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Homework #4

MATH 550
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2.B # 10

Show that the set of real numbers that have a decimal expansion with the digit 5 appearing infinitely often is a Borel set.

Proof. First define the following shorthand

$$A = \{\text{the set of all real numbers that have a decimal representation where 5 occurs an infinite number of times}\},$$

then note that $R \setminus A = A^C =$

$$\{\text{The set of all real numbers that have a decimal representation where 5 occurs a finite numbers of times}\}.$$

So first, I'll show that A^C is Borel, implying that A is Borel. But I'll do this on the interval $[0,1]$, so that through translation invariance of Borel sets (Exercise $2.B.7$) we will get that all of A^C is Borel.

So let's note now that we can construct A^C in the following manner:

$$\begin{aligned} A^C \cap [0,1] &= \bigcup_{n=1}^{\infty} \{x \in \mathbb{R} : x = 0.\ \ldots\ a_{i_1-1}5a_{i_1+1}\ldots a_{i_n-1}5a_{i_n+1}\ldots,\ \text{and } a \in \{0,1,2,3,4,6,7,8,9\}\} \\ &= \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{n} \{x \in \mathbb{R} : x = 0.\ \ldots\ a_{i_k-1}5a_{i_k+1}\ \ldots, a \in \{0,1,2,3,4,5,6,7,8,9\}\}. \end{aligned}$$

So notice that the a's can attain any value in $\{0,1,2,3,4,5,6,7,8,9\}$, so that this set is actually an interval, so we'll show this. First note that a terminating decimal representation such as $0.\ \ldots\ a_{i_k-1}5$ can just be represented as: $0.\ \ldots\ a_{i_k-1}500\ \ldots.$ Take for any $0 \leq k \leq n$, $x \in [0.\ \ldots\ a_{i_k-1}5, 0.\ \ldots\ a_{i_k-1}6]$, then $0.\ \ldots\ a_{i_k-1}5 \leq x \leq 0.\ \ldots\ a_{i_k-1}6$ so that $x = 0.\ \ldots\ a_{i_k-1}5a_{i_k+1}\ \ldots$, hence $x \in \{x \in \mathbb{R} : x = 0.\ \ldots\ a_{i_k-1}5a_{i_k+1}\ \ldots, a \in \{0,1,2,3,4,5,6,7,8,9\}\}$.

Conversely, take $x \in \{x \in \mathbb{R} : x = 0.\ \ldots\ a_{i_k-1}5a_{i_k+1}\ \ldots, a \in \{0,1,2,3,4,5,6,7,8,9\}\}$, so then $x = 0.\ \ldots\ a_{i_1-1}5a_{i_1+1}\ \ldots\ a_{i_n-1}5a_{i_n+1}\ \ldots$ for some integers $a \in \{0,\ldots,9\}$. Notice that $0 \leq a \leq 9$ so that: $x = 0.\ \ldots\ a_{i_1-1}500\ \ldots\ a_{i_n-1}5a_{i_n+1}\ \ldots \leq 0.\ \ldots\ a_{i_1-1}5a_{i_1+1}\ \ldots\ a_{i_n-1}599\ \ldots.$ But notice that $0.\ \ldots\ a_{i_1-1}5a_{i_1+1}\ \ldots\ a_{i_n-1}599\ \ldots\ = 0.\ \ldots\ a_{i_k-1}600\ldots.$ So we have $0.\ \ldots\ a_{i_k-1}5 \leq x \leq 0.\ \ldots\ a_{i_k-1}6$, and hence $x \in [0.\ \ldots\ a_{i_k-1}5, 0.\ \ldots\ a_{i_k-1}6]$ and thus: $\{x \in \mathbb{R} : x = 0.\ \ldots\ a_{i_k-1}5a_{i_k+1}\ \ldots, a \in \{0,1,2,3,4,5,6,7,8,9\}\} = [0.\ \ldots\ a_{i_k-1}5, 0.\ \ldots\ a_{i_k-1}6]$.

Well, note that this is an interval! Intervals are Borel (i.e all half-closed, closed and open intervals) so that the finite and then countable union we got from the previous expression is Borel. Thus

$$A^C \cap [0,1] = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{n} \{x \in \mathbb{R} : x = 0.\ \ldots\ a_{i_k-1}5a_{i_k+1}\ \ldots, a \in \{0,1,2,3,4,5,6,7,8,9\}\}$$

is a Borel set.

