Joseph C. McGuire | MATH 530
Dr. Anthony Mendes | **Homework # 9** | October 19, 2024

43.

Use the function $x^2 \prod_{n=1}^{\infty}(1-x^n)^6$ to show that the number of integer partitions of $7n+5$ is divisible by 7.

Proof. Let $p(n) = \#$ of $(\lambda \vdash n)$. Then we'll show that $p(7n+5) \equiv_7 0$, for $n \in \mathbb{N}$.

Note that $\sum_{n=1}^{\infty} p(n-1)x^{n+1} = x^2 \sum_{n=0}^{\infty} p(n)x^n = x^2 \prod_{n=1}^{\infty} \frac{1}{1-x^n} = x^2 \prod_{n=1}^{\infty}(1-x^n)^6 \prod_{n=2}^{\infty}\left(\frac{1}{1-x^n}\right)^7.$
Define the function $F(x) = x^2 \prod_{n=1}^{\infty}(1-x^n)^6 = \sum_{k=0}^{\infty} a_n x^n$, then we have the following:

$$\begin{aligned}
\sum_{n=2}^{\infty} p(n-2)x^n &= \sum_{n=1}^{\infty} p(n-1)x^{n+1} \\
&= x^2 \prod_{n=1}^{\infty}(1-x^n)^6 \prod_{n=1}^{\infty}\left(\frac{1}{1-x^n}\right)^7 \\
&= F(x) \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^7} \\
&\equiv_7 F(x) \prod_{n=1}^{\infty} \frac{1}{1-x^{7n}} \\
&= \sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} p\left(\frac{n}{7}\right) x^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k p\left(\frac{n-k}{7}\right) x^n.
\end{aligned}$$

The congruence above came from the fact that $f(x^p) \equiv_p (f(x))^p$ for prime's p and generating functions f, the second to last equality comes from noting that $\prod_{n=1}^{\infty} \frac{1}{1-x^{7n}}$ is the generating function for integer partitions with parts that are divisible by 7 so we define $p\left(\frac{n}{7}\right) = 0$ whenever $n \not\equiv_7 0$, the final equality comes from the sum product theorem. Plugging in $7n$ for n and matching coefficients we see that:

$$p(7n-2) \equiv_7 p(7n+5) \equiv_7 \sum_{k=0}^{7n} a_k p\left(\frac{7n-k}{7}\right).$$

So to finish the proof we'll need to show that $a_k \equiv_7 0$ when $n \equiv_7 0$. So we need to determine the a_k's to do this:

$$
\begin{aligned}
F(x) &= x^2 \prod_{n=1}^{\infty}(1-x^n)^6 \\
&= \left(x \prod_{n=1}^{\infty}(1-x^n)^3 \right)^2 \\
&= \left(\sum_{n=0}^{\infty}(-1)^n(2n+1)x^{\binom{n+1}{2}+1} \right) \left(\sum_{k=0}^{\infty}(-1)^k(2k+1)x^{\binom{k+1}{2}} \right) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^n + k(2n+1)(2k+1)x^{\binom{n+1}{2}+\binom{k+1}{2}+1}.
\end{aligned}
$$

Note that testing the exponent $\binom{n+1}{2}+\binom{k+1}{2}+1$ for $n,k \in \{0,1,2,3,4,5,6\}$ computing these we get that the only n,k pair that gives us an exponent congruent to $0 \mod 7$ is $n = k = 3$. Additionally, we get that $(2n+1)(2k+1) \equiv_7 0$, hence $a_k \equiv_7 0$. So we get that $a_k \equiv_7 0$ whenever $k \equiv_7 0$. $\square$

44.

This exercise process a finite version of Jacobi's triple product identity.

a.

Prove $\prod_{i=1}^{n}(1+yq^{i-1})(1+y^{-1}q^{i}) = \sum_{k=-n}^{n} y^k q^{k(k-1)/2} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q$.

Proof. We'll show this by showing that the coefficient of y^k in $\prod_{i=1}^{n}(1+yq^{i-1})(1+y^{-1}q^{i})$ are $q^{k(k-1)/2} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q$.

First, note $\begin{bmatrix} 2n \\ n+k \end{bmatrix}_q$ counts the number of integer partitions of $2n$, $\lambda \vdash 2n$, with $\max(\lambda) \leq n+k$ and $l(\lambda) \leq n-k$. This result was obtained in Video 30 on Integer Partitions in a Box. So that the quantity $q^{k(k-1)/2} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q$ counts the number of integer partitions of $2n$, $\lambda \vdash 2n$, with $\max(\lambda) \leq n+k$ and $l(\lambda) \leq n-k$ with "stairs" of height $k-1$.

