Elements of Analysis Notes

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0 Introduction

0.1 The Algebra of Sets

Definition 1 (Set Equality) *Two sets are equal if the contain the same elements.* $A = B$

Definition 2 (Intersection of Sets) *If* A, B *are sets, then their intersection is the set of all elements that belong to both* A *and* B. $A \cap B$

Definition 3 (Union of Sets) *If* A, B *are sets then their union is the set of all elements which belong either to* A *or* B *or to both* A, B. $A \cup B$

Definition 4 (Empty Set) *The empty set* $\emptyset$ *is the set that has no elements. Two sets* A, B *are disjoint if* $A \cap B = \emptyset$.

Theorem 5 *Let* A, B, C *be any sets, then*

1. $A \cap A = A$, $A \cup A = A$;
2. $A \cap B = B \cap A$, $A \cup B = B \cup A$;
3. $(A \cap B) \cap C = A \cap (B \cap C)$, $(A \cup B) \cup C = A \cup (B \cup C)$;
4. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Definition 6 (Set Complement) *If* A, B *are sets, then the complement of* B *relative to* A *is the set of all elements of* A *which do not belong to* B.

$$A \setminus B = \{x \in A : x \notin B\}$$

Theorem 7 *The sets* $A \cap B$ *and* $A \setminus B$ *are disjoint and* $A = (A \cap B) \cup (A \setminus B)$.

Theorem 8 *If* A, B, C *are any sets, then*

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

Definition 9 (Cartesian Products) *If* A, B *are two non-empty sets, then the Cartesian product* $A \times B$ *is the set of all ordered pairs* (a, b) *with* $a \in A$ *and* $b \in B$.

0.2 Functions

Definition 10 (Functions, Domain, Range) *Let* A, B *be two sets not necessarily distinct. A function from* A *to* B *is a set* f *of ordered pairs in* $A \times B$ *with the property that if* $(a, b), (a, b') \in f$, *then* $b = b'$. *The set of all elements of* A *that can occur as first members is called the domain of* f *denoted* $D(f)$. *The set of all elements* B *that occur in* f *is called the range of* f. $f : A \to B$ *is a function from* A *into* B. *When* B *is equal to the range of* f, *then* f *maps onto* B.

Definition 11 (Composition of Functions) *Let* f *be a function with domain* $D(f)$ *in* A *and range* $R(f)$ *in* B *and let* g *be a function with domain* $D(g)$ *in* B *and range* $R(g)$ *in* C. *The composition* $g \circ f = \{(a, c) \in A \times C : \exists b \in B \text{ such that } (a, b) \in f, (b, c) \in g\}$

Theorem 12 *If f and g are functions, the composition of $g \circ f$ is a function with*

$$D(g \circ f) = \{x \in D(f) : f(x) \in D(g)\}$$

$$R(g \circ f) = \{g(f(x)) : x \in D(g \circ f)\}.$$

Definition 13 (One-One) *Let f be a function with domain $D(f)$ in A and range $R(f)$ in B. We say that f is one-to-one if and only if when $a, a' \in D(f)$ and $a \neq a'$, then $f(a) \neq f(a')$.*

Definition 14 (Inverse) *Let f be a one-to-one function with domain $D(f)$ in A and range $R(f)$ in B. If $g = \{(b, a) \in B \times A : (a, b) \in f\}$, then g is a one-to-one function with domain $D(g) = R(f)$ in B and with range $D(g) = D(f)$ in A. The function g is called the function inverse to f and we ordinarily denote g by f^{-1}.*

Definition 15 (Direct Image) *If E is a subset of A, then the direct image of E under f is the subset of $R(f)$ given by $\{f(x) : x \in E \cap D(f)\}$.*

Theorem 16 *Let f be a function with domain in A and range in B and let E, F be subsets of A.*

1. *If $E \subseteq F$, then $f(E) \subseteq f(F)$;*
2. *$f(E \cap F) \subseteq f(E) \cap f(F)$;*
3. *$f(E \cup F) = f(E) \cup f(F)$;*
4. *$f(E \setminus F) \subseteq f(E)$.*

Definition 17 (Inverse Image) *If $H \subseteq B$, then the inverse image of H under f is the subset of $D(f)$ given by $f^{-1}(H) = \{x : f(x) \in H\}$.*

Theorem 18 *Let $f : A \to B$ be a functions and let $G, H \subseteq B$.*

1. *If $G \subseteq H$, then $f^{-1}(G) \subseteq f^{-1}(H)$;*
2. *$f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$;*
3. *$f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$;*
4. *$f^{-1}(G \setminus H) = f^{-1}(G) \setminus f^{-1}(H)$.*

Theorem 19 (Interchanging of Derivative and Limit) *Let (f_n) be a sequence defined on an interval $J \subseteq \mathbb{R}$ and with values in $\mathbb{R}$. Suppose that there's a point $x_0 \in J$ at which the sequence $(f_n(x_0))$ converges, that the derivatives f_n' exist on J, and that the sequence (f_n') converges uniformly on J to a function g. Then the sequence (f_n) converges uniformly on J to a function f which has a derivative at every point of J and $f' = g$.*

0.3 Finite and Infinite Sets

Theorem 20 *Any subset of a finite set is finite. Any subset of a countable set is countable.*

Theorem 21 *The union of a finite collection of finite sets is a finite set. The union of a countable collection of countable sets is a countable sets is a countable set.*

1 The Real Numbers

1.1 Fields

1.2 Ordered Fields

Corollary 22 *If $a > b$, then $a > \frac{a+b}{2} > b$*

Theorem 23 *1. $|a| = 0$ iff $a = 0$;*

2. *$|-a| = |a|$ for all $a \in F$;*
3. *$|ab| = |a||b|$ for all $a, b \in F$;*
4. *If $c \geq 0$, then $|a| \leq c$ if and only if $-c \leq a \leq c$;*
5. *$-|a| \leq a \leq |a|$ for all $a \in F$.*

Theorem 24 *Let $a, b \in F$, then $||a| - |b|| \leq |a \pm b| \leq |a| + |b|$.*

Corollary 25 *Let $x_1, ..., x_n \in F$, then $|x_1 + x_2 + ... + x_n| \leq |x_1| + |x_2| + ... + |x_n|$.*

Theorem 26 *Let F be an Archimedean field.*

1. *If $y > 0$ and $z > 0$, there's a $n \in \mathbb{N}$ such that $ny > z$;*
2. *If $z > 0$, there's a $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < z$;*
3. *If $y > 0$, there's a $n \in \mathbb{N}$ such that $n - 1 \leq y < n$.*

Theorem 27 *Let F be an Archimedean field containing a positive irrational element ζ. If z is a positive element in F, then there's a natural number m such that the positive irrational element $\frac{\zeta}{m}$ satisfies $0 < \frac{\zeta}{m} < z$.*

Theorem 28 *If $y, z \in F$ and $y < z$, then there's a rational number $r \in F$ such that $y < r < z$.*

Theorem 29 *If the Archimedean field F contains an irrational elements ζ and if $y < z$, then there's a rational number r such that the irrational number $r\zeta$ satisfies $y < r\zeta < z$.*

Theorem 30 *Let $x \in F$. For each integer $n \in \mathbb{N}$, there's a closed interval $I_n = [a_n, a_n + \frac{1}{2^n}]$ containing the point x, where a_n is a rational element and $I_{n+1} \subseteq I_n$ for $n \in \mathbb{N}$.*

1.3 The Real Number System

Definition 31 *An Archimedean field R is said to be complete if each sequence of non-empty intervals $I_n = [a_n, b_n], n \in \mathbb{N}$, of R which is nested such that $I_1 \supseteq I_2 \supseteq ... \supseteq I_n \supseteq ...$, has an element which belongs to all of the intervals $I_n, n \in \mathbb{N}$.*

Definition 32 (Upper/Lower Bound) *Let S be a subset of $\mathbb{R}$. An element $u \in \mathbb{R}$ is said to be an upper bound of S if $s \leq u$ for all $s \in S$. Similarly, an element $w \in \mathbb{R}$ is said to be a lower bound of S if $w \leq s$ for all $s \in S$.*

Definition 33 (Supremum/Infimum) *Let $S \subseteq \mathbb{R}$ be bounded above. An upper bound of S is said to be a supremum of S if it's less than any other upper bound of S. Similarly, if S is bounded below, then a lower bound of S is an infimum of S if it's greater than any other lower bound of S.*

Theorem 34 *A number u is the supremum of a non-empty set S of real numbers if and only if it has the following two properties:*

1. *There are no elements $s \in S$ with $u < s$;*
2. *If $v < u$, then there's an element $s \in S$ such that $v < s$.*

Theorem 35 (Supremum Principle) *Every non-empty subset of real numbers which has an upper bound also has a supremum.*

Corollary 36 *Every non-empty subset of real numbers which has a lower bound has an infimum.*

Definition 37 (Cuts) *Let F be an ordered field. An ordered pair of non-void subsets $A, B \subseteq F$ is said to form a cut in F if $A \cap B = \emptyset$, $A \cup B = F$, and if whenever $a \in A$ and $b \in B$, then $a < b$.*

Theorem 38 (Cut Principle) *If the pair A, B form a cut in $\mathbb{R}$, then there exists a real number ζ such that every element $a \in A$ satisfies $a \leq \zeta$ and every element $b \in B$ satisfies $b \geq \zeta$.*

2 The Topology of Cartesian Spaces

2.1 Cartesian Spaces

Theorem 39 (Cauchy-Bunyakovskii-Schwarz Inequality) *If $x, y \in \mathbb{R}^p$, then $x \cdot y \leq |x||y|$.*

Corollary 40 *If $x, y \in R^p$, then $|x \cdot y| \leq |x||y|$. Moreover, if $|x \cdot y| = |x||y|$ if and only if there exists $c \in \mathbb{R}$ such that $x = cy$ for non-zero $x, y \in \mathbb{R}^p$.*

Theorem 41 (Norm Properties) *Let $x, y \in \mathbb{R}^p$ and let $c \in \mathbb{R}$, then*

1. *$|x| \geq 0$;*

2. $|x| = 0$ *if and only if* $x = 0$;

3. $|cx| = |c||x|$;

4. $||x| - |y|| \le |x \pm y| \le |x| + |y|$.

Definition 42 (Open/Closed Balls and Spheres) *Let* $x \in \mathbb{R}^p$ *and* $r > 0$. *Then the set* $\{y \in \mathbb{R}^p : |x-y| < r\}$ *is an open ball with center* x *and radius* r, *with* $\le r$ *this is a closed ball, with* $= r$ *this is a sphere.*

Theorem 43 (Parallelogram Identity) *If* $x, y \in \mathbb{R}^p$, *then* $|x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2)$

Theorem 44 *If* $x = (\zeta_1, ..., \zeta_p) \in \mathbb{R}^p$, *then* $|\zeta_j| \le |x| \le \sqrt{p}\sup\{|\zeta_1|, ..., |\zeta_p|\}$ *for all* $j \in \{1, ..., p\}$.

2.2 Elementary Topological Concepts

Definition 45 (Open Set) *A set* $G \subset \mathbb{R}^p$ *is said to be open in* $\mathbb{R}^p$ *if, for each point* $x \in G$, *there exists a positive real number* r *such that every point* $y \in \mathbb{R}^p$ *satisfying* $|x - y| < r$ *also belongs to the set* G.

Theorem 46 (Open Set Properties) 1. *The empty set* $\emptyset$ *and the entire space* $\mathbb{R}^p$ *are open in* $\mathbb{R}^p$;

2. *The intersection of any two open sets (any finite number) is open in* $\mathbb{R}^p$;

3. *The union of any collection of open sets is open in* $\mathbb{R}^p$.

Definition 47 (Closed Sets) *A set* $F \subseteq \mathbb{R}^p$ *is said to be closed in* $\mathbb{R}^p$ *in case its complement* $C(F) = \mathbb{R}^p \setminus F$ *is open in* $\mathbb{R}^p$.

