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# Homework # 13

MATH 530
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## 64.

Define a ring homomorphism $\varphi$ on $\Lambda$ by $\varphi(e_n) = \begin{cases} 1 & \text{if } n = 0 \text{ or } n = 2, \\ 2x & \text{if } n = 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$

### a.

Recall from exercise (23) the definitions of the Chebyshev polynomial of the first kind $T_n(x)$ and the Chebyshev polynomial of the second kind $U_n(x)$. ($\sum_{n=0}^{\infty} T_n(y)x^n = \dfrac{1 - yx}{1 - 2yx + x^2}$, $\sum_{n=1}^{\infty} U_n(y)x^n = \dfrac{1}{1 - 2xy + x^2}$) Show that $\varphi(p_n) = 2T_n(x)$ for $n \geq 1$ and $\varphi(h_n) = U_n(x)$ for $n \geq 0$. It may be help to use the identity found in Exercise (60). ($\sum_{n=0}^{\infty} h_n t^n = H(t)$, $\sum_{n=1}^{\infty} \frac{p_n}{n} t^n = \ln(H(t))$, $\sum_{n=1}^{\infty} p_n t^n = \frac{tH'(t)}{H(t)}$.)

*Proof.* We'll show $\varphi(p_n) = 2T_n(x)$ for $n \geq 1$ by showing that the generating function for $\varphi(p_n)$ is equal to $\frac{1-xt}{1-2xt+t^2}$. Note the following to be used in this proof and the other equality to be used:

$$\begin{aligned} H(t) &= \frac{1}{E(-t)} \\ H'(t) &= \frac{1}{E'(-t)} \\ E(t) &= \sum_{n=0}^{\infty} e_n t^n \\ &= \prod_{n=1}^{\infty} (1 + t^n). \end{aligned}$$

Using these consider the following:

$$\begin{aligned}
\varphi\left(\sum_{n=1}^{\infty} p_n t^n\right) &= \sum_{n=1}^{\infty} \varphi(p_n) t^n \\
&= \varphi\left(t\frac{H'(t)}{H(t)}\right) \\
&= \frac{\sum_{n=1}^{\infty}(-1)^{n-1} n e_n t^n}{\sum_{n=0}^{\infty} \varphi(e_n)(-t)^n} \\
&= \frac{\varphi(e_1)t - 2\varphi(e_2)t^2}{\varphi(e_0) - \varphi(e_1)t + \varphi(e_2)t^2} \\
&= \frac{2xt - 2t^2}{1 - 2xt + t^2}.
\end{aligned}$$

We're almost done, note that we want to show that the generating functions agree for all $n \geq 1$, so divide the above by 2 and add in 1 to account for the extra $T_0(x)$ term in the Chebyshev polynomials of the first kind's generating function. This gives us what we wanted to show:

$$\frac{xt - t^2}{1 - 2xt + t^2} + 1 = \frac{xt - t^2 + 1 - 2xt + t^2}{1 - 2xt + t^2} = \frac{1 - xt}{1 - 2xt + t^2},$$

the exact form of the generating function of the Chebyshev polynomials of the first kind. Hence we have $\varphi(p_n) = 2T_n(x)$ for $n \geq 1$.

To show $\varphi(h_n) = U_n(x)$ for $n \geq 0$, we'll follow a similar process using $H(t) = \dfrac{1}{E(-t)}$:

$$\begin{aligned}
\sum_{n=0}^{\infty} \varphi(h_n) t^n &= \frac{1}{\varphi(E(-t))} \\
&= \frac{1}{\sum_{n=0}^{\infty} \varphi(e_n)(-t)^n} \\
&= \frac{1}{\varphi(e_0) - \varphi(e_1)t + \varphi(e_2)t^2} \\
&= \frac{1}{1 - 2xt + t^2}.
\end{aligned}$$

This is exactly the generating function for the Chebyshev polynomials of the second kind, hence we can conclude for $n \geq 0$:

$$\varphi(h_n) = U_n(x).$$

□

# b.

Use previously established relationships between $e_n, h_n$, and $p_n$ (such as those in exercise 59) to show these identities hold for $n \geq 3$:

