ORBITAL PREDICTION

MATH 104B FINAL PROJECT

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Math 104B: Numerical Analysis
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1 *Introduction*

The motivation of this project is to be able to calculate the Earth's orbit around the Sun at any given time in a 2D format. To be able to implement this, we would first look into using an exact method of being able to calculate this and then using 2^{nd} and 4^{th} Order Runge Kutta Methods to see which approximation would be more accurate. Our exact method of finding the orbit is not completely analytical as to be able to find one value that is used throughout the calculations, we would have to implement Newtons method. Thus, we will be comparing these two methods with the 'exact' method being the one that is being used for the comparison.

2 *Theoretical & Methodical Foundations*

To be able to calculate the orbit of Earth around the Sun you first have to know some basic information. The orbit of Earth is an ellipse, this means that there will be a point where it is closest to the Sun and furthest from the Sun. The point closest to the Sun will be called the Perihelion and point furthest to the Sun will be called Aphelion. The center of the ellipse will not be where we mark the origin at but instead the Sun will be our origin (a.k.a the foci). In addition, we will need to know horizontal distance from the edge to the center which will be denoted as the semi-major axis, how much bend there is to the ellipse which is known as the eccentricity, initial position vector, initial velocity vector , initial acceleration vector, mass of the Earth, mass of the Sun, and Gravitational constant. Then to begin with our exact solution:

a = Semi-Major Axis, M1 = Mass of Sun, M2 = Mass of Earth, e = eccentricity, t = time, t_p = initial time

Step 1: Calculate the mean motion: $n = \sqrt{\frac{G(M1)(M2)}{a^3}}$

Step 2: Calculate the Mean anomaly: $M = n(t - t_p)$

Step 3: Solve for E(Eccentric anomaly): where $M = E - esinE$ becomes $f(E) = E - esinE - M$

Step 4: Find rectangular coordinates: $X = a(CosE - e), Y = a(\sqrt{1 - e^2}sinE)$

Where Mean motion is the angular speed required for Earth to complete one orbit. Mean anomaly is the fraction of the elliptical orbit's period that has elapsed since Earth has passed perihelion which is giving us angular distance from the perihelion at time t. Eccentric anomaly is an angular parameter that defines the position of Earth moving through an elliptic orbit. This is found numerically by using Newtons method as it cannot be computed analytically. Thus, from the above steps this will produce the 'exact' solution. Now to determine how to compute the 2^{nd} and 4^{th} order Runge Kutta methods. Since this problem is only a 2-body problem this means we are not accounting for the forces of the other planets acting on the Earth, then we have our equation of motion:

$\vec{r}$ = position vector

$$\ddot{r} = \frac{-G(M1 * M2)}{|r|^3}\vec{r}$$

Where the initial position vector will be the distance from the sun to the perihelion. Then we define a constant Q as:

T = Period, R = Distance from Sun to Perihelion

$$Q = \frac{G * M1 * T^2}{R^3}$$

Which comes from manipulating the equation of motion. Since we are using the Runge Kutta method we want to turn this 2nd order ODE into a 1st order ODE but since we are doing this in regard to position, we will want both the x and y equation for both so we will then have four 1st order ODEs:

$$x = \frac{r_x}{|r|}, y = \frac{r_y}{|r|}$$

$\dot{x} = V_x$ Velocity for x component

$\dot{y} = V_y$ Velocity for y component

$\dot{V}_x = -Q * \frac{x}{\sqrt{x^2+y^2}^3}$ Acceleration for x component

$\dot{V}_y = -Q * \frac{y}{\sqrt{x^2+y^2}^3}$ Acceleration for y component

Now that we have our first order ODE's we will be able to implement the 2nd Order Runge Kutta Method. Our h variable will be our time step.