Following a similar but finite version of the process in Video 37, of tracing the diagonal of Young diagram and then partitioning that into two distinct integer partitions this produces two distinct integer partitions μ and λ, with maximum part being n, thus these correspond to the products:

$$\prod_{i=0}^{n}(1+yq^{i}) \prod_{i=1}^{n}\left(1+\frac{q^i}{y}\right).$$

Thus following the example of Video 37 on The Jacobi Triple Product, we can adjust this argument for $k<0$ with $\frac{k(k-1)}{2}$ becoming $\frac{|k|(|k|+1)}{2}$. Hence summing over the possible values of k (that is, only up to $k = \pm n$):

$$\prod_{i=1}^{n}\left(1+\frac{q^i}{y}\right) \prod_{i=0}^{n}\left(1+yq^{i}\right) = \sum_{k=-n}^{n} y^k q^{k(k-1)/2} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q.$$

$\square$

b.

Take $\lim\limits_{n\to\infty}$ of the above expression to find the full Jacobi triple product.

Proof. Note that the left hand side of the equation doesn't change other than the limits of the product so we'll work with the right hand side of the equation first. In particular the piece $\begin{bmatrix} 2n \\ n+k \end{bmatrix}_q$:

$$\begin{aligned}
\lim_{n\to\infty} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q &= \lim_{n\to\infty} \frac{\frac{(1-q^{2n})}{1-q}\frac{1-q^{2n-1}}{1-q}\cdots\frac{1-q}{1-q}}{\left(\frac{1-q^{n+k}}{1-q}\frac{1-q^{n+k-1}}{1-q}\cdots\frac{1-q}{1-q}\right)\left(\frac{1-q^{n-k}}{1-q}\frac{1-q^{n-k-1}}{1-q}\cdots\frac{1-q}{1-q}\right)} \\
&= \lim_{n\to\infty} \frac{q^{n+k+(n-k)}}{q^{2n}} \frac{\prod\limits_{i=n-k+1}^{n}(1-q^i)}{\prod\limits_{i=1}^{n-k}(1-q^i)} \\
&= \frac{1}{\prod\limits_{i=1}^{\infty}(1-q^i)}.
\end{aligned}$$

Where the last equality comes from the fact that the limits of product in the numerator are both going to ∞ so that the product is eliminated. The equation as $n \to \infty$ is:

$$\prod_{i=0}^{\infty}(1+yq^i)\prod_{i=1}^{\infty}(1+y^{-1}q^i) = \sum_{n=-\infty}^{\infty} y^n q^{n(n-1)/2} \frac{1}{\prod\limits_{i=1}^{\infty}(1-q^i)}$$

multiplying both sides by the infinite product on the right gives us the Jacobi Triple Identity:

$$\prod_{i=1}^{\infty}(1-q^i)\prod_{i=0}^{\infty}(1+yq^i)\prod_{i=1}^{\infty}(1+y^{-1}q^i) = \sum_{k\in\mathbb{Z}} y^k q^{k(k-1)/2}$$

□

46.

Let $F(x,y)$ be the function that satisfies the recursion $F(x,y) = F(x,xy) + xyF(x,x^2y)$ and $F(x,0) = 1$.

a.

Use Euler's device to show $F(x,y) = \sum_{n=0}^{\infty} \frac{y^n x^{n^2}}{(1-x)\ldots(1-x^n)}$.

Proof. We have $F(x,y) = F(x,xy) + xyF(x,x^2y)$, assuming $F(x,y) = \sum_{n=0}^{\infty} a_n(x)y^n$ will give us the following:

$$\begin{aligned}\sum_{n=0}^{\infty} a_n(x)y^n &= \sum_{n=0}^{\infty} a_n(x)x^n y^n + \sum_{n=0}^{\infty} a_n(x)x^{2n+1}y^{n+1} \\ &= \sum_{n=0}^{\infty} a_n(x)x^n y^n + \sum_{n=1}^{\infty} a_{n-1}(x)x^{2n-1}y^n \\ &= a_0(x) + \sum_{n=1}^{\infty} y^n(a_n(x)x^n + x^{2n-1}a_{n-1}(x)).\end{aligned}$$

