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Homework #9

MATH 550
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4.A.1

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $h : X \to \mathbb{R}$ is an $\mathcal{S}-$measurable function. Prove that

$$\mu(\{x \in X : |h(x)| \geq c\}) \leq \frac{1}{c^p} \int |h|^p \, d\mu$$

for all positive numbers c and p.

Proof. Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $h : X \to \mathbb{R}$ is an $\mathcal{S}-$measurable function. We'll break this into 3 cases (#1) $c \geq 1$ and $p > 0$, (#2) $0 < c < 1$ and $0 < p < 1$, and (#3) $0 < c < 1$ and $p \geq 1$.

1. ($c \geq 1$ and $p > 0$)

   Note in this case $1 < c \leq c^p$, for all c, p being consider in this case, with that consider the following:

$$\begin{aligned}\mu(\{x \in X : |h(x)| \geq c\} &= \frac{1}{c^p} \int_{\{x \in X : |h(x)| \geq c\}} c^p \, d\mu \\ &\leq \frac{1}{c^p} \int_{\{x \in X : |h(x)| \geq c\}} |h(x)|^p \, d\mu \\ &\leq \frac{1}{c^p} \int |h|^p \, d\mu\end{aligned}$$

   the second-to-last inequality comes from $|h| \geq c > 1$ implying $|h|^p \geq c^p > 1$. This is our result when $c \geq 1$ and $p > 0$.

2. ($0 < c < 1$ and $0 < p < 1$)

   Note that in this case we also have $c \leq c^p \leq 1$, for all c, p being considered in this case, with that consider the following:

$$\begin{aligned}\mu(\{x \in X : |h(x)| \geq c\}) &= \frac{1}{c^p} \int_{\{x \in X : |h(x)| \geq c\}} c^p \, d\mu \\ &= \frac{1}{c^p} \int_{\{x \in X : 1 > |h(x)| \geq c\}} c^p \, d\mu + \frac{1}{c^p} \int_{\{x \in X : |h(x)| \geq 1 \geq c\}} c^p \, d\mu \\ &\leq \frac{1}{c^p} \int_{\{x \in X : 1 > |h(x)| \geq c\}} |h|^p \, d\mu + \frac{1}{c^p} \int_{\{x \in X : |h(x)| \geq 1\}} |h|^p \, d\mu \\ &\leq \frac{1}{c^p} \int |h|^p \, d\mu,\end{aligned}$$

the second equality comes from disjoint additivity of integrals, the second-to-last inequality comes from $|h(x)| \geq c$ implying $|h(x)|^p \geq c^p$ for the first term, for the second term if $|h(x)| \geq 1$ then $|h(x)|^p \geq 1$ giving us $|h(x)|^p \geq c^p$, the final inequality comes from the two set's being disjoint, hence we can employ additivity over the domain of integration. This proves our result when $0 < c < 1$ and $0 < p < 1$.

3. $(0 < c < 1$ and $p \geq 1)$

   In this case we have $0 < c^p \geq c < 1$ for all c, p considered in this case, with that in mind consider the following:

$$\begin{aligned}
\mu(\{x \in X : |h(x)| \geq c\}) &= \frac{1}{c^p} \int_{\{x \in X : |h(x)| \geq c\}} c^p \, d\mu \\
&= \frac{1}{c^p} \left(\int_{\{x \in X : |h(x)| \geq 1 > c\}} c^p \, d\mu + \int_{\{x \in X : 1 > |h(x)| \geq c\}} c^p \, d\mu \right) \\
&\leq \frac{1}{c^p} \left(\int_{\{x \in X : |h(x)| \geq 1 > c\}} |h| \, d\mu + \int_{\{x \in X : 1 > |h(x)| \geq c\}} |h|^p \right) \\
&\leq \frac{1}{c^p} \left(\int_{\{x \in X : |h(x)| \geq 1 > c\}} |h|^p \, d\mu + \int_{\{x \in X : 1 > |h(x)| \geq c\}} |h|^p \right) \\
&\leq \frac{1}{c^p} \int |h^p| \, d\mu,
\end{aligned}$$

   the second equality comes from disjoint additivity of integrals, the second inequality comes from $|h(x)| \geq 1 > c \geq c^p$ in the first term and the second term is using the fact that $|h(x)| \geq c$ implies $|h(x)|^p \geq c^p$, the third inequality uses the fact that for the first term $|h(x)| \leq |h(x)|^p$ for $|h(x)| \geq 1$, the final inequality is disjoint additivity of integrals. This gives us our result when $0 < c < 1$ and $p \geq 1$.

