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Homework #2

MATH 495
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Exercise 1.

Verify the equality,

$$\mathbb{V}(\{F_i\}_{i\in I}) \cap \mathbb{V}(\{F_j\}_{j\in J}) = \mathbb{V}(\{F_i\}_{i\in I\cup J}).$$

Give some specific examples or intersections of varieties (with equations) and draw pictures to accompany your work.

Solution:

Let $x \in \mathbb{V}(\{F_i\}_{i\in I}) \cap \mathbb{V}(\{F_j\}_{j\in J})$. Then $x \in \mathbb{V}(\{F_i\}_{i\in I})$ and $x \in \mathbb{V}(\{F_j\}_{j\in J}$, so by def. for all $i \in I$ and $j \in J$, we have $F_i(x) = 0$ and $F_j(x) = 0$. Hence for all $i \in I \cup J, F_i(x) = 0$ and thus $x \in \mathbb{V}(\{F_i\}_{i\in I\cup J})$.

$$\mathbb{V}(\{F_i\}_{i\in I}) \cap \mathbb{V}(\{F_j\}_{j\in J}) \subset \mathbb{V}(\{F_i\}_{i\in I\cup J})$$

Let $x \in \mathbb{V}(\{F_i\}_{i\in I\cup J})$. Then for all $i \in I \cup J$ we have $F_i(x) = 0$. Hence for all $i \in I$ and $j \in J$ we have $F_i(x) = 0$ and $F_j(x) = 0$. Thus $x \in \mathbb{V}(\{F_i\}_{i\in I}) \cap \mathbb{V}(\{F_j\}_{j\in J})$.

$$\mathbb{V}(\{F_i\}_{i\in I\cup J}) \subset \mathbb{V}(\{F_i\}_{i\in I}) \cap \mathbb{V}(\{F_j\}_{j\in J})$$

Finally, we have

$$\mathbb{V}(\{F_i\}_{i\in I}) \cap \mathbb{V}(\{F_j\}_{j\in J}) = \mathbb{V}(\{F_i\}_{i\in I\cup J})$$

Exercise 3.

Describe the set Z introduced in this section. Show that this set turns $\mathbb{C}^n$ into a topological space (i.e., show that Z satisfies the four axioms defining a topology in $\mathbb{C}^n$ with Z the open sets.)

Solution:

To show that Z forms a topology on $\mathbb{C}^n$ we will show that:

1. Finite union of closed sets are closed
2. Arbitrary intersections of closed sets are closed
3. $\mathbb{C}^n$ is closed
4. $\emptyset$ is closed.

Note that these are the topological axioms in terms of closed sets, in the case of Z these are affine algebraic varieties.

1. We know that the union of two algebraic varieties $\mathbb{V}(\{F_i\}_{i\in I}) \cup \mathbb{V}(\{F_j\}_{j\in J}) = \mathbb{V}(\{F_iF_j\}_{(i,j)\in I\times J})$, by exercise 2. If we repeat this process n-times, then we end up with

$$\bigcup_{k=1}^{n} \mathbb{V}(\{F_i\}_{i\in I_k}) = \mathbb{V}(\{F_{I_1}F_{I_2}...F_{I_n}\}_{(i_1,i_2,...,i_n)\in I_1\times I_2\times...\times I_n})$$

   So a finite union of affine algebraic varieties is an affine algebraic variety.

2. We have an arbitrary intersection of affine algebraic varieties is an affine algebraic variety, by p.6.
3. $\mathbb{C}^n$ is an affine algebraic variety, because $\mathbb{V}(0) = \mathbb{C}^n$.
4. $\emptyset$ is an affine algebraic variety because $\mathbb{V}(1) = \emptyset$.

Thus Z is a topology on $\mathbb{C}^n$.

Exercise 4.

What is the name of the topology on $\mathbb{C}^n$ discussed in Exercise 3? Consider how this topology is different than the usual Euclidean topology on $\mathbb{C}^n$. In particular, why is this topology coarser than the usual Euclidean topology.

Solution:

The Zariski topology is the name of this particular topology. This topology isn't a very well-behaved topology as discussed in the text, as it isn't Hausdorff then we don't have the Euclidean notion of compactness. Additionally, every open set in Z is unbounded in Euclidean topology, a strange behavior. Partly due to this coarseness of this topology, all the non-trivial open sets are dense in Euclidean topology and Z topology, another strange behavior. Additionally, because Z isn't Hausdorff, limits in the Z topology don't need to be unique. And while every Z-closed set is Euclidean-closed, the converse isn't true.

One can show why the Z topology is coarser than Euclidean topology. In Euclidean topology base sets are open balls in n-space. Using the definition of Z-topology, as open sets being the complement of affine algebraic varieties, then you can build every open set in Z using open sets in $\mathbb{C}^n$. One can imagine having unions of open balls that have a boundary that is formed by the affine algebraic variety.

Exercise 5.

Explain and justify the statement: No nontrivial algebraic variety in $\mathbb{C}^n$ can have interior points.

Solution:

Take any subset of an affine algebraic variety V, call it A, where $V = \mathbb{V}(\{F_i\}_{i\in I})$. Define f_i as the restriction of F_i on A, so then since the restriction of any continuous functions is continuous, by topology results, then we know that $f_i^{-1}(\{0\}) = A$ is closed, since $\{0\}$ is closed in $\mathbb{C}$. Thus every subset of V is a closed set. So then, by def., $Int(V) = \emptyset$ and hence there are now interior points of V.

Exercise 1.2.3.

Show that the twisted cubic curve depicted in Figure 1.5 consists of all points in $\mathbb{A}^3$ of the form (t, t^2, t^3), where $t \in \mathbb{C}$.

Solution:

The equation for this curve is given by $\mathbb{V}(x^2 - y, x^3 - z)$.

This is equivalent to the solutions of simultaneous set of equations:

$$x^2 - y = 0 \qquad\qquad x^3 - z = 0$$

So then the curve is given:

$$y = x^2 \qquad\qquad z = x^3$$

So the only independent variable present is x, so then our only solution is: $(x, y, z) = (x, x^2, x^3)$ for $x \in \mathbb{C}$.