Matching coefficients we find that $a_n(x) = a_n(x)x^n + x^{2n-1}a_{n-1}(x)$ for integers $n \geq 1$. Then $a_n(x) = \frac{x^{2n-1}a_{n-1}(x)}{1-x^n}$, upon iterating this we see that: $a_n(x) = \frac{x^{2n-1}}{1-x^n}\frac{x^{2n-3}}{1-x^{n-1}}a_{n-2}(x)$. Repeating this n times we'll get:

$$a_n(x) = \frac{x^{2n-1}x^{2n-3}\ldots x^3x^1a_0(x)}{(1-x^n)(1-x^{n-1})\ldots(1-x^2)(1-x)} = \frac{x^{(2n-1)+(2n-3)+\ldots+3+1}a_0(x)}{(1-x)(1-x^2)\ldots(1-x^n)}.$$

Using the fact that the first n odd numbers sum to n^2 we get our almost our result:

$$F(x,y) = \sum_{n=0}^{\infty} a_n(x)y^n = a_0(x) + \sum_{n=1}^{\infty} \frac{y^n x^{n^2}}{(1-x)\ldots(1-x^n)}.$$

To resolve $a_0(x)$ note that $F(x,0) = 1$ gives us: $F(x,0) = \sum_{n=0}^{\infty} a_n(x)0^n = a_0(x)0^0 + 0 + \ldots$. Assume that $0^0 = 1$, and this gives us $a_0(x) = 1$. Hence we have that $F(x,y) = \sum_{n=0}^{\infty} \frac{y^n x^{n^2}}{(1-x)\ldots(1-x^n)}$, our result! $\square$

b.

Use the result in the theorem on the next page to show $F(q,1) = \left(\sum_{k\in\mathbb{Z}} (-1)^k q^{k(5k-1)/2}\right) \prod_{i=1}^{\infty} \frac{1}{1-q^i}$.

(The function

$$F(x,y) = \left(1 + \sum_{n=1}^{\infty} (-1)^n y^{2n} x^{n(5n-1)/2} (1-yx^{2n}) \frac{(1-yx)\dots(1-yx^{n-1})}{(1-x)\dots(1-x^n)}\right) \prod_{i=1}^{\infty} \frac{1}{1-yx^i}$$

satisfies $F(x,y) = F(x,xy) + xyF(x,x^2y)$ and $F(x,0) = 1$.)

Proof. Consider the following calculations:

$$\begin{aligned}
F(q,1) &= \left(1 + \sum_{n=1}^{\infty} (-1)^n q^{n(5n-1)/2} (1-q^{2n}) \frac{(1-q)\dots(1-q^{n-1})}{(1-q)\dots(1-q^n)}\right) \prod_{i=1}^{\infty} \frac{1}{1-q^i} \\
&= \left(1 + \sum_{n=1}^{\infty} (-1)^n q^{n(5n-1)/2} \frac{(1-q^n)(1+q^n)}{1-q^n}\right) \prod_{i=1}^{\infty} \frac{1}{1-q^i} \\
&= \left(1 + \sum_{n=1}^{\infty} (-1)^n q^{n(5n-1)/2} (1+q^n)\right) \prod_{i=1}^{\infty} \frac{1}{1-q^i} \\
&= \left(1 + \sum_{n=1}^{\infty} (-1)^n q^{n(5n-1)/2} + \sum_{n=1}^{\infty} (-1)^{n+1} q^{(n(5n-1)+2n)/2}\right) \prod_{i=1}^{\infty} \frac{1}{1-q^i} \\
&= \left(1 + \sum_{n=1}^{\infty} (-1)^n q^{n(5n-1)/2} + \sum_{n=1}^{\infty} (-1)^{n+1} q^{n(5n+1)/2}\right) \prod_{i=1}^{\infty} \frac{1}{1-q^i}.
\end{aligned}$$

This is actually our result, notice that the "+1" accounts for $k = 0$, and the second series accounts for integers $k < 0$. Hence we can rewrite this as the thing we wanted to show:

$$F(q,1) = \left(\sum_{k\in\mathbb{Z}} (-1)^k q^{k(5k-1)/2}\right) \prod_{i=1}^{\infty} \frac{1}{1-q^i}.$$

□

c.

Take $y = -q^2$ and $x = q^5$ in Jacobi's triple product to show $F(q,1) = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$.

