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Homework # 2

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8.

Let $b_{n,k}$ be the number of set partitions of n into k sets and $T_n(y) = \sum_{k=0}^{n} b_{n,k}y^k$. We have $\sum_{n=0}^{\infty} T_n(y)\frac{x^n}{n!} = e^{y(e^x-1)}$.

a.)

Show that $T_n(a+b) = \sum_{k=0}^{n} \binom{n}{k} T_k(a)T_{n-k}(b)$.

Proof. By the above we have $\sum_{n=0}^{\infty} T_n(a+b)\frac{x^n}{n!} = e^{(a+b)(e^x-1)} = e^{a(e^x-1)+b(e^x-1)} = e^{a(e^x-1)}e^{b(e^x-1)}$. This then is reduced to the a problem of multiplying two series:

$$\begin{aligned}\sum_{n=0}^{\infty} T_n(a+b)\frac{x^n}{n!} &= e^{a(e^x-1)}e^{b(e^x-1)} \\ &= \left(\sum_{n=0}^{\infty} T_n(a)\frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} T_n(b)\frac{x^n}{n!}\right) \\ &= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \frac{T_k(a)}{k!}\frac{T_{n-k}(b)}{(n-k!)}\end{aligned}$$

The coefficents of these power series must then be equal, giving us:

$$\frac{T_n(a+b)}{n!} = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} T_k(a)T_{n-k}(b).$$

Multiplying both sides by $n!$ gives us the binomial coefficient so that $T_n(a+b) = \sum_{k=0}^{n} \binom{n}{k} T_k(a)T_{n-k}(b)$. □

b.)

Show that $\sum_{n=0}^{\infty} t_{n+1}(y)\frac{x^n}{n!} = e^{y(e^x-1)}ye^x$ and use this to show $T_{n+1}(y) = y\sum_{k=0}^{n} \binom{n}{k} T_k(y)$.

Proof. To do this we'll take the partial derivative of both sides of $\sum_{n=0}^{\infty} T_n(y)\frac{x^n}{n!} = e^{y(e^x-1)}$.

$$\begin{aligned}
\sum_{n=1}^{\infty} T_n(y)\frac{nx^{n-1}}{n!} &= e^{y(e^x-1)}\frac{\partial}{\partial x}(y(e^x-1)) \\
\sum_{n=1}^{\infty} T_n(y)\frac{x^{n-1}}{(n-1)!} &= e^{y(e^x-1)}ye^x \\
\sum_{n=0}^{\infty} T_{n+1}(y)\frac{x^n}{n!} &= e^{y(e^x-1)}ye^x
\end{aligned}$$

To show the other piece, consider the following, we'll use the fact that $\sum_{n=0}^{\infty} T_n(y)\frac{x^n}{n!} = e^{y(e^x-1)}$ and $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

$$\begin{aligned}
\sum_{n=0}^{\infty} T_{n+1}(y)\frac{x^n}{n!} &= e^{y(e^x-1)}ye^x \\
&= \left(\sum_{n=0}^{\infty} T_n(y)\frac{x^n}{n!}\right) y \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \\
&= y\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} T_k(y)\frac{1}{k!}\frac{1}{(n-k)!}.
\end{aligned}$$

Similar to (a) we have to have equality in the coefficients of these two so that:

$$\begin{aligned}
T_{n+1}(y)\frac{1}{n!} &= \sum_{k=0}^{n} T_k(y)\frac{1}{k!(n-k)!} \\
T_{n+1}(y) &= y\sum_{k=0}^{n}\binom{n}{k}T_k(y)
\end{aligned}$$

$\square$

9.)

