EGM0004

Sistemas Não Lineares

Prof. **Josenalde Barbosa de Oliveira – UFRN**

josenalde.oliveira@ufrn.br

Programa de Pós-Graduação em Engenharia Mecatrônica

[LaSalle, J.P.](#) *Some extensions of Liapunov's second method,* IRE Transactions on Circuit Theory, CT-7, pp. 520–527, 1960.

Teoremas sobre estabilidade

Teorema 6: Teorema de La Salle (princípio da invariância)
Pode ocorrer de $\dot{V} \leq 0$ e ainda assim concluir sobre estabilidade assintotica

Definições:

Positively invariant set

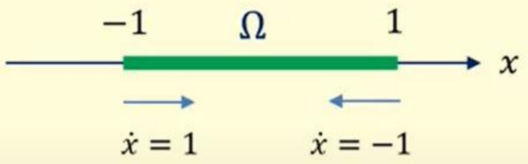
A set Ω is said to be a ***positively invariant set*** with respect to $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ if

$$\boldsymbol{x}(0) \in \Omega \quad \Rightarrow \quad \boldsymbol{x}(t) \in \Omega, \qquad \forall\, t \geq 0$$

Example 1.

The set $\Omega = \{x \in \mathbb{R} : |x| \leq 1\}$ is a positively invariant set with respect to $\dot{x} = -x$.

To check if Ω is positively invariant w.r.t the system dynamics, we *just check the boundaries*.

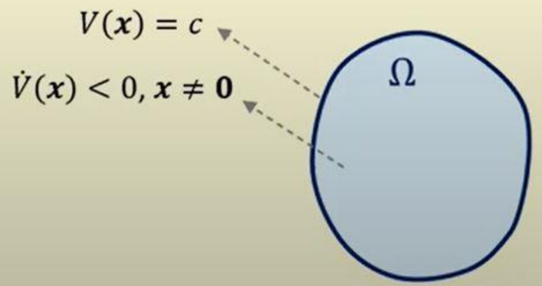


Example 2.

Let V be a positive definite.
and $\dot{V}(\boldsymbol{x})$ be negative definite for $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$.

Define Ω as $\Omega = \{\boldsymbol{x} \in \mathbb{R}^n : V(\boldsymbol{x}) \leq c\}$.

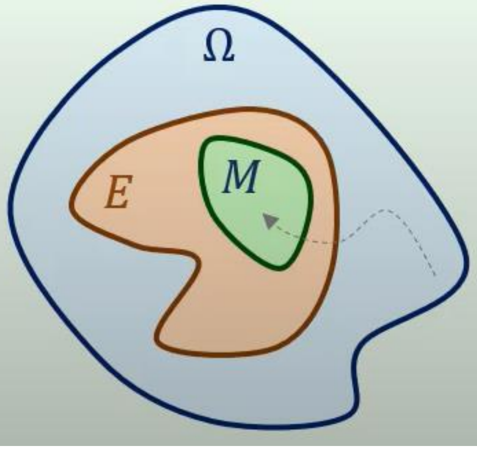
Then Ω is a positively invariant set w.r.t. $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$. We ***check the boundaries***.



La Salle

- Let Ω be a compact set that is ***positively invariant*** with respect to $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$
- Let V be a **continuously differentiable** function on Ω such that $\dot{V}(\boldsymbol{x}) \leq 0$ in Ω
- Let $E \subset \Omega$ be the set of all points in Ω such that $\dot{V}(\boldsymbol{x}) = 0$
- Let M be the ***largest positively invariant*** set in E

Then every solution starting in Ω approaches M as $t \to \infty$



La Salle local e global

Teorema 6: Teorema de La Salle (princípio da invariância)

Se para um sistema $\dot{x} = f(x), f(0) = 0,$ existe uma função $V(x)$ tal que

a) $V(x) > 0$

b) $V(x)$ é continuamente diferenciável

c) $\dot{V}(x) \leq 0$

d) $\nexists x \neq 0$ tal que $\dot{V}(x) = 0, \forall t \geq 0$ Então é assintoticamente estável

O conjunto definido pela derivada de V(x)=0 não contenha trajetórias além da trajetória identicamente nula

