1 Recurrence Relations and Generating Functions

Method 1 (How to solve for a recurrence relation.). *1. Given a recurrence relation a_n*

2. *Plug into generating function $A(x)$ series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and expand*
3. *Break up into things that are $A(x)$ on the right*
4. *Solve for $A(x)$'s generating function*
5. *Rational Function to the right, break up with partial fractions*
6. *replace the known generating functions with their series*
7. *match coefficients*

Method 2 (Finding path recurrence relations.). *Determine the first choices you have. These should be distinct and independent.*

Identity 1 (Series Multiplication).

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{n=0}^{k} a_n b_{n-k}\right) x^n$$

Identity 2 (Taking Power of Series).

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)^k = \sum_{n=0}^{\infty}\left(\sum_{\substack{i_1,\ldots,i_k \geq 0\\ i_1+\ldots+i_k=n}} a_{i_1}\ldots a_{i_k}\right) x^n$$

Method 3 (Lookout for Singularities). *If you have 2 solutions of $c(x)$, lookout for singularities caused by x in the denominator.*

Generating Function 1 (Catalan Numbers). *How many paths above the diagonal to get from $(0,0)$ to (n,n) with steps of $(1,0)$ and $(0,1)$? The answer c_n has generating function:*

$$C(x) = \sum_{n=0}^{\infty} c_n x^n = \frac{1-\sqrt{1-4x}}{2x}$$

and satisfies the relationship $xC(x)^2 - C(x) + 1 = 0$. We rejected the positive solution because of $C(x) \to \infty$ at $x \to 0^+$.

The closed form of the sequence is

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

Generating Function 2 (The Bell Numbers)**.** *Let the sequence b_n be the number of set partitions of n, and $B(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}$.*

$$b_{n+1} = \sum_{k=0}^{n} \binom{n}{k} b_{n-k}$$

and

$$B'(x) = B(x)e^x$$

with $B(0) = b_0 = 1$. Solving this with: $B(x) = e^{e^x - 1}$.

The recurrence relation has a closed form

$$b_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}$$

Generating Function 3 (The Bells Numbers into k Sets)**.** *Let $b_{n,k}$ be the number of set partitions of n into k sets. Then $b_{n+1,k} = b_{n,k-1} + b_{n,k}k$.*

$$B(x,y) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} b_{n,k} y^k \right) \frac{x^n}{n!}$$

$$B(x,1) = e^{e^x} e^{-1}$$

comes from the original Bell numbers. Satisfies $B_x = yB + yB_y$ and

$$B(x,y) = e^{ye^x} e^{-y}$$

Generating Function 4 (Cards and Hands)**.** *Let C_n be the set of cards with weights n so*

$$C(x) = \sum_{n=1}^{\infty} |C_n| \frac{x^n}{n!}$$

and where H_n is the set of hands with weight n

$$\sum_{n=0}^{\infty} \sum_{h \in H_n} y^{\#\text{ of cards in hand } h} \frac{x^n}{n!} = e^{yC(x)}$$

and

$$|C_n| = (n-1)!$$

so that

$$C(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = \ln\left(\frac{1}{1-x}\right).$$

Theorem 1.1 (The Exponential Formula)**.**

$$\sum_{n=0}^{\infty} \left(\sum_{\sigma \in S_n} y^{\#\text{ of cycles in } \sigma} \right) \frac{x^n}{n!} = e^{y \ln\left(\frac{1}{1-x}\right)}$$

Example 1 (Finding the Expectation Value of Cycles in a Permutation of n)**.**

$$\begin{aligned}\frac{\partial}{\partial y}\left(e^{y\ln\left(\frac{1}{1-x}\right)}\right)\Bigg|_{y=1} &= \ln\left(\frac{1}{1-x}\right)e^{y\ln\left(\frac{1}{1-x}\right)}\Bigg|_{y=1}\\ &= \frac{1}{1-x}\ln\left(\frac{1}{1-x}\right)\\ &= \left(\sum_{n=0}^{\infty}x^n\right)\left(\sum_{n=1}^{\infty}\frac{x^n}{n}\right)\\ &= \sum_{n=0}^{\infty}\sum_{k=1}^{n}\frac{1}{k}x^n.\end{aligned}$$

