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Homework #1

MATH 550
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1.A: #2

Suppose $a \leq s < t \leq b$. Define $f : [a,b] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } s < x < t, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is Riemann integrable on $[a,b]$ and that $\int_a^b f = t - s$.

Proof. We'll use the fact that $U(f,[a,b]) \leq U(P,f,[a,b])$ and $L(P,f,[a,b]) \leq L(f,[a,b])$ to show that $U(f,[a,b]) = L(f,[a,b]) = t - s$.

Consider the partition P of $[a,b]$ with points $x_0 = a, x_1 = s, x_2 = t, x_3 = b$. Then we have

$$U(P,f,[a,b]) = 0(s-a) + 1(t-s) + 0(b-t) = t - s.$$

So $t - s \geq U(f,[a,b])$.

Then let $\epsilon > 0$ be large enough so that $a < s - \frac{1}{\epsilon} < s + \frac{1}{\epsilon} < t - \frac{1}{\epsilon} < t + \frac{1}{\epsilon} < t$. Let this be are partition of $[a,b]$, call it P_ϵ; namely, $a, s - \frac{1}{\epsilon}, s + \frac{1}{\epsilon}, t - \frac{1}{\epsilon}, t + \frac{1}{\epsilon}, b$. Then:

$$\begin{aligned} L(P_\epsilon, f, [a,b]) &= (s - \frac{1}{\epsilon} - a)0 + (s + \frac{1}{\epsilon} - (s - \frac{1}{\epsilon}))0 + (t - \frac{1}{\epsilon} - (s + \frac{1}{\epsilon}))1 \\ &+ (t + \frac{1}{\epsilon} - (t - \frac{1}{\epsilon}))0 + (b - t - \frac{1}{\epsilon})0 \\ &= t - s - \frac{2}{\epsilon} \end{aligned}$$

For any $\epsilon > 0$ with the previously stated conditions this is a valid partition of $[a,b]$, giving us:

$$\sup_{\epsilon > 0} t - s - \frac{2}{\epsilon} = t - s \leq L(f,[a,b])$$

Thus $t - s \leq L(f,[a,b]) \leq U(f,[a,b]) \leq t - s$. Hence $L(f,[a,b]) = U(f,[a,b]) = t - s$ so f is Riemann integrable with $\int\limits_a^b f = t - s$. $\square$

1.A: # 6

Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable. Suppose $g : [a,b] \to \mathbb{R}$ is a function such that $g(x) = f(x)$ for all except finitely many $x \in [a,b]$. Prove that g is Riemann integrable on $[a,b]$ and

$$\int_a^b g = \int_a^b f$$

Proof. Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function, and $g : [a,b] \to \mathbb{R}$ be a function such that $f(x) = g(x)$ for all $x \in [a,b] \backslash \{d_1, ..., d_n\}$ where d_i are the points where $f(x) \neq g(x)$.

Then since f is Riemann integrable we have f is bounded, by definition. So there exists a $M_1 \in \mathbb{R}$ such that $M_1 \geq |f(x)|$ for all $x \in [a,b]$. Moreover since f and g agree everywhere except for $x_1, \ldots, x_n$, and $M_2 = \sup\{|g(x_1)|, ..., |g(x_n)|\}$ exists since any finite set has a supremum and a infimum. Hence $|g(x)| \leq \sup\{M_1, M_2\}$ for all $x \in [a,b]$ and g is bounded. This is important, since we can now use the ϵ-definition of Riemann integrability to show g is Riemann integrable. That is, we'll show for all $\epsilon > 0$ there exists a partition P_ϵ such that $U(P_\epsilon, g, [a,b]) - L(P_\epsilon, g, [a,b]) < \epsilon$.

Since f is Riemann integrable, we have for all $\epsilon > 0$, there exists a P_ϵ such that $U(P_\epsilon, f, [a,b]) - L(P_\epsilon, f, [a,b]) < \epsilon$.

Let $\epsilon > 0$ and choose the partition P_ϵ such that $U(P_\epsilon, f, [a,b]) - L(P_\epsilon, f, [a,b]) < \frac{\epsilon}{2}$. Then we'll break the partition into two sets, note that for each d_i there are elements $x_{j_i} \in P$ such that $x_{j_i - 1} \leq d_i \leq x_{j_i}$, let the set A denote the set of all least such x_{j_i}, i.e $A = \{x_{j_1}, \ldots, x_{j_n}\}$ for any partition P. Let $A^C = P \setminus A$, for any partition P.

