Math 561 Homework 9

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$\boxed{\textbf{1} \text{ (a)}}$ Let $A \in \mathrm{M}_n(k)$ be a nilpotent matrix for some field k. This means that $A^b = 0$ for some $b \geq 1$. We wish to list the possible Jordan canonical forms of A. First, since $A^b = 0$, it follows that $m_A(x) \mid x^b$. Thus, $m_A(x) = x^j$ for some $1 \leq j \leq \min(b, n)$ (since $m_A(x)$ can be at most degree n). Since the invariant factors must divide $m_A(x)$, for each i we have $d_i(x) = x^\ell$ for some $1 \leq \ell \leq j$. Thus, the Jordan blocks for each invariant factor will look like

$$J_i(0) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & \dots & \dots & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}$$

Thus, the Jordan canonical form for A will look like

$$\mathrm{JCF}(A) = \begin{bmatrix} [J_1(0)] & 0 & \dots & 0 \\ 0 & [J_2(0)] & \dots & 0 \\ \vdots & & \ddots & 0 \\ 0 & \dots & \dots & [J_r(0)] \end{bmatrix}$$

In other words, the main diagonal will be all zeros, the upper diagonal may have some ones, and everything else will be zero. $\square$

$\boxed{\textbf{1 (b)}}$ Let us suppose that A is nilpotent. This means that $A^i = 0$ for some $i \geq 1$. We wish to prove that $A^j = 0$ for some $j \leq n$. If $i \leq n$, then we are done. So let us suppose that $i > n$. First, from part (a) above, we know that all the invariant factors are of the form x^ℓ for some integer ℓ. Since the product of all the invariant factors is $p_A(x)$ and $\deg(p_A(x)) = n$, then we can conclude that $p_A(x) = x^n$. Thus, since $p_A(A) = 0$, we have that

$$0 = p_A(A) = A^n$$

Since $n \leq n$, we have found $j \leq n$ such that $A^j = 0$, as desired. $\square$

$\boxed{\textbf{1} \text{ (c)}}$ We wish to show that if A is nilpotent, then $\text{tr}(A) = 0$. By Example 5.11.3 in Handout 9, we know that $p_A(x) = x^n - c_{n-1}x^{n-1} + \cdots + (-1)^n c_0$, where c_{n-1} is $\text{tr}(A)$. Of course, since $p_A(x) = x^n$, the coefficient of $x^{n-1} = 0$, meaning that $0 = c_{n-1} = \text{tr}(A)$, as desired. $\square$

$\boxed{\mathbf{2}}$ We wish to diagonalize $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ over $\mathbb{C}$. By Corollary 5.7 of Handout 9, the characteristic polynomial of A is

$$\begin{aligned} p_A(x) &= \det(xI - A) \\ &= \det\left(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} x - \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & x - \cos(\theta) \end{bmatrix}\right) \\ &= x^2 - 2x\cos(\theta) + \cos^2(\theta) + \sin^2(\theta) \\ &= x^2 - 2x\cos(\theta) + 1 \\ &= (x - e^{i\theta})(x - e^{-i\theta}) \end{aligned}$$

Since our characteristic polynomial split into distinct linear factors, then it must be the case that our minimum polynomial is equal to the characteristic polynomial. In other words,

$$m_A(x) = p_A(x) = (x - e^{i\theta})(x - e^{-i\theta})$$

Since the minimum polynomial splits into distinct linear factors, then we know A is diagonalizable and that

$$\boxed{A \sim \operatorname{diag}\{e^{i\theta}, e^{-i\theta}\} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}}$$

$\boxed{\mathbf{3}}$ Recall that $\mathrm{SO}_3 = \{A \in \mathrm{GL}_3(\mathbb{R}) : A^tA = I \text{ and } \det(A) = 1\}$. For an arbitrary $A \in \mathrm{SO}_3$, we wish to show that A is diagonalizable. First, by these definitions, we know that A preserves the dot product and thus fixes lengths and angles of vectors. Because of this, for any arbitrary vector $v \in \mathbb{R}^3$, we can say that

$$|v| = |Av|$$

Let us consider an arbitrary eigenvalue λ of A. By definition,

$$Av = \lambda v$$

for some eigenvector v. Using our result from above, we have that

$$\begin{aligned} |v| &= |Av| && \text{Using fact that } A \text{ preserves lengths.} \\ |v| &= |\lambda v| && \text{Using fact that } Av = \lambda v. \\ |v| &= |\lambda||v| && \text{Factor out } |\lambda|. \end{aligned}$$

From this, it follows that $|\lambda| = 1$, meaning that $\lambda = 1$ or $\lambda = -1$. Let us first assume that $\lambda = 1$. This means that A fixes a line through the origin in $\mathbb{R}^3$, which means that it must also fix the plane perpendicular to this line that goes through the origin, which we will denote P.