$$k_{x1} = h\dot{x}(x, y, v_x, v_y)$$

$$k_{y1} = h\dot{y}(x, y, v_x, v_y)$$

$$k_{v_{x1}} = h\dot{V}_x(x, y, v_x, v_y)$$

$$k_{v_{y1}} = h\dot{V}_y(x, y, v_x, v_y)$$

$$k_{x2} = h\dot{x}(x + \frac{k_{x1}}{2}, y + \frac{k_{y1}}{2}, v_x + \frac{k_{v_{x1}}}{2}, v_y + \frac{k_{v_{y1}}}{2})$$

$$k_{y2} = h\dot{y}(x + \frac{k_{x1}}{2}, y + \frac{k_{y1}}{2}, v_x + \frac{k_{v_{x1}}}{2}, v_y + \frac{k_{v_{y1}}}{2})$$

$$k_{v_{x2}} = h\dot{V}_x(x + \frac{k_{x1}}{2}, y + \frac{k_{y1}}{2}, v_x + \frac{k_{v_{x1}}}{2}, v_y + \frac{k_{v_{y1}}}{2})$$

$$k_{v_{y2}} = h\dot{V}_y(x + \frac{k_{x1}}{2}, y + \frac{k_{y1}}{2}, v_x + \frac{k_{v_{x1}}}{2}, v_y + \frac{k_{v_{y1}}}{2})$$

$$x_{i+1} = x_i + k_{x2}$$

$$y_{i+1} = y_i + k_{y2}$$

$$V_{x_{i+1}} = V_{x_i} + k_{v_{x2}}$$

$$V_{y_{i+1}} = V_{y_i} + k_{v_{y2}}$$

Now for the 4th Order Runge Kutta Method:

$$k_{x1} = h\dot{x}(x, y, v_x, v_y)$$

$$k_{y1} = h\dot{y}(x, y, v_x, v_y)$$

$$k_{v_{x1}} = h\dot{V}_x(x, y, v_x, v_y)$$

$$k_{v_{y1}} = h\dot{V}_y(x, y, v_x, v_y)$$

$$k_{x2} = h\dot{x}(x + \frac{k_{x1}}{2}, y + \frac{k_{y1}}{2}, v_x + \frac{k_{v_{x1}}}{2}, v_y + \frac{k_{v_{y1}}}{2})$$

$$k_{y2} = h\dot{y}(x + \frac{k_{x1}}{2}, y + \frac{k_{y1}}{2}, v_x + \frac{k_{v_{x1}}}{2}, v_y + \frac{k_{v_{y1}}}{2})$$

$$k_{v_{x2}} = h\dot{V}_x(x + \frac{k_{x1}}{2}, y + \frac{k_{y1}}{2}, v_x + \frac{k_{v_{x1}}}{2}, v_y + \frac{k_{v_{y1}}}{2})$$

$$k_{v_{y2}} = h\dot{V}_y(x + \frac{k_{x1}}{2}, y + \frac{k_{y1}}{2}, v_x + \frac{k_{v_{x1}}}{2}, v_y + \frac{k_{v_{y1}}}{2})$$

$$k_{x3} = h\dot{x}(x + \frac{k_{x2}}{2}, y + \frac{k_{y2}}{2}, v_x + \frac{k_{v_{x2}}}{2}, v_y + \frac{k_{v_{x2}}}{2})$$

$$k_{y3} = h\dot{y}(x + \frac{k_{x2}}{2}, y + \frac{k_{y2}}{2}, v_x + \frac{k_{v_{x2}}}{2}, v_y + \frac{k_{v_{x2}}}{2})$$

$$k_{v_{x3}} = h\dot{V}_x(x + \frac{k_{x2}}{2}, y + \frac{k_{y2}}{2}, v_x + \frac{k_{v_{x2}}}{2}, v_y + \frac{k_{v_{x2}}}{2})$$

$$k_{v_{y3}} = h\dot{V}_y(x + \frac{k_{x2}}{2}, y + \frac{k_{y2}}{2}, v_x + \frac{k_{v_{x2}}}{2}, v_y + \frac{k_{v_{x2}}}{2})$$