Theorem 48 (Closed Set Properties) 1. *The empty set* $\emptyset$ *and the entire space* $\mathbb{R}^p$ *are closed in* $\mathbb{R}^p$;

2. *The union of any two closed sets (any finite number) is closed in* $\mathbb{R}^p$;

3. *The intersection of any collection of closed sets is closed in* $\mathbb{R}^p$.

Definition 49 (Neighborhoods/Interior Points/Limit Points) *If* $x \in \mathbb{R}^p$, *then any set which contains an open set containing* x *is called a neighborhood of* $x \in \mathbb{R}^p$. *A point* x *is said to be an interior point of a set* A *in case* A *is a neighborhood of the point* x. *A point* x *is said to be a cluster point of a set* A *in case every neighborhood of* x *contains at least one point of* A *distinct from* x.

Theorem 50 *Let* $B \subseteq \mathbb{R}^p$, *then the following statements are equivalent:*

1. B *is open.*

2. *Every point of* B *is an interior point of* B.

3. B *is a neighborhood of each of its points.*

Theorem 51 *A set* F *is closed in* $\mathbb{R}^p$ *if and only if it contains all of its cluster points.*

Theorem 52 (Nested Intervals Theorem) *Let* (I_k) *be a sequence of non-empty closed intervals in* $\mathbb{R}^p$ *which is nested in the sense that* $I_1 \supseteq I_2 \supseteq ... \supseteq I_k \supseteq ....$ *Then there exists a point in* $\mathbb{R}^p$ *which belongs to all of the intervals.*

Theorem 53 (Bolzano-Weierstrass Theorem.) *Every bounded infinite subset of* $\mathbb{R}^p$ *has a cluster point.*

Definition 54 (Disconnected/Connected) *A subset* $D \subseteq \mathbb{R}^p$ *is said to be disconnected if there exist two open sets* A, B *such that* $A \cap D$ *and* $B \cap D$ *are disjoint non-empty sets whose union is* D. *In that case* A, B *is said to form a disconnection of* D. *A subset* $C \subseteq \mathbb{R}^p$ *which is not disconnected is said to be connected.*

Theorem 55 *The closed unit interval* $\mathbf{I} = [0, 1]$ *is a connected subset of* $\mathbb{R}$

Theorem 56 *The entire space* $\mathbb{R}^p$ *is connected.*

Corollary 57 *The only subsets of* $\mathbb{R}^p$ *which are both open and closed are* $\emptyset$ *and* $\mathbb{R}^p$.

Theorem 58 *Let* G *be an open set in* $\mathbb{R}^p$. *Then* G *is connected if and only if any pair of points* x, y *in* G *can be joined by a polygonal curve lying entirely in* G.

2.3 The Theorems of Heine-Borel and Baire

Definition 59 (Compact) *A set K is said to be compact if, whenever it's contained in the union of a collection $G = \{G_\alpha\}$ of open sets, then it's also contained in the union of some finite number of the sets in G.*

Theorem 60 (Heine-Borel Theorem) *A subset of $\mathbb{R}^p$ is compact if and only if it's closed and bounded.*

Theorem 61 (Cantor Intersection Theorem) *Let F_1 be a non-empty subset of $\mathbb{R}^p$ and let*

$$F_1 \supseteq F_2 \supseteq ... \supseteq F_n \supseteq ...$$

be a sequence of non-empty closed sets. Then there exists a point belonging to all of the sets $\{F_k : k \in \mathbb{N}\}$

Theorem 62 (Lebesque Covering Theorem) *Suppose $G = \{G_\alpha\}$ is a covering of a compact subset $K \subseteq \mathbb{R}^p$. There exists a positive number λ such that if $x, y \in K$ and $|x - y| < \lambda$, then there's a set G containing both x and y.*

Theorem 63 (Nearest Point Theorem) *Let F be a non-void closed subset of $\mathbb{R}^p$ and let x be a point outside of F. Then there exists at least one point y belonging to F such that $|z - x| \geq |y - x|$ for all $x \in F$.*

Theorem 64 (Circumscribing Contour Theorem) *Let F be a closed and bounded set in $\mathbb{R}^p$ and let G be an open set which contains F. Then there exists a closed curve C, lying entirely in G and made up of arcs of a finite number of circles, such that F is surrounded by C.*

Theorem 65 (Baire's Theorem) *If $\{H_k : k \in \mathbb{N}\}$ is a countable family of closed subsets of $\mathbb{R}^p$ whose union contains a non-void open sey, then at least one of the sets H_k contains a non-void open set.*

Corollary 66 *The space R^2 isn't the union of a countable number of lines.*

Corollary 67 *The set of irrational numbers in $\mathbb{R}$ isn't the union of a countable family of closed sets, none which contains a non-empty open set.*

3 Convergence

3.1 Introduction to Sequences

Definition 68 (Sequence in $\mathbb{R}^p$) *A sequence in $\mathbb{R}^p$ is a function whose domain is the set $\mathbb{N}$ of natural numbers and whose range is contained in $\mathbb{R}^p$.*

Definition 69 (Sequence Operations) *If $X = (x_n)$ and $Y = (y_n)$ are sequences in $\mathbb{R}^p$ then we define their sum to be the sequence $X + Y = (x_n + y_n)$ in $\mathbb{R}^p$, their difference to be the sequence $X - Y = (x_n - y_n)$, and their inner product to be the sequence $X \cdot Y = (x_n \cdot y_n)$ in $\mathbb{R}^p$ which is obtained by taking the inner product of corresponding terms. Similarly, if $X = (x_n)$ is a sequence in $\mathbb{R}$ and $Y = (y_n)$ is a sequence in $\mathbb{R}^p$, we define $XY = (x_n y_n)$. Finally, if $Y = (y_n)$ is a sequence in $\mathbb{R}$ with $y_n \neq 0$ we definite the quotient of a sequence $X = (x_n)$ in $\mathbb{R}^p$ by Y to be the sequence $X/Y = (x_n/y_n)$.*

Definition 70 (Limit of a Sequence) *Let $X = (x_n)$ be a sequence in $\mathbb{R}^p$. An element $x \in \mathbb{R}^p$ is said to be a limit of X if, for each neighborhood V of x there's a natural number K_V such that if $n \geq K_V$, then $x_n \in V$. If x is a limit of X, we also say that X converges to x. If a sequence has a limit, we say that the sequence is convergent. If a sequence has no limit then we say that it's divergent.*

Theorem 71 *Let $X = (x_n)$ be a sequence in $\mathbb{R}^p$. An element $x \in \mathbb{R}^p$ is a limit of X if and only if for each positive real number ϵ there is a natural number $K(\epsilon)$ such that if $n \geq K(\epsilon)$, then $|x_n - x| < \epsilon$.*

Theorem 72 (Uniqueness of Limits) *A sequence in $\mathbb{R}^p$ can have at most one limit.*

Lemma 73 *A convergent sequence in $\mathbb{R}^p$ is bounded.*

Theorem 74 *A sequence (x_n) in $\mathbb{R}^p$ with $x_n = (\zeta_{1n}, ..., \zeta_{pn}), n \in \mathbb{N}$ converges to an element $y = (\eta_1, ..., \eta_p)$ if and only if the corresponding p sequences of real numbers $(\zeta_{1n}, ..., \zeta_{pn})$, converge to $\eta_1, ..., \eta_p$ respectively.*

Definition 75 (Subsequences) *If $X = (x_n)$ is a sequence in $\mathbb{R}^p$ and if $r_1 < r_2 < ... < r_n < ...$ is a strictly increasing sequence of natural numbers, then the sequence X' in $\mathbb{R}^p$ is given by $(x_{r_1}, ..., x_{r_n}, ...)$, is called a subsequence of X.*

Lemma 76 *Is a sequence X in $\mathbb{R}^p$ converges to an element x, then any subsequence of X also converges to x.*

Corollary 77 *If $X = (x_n)$ is a sequence which converges to an element $x \in \mathbb{R}^p$ and if $m \in \mathbb{N}$, then the sequence $X' = (x_{m+1}, x_{m+2}, ...)$ also converges to x.*

Theorem 78 *If $X = (x_n)$ is a sequence in $\mathbb{R}^p$, then the following statements are equivalent:*

1. *X doesn't converge to x.*
2. *Then there exists a neighborhood V of x such that if $n \in \mathbb{N}$, then there's a natural number $m = m(n) \geq n$ such that $x_m \notin V$*
3. *There exists a neighborhood V of x and a subsequence X' of X such that none of the elements of X' belong to V.*

Theorem 79

1. *Let X and Y be sequences in $\mathbb{R}^p$ which converge to x, y, respectively. Then the sequences $X + Y, X - Y$, and $X \cdot Y$ converge to $x + y, x - y, x \cdot y$, respectively.*
2. *Let $X = (x_n)$ be a sequences in $\mathbb{R}^p$ which converges to x and let $A = (a_n)$ be a sequence in $\mathbb{R}$ which converges to a. Then the sequence $(a_n x_n)$ in $\mathbb{R}^p$ converges to ax.*
3. *Let $X = (x_n)$ be a sequence in $\mathbb{R}^p$ which converges to x and let $B = (b_n)$ be a sequence of non-zero real numbers which converges to a non-zero number b. Then the sequence $(b_n^{-1} x_n)$ in $\mathbb{R}^p$ converges to $b^{-1}x$*

Lemma 80 *Suppose that $X = (x_n)$ is a convergent sequence in $\mathbb{R}^p$ with limit x. If there exists an element $c \in \mathbb{R}^p$ and a number $r > 0$ such that $|x_n - c| \leq r$ for n sufficiently large, then $|x - c| \leq r$.*

3.2 Criteria for the Convergence of Sequences

Theorem 81 (Monotone Convergence Theorem) *Let $X = (x_n)$ be a sequence of real numbers which is monotone increasing in the sense that $x_1 \leq x_2 \leq ... \leq x_n \leq x_{n+1} \leq ...$ Then the sequence X converges if and only if it's bounded, in which case $\lim(x_n) = \sup\{x_n\}$.*

Corollary 82 *Let $X = (x_n)$ be a sequence of real numbers which is monotone decreasing in the sense that $x_1 \geq x_2 \geq ... \geq x_n \geq x_{n+1} \geq ....$ Then the sequence X converges if and only if it's bounded, in which case $\lim(x_n) = \inf\{x_n\}$*

Theorem 83 (Bolzano-Weierstrass Theorem) *A bounded sequence in $\mathbb{R}^p$ has a convergent subsequence.*

Corollary 84 *If $X = (x_n)$ is a sequence in $\mathbb{R}^p$ and x^* is a cluster point of the set $\{x_n : n \in \mathbb{N}\}$, then there's a subsequence X' of X which converges to x^*.*

Definition 85 (Cauchy Sequence) *A sequence $X = (x_n)$ in $\mathbb{R}^p$ is said to be a Cauchy sequence in case for every positive real number ϵ there's a natural number $M(\epsilon)$ such that if $m, n \geq M(\epsilon)$, then $|x_m - x_n| < \epsilon$.*

Lemma 86 *If $X = (x_n)$ is a convergent sequence in $\mathbb{R}^p$, then X is a Cauchy sequence.*

Lemma 87 *If a subsequence X' of a Cauchy sequence X in $\mathbb{R}^p$ converges to an element x, then the entire sequence X converges to x.*

Theorem 88 (Cauchy Convergence Criterion) *A sequence in $\mathbb{R}^p$ is convergent if and only if it's a Cauchy sequence.*

3.3 Sequences of Functions

Definition 89 (Convergence of Sequences of Functions) *Let (f_n) be a sequence of functions with common domain $D \subseteq \mathbb{R}^p$ and with range in $\mathbb{R}^q$, let $D_0 \subseteq D$, and let $f : D_0 \to \mathbb{R}^q$. We say that the sequence (f_n) converges on D_0 to f if, for each $x \in_0$ the sequence $(f_n(x))$ converges in $\mathbb{R}^q$ to $f(x)$. In this case we call the function f the limit on D_0 of the sequence (f_n). When such a function f exists we say that the sequence (f_n) converges t o f on D_0, or simply that the sequence is convergent on D_0.*

