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Homework # 14

67.

How many ways are there to color the edges of of a cube using N colors (two coloring's are the same if the cube can be rotated to turn one coloring into another)? How many ways are there to color the vertices if 6 edges must be red and 6 must be black?

Proof. Call the group of rotations on a cube C_4. It should be clear that this is a group, since every rotation can be undone, and two rotations should produce another rotation, and we can remain still to produce the identity rotation. We begin by finding the cycle index polynomial of C_4:

$$Z_{C_4} = \frac{1}{|C_4|} \sum_{g \in C_4} p_{\lambda(g)}.$$

First, note that we have the identity rotation of the edges. The cycle type of this is (1^{12}) Note then that we can pair any vertex with another opposite vertex, there 8 such lines, rotating about these will leave no edges fixed and move the edges by 3. Since there are 12 edges in total, this gives us of p_3^4 with 8 of these. We can also fix an axes of rotation about opposite faces. There are 3 such pairs of axes. I can perform a quarter turn on each axes, this gives us p_4^3 since there are $12 edges$. There are 2 distinct quarter turns on this axes. But I can also perform a 180^o rotation about these axes, giving us 6 transpositions, so that we have p_2^6. Finally, I can draw an axes of rotation through pairs of edges, there are 6 such unique pairings. Then rotating about these edges will produce 2 fixed edges and 5 transpositions, giving us $p_1^2 p_2^5$. Combining all of these we find that there are 24 in total, with their cycle types giving us the cycle index polynomial of:

$$Z_{C_3} = \frac{1}{24}(p_1^{12} + 8p_3^4 + 6p_4^3 + 3p_2^6 + 6p_1^2 p_2^5).$$

This gives us there are $\frac{1}{24}(N^{12} + 8N^4 + 6N^3 + 3N^6 + 6N^7)$ total in-equivalent coloring's of the edges of a cube.

Now we need to find the coefficient of $x_1^6 x_2^6$ in the expansion of Z_{C_3}. For which we'll turn to Mathematica. Doing this with the coefficient function in Mathematica, we find there are in total:

$$\frac{1}{24}(924 + 8 \cdot 6 + 6 \cdot 0 + 3 \cdot 20 + 6 \cdot 20) = 48$$

in-equivalent ways of coloring a cube's edges with 6 being red and 6 being black. □

70.

Let C_n be the cyclic group of order n (the group generated by rotations of an n-sided regular polygon) and let D_n be the dihedral group of order $2n$ (the group generated by rotations and reflections of an n-sided regular polygon). Show that the cycle index polynomials for these groups are

$$Z_{C_n} = \frac{1}{n}\sum_{i=1}^{n}(p_{n/\gcd(i,n)})^{\gcd(i,n)} \text{ and } Z_{D_n} = \frac{1}{2}Z_{C_n} + \begin{cases} p_1 p_2^{(n-1)/2}/2 & \text{if n is odd,} \\ (p_2^{n/2} + p_1^2 p_2^{(n-2)/2}/4 & \text{if n is even.} \end{cases}$$

where $\gcd(i, n)$ is the greatest common divisor of i and n.

Proof. For any group G we have that the cyclic index polynomial generated by G is:

$$Z_G = \frac{1}{|G|}\sum_{g\in G} p_{\lambda(g)},$$

where $\lambda(g)$ is the cycle-type generated by the element g.

For C_n, we have $|C_n| = n$, by definition of C_n. Furthermore, it is cyclic, meaning that there exists some $a \in C_n$, so that if $g \in C_n$, then $g = a^k$ for some $k \in \{1, \dots, n\}$. Meaning that the sum over all elements of C_n can be index by the set $\{1, \dots, n\}$, iterating over a^k for k in this set.

To determine the cycle type of a^k, that is $\lambda(a^k)$, we'll use some abstract algebra facts. In particular, note that in a cyclic group the order of a^k is given by $n/\gcd(n, k)$; that is, $|a^k| = n/\gcd(n, k)$ for all $k \in \{1, \dots, n\}$. This gives us that $|a^k|\gcd(n, k) = n$ for all $k \in \{1, \dots, n\}$, this in conjunction with the interpretation of C_n's group action being rigid rotations causing "closed loops" from vertices being sent to other vertices imply that a^k splits into same sized cycles. In particular there will be $\gcd(n, k)$ of them, the length of each being $n/\gcd(n, k)$, since the lengths of the cycles must add up to n. Giving us for a given $a^k \in C_n$,

$$p_{\lambda(k)} = p_{(n/\gcd(n,k), n/\gcd(n,k), \dots, n/\gcd(n.k))} = (p_{n/\gcd(n,k)})^{\gcd(n,k)}.$$