$$k_{x4} = h\dot{x}(x + k_{x3}, y + k_{y3}, v_x + k_{v_{x3}}, v_y + k_{v_{y3}})$$

$$k_{y4} = h\dot{y}(x + k_{x3}, y + k_{y3}, v_x + k_{v_{x3}}, v_y + k_{v_{y3}})$$

$$k_{v_{x4}} = h\dot{V}_x(x + k_{x3}, y + k_{y3}, v_x + k_{v_{x3}}, v_y + k_{v_{y3}})$$

$$k_{v_{y4}} = h\dot{V}_y(x + k_{x3}, y + k_{y3}, v_x + k_{v_{x3}}, v_y + k_{v_{y3}})$$

$$x_{i+1} = x_i + \frac{1}{6}(k_{x1} + 2k_{x2} + 2k_{x3} + k_{x4})$$

$$y_{i+1} = y_i + \frac{1}{6}(k_{y1} + 2k_{y2} + 2k_{y3} + k_{y4})$$

$$V_{x_{i+1}} = V_{x_i} + \frac{1}{6}(k_{v_{x1}} + 2k_{v_{x2}} + 2k_{v_{x3}} + k_{v_{x4}})$$

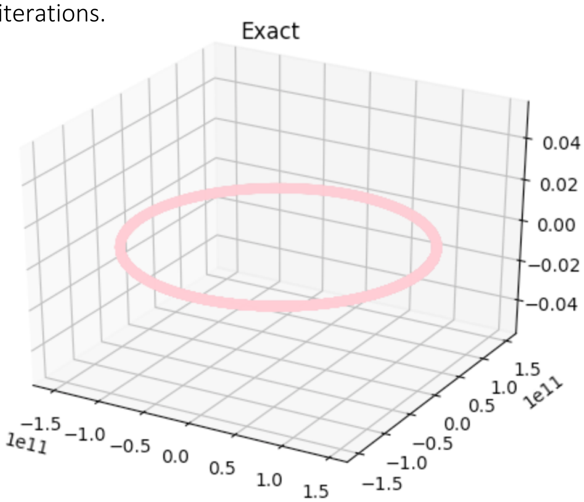
$$V_{y_{i+1}} = V_{y_i} + \frac{1}{6}(k_{v_{y1}} + 2k_{v_{y2}} + 2k_{v_{y3}} + k_{v_{y4}})$$

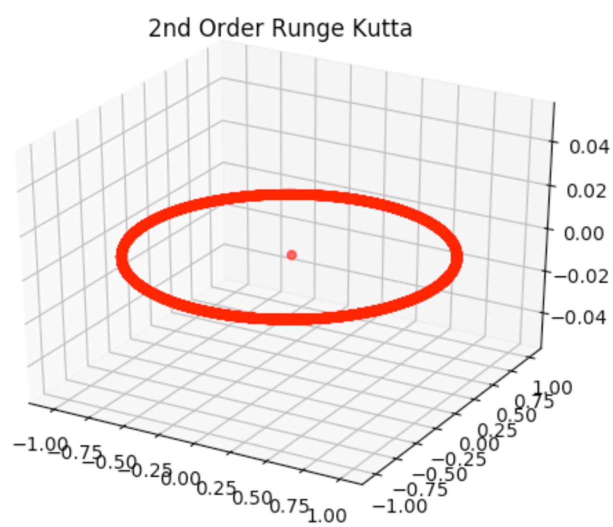
Thus, using these two methods will give us our approximations when given initial conditions. The initial conditions will be for position and velocity. We see that the 2nd Order Runge Kutta method has local truncation error $O(h^2)$ and our 4th Order Runge Kutta method has local truncation error $O(h^4)$. Thus, from this we can expect our 4th order method to be more accurate and closer to our exact solution. The reason for using the Runge Kutta method versus the Euler method is because of the fact that Euler can be unstable as it does not take into account for curvature as Runge Kutta does.

