JOSE ANTONIO PADIAL MOLINA

CA

Teoría

OPERACIONES FRACCIONES ALGEBRAICAS

$$\frac{7}{2}+\frac{5}{6}-\frac{5}{12}=\frac{42}{12}+\frac{10}{12}-\frac{5}{12}=\frac{47}{12}$$

$m.c.m(2,6,12)=12$

$2=2 \qquad 6=2\cdot 3 \qquad 12=2^2\cdot 3$

RACIONALIZACIÓN FRACCIONES ALGEBRAICAS

$$\frac{x}{x-\sqrt{x}} \Rightarrow \frac{x}{(x-\sqrt{x})}\cdot\frac{(x+\sqrt{x})}{(x+\sqrt{x})}=\frac{x\cdot(x+\sqrt{x})}{x^2-x}=\frac{x+\sqrt{x}}{x-1}$$

$$\frac{x}{\sqrt{x+1}-\sqrt{x-1}}=\frac{x(\sqrt{x+1}+\sqrt{x-1})}{(x+1)-(x-1)}=\frac{x(\sqrt{x+1}+\sqrt{x-1})}{2}$$

$\mathbb{R}$

$\left.\begin{array}{l} a\leq b \\ c\in\mathbb{R}\end{array}\right\} a+c\leq b+c$

$\left.\begin{array}{l} a\leq b \\ c\geq 0\end{array}\right\} \Rightarrow a\cdot c\leq b\cdot c$

$\mathbb{R}^+=\{x\in\mathbb{R} : x>0\}$

$\mathbb{R}^-=\{x\in\mathbb{R} : x<0\}$

$\mathbb{R}=\mathbb{R}^-\cup\mathbb{R}^+\cup\{0\}$

$\mathbb{R}^*=\mathbb{R}-\{0\}=\mathbb{R}^+\cup\mathbb{R}^-$

$\mathbb{R}^+_0=\mathbb{R}^+\cup\{0\}=\{x\in\mathbb{R}:x\geq 0\}$

$\mathbb{R}^-_0=\mathbb{R}^-\cup\{0\}=\{x\in\mathbb{R}:x\leq 0\}$

INTERVALOS $\mathbb{R}$

$a, b \in \mathbb{R} \quad a < b \rightarrow [a, b]$

$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

$]a, b[= \{x \in \mathbb{R} : a < x < b\}$

$[a, b[= \{x \in \mathbb{R} : a \leq x < b\}$

$]a, b] = \{x \in \mathbb{R} : a < x \leq b\}$

$[a, +\infty[= \{x \in \mathbb{R} : a \leq x\}$

$]a, +\infty[= \{x \in \mathbb{R} : a < x\}$

$]-\infty, a] = \{x \in \mathbb{R} : a \geq x\}$

$]-\infty, a[= \{x \in \mathbb{R} : a > x\}$

$]-\infty, +\infty[= \mathbb{R}$

INECUACIONES

$2x + 5 = x - 5 \; ; \; 2x - x + 5 = -5 \; ; \; x = -10$

$2x + 5 \geq x - 5 \; ; \; 2x - x + 5 \geq -5 \; ; \; x \geq -10$

-10

$\frac{3x}{5} - \frac{11x}{10} \geq 1 - \frac{6-x}{4} \qquad m.c.m(5, 10, 4) = 20$

$\frac{12x}{20} - \frac{22x}{20} \geq \frac{20}{20} - \frac{30-5x}{20} \; ; \; \frac{-10x}{20} \geq \frac{-10+5x}{20} \; ;$

$-10x \geq -10 + 5x \; ; \; -15x \geq -10 \; ; \; x \leq \frac{10}{15} \; ; \; x \leq 2/3$

VALOR ABSOLUTO $|X|$

$X \in \mathbb{R}$

$$|X| = \begin{cases} X & \text{si } X \geq 0 \\ -X & \text{si } X < 0 \end{cases}$$

$|X| \geq 0 \quad \forall x \in \mathbb{R}$

$|X| = 0 \iff X = 0$

$|X \cdot y| = |X| \cdot |y|$

$\left|\frac{X}{y}\right| = \frac{|X|}{|y|} \quad (\forall y \neq 0)$

$|x+y| \leq |x| + |y|$

$|X| \leq a \iff -a \leq X \leq a$

$$|X| \geq 0 \iff \begin{cases} X \geq a \\ X \leq -a \end{cases}$$

OPERACIONES FUNCIONES

$(f+g)(x) = f(x) + g(x)$

$(f \cdot g)(x) = f(x) \cdot g(x)$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$f(x) = |x^2 - 1| \quad \forall x \geq 1$

$\mathrm{Dom}(f) = [1, +\infty)$

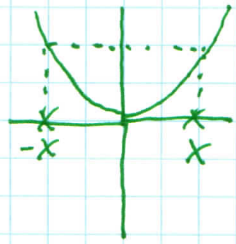
FUNCIONES INYECTIVAS, SOBREYECTIVAS, BIYECTIVAS

$f: A \to \mathbb{R}$, f es inyectiva $\Leftrightarrow [x \neq y \Leftrightarrow f(x) \neq f(y)]$

$A \in \mathbb{R}$

$f: [-1, 1] \to \mathbb{R}$, $f(x) = x^2$ No inyectiva

$f: [0, 1] \to \mathbb{R}$, $f(x) = x^2$ Sí inyectiva



f es sobreyectiva $\Leftrightarrow f(A) = \mathbb{R}$

$f: [-1, 1] \to \mathbb{R}$ $f(x) = x^2$ f no sobreyectiva

$f: [0, 1]$ $f(x) = x^2$ f es sobreyectiva

f es biyectiva $\Leftrightarrow$ f es inyectiva y sobreyectiva

$f: [0, 1]$, $f(x) = x^2$ es sobreyectiva, inyectiva con lo cual es biyectiva.

Función Inversa

$$\left.\begin{array}{l} f: A \longrightarrow f(A) \quad \text{biyectiva} \\ \quad x \longmapsto f(x) \end{array}\right\} \Rightarrow \exists f^{-1}: f(A) \rightarrow A$$

$$(f^{-1} \cdot f) \cdot (x) = x$$

Función Par Impar

f es par $\Leftrightarrow f(-x) = f(x) \quad \forall x \in A$

$f(x) = x^2$

$f(-x) = (-x)^2 = x^2 = f(x)$

f es impar $\Leftrightarrow f(-x) = -f(x) \quad \forall x \in A$

$f(x) = x$; $\quad f(x) = sen(x)$

$sen(x) = -sen(x)$

FUNCIONES ELEMENTALES

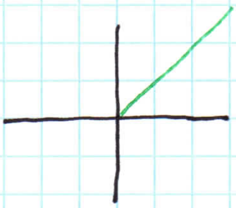
* Funciones Potenciales

$f: \mathbb{R}^+ \to \mathbb{R}$ $\quad f(x) = x^b \quad (b \in \mathbb{R})$

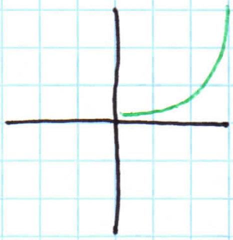
$f(x) = x^b \quad (b>0) \to$ estrictamente creciente

$f(x) = x^b \quad (b<0) \to$ estrictamente decreciente

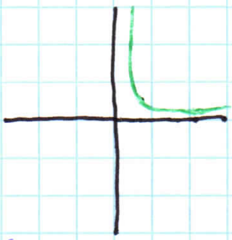
$f(x) = x$



$f(x) = x^2$

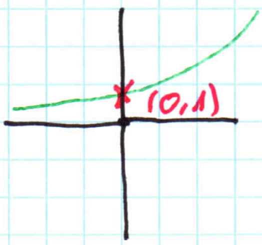


$f(x) = 1/x = x^{-1}$



* Función Exponencial

$f: \mathbb{R} \to \mathbb{R} \quad f(x) = e^x \quad e^x > 0, \ \forall x \in \mathbb{R}$



f es estrictamente creciente;

$f(0) = 1$; $y = 0 \to$ asíntota cuando $x \to -\infty$

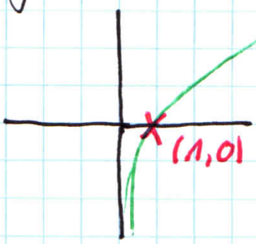
$\lim_{x \to -\infty} e^x = 0 \qquad \lim_{x \to +\infty} e^x = +\infty$

$e^x \cdot e^y = e^{x+y}$

$\frac{e^x}{e^y} = e^{x-y}$

* Función Logaritmo (Neperiano)

$$f: \mathbb{R} \xrightarrow{f(x)=e^x} \mathbb{R}^+ \Rightarrow \exists f^{-1} = \log : \mathbb{R}^+ \to \mathbb{R} \qquad \log(x) \quad (x>0)$$



$x=0 \to$ asíntota vertical

$$\lim_{x\to 0} \log(x) = -\infty$$

$$\lim_{x\to +\infty} \log(x) = +\infty$$

$$\log(e^x) = x \; ; \; e^{\log(x)} = x$$

$$\log(x \cdot y) = \log(x) + \log(y)$$

$$\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$$

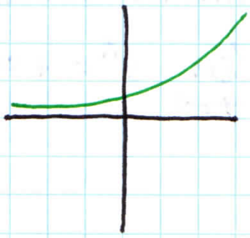
$$\log(x^y) = y \log(x)$$

* Fórmula del número e

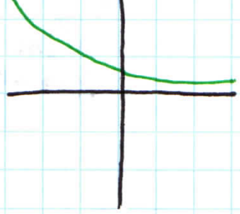
Sean $a \in \mathbb{R}^+$, $b \in \mathbb{R}$

$$a^b = e^{\log(a^b)} = e^{b\log(a)}$$

$$2^x = e^{\log(2)x}$$



$a^x \; (a>1)$



$a^x \; (a<1)$

* Otras funciones logarítmicas

$a \neq 1 \qquad \mathbb{R} \xrightarrow[\text{biyectiva}]{a^x} \mathbb{R}^+ \Rightarrow \exists \log_a : \mathbb{R}^+ \to \mathbb{R}$

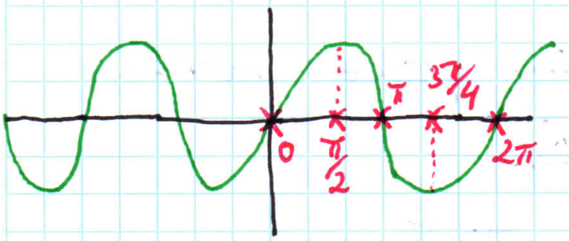
$\log_a(a^x) = \qquad a^{\log_a(x)} = x$

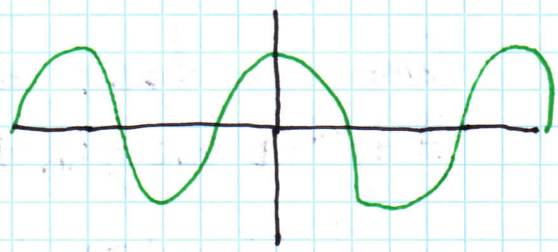
$$\log_a(x) = \frac{\log(x)}{\log(a)} \qquad \forall x > 0;\ \forall a > 0;\ a \neq 1$$

* Funciones Seno y Coseno

$\text{sen}: \mathbb{R} \to \mathbb{R}$ $\qquad\qquad$ $\cos: \mathbb{R} \to \mathbb{R}$

$x \mapsto \text{sen}(x)$ $\qquad\qquad$ $x \mapsto \cos(x)$





$\text{sen}(-x) = -\text{sen}(x) \quad \forall x \quad$ (es impar)

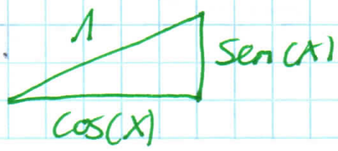
$\cos(-x) = \cos(x) \quad \forall x \quad$ (es par)

$$\left.\begin{array}{l} \text{sen}(x+2\pi) = \text{sen}(x) \quad \forall x \\ \cos(x+2\pi) = \cos(x) \quad \forall x \end{array}\right\} \begin{array}{l} \nexists \lim\limits_{x\to\pm\infty} \text{sen}(x) \\ \nexists \lim\limits_{x\to\pm\infty} \cos(x) \end{array}$$

$|\text{sen}(x)| \leq 1 \Leftrightarrow -1 \leq \text{sen}(x) \leq 1 \quad \forall x \in \mathbb{R}$

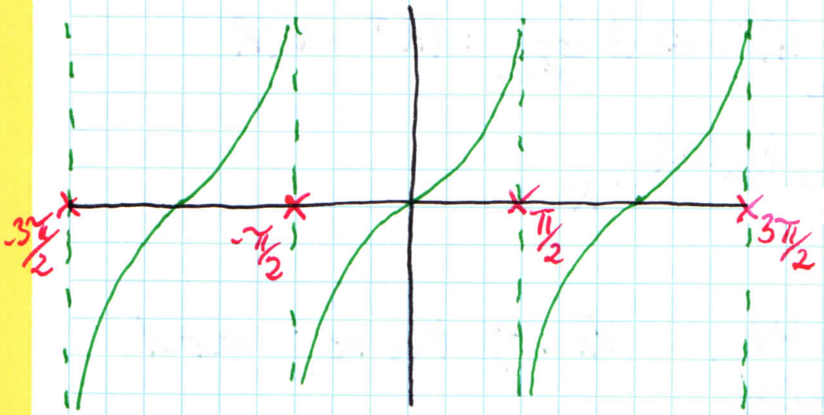
$|\cos(x)| \leq 1 \Leftrightarrow -1 \leq \cos(x) \leq 1 \quad \forall x \in \mathbb{R}$

$\cos^2(x) + \text{sen}^2(x) = 1$



* Función tangente

$$tg(x) = \frac{sen(x)}{cos(x)}, \forall x \in \mathbb{R} \setminus \{\tfrac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$$



$$tg(x+\pi) = tg(x) \quad \forall x \in \mathbb{R} \setminus \{\tfrac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$$

$$\lim_{x \to \frac{\pi}{2}^-} tg(x) = +\infty \qquad \lim_{x \to \frac{\pi}{2}^+} tg(x) = -\infty$$

* Función arcotangente

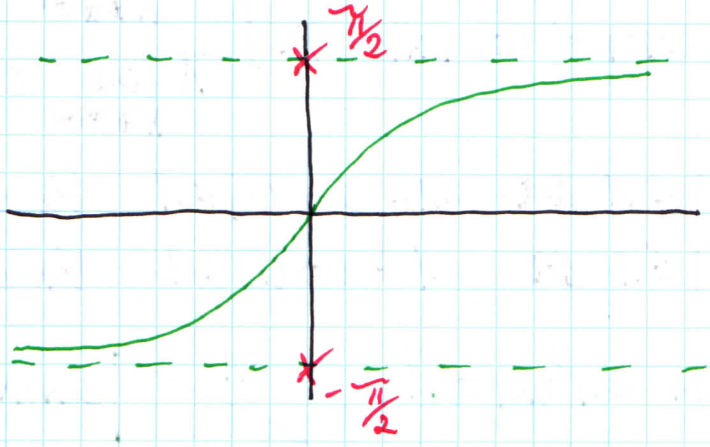
Como la $tg(x)$ es biyectiva posee inversa

$\exists\, tg^{-1} \to arctg : \mathbb{R} \to]-\tfrac{\pi}{2}, \tfrac{\pi}{2}[$

$arctg(tg(x)) = x \quad \forall x \in]-\tfrac{\pi}{2}, \tfrac{\pi}{2}[$

$tg(arctg(x)) = x \quad \forall x \in \mathbb{R}$

$$\lim_{x \to +\infty} arctg(x) = \tfrac{\pi}{2} \qquad \lim_{x \to -\infty} arctg(x) = -\tfrac{\pi}{2}$$



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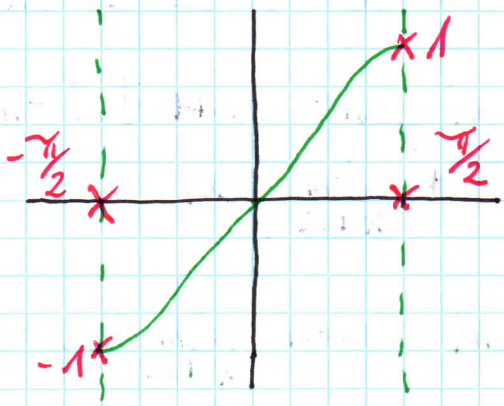
* Función arcoseno

La función seno es biyectiva, con lo cual posee inversa

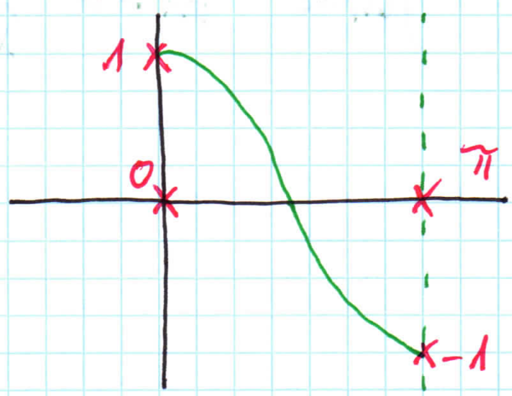
$\exists\ sen^{-1}(x) = arcsen : [-1, 1] \to [-\pi/2, \pi/2]$



* Función arcocoseno

La función coseno es biyectiva, con lo cual posee inversa

$\exists\ cos^{-1} = arccos : [-1, 1] \to [0, \pi]$



LÍMITE FUNCIONAL Y CONTINUIDAD

$f: I \to \mathbb{R}$ $\quad a \in I$

f es continua en $a \Leftrightarrow \exists \lim_{x \to a} f(x) = f(a)$

* Definición, límite de una función en un punto

Sea $f: I \to \mathbb{R}$ $\quad a \in I$ ó a es un extremo de I

f tiene límite en el punto $a \Leftrightarrow \exists L \in \mathbb{R}$ t.q.: $x \to a \Rightarrow$

$\Rightarrow f(x) \to L \Rightarrow \lim_{x \to a} f(x)$

* $f(x) = 5 \quad \forall x \in \mathbb{R} \quad a = 1 \quad \lim_{x \to a} f(x) = 5$
* $f(x) = x \quad a \in \mathbb{R} \quad \lim_{x \to a} f(x) = \lim_{x \to a} x = a$
* $f: \mathbb{R} \to \mathbb{R} \quad f(x) \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} 0 = 0$$

* $f: \mathbb{R}^* \to \mathbb{R} \quad f(x) = 0 \quad \forall x \neq 0 \quad \lim_{x \to 0} f(x) = 0$

* Álgebra de límites

Sean $f, g: I \to \mathbb{R}$, $a \in I$ (ó a es un extremo)
$\exists \lim_{x \to a} f(x)$ y $\exists \lim_{x \to a} g(x)$

* $\lim_{x \to a} (f+g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
* $\lim_{x \to a} (f \cdot g)(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
* $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ } Si $\lim_{x \to a} g(x) \neq 0$
* $\left.\begin{array}{l} \lim_{x \to 0} f(x) = 0 \\ g \text{ es acotada} \end{array}\right\} \Rightarrow \lim_{x \to a} (f \cdot g)(x) = 0$ "0 · acotada = 0"

* $\lim_{x \to 0} (x^2 + 3x - 5) = 0 \cdot 0 + 0 \cdot 3 - 5 = -5$
* $\lim_{x \to 1} (3x^3 - 2x + 7) = 8$
* $\lim_{x \to -1} \frac{x^2 - 1}{x - 1} = \frac{0}{-2} = 0$

* Límites Laterales

$f: I \to \mathbb{R}$

$$\lim_{x \to a^+} f(x) = \lim_{\substack{x \to a \\ (x > a)}} f(x)$$

$$\lim_{x \to a^-} f(x) = \lim_{\substack{x \to a \\ (x < a)}} f(x)$$

$\exists \lim_{x \to a} f(x) = l \iff \exists \lim_{x \to a^+} f(x)$ y $\exists \lim_{x \to a^-} f(x)$ y coinciden con l

* $\lim_{x \to 0} |x|$

$$\left.\begin{array}{l} \lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0 \\ \lim_{x \to 0^-} = \lim_{x \to 0^-} -x = 0 \end{array}\right\} \exists \lim_{x \to 0} |x| = 0$$

* $f: \mathbb{R}^* \to \mathbb{R}$ $\quad f(x) = \dfrac{|x|}{x} \quad \forall x \in \mathbb{R}^*$ ¿$\exists \lim_{x \to 0} f(x)$?

$$f(x) \begin{cases} \dfrac{x}{x} = 1 & x > 0 \\ -\dfrac{x}{x} = -1 & x < 0 \end{cases}$$

$$\left.\begin{array}{l} \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1 \\ \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -1 = -1 \end{array}\right\} \nexists \lim_{x \to 0} f(x)$$

* Ejemplos

* $\lim_{x\to 1} |x^2 - x + 5| = |5| = 5$

* $\lim_{x\to 0} e^{x^2+2x-1} = e^{-1} = 1/e$

* $\lim_{x\to 0} \cos(\pi + x) = \cos(\pi) = -1$

* $\lim_{x\to 0} \operatorname{arctg}(x^2+1) = \operatorname{arctg}(1) = \pi/4$

* $f : \mathbb{R} \to \mathbb{R}$

$$f(x) \begin{cases} x^2+1 & x \leq 0 \\ e^{x^2} & 0 < x < 1 \\ \cos(x+1) & x \geq 1 \end{cases}$$

$\boxed{a=0}$

f continua en $0 \overset{?}{\Leftrightarrow} \exists \lim_{x\to 0} f(x) = f(0) = 1$

$$\left.\begin{array}{l} \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{x^2} = 1 \\ \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x^2+1) = 1 \end{array}\right\} \exists \lim_{x\to 0} f(x) = 1$$

$\boxed{a=1}$

f continua en $1 \overset{?}{\Leftrightarrow} \exists \lim_{x\to 1} f(x) = f(1) = \cos(2)$

$$\left.\begin{array}{l} \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \cos(x+1) = \cos(2) \\ \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} e^{x^2} = e \end{array}\right\} \nexists \lim_{x\to 1} f(x)$$

* $f: \mathbb{R}^* \to \mathbb{R}$

$f(x) = 1/x$ ¿f es continua en 0? $\nexists f(0)$

* $f: \mathbb{R} \to \mathbb{R}$

$$f(x) \begin{cases} 1/x & x \neq 0 \\ 18 & x = 0 \end{cases}$$

* Límites infinitos

$$\lim_{\substack{x \to a \\ (x \neq a)}} f(x) = +\infty \qquad \lim_{x \to a} f(x) = -\infty$$

* Límites en infinito

$$\lim_{x \to +\infty} f(x) = L \qquad \lim_{x \to -\infty} f(x) = L$$

* Límites infinitos en infinito

$$\lim_{x \to +\infty} f(x) = +\infty \qquad \lim_{x \to +\infty} f(x) = -\infty$$

$$\lim_{x \to -\infty} f(x) = +\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

$$\lim_{x \to +\infty} e^x = +\infty \qquad \lim_{x \to +\infty} \operatorname{arctg}(x) = \pi/2$$

$$\lim_{x \to 0} \log(x) = -\infty$$

* Propiedad

$$f(x)^{g(x)} = e^{g(x)\cdot \log(f(x))}$$

$$\lim_{x\to a} f(x) = 1 \qquad \lim_{x\to a} g(x) = \infty$$

$$\lim_{x\to\infty} x^x = +\infty$$

$$x^x = e^{x\log(x)} = e^{+\infty} = +\infty$$

* Escala de infinitos

$$\log(x) < x^b < a^x < x^x$$

* Aplicaciones de la escala

$$\lim_{x\to+\infty} \frac{\log(x)}{x} = 0 \qquad \lim \frac{2^x}{x^2} = +\infty$$

$$\lim_{x\to+\infty} \frac{\sqrt{x}}{x^x} = 0$$

$$\lim_{x\to+\infty} (e^x - \sqrt{x}) = \infty - \infty = +\infty$$

$$e^x\left[1 - \frac{\sqrt{x}}{e^x}\right] = e^x[1-0] = e^x = +\infty$$

$$\lim_{x\to 0} x^x = 1$$

$$x^x = e^{x\log(x)} = e^{0\cdot\infty}$$

$$x^x = \left\{\begin{matrix} y = 1/x \Leftrightarrow x = 1/y \\ x\to 0 \Rightarrow y\to+\infty \end{matrix}\right\} (1/y)^{(1/y)} = \frac{1^{1/y}}{y^{1/y}} = 1/1 = 1$$

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$$\lim_{x\to+\infty} y^{1/y} = \infty^0 = 1$$

$$y^{1/y} = e^{\frac{1}{y}\log(y)} = e^{0\cdot\infty} = e^{\frac{\log(y)}{y}} = e^0 = 1$$

$$\lim_{x\to 1} (x-1)^{x-1} = 0^0 = 1$$

$$\lim_{x\to 0} \operatorname{sen}(x)^{\operatorname{sen}(x)} = 0^0 = 1$$

$$\lim_{x\to+\infty} \log(x)^{\frac{1}{\log(x)}} = 1$$

$$\lim_{x\to+\infty} (x^2+1)^x = e^{x\cdot\log(x^2+1)} = e^{\infty\cdot\infty} = +\infty$$

$$\left.\begin{array}{l} \text{“}\infty^0\text{”} \\ \text{“}0^0\text{”} \end{array}\right\} f(x)^{g(x)} = e^{g(x)\cdot\log(f(x))}$$

* Regla del número "e" (solo par "1^∞")

$\lim_{x\to a} f(x)^{g(x)}$, donde $\lim_{x\to a} f(x) = 1$, $\lim_{x\to a} g(x) = \infty$

* $\lim_{x\to a} g(x)\cdot[f(x)-1] = L \iff \lim_{x\to a} f(x)^{g(x)} = e^L$

* $\lim_{x\to a} g(x)\cdot[f(x)-1] = +\infty \iff \lim_{x\to a} f(x)^{g(x)} = +\infty$

* $\lim_{x\to a} g(x)\cdot[f(x)-1] = -\infty \iff \lim_{x\to a} f(x)^{g(x)} = 0$

$$\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^x = e^1 = e$$

- $f(x) = (1+1/x)$; $g(x) = x$

$$g(x)[f(x)-1] = x\left[1+\frac{1}{x}-1\right] = \frac{x}{x} = 1 \to L = 1$$

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TEOREMAS SOBRE F CONTINUAS

* Teorema de los ceros de Bolzano

Sea $f:[a,b] \to \mathbb{R}$ continua
$f(a)\cdot f(b) < 0$

Versión general del teorema de Bolzano

$$\left.\begin{array}{l} f:]a,b[\to \mathbb{R} \text{ continua} \\ \lim_{x\to a} f(x) \; \lim_{x\to b} f(x) < 0 \end{array}\right\} \Rightarrow \exists\, c \in]a,b[,\ f(c)=0$$

En particular

$$\left.\begin{array}{l} \lim_{x\to a} f(x) = -\infty \text{ y } \lim_{x\to b} f(x) = +\infty \\ \lim_{x\to a} f(x) = +\infty \text{ y } \lim_{x\to b} f(x) = -\infty \end{array}\right\} \exists\, c \in]a,b[,\ f(c)=0$$

JOSE ANTONIO PADIAL MOLINA

* Teorema del Valor Intermedio

$f: I \to \mathbb{R}$ continua $\Rightarrow f(I)$ es un intervalo.

$f:]0,1[\to \mathbb{R}$ $\quad f(x) = \dfrac{2x-1}{x(x-1)}$

* ¿f continua en $]0,1[$? Sí con denominador $\neq 0$

* $\lim_{x \to 0} f(x) = \lim_{x \to 0} \dfrac{2x-1}{x(x-1)} = \dfrac{-1}{0} = +\infty$

* $\lim_{x \to 1} f(x) = \lim_{x \to 1} \dfrac{2x-1}{x(x-1)} = \dfrac{1}{0} = -\infty$

* $f(]0,1[) = (\text{intervalo}) = \mathbb{R}$

* Teorema de Weierstrass (Propiedad de Compacidad)

$f[a,b] \to \mathbb{R}$ continua $\Rightarrow f([a,b])$ otro intervalo compacto.

$f([a,b]) = [m, M]$

mínimo absoluto $\quad$ máximo absoluto

$f: [a,b] \to \mathbb{R} \Rightarrow \exists\, x_0, y_0 \in [a,b]$ t.q $f([a,b]) = [f(x_0), f(y_0)]$

CALCULO DE UNA IMAGEN CONTINUA Y MONÓTONA DEFINIDA INTERVALO

1.- $f:[a,b]\to\mathbb{R}$ continua y $\nearrow \Rightarrow f([a,b])=[f(a),f(b)]$

2.- $f:[a,b]\to\mathbb{R}$ continua y $\searrow \Rightarrow f([a,b])=[f(b),f(a)]$

3.- $f:[a,b[\to\mathbb{R}$ continua y $\nearrow \Rightarrow f([a,b[)=[f(a),\lim_{x\to b} f(x)[$

4.- $f:[a,b[\to\mathbb{R}$ continua y $\searrow \Rightarrow f([a,b[)=]\lim_{x\to b} f(x), f(a)]$

5.- $f:]a,b[\to\mathbb{R}$ continua y $\nearrow \Rightarrow f(]a,b[)=]\lim_{x\to a} f(x), \lim_{x\to b} f(x)[$

6.- $f:]a,b[\to\mathbb{R}$ continua y $\searrow \Rightarrow f(]a,b[)=]\lim_{x\to b} f(x), \lim_{x\to a} f(x)[$

DERIVACIÓN

Sea $f: I\to\mathbb{R}$; $a\in I$

Definición

f es derivable en $a \Leftrightarrow \exists \lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a)\in\mathbb{R}$

f es derivable en $I \Leftrightarrow \exists f'(a)\ \forall a\in I$

$f': I\to\mathbb{R}$
$x\to f'(x)$

* $f(x)=5\quad \forall x\in\mathbb{R} \Rightarrow \exists f'(a)=0 \quad \forall a\in\mathbb{R}$

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{5-5}{x-a}=\lim_{x\to a}\frac{0}{x-a}=\lim_{x\to a} 0=0$$

* $f(x)=x\quad \forall x\in\mathbb{R} \Rightarrow \exists f'(a)=1 \quad \forall a\in\mathbb{R}$

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{x-a}{x-a}=1$$

* $f(x) = x^2 \ \forall x \in \mathbb{R} \Rightarrow \exists f'(a) = 2x \ \forall a \in \mathbb{R}$

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a} \frac{x^2-a^2}{x-a} = \lim_{x \to a} \frac{(x+a)(x-a)}{x-a} = 2a$$

* $f(x) = |x|$ ¿$\exists f'(0)$?

$$\lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0} \frac{|x|}{x} \begin{cases} \lim_{x \to 0^+} \frac{x}{x} = 1 \\ \lim_{x \to 0^-} \frac{x}{x} = -1 \end{cases} \Rightarrow \nexists f'(0)$$

Derivada Lateral

$$\exists f'(a^+) = \lim_{x \to a^+} \frac{f(x)-f(a)}{x-a}$$

$$\exists f'(a^-) = \lim_{x \to a^-} \frac{f(x)-f(a)}{x-a}$$

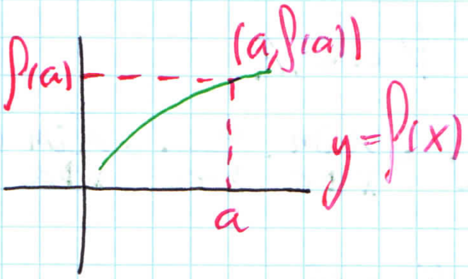
$\exists f'(a) \Leftrightarrow \exists f'(a^+)$ y $\exists f'(a^-)$ y coincidan

Condición Necesaria De Derivabilidad

$\exists f'(a) \Rightarrow f$ tiene que ser continua en a.

~~1.- $f: \mathbb{Z} \to \mathbb{R}$, $a \in \mathbb{Z}$, Estudiar la deriva~~

Recta Tangente



La ecuación de la recta tangente de una función en un punto "a" es la siguiente

$$y = f(a) + f'(a) \cdot (x-a)$$

La pendiente es el valor que acompaña a la x, solo grado 1.

$f'(a) \Rightarrow$ es la pendiente de la recta tangente.

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REGLAS DE CÁLCULO DE DERIVADAS

$(f+g)'(x) = f'(x) + g'(x)$

$(f\cdot g)'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

$(f/g)'(x) = \dfrac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$ "$g(x)\neq 0$"

$(f\circ g)'(x) = f'(g(x))\cdot g'(x)$ "Regla de la cadena"

DERIVADAS DE LAS F ELEMENTALES

* $f(x) = x^b \Rightarrow f'(x) = bx^{b-1}$

* $f(x) = 4x^3 + 2x + 8$ $\quad f'(x) = 12x^2 + 2$
* $f(x) = \dfrac{x^3 - 2x + 8}{x^2+1}$ $\quad f'(x) = \dfrac{(3x-2)\cdot(x^2+1) - (x^3-2x+8)\cdot(2x)}{(x^2+1)^2}$
* $f(x) = (2x^2+8x)^{1000}$ $\quad f'(x) = 1000.(2x^2+8x)^{999}\cdot(4x+8)$
* $f(x) = 1/x = x^{-1}$ $\quad f'(x) = (-1)\cdot x^{-2} = -1/x^2$
* $f(x) = \sqrt{x} = x^{1/2}$ $\quad f'(x) = 1/2 \cdot x^{-1/2} = \dfrac{1}{2\sqrt{x}}$
* $f(x) = \sqrt{\dfrac{x^2+1}{x}}$ $\quad f'(x) = \dfrac{1}{2\sqrt{\dfrac{x^2+1}{x}}} \cdot \dfrac{2xx - (x^2+1)}{x^2} =$

$$= \frac{x^2-1}{2\sqrt{x^3}(x^2+1)} = \frac{x^2-1}{2x\sqrt{x}(x^2+1)}$$

* $f(x) = x/7 = 1/7\, x$ $\quad f'(x) = 1/7$

* $f(x) = e^x \Rightarrow f'(x) = e^x$ // $f(x) = e^{g(x)} \Rightarrow f'(x) = e^{g(x)} \cdot g'(x)$

* $f(x) = e^{1/x}$. $f'(x) = e^{1/x} \cdot (-1/x^2)$

* $f(x) = a^x \Rightarrow f'(x) = a^x \cdot \log(a)$

* $f(x) = 2^x$ $f'(x) = \log(2) \cdot 2^x$

* $f(x) = 5^{\sqrt{x}}$ $f'(x) = \log(5) \cdot 5^{\sqrt{x}} \cdot \dfrac{1}{2\sqrt{x}}$

* $f(x) = \log(x) \Rightarrow f'(x) = 1/x$ // $f(x) = \log(g(x)) \Rightarrow f'(x) = \dfrac{g'(x)}{g(x)}$

* $f(x) = \log(x^2+1)$ $f'(x) = \dfrac{2x}{x^2+1}$

* $f(x) = \log(x^2+e^{x^2})$ $f'(x) = \dfrac{2x + 2x \cdot e^{x^2}}{x^2+e^{x^2}}$

* $f(x) = \mathrm{sen}(x) \Rightarrow f'(x) = \cos(x)$

$f(x) = \cos(x) \Rightarrow f'(x) = -\mathrm{sen}(x)$

* $f(x) = \mathrm{sen}(e^{1/x} + 5x)$ $f'(x) = \cos(e^{1/x}+5x) \cdot (5 + e^{1/x} \cdot (-1/x^2))$

* $f(x) = \cos(\mathrm{sen}(x^3))$ $f'(x) = -\mathrm{sen}(\mathrm{sen}(x^3)) \cdot \cos(x^3) \cdot 3x^2$

* $f(x) = \mathrm{tg}(x) = \dfrac{\mathrm{sen}(x)}{\cos(x)} \Rightarrow f'(x) = \dfrac{1}{\cos^2(x)} = \sec^2(x) = 1 + \mathrm{tg}^2(x)$

$f(x) = \mathrm{cotg}(x) = \dfrac{\cos(x)}{\mathrm{sen}(x)} \Rightarrow f'(x) = -\dfrac{1}{\mathrm{sen}^2(x)} = -\mathrm{cosec}^2(x) =$

$= -[1 + \mathrm{tg}^2(x)]$

* $f(x) = \mathrm{arcsen}(x) \Rightarrow f'(x) = \dfrac{1}{\sqrt{1-x^2}}$ $\forall x \in \,]-1,1[$

$f(x) = \arccos(x) \Rightarrow f'(x) = \dfrac{-1}{\sqrt{1-x^2}}$ $\forall x \in \,]-1,1[$

$f(x) = \mathrm{arc\,tg}(x) \Rightarrow f'(x) = \dfrac{1}{1+x^2}$ $\forall x \in \mathbb{R}$

* $f(x) = \mathrm{arctg}(\log(x))$ $f'(x) = \dfrac{1}{1+(\log(x))^2} \cdot 1/x$

REGLAS DE L'HÔPITAL

* "$\frac{0}{0}$"

$$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x) \quad \exists f'(x), g'(x)$$

$$\exists \lim_{x\to a} \frac{f'(x)}{g'(x)} = L \Rightarrow \exists \lim_{x\to a} \frac{f(x)}{g(x)} = L$$

* "$\frac{\infty}{\infty}$", "$\frac{?}{\infty}$"

$$\exists \lim_{x\to a} \frac{f'(x)}{g'(x)} = \begin{matrix} L \\ +\infty \\ -\infty \end{matrix} \Rightarrow \exists \lim_{x\to a} \frac{f(x)}{g(x)} = \begin{matrix} L \\ +\infty \\ -\infty \end{matrix}$$

CONSECUENCIAS T.V.M.

$f: I \to \mathbb{R} \quad \exists f'(x) \quad \forall x \in I$

* f es creciente $\Leftrightarrow f'(x) \geq 0, \forall x \in I$
* f es decreciente $\Leftrightarrow f'(x) \leq 0, \forall x \in I$
* f es constante $\Leftrightarrow f'(x) = 0, \forall x \in I$
* $f'(x) \neq 0 \quad \forall x \in I \Rightarrow \begin{cases} f'(x) > 0 \Rightarrow f \text{ es estrictamente creciente} \\ f'(x) < 0 \Rightarrow f \text{ es estrictamente decreciente} \end{cases}$

Condición Suficiente Extremo Relativo

Sea $f: I \to \mathbb{R}$, $a \in \mathring{I}$ $\exists f'(x), \forall x \in I$, $f'(a) = 0$
$\exists\, \delta > 0$, $]a-\delta, a+\delta[\ni a$

* $x \in \,]a-\delta, a+\delta[$ $\left\{ \begin{array}{l} x < a \Rightarrow f'(x) \geq 0 \\ x > a \Rightarrow f'(x) \leq 0 \end{array} \right\} \Rightarrow f$ alcanza un máximo relativo

* $x \in \,]a-\delta, a+\delta[$ $\left\{ \begin{array}{l} x < a \Rightarrow f'(x) \leq 0 \\ x > a \Rightarrow f'(x) \geq 0 \end{array} \right\} \Rightarrow f$ alcanza un mínimo relativo

* Si $f'(x)$ no cambia de signo en torno a "a" f no alcanza en "a" un extremo relativo.

Candidatos A Extremo Relativo

* Si f derivable en I, los puntos críticos
* $\nexists f'(a)$ puntos de no derivabilidad.

Aplicación Cálculo Diferencial

* Cálculo de límites (con L'Hôpital)
* Cálculo de imagen de funciones
* Comprobar desigualdades entre funciones
* Determinar el número de soluciones de una ecuación (ó el número de "0" de una función)
* Optimización

TAYLOR

función de orden "n" en el $\underbrace{\text{punto "a"}}_{\text{centro de desarrollo de Taylor}}$

$f: I \to \mathbb{R}$, $a \in I$, $n \in \mathbb{N}$, $\exists f^n(a)$

$$P_n(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^n(a)}{n!}(x-a)^n$$

$$\left.\begin{matrix} f(x)=e^x \\ a=0 \\ n \in \mathbb{N} \end{matrix}\right\} P_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n$$

$$e = f'(x) = f''(x) = f'''(x) = \dots = f^n(x) \Rightarrow$$

$$\Rightarrow f'(0) = f''(0) = f'''(0) = \dots = f^n(0) = e^0 = 1$$

CONDICIÓN SUFICIENTE DE EXTREMO RELATIVO

$f: I \to \mathbb{R}$, $a \in \overset{\circ}{I}$, $n \in \mathbb{N}$, $n \geq 2$ $\exists f^n(a)$

$0 = f'(a) = f''(a) = f'''(a) = \dots = f^n(a)$ y $f^n(a) \neq 0$

Entonces: $n \to$ par $\to$ extremo $\to f^n(a) > 0 \to$ Mínimo relativo
$\to f^n(a) < 0 \to$ Máximo relativo

$n \to$ impar $\to$ no hay extremo.

Caso particular (n=2) (Test de f'')

$f'(a)=0$ y $f''(a)\neq 0$

* $f''(a)>0 \Rightarrow$ Mínimo relativo.
* $f''(a)<0 \Rightarrow$ Máximo relativo.

RESTO DE TAYLOR DE F

$f(x) \sim P_n(x)$

valor exacto — valor aproximado

$$\text{error} = |f(x)-P_n(x)| = |R_n(x)|$$

$$f(x)-P_n(x) = \text{resto Taylor} = R_n(x)$$

Sea $f: I \to \mathbb{R}$, $a\in I$, $n\in\mathbb{N}$ $\exists f^{n+1}(x)$ $\forall x\in I$

Sea P_n (polinomio de Taylor de f en a, orden n)

Entonces: Si $x\in I$, entonces $\exists\, c \in\,]a,x[$ ó $]x,a[$

$x=a$

$$f(x) = P_n(x) + \boxed{\frac{f^{n+1}(c)}{n+1!}(x-a)^{n+1}} \Rightarrow R_n(X)$$

INTEGRACIÓN

$\int f(x)\,dx$

$f:[a,b]\to\mathbb{R}$ y $F:[a,b]\to\mathbb{R}$

F es una primitiva de $f \overset{def}{\Leftrightarrow} \exists\, F'(x)=f(x),\ \forall x$

$f(x)=0 \Rightarrow F(x)=C$

$f(x)=x \Rightarrow F(x)=\frac{x^2}{2}+C \Rightarrow F'(x)=\frac{2x}{2}=x$

$f(x)=x^m,\ m\neq 1 \Rightarrow F(x)=\frac{x^{m+1}}{m+1}+C$

$f(x)=\frac{1}{x} \Rightarrow F(x)=\log|x|+C$

$f:[a,b]\to\mathbb{R}$ y continua y F es una primitiva de f

$$\int_a^b f(x)\,dx \overset{def}{\Rightarrow} F(b)-F(a)$$

$$\int f(x)\,dx = \text{una primitiva de } f$$

$f(x)=0 \Rightarrow \int_0^1 0\,dx = C-C=0$

$f(x)=x \Rightarrow \int_1^2 x\,dx = F(2)-F(1) = \frac{4}{2}+C-\left(\frac{1}{2}+C\right)=\frac{4}{2}-\frac{1}{2}=\frac{3}{2}$

$f(x)=\frac{1}{x} \Rightarrow \int_1^2 \frac{1}{x}\,dx = F(2)-F(1)=\log(2)-\log(1)=\log|x|\Big]_1^2$

$\Downarrow$

$F(x)=\log(|x|)$ como va del 1 al 2, siempre positiva se anula el valor absoluto.

$f:]a,b[\to \mathbb{R}$ continua y F primitiva de f

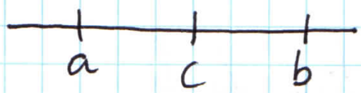
$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = \lim_{x \to b} (F(x)) - \lim_{x \to a} (F(x))$$

$$\int_0^{+\infty} \frac{1}{1+x^2}\,dx = \operatorname{arctg}(x)\Big]_0^{+\infty} = \lim_{x \to +\infty} \operatorname{arctg}(x) - \operatorname{arctg}(0) =$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

PROPIEDADES DE LAS INTEGRALES

* $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
* $\int_a^b \alpha \cdot f(x)\,dx = \alpha \cdot \int_a^b f(x)\,dx$
* $f(x) \geq g(x), \ \forall x \in [a,b] \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
* Propiedad de aditividad.



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

TEOREMA FUNDAMENTAL DEL CÁLCULO

Sea $f: I \to \mathbb{R}$, f continua en I y $a \in I$. Se puede definir:

$$F: I \to \mathbb{R}$$
$$x \to F(x) = \int_a^x f(t)\,dt$$

Tesis: $\exists F'(x) = f(x) \quad \forall x \in I \quad \left(\int_a^x f(t) \cdot dt\right)' = f(x)$

CÁLCULO DE PRIMITIVAS

0.- Primitivas inmediatas

1.- Cambio de variable o sustitución

$$\int f(x)\,dx = \begin{bmatrix} x = g(t) \\ dx = g'(t)\,dt \end{bmatrix} = \int f(g(t))\cdot g'(t)\,dt$$

2.- Integración por partes

$$\int u\,dv = uv - \int v\,du \qquad (uv)' = u\,dv + v\,du$$

Arcos
Logaritmos
Potenciales
Exponenciales
Senos

3.- Integración de $\int$ racionales

$$\int \frac{P(x)}{Q(x)}\,dx, \quad gr(P) < gr(Q)$$

Si $\int \frac{P(x)}{Q(x)}\,dx$, con $gr(P) \geq gr(Q)$,

$$\int \frac{P(x)}{Q(x)}\,dx = \int C(x)\,dx + \int \frac{R(x)}{Q(x)}\,dx \qquad gr(R) < gr(Q)$$

3.1.- $\int \frac{Mx+N}{x^2+bx+c}\,dx$, con $x^2+bx+c \neq 0 \quad \forall x$

$$x^2+bx+c = (x+a)^2 + K = [t = x+a] = t^2 + K$$

3.2.- $\int \frac{P(x)}{Q(x)}$ $(gr(P) < gr(Q))$

(Q(x) tiene raíces reales o/y imaginarias)

$$Q(x) = (x-a_1)\cdot(x-a_2)\cdots(x-a_n)\cdot(x^2+bx+c)$$

3.3.- $\int \frac{P(x)}{Q(x)}\,dx$, con Q(x) con raíces múltiples

$$Q(x) = (x-a)^n\cdot(x-b)^m$$

Si $n=2$, $m=3$

$$\frac{P(x)}{Q(x)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b} + \frac{D}{(x-b)^2} + \frac{E}{(x-b)^3}$$

3.4.- $\int \frac{P(x)}{Q(x)}\,dx$ con Q(x) con raíces reales y/o imaginarias múltiples

$$Q(x) = (x-a)^n (x^2+\alpha x+\beta)^m$$

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \cdots + \frac{A_n}{(x-a)^n} + \frac{M_1x+N_1}{x^2+\alpha+\beta} + \frac{M_2x+N_2}{(x^2+\alpha+\beta)^2} + \cdots + \frac{M_m+N_m}{(x^2+\alpha+\beta)^m}$$

4.- Integración de funciones trigonométricas

$$\int \operatorname{sen}(x)\,dx = -\cos(x) + c \qquad \int \cos(x)\,dx = \operatorname{sen}(x) + c$$

$$\int \operatorname{sen}(2x)\,dx = \frac{-\cos(2x)}{2} \qquad \int \operatorname{sen}(ax)\,dx = \frac{-\cos(ax)}{a}$$

$$\int \cos(2x)\,dx = \frac{\operatorname{sen}(2x)}{2} \qquad \int \cos(ax)\,dx = \frac{\operatorname{sen}(ax)}{a}$$

$$\int tg(x)\,dx = \int \frac{\operatorname{sen}(x)}{\cos(x)}\,dx = -\int \frac{-\operatorname{sen}(x)}{\cos(x)}\,dx = -\log|\cos(x)| + c$$

$$\int (1+tg^2(x))\,dx = \int \sec^2(x)\,dx = \int \frac{1}{\cos^2(x)}\,dx = tg(x) + c$$

$$\int \cos^2(x)\,dx = \int \frac{1+\cos(2x)}{2}\,dx = \int \frac{1}{2}\,dx + \int \frac{\cos(2x)}{2}\,dx =$$

$$= \frac{x}{2} + \frac{\operatorname{sen}(2x)}{4} + c$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\int \operatorname{sen}^2(x)\,dx = \int \frac{1-\cos(2x)}{2}\,dx = \frac{x}{2} - \frac{\operatorname{sen}(2x)}{4} + c$$

$$\operatorname{sen}^2(x) = \frac{1-\cos(2x)}{2}$$

$$\int \frac{1}{\operatorname{sen}(x)}\,dx = \int \frac{\operatorname{sen}(x)}{\operatorname{sen}^2(x)}\,dx = \int \frac{\operatorname{sen}(x)}{1-\cos^2(x)}\,dx = \begin{bmatrix} t = \cos(x) \\ dt = -\operatorname{sen}(x)\,dx \end{bmatrix} =$$

$$= -\int \frac{1}{1-t^2}\,dt = \int \frac{1}{t^2+1}\,dt$$

4.1.- $\int sen^m(x) \cdot cos^n(x)\, dx$ (m o n es entero impar)

4.2.- $\int sen^m(x) \cdot cos^n(x)\, dx$ (m y n son enteros pares)

$$cos^2(x) = \frac{1+cos(x)}{2} \qquad sen^2(x) = \frac{1-cos(x)}{2}$$

4.3.- $\int \mathcal{R}(sen(x), cos(x))\, dx$ cambio "estandard"

$t = tg\left(\frac{x}{2}\right)$

$$\frac{P(x)}{Q(x)} = \mathcal{R}(x) \qquad \mathcal{R}(x,y) = \frac{1+2xy^2+3x^2+4xy^2}{7xy-8xy^4}$$

$t = tg\left(\frac{x}{2}\right)$

$\Rightarrow arctg(t) = \frac{x}{2}$; $2arctg(t) = x \Leftrightarrow dx = \frac{2}{1+t^2}\, dt$

$\Rightarrow cos(x) = \frac{1-t^2}{1+t^2}$

$\Rightarrow sen(x) = \frac{2t}{1+t^2}$

4.4.- $\int \mathcal{R}(sen(x), cos(x))\, dx$ $\mathcal{R}$ es par en sen (x), y cos(x)

$$(\mathcal{R}(-sen(x), -cos(x)) = (\mathcal{R}(sen(x), cos(x))$$

$t = tg(x)$

$\Rightarrow x = arctg(t) \Rightarrow dx = \frac{1}{1+t^2}\, dt$

$\Rightarrow cos(x) = \frac{1}{sec(x)} = \frac{1}{\sqrt{1+t^2}}$

$\Rightarrow sen(x) = tg(x) \cdot cos(x) = \frac{t}{\sqrt{1+t^2}}$

SUCESIONES Y SERIES NUMÉRICAS

* Definición de Sucesión de N° Reales

Una sucesión de n° reales es una función de $\mathbb{N}$ en $\mathbb{R}$

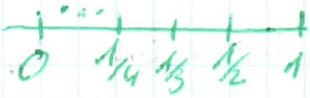
$$\{a_n\} = f : \mathbb{N} \longrightarrow \mathbb{R}$$
$$n \longmapsto f(n) = a_n = \text{término general}$$

* $a_n = 14$, $\forall n \in \mathbb{N}$

* $a_n = n$, $\forall n \in \mathbb{N}$

... $\{n\} \nearrow$ (monó creciente)

* $a_n = 1/n$, $\forall n \in \mathbb{N}$



$\{1/n\} \searrow$ (monó decreciente)

* $a_n = (-1)^n$, $\forall n \in \mathbb{N}$

* Por "recurrencia" $x_1 = 1$, $x_2 = 2$ $\quad x_{n+2} = x_{n+1}$ $\quad \forall n \in \mathbb{N}$

* $x_1 = 1$ $\quad x_{n+1} = \sqrt{3x_n}$ $\quad \forall n \in \mathbb{N}$

* Definición de Sucesión divergente

$\{a_n\}$ divergente positiva $\overset{def}{\Longleftrightarrow}$ $\{a_n\} \to +\infty$

$\{a_n\}$ divergente negativa $\overset{def}{\Longleftrightarrow}$ $\{a_n\} \to -\infty$

No aplicar L'Hôpital

* $a_n = n$ $\quad \forall n \in \mathbb{N} \Rightarrow \{n\} \to +\infty$

* $a_n = 1/n$ $\quad \forall n \in \mathbb{N} \Rightarrow \{1/n\} \to 0$

* $a_n = (-1)^n \quad \forall n \in \mathbb{N} \Rightarrow$ No convergente, no divergente

* $\lim \dfrac{n^2 - 2n + 1}{3n^2 + 5} = 1/3$

* $\lim e^{-n} = 0$

* $\lim \operatorname{arctg}(n) = \pi/2$

* $\lim \left(\dfrac{n^2+1}{n^2+5}\right)^n = 1^\infty =$ Regla del n° e" $=$

$$= n \cdot \left[\frac{n^2+1}{n^2+5} - 1\right] = n \cdot \left[\frac{n^2+1-n^2-5}{n^2+5}\right] = \frac{-4n}{n^2+5} = 0 \to \lambda$$

$$e^0 = 1 \to \lim = 1$$

* Definición de Sucesión convergente

Sea $\{a_n\} \quad a_n = f(n)$

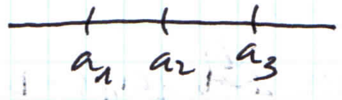
$\{a_n\}$ es convergente a "a" $\overset{def}{\Leftrightarrow} \exists \lim\limits_{n\to\infty} f(n) = a$

$\{a_n\} \to a = \lim a_n$

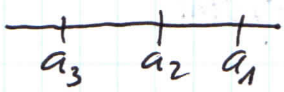
$a_n = c \quad \forall n \in \mathbb{N} \Rightarrow \{a_n\} = \{c\} \to c$

* Definición de sucesión monótona

$\{a_n\} \nearrow$ monótona creciente $\overset{def}{\Leftrightarrow} a_n \leq a_{n+1} \quad \forall n \in \mathbb{N}$



$\{a_n\} \searrow$ monótona decreciente $\overset{def}{\Leftrightarrow} a_n \geq a_{n+1} \quad \forall n \in \mathbb{N}$



CA
Teoría

* Definición acotable

$\{a_n\}$ acotada $\overset{def}{\Leftrightarrow} \exists\, m, M$ t.q. $m \le a_n \le M \ \forall n \in \mathbb{N}$

* $a_n = C, \ \forall n \in \mathbb{N}$ $\begin{cases} \text{convergente} \\ \text{acotada} \\ \text{monótona} \end{cases}$

* $a_n = n, \ \forall n \in \mathbb{N}$ $\begin{cases} \{n\} \nearrow +\infty \\ 1 \le n, \text{ no acotada.} \end{cases}$

* $\{1/n\} \searrow 0$ $\quad 0 \le 1/n \le 1 \ \forall n \in \mathbb{N} \Rightarrow$ Acotada

* $\{(-1)^n\}$ no convergente, ni divergente, no monótona, sí acotada

* Teorema

$$\left.\begin{array}{c} \text{Monótona} \\ + \\ \text{Acotada} \end{array}\right\} \Rightarrow \text{Convergente}$$

$$\text{Convergente} \underset{\not\Leftarrow}{\Rightarrow} \text{Acotada}$$

Convergente $\not\Rightarrow$ Monótona, Monótona $\not\Rightarrow$ Convergente

Acotada $\not\Rightarrow$ Monótona, Monótona $\not\Rightarrow$ Acotada

Monótona

* Principio método inducción

¿P_n se verifica, $\forall n, \mathbb{N}$?

1.- $n = 1$ Comprobar que P_1 es cierto

2.- Hipótesis de inducción, Se asume que P_n es cierto

3.- Se demuestra para $n+1$ (P_{n+1} es cierto)

* Definición de Series de Nº Reales

Una serie es: dado $\{a_n\}$

$$\{a_1 + a_2 + \cdots + a_n\} \equiv \sum a_n$$

* Definición de serie convergente

$\sum a_n$ es convergente $\overset{def}{\Leftrightarrow} \exists \lim \{a_1 + a_2 + \cdots + a_n\} \in \mathbb{R}$

(Suma de la serie) $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \{a_1 + a_2 + \cdots + a_n\} = a_1 + a_2 + \cdots \in \mathbb{R}$

$\sum a_n$ divergente $\Leftrightarrow \lim_{n\to\infty} \{a_1 + a_2 + \cdots + a_n\} = +\infty$

* $\sum x^n$ es convergente $\Leftrightarrow |x| < 1$
(serie geométrica de razón "x")

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$$

* Caso particular

$\sum \frac{1}{2^n} = \sum \left(\frac{1}{2}\right)^n$ es convergente

$$x = \frac{1}{2} ; \left|\frac{1}{2}\right| < 1$$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = \frac{1}{1-\frac{1}{2}} = 2$$

* $\sum 1 \equiv \{1+1+1+\cdots+1\} = \{n\} \to +\infty \Rightarrow$ Divergente

$$\boxed{\sum a_n \text{ convergente} \Rightarrow \{a_n\} \to 0}$$

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Teoría

* $\sum 1/n \quad \left(\sum_{n=1}^{\infty} 1/n = +\infty\right)$ y $\{1/n\} \to 0$

(S. Armónica) $1 + 1/2 + 1/3 + \dots \to +\infty$

* $\sum 2^n$ divergente

* $\sum n \to \{n\} \to +\infty$ divergente

* $\sum \frac{n}{n^2+1}$ ¿?

$\sum \alpha a_n = \alpha \sum_{n=1}^{\infty} a_n$ // $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

$\sum a_n = \sum_{n\geq 1} a_n$ // $\sum x^n = \sum_{n\geq 0} x^n = 1 + x + x^2 + \dots$

$\sum 1/\log(n) = \sum_{n=2}^{\infty} \frac{1}{\log(n)}$

* $\sum_{n\geq 1} 1/3^n$ convergente $\sum_{n=1}^{\infty} (1/3)^n = \sum_{n=0}^{\infty} (1/3)^n - 1$

* $\sum_{n\geq 0} 5/7^n = 5 \sum_{n\geq 0} (1/7)^n$ convergente

$\sum_{n=0}^{\infty} 5/7^n = 5 \sum_{n=0}^{\infty} (1/7)^n = 5 \cdot \frac{1}{1 - 1/7} = \frac{35}{6}$

* **Criterio de convergencia ($\sum a_n$, $a_n \geq 0$)**

* **Criterio de la raíz (n-sima) (C. de Cauchy)**

Sea $\sum a_n$, $a_n \geq 0$: $\lim \sqrt[n]{a_n} = L$
- $L < 1 \Rightarrow \sum a_n$ convergente
- $L > 1 \Rightarrow \sum a_n$ divergente
- $L = 1 \Rightarrow$???

* **Criterio del cociente (o de D'Alambert)**

Sea $\sum a_n$, $a_n > 0$: $\lim \frac{a_{n+1}}{a_n} = L$
- $L < 1 \Rightarrow$ Convergente
- $L > 1 \Rightarrow$ Divergente
- $L = 1 \Rightarrow$???

* Criterio de la comparación

$\sum a_n$ y $\sum b_n$ ($a_n > 0$, $\forall n$; $b_n > 0$, $\forall n$)

* $\lim \frac{a_n}{b_n} = L \neq 0 \longrightarrow \begin{cases} \sum a_n \text{ convergente} \Leftrightarrow \sum b_n \text{ conver} \\ \sum a_n \text{ divergente} \Leftrightarrow \sum b_n \text{ divergente} \end{cases}$

* $\lim \frac{a_n}{b_n} = L = 0 \longrightarrow \begin{cases} \sum a_n \text{ divergente} \Rightarrow \sum b_n \text{ divergente} \\ \sum b_n \text{ convergente} \Rightarrow \sum a_n \text{ convergente} \end{cases}$

* $\lim \frac{a_n}{b_n} = L = +\infty \longrightarrow \begin{cases} \sum a_n \text{ convergente} \Rightarrow \sum b_n \text{ convergente} \\ \sum b_n \text{ divergente} \Rightarrow \sum a_n \text{ divergente} \end{cases}$

CA

Ejercicios

1) Simplificar elementales

* $\frac{4x^5}{2x^3} = 2x^2$

* $\frac{2x^3}{8x^{-4}} = \frac{1}{4} \cdot x^7 = \frac{x^7}{4}$

* $\frac{2x(x-4)}{(x-4)^2 y} = \frac{2x}{(x-4)y}$

* $\frac{2x^2+8x}{4x^2-4x} = \frac{2x(x+4)}{4x(x-1)} = \frac{x+4}{2x-2}$

* $\frac{x^2-xy}{xy-y^2} = \frac{x(x-y)}{y(x-y)} = \frac{x}{y}$

* $\frac{x^3+x}{2x^2+9x} = \frac{x(x^2+1)}{x(2x+9)} = \frac{x^2+1}{2x+9}$

* $\frac{4a^2+a}{ab+a^2} = \frac{a(4a+1)}{a(b-a)} = \frac{4a+1}{b-a}$

* $x^3 = x$

  $x^3 - x = 0 \; ; \quad x \cdot (x^2-1) = 0 \; \begin{cases} x = 0 \\ x = \pm 1 \end{cases}$

* $\frac{x^2+4x+4}{x^2-4} = \frac{(x+2)^2}{(x+2)(x-2)} = \frac{x+2}{x-2}$

$(a+b)^2 = a^2 + b^2 + 2ab$

$(a-b)^2 = a^2 + b^2 - 2ab$

$a^2 - b^2 = (a+b) \cdot (a-b)$

* $\frac{5x^2-5}{x^2-x} = \frac{5(x^2-1)}{x(x-1)} = \frac{5\cdot(x+1)\cdot(x-1)}{x(x-1)} = \frac{5x+5}{x}$

* $\frac{3x^3-3x^2}{6(x^2-2x+1)} = \frac{3x^2(x-1)}{6(x-1)^2} = \frac{3x^2}{6(x-1)} = \frac{x^2}{2(x-1)}$

* $\sqrt{a} = a^{1/2}$

* $\sqrt{a\cdot b} = \sqrt{a}\cdot\sqrt{b}$

* $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

2) Operaciones con fracciones algebraicas

* $\frac{1}{x-1} + \frac{2x-1}{x^2-1} - \frac{x+2}{x+1} = \frac{(x+1)+(2x-1)-(x+2)\cdot(x-1)}{x^2-1} =$

$= \frac{3x-(x^2+x-2)}{x^2-1} = \frac{-x^2+2x+2}{x^2-1}$

$x-1 = x-1$; $x^2-1 = (x+1)\cdot(x-1)$; $x+1 = x+1$

* $\frac{3x+1}{x} - \frac{1}{x^2-x} = \frac{(3x+1)(x-1)-1}{x(x-1)} = \frac{3x^2-2x-2}{x^2-x}$

* $x + \frac{2x+1}{x-3} = \frac{x^2-3x+2x+1}{x-3} = \frac{x^2-x+1}{x-3}$

* $\frac{x+1}{x-1}\cdot\frac{x^2-x}{x^2+2x+1} = \frac{x^3-x}{x^3+x^2-x+1}$

* $\frac{\frac{x-2}{x+2}}{\frac{x}{(x+2)^2}} = \frac{(x-2)(x+2)^2}{(x+2)\,x} = \frac{x^2-4}{x}$

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R1.1 $\frac{2x-3}{x+2} < 1/3$

$\frac{2x-3}{x+2} - \frac{1}{3} < 0$; $\frac{5x-11}{3(x+2)} < 0$

1.- $\left.\begin{array}{ll} 5x-11>0 ; & x > 11/5 \\ x+2<0 ; & x<-2 \end{array}\right\}$ No es posible

2.- $\left.\begin{array}{ll} 5x-11<0 ; & x < 11/5 \\ x+2>0 ; & x>-2 \end{array}\right\}$ $x \in]-2, 11/5[$

$x^2-3x-3 < -4x-1$

$x^2+x-2<0$; $(x-1)\cdot(x+2)<0$

$x = \begin{cases} \to 1 \\ \to -2 \end{cases}$

Modo 1

1.- $\left.\begin{array}{l} x-1<0 \Rightarrow x<1 \\ x+2>0 \Rightarrow x>-2 \end{array}\right\}$ $x \in]-2, 1[$

2.- $\left.\begin{array}{l} x-1>0 \Rightarrow x>1 \\ x+2<0 \Rightarrow x<-2 \end{array}\right\}$ No es posible

Modo 2

	$x<-2$	$-2<x<1$	$x>1$
$x-1$	$-$	$-$	$+$
$x+2$	$-$	$+$	$+$
$(x-1)(x+2)$	$+$	$-$	$+$

$x \in]-2, 1[$

porque $(x-1)(x+2)<0$

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1) $|x|=2 \Rightarrow x=2$ ó $x=-2$

$$|x|=2 \begin{cases} x\geq 0 & |x|=x=2 \\ x<0 & |x|=-x=-2 \end{cases}$$

2) $|x-1|=4$

$$|x-1|=4 \begin{cases} x-1=4 \Rightarrow x=5 \\ x-1=-4 \Rightarrow x=-3 \end{cases}$$

3) $|x|=4$ $\quad x=-4 \quad x=4$

4) $|x-7|=5 \begin{cases} x-7=5 \Rightarrow x=12 \\ x-7=-5 \Rightarrow x=2 \end{cases}$

5) $|x|\leq 4$

$$-4\leq x\leq 4 \Leftrightarrow x\in[-4,4]$$

6) $|x|>2 \begin{cases} x>2 \\ x<-2 \end{cases}$ $\quad x\in(-\infty,2)\cup(2,+\infty)$

7) $|x-1|<2$ $\quad -2<x-1<2$

$$|x-1|<2 \Leftrightarrow \left.\begin{cases} x-1<2 \Leftrightarrow x<3 \\ x-1>-2 \Leftrightarrow x>-1 \end{cases}\right\} -1<x<3$$

$$\Downarrow$$

$$x\in(-1,3)$$

8) $|x+4|\geq -2$ $\quad \forall x\in\mathbb{R}$ $\qquad |x+4|\leq -2$ $\quad \nexists x\in\mathbb{R}$

9) $|x+4|\geq 2 \left.\begin{cases} x+4\geq 2 \Leftrightarrow x\geq -2 \\ x+4\leq 2 \Leftrightarrow x\leq -6 \end{cases}\right\} x\in(-\infty,-6]\cup[-2,+\infty)$

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R1.2)

a) $1/x + 1/(1-x) > 0$ $\quad \frac{1-x+x}{x\cdot(1-x)} > 0$; $\frac{1}{x(1-x)} > 0$;

$x(1-x) > 0 \quad x \neq 0$ y $x \neq 1$

	$x<0$	$0<x<1$	$x>1$
x	$-$	$+$	$+$
$1-x$	$+$	$+$	$-$
$x(1-x)$	$-$	$+$	$-$

$x \in (0,1)$

b) $x^2 - 5x + 9 > x$ $\quad x^2 - 6x + 9 > 0$; $(x-3)(x-3) > 0$

$x \neq 3 \quad \forall x \in \mathbb{R} \quad (-\infty, 3) \cup (3, +\infty)$

c) $x^3(x-2)(x+3)^2 < 0$

$x \neq 0 \quad x \neq 2 \quad x \neq -3 \qquad x \in (0,2)$

	$x<-3$	$-3<x<0$	$0<x<2$	$x>2$
x^3	$-$	$-$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
$(x+3)^2$	$+$	$+$	$+$	$+$
$x^3(x-2)(x+3)^2$	$+$	$+$	$-$	$+$
	NO	NO	SI	NO

d) $x^2 \leq x$ $\quad x^2 - x \leq 0$; $x(x-1) \leq 0$; $x \neq 0$, $x \neq 1$

	$x<0$	$0<x<1$	$x>1$
x	$-$	$+$	$+$
$x-1$	$-$	$-$	$+$
$x(x-1)$	$+$	$-$	$+$
	NO	Sí	NO

$x \in [0,1]$

R1.3)

a) $|x-1| \cdot |x+2| = 3$; $|(x-1)\cdot(x+2)| = 3$

$$\begin{cases} (x-1)(x+2) = 3 \; ; \; x = \dfrac{-1 \pm \sqrt{21}}{2} \\ (x-1)(x+2) = -3 \quad x = \nexists \end{cases}$$

d) $|x+1| < |x+3|$

R1.5)

a) $f(x) = \sqrt{\dfrac{x-2}{x+2}}$

Composición de una racional con una raíz cuadrada

$\exists f(x) \Rightarrow \dfrac{x-2}{x+2} \geq 0$

	$x<-2$	$-2<x<2$	$x>2$
$x-2$	$-$	$-$	$+$
$x+2$	$-$	$+$	$+$
$\frac{x-2}{x+2}$	$+$	$-$	$+$

$]-\infty, -2[\cup [2, +\infty[$

b) $f(x) = \log\left(\dfrac{x^2-5x+6}{x^2+4x+6}\right)$

Composición de una racional con un logaritmo

$x^2-5x+6=0 \Rightarrow (x-2)(x-3)$

$x^2+4x+6=0 \Rightarrow \nexists$

$]-\infty, 2[\cup]3, +\infty[$

c) $f(x) = \sqrt{\dfrac{x}{1-|x|}}$

$\exists f(x) \iff x \neq \pm 1 \qquad \dfrac{x}{1-|x|} \geq 0$

	$x<-1$	$-1<x<0$	$0<x<1$	$x>1$
x	$-$	$-$	$+$	$+$
$1-\lvert x\rvert$	$-$	$+$	$+$	$-$
$\frac{x}{1-\lvert x\rvert}$	$+$	$-$	$+$	$-$

$\mathrm{Dom}(f) = \left]-\infty, -1\right[\cup \left[0, 1\right[$

R1.6) $f(x) = 1/x \qquad g(x) = 1/\sqrt{x}$

$\mathrm{Dom}(f) = \mathbb{R}^* \qquad \mathrm{Dom}(g) = \mathbb{R}^+$

$(f+g)(x) = f(x) + g(x) = 1/x + 1/\sqrt{x} = \dfrac{1+\sqrt{x}}{x} \Rightarrow \mathbb{R}^+$

$(fg)(x) = f(x) \cdot g(x) = 1/x \cdot 1/\sqrt{x} = \dfrac{1}{x\sqrt{x}} \Rightarrow \mathbb{R}^+$

$(f \circ g)(x) = f(g(x)) = f(1/\sqrt{x}) = \sqrt{x} \Rightarrow \mathbb{R}^+$

R1.7) $f(x) = |x+1| - |x-1|$

$f(-x) = |-x+1| - |-x-1| = |x-1| - |x+1| \Rightarrow -f(x)$

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Ejercicios

5

R1.5)

e) $f(x) = \log(\operatorname{sen}(x))$ ¿Dom(f)?

$\exists f(x) \Leftrightarrow \exists \log(\operatorname{sen}(x)) \Rightarrow \operatorname{sen}(x) > 0$

$$\operatorname{Dom} f(x) = \bigcup_{k \in \mathbb{Z}} \left]2k\pi, 2k\pi + \pi\right[$$

f) $f(x) = \sqrt{\log(\operatorname{sen}(x))}$ ¿Dom(f)?

$\exists f(x) \Leftrightarrow \exists \log \sqrt{\log(\operatorname{sen}(x))} \Leftrightarrow \log(\operatorname{sen}(x)) \geq 0 \Leftrightarrow$

$\Leftrightarrow \operatorname{sen}(x) \geq 1$

$x = \pi/2 + 2\pi k, \ k \in \mathbb{Z}$

R1.7)

b) $f(x) = \log\left(\frac{1+x}{1-x}\right)$ ¿Par / Impar / Nada?

$f(-x) = \log\left(\frac{1-x}{1+x}\right)$

$f(x) = \log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$

$f(-x) = \log(1-x) - \log(1+x) = -f(x) \Rightarrow$ Impar

R1.8) $e^{3x+8}(x+7)>0$

$e^{3x+8} > 0 \Leftrightarrow$ Siempre

$(x+7)>0 \Leftrightarrow \boxed{x>-7}$

R1.9) Sea $a,b \in \mathbb{R}^+$

$¿\underbrace{a^{\log(b)}}_{x} = \underbrace{b^{\log(a)}}_{y}?$

$$\left.\begin{array}{l} \log(x) = \log\left(a^{\log(b)}\right) = \log(b)\cdot\log(a) \\ \log(y) = \log\left(b^{\log(a)}\right) = \log(b)\cdot\log(a) \end{array}\right\} \log(x) = \log(y)$$

$$e^{\log(x)} = e^{\log(y)} \; ; \; x = y$$

R1.10) $\dfrac{1}{\log_x(a)} = \dfrac{1}{\log_b(a)} + \dfrac{1}{\log_c(a)} + \dfrac{1}{\log_d(a)}$

$a>0 \; ; \; a \neq 1 \; ; \; b,c,d \in \mathbb{R}^+ \; ; \; b,c,d \neq 1$

$$\frac{\log(x)}{\log(a)} = \frac{\log(b)}{\log(a)} + \frac{\log(c)}{\log(a)} + \frac{\log(d)}{\log(a)} \; ;$$

$$\log(x) = \log(b) + \log(c) + \log(d) \; ;$$

$$\log(x) = \log(b\cdot c\cdot d) \; ;$$

$$e^{\log(x)} = e^{\log(b\cdot c\cdot d)} \; ; \; x = b\cdot c\cdot d$$

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Ejercicios

R1.12) $\log(x+\sqrt{1+x^2}) + \log(\sqrt{1+x^2}-x) \stackrel{?}{=} 0$

$\log((x+\sqrt{1+x^2})\cdot(\sqrt{1+x^2}-x)) = 0;$

$\log((1+x^2)-x^2) = 0; \quad \log(1) = 0$

R1.7)

e) $f(x) = sen(|x|)$ ¿Par / Impar / Nada?

$f(-x) = sen(|-x|) = sen(|x|) = f(x) \Rightarrow PAR$

R1.13) $x^{\sqrt{x}} = (\sqrt{x})^x$

$\log(x^{\sqrt{x}}) = \log(\sqrt{x})^x$

$\sqrt{x}\log(x) = x\log(\sqrt{x})$

$\sqrt{x}\log(x) = x\log(x^{1/2})$

$\sqrt{x}\log(x) = \frac{x}{2}\log(x)$

$(\sqrt{x}\log(x)) - (\frac{x}{2}\log(x)) = 0$

$\log(x)(\sqrt{x}-\frac{x}{2}) = 0 \begin{cases} 1.- \log(x) = 0 \Rightarrow \boxed{x=1} \\ 2.- \sqrt{x}-\frac{x}{2} = 0 \Rightarrow \sqrt{x} = \frac{x}{2}; \\ \quad 2\sqrt{x} = x \Rightarrow \boxed{x=4} \end{cases}$

R 1.14)

a) $x = a^{\frac{\log(\log(a))}{\log(a)}}$

$$\log(x) = \log\left(a^{\frac{\log(\log(a))}{\log(a)}}\right) = \frac{\log(\log(a))}{\log(a)} \cdot \log(a) =$$

$$= \log(\log(a))$$

$$x = \log(a)$$

b) $\log_a\left(\log_a\left(a^{a^x}\right)\right)$

$$\log_a\left(a^x \underbrace{\log_a(a)}_{1}\right) = \log_a(a^x) = x$$

Ejemplos)

* $\lim_{x \to +\infty} x^2 - x = +\infty$

$$x^2 - x = x^2 \cdot [1 - 1/x] = +\infty \cdot 1 = +\infty$$

* $\lim_{x \to +\infty} -7x^3 + 8x^2 - 5x = -\infty$

* $\lim_{x \to -\infty} x^3 - 2x = -\infty$

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R2.1)

a) $\lim_{x\to+\infty} \frac{x}{7x+4} = 1/7$

b) $\lim_{x\to+\infty} \frac{5x+3}{2x^2+1} = 0$

c) $\lim_{x\to+\infty} \frac{x^3+3x}{2x^2+5} = +\infty$

d) $\lim_{x\to 2} \frac{x^2-4}{x-2} = \frac{(x+2)(x-2)}{(x-2)} = x+2 = 4$

R2.2)

a) $\lim_{x\to 4} \left(1/x - 1/4\right)\cdot\left(1/x-4\right) = 0\cdot\infty = -1/16$

$\left(\frac{4-x}{4x}\right)\cdot\left(\frac{1}{x-4}\right) = \frac{4-x}{4x\cdot(x-4)} = \frac{-1}{4x} = -1/16$

b) $\lim_{x\to 0} \frac{x^4}{3x^3+2x^2+x} = 0$

$\frac{x^4}{x(3x^3+2x+1)} = \frac{x^3}{3x^2+2x+1} = \frac{0}{1} = 0$

c) $\lim_{x\to 1} \frac{\sqrt{x}-1}{|x-1|} = \frac{0}{0} \Rightarrow \nexists \lim_{x\to 1}$

$* \lim_{x\to 1^+} \frac{\sqrt{x}-1}{x-1} = 1/2$

$\frac{\sqrt{x}-1}{x-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1} = \frac{x-1}{(x-1)\cdot(\sqrt{x}+1)} = \frac{1}{\sqrt{x}+1} = 1/2$

$* \lim_{x\to 1^-} \frac{\sqrt{x}-1}{-(x-1)} = -1/2$

d) $\lim_{x\to+\infty} \sqrt{x+\sqrt{x}} - \sqrt{x} = \text{“}\infty - \infty\text{”}$

$$\frac{(\sqrt{x+\sqrt{x}} - \sqrt{x})\cdot(\sqrt{x+\sqrt{x}} + \sqrt{x})}{\sqrt{x+\sqrt{x}} + \sqrt{x}} = \frac{x+\sqrt{x}-x}{\sqrt{x+\sqrt{x}}+\sqrt{x}} =$$

$$= \frac{\sqrt{x}}{\sqrt{x+\sqrt{x}}+\sqrt{x}} = \frac{\infty}{\infty} = \frac{\frac{\sqrt{x}}{\sqrt{x}}}{\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x}} + \frac{\sqrt{x}}{\sqrt{x}}} = \frac{1}{\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x}} + 1} =$$

$$= \frac{1}{\sqrt{1+1/\sqrt{x}}+1} = 1/2$$

R2.5

a) $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \begin{cases} \frac{1}{1+e^{1/x}} & x \neq 0 \\ 0 & x = 0 \end{cases}$

¿Continuidad? ¿Comportamiento $+\infty, -\infty$?

$\mathbb{R}^*$, $f(x) = \frac{1}{1+e^{1/x}}$ es continua

$1+e^{1/x} > 1$

¿f es continua en 0? $\Leftrightarrow$ ¿$\exists \lim_{x\to 0} f(x) = f(0) = 0$?

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{1+e^{1/x}} = 0$$

$$x \to 0 = 1/x = \pm\infty \left[\begin{matrix} x\to 0^+ \Rightarrow 1/x = +\infty \\ x \to 0^- \Rightarrow 1/x \to -\infty \end{matrix}\right]$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{1+e^{1/x}} = 1/1 = 1$$

No existe límite con lo cual no es continua

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{1}{1+e^{1/x}} = 1/2$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{1+e^{1/x}} = 1/2$$

CA
Ejercicios

Ejemplo) $\lim_{x\to+\infty} x^2 \cdot \log\left(\frac{2x^2+5}{2x^2+7}\right) = \text{"}\infty \cdot 0\text{"}$

$$x^2 \cdot \log\left(\frac{2x^2+5}{2x^2+7}\right) = \log\left(\frac{2x^2+5}{2x^2+7}\right)^{x^2} =$$

$\lim_{x\to\infty} \left(\frac{2x^2+5}{2x^2+7}\right)^{x^2} = \text{"}1^\infty\text{"} \to$ Regla del número e

$$x^2\left[\frac{2x^2+5}{2x^2+7} - 1\right] = \frac{-2x^2}{2x^2+7} \to L = -1$$

$$\lim_{x\to+\infty} \left(\frac{2x^2+5}{2x^2+7}\right)^{x^2} = e^{-1}$$

$$\lim_{x\to+\infty} \log\left(\frac{2x^2+5}{2x^2+7}\right)^{x^2} = \log e^{-1} = -1$$

R2.1d) $\exists x \in \mathbb{R}^*$ t.q. $\log(x) + \sqrt{x} = 0$

$f(x) = \log(x) + \sqrt{x}$ $\quad f:]0, +\infty[\to \mathbb{R}$

(f es continua en $\mathbb{R}^+$)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \log(x) + \sqrt{x} = -\infty + 0 = -\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \log(x) + \sqrt{x} = +\infty + \infty = +\infty$$

$\exists\, c \in]0, +\infty[\ /\ f(c) = 0$

R2.11) $x+e^x+\operatorname{arctg}(x)=0$ ¿Solo una solución real?

¿$\exists\, c\in\mathbb{R},\ c+e^c+\operatorname{arctg}(c)=0$?

$f:\mathbb{R}\to\mathbb{R}\quad f(x)=x+e^x+\operatorname{arctg}(x)$

$$\left.\begin{array}{l}\lim\limits_{x\to+\infty} f(x)=+\infty\\ \lim\limits_{x\to-\infty} f(x)=-\infty\end{array}\right\}\Rightarrow \exists\, c\in\mathbb{R}\ \ f(c)=0 \text{ ¿único?}$$

R2.12) $f:\mathbb{R}^*\to\mathbb{R}\quad f(x)=\operatorname{arctg}(\log|x|)$ ¿$f(\mathbb{R}^*)$?

$\mathbb{R}^*=\,]-\infty,0[\,\cup\,]0,+\infty[$

$f(\mathbb{R}^*)=f(]-\infty,0[)\cup f(]0,+\infty[)$

1.- $f(]0,+\infty[)$

$f(x)=\operatorname{arctg}(\log(x))\qquad 0<x<y\Rightarrow \log(x)<\log(y)$

$\operatorname{arctg}(\log(x))<\operatorname{arctg}(\log(y))$

$\lim\limits_{x\to0} f(x)=\lim\limits_{x\to0^+} f(x)=-\frac{\pi}{2}$

$\lim\limits_{x\to+\infty} f(x)=\frac{\pi}{2}$

$f(]0,+\infty[)=\,]\lim\limits_{x\to0} f(x),\lim\limits_{x\to+\infty} f(x)[\,=\,]-\frac{\pi}{2},\frac{\pi}{2}[$

2.- $f(]-\infty,0[)$

$f(x)=\operatorname{arctg}(\log(-x))$

$f(]-\infty,0[)=\,]\lim\limits_{x\to0} f(x),\lim\limits_{x\to-\infty} f(x)[$

$\lim\limits_{x\to0^-} f(x)=-\frac{\pi}{2}\qquad \lim\limits_{x\to-\infty} f(x)=\frac{\pi}{2}$

$f(]-\infty,0[)=\,]-\frac{\pi}{2},\frac{\pi}{2}[$

Por lo tanto, $f(\mathbb{R}^*)=\,]-\frac{\pi}{2},\frac{\pi}{2}[$

CA
Ejercicios

R2.9) $f: \mathbb{R}^* \to \mathbb{R}$ ¿comportamiento de f en 0?

$f(x) = \operatorname{arctg}(7/x) - \operatorname{arctg}(-5/x)$ $\qquad g(x) = x f(x)$

$$\left.\begin{array}{l} \lim_{x \to 0^+} = \pi/2 - (-\pi/2) = \pi \\ \lim_{x \to 0^+} = -\pi/2 - \pi/2 = -\pi \end{array}\right\} \nexists \lim_{x \to 0} f(x)$$

$$\lim_{x \to 0} g(x) = \left\{\begin{array}{l} \lim_{x \to 0^+} = 0 \cdot \pi = 0 \\ \lim_{x \to 0^-} = 0 \cdot (-\pi) = 0 \end{array}\right\} = 0$$

1) $f: I \to \mathbb{R}$, $a \in I$, Estudia la derivabilidad de f en a

* Estudiar la continuidad de f en a
  - a) f no continua en $a \Rightarrow \nexists f'(a)$
  - b) f continua en $a \Rightarrow$????

* $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

R3.1)

c) $f(x) = x^2+1$ ¿recta tangente en $(3,10)$?

$a=3$; $f(a) = f(3) = 10$

$f'(x) = 2x \Rightarrow f'(3) = 6$

$$\boxed{y = 10 + 6\cdot(x-3)}$$

a) $f(x) = \dfrac{x}{x^2+1}$ ¿recta tangente en el origen $(0,0)$?

$f(0) \rightarrow (0, f(0)) = (0,0)$

$f'(x) = \dfrac{x^2+1-2x^2}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} \Rightarrow f'(0) = \dfrac{1}{1} = 1$

$y = 0 + 1\cdot(x-0) = x$

d) $f(x) = |x|$ ¿Recta tangente en el $(1,1)$?

$f(x) \begin{cases} x & \text{si } x>0 \\ -x & \text{si } x<0 \end{cases}$ $\quad f'(x) \begin{cases} 1 & \text{si } x>0 \\ -1 & \text{si } x<0 \end{cases}$

$(1, f'(1)) \rightarrow f'(1) = 1$

$y = 1 + 1(x-1) = x$

R3.2)

a) $f(x) = sen(x+3) \qquad f'(x) = \cos(x+3)\cdot 1$

b) $f(x) = \cos^2(x) \qquad f'(x) = 2\cos(x)\cdot(-sen(x)) = -2sen(2x)$

c) $f(x) = \dfrac{1}{\cos(x)} = \sec(x) \qquad f'(x) = \dfrac{sen(x)}{\cos^2(x)} = \dfrac{sen(x)}{\cos(x)}\cdot\dfrac{1}{\cos(x)} = tg(x)\cdot\sec(x)$

d) $y = \sec(x) \qquad f'(x) = tg(x)\cdot\sec(x)$

e) $f(x) = \sqrt{\dfrac{1+x}{1-x}} \qquad f'(x) = \dfrac{1}{2\sqrt{\dfrac{1+x}{1-x}}}\cdot\left(\dfrac{1+x}{1-x}\right)' = \dfrac{1}{\sqrt{\dfrac{1+x}{1-x}}\cdot(1-x)^2} =$

$= \dfrac{1}{\sqrt{\dfrac{(1+x)(1-x)^4}{(1-x)}}} = \dfrac{1}{\sqrt{(1+x)(1-x)^3}} = \dfrac{1}{(1-x)\sqrt{(1+x)(1-x)}} = \dfrac{1}{(1-x)\sqrt{(1-x^2)}}$

$\left(\dfrac{1+x}{1-x}\right)' = \dfrac{2}{(1-x)^2}$

f) $f(x) = \sqrt[3]{x^2+1} = (x^2+1)^{1/3}$

$f'(x) = \frac{1}{3}(x^2+1)^{1/3-1}\cdot 2x = \frac{2}{3}x\cdot\dfrac{1}{\sqrt[3]{(x^2+1)^2}}$

R3.3)

a) $f(x) = \left(\sqrt[5]{x} - \dfrac{1}{\sqrt[5]{x}}\right)^5 = \left(x^{1/5} - x^{-1/5}\right)^5$

$f'(x) = 5\left(x^{1/5} - x^{-1/5}\right)^4\cdot\left(\frac{1}{5}x^{-4/5} + \frac{1}{5}x^{-6/5}\right)$

b) $f(x) = \cos(\cos(\cos(x)))$

$f'(x) = -sen(\cos(\cos(x)))\cdot(-sen(\cos(x)))\cdot(-sen(x))$

c) $f(x) = x^4\cdot(e^x\cdot\log(x))$

$f'(x) = 4x^3\cdot(e^x\cdot\log(x)) + x^4\cdot(e^x\cdot\log(x) + e^x\cdot 1/x) =$

$= 4x^3e^x\cdot\log(x) + x^4\cdot e^x\cdot\log(x) + x^4e^x\cdot\frac{1}{x} =$

$= x^3e^x[4\log(x) + x\log(x) + 1]$

d) $f(x) = x^x = e^{x\log(x)}$

$$f'(x) = e^{x\log(x)} \cdot (\log(x) + x \, 1/x) = e^{x\log(x)}[\log(x)+1] =$$

$$= x^x[\log(x)+1]$$

e) $f(x) = \sqrt{x}^{\sqrt{x}} = e^{\sqrt{x}\log(\sqrt{x})} = e^{\sqrt{x}\cdot\log(x^{1/2})} = e^{\frac{\sqrt{x}}{2}\cdot\log(x)} =$
$= e^{1/2\cdot\sqrt{x}\cdot\log(x)}$

$$f'(x) = e^{1/2\cdot\sqrt{x}\cdot\log(x)} \cdot 1/2\left[\frac{1}{2\sqrt{x}}\cdot\log(x) + \sqrt{x}\cdot 1/x\right] =$$

$$= 1/2\,\sqrt{x}^{\sqrt{x}}\cdot\left[\frac{\log(x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right] = \frac{1}{2}\,\frac{\sqrt{x}^{\sqrt{x}}}{\sqrt{x}}\left[\frac{\log(x)}{2}+1\right] =$$

$$= 1/2\,\sqrt{x}^{\sqrt{x}-1}[\log(\sqrt{x})+1]$$

f) $f(x) = 1/2\, x|x|$ $\quad \exists f'(x) \quad \forall x \in \mathbb{R}^* \quad ¿\exists f'(0)?$

$$\mathbb{R}^* \begin{cases} x>0 \Rightarrow f'(x) = 1/2\cdot[|x|+x\cdot 1] = 1/2\,(|x|+x) = 1/2\cdot 2x = x \\ x<0 \Rightarrow f'(x) = 1/2\,[|x|+x\cdot 1] = 1/2\,(|x|+x) = 1/2\cdot(-2x) = -x \end{cases}$$

$¿\exists f'(0)?$

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{1/2\; x|x|}{x} = \text{Indeterminada} =$$

$$= \text{"}\frac{0}{0}\text{"} = \lim_{x\to 0} 1/2\,|x| = 0 \Rightarrow \exists f'(0) = 0$$

CA
Ejercicios

4) $f: \mathbb{R} \to \mathbb{R}$; $f(x) = \begin{cases} 2x & x<0 \\ 3x^2 & x \geq 0 \end{cases}$ ¿Estudia derivabilidad?

* Continuidad

$x<0 \Rightarrow f$ continua
$x>0 \Rightarrow f$ continua

$$\left.\begin{array}{l} \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2x = 0 \\ \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 3x^2 = 0 \end{array}\right\} \exists \lim_{x\to 0} f(x) = 0 = f(0)$$

$\Downarrow$

f es continua en 0

* Derivabilidad

$$\mathbb{R}^* \begin{cases} x<0 \Rightarrow f(x) = 2x \Rightarrow f'(x) = 2 \\ x>0 \Rightarrow f(x) = 3x^2 \Rightarrow f'(x) = 6x \end{cases}$$

$$\nexists f'(0) \begin{cases} f'(0^-) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{2x}{x} = \lim_{x\to 0^-} 2 = 2 \\ f'(0^+) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{3x^2}{x} = \lim_{x\to 0^+} 3x = 0 \end{cases}$$

Es continua en 0 pero no derivable.

R3.6) $f:]-\frac{\pi}{2}, \frac{\pi}{2}[\to \mathbb{R}$, $f(x) = \begin{cases} \frac{\log(1-\mathrm{sen}(x)) - 2\log(\cos(x))}{\mathrm{sen}(x)} & x \neq 0 \\ a & x=0 \end{cases}$

¿a? para que f sea continua en 0

$$f(0) = a = \lim_{x\to 0} f(x)$$

$$\lim_{\substack{x\to 0 \\ x\neq 0}} f(x) = \lim_{x\to 0} \frac{\log(1-\mathrm{sen}(x)) - 2\log(\cos(x))}{\mathrm{sen}(x)} = \frac{0}{0} = \text{L'Hôpital} =$$

$$= \frac{\frac{-\cos(x)}{1-\mathrm{sen}(x)} - 2\frac{-\mathrm{sen}(x)}{\cos(x)}}{\cos(x)} = -1 \qquad \boxed{a=-1}$$

R3.5) $f(x) = 2x^3 - 3x^2 - 12x + 40$

¿Puntos en los que la recta tangente paralela al eje OX?

$\hookrightarrow y = 0 \rightarrow$ pendiente $= 0 \rightarrow f'(x) = 0$

¿Puntos en los que la recta tangente es paralela a la bisectriz del primer cuadrante?

$y = x \rightarrow$ pendiente $= 1 \rightarrow f'(x) = 1$

R3.14 $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = e^{-x^2}(x^2-3)$ ¿Imagen de f? $f(\mathbb{R}) = ?$

* Punto crítico $f'(x) = 0$

$$f'(x) = e^{-2x}(x^2-3)\cdot(2x) + e^{-x^2}\cdot(2x)$$

$$f'(x) = 2xe^{-x^2}\cdot[-(x^2-3)+1] = 2xe^{-x^2}[4-x^2]$$

$$f'(x) = 2xe^{-x^2}(2-x)(2+x)$$

$$f'(x) = 0 \iff \begin{cases} x=-2 \\ x=0 \\ x=2 \end{cases}$$

* Intervalos de monotonía

$]-\infty, -2[\ : f'(x) > 0 \Rightarrow f \nearrow$

$]-2, 0[\ : f'(x) < 0 \Rightarrow f \searrow$

$]0, 2[\ : f'(x) > 0 \Rightarrow f \nearrow$

$]2, +\infty[\ : f'(x) < 0 \Rightarrow f \searrow$

en -2 max relativo

en 0 min relativo

en 2 max relativo.

R3.9 Demostrar: $\underbrace{\frac{x}{1+x^2} < \operatorname{arctg}(x)}_{a} \underbrace{< x}_{b} \qquad \forall x > 0$

a) $\frac{x}{1+x^2} < \operatorname{arctg}(x)$

$$0 < \operatorname{arctg}(x) - \frac{x}{1+x^2} \qquad \forall x > 0$$

$f(x) = \operatorname{arctg}(x) - \frac{x}{1+x^2} \qquad f: \mathbb{R}^+ \to \mathbb{R}$ ¿$0 < f(x), \forall x > 0$?

f es continua por diferencia de continuas

f es derivable por diferencia de derivables.

$$f'(x) = \frac{1}{1+x^2} - \frac{1+x^2-2x\cdot x}{(1+x^2)^2}$$

$$f'(x) = \frac{1}{1+x^2} - \frac{(1-x^2)}{(1+x^2)^2} = \frac{1+x^2-(1-x^2)}{(1+x^2)^2} = \frac{2x^2}{(1+x^2)^2}$$

$f'(x)=0 \to$ Calcular puntos críticos.

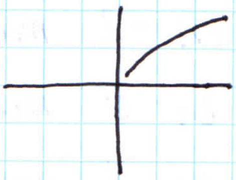
$2x^2 = 0 \Leftrightarrow x=0 \quad 0 \notin \mathbb{R}^+$

con lo cual nunca se anula ($f'(x) \neq 0 \ \forall x \in \mathbb{R}^+$)

f es estrictamente monótona en $\mathbb{R}^+$

$f'(x) = \frac{2x^2}{(1+x^2)^2} > 0 \quad \forall x \in \mathbb{R}^+ \Rightarrow$ estrictamente creciente.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \left(\operatorname{arctg}(x) - \frac{x}{1+x^2}\right) = 0-0=0$$



$f(x) > 0, \ \forall x > 0$

b) $\operatorname{arctg}(x) < x \quad \forall x > 0$

$0 < x - \operatorname{arctg}(x) \quad \forall x > 0$

$f(x) = x - \operatorname{arctg}(x) \quad \forall x > 0 \quad$ ¿ $0 < f(x), \ \forall x > 0$?

f es continua; f es derivable; $f'(x) = 1 - \frac{1}{1+x^2}$

$f'(x) = \frac{x^2}{1+x^2}$; $f'(x) = 0 \Leftrightarrow x^2 = 0 \Leftrightarrow x = 0 \quad 0 \notin \mathbb{R}^+$

f es estrictamente monótona.

$f'(x) = \frac{x^2}{1+x^2} > 0 \quad \forall x \in \mathbb{R}^+ \Rightarrow f$ estrictamente creciente.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \left(x - \operatorname{arctg}(x)\right) = 0-0=0 \quad f(x) > 0 \quad \forall x \in \mathbb{R}^+$$

La cadena de desigualdad es cierta en todo su dominio.

JOSE ANTONIO PADIAL MOLINA

R3.10 ¿ nº de soluciones? $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x + e^{-x} - 2$

$x + e^{-x} = 2$; $x + e^{-x} - 2 = 0$ ¿ nº de ceros de f?

nº de x, $f(x) = 0$

f es continua y derivable.

Teorema de Bolzano

$\lim_{x \to -\infty} f(x) = +\infty$ $\lim_{x \to +\infty} f(x) = +\infty$

$f'(x) = 1 - e^{-x}$

* Puntos críticos

$f'(x) = 0 \Leftrightarrow 1 - e^{-x} = 0 \Leftrightarrow e^{-x} = 1 \Leftrightarrow x = 0$

Según el teorema de Rolle

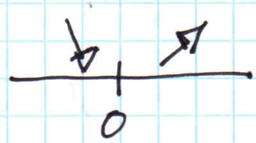
nº de ceros de f' es 1 $\Rightarrow$ nº de ceros de f es ≤ 2

nº de ceros de f' es k $\Rightarrow$ nº de ceros de f es $\leq k+1$

* Intervalos de monotonía

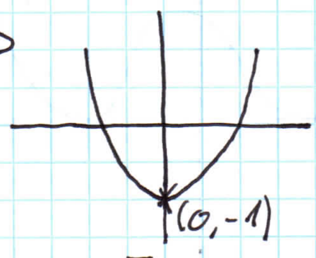
$x < 0 \Rightarrow f'(x) = 1 - e^{-x} < 0 \Rightarrow f \searrow$

$x > 0 \Rightarrow f'(x) = 1 - e^{-x} > 0 \Rightarrow f \nearrow$



En $x = 0$ mínimo absoluto

$f(0) = -1 \Rightarrow$



$\Rightarrow$ 2 ceros $\Rightarrow$ 2 soluciones

$f(\mathbb{R}) = [-1, +\infty[$

R3.11 ¿ nº de ceros? ¿ Imagen? $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = x^6 - 3x^2 + 2$$

$$\lim_{x \to +\infty} f(x) = +\infty = \lim_{x \to -\infty} f(x)$$

$$f'(x) = 6x^5 - 6x = 6x \cdot (x^4 - 1) = 6x(x^2-1) \cdot (x^2+1) =$$

$$= 6x(x-1) \cdot (x+1) \cdot (x^2+1)$$

* Puntos críticos

$\{0, 1, -1\} \Rightarrow$ nº de ceros de $f \leq 4$ función par.

* Intervalos de monotonía

$$\begin{array}{ccccccc} & \searrow & & \nearrow & & \searrow & & \nearrow \\ \hline & & -1 & & 0 & & 1 & \end{array}$$

$]-\infty, -1[\Rightarrow f'(x) = 6x(x-1)(x+1)(x^2+1) < 0 \Rightarrow f \searrow$

$]-1, 0[\Rightarrow f'(x) = 6x(x-1)(x+1)(x^2+1) > 0 \Rightarrow f \nearrow$

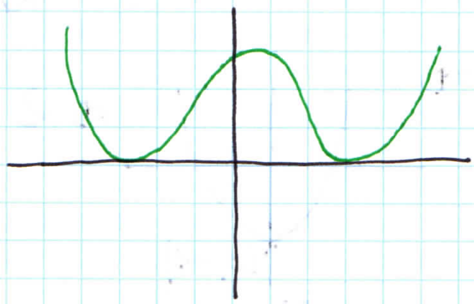
$]0, 1[\Rightarrow f'(x) = 6x(x-1)(x+1)(x^2+1) < 0 \Rightarrow f \searrow$

$]1, +\infty[\Rightarrow f'(x) = 6x(x-1)(x+1)(x^2+1) > 0 \Rightarrow f \nearrow$

-1 y 1 mínimos relativos $\quad$ 0 máximo relativo

$f(0) = 2 \quad f(1) = f(-1) = 0 \to -1$ y 1 son mín absolutos

Nº de ceros $= 2 \quad f(\mathbb{R}) = [0, +\infty[$



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R3:12 $f: \mathbb{R} \setminus \{-1\} \to \mathbb{R}$ $\quad f(x) = \operatorname{arctg}\left(\frac{1-x}{1+x}\right) + \operatorname{arctg}(x)$

¿ $f(\mathbb{R} \setminus \{-1\})$?

$$f(\mathbb{R} \setminus \{-1\}) = f(]-\infty,-1[\cup]-1,+\infty[) =$$
$$= f(]-\infty,-1[) \cup f(]-1,+\infty[)$$

f es continua.

$$f'(x) = \frac{1}{1+\left(\frac{1-x}{1+x}\right)^2} \cdot \frac{-(1+x)-(1-x)}{(1+x)^2} + \frac{1}{1+x^2}$$

$$f'(x) = \frac{1}{1+\frac{(1-x)^2}{(1+x)^2}} \cdot \frac{-2}{(1+x)^2} + \frac{1}{1+x^2}$$

$$f'(x) = \frac{1}{\frac{(1+x)^2+(1-x)^2}{(1+x)^2}} \cdot \frac{-2}{(1+x)^2} + \frac{1}{1+x^2}$$

$$f'(x) = \frac{-2}{1+2x+x^2+1-2x+x^2} + \frac{1}{1+x^2} = \frac{-2}{2+2x^2} + \frac{1}{1+x^2}$$

$$f'(x) = \frac{-1}{1+x^2} + \frac{1}{1+x^2} = 0 \quad \forall x \to \begin{cases} f(x) = C \text{ en }]-\infty,-1[\\ f(x) = C_1 \text{ en }]-1,+\infty[\end{cases}$$

$$f'(x) = 0 \ \forall x \in I \Leftrightarrow f(x) = C$$

* $x \in]-\infty,-1[$, $f(x) = C = ?$

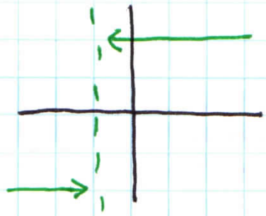
$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \overbrace{\operatorname{arctg}\underbrace{\left(\frac{1-x}{1+x}\right)}_{2/0}}^{-\pi/2} + \underbrace{\operatorname{arctg}(x)}_{-\pi/4} =$$

$x < -1$; $x+1 < 0$ es negativo $\Rightarrow -\infty$

$$= -\frac{\pi}{2} - \frac{\pi}{4} = -3\frac{\pi}{4}$$

* $x \in]-1, +\infty[\Rightarrow f(x) = C_1 = ?$

$$\lim_{x \to -1^+} f(x) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$



$$f(\mathbb{R} \setminus \{-1\}) = \left\{ \frac{\pi}{4}, \frac{-3\pi}{4} \right\}$$

R3.20)

a) $\lim_{x \to 0^+} (\cos(x) + 2\text{sen}(3x))^{1/x} = 1^\infty = \text{Indeterminada} =$

$= \text{"Regla del } n = e\text{"} = \lim_{x \to 0^+} \frac{1}{x} [\cos(x) + 2\text{sen}(3x) - 1] =$

$= \lim_{x \to 0^+} \frac{\cos(x) + 2\text{sen}(3x) - 1}{x} = \text{"}\frac{0}{0}\text{"} = \text{L'Hôpital} =$

$= \lim_{x \to 0^+} -\text{sen}(x) + 6\cos(3x) = 0 + 6 = 6 = L$

$$\lim_{x \to 0^+} (\cos(x) + 2\text{sen}(3x))^{1/x} = e^6$$

b) $\lim_{x \to 0} \frac{(1-\cos(x))\,\text{sen}(4x)}{x^3 \cdot \cos(\frac{\pi}{4} - x)} = \text{"}\frac{0}{0}\text{"} = \text{L'Hôpital} =$

$\lim_{x \to 0} \frac{tg(x)}{x} = 1$

$= \lim_{x \to 0} \frac{1}{\cos(\frac{\pi}{4} - x)} \cdot \lim_{x \to 0} \frac{(1-\cos(x))\,\text{sen}(4x)}{x^3} =$

$= \sqrt{2} \cdot \lim_{x \to 0} \frac{(1-\cos(x))\,\text{sen}(4x)}{x^3} = \sqrt{2} \cdot 4 \cdot \frac{1}{2} = 2\sqrt{2}$

$\lim_{x \to 0} \frac{\text{sen}(x)}{x} = 1$

$\frac{(1-\cos(x))\,\text{sen}(4x)}{x^3} = \frac{(1-\cos(x))}{x^2} \cdot \frac{\text{sen}(4x)}{x} =$

$= \frac{1-\cos(x)}{x^2} \cdot 4\,\frac{\text{sen}(4x)}{4x} \rightarrow \lim_{x \to 0} f(x) = 1$

$\lim_{x \to 0} \frac{1-\cos(x)}{x^2} = \text{"}\frac{0}{0}\text{"} = \text{L'Hôpital} = \frac{\text{sen}(x)}{2x} \rightarrow \lim_{x \to 0} f(x) = 1$

$4 \cdot 1 \cdot \frac{1}{2}$

R3.22)

a) $$\lim_{x\to+\infty} \frac{\log(2+3e^x)}{\sqrt{2+3x^2}} = \text{"}\frac{\infty}{\infty}\text{"} = L'Hôpital =$$

$$= \frac{\frac{3e^x}{2+3e^x}}{\frac{6x}{2\sqrt{2+3x^2}}} = \frac{6e^x\cdot\sqrt{2+3x^2}}{(2+3e^x)\cdot 6x} = \frac{e^x\sqrt{2+3x^2}}{(2+3e^x)x} = \text{"}\frac{\infty}{\infty}\text{"} =$$

$$= \text{Indeterminada} = \frac{e^x}{2+3e^x}\cdot\frac{\sqrt{2+3x^2}}{x} =$$

$$= \frac{1}{\frac{2}{e^x}+3}\cdot\frac{\sqrt{2+3x^2}}{x} = \frac{\sqrt{3}}{3}$$

$\to 1/3$ $\quad\to\sqrt{3}$

R3.24)

$$\lim_{x\to 0}\left(\frac{3\,\text{sen}(x)-3x\cos(x)}{x^3}\right)^{1/x}$$

$$\text{base} = \lim_{x\to 0}\frac{3\,\text{sen}(x)-3x\cos(x)}{x^3} = \text{"}\frac{0}{0}\text{"} = L'Hôpital =$$

$$= \frac{3\cos(x)-3[\cos(x)-x\,\text{sen}(x)]}{3x^2} = \frac{3\cos(x)-3\cos(x)+3x\,\text{sen}(x)}{3x^2} =$$

$$= \frac{\text{sen}(x)}{x}$$

$$\lim_{x\to 0} f(x) = \text{"}1^\infty\text{"} = \text{Indeterminada} = \text{"Regla del nº e"} =$$

$$= \frac{1}{x}\left[\frac{3\,\text{sen}(x)-3x\cos(x)}{x^3}-1\right] = \frac{1}{x}\left[\frac{3\,\text{sen}(x)-3x\cos(x)-x^3}{x^3}\right] =$$

$$= \frac{3\,\text{sen}(x)-3x\cos(x)-x^3}{x^4} = \text{"}\frac{0}{0}\text{"} = L'Hôpital =$$

$$= \frac{3x\,\text{sen}(x)-3x^2}{4x^3} = \frac{3}{4}\cdot\frac{\text{sen}(x)-x}{x^2} = \frac{0}{0} = L'Hôpital =$$

$$= \frac{\cos(x)-1}{2x} = \text{"}\frac{0}{0}\text{"} = L'Hôpital = \frac{-\text{sen}(x)}{2} = 0 = L$$

$$\lim_{x\to 0} f(x) = e^0 = 1$$

R3.26)

$x, y \in \mathbb{R}^+ \begin{cases} \to \text{Suma} = 20 \\ \to \text{producto sea máximo} \end{cases}$

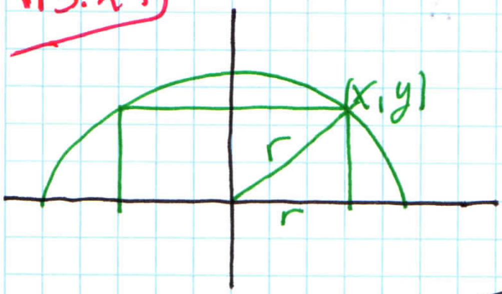
$x+y=20 \;;\; y = 20-x$

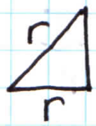
$x \cdot y = \max \to x\cdot(20-x) \to f(x) = x(20-x)$

$f(x) = 20x - x^2 \qquad f'(x) = 20-2x \qquad f'(x)=0 \to x=10$

$f: \,]0, 20[\, \to \mathbb{R}$

R3.27)





area = base · altura

base = $2x$

altura = y

area = $2xy$

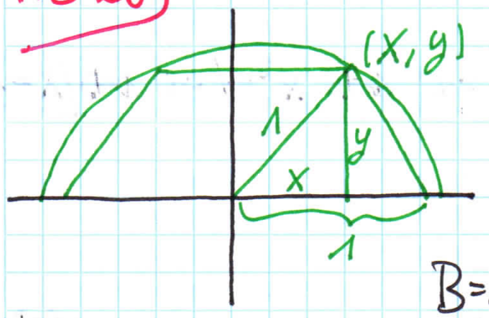
$x^2+y^2 = r^2 \;;\; y = \sqrt{r^2-x^2}$

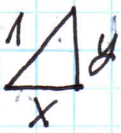
$f(x) = 2x\sqrt{r^2-x^2}$

$$f'(x) = 2\left[\sqrt{r^2-x^2} + x\,\frac{(-2x)}{2\sqrt{r^2-x^2}}\right] = 2\left[\frac{r^2-2x^2}{\sqrt{r^2-x^2}}\right]$$

$$f'(x)=0 \Leftrightarrow r^2-2x^2=0 \Leftrightarrow x^2 = \frac{r^2}{2} \to x = \frac{r}{\sqrt{2}}$$

R3.28)





$$\text{área} = \left(\frac{b+B}{2}\right)\cdot h =$$

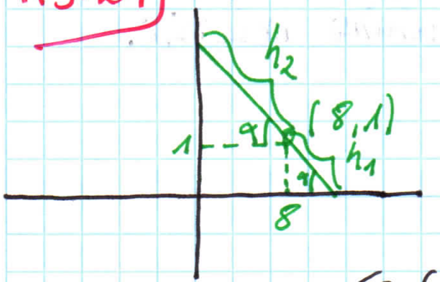
$B=2 \quad b=2x \quad h=y \qquad = \left(\frac{2+2x}{2}\right)\cdot y = (x+1)\,y$

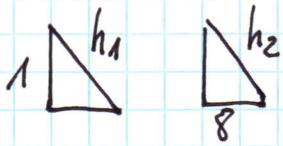
$$x^2+y^2=1 \Rightarrow y=\sqrt{1-x^2}$$

$$f:[0,1]\to \mathbb{R} \qquad f(x)=(x+1)\cdot\sqrt{1-x^2}$$

$f'(x)=0$ en $]0,1[$

R3.29)





$h_1+h_2 \Rightarrow$ ¿ mínimo?

$$sen(\alpha)=\frac{1}{h_1} \Rightarrow h_1=\frac{1}{sen(x)}$$

$$cos(\alpha)=\frac{8}{h_2} \Rightarrow h_2=\frac{8}{cos(\alpha)}$$

$$f(x)=\frac{1}{sen(\alpha)}+\frac{8}{cos(\alpha)} \qquad \alpha\in\,]0,\pi/2[$$

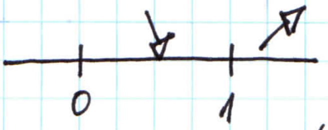
R3.30) $x + 1/x \geq 2$ $\quad f: \mathbb{R}^+ \to \mathbb{R}$ $\quad f(x) = x + 1/x$

¿$f(x) \geq 2, \forall x \in \mathbb{R}^+$?

$$f'(x) = \frac{x^2-1}{x^2} \; ; \; f'(x) = 0 \Leftrightarrow \frac{x^2-1}{x^2} = 0 \Leftrightarrow x^2 - 1 = 0 \Leftrightarrow \boxed{x=1}$$

<u>Opción 1</u>

- Intervalos de monotonía



$$f'(x) = \frac{(x-1)(x+1)}{x^2} \begin{cases} 0<x<1 \Rightarrow f'(x)<0 \\ x>1 \Rightarrow f'(x)>0 \end{cases}$$

mínimo relativo

$\Downarrow$

mínimo absoluto $\quad f(1) = 1+1 = 2 \Rightarrow f(x) \geq 2 \; \forall x \in \mathbb{R}^+$

<u>Opción 2</u>

$$f''(x) = \frac{2}{x^3} \; ; \; f''(1) = 2 > 0 \to \text{mínimo relativo.}$$

R3.35) $f(x) = x^4 - 5x^3 - 3x^2 + 7x + 6$, en potencias de $(x-2)$

$a = 2 \qquad n = 4$

$$f(x) = x^4 - 5x^3 - 3x^2 + 7x + 6 \Rightarrow f(2) = -16$$
$$f'(x) = 4x^3 - 15x^2 - 6x + 7 \Rightarrow f'(2) = -33$$
$$f''(x) = 12x^2 - 30x - 6 \Rightarrow f''(2) = -18$$
$$f'''(x) = 24x - 30 \Rightarrow f'''(2) = 18$$
$$f^{iv}(x) = 24$$

$$P_4(x) = -16 - 33\cdot(x-2) - \frac{18}{2}(x-2)^2 + \frac{18}{6}(x-2)^3 + \frac{24}{24}(x-2)^4$$

R3.39) f una función cuyo P_3 centrado en 0.

$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3}$$

¿Calcular P_3 centrado en 0 de $g(x) = x f(x)$?

$$f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3}$$

$$f(0) = 1$$

$$f'(0) = 1$$

$$\frac{f''(0)}{2!} = \frac{1}{2} \Rightarrow f''(0) = 1$$

$$\frac{f'''(0)}{3!} = \frac{1}{3!} = f'''(0) = 2$$

$$G_3(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3$$

$$g(x) = x f(x) \Rightarrow g(0) = 0\cdot 1 = 0$$

$$g'(x) = f(x) + x f'(x) \Rightarrow g'(0) = 1 + 0\cdot 1 = 1$$

$$g''(x) = f'(x) + f'(x) + x f''(x) \Rightarrow g''(0) = 1 + 1 + 0\cdot 1 = 2$$

$$g'''(x) = f''(x) + f''(x) + f''(x) + x f'''(x) = 1 + 1 + 1 + 0\cdot 2 = 3$$

$$G_3(x) = 0 + x + x^2 + \frac{1}{2}x^3$$

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1) Acota el error que se comete cuando aproximamos el valor de e a través del polinomio de Taylor de la f exponencial en el 0 y de orden 3

$f(x) = e^x \quad a=0 \quad n=3$

$e = e^1 = f(1) \sim P_3(1) \rightarrow \text{error} = |R_3(1)| = \left| \frac{f^{IV}(c)}{4!} \cdot 1^4 \right|$

$c \in]0,1[\rightarrow 0 < c < 1$

$\text{error} = \left| \frac{e^c}{4!} \right| = \frac{e^c}{4!} \qquad e^c < e < 3$ cuando acotamos es solo por arriba

$\Downarrow$

$\frac{e^c}{4!} < \frac{3}{4!}$

$\text{error} = \frac{e^c}{4!} < \frac{3}{4!} = \frac{3}{4!} = 1/8 = 0'125$

$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \qquad e \sim P_3(1) = 1 + 1 + 1/2 + 1/6$

2) Acote el error que se comete cuando aproximamos "$\sqrt{e}$" a través del P_4 de f exponencial con centro "0"

$f(x) = e^x \quad a=0 \quad n=4$

$\sqrt{e} = e^{1/2} = f(1/2) \quad \boxed{x = 1/2}$

$\text{error} = |f(1/2) - P_4(1/2)| = |R_4(1/2)| = \left| \frac{f^{V}(c)}{5!} (1/2)^5 \right|$

$c \in]0, 1/2[\rightarrow 0 < c < 1/2 \qquad e^c < e^{1/2} < 2$

$\text{error} = \left| \frac{e^c}{5!} \cdot \frac{1}{2^5} \right| = \frac{e^c}{5!} \cdot \frac{1}{2^5} = \frac{e^c}{5! 2^5} < \frac{2}{5! 2^5} = 0'00052$

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3) Acota el error que se comete al aproximar $\text{sen}(1/2)$ por P_3 de la f sen en $a=0$

$f(x) = \text{sen}(x) \qquad a=0 \qquad n=3$

$f(1/2) = \text{sen}(1/2)$

$\text{error} = |f(1/2) - P_3(1/2)| = |R_3(1/2)| = \left|\frac{f^{IV}(c)}{4!} \cdot (1/2)^4\right|$

$0 < c < 1/2$

$\text{error} = \frac{1}{2^4 \cdot 4!} \cdot |f^{IV}(c)| \leq \frac{1}{2^4 4!}$

$\underbrace{\text{sen}(c);\ -\text{sen}(c);\ \cos(c);\ -\cos(c)}_{|f^{IV}(c)| \leq 1}$

R3.36)

a) $\alpha = \sqrt{e}$ error $< 10^{-2}$ $f(x) = e^x$

$\alpha = \sqrt{e} = f(1/2)$ $a=0$ ¿n? $\Rightarrow$ para error $< 10^{-2}$

$$\text{error} = |f(x) - P_n(x)| = |R_n(x)| = \left|\frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}\right|$$

$$\text{error} = |f(1/2) - P_n(1/2)| = |R_n(1/2)| = \left|\frac{f^{n+1}(c)}{(n+1)!}\left(\frac{1}{2}\right)^{n+1}\right|$$

$$0 < c < 1/2 \qquad e^c < 2$$

$$\text{error} = \frac{e^c}{(n+1)!} \cdot \frac{1}{2^{n+1}} < \frac{2}{(n+1)!\,2^{n+1}} = \frac{1}{(n+1)!\,2^n} < 10^{-2} =$$

$$= (n+1)!\,2^n > 10^2 = 100$$

$n=1 \Rightarrow 4 > 100$ ✗

$n=2 \Rightarrow 24 > 100$ ✗

$n=3 \Rightarrow 192 > 100$ ✓

$$f(1/2) \sim P_3(1/2) \qquad P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$P_3(1/2) = 1 + \frac{1}{2} + \frac{1}{2!\,2^2} + \frac{1}{3!\,2^3}$$

Ejercicios

1) $\int x^2 dx = \frac{x^3}{3} + c$

2) $\int sen^2(x) \cdot \cos(x) dx = \frac{sen^3(x)}{3} + c$

3) $\int (x^2+1)^{1000} 2x\, dx = \frac{(x^2+1)^{1001}}{1001} + c$

4) $\int \sqrt{x^3+1}\, x^2 dx = \frac{1}{3} \int (x^3+1)^{1/2} \cdot 3x^2 dx = \frac{1}{3} \cdot \frac{(x^3+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

5) $\int \frac{x^2}{\sqrt[4]{x^3+2}} dx = \int (x^3+2)^{-1/4} x^2 dx = \frac{1}{3} \int (x^3+2)^{-1/4} 3x^2 dx =$

$= \frac{1}{3} \frac{(x^3+2)^{-\frac{1}{4}+1}}{-\frac{1}{4}+1} + c$

$\int \frac{1}{x} dx = \log|x| + c \; ; \int \frac{g'(x)}{g(x)} dx = \log|g(x)|$

6) $\int \frac{x}{x^2+1} dx = \frac{1}{2} \int \frac{2x}{x^2+1} dx = \frac{1}{2} \log|x^2+1| + c = \frac{1}{2} \log(x^2+1) + c$

7) $\int \frac{\cos(x)}{sen(x)} dx = \log|sen(x)| + c$

8) $\int tg(x) dx = \int \frac{sen(x)}{\cos(x)} dx = -1 \cdot \int \frac{-sen(x)}{\cos(x)} dx = -1 \cdot \log|\cos(x)| + c$

9) $\int \sqrt{\cos(x)} \cdot sen(x) dx = \int (\cos(x))^{1/2} \cdot sen(x) dx = -\int (\cos(x))^{1/2} \cdot -sen(x) dx =$

$= -\frac{(\cos(x))^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

10) $\int \frac{x^2}{1-2x^3} dx = -\frac{1}{6} \cdot \int \frac{-6x^2}{1-2x^3} dx = -\frac{1}{6} \cdot \log|1-2x^3| + c$

11) $\int \frac{1}{x \log(x)} dx = \int \frac{1/x}{\log(x)} dx = \log|\log(x)| + c$

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12) $\int \frac{e^{1/x}}{x^2}\,dx = \int e^{1/x}\cdot \frac{1}{x^2}\,dx = -\int e^{1/x}\cdot\frac{(-1)}{x^2}\,dx = -e^{1/x}+c$

13) $\int \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx = \int 2^{\sqrt{x}}\cdot \frac{1}{\sqrt{x}}\,dx = 2\int 2^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}\,dx = 2\cdot\frac{2^{\sqrt{x}}}{\log(2)}+c =$

$= \frac{2^{\sqrt{x}+1}}{\log(2)}+c$

14) $\int e^{\cos(x)}\cdot \operatorname{sen}(x)\,dx = -e^{\cos(x)}+c$

$$\int_a^a f(x)\,dx = 0\,; \quad a<b \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_3^2 \frac{1}{x}\,dx = -\int_2^3 \frac{1}{x}\,dx$$

1) $F(x)=\int_1^x \operatorname{sen}(t)\,dt \Leftrightarrow \exists F'(x) = \left(\int_1^x \operatorname{sen}(t)\right)' = \operatorname{sen}(x)$

2) $F(x)=\int_0^x e^{-t^2}\,dt \Leftrightarrow \exists F'(x) = \left(\int_0^x e^{-t^2}dt\right)' = e^{-x^2}$

3) $F(x)=\int_1^{x^2} e^{-t^2}\,dt \qquad x \xrightarrow{\text{derivable}} x^2 \xrightarrow{\text{derivable}} \int_1^{x^2} e^{-t^2}\,dt$

$$\exists F'(x) = \left(\int_1^{x^2} e^{-t^2}\,dt\right) = e^{-(x^2)^2}\cdot 2x$$

$$\left(\int_a^{g(x)} f(t)\,dt\right)' = f(g(x))\cdot g'(x)$$

4) $F(x)=\int_{\operatorname{sen}(x)}^0 e^{-t^2}\,dt = -\int_0^{\operatorname{sen}(x)} e^{-t^2}\,dt \Rightarrow \exists F'(x) = -e^{-\operatorname{sen}^2(x)}\cdot\cos(x)$

$$\left(\int_{h(x)}^a f(t)\,dt\right)' = -f(h(x))\cdot h'(x)$$

5) $F(x)=\int_{x^2}^{x^3}\log(1+t^2)\,dt=\int_{x^2}^{0}\log(1+t^2)\,dt+\int_{0}^{x^3}\log(1+t^2)\,dt=$

$$=\int_0^{x^3}\log(1+t^2)\,dt-\int_0^{x^2}\log(1+t^2)\,dt \Rightarrow \exists\, F'(x)=\log(1+(x^3)^2)\cdot 3x^2-$$
$$-\log(1-(x^2)^2)\cdot 2x$$

$$\left(\int_{h(x)}^{g(x)} f(t)\,dt\right)'=f(g(x))\cdot g'(x)-f(h(x))\cdot h'(x)$$

6) Calcular $\lim_{x\to 0}\dfrac{\int_0^{x^2}e^{-t^2}\,dt}{\operatorname{sen}^2(x)}$

$$\lim_{x\to 0}\int_0^{x^2}e^{-t^2}\,dt=\int_0^0 e^{-t^2}\,dt=0$$

L'Hôpital: $\dfrac{\left(\int_0^{x^2}e^{-t^2}\,dt\right)'}{2\operatorname{sen}(x)\cdot\cos(x)}=\dfrac{e^{-x^4}\cdot 2x}{2\operatorname{sen}(x)\cdot\cos(x)}=\dfrac{e^{-x^4}}{\cos(x)}\cdot\dfrac{x}{\operatorname{sen}(x)}$

$$\lim_{x\to 0}\frac{e^{-x^4}}{\cos(x)}\cdot\frac{x}{\operatorname{sen}(x)}=1\cdot 1=1$$

7) Calcular $\lim_{x\to+\infty}\dfrac{\int_0^{\log(x^2)}e^{-(t-1)^2}\,dt}{\log(x)}$

L'Hôpital: $\dfrac{e^{-(\log(x^2)-1)^2}\cdot\frac{2x}{x^2}}{1/x}=e^{-\log(\log(x^2)-1)^2}\cdot\dfrac{2/x}{1/x}=$

$$=2\,e^{-(\log(x)^2-1)^2}\xrightarrow{x\to+\infty}2\cdot 0=0$$

8) $\int x e^{ax^2+b} = \frac{1}{2a} \cdot \int 2ax e^{ax^2+b} dx = \frac{1}{2a} \cdot e^{ax^2+b} + c$

$(a, b \in \mathbb{R})$ $(\forall a \neq 0, \forall b \in \mathbb{R})$

$\boxed{a=0} \Rightarrow \int x e^{b} dx = e^{b} \int x dx = e^{b} \cdot \frac{x^2}{2} + c$

9) $\int \frac{1}{1+e^x} dx = \int \frac{1+e^x-e^x}{1+e^x} dx = \int \left(\frac{1+e^x}{1+e^x} - \frac{e^x}{1+e^x} \right) dx =$

$= \int \left(1 - \frac{e^x}{1+e^x}\right) dx = \int 1 dx - \int \frac{e^x}{1+e^x} dx = x - \log|1+e^x| + c$

10) $\int \frac{1+\operatorname{sen}(x)}{\cos^2(x)} dx = \int \frac{1}{\cos^2(x)} dx + \int \frac{\operatorname{sen}(x)}{\cos^2(x)} dx = \operatorname{tg}(x) +$

$+ \int \operatorname{sen}(x) \cdot \cos^{-2}(x) dx = \operatorname{tg}(x) + \frac{1}{\cos(x)} + c$

1) Ejemplo cambio de variable o sustitución

* $\int \frac{\cos(x)}{\sqrt[3]{\operatorname{sen}(x)}} dx = \begin{bmatrix} \operatorname{sen}(x) = t \\ \cos(x) dx = dt \end{bmatrix} = \int \frac{1}{\sqrt[3]{t}} dt = \int t^{-1/3} dt =$

$= \frac{t^{-1/3+1}}{-\frac{1}{3}+1} + c = \frac{3}{2} \cdot t^{2/3} + c = \frac{3}{2} \operatorname{sen}(x)^{\frac{2}{3}} + c$

* $\int \frac{x^2}{\sqrt{1-x^6}} dx = \begin{bmatrix} x^3 = t \\ 3x^2 dx = dt \end{bmatrix} = \begin{bmatrix} x^3 = t \\ x^2 dx = \frac{dt}{3} \end{bmatrix} = \int \frac{\frac{dt}{3}}{\sqrt{1-t^2}} =$

$= \frac{1}{3} \int \frac{1}{\sqrt{1-t^2}} dt = \frac{1}{3} \operatorname{arcsen}(t) + c = \frac{1}{3} \operatorname{arcsen}(x^3) + c$

CA

Ejercicios

2) Ejemplo integración por partes

* $\int \log(x)\,dx = \begin{bmatrix} u=\log(x) \Rightarrow du = \frac{1}{x}dx \\ dv = dx \Rightarrow v = x \end{bmatrix} =$

$= \log(x)\cdot x - \int x \cdot \frac{1}{x}dx = \log(x)\cdot x - \int 1\,dx = x\log(x) - x + c$

* $\int x e^x dx = \begin{bmatrix} u = x \Rightarrow du = dx \\ dv = e^x dx \Rightarrow v = e^x \end{bmatrix} = xe^x - \int e^x dx = xe^x - e^x + c$

* $\int x^2 e^x dx = \begin{bmatrix} u = x^2 \Rightarrow du = 2x\,dx \\ dv = e^x dx \Rightarrow v = e^x \end{bmatrix} = x^2 e^x - 2\int x e^x dx =$

$= x^2 e^x - 2x e^x + 2e^x + c$

* $\int e^x \operatorname{sen}(x)\,dx = \begin{bmatrix} u = e^x \Rightarrow du = e^x dx \\ dv = \operatorname{sen}(x)dx \Rightarrow v = -\cos(x) \end{bmatrix} =$

$= -e^x\cdot\cos(x) + \int e^x\cdot\cos(x)\,dx = \begin{bmatrix} u = e^x \Rightarrow du = e^x dx \\ dv = \cos(x)dx \Rightarrow v = \operatorname{sen}(x) \end{bmatrix} =$

$= -e^x\cdot\cos(x) + e^x\cdot\operatorname{sen}(x) - \int e^x \operatorname{sen}(x)\,dx$

$\alpha = e^x(\operatorname{sen}(x) - \cos(x)) - \alpha$

$\alpha = \frac{1}{2} e^x(\operatorname{sen}(x) - \cos(x))$

* $\int \operatorname{arctg}(x)\,dx = \begin{bmatrix} u = \operatorname{arctg}(x) \Rightarrow du = \frac{1}{1+x^2}dx \\ dv = dx \Rightarrow v = x \end{bmatrix} =$

$= x\operatorname{arctg}(x) - \int x\cdot\frac{1}{1+x^2}dx = x\operatorname{arctg}(x) - \int \frac{x}{1+x^2}dx =$

$= x\operatorname{arctg}(x) - \frac{1}{2}\log(1+x^2) + c$

* $\int \frac{x}{(x^2+1)^2}dx = \frac{1}{2}\int 2x(x^2+1)^{-2}dx = \frac{1}{2}\frac{(x^2+1)^{-2+1}}{-2+1}+c =$

$= -\frac{1}{2}\cdot\frac{1}{x^2+1}+c$

* $\int \frac{x^2}{(x^3+7)^3} = \int x^2\cdot(x^3+7)^{-3}dx = \frac{1}{3}\cdot\int(3x^2)\cdot(x^3+7)^{-3}dx =$

$= \frac{1}{3}\frac{(x^3+7)^{-2}}{-2}+c$

3.1.) Ejemplo de integración de $\int$ racionales

* $\int \frac{2x+3}{x^2+2x+2}dx \qquad x^2+2x+2=(x^2+1)^2+1=[t=x+1]=t^2+1$

$$\int\frac{2x+3}{x^2+2x+2}dx = \left[\begin{array}{l} t=x+1 \Rightarrow x=t-1 \\ dt=dx\end{array}\right] = \int\frac{2\cdot(t-1)+3}{t^2+1}dt =$$

$$=\int\frac{2t+1}{t^2+1}dt = \int\frac{2t}{t^2+1}dt+\int\frac{1}{t^2+1}dt =$$

$$= \log|t^2+1| + arc\,tg(t)+c = \log(x^2+2x+2)+arc\,tg(x+1)+c$$

3.2) Ejemplo de integración de $\int$ racionales

* $\int\frac{x+7}{x(x^2-1)}dx \qquad Q(x)=x(x^2-1)=x(x+1)\cdot(x-1)$

$$\frac{x+7}{x(x+1)(x-1)} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-1} = \frac{A(x^2-1)+Bx(x+1)+Cx(x-1)}{x(x-1)(x+1)}$$

$x+7 = A(x^2-1)+Bx(x+1)+Cx(x-1)$

$x=0 \Rightarrow 7=-A$; $A=-7$

$x=1 \Rightarrow 8=2B$; $B=4$

$x=-1 \Rightarrow 6=2C$; $C=3$

CA

Ejercicios

$$\int \frac{x+7}{x(x^2-1)}\,dx = \int \left(\frac{-7}{x} + \frac{4}{x-1} + \frac{3}{x+1} \right) dx =$$

$$= -7 \cdot \log|x| + 4\log|x-1| + 3\log|x+1| + C$$

* $\int \frac{1}{x^4-1}\,dx \qquad x^4-1 = (x^2+1)\cdot(x^2-1) = (x-1)\cdot(x+1)\cdot(x^2+1)$

$$\frac{1}{x^4-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} =$$

$$= \frac{A(x+1)\cdot(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2+1)}{(x-1)(x+1)\cdot(x^2+1)}$$

$$1 = A(x^3+x^2+x+1) + B(x^3-x^2+x-1) + Cx^3 - Cx + Dx^2 - D$$

(coeficiente grado 3) $A+B+C=0$

(coeficiente grado 2) $A-B+D=0$

(coeficiente grado 1) $A+B-C=0$

(coeficiente grado 0) $A-B-D=1$

$2(A+B)=0 \Rightarrow A+B=0 \Rightarrow B=-A$

$2(A-B)=1 \Rightarrow 2(A+A)=1 \Rightarrow$

$\Rightarrow 4A=1 \Rightarrow A=1/4$

$$B=-1/4 \qquad C=0 \qquad D=-1/2$$

$$\frac{1}{x^4-1} = \frac{1/4}{x-1} + \frac{-1/4}{x+1} + \frac{-1/2}{x^2+1}$$

$$\int \frac{1}{x^4-1}\,dx = \frac{1}{4}\log|x-1| - \frac{1}{4}\log|x+1| - \frac{1}{2}\operatorname{arctg}(x) + C$$

$$* \int_2^{+\infty} \frac{1}{x^4-1}\,dx = \frac{1}{4}\log|x-1| - \frac{1}{4}\log|x+1| - \frac{1}{2}\operatorname{arctg}(x)\Big]_2^{+\infty} =$$

$$= \frac{1}{4}\left(\log|x-1| - \log|x+1|\right) - \frac{1}{2}\operatorname{arctg}(x)\Big]_2^{+\infty} =$$

$$= \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\operatorname{arctg}(x)\Big]_2^{+\infty} = \left(0 - \frac{\pi}{4}\right) - \left(\frac{1}{4}\log(1/3) -\right.$$

$$\left. - \frac{1}{2}\operatorname{arctg}(2)\right) = -\frac{\pi}{4} + \log|\sqrt[4]{3}| + \frac{1}{2}\operatorname{arctg}(2)$$

3.3) Ejemplo de integración de ∫ racionales

$$* \int \frac{1}{(x-1)\cdot(x+1)^3}\,dx$$

$$(x+y)^3 = x^3 + 3x^2 + y + 3xy^2 + y^3$$

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3} =$$

$$= \frac{A(x+1)^3 + B(x-1)(x+1)^2 + C(x-1)(x+1) + D(x-1)}{(x-1)\cdot(x+1)^3}$$

$$1 = A(x^3+3x^2+3x+1) + B(x^3+x^2-x-1) + C(x^2-1) + Dx - D$$

$$\left.\begin{array}{l} A+B=0 \\ 3A+B+C=0 \\ 3A-B+D=0 \\ A-B-C-D=1 \end{array}\right\} \quad \begin{array}{ll} A=\frac{1}{8} & B=-\frac{1}{8} \\ C=-\frac{1}{4} & D=-\frac{1}{2} \end{array}$$

$$\int \frac{1}{(x-1)(x+1)^3}\,dx = \frac{1}{8}\int \frac{1}{x-1}\,dx - \frac{1}{8}\int \frac{1}{x+1}\,dx - \frac{1}{4}\int \frac{1}{(x+1)^2}\,dx -$$

$$- \frac{1}{2}\int \frac{1}{(x+1)^3}\,dx = \frac{1}{8}\log|x-1| - \frac{1}{8}\log|x+1| + \frac{1}{4}\frac{1}{x+1} + \frac{1}{4}\frac{1}{(x+1)^2} + C$$

CA

Ejercicios

3.4) Ejemplo de integración de f racional

* $\int \frac{1}{x^2(x^2+1)^2}\,dx \qquad Q(x) = x^2(x^2+1)^2$

$$\frac{1}{x^2(x^2+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$$

$$\frac{1}{x^2(x^2+1)^2} = \frac{Ax(x^2+1)^2 + B(x^2+1)^2 + (Cx+D)x^2(x^2+1) + (Ex+F)x^2}{x^2(x^2+1)^2} =$$

$$= \frac{A(x^5+2x^3+x) + B(x^4+2x^2+1) + Cx^5 + Cx^3 + Dx^4 + Dx^2 + Ex^3 + Fx^2}{x^2(x^2+1)^2}$$

$A + C = 0 \Rightarrow -A = C$

$B + D = 0 \Rightarrow D = -B$

$2A + C + E = 0 \Rightarrow A + E = 0 \Rightarrow E = -A$

$2B + D + F = 0$

$A = 0 = C \qquad B = 1 \qquad D = 1 \qquad E = 0 \qquad F = -1$

$$\int \frac{1}{x^2(x^2+1)^2}\,dx = \int \frac{1}{x^2} + \int \frac{-1}{x^2+1} + \int \frac{-1}{(x^2+1)^2} =$$

$$= -\frac{1}{x} - \operatorname{arctg}(x) - \int \frac{1}{(x^2+1)^2}\,dx$$

$$\int \frac{1}{(x^2+1)^2}\,dx = \int \frac{1+x^2-x^2}{(x^2+1)^2}\,dx = \int \frac{1+x^2}{(x^2+1)^2}\,dx - \int \frac{x^2}{(x^2+1)^2}\,dx =$$

$$= \operatorname{arctg}(x) - \int \frac{x}{(x^2+1)^2} \cdot x\,dx$$

$$\int \frac{x}{(x^2+1)^2}\cdot x\,dx = \left[\begin{array}{l} u=x \Rightarrow du=dx \\ dv=\frac{x}{(x^2+1)^2} \Rightarrow v=\int\frac{x}{(x^2+1)^2}dx=\frac{-1}{2(x^2+1)}\end{array}\right] =$$

$$= \frac{-x}{2(x^2+1)} + \frac{1}{2}\int\frac{1}{(x^2+1)} = \frac{-x}{2(x^2+1)} + \frac{1}{2}\operatorname{arctg}(x)$$

<u>Solución</u>: $-\frac{1}{x} - \operatorname{arctg}(x) - \left(\operatorname{arctg}(x) - \left(\frac{-x}{2(x^2+1)} + \frac{1}{2}\operatorname{arctg}(x)\right)\right)$

R4.1)

b) $F(x) = \int_x^b \frac{1}{1+t^2+\operatorname{sen}^2(t)}\,dt = -\int_b^x \frac{1}{1+t^2+\operatorname{sen}^2(t)}\,dt$

$$f(t) = \frac{1}{1+t^2+\operatorname{sen}^2(t)} \overset{T.F.C.}{\Rightarrow} \exists F'(x) = -\frac{1}{1+x^2+\operatorname{sen}^2(x)}$$

c) $F(x) = \int_a^b \frac{x}{1+t^2+\operatorname{sen}^2(t)}\,dx = x\int_a^b \frac{1}{1+t^2+\operatorname{sen}^2(t)}\,dt = \alpha\cdot x$

$$\exists F'(x) = \alpha = \int_a^b \frac{1}{1+t^2+\operatorname{sen}^2(t)}\,dt$$

CA

Ejercicios

R4.5) $f:[1,+\infty[\rightarrow \mathbb{R}$; $f(x)=\int_0^{x-1}(e^{-t^2}-e^{-2t})\,dt$

a) Calcular el máximo absoluto de f

b) Sabiendo que $\lim_{x\to+\infty} f(x)=\frac{1}{2}(\sqrt{\pi}-1)$, calcula el máximo

a) f es derivable $\Rightarrow$ es continua en $[1,+\infty[$

$$f'(x)=e^{-(x-1)^2}-e^{-2(x-1)}$$

$$f'(x)=0 \Leftrightarrow ?$$

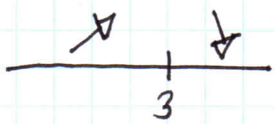
$$e^{-(x-1)^2}=e^{-2(x-1)};\quad -(x-1)^2=-2(x-1);$$

$$(x-1)\left[-(x-1)+2\right]=0 \begin{cases} \rightarrow x=1 \\ \rightarrow x=3 \end{cases}$$

P. Crítico $\rightarrow x=3$ $\qquad f'(3)=0$

$1<x<3 \Rightarrow f'(x)>0 \Rightarrow f \nearrow$

$3<x \Rightarrow f'(x)<0 \Rightarrow f \searrow$



$x=3 \Rightarrow$ Máximo absoluto.

b) Mínimo absoluto en $x=1$ porque vale 0

$$f(1)=\int_0^0 t\,dt=0$$

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R4.8)

f) $\int \frac{x^2+1}{x-1}\,dx$

$$x^2+1 \;\underline{\lfloor x-1}\qquad x+1 \qquad 2/$$

$$2+(x-1)\cdot(x+1) = 2+(x^2+x-x-1) = x^2+1$$

$$\int \frac{x^2+1}{x-1}\,dx = \int \left(x+1 + \frac{2}{x-1}\right)dx = \int x+1\,dx + 2\int \frac{1}{x-1}\,dx =$$

$$= \frac{x^2}{2} + x + 2\log|x-1| + c$$

R4.10)

$$\int \operatorname{arctg}(x)\,dx = \begin{bmatrix} u = \operatorname{arctg}(x) \Rightarrow du = \frac{1}{1+x^2}\,dx \\ dv = dx \Rightarrow v = x \end{bmatrix} =$$

$$= x\operatorname{arctg}(x) - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx = x\operatorname{arctg}(x) - \frac{1}{2}\log|1+x^2| + c$$

4.1) Ejemplo integración funciones trigonométricas

$*$ $\int \cos^2(x)\cdot \operatorname{sen}(x)\,dx = -\int \cos^2(x)\cdot(-\operatorname{sen}(x))\,dx = \frac{-\cos^3(x)}{3} + c$

$*$ $\int \cos^3(x)\,dx = \int \cos^2(x)\cdot\cos(x)\,dx = \int (1-\operatorname{sen}^2(x))\cdot\cos(x)\,dx =$

$$= \begin{bmatrix} t = \operatorname{sen}(x) \\ dt = \cos(x)\,dx \end{bmatrix} = \int (1-t^2)\,dt = t - \frac{t^3}{3} + c = \operatorname{sen}(x) - \frac{\operatorname{sen}^3(x)}{3} + c$$

$*$ $\int \operatorname{sen}^5(x)\,dx = \int \operatorname{sen}^4(x)\cdot\operatorname{sen}(x)\,dx = \int (1-\cos^2(x))^2\cdot\operatorname{sen}(x)\,dx =$

$$= \begin{bmatrix} t = \cos(x) \\ -dt = \operatorname{sen}(x)\,dx \end{bmatrix} = -\int (1-t^2)^2\,dt = -\int (1-2t^2+t^4)\,dt =$$

$$= \int (-1+2t^2-t^4)\,dt = -t + 2\frac{t^3}{3} - \frac{t^5}{5} + c$$

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$$* \int sen^2(x)\cdot cos^3(x)\,dx = \int sen^2(x)\cdot cos^2(x)\cdot cos(x)\,dx$$

$$= \int sen^2(x)\cdot(1-sen^2(x))\cdot cos(x)\,dx = \begin{bmatrix} t = sen(x) \\ dt = cos(x)\,dx \end{bmatrix} =$$

$$= \int t^2(1-t^2)\,dt = \int (t^2-t^4)\,dt = \frac{t^3}{3} - \frac{t^5}{5} + c =$$

$$\frac{sen^3(x)}{3} - \frac{sen^5(x)}{5} + c$$

4.2) Ejemplo integración funciones trigonométricas.

$$* \int sen^2(x)\cdot cos^2(x)\,dx = \int \left(\frac{1-cos(2x)}{2}\right)\cdot\left(\frac{1+cos(2x)}{2}\right)dx =$$

$$= \frac{1}{4}\int (1-cos(2x))\cdot(1+cos(2x))\,dx =$$

$$= \frac{1}{4}\int (1-cos^2(2x))\,dx = \frac{1}{4}\int 1\,dx - \frac{1}{4}\int cos^2(2x)\,dx =$$

$$= \frac{x}{4} - \frac{1}{4}\int \frac{1+cos(4x)}{2}\,dx = \frac{x}{4} - \frac{1}{4}\int \frac{1}{2}\,dx - \frac{1}{4}\int \frac{cos(4x)}{2}\,dx =$$

$$= \frac{x}{4} - \frac{x}{8} - \frac{1}{8}\int cos(4x)\,dx = \frac{x}{8} - \frac{1}{32}\cdot sen(4x) + c$$

4.3) Ejemplo de integración funciones trigonométricas

$$* \int \frac{1}{\operatorname{sen}(x) - tg(x)}\,dx = \int \frac{1}{\operatorname{sen}(x) - \frac{\operatorname{sen}(x)}{\cos(x)}}\,dx = \begin{bmatrix} t = tg\left(\frac{x}{2}\right) \Rightarrow dx = \frac{2}{1+t^2}\,dt \\ \cos(x) = \frac{1-t^2}{1+t^2}\ ;\ \operatorname{sen}(x) = \frac{2t}{1+t^2} \end{bmatrix} =$$

$$= \int \frac{1}{\frac{\operatorname{sen}(x)\cdot\cos(x) - \operatorname{sen}(x)}{\cos(x)}}\,dx = \int \frac{\cos(x)}{\operatorname{sen}(x)\cdot\cos(x) - \operatorname{sen}(x)}\,dx =$$

$$= \int \frac{\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}\cdot\frac{(1-t^2)}{1+t^2} - \frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$* \int \frac{\cos(x)}{\operatorname{sen}(x)\cdot\cos(x) - \operatorname{sen}(x)}\,dx = 2\int \frac{1-t^2}{2t-2t^3-2t-2t^3}\,dt =$$

$$= 2\int \frac{1-t^2}{-4t^3} = \frac{2}{4}\int \frac{t^2-1}{t^3} = \frac{1}{2}\int \left(\frac{1}{t} - \frac{1}{t^2}\right) dt =$$

$$= \frac{1}{2}\left(\log|t| - \frac{t^{-2}}{-2}\right) = \frac{1}{2}\log|t| + \frac{1}{4}\cdot\frac{1}{t^2} + C =$$

$$= \frac{1}{2}\cdot\log\left|tg\left(\frac{x}{2}\right)\right| + \frac{1}{4}\cdot\frac{1}{tg\left(\frac{x}{2}\right)} + C$$

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4.4.) Ejemplo integración funciones trigonométricas

* $$\int \frac{1}{sen^3(x)\cdot cos^5(x)}\,dx = \left[\begin{array}{l} t = tg(x) \Leftrightarrow dx = \frac{1}{1+t^2}\,dt \\ cos(x) = \frac{1}{\sqrt{1+t^2}}\ ;\ sen(x) = \frac{t}{\sqrt{1+t^2}} \end{array}\right] =$$

$$= \int \frac{1}{\frac{t^3}{(\sqrt{1+t^2})^3}\cdot\frac{1}{(\sqrt{1+t^2})^5}}\cdot\frac{1}{1+t^2}\,dt =$$

$$= \int \frac{(\sqrt{1+t^2})^8}{t^3}\cdot\frac{1}{1+t^2}\,dt = \int \frac{(1+t^2)^4}{t^3}\cdot\frac{1}{1+t^2}\,dt =$$

$$= \int \frac{(1+t^2)^3}{t^3}\,dt = \int \frac{1+3t^2+3t^4+t^6}{t^3}\,dt =$$

$$= \int 1/t^3\,dt + 3\int \frac{1}{t}\,dt + 3\int t\,dt + \int t^3\,dt =$$

$$= \frac{t^{-2}}{-2} + 3\log|t| + 3\,\frac{t^2}{2} + \frac{t^4}{4} + C =$$

$$= \frac{tg(x)^{-2}}{-2} + 3\log|tg(x)| + \frac{3\,tg(x)^2}{2} + \frac{tg(x)^4}{4} + C$$

R4.14)

a) $\int \frac{\cos(x)}{1+\cos(x)}\,dx = \left[\begin{array}{l} t=tg(x/2) \Leftrightarrow dx=\frac{2}{1+t^2}\,dt \\ \cos(x)=\frac{1-t^2}{1+t^2}\ ;\ \operatorname{sen}(x)=\frac{2t}{1+t^2}\end{array}\right] =$

$$= \int \frac{\frac{1-t^2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt = \int \frac{\frac{1-t^2}{1+t^2}}{\frac{(1+t^2)+(1-t^2)}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt =$$

$$= \int \frac{(1-t^2)\cdot(1+t^2)}{(1+t^2)\cdot(1+t^2)}\,dt = \int \frac{1-t^2}{1+t^2}\,dt = -\int \frac{t^2-1}{t^2+1}\,dt =$$

$$= -\int \frac{t^2+1-1-1}{t^2+1} = -\int \left(1-\frac{2}{t^2+1}\right)dt = t+2\operatorname{arctg}(t)+C =$$

$$= -tg(x/2) + 2\operatorname{arctg}(tg(x/2))+C = -tg(x/2)+x+C$$

b) $\int \frac{1+tg(x)}{1-tg(x)}\,dx = \left[\begin{array}{l} t=tg(x) \Leftrightarrow dx=\frac{1}{1+t^2}\,dt \\ \cos(x)=\frac{1}{\sqrt{1+t^2}}\ ;\ \operatorname{sen}(x)=\frac{t}{\sqrt{1+t^2}}\end{array}\right] =$

$$= \int \frac{1+t}{1-t}\cdot\frac{1}{1+t^2}\,dt = -\int \frac{t+1}{(t-1)\cdot(t^2+1)}\,dt$$

$$\frac{t+1}{(t-1)(t^2+1)} = \frac{A}{t-1} \cdot \frac{Bt+C}{t^2+1}$$

$A=-1$

$B=1$

$C=0$

d) $\displaystyle\int \frac{1}{3\,sen^2(x)+5\cos^2(x)}\,dx = \int \frac{1}{3+2\cos^2(x)}\,dx =$

$$= \left[\begin{array}{l} t = tg(x) \Rightarrow dx = \dfrac{1}{1+t^2}\,dx \\ \cos(x) = \dfrac{1}{\sqrt{1+t^2}} \;;\; sen(x) = \dfrac{t}{\sqrt{1+t^2}} \end{array}\right] = \int \frac{1}{3+\dfrac{2}{\left(\sqrt{1+t^2}\right)^2}} \cdot \frac{1}{1+t^2}\,dt =$$

$$= \int \frac{1}{\dfrac{3(1+t^2)+2}{1+t^2}} \cdot \frac{1}{1-t^2}\,dt = \int \frac{1}{3t^2+5}\,dt = \int \frac{1}{5\left(\frac{3}{5}t^2+1\right)}\,dt =$$

$$= \frac{1}{5}\int \frac{1}{\left(\sqrt{\frac{3}{5}}\,t\right)^2+1}\,dt = \sqrt{\frac{5}{3}} \cdot \frac{1}{5} \cdot \int \frac{\sqrt{\frac{3}{5}}}{\left(\sqrt{\frac{3}{5}}\,t\right)^2+1}\,dt =$$

$$= \frac{1}{\sqrt{15}} \cdot arctg\left(\sqrt{\tfrac{3}{5}}\,t\right) + C = \frac{1}{\sqrt{15}} \cdot arctg\left(\sqrt{\tfrac{3}{5}} \cdot tg(x)\right) + C$$

R4.13)

f) $\displaystyle\int \frac{\cos^5(x)}{sen^3(x)}\,dx = \int \frac{\cos^4(x)\cdot\cos(x)\,dx}{sen^3(x)} = \left[\begin{array}{l} t = sen(x) \\ dt = \cos(x)\,dx \end{array}\right] =$

$$= \int \frac{(1-t^2)}{t^3}\,dt = \int \frac{1-2t+t^4}{t^3}\,dt = \int \frac{1}{t^3}\,dt - 2\int \frac{1}{t}\,dt + \int t\,dt =$$

$$= \frac{t^{-2}}{-2} - 2\log|t| + \frac{t^2}{2} + C = \frac{sen^{-2}(x)}{-2} - 2\log|sen(x)| + \frac{sen^2(x)}{2} + C$$

R4.7) $f(x) = \int_0^{x^3-x^2} e^{-t^2}\,dt \quad \forall x \in \mathbb{R}$

a) Intervalos de monotonía de f en $\mathbb{R}$

b) Extremos relativos de f

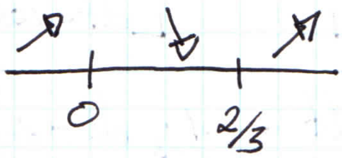
c) $\lim\limits_{x \to 0} \dfrac{f(x)}{\operatorname{sen}(x^3-x^2)}$

a) $x \longmapsto x^3 - x^2 \longrightarrow \int_0^{x^3-x^2} e^{-t^2}\,dt$

f derivable ($\Rightarrow$ continua) en $\mathbb{R}$ por ser composición de f derivables.

$$f'(x) = e^{-(x^3-x^2)^2}(3x^2-2x)$$

$$f'(x) = 0 \Leftrightarrow 3x^2 - 2x = 0 \begin{cases} \rightarrow x = 0 \\ \rightarrow x = 2/3 \end{cases}$$



$x < 0 \Rightarrow f'(x) > 0$

$0 < x < 2/3 \Rightarrow f'(x) < 0$

$x > 2/3 \Rightarrow f'(x) > 0$

b) $0 \rightarrow$ máximo relativo $\qquad 2/3 \rightarrow$ mínimo relativo

c) $\lim\limits_{x \to 0} \dfrac{f(x)}{\operatorname{sen}(x^3-x^2)} = \text{"}\dfrac{0}{0}\text{"} = \text{L'Hôpital} =$

$$= \lim_{x \to 0} \frac{e^{-(x^3-x^2)^2} \cdot (3x^2-2x)}{\cos(x^3-x^2) \cdot (3x^2-2x)} = \lim_{x \to 0} \frac{e^{-(x^3-x^2)^2}}{\cos(x^3-x^2)} =$$

$$= \frac{e^0}{1} = 1$$

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R5.1) $x_n = 1 \quad x_{n+1} = \sqrt{3x_n} \quad \forall n \in \mathbb{N}$

(1) Monotonía

$x_n = 1 \leq x_2 = \sqrt{3}$

¿$x_n \leq x_{n+1}, \forall n \in \mathbb{N}$? $x_1 \leq x_2$ (es cierto)

Suponemos que $x_n \leq x_{n+1}$

¿$x_{n+1} \leq x_{n+2}$?

$$x_n \leq x_{n+1} \Rightarrow 3x_n \leq 3x_{n+1} \Rightarrow \underbrace{\sqrt{3x_n}}_{x_{n+1}} \leq \underbrace{\sqrt{3x_{n+1}}}_{x_{n+2}}$$

$\{x_n\} \nearrow$

(2) Acotación

$1 = x_1 \leq x_n \overset{?}{\leq} 3 \quad \forall n \in \mathbb{N}$

¿$x_n \leq 3 \; \forall n \in \mathbb{N}$? $n=1$; $x_1 = 1 \leq 3$ (cierto)

Se supone que $x_n \leq 3$

¿$n+1$? $\Leftrightarrow$ ¿$x_{n+1} \leq 3$?

$$x_n \leq 3 \Rightarrow 3x_{n+1} \leq 9 \Rightarrow \underbrace{\sqrt{3x_n}}_{x_{n+1}} \leq \sqrt{9}$$

$x_{n+1} \leq 3$

$\{x_n\}$ acotada $\quad 1 \leq x_n \leq 3 \; \forall n \in \mathbb{N}$

¿$\exists \lim x_n = x$? $\quad 1 \leq x_n \leq 3 \quad \forall n \in \mathbb{N}$

$$\Downarrow$$

$$1 \leq x \leq 3$$

$$(x_{n+1})^2 = 3x_n \quad \forall n \in \mathbb{N}$$

$$\underbrace{(x_{n+1}) \cdot (x_{n+1})}_{x^2} = 3x_n$$

$$x^2 = 3x \Rightarrow x = \begin{cases} x=0 & \text{No vale} \\ x=3 & \text{Sí vale} \end{cases}$$

Con lo cual el lim vale 3

R5.4) $\{x_n\}$ $\quad x_1 = 1/2 \quad x_{n+1} = x_n^2 + 4/25 \quad \forall n \in \mathbb{N}$

(1) Monótona

$$x_1 = 1/2 \quad x_2 = \frac{41}{100} < 1/2 = x_1$$

¿$\{x_n\} \searrow$? ¿$x_n \geq x_{n+1}$? $\forall n \in \mathbb{N}$

(2) Inducción

$n=1 \quad x_1 \geq x_2$ (cierto)

Se supone que $x_n \geq x_{n+1}$

¿$n+1$? $\Leftrightarrow$ ¿$x_{n+1} \geq x_{n+2}$?

$$x_n \geq x_{n+1} \Rightarrow x_n^2 \geq x_{n+1}^2 \Rightarrow \underbrace{x_n^2 + 4/25 \geq x_{n+1}^2 + 4/25}_{x_{n+1} \geq x_{n+2}}$$

Monótona decreciente $\{x_n\} \searrow$

CA
Ejercicios

$0 \leq x_n \leq x_1 = 1/2 \quad \forall n \in \mathbb{N} \quad ¿\exists \lim x_n = x?$

$x_{n+1} = x_n^2 + 4/25 \quad \forall n \in \mathbb{N}$

$\Downarrow$

$x = x^2 + 4/25 \longrightarrow x = \begin{cases} 0'8 \\ 1/5 = 0'2 \end{cases} \qquad 0 \leq x_n \leq 1/2 = 0'5$

1) Sea $x \in \mathbb{R}$, $1 + x + x^2 + \cdots + x^n = \dfrac{x^{n+1} - 1}{x - 1}$, $\forall n \in \mathbb{N}$
$|x| < 1$

$$\lim (1 + x + x^2 + \cdots x^n) = \lim \frac{x^{n+1} - 1}{x - 1}$$

(Progresión geométrica)

RS. 4) $\displaystyle\sum_{n\geq 1} \frac{(-1)^n}{4^{n+1}} = \sum_{n\geq 1} \frac{(-1)^n}{4 \cdot 4^n} = \frac{1}{4} \sum_{n\geq 1} \left(-\frac{1}{4}\right)^n$ convergente

$$\frac{1}{4} \sum_{n=1}^{\infty} \left(-\frac{1}{4}\right)^n = \frac{1}{4} \cdot \left[\frac{1}{1 + 1/4} - 1\right] = -\frac{1}{20}$$

RS.5) $\displaystyle\sum_{n\geq 0}^{\infty} \frac{(-1)^{n+1} + 2^n}{8^{n+2}} = \sum_{n\geq 0}^{\infty} \left(\frac{-1}{8^{n+2}}\right) + \sum_{n\geq 0}^{\infty} \frac{2^n}{8^{n+2}} =$

$$= \sum_{n\geq 0}^{\infty} \frac{-1}{8^2} \cdot \frac{(-1)^n}{8^n} + \frac{1}{8^2} \sum_{n\geq 0}^{\infty} \frac{2^n}{8^n} = -\frac{1}{8^2} \sum_{n\geq 0}^{\infty} \left(-\frac{1}{8}\right)^n + \frac{1}{8} \sum_{n\geq 0}^{\infty} \left(\frac{1}{4}\right)^n$$

convergente

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} + 2^n}{8^{n+2}} = -\frac{1}{8} \sum_{n=0}^{\infty} \left(-\frac{1}{8}\right)^n + \frac{1}{8^2} + \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n =$$

$$= \frac{1}{8^2} \cdot \frac{1}{1 + 1/8} + \frac{1}{8^2} \cdot \frac{1}{1 - 1/4} = -\frac{1}{72} + \frac{1}{48}$$

R5.6)

a) $\sum \left(\frac{n+1}{3n-1}\right)^n \to$ Convergente

$$\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{n+1}{3n-1}\right)^n} = \frac{n+1}{3n-1} \to \frac{1}{3} < 1$$

b) $\sum \left(\frac{n}{3n-2}\right)^{2n-1} \to$ Convergente

$$\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{n}{3n-2}\right)^{2n-1}} = \left(\frac{n}{3n-2}\right)^{\frac{2n-1}{n}} \to \left(\frac{1}{3}\right)^2 = \frac{1}{9} < 1$$

c) $\sum \frac{n^n}{(2n+1)^n} \to$ Convergente

$$\sqrt{a_n} = \sqrt{\frac{n^n}{(2n+1)^n}} = \frac{n}{2n+1} \to \frac{1}{2} < 1$$

d) $\sum \left(1+\frac{1}{n}\right)^{-n^2} \to$ Convergente

$$\sqrt[n]{a_n} = \sqrt[n]{\left(1+\frac{1}{n}\right)^{-n^2}} = \left(1+\frac{1}{n}\right)^{-n} =$$

$$= -n\left[1+\frac{1}{n}-1\right] = L \Rightarrow -n\,\frac{1}{n} = \frac{-n}{n} = -1 \to e^{-1} = \frac{1}{e} < 1$$

R5.7)

a) $\sum \frac{1}{n2^n} \to$ Convergente

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)2^{n+1}}}{\frac{1}{n2^n}} = a_{n+1} \cdot \frac{1}{a_n} = \frac{1}{(n+1)2^{n+1}} \cdot n \cdot 2^n$$

$$= \frac{n\,2^n}{(n+1)\,2^{n+1}} = \frac{n2^n}{(n+1)\,2^n\,2} = \frac{n}{2n+1} \to \frac{1}{2} < 1$$

d) $\sum \frac{2^n \cdot n!}{n^n}$

$$\frac{a_{n+1}}{a_n} = a_{n+1} \cdot \frac{1}{a_n} = \frac{2^{n+1} \cdot (n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n \cdot n!} =$$

$$= 2\,\frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} = 2\,\frac{(n+1)n!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} =$$

$$= \frac{2(n+1)\,n^n}{(n+1)^{n+1}} = \frac{2n^n}{(n+1)^n} = 2\cdot\left(\frac{n}{n+1}\right)^n = 2\cdot\frac{1}{e} = \frac{2}{e} < 1$$

CA

Ejercicios

1) $\sum \frac{1}{n+8} \to$ divergente $\sim \sum 1/n$

$$\frac{\frac{1}{n+8}}{\frac{1}{n}} = \frac{1}{n+8} \cdot n = \frac{n}{n+8} \to 1 \neq 0$$

2) $\sum \frac{1}{2^n+18} \sim \sum \frac{1}{2^n}$

$$\frac{\frac{1}{2^n+18}}{\frac{1}{2^n}} = \frac{2^n}{2^n+18} = \frac{1}{1+\frac{18}{2^n}} \to 1 \neq 0$$

3) $\sum \frac{n}{2n^2+5n-7} \sim \sum 1/n$ divergente

$$\frac{\frac{n}{2n^2+5n-7}}{\frac{1}{n}} = \frac{n^2}{2n^2+5n-7} = \frac{1}{2} \neq 0$$

4) $\sum \frac{3n^2-7n+5}{8n^5-2n+14} \sim \sum \frac{1}{n^3}$ Convergente

$$\frac{\frac{3n^2-7n+5}{8n^5-2n+14}}{\frac{1}{n^3}} = \frac{3n^5-7n^4+5n^3}{8n^5-2n+14} = \frac{3}{8} \neq 0$$

5) $\sum \frac{1}{\sqrt{n(n+1)}} \sim \sum \frac{1}{n}$ divergente

6) $\sum \frac{1}{2^n-n} \sim \sum \frac{1}{2^n}$ convergente

$$\frac{\frac{1}{2^n-n}}{\frac{1}{2^n}} = \frac{2^n}{2^n-n} = \frac{1}{2-\frac{n}{2^n}} = 1 \neq 0$$

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7) $\sum \frac{1}{\log(n)} \sim \sum 1/n$ divergente

$$\frac{\frac{1}{\log(n)}}{1/n} = \frac{n}{\log(n)} \to +\infty$$

8) a) $\sum \frac{\log(n)}{n} \sim \sum 1/n$ divergente

$$\frac{\frac{\log(n)}{n}}{1/n} = \frac{n \log(n)}{n} = \log(n) = +\infty$$

e) $\sum \frac{1}{(2n-1)2n} \sim \sum \frac{1}{n^2}$ Convergente

9) a) $\sum \frac{2^n}{n} \to +\infty$ divergente

b) $\sum \frac{n+1}{2n+1} \to 1/2 \neq 0$ divergente

c) $\sum \frac{1}{n^2 \log(n)} \sim \sum \frac{1}{n^2}$

$$\frac{n^2}{n^2 \log(n)} = \frac{1}{\log(n)} \to 0$$

d) $\sum \frac{3n-1}{(\sqrt{2})^n}$ convergente

$$\frac{a_{n+1}}{a_n} = a_{n+1} \cdot \frac{1}{a_n} = \frac{3(n+1)-1}{(\sqrt{2})^{n+1}} \cdot \frac{(\sqrt{2})^n}{3n-1} = \frac{1}{\sqrt{2}} < 1$$