Lemma 90 (Epsilon Def. of Seq. of Functions Convergence) *A sequence (f_n) of functions on $D \supseteq \mathbb{R}^p$ to $\mathbb{R}^q$ converges to a function f on a set $D_0 \subseteq D$ if and only if for each $\epsilon > 0$ and each $x \in D_0$ there is a natural number $K(\epsilon, x)$ such that if $n \geq K(\epsilon, x)$, then $|f_n(x) - f(x)| < \epsilon$.*

Definition 91 (Uniform Convergence of Seq. of Functions) *A sequence (f_n) of functions on $D \subseteq \mathbb{R}^p$ to $\mathbb{R}^q$ converges uniformly on a subset $D_0 \subseteq D$ to a function f in case for each $\epsilon > 0$ there's a natural number $K(\epsilon)$ such that if $n \geq K(\epsilon)$ and $x \in D_0$, then $|f_n(x) - f(x)| < \epsilon$. In this case the sequence is called uniformly convergent on D_0.*

Lemma 92 (Criterion for non-uniform convergence) *A sequence (f_n) doesn't converge uniformly on D_0 to f if and only if for some $\epsilon_0 > 0$ there exists a subsequence (f_{n_k}) of (f_n) and a sequence (x_k) in D_0 such that $|f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0$ for $k \in \mathbb{N}$.*

Definition 93 (D-Norm of f) *If f is a bounded function defined on a subset $D \subseteq \mathbb{R}^p$ to $\mathbb{R}^q$, then the D-norm of f is the real number given by: $||f||_D = \sup\{|f(x) : x \in D|\}$.*

Lemma 94 (D-Norm Properties) *If f, g are bounded functions defined on $D \subseteq \mathbb{R}^p$ to $\mathbb{R}^q$, then the D-norm satisfies:*

1. $||f|| = 0$ *if and only if* $f(x) = 0$ *for all* $x \in D$.
2. $||cf|| = |c|||f||$ *for any real number* c.
3. $|||f|| - ||g||| \leq ||f \pm g|| \leq ||f|| + ||g||$.

Lemma 95 (D-Norm Uniform Convergence) *A sequence (f_n) of bounded functions on $D \subseteq \mathbb{R}^p$ to $\mathbb{R}^q$ converges uniformly on D to a function f if and only if $||f_n - f|| \to 0$.*

Theorem 96 (Cauchy Criterion for Uniform Convergence) *Let (f_n) be a sequence of bounded functions on $D \subseteq \mathbb{R}^p$ with values in $\mathbb{R}^q$. Then there's a function to which (f_n) is uniformly convergent on D if and only if for each $\epsilon > 0$ there's a number $M(\epsilon)$ such that if $m, n \geq M(\epsilon)$, then the D-norm satisfies $||f_m - f_n|| < \epsilon$.*

3.4 Some Extensions and Applications

Definition 97 (Limit Superior/Inferior) *If $X = (x_n)$ is a sequence of real number which is bounded above (below), then the limit superior (inferior) of $X = (x_n)$ which we denote $\limsup X = \limsup(x_n)$ is the infimum (supremum) of real numbers v with the property that there are only a finite number of natural numbers n such that $v < x_n$ ($x_n < v$).*

Lemma 98 (Uniqueness and Existence of Limit Superior/Inferior) *Let $X = (x_n)$ be a sequence of real numbers which is bounded. Then the limit superior of X exists and is uniquely determined.*

Theorem 99 (Limit Superior/Inferior Connection to Sequence Limits) *If $X = (x_n)$ is a sequence of real numbers which is bounded above, then the following statements are equivalent:*

1. $x^* = \limsup(x_n)$.
2. *If $\epsilon > 0$, there are only a finite number of natural numbers n such that $x^* + \epsilon < x_n$ but there are an infinite number such that $x^* - \epsilon < x_n$.*
3. *If $v_m = \sup\{x_n : n \geq m\}$, then $x^* = \inf\{v_m : m \geq 1\}$.*
4. *If $v_m = \sup\{x_n : n \geq m\}$, then $x^* = \lim(v_m)$.*
5. *If V is the set of real numbers v such that there's a subsequence of X which converges to v, then $x^* = \sup V$.*

Theorem 100 (Limit Superior/Inferior Algebraic Properties) *Let $X = (x_n)$ and $Y = (y_n)$ be bounded sequences of real numbers. Then the following relations hold:*

1. $\liminf(x_n) \leq \limsup(x_n)$.
2. *If $c \geq 0$, then $\liminf(cx_n) = c\liminf(x_n)$ and $\limsup(cx_n) = c\limsup(x_n)$.*
3. *If $c \leq 0$, then $\liminf(cx_n) = c\limsup(x_n)$ and $\limsup(cx_n) = c\liminf(x_n)$.*
4. $\liminf(x_n) + \liminf(y_n) \leq \liminf(x_n + y_n)$.
5. $\limsup(x_n + y_n) \leq \limsup(x_n) + \limsup(y_n)$.
6. *If $x_n \leq y_n$ for all n, then $\liminf(x_n) \leq \liminf(y_n)$ and $\limsup(x_n) \leq \limsup(y_n)$.*

Lemma 101 (Limit Superior/Inferior as Limit of Sequence) *Let X be a bounded sequence of real numbers. Then X is convergent if and only if $\liminf X = \limsup X$ in which case $\lim X$ is the common value.*

Definition 102 (Big O and little o Notation) *Let $X = (x_n)$ be the sequence of $\mathbb{R}^p$ and let $Y = (y_n)$ be a non-zero sequence in $\mathbb{R}^p$. We say that they are equivalent and write X Y or (x_n) (y_n) in case $\lim\left(\frac{|x_n|}{|y_n|}\right) = 1$. We say X is of lower order of magnitude than Y and write $X = o(Y)$ or $x_n = o(y_n)$, in case $\lim\left(\frac{|x_n|}{|y_n|}\right) = 0$. Finally we say that X is dominated by Y and write $X = O(Y)$ or $x_n = O(y_n)$ in case there's a positive constant K such that $|x_n| \leq K|y_n|$ for all n sufficiently large.*

Definition 103 (Arithmetic Mean of a sequence) *If $X = (x_n)$ is a sequence of elements in $\mathbb{R}^p$, then the sequence $S = (\sigma_n)$ is defined by: $\sigma_1 = x_1, \sigma_2 = \frac{x_1+x_2}{2}, ..., \sigma_n = \frac{x_1+x_2+...+x_n}{n}, ...$ is called the sequence of arithmetic means of X.*

Theorem 104 (Convergence $\to$ Arithmetic Convergence) *If the sequence $X = (x_n)$ converges to x, then the sequence $S = (\sigma_n)$ of arithmetic means of X also converges to x.*

4 Continuous Functions

4.1 Local Properties of Continuous Functions

Definition 105 (Continuity at a point) *Let $a \in D$ of the function f. We say that f is continuous at a if for every neighborhood V of $f(a)$ there's a neighborhood U of a (depending on V) such that if $x \in D \cap U$, then $f(x) \in V$. If D_1 is a subset of D, we say that f is continuous on D_1 in case it's continuous at every point in D_1.*

Theorem 106 (Equivalent Definitions of Continuity at a Point) *Let $a \in D$ of the function f. The following statements are equivalent:*

1. *f is continuous at a.*
2. *If $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that if $x \in D$ and $|x - a| < \delta(\epsilon)$, then $|f(x) - f(a)| < \epsilon$.*
3. *If (x_n) is any sequence of elements in D which converge to a, then the sequence $(f(x_n)) \to f(a)$.*

Theorem 107 (Discontinuity Criterion) *The function f is not continuous at a point $a \in D$ if and only if there's a sequence $(x_n) \to a$ in D but $(f(x_n)) \not\to f(a)$*

Theorem 108 (Open Sets in Codomain Correspond Open Sets in the Domain Under Continuous Maps) *The function f is continuous at a point in D if and only if for every neighborhood V of $f(a)$ there's a neighborhood V_1 of a such that $V_1 \cap D = f^{-1}(V)$.*

Theorem 109 (Algebraic Properties of Continuous Functions) *If the function f, g, ϕ are continuous at a point then their algebraic combinations $f + g, f - g, f \cdot g, cf, \phi f$, and f/ϕ are also continuous at this point.*

Theorem 110 (Absolute Value of a Continuous Function is Continuous) *If f is continuous at a point a, then $|f|$ is also continuous there.*

Theorem 111 (Composition of Continuous Functions are Continuous) *If f is continuous at a and g is continuous at $b = f(a)$, then their composition $g \circ f$ is continuous at a.*

Definition 112 (Linear Functions) *A function with domain $\mathbb{R}^p$ and range in $\mathbb{R}^q$ is a linear transformation if for all $x, y \in \mathbb{R}^p$ and $c \in \mathbb{R}$, $f(x + y) = f(x) + f(y), f(cx) = cf(x)$.*

Theorem 113 (Linear Functions are Lipshitz) *If f is a linear function with domain $\mathbb{R}^p$ and range in $\mathbb{R}^q$, then there exist a positive constant A such that if $u, v \in \mathbb{R}^p$, $|f(u) - f(v)| \leq A|u - v|$. Therefore, a linear function on $\mathbb{R}^p$ to $\mathbb{R}^q$ is continuous at every point.*

4.2 Global Properties of Continuous Functions

Theorem 114 (Global Continuity Theorem) *The following statements are equivalent:*

1. *f is continuous on its domain D.*
2. *If G is any open(closed) set in $\mathbb{R}^q$, then there exists an open(closed) set G_1 in $\mathbb{R}^p$ such that $G_1 \cap D = f^{-1}(G)$.*

Corollary 115 (Inverse Images of Open/Closed Sets are Preserved Under Continuous Maps) *Let f be defined on all of $\mathbb{R}^p$ and with range in R^q. Then the following statements are equivalent:*

1. *f is continuous on $\mathbb{R}^p$.*
2. *If G is open (closed) in $\mathbb{R}^q$, then $f^{-1}(G)$ is open in $\mathbb{R}^p$.*

Theorem 116 (Preservation of Connectedness) *If H is connected and f is continuous on H, then $f(H)$ is connected.*

Theorem 117 (Bolzano's Intermediate Value Theorem) *Let H be a connected subset of $\mathbb{R}^p$ and let f be bounded and continuous on H and with values in R. If $k \in \mathbb{R}$ satisfies $\inf\{f(x) : x \in H\} < k < \sup\{f(x) : x \in H\}$, then there's at least one point in H where f takes the value of k.*

Theorem 118 (Preservation of Compactness) *If K is compact and f is continuous on K, then $f(K)$ is compact.*

Theorem 119 (Maximum and Minimum Value Theorem) *Let f be continuous on a compact set $K \subseteq \mathbb{R}^p$ and with values in $\mathbb{R}^q$. Then there are points x^* and x_* in K such that: $|f(x^*)| = \sup\{|f(x)| : x \in K\}$ and $|f(x_*)| = \inf\{|f(x)| : x \in K\}$*

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Corollary 120 (More Useful Max/Min Value Theorem) *If f is continuous on a compact subset K of $\mathbb{R}^p$ and has real values, then there exists $x^*, x_* \in K$ such that: $f(x^*) = \sup\{f(x) : x \in K\}$ and $f(x_*) = \inf\{f(x) : x \in K\}$.*

Corollary 121 (Linear Functions are Greater than Some Tangent Line) *If f is a one-one linear function on $\mathbb{R}^p$ to $\mathbb{R}^q$, then there's a positive number m such that $|f(x)| \geq m|x|$ for all $x \in \mathbb{R}^p$.*

