Section 11.A: Fourier Coefficients and The Riemann-Lebesgue Lemma

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Fourier Coefficients and Outline

For $k \in Z$ we define the family $e_k : (-\pi, \pi] \to R$ defined by

$$e_k(t) = \begin{cases} \frac{1}{\sqrt{\pi}} \sin(kt) & \text{if } k > 0 \\ \frac{1}{\sqrt{2\pi}} & \text{if } k = 0 \\ \frac{1}{\sqrt{\pi}} \cos(kt) & \text{if } k < 0 \end{cases}. \tag{1}$$

These are easily shown to be an orthonormal family on $L^2((-\pi, \pi])$. The difficulty comes in showing that it's a basis on $L^2((-\pi, \pi])$. One way to do this is with the Spectral Theorem for Compact Operators, we won't be doing this though.

The Unit Circle

To show that this is a basis of $L^1((-\pi, \pi])$ we'll use the map

$$t \mapsto e^{it} = \cos(t) + i\sin(t), \tag{2}$$

for $t \in (-\pi, \pi]$ to establish a bijection between the interval $(-\pi, \pi]$ and the unit circle.

Definition (D; ∂D)

- D denotes the open disk in the complex plane:

$$D = \{w \in \mathbf{C} : |w| < 1\}.$$

- ∂D is the unit cirlce in the complex plane:

$$\partial D = \{z \in \mathbf{C} : |z| = 1\}.$$

The Unit Circle

We know a lot about the interval $(-\pi, \pi]$ because of our work in Chapter 2 and background in Lebesgue integration. We'll use the bijection between ∂D and $(-\pi, \pi]$ to define measurable sets and a measure on the unit circle, ∂D.

Definition (Measurable Subsets of ∂D; σ)

- A subset E of ∂D is measurable if $\{t \in (-\pi, \pi] : e^{it} \in E\}$ is a Borel subset of R.
- σ is the measure on the measurable subsets of ∂D obtained by transforming Lebesgue measure from $(-\pi, \pi]$ to ∂D, normalized so that $\sigma(\partial D) = 1$. If E is measurable, then

$$\sigma(E) = \frac{|\{t \in (-\pi, \pi] : e^{it} \in E\}|}{2\pi}.$$

Integration on the Unit Circle

Definition (The Lebesgue Integral on ∂D)

Let $f : \partial D \to \mathbb{C}$ be a measurable function, then, if it makes sense, define:

$$\int_{\partial D} f \, d\sigma = \int_{\partial D} f(z) \, d\sigma(z) = \int_{-\pi}^{\pi} f(e^{it}) \frac{dt}{2\pi}.$$

Definition ($L^p(\partial D)$)

For $1 \leq p \leq \infty$, define $L^p(\partial D)$ to mean the complex version of $L^p(\sigma)$, where σ is the normalized Lebesgue measure on $(-\pi, \pi]$.

A quick note, assume $t \in \mathbb{R}$ and $n \in \mathbb{Z}$, and $z = e^{it}$

- $\bar{z} = e^{-it}$
- $z^n = e^{int}$
- $\bar{z}^n = e^{-int}$

Polynomials on the Unit Circle

Theorem (Orthonormal Family in $L^2(\partial D)$)

$\{z^n\}_{n\in Z}$ is an orthonormal family in $L^2(\partial D)$.

Proof.

In $n \in Z$, then:

$$\begin{aligned} < z^n, z^n > &= \int_{\partial D} z^n \bar{z}^n \; dz \\ &= \int_{-\pi}^{\pi} e^{itn} e^{\bar{i}tn} \; \frac{dt}{2\pi} \\ &= \int_{-\pi}^{\pi} |e^{itn}|^2 \frac{dt}{2\pi} \\ &= \int_{-\pi}^{\pi} \frac{dt}{2\pi} = 1. \end{aligned}$$

$\square$

Monomials on the Unit Circle

Proof. (Cont.)

If $m, n \in Z$ and $m \neq n$, then:

$$\begin{aligned} < z^n, z^m > &= \int_{\partial D} z^n \bar{z^m}\ dz \\ &= \int_{-\pi}^{\pi} e^{int} e^{-imt}\ \frac{dt}{2\pi} \\ &= \int_{-\pi}^{\pi} e^{i(n-m)t}\ \frac{dt}{2\pi} \\ &= \frac{e^{i(m-n)t}}{2\pi i(m-n)}\Bigg|_{t=-\pi}^{t=\pi} \\ &= \frac{e^{i(m-n)\pi} - e^{-i(m-n)\pi}}{2\pi i(m-n)} = 0. \end{aligned}$$

As desired. □

Polynomials as a Basis on the Unit Circle

We'll rigourously show that the monomials form a basis for $L^2(\partial D)$ in the next section, until then recall that from Hilbert space theory if f is in the closure of the span of $\{z^n\}_{n\in Z}$ and in $L^2(\partial D)$ then we'll have:

$$f = \sum_{n\in Z} < f, z^n > z^n$$

where the infinite sum converges as an unordered sum in the norm of $L^2(\partial D)$, where

$$< f, z^n >= \int_{\partial D} f(z)\bar{z}^n dz = \int_{-\pi}^{\pi} f(e^{it})e^{-int}\frac{dt}{2\pi}.$$

So since $|z| = 1$ for $z \in \partial D$, we'll have $|z^n| = 1$ for every $z \in \partial D$, this integral makes since even when $f \in L^1(\partial D)$.

Fourier Coefficients and Series

Definition (Fourier Coefficients; $\hat{f}(n)$; Fourier Series)

Suppose $f \in L^1(\partial D)$.

- For $n \in \mathbb{Z}$, the n^{th} Fourier Coefficient of f is denoted $\hat{f}(n)$ and is defined by:

$$\hat{f}(n) = \inf_{\partial D} f(z)\bar{z}^n \, d\sigma(z) = \int_{-\pi}^{\pi} f(e^{it})e^{-int} \frac{dt}{2\pi}.$$

- The Fourier series of f is the formal sum:

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)z^n.$$

We will formalize in what sense f is equal to this sum later.

Examples of Fourier Coefficients

Suppose h is an analytic function on an open set that contains $\bar{D}$ (The closure of the open unit disk). Then h has a power series representation:

$$h(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where the sum on the right converges uniformly on $\bar{D}$ to h. Because uniform convergence on ∂D implies convergence in $L^2(\partial D)$, 8.58(b) and 11.6 imply that

$$(h|_{\partial D})^{\wedge}(n) = \begin{cases} a_n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases},$$

for all $n \in Z$. In other words, functions analytic on an open set containing the closure of the open unit disk $\bar{D}$, the Fourier series is the same as a Taylor series.

Examples of Fourier Coefficients

Suppose $f : \partial D \to \mathrm{R}$ is defined by

$$f(z) = \frac{1}{|3 - z|^2}.$$

Note that $|z| = 1$, so that $3 \notin \partial D$. Then for $z \in \partial D$ we have that:

$$\begin{aligned} f(z) &= \frac{1}{3 - z}\frac{1}{3 - \bar{z}} \\ &= \frac{1}{8}\left(\frac{z}{3 - z} + \frac{3}{3 - \bar{z}}\right) \\ &= \frac{1}{8}\left(\frac{\frac{z}{3}}{1 - \frac{z}{3}} + \frac{1}{1 - \frac{\bar{z}}{3}}\right) \\ &= \frac{1}{8}\left(\frac{z}{3}\sum_{n=0}^{\infty}\frac{z^n}{3^n} + \sum_{n=0}^{\infty}\frac{(\bar{z})^n}{3^n}\right) \end{aligned}$$

Examples of Fourier Coefficients

$$= \frac{1}{8}\left(\frac{z}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n + \sum_{n=0}^{\infty}\frac{z^{-n}}{3^n}\right)$$
$$= \frac{1}{8}\sum_{n=-\infty}^{\infty}\frac{z^n}{3^{|n|}},$$

where the infinite sum converges uniformly on ∂D. Thus we see that

$$\hat{f}(n) = \frac{1}{8}\frac{1}{3^{|n|}} \qquad n \in \mathbb{Z}.$$

Algebraic Properties of Fourier Coefficients

Theorem (Algebraic Properties of Fourier Coefficients)

Suppose $f, g \in L^1(\partial D)$ and $n \in \mathbb{Z}$. Then

(a) $(f+g)^\wedge(n) = \hat{f}(n) + \hat{g}(n)$;

(b) $(\alpha f)^\wedge(n) = \alpha \hat{f}(n)$ *for all* $\alpha \in \mathbb{C}$;

(c) $|\hat{f}(n)| \leq \|f\|_1$.

Suppose $f, g \in L^1(\partial D)$ and $n \in \mathbb{Z}$. Then

$$\begin{aligned}
(f+g)^\wedge(n) &= \int_{\partial D} (f+g)(z)\bar{z}^n \, dz \\
&= \int_{-\pi}^{\pi} \left(f(z)e^{-int} + g(z)e^{-int}\right) \frac{dt}{2\pi} \\
&= \hat{f}(n) + \hat{g}(n)
\end{aligned}$$

Algebraic Properties of Fourier Coefficients

Then we'll also have

$$\begin{aligned}(\alpha f)^{\wedge}(n) &= \int_{\partial D} (\alpha f)(z)\bar{z^n}\ dz \\ &= \alpha \hat{f}(n).\end{aligned}$$

To show the last, consider the following:

$$\begin{aligned}\left|\hat{f}(n)\right| &= \left|\int_{\partial D} f(z)\bar{z^n}\ dz\right| \\ \text{(Def. of Fourier Coef.)} &= \left|\int_{-\pi}^{\pi} f(e^{it})e^{-int}\frac{dt}{2\pi}\right| \\ \text{(Theorem 3.23)} &\leq \int_{-\pi}^{\pi} |f(e^{it})e^{-int}|\frac{dt}{2\pi} \\ (|e^{-int}| = 1) &= \int_{-\pi}^{\pi} |f(e^{it}|\frac{dt}{2\pi} \\ &= \|f\|_1,\end{aligned}$$

where we use the normalized-norm on $(-\pi, \pi]$ where we divide by 2π.

The Riemann-Lebesgue Lemma, Example

Combining (a) and (b) this shows that the map $f \mapsto \hat{f}(n)$ between $L^1(\partial D) \to \mathbf{C}$ is a linear functional. Then (c) gives us that this is indeed a bounded linear functional with norm 1. (c) also gives us that $\{\hat{f}(n)\}_{n\in\mathbf{Z}}$ is a bounded family for all $f \in L^1(\partial D)$.

By the example we did earlier, 11.8, we have that this family also has a stronger property namely that $\lim_{n\to\pm\infty} \hat{f}(n) = 0$.

That is, since

$$(h|_{\partial D})^{\wedge}(n) = \begin{cases} a_n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases},$$

where $h(z) = \sum_{n=0}^{\infty} a_n z^n$ is some analytic function on an open set containing $\bar{D}$ (the closure of the open unit disk). Then sense this is a uniformly convergent series, we must have $\lim_{n\to\infty} a_n = 0$ and clearly $\lim_{n\to\infty} 0 = 0$.

The Riemann-Lebesgue Lemma

Theorem

Suppose $f \in L^1(\partial D)$. Then $\lim_{n\to\pm\infty} \hat{f}(n) = 0$.

Proof.

Suppose $\epsilon > 0$. Then there exists a $g \in L^2(\partial D)$ such that $\|f - g\|_1 < \epsilon$ (by 3.44). By 11.6 and Bessel's Inequality (8.57) we have that:

$$\sum_{n=-\infty}^{\infty} |\hat{g}(n)|^2 \leq \|g\|_2^2 < \infty.$$

That is, there exists a $M \in \mathbb{Z}^+$ such that $|\hat{g}(n)| < \epsilon$ for all $n \in \mathbb{Z}$ with $|n| \geq M$. $\square$

The Riemann-Lebesgue Lemma

Proof. Cont.

Now if $n \in \mathbb{Z}$ and $|n| \geq M$, then:

$$\begin{aligned} |\hat{f}(n)| &\leq |\hat{f}(n) - \hat{g}(n)| + |\hat{g}(n)| \\ \text{(11.9(a) and work)} &< |(f-g)^{\wedge}(n)| + \epsilon \\ \text{(11.9(c))} &\leq \|f-g\|_1 + \epsilon \\ \text{(3.44 and work)} &< 2\epsilon. \end{aligned}$$

Giving us $\lim_{n\to\pm\infty} \hat{f}(n) = 0$, as desired. □