Let us consider the restriction of A to this plane. We need our columns to still be mutually perpendicular with length 1. For the $(1,1)$ entry of $A\,|_P$, if set it to $\cos(\theta)$, then our $(2,1)$ entry must be $\pm\sqrt{1-\cos^2(\theta)} = \pm\sin(\theta)$. To make sure our columns our mutually perpendicular, we possible choices for the second column are

$$\begin{bmatrix} \mp\sin(\theta) \\ \cos(\theta) \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} \pm\sin(\theta) \\ -\cos(\theta) \end{bmatrix}$$

If it is the first case, then our restricted matrix looks like

$$A\,|_P = \begin{bmatrix} \cos(\theta) & \mp\sin(\theta) \\ \pm\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Regardless of the signs on $\sin(\theta)$, we can simply apply our work from Problem 2 above and conclude that the other eigenvalues are $e^{i\theta}$ and $e^{-i\theta}$. If $\theta \neq 0, \pi$, then our eigenvalues (including the $\lambda = 1$ from earlier) are all distinct, which forces the minimum polynomial to have distinct factors. Thus, by Theorem 6.4 of Handout 9, it will be diagonalizable.

Now, still assuming that $\lambda = 1$, we know that

$$A \sim \begin{bmatrix} 1 & 0 \\ 0 & [A\,|_P] \end{bmatrix}$$

If $\theta = 0$, then $A\,|_P = I_2$, which means that A would be similar to I_3, a diagonal matrix. Thus A is diagonalizable. However, if $\theta = \pi$, then $A\,|_P = -I$, which means that

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

another diagonal matrix. So again, A is diagonalizable. Now, let us consider the second possibility for the second column of $A\,|_P$. In this case, we have

$$\det(A\,|_P) = -\cos^2(\theta) - \sin^2(\theta) = -1$$

However, recall that $\det(A) = 1$. If we assume this second possibility for the second column of $A\,|_P$, then we would have that $\det(A) = -1$, a contradiction. So only the first possibility for the second column is possible. Therefore in any valid case from assuming $\lambda = 1$, we have that A is diagonalizable.

Let us now assume that $\lambda = -1$. Now, A reflects vectors on a line across the origin. Still, A will preserve the plane perpendicular to this line that goes through the origin, which we will again denote P. Similar to before, we have that

$$A \sim \begin{bmatrix} -1 & 0 \\ 0 & [A\,|_P] \end{bmatrix}$$

Again recall that $\det(A) = 1$. Thus, for this to be true, we must have $\det(A\mid_P) = -1$. This forces

$$A\mid_P = \begin{bmatrix} \cos(\theta) & \pm\sin(\theta) \\ \pm\sin(\theta) & -\cos(\theta) \end{bmatrix},$$

which has eigenvalues 1 and -1. Thus, we have that

$$A\mid_P \sim \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which of course implies that

$$A \sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

a diagonal matrix. So A is diagonalizable. Therefore, in any case, we have shown that A is diagonalizable over $\mathbb{C}$, as required.

$\boxed{\mathbf{4}}$ ($\Longrightarrow$) Suppose V is an n-dimensional k-vector space and $T \in \mathrm{End}_k(V)$. Furthermore, suppose that λ is an eigenvalue of T. We wish to show that $x - \lambda \mid m_T(x)$. First, since λ is an eigenvalue of T, then there exists non-zero $v \in V$ such that

$$T(v) = \lambda v$$

By Theorem 3.4 of Handout 9, we know that

$$V \simeq \coprod_{i=1}^{r} \frac{k[x]}{(d_i(x))}$$

Thus, there exists non-zero $\bigoplus_{i=1}^{r} (f_i(x) + (d_i(x)))$ such that

$$x \left(\bigoplus_{i=1}^{r} (f_i(x) + (d_i(x))) \right) = \lambda \left(\bigoplus_{i=1}^{r} (f_i(x) + (d_i(x)))) \right)$$

This of course implies that

$$(x - \lambda) \left(\bigoplus_{i=1}^{r} (f_i(x) + (d_i(x))) \right) = 0$$

While $\bigoplus_{i=1}^{r} (f_i(x) + (d_i(x)))$ is non-zero, each component may not necessarily be non-zero. However, we do know that at least one of them will be non-zero. Without loss of generality, suppose that $f_i(x)$ is non-zero (meaning $f_i(x) + (d_i(x)) \neq (d_i(x))$). Once we multiply by $(x - \lambda)$, we have for that component that $(x - \lambda)f_i(x) + (d_i(x)) = (d_i(x))$. Using the argument from Problem 5 of HW 8, we can conclude that $(x - \lambda) \mid d_i(x)$.