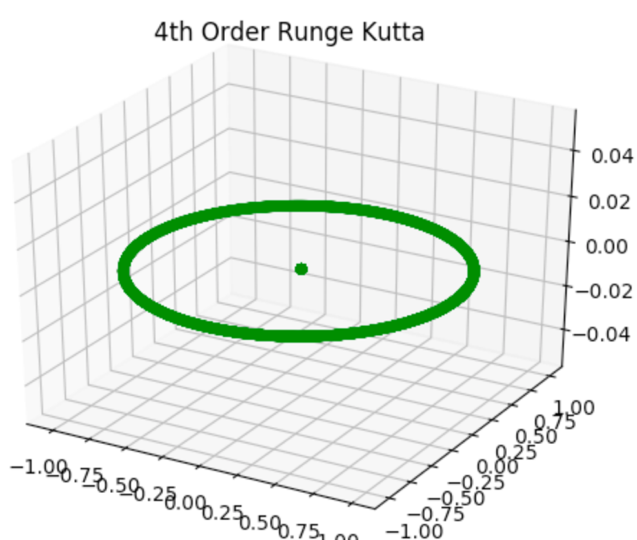
3 *Solutions*

Iteration	*Exact*	*Runge Kutta 2nd*	*Runge Kutta 4th*
8363	*1.0105177803167822*	*1.0072179456947816*	*1.007217893657691*
8364	*0.9835820988327677*	*1.0072089525431152*	*1.007208900582322*
8365	*0.9834151684188706*	*1.0071999634186246*	*1.0071999115340864*
8366	*1.0027466698750054*	*1.0071909783251936*	*1.0071909265168688*
8367	*1.0067986306323147*	*1.007181997266705*	*1.0071819455345519*
8368	*0.9837011722624575*	*1.0071730202470404*	*1.0071729685910167*
8369	*0.9838820164564*	*1.0071640472700787*	*1.0071639956901426*

This chart is showing the differences in radius (AU) between the exact and approximate solutions. Every iteration is 52 minutes apart and from the chart above we can see that the 4th Order Runge Kutta is truly more accurate than the 2nd Order Runge Kutta. For every iteration the 4th Order is closer to the Exact than the 2nd Order. Since the time difference is 52 minutes, we will be getting small differences in between iterations.