Teorema 7: Teorema de La Salle global

Se para um sistema $\dot{x} = f(x), f(0) = 0,$ existe uma função $V(x)$ tal que

a) $V(x) > 0$

b) $V(x)$ é continuamente diferenciável

c) $\dot{V}(x) \leq 0$

d) $\nexists x \neq 0$ tal que $\dot{V}(x) = 0, \forall t \geq 0$

e) $V(x) \to \infty$ quando $||x|| \to \infty$ Então é globalmente assintoticamente estável

La Salle local e global

Teorema 6: Teorema de La Salle (princípio da invariância)

Exemplo:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2$$
$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 > 0$$

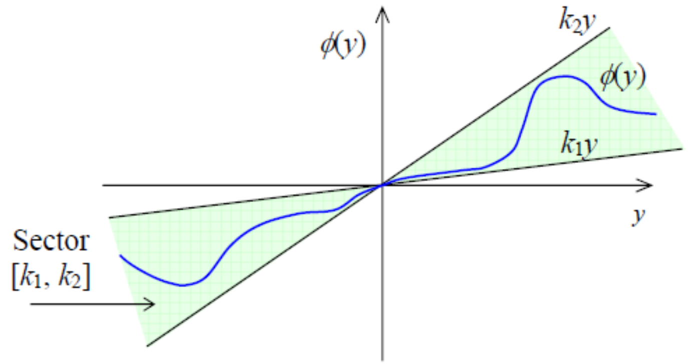


Fig. 3.13 – Função pertencente a um sector.

Exemplo:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\phi(x_1) - \mu x_2, \mu > 0$$
$$V(x) = \frac{1}{2}x_2^2 + \int_0^{x_1} \phi(t)dt > 0$$

$\phi(x_1)x_1 > 0$ se $x_1 \neq 0$ Função setorial nos quadrantes impares

$x_1 = 0 \implies \phi(x_1) = 0$

La Salle local e global

Teorema 7: Teorema de La Salle global

Exemplo:

$$\dot{x}_1 = x_2$$

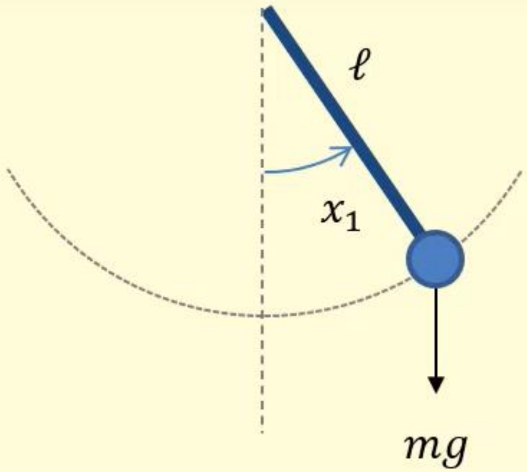
$$\dot{x}_2 = -x_1 + x_3$$

$$\dot{x}_3 = -x_2 - x_3$$

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 > 0$$

La Salle

Example 3 (pendulum with friction): Show the asymptotic stability of the origin $x = (0,0)$ using LaSalle's Theorem for the general case where $k \neq 0$. For simplicity, let $\ell = g$ and $m = 1$.



State equations:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2$$

Let $V(\boldsymbol{x}) = 1 - \cos x_1 + \frac{1}{2}x_2^2$

$$V(x) = mgl(1 - cosx_1) + \frac{1}{2}ml^2x_2^2$$

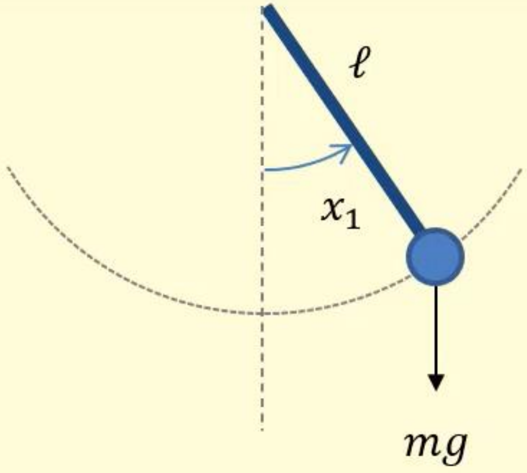
m: mass of the bob
ℓ: length of the rod
k: friction coefficient

$$\dot{V}(\boldsymbol{x}) = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = (\sin x_1)(x_2) + (x_2)(-\sin x_1 - kx_2) = -kx_2^2 \leq 0$$

So $\dot{V}(\boldsymbol{x})$ is negative semi-definite.