Now, recalling that $d_i(x) \mid d_j(x)$ if $i < j$ and that $m_T(x) = d_r(x)$, we have that $d_i(x) \mid m_T(x)$. Therefore, since $(x - \lambda) \mid d_i(x)$, it follows that $(x - \lambda) \mid m_T(x)$ as well. Thus we have proven our desired result.

($\Longleftarrow$) Now, let us suppose that $(x - \lambda) \mid m_T(x)$. We wish to show that λ is an eigenvalue for T. Since $(x - \lambda) \mid m_T(x)$, this means that $(x - \lambda)f(x) = m_T(x)$ for some $f(x) \in k[x]$. Noting that $m_T(x) = d_r(x)$, we can write

$$(x - \lambda)(f(x) + (d_r(x))) = (d_r(x))$$

This means that

$$x(f(x) + (d_r(x))) = \lambda(f(x) + (d_r(x)))$$

Since $(f(x) + d_r(x)) = 0 \oplus \ldots 0 \oplus (f(x) + (d_r(x))) \in \coprod_{i=1}^{r} \frac{k[x]}{(d_i(x))}$ and $V \simeq \coprod_{i=1}^{r} \frac{k[x]}{(d_i(x))}$, it follows that there exists $v \in V$ such that $xv = \lambda v$, which of course implies

$$T(v) = \lambda v$$

By definition, λ is an eigenvalue for T, as desired. $\Diamond$

Now, suppose the invariants of T are $d_1(x), \ldots, d_r(x)$ and $d_s(x)$ is the smallest one divisible by $x - \lambda$. We wish to compute $\dim_k E_\lambda$. First, since $(x - \lambda)$ divides $d_s(x)$, then since $d_s(x)$ divides all of the larger invariant factors, then $(x - \lambda)$ will divide those larger invariant factors as well. In particular, $(x - \lambda) \mid d_i(x)$ for $s \leq i \leq r$. Thus, for each of these $d_i(x)$'s, we will acquire a Jordan block with λ. Knowing that $\dim_k E_\lambda$ is equal to the number of Jordan blocks with λ, we can say that

$$\boxed{\dim_k E_\lambda = (r - s + 1)},$$

which is equal to the number of Jordan blocks involving λ in the Jordan canonical form for T.

$\boxed{\mathbf{5}}$ We wish to determine the number of similarity classes in $\mathrm{M}_6(\mathbb{C})$ with characteristic polynomial $(x^4-1)(x^2-1)$. First, by Theorem 5.11, similar matrices have the same invariant factors. Thus, to find the number of similarity classes in $\mathrm{M}_6(\mathbb{C})$ with characteristic polynomial $(x^4-1)(x^2-1)$, it is sufficient to find the number of possible minimum polynomials (which determines the invariant factors).

Note that $(x^4-1)(x^2-1) = (x-1)^2(x+1)^2(x-i)(x+i)$. For a matrix A with this characteristic polynomial, there are four possibilities for $m_A(x)$. They are:

1. $m_A(x) = (x-1)^2(x+1)^2(x-i)(x+i) = p_A(x)$ (the only invariant factor)
2. $m_A(x) = (x-1)^2(x+1)(x-i)(x+i)$ (with invariant factor $d_1(x) = (x+1)$)
3. $m_A(x) = (x-1)(x+1)^2(x-i)(x+i)$ (with invariant factor $d_1(x) = (x-1)$)
4. $m_A(x) = (x-1)(x+1)(x-i)(x+i)$ (with invariant factor $d_1(x) = (x-1)(x+1)$)

Thus, since there are 4 possible minimum polynomials, there are $\boxed{\text{4 similarity classes}}$ in $\mathrm{M}_6(\mathbb{C})$ with characteristic polynomial $(x^4-1)(x^2-1)$. $\square$

$\boxed{\mathbf{6}}$ Let

$$A = \sum_{\substack{1 \le i \le n \\ 1 \le j \le n}} e_{ij} \in \mathrm{M}_n(\mathbb{Q}),$$

the matrix of all 1's. We wish to find JCF(A). First, we will find the characteristic polynomial of A, $p_A(x)$. Using the trick from Example 5.11.3 in Handout 9, we know that the coefficient of x^{n-1} will be $-\mathrm{tr}(A) = -(1 + \cdots + 1) = -n$.

