Joseph C. McGuire | MATH 530
Dr. Anthony Mendes | **Homework # 12** | October 19, 2024

56.

Prove that the coefficient of m_λ in h_μ is the number of matrices with nonnegative integer entries with row sum λ and column sum μ.

Proof. We'll show this by showing that we can form any monomial $x_1^{\lambda_1} \dots x_k^{\lambda_k}$ by multiplying out $h_\mu = h_{\mu_1} \dots h_{\mu_l}$. This last breaking up of the product works since we have that the rows are independent, so their weights can be broken up and seperated. We'll show this by the same method used to show this for h_μ's coefficient. Label the table with rows indexed by $x_1, \dots, x_k$ and columns indexed by $h_{\mu_1}, \dots, h_{\mu_l}$. Place a 1 in the x_i row and h_{u_j} column if you select the monomial from the h_{μ_j} to contribute to the final product, and 0 if you don't. The row sum's will give us the λ partition and the column sums will give us μ. Hence the ways to form these is the coefficient of m_λ in h_μ. The number of such ways to do this is exactly $\mathbb{Z}_2 M_{\lambda,\mu}$. This is our result! □

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57.

Let $\mu \vdash n$.

(a)

Use a similar proof as used to prove $h_\mu = \sum_{\lambda \vdash n} (-1)^{n-l(\lambda)} |B_{\lambda,\mu}| e_\lambda$ to prove $e_\mu = \sum_{\lambda \vdash n} (-1)^{n-l(\lambda)} |B_{\lambda,\mu}| h_\lambda$.

Proof. To show this combinatorially, we'll take an object corresponding to $(-1)^{n-l(\lambda)}|B_{\lambda,\mu}|h_\lambda$ and define a sign-reversing involution around this. These objects will correspond to a brick tabloid with a labeling inside of each brick being a weakly decreasing sequence. Define the sign of this object to be number of cells in brick tabloid subtracted by number of bricks in Tabloid. Then we'll define the same involution to proof the mentioned identity in the problem statement. That is an involution starting at the top row and scanning left to right look for bricks where the length is greater than or equal to 2 or brick length 1 and it's followed by another brick such that the integer labels weakly decrease. If the length of the brick is greater than or equal to 2, then chop off the first cell. If the length of the brick is 1 and the next brick will form a weakly decreasing sequence, glue them together. Otherwise leave it as a fixed point. The fixed points will then be all the brick tabloids where all bricks are of length 1 and not weakly decreasing, hence strictly increasing. Reflecting this tableaux gives us exactly an object counted by e_μ. Thus we have our result! □

(b)

Let $p(n)$ be the number of integer partitions of n and let B_n be the $p(n) \times p(n)$ matrix with row μ, column λ entry equal to $(-1)^{n-l(\lambda)}|B_{\lambda,\mu}|$. Why does part $(a.)$ imply $B_n^{-1} = B_n$?

Proof. First, note that since $\{e_\lambda : \lambda \vdash n\}$ is a basis of Λ_n, the above shows us that $\{h_\lambda : \lambda \vdash n\}$ is also a basis of Λ_n. So that this matrix is the transition matrix from a basis of e_λ to h_λ. So that since (a) tells us that converting to the respective bases is the same process, we can deduce that the transition matrix is its own matrix. This is our result! $\square$

58.

A weighted brick tabloid of content λ and shape μ is the usual brick tabloid of content λ and shape μ but with one cell in the final brick in each row shaded. Let $WB_{\lambda,\mu}$ be the set of all weighted brick tabloid of content λ and shape μ.

(c)

By counting weighted brick tabloids, find the 5×5 matrix with row and columns indexed by integer partitions of 4 and with row μ and column λ entry equal to $(-1)^{n-l(\lambda)}|WB_{\lambda,\mu}|$. Why does this matrix verify that $\{p_\lambda : \lambda \vdash 4\}$ is a basis for Λ_4? More generally, why is $\{p_\lambda : \lambda \vdash n\}$ a basis for Λ_n?