9. An ordered set partition of n is an ordered list of disjoint nonempty sets with union $\{1,\ldots,n\}$. For example, there are 13 ordered set partitions of 3:

$(\{1,2,3\})$,
$(\{1\},\{2,3\})$, $(\{2,3\},\{1\})$, $(\{2\},\{1,3\})$, $(\{1,3\},\{2\})$, $(\{3\},\{1,2\})$, $(\{1,2\},\{3\})$,
$(\{1\},\{2\},\{3\})$, $(\{1\},\{3\},\{2\})$, $(\{2\},\{1\},\{3\})$, $(\{2\},\{3\},\{1\})$, $(\{3\},\{1\},\{2\})$, $(\{3\},\{2\},\{1\})$.

Let a_n be the number of ordered set partitions of n and let $A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.

a.)

Show that $a_0 = 1$ and $a_n = \sum_{k=1}^{n} \binom{n}{k} a_{n-k}$ for $n \geq 1$.

Proof. We'll take the convention that there's one way to partition the empty set, that is no way to do it. So $a_0 = 1$.

For $n \geq 1$, first we'll let $k \in \{1,\ldots,n\}$, choose k elements from $\{1,\ldots,n\}$. No matter the elements, we have $\binom{n}{k}$ choices for elements in this set, the order inside of the set doesn't matter. This first set will be our first set in the ordered partition. Once we do this it's the problem of a_{n-k} that is how many ordered partitions of size $n-k$ can we make. Hence we have after summing over all possible $k \in \{1,\ldots,n\}$: $\sum_{k=1}^{n} \binom{n}{k} a_{n-k}$. □

b.)

Show that $A(x) = \frac{1}{2-e^x}$.

Proof. Let $A(x) = \sum\limits_{n=0}^{\infty} a_n \frac{x^n}{n!}$. Then consider the following:

$$\begin{aligned}
A(x) &= 1 + \sum_{n=1}^{\infty} a_n \frac{x^n}{n!} \\
&= 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k=1}^{n} \binom{n}{k} a_{n-k} \\
&= 1 + \sum_{n=1}^{\infty} x^n \sum_{k=1}^{n} \frac{a_{n-k}}{k!(n-k)!} \\
&= 1 + \left(\sum_{n=1}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=1}^{\infty} \frac{x^n a_n}{n!}\right) \\
&= 1 + (e^x - 1)A(x) \\
A(x) - (e^x - 1)A(x) &= 1 \\
A(x)(2 - e^x) &= 1 \\
A(x) &= \frac{1}{2 - e^x}.
\end{aligned}$$

□

c.)

Expand $A(x)$ as a geometric series to show that $a_n = \frac{1}{2} \sum\limits_{k=0}^{\infty} \frac{k^n}{2^k}$.

Proof. Consider the following:

$$\begin{aligned} A(x) = \sum_{n=0}^{\infty} \frac{x^n a_n}{n!} &= \frac{1}{2 - e^x} \\ &= \frac{1}{2} \frac{1}{1 - \frac{e^x}{2}} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{e^x}{2} \right)^n \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{e^{xn}}{2^n} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{k=0}^{\infty} \frac{(xn)^k}{k!} \\ &= \sum_{k=0}^{\infty} x^k \frac{1}{2} \sum_{n=0}^{\infty} \frac{n^k}{k!} \frac{1}{2^n}. \end{aligned}$$

After a relabeling we get that the coefficients must again be equivalent:

$$\frac{a_n}{n!} = \sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}} \frac{1}{n!}$$

Giving us our result: $a_n = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k^n}{2^k}$ □

d.)

Let $a_{n,k}$ be the number of ordered set partitions of n into exactly k sets. Show that $a_{n+1,k} = ka_{n,k} + ka_{n,k-1}$.

Proof. We'll break this into two independent cases. Take the set $\{1, \ldots, n, n+1\}$. So either $\{n+1\}$ will exist in the ordered partition or it will not.

If $\{n+1\}$ is in the partition, then we'll have exactly k places to choose to put this singleton in the partition. The remaining problem is then just $a_{n,k-1}$, because we already have 1 set in the partition $\{n+1\}$, but we can only choose from the set $\{1, \ldots, n\}$. Because the two steps above depend on each other we must multiply the total choice so we have $ka_{n,k-1}$ choices here.

