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Write Up #3

Exercise 1.

Re-read the definition of a *morphism of algebraic varieties.* Describe this object in our own words. Give an example to help someone else understand.

Solution:

A morphism of algebraic varieties is a map that sends the vanishing of complex polynomials from some affine space $\mathbb{A}^n$, to the vanishing of other complex polynomials in $\mathbb{A}^m$.

Example of one such map would be:

$$f : \mathbb{V}(xy^2 - x) \to \mathbb{V}(x^2 - y)$$

$$f(t_1, 1) = (t_1^2, t_1)$$

So this maps the vanishing of $xy^2 = x$ to the parabola $y = x^2$.

Exercise 4.

Give an example of an isomorphism of varieties (that is *not* a change of coordinate map).

Hint: See the example of the map $\mathbb{A}^1 \to \mathbb{C}$ in the text. Show that this map is an isomorphism.

Solution:

Consider the morphism: $\mathbb{A}^1 \to C$, where $C = \mathbb{V}(y - x^2)$, given by $f(t) = (t, t^2)$. Then, since $\mathbb{A}^1$ and C are both algebraic varieties, this is a morphism between algebraic varieties. So now all we have left to show is that f is an isomorphism, which is equivalent to showing that it is a bijective map.

(1-1)

Let $f(t_1) = f(t_2)$ for some $t_1, t_2 \in \mathbb{A}^1$. Then we have: $(t_1, t_1^2) = (t_2, t_2^2)$. This gives us the system of equations:

$$\begin{aligned} t_1 &= t_2 \\ t_1^2 &= t_2^2 \end{aligned}$$

Which gives us that $t_1 = t_2$, hence f is 1-1.

(Onto)

Let $(x, y) \in C$, then by the definition of C we have that $y - x^2 = 0$. Then choose $t \in \mathbb{A}^1$ such that $t = y$ and $t = \sqrt{x}$. This is acceptable, only because t lies on the curve C and such points exist for all points $(x, y) \in C$. So that: $f(t) = (t, t^2) = ((\sqrt{x})^2, y) = (x, y)$.

Hence the map f is bijective and a morphism of algebraic varieties, hence f is an isomorphism.

Exercise 5.

What is an *automorphism*? Give a detailed example.

Solution:

An automorphism is a morphism on mapping to and from the same set.

An example of such a map is:

$$f : V \to V$$

where $V = \mathbb{V}(xy - 1)$.

$$f(t) = (t, 2t)$$

Exercise 6.

What is *affine n-space*?

Solution:

Affine n-space is the set of zeros of all complex polynomials with n variables, or the vanishing of those polynomials.

Exercise 7.

Is the Zariski topology a Hausdorff topology? Why or why not? What does this even mean? What are the implications (intuitively) of this?

Solution: No. The reasoning behind is that given any two points in the Zariski topology, there won't be any distinct open neighborhoods of one point that doesn't overlap with a neighborhood of the other point. Essentially, the reason Zariski is not Hausdorff because of how large the open sets are in this topology.

The implications of this mean that any property involving open neighborhoods of points is some what muddled because of this, e.g. limits using the topology, do not have to be unique. My reasoning for this is, informally, that converging to a open set in this topology will yield more than one point. So as in the standard topology on the real line, limits revolve utilize the idea of contraction around an open set around the point to define a limit. So that in the Zariski topology if the open sets are very large, then the idea of contraction of open sets isn't well-defined, i.e. what does it mean to contract sets that are as unusually large as the Zariski open sets.