Thus we have shown the result holds for all $c > 0$ and $p > 0$. $\square$

4.A.2

Suppose $(X, \mathcal{S}, \mu)$ is a measure space with $\mu(X) = 1$ and $h \in \mathcal{L}^1(\mu)$. Prove that

$$\mu\left(\left\{x \in X : \left|h(x) - \int h \, d\mu\right| \geq c\right\}\right) \leq \frac{1}{c^2}\left(\int h^2 \, d\mu - \left(\int h \, d\mu\right)^2\right)$$

for all $c > 0$.

Proof. Suppose $(X, \mathcal{S}, \mu)$ is a measure space with $\mu(X) = 1$ and $h \in \mathcal{L}^1(\mu)$.
First, note that since $h \in \mathcal{L}^1(\mu)$ we have that $\int |h| \, d\mu < \infty$ and that h is $\mathcal{S}$-measurable, implying both $\int h \, d\mu$ is defined and $\left|\int h \, d\mu\right| < \infty$ (3.23). Since we have that $\left|\int h \, d\mu\right| < \infty$ we can conclude that $\int h \, d\mu < \infty$. So then we have that the function $|h(x) - \int h \, d\mu| \in \mathcal{L}^1(\mathbb{R})$. Furthermore, since $\int |h| \, d\mu < \infty$ an application of the triangle inequality gives us:

$$\int \left|h(x) - \int |h| \, d\mu\right| \, d\mu < \infty.$$

Consider the case now where $\int |h|^2 \, d\mu = \int h^2 \, d\mu = \infty$. In which case, the inequality should immediately follow, since we have shown that $\int h \, d\mu < \infty$.
Now assume that $\int |h|^2 \, d\mu = \int h^2 \, d\mu < \infty$. Hence this function meets the hypotheses of $4.A.1$, giving us the following:

$$\begin{aligned}
\mu(\{x \in X : \left|h(x) - \int h \, d\mu\right| \geq c\}) &\leq \frac{1}{c^2}\int \left|h(x) - \int h \, d\mu\right|^2 \, d\mu \\
&= \frac{1}{c^2}\int \left(h(x) - \int h \, d\mu\right)^2 \, d\mu \\
&= \frac{1}{c^2}\int \left(h^2(x) - 2h(x)\int h \, d\mu + \left(\int h \, d\mu\right)^2\right) \, d\mu \\
&= \frac{1}{c^2}\left(\int h^2 \, d\mu + \int (-2h \int h \, d\mu) + \int \left(\int h \, d\mu\right)^2\right) \\
&= \frac{1}{c^2}\left(\int h^2 \, d\mu - 2\left(\int h \, d\mu\right)^2 + \mu(X)\left(\int h \, d\mu\right)^2\right) \\
&= \frac{1}{c^2}\left(\int h^2 \, d\mu - 2\left(\int h \, d\mu\right)^2 + \left(\int h \, d\mu\right)^2\right) \\
&= \frac{1}{c^2}\left(\int h^2 \, d\mu - \left(\int h \, d\mu\right)^2\right),
\end{aligned}$$

the inequality is an application of $4.A.1$, the third equality uses integration distributing over addition and scalar multiplication and the fact that since $\int |h|^2 < \infty$ and $\int |h| < \infty$ any multiple of it must then also must be clearly finite when that scalar is finite. Combining the first line with last is the result we wished to prove! $\square$

4.A.4

Show that the constant 3 in the Vitali Covering Lemma (4.4) cannot be replaced by a smaller positive constant.

Lemma. *If $I = (a, b)$ is an open interval and $c > 0$, then $c * I = (-\frac{c(b-a)}{2} + \frac{b+a}{2}, \frac{b+a}{2} + \frac{c(b-a)}{2})$ has the same center as I and c-times the length of I; that is $|I| = b - a$ and $|c * I| = c(b-a)$.*

Proof. Let $I = (a, b)$ and $c > 0$ be fixed. Note that the mid-point of I is $\frac{a+b}{2}$. Moreover I is defined by all $x \in \mathbb{R}$ such that: $a < x < b$. We can rewrite this as all $x \in \mathbb{R}$ such that

$$|x - \frac{b+a}{2}| < \frac{b-a}{2}$$

. So that scaling this open ball by c is as simple as

$$|x - \frac{b+a}{2}| < \frac{c(b-a)}{2} \iff -\frac{c(b-a)}{2} + \frac{a+b}{2} < x < \frac{a+b}{2} + \frac{c(b-a)}{2}.$$