Alternatively, $\{n+1\}$ isn't in the partition. If this is the case, then $n+1$ is in one of the sets in the partition. So forgetting about $n+1$ for a moment, creating the partition without

$n+1$ is just the problem $a_{n,k}$. Once we have this partition, place $n+1$ into one of these sets, you have exactly k sets to choose from. Because these steps depend on each other we multiply the choice, hence we have in total $a_{n,k}k$ choices in this case.

Considering the two independent cases we have in total:

$$a_{n+1,k} = ka_{n,k} + ka_{n,k-1}.$$

□

e.)

Let $A(x,y) = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} a_{n,k} y^k\right) \frac{x^n}{n!}$.

Proof. Let $A(x,y) = \sum_{n=0}^{\infty}\sum_{k=1}^{n} a_{n,k} y^k \frac{x^n}{n!}$. Note we can then take the outer sum to start at 0 or 1, since $a_{0,k} = 0$ for any $k \neq 0$

Note that we'll use the solution of $(d.)$ in the following proof. Additionally note that if $k > n$ then $a_{n,k} = 0$. Then consider the following:

$$\begin{aligned}
A_x(x,y) &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} a_{n,k} y^k \frac{x^{n-1}}{(n-1)!} \\
&= \sum_{n=0}^{\infty}\sum_{k=1}^{n+1} a_{n+1,k} y^k \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=1}^{n+1} (ka_{n,k} + ka_{n,k-1}) y^k \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=1}^{n+1} ka_{n,k} y^k \frac{x^n}{n!} + \sum_{n=0}^{\infty}\sum_{k=1}^{n+1} ka_{n,k-1} y^k \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=1}^{n} ka_{n,k} y^k \frac{x^n}{n!} + \sum_{n=0}^{\infty}\sum_{k=0}^{n+1} (k+1)a_{n,k} y^{k+1} \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=1}^{n} ka_{n,k} y^k \frac{x^n}{n!} + \sum_{n=0}^{\infty}\sum_{k=1}^{n} ka_{n,k} y^{k+1} \frac{x^n}{n!} + \sum_{n=0}^{\infty}\sum_{k=1}^{n} a_{n,k} y^{k+1} \frac{x^n}{n!}.
\end{aligned}$$

Note then that:

$$\begin{aligned} yA &= y\sum_{n=0}^{\infty}\sum_{k=1}^{n} a_{n,k}y^k\frac{x^n}{n!} \\ yA_y &= \sum_{n=0}^{\infty}\sum_{k=1}^{n} a_{n,k}ky^k\frac{x^n}{n!} \\ y^2A_y &= \sum_{n=0}^{\infty}\sum_{k=1}^{n} a_{n,k}ky^{k+1}\frac{x^n}{n!}. \end{aligned}$$

Looking at the end of the first chain of equations we find that $A_x = yA + yA_y + y^2A_y$ our result!

Finally, for $y = 1$ we get:

$$\begin{aligned} A(x,1) &= \sum_{n=0}^{\infty}\sum_{k=1}^{n} a_{n,k}1^k\frac{x^n}{n!} \\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} a_{n,k}\frac{x^n}{n!}. \end{aligned}$$

To show that $A(x,1) = \frac{1}{2-e^x}$ is then equivalent to show that $\sum_{k=1}^{n} a_{n,k} = a_n$. This holds since to get all the ordered partitions of $\{1,\ldots,n\}$ we have to include all partitions of size k for $\{1,\ldots,n\}$. That is, getting all the ordered partitions of the n-set, we can break this down into $n-1$ independent problems, of how to break the n-set down into a partition of size k. There would exist no partitions of size greater n or less than 1, so that we get $a_n = \sum_{k=1}^{n} a_{n,k}$. Thus $A(x,1) = A(x) = \frac{1}{2-e^x}$. □

f.)

