Theorem 1.1.1 (De Morgan's Laws) If A, B, and C are sets, then

(a.) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(b.) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(c.) $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$

(d.) $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$

Definition 1.2.1 Let A and B be any two sets. A function f from A into B is a subset of $A \times B$ with the property that each $x \in A$ is the first component of precisely one ordered pair $(x, y) \in f$; that is, for every $x \in A$ there exists $y \in B$ such that $(x, y) \in f$, and if (x, y) and (x, y') are elements of f, then $y = y'$. The set A is called the domain of f, denoted $Dom(f)$. The range of f, denoted $Range(f)$, is defined by $Range(f) = \{y \in B : (x, y) \in f \text{ for some } x \in A\}$. If $Range(f) = B$, then the function f is said to be onto B.

Definition 1.2.3 Let f be a function from A into B. If $E \subset A$, then $f(E)$, the image of E under f, is defined by $f(E) = \{f(x) : x \in E\}$. If $H \subset B$, the inverse image of H, denoted $f^{-1}(H) = \{x \in A : f(x) \in H\}$. If $H = \{y\}$, we will write $f^{-1}(y)$ instead of $f^{-1}(\{y\})$. Thus for $y \in B$, $f^{-1}(y) = \{x \in A : f(x) = y\}$.

Theorem 1.2.5 Let f be a function from A into B. If A_1 and A_2 are subsets of A, then

(a.) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$,

(b.) $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$.

Theorem 1.2.6 Let f be a function from A to B. If B_1 and B_2 are subsets of B, then

(a.) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$,

(b.) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$,

(c.) $f^{-1}(B \setminus B_1) = A \setminus f^{-1}(B_1)$.

Definition 1.2.7 A function from A into B is said to be one-to-one if whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Definition 1.2.8 If f is a one-to-one function from A onto B, let $f^{-1} = \{(y, x) \in B \times A : f(x) = y\}$. The function f^{-1} from B onto A is called the inverse function of f. Furthermore, for each $y \in B$, $x = f^{-1}(y)$ if and only if $f(x) = y$.

Definition 1.2.10 If f is a function from A to B and g is a function from B to C, then the function $g \circ f : A \to C$, defined by $g \circ f = \{(x, z) \in A \times C : z = g(f(x))\}$

Theorem 1.3.1 (Principle of Mathematical Induction) For each $n \in \mathbb{N}$, let $P(n)$ be a statement about the positive integer n. If

(a.) $P(1)$ is true, and

(b.) $P(k + 1)$ is true whenever $P(k)$ is true,

then P(n) is true for all $n \in \mathbb{N}$.

Well-Ordering Principle 1.3.2 Every nonempty subset of $\mathbb{N}$ has a smallest element.

Theorem 1.3.4 (Second Principle of Mathematical Induction) For each $n \in \mathbb{N}$, let $P(n)$ be a statement about the positive integer n. If

(a.) $P(1)$ is true, and

(b.) for $k > 1$, $P(k)$ is true whenever $P(j)$ is true for all positive integers $j < k$,

then $P(n)$ is true for all $n \in \mathbb{N}$.

Definition 1.4.1 A subset E of $\mathbb{R}$ is bounded above if there exists $\beta \in \mathbb{R}$ such that $x \leq \beta$ for every $x \in E$. Such a β is called an upper bound of E.

Definition 1.4.3 Let E be a nonempty subset of $\mathbb{R}$ that is bounded above. An element $\alpha \in \mathbb{R}$ is called the least upper bound or supremum of E if

(i.) α is an upper bound of E, and

(ii.) if $\beta \in \mathbb{R}$ satisfies $\beta < \alpha$, then β is not an upper bound of E.

Theorem 1.4.4 Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. An upper bound α of A is the supremum of A if and only if for $\beta < \alpha$, there exists an element $x \in A$ such that $\beta < x \leq \alpha$.

Supremum or Least Upper Bound Property of $\mathbb{R}$ Every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

Infimum or Greatest Lower Bound Property of $\mathbb{R}$ Every nonempty subset of $\mathbb{R}$ that is bounded below has an infimum in $\mathbb{R}$.

Definition 1.4.9 If E is a nonempty subset of $\mathbb{R}$, we set

$sup(E) = \infty$, if E is not bounded above, and

$inf(E) = -\infty$, if E is not bounded below.