Theorem 122 (Continuity of the Inverse Function) *Let K be a compact subset of $\mathbb{R}^p$ and let f be a continuous one-one function with domain K and range $f(K)$ in $\mathbb{R}^q$. Then the inverse function is continuous with domain $f(K)$ and range K.*

Theorem 123 (Pointwise Continuity) *If f is defined on a subset $D \subseteq \mathbb{R}^p$ and range in $\mathbb{R}^q$, then it's readily seen that the following statements are equivalent:*

1. *f is continuous on D.*
2. *Given $\epsilon > 0$ and $u \in D$, there's a $\delta(\epsilon, u) > 0$ such that if $x \in D$ and $|x - u| < \delta$, then $|f(x) - f(u)| < \epsilon$.*

Definition 124 (Uniform Continuity) *Let f have domain $D \subseteq \mathbb{R}^p$ and range in $\mathbb{R}^q$. We say that f is uniformly continuous on D if for each $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that if $x, u \in D$ and $|x - u| < \delta(\epsilon)$, then $|f(x) - f(u)| < \epsilon$.*

Lemma 125 (Dis-Uniform Continuity Criterion) *A necessary and sufficient condition that the function f isn't uniformly continuous on its domain is that there exists a positive $\epsilon_0 > 0$ and two sequences $X = (x_n), Y = (y_n)$ in D such that if $n \in \mathbb{N}$, then $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$.*

4.3 Sequences of Continuous Functions

Theorem 126 (Uniformly Convergent Sequence of Functions Converge to Continuous Functions) *Let $F = (f_n)$ be a sequence of continuous functions with domain $D \subseteq \mathbb{R}^p$ and range in $\mathbb{R}^q$ and let this sequence converge uniformly on D to a function f. Then f is continuous on D.*

Definition 127 (Step Functions) *A function g with domain $\mathbb{R}^p$ and range in $\mathbb{R}^q$ is called a step function if it assumes only a finite number of distinct values in $\mathbb{R}^q$, each non-zero value being taken on an interval in $\mathbb{R}^p$.*

Theorem 128 (Step Function Approx. on Compact Sets) *Let f be a continuous function whose domain D is a compact interval in $\mathbb{R}^p$ and whose values belong to $\mathbb{R}^q$. Then f can be uniformly approximated on D by step functions.*

Theorem 129 (Peicewise Approx. Theorem) *Let f be a continuous function whose domain is a compact interval J in $\mathbb{R}$. Then f can be uniformly approximated on J by continuous piecewise linear functions.*

Definition 130 (Bernstein Polynomial) *Let f be a function with domain $\mathbb{I} = [0, 1]$ and range in $\mathbb{R}$. The nth Bernstein polynomial for f is defined to be*

$$B_n(x) = B_n(x; f) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1 - x)^{n-k}$$

Theorem 131 (Bernstein Approximation Theorem) *Let f be continuous on $\mathbb{I} = [0, 1]$ with values in $\mathbb{R}$. Then the sequence of Bernstein polynomials for f converges uniformly on $\mathbb{I} = [0, 1]$ to f.*

Theorem 132 (Weierstrass Approximation Theorem) *Let f be a continuous function on a compact interval on $\mathbb{R}$ and with values in $\mathbb{R}$. Then f can be uniformly approximated by polynomials.*

Theorem 133 (Stone Approximation Theorem) *Let K be a compact subset of $\mathbb{R}^p$ and let G be a collection of continuous functions on K to $\mathbb{R}$ with the properties:*

1. *If $f, g \in G$, then $\sup\{f, g\}$ and $\inf\{f, g\}$ belong in G.*
2. *If $a, b \in \mathbb{R}$ and $x \neq y \in K$, then there exists a function $f \in G$ such that $f(x) = a, f(y) = b$.*

Then any continuous function on K to $\mathbb{R}$ can be uniformly approximated on K by functions in G.

Theorem 134 (Stone-Weierstrass Theorem) *Let K be a compact subset of $\mathbb{R}^p$ and let G be a collection of continuous functions on K to $\mathbb{R}$ with the properties:*

1. *The constant function $c(x) = 1$, $x \in K$ belongs to G.*
2. *If $f, g \in G$, then $\alpha f + \beta b \in G$.*
3. *If $f, g \in G$, then $fg \in G$.*
4. *If $x \neq y$ are two points of K, there exists a function $f \in G$ such that $f(x) \neq f(y)$.*

Then any continuous function on K to $\mathbb{R}$ can be uniformly approximated on K by functions in G.

4.4 Limits of Functions

Definition 135 (Deleted/Non-Deleted Limits)

1. *An element $b \in \mathbb{R}^q$ is said to be the deleted limit of f at c if for every neighborhood V of b there's a neighborhood U of c such that if $x \in U \cap D$ and $x \neq c$, then $f(x) \in V$. In this case we write $b = \lim_c f$ or $b = \lim_{x \to c} f(x)$*
2. *An element $b \in \mathbb{R}^q$ is said to be the non-deleted limit of f at c if for every neighborhood V of b there's a neighborhood U of c such that if $x \in U \cap D$, then $f(x) \in V$. In this case, $b = Lim_c f$ or $b = Lim_{x \to c} f(x)$*

Lemma 136

1. *If either of the limits $\lim_c f, Lim_c f$ exists, it's uniquely determined.*
2. *If the non-deleted limit exists, then the deleted limit exists and $\lim_c f = Lim_c f$.*
3. *If $c \notin D$ of f, then the deleted limit exists if and only if the non-deleted limit exists.*

Theorem 137 (Equivalent Definitions of Deleted Limits) *The following statements, pertaining to the deleted limit, are equivalent.*

1. *The deleted limit $b = \lim_c f$ exists.*
2. *If $\epsilon > 0$, there's a $\delta > 0$ such that if $x \in D$ and $0 < |x - c| < \delta$, then $|f(x) - b| < \epsilon$.*
3. *If (x_n) is any sequence in D such that $x_n \neq c$ and $c \neq \lim(x_n)$, then $b = \lim(f(x_n))$*

Theorem 138 (Equivalent Definitions of Non-Deleted Limits) *The following statements, pertaining to the non-deleted limit, are equivalent.*

1. *The non-deleted limit $b = Lim_c f$ exists.*
2. *If $\epsilon > 0$, there's a $\delta > 0$ such that if $x \in D$ and $0 \leq |x - c| < \delta$, then $|f(x) - b| < \epsilon$.*
3. *If (x_n) is any sequence in D such that $c = \lim(x_n)$ then we have $b = \lim(f(x_n))$.*

Theorem 139 (Cluster Points and Deleted/Non-Deleted Limits) *If $c \in D$ is a cluster point, then the following statements are equivalent:*

1. *The function f is continuous at c.*
2. *The deleted limit $\lim_c f$ exists and equals $f(c)$ (Deleted limits are the typical limit)*
3. *The non-deleted limit $Lim_c f$ exists.*

Theorem 140 (Composition of Limits) *Suppose that f has domain $D(f)$ in $\mathbb{R}^p$ and range in $\mathbb{R}^q$ has domain $D(g)$ in $\mathbb{R}^q$ and range in $\mathbb{R}^r$. Let $g \circ f$ be the composition of g and f and let c be a cluster point of $D(g \circ f)$.*

1. *If the deleted limits $b = \lim_c f$, $a = \lim_b g$ both exist and if either g is continuous at b or $f(x) \neq b$ for x in a neighborhood of c, then the deleted limit of $g \circ f$ exists at c and $a = \lim_c(g \circ f)$.*

2. *If the non-deleted limits $b = Lim_a f, a = Lim_b g$ both exist, then the non-deleted limit of $g \circ f$ exists at c and $a = Lim(g \circ f)$*

Definition 141 (Deleted/Non-Deleted Limit Superiors) *Suppose that f is bounded on a neighborhood of the point c. If $r > 0$, define $\phi(r)$ and $\Phi(r)$ by:*

$$\phi(r) = \sup\{f(x) : 0 < |x - c| < r, x \in D\}$$

$$\Phi(r) = \sup\{f(x) : 0 \leq |x - c| < r, x \in D\}$$

and the set

$$\limsup f_{x \to c} = \inf\{\phi(r) : r > 0\}$$

$$Lim \sup_{x \to c} f = \inf\{\Phi(r) : r > 0\}$$

These quantities are called the deleted limit superior and the non-deleted limit superior of f at c, respectively.

Lemma 142 (Equivalent Limit Superior Definitions) *If ϕ, Φ are as defined above, then $\limsup_{x \to c} f = \lim_{r \to 0} \phi(r)$, $Lim \sup_{x \to c} f = \lim_{r \to 0} \Phi(r)$.*

Lemma 143 1. *If $M > \limsup_{x \to c} f$, then there exists a neighborhood U of c such that $f(x) < M, c \neq x \in D \cap U$.*

2. *If $M > Lim \sup_{x \to c} f$, then there exists a neighborhood U of c such that $f(x) < M, x \in D \cap U$.*

Lemma 144 (Triangle Inequality for Limit Superiors) *Let f, g be bounded on a neighborhood of c and suppose that c is a cluster point of $D(f + g)$. Then*

$$\limsup_{x \to c}(f + g) \leq \limsup_{x \to c} f + \limsup_{x \to c} g$$

$$Lim \sup_{x \to c}(f + g) \leq Lim \sup_{x \to c} f + Lim \sup_{x \to c} g$$

Definition 145 (Upper Semi-Continuous) *A function f on D to R is said to be upper semi-continuous at a point c in D in case $f(c) = Lim \sup_{x \to c} f$. It's said to be upper semi-continuous on D if it's upper semi-continuous at every point of D.*

Lemma 146 (Upper Semi-Continuous "Splitting" of the Range) *Let f be an upper semi-continuous function with domain $D \subseteq \mathbb{R}^p$ and let $k \in \mathbb{R}$. Then there exists an open set G and closed set K such that: $G \cap D = \{x \in D : f(x) < k\}$ and $F \cap D = \{x \in D : f(x) \geq k\}$*

5 Differentiation

5.1 The Derivative in $\mathbb{R}$

Definition 147 (Def. of a Derivative) *If c is a cluster point of D and belongs to D, we say that a real number L is the derivative of f at c if for every positive ϵ there's a positive $\delta(\epsilon) > 0$ such that if $x \in D$ and $0 < |x - c| < \delta(\epsilon)$, then*

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| < \epsilon.$$

In this case we write $f'(c) = L$.