Moreover the length of this interval is $c(b-a)$ and has midpoint $\frac{a+b}{2}$. This is what we wished to prove! □

Proof. Let $0 < c < 3$ be fixed and $\epsilon = 1 - \frac{c}{3}$ (notice $\epsilon > 0$). Let $I_1 = (-1, +1)$ so that $c * I = (-c, c)$. Now let $I_2 = (1 - \epsilon, c + \epsilon)$. So that $c * I_2 = (-\frac{c(c + 2\epsilon - 1)}{2} + \frac{c+1}{2}, \frac{c+1}{2} + \frac{c(c - 1 + 2\epsilon)}{2})$. Now note that $I_1 \cup I_2 = (-1, c + \epsilon)$ and that $I_1 \cap I_2 \neq \emptyset$. So because these are not disjoint the only list of disjoint sets are (I_1) and (I_2). So since $c < c + \epsilon$ we have $I_1 \cup I_2 \not\subset c * I_1$. To finish the proof, we'll show that $-1 < \frac{-c(c + 2\epsilon - 1)}{2} + \frac{c+1}{2}$, to do this in a somewhat natural way we'll show that this statement is logically equivalent to $c^2 < 9$

which is true for all $0 < c < 3$. So consider the following:

$$
\begin{aligned}
& -1 < \frac{-c(c+2\epsilon-1)+c+1}{2} \\
\iff & -2 < -c^2 - 2c\epsilon + 2c + 1 \\
\iff & 0 < -c^2 + 2c(1-\epsilon) + 3 \\
\iff & 0 < -c^2 + 2c\frac{c}{3} + 3 \\
\iff & 0 < \frac{-c^2}{3} + 3 \\
\iff & c^2 < 9,
\end{aligned}
$$

because this last statement is true for all $0 < c < 3$, we have that are first statement is true. This implies $I_1 \cup I_2 \not\subset c * I_2$.

So since $0 < c < 3$ is arbitrary, we have that the Vitali Covering Lemma cannot hold for anything less than 3, so 3 is the least constant that works for the lemma. $\square$

4.A.6

Verify the formula in Example 4.7 for the Hardy-Littlewood maximal function of $\chi_{[0,1]}$.

Proof. First, note that we have for any $b \in \mathbb{R}$ and $h \in \mathcal{L}^1(\mathbb{R})$

$$h^*(b) = \sup_{t>0} \frac{1}{2t} \int_{b-t}^{b+t} |h| \leq \sup_{t>0} \sup_{[b-t,b+t]} |h| = \sup_{\mathbb{R}} |h|.$$

In this particular case of $h = \chi_{[0,1]}$ we'll have $\chi^*_{[0,1]}(b) \leq 1$ for all $b \in \mathbb{R}$.
Next note that we get the following from the definition of the integral over a subset and the definition of a characteristic function:

$$\chi^*_{[0,1]} = \sup_{t>0} \frac{1}{2t} \int_{b-t}^{b+t} \chi_{[0,1]} = \sup_{t>0} \frac{1}{2t} \int \chi_{[b-t,b+t]}\chi_{[0,1]} = \sup_{t>0} \frac{1}{2t} \int \chi_{[0,1]\cap[b-t,b+t]} = \sup_{t>0} \frac{1}{2t} \lambda([0,1]\cap[b-t,b+t]).$$

We'll then determine the values of this on the intervals: $(0,1), (-\infty,0], [1,\infty)$. Note that we'll only go through the explanation for $(-\infty, 0]$, but that the same reasoning will apply to $[1,\infty)$ instead with a $b+t=1$ replaced with $b-t=0$.

1. $(b \in (0,1))$

   Note that in this case whatever the value of b, we may choose a $t > 0$ such that $[b-t, b+t] \subseteq [0,1]$, so that by the preceding paragraph:

   $$\frac{1}{2t}\lambda([b-t,b+t] \cap [0,1]) = \frac{1}{2t}(b+t-b+t) = 1.$$

   Since we have that $\chi^*_{[0,1]}(b)$ is the sup of all such values determined in this way, and we have $\chi^*_{[0,1]}(b) = 1$ for all $0 < b < 1$.