To show that all of A^C is Borel, note that we can use translation invariance of Borel sets to get that $(A^C \cap [0,1]) + t$ is Borel for any $t \in \mathbb{R}$. So that the set

$$\bigcup_{n=1}^{\infty}(((A^C \cap [0,1]) + n) \cup (((A^C \cap [0,1]) - n),$$

must be Borel. I'll show that $A^C = \bigcup\limits_{n=1}^{\infty}(((A^C \cap [0,1]) + n) \cup (((A^C \cap [0,1]) - n)$. Let $x \in A^C$ then x has a decimal representation that has a finite number of $5'$s in it. So that there must exist a n or $-n$ such that either $-n \leq x$ or $x \leq n$ so that $x \in (A^C \cap [0,1]) + n = \{x + n \in \mathbb{R} :$ has a decimal expansion with a finite number of 5's$\}$ or $x \in (A^C \cap [0,1]) - n = \{x - n \in \mathbb{R} :$ has a decimal expansion with a finite number of 5's$\}$.

Conversely, take $x \in \bigcup_{n=1}^{\infty}(((A^C \cap [0,1]) + n) \cup (((A^C \cap [0,1]) - n)$ so that for some $n \in \mathbb{N}$ we have either $x \in (A^C \cap [0,1]) + n = \{x + n \in \mathbb{R} :$ has a decimal expansion with a finite number of 5's$\}$ or $x \in (A^C \cap [0,1]) - n = \{x - n \in \mathbb{R} :$ has a decimal expansion with a finite number of 5's$\}$. Clearly adding (or subtracting) a natural number will only add or subtract a finite number of $5'$s from a decimal expansion thus $x \in A^C$.

Thus, finally, we have that A^C is Borel and hence A is Borel. $\square$

2.B # 21

Prove 2.52.

Theorem. 1 (2.52: Condition for Measurable function). *Suppose (X, S) is a measurable space and $f : X \to [-\infty, \infty]$ is a function such that*

$$f^{-1}((a, \infty]) \in S$$

for all $a \in \mathbb{R}$. Then f is an $S-$measurable function.

Proof. Let (X, S) be a measurable space and $f : X \to [-\infty, \infty]$ is a function such that

$$f^{-1}((a, \infty])$$

for all $a \in \mathbb{R}$. Then we'll follow a similar proof to 2.39 translated to extended Borel sets.
Let $\mathrm{T} = \{A \subset [-\infty, \infty] : f^{-1}(A) \in S\}$. Now to show that T is a $\sigma-$algebra of $[-\infty, \infty]$.

- We have $f^{-1}(\emptyset) = \emptyset$, so that $\emptyset \in \mathrm{T}$.
- Let $\hat{A} \in \mathrm{T}$, then $f^{-1}(\hat{A}) \in S$; so that because of S being a $\sigma-$algebra we have it's closed under complementation:

  $$f^{-1}([\infty, \infty] \setminus \hat{A}) = X \setminus f^{-1}(\hat{A}) \in S.$$

  With the last equality coming from (2.33(a.)) which holds for any sets including the extended real line. So by the definition of T we have $[-\infty, \infty] \setminus \hat{A} \in \mathrm{T}$.
- Now let $\hat{A}_1, \hat{A}_2, \ldots \in \mathrm{T}$ so that $f^{-1}(\hat{A}_1), f^{-1}(\hat{A}_2), \ldots \in S$; with S being a $\sigma-$algebra we have that and by (2.33(b)) :

  $$f^{-1}\left(\bigcup_{k=1}^{\infty} \hat{A}_k\right) = \bigcup_{k=1}^{\infty} f^{-1}(\hat{A}_k) \in S.$$

  So that $\bigcup\limits_{k=1}^{\infty} \hat{A}_k \in \mathrm{T}$.

Thus T is a $\sigma-$algebra on $[-\infty, \infty]$.
Then note that T contains $(a, \infty]$ for all $a \in \mathbb{R}$ so that it contains $[-\infty, b]$ for all $b \in \mathbb{R}$ so that it contains any open interval (a, b) for $a, b \in [-\infty, \infty]$. So that it must contain $(-\infty, \infty) = \mathbb{R}$. That is, it must contain (a, b) when $a = -\infty$ and $b = \infty$. Using Problem 11 we have that T being a $\sigma-$Algebra implies that $\{F \cap \mathbb{R} : F \in \mathrm{T}\}$ is a $\sigma-$algebra on $\mathbb{R}$.

