550 GA 2

Montserrat Dabkowski, Joseph McGuire, and Karl Zieber

10/18/2020

Problems Picked: 2D # 17, 18.

2.D.14

Show that $\frac{1}{4}$ and $\frac{9}{13}$ are both in the Cantor set.

Proof. We'll show this by showing that there exists ternary (base 3) decimal representations of these such that there are no 1's occurring in them, from (2.75) we then have the two are in the Cantor set.

We'll build the ternary expansion for these two in the following way:

- Take the number $\frac{a}{b} \in \mathbb{Q}$;
- Multiply by our base 3, so that $\frac{a}{b} \cdot 3$;
- Write this product as $k + \frac{r}{b}$ for some $k \in \{0, 1, 2\}$ where $\frac{r}{b} < 1$
- Add k to the ternary decimal representation giving us: $0.k$
- Take the term $\frac{r}{b}$ and multiply by 3 and repeat the process above.

Doing this for $\frac{1}{4}$:

$$\begin{aligned}
\frac{3}{4} &= 0 + \frac{3}{4} \\
\frac{3}{4}(3) &= 2 + \frac{1}{4} \\
\frac{1}{4}(3) &= 0 + \frac{3}{4} \\
\frac{3}{4}(3) &= 2 + \frac{1}{4} \\
\vdots \quad & \quad \vdots
\end{aligned}$$

So from this chain gives us a ternary decimal representation:

$$\frac{1}{4} = \overline{02}_3.$$

This contains no 1's so that by (2.75) this implies $\frac{1}{4} \in C$.

Repeating this process for $\frac{9}{13}$ (with fractions we adjust this process by taking remainder of product)

$$\begin{aligned}
\frac{27}{13} &= 2 + \frac{1}{13} \\
\frac{3}{13} &= 0 + \frac{3}{13} \\
\frac{9}{13} &= 0 + \frac{9}{13} \\
\frac{27}{13} &= 2 + \frac{1}{13} \\
\vdots \quad & \quad \vdots
\end{aligned}$$

Giving us a ternary representation of $\frac{9}{13} = \overline{0.200}$. This too has no 1's in this representation, by (2.75) this implies $\frac{9}{13} \in C$. □

2.D.15

Show that $\frac{13}{17}$ is not in the Cantor set.

Proof. Note that $\frac{13}{17} \approx 0.7647\cdots$. Further, we can also calculate that

$$\frac{61}{81} \approx 0.7350\cdots, \quad \frac{62}{81} \approx 0.7654\cdots$$

This illustrates that $\frac{61}{81} < \frac{13}{17} < \frac{62}{81}$, that is,

$$\frac{13}{17} \in \left(\frac{61}{81}, \frac{62}{81}\right) \subseteq G_4$$

which implies $\frac{13}{17} \notin C$. □

2.D.16

List the eight open intervals whose union is G_4 in the definition of the Cantor set (2.74).

Proof. We'll start off with $[0,1] \setminus G_3 = [0, \frac{1}{27}] \cup [\frac{2}{27}, \frac{7}{27}] \cup [\frac{8}{27}, \frac{19}{27}] \cup [\frac{20}{27}, \frac{25}{27}] \cup [\frac{26}{27}, 1]$, we'll then break each interval into 3 closed intervals written with a denominator of $3^4 = 81$:

$$\begin{aligned}
[0, 1/27] &= [0, 1/81] \cup [1/81, 2/81] \cup [2/81, 3/81] \\
[2/27, 1/9] &= [6/81, 7/81] \cup [7/81, 8/81] \cup [8/81, 9/81] \\
[2/9, 7/27] &= [18/81, 19/81] \cup [19/81, 20/81] \cup [8/81, 9/81 \\
[8/27, 1/3] &= [24/81, 25/81] \cup [25/81, 26/81] \cup [26/81, 27/81] \\
[2/3, 19/27] &= [54/81, 55/81] \cup [55/81, 56/81] \cup [56/81, 57/81] \\
[20/27, 7/9] &= [60/81, 61/81] \cup [61/81, 62/81] \cup [62/81, 63/81] \\
[8/9, 26/27] &= [72/81, 73/81] \cup [73/81, 74/81] \cup [74/81, 75/81] \\
[26/27, 1] &= [78/81, 79/81] \cup [79/81, 80/81] \cup [80/81, 1].
\end{aligned}$$