2. $(b \in [1,\infty))$

   In this case we can rule out that the supremum occurs for any $t > 0$ such that $b+t \leq 0$, since the value of the integral will be 0 for such $b+t$. So consider $b+t > 0$. Then the maximization problem becomes:

   $$\sup_{t>0} \frac{1}{2t}\lambda([0,b+t] \cap [0,1]).$$

   Note that we can rule out that the supremum occurs to the right of 1, since the integral won't become larger for $b+t > 1$. So consider $0 < b+t \leq 1$, then $\lambda([0,b+t] \cap [0,1]) = b+t \leq 1$. Then the problem becomes:

   $$\sup_{t>0} \frac{b+t}{2t} = \sup_{t>0} \frac{b}{2t} + \frac{1}{2}.$$

Since we have $b \in (-\infty, 0]$, the quantity $\dfrac{b}{2t}$ will only become larger as $t > 0$ increases. Meaning that it will attain a maximum at $b + t = 1$, that is when $t = 1 - b$. Hence the maximization problem is solved for this value of $t > 0$, giving us:

$$\chi^*_{[0,1]}(b) = \frac{1}{2(1-b)}$$

for all $b \in (-\infty, 0]$.

3. ($b \in [1, \infty)$)

   Almost identical reasoning can be applied here as was employed in the previous case. Except that $b - t$ is the variable in question, and the quantity in question will similarly attain a maximum at $b - t = 0$, giving us

$$\chi^*_{[0,1]}(b) = \frac{1}{2b}$$

   for all $b \in [1, \infty)$.

Hence we have:

$$\chi^*_{[0,1]}(b) = \begin{cases} 1 & \text{if } b \in (0,1) \\ \dfrac{1}{2(1-b)} & \text{if } b \in (-\infty, 0] \\ \dfrac{1}{2b} & \text{if } b \in [1, \infty) \end{cases}.$$

□

4.A.7

Find a formula for the Hardy-Littlewood maximal function of the characteristic function of $[0,1] \cup [2,3]$.

Proof. Let $h : \mathbb{R} \to \{0,1\}$ be defined by $h(x) = \chi_{[0,1]}(x) + \chi_{[2,3]}(x)$. Define $T(b,t) = \frac{1}{2t}\int_{b-t}^{b+t} |h|$. Then I claim then that

$$h^*(b) = \begin{cases} \dfrac{1}{3-b} & -\infty < b < -1 \\ \dfrac{1}{2(1-b)} & -1 \leq b \leq 0 \\ 1 & 0 < b < 1 \text{ or } 2 < b < 3 \\ 1 - \dfrac{1}{2b} & 1 \leq b \leq \dfrac{3}{2} \\ \dfrac{5-2b}{2(3-b)} & \dfrac{3}{2} \leq b \leq 2 \\ \dfrac{1}{2(b-2)} & 3 \leq b \leq 4 \\ \dfrac{1}{b} & 4 < b < \infty \end{cases}$$

1. $(-\infty < b < -1)$

   For such a b, note that $\frac{1}{2t}\int_{b-t}^{b+t} |h| = 0$ until $b + t > 0$. So that for $b + t < 1$, we'll have

   $$A(b,t) = \frac{1}{2t}\int_0^{b+t} 1 = \frac{b+t}{2t} = \frac{b}{2t} + \frac{1}{2},$$

   since $b < 0$ this quantity increases as t increases, so that we must have at least $b + t \geq 1$. Note that $1 \leq b + t < 2$ would gain us no area under the integral and would only decrease the value of $A(b,t)$, so we'll consider when $b + t > 2$, giving us: $\frac{1}{2t}(1 + \int_2^{b+t} 1) = \frac{1}{2t}(b + t - 1) = \frac{b}{t} + \frac{1}{2} - \frac{1}{2t}$. Clearly the area gained by the integral will increase until $b + t = 3$, so that $A(b, 3-b) = \frac{1}{3-b}$. Notice that this is greater than $A(b, 1-b) = \frac{1}{2(1-b)} = \frac{1}{2-2b}$, to show this concretely notice that for $b < -1$:

   $$3 - b < 2 - 2b \iff 1 - b < -2b \iff \frac{1}{-b} + 1 < 2,$$

with this last statement being true we must have $3-b<2-2b$ hence $A(b,1-b)<A(b,3-b)$. Moreover, notice that it was at these two points that the most area was accumulated in the preceding intervals. Implying that the supremum over t of $A(b,t)$ must occur either when $t=3-b$ or $t=1-b$. By the preceding argument, we have then that $h^*(b)=\dfrac{1}{3-b}$ for all $-\infty<b<-1$.

2. $(4<b<\infty)$

   Notice that in this case, we can apply the same reasoning as in case 1. Except the maximum will occur when $b-t=0$ giving us the value of $\dfrac{1}{2b}(1+1)=\dfrac{1}{b}$.