Definition 1.4.10 For $a, b \in \mathbb{R}$, $a \leq b$, the open interval (a,b) is defined as $(a, b) = \{x \in \mathbb{R} : a < x < b\}$, whereas the closed interval $[a, b]$ is defined as $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$. In addition, we also have the half-open (half-closed) intervals $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$, $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$, and the infinite intervals $(a, \infty) = \{x \in \mathbb{R} : a < x < \infty\}$, $[a, \infty) = \{x \in \mathbb{R} : a \leq x < \infty\}$, with analogous definitions for $(-\infty, b)$ and $(-\infty, b]$. The intervals $(a, \infty), (-\infty, b), and (-\infty, \infty) = \mathbb{R}$ are also referred to as open intervals, whereas the intervals $[a, \infty$ and $(-\infty, b]$ are called closed intervals.

Definition 1.4.11 A subset J of $\mathbb{R}$ is an interval if whenever $x, y \in J$ with $x < y$, then every $x < t < y$ is in J.

Theorem 1.5.1 (Archimedian Property) If $x, y \in \mathbb{R}$ and $x > 0$, then there exists a positive integer n such that $nx > y$.

Theorem 1.5.2 If $x, y \in \mathbb{R}$ and $x < y$, then there exists $r \in \mathbb{Q}$ such that $x < r < y$.

Theorem 1.5.3 For every real number $x > 0$ and every positive integer n, there exists a unique positive real number y so that $y^n = x$.

Corollary 1.5.4 If a, b are positive real numbers, and n is a positive integer, then $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$.

Definition 1.7.1 Two sets A and B are said to be equivalent (or to have the same cardinality), denoted $A\ B$, if there exists a one-to-one function of A onto B. The notion of equivalence of sets satisfies the following:

(i.) $A \sim A$, (Reflexive)

(ii.) If $A \sim B$, then $B \sim A$ (Symmetric)

(iii.) If $A \sim B$ and $B \sim C$, then $A \sim C$. (Transitive)

Definition 1.7.2 For each positive integers n, let $\mathbb{N}_n = \{1, 2, ..., n\}$. As in Section 1.1, $\mathbb{N}$ denotes the set of all positive integers. If A is a set, we say:

(a.) A is finite is $A\ \mathbb{N}_n$ for some n, or if $A = \emptyset$.

(b.) A is infinite if A is not finite.

(c.) A is countable if $A\ \mathbb{N}$.

(d.) A is uncountable if A is neither finite nor countable.

(e.) A is at most countable if A is finite or countable.

Theorem 1.7.4 $\mathbb{N} \times \mathbb{N}$ is countable.

Definition 1.7.5 If A is a set, by a sequence in A we mean a function f from $\mathbb{N}$ into A. For each $n \in \mathbb{N}$, let $x_n = f(n)$. Then x_n is called the n-th term of the sequence f.

Theorem 1.7.6 Every infinite subset of a countable set is countable.

Theorem 1.7.7 If f maps $\mathbb{N}$ onto A, then A is at most countable.

Definition 1.7.8 Let A and X be nonempty sets. An indexed family of subsets of X with index set A is a function from A into $P(A)$.

Definition 1.7.10 Suppose $\{E_\alpha\}_{\alpha\in A}$ is an indexed family of subsets of X. The union of the family of sets $\{E_\alpha\}_{\alpha\in A}$ is defined to be $\bigcup_{\alpha\in A} E_\alpha = \{x \in X : x \in E_\alpha \text{ for some } \alpha \in A\}$. The intersection of the family of sets $\{E_\alpha\}_{\alpha\in A}$ is defined as $\bigcap_{\alpha\in A}\{E_\alpha\}_{\alpha\in A} = \{x \in X : x \in E_\alpha \text{for all } \alpha \in A\}$.

Theorem 1.7.12 (Distributive Laws) If E_α , $\alpha \in A$, and E are subsets of a set X, then

(a.) $E \cap (\bigcup_{\alpha\in A} E_\alpha) = \bigcup_{\alpha\in A}(E \cap E_\alpha)$,

(b.) $E \cup (\bigcap_{\alpha\in A} E_\alpha) = \bigcap_{\alpha\in A}(E \cup E_\alpha)$.