Furthermore, the coefficients c_i of x^i for $0 \le i \le n-2$ are found by adding together particular diagonal minors of A. However, since all of our entries in A are identical, then these minor computations will simply yield 0. This is because repeated cofactor expansions of these minor computations will eventually to some expression of the following form:

$$(a_1)\det\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) + \cdots + (a_\ell)\det\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$$

for some coefficients $a_1, \ldots, a_\ell$. However, $\det\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = 1 - 1 = 0$. Thus, it follows that $c_0 = \cdots = c_{n-2} = 0$. Hence, our characteristic polynomial for A is

$$p_A(x) = x^n - nx^{n-1} = x^{n-1}(x - n)$$

Now, let us consider $x(x-n)$ for $x = A$. This gives us

$$A(A - nI) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \dots & \dots & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \dots & \dots & 1 \end{bmatrix} - \begin{bmatrix} n & 0 & \dots & 0 \\ 0 & n & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & n \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \dots & \dots & 1 \end{bmatrix} \begin{bmatrix} 1-n & 1 & \dots & 1 \\ 1 & 1-n & & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \dots & \dots & 1-n \end{bmatrix}$$

To calculate the (i, j) entry of this matrix multiplication, we multiply the i-th row of the left matrix by the j-th column of the right matrix component-wise, then sum the result. In this particular case, every entry will be of the form

$$\left(\sum_{i=1}^{n-1} 1 * 1 \right) + 1 * (1 - n) = n - 1 + (1 - n) = 0$$

Thus, since $A(A - nI) = 0$, we can conclude that the minimum polynomial of A, $m_A(x)$, must divide $x(x-n)$. Furthermore, since $A \neq 0$ and $(A - nI) \neq 0$, this means that $m_A(x) = x(x-n)$. Lastly, since our invariant factors must divide each other sequentially and $m_A(x)$ is the "largest" invariant factor, then the only possibility that our invariant factors could be is

$$d_1(x) = x, \qquad d_2(x) = x, \qquad \ldots \qquad d_{n-2}(x) = x, \qquad d_{n-1} = m_A(x) = x(x-n)$$

Therefore, with our invariant factors found, we can easily construct the Jordan canonical form for A. We have that

$$\boxed{\mathrm{JCF}(A) = \begin{bmatrix} n & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{bmatrix}}$$

$\boxed{7}$ We wish to compute the characteristic polynomial and the Jordan canonical form for the matrix

$$A = \begin{bmatrix} -8 & -10 & -1 \\ 7 & 9 & 1 \\ 3 & 2 & 0 \end{bmatrix}$$

Using Corollary 5.7 from Handout 9, the characteristic polynomial $p_A(x)$ is given by

$$\begin{aligned}
p_A(x) &= \det(xI - A) \\
&= \det\left(\begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix} - \begin{bmatrix} -8 & -10 & -1 \\ 7 & 9 & 1 \\ 3 & 2 & 0 \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} x+8 & 10 & 1 \\ -7 & x-9 & -1 \\ -3 & -2 & x \end{bmatrix}\right) \\
&= (1)\det\left(\begin{bmatrix} -7 & x-9 \\ -3 & -2 \end{bmatrix}\right) - (-1)\det\left(\begin{bmatrix} x+8 & 10 \\ -3 & -2 \end{bmatrix}\right) + (x)\det\left(\begin{bmatrix} x+8 & 10 \\ -7 & x-9 \end{bmatrix}\right) \\
&= (3x-13) - (2x-14) + (x^3 - x^2 - 2x) \\
&= \boxed{x^3 - x^2 - x + 1 = (x-1)^2(x+1)}
\end{aligned}$$