The average number of cycles in a permutation of n will be:

$$\sum_{k=1}^{n}\frac{1}{k}.$$

Example 2 (How many permutation of n will have only even-sized cycles)**.**

$$\begin{aligned}\sum_{n=0}^{\infty}(\#\ \textit{of}\ \sigma\in S_n\ \textit{with only even cycles})\frac{x^n}{n!} &= e^{\sum\limits_{n=1}^{\infty}|C_{2n}|\frac{x^{2n}}{2n!}}\\ &= e^{\sum_{n=1}^{\infty}\frac{x^{2n}}{2n}}\\ &= e^{\frac{1}{2}\sum_{n=1}^{\infty}\frac{(x^2)^n}{n}}\\ &= e^{\frac{1}{2}\ln\left(\frac{1}{1-x^2}\right)}\\ &= \frac{1}{\sqrt{1-x^2}}\\ &= (1-x)^{-1/2}\\ &= \sum_{k=0}^{\infty}\binom{-1/2}{k}(-x^2)^k.\end{aligned}$$

Matching coefficients

$$\begin{cases}(2k)!\binom{-1/2}{k}(-1)^k & \textit{if } n=2k\\ 0 & \textit{otherwise}\end{cases}$$

Example 3 (The Number of Labeled Graphs of n)**.** *# of labeled graphs on n vertices* $=$

$2^{\binom{n}{2}}$. *Then* $|C_n| = 2^{\binom{n}{2}}$ *so that*

$$\sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{x^n}{n!} = e^{C(x)} = e^{\sum_{n=1}^{\infty} |C_n| \frac{x^n}{n!}}$$

taking the natural logarithm of both sides

$$C(x) = \ln\left(\sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{x^n}{n!}\right)$$

Example 4 (How Many 2−regular graphs on vertices are there?)**.** 2*-regular graphs every node has degree* 2*. So* $|C_n| = 0$ *for* $n = 1, 2$ *and* $|C_n| = \dfrac{(n-1)!}{2}$ *for* $n \geq 3$*. This is because of cycle representations of the graph, is just* $|S_{n-1}|$ *since these are transpositions, we over-count by* 2 *however. Finally*

$$\begin{aligned}
\sum_{n=0}^{\infty}(\#\textit{ of 2-regular graphs on } n\frac{x^n}{n!} &= e^{\sum_{n=1}^{\infty} |C_n| \frac{x^n}{n!}} \\
&= e^{\sum_{n=3}^{\infty} \frac{(n-1)!}{2} \frac{x^n}{n!}} \\
&= e^{\frac{1}{2}(\ln\left(\frac{1}{1-x}\right) - x - \frac{x^2}{4})} \\
&= \frac{e^{-x-\frac{x^2}{4}}}{\sqrt{1-x}}
\end{aligned}$$

2 Asymptotics

Definition 1.

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x}\ dx$$

for $\alpha > 0$*.*

Theorem 2.1.

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$$

for $\alpha > 0$*.*

Corollary 2.1.1.

$$\Gamma(n) = (n-1)!$$

for $n \in \mathbb{Z}^+$*.*

Identity 3.