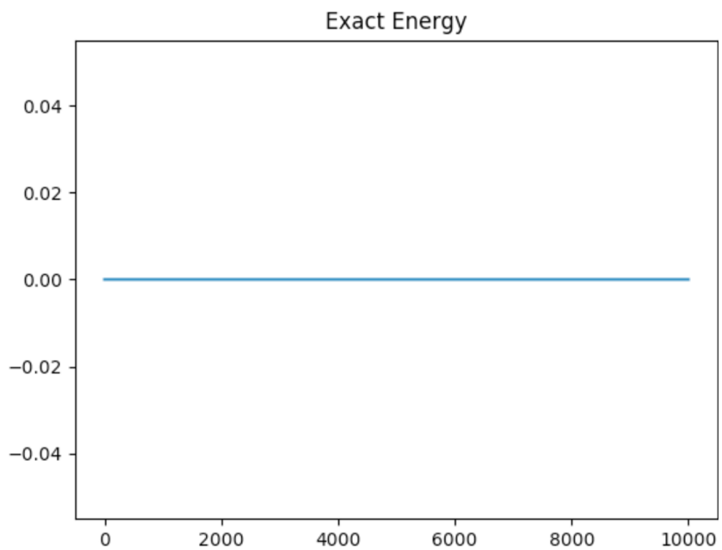


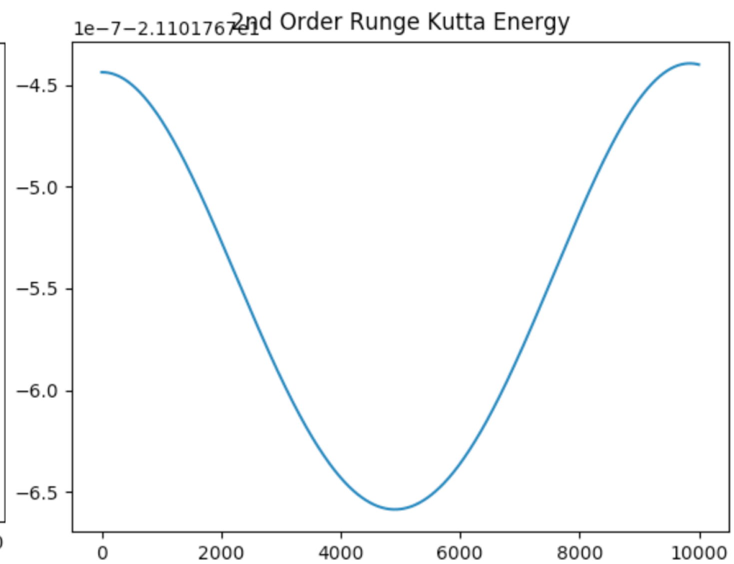


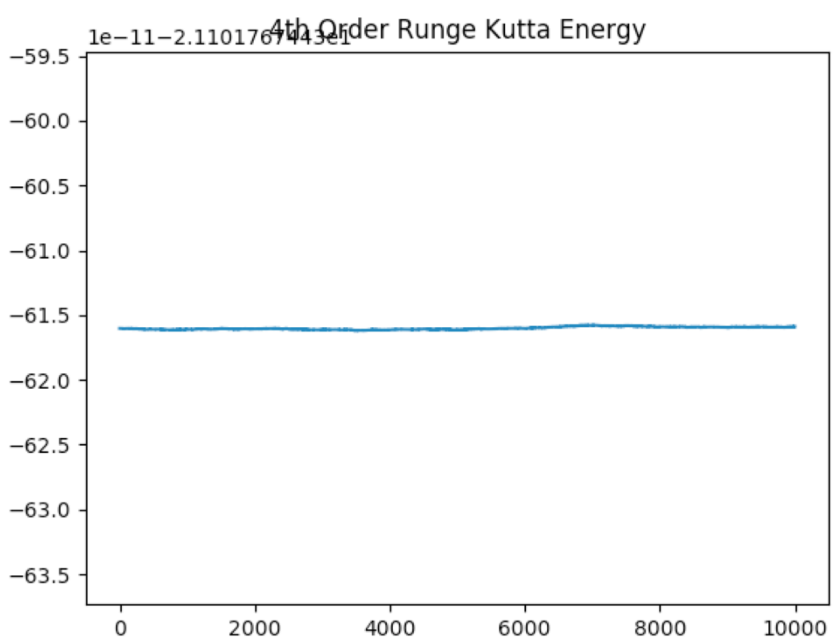


From these values it seems as if both methods are very close to each other, but you can see more of the difference when you use the Energy equation. The equation of Energy for the exact equation will be $Energy = \sqrt{G * M1 * M2(\frac{2}{r} - \frac{1}{a})}$ and our equation for the approximations will be

$Energy = .5\left(v^2{}_x + v^2{}_y\right) - \frac{Q}{\sqrt{x^2+y^2}}$ and from these graphs you can see the difference in accuracy a lot clearer. Hence, we can see how much better 4th Order Runge Kutta is.







3 *References*

Implementing a Fourth Order Runge-Kutta Method for Orbit Simulation. spiff.rit.edu/richmond/nbody/OrbitRungeKutta4.pdf.

"Kepler Orbit." *Wikipedia*, Wikimedia Foundation, 5 May 2019, en.wikipedia.org/wiki/Kepler_orbit.

NASA, NASA, nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html.