To determine the Jordan canonical form, we must first determine the invariant factors. Noting that $(A-1)$, $(A+1)$, $(A-1)^2$, and $(A-1)(A+1)$ are all non-zero, we can conclude that the minimum polynomial $m_A(x)$ is equal to $(x-1)^2(x+1) = p_A(x)$. Thus, it follows that the Jordan canonical form for A is

$$\boxed{\mathrm{JCF}(A) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

We also wish to compute the characteristic polynomial and the Jordan canonical form for the matrix

$$B = \begin{bmatrix} -3 & 2 & -4 \\ 4 & -1 & 4 \\ 4 & -2 & 5 \end{bmatrix}$$

Using Corollary 5.7 from Handout 9, the characteristic polynomial $p_B(x)$ is given by

$$\begin{aligned}
p_B(x) &= \det(xI - B) \\
&= \det\left(\begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix} - \begin{bmatrix} -3 & 2 & -4 \\ 4 & -1 & 4 \\ 4 & -2 & 5 \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} x+3 & -2 & 4 \\ -4 & x+1 & -4 \\ -4 & 2 & x-5 \end{bmatrix}\right) \\
&= (x+3)\det\left(\begin{bmatrix} x+1 & -4 \\ 2 & x-5 \end{bmatrix}\right) - (-2)\det\left(\begin{bmatrix} -4 & -4 \\ -4 & x-5 \end{bmatrix}\right) + (4)\det\left(\begin{bmatrix} -4 & x+1 \\ -4 & 2 \end{bmatrix}\right) \\
&= (x^3 - x^2 - 9x + 9) - (8x - 8) + (16x - 16) \\
&= \boxed{x^3 - x^2 - x + 1 = (x-1)^2(x+1)}
\end{aligned}$$

To determine the Jordan canonical form, we must first determine the invariant factors. Noting that $(B-1)$, $(B+1)$ are non-zero but $(B-1)(B+1) = 0$, we can conclude that the minimum polynomial $m_B(x)$ is equal to $(x-1)(x+1)$. Therefore, our invariant factors are $d_1(x) = (x-1)$ and $d_2(x) = m_B(x) = (x-1)(x+1)$. Thus, it follows that the Jordan canonical form for B is

$$\boxed{\mathrm{JCF}(B) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$\boxed{\mathbf{8}}$ We wish to determine all Jordan canonical forms for the matrices that have characteristic polynomial $(x-2)^3(x-3)^2$. For a matrix A with this characteristic polynomial, there are six possibilities for $m_A(x)$. They are:

1. $m_A(x) = (x-2)^3(x-3)^2 = p_A(x)$ (the only invariant factor)
2. $m_A(x) = (x-2)^3(x-3)$ (with invariant factor $d_1(x) = (x-3)$)
3. $m_A(x) = (x-2)^2(x-3)^2$ (with invariant factor $d_1(x) = (x-2)$)
4. $m_A(x) = (x-2)^2(x-3)$ (with invariant factor $d_1(x) = (x-2)(x-3)$)
5. $m_A(x) = (x-2)(x-3)^2$ (with invariant factor $d_1(x) = (x-2)$ and $d_2(x) = (x-2)$)
6. $m_A(x) = (x-2)(x-3)$ (with invariant factors $d_1(x) = (x-2)$ and $d_2(x) = (x-2)(x-3)$)

These possible minimum polynomials correspond to the following Jordan canonical forms:

$$1: \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \qquad 2: \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \qquad 3: \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$$4: \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \qquad 5: \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \qquad 6: \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$\boxed{\textbf{9}}$ Suppose $A \in \mathrm{M}_2(\mathbb{Q})$, $A^3 = I$, and $A \neq I$. Since $A^3 = I$ (meaning $A^3 - I = 0$), then it follows that $m_A(x) \mid x^3 - 1 = (x-1)(x^2+x+1)$. Additionally, since $A \neq I$ by assumption, then $m_A(x) \neq x-1$. Furthermore, since $A \in M_2(\mathbb{Q})$, then it follows that $\deg(p_A(x)) = 2$. Thus, since $m_A(x) \mid p_A(x)$, then $\deg(m_A(x)) \leq \deg(p_A(x)) = 2$.

Combining these results, since $m_A(x) \mid (x-1)(x^2+x+1)$, $\deg(m_A(x)) \leq 2$, and $m_A(x) \neq x-1$, then it follows that $m_A(x) = x^2+x+1$. Since $m_A(x)$ and $p_A(x)$ have the same degree, they must be equal, meaning there is only one invariant factor. Over $\mathbb{C}$, $m_A(x)$ factors into

$$m_A(x) = x^2 + x + 1 = \left(x - \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\right)\left(x - \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\right)$$

Thus, we can say that the Jordan canonical form of A over $\mathbb{C}$ is

$$\mathrm{JCF}(A) = \begin{bmatrix} \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) & 0 \\ 0 & \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) \end{bmatrix}$$