Then define the refinement $Q_\epsilon \supseteq P_\epsilon$ such that for all $x_{j_i} \in A$ we have $x_{j_i} - x_{j_i - 1} = \frac{1}{2(M-m+1)\#A}$ where $M \geq g(x) \geq m$ for all $x \in [a,b]$ (g is bounded) and $\#A$ denotes the number of elements in A (A is finite and non-empty so $\#A > 0$).

Then we have the following:

$$\begin{aligned}
U(Q_\epsilon, g, [a,b]) - Ł(Q_\epsilon, g, [a,b]) &= \sum_{x_j \in A \cup A^C} \sup_{[x_{j-1},x_j]} (g(x))(x_j - x_{j-1}) - \sum_{x_j \in A \cup A^C} \inf_{[x_{j-1},x_j]} (g(x))(x_j - x_{j-1}) \\
(A, A^C \text{ are disjoint}) &= \sum_{x_j \in A} \sup_{[x_{j-1},x_j]} (g(x))(x_j - x_{j-1}) - \sum_{x_j \in A} \inf_{[x_{j-1},x_j]} (g(x))(x_j - x_{j-1}) \\
&+ \sum_{x_j \in A^C} \sup_{[x_{j-1},x_j]} (g(x))(x_j - x_{j-1}) - \sum_{x_j \in A^C} \inf_{[x_{j-1},x_j]} (g(x))(x_j - x_{j-1}) \\
&= \frac{\epsilon}{2(M-m+1)\#A} \sum_{x_j \in A} \sup_{[x_{j-1},x_j]} (g(x)) \\
&- \frac{\epsilon}{2(M-m+1)\#A} \sum_{x_j \in A} \inf_{[x_{j-1},x_j]} (g(x)) \\
(f \text{ agrees with } g \text{ outside of } A) &+ \sum_{x_j \in A^C} \sup_{[x_{j-1},x_j]} (f(x))(x_j - x_{j-1}) - \sum_{x_j \in A^C} \inf_{[x_{j-1},x_j]} (f(x))(x_j - x_{j-1}) \\
&\leq \frac{\epsilon M \#A}{2(M-m+1)\#A} - \frac{\epsilon m \#A}{2(M-m+1)\#A} \\
(A^C \text{ is a subset of } P_\epsilon) &+ U(P_\epsilon, f, [a,b]) - L(P_\epsilon, f, [a,b]) \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Thus $U(Q_\epsilon, g, [a,b]) - L(Q_\epsilon, g, [a,b]) < \epsilon$. Hence by 1.3, g is Riemann integrable on $[a,b]$. □

1.A: # 7

Suppose $f : [a,b] \to \mathbb{R}$ is a bounded function. For $n \in \mathbb{Z}^+$, let P_n denote the partition that divides $[a,b]$ into 2^n intervals of equal size. Prove that

$$L(f, [a,b]) = \lim_{n\to\infty} L(f, P_n, [a,b]) \text{ and } U(f, [a,b]) = \lim_{n\to\infty} U(f, P_n, [a,b])$$

Proof. Let $f : [a,b] \to \mathbb{R}$ be a bounded function and For $n \in \mathbb{Z}^+$, let P_n denote the partition that divides $[a,b]$ into 2^n intervals of equal size. Additionally for any partition P, let $\|P\| = \sup\{x_j - x_{j-1} : x_j \in P\}$. Finally, let $\mathbb{N} = \mathbb{Z}^+$.

First, note that the lower and upper sums with this partition P_n can be written as $\sum_{j=1}^{2^n} \inf_{[x_{j-1},x_j]} f \frac{1}{2^n}$ for any $n \in \mathbb{N}$ and $\sum_{j=1}^{2^n} \sup_{[x_{j-1},x_j]} f \frac{1}{2^n}$, respectively. Moreover, since $L(f, P_{n+1}, [a,b])$ and $U(f, P_{n+1}, [a,b])$

produce finer refinements for all $n \in \mathbb{N}$, we have the sequence $\{L(f, P_n, [a,b])\}_{n=1}^{\infty}$ and $\{U(f, P_n, [a,b])\}_{n=1}^{\infty}$ are increasing and decreasing sequences, respectively. And since f is bounded on $[a,b]$, it's lower and upper sums are bounded, meaning that $\{L(f, P_n, [a,b])\}_{n=1}^{\infty}$ and $\{U(f, P_n, [a,b])\}_{n=1}^{\infty}$ will converge to their respective supremum and infimum.