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Definition 2 (Asymptotics)**.**

$$a_n \;\; b_n \iff \lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

Theorem 2.2 (Stirling's Approximation)**.**

$$\Gamma(n+1) \;\; \sqrt{2\pi n}\frac{n^n}{e^n}$$

Identity 4.

$$\sqrt{2\pi} = \int_{-\infty}^{+\infty} e^{-x^2/2}\ dx$$

Theorem 2.3.

$$(-1)^n\binom{-\alpha}{n} = \binom{n+\alpha-1}{n} \;\; \frac{n^{\alpha-1}}{\Gamma(\alpha)}$$

Corollary 2.3.1. *If $0 < \beta < \alpha$, then*

$$\lim_{n\to\infty} \frac{\binom{-\beta}{n}}{\binom{-\alpha}{n}} = 0.$$

Theorem 2.4. *Let b_n have generating function*

$$g(x) = \frac{1}{(1-x)^\alpha} + \frac{C_1}{(1-x)^{\alpha-1}} + \frac{C_2}{(1-x)^{\alpha-2}} + \ldots + \frac{C_k}{(1-x)^{\alpha-k}}$$

with $\alpha > k$. Then $b_n \;\; \frac{n^{\alpha-1}}{\Gamma(\alpha)}$.

Identity 5.

$$\int_0^1 x^{\alpha-1}(1-x)^n\ dx = \frac{n!}{\alpha(\alpha+1)\ldots(\alpha+n)}$$

Identity 6.

$$\Gamma(\alpha) = \lim_{n\to\infty} \frac{n^\alpha (n-1)!}{\alpha(\alpha+1)\ldots(\alpha+n)}$$

Corollary 2.4.1. *Let b_n have generating function*

$$g(x) = \frac{C}{(R-x)^\alpha} + \frac{C_1}{(R-x)^{\alpha-1}} + \ldots + \frac{C_k}{(R-x)^{\alpha-k}}$$

with $\alpha > k$. Then

$$b_n \;\; \frac{Cn^{\alpha-1}}{R^{n+\alpha}\Gamma(\alpha)}$$

3 Singularities

Definition 3 (Singularity and Analytic)**.** *A complex valued function is analytic at* x_0 *if* $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ *for all* $|x-x_0| < \epsilon$ *for some* $\epsilon > 0$*, otherwise* $f(x)$ *has a singularity at* x_0*.*

Example 5 (Singularities and Analyticity)**.**

- $e^x, \sin(x), x^6 - 3x + 1$ *have no singularities, their derivatives are smooth on all of* $\mathbb{R}$
- $\dfrac{x}{(x-2)(x-3)^2}$ *has singularities at* $x = 2, 3$ *because their derivatives are not smooth in any neighborhood of* 2 *or* 3
- $\sqrt{4-x^2}$ *has singularities at* $x = \pm 2$ *since it's derivative doesn't exist at these points*
- $\dfrac{1}{1-\sin(x)}$ *has singularities at* $x = \dfrac{\pi}{2} + 2k\pi$ *for all* $k \in \mathbb{Z}$

Example 6 (Removing Singularities)**.**

- *To remove the singularity of* $g(x) = \dfrac{x}{(x-2)(x-3)^2}$ *at* $x = 2$*, we'll multiply by* $(2-x)$*.* $\lim_{x\to 2}(2-x)g(x) = \lim_{x\to 2} \dfrac{-x}{(x-3)^2}$*, this limit exists because of L'Hopitals rule.*
- $h(x) = \dfrac{1}{1-\sin(x)}$

$$\begin{aligned}
\lim_{x\to\pi/2}\left(\frac{\pi}{2} - x\right)h(x) &= \lim_{x\to\pi/2} \frac{\frac{\pi}{2} - x}{1-\sin(x)} \\
&= \lim_{x\to\pi/2} \frac{-1}{-\cos(x)} = \infty
\end{aligned}$$

but

$$\begin{aligned}
\lim_{x\to\pi/2}\left(\frac{\pi}{2} - x\right)^2 h(x) &= \lim_{x\to\pi/2} \frac{\left(\frac{\pi}{2} - x\right)^2}{1-\sin(x)} \\
&= \lim_{x\to\pi/2} \frac{-2\left(\frac{\pi}{2} - x\right)}{-\cos(x)} \\
&= \lim_{x\to\pi/2} \frac{-2(-1)}{\sin(x)} \\
&= 2
\end{aligned}$$