Hence the cycle index polynomial generated by G is:

$$Z_{C_n} = \frac{1}{n}\sum_{k=1}^{n}(p_{n/\gcd(n,k)})^{\gcd(n,k)}.$$

For D_n, note that the definition of this group as given above we have $|D_n| = 2n$. Furthermore, this group is fully generated by $n+1$ elements. That is, the rotational generator discussed in the first part of this proof, and then the n total reflections that exist in D_n (noting that each reflection is its own inverse). So then the sum iterating over $g \in D_n$ can then be partitioned into two sums, one over reflections, the other over rotations. The rotational sum was found in the first part of this proof, so that we need to just discover the cycle index polynomial for the set of reflections in D_n.

Note that the reflections in D_n are based on the parity of n; that is, if n is odd, the axes of symmetries will be centered solely on the vertices of shape being acted upon by D_n, if n is even, the axes of symmetries will be through lines that pass through paired vertices as well as through lines that bisect the shape. For a better explanation of this see the attached drawings showing these for D_8 and D_9.

The result being when n is odd, the reflections will leave one vertex fixed and we'll have $(n-1)/2$ transpositions, giving us cycle type $(2^{(n-1)/2}, 1)$. There will be n of these as we saw with D_9. Giving us:

$$p_{\lambda(g)} = np_2^{(n-1)/2} \text{ when n is odd.}$$

With n being even, the reflections will be split as those that leave none of the vertices fixed and all of the actions are transpositions, so that this will have cycle type $(2^{n/2})$, and those that leave two elements fixed and transposing the $n-2$ remaining elements, this will have cycle type $(2^{(n-2)/2}, 1^2)$. As we see in the case of D_8, there will be exactly $\frac{n}{2}$ of these cycles types for the even permutations. This will give us:

$$p_{\lambda(g)} = \frac{n}{2}p_2^{n/2} + \frac{n}{2}p_1^2 p_2^{(n-2)/2} \text{ when n is even.}$$

Combining these gives us our result:

$$Z_{D_n} = \frac{1}{2}Z_{C_n} + \frac{1}{2}\begin{cases} p_1 p_2^{(n-1)/2} & \text{if n is odd,} \\ (p_2^{n/2} + p_1^2 p_2^{(n-2)/2}/2 & \text{if n is even.} \end{cases}$$