Then note that for T to contain all the extended Borel sets, we need to just show that this $\sigma-$algebra contains all of the regular Borel sets of $\mathbb{R}$. That is, a set A is Borel in $[-\infty, \infty]$ if and only if $A \cap \mathbb{R}$ is Borel in $\mathbb{R}$. Since this is a $\sigma-$algebra and (a, b) is contained in it, we have that every open set is contained in this $\sigma-$algebra. Due to open sets in $\mathbb{R}$ being countable unions of open intervals. Meaning that S contains every Borel set of $\mathbb{R}$, hence T contains every extended Borel set of $[-\infty, \infty]$. That is hence f is $S-$measurable. $\square$

Homework #4

2.B # 27

Prove or give a counterexample: If (X, S) is a measurable space and

$$f : X \to [-\infty, \infty]$$

is a function such that $f^{-1}((a, \infty)) \in S$ for every $a \in \mathbb{R}$, then f is an $S-$measurable function.

Counterexample. Using the space $(\mathbb{R}, S)$ with $S = \{\emptyset, \mathbb{R}, (-\infty, 0), [0, \infty)\}$, we have that a function is S-measurable on this space if and only if that function is constant on both $(-\infty, 0)$ and $[0, \infty)$. So define the function $f : \mathbb{R} \to [-\infty, \infty]$ given by:

$$f(x) = \begin{cases} -\infty & x = 0 \\ +\infty, & x \neq 0 \end{cases}.$$

So that for any $a \in \mathbb{R}$ we have:

$$f^{-1}((a, \infty)) = \emptyset.$$

But of course f isn't constant on $[0, \infty)$ so that f isn't $S-$measurable in this space. $\square$

2.B # 29

Give an example of a measurable space (X, S) and a family $\{f_t\}_{t\in\mathbb{R}}$ such that each f_t is an $S-$measurable function from X to $[0,1]$, but the function $f : X \to [0,1]$ defined by

$$f(x) = \sup\{f_t(x) : t \in \mathbb{R}\}$$

isn't $S-$measurable. [Compare this exercise to 2.53, where the index set is $\mathbb{Z}^+$ rather than $\mathbb{R}$.]

Proof. Let $X = \mathbb{R}$, with S being the collection of Borel sets of $\mathbb{R}$. Then since the collection of Borel sets isn't equal to all subsets of the real line, this was the defining feature of why Borel sets are needed instead of the power set of $\mathbb{R}$, we can choose a non-Borel set $E \subset \mathbb{R}$. Then, as with any set, we may write it as the union of singletons:

$$E = \bigcup_{t\in E}\{t\}.$$

Now define consider the characteristic function on the singletons:

$$\chi_{\{t\}}(x) = \begin{cases} 1 & x = t \\ 0 & \text{otherwise} \end{cases}.$$

We have that since any singleton is Borel, because it's complement is a union of open intervals, so that by Example (2.38) that each $\chi_{\{t\}}$ is individually $S-$measurable from $\mathbb{R}$ to $[0,1]$. Now define the collection of function $\{f_t\}_{t\in\mathbb{R}}$ by $f_t : \mathbb{R} \to [0,1]$ given by:

$$f_t = \begin{cases} \chi_{\{t\}} & \text{if, } t \in E \\ 0 & \text{otherwise.} \end{cases}$$

Since 0 is continuous on all of $\mathbb{R}$ we have that f_t when $t \notin E$ that f_t is Borel measurable. So that in either case f_t is Borel-measurable for all $t \in \mathbb{R}$.

But taking the pointwise supremum of this collection, that is the function $f : \mathbb{R} \to [0,1]$ given by:

$$f(x) = \sup\{f_t(x) : t \in \mathbb{R}\}$$

will be $f(x) = \chi_E(x)$.

To show this, note that $\sup\{f_t(x) : t \in \mathbb{R}\}$ is going to be 0 for all $x \notin E$ and 1 for all $x \in E$, because the only points where this supremum will be 1 is on the points of E, hence

$$f(x) = \chi_E(x).$$

Note that from Example 2.38 we have that χ_E is Borel measurable if and only if E is Borel. By our construction E is not Borel and thus χ_E isn't Borel. So that while the collection is individually $S-$measurable, but the pointwise supremum f over this collection isn't $S-$measurable.