Theorem 3.1. *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *if there's a* $\alpha > 0$ *such that*

- $(R-x)^\alpha$ *is analytic at all* x_0 *with* $|x_0| < R'$ *for some* $R' > R$*, and*

- $\lim\limits_{x\to R}((R-x)^\alpha f(x)) = C$ *for* $C \neq 0, \infty, -\infty$,

then

$$a_n \quad \frac{n^{\alpha-1}}{\Gamma(\alpha)R^{n+\alpha}}.$$

Note that (1) *will ensure that we can remove singularities at* R' *that the new singularity will be greater than* R', (2) *not multiplying by an unnecessarily large power.*

Theorem 3.2 (The Asymptotic Theorem)**.** *If* $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ *and*

- $(R-x)^\alpha f(x)$ *is analytic for* $|x_0| > R$
- $C = \lim\limits_{x\to R}(R-x)^\alpha f(x)$ *isn't* $0, \pm\infty$,

then

$$a_n \quad \frac{Cn^{\alpha-1}}{\Gamma(\alpha)R^{n+\alpha}}.$$

Example 7 (Using the Asymptotic Theorem)**.** $A(x) = \dfrac{1}{(1-3x)(1+2x)} = \sum\limits_{n=0}^{\infty} a_n x^n$. *We found that* $a_n = \dfrac{1}{3}\left(3^{n+1} + (-1)^n 2^{n+1}\right)$.

We can remove the singularities with $x = \dfrac{-1}{2}, \dfrac{1}{3}$ *with* $\alpha = 1$*:*

$$\lim_{x\to\frac{1}{3}} \left(\frac{1}{3} - x\right) A(x) = \lim_{x\to\frac{1}{3}} \frac{1}{3(1+2x)} = \frac{1}{5}.$$

By the asymptotic theorem

$$a_n \quad \frac{1}{5}\frac{n^0}{\Gamma(1)\frac{}{3^{n+1}}} = \frac{3^{n+1}}{5}$$

Example 8 (Approximating the Probability of a Permutation of n has no cycle of length 1)**.**

$$\sum_{n=0}^{\infty}(\#\sigma \textit{ with no 1 cycles})\frac{x^n}{n!} = e^{\sum_{n=2}^{\infty}}\frac{x^n}{n} = \frac{1}{1-x}e^{-x}.$$

We can find an asmyptotic relationship by noting that $R = 1$ *is a singularity nearest to* 0, *so that*

$$C = \lim_{x\to 1}(1-x)\frac{e^{-x}}{1-x} = e^{-1}.$$

Meaning $C = e^{-1}, \alpha = 1, R = 1$ *giving us*

$$a_n \quad \frac{e^{-1}n^{1-1}}{\Gamma(1)1^{n+1}} = \frac{1}{e}$$

Example 9 (When The Asmyptotic Theorem Fails to Work). *With* $\frac{1}{\sqrt{1-x^2}}$ *the singularities are equally space from* 0*, meaning there is no nearest singularity. So we actually cannot apply the asymptotic theorem here.*

4 Permutation Statistics

Definition 4 (Permutation Statistic). *A permutation statistic is simply a function* $s : S_n \to \mathbb{R}$. *Let* $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, *then*

1. *(# of Descents)*
   $des(\sigma) = (\#\text{ of indicies } i \text{ with } \sigma_i > \sigma_{i+1})$
2. *(Exceedences)*
   $exc(\sigma) = (\#\text{ with } \sigma_i > 1)$
3. *(Inversions)*
   $inv(\sigma) = (\#i < j \text{ with } \sigma_i > \sigma_j)$
4. *(Major Index)*
   $maj(\sigma) = \sum_{i \text{ with } \sigma_i > \sigma_{i+1}} i.$

Theorem 4.1. *Descents* $(\#i \text{ with } \sigma_i > \sigma_{i+1})$ *and excedences* $(\#i \text{ with } \sigma_i > i)$ *are equidistributed.*

Definition 5 (Q-Analogeue).