## i.

$$U_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} (-1)^i (2x)^{n-2i}$$

*Proof.* First, note that since we have $h_\mu = \sum_{\lambda \vdash n} (-1)^{n-l(\lambda)} |B_{\lambda,\mu}| e_\lambda$ we can deduce the following:

$$\begin{aligned}
\varphi(h_n) &= \sum_{\lambda \vdash n} (-1)^{n-l(\lambda)} |B_{\lambda,(n)}| \varphi(e_\lambda) \\
&= \sum_{\lambda \vdash n} (-1)^{n-l(\lambda)} |B_{\lambda,(n)}| \varphi(e_{\lambda_1}) \dots \varphi(e_{\lambda_{l(\lambda)}}) \\
&= \sum_{\substack{\lambda \vdash n \\ \lambda \text{ has no parts greater than } 2}} (-1)^{n-l(\lambda)} |B_{\lambda,(n)}| (2x) \dots (2x).
\end{aligned}$$

Note then that this is a sum over all $\lambda$ partitions of $n$ where $\lambda$ has only parts of 1 and 2. We can count these by counting the number of 2's and 1's that occur in each integer partition. We'll count the integer partitions with $i$ 2's, moreover, the largest number of 2's that occur in the integer partition will be $\lfloor n/2 \rfloor$. Hence we can take on the following values $i \in \{0, 1, \dots, \lfloor n/2 \rfloor\}$. The number of 1 that occur in the integer partitions will occur is then $n - 2i$. This is accounted for with $(2x)^{n-2i}$. Finally, counting brick tabloids of content $\lambda$ and shape $(n)$, will be placing $i$ bricks of length 2 and $n - i$ bricks of length 1. Treating this as a balls-and-bars problem we can determine placing the $n - i$ 1's will leave us with $i$ choices positions of 2's, hence $\binom{n-i}{i}$ choices in total. Then the sign for each of these objects is obtained by the number of 2's in the integer partition, that is $(-1)^i$. Hence we have:

$$\varphi(h_n) = \sum_{\lambda \vdash n} (-1)^{n-l(\lambda)} |B_{\lambda,(n)} \varphi(e_n) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} (-1)^i (2x)^{n-2i}.$$

□

**ii.**

$$U_n(x) = \frac{2}{n}\sum_{i=0}^{n-1} U_i(x)T_{n-i}(x)$$

*Proof.* Utilizing $(59.b)$:

$$\begin{aligned}\varphi(h_n) &= \frac{1}{n}\sum_{i=0}^{n-1}\varphi(h_i)\varphi(p_{n-i}) \\ &= \frac{2}{n}\sum_{i=0}^{n-1} T_i(x)U_{n-i}(x).\end{aligned}$$

□

**iii.**

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0$$

*Proof.* Applying the fact that $H(z)E(-z) = 1$ gives us:

$$\sum_{n=0} \left( \sum_{i=0}^{n} (-1)^i e_i h_{n-i} \right) t^n.$$

Giving us that $\sum_{i=0}^{n} e_i h_{n-i} = 0$ for all $n \geq 1$. So with that in mind consider the following:

$$\begin{aligned}
\sum_{i=0}^{n-1} (-1)^i \varphi(e_i)\varphi(h_{n-i}) &= \varphi(e_0)U_n(x) - \varphi(e_1)U_{n-1}(x) + \varphi(e_2)U_{n-2}(x) \\
&= U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0.
\end{aligned}$$

□

**iv.**

$$T_n(x) - 2xT_{n-1}(x) + T_{n-2}(x) = 0$$

*Proof.* Applying (59.$c$):

$$\begin{aligned}\sum_{i=0}^{n-1}(-1)^i\varphi(e_i)\varphi(p_{n-i}) &= (-1)^{n-1}n\varphi(e_n)\\ \varphi(e_0)2T_n(x) - \varphi(e_1)2T_{n-1}(x) + \varphi(e_2)2T_{n-2}(x) &= 0\\ T_n(x) - 2xT_{n-1}(x) + T_{n-2}(x) &= 0\checkmark\end{aligned}$$

$\square$

# 65.

Define a ring homomorphism $\varphi$ on $\Lambda$ by $\varphi(e_n) = \begin{cases} (-1)^{k+k(3k-1)/2} & \text{if } n = k(3k-1)/2 \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{if not.} \end{cases}$

## a.

Show that $\varphi(h_n) = p(n)$ where $p(n)$ is the number of integer partitions of $n$.