Theorem 1.7.13 (De Morgan's Laws) If $\{E_\alpha\}_{\alpha\in A}$ is a family of subsets of X, then

(a.) $(\bigcup_{\alpha\in A} E_\alpha)^c = \bigcap_{\alpha\in A} E_\alpha^c$,

(b.) $(\bigcap_{\alpha\in A} E_\alpha)^c = \bigcup_{\alpha\in A} E_\alpha^c$,

Theorem 1.7.14 Let f be a function from X into Y.

(a.) If $\{E_\alpha\}_{\alpha\in A}$ is a family of subsets of X, then

$f(\bigcup_{\alpha\in A} E_\alpha) = \bigcup_{\alpha\in A} f(E_\alpha)$,

$f(\bigcap_{\alpha\in A} E_\alpha) \subset \bigcap_{\alpha\in A} f(E_\alpha)$.

(b.) If $\{B_\alpha\}_{\alpha\in A}$ is a family of subsets of Y, then

$f^{-1}(\bigcup_{\alpha\in A} B_\alpha) = \bigcup_{\alpha\in A} f^{-1}(B_\alpha)$

$f^{-1}(\bigcap_{\alpha\in A} B_\alpha) = \bigcap_{\alpha\in A} f^{-1}(B_\alpha)$

Theorem 1.7.15 If $\{E_\alpha\}_{n=1}^{\infty}$ is a sequence of countable sets and $S = \bigcup_{n=1}^{\infty} E_n$, then S is countable.

Corollary 1.7.16 $\mathbb{Q}$ is countable.

Theorem 1.7.17 The closed interval $[0,1]$ is uncountable.

Theorem 1.7.18 If A is the set of all sequences whose element are 0 or 1, then A is uncountable.

Definition 2.1.1 For a real number x, the absolute value of x, denoted $|x|$, is defined by $\begin{cases} x & x < 0 \\ -x & x \leq 0 \end{cases}$.

Theorem 2.1.2

(a.) $|-x| = |x|$ for all $x \in \mathbb{R}$.

(b.) $|xy| = |x||y|$ for all $x, y \in \mathbb{R}$.

(c.) $|x| = \sqrt{x^2}$ for all $x \in \mathbb{R}$.

(d.) if $r > 0$, then $|x| < r$ if and only if $-r < x < r$.

(e.) $-|x| \leq x \leq |x|$ for all $x \in \mathbb{R}$.

Theorem 2.1.3 (Triangle Inequality) For all $x, y \in \mathbb{R}$, we have $|x + y| \leq |x| + |y|$.

Corollary 2.1.4 For all $x, y, z \in \mathbb{R}$, we have

(a.) $|x - y| \leq |x - z| + |z - y|$.

(b.) $||x| - |y|| \leq |x - y|$.

Definition 2.1.6 Let $p \in \mathbb{R}$ and $\epsilon > 0$. The set $N_\epsilon(p) = \{x \in \mathbb{R} : |x - p| < \epsilon\}$.

Definition 2.1.7 A sequence $\{p_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is said to converge if there exists a point $p \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists a positive integer n_0 such that $p_n \in N_\epsilon(p)$ for all $n \geq n_0$. If this is the case, we say that $\{p_n\}$ converges to p, or that p is the limit of the sequence $\{p_n\}$, and we write $lim_{n\to\infty} p_n = p$ or $p \to p$. If $\{p_n\}$ doesn't converge, then $\{p_n\}$ is said to diverge.

Definition 2.1.9 A sequence $\{p_n\}$ in $\mathbb{R}$ is said to be bounded if there exists a positive constant M such that $|p_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem 2.1.10

(a.) If a sequence $\{p_n\}$ in $\mathbb{R}$ converges, then its limit is unique.

(b.) Every convergent sequence in $\mathbb{R}$ is bounded.

Theorem 2.2.1 If $\{a_n\}$ and $\{b_n\}$ are convergent sequences of real numbers with $lim_{n\to\infty}a_n = a$ and $lim_{n\to\infty}b_n = b$, then

(a.) $lim_{n\to\infty}(a_n + b_n) = a + b$, and

(b.) $lim_{n\to\infty}a_n b_n = ab$.