Lemma 148 (Differentiability Implies Continuity) *If f has a derivative at c, then f is continuous there.*

Lemma 149 (Signs of Derivative Imply Maximum on a Open Interval) 1. *If f has a derivative at c and $f'(c) > 0$, there exists a positive number δ such that if $x \in D$ and $c < x < c + \delta$, then $f(c) < f(x)$.*

2. *If $f'(c) < 0$, there exists a positive number δ such that if $x \in D$ and $c\delta < x < c$, then $f(c) < f(x)$.*

Theorem 150 (Interior Maximum Theorem) *Let c be an interior point of D at which f has a relative maximum. If the derivative of f at c exists, then it must equal zero.*

Theorem 151 (Rolle's Theorem) *Suppose that f is continuous on a closed interval $J = [a, b]$, that the derivative f' exists in the open interval (a, b), and that $f(a) = f(b) = 0$. Then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.*

Theorem 152 (Mean Value Theorem) *Suppose that f is continuous on a closed interval $J = [a, b]$ and has a derivative on (a, b). Then there exists a point $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.*

Corollary 153 (Mean Value Theorem + Differentiability $\rightarrow$ Continuity) *If f has a derivative on $J = [a, b]$, then there exists a point $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.*

Theorem 154 (Cauchy Mean Value Theorem) *Let f, g be continuous on $J = [a, b]$ and has derivative on (a, b). Then there exists a point $c \in (a, b)$ such that $f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)]$.*

Theorem 155 (Taylor's Theorem) *Suppose that $n \in \mathbb{N}$, that f has derivatives $f', f'', ..., f^{(n-1)}$ are defined and continuous on $J = [a, b]$, and that $f^(n)$ exists in (a, b). If $\alpha, \beta \in J$, then there exists a $\gamma \in \mathbb{N}$ between α and β such that:*

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + \frac{f^{(n)}(\gamma)}{n!}(\beta - \alpha)^n$$

Corollary 156 (Applications of the Mean Value Theorem) *Assume that f is continuous on $J = [a, b]$ and its derivative exists on (a, b).*

1. *If $f'(x) = 0$ for $a < x < b$, then f is constant on J.*
2. *If $f'(x) = g'(x)$ for $a < x < b$, then f, g differ on J by a constant.*
3. *If $f'(x) \geq 0$ for $a < x < b$ and if $x_1 \leq x_2$ belong to J, then $f(x_1) \leq f(x_2)$.*
4. *If $f'(x) > 0$ for $a < x < b$ and if $x_1 < x_2$ belong to J, then $f(x_1) < f(x_2)$.*
5. *If $f'(x) \geq 0$ for $a < x < a + \delta$, then a is a relative minimum point of J.*
6. *If $f'(x) \geq 0$ for $b - \delta < x < b$, then b is a relative maximum point of f.*
7. *If $|f'(x)| \leq M$ for $a < x < b$, then f satisfies the Lipschtiz condition: $|f(x_1)f(x_2)| \leq |x_1 - x_2|, x_1, x_2 \in J$.*

5.2 The Derivative in $\mathbb{R}^p$

Definition 157 (Directional Derivative) *Let f be defined on $D \subseteq \mathbb{R}^p$ and have values in $\mathbb{R}^q$, let $c \in D$ be an interior point of D, and u be any point in $\mathbb{R}^p$. A vector L_u in $\mathbb{R}^q$ is said to be the directional derivative of f at c in the direction of u if for each positive real number $\epsilon > 0$, there exists a positive number $\delta(\epsilon) > 0$ such that if $0 < |t| < \delta(\epsilon)$, then $\left|\frac{f(c+tu)-f(c)}{t} - L_u\right| < \epsilon$.*

Definition 158 (Differentiable at a Point) *Let f have domain $D \subseteq \mathbb{R}^p$ and range in $\mathbb{R}^q$ and let c be an interior point of D. We say that f is differentiable at c if there exists a linear function L on $\mathbb{R}^p$ to $\mathbb{R}^q$ such that for every positive number $\epsilon > 0$ there exist $\delta(\epsilon) > 0$ such that if $|x - c| < \delta(\epsilon)$, then $x \in D$ and $|f(x) - f(c) - L(x - c)| \leq \epsilon|x - c|$. The linear function L is called the derivative of f at c. (Also called differential)*

Lemma 159 (Differential is unique, if it Exists) *A function has at most one derivative at a point.*

Lemma 160 (Differentiability implies Lipschitz) *If f is differentaible at a point c, then there exist positive real number δ, K such that if $|x - c| < \delta$, then $|f(x) - f(c)| \leq K|x - c|$.*

Theorem 161 (Differential $\implies$ Directional Derivative) *Let f be defined on $D \subseteq \mathbb{R}^p$ and have range in $\mathbb{R}^q$. If f is differentiable at the point $c \in D$ and u is any point in $\mathbb{R}^p$, then the directional derivative of f at c in the direction of u exists and equals $Df(c)(u)$.*

Theorem 162 (Partial Derivative in a Neighborhood $\implies$ Differential in Neighborhood) *If the partial derivatives of f exist in a neighborhood of c and are continuous at c, then f is differentiable at c.*

Theorem 163 (Algebraic Properties of Differential)

1. *If f, g are differentiable at a point $c \in \mathbb{R}^p$ and have values in $\mathbb{R}^q$ and if $\alpha, \beta \in \mathbb{R}$, then the function $h = \alpha f + \beta g$ is differentiable at c and $Dh(c) = \alpha Df(c) + \beta Dg(c)$.*
2. *If f, g are as above, then their inner product $k = f \cdot g$ is differentiable at c and $Dk(c)(u) = Df(c)(u) \cdot g(c) + f(c) \cdot Dg(c)(u)$.*
3. *If ϕ is differentiable at $c \in \mathbb{R}^p$ and has values in $\mathbb{R}$, then the product ϕf is differentiable at c and $D(\phi f)(c)(u) = D\phi(c)(u)f(c) + \phi(c)Df(c)(u)$.*

Theorem 164 (Chain Rule) *Let f be a function with domain $D(f) \subseteq \mathbb{R}^p$ and range in $\mathbb{R}^q$ and let g have domain $D(g) \subseteq \mathbb{R}^q$ and range in $\mathbb{R}^r$. Suppose that f is differentiable at point c and that g is differentiable at $f(c)$. Then the composition $g \circ f$ is differentiable at c and is $Dg(b) \circ Df(c)$.*

Theorem 165 (Mean Value Theorem) *Let f be defined on a subset $D \subseteq \mathbb{R}^p$ and have values in $\mathbb{R}$. Suppose that the set D contains the points a, b and the line segment joining them and that f is differentiable at every point of the line segment. Then there exists a point c on this line segment such that $f(b) - f(a) = Df(c)(b-a)$.*

Corollary 166 (Mean Value Theorem in $\mathbb{R}^n$) *Let $f$be defined on $D \subseteq \mathbb{R}^p$ and with values in $\mathbb{R}^q$. Supposed that D contains the points a, b and the line segment joining them and that f is differentiable at every point of this segment. If $y \in \mathbb{R}^p$, then there exists a point c on this line segment such that $\{f(b)-f(a)\} \cdot y = \{Df(c)(b-a)\} \cdot y$.*

Corollary 167 (Mean Value Theorem with Linear Function) *Let f be defined on $D \subseteq \mathbb{R}^p$ and with values in $\mathbb{R}^q$. Suppose that the set D contains points a, b and the line segment joining them and that f is differentiable at every point of this segment. Then there exists a linear function L of $\mathbb{R}^p \to \mathbb{R}^q$ such that $f(b)-f(a) = L(b-a)$.*

Lemma 168 (Limit Definition of Partial Derivative at $(0,0)$) *Suppose that f is defined on a neighborhood U of the origin in $\mathbb{R}^2$ with values in $\mathbb{R}$, that the partial derivatives of f_x and f_{xy} exist in U, and that f_{xy} is continuous at $(0,0)$. If $A(h,k) = f(h,k) - f(h,0) - f(0,k) + f(0,0)$, then we have $f_{xy}(0,0) = \lim\limits_{(k,k)\to(0,0)} \frac{A(h,k)}{hk}$*

Theorem 169 (Commutativity of Second Derivatives) *Suppose that f is defined on a neighborhood U of a point $(x,y) \in \mathbb{R}^2$ with values in R. Suppose that the partial derivatives f_x, f_y, f_{xy} exist in U and that f_{xy} is continuous at (x,y). Then the partial derivative f_{yx} exists at (x,y) and $f_{yx}(x,y) = f_{xy}(x,y)$*

5.3 Mapping Theorems and Extremum Problems

Definition 170 (Definition of Continuity Class of Functions) *If the partial derivatives of f exists and are continuous at a point c interior to D, then we say that f belongs to Class C' at c. If $D_0 \subseteq D$ and if $f \in C'$ at every point in D_0, we say that $f \in C'$ on D_0.*

Lemma 171 (Sort of Lipschitz for Continuous Differentiable Functions) *If $f \in C'$ on a neighborhood of a point c and $\epsilon > 0$, then there exists a $\delta(\epsilon) > 0$ such that if $|x-c| < \delta(\epsilon)$, then $|Df(x)(z) - Df(c)(z)| \leq \epsilon|z|$, for all $z \in \mathbb{R}^p$.*

Theorem 172 (Approximation Lemma) *If $f \in C'$ on a neighborhood of a point c and if $\epsilon > 0$, then there exists a $\delta(\epsilon) > 0$ such that if $|x_i - c| < \delta(\epsilon), i = 1, 2$, then $|f(x_1) - f(x_2) - Df(c)(x_1 - x_2)| \leq \epsilon|x_1 - x_2|$.*

Theorem 173 (Locally One-One Mapping) *If $f \in C'$ on a neighborhood of c and the derivative $Df(c)$ is one-to-one, then there exists a positive $\delta > 0$ such that the restriction of f to $U = \{x \in \mathbb{R}^p : |x-c| \leq \delta\}$ is one-to-one.*

Lemma 174 (Surjective Linear Image Inequality) *If L is a linear function of $\mathbb{R}^p$ onto all of $\mathbb{R}^q$, then there's a positive constant m such that every element $y \in \mathbb{R}^q$ is the image under L of an element $x \in \mathbb{R}^p$ such that $|x| \leq m|y|$.*

Lemma 175 *Let g be continuous on $D(g) = \{x \in \mathbb{R}^p : |x| < \alpha\}$ with values in$\mathbb{R}^q$ and that $g(0) = 0$. Let L be linear and map $\mathbb{R}^p$ onto all $\mathbb{R}^q$ and let $m > 0$ as in the preceding lemma. Suppose that $|g(x_1)-g(x_2)-L(x_2-x_1)| \leq \frac{1}{2m}|x_1 - x_2|$ for $|x_i| < \alpha$. Then any vector $y \in \mathbb{R}^q$ satisfying $|y| < \beta = \frac{\alpha}{2m}$ is the image under g of an element in $D(g)$.*

Theorem 176 (Local Solvability Theorem) *Suppose that $f \in C'$ on a neighborhood of c and that the derivative $Df(c)$ maps $\mathbb{R}^p$ onto all of $\mathbb{R}^q$. There are positive numbers α, β such that if $y \in \mathbb{R}^q$ and $|y - f(c)| < \beta$, then there is an element $x \in \mathbb{R}^p$ with $|x-c| < \alpha$ such that $f(x) = y$.*

Theorem 177 (Open Mapping Theorem) *Let D be an open subset $\mathbb{R}^p$ and let $f \in C'(D)$. If, for each $x \in D$, the derivative $Df(x)$ maps $\mathbb{R}^p$ onto $\mathbb{R}^q$, then $f(D)$ is open in $\mathbb{R}^q$. Moreover, if G is any subset of D, then $f(G)$ is open in $\mathbb{R}^q$.*

Theorem 178 (Inversion Theorem) *Suppose that f is in class C' on a neighborhood of $c \in \mathbb{R}^p$ with values in $\mathbb{R}^p$ and the derivative $Df(c)$ is a one-one map of $\mathbb{R}^p$ onto $\mathbb{R}^p$. Then there exists a neighborhood U of c such that $V = f(U)$ is a neighborhood of $f(c)$, f is a one-one mapping of U onto V, and f has a continuous inverse function g defined on V to U. Moreover, $g \in C'$ on V and if $y \in V$ and $x = g(y) \in U$, then the linear function $Dg(y)$ is the inverse of the linear function $Df(x)$.*

Theorem 179 (Implicit Function Theorem) *Suppose that $F \in C'$ on a neighborhood of $(0,0)$ in $\mathbb{R}^p \times \mathbb{R}^q$ and has values in $\mathbb{R}^p$. Suppose that $F(0,0) = 0$ and that the linear function L, defined by $L(u) = DF(0,0)(u,0), u \in \mathbb{R}^p$, is a one-to-one function of $\mathbb{R}^p$ onto $\mathbb{R}^p$. Then there exists a function $\phi \in C'$ on a neighborhood W of 0 in $\mathbb{R}^q$ to $\mathbb{R}^p$ such that $\phi(0) = 0$ and $F[\phi(y), y] = 0, y \in W$.*

Corollary 180 (Corollary to Implicit Function Theorem) *With the hypothesis just stated and notation above, the derivative of ϕ at $y \in W$ is the linear function on $\mathbb{R}^q$ to $\mathbb{R}^p$ given by $D\phi(y) = -(D_x F)^{-1} \circ (D_y F)$.*

