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Group Project #1

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Problem

Give an example of a non-Riemann integrable function, that is unbounded.

Solution. Consider the function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} p & \text{if } x \in \mathbb{Q} \setminus \{0\} \text{ and } x = \frac{p}{q} \text{ with this being in irreducible terms} \\ 0 & \text{otherwise} \end{cases}$$

This function is well-defined on $[0,1]$ because $x \in \mathbb{Q}$ has a unique representation in terms of irreducible integers p, q so that $x = \frac{p}{q}$.

Then take P to be any partition of $[0,1]$. Sense we have $\mathbb{R} \setminus \mathbb{Q}$ is dense in $[0,1]$ we have any infinite subinterval $[a,b] \subset [0,1]$ we'll have at least one $x \in [a,b]$ such that $f(x) = 0$. So for any partition P we'll have: $L(f, P, [0,1]) = \sum_{j=1}^{n} \inf_{[x_{j-1}, x_j]} f \ (x_j - x_{j-1}) = \sum_{j=1}^{n} 0(x_j - x_{j-1}) = 0$. So this is true for any partition P, hence $L(f, [0,1]) = 0$.

Note then for any $n \in \mathbb{N}$ we'll have $\frac{n}{n+1} \in [0,1]$. So $f\left(\frac{n}{n+1}\right) = n$, but $\lim_{n \to \infty} n = \infty$ and $\lim_{n \to \infty} \frac{n}{n+1} = 0$. So for $x_1 \in P$, being the first partition point after $x_0 = 0$, we'll have at some point $\frac{n}{n+1} \in [x_0 = 0, x_1]$. Hence $\sup_{x \in [x_0, x_1]} f = \infty$, hence we'll have $U(P, f, [0,1]) = x_1 \sup_{x \in [x_0, x_1]} f + \sum_{j=2}^{m} (x_j - x_{j-1}) \sup_{x \in [x_{j-1}, x_j]} f = \infty$. Since this is true for any partition P of $[0,1]$ we'll have $U(f, [0,1]) = \infty$ and hence $L(f, [0,1]) = 0 \neq \infty = U(f, [0,1])$ so that f isn't Riemann integrable, even though there are only a countable number of discontinuities. Riemann integration fails here, because f is unbounded on $[0,1]$. □