Joseph C. McGuire
Dr. Linda Patton MATH 550
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Homework #2

2.A # 1

Prove that if A and B are subsets of $\mathbb{R}$ and $|B| = 0$, then $|A \cup B| = |A|$.

Proof. Let $A, B \subseteq \mathbb{R}$ with $|B| = 0$.

Since we have $A \subseteq A \cup B$, we get $|A| \leq |A \cup B|$ for free, since outer measure preserves order.

Conversely, define the sequence of subsets of $\mathbb{R}$, $(A_k)_{k=1}^{\infty}$ such that $A_1 = A, A_2 = B, A_n = \emptyset$ for all $n \geq 3$.

Then by the countable subadditivity theorem for outer measure, we have

$$\left|\bigcup_{k=1}^{\infty} A_k\right| \leq \sum_{k=1}^{\infty} |A_k|.$$

But with our definition of (A_k), we get that $\bigcup\limits_{k=1}^{\infty} A_k = A \cup B \cup \bigcup\limits_{k=3}^{\infty} \emptyset = A \cup B$ and similarly for the sum $\sum\limits_{k=1}^{\infty} |A_k| = |A| + |B| + \sum\limits_{k=3}^{\infty} |\emptyset| = |A| + |B| = |A|$. Hence we have $|A \cup B| \leq |A|$.

Giving us our result $|A \cup B| = |A|$. $\square$

2.A # 3

Prove that if $A, B \subseteq \mathbb{R}$ and $|A| < \infty$, then $|B \setminus A| \geq |B| - |A|$.

Proof. Let $A, B \subseteq \mathbb{R}$ and $|A| < \infty$. And note that $B \setminus A = B \cap A^C$ where $A^C = \mathbb{R} \setminus A$.

Then if $|B| = \infty$, then the result follows since $|B| - |A| = \infty$ if $A < \infty$ and everything is less than or equal to ∞ in the extended real numbers.

So $|B| < \infty$. Then if $|B| \leq |A|$, then $|B| - |A| \leq 0 \leq |B \setminus A|$, the last inequality coming from the fact that outer measure is always nonnegative.

So assume $|B| < \infty$ and $|B| > |A|$. Then note that from basic set theory we have $A^C \cup A = \mathbb{R}$ for any set $A \subseteq \mathbb{R}$ where $A^C = \mathbb{R} \setminus A$. So that $B = \mathbb{R} \cap B = (A^C \cup A) \cap B = (A^C \cap B) \cup (A \cap B)$, the last equality coming from De Morgan's Laws. Also note that $A \cap B \subseteq A$.

Hence $|B| = |(A \cap B) \cup (A^C \cap B)| \leq |A \cap B| + |A^C \cap B| \leq |A| + |A^C \cap B|$, the first inequality coming from an application of the countable subadditivity theorem, similar to the application in #1, and the second inequality coming from outer measure preserving order of subsets. So we have $|B| \leq |A| + |A^C \cap B| = |A| + |B \setminus A|$, thus $|B| - |A| \leq |B \setminus A|$. $\square$

2.A # 6

Prove that if $a, b \in \mathbb{R}$ and $a < b$, then

$$|(a,b)| = |[a,b)| = |(a,b]| = b - a.$$

Proof. Let $a, b \in \mathbb{R}$ and $a < b$.

Assume $a < b < \infty$. If this is the case, then we may write $(a,b) \cup \{b\} = (a,b]$ and similarly $(a,b) \cup \{a\} = [a,b)$. Additionally, for these sets we have $[a,b) \subset [a,b]$ and $(a,b] \subset [a,b]$ and $(a,b) \subset [a,b]$. From this we get that through the fact that outer measure preserves order that $|(a,b]| \leq |[a,b]| = b-a$ and $|[a,b)| \leq |[a,b]| = b-a$ and $|(a,b)| \leq |[a,b]| = b-a$. Moreover, since $[a,b] \setminus \{b\} = [a,b)$ and that $[a,b] \setminus \{a\}$ using this fact and $2.A\#3$ we get that $|[a,b)| \geq |[a,b]| - |\{b\}| = b - a - 0 = b - a$ and $|(a,b]| \geq b - a - 0 = b - a$, the last piece coming from the fact that at most countable sets have measure 0 and 2.14 for the measure of the closed interval.