G_4 will then be the union of the open middle intervals listed above giving us:

$$\begin{aligned}
G_4 = {} & (1/81, 2/81) \cup (7/81, 8/81) \cup (19/81, 20/81) \cup (25/81, 26/81) \\
& \cup (55/81, 56/81) \cup (61/81, 62/82) \cup (73/81, 74/81) \cup (80/81)
\end{aligned}$$

□

2.D.17

Proof. ($\subseteq$) Let $x \in \frac{1}{2}C + \frac{1}{2}C$. Thus $x = \frac{1}{2}a + \frac{1}{2}b$ for some $a, b \in C$. Since $C \subseteq [0,1]$, we know $0 \leq a, b \leq 1$, which implies $0 \leq \frac{1}{2}a, \frac{1}{2}b \leq \frac{1}{2}$. So $0 \leq \frac{1}{2}a + \frac{1}{2}b \leq 1$, thus $x = \frac{1}{2}a + \frac{1}{2}b \in [0,1]$.

($\supseteq$) Let $x \in [0,1]$. Consider x expanded in base-3 as a decimal. If $x = 1$, then since $1 \in C$, we have

$$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1 = x$$

so $x = 1 \in \frac{1}{2}C + \frac{1}{2}C$. Suppose $x \neq 1$, and we wish to find $a, b \in C$ such that $\frac{1}{2}a + \frac{1}{2}b = x$. Note then the number left of the decimal place is 0. So consider the i^{th} decimal place. There is either a 0, 1, or 2 there:

Case 1: Suppose the i^{th} decimal spot for x is a 0. Then let a, b be such that a and b both have 0 in the i^{th} decimal place. Note that when we divide by 2, a and b still have 0 in the i^{th} decimal place, and when we add them, their sum will also have a 0 in the i^{th} spot, as desired.

Case 2: Suppose the i^{th} decimal place for x is a 1. Then let a be such that it has a 2 in the i^{th} decimal spot an b be such that it has a 0 in the i^{th} decimal spot. Then $\frac{1}{2}a$ has a 1 in the i^{th} decimal spot and $\frac{1}{2}b$ has a 0 in the i^{th} decimal spot. So $\frac{1}{2}a + \frac{1}{2}b$ has a 1 in the i^{th} decimal spot, as desired.

Case 3: Suppose the i^{th} decimal spot of x is a 2. Then let a, b both be such that they have 2's in the i^{th} decimal spot. Then $\frac{1}{2}a$ and $\frac{1}{2}b$ both have 1's in the i^{th} decimal place, and so $\frac{1}{2}a + \frac{1}{2}b$ has a 2 in the i^{th} decimal place, as desired.

So, in all cases, we can entry-wise select appropriate $a, b \in C$ such that $x = \frac{1}{2}a + \frac{1}{2}b$, so $x \in \frac{1}{2}C + \frac{1}{2}C$. □

2.D.18

Proof. For this proof, we will consider the construction of C as we defined it in MATH 412, i.e., $C = \bigcap_{n=1}^{\infty} C_n$, where each C_n is a union of 2^n closed intervals with length $\frac{1}{3^n}$. So $x \in C$ implies $x \in C_n$ for all $n \in \mathbb{N}$. So for each n, x is in a closed interval I_n of length $\frac{1}{3^n}$. Define x_n to be the left endpoint of I_n, unless x itself is the left endpoint, in which case define x_n as the right endpoint of I_n. By construction, we have that $|x_n - x| \leq \frac{1}{3^n}$. Further, $\{x_n\} \subseteq C$ since the endpoints of the intervals are all in C (by MATH 412 facts). Further, $x_n \neq x$ for all $n \in \mathbb{N}$, and

$$\lim_{n \to \infty} x_n = x$$

by the fact that $\frac{1}{3^n} \to 0$ as $n \to \infty$, coupled with OLT. So x is a limit point of C.

This shows that any point in C is a limit point of C. So if we have an open interval A that contains some $x \in C$, by the definition of open, there exists some $\epsilon > 0$ such that $B(x, \epsilon) \subseteq A$. Since x is a limit point, there is a sequence $(x_n) \to x$ where $x_n \in C$ and $x_n \neq x$ for all $n \in N$, and by the definition of convergence, there exists some $N \in \mathbb{N}$ such that $n \geq N$ implies $|x - x_n| < \epsilon$, that is, $x_n \in B(x, \epsilon)$. So there is an infinite number of points in $C \cap B(x, \epsilon)$, and since $B(x, \epsilon) \subseteq A$, there are also an infinite number of points in $C \cap A$ as desired. $\square$