La Salle

Example 3 (pendulum with friction): Show the asymptotic stability of the origin $x = (0,0)$ using LaSalle's Theorem for the general case where $k \neq 0$. For simplicity, let $\ell = g$ and $m = 1$.



m: mass of the bob
ℓ: length of the rod
k: friction coefficient

State equations:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2$$

Let $V(\boldsymbol{x}) = 1 - \cos x_1 + \frac{1}{2}x_2^2$ $\quad \dot{V}(x) = \nabla V \cdot f(x)$

$$\dot{V}(\boldsymbol{x}) = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = (\sin x_1)(x_2) + (x_2)(-\sin x_1 - kx_2) = -kx_2^2 \leq 0$$

So $\dot{V}(\boldsymbol{x})$ is negative semi-definite.

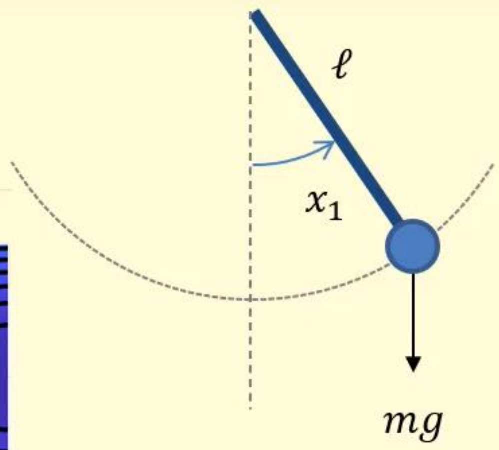
The set Ω is a compact set that is ***positively invariant*** with respect to $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$.

The set Ω is usually chosen as a level set of the V if V is a positive definite function.

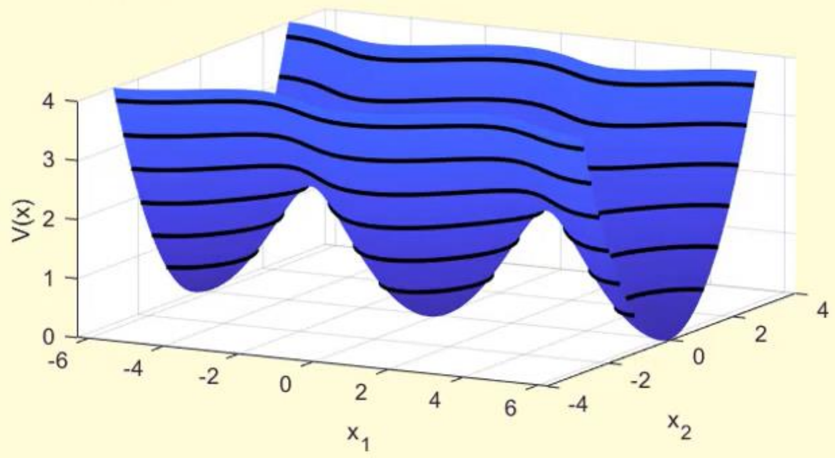
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\sin x_1 - kx_2$$