Finally, we may write $(a,b) = [a,b] \setminus \{a,b\}$. So that using the same reasoning above, we get that $|(a,b)| \geq b - a$.

So we have $|(a,b)| = |[a,b)| = |(a,b]| = b - a = |[a,b]|$. $\square$

2.A # 7

Suppose $a, b, c, d \in \mathbb{R}$ and $a < b$ and $c < d$. Prove that

$$|(a,b) \cup (c,d)| = b - a + d - c \text{ if and only if } (a,b) \cap (c,d) = \emptyset$$

Proof. Let $a, b, c, d \in \mathbb{R}$ and $a < b$ and $c < d$. Suppose $|(a,b) \cup (c,d)| = b - a + d - c$. Then note that $(a,b) \cup (c,d) \subset (a,d)$, since outer measure preserves order, we have $|(a,b) \cup (c,d)| = b - a + d - c \leq |(a,d)| = d - a$, the last equality coming from 2.14. This inequality gives us $b - c \leq 0 \implies b \leq c$. Taking $x \in (a,b) \cap (c,d)$ we find $a < x < b$ and $c < x < d$. Since $b \leq c$ this implies $x < x$, a contradiction. So that $(a,b) \cap (c,d) = \emptyset$.

Conversely, assume that $(a,b) \cap (c,d) = \emptyset$. Since these two are disjoint without loss of generality, assume $b < c$ that is the interval (a,b) is to the left of (c,d). Note that by $2.A\#6$ we have $|(a,b)| = b - a$ and similarly $|(c,d)| = d - c$,s so that by subadditivity of outer measure we have that $|(a,b) \cup (c,d)| \leq |(a,b)| + |(c,d)| = b - a + d - c$. For the other side of the inequality, note that $(a,b) \cup (c,d) = (a,d) \setminus [b,c]$. So that $2.A\#3$ we get $|(a,b) \cup (c,d)| = |(a,d) \setminus [b,c]| \geq |(a,d)| - |[b,c]| = d - a - (c - b) = d - c + b - a$. Thus we have $|(a,b) \cup (c,d)| = d - c + b - a$.

Hence $|(a,b) \cup (c,d)| = d - c + b - a$ if and only if $(a,b) \cap (c,d) = \emptyset$. $\square$

2.A # 9

Prove that $|A| = \lim_{t\to\infty} |A \cap (-t,t)|$ for all $A \subset \mathbb{R}$.

Proof. Let $A \subset \mathbb{R}$ and $t > 0$. Let $(-t,t)^C = \mathbb{R} \setminus (-t,t)$.

Then note that $|A\cap(-t,t)| = \inf\{\sum_{n=1} l(I_n) : \{I_n\}_{n=1}^{\infty}$ is a sequence of open intervals such that $\bigcup_{n=1}^{\infty} I_n \supseteq A\cap(-t,t)\}$ and $|A\cap(-t,t)^C| = \inf\{\sum_{n=1} I_n : \{I_n\}_{n=1}^{\infty}$ is a sequence of open intervals such that $\bigcup_{n=1}^{\infty} I_n \supseteq A \cap (-t,t)^C\}$. Additionally, note that $(-t,t)^C = (-\infty,-t] \cup [t,\infty)$. As we take the limit: $\lim_{t\to\infty}(|A\cap(-t,t)| + |A\cap(-t,t)^C|)$, we'll have $t \to \infty$ and $-t \to -\infty$, meaning that the $(-t,t)$ will tend towards $(-\infty,\infty)$. Note that $A\cap(-\infty,\infty) = A$ and that $(-\infty,\infty)^C = \emptyset$, so that the infimums become the infimum over the set $\{\sum_{n=1}^{\infty} l(I_n) :$ is a sequence of open intervals such that $\bigcup_{n=1}^{\infty} I_n \supseteq A\cap(-\infty,\infty)\}$ and $\{\sum_{n=1}^{\infty} l(I_n) :$ is a sequence of open intervals such that $\bigcup_{n=1}^{\infty} I_n \supseteq A\cap(-\infty,\infty)^C\}$.
Taking $\mathbb{R} = (-\infty,\infty)$ we get that $(-\infty,\infty)^C = \emptyset$, and so $A \cap (-\infty,\infty) = A$ and $A \cap \emptyset = \emptyset$.
So we get $\lim_{t\to\infty} |A \cap (-t,t)| + |A \cap (-t,t)^C| = |A| + |\emptyset| = |A|$. □

2.A # 10

Prove that $|[0,1] \setminus \mathbb{Q}| = 1$.

