Efficient Indoor Heating: a Finite Element Analysis of Room Temperature

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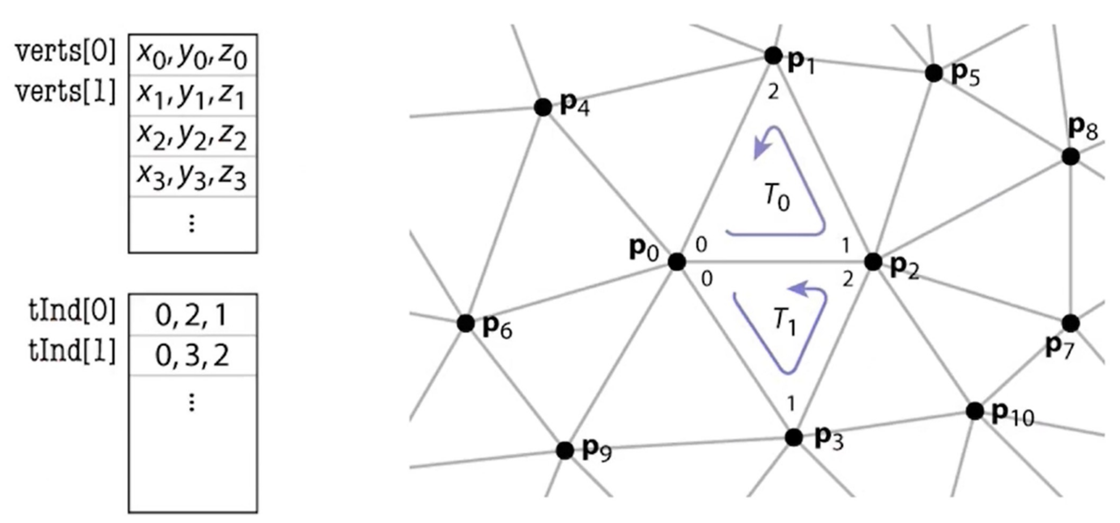
Abstract

Efficient room heating aims to create stable indoor temperature while minimizing electricity expenses. By analyzing an external heat source's capability to maintain warm temperatures in a prism shaped room, this research aims to create a computational mesh representing heat flow in real time. This mathematical analysis will provide insights into efficient heating strategies for stable indoor environments.

Background: the Finite Element Method (FEM)

The Finite Element Method (FEM) is a powerful numerical technique used for solving partial differential equations in engineering and mathematics. It breaks down complex problems into smaller, discrete parts called finite elements. By meshing these elements together, we will represent this heat flux problem as discrete algebraic equations with boundary conditions.



Problem Statement

Consider a square room section bounded by 4 conditioned borders

1. **Neumann Condition**: The roof (τ_n) is isolated: $\partial T/\partial n\,|_{\tau\_n} = 0$
2. **Dirichlet Condition**: floor temperature is constant : $T = T_i$
3. **Robin Condition** (left, right): $k\,\partial T/\partial n = -\lambda(T - T\text{a})$

k: thermal conductivity coefficient λ: heat function

T: boundary temperature Ta: ambient temperature

4. Inner Flow of Heat: $c_p\rho\frac{\partial T}{\partial t} = k\Delta T + f$

c_p : specific heat, k: thermal conductivity coefficient

ρ : density T: room temperature

ΔT: Laplacian operator f: external heat source

Method: Spatial discretization

To discretize the equation over space, we need to find $\vec{T}(t)$ such that:

$$c_p\rho M\frac{d\vec{T}}{dt}(t) + kS\vec{T}(t) + \lambda R\vec{T}(t) = \lambda T_a\vec{r} + \vec{f}$$

M: Mass S : Stiffness

Defining the matrix and the vector:

$K = kS + \Delta R$ $\vec{b} = \lambda Ta\vec{r} + \vec{f}$

leads us to the equation:

$$c_p\rho M\frac{d\vec{T}}{dt}(t) + K\vec{T}(t) = \vec{b}$$

Method: Time discretization

Using discrete time steps: $t_0 = 0, t_1 = \Delta t, t_2 = 2\Delta t, \cdots, t_n = n\Delta t$, we approximate $\vec{T}^n = T_1^n, \cdots, T_J^n$ as the vector of vertex temperatures at t_n.

For the forward approximation of the derivative, we use $\frac{d\vec{T}}{dt}(t) \approx \left(\frac{T^{n+1} - T^n}{\Delta t}\right)$, with Wilson's theta method: $\vec{T}(t) \approx (1 - \theta)T^n + \theta T^{n+1}$

This leads to the linear system:

$$(c_p\rho M + \theta\Delta t\,K)T^{n+1} = (c_p\rho M - (1 - \theta)\,\Delta t\,K + \Delta t\,\vec{b})$$

We choose θ = ½, known as the Crank-Nicolson method:

✓ Converges unconditionally (no CFL condition)
✓ Smallest error convergence.
x The calculation of the inverse is computationally expensive and not guaranteed to exist

We can now calculate T^{n+1} recursively using the known intial temperature T^0 and the previous heat coefficients to update each node

Modeling: A vertex approach

To isolate the value of each node we will combine the temperature equation $T_j(t)$ with the with the canonic function $\xi(x_j)$, which evaluates to 1 at x_j and 0 at all other nodes.