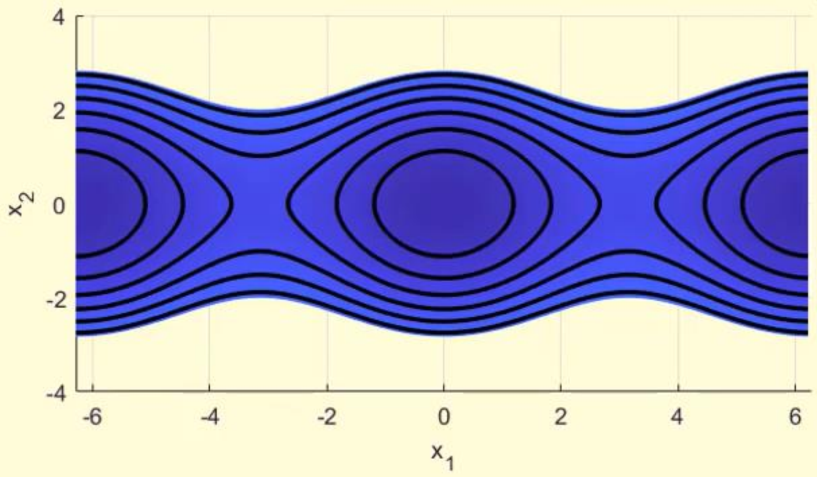
$$V(\boldsymbol{x}) = 1 - \cos x_1 + \frac{1}{2}x_2^2$$

$$\dot{V}(\boldsymbol{x}) = -kx_2^2 \leq 0$$



The level curves:

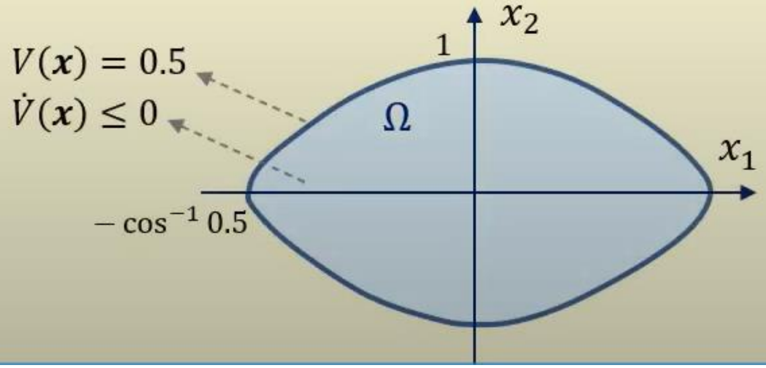




$$\text{ou..}\ \Omega = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T : -\pi < x_1 < \pi, |x_2| < k$$

We choose the set Ω as:

$$\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 : V(\boldsymbol{x}) \leq 0.5\}$$



$\dot{x}_1 = x_2$
$\dot{x}_2 = -\sin x_1 - kx_2$

$$V(\boldsymbol{x}) = 1 - \cos x_1 + \frac{1}{2}x_2^2 \qquad \dot{V}(\boldsymbol{x}) = -kx_2^2 \leq 0$$

$$E := \{x : \dot{V} = 0\}$$

The set $E \subset \Omega$ is the set of all points in Ω such that $\dot{V}(\boldsymbol{x}) = 0$.

$$\dot{V}(\boldsymbol{x}) = 0 \quad \Rightarrow \quad -kx_2^2 = 0 \quad \Rightarrow \quad x_2 = 0$$

$$\boxed{E = \{(x_1, x_2) \in \Omega:\ x_2 = 0\}}$$

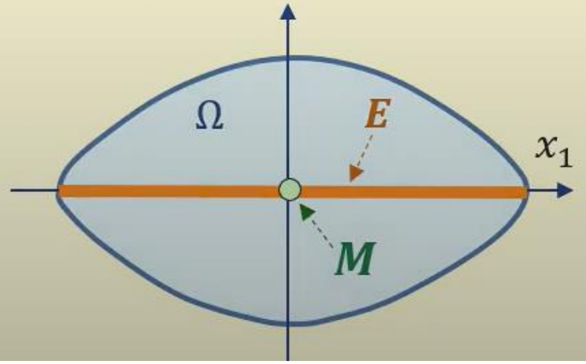
The set M is the *largest positively invariant* set in E.

To find M, we let $x_2 = 0$ for all $t \geq 0$. So $\dot{x}_2(t) = 0$.