Theorem 181 (Relative Extremum Theorem) *Let f be a function with domain $D \subseteq \mathbb{R}^P$ and with range in $\mathbb{R}$. If C is an interior point of D at which f is differentiable and has a relative extremum, then $Df(c) = 0$.*

Corollary 182 (Concavity Extremization) *Let the real-valued function f have continuous second partial derivatives in a neighborhood of a critical point $c \in \mathbb{R}^2$, and let $\Delta = f_{\zeta\zeta}(c) f_{\eta\eta}(c) - [f_{\zeta\eta}(c)]^2$.*

1. *If $\Delta > 0$ and if $f_{\zeta\zeta}(c) > 0$, then f has a relative minimum at c.*
2. *If $\Delta > 0$ and if $f_{\zeta\zeta}(c) < 0$, then f has relative maximum at c.*
3. *If $\Delta < 0$, then the point c is a saddle point of f.*

Theorem 183 (Lagrange's Method) *Let $f, g \in C'$ on a neighborhood of a point $c \in \mathbb{R}^p$ and with values in $\mathbb{R}$. Suppose that there exists a neighborhood of c such that $f(x) \geq f(c)$ or $f(x) \leq f(c)$ for all x in this neighborhood which also satisfy the constraint $g(x) = 0$. If $Dg(c) \neq 0$, then there exists a real number λ such that $Df(c) = \lambda Dg(c)$.*

6 Integration

6.1 The Riemann-Stieltjes Integral

Definition 184 (Partition/Partition Points/Refinements) *A partition of $J = [a,b]$ is a finite collection of non-overlapping intervals whose union is J. Described as $P = (x_0, x_1, ..., x_n)$ such that $a = x_0 \leq x_1 \leq ... \leq x_n = b$ and such that the subintervals occuring are the intervals $[x_{k-1}, x_k], k \in \{1, ..., n\}$. $x_k's$ are the partition points corresponding to P. If P, Q are partitions of J, we say that Q is a refinement of P or that Q is finer than P in case every subinterval in Q is contained in some subinterval in P. Equivalent to every partition point in P is also a partition point in Q. Written $P \subseteq Q$*

Definition 185 (Riemann-Stieltjes Sum) *If P is a refinement of J, then a Reimann-Stieltjes sum of f with respect to g and corresponding to $P = (x_0, x_1, ..., x_n)$ is a real number*

$$S(P; f, g) = \sum_{k=1}^{n} f(\zeta_k)\{g(x_k) - g(x_{k-1})\}$$

Here we have selected numbers ζ_k satisfying $x_{k-1} \leq \zeta_k \leq x_k, k \in \{1, ..., n\}$.

Definition 186 (Reimann Integratable + The Integral) *We say that f is integratable with respect to g on J if there exists a real number I such that for every $\epsilon > 0$ there's a partition P_ϵ of J such that if P is any refinement of P_ϵ and $S(P; f, g)$ is any Reimann-Stieltjes sum corresponding to P, then $|S(P; f, g) - I| < \epsilon$. In case that I is uniquely determined then we denote $I = \int_a^b f \, dg = \int_a^b f(t) \, dg(t)$; it's called the Riemann-Stieltjes integral of f with respect to g over $J = [a,b]$. We call the function f, the integrand, g the integrator. In the special case $g(x) = x$, if f is integratable with respect to g, we usually say that f is Reimann integratable.*

Theorem 187 (Cauchy Criterion for Integratability) *The function f is integratable with respect to g over $J = [a,b]$ if and only if for each positive real number ϵ there's a partition Q_ϵ of J such that if P and Q are refinements of Q_ϵ and if $S(P; f, g), S(Q; f, g)$ are any corresponding Riemann Stieltjes sums, then $|S(P; f, g) - S(Q; f, g)| < \epsilon$.*

Theorem 188 (Algebraic Properties of the Integral)

1. *If f_1, f_2 are integratable with respect to g on J and $\alpha, \beta \in \mathbb{R}$, then $\alpha f_1 + \beta f_2$ is integratable with respect to g on J and*

$$\int_a^b (\alpha f_1 + \beta f_2) \, dg = \alpha \int_a^b f_1 \, dg + \beta \int_a^b f_2 \, dg$$

2. *If f is integratable with respect to g_1, g_2 on J and $\alpha, \beta \in \mathbb{R}$, then f is integrable with respect to $g = \alpha g_1 + \beta g_2$ on J and*

$$\int_a^b f \, dg = \alpha \int_a^b f \, dg_1 + \beta \int_a^b f \, dg_2$$

Theorem 189 (Breaking Up the Domain of Integration) *1. Suppose that $a < c < b$ and that f is integratable with respect to g over both of the subintervals $[a, c], [c, b]$. Then f is integratable with respect to g on the interval $[a, b]$ and*

$$\int_a^b f\ dg = \int_a^c f\ dg + \int_c^b f\ dg$$

2. Let f be integratable with respect to g on the interval $[a, b]$ and let c satisfy $a < c < b$. Then f is integratable with respect to g on the subintervals $[a, c], [c, b]$ and the above formula also holds.

Theorem 190 (Integration by Parts) *A function f is integratable with respect to g over $[a, b]$ if and only if g is integratable with respect to f over $[a, b]$. In this case,*

$$\int_a^b f\ dg + \int_a^b g\ df = f(b)g(b) - f(a)g(a)$$

Theorem 191 (Integratability Theorem) *If f is continuous on J and g is monotone increasing, then f is integratable with respect to g over J.*

Corollary 192 *If f is monotone and g is continuous on J, then f is integratable with respect to g over J.*

Lemma 193 (Absolute Values over Inequalities, Integrals) *Let f be continuous and let g be monotone increasing on J. Then we have the estimate:*

$$\left|\int_a^b f\ dg\right| \le \int_a^b |f|\ dg \le ||f||\{g(b) - g(a)\}$$

If $m \le f(x) \le M$ for all $x \in J$, then

$$m\{g(b) - g(a)\} \le \int_a^b f\ dg \le M\{g(b) - g(a)\}$$

Theorem 194 (Interchanging Limits and Integrals) *Let g be a monotone increasing function on J and let (f_n) be a sequence which are integratable with repect to g over J. Suppose that the sequence (f_n) converges uniformly on J to a limit function f. Then f is integratable with respect to g and*

$$\int_a^b f\ dg = \lim \int_a^b f_n\ dg$$

Lemma 195 *Let f be a non-negative Riemann integratable function on $J = [0, 1]$, and suppose that $\alpha = \int_0^1 f > 0$. Then the set $E = \{x \in J : f(x) \ge \frac{\alpha}{3}\}$ contains a finite number of intervals of total length exceeding $\frac{\alpha}{3||f||}$.*

Theorem 196 (Bounded Convergence Theorem) *Let (f_n) be a sequence of functions which are Riemann integratable on $J = [a, b]$ and such that $||f_n|| \le B, n \in \mathbb{N}$. If the sequence converges at each point on J to a Riemann integratable function f, then*

$$f_a^b f = \lim \inf_a^b f_n$$

Theorem 197 (Monotone Convergence Theorem) *Let (f_n) be a monotone sequence of Riemann integratable functions which converges at each point of $J = [a, b]$ to a Riemann integratable function f. Then*

$$\int_a^b f = \lim \int_a^b f_n.$$

Definition 198 (Linear Functional) *Denote the collection of all real-valued continuous functions defined on $C_{\mathbb{R}}(J)$ and write $||f|| \sup\{|f(x)| : x \in J\}$. A linear functional on $C_{\mathbb{R}}(J)$ is a real-valued function G defined for each function in $C_{\mathbb{R}}(J)$ such that if $f_1, f_2 \in C_{\mathbb{R}}(J)$ and $\alpha, \beta \in \mathbb{R}$, then $G(\alpha f_1 + \beta f_1) = \alpha G(f_1) + \beta G(f_2)$. The linear functional G on $C_{\mathbb{R}}(J)$ is positive if, for each $f \in C_{\mathbb{R}}(J)$ such that $f(x) \ge 0$ for $x \in J$, then $G(f) \ge 0$. The linear functional G on $C_{\mathbb{R}}(J)$ is bounded if there exists a constant M such that $|G(f)| \le M||f||, f \in C_{\mathbb{R}}(J)$.*

Lemma 199 (Monotonic Integrals $\implies$ Functional is Bounded) *If g is a monotone increasing function and G is defined for $f \in C_{\mathbb{R}}(J)$ by $G(f) = \int_a^b f\ dg$, then G is bounded positive linear functional on $C_{\mathbb{R}}(J)$.*

Theorem 200 (Riesz Representation Theorem) *If G is a bounded positive linear functional on $C_{\mathbb{R}}(J)$, then there exists a monotone increasing function g on J such that*

$$G(f) = \int_a^b f\ dg, f \in C_{\mathbb{R}}(J)$$

6.2 The Main Theorems of Integral Calculus

Theorem 201 (First Mean Value Theorem) *If g is increasing on $J = [a, b]$ and f is continuous on J to $\mathbb{R}$, then there exists a number $c \in J$ such that*

$$\int_a^b f\ dg = f(c) \int_a^b = f(c)\{g(b) - g(a)\}$$

Theorem 202 (Differentiation Theorem) *Suppose that f is continuous on J and that g is increasing on J and has derivative at a point $c \in J$. Then the function F, defined for $x \in J$ by:*

$$F(x) = \int_a^x f\ dg$$

Theorem 203 (Fundamental Theorem of Integral Calculus) *Let f be continuous on $J = [a, b]$. A function F on J satisfies*

$$F(x) - F(a) = \int_a^x f, x \in J \iff F' = f \text{ on } J$$

Theorem 204 *If the derivative $g' = h$ exists and is continuous on J and if f is integratable with respect to g, then the product fh is Riemann integratable and*

$$\int_a^b f\ dg = \int_a^b fh$$

.

Theorem 205 (First Mean Value Theorem) *If f, h are continuous on J and h is non-negative, then there exists a point c in J such that*

$$\int_a^b f(x)h(x)\ dx = f(c) \int_a^b h(x)\ dx$$

Theorem 206 (Integration by Parts) *If f, g have continuous derivatives on $[a, b]$, then*

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g$$

Theorem 207 (Second Mean Value Theorem)

1. *If f is increasing and g is continuous on $J = [a, b]$, then there exists a point $c \in J$ such that*

$$\int_a^b f\ dg = f(a) \int_a^c dg + f(b) \int_c^b dg$$

2. *If f is increasing and h is continuous on J, then there exists a point $c \in J$ such that*

$$\int_a^b fh = f(a) \int_a^c h + f(b) \int_c^b h$$

3. *If f is non-negative and increasing and h is continuous on J, then there exists a point $c \in J$ such that*

$$\int_a^b fh = f(b) \int_c^b h$$

Theorem 208 (Change of Variable Theorem) *Let ϕ be defined on an interval $[\alpha, \beta]$ to $\mathbb{R}$ with continuous derivative and suppose that $a = \phi(\alpha) < b = \phi(\beta)$. If f is continuous on the range of ϕ, then*

$$\int_a^b f(x)\ dx = \int_\alpha^\beta f[\phi(t)]\phi'(t)\ dt$$

Theorem 209 *If f is continuous on D to $\mathbb{R}$ and if $F(t) = \int\limits_a^b f(x, t)\ dx$, then $F(t)$ is continuous on $[c, d]$ to $\mathbb{R}$.*

Theorem 210 *If f and its partial derivative f_t are continuous on D to $\mathbb{R}$, then the function $F(t) = \int\limits_a^b f(x, t)\ dx$ has a derivative on $[c, d]$ and $F'(t) = \int\limits_a^b f_t(x, t)\ dx$.*

Theorem 211 (Leibniz's Formula) *Suppose that f, f_t are continuous on D to $\mathbb{R}$ and that α, β are functions which are differentiable on the interval $[c,d]$ and have values in $[a,b]$. If ϕ is defined on $[c,d]$ by: $\phi(t) = \int\limits_{\alpha(t)}^{\beta(t)} f(x,t)\ dx$, then ϕ has a derivative for each $t \in [c,d]$ which is given by:*

$$\phi'(t) = f[\beta(t),t]\beta'(t) - f[\alpha(t),t]\alpha'(t) + \int_{\alpha(t)}^{\beta(t)} f_t(x,t)\ dx$$

Theorem 212 (Interchange Theorem) *If f is continuous on D with values in $\mathbb{R}$, then*

$$\int_c^d \int_a^b f(x,t)\ dx\ dt = \int_a^b \int_c^d f(x,t)\ dt\ dx$$

Theorem 213 (Taylor's Theorem) *Suppose that f and its derivative $f', f'', ..., f^{(n)}$ are continuous on $[a,b] \subseteq \mathbb{R}$. Then*

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + ... + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + R_n$$

with $R_n = \frac{1}{(n-1)!}\int_a^b (b-t)^{n-1} f^{(n)}(t)\ dt$.

