Kellogg School of Management
FINC 585-3: Asset Pricing
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# Assignment 1

The objective of this assignment is to gain familiarity with basic properties of univariate equityindex return series and some standard test statistics gauging the existence of return dependence. Both the daily S&P 500 and CRSP return series are available under `data/` on the Canvas site.

**Problem 1.**

Provide a sketch of a proof for the Hausman Principle, given on page 11.

**Proposition.** *Consider the following estimators:*

- $\widehat{\sigma}_a^2$ *is Efficient (MLE) Estimator under the Null Hypothesis, i.e, it has Smallest Asymptotic Variance amongst consistent estimators*
- $\widehat{\sigma}_b^2$ *is Consistent, albeit NOT Efficient Estimator under the Null*

***Hausman Principle:*** $\mathbb{C}(\widehat{\sigma}_a^2, \widehat{\sigma}_b^2) = \mathbb{V}(\widehat{\sigma}_a^2)$

***Solution.***

Let $\widehat{\mu}$ and $\widetilde{\mu}$ be two consistent estimators of a parameter $\mu$ in the real line. Assume that

$$\mathbb{P}\left\{\widehat{\mu} \neq \widetilde{\mu}\right\} > 0$$

(so they are not the same) and that $\widehat{\mu}$ is efficient (i.e., it has the smallest variance). Fix $\lambda \in [0, 1]$ and let $\widetilde{\mu}(\lambda)$ be a convex combination between these two estimators:

$$\bar{\mu}(\lambda) = \lambda\widehat{\mu} + (1-\lambda)\widetilde{\mu}$$

By definition it is also consistent and its variance is given by:

$$\mathbb{V}(\bar{\mu}(\lambda)) = \lambda^2\,\mathbb{V}(\widehat{\mu}) + (1-\lambda)^2\,\mathbb{V}(\widetilde{\mu}) + 2\lambda(1-\lambda)\,\mathbb{C}(\widehat{\mu}, \widetilde{\mu})$$

Lets find the value of $\lambda$ that minimizes the variance of $\bar{\mu}(\lambda)$. The first order condition of the minimization of the equation above is:

$$\frac{\partial\,\mathbb{V}(\bar{\mu})}{\partial\lambda}(\lambda) = 2\lambda\,\mathbb{V}(\widehat{\mu}) - 2(1-\lambda)\,\mathbb{V}(\widetilde{\mu}) + 2(1-2\lambda)\,\mathbb{C}(\widehat{\mu}, \widetilde{\mu}) = 0$$

And the second order condition is:

$$\frac{\partial^2\,\mathbb{V}(\bar{\mu})}{\partial\lambda^2}(\lambda) = 2\,\mathbb{V}(\widehat{\mu}) - 2\,\mathbb{V}(\widetilde{\mu}) + -4\,\mathbb{C}(\widehat{\mu}, \widetilde{\mu}) = 2\,\mathbb{V}(\widehat{\mu} - \widetilde{\mu}) > 0$$

where the strictly positivity follows from the fact that $\widehat{\mu}$ and $\widetilde{\mu}$ are not the same estimate. These two conditions characteris a global minimizer $\lambda^*$ of the variance of $\bar{\mu}(\lambda)$. Since $\widehat{\mu}$ is efficient, it must be the case that $\lambda^* = 1$. Therefore, from the FOC:

$$\frac{\partial \mathbb{V}(\bar{\mu})}{\partial \lambda}(1) = 2\,\mathbb{V}(\widehat{\mu}) - 2\,\mathbb{C}(\widehat{\mu}, \widetilde{\mu}) = 0$$

Therefore we get that:

$$\mathbb{V}(\widehat{\mu}) = \mathbb{C}(\widehat{\mu}, \widetilde{\mu})$$

$\square$

**Problem 2.**

Derive the asymptotic distribution for the VD statistic in page 12.