Verify that $A(x,y) = \frac{1}{1-y(e^x-1)}$.

Proof. I'll assume the Cauchy problem:

$$\begin{cases} A_x = yA + yA_y + y^2A_y \\ A(x,1) = \frac{1}{2-e^x} \end{cases}$$

gives us a unique solution: $A(x,y)$.(Not 100% this is a good assumption.) So that verifying $A(x,y) = \frac{1}{1-y(e^x-1)}$ is equivalent to showing $\frac{1}{1-y(e^x-1)}$ satisfies the above problem.

For convenience call $f(x, y) = \frac{1}{1-y(e^x-1)}$ and we'll show that $f(x, y)$ is also a solution to the above boundary condition problem.

$$\begin{aligned} f_x &= D_x\{(1 - y(e^x - 1))^{-1}\} \\ &= -1(-ye^x)(1 - y(e^x - 1))^{-2} \\ &= \frac{ye^x}{(1 - y(e^x - 1))^2} \\ f_y &= D_y\{(1 - y(e^x - 1))^{-1}\} \\ &= -1(-e^x + 1)(1 - y(e^x - 1))^{-2} \\ &= \frac{e^x - 1}{(1 - y(e^x - 1))^2} \end{aligned}$$

So we have $yf = \frac{y}{1-y(e^x-1)} = \frac{y-y^2(e^x-1)}{(1-y(e^x-1))^2}$, $yf_y = \frac{ye^x-y}{(1-y(e^x-1))^2}$ and $y^2 f_y = \frac{y^2e^x-y^2}{(1-y(e^x-1))^2}$. So combining these we get:

$$\begin{aligned} yf + yf_y + y^2 f_y &= \frac{y - y^2(e^x - 1) + e^x y - y + y^2 e^x - y^2}{(1 - y(e^x - 1))^2} \\ &= \frac{-y^2 e^x + y^2 + e^x y + y^2 e^x - y^2}{(1 - y(e^x - 1))^2} \\ &= \frac{e^x y}{(1 - y(e^x - 1))^2} = f_x. \end{aligned}$$

Additionally $f(x, 1) = \frac{1}{1-e^x+1} = \frac{1}{2-e^x}$. So making the stretch that the boundary value problem has a unique solution we have $f(x, y) = \frac{1}{1-y(e^x-1)} = A(x, y)$. □

g.)

Let t_n be the total number of sets in all ordered set partitions of n. Find a generating function for $\sum_{n=0}^{\infty} t_n \frac{x^n}{n!}$.

Proof. We'll get two distinct answers if we double count sets, say for instance do we count $\{1\}$ once or as many times as it occurs in the partitions.

If it's the former, then there are $\sum_{k=1}^{n} \binom{n}{k}$ ways to do so. That is, if we don't repeatedly count the same set, and only count it once for all the ordered partitions of $\{1, \ldots, n\}$. This holds because we break the problem down, given a k in this set then there are $\binom{n}{k}$ distinct ways of building a set. Each one of these sets will occur at least once, for all the ordered partitions of $\{1, \ldots, n\}$, then we must sum over all possible k.

Then we have the following:

$$\begin{aligned}
T(x) &= \sum_{n=0}^{\infty} t_n \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{k=1}^{n} \binom{n}{k} \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{k=1}^{n} \frac{1}{k!(n-k)!} x^n \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{x^n}{k!(n-k)!} \\
&= \left(\sum_{n=1}^{\infty} \frac{x^n}{n!} \right) \left(\sum_{n=1}^{\infty} \frac{x^n}{n!} \right) \\
&= (e^x - 1)(e^x - 1).
\end{aligned}$$

So the exponential generating function for the number of distinct sets in the ordered partitions of $\{1, \ldots, n\}$ is $(e^x - 1)(e^x - 1)$. $\square$