Efficient Indoor Heating: a Finite Element Analysis of Room Temperature

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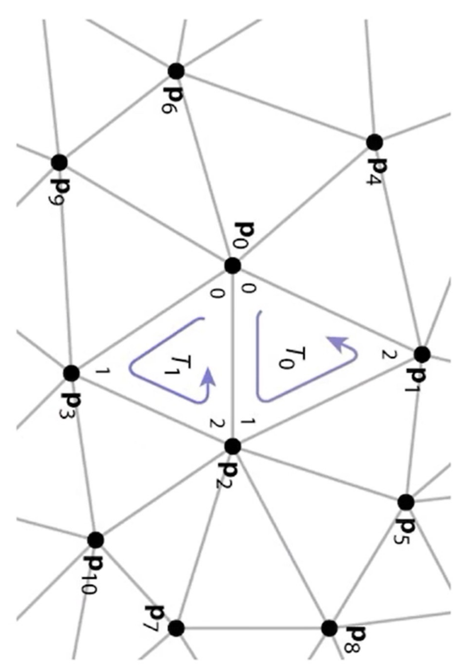
Abstract

This research aims to minimize household electricity expenses by analyzing room temperature with a simplified heat equation. The proposed numerical approach creates a computational mesh for real-time heat flow analysis, utilizing the finite element method for space discretization and finite differences method for time discretization.

Background: the Finite Element Method (FEM)

- Powerful numerical technique
- Solve partial differential equations
- Engineering/mathematics problems
- Divide and conquer
- discrete algebraic equations
- boundary conditions



Problem Statement

Consider a square room section with 4 conditioned borders:

1. **Neumann Condition**: The roof (Γ_N) is isolated: $\partial T/\partial n\,|_{\Gamma_N} = 0$
2. **Dirichlet Condition**: Constant floor temperature (Γ_D) : $T|_{\Gamma_D} = T_F$
3. **Robin Condition** (left, right): $k\,\partial T/\partial n = -\lambda(T - Ta)$

In the interior, the temperature is given by the PDE:

$$c_p\rho\frac{\partial T}{\partial t} = k\Delta T + f$$

T: Temperature — Ta: Ambient temperature

k: Thermal conductivity coefficient — λ: Diffusive constant

c_p : Specific heat, — ρ : Air density

ΔT: Laplacian operator — f: External heat source

Method: Spatial discretization

- Space discretization transforms the problem into finding $\vec{T}(t)$, a nodal function defined uniquely in terms of the vertices of the mesh triangulation, such that:

$$c_p\rho M\frac{d\vec{T}}{dt}(t) + kS\vec{T}(t) + \lambda R\vec{T}(t) = \lambda T_a\vec{r} + \vec{f}$$

M: Mass matrix — S : Stiffness matrix

R, $\vec{r}$: Obtained from boundary conditions

$\vec{f}$: Obtained from external heat source

Defining the matrix and the vector:

$$K = kS + \Delta R \qquad \vec{b} = \lambda Ta\vec{r} + \vec{f}$$

leads us to the equation:

$$c_p\rho M\frac{d\vec{T}}{dt}(t) + K\vec{T}(t) = \vec{b}$$

Method: Time discretization

Using discrete time steps: $t_0 = 0, t_1 = \Delta t, t_2 = 2\Delta t, \cdots, t_n = n\Delta t$, we approximate $\vec{T}^n = T_1^n, \cdots, T_J^n$ as the vector of vertex temperatures at t_n.

For the forward approximation of the derivative, we use

$$\frac{d\vec{T}}{dt}(t) \approx \left(\frac{T^{n+1} - T^n}{\Delta t}\right)$$

Wilson's theta method:

$$\vec{T}(t) \approx (1-\theta)T^n + \theta T^{n+1}, 0 \le \theta \le 1$$

This leads to the linear system:

$$(c_p\rho M + \theta\Delta t\, K)T^{n+1} = (c_p\rho M - (1-\theta)\,\Delta t\, K + \Delta t\,\vec{b})T^n$$

We choose θ = ½, known as the Crank-Nicolson method:

✓ Unconditionally convergent and stable (no CFL condition).

✓ Smallest errors and good convergence rates.