Proof. The Triple Jacobi Identity gives us:

$$(1+y)\prod_{n=1}^{\infty}(1-x^n)(1+yx^n)(1+\frac{x^n}{y}) = \sum_{k\in\mathbb{Z}} y^k x^{k(k-1)/2}.$$

Plugging in $y=-q^2$ and x^5 into the right-hand side:

$$\sum_{k\in\mathbb{Z}}(-1)^k q^{2k} q^{5k(k-1)/2} = (-1)q^{(4k+5k(k-1))/2} = \sum_{k\in\mathbb{Z}}(-1)^k q^{k(5k-1)/2}.$$

Multiplying in $\prod\limits_{i=1}^{\infty}\frac{1}{1-x^i}$ gives us $F(q,1)$ from part (b). The left hand-side can be manipulated as follows:

$$\begin{aligned}(1-q^2)\prod_{n=1}^{\infty}(1-q^{5n})(1-q^{5n+2})(1-q^{5n-2})\prod_{n=1}^{\infty}\frac{1}{1-q^n}\\ = \frac{(1-q^2)(1-q^3)(1-q^5)(1-q^7)\ldots}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)(1-q^6)(1-q^7)\ldots}\\ = \frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})\ldots}\\ = \prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}.\end{aligned}$$

By the Jacobi Triple Product identity we then have our result:

$$F(q,1) = \prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

□

d.

Show that (the number of integer partitions of n with parts differing by at least 2) is equal to (the number of integer partitions of n with parts congruent to $\pm 1 \mod 5$. (Hint: $1+3+5+\ldots+(2n-1)=n^2$.)

Proof. Taking the result from $(a.)$ and $(c.)$ we have that $F(q,1) = \sum\limits_{n=0}^{\infty}\frac{q^{n^2}}{(1-q)\ldots(1-q^n)} = \prod\limits_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$.

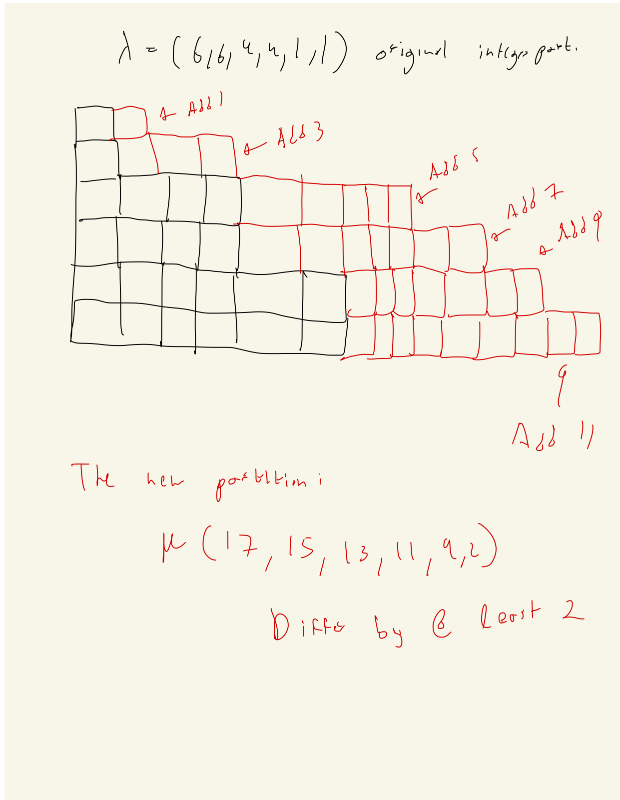

Figure 1: More poorly drawn Young diagrams

So we have that the product above has a generating function that counts the integer partitions of n with parts congruent to $\pm 1 \equiv_5 1, 4$, so that all we need to show is that the sum above counts integer partitions with parts that differ by at least 2. Take an integer partition with parts that are congruent to $\pm 1 \mod 5$, then we have a Young diagram whose parts differ by $0, 2, 4 \mod 5$. We'll then construct any integer partition whose parts differ by at least 2 by padding this integer partition in way illustrated in 1. That is, we pad this integer partition with odd numbers to produce an integer partition whose elements differ by at least 2, this is $q^{1+3+5+\ldots+(2n-1)} = q^{n^2}$ in the sum, along with existing Young Diagram this gives us that this counts generating function counts integer partitions whose parts differ by at least 2. Hence we have the number of integer partitions of n with parts differing by at least 2 is equal to the number of integer partitions of n with parts congruent to $\pm 1 \mod 5$. $\square$