3. $(-1\le b\le 0)$

   For $b\in[-1,0]$, note that the value of $A(b,t)$ will be 0 until $b+t>0$. In fact for $b+t\le 1$ we'll have
$$\frac{1}{2t}\int_0^{b+t}1=\frac{(b+t)}{2t}=\frac{b}{2t}+\frac{1}{2},$$
   since $b\le 0$ we'll have that as t increases $b+t$ will increase. So that $A(b,t)$ for $1\ge b+t>0$ will attain a maximum at $b+t=1$; that is when $t=1-b$. Now, note that when $1<b+t\le 2$, $A(b,t)$ will only decrease, so we'll consider when $b+t>2$. Consider now $2<b+t\le 3$
$$\frac{1}{2t}(1+\int_2^{b+t}1)=\frac{b+t-1}{2t}=\frac{b}{2t}+\frac{1}{2}-\frac{1}{2t},$$
   we'll also have that since $b\le 0$ that this quantity will only increase $b+t\le 3$ increases. Implying a maximum occurs for $2<b+t\le 3$ occurs when $b+t=3$, with a maximum of $A(b,3-b)=\dfrac{1}{3-b}$. Now notice the similarity to the first case, however here we have that $-1\le b\le 0$. So consider
$$2-2b\le 3-b \iff -b\le 1,$$
   this last statement is certainly true for $-1\le b\le 0$, giving us that $A(b,1-b)\ge A(b,3-b)$. By our previous arguments, we've seen that these are the only two possibly points for the maximum. The above inequality implies $h^*(b)=\dfrac{1}{2(1-b)}$ for all $b\in[-1,0]$.

4. $(3\le b\le 4)$

   Notice that the same reasoning will apply to this case as the previous, save for $b+t$ being replaced with $b-t$ and the maximum occurring at $b-t=2$, giving us the value of $h^*(b)=\dfrac{1}{2(b-2)}$ for all $b\in[3,4]$.

5. $(0 < b < 1)$

   Notice that by our work in $4.A.4$ we showed the bound $h^*(b) \leq \sup_{\mathbb{R}} |h|$. Here $\sup_{\mathbb{R}} |h| = 1$. So notice that for any $b \in (0,1)$ we can choose $t > 0$ such that $b + t < 1$ and $0 < b - t$. So that $A(b,t) = 1$. Implying that $h^*(b) = 1$ for all $0 < b < 1$.

6. $(2 < b < 3)$

   By the same argument in the previous case we get the same value, $h^*(b) = 1$ for all $b \in (2,3)$.

7. $(1 \leq b \leq \frac{3}{2})$

   For such b, note that $A(b,t) = 0$ until $b - t < 1$. Since b is the left of $\frac{1}{2}$, this is guaranteed to occur before $b + t > 2$. So that for $b - t < 1$ notice that previous arguments show that $A(b,t)$ will increase as $t > 0$ increases. So then consider when both $b - t < 1$ and $b + t > 2$. Giving us

   $$A(b,t) = \frac{1}{2t}\left(\int_{b-t}^{1} 1 + \int_{2}^{b+t} 1\right) = \frac{1 - b + t + b + t - 2}{2t} = \frac{2t-1}{2t}.$$

   Since $b > 0$, this quantity will only decrease as t increases, implying as soon as $b - t = 0$ we should stop increasing t. Giving us the value of $h^*(b) = \frac{2b-1}{2b}$ for all $b \in [1, \frac{3}{2}]$.

8. $(\frac{3}{2} \leq b \leq 2)$

   The same reasoning as the previous case applies here save for $b + t = 3$, giving us the value of $h^*(b) = \frac{5-2b}{2(3-b)}$ for all $\frac{3}{2} \leq b \leq 2$.

These values agree with our claim, this is the result we wished to prove! $\square$

4.A.8

Find a formula for the Hardy-Littlewood maximal function of the function $h : \mathbb{R} \to [0, \infty)$ defined by

$$h(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $h : \mathbb{R} \to [0, \infty)$ defined by

$$h(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

1. $(-\infty < b \leq 0)$

   Notice that for such b, the value of the integral will be 0 until $b + t < 0$. At which point we'll have

   $$\frac{1}{2t} \int_0^{b+t} x \, dx = \frac{1}{2t} \frac{x^2}{2} \bigg|_{x=0}^{b+t} = \frac{(b+t)^2}{4t} = \frac{b^2 + 2bt + t^2}{4t} = \frac{b^2}{4t} + \frac{b}{2} + \frac{t}{4}.$$