Then note that $\lim\limits_{n\to\infty} L(f, P_n, [a,b]) \leq \sup\limits_{P} L(f, P, [a,b])$, because if $\lim\limits_{n\to\infty} L(f, P_n, [a,b]) > \sup\limits_{P} L(f, P, [a,b])$, then since $\lim\limits_{n\to\infty} L(f, P_n, [a,b])$ is a cluster point of the set $\{L(f, P_n, [a,b]) : n \in \mathbb{N}\}$ we would have a partition P_m such that $\sup\limits_{P} L(f, P, [a,b]) < L(f, P_m, [a,b])$, a contradiction. So that $\lim\limits_{n\to\infty} L(f, P_n, [a,b]) \leq \sup\limits_{P} L(f, P, [a,b])$. Using similar reasoning we get that $\lim\limits_{n\to\infty} U(f, P_n, [a,b]) \geq \inf\limits_{P} U(f, P, [a,b])$.

To show the converse we'll show that $\lim\limits_{n\to\infty} L(f, P_n, [a,b]) = \sup\limits_{n\in\mathbb{N}} L(f, P_n, [a,b])$ is an upper bound of $\{L(f, P, [a,b]) : P \text{ is any partition of } [a,b]\}$.

Let P be any partition of $[a,b]$, then we can always define a refinement Q of P such that $\|Q\| \leq \frac{1}{2^m}$ for any $m \in \mathbb{N}$. This is just equivalent to saying we can make a refinement of P whose maximum width is arbitrarily small. So then we we'll have the following:

$$\begin{aligned}
L(f, P, [a,b]) &\leq L(f, Q, [a,b]) \\
&= \sum_{j=1}^{n} \inf_{[x_{j-1}, x_j]} f(x_j - x_{j-1}) \\
&\leq \sum_{j=1}^{n} \inf_{[x_{j-1}, x_j]} f\|Q\| \\
&\leq \frac{1}{2^m} \sum_{j=1}^{n} \inf_{[x_{j-1}, x_j]} f \\
&\leq \frac{1}{2^m} \sum_{j=1}^{2^m} \inf_{[x_{j-1}, x_j]} f \\
&\leq L(f, P_m, [a,b]) \\
&\leq \sup_{m\in\mathbb{N}} L(f, P_m, [a,b]) \\
&= \lim_{m\to\infty} L(f, P_m, [a,b])
\end{aligned}$$

Hence $\lim\limits_{m\to\infty} L(f, P_m, [a,b])$ is an upper bound of $\{L(f, P, [a,b] : P \text{ is any partition of } [a,b])\}$. Thus we have $\lim\limits_{n\to\infty} L(f, P_n, [a,b]) = \sup\limits_{P} L(f, P, [a,b]) = L(f, [a,b])$.

I'll go through the steps above to show that $\lim\limits_{n\to\infty} U(f, P_n, [a,b]) = \inf\limits_{n\in\mathbb{N}} U(f, P_n, [a,b])$ is a lower bound of $\{U(f, P, [a, b]) : P$ is any partition of $[a, b]\}$.

First, define $\|Q\|^* = \inf\{x_j - x_{j-1} : x_j \in Q\}$ for any partition Q of $[a, b]$.

Take P to be any partition of $[a, b]$, then note that because taking refinements Q of P causes $U(f, Q, [a, b])$ to get larger, we can produce a refinement such that $\|Q\|^* \geq \frac{1}{2^m}$ for some $m \in \mathbb{N}$. So consider the following:

$$\begin{aligned}
U(f, P, [a, b]) &\geq U(f, Q, [a, b]) \\
&= \sum_{j=1}^{n} \sup_{[x_{j-1}, x_j]} f(x_j - x_{j-1}) \\
&\geq \|Q\|^* \sum_{j=1}^{n} \sup_{[x_{j-1}, x_j]} f \\
&\geq \frac{1}{2^m} \sum_{j=1}^{2^m} \sup_{[x_{j-1}, x_j]} f \\
&= U(f, P_m, [a, b]) \\
&\geq \inf_{m \in \mathbb{N}} U(f, P_m, [a, b]) \\
&= \lim_{m \to \infty} U(f, P_m, [a, b])
\end{aligned}$$

Hence $\lim_{m \to \infty} U(f, P_m, [a, b])$ is a lower bound of $\{U(f, P, [a, b]) : P$ is any partition of $[a, b]\}$ so that we get our result:

$$\lim_{n \to \infty} U(f, P_n, [a, b]) = \sup_{P} U(f, P, [a, b]) = U(f, [a, b])$$

□

1.B # 2

Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Prove that f is Riemann integrable if and only if $L(-f, [a, b]) = -L(f, [a, b])$.