*Proof.* Consider the following:

$$\begin{aligned}
\sum_{n=0}^{\infty} \varphi(h_n)t^n &= \varphi\left(\frac{1}{E(-t)}\right) \\
&= \frac{1}{\sum_{n=0}^{\infty} \varphi(e_n)(-t)^n} \\
&= \frac{1}{\sum_{k=0}^{\infty} (-1)^{k+k(3k-1)/2}(-t)^{k(3k-1)/2}} \\
&= \frac{1}{\sum_{k=0}^{\infty} (-1)^k t^{k(3k-1)/2}} \\
&= \frac{1}{\prod_{n=1}^{\infty}(1-t^n)} \\
&= \prod_{n=1}^{\infty} \frac{1}{1-t^n} \\
&= \sum_{n=0}^{\infty} p(n)t^n,
\end{aligned}$$

the first equality uses the fact $H(t) = \dfrac{1}{E(-t)}$, the third equality uses the fact that for $n$ that aren't pentagonal will be 0, the fifth equality uses the pentagonal number theorem, and then the last equality uses the fact that the product uses is the generating function for $p(n)$. Thus we have our result! $\square$

## b.

Apply $\varphi$ to the generating function for $p_n/n$ in Exercise 60 to show that $\varphi(p_n) = \sigma(n)$ is the sum of the positive integer divisors of $n$.

*Proof.* Consider the following:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\varphi(p_n)}{n} t^n &= \varphi(\ln(H(t))) \\
&= \varphi\left(\ln\left(\prod_{i=1}^{\infty} \frac{1}{1-t^i}\right)\right) \\
&= \varphi\left(\sum_{i=1}^{\infty} \ln \frac{1}{1-t^i}\right) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \frac{z^{ik}}{k}.
\end{aligned}$$

We can compare the coefficients of these series for matching $n$, this only occurs when $ik = n$ in symbols this gives us:

$$\varphi(p_n) = n \sum_{ik=n} \frac{1}{k} = \sum_{ik=n} i.$$

The $i$'s account for all of the positive divisors of $n$, giving us $\varphi(p_n) = \sigma(n)$. $\square$

## c.

Use an identity in Exercise 59 to show that $p(n) = \dfrac{1}{n}\sum_{i=1}^{n}\sigma(i)p(n-i)$, thereby giving us a recursion for the number of integer partitions of $n$. Calculate $p(7)$ using this recursion.

*Proof.* Using (59.$b$):

$$\begin{aligned}\varphi(h_n) &= \frac{1}{n}\sum_{i=0}^{n-1}\varphi(h_i)\varphi(p_{n-i}) \\ &= \frac{1}{n}\sum_{i=0}^{n-1}p(i)\sigma(n-i)\end{aligned}$$

This is our result after noting $\varphi(h_n) = p(n)$, $\sigma(0) = 0$ and reindexing. □

*Calculation.*

$$\begin{aligned}p(7) &= \frac{1}{7}\sum_{i=1}^{n}p(i)\sigma(n-i) \\ &= \frac{1}{7}(p(1)\sigma(6) + p(2)\sigma(5) + p(3)\sigma(4) + p(4)\sigma(3) + p(5)\sigma(2) + p(6)\sigma(1) + p(7)\sigma(0)) \\ &= \frac{1}{7}(\sigma(6) + 2\sigma(5) + 3\sigma(4) + 5\sigma(3) + 7\sigma(2) + 11\sigma(1)) \\ &= \frac{1}{7}(12 + 12 + 21 + 20 + 14 + 11) = 15.\end{aligned}$$

□