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Homework # 5

21.

Let a_n be the number of permutations of n with ordered cycles.

a.

Use the generating function is Set 3 Exercise 13c to find an asymptotic formula for a_n.

Proof. The generating function in question being $\frac{1}{1-\ln\left(\frac{1}{1-x}\right)}$. Note that the singularity here occurs when $\ln\left(\frac{1}{1-x}\right) = 1$ i.e when x satisfies: $\frac{1}{1-x} = e \implies e^{-1} = 1 - x \implies x = 1 - e^{-1}$. There might be a singularity at $x = 1$ as well, it would require further investigation, but regardless $0 < 1 - \frac{1}{e} < 1$ so that $1 - \frac{1}{e}$ is the closest singularity to 0. So plugging this into the asymptotic formula, with a test value of $\alpha = 1$

$$\begin{aligned}\lim_{x \to 1-(1/e)} \frac{(x - (1 - (1/e))}{1 - \ln\left(\frac{1}{1-x}\right)} &= \frac{\lim_{x \to 1-(1/e)} 1}{\lim_{x \to 1-(1/e)} \frac{1}{1-x}} \\ &= \frac{1}{e} = C.\end{aligned}$$

Where the second equality follows from L'Hopital's Rule. So that for asymptotic formula we have $C = e^{-1}$, $\alpha = 1$ and $R = \frac{e-1}{e}$, giving us:

$$a_n \sim \frac{Cn^{\alpha-1}}{R^{n+\alpha}\Gamma(\alpha)} = \frac{e^{-1}n^0}{1\left(\frac{e-1}{e}\right)^{n+1}} = \frac{e^n}{(e-1)^{n+1}}.$$

□

b.

Find an asymptotic formula for the average number of cycles in a permutation of n with ordered cycles.

Proof. Note that the average will be given by:

$$\frac{\text{Total number of permutations with ordered cylces}}{\text{Total number of permutations}} = \frac{a_n}{n!}.$$

That is we got the numerator from part $(a.)$ of this problem, and that the total number of permutations of n is always $n!$, so using the asymptotic result from the above and that multiplication preserves the asymptotic relation:

$$\frac{a_n}{n!} \sim \frac{e^n}{n!(e-1)^{n+1}}.$$

$\square$

22.

Find an asymptotic formula for the probability that a permutation of n doesn't have a cycle of length $1, 2$, or 3.

Proof. Note that we have the generating function of the number of permutations of n that don't contain 1-cycles as being:

$$e^{\ln\left(\frac{1}{1-x}\right)-x}.$$

Following the same reasoning using the exponential formula we get that the generating function for the number of permutations of n that don't contain $1, 2$ or 3 cycles is:

$$e^{\ln\left(\frac{1}{1-x}\right)-x-\frac{x^2}{2}-\frac{x^3}{3}} = \frac{e^{-x-\frac{x^2}{2}-\frac{x^3}{3}}}{1-x}.$$

The only singularity is 1 so that we get that:

$$\begin{aligned}\lim_{x\to 1}\frac{(1-x)e^{-x-\frac{x^2}{2}-\frac{x^3}{3}}}{1-x} &= \frac{\lim_{x\to 1} e^{-x-\frac{x^2}{2}-\frac{x^3}{3}}}{\lim_{x\to 1} 1} \\ &= e^{-1-\frac{1}{2}-\frac{1}{3}} \\ &= e^{-\frac{3}{2}-\frac{1}{3}} \\ &= e^{-\frac{11}{6}}.\end{aligned}$$

This is our C value in the asymptotic theorem and we have $R = 1$ and $\alpha = 1$ from the singularity, so that we get the following asymptotic result (with a_n being the total number of permutations of n with no $1, 2, 3$ cycles):

$$\begin{aligned}\frac{\text{The number of permutations of n with no 1,2,3 cycles}}{\text{The total number of permutations of n}} &= \frac{a_n}{n!} \\ &\sim \frac{Cn^{\alpha-1}}{\Gamma(\alpha)R^{n+\alpha}} \\ &= e^{-11/6}\frac{n^{1-1}}{\Gamma(1)1^{n+1}} \\ &= e^{\frac{-11}{6}}.\end{aligned}$$

So the probability of this is approximately $e^{-11/6} \approx 0.1598$. □

23.

