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Homework # 10

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4.B.7

Give an example of a Borel subset of $\mathbb{R}$ whose density at 0 isn't defined.

Proof. Define the sequence of sets $A_n = \left(\frac{1}{2^n+1}, \frac{1}{2^n}\right)$ and $A = \bigcup_{n=1}^{\infty} A_n$. Note that each A_n is disjoint as $2^{n+1} < 2^{n+2}$ for all $n \in N$ and hence $\frac{1}{2^{n+2}} < \frac{1}{2^n+1}$. So that by the disjoint additivity of outer measure:

$$|A| = \left|\bigcup_{n=1}^{\infty} A_n\right| = \sum_{n=1}^{\infty} |A_n| = \sum_{n=1}^{\infty} \frac{1}{2^n} - \frac{1}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{2^{n+1}-2^n}{2^n 2^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{2^n} = (2-1) = 1.$$

Note that from the above we had that

$$|A_n| = \frac{1}{2^n}.$$

So that we'll take a sequential limit for $t \downarrow 0$, by taking $\frac{1}{2^N} \downarrow 0$, that is by taking $N \to \infty$ we'll look be looking at $\left(\frac{-1}{2^N}, \frac{1}{2^N}\right)$ instead of the $(-t, t)$ in the definition of the density at $x = 0$. Then consider the following

$$\begin{aligned}
\lim_{t \downarrow 0} \frac{|A \cap (-t, t)|}{2t} &= \lim_{N \to \infty} \frac{\left|A \cap \left(-\frac{1}{2^N}, \frac{1}{2^N}\right)\right|}{\frac{1}{2^N}} \\
&= \lim_{N \to \infty} 2^N \left|\bigcup_{n=1}^{\infty} \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right) \cap \left(\frac{-1}{2^N}, \frac{+1}{2^N}\right)\right| \\
&= \lim_{N \to \infty} 2^N \left|\bigcup_{n=N}^{\infty} \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right)\right| \\
&= \lim_{N \to \infty} 2^N \sum_{n=N}^{\infty} \frac{1}{2^n} \\
&= \lim_{N \to \infty} \sum_{n=0}^{\infty} \frac{1}{2^n} = 2.
\end{aligned}$$

So to show the limit does not exist, we'll show that there's another sequential limit that doesn't converge to 2. With that in mind, define $M \in \mathbb{N}$ to be the largest integer such that

$\dfrac{1}{2^N} < \dfrac{1}{M}$, so note that as $t \downarrow 0$, that $M \uparrow \infty$.

$$\begin{aligned}
\lim_{t\downarrow 0} \frac{|A \cap (-t,t)|}{2t} &= \lim_{N\uparrow\infty} N \left| \bigcup_{n=1}^{\infty} \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right) \cap \left(\frac{-1}{N}, \frac{1}{N}\right) \right| \\
&= \lim_{N\uparrow\infty} N \left| \bigcup_{n=M}^{\infty} \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right) \right| \\
&= \lim_{N\uparrow\infty} N \sum_{n=M}^{\infty} \frac{1}{2^n} \\
&= \lim_{N\uparrow\infty} \frac{N}{2^M} = 0,
\end{aligned}$$

this last equality follows since $2^M \uparrow \infty$ as $N \uparrow \infty$ at a much quicker rate than $N \uparrow \infty$, that is an exponential will dominate a linear function.

So since there are two sequential limits that do not agree, we may conclude that the limit that defines the density of A is undefined at 0. $\square$

4.B.8

Give an example of a Borel subset of $\mathbb{R}$ whose density at 0 is $\dfrac{1}{3}$.

Proof. Define the sequence of sets $A_n = \left(\frac{1}{n}, \frac{1}{n} + \frac{2}{3}\left(\frac{1}{n} - \frac{1}{n+1}\right)\right)$ for all $n \in \mathbb{N}$ and the set $A = \bigcup_{n=1}^{\infty} A_n$. A is hence a Borel measurable subset of $\mathbb{R}$ since it's the countable union of open intervals.

First, we'll show that for all $n \in \mathbb{N}$,

$$\frac{1}{n+1} + \frac{2}{3}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) < \frac{1}{n}.$$

This will imply that the collection $\{A_n : n \in \mathbb{N}\}$ is pairwise disjoint. We'll show this inequality by reducing it to something that is obviously true:

$$\begin{aligned}
\frac{1}{n+1} + \frac{2}{3}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) &< \frac{1}{n} \\
\frac{1}{n+1} - \frac{3}{2(n+2)} &< \frac{3}{n} - \frac{3}{2(n+1)} \\
\frac{n+2-n-1}{(n+2)(n+1)} &< \frac{3(n+1-n)}{2n(n+1)} \\
n^2 + n &< \frac{3}{2}(n+2)(n+1) \\
n^2 + n &< \frac{3}{2}(n^2 + 2n + 1),
\end{aligned}$$

this last statement is obviously true for all $n \in \mathbb{N}$, giving us that the collection $\{A_n : n \in \mathbb{N}\}$ is pairwise disjoint.

Now let $0 < t < 1$ and $N \in \mathbb{N}$ to be the smallest integer $\dfrac{1}{N} + \dfrac{1}{3}\left(\dfrac{1}{N} - \dfrac{1}{N+1}\right) < t$. Then consider the following:

$$\begin{aligned}
\lim_{t\downarrow 0}\frac{|A\cap(-t,t)|}{2t} &= \lim_{t\downarrow 0}\frac{\left|\bigcup_{n=1}^{\infty}\left(\frac{1}{n},\frac{1}{n}+\frac{2}{3}\left(\frac{1}{n}-\frac{1}{n+1}\right)\right)\cap(-t,t)\right|}{2t}\\
&= \lim_{t\downarrow 0}\frac{\left|\bigcup_{n=1}^{\infty}\left(\left(\frac{1}{n},\frac{1}{n}+\frac{2}{3}\left(\frac{1}{n}-\frac{1}{n+1}\right)\right)\cap(-t,t)\right)\right|}{2t}\\
&= \lim_{t\downarrow 0}\frac{\left|\bigcup_{n=N}^{\infty}\left(\frac{1}{n},\frac{1}{n}+\frac{2}{3}\left(\frac{1}{n}-\frac{1}{n+1}\right)\right)\right|}{2t}\\
&= \lim_{t\downarrow 0}\frac{\sum_{n=N}^{\infty}\left|\left(\frac{1}{n},\frac{1}{n}+\frac{2}{3}\left(\frac{1}{n}-\frac{1}{n+1}\right)\right)\right|}{2t}\\
&= \lim_{t\downarrow 0}\frac{\sum_{n=N}^{\infty}\frac{2}{3}\left(\frac{1}{N}-\frac{1}{N+1}\right)}{2t}\\
&= \lim_{t\downarrow 0}\frac{\frac{1}{N}}{3t}=\frac{1}{3},
\end{aligned}$$

the second equality is De Morgan's Laws, the third equality is by our definition of N, the fourth is by the fact that A_n's are pairwise disjoint, and the second to last equality uses the fact that the series in question is telescoping and the final equality uses the fact that when t is small $\frac{1}{N}\sim t$.

Thus the set A has density $\frac{1}{3}$ at 0. $\square$

4.B.10

Suppose E is a Lebesgue measurable subset of $\mathbb{R}$ such that the density of E equals 1 at every element of E and equals 0 at every element of $\mathbb{R} \setminus E$. Prove that $E = \emptyset$ or $E = \mathbb{R}$.

Proof. I have a feeling this is wrong, not sure where it goes wrong though.

Suppose E is a Lebesgue measurable subset of $\mathbb{R}$ such that the density of E equals 1 at every element of E and equals 0 at every element of $\mathbb{R} \setminus E$.

Let $x, y \in \mathbb{R}$ and $0 < t < 1$

$$
\begin{aligned}
\left|\lim_{t \downarrow 0} \frac{|E \cap (x-t, x+t)|}{2t} - \lim_{t \downarrow 0} \frac{|E \cap (y-t, y+t)|}{2t}\right| &= \left|\lim_{t \downarrow 0} \left(\frac{|E \cap (x-t, x+t)|}{2t} - \frac{|E \cap (y-t, y+t)|}{2t}\right)\right| \\
&= \lim_{t \downarrow 0} \left|\frac{|E \cap (x-t, x+t) \setminus E \cap (y-t, y+t)|}{2t}\right| \\
&= \lim_{t \downarrow 0} \frac{|E \cap ((x-t, x+t) \setminus (y-t, y+t))|}{2t} \\
&\leq \lim_{t \downarrow 0} \frac{|(x-t, x+t) \setminus (y-t, y+t)|}{2t} \\
&= \lim_{t \downarrow 0} \frac{|(x-t, x+t)| - |(y-t, y+t)|}{2t} \\
&= 0.
\end{aligned}
$$

Some explanation, the first line can only be applied since the limit exists everywhere in $\mathbb{R}$, the second line uses the fact that both sets have finite measure and 2.57, the third line is an application of De Morgan's Laws, the fourth line uses the fact that outer measure preserves order of sets, the fifth line uses (2.57) again, the final line uses the fact that the sets have the same measure.

So this gives us that the density is constant everywhere in $\mathbb{R}$, that is, either $E = \mathbb{R}$ or $E = \emptyset$. □