Distributing the 1/2 gives us the formula we wished to show. □

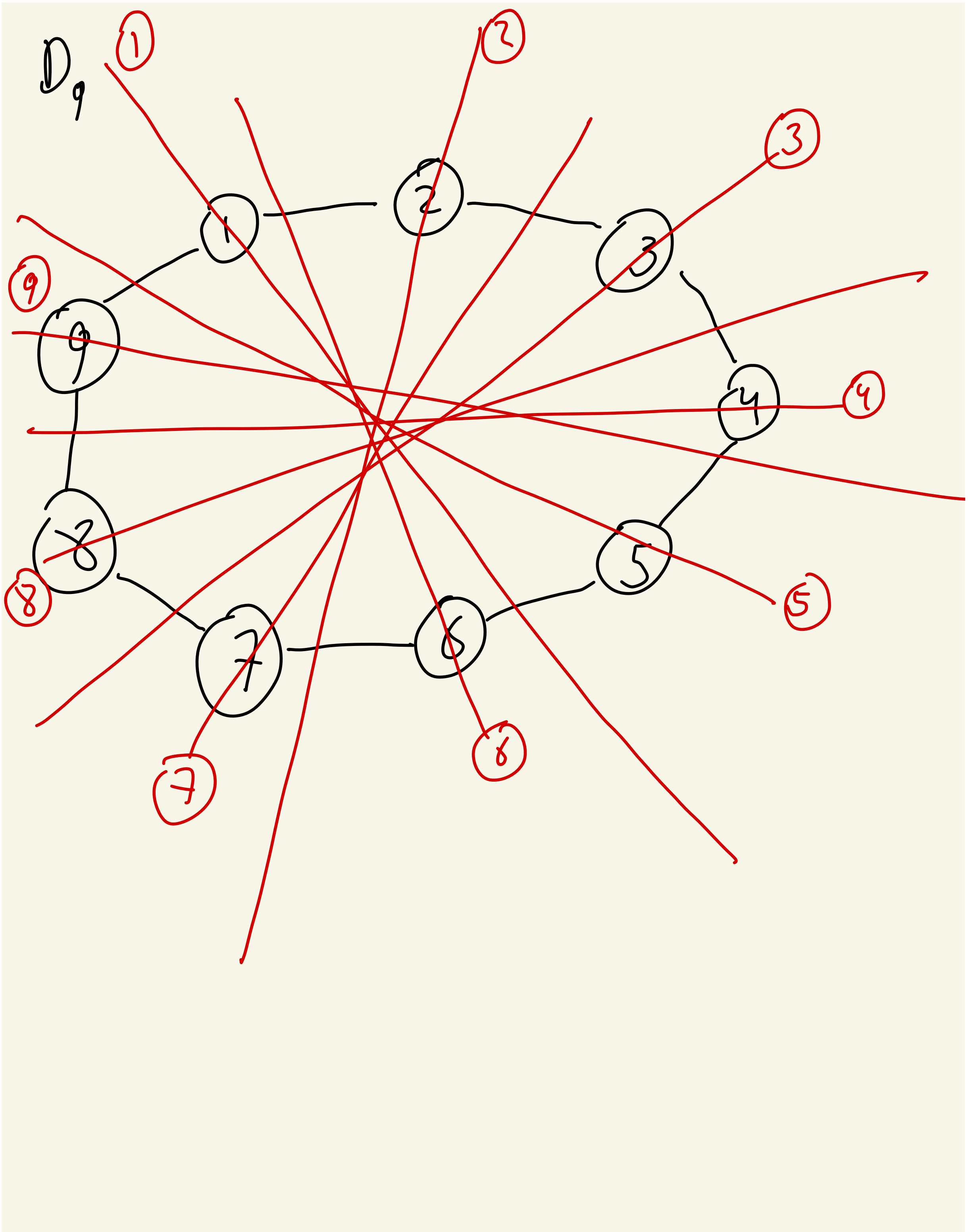

Figure 1: D_9

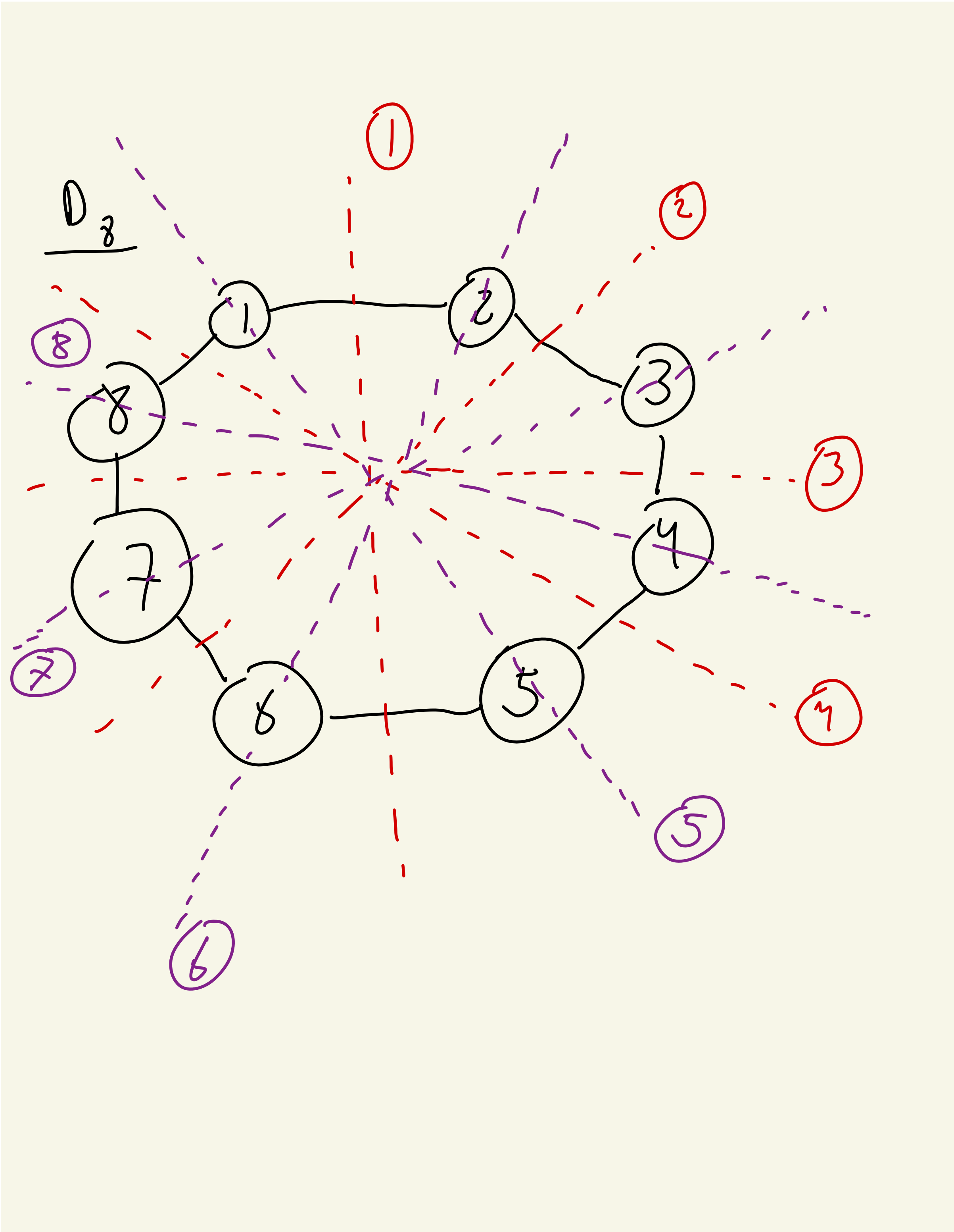

Figure 2: D_8

71.

Show the cycle index polynomial for the symmetric group S_n is the homogeneous symmetric function h_n.

Proof. Note the definition of the cycle index polynomial for a group G is:

$$Z_G = \frac{1}{|G|} \sum_{g \in G} p_{\lambda(g)}.$$

Here the order S_n is $n!$, additionally, since the group elements of S_n are just all of the permutations of $\{1, \ldots, n\}$, will will every single integer partition of n with $\lambda(g)$. That is, the summation can be indexed over integer partitions of n; $\lambda \vdash n$. For a given $\lambda \vdash n$, there will be multiple permutations that have the cycle type λ. But note that from reference set 6 (Mendes and Remmel page 11), we have already found that the number of permutations with this cycle type is exactly:

$$\frac{n!}{1_1^m 2^{m_2} \ldots m_1! m_2! \ldots} = \frac{n!}{z_\lambda},$$

where $\lambda = 1^{m_1} 2^{m_2} \ldots$ in multiplicative notation. Hence this gives us that that the cycle index polynomial for S_n is exactly:

$$Z_{S_n} = \frac{1}{n!} \sum_{\lambda \vdash n} \frac{n!}{z_\lambda} p_\lambda.$$

We will now show through a bijective proof that this corresponds to a homogeneous symmetric polynomial h_n. In particular, we'll show that $n! h_n = \sum_{\lambda \vdash n} \frac{n!}{z_\lambda} p_\lambda$.