Proof. The table can be seen in figure 1 . Note that by $(57)(b)$ we have that $\{e_\lambda : \lambda \vdash n\}$ is a basis for Λ_n, so that since this matrix is upper-triangular and that this matrix is the coefficients given from (a) we have that $\{p_\lambda : \lambda \vdash 4\}$ is a basis of Λ_n because the transition in invertible. Since we can determine such a matrix for any $n \in \mathbb{N}$ it follows we can use this matrix to determine the representation of an element of Λ_n given in the basis of $\{e_\lambda : \lambda \vdash n\}$, and convert that to a span of $\{p_\lambda : \lambda \vdash n\}$. Hence $\{p_\lambda : \lambda \vdash n\}$ is a basis of Λ_n. $\square$

$(4), (3,1), (2^2), (2,1^2), (1^4)$

$\mu \backslash \lambda$	(4)	$(3,1)$	(2^2)	$(2,1^2)$	(1^4)
(4)	-4	4	2	-4	1
$(3,1)$	0	3	0	-3	1
(2^2)	0	0	$+4$	-4	1
$(2,1^2)$	0	0	0	-2	1
(1^4)	0	0	0	0	$+1$

59.

Prove these identities are true for $n \geq 1$ using bijections or sign reversing involutions.

(a)

$$p_n = \sum_{i=0}^{n-1} (-1)^i s_{(n-i,1^i)}.$$

Proof. We'll show this by an involution. Note that the right side of the equation can correspond to Column Strict Tableaux with shape $(n-i, 1^i)$. Define the sign to be determined by 1^i. Define the involution to be searching for the largest integer in appearing in T. If it appears in the first column, then the column has more than one cell, so move the integer to the right of the bottom row. If the largest integer appears at bottom row of T and m is larger than any cell in the first column, move m to the first column. Fixed points of this will have the largest integer in the bottom row and the first column of T. Furthermore T's first column has 1 cell. Hence T has 1 row, and since the bottom row must be weakly decreasing, we must have that T agree with the largest integer everywhere. That is T is row constant. This is an object in p_n. This is our result! $\square$

(b)

$$\sum_{i=0}^{n-1} h_i p_{n-i} = nh_n.$$

Proof. We'll show this by a bijection. We'll match objects represented by $h_i p_{n-i}$ to h_n, for all $i \in \{1, \ldots, n\}$. Note that, we can take a weakly decreasing tableaux, as represented by h_i. Adjoin the row constant tableaux to two strictly decreasing elements in h_i, otherwise adjoin it to the tableaux so that it remains weakly decreasing. We can undo this process by undoing the process we just defined. Moreover, this new tableaux is a weakly decreasing tableaux of length n. Hence we get that this pairing is bijective. Meaning we get that for all $h_i p_{n-i}$ there's a corresponding h_n for all $i \in \{1, \ldots, n\}$. Hence we have our result! $\square$

60.

Let $p_n = p_{(n)}(x_1, \ldots, x_N)$ be the power symmetric polynomial in $x_1, \ldots, x_N$, let $h_n = h_{(n)}(x_1, \ldots, x_N)$ be the homogeneous symmetric polynomial, and let $H(t) = \sum_{n=0}^{\infty} h_n t^n$. Show $\sum_{n=1}^{\infty} \frac{p_n}{n} t^n = \ln(H(t))$ and $\sum_{n=1}^{\infty} p_n t^n = \frac{tH'(t)}{H(t)}$.

Proof. We'll show the latter equality:

$$\begin{aligned}
H(t)\sum_{n=1}^{\infty} p_n t^n &= \left(\sum_{n=1}^{\infty} h_{n-1}t^{n-1}\right)\left(\sum_{n=1}^{\infty} p_n t^{n-1}\right)t \\
&= t\sum_{n=1}^{\infty}\sum_{k=1}^{n} h_{k-1}p_{n-k+1}t^{n-1} \\
&= t\sum_{n=1}^{\infty}\sum_{k=0}^{n-1} h_k p_{n-k}t^{n-1} \\
&= t\sum_{n=1}^{\infty} nh_n t^{n-1} \\
&= tH'(t),
\end{aligned}$$

note the second to last equality came from reindexing and also noting that $p_0 = 0$. So we'll show the other equality by noting an initial condition of $H(t)$ as $H(0) = 1$. Hence solving the ODE

$$\sum_{n=1}^{\infty} p_n t^{n-1} = \frac{H'(t)}{H(t)}$$

as a seperable one:

$$\begin{aligned}
\int \sum_{n=1}^{\infty} p_n t^{n-1}\ dt &= \ln(H(t)) + C \\
\sum_{n=1}^{\infty}\int p_n t^{n-1}\ dt &= \ln(H(t)) + C \\
\sum_{n=1}^{\infty} \frac{p_n}{n} t^n &= \ln(H(t)) + C,
\end{aligned}$$

imposing our condition gets us our result:

$$\sum_{n=1}^{\infty} \frac{p_n}{n} t^n = \ln(H(t)).$$

$\square$