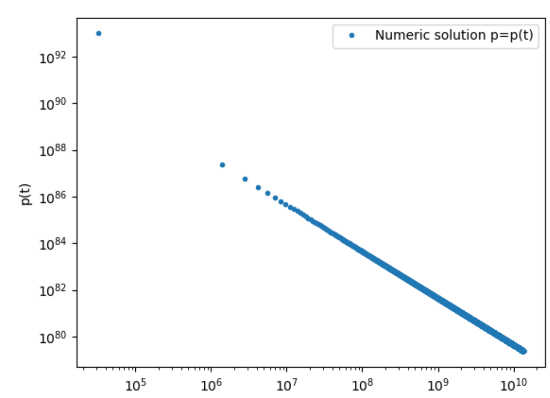

Figure 10: p(t)

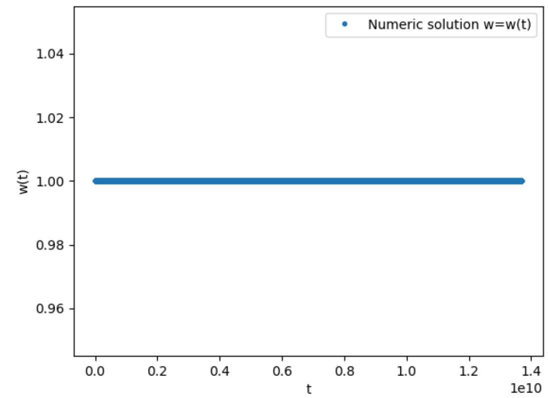

Figure 11: w(t)

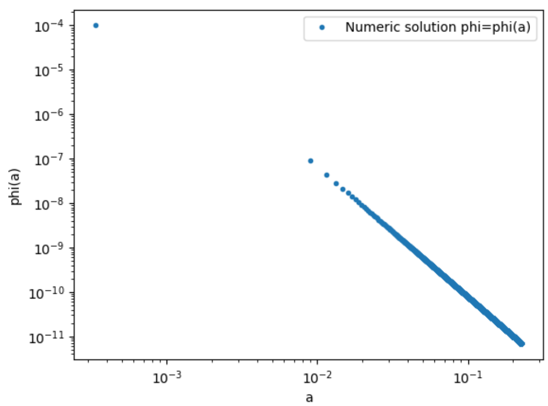

Figure 12: $\dot{\phi}(a)$

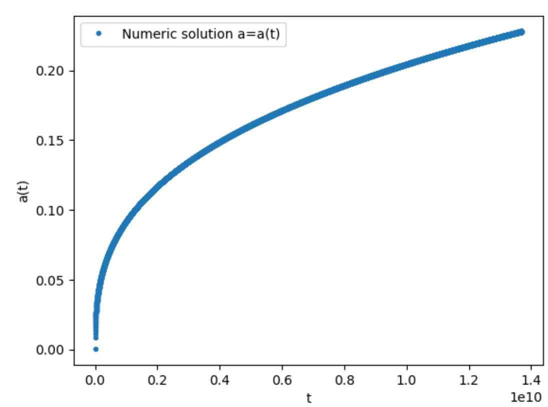

Figure 13: a(t)

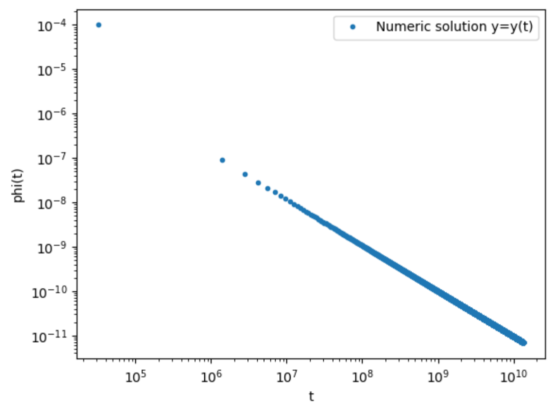

Figure 14: $\dot{\phi}(t)$

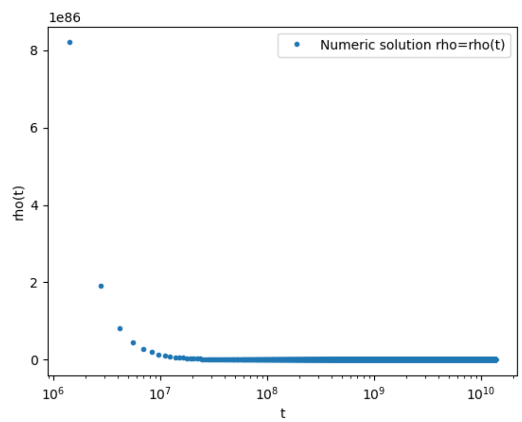

Figure 15: $\rho(t)$

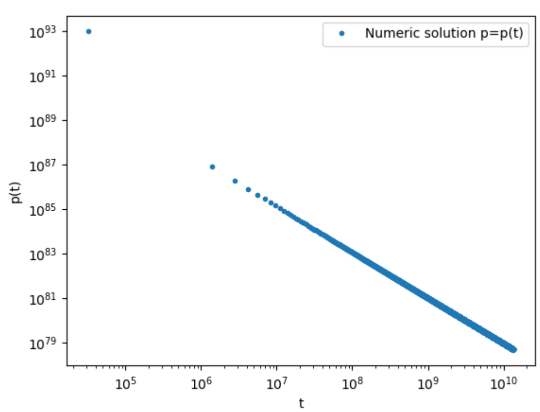

Figure 16: p(t)

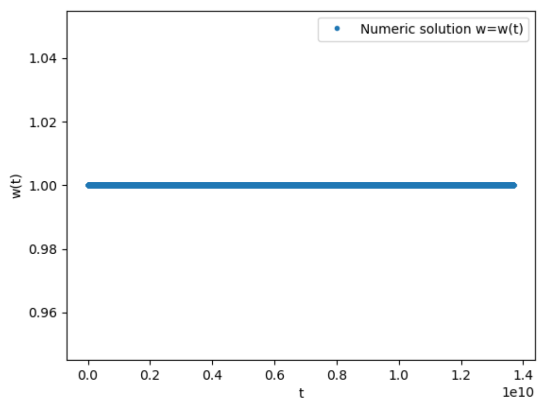
Numeric solution w=w(t)
1.04
1.02
1.00
0.98
0.96
w(t)
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
t
1e10

Figure 17: w(t)