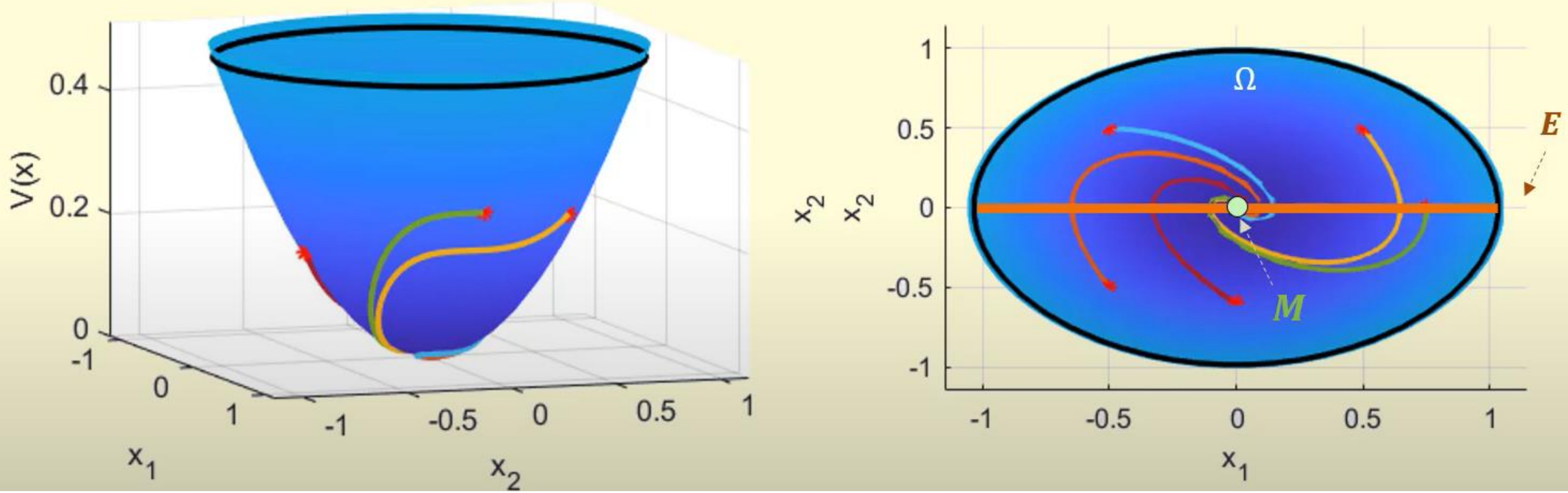
Using the system's dynamical model we have:

$$\dot{x}_2(t) = 0 \quad \Rightarrow \quad -\sin x_1 - kx_2 = 0 \quad \Rightarrow \quad \sin x_1 = 0 \quad \Rightarrow \quad x_1 = 0$$

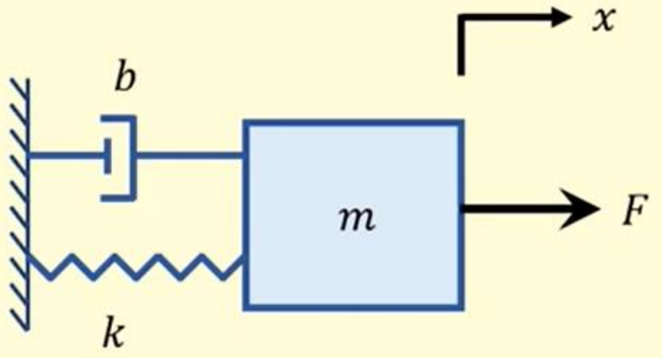
So the set M is: $\boxed{M = \{(x_1, x_2) = (0,0)\}}$



LaSalle's Invariance principle: Every solution starting in Ω approaches M as $t \to \infty$



Example 4: Using LaSalle's Invariance Principle, show the origin is an asymptotic stability equilibrium of the mass-spring-damper system. For simplicity, let $u = 0,\ k = b = m$.