(c.) Furthermore, if $a \neq 0$, and $a_n \neq 0$ for all n, then $lim_{n\to\infty}\frac{b_n}{a_n} = \frac{b}{a}$.

Corollary 2.2.2 If $\{a_n\}$ is a convergent sequence of real numbers with $lim_{n\to\infty}a_n = a$, then for any $c \in \mathbb{R}$,

(a.) $lim_{n\to\infty}(a_n + c) = a + c$, and

(b.) $lim_{n\to\infty}ca_n = ca$.

Theorem 2.2.3 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. If $\{b_n\}$ is bounded and $lim_{n\to\infty}a_n = 0$, then $lim_{n\to\infty}a_n b_n = 0$.

Theorem 2.2.4 Suppose $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences of real numbers for which there exists $n_0 \in \mathbb{N}$ such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$, $n \geq n_0$, and that $lim_{n\to\infty}a_n = lim_{n\to\infty}c_n = L$. Then the sequence $\{b_n\}$ converges and $lim_{n\to\infty}b_n = L$.

Theorem 2.2.5 For $a \in \mathbb{R}$, $n \in \mathbb{N}$, $(1 + a)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} a^k$.

Theorem 2.2.6

(a.) If $p > 0$, then $lim_{n\to\infty}\frac{1}{n^p} = 0$.

(b.) If $p > 0$, then $lim_{n\to\infty}\sqrt[n]{p} = 1$.

(c.) $lim_{n\to\infty}\sqrt[n]{n} = 1$.

(d.) If p ¿ 1 and α is real, then $lim_{n\to\infty}\frac{n^\alpha}{p^n} = 0$.

(e.) If $|p| < 1$, then $lim_{n\to\infty}p^n = 0$.

(f.) For all $p \in \mathbb{R}$, $lim_{n\to\infty}\frac{p^n}{n!} = 0$.

Definition 2.3.1 A sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is said to be:

(a.) monotone increasing (or nondecreasing) if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$;

(b.) monotone decreasing (or nonincreasing) if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$;

(c.) monotone if it is either monotone increasing or monotone decreasing.

Theorem 2.3.2 If $\{a_n\}_{n=1}^{\infty}$ is monotone and bounded, then $\{a_n\}_{n=1}^{\infty}$ converges. Moreover, $lim_{n\to\infty}a_n = sup\{a_n\}$.

Corollary 2.3.3 If $\{I_n\}_{n=1}^{\infty}$ is a sequence of closed and bounded intervals with $I_n \supset I_{n+1}$ for all $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Definition 2.3.6 Let $\{a_n\}$ be a sequence of real numbers. We say that $\{a_n\}$ approaches infinity, or that $\{a_n\}$ diverges to ∞, denoted $a_n \to \infty$, if for every positive real number M, there exists an integer $n_0 \in \mathbb{N}$ such that $a_n > M$ for all $n \geq n_0$.

Theorem 2.3.7 If $\{a_n\}$ is monotone increasing and not bounded above, then $a_n \to \infty$ as $n \to \infty$.

Definition 2.4.1 Given a sequence $\{p_n\}$ in $\mathbb{R}$, consider a sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers such that $n_1 < n_2 < n_3 < ....$ Then the sequence $\{p_{nk}\}_{k=1}^{\infty}$ is called a subsequence of the sequence $\{p_n\}$.

Theorem 2.4.3 Let $\{p_n\}$ be a sequence in $\mathbb{R}$. If $\{p_n\}$ converges to p, then every subsequence of $\{p_n\}$ also converges to p.

Definition 2.4.5 Let E be a subset of $\mathbb{R}$.

(a.) A point $p \in \mathbb{R}$ is a limit point of E if every ϵ-neighborhood $N_\epsilon(p)$ of p contains a point $q \in E$ with $q \neq p$.

(b.) A point $p \in E$ that is not a limit point of E is called an isolated point of E. In the definition of limit point it is not required that p is a point of E. Also, a point $p \in E$ is an isolated point of E if there exists an $\epsilon > 0$ such that $N_\epsilon(p) \cap E = \{p\}$.

Theorem 2.4.7 Let E be a subset of $\mathbb{R}$.