**VD Statistic:**

$$VD = \widehat{\sigma}_a^2 - \widehat{\sigma}_b^2$$

***Solution.***

Recall from the lecture notes that the Variance Difference (VD) statistic is defined as:

$$VD(q) = \widehat{\sigma}_a^2 - \widehat{\sigma}_b^2$$

Also, recall that the asymptotic distribution of the estimators is given by:

$$\sqrt{nq}\begin{pmatrix} \widehat{\sigma}_a^2 - \sigma^2 \\ \widehat{\sigma}_b^2 - \sigma^2 \end{pmatrix} \sim N\left(0, \begin{pmatrix} 2\sigma^4 & 2\sigma^4 \\ 2\sigma^4 & 2q\sigma^4 \end{pmatrix}\right)$$

Using the Delta Method we have that:

$$\sqrt{nq}VD(q) \xrightarrow{d} \nabla VD|_{\sigma^2,\sigma^2} N\left(0, \begin{pmatrix} 2\sigma^4 & 2\sigma^4 \\ 2\sigma^4 & 2q\sigma^4 \end{pmatrix}\right)$$

Taking the derivatives of VD w.r.t $\widehat{\sigma}_a^2$ and $\widehat{\sigma}_b^2$ we get:

$$\nabla VD = \begin{pmatrix} 1 & -1 \end{pmatrix}$$

Therefore the the asymptotic variance of VD is:

$$\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 2\sigma^4 & 2\sigma^4 \\ 2\sigma^4 & 2q\sigma^4 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2\sigma^4(q-1)$$

□

**Problem 3.**

Do the exercise on page 13 (on the VR statistic with overlapping returns). Hint: If it seems tricky, you can check the literature for inspiration.

**Exercise: (Overlapping returns)**

Under the Null Hypothesis,

$$\sqrt{nq}\,(\widehat{\rho}_1, \ldots, \widehat{\rho}_{q-1}) \xrightarrow{d} N\left(0, I_{(q-1)\times(q-1)}\right)$$

We also saw,

$$\widehat{VR}(q) = 1 + 2\sum_{k=1}^{(q-1)}\left(1 - \frac{k}{q}\right)\widehat{\rho}_k$$

Use the Delta method to show that, if we use overlapping returns to estimate the higher order return auto-covariances then,

$$\sqrt{nq}\left(\widehat{VR}(q) - 1\right) \xrightarrow{d} N\left(0, \frac{2(q-1)(2q-1)}{3q}\right)$$

***Solution.***

This exercise corresponds to the Proof of Theorem 2 in Lo and MacKinlay (1988) (p.p. 62-64). Since we can express the VR statistic as a sum of the correlations we can use the Delta Method to derive its asymptotic distribution. Letting

$$\begin{aligned}\nabla\widehat{VR}(q) &\coloneqq 2\left(\frac{\partial\widehat{VR}(q)}{\partial\rho_q}, \ldots, \frac{\partial\widehat{VR}(q)}{\partial\rho_{q-1}}\right)\\ &= 2\left(1 - \frac{1}{q}, 1 - \frac{2}{q}, \ldots, 1 - \frac{q-1}{q}\right)\end{aligned}$$

Therefore the asymptotic variance of the VR statistic can be derived as:

$$\begin{aligned}\Omega &\coloneqq \nabla\widehat{VR}(q) I_{(q-1)\times(q-1)} \nabla\widehat{VR}(q)^{\intercal}\\ &= 4\sum_{k=1}^{q-1}\left(1 - \frac{k}{q}\right)^2\\ &= \frac{2(q-1)(2q-1)}{3q}\end{aligned}$$

So we can infer that the asymptotic distribution of $\widehat{VR}(q)$ is given by:

$$\sqrt{nq}\left(\widehat{VR}(q) - 1\right) \xrightarrow{d} N\left(0, \frac{2(q-1)(2q-1)}{3q}\right)$$

$\square$

**Problem 4.**

Hong, Linton, and Zhang (2017) (HLZ) provide a correction and extension to the Variance Ratio statistic explored by Lo and MacKinlay (1988) (no formal proofs needed below).

a) Identify the changes in assumptions they introduce relative to Lo & MacKinlay.

b) What are the advantages of their proposed new assumptions?

c) State the limiting distribution result for the univariate VR statistic in their paper.