State space model: $\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$

$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$

m:	Mass
k:	Spring constant
b:	Damping constant
x:	Displacement of the mass
$\dot{x} = v$:	Velocity of the mass
F:	Input force

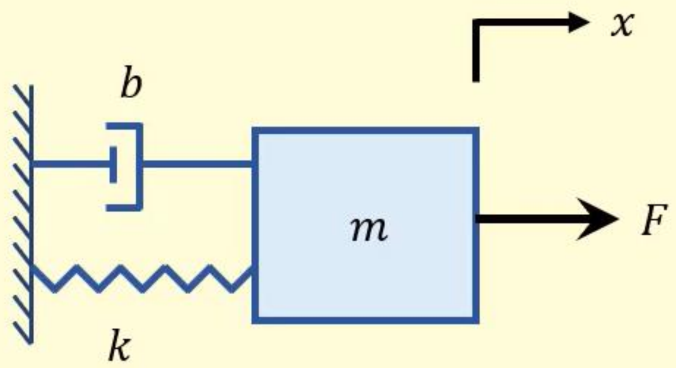
State equations: $\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ -x - v \end{bmatrix}$

Consider $V(x, v) = \frac{1}{2}x^2 + \frac{1}{2}v^2$

$$\dot{V}(x, v) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial v}\dot{v} = (x)(v) + (v)(-x - v) = -v^2 \leq 0$$

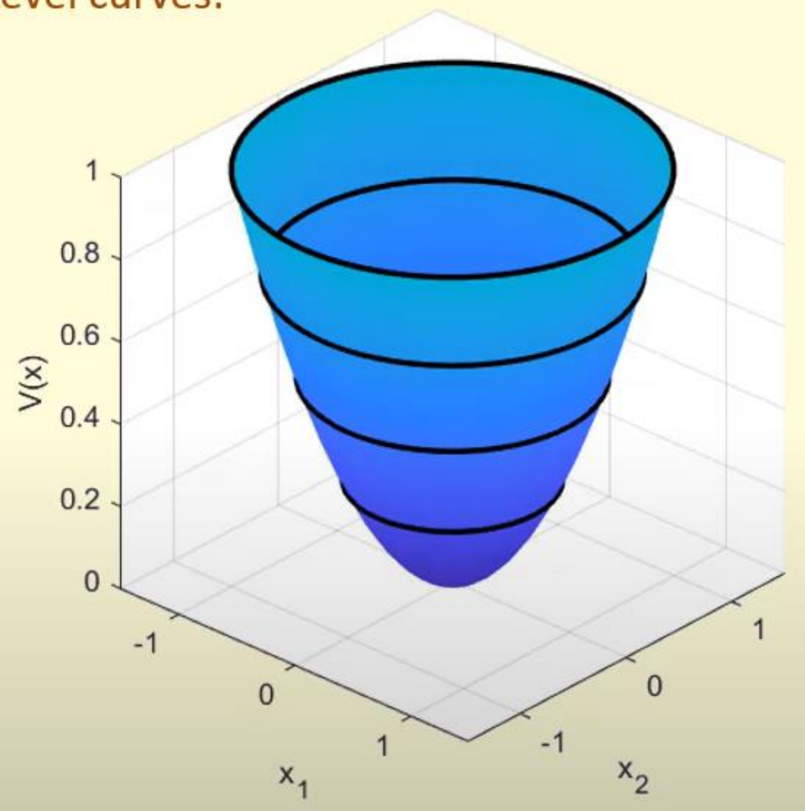
So $\dot{V}(\boldsymbol{x})$ is negative semi-definite.

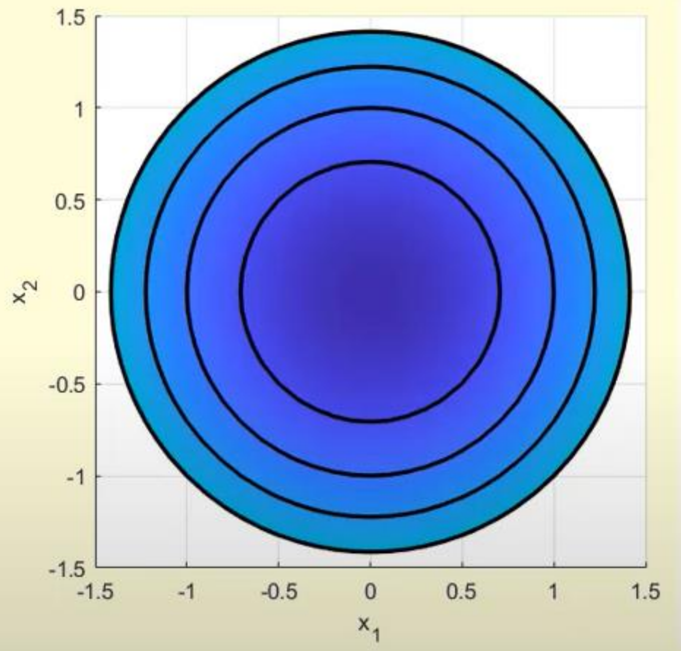
$$\begin{bmatrix}\dot{x}\\ \dot{v}\end{bmatrix} = \begin{bmatrix} v \\ -x - v\end{bmatrix} \qquad V(x,v) = \frac{1}{2}x^2 + \frac{1}{2}v^2 \qquad \dot{V}(x,v) = -v^2 \leq 0$$