Note that this would not work if E were countable, then E would be Borel, so that in the case where the index set is $\mathbb{Z}^+$ this example wouldn't work. $\square$

2.C # 1

Explain why there doesn't exist a measure space (X, S, μ) with the property that $\{\mu(E) : E \in S\} = [0, 1)$.

Proof. Let (X, S, μ) be a measure space and for sake of contradiction assume that we have the following property of the measure:

$$\{\mu(E) : E \in S\} = [0, 1).$$

So since $\mu : S \to [0, 1)$ is well-defined, this implies that S is uncountable. Since I can take any $x \in [0, 1)$ (an uncountable set in $\mathbb{R}$) and then there must be a set E such that $\mu(E) = x$, so μ is surjective onto $[0, 1)$ meaning that S must be at least uncountable.

So then for any $0 < \epsilon \leq 1$ there must exist a $E_\epsilon \in S$ such that $\mu(E_\epsilon) = 1 - \epsilon$. So that there must a collection in S such that $\{E_n\}_{n=1}^{\infty}$ with $\mu(E_n) = 1 - \frac{1}{n}$. So then define the sequence $G_1 = E_1, G_2 = G_1 \cup E_2, \ldots, G_n = G_{n-1} \cup E_n, \ldots$, so that clearly this is an increasing sequence of sets, i.e $G_1 \subset G_2 \subset \ldots \subset G_n \subset \ldots$. Since S is a $\sigma-$algebra each $G_n \in S$ and so that the big union of $\{G_n\}$ is also in S. So that applying (2.59) we have that:

$$\mu\left(\bigcup_{k=1}^{\infty} G_k\right) = \lim_{k \to \infty} \mu(G_k).$$

But notice that $\mu(G_k) = \mu(E_1 \cup \ldots \cup E_k) \geq \mu(E_k) = 1 - \frac{1}{k}$. So that $\lim_{k \to \infty} \mu(G_k) \geq \lim_{k \to \infty} 1 - \frac{1}{k} = 1$. So that $\mu\left(\bigcup_{k=1}^{\infty} G_k\right) \geq 1$, a contradiction. □

2.C # 2

Let $2^{\mathbb{Z}^+}$ denote the $\sigma-$algebra on $\mathbb{Z}^+$ consisting of all subsets of $\mathbb{Z}^+$. Suppose μ is a measure on $(\mathbb{Z}^+, 2^{\mathbb{Z}^+})$. Prove that there's a sequence $w_1, w_2, \dots$ in $[0, \infty]$ such that

$$\mu(E) = \sum_{k \in E} w_k$$

for every set $E \subset \mathbb{Z}^+$.

Proof. Let $2^{\mathbb{Z}^+}$ denote the $\sigma-$algebra on $\mathbb{Z}^+$ consisting of all subsets of $\mathbb{Z}^+$. Suppose μ is a measure on $(\mathbb{Z}^+, 2^{\mathbb{Z}^+})$. That is $\mu : 2^{\mathbb{Z}^+} \to [0, \infty]$ is a well-defined function.

Let $E \subseteq \mathbb{Z}^+$. Note then that we can write E as a countable union of singletons, because any subset of $\mathbb{Z}$ is either finite or countable, thus:

$$E = \bigcup_{t \in E} \{t\}.$$

Moreover because μ is a measure (has additivity) we have that:

$$\mu(E) = \mu\left(\bigcup_{t \in E} \{t\}\right) = \sum_{t \in E} \mu(\{t\}).$$

This must hold for any $E \subseteq \mathbb{Z}^+$.

So define the sequence $w_1, w_2, \dots$ by $w_1 = \mu(\{1\}), w_2 = \mu(\{2\}), \dots, w_n = \mu(\{n\}), \dots$. Then we have that for any $E \subset \mathbb{Z}^+$

$$\mu(E) = \sum_{t \in E} \mu(\{t\}) = \sum_{t \in E} w_t.$$

And we have our result! $\square$

2.C # 3

Give an example of a measure μ on $(\mathbb{Z}^+, 2^{\mathbb{Z}^+})$ such that

$$\{\mu(E) : E \subset \mathbb{Z}^+\} = [0, 1].$$

Example. Let $\mu : 2^{\mathbb{Z}^+} \to [0, \infty]$ be the function given by $\mu(E)$ =for all $k \in E$ place a 1 in that place in a decimal expansion and 0 everywhere else divided by $0.11111\ldots$, e.g. $\mu(\{1\}) = \frac{0.1}{0.1111\ldots}\ldots$, $\mu(\{1,3\}) = \frac{0.101000}{0.111111111\ldots}\ldots$. Note the reason for this choice is that without dividing by $0.111\ldots$ we get that this function is one-to-one and onto the interval $[0, 0.1111\ldots]$ so we extended this to $[0,1]$ by dividing by $0.111\ldots$. So to show that this is a measure on $(\mathbb{Z}^+, 2^{\mathbb{Z}^+})$, note that $\mu(\emptyset)$ places no 1's so that $\mu(\emptyset) = \frac{0.0000\ldots}{0.11111\ldots} = 0$.