- $$[n]_q = 1 + q + q^2 + \ldots + q^{n-1} = \frac{1-q^n}{1-q}$$
- $$[n]_q! = [n]_q \cdot [n-1]_q \cdot \ldots \cdot [1]_q$$
- $$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

Theorem 4.2. $[n]_q! = \sum_{\sigma \in S_n} q^{inv(\sigma)} = \sum_{\sigma \in S_n} q^{maj(\sigma)}$ *can be shown with a bijection* $\phi : S_n \to S_n$ *such that* $inv(\sigma) = maj(\phi(\sigma))$.

Definition 6 (Rearrangements). $R(0^k, 1^{n-k})$ *is the set of rearrangements of* k $0's$ *and* $n-k$, 1*'s.*

Theorem 4.3.

$$\sum_{r \in R(0^k, 1^{n-k})} q^{inv(r)} = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

Corollary 4.3.1. $\begin{bmatrix} n \\ k \end{bmatrix}_q$ *is a polynomial in* q

5 Integer Partitions

Definition 7. *An integer partition is a strictly decreasing list of integers which sum to n. The* $\max(\lambda)$ *is the largest number appearing in the list,* $l(\lambda)$ *is the length of the list.* λ' *is the conjugate that is obtained by transposing rows with columns in a young tableaux.*

Theorem 5.1.

$$\sum_{\lambda \vdash n} q^{l(\lambda)} = \sum_{\lambda \vdash n} q^{\max(\lambda)}$$

, shows that $l(\lambda) = \max(\lambda')$.

Theorem 5.2.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\substack{\lambda \text{ with } l(\lambda) \leq k \\ \text{and } \max(\lambda}} {\leq n-k}$$

This is how many Young Diagrams that can fit into a box of dimension k (height) and $n-k$ (width).

Theorem 5.3.

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \sum_{j=0}^{n} q^{k-j} \begin{bmatrix} n-j \\ k-j \end{bmatrix}_q = \sum_{j=0}^{n} q^{k-j} \begin{bmatrix} n-j \\ n-k \end{bmatrix}$$

Theorem 5.4.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ m \end{bmatrix}_q \begin{bmatrix} n-m \\ k-m \end{bmatrix}_q$$

6 Generating Functions for Integer Partitions

Theorem 6.1.

$$\sum_{n=0}^{\infty} \left(\sum_{\substack{\lambda \vdash n \\ l(\lambda) \leq k}} y^{l(\lambda)} \right) x^n = \frac{1}{(1-yx)(1-yx^2)\dots(1-yx^n)} = \prod_{i=1}^{n} \frac{1}{1-yx^i}$$

Theorem 6.2.

$$\prod_{i=1}^{\infty} \frac{1}{1-yx^i} = \sum_{n=0}^{\infty} \frac{y^n x^n}{(1-x)(1-x^2)\dots(1-x^n)}$$

Theorem 6.3.

$$\prod_{i=1}^{\infty} \frac{1}{1-x^i} = \sum_{n=0}^{\infty} \frac{x^{n^2}}{(1-x)\dots(1-x^n)(1-x)\dots(1-x^n)}$$

the left-hand side is the generating function for all integer partitions of n, the right-hand side is the generating function for # of cells in Young Diagram.

Theorem 6.4.

$$\sum_{n=0}^{\infty}(\#\lambda \vdash n \textit{ with distinct parts})x^n = \prod_{n=1}^{\infty}(1+x^n) = \prod_{n=1}^{\infty}\frac{1}{1-x^{2n-1}} = \sum_{n=0}^{\infty}(\#\lambda \vdash n \textit{ with only odd parts })x^n$$

Definition 8 (A Pentagonal Number). $\dfrac{k(3k-1)}{2}$

Theorem 6.5.