6.3 Integration in Cartesian Spaces

6.4 Improper and Indefinite Integrals

7 Infinite Series

7.1 Convergence of Infinite Series

Definition 214 (Infinite Series/ Partial Sums) *If $X = (x_n)$ is a sequence in $\mathbb{R}^p$, then the infinite series generated by X is the sequence $S = (s_k)$ defined by $s_1 = x_1, s_2 = s_1 + x_2, ..., s_k = s_{k-1} + x_k$. If S converges, we refer to $\lim S$ as the sum of the infinite series. The elements x_n are called the terms and the elements s_k are called the partial sums of this infinite series*

Theorem 215 (Algebraic Properties of Infinite Series)

1. *If the series $\sum(x_n)$ and $\sum(y_n)$ converge, then the series $\sum(x_n+y_n)$ converges and the sums are related by the formula $\sum(x_n+y_n) = \sum(x_n)+\sum(y_n)$. Similarly for $X - Y$.*
2. *If the series $\sum(x_n)$ is convergent, $c \in \mathbb{R}$, and w is a fixed element of $\mathbb{R}^p$, then the series $\sum(cx_n)$ and $\sum(w \cdot x_n)$ converge and $\sum(cx_n) = c\sum(x_n), \sum(w \cdot x_n) = w \cdot \sum(x_n)$.*

Lemma 216 *If $\sum(x_n)$ converges in $\mathbb{R}^p$, then $\lim(x_n) = 0$.*

Theorem 217 (Non-Negative Series Convergent $\iff$ Partial Sums Converge) *Let (x_n) be a sequence of non-negative real numbers. Then $\sum(x_n)$ converges if and only if the sequence $S = (s_k)$ of partial sums is bounded. In this case,$\sum x_n = \lim(s_k) = \sup\{s_k\}$*

Definition 218 (Absolute Convergence) *Let $X = (x_n)$ be a sequence in $\mathbb{R}^p$. We say that the series $\sum(x_n)$ is absolutely convergent if the series $\sum(|x_n|)$ is convergent in $\mathbb{R}$. A series is said to be conditionally convergent if it's convergent but not absolutely convergent.*

Theorem 219 (Absolute Convergent $\implies$ Convergence) *If a series in $\mathbb{R}^p$ is absolutely convergent, then it's convergent.*

Theorem 220 (Rearrangement Theorem) *Let $\sum(x_n)$ be an absolutely convergent series in $\mathbb{R}^p$. Then any rearrangement of $\sum(x_n)$ converges absolutely to the same value.*

Lemma 221 *Suppose that the iterated series $\sum\limits_{j=1}^{\infty}\sum\limits_{i=1}^{\infty}(|x_{ij}|)$ converges. Then the double series $\sum(x_{ij})$ is absolutely convergent.*

Theorem 222 *Suppose that the double series $\sum(x_{ij})$ converges absolutely to $x \in \mathbb{R}^p$. Then both of the iterated series $\sum\limits_{j=1}^{\infty}\sum\limits_{i=1}^{\infty} x_{ij} = \sum\limits_{i=1}^{\infty}\sum\limits_{j=1}^{\infty}(x_{ij}) = x$.*

7.2 Tests of Convergence

Theorem 223 (Comparison Test) *Let $X = (x_n), Y = (y_n)$ be non-negative real sequences and suppose that for some natural number K, $x_n \leq y_n, n \geq K$. Then the convergence of $\sum(y_n) \implies$ the convergence of $\sum(x_n)$.*

Theorem 224 (Limits Comparison Test) *Suppose that $X = (x_n), Y = (y_n)$ are non-negative real sequences.*

1. *If the relation $\lim \frac{X}{Y} \neq 0$ holds, then $\sum(x_n)$ is convergent if and only if $\sum(y_n)$ is convergent.*
2. *If $\lim \frac{X}{Y} = 0$ and $Y = (y_n)$ is convergent, then $\sum(x_n)$ is convergent.*

Theorem 225 (Root Test)
1. *If $X = (x_n)$ is a sequence in $\mathbb{R}^p$ and there exists a non-negative number $r < 1$ and $K \in \mathbb{N}$ such that $|x_n|^{1/n} \leq r, n \geq K$, then the series $\sum(x_n)$ is absolutely convergent.*
2. *If there exists a number $r > 1$ and a natural number K such that $|x_n|^{1/n} \geq r, n \geq K$, then the series $\sum(x_n)$ is divergent.*

Corollary 226 *If r satisfies $0 < r < 1$ and if the sequence $X = (x_n)$ satisfies the root test (a), then the partial sums $s_n, n \geq K$, approximate the sum $s = \sum(x_n)$ according to the estimate*

$$|s - s_n| \leq \frac{r^{n+1}}{1-r}, n \geq K$$

Corollary 227 *Let $X = (x_n)$ be a sequence in $\mathbb{R}^p$ and set $r = \lim(|x_n|^{1/n})$ whenever this limit exists. Then $\sum(x_n)$ is absolutely convergent when $r < 1$ and is divergent when $r > 1$.*

Theorem 228 (Ratio Test)
1. *If $X = (x_n)$ is a sequence of non-zero elements of $\mathbb{R}^p$ and there's a positive number $r < 1$ and natural number K such that $\frac{|x_{n+1}|}{x_n} \leq r, n \geq K$, then the series $\sum(x_n)$ is divergent.*
2. *If there exists a number $r > 1$ and $K \in \mathbb{N}$ such that $\frac{|x_{n+1}|}{|x_n|} \geq r, n \geq K$, then the series $\sum(x_n)$ is divergent.*

Corollary 229 *If r satisfies $0 \leq r < 1$ and if the sequence $X = (x_n)$ the convergent root test for $n \geq K$, then the partial sums approximate the sum $s = \sum(x_n)$ according to the estimate $|s - s_n| \leq \frac{r}{1-r}|x_n|, n \geq K$.*

Corollary 230 *Let $X = (x_n)$ be a sequence in $\mathbb{R}^p$ and set $r = \lim\left(\frac{|x_{n+1}|}{|x_n|}\right)$ whenever the limit exists. Then the series $\sum(x_n)$ is absolutely convergent when $r < 1$ and divergent when $r > 1$.*

Theorem 231 (Raabe's Test)
1. *If $X = (x_n)$ is a sequence of non-zero elements in $\mathbb{R}^p$ and there's a real number $a > 1$ and a natural number K such that $\frac{|x_{n+1}|}{|x_n|} \leq 1 - \frac{a}{n}, n \geq K$, then the series $\sum(x_n)$ is absolutely convergent.*
2. *If there's a real number $a \leq 1$ and $K \in \mathbb{N}$ such that $\frac{|x_{n+1}|}{|x_n|} \geq 1 - \frac{a}{n}, n \geq K$, then the series $\sum(x_n)$ is not absolutely convergent.*

Corollary 232 *If $a > 1$ and if the sequence $X = (x_n)$ satisfies Raabe's Test for convergent series, then the partial sums approximate the sum s of $\sum(x_k)$ according to the estimate $|s - s_n| \leq \frac{n}{a-1}|x_{n+1}|, n \geq K$.*

Corollary 233 *Let $X = (x_n)$ be a sequence of non-zero elements of $\mathbb{R}^P$ and set $a = \lim\left(n\left(a - \frac{|x_{n+1}|}{|x_n|}\right)\right)$, whenever this limit exists. Then $\sum(x_n)$ is absolutely convergent when $a > 1$ and is not absolutely convergent when $a < 1$.*

Theorem 234 (Integral Test) *Let f be a positive, non-increasing continuous function on $\{t : t \geq 1\}$. Then the series $\sum(f(n))$ converges if and only if the infinite integral $\int\limits_1^\infty f(t)\ dt = \lim\limits_n \left(\int\limits_1^n f(t)\ dt\right)$ exists. In the case of convergence, the partial sums $s_n = \sum\limits_{k=1}^n (f(k))$ and the sum s of $\sum\limits_{k=1}^\infty (f(k))$ satisfy the estimate:*

$$\int_{n+1}^\infty f(t)\ dt \leq s - s_n \leq \int_n^\infty f(t)\ dt$$

Lemma 235 (Abel's Lemma) *Let $X = (x_n)$ and $Y = (y_n)$ be sequences and let the partial sums of $\sum(y_n)$ be denoted by (s_k). If $m \geq n$, then $\sum\limits_{j=n}^m x_j y_j = (x_{m+1}s_n - x_n s_{n-1}) + \sum\limits_{j=n}^m (x_j - x_{j+1})s_j$.*

Theorem 236 (Dirichlet's Test) *Suppose that the partial sums of $\sum(y_n)$ are bounded.*

1. *If the sequence $X = (x_n)$ converges to zero, and if $\sum |x_n - x_{n+1}|$ is convergent, then the series $\sum(x_n y_n)$ is convergent.*
2. *In particular, if $X = (x_n)$ is a decreasing sequence of positive real numbers which converge to 0, then the series $\sum(x_n y_n)$ is convergent.*

Corollary 237 *In part (b) of Dirichlet's Test, the error estimate $\left|\sum\limits_{j=1}^{\infty} x_j y_j - \sum\limits_{j=1}^{n} x_j y_j\right| \leq 2|x_{n+1}|B$, with B an upper bound for the partial sums of $\sum(y_j)$.*

Theorem 238 (Abel's Test) *Suppose that the series $\sum(y_n)$ is convergent in $\mathbb{R}^p$ and that $X = (x_n)$ is a convergent monotone sequence in $\mathbb{R}$. Then the series $\sum(x_n y_n)$ is convergent.*

Definition 239 (Alternating Series) *A sequence $X = (x_n)$ of non-zero real numbers is alternating if the terms $(-1)^n x_n, n \in \mathbb{N}$ are all positive (or all negative) real numbers. If a sequence $X = (x_n)$ is alternating, we say that the series $\sum(x_n)$ it generates an alternating series.*

Theorem 240 (Alternating Series Test) *Let $Z = (z_n)$ be a non-increasing sequence of positive numbers with $\lim(z_n) = 0$. Then the alternating series $\sum((-1)^n z_n)$ is convergent. Moreover, if s is a sum of this series and s_n is the n^{th} partial sum, then we have the estimate $|s - s_n| \leq z_{n+1}$ for rapidity of convergence.*

7.3 Series of Functions

Definition 241 *If (f_n) is a sequence of functions defined on $D \subseteq \mathbb{R}^p$ with values in $\mathbb{R}^q$, the sequence of partial sums (s_n) of the infinite series $\sum(f_n)$ is defined for $x \in D$ by: $s_1(x) = f_1(x), s_2(x) = s_1(x) + f_2(x), ..., s_{n+1}(x) = s_n(x) + f_{n+1}(x)$. In the case the sequence (s_n) converge on D to a function f, we say that the infinite series of functions $\sum(f_n)$ converges to f on D. We write $\sum(f_n) = \sum\limits_{n=1}^{\infty}(f_n) = \sum\limits_{n=1}^{\infty} f_n$ or denote either the series or the limits function, when it exists. When $\sum(|f_n(x)|)$ converges for each $x \in D$, then we say that $\sum(f_n)$ is absolutely convergent on D. If the sequence (s_n) is uniformly convergent on D to f, then we say that $\sum(f_n)$ is uniformly convergent on D, or that it converges to f uniformly on D.*