Take a weakly increasing tableaux corresponding to h_n. Label each entry in the row with a unique integer from the set $\{1, 2, \ldots, n\}$. The ways to do this for a given weakly increasing single row tableaux is exactly $n! h_n$. We'll show that we can bijectively pair this with objects counted by $\sum_{\lambda \vdash n} \frac{n!}{z_\lambda} p_\lambda$. We define the bijection in this way, for the largest integer appearing in the weakly increasing sequence search for the largest label, this is guaranteed to exist by our unique labeling, at the cell labeled with the largest integer cut the row into two with everything the right of the cell, including the cell itself, and everything to the left of the cell. Repeat this process on the left block created by this cut, and continue this until there is nothing to the left. The labels of the separated blocks create a permutation in disjoint cycle notation. Reorder the created blocks into an tableaux of shape λ. Note the blocks created in this way will be row constant. That is for any given row constant tableaux created in this fashion, they will be labeled with the cycle type of λ and so there $\frac{n!}{z_\lambda}$ possible integer labelling associated with the Tableaux formed. This is counted exactly by the sum

$$\sum_{\lambda \vdash n} \frac{n!}{z_\lambda} p_\lambda.$$

To show that this is a bijective process, note that if we have a labeled row constant tableaux counted by this sum, that we can gather up all of the rows with entries i. To bring these back together into a single block, glue them together based on the order of the first labels that appear in each block. Recombining all of the blocks of i in a weakly increasing order, will create our original object. Hence this process is reversible and thus bijective. Therefore, we can conclude that

$$Z_{S_n} = \frac{1}{n!} \sum_{\lambda \vdash n} \frac{n!}{z_\lambda} p_\lambda = h_n.$$

This is the result we wished to prove! □