Proof. Let $f : [a, b] \to \mathbb{R}$ be a bounded function and Riemann integrable on $[a, b]$. Then we have $L(f, [a, b]) = U(f, [a, b])$.

Additionally, we have $-f$ is also Riemann integrable, since finite linear combinations of Riemann integrable functions are Riemann integrable. That is, if f, g are Riemann integrable, then $af + bg$ is also Riemann integrable for $a, b \in \mathbb{R}$. So we have $L(-f, [a, b]) = U(-f, [a, b])$.

Note that over any closed interval $[a, b]$, $\sup_{[a,b]}(-f) = -\inf_{[a,b]}(f)$, since the $\sup(-f)$ is the maximum such value of $-f$ on $[a, b]$ meaning that it's the smallest value of the function f on $[a, b]$ inverted by a sign. So then since $U(f, [a, b]) = L(f, [a, b]) = \sup_P L(f, P, [a, b])$ so $L(f, [a, b]) \geq L(f, P, [a, b])$ for any partition P and furthermore this is the least upper bound. With that consider the following:

$$\begin{aligned}
L(f, P, [a, b]) &\leq L(f, [a, b]) \\
-L(f, [a, b]) &\leq -L(f, P, [a, b]) \\
-L(f, [a, b]) &\leq -\sum_{j=1}^{n} \inf_{[x_{j-1}, x_j]} f(x_j - x_{j-1}) \\
-L(f, [a, b]) &\leq \sum_{j=1}^{n} - \inf_{[x_{j-1}, x_j]} f(x_j - x_{j-1}) \\
-L(f, [a, b]) &\leq \sum_{j=1}^{n} \sup_{[x_{j-1}, x_j]} -f(x_j - x_{j-1}) \\
&\leq U(-f, [a, b]) \\
&= L(-f, [a, b])
\end{aligned}$$

So $-L(f, [a, b]) \leq L(-f, [a, b])$, we'll show the other direction of the inequality to get equality:

$$\begin{aligned}
\sum_{j=1}^{n} \inf_{[x_{j-1}, x_j]} -f(x_j - x_{j-1}) &= -\sum_{j=1}^{n} \sup_{[x_{j-1}, x_j]} f(x_j - x_{j-1}) \\
&= -U(f, P, [a, b]) \\
&\leq -U(f, [a, b]) = -L(f, [a, b])
\end{aligned}$$

Since $-L(f, [a, b])$ is an upper bound of $L(-f, P, [a, b])$ we have $L(-f, [a, b]) \leq -L(f, [a, b])$. Thus $L(-f, [a, b]) = -L(f, [a, b])$ whenever f is Riemann integrable on $[a, b]$.

Conversely, assume $L(-f, [a, b]) = -L(f, [a, b])$. Then since f is bounded we have $L(f, [a, b]) \leq$

$U(f,[a,b])$. Moreover, for any partition P we have:

$$\begin{aligned}
U(f,[a,b]) &\leq U(f,P,[a,b]) \\
&= \sum_{j=1}^{n} \sup_{[x_{j-1},x_j]} f(x_j - x_{j-1}) \\
&= -\sum_{j=1}^{n} \inf_{[x_{j-1},x_j]} f(x_j - x_{j-1}) \\
&= -L(-f,P,[a,b]) \\
&= -(-L(f,P,[a,b])) = L(f,P,[a,b]) \\
&\leq L(f,[a,b])
\end{aligned}$$

Thus $L(f,[a,b]) = U(f,[a,b])$ hence f is Riemann integrable on $[a,b]$. $\square$

1.B # 5

Give an example of a sequence of continuous real-valued function $f_1, f_2, \ldots$ on $[0,1]$ and a continuous real-valued function f on $[0,1]$ such that

$$f(x) = \lim_{k\to\infty} f_k(x)$$

for each $x \in [0,1]$ but

$$\int_0^1 f \;\neq \lim_{k\to\infty} \int_0^1 f_k(x).$$

Solution. Consider the sequence of functions $\{f_n\}_{n=1}^{\infty}$ such that $f_n : [0,1] \to \infty$ given by $f_n(x) = (1+x)^{1/n}$. Then $\lim\limits_{n=1}(1+x)^{1/n} = 1$ for all $x \in [0,1]$. We can show this since $(1+x)^{1/n} \geq 1$ for all $x \in [0,1]$ and $n \in \mathbb{N}$ and the fact that $2^{1/n} \geq (1+x)^{1/n}$ for all $x \in [0,1]$ and $\lim\limits_{n\to\infty} 2^{1/n} = 1$.