   Notice that b^2 is positive, moreover $\dfrac{b^2}{4t} < \dfrac{t}{4} \iff b^2 > t^2 < 0 \iff (b-t)(b+t) < 0$, this last statement is true for all $b + t > 0$, hence we wish to maximize $\dfrac{t}{4}$ primarily. This will occur when $b + t = 1 \implies t = 1 - b$. Giving us the maximum of $\dfrac{1}{4(1-b)}$. Hence $h^*(b) = \dfrac{1}{4(1-b)}$ for all $b \in (-\infty, -\dfrac{1}{2})$

2. $(\dfrac{1}{2} \leq b < 1)$

   Notice that in this case we can always choose a $b + t < 1$ and $0 < b - t$ so that

   $$\frac{1}{2t} \int_{b-t}^{b+t} x \, dx = \frac{1}{4t}((b+t)^2 - (b-t)^2) = \frac{4bt}{4t} = b,$$

   so that since this value doesn't depend on t, and the integral gains no value for $b+t > 1$ or $b - t < 0$, we can conclude that $h^*(b) = b$ for all $\dfrac{1}{2} < b < 1$.

3. $(b = 1)$

For this case consider when $1 - t > 0$, so that

$$\frac{1}{2t}\int_{1-t}^{1} x\ dx = \frac{1}{4t}(1 - (1-t)^2) = \frac{1 - (1^2 - 2t + t^2)}{4t} = \frac{2-t}{4}.$$

It should be clear this is a maximum when $t \to 0$, giving us the value $h^*(1) = \dfrac{1}{2}$.

4. $(b > 1)$

   For this b, this will be 0 until $0 < b - t < 1$. So that at this point we'll have:

$$\frac{1}{2t}\int_{b-t}^{1} x\ dx = \frac{1}{4t}(1^2 - (b-t)^2) = \frac{1}{4t}(1 - (b^2 - 2bt - t^2)) = \frac{1 - b^2 + 2bt - t^2}{4t}.$$

   We'll find the critical value of this over $t > 0$, taking the deriviative and setting this to zero:

$$\frac{b^2 - 1}{4t^2} - \frac{1}{4} = 0 \iff b^2 = t^2 - 1 \iff t = \pm\sqrt{b^2 - 1},$$

   we can reject the negative value giving an optimum at $t = \sqrt{b^2 - 1}$. We'll find by plugging this in that:

$$h^*(b) = \frac{1}{2(\sqrt{b^2 - 1})}\int_{\sqrt{b^2-1}}^{1} x\ dx = \frac{1}{4\sqrt{b^2 - 1}}(1 - (b^2 - 1)) = \frac{-b^2}{4\sqrt{b^2 - 1}}.$$

   This is wrong...

This covers all cases for $b \in \mathbb{R}$. $\square$

4.B.1

Suppose $f \in \mathcal{L}^1(\mathbb{R})$. Prove that

$$\lim_{t\downarrow 0} \frac{1}{2t}\int_{b-t}^{b+t} |f - f_{[b-t,b+t]}| = 0$$

for almost every $b \in \mathbb{R}$.

Proof. Suppose $f \in \mathcal{L}^1(\mathbb{R})$. Define $f_{[b-t,b+t]} = \dfrac{1}{2t}\int\limits_{b-t}^{b+t} f$.

Let $t \neq 0$. Consider the following use of the triangle inequality

$$\begin{aligned}\frac{1}{2t}\int_{b-t}^{b+t} |f - f_{[b-t,b+t]}| &= \frac{1}{2t}\int_{b-t}^{b+t} |f - f(b) + f(b) - f_{[b-t,b+t]}| \\ &\leq \frac{1}{2t}\int_{b-t}^{b+t} |f - f(b)| + \frac{1}{2t}\int_{b-t}^{b+t} |f(b) - f_{[b-t,b+t]}| \\ &= \frac{1}{2t}\int_{b-t}^{b+t} |f - f(b)| + \frac{(b+t-b+t)}{2t}|f(b) - f_{[b-t,b+t]}|,\end{aligned}$$

where we used the fact that since $t \neq 0$ and $f \in \mathcal{L}^1(\mathbb{R})$ to break up the integral in the first inequality. Finally taking the limit as $t \downarrow 0$ of the first and last line's:

$$\lim_{t\downarrow 0} \frac{1}{2t}\int_{b-t}^{b+t} |f - f_{[b-t,b+t]}| \leq \lim_{t\downarrow 0}\left(\frac{1}{2t}\int_{b-t}^{b+t} |f - f(b)| + |f(b) - f_{[b-t,b+t]}|\right),$$

by 4.21 the second term on the left goes to 0 and the first term goes to zero by the first version of Lebesgue's differentiation theorem. Giving us our result:

$$\lim_{t\downarrow 0} \frac{1}{2t}\int_{b-t}^{b+t} |f - f_{[b-t,b+t]}| = 0.$$

□

4.B.3

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function such that $f^2 \in \mathcal{L}^1(\mathbb{R})$. Prove that

$$\lim_{t \downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)|^2 = 0$$

for almost every $b \in \mathbb{R}$.