We define

1.

The Chebyshev polynomial of the first kind $T_n(y)$ by $\sum_{n=0}^{\infty} T_n(y)x^n = \frac{1-yx}{1-2yx+x^2}$,

Proof. Note that for $y = 5/4$ we have:

$$\sum_{n=0}^{\infty} T_n(5/4)x^n = \frac{1-(5/4)x}{1-(5/2)x+x^2}.$$

Using the quadratic formula we get that the zeros of the denominator polynomial to be $x = 2, \frac{1}{2}$. So that the nearest singularity to 0 is 1/2. So that $R = \frac{1}{2}$ and we get the following:

$$\begin{aligned}\lim_{x\to 1/2} \frac{(x-1/2)(1-(5/4)x)}{1-(5/4)x+x^2} &= \lim_{x\to 1/2} \frac{(-5/4)x^2+(13/8)x-(1/2)}{1-(5/4)x+x^2} \\ &= \frac{\lim_{x\to 1/2}(-5/2)x+(13/8)}{\lim_{x\to 1/2} 2x-(5/2)} \\ &= -\frac{1}{4}.\end{aligned}$$

So that we have $\alpha = 1$ and $R = \frac{1}{2}$ and $C = -\frac{1}{4}$ in the asymptotic formula, so we get: $T_n(5/4) \sim \frac{Cn^{\alpha-1}}{\Gamma(\alpha)R^{n+\alpha}} = \frac{-1}{4}\frac{n^0}{\Gamma(1)(1/2)^{n+1}} = -\frac{2^{n+1}}{4} = -2^{n-1}$ □

2.

The Chebyshev Polynomial of the second kind $U_n(y)$ by $\sum_{n=0}^{\infty} U_n(y)x^n = \frac{1}{1-2yx+x^2}$, and

Proof. We start off with $\sum_{n=0}^{\infty} U_n(5/4)x^n = \frac{1}{1-2(5/4)x+x^2} = \frac{1}{1-(5/2)x+x^2}$. Additionally the denominator polynomial is the same as (1.) so that we get singularities of $x = 2, 1/2$.

$$\lim_{x\to 1/2} \frac{(x-1/2)}{x^2-(5/2)x+1} = \frac{\lim\limits_{x\to 1/2} 1}{\lim\limits_{x\to 1/2} 2x-(5/2)}$$
$$= \frac{1}{-3/2} = \frac{-2}{3}.$$

So that we have $R = \frac{1}{2}$, $C = -\frac{2}{3}, \alpha = 1$. So that the asymptotic formula gives us:

$$\begin{aligned} U_n(5/4) &\sim \frac{Cn^{\alpha-1}}{\Gamma(\alpha)R^{n+\alpha}} \\ &= -\frac{2}{3}\frac{n^{1-1}}{\Gamma(1)(1/2)^{n+1}} \\ &= -\frac{2^{n+2}}{3}. \end{aligned}$$

□

3.

The Legendre polynomial $P_n(y)$ by $\sum\limits_{n=0}^{\infty} P_n(y)x^n = \frac{1}{\sqrt{1-2yx+x^2}}$.

Proof. The polynomial in the denominator now has a square root around it, but the singularities remain at $x = 2, \frac{1}{2}$. So evaluating the limit with WolframAlpha we get that:

$$\lim_{x\to\frac{1}{2}} (\frac{1}{2}-x)^{1/2} \frac{1}{(x-1)^{1/2}(x-1/2)^{1/2}} = \sqrt{\frac{2}{3}}.$$

So that $C = \sqrt{\frac{2}{3}}$ with $\alpha = 1/2, R = 1/2$ so the asymptotic formula gets us:

$$\begin{aligned} P_n(5/4) &\sim \frac{Cn^{\alpha-1}}{\Gamma(\alpha)R^{n+\alpha}} \\ &= \sqrt{\frac{2}{3}}\frac{n^{-1/2}}{\Gamma(1/2)(1/2)^{n+1/2}} \\ &= \frac{2^{n+1}}{n^{1/2}\sqrt{3\pi}}. \end{aligned}$$

□

Find asymptotic formulas for $T_n(5/4), U_n(5/4)$, and $P_n(5/4)$.