We can now calculate T^{n+1} recursively using the known intial temperature T^0 and the previous heat coefficients to update each node

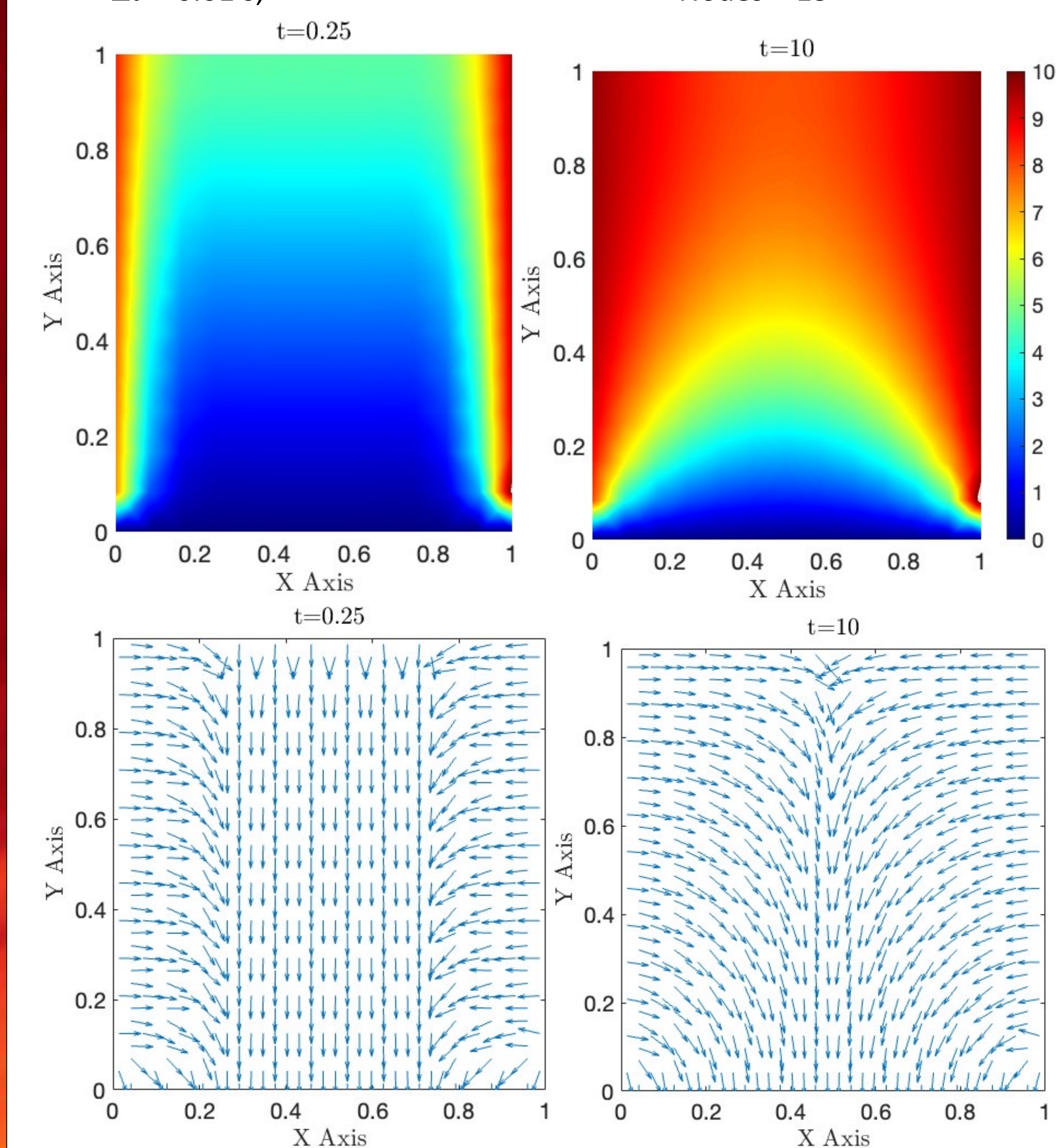
Experimental Results

$T_0(x,y) = 5y$ °C (Air) — $k = 0.02435\,\frac{W}{\text{m °C}}$ (Air)

T_a = 10 °C (Air) — $c_p = 1.012\,\frac{J}{\text{kg °C}}$(Air)

$\rho = 1.204\,\frac{Kg}{m^3}$ (Air) — $\lambda = 0.6, 2.0\,\frac{W}{\text{m °C}}$(Concrete)

Δt = 0.01 s; — Nodes = 13



Room temperature (°C) and heat flux: $q = -\kappa\nabla T\ \left(\frac{\text{W}}{\text{m}^2}\right)$ for t = 0.25 s and t = 10s.

Conclusions

When exposed to an even heat source from the sides, room heat spreads from top to bottom. This suggest to apply heat in an uneven manner, prioritizing initial heat delivery to the bottom.

References

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