Lemma. *If $f^2 \in \mathcal{L}^1(\mathbb{R})$ and S is a bounded subset of $\mathbb{R}$, then $\chi_S f \in \mathcal{L}^1(\mathbb{R})$.*

Proof. Let $f^2 \in \mathcal{L}^1(\mathbb{R})$ and S be a bounded subset of $\mathbb{R}$. Define $A = \{x \in \mathbb{R} : |f(x)| \geq 1\}$ and $\mathbb{R} \setminus A = \{x \in \mathbb{R} : |f(x)| < 1\}$.

Note then for all $x \in A$ we'll have $|f(x)| \leq |f(x)|^2$ so that $\int_A |f| \leq \int_A |f|^2 < \infty$. This implies that $\int_A \chi_S |f| \leq \int_A \chi_S |f|^2 < \infty$. To show that $\chi_S |f|$ has a finite integral on the rest of $\mathbb{R}$, consider the following:

$$\begin{aligned}
\int_{\mathbb{R}\setminus A} \chi_S |f| &= \int_{\mathbb{R}} \chi_{\mathbb{R}\setminus A} \chi_S |f| \\
&= \int \chi_{(\mathbb{R}\setminus A)\cap S} |f| \\
&= \int_{\mathbb{R}\setminus A \cap S} |f| \\
&\leq \int_S 1 \\
&= |S| < \infty,
\end{aligned}$$

the first equality comes from the definition of the integral over a subset, the second equality comes from the definition of a characteristic function, the third equality again comes from the definition of the integral over a subset, and the inequality comes from the fact that $|f(x)| \leq 1$ for all $x \in \mathbb{R} \setminus A$. Hence combining $\int_A \chi |f| < \infty$ and $\int_{\mathbb{R}\setminus A} \chi_S |f| < \infty$ we have that $\chi_S f \in \mathcal{L}^1(\mathbb{R})$, this is our desired result. $\square$

Proof. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function such that $f^2 \in \mathcal{L}^1(\mathbb{R})$.

Let $0 < t < 1$, and $b \in \mathbb{R}$, then define $S = [b - t - 1, b + t + 1]$ so that $[b - t, b + t] \subseteq$

$[b-t-1, b+t+1]$,

$$
\begin{aligned}
\frac{1}{2t}\int_{b-t}^{b+t} |f - f(b)|^2 &\leq \frac{1}{2t}\int_{b-t}^{b+t} (f - f(b))^2 \\
&\leq \frac{1}{2t}\int_{b-t}^{b+t} (f^2 - 2f(b)f + f(b)^2) \\
&\leq \frac{1}{2t}\int_{b-t}^{b+t} (f^2 - 2f(b)f + f(b)^2) \\
&= \frac{1}{2t}\int_{b-t}^{b+t} (f^2 - 2f(b)f\chi_S + f(b)^2) \\
&= \frac{1}{2t}\left(\int_{b-t}^{b+t} f^2 - 2f(b)\int_{b-t}^{b+t} f\chi_S + \int_{b-t}^{b+t} f(b)^2\right) \\
&= \frac{1}{2t}\left(\int_{b-t}^{b+t} f^2 - 2f(b)\int_{b-t}^{b+t} \chi_S f + 2tf(b)^2\right) \\
&= \frac{1}{2t}\int_{b-t}^{b+t} f^2 - 2f(b)\frac{1}{2t}\int_{b-t}^{b+t} \chi_S f + f(b)^2,
\end{aligned}
$$

in the fourth line we used the fact that the function $f\chi_S = f$ on $[b-t, b+t]$, in the fifth line we used that fact that f is a finite-valued function (there's no $x \in \mathbb{R}$ such that $f(x) = \infty$) and that $\chi_S f \in \mathcal{L}^1(\mathbb{R})$ and $f^2 \in \mathcal{L}^1(\mathbb{R})$ to split up the integral in line 5. Now we'll take the limit as $t \downarrow 0$ of the last and first lines to give us our desired result:

$$
\lim_{t\downarrow 0} \frac{1}{2t}\int_{b-t}^{b+t} |f - f(b)|^2 \leq \lim_{t\downarrow 0}\left(\frac{1}{2t}\int_{b-t}^{b+t} f^2 - 2f(b)\int_{b-t}^{b+t} f + f(b)^2\right).
$$