24.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a complex valued function with nonnegative real coefficients $a_n \geq 0$. Suppose that a singularity of f with smallest complex magnitude R (this means that R is the radius of convergence of $f(z)$ and that the series $f(z_0)$ diverges for all z_0 with $|z_0| > R$). This exercise will show that R is a singularity of f.

a.

Show that $f(z) = \sum_{k=0}^{\infty} \left(\sum_{n=k} \binom{n}{k} a_n (R/2)^{n-k} \right) \left(z - \frac{R}{2}\right)^k$ in some neighborhood of $R/2$.

Proof. We'll show this by using the generalized binomial theorem plus the swapping of two sums:

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} a_n z^n \\
&= \sum_{n=0}^{\infty} a_n \left(z - \frac{R}{2} + \frac{R}{2}\right)^n \\
&= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{\infty} \binom{n}{k} \left(\frac{R}{2}\right)^{n-k} \left(z - \frac{R}{2}\right)^k \\
&= \sum_{k=0}^{\infty} \left(z - \frac{R}{2}\right)^k \sum_{n=0}^{\infty} a_n \binom{n}{k} \left(\frac{R}{2}\right)^{n-k} \\
&= \sum_{k=0}^{\infty} \left(z - \frac{R}{2}\right)^k \sum_{n=k}^{\infty} a_n \binom{n}{k} \left(\frac{R}{2}\right)^{n-k}.
\end{aligned}$$

Third equality follows from the generalized binomial theorem of series, Fourth equality swaps the two sums (assuming that its convergent in the neighborhood of $\frac{R}{2}$), and the final takes advantage that in the generalized binomial coefficient that if $n < k$ then $\binom{n}{k} = 0$. □

b.

Looking for a contradiction, assume that R is not a singularity of f. This means that there's an $\epsilon > 0$ such that the above expression is valid for $R+\epsilon$. Take $R+\epsilon$ in the above expression and prove that

$$f(R + \epsilon) = \sum_{n=0}^{\infty} a_n (R + \epsilon)^n.$$

Why is this a contradiction? Where was the hypothesis that $a_n \geq 0$ used?

Proof. Using the above result we get that:

$$\begin{aligned}
f(R+\epsilon) &= \sum_{k=0}^{\infty} \left(R+\epsilon-\frac{R}{2}\right)^k \sum_{n=k}^{\infty} a_n \binom{n}{k} \left(\frac{R}{2}\right)^{n-k} \\
&= \sum_{k=0}^{\infty} \left(\frac{R}{2}+\epsilon\right)^k \sum_{n=k}^{\infty} a_n \binom{n}{k} \left(\frac{R}{2}\right)^{n-k} \\
&= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{\infty} \binom{n}{k} \left(\frac{R}{2}\right)^{n-k} \left(\frac{R}{2}+\epsilon\right)^k \\
&= \sum_{n=0}^{\infty} a_n \left(\frac{R}{2}+\frac{R}{2}+\epsilon\right)^n \\
&= \sum_{n=0}^{\infty} a_n \left(R+\epsilon\right)^n.
\end{aligned}$$

The first inequality coming from plugging in $\frac{R}{2}+\epsilon$ into the above expression, the third we swap the sums, and then the fourth comes from identifying the binomial theorem for series.

Assuming that R isn't a singularity of f, then the above chain is valid. But we have that any $|z_0| > R$ that f diverges for such z_0's. But note that no matter $\epsilon > 0$ we have that $R+\epsilon > R$ and thus $f(R+\epsilon)$ diverges, a contradiction of the above, since we say that this function is well-defined on $\mathbb{C}$ (i.e not the extended real line, so $f(R+\epsilon) \neq \infty$).

The $a_n \geq 0$ comes into play since otherwise it's possible we might have something like a telescoping series, where even though $R+\epsilon > R$ the value of the series could very well be finite such as with a telescoping series. $\square$