So both $f_n(x)$ and $f(x) = 1$ are continuous on $[0,1]$ and so are integrable $[0,1]$ with $\int_0^1 f\, dx = 1$. But

$$\int_0^1 f_n(x) = \int_0^1 (1+x)^{1/n} = \frac{(1+x)^{n+1} n}{n+1}$$

and $\lim\limits_{n\to\infty} \frac{n}{n+1} = 0$ and $\lim\limits_{n\to\infty}(1+x)^{1/n} = 1$ so that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\; dx = 0 \neq \int_0^1 f(x)\; dx = 1$$

$\square$

2.A # 2

Suppose $A \subset \mathbb{R}$ and $t \in \mathbb{R}$. Let $tA = \{ta\} : a \in A$. Prove that $|tA| = |t||A|$. Assume that $0 \cdot \infty$ is defined to be 0.

Proof. Let $I = (a, b)$ then $cI = (ca, cb)$ so that $l(cI) = cb - ca = c(b - a)$. So for an open interval $|cI| = |c||I|$.

So let $I_1, I_2, ...$ be a sequence of open intervals whose union contains A. Then for any $t \in \mathbb{R}$ we have $tI_1, tI_2, \ldots$ is a set of open intervals whose union contains $|tA|$ Then clearly if $|tA| \leq \sum_{k=1}^{\infty} l(tI_k) \leq \sum_{k=1}^{\infty} |tI_k| = \sum_{k=1}^{\infty} |t||I_k| = |t| \sum_{k=1}^{\infty} |I_k|$. Taking the infimum of the above: $|tA| \leq |t||A|$. So show the over side of the inequality:

$$
\begin{aligned}
|t||A| &\leq |t| \sum_{k=1}^{\infty} l(I_k) \\
&\leq \sum_{k=1}^{\infty} |t||I_k| \\
&\leq \sum_{k=1}^{\infty} |tI_k|
\end{aligned}
$$

Finally taking the infimum of both sides we get $|t||A| \leq |tA|$.

Thus $|tA| = |t||A|$. □

2.A # 14

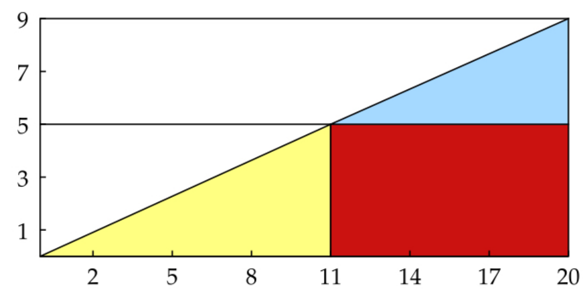
Solution. This is just a triangle with base 20 and height 9, so that $20(0.5)9 = 90$.

The two triangles areas are: $(9 - 5)(20 - 11)0.5 = 4(9)0.5 = 18$ and $(11)(5)0.5 = 27.5$, for the blue and yellow one's respectively. And the red rectangle has area $(20 - 11)5 = 45$. So adding those up we clearly get 90.5 in total.

This is a tricky question, because the slope of triangles differ very subtly. So we were in correct to assume the formula of a right triangle in part (a). Notice that the slope of the line at $(20, 9)$ is $\frac{9-5}{20-11} = \frac{4}{9}$, while at $(11, 5)$ it is $\frac{5-0}{11-0} = \frac{5}{11}$. So this is a trick of the eye, this isn't actually a right triangle. □

Homework #1

Consider the following figure, which is drawn accurately to scale.



(a) Show that the right triangle whose vertices are $(0,0)$, $(20,0)$, and $(20,9)$ has area 90.
[*We have not defined area yet, but just use the elementary formulas for the areas of triangles and rectangles that you learned long ago.*]

(b) Show that the yellow (lower) right triangle has area 27.5.

(c) Show that the red rectangle has area 45.

(d) Show that the blue (upper) right triangle has area 18.

(e) Add the results of parts (b), (c), and (d), showing that the area of the colored region is 90.5.

(f) Seeing the figure above, most people expect parts (a) and (e) to have the same result. Yet in part (a) we found area 90, and in part (e) we found area 90.5. Explain why these results differ.
[*You may be tempted to think that what we have here is a two-dimensional example similar to the result about the nonadditivity of outer measure* (*2.18*). *However, genuine examples of nonadditivity require much more complicated sets than in this example.*]