(a.) If p is a limit point of E, then every neighborhood of p contains infinitely many points of E.

(b.) If p is a limit point of E, then there exists a sequence $\{p_n\}$ in E with $p_n \neq p$ for all $n \in \mathbb{N}$, such that $lim_{n\to\infty} p_n = p$.

Corollary 2.4.8 A finite set has no limit points.

Theorem 2.4.10 (Bolzano-Weierstrass) Every bounded infinite subset of $\mathbb{R}$ has a limit point.

Corollary 2.4.11 (Bolzano-Weierstrass) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Theorem 2.4.12 Let $\{p_n\}$ be a sequence in $\mathbb{R}$. If p is a limit point of $\{p_n : n \in \mathbb{N}\}$, then there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $p_{n_k} \to \infty$ as $k \to \infty$.

Definition 2.6.1 A sequence $\{p_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is a Cauchy sequence if for every $\epsilon > 0$, there exists a positive integer n_0 such that $|p_n - p_m| < \epsilon$ for all integers $n, m \geq_0$.

Theorem 2.6.2

(a.) Every convergent sequence in $\mathbb{R}$ is a Cauchy sequence.

(b.) Every Cauchy sequence is bounded.

Theorem 2.6.3 If $\{p_n\}$ is a Cauchy sequence in $\mathbb{R}$ that has a convergent subsequence, then the sequence $\{p_n\}$ converges.

Theorem 2.6.4 Every Cauchy sequence of real numbers converges.

Definition 2.7.1 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Let $\{s_n\}_{n=1}^{\infty}$ be the sequence obtained from $\{a_n\}$. where for each $n \in \mathbb{N}$, $s_n = \sum_{k=1}^{n} a_k$. The sequence $\{s_n\}$ is called an infinite series, or series, and is denoted either as $\sum_{k=1}^{\infty} a_k$ or $a_1 + ... + a_n + ...$. For each $n \in \mathbb{N}$, s_n is called the n-th partial sum of the series and a_n is called the n-th term of the series.

The series $\sum_{k=1}^{\infty} a_k$ converges if and only if the sequence $\{s_n\}$ of n-th partial sums converges in $\mathbb{R}$. If $lim_{n\to\infty} s_n = s$, then s is called the sum of the series, and we write $s = \sum_{k=1}^{\infty} a_k$. If the sequence $\{s_n\}$ diverges, then the series $\sum_{k=1}^{\infty} a_k$ is said to diverge.

Theorem 2.7.3 The series $\sum_{k=1}^{\infty} a_k$ converges if and only if given $\epsilon > 0$, there exists a positive integer n_0, such that $\sum_{k=n+1}^{m} a_k < \epsilon$ for all $m > n \geq n_0$.

Corollary 2.7.5 If $\sum_{k=1}^{\infty} a_k$ converges if and only if $lim_{k\to\infty} a_k = 0$.

Theorem 2.7.6 Suppose $a_k \geq 0$ for all $k \in \mathbb{N}$. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\{s_n\}$ is bounded above.

Definition 3.1.1 Let E be a subset of $\mathbb{R}$. A point $p \in E$ is called an interior point of E if there exists an $\epsilon > 0$ such that $N_\epsilon(p) \subset E$. The set of interior points of E is denoted $Int(E)$, and is called the interior of E.

Definition 3.1.3

(a.) A subset O of $\mathbb{R}$ is open if every point of O is an interior point of O.

(b.) A subset F of $\mathbb{R}$ is closed if $F^c = \mathbb{R} \setminus F$ is open. From the definition of an interior point it should be clear that a set $O \subset \mathbb{R}$ is open if and only if for every $p \in O$ there exists an $\epsilon > 0$ (depending on p) so that $N_\epsilon(p) \subset O$.

Theorem 3.1.5 Every open interval in $\mathbb{R}$ is an open subset of $\mathbb{R}$.

Theorem 3.1.6

(a.) For any collection $\{O_\alpha\}_{\alpha\in A}$ of open subsets of $\mathbb{R}$, $\bigcup_{\alpha\in A} O_\alpha$ is open.

(b.) For any finite collection $\{O_1, ..., O_n\}$ of open subsets of $\mathbb{R}$, $\bigcap_{j=1}^{n} O_j$ is open.