Linear Algebra Done Right Notes

Joseph McGuire

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1 Vector Spaces

1.1 $\mathbb{R}^n$ and $\mathbb{C}^n$

Theorem 1 (Properties of Complex Arithmetic) *1. Commutativity* $\alpha + \beta = \beta + \alpha$ *and* $\alpha\beta = \beta\alpha$ *for all* $\alpha, \beta \in \mathbb{C}$*;*

2. *Associativity* $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ *and* $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ *for all* $\alpha, \beta, \lambda \in \mathbb{C}$*;*

3. *Identities* $\lambda + 0 = \lambda$ *and* $\lambda 1 = \lambda$ *for* $\lambda \in \mathbb{C}$*;*

4. *Additive Inverse for every* $\alpha \in \mathbb{C}$*, there exists a unique* $\beta \in \mathbb{C}$ *such that* $\alpha + \beta = 0$*;*

5. *Multiplicative Inverse for every* $\alpha \in$ *with* $\alpha \neq 0$*, there exists a unique* $\beta \in \mathbb{C}$ *such that* $\alpha\beta = 1$*;*

6. *Distributive Property* $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ *for all* $\lambda, \alpha, \beta \in \mathbb{C}$.

1.2 Definition of Vector Spaces

Theorem 2 (Unique Additive Identity) *A vector space has a unique additive identity.*

Theorem 3 (Unique Additive Identity) *Every element in a vector space has a unique additive inverse.*

Theorem 4 $0v = 0$ *for every* $v \in V$.

Theorem 5 $a0 = 0$ *for every* $a \in F$

Theorem 6 $(-1)v = -v$ *for every* $v \in V$.

1.3 Subspaces

Definition 1 *A subset U of V is called a subspace of V if U is also a vector space (using the same addition and scalar multiplication as on V).*

Theorem 7 (Conditions for a Subspace) *A subset U of V is a subspace of V iff U satisfies the following three conditions:*

- *(Additive Identity)* $0 \in U$;
- *(Closed Under Addition)* $u, w \in U$ *implies* $u + w \in U$;
- *(Closed Under Scalar Multiplication)* $a \in F$ *and* $u \in U$ *implies* $au \in U$.

Example 1 (Subspaces)

1. *The set of continuous real-valued functions on the interval* $[0, 1]$ *is a subspace of* $\mathbb{R}^{[0,1]}$ *(the set of all real-valued functions on* $[0, 1]$*).*
2. *The set of differentiable real-valued functions on* $\mathbb{R}$ *is a subspace of* $\mathbb{R}^{\mathbb{R}}$ *(the set of all real-valued functions on* $\mathbb{R}$*).*
3. *The set of all sequences of complex numbers with limit 0 is a subspace of* $\mathbb{C}^\infty$ *(the set of infinite sequences with entries in* $\mathbb{C}$*).*

Definition 2 (Sum of Subsets) *Suppose $U_1, ..., U_m$ are subsets of V. The* ***sum*** *of $U_1, ..., U_m$, denoted $U_1 + ... + U_m$, is the set of all possible sums of elements of $U_1, ..., U_m$. More precisely,*

$$U_1 + ... + U_m = \{u_1 + ... + u_m : u_i \in U_i\}$$

Theorem 8 (Sum of subspaces is the smallest containing subspace) *Suppose $U_1, ..., U_m$ are subspaces of V. Then $U_1 + ... + U_m$ is the smallest subspace of V containing $U_1, ..., U_m$.*

Definition 3 (Direct Sum) *Suppose $U_1, ..., U_m$ are subspaces of V.*

- *The sum $U_1 + ... + U_m$ is called a* ***direct sum*** *if each element of $U_1 + ... + U_m$ can be written in only one way as a sum $u_1 + ... + u_m$, where each $u_j \in U_j$.*
- *If $U_1 + ... + U_m$ is a direct sum, then $U_1 \oplus ... \oplus U_m$ denotes $U_1 + ... + U_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.*

Theorem 9 (Condition for a Direct Sum) *Suppose $U_1, ..., U_m$ are subspaces of V. Then $U_1 + ... + U_m$ is a direct sum iff the only way to write 0 as a sum $u_1 + ... + u_m$, where each $u_j \in U_j$, is by taking for each $u_j = 0$.*

Theorem 10 (Direct Sum of Two Subspaces) *Suppose U and W are subspaces of V. Then $U + W$ is a direct sum iff $U \cap W = \{0\}$.*

2 Finite-Dimensional Vector Spaces

2.1 Span and Linear Independence

Definition 4 (Linear Combination) *A linear combination of a list $v_1, ..., v_m$ of vectors in V is a vector of the form*

$$a_1 v_1 + ... + a_m v_m$$

where $a_i \in F$

Definition 5 (Span) *The set of all linear combinations of a list of vectors $v_1, ..., v_m \in V$ is called the span of $v_1, ..., v_m$, denoted span$(v_1, ..., v_m)$. In other words,*

$$(v_1, ..., v_m) = \{a_1 v_1 + ... + a_m v_m : a_1, ..., a_m \in F\}.$$

The span of the empty list () is defined to be $\{0\}$.

Theorem 11 (Span is the Smallest Containing Subspace) *The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list.*

Definition 6 (Spans) *If $(v_1, ...v_m) = V$, we say that $v_1, ..., v_m$ spans V.*

Definition 7 (finite-dimensional vector space) *A vector space is called finite-dimensional if some list of vectors in it spans the space.*

Definition 8 (Polynomial, $P(F)$)
- *A Function $p : F \to F$ is called a polynomial with coefficients in F if there exists $a_0, ..., a_m \in F$ such that:*

  $$p(z) = a_0 + a_1 z + ... + a_m z^m$$

  for all $z \in F$.
- *The set $P(F)$ is the set of all polynomials with coefficients in F.*

Definition 9 (Degree of a Polynomial, $\deg(p)$)
- *A polynomial $p \in P(F)$ is said to have degree m if there exists scalars $a_0, a_1, ..., a_m \in F$ with $a_m \neq 0$ such that:*

  $$p(z) = a_0 + a_1 z + ... + a_m z^m$$

  for all $z \in F$. If p has degree m, we write $\deg(p) = m$.
- *The polynomial that is identically 0 is said to have degree $-$*

Definition 10 ($P_m(F)$) *For m a nonnegative integer, $P_m(F)$ denotes the set of all polynomials with coefficients in F and degree at most m.*

Definition 11 (Infinite-Dimensional Vector Space) *A vector space is called infinite-dimensional if it not finite-dimensional.*

Definition 12 (Linearly Independent)
- *A list $v_1, ..., v_m$ of vectors in V is called linearly independent if the only choice of $a_1, ..., a_m \in F$ that makes $a_1 v_1 + ... + a_m v_m = 0$ is $a_1 = ... = a_m = 0$*
- *The empty list () is also declared to be linearly independent. .*

Example 2
- *A list of one vector $v \in V$ is linearly independent iff $v \neq 0$.*
- *A list of two vectors in V is linearly independent iff neither vector is a scalar multiple of the other.*
- *The list $1, z, ..., z_m$ is linearly independent in $P(F)$ for each nonnegative integer m.*

Definition 13 (Linearly Dependent)
- *A list of vectors in V is called linearly dependent if it's not linearly independent.*
- *In other words, a list $v_1, ..., v_m$ of vectors in V is linearly dependent if there exists $a_1, ..., a_m \in F$, not all 0, such that $a_1 v_1 + ... + a_m v_m = 0$.*

Theorem 12 (Linear Dependence Lemma) *Suppose $v_1, ..., v_m$ is a linear*

2.2 Bases

2.3 Dimension