Now take a disjoint sequence in $2^{\mathbb{Z}^+}$, $\{E_k\}_{k=1}^{\infty}$. Then note for them to be disjoint, 1's will only occur in distinct "slots" of the decimal expansion. Then $\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = x$, where x has a decimal expansion that is 0 everywhere except where it's 1 at all places k in it's decimal expansion where $k \in \bigcup_{k=1}^{\infty}$ then all divided by $0.111\ldots$. Then adding these $\mu(E_k)$'s up, because they are only 1 and 0 we don't have to worry about carrying anything over, additionally, because E_k's are all distinct they add up to get us precisely the same thing as $\mu(\bigcup_{k=1}^{\infty} E_k)$. So we get that μ is a measure on $(\mathbb{Z}^+, 2^{\mathbb{Z}^+})$.

To show that $\{\mu(E) : E \subset \mathbb{Z}^+\} \subseteq [0,1]$, note that $0 \leq \mu(E)$ for all $E \subset \mathbb{Z}^+$ moreover, note that the largest output of μ is: $\mu(\mathbb{Z}^+) = \frac{0.111\ldots}{0.111\ldots} = 1$. So that we have μ is bounded between 0 and 1 hence $\{\mu(E) : E \subset \mathbb{Z}^+\} \subseteq [0,1]$.

Conversely, take $x \in [0,1]$. Note then that x has a decimal expansion such that $x = 0.a_1a_2\ldots a_n\ldots$. Multiplying by $0.1111\ldots$: $x(0.111\ldots) = (0.a_1a_2\ldots)(0.111\ldots)$ this will put us in the interval $[0, 0.111\ldots]$, since x is at most 1. (This is a stretch) Note that by dividing by $0.111\ldots$ of any binary decimal representation that μ produces will give us a value in $[0,1]$ so that there will exist a $E \subseteq \mathbb{Z}^+$ such that $\mu(E) = x$. Thus $\{\mu(E) : E \subseteq \mathbb{Z}^+\} = [0,1]$. $\square$

2.C # 8

Give an example of a set X, a $\sigma-$algebra S of subsets of X, a set A of subsets of X such that the smallest $\sigma-$algebra on X containing $\mathcal{A}$ is S, and two measures μ and ν on (X, S) such that $\mu(A) = \nu(A)$ for all $A \in \mathcal{A}$ and $\mu(X) = v(X) < \infty$, but $\mu \neq v$.

Example. Let $X = \{1, 2, 3, 4\}$, $\mathcal{A} = \{\{1, 2, 3\}, \{2\}\}$ so that the smallest $\sigma-$algebra containing this collection can be constructed using complements, intersections and unions(we need closure under all these things): $\{1, 2, 3\}^C = \{4\}$, $\{2\}^C = \{1, 3, 4\}$. $\{4\} \cup \{2\} = \{2, 4\}$ and $\{1, 3, 4\} \cap \{1, 2, 3\} = \{1, 3\}$. These last two sets will be the crux of the example. Define the measure

E	$\mu(E)$	$\nu(E)$
$\emptyset$	0	0
$\{1, 2, 3, 4\}$	1	1
$\{1, 2, 3\}$	$\frac{1}{2}$	$\frac{1}{2}$
$\{2\}$	$\frac{1}{2}$	$\frac{1}{2}$
$\{1, 3, 4\}$	$\frac{1}{2}$	$\frac{1}{2}$
$\{4\}$	$\frac{1}{2}$	$\frac{1}{2}$
$\{2, 4\}$	$\frac{3}{4}$	$\frac{9}{10}$
$\{1, 3\}$	$\frac{1}{4}$	$\frac{1}{10}$

It can be checked that this preserves additivity and clearly that $\emptyset \mapsto 0$. Note that the only reason this works is that we produced a σ-algebra with an odd number of elements, so that the requirement that they agree on $\mathcal{A}$ forces that they agree on their complements as well, we got around this by the extra sets that came through the intersections. $\square$

2.C # 9

Suppose μ and ν are measures on a measurable space $(X, \mathcal{S})$. Prove that $\mu+\nu$ is a measure on $(X, \mathcal{S})$. [Here $\mu+\nu$ is the usual sum of two functions: if $E \in \mathcal{S}$, then $(\mu+\nu)(E) = \mu(E)+\nu(E)$.]