Now we'll use the fact that $f(x) = (\chi_S f)(x)$ for all $x \in [b-t, b+t]$ and then $\chi_S f, f^2 \in \mathcal{L}^1(\mathbb{R})$ to finally use 4.21 to get:

$$
\begin{aligned}
\lim_{t\downarrow 0} \frac{1}{2t}\int_{b-t}^{b+t} |f - f(b)|^2 &\leq \lim_{t\downarrow 0}\frac{1}{2t}\int_{b-t}^{b+t} f^2 - 2f(b)\lim_{t\downarrow 0}\int_{b-t}^{b+t} \chi_S f + \lim_{t\downarrow 0} f(b)^2 \\
&= f^2(b) - 2f(b)(\chi_S f)(b) + f(b)^2 \\
&= 0.
\end{aligned}
$$

This is our desired result! $\square$

4.B.4

Prove that the Lebesgue Differentiation Theorem (4.19) still holds if the hypothesis that $\int\limits_{-\infty}^{\infty} |f| < \infty$ is weakened to the requirement that $\int\limits_{-\infty}^{x} |f| < \infty$ for all $x \in \mathbb{R}$.

Theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be an $\mathcal{S}-$measurable function. Suppose $\int\limits_{-\infty}^{x} |f| < \infty$ for all $x \in \mathbb{R}$. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \int_{-\infty}^{x} f.$$

Then $g'(b) = f(b)$ for almost every $b \in \mathbb{R}$. □

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be an $\mathcal{S}-$measurable function. Suppose $\int\limits_{-\infty}^{x} |f| < \infty$ for all $x \in \mathbb{R}$. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \int_{-\infty}^{x} f.$$

Then note that we have $\int\limits_{-\infty}^{x} |f| = \int\limits_{-\infty}^{\infty} \chi_{(-\infty,x)}|f| < \infty$, hence $\chi_{(-\infty,x)}f \in \mathcal{L}^1(\mathbb{R})$ for every $x \in \mathbb{R}$. Importantly, for any $b \in \mathbb{R}$ and $t \neq 0$, we can choose an $x \in \mathbb{R}$ such that $b + t < x$. Let $t \neq 0$. With that consider the following

$$\begin{aligned}
\left|\frac{g(b+t) - g(b)}{t} - f(b)\right| &= \left|\frac{\int_b^{b+t}(f - f(b)}{t}\right| \\
&\leq \frac{1}{t}\int_b^{b+t} |f - f(b)| \\
&\leq \frac{1}{t}\int_{b-t}^{b+t} |f - f(b)|
\end{aligned}$$

for all $b \in \mathbb{R}$. Since $\chi_{(-\infty,x)}f \in \mathcal{L}^1(\mathbb{R})$, we can adjust this without any change to the value in the following manner and let $b + t < x$ be fixed:

$$\frac{1}{t}\int_{b-t}^{b+t} |f\chi_{(-\infty,x)} - f(b)\chi_{(-\infty,x)}| = \frac{1}{t}\int_{b-t}^{b+t} \chi_{(-\infty,x)}|f - f(b)| = \frac{1}{t}\int_{b-t}^{b+t} |f - f(b)|,$$

so that by 4.10, this last quantity has a limit of 0 as $t \downarrow 0$ for almost every $b \in \mathbb{R}$. Thus $g'(b) = f(b)$ for almost every $b \in \mathbb{R}$. □

Homework #9

4.B.6

Prove that if $h \in \mathcal{L}^1(\mathbb{R})$ and $\int_{-\infty}^{s} h = 0$ for all $s \in \mathbb{R}$, then $h(s) = 0$ for almost every $s \in \mathbb{R}$.

Proof. Let $h \in L^1(\mathbb{R})$ and define $g : \mathbb{R} \to \mathbb{R}$ by $\int\limits_{-\infty}^{s} h$ for $s \in \mathbb{R}$. Let $\int\limits_{-\infty}^{s} h = 0$ for all $s \in \mathbb{R}$. Then by (4.19), the second version of Lebesgue's differentiation theorem tells us that $g'(x) = f(x)$ for almost every $x \in \mathbb{R}$. Notice that the derivative of $g(x) = 0$ is $g'(x) = 0$, hence $f(x) = 0$ for almost every $x \in \mathbb{R}$. $\square$