Theorem 242 *If f_n is continuous on $D \subseteq \mathbb{R}^p$ to $\mathbb{R}^q$ for each $n \in \mathbb{N}$ and if $\sum(f_n)$ converges to f uniformly on D, then f is continuous on D.*

Theorem 243 *Suppose that the real-valued functions f_n are Riemann-Stieltjes integratable with respect to g on the interval $J = [a, b]$ for each $n \in \mathbb{N}$. If the series $\sum(f_n)$ converges to f uniformly on D, then f is Riemann-Stieltjes integratable with respect to g and $\int\limits_a^b f\ dg = \sum\limits_{n=1}^{\infty} \int\limits_a^b f_n\ dg$.*

Theorem 244 *If f_n are non-negative Riemann integratable functions on $J = [a, b]$ and if their sum $f = \sum(f_n)$ is Riemann integratable, then $\int_a^b f = \sum\limits_{n=1}^{\infty} \int_a^b f_n$.*

Theorem 245 *For each $n \in \mathbb{N}$, let f_n be a real-valued function on $J = [a, b]$ which has a derivative f'_n on J. Suppose that the infinite series $\sum(f_n)$ converges for at least one point of J and that the series of derivatives $\sum(f'_n)$ converges uniformly on J. Then there exists a real-valued function f on J such that $\sum(f_n)$ converges uniformly on J to f. In addition, f has a derivative on J and $f' = \sum f'_n$.*

Theorem 246 (Cauchy's Criterion) *Let (f_n) be a sequence of functions on $D \subseteq \mathbb{R}^p$ to $\mathbb{R}^q$. The infinite series $\sum(f_n)$ is uniformly convergent on D if and only if for all $\epsilon > 0$ there exists a $M(\epsilon)$ such that if $m \geq n \geq M(\epsilon)$, then $||f_n + f_{n+1} + ... + f_m||_D < \epsilon$.*

Theorem 247 (Weierstrass M-Test) *Let (M_n) be a sequence of non-negative real numbers such that $||f_n|| \leq M_n, n \in \mathbb{N}$. If the infinite series $\sum(M_n)$ is convergent, then $\sum(f_n)$ is uniformly convergent on D.*

Theorem 248 (Dirichlet's Test) *Let (f_n) be a sequence of functions on $D \subseteq \mathbb{R}^p$ to $\mathbb{R}^q$ such that the partial sums $s_n = \sum\limits_{j=1}^{n} f_j, n \in \mathbb{N}$, are all bounded in D-norm. Let (ϕ_n) be a decreasing sequence of functions on D to $\mathbb{R}$ which converges uniformly on D to 0. Then the series $\sum(\phi_n f_n)$ converges uniformly on D.*

Theorem 249 (Abel's Test) *Let $\sum(f_n)$ be a series of functions on $D \subseteq \mathbb{R}^p$ to $\mathbb{R}^q$ which is uniformly convergent on D. Let (ϕ_n) be a bounded and monotone sequence of real-valued functions on D. Then the series $\sum(\phi_n f_n)$ converges uniformly on D.*

Definition 250 (Power Series) *A series of real functions $\sum(f_n)$ is said to be a power series around $x = c$ if the function f_n has the form $f_n(x) = a_n(x-c)^n$, with $a_n, c \in \mathbb{R}, n \in \mathbb{N} \cup \{0\}$.*

Definition 251 (Radius/Interval of Convergence) *Let $\sum(a_n x^n)$ (a power series around $c = 0$) be a power series. If the sequence $(|a_n|^{1/n})$ is bounded, we set $\rho = \limsup(|a_n|^{1/n})$; if this sequence is not bounded we set $\rho = +\infty$. We define the radius of convergence of $\sum(a_n x^n)$ to be given by $R =$*

$$0, \rho = +\infty$$
$$\frac{1}{\rho}, 0 < \rho < +\infty$$
$$+\infty$$

The interval of convergence is the open interval $(-R, R)$.

Theorem 252 (Cauchy-Hadaman Theorem) *If the radius of convergence R for the power series $\sum(a_n x^n)$, then the series is absolutely convergent if $|x| < R$ and divergent if $|x| > R$.*

Theorem 253 *Let R be the radius of convergence of $\sum(a_n x^n)$ and let K be a compact subset of the interval of convergence $(-R, R)$. Then the power series converges uniformly on K.*

Theorem 254 *The limit of a power series is continuous on the interval of convergence. A power series can be integrated term-by-term over any compact interval contained in the interval of convergence.*

Theorem 255 (Differentiataion Theorem) *A power series can be differentiated term-by-term within the interval of convergence. In fact, if $f(x) = \sum\limits_{n=0}^{\infty}(a_n x^n)$, then $f'(x) = \sum\limits_{n=1}^{\infty}(n a_n x^{n-1})$. Both series have the same radius of convergence.*

Theorem 256 (Uniqueness Theorem) *If $\sum(a_n x^n)$ and $\sum(b_n x^n)$ converge on some interval $(-r, +r), r > 0$, to the same function f, then $a_n = b_n, n \in \mathbb{N}$.*

Theorem 257 (Multiplication Theorem) *If f, g are given on the interval $(-r, r)$ by the power series $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum\limits_{n=0}^{\infty} b_n x^n$, their product is given on this interval by the series $\sum(c_n x^n)$, where the coefficients (c_n) are $c_n = \sum\limits_{k=0}^{\infty} a_k b_{n-k}, n \in \{0, 1, 2, ...\}$.*

Theorem 258 (Bernstein's Theorem) *Let f be defined and possess derivatives of all order on an interval $[0, r]$ and suppose that f and all of its derivatives are non-negative on the interval $[0, r]$. If $0 \leq x < r$, then $f(x) = \sum\limits_{n=0}^{\infty} \frac{f^(n)(0)}{n!} x^n$.*

Theorem 259 (Abel's Theorem) *Suppose that the power series $\sum\limits_{n=0}^{\infty}(a_n x^n)$ converges to $f(x)$ for $|x| < 1$ and that $\sum_{n=0}^{\infty}(a_n)$ converges to A. Then the power series converges uniformly in $\mathbb{I} = [0, 1]$ and $\lim\limits_{x \to 1-} f(x) = A$.*

Theorem 260 (Tauber's Theorem) *Suppose that the power series $\sum(a_n x^n)$ converges to $f(x)$ for $|x| < 1$ and that $\lim(n a_n) = 0$. If $\lim f(x) = A$ as $x \to 1-$, then the series $\sum(a_n)$ converges to A.*

8 Alternatives from "An Introduction to Analysis" by Wade

8.1 The Riemann Integral

Definition 261 *Let $[a, b]$ be a closed and bounded interval.*

1. *A partition of $[a, b]$ is a set of points $P = \{x_0, x_1, ..., x_n\}$ such that $a < x_0 < x_1 < ... < x_n = b$.*
2. *The norm of a partition $P = \{x_0, x_1, ..., x_n\}$ is the number $||P|| = \max\limits_{1 \leq j < \leq n} |x_j - x_{j-1}|$.*

3. *A refinement of a partition $P = \{x_0, x_1, ..., x_n\}$ is a partition Q of $[a, b]$ which satisfies $Q \supseteq P$. In this case we say that Q is finer than P.*

Definition 262 *Let $[a, b]$ be a closed bounded interval, $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$, and $f : [a, b] \to \mathbb{R}$ be bounded.*

1. *The upper Riemann sum of f over P is the number $U(f, P) := \sum\limits_{j=1}^{n} M_j(f)(x_j - x_{j-1})$ where $M_j(f) := \sup\limits_{x\in[x_{j-1},x_j]} f(x)$.*

2. *The lower Riemann sum of f over P is the number $L(f, P) := \sum\limits_{j=1}^{n} m_j(f)(x_j - x_{j-1})$ where $m_j(f) := \inf\limits_{x\in[x_{j-1},x_j]} f(x)$.*

Lemma 263 *If $g : \mathbb{N} \to \mathbb{R}$, then $\sum\limits_{k=m}^{n} (g(k + 1) - g(k)) = g(n + 1) - g(m)$ for all $n \geq m$ in $\mathbb{N}$*

Remark* 1 *If $f(x) = \alpha$ is constant on $[a, b]$, then $U(f, P) = L(f, P) = \alpha(b - a)$ for all partitions P on $[a, b]$.*

Remark* 2 *$L(f, P) \leq U(f, P)$ for all partitions P and all bounded functions f.*

Remark* 3 *If $P = \{x_0, x_1, ..., x_n\}$ is a partition of $[a, b]$ and $Q = \{c\} \cup P$ for some $c \in (a, b)$, then $L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$.*

Lemma 264

1. *If P is any partition of $[a, b]$ and Q is a refinement of P, then $L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$.*

2. *If P, Q are any partitions of $[a, b]$ then $L(f, P) \leq U(f, Q)$.*

Theorem 265 *If f is continuous on a closed bounded interval $[a, b]$, then f is integratable on $[a, b]$.*

Definition 266 *Let $[a, b]$ be a closed and bounded nondegenerate interval and $f : [a, b] \to \mathbb{R}$ be bounded.*

1. *The upper integral of f on $[a, b]$ is the number $(U) \int\limits_a^b f(x)\ dx := \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}$.*

2. *The lower integral of f on $[a, b]$ is the number $(L) \int\limits_a^b f(x)\ dx := \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}$.*

Remark* 4 *If $f : [a, b] \to \mathbb{R}$ is bounded, then its upper and lower integrals exist and are finite, and satisfy $(L) \int\limits_a^b f(x)\ dx \leq (U) \int\limits_a^b f(x)\ dx$.*

Theorem 267 *Let $[a, b]$ be a closed bounded nondegenerate interval and $f : [a, b] \to \mathbb{R}$ be bounded. Then f is integratable on $[a, b]$ if and only if*

$$(L) \int_a^b f(x)\ dx = (U) \int_a^b f(x)\ dx,$$

in which case we define the integral of f on $[a, b]$ to be the number

$$\int_a^b f(x)\ dx := (U) \int_a^b f(x)\ dx = (L) \int_a^b f(x)\ dx.$$

Theorem 268 *If $f(x) = \alpha$ on $[a, b]$, then $\int_a^b f(x)\ dx = \alpha(b - a)$.*

Definition 269 *Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$ and $f : [a, b] \to \mathbb{R}$*

1. *A Riemann sum of f with respect to P is a sum of the form $\sum_{j=1}^{n} f(t_j)(x_j - x_{j-1})$, where the choice of $t_j \in [x_{j-1}, x_j]$ is arbitrary.*

2. *The Riemann sum of f is said to converge to $I(f)$ as $||P|| \to 0$ if given for all $\epsilon > 0$ there's a partition P_ϵ of $[a, b]$ such that $P \supseteq P_\epsilon$ implies $\left|\sum\limits_{j=1}^{n} f(t_j)(x_j - x_{j-1}) - I(f)\right| < \epsilon$ for all choices of $t_j \in [x_{j-1}, x_j]$, $j = 1, ..., n$. In this case we shall use the notation $I(f) = \lim\limits_{||P||\to 0} \sum\limits_{j=1}^{n} f(t_j)(x_j - x_{j-1})$.*

Theorem 270 *Let $[a,b]$ be a closed bounded nondegenerate interval and $f : [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann integratable on $[a,b]$ if and only if $I(f) = \lim_{||P|| \to 0} \sum f(t_j)(x_j - x_{j-1})$ exists, in which case $I(f) = \int_a^b f(x)\ dx$.*

Theorem 271 *If $f : [a,b] \to \mathbb{R}$ is bounded and has at most a countable number of discontinuities, then f is Reimann integratable.*