Proof. Let $(X, \mathcal{S})$ be a measurable space and μ, ν be measures on this space. To show that $\mu+\nu$ is a measure, note that $(\mu+\nu)(\emptyset) = \mu(\emptyset)+\nu(\emptyset) = 0+0 = 0$, the second equality coming from function addition def., the third coming from that both μ and ν are measures so that they send the empty set to 0.

To complete the proof, take a disjoint sequence of sets in $\mathcal{S}$ call this $\{E_k\}_{k=1}^{\infty}$.

So note that since both μ and ν are measures we have the following:

$$\begin{aligned}(\mu+\nu)\left(\bigcup_{k=1}^{\infty} E_k\right) &= \mu\left(\bigcup_{k=1}^{\infty} E_k\right)+\nu\left(\bigcup_{k=1}^{\infty} E_k\right)\\ &= \sum_{k=1}^{\infty}\mu(E_k)+\sum_{k=1}^{\infty}\nu(E_k)\\ &= \sum_{k=1}^{\infty}\mu(E_k)+\nu(E_k)\\ &= \sum_{k=1}^{\infty}(\mu+\nu)(E_k).\end{aligned}$$

The first and last equality come from definition of function addition, second equality comes from μ and ν being countable additivity, and the third equality only holds if both series are convergent. So this holds and our result follows if both are convergent. Otherwise assume that at least one of the series diverges, without loss of generality, assume that $\sum_{k=1}^{\infty}\mu(E_k)$. So then regardless of $\nu(E_k)$ we'll get that $\sum_{k=1}^{\infty}\mu(E_k)+\sum_{k=1}^{\infty}\nu(E_k) = \infty+\sum_{k=1}^{\infty}\nu(E_k) = \infty$ (Note that $\nu(E_k) \geq 0$ so that the series can't equal $-\infty$). Well with the countable additivity of μ we get that $\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \infty$. So that $(\mu+\nu)\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu\left(\bigcup_{k=1}^{\infty} E_k\right)+\nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \infty+\nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \infty$(again $\nu(E_k) \geq 0$ so that the series can't equal $-\infty$). Thus the result holds if one of the series is divergent.

Thus our result holds for both cases and we're done. $\square$

2.C # 10

Give an example of a measure space $(X, \mathcal{S}, \mu)$ and a decreasing sequence $E_1 \supset E_2 \supset \ldots$ of sets in $\mathcal{S}$ such that

$$\mu\left(\bigcap_{k=1}^{\infty} E_k\right) \neq \lim_{k\to\infty} \mu(E_k).$$

Proof. Take $X = \mathbb{R}$ and $\mathcal{S}$ to be the Borel collection on $\mathbb{R}$. Then define the function $\mu : \mathcal{S} \to [0, \infty]$ to be the counting measure discussed in (2.55). Note that $\mu(\emptyset) = 0$. By that same example we have that this is a measure on any space $(X, \mathcal{S})$ so that $(\mathbb{R}, \mathcal{S}, \mu)$ is a measure space.

Now define the sequence $\{E_n\}_{n=1}^{\infty}$ by $E_n = \{n, n+1, \ldots\}$ for all $n \in \mathbb{N}$. Moreover, note that these are all Borel sets because they are at most countable. Note that this is a decreasing sequence: $\{1, 2, \ldots\} \supset \{2, 3, \ldots\} \supset \ldots \supset \{n, n+1, \ldots\} \supset \ldots$. But for any $E_n = \{n, n+1, \ldots\}$ we have that $\mu(E_n) = \infty$ because E_n is countable for all $n \in \mathbb{N}$, so that $\lim\limits_{n\to\infty} \mu(E_n) = \lim\limits_{n\to\infty} \infty = \infty$. However, note that

$$\bigcap_{n=1}^{\infty} E_n = \bigcap_{n=1}^{\infty} \{n, n+1, \ldots\} = \emptyset.$$