***Solution.***

a) Let $X_t$ be the series of the returns and $\widetilde{X}_t := X_t - \mathbb{E}(X_t)$. Hong et al. (2017) note that the assumption that $\widetilde{X}_t\widetilde{X}_{t-j}$ and $\widetilde{X}_s\widetilde{X}_{s-j}$ are uncorrelated is missing in the original work of Lo and MacKinlay (1988). According to them, this assumption does not follow from the assumption that $X_t$ is an uncorrelated sequence. In other words, this means that there is no "correlation between correlations" of the returns. This is necessary for the analysis of the asymptotic distribution of the Variance Ratio test as the fourth moments appear in the derivation of the asymptotic variance.
This assumption is contained in the Assumption MH* of Hong et al. (2017) p.p. 184 as:

MH1. (i) For all $t$ $\widetilde{X}_t$ satisfies $\mathbb{E}\left(\widetilde{X}_t\right) = 0$, $\mathbb{E}\left(\widetilde{X}_t\widetilde{X}_{t-j}^\intercal\right) = 0$ for all $j \neq 0$; (ii) for all $t, s$ which $s \neq t$ and all $j, k = 1, \dots, K$ $\mathbb{E}\left(\widetilde{X}_t\widetilde{X}_{t-j}^\intercal \otimes \widetilde{X}_s\widetilde{X}_{s-k}^\intercal\right) = 0$.

b) The assumptions on HLZ are weaker than the ones in LM. Moreover, they show that the assymptions in LM are not necessary to guarantee the correct asymptotic distribution of the autocorrelation estimates.

c) According to Theorem 1. of HLZ (p.p. 185), the limiting distribution of the (here simplified to univariate) Variance Ratio statistic is given by:

$$\sqrt{T}\left(\widehat{VR}(q) - 1\right) \xrightarrow{d} N\left(0, \Omega(q)\right)$$

with

$$\Omega(q) := 4\sum_{j=1}^{q-1}\sum_{k=1}^{q-1}\left(1 - \frac{j}{q}\right)\left(1 - \frac{k}{q}\right)\frac{\Xi_{k,j}}{\sigma^4}$$

where $\Xi_{k,j} := \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left(\widetilde{X}_t\widetilde{X}_{t-j}\widetilde{X}_{t-k}\widetilde{X}_{t-j-k}\right)$.

□

**Problem 5.**

Apply the Spearman Corerlation Test on page 4 of the notes to explore dependences for the daily S&P 500 index returns.

a) First, do the test as suggested for January 1991-December 2006 period

b) Second, do the same test for the sample period 2007-2022

c) Repeat the above, but using the absolute returns in lieu of the returns

***Solution.***

I have calculated the results for the Spearman Rank Correlation Test for the samples using the Value and Equal Weighted S&P 500. The results are shown in Table 1 and Table 2, respectively. As we can see, there is evidence pointing for the presence of autocorrelation for the Equal Weighted returns for the whole sample. The Value Weighted returns, however, fail to present any autocorrelation in the first part of the sample. Therefore we may conclude that the autocorrelation may be driven by the small cap stocks which receive more weight on the equal weighted portfolio. For the absolute returns, the results show that there is significant autocorrelation for the whole sample for both portfolios. This is in lign with the literature of volatility clustering.

□

Table 1: Spearman Rank Correlation Test for Value Weighted S&P 500 Returns

| | Z | pval | [{ 95 %CI} | {95 %CI}] |
|---|---|---|---|---|
| 1991-2006 | −0.223 | 0.824 | [ −1.868; | 1.422] |
| 2007- | 3.613 | 0.000 | [ 1.969; | 5.258] |
| 1991-2006 (Abs) | −3.576 | 0.000 | [ −5.221; | −1.931] |
| 2007- (Abs) | −8.024 | 0.000 | [ −9.668; | −6.379] |

Table 2: Spearman Rank Correlation Test for Equal Weighted S&P 500 Returns

| | Z | pval | [{ 95 %CI} | {95 %CI}] |
|---|---|---|---|---|
| 1991-2006 | −2.916 | 0.004 | [ −4.560; | −1.271] |
| 2007- | 2.404 | 0.016 | [ 0.759; | 4.049] |
| 1991-2006 (Abs) | −1.926 | 0.054 | [ −3.570; | −0.281] |
| 2007- (Abs) | −8.660 | 0.000 | [−10.305; | −7.016] |

**Problem 6.**

Implement the Lo and MacKinlay (1988) ratio test for the CRSP value-weighted and equal-weighted indices over their sample period, and for the periods 1991-2006 and 2007-2022. Try to assess significance according to the Lo & Mackinlay and HLZ robust limiting distributions.

***Solution.***

I calculated the Variance Ratio statistic using up to 20 lags for both the value and the equal-weighted indices for the periods required and the results are shown on Figures 1-4.
Recall that the VR statistic can be written as a positive linear combination of the autocorrelation coefficients:

$$VR(q) \approx 1 + 2\sum_{k=1}^{q-1}\left(1 - \frac{k}{q}\right)\rho_k$$

Therefore we may interpret the direction of the autocorrelation (specially on $q = 2$ as the sign of the coefficient.
For the value weighted portfolio, there is evidence pointing to the existence of **negative** autocorrelation for the sample of 2007-2022 only. The results are significant up until $q = 11$ when using the HLZ std. dev. and $q = 8$ when using the LM std. dev. as seen on Figure 2. For the sample of 1991-2006, there is no evidence of autocorrelation.
The results are different for the equal-weighted portfolio. As we can see in Figures 3 and 4, the period between 1991 and 2006 shows strong signs of positive autocorrelation. The results are significant for the 20 values of $q$ that were tested and the Variance Ratio statistic is always increasing, indicating that each one of the autocorrelations may be positive. For the sample of 2007-2022, the results are not significant for any of the values of $q$ tested. All these results are consistent with either the HLZ or the LM std. deviations.

□

Figure 1: Variance Ratio Test for the Value Weighted Index for 1991-2006

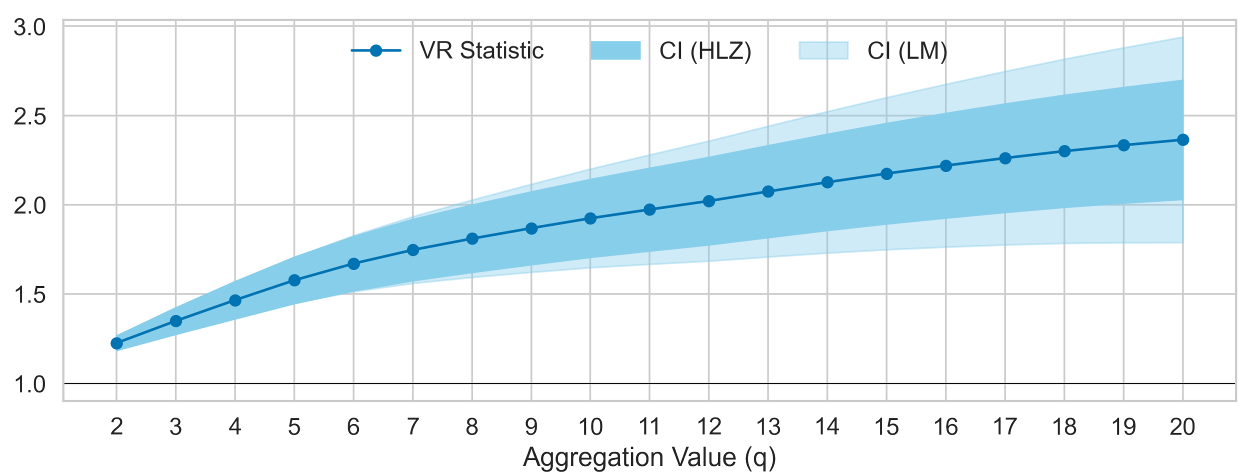


Figure 2: Variance Ratio Test for the Value Weighted Index for 2007-

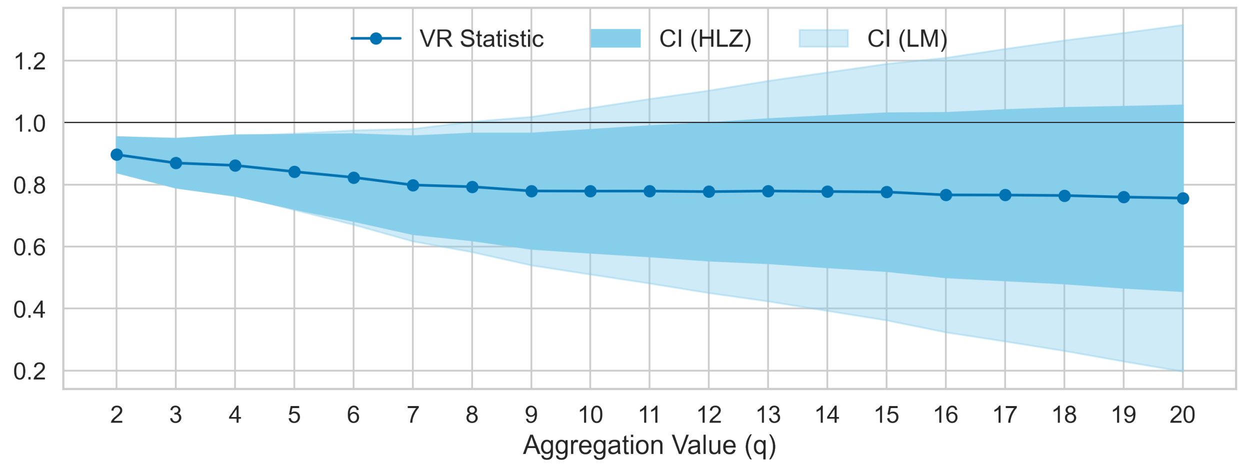



Figure 3: Variance Ratio Test for the Equal Weighted Index for 1991-2006

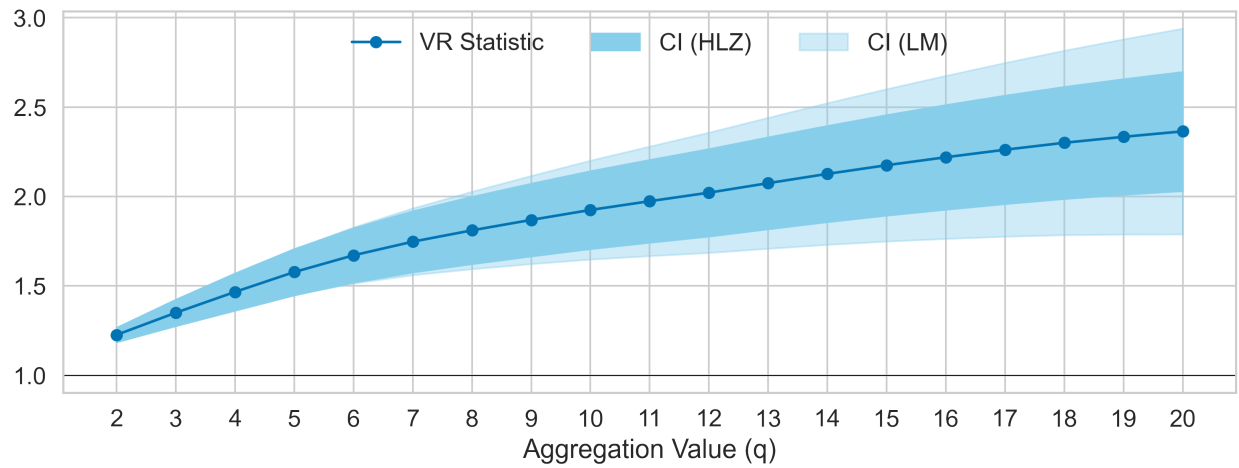
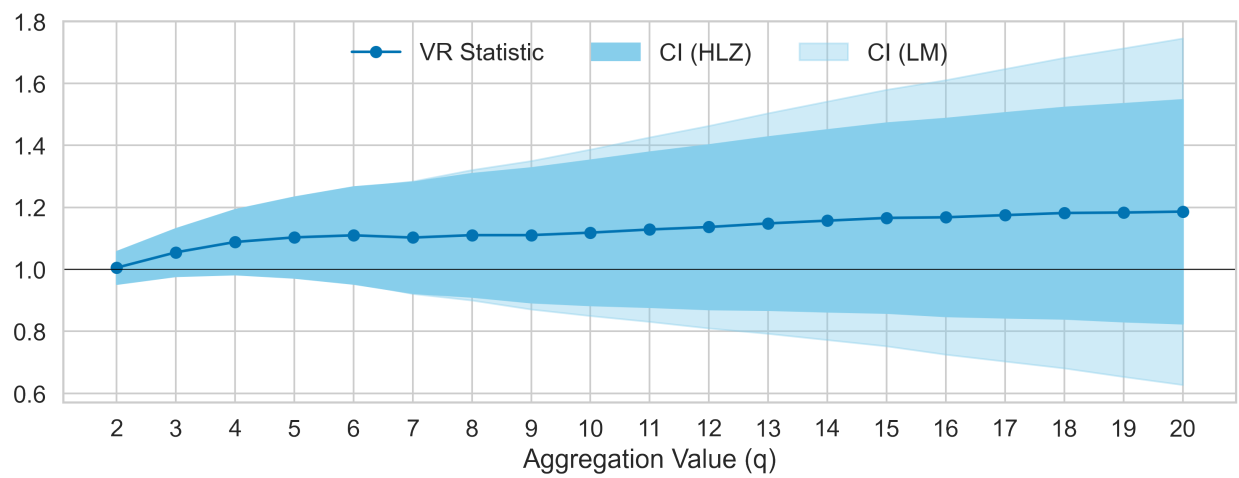



Figure 4: Variance Ratio Test for the Equal Weighted Index for 2007-



**Problem 7.**

Provide a proof of the Roll estimator for the bid-ask spread based on the return autocorrelation. Using the original paper (Roll, 1984), it should be straightforward to pin down assumptions that will allow for a proof of the results.

***Solution.***

Following Roll (1984), assume that:

1) The asset is traded in a informativelly efficient market;

2) The probability distribution of the asset return is stationary;

3) There are no trading costs;

4) There is a market maker providing liquidity which knows the efficient value of the asset.

The market maker operates by choosing a bid-ask spread $s$ symmetric around the efficient price $p$. Under these assumptions, the price conveil all the information available about the market and any changes to the price must be due to unantecipated information. Therefore, the price path can be described by Figure 5 where each possible path has equal probability.

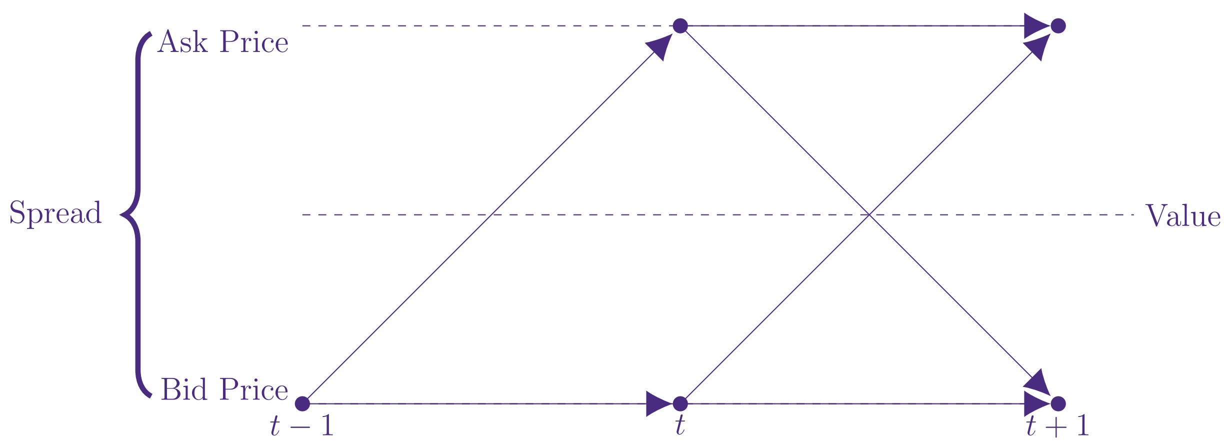



Figure 5: Price path in the Roll model

In other words, the joint distribution of the price change ($\Delta p$) at $t$ and $t+1$ conditional on $p_{t-1}$ being at the ask ($A$) or bid ($B$) price is given by:

$$p_{t-1} = B \qquad\qquad p_{t-1} = A$$

$$
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{\Delta p_t} \\
 & & 0 & +s \\ \hline
 & -s & 0 & 1/4 \\ \hline
\Delta p_{t+1} & 0 & 1/4 & 1/4 \\ \hline
 & +s & 1/4 & 0 \\ \hline
\end{array}
\qquad
\begin{array}{c|c|c|c}
 \multicolumn{3}{c}{\Delta p_t} & \\
 & -s & 0 & \\ \hline
-s & 0 & 1/4 & \\ \hline
0 & 1/4 & 1/4 & \Delta p_{t+1} \\ \hline
+s & 1/4 & 0 & \\ \hline
\end{array}
$$

Since $p_{t-1}$ is equally likely to be either in the bid or the ask and the expected value of $\Delta p_t$ is zero, we can calculate the covariance:

$$
\begin{aligned}
\mathbb{C}(\Delta p_t, \Delta p_{t+1}) &= \mathbb{P}(p_{t-1} = B)\,\mathbb{E}(\Delta p_{t+1}\Delta p_t | p_{t-1} = B) + \mathbb{P}(p_{t-1} = A)\,\mathbb{E}(\Delta p_{t+1}\Delta p_t | p_{t-1} = A) \\
&= \frac{1}{2} \times \frac{1}{4}(-s) \times (+s) + \frac{1}{2} \times \frac{1}{4}(-s) \times (+s) \\
&= -\frac{1}{4}s^2
\end{aligned}
$$

and construct an estimator of the spread $s$ as:

$$\widehat{s} = \sqrt{-\frac{1}{4}\widehat{\mathbb{C}}\left(\Delta p_t, \Delta p_{t+1}\right)}$$

For this estimation to be consistent, we need to impose the assumptions from Theorem 7.2.2 in Brockwell and Davis (2009) that the returns are stationary, i.e.,

$$\Delta p_t = \sum_{j \in \mathbb{Z}} \psi_j Z_{t-j} \qquad Z_t \sim I.I.D\left(0, \sigma^2\right)$$

with $\sum_{j \in \mathbb{Z}} |\psi_j| < \infty$ and $\sum_{j \in \mathbb{Z}} \psi_j^2\,|j| < \infty$. □

**Problem 8.**

Prove the initial expressions for $\beta$ and $\lambda$, and then all the formulas displayed under implications 1)-4) in the Kyle Model (Market Microstructure Notes).

***Solution.***

Consider the one period Kyle model from the Market Microstructure Notes. In this model there is only one asset, informed and noise traders submit size orders to a market maker who sets the prices. The terminal value of the asset, denoted by $V$ is known by informed traders, this value follows a normal distribution with mean $\mu_0$ and variance $\sigma_0^2$.

$$V \sim N\left(0, \sigma_0^2\right)$$

Independently from that, noise traders submit orders of size $Z$, which is also normally distributed with mean 0 and variance $\sigma_Z^2$.

$$Z \sim N\left(0, \sigma_Z^2\right)$$

We look for a linear equilibrium defined below

**Definition** (Linear Equilibrium)**.** A linear equilibrium is defined as a tuple $(P, X)$ such that

1. The market maker's price is a linear function of the total order flow $X + Z$, i.e., for some $\mu^\star, \lambda^\star \in \mathbb{R}$:
$$P = \mu^\star + \lambda^\star(X + Z)$$

2. The informed trader's demand is a linear function of the value of the asset, i.e., for some $\alpha^\star, \beta^\star \in \mathbb{R}$:
$$X = \alpha^\star + \beta^\star V$$

3. The informed trader's demand satisfies profit maximization;

4. The market maker strategy satisfies both conditional zero profit condition (CZPC):
$$P = \mathbb{E}(V|X + Z) \tag{CZPC}$$

We start by deriving the informed trader demand. Knowing the terminal value $V$, the informed trader's profit condition is a random variable given by

$$\Pi = X\left(V - P\right)$$

Under common knowledge of rationality, the informed trader considers that the market maker will set a linear price function, so we can write that

$$\Pi = X\left(V - \mu - \lambda(X + Z)\right)$$

for some $\mu, \lambda \in \mathbb{R}$. The optimality condition for the informed trader than is given by

$$\max_X \mathbb{E}(\Pi|X, V) \tag{IT Optimality}$$

using the formula for $\Pi$ and $X$ and considering that $V$ is known to the insider, this can be rewritten as

$$\begin{aligned}\mathbb{E}(\Pi|X, V) &= \mathbb{E}(X\,(V - \mu - \lambda(X+Z))\,|X, V) \\ &= XV - (\mu + \lambda X)\,X - \lambda X\,\mathbb{E}(Z) \\ &= XV - (\mu + \lambda X)\,X\end{aligned}$$

Taking the first order condition w.r.t X we get:

$$V - \mu - 2\lambda X = 0 \tag{FOC}$$

$$X = -\frac{\mu}{2\lambda} + \frac{V}{2\lambda}$$

Which is a linear solution for the informed trader's demand with $\alpha = -\frac{\mu}{2\lambda}$ and $\beta = \frac{1}{2\lambda}$ as a function of the market maker's price function.

Next we derive the market maker's rule. On equilibrium, the market maker knows the solution to the informed trader's problem and sets the price function accordingly. From the conditional zero profit condition (CZPC), we can derive the unconditional zero profit condition (UZPC):

$$\begin{aligned}\mathbb{E}(P) &= \mathbb{E}(V) \qquad \text{(UZPC)}\\ \mathbb{E}(\mu + \lambda(X+Z)) &= \mu_0 \\ \mu + \lambda\,\mathbb{E}\left[-\frac{\mu}{2\lambda} + \frac{V}{2\lambda} + Z\right] &= \mu_0 \\ \mu - \frac{\mu}{2} + \frac{\mu_0}{2} &= \mu_0 \\ \mu &= \mu_0\end{aligned}$$

Plugging it back to the formula of $X$ we get that

$$X = -\frac{\mu_0}{2\lambda} + \frac{V}{2\lambda}$$

Now observe that, since $V$, the terminal value, is normally distributed, this implies that $X$ itself is also normally distributed with mean

$$\mathbb{E}(X) = -\frac{\mu_0}{2\lambda} + \frac{\mu_0}{2\lambda} = 0$$

and variance

$$\mathbb{V}(X) = \frac{\sigma_0^2}{4\lambda^2}$$

Moreover, since $V$ and $Z$ are independent, this implies that $X$ and $Z$ are jointly normal distributed, which, in turn, implies that $X+Z$ is also normal. Using the formula for $X$ and the UZPC we can than write that

$$X + Z \sim N\left(0, \frac{\sigma_0^2}{4\lambda^2} + \sigma_Z^2\right)$$

Finally, $X+Z$ and $V$ are jointly normal, so making the projection of $V$ onto $X+Z$ allow us to derive the conditional expectation for the CZPC:

$$\begin{aligned}
\mathbb{E}\left[V|X+Z\right] &= \mathbb{E}(V) + \frac{\mathbb{C}(V, X+Z)}{\mathbb{V}(X+Z)}\left[(X+Z) - \mathbb{E}(X+Z)\right] \\
&= \mu_0 + \frac{\frac{\sigma_0^2}{2\lambda}}{\frac{\sigma_0^2}{4\lambda^2} + \sigma_Z^2}(X+Z)
\end{aligned}$$

Equating this to $P$ from CZPC gives us the final price function:

$$P = \mu_0 + \frac{2\lambda\sigma_0}{\sigma_0^2 + 4\lambda^2\sigma_Z^2}(X+Z) \qquad \text{(Price Function)}$$

So we implicitly derived that

$$\begin{aligned}
\lambda &= \frac{2\lambda}{2\sigma_0^2 + 4\lambda^2\sigma_Z^2} \\
4\lambda^3\sigma_Z^2 + \lambda\sigma_0^2 &= 2\lambda\sigma_0^2 \\
4\lambda^2\sigma_Z^2 &= \sigma_0^2 \\
\lambda &= \frac{\sigma_0}{2\sigma_Z}
\end{aligned}$$

Plugging this back to the insider's demand function gives

$$\beta = \frac{1}{2\lambda} = \frac{\sigma_Z}{\sigma_0}$$

Which completes the characterization of the equilibrium in our model with the price and demand functions given by

$$\begin{aligned}
X &= -\mu_0\frac{\sigma_Z}{\sigma_0} + \frac{\sigma_Z}{\sigma_0}V \\
P &= \mu_0 + \frac{\sigma_0}{2\sigma_Z}(X+Z)
\end{aligned} \qquad \text{(Eq.)}$$

□

## References

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