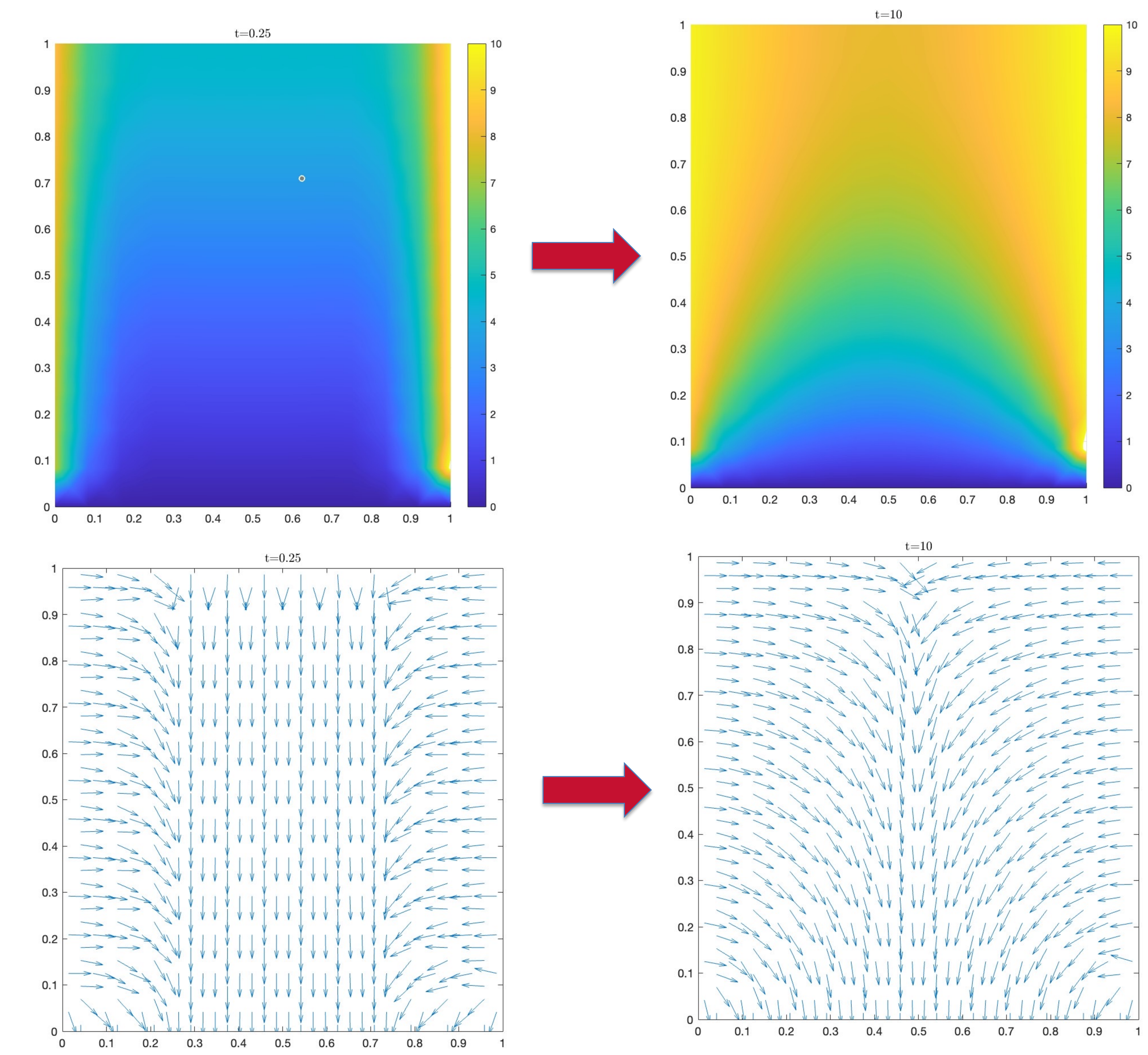
$$T_n^j = \sum_{j=i}^{J} T_j(t)\xi_j(x)$$

Results

T^0 = 10 ° C ρ = 1.204 kg/m³; c_p : 1.012 J/(kg·° C)

Δt = 0.01 s; k = 0.02435 W/(m·° C)

t_{max}= 20 s:



Conclusions

The experimental analysis of the room's heat distribution when exposed to an even heat source reveals a distinct pattern, with heat spreading from the top to the bottom. As a result, the upper portions experience faster temperature growth, while the lower parts are relatively slower to heat up due to heat stratification. To effectively maintain a consistent and comfortable temperature within the room, it is recommended to apply the heat source in an uneven manner, prioritizing initial heat delivery to the bottom.

References

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Moës, N., Béchet, E., & Tourbier, M. (2006). Imposing Dirichlet boundary conditions in the extended finite element method. *International Journal of Numerical Methods in Engineering*, 67, 1641-1669. doi:10.1002/nme.1675

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