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{k\in\mathbb{Z}}(-1)^k x^{k(3k-1)/2}$$

Corollary 6.5.1.

$$\prod_{i=1}^{\infty}\frac{1}{1-x^i} = \sum_{n=0}^{\infty}p(n)x^n = \sum_{k\in\mathbb{Z}}(-1)^k x^{k(3k-1)/2}$$

Identity 7.

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \dots$$

where we're subtracting the pentagonal numbers and the sign pattern repeats as seen above.

Theorem 6.6. *The coefficient of y^k in*

$$\prod_{n=0}^{\infty}(1+yx^n)\prod_{n=1}^{\infty}\left(1+\frac{x^n}{y}\right)$$

is $x^{k(k-1)/2}\prod_{i=1}^{\infty}\dfrac{1}{1-x^i}$

Theorem 6.7 (Jacobi's Triple Product Identity)**.**

$$\sum_{k\in\mathbb{Z}}y^k x^{k(k-1)/2} = \prod_{i=1}^{\infty}(1-x^i)\prod_{n=0}^{\infty}(1+yx^n)\prod_{n=1}^{\infty}\left(1+\frac{x^n}{y}\right)$$

$$= (1+y)\prod_{n=1}^{\infty}(1-x^n)(1+yx^n)\left(1+\frac{x^n}{y}\right)$$

Theorem 6.8.

$$\prod_{i=1}^{\infty}(1-x^i)^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)x^{k(k+1)/2}$$

Theorem 6.9 (Fermat's Little Theorem)**.** *Let p be prime,*

$$a^p \equiv_p a$$

for $a \in \mathbb{Z}^+$

Corollary 6.9.1 (Freshman's Dream)**.**

$$(a_1 + \ldots + a_k)^p \equiv_p (a_1 + \ldots + a_k) \equiv_p (a_1^p + \ldots + a_k^p)$$

Theorem 6.10. *$p(n) = \#\lambda \vdash n$, then $p(5n-1) \equiv_5 0$*

Example 10 (Ramunajan Congruence)**.**

$$\begin{aligned}
\sum_{n=1}^{\infty} p(n-1)x^n &= x \prod_{n=1}^{\infty} \frac{1}{1-x^n} \\
&= x \prod_{i=1}^{\infty} (1-x^i)^4 \prod_{i=1}^{\infty} \frac{1}{(1-x^i)^5} \\
&\equiv_5 F(x) \prod_{n=1}^{\infty} \frac{1}{1-x^{5n}} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k p\left(\frac{n-k}{5}\right) \right) x^n
\end{aligned}$$

we need to show $p(5n-1) \equiv_5 \sum_{k=0}^{5n} a_k p\left(\frac{5n-k}{5}\right)$ thus we need to show $a_k \equiv_5 0$ when $k|5$.

$$\begin{aligned}
F(x) &= x \prod_{n=1}^{\infty} (1-x^n) \prod_{n=1}^{\infty} (1-x^n)^2 \\
&= \sum_{n=0}^{\infty} \sum_{k \in \mathbb{Z}} (-1)^{n+k} (2n+1) x^{k(3k-1)/2 + \binom{n+1}{2} + 1}
\end{aligned}$$

by calculating $k(3k-1)/2 + \binom{n+1}{2} + 1$ for values modulo 5, we find only 1 0 at $n = 2$ and $k = 1$.