To show this, for sake of contradiction suppose that $x \in \bigcap\limits_{n=1}^{\infty} E_n$, then $x \in \{n, n+1, \ldots\}$ for all $n \in \mathbb{N}$. So that for all $n \in \mathbb{N}$ we have $n \leq x$ a contradiction, because $\mathbb{N}$ is well-ordered and countably infinite. So that we have that $\mu\left(\bigcap\limits_{n=1}^{\infty} E_n\right) = \mu(\emptyset) = 0$. Of course $0 \neq \infty$ so that we're done we have that this sequence satisfies $\mu\left(\bigcap\limits_{k=1}^{\infty} E_k\right) \neq \lim\limits_{k\to\infty} \mu(E_k)$. ☐

2.C # 12

Suppose X is a set and $\mathcal{S}$ is the $\sigma-$algebra of all subsets E of X such that E is countable or $X \setminus E$ is countable. Give a complete description of the set of all measures on $(X, \mathcal{S})$.

Proof. Let X be a set and $\mathcal{S}$ is the $\sigma-$algebra of all subsets of X such that E is countable or $X \setminus E$ is countable.

To provide such a characterization, I'll show in general what restrictions we have on the measure for sets in $\mathcal{S}$, and what restrictions we have on the measure of their complements and the measure of the whole space X.

Let $E \in \mathcal{S}$, then either E is countable or $X \setminus E$ is countable. Let μ be any measure on the space $(X, \mathcal{S})$.

If E is countable, then by additivity of measure we have that $\mu(E) = \mu\left(\bigcup_{k\in E}\{k\}\right) = \sum_{k\in E}\mu(\{k\})$.

Now consider the case where $X \setminus E$ is countable, in which case the entire space X is countable. So we don't have a lot of new information here just:

$$\mu(X \setminus E) = \sum_{k\in X\setminus E}\mu(\{k\})$$

and

$$\mu(X) = \sum_{k\in X}\mu(\{k\}) + \sum_{k\in X\setminus E}\mu(\{k\}) = \sum_{k\in X}\mu(\{k\}).$$

Alternatively, suppose that $X \setminus E$ is uncountable. Then the entire space X must also be uncountable. So then either $\mu(E) = \infty$ or $\mu(E) < \infty$. In the case that $\mu(E) = \infty$, there are no restrictions on $\mu(X \setminus E)$; that is, imposing additivity of disjoint unions we have that $\mu(E) + \mu(X \setminus E) = \mu(X)$ and with $\mu(E)$ we can either have $\mu(X \setminus E) < \infty$ or $\mu(X \setminus E) = \infty$ either giving us $\mu(X) = \infty$. If $\mu(E) < \infty$, then we can fully characterize the measure of every complement by:

$$\mu(X \setminus E) = \mu(X) - \sum_{k\in E}\mu(\{k\}).$$

Note $\mu(X)$ could possibly be infinite, in which case $\mu(X \setminus E) = \infty$.

So we have fully characterized the measure of every set in $\mathcal{S}$ as well as its complement if E is countable, and we have seen that this always involves the measure of singletons and the entire space itself.

Now for completeness, consider the case where $E \in \mathcal{S}$ is uncountable. So that $X \setminus E$ is countable. Similarly to how we dealt with E from the previous part of the proof, we get that

$$\mu(X \setminus E) = \sum_{k \in X \setminus E} \mu(\{k\}).$$

Now either $\mu(E) = \infty$ or $\mu(E) < \infty$.

If $\mu(E) = \infty$, then $\mu(X) = \infty$(measure preserving order) and that $\mu(X \setminus E)$ could be finite or infinite; that is, the above series could converge or diverge. And that's all the information we can ascertain about this measure, since we don't have the ability to defining $\mu(X \setminus E)$ in terms of $\mu(E)$ without getting an indeterminate form such as $\infty - \infty$.

If $\mu(E) < \infty$, then we have that $\mu(X \setminus E) = \mu(X) - \mu(E)$ so that:

$$\sum_{k \in X \setminus E} \mu(\{k\}) = \mu(X) - \mu(E)$$

With $\mu(X)$ again possibly being infinite. In the case that $\mu(X \setminus E) = \infty$, then the series above diverges and we can't characterize $\mu(E)$ any better, since both $\mu(X) = \infty$ and $\mu(X \setminus E) = \infty$. All we can say is that $\mu(E)$ is finite. In the case that $\mu(X \setminus E) < \infty$, then we can fully characterize $\mu(E) