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Homework # 11

53.

Let RCS_λ denote the set of reverse column strict tablueax; that is, all tablueax where integer labeling weakly decreases in rows and strictly decreases up columns. Show that $s_\lambda = \sum_{RCS_\lambda} w(T)$ for any $\lambda \vdash n$.

Proof. To show this note that we've shown that $s_\lambda = \sum_{CS_\lambda} w(T)$ where CS_λ is the set of column strict tableux with shape $\lambda \vdash n$. So to show $s_\lambda = \sum_{RCS_\lambda} w(T)$, we'll show that there's a bijection between RCS_λ and CS_λ.

We'll define a bijective process, call it $\phi : RCS_\lambda \to CS_\lambda$ in the following way. Take an element of RCS_λ, replace each integer in the tableaux k and replace it with $n + 1 - k$. This will reverse the column strict decreasing to column strict increasing, and the weakly decreasing rows to weakly increasing. To show that this is a well-defined bijection between the two sets, note that the above operation can be performed on any tableaux with shape $\lambda \vdash n$ and will produce a unique result for any given tableaux. Hence the ϕ is well-defined from RCS_λ to CS_λ.

To show that ϕ^{-1} is well-defined from CS_λ to RCS_λ, note that we can undo this process by doing over again on any $\phi(T)$. That is, take a $\phi(T)$ tableaux, then this is a column strictly increasing and row weakly increasing tableaux. Take any k in this tableaux and replace it with $n+1-k$. This will turn $\phi(T)$ from a column strict tableaux to a reverse column strict tableaux, and moreover since we created $\phi(T)$ by this same process then this will simply undo ϕ and just leave us with T. Hence $\phi^{-1} : CS_\lambda \to RCS_\lambda$ is a well-defined map and is the inverse of ϕ.

So that $\phi : RCS_\lambda \to CS_\lambda$ is a bijective process and moreover an involution. Hence $s_\lambda = \sum_{CS_\lambda} w(T) = \sum_{RCS_\lambda} w(T)$ and we have our result! $\square$

54.

Prove that the power symmetric polynomial $p_\lambda(x_1, \ldots, x_N)$, the homogeneous symmetric polynomial $h_\lambda(x_1, \ldots, x_N)$, and the elementary symmetric polynomial $e_\lambda(x_1, \ldots, x_N)$ are indeed symmetric polynomials.

Proof. Note that we'll need to show that for any $\sigma \in S_n$ that $p_\lambda(x_1, \ldots, x_N) = p_\lambda(x_{\sigma(1)}, \ldots, x_{\sigma(N)})$ and similarly for the homogeneous and elementary polynomials. To do this though, note that every permutation $\sigma \in S_n$ is the product of transpositions, but moreover is the product of adjacent transpositions. So if we can show that for $p_\lambda(x_1, \ldots, x_p, x_{p+1}, \ldots, x_N) = p_\lambda(x_1, \ldots, x_{p+1}, x_p, \ldots, x_N)$ for any $p \in \{1, \ldots, N-1\}$ then we'll have that p_λ is symmetric and similarly for elementary and homogeneous. Moreover, this means that we need to bijectively map a tableaux with j occurrences of i and k occurrences of $i+1$ to a tableaux with k occurrences of i and j occurrences of $i+1$. So we'll show this for the power symmetric, homogeneous and elementary.

- **Power Symmetric Polynomial.** Take a row constant tableaux with j occurrences of i and k occurrences of $i+1$. By simply swapping each i with $i+1$ and each $i+1$ with i this tableaux will remain row constant and now have j occurrences of i and k occurrences of $i+1$. Moreover we can reverse this process by repeating it. Hence this is an involution that produces a row constant tableaux. Thus we have a bijective map that takes a row constant tableaux with j occurrences of i and k occurrences of $i+1$ and maps it to a row constant tableaux with k occurrences of i and j occurrences of $i+1$. Hence we have that p_λ is symmetric for any adjacent transposition, and hence must be symmetric for all $\sigma \in S_n$.

- **Homogeneous Symmetric Polynomial.** Similar to the previous case, take a row weakly increasing tableaux T of shape λ with j occurrences of i and k occurrences of $i+1$. Take all sequences of i and replace them with $i+1$, similarly take all the sequences of $i+1$ and replace them with i. Since for any sequence of i's, the element to the right of this sequence must be at least $i+1$, replacing the sequence with $i+1$ will not break the weakly increasing condition on the tableaux. The same reasoning holds for the sequences of $i+1$ except the greatest element to the left of any such sequence would be i. So by swapping all the occurrences of i with $i+1$ we get a weakly increasing tableaux, furthermore just repeat the process to undo it. Hence this process is reversible and hence bijective. Thus we have that the homogeneous symmetric polynomials remain the same after adjacent transpositions, and hence must be symmetric for any $\sigma \in S_n$.

- **Elementary Symmetric Polynomials.** Take any row increasing tableaux T with j

occurrences of i and k occurrences of $i + 1$. In this case we must adjust our strategy slightly. For case where i is the left of $i + 1$, leave these fixed. Otherwise, swap i and $i+1$. This will remain row increasing, since for occurrence of i the left adjacent element could at most be $i-1$ and the right adjacent must be at least $i+2$, by our restriction, hence swapping i with $i + 1$ will leave the tableaux to be row increasing. Similarly for $i+1$, except to the left the most the element could be is $i-1$, by our restriction, but the swap will leave it row increasing. Moreover, we can undo this with the same process applied again. Hence this process is reversible and thus bijective. So we can bijectively map a row increasing tableaux with j occurrences of i and k occurrences of $i + 1$ to a row increasing tableaux with k occurrences of i and j occurrences of $i + 1$. Hence e_λ remains the same under adjacent transpositions and hence must be symmetric for any $\sigma \in S_n$.

Thus we have shown that power, homogeneous and elementary symmetric polynomials are indeed symmetric polynomials. We have our result! $\square$

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55.

An alternating polynomial f in $x_1, \dots, x_n$ is a polynomial such that for all $\sigma = \sigma_1 \dots \sigma_n \in S_n$,

$$f(x_1, \dots, x_n) = \text{sign}(\sigma) f(x_{\sigma_1}, \dots, x_{\sigma_n}).$$

a.

Show that an alternating polynomial is divisible by $\Delta = \prod_{i<j} (x_i - x_j)$.

Proof. Take any transposition in S_n, namely $(i\ j)$ in cycle notation where $i < j$. Then by the definition of an alternating polynomial we have that $f(x_1, \dots, x_i, \dots, x_j, \dots, x_n) = -1 f(x_1, \dots, x_j, \dots, x_i, \dots, x_n)$; that is, any transposition has a sign of -1. Notice though that this means that for any $i < j$ that evaluating the above at $x_j = x_i$ we'll get:

$$f(x_1, \dots, x_i, \dots, x_i, \dots, x_n) = -f(x_1, \dots, x_i, \dots, x_i, \dots, x_n).$$

Hence we have that $(x_i - x_j)$ for all $i < j$ are factors of the alternating polynomial $f(x_1, \dots, x_n)$. Thus $\prod_{i<j} (x_i - x_j)$ is a factor of $f(x_1, \dots, x_n)$. $\square$

b.

Let $\mathcal{A}_k$ be the vector space of alternating polynomials with every term degree k. Show that division by Δ is a vector space isomorphism between $\mathcal{A}_{n+\binom{n}{2}}$ and Λ_n. (Therefore understanding Λ_n is the same as understanding $\mathcal{A}_{n+\binom{n}{2}}$.)

Proof. Define the map $L : \mathcal{A}_{n+\binom{n}{2}} \to \Lambda_n$ by $L(a) = \frac{a}{\Delta}$ for all $a \in \mathcal{A}_{n+\binom{n}{2}}$. To show that this is well-defined, note that the degree of Δ is $\binom{n}{2}$, since for any $1 \leq j \leq n$ we have that the degree of $\prod_{i=1}^{j}(x_i - x_j)$ will be j, so that we just add up over all of the j's from 1 to n, giving us $1+2+\ldots+(n-1) = \binom{n}{2}$. Additionally, a has degree $n+\binom{n+2}{2}$, and moreover has a factor of Δ by $(a.)$, hence the quotient of these two will have degree n. Moreover, Δ is an alternating polynomial, so that $\Delta(x_1, \ldots, x_n) = \text{sign}(\sigma)\Delta(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. To show this note transposing only two elements so for a given $\Delta(x_1, \ldots, x_k, \ldots, x_m, \ldots, x_n)$ looking at $\Delta(x_1, \ldots, x_m, \ldots, x_k, \ldots, x_n)$ we'll get $\Delta(x_1, \ldots, x_k, \ldots, x_m, \ldots, x_n) = -1\Delta(x_1, \ldots, x_m, \ldots, x_k, \ldots, x_n)$. Thus for any permutation $\sigma \in S_n$, $\Delta(x_1, \ldots, x_n) = \text{sign}(\sigma)\Delta(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Hence Δ is an alternating polynomial and moreover it's quotient with a is simply symmetric since the sign of the permutation will cancel leaving us with $\frac{a}{\Delta}$ being symmetric under any permutation in S_n. Thus this map is well-defined.

To show it's linear suppose that $a, b \in \mathcal{A}_{n+\binom{n}{2}}$ and $c, d \in \mathbb{F}$ for any field $\mathbb{F}$.

$$\begin{aligned}
L(ca + db) &= \frac{ca + db}{\Delta} \\
&= c\frac{a}{\Delta} + d\frac{b}{\Delta} \\
&= cL(a) + dL(b).
\end{aligned}$$

Hence $L : \mathcal{A}_{n+\binom{n}{2}} \to \Lambda_n$ is a linear map. To finish this off, note that L is clearly $1-1$ since if $L(a) = L(b)$ for $a, b \in \mathcal{A}_{n+\binom{n}{2}}$ we'll have $\frac{a}{\Delta} = \frac{b}{\Delta}$ implying $a = b$, hence L is $1-1$.

Moreover, let $y \in \Lambda_n$ be any symmetric polynomial of degree n Then $\Delta y \in \mathcal{A}_{n+\binom{n}{2}}$ since Δ is alternating and has degree $\binom{n}{2}$. Notice though that $L(\Delta y) = \frac{\Delta y}{\Delta} = y$. Hence L is a linear 1-1 map from $\mathcal{A}_{n+\binom{n}{2}}$ onto Λ_n, and thus L is a vector space isomorphism between the two spaces. That is our result! $\square$