Homework #2

Joseph C. McGuire
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8.) Test the series $\sum_{k=1}^{\infty} \frac{7^k}{k!}$ for convergence.

To prove this converges we will use the ratio test:

Proof. Let $a_k = \frac{7^k}{(k+1)!}$ and then consider the series $\sum_{k=1}^{\infty} a_k$. To show that this converges, we will use the ratio test:

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{7^{k+1}}{(k+1)!}}{\frac{7^k}{k!}} = \lim_{k\to\infty} \frac{7^{k+1}k!}{7^k(k+1)!} = \lim_{k\to\infty} \frac{7}{k+1} = 7 \cdot \lim_{k\to\infty} \frac{1}{k+1} = 7 \cdot 0 = 0$$

The last limit follow from previous results. Thus since $R = 0 < 1$, we have that the series, $\sum_{k=1}^{\infty} a_k$, is absolutely convergent. Hence is convergent. $\square$

9.) Test the series $\sum_{k=1}^{\infty} \frac{sin(k)}{2k^2+1}$ for convergence.

To prove this we will use the comparison test on the sequence $|\{a_k\}|$ to show it converges absolutely.

Proof. Let $\{a_k\}_{k=1}^{\infty} = \left\{\frac{sin(k)}{2k^2+1}\right\}$, then consider the series $\sum_{k=1}^{\infty} a_k$. Let $\{b_k\}_{k=1}^{\infty} = \left\{\frac{1}{2k^2+1} : k \in \mathbb{N}\right\}$. Then since $|sin(x)| \leq 1$ for all $x \in \mathbb{R}$, we have $|a_k| \leq b_k$ for all $k \in \mathbb{R}$. Also both $|a_k| > 0$ and $b_k > 0$.
Then we know that $\frac{1}{2k^2+1} \leq \frac{1}{2k^2}$, and that the series $\sum_{k=1}^{\infty} \frac{1}{2k^2}$ is a convergent p-series. Thus, by the Comparison test, we have that the the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent. $\square$

10.) Let a and b be fixed real numbers satisfying $0 < b < a$. For what values a and b (satisfying the previous condition) does the series $\sum_{k=1}^{\infty}(\frac{1}{a^k - b^k})$ converge? diverge?

Proof. Let a and b be fixed such that $0 < b < a$ and $a, b \in \mathbb{R}$. Then consider the series $\sum_{k=1}^{\infty} \frac{1}{a^k - b^k}$.
(Divergent Series)
Thus the values for which this is true are when $0 < b < a < 1$. To show this, note that $b^k < a^k < 1$ and $0 < a^k - b^k < 1$. Hence $0 < a^k - b^k < 1$. So the sequence $\frac{1}{a^k - b^k} > \frac{1}{a^k}$. Since $\sum_{k=1}^{\infty} \frac{1}{a^k}$ is a geometric series such that $|\frac{1}{a}| \geq 1$, hence is a divergent geometric series. Thus by the Comparison Test, we have that $\sum_{k=1}^{\infty} \frac{1}{a^k - b^k}$ is a divergent series.
(Convergent Series)
Let $1 \leq b^k \leq a^k$, then we will show that it converges. Here we will use the Limit Comparison Test. Then take the sequence $b_k = \frac{1}{a^k}$, then all terms are positive. Taking the ratio

$\lim_{k\to\infty} \frac{\frac{1}{a^k}}{\frac{1}{a^k-b^k}} = \lim_{k\to\infty} \frac{a^k-b^k}{a^k} = \lim_{k\to\infty} \frac{a^k-b^k}{a^k} = \lim_{k\to\infty} 1-(\frac{b}{a})^k = 1$. Hence by the Limit Comparison Test, since $\sum_{k=1}^{\infty}(\frac{1}{a})^k$ is a convergent geometric series, we have that the series $\sum_{k=1}^{\infty} \frac{1}{a^k-b^k}$ is convergent. Thus for the interval $a, b \in (0,1)$ we have that the series is divergent, and in the interval $a, b \in [b, \infty)$. □

11.)

a. Does the series $\sum_{k=1}^{\infty}(-1)^{k-1}b_k$ satisfy the hypotheses of the Alternating Series Test? Explain.

Note that for b_k to satisfy the hypothesis, we need b_k to be monotonic decreasing. But note that for k = 6 and k = 7, we have $b_6 = \frac{1}{216} > b_7 = \frac{1}{49}$. Hence the sequence b_k is not monotonic decreasing, and thus doesn't meet all of the hypothesis for the Alternating Series Test.

b. Does $\sum_{k=1}^{\infty}(-1)^{k-1}b_k$ converge or diverge?

Proof. Consider the sequence $|(-1)^{k-1}b_k| = |b_k| = b_k$. This follows from the fact that all terms are positive. Consider the following:

$$\sum_{k=0}^{\infty} b_k = 1 + \frac{1}{8} + \frac{1}{9} + ... = \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k)^3}.$$

Both of these are convergent p-series, hence our series $\sum_{k=1}^{\infty}(-1)^{k-1}b_k$ is absolutely convergent, thus is convergent. □

12.) Prove that the series converges either or diverges. If it diverges, give an example.

$\sum_{k=1}^{\infty} a_k^2$

I claim that the series is convergent.

Proof. Let $a_k \geq 0$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} a_k < \infty$.
Then consider the series $\sum_{k=1}^{\infty} a_k^2$. We proceed by the Limit Comparison test, not that for our sequences a_k^2 and a_k have all positive terms. $\lim_{k\to\infty} \frac{a_k^2}{a_k} = \lim_{k\to\infty} a_k = 0$, by Corollary 2.7.5, since we have that $\sum a_k < \infty$. Thus by Corollary 7.1.3, we have that $\sum_{k=1}^{\infty} a_k^2$ is convergent. □

13.) Prove that the series converges either or diverges. If it diverges, give an example.

$\sum_{k=1}^{\infty} \frac{a_k^2}{1+a_k}$.

Proof. Let $a_k \geq 0$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} a_k < \infty$.
Note that since $a_k \geq 0$, we have $0 \leq \frac{a_k^2}{1+a_k^2} \leq \frac{a_k^2}{a_k} = a_k$. Again, since we have that the series $\sum_{k=1}^{\infty} a_k < \infty$, by the Comparison Test, we have the series $\sum_{k=1}^{\infty} \frac{a_k^2}{1+a_k}$ is convergent. □