Proof. So first note that $\mathbb{Q}$ is countable. Hence $|\mathbb{Q}| = 0$, by 2.4.
Then since $[0,1] \supset [0,1] \setminus \mathbb{Q}$ we have by outer measure preserving order, $|[0,1]| = 1 \geq |[0,1]\setminus\mathbb{Q}|$. In conjunction with $2.A\#3$ we have our result $|[0,1]\setminus\mathbb{Q}| \geq |[0,1]| - |\mathbb{Q}| = 1 - 0 = 1$.
Thus $|[0,1] \setminus \mathbb{Q}| = 1$. □

2.A # 11

Prove that if $I_1, I_2, \ldots$ is a disjoint sequence of open intervals, then

$$\left|\bigcup_{k=1}^{\infty} I_k\right| = \sum_{k=1}^{\infty} l(I_k).$$

Proof. Let $I_1, I_2, \ldots$ be a sequence of disjoint open intervals.

Then note that combining the subadditivity of outer measure and the result from 2.A#6 we get that:

$$\left|\bigcup_{k=1}^{\infty} I_k\right| \leq \sum_{k=1}^{\infty} |I_k| = \sum_{k=1}^{\infty} l(I_k)$$

First, we'll show that $\left|\bigcup_{k=1}^{n} I_k\right| = \sum_{k=1}^{n} l(I_k)$, by induction of n.

- (Basis) Note for $n = 2$ this follows from 2.A#7.
- (Inductive Hypothesis) Suppose for some $n \in \mathbb{N}$ that we have

$$\left|\bigcup_{k=1}^{n} I_k\right| = \sum_{k=1}^{n} l(I_n)$$

- So then consider the following:

$$\begin{aligned}\sum_{k=1}^{n+1} l(I_k) &\leq \sum_{k=1}^{n} l(I_k) + l(I_{n+1}) \\ &\leq \left|\bigcup_{k=1}^{n} I_k\right| + l(I_{n+1}) \\ &\leq \sum_{k=1}^{n} l(I_k) + l(I_{n+1}) = \sum_{k=1}^{n+1} l(I_k).\end{aligned}$$

The second inequality coming from the inductive hypothesis, the last inequality coming from subadditivty of measure and the fact that for open intervals $l(I) = |I|$, the result from 2.A#6. So that we get $\left|\bigcup_{k=1}^{n} I_k\right| = \sum_{k=1}^{n} l(I_k)$ for any $n \in \mathbb{N}$ by the principle of mathematical induction

So then using 2.A#9 we get that $\lim_{n\to\infty} \left|\left(\bigcup_{k=1}^{n} I_k\right) \cap (-n, n)\right| = \left|\bigcup_{k=1}^{n} I_k\right|$. So that, we have our result:

$$\lim_{n\to\infty} \left|\bigcup_{k=1}^{n} I_k\right| = \lim_{n\to\infty} \lim_{n\to\infty} \left|\left(\bigcup_{k=1}^{n} I_k\right) \cap (-n, n)\right| = \lim_{n\to\infty} \sum_{k=1}^{n} l(I_k).$$

The second equality being needed so that we just take the limit of a number and not technically the limit over a bunch of unions. $\square$

2.B # 1

Show that $S = \{\bigcup_{n\in K}(n, n+1] : K \subset \mathbb{Z}\}$ is a $\sigma-$algebra on $\mathbb{R}$.

Proof. To show that S on $\mathbb{R}$ is a σ-algebra we'll show that (1) $\emptyset \in S$; (2) if $E \in S$, then $\mathbb{R} \setminus E \in S$; and (3) if $\{E_n\}_{n=1}^{\infty}$ is a sequence of elements in S, then $\bigcup\limits_{k=1}^{\infty} E_k \in S$.

- Just take $K = \emptyset \subset \mathbb{Z}$ so that you're union-ing nothing i.e. $\bigcup\limits_{\emptyset}(n, n+1] = \emptyset$.

- Suppose $E \in S$. Then for some $K \subset \mathbb{Z}$ we have $E = \bigcup\limits_{n\in K}(n, n+1]$. Note that $\mathbb{Z}$ is countable, so then any subset of $\mathbb{Z}$ is at most countable. Now we enumerate K such that $K = \{n_1, n_2, \ldots\}$. Note if K is finite, just allow this labeling to terminate at some index. Additionally note that if K isn't finite, then K is unbounded, because $\mathbb{Z}$ has no bounded infinite subsets, because any element in $\mathbb{Z}$ is $n + 1$ for some other $n \in \mathbb{Z}$.

  So then we will write $E = \bigcup\limits_{k=1}^{\infty}(n_k, n_k+1] = (n_1, n_1+1] \cup (n_2, n_2+1] \cup \ldots$. Note then that for any distinct integers n_i, n_j with $n_i < n_j$ we have $(n_i, n_i+1] \cap (n_j, n_j{+}_1] = \emptyset$. Otherwise there would be a x such that $n_i < x \leq n_i + 1 < x \leq n_j + 1$ giving us the contradiction $x < x$. With that we can then take $\mathbb{R} \setminus E = \mathbb{R} \setminus \bigcup\limits_{k=1}^{\infty}(n_k, n_k+1] = \bigcap\limits_{k=1}^{\infty} \mathbb{R}\setminus(n_k, n_k+1] = \bigcap\limits_{k=1}^{\infty}(-\infty, n_k] \cup (n_k+1, \infty) = \left(\bigcap\limits_{k=1}^{\infty}(-\infty, n_k]\right) \cup \left(\bigcap\limits_{k=1}^{\infty}(n_k+1, +\infty)\right)$. Note that since $K \subseteq \mathbb{Z}$ and $\mathbb{Z}$ has the well-ordering principle then there exists $n_j \in K$ such that $|n_j| < |n_k|$ for all $k \in K \setminus \{n_j\}$. Then take $M = \mathbb{Z} \setminus \{n_j\}$ So then we have the following:

$$\begin{aligned}\left(\bigcap_{k=1}^{\infty}(-\infty, n_k]\right) \cup \left(\bigcap_{k=1}^{\infty}(n_k+1, +\infty)\right) &= (-\infty, n_j] \cup (n_j+1, \infty) \\ &= \bigcup_{n_k\in M}(n_k, n_k+1].\end{aligned}$$

  Thus $\mathbb{R} \setminus E \in S$.

- Let $\{E_k\}_{k=1}^{\infty}$ be a sequence of sets in S, where $E_k = \bigcup\limits_{n\in K_k}(n, n+1]$, for $K_k \subseteq \mathbb{Z}$ for all $k \in \mathbb{N}$. Let $\{n_{k_1}, n_{k_2}, \ldots\}$ be an enumeration of each K_k, again this is possible because

$\mathbb{Z}$ is a countable set. So that we have the following:

$$\begin{aligned}
\bigcup_{k=1}^{\infty} E_k &= \bigcup_{k=1}^{\infty} \bigcup_{n \in K_k} (n, n+1] \\
&= \bigcup_{k=1}^{\infty} (n_{k_1}, n_{k_1}] \cup (n_{k_2}, n_{k_2}] \cup \ldots \\
&= ((n_{1_1}, n_{1_1}+1] \cup (n_{2_1}, n_{2_1}+1] \cup \ldots) \cup ((n_{1_2}, n_{1_2}+1] \cup (n_{2_2}, n_{2_2}+1] \cup \ldots) \cup \ldots
\end{aligned}$$

Since we have that for two distinct integers n_i, n_j with $n_i < n_j$ that $(n_i, n_i+1] \cap (n_j, n_j+1] = \emptyset$, and that the above union is at most countable, so that just define the index set to be the indices as seen in the last line, call this K so that we have:

$$\bigcup_{k=1}^{\infty} E_k = \bigcup_{n \in K} (n_k, n_k+1].$$

Thus $\bigcup_{k=1}^{\infty} E_k \in S$.

Finally, we're done and we have S is a $\sigma-$algebra over $\mathbb{R}$.

$\square$