The set Ω is a compact set that is ***positively invariant*** with respect to $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$.

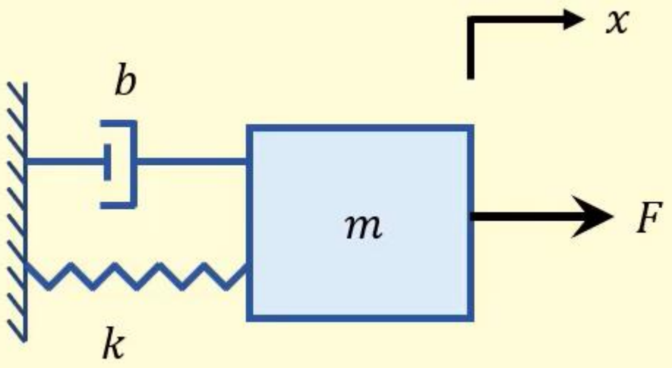
Level curves:





We choose the set Ω as: $\Omega = \{(x, v)\colon\ V(x,v) \leq 1\}$

$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ -x - v \end{bmatrix}$ $\qquad V(x,v) = \frac{1}{2}x^2 + \frac{1}{2}v^2$ $\qquad \dot{V}(x,v) = -v^2 \le 0$



The set $E \subset \Omega$ is the set of all points in Ω such that $\dot{V}(x,v) = 0$.

$\dot{V}(x,v) = 0 \;\Rightarrow\; -kv^2 = 0 \;\Rightarrow\; v = 0$

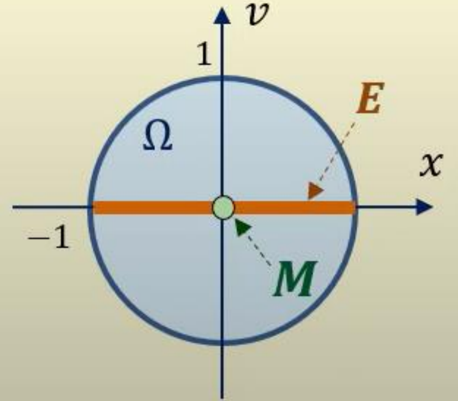
$E = \{(x,v) \in \Omega\colon\ v = 0\}$

The set M is the *largest positively invariant* set in E.

In the set M we have $v = 0$ for all t. So $\dot{v}(t) = 0$. Using the system dynamics we have:

$\dot{v}(t) = 0 \;\Rightarrow\; -x - v = 0 \;\Rightarrow\; x = 0$

So the set M is $\qquad M = \{(x,v) = (0,0)\}$



LaSalle's Invariance Principle: Every solution starting in Ω approaches M as $t \to \infty$

Lyapunov Stability Theorem

$V(x)$ is (continuously differentiable and) **positive definite** on $B_r(0)$

$\dot{V}\,(x)$ is **negative definite** on $B_r(0)$

Discovered by Aleksandr Lyapunov - 1892

LaSalle's Invariance Principle

$V(x)$ is continuously differentiable on Ω
The set Ω is positively invariant with respect to $\dot{x} = f(x)$

$\dot{V}\,(x)$ is **negative semidefinite** on Ω

Discovered independently by

- *Nikolay Krasovsky* – 1959 (An extension of a result in 1952 by *Barbashin* & *Krasovsky)*
- *Joseph LaSalle* – 1960