7 Bijection Machine

Method 4. *$f : A \to B$ swaps the A_i diseases listed in D for B_i diseases, $\alpha : A \to A$ by adding/removing the smallest index i from D, $\beta : B \to B$ is defined by adding/removing the smallest index i from D. In this order repeat α, f, β, f^{-1} until we have no B.*

8 Tableaux and Symmetric Functions

Definition 9. *A tableaux of shape $\lambda \vdash n$ is a filling of the Young diagram of λ with elements in $\{1, \ldots, N\}$ (here $N >> n$).*

- *Row constant if the rows of T are constant*

- *Row nondecreasing if the rows of T are nondecreasing read left to right*
- *Row increasing similar to above*
- *Column strict if T is row nondecreasing and column increasing*

Finally the weight of a Tableaux is defined to be $w(T) = \prod_{c \in T} x_{\#inc}$ *where c is an entry in the tableaux.*

Definition 10 (Class of Symmetric Function).

- *The power symmetric polynomial* $P_\lambda(x_1, \ldots, x_N) \equiv \sum_{\textit{Row Constant } T \textit{ of shape } \lambda} w(T)$
- *The homogeneous symmetric polynomial* $h_\lambda(x_1, \ldots, x_N) \equiv \sum_{\textit{Row nondecreasing Tableaux } T \textit{ of shape } \lambda} w(T)$
- *The elementary symmetric polynomial* $e_\lambda(x_1, \ldots, x_N) \equiv \sum_{\textit{Row increasing tableaux } T \textit{ of shape } \lambda} w(T)$
- *The Schur polynomial* $s_\lambda(x_1, \ldots, x_N) = \sum_{\textit{Column strict } T \textit{ of shape } \lambda} w(T)$

Definition 11 (Symmetric Functions). *A polynomial f in $x_1, \ldots, x_N$ is a symmetric polynomial/function if*

$$f(x_1, \ldots, x_N) = f(x_{\sigma(1)}, \ldots, x_{\sigma(N)})$$

for all $\sigma \in S_n$.

Theorem 8.1.

$$s_\lambda(x_1, \ldots, x_N) = \sum_{\textit{Column strict } T \textit{ of shape } \lambda} w(T)$$

is a symmetric polynomial.

Definition 12 (The Monomial Symmetric Polynomial).

$$m_\lambda(x_1, \ldots, x_N) - \sum_{\sigma \in S_n} x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \ldots x_{\sigma(N)}^{\lambda_N}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ and possibly contains 0's. These will be the smallest symmetric polynomials that contain the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_N^{\lambda_N}$

Theorem 8.2. *The set Λ_n of all symmetric polynomials of degree n has a basis that is $\{m_\lambda : \lambda \vdash n\}$.*

Corollary 8.2.1. $\dim(\Lambda_n) = p(n)$

Definition 13 ($0,1$ Matrices). *Let $\mathbb{Z}_2 M_{\mu,\lambda}$ be the $\#$ of $0,1$ matrices of row sum μ and column sum λ.*

Identity 8.

- $h_{(\lambda_1, \ldots, \lambda_N)} = h_{\lambda_1} \ldots h_{\lambda_N}$

- $e_{(\lambda_1,\ldots,\lambda_N)} = e_{\lambda_1} \ldots e_{\lambda_N}$
- $p_{(\lambda_1,\ldots,\lambda_N)} = p_{\lambda_1} \ldots p_{\lambda_N}$

Theorem 8.3. *The coefficients of m_λ in e_μ is $\mathbb{Z}_2 M_{\mu,\lambda}$.*

Corollary 8.3.1. *$\{e_\lambda : \lambda \vdash n\}$ is a basis of the vector space Λ_n.*

Definition 14. *Let $B_{\lambda,\mu}$ be the set of all Brick Tabloids of content λ and shape μ. These are fillings of Young diagrams of μ with horizontal bricks of shape contained in λ.*

Theorem 8.4.

$$h_\mu = \sum_{\lambda \vdash n} (-1)^{n-l(\lambda)} |B_{\lambda,\mu}| e_\lambda$$

Method 5 (Sign-Reversing Involution)**.** *Take an object with shape μ and content λ filling each brick with a strictly increasing sequence. Scan from top to bottom, left to right, to find if we can combine two bricks. Fixed points will all have length* 1 *and only weakly decreasing sequence in rows*

9 Generating Functions for Symmetric Polynomials

Generating Function 5.

$$E(t) = \sum_{n=0}^{\infty} e_n(x_1, \ldots, x_N) t^n = \prod_{i=1}^{N} (1 + x_i t)$$

Generating Function 6.

$$H(t) = \sum_{n=0}^{\infty} h_n(x_1, \ldots, x_N) t^n = \prod_{i=1}^{N} \frac{1}{1 - x_i t}$$

Theorem 9.1.

$$E(t) = \frac{1}{H(-t)}$$

Method 6 (Sign-Reversing Involution)**.** *Take a $2-$tuple of strictly increasing row of k cells in the first entry, and $n-k$ sized row of a weakly decreasing sequence. Define the involution to simply move the largest cell in the first to the right. Fixed points only occur when $n = 0$. Done.*

10 Ring Homomorphisms

Definition 15 (Ring Homomorphism)**.** *A ring R is a set with operations $+, -, *$ that satisfies linearity in addition and $\phi(ab) = \phi(a)\phi(b)$.*

Theorem 10.1. *If $\phi : \Lambda \to \mathbb{Q}[x]$ is a ring homomorphism defined by $\phi(1) = 1$ and $\phi(e_n) = (-1)^{n-1}\frac{(x-1)^{n-1}}{n!}$ for all $n \geq 1$, then $\phi(n!h_n) = \sum_{\sigma \in S_n} x^{des(\sigma)}$.*

Theorem 10.2.

$$\sum_{n=0}^{\infty}\left(\sum_{\sigma \in S_n} x^{des(\sigma)}\right)\frac{e^n}{n!} = \frac{x-1}{x-e^{t(x-1)}}$$

Definition 16 (Word)**.** *A word of length n with letters $\{1,\ldots,k\}$ is a finite sequence with $w_k \in \{1,\ldots,k\}$. Let $\{1,\ldots,k\}^n$ denote the set of all words with length n.*

Theorem 10.3. *If $\phi(e_n) = (-1)^{n+1}\binom{k}{n}(x-1)^{n-1}$, then $\phi(h_n) = \sum_{w \in \{1,\ldots,k\}^n} x^{des(w)}$*

11 Polya's Enumeration Theorem

Definition 17 (Cycle Index Polynomial)**.** *The cycle index polynomial $Z_G = \frac{1}{|G|}\sum_{\sigma \in G} p_{\lambda(\sigma)}$ where that is the power symmetric polynomial.*

Definition 18 (Coloring)**.** *Is a function $f : \{1,\ldots,n\} \to \{1,2,\ldots\}$. The weight of the coloring $w(f) \equiv \prod_{i=1}^{\infty} x_i^{|f^{-1}\{i\}|}$, that is $|f^{-1}(\{i\})|$ is the number of times the color i was used*

Definition 19 (Coloring Polynomial)**.**

$$F_G = \sum_{equivalence\ class\ of\ [f]} w(f),$$

where the equivalence classes are determined by what we consider to be an equivalent coloring.

Theorem 11.1.

$$Z_G = F_G$$

Example 11 (Using Polya's Enumeration Theorem)**.** *How many ways can we color the vertices of a cube? What if 2 vertices are red and 2 are black?*

$Z_{D_4} = \frac{1}{8}(p_1^4 + 2p_1^2p_2 + 3p_2^2 + 2p_4) = F_{D_4}$ *with* $p_n = (x_1^n + x_2^n + \ldots + x_N^n)$ *if we take* $x_i = 1$ *we get*

$$\frac{1}{8}(N^6 + 2N^3 + 3N^2 + 2N)$$

total colorings for N colors. To find how many are red, black, then find the coefficient of $x_1^2x_2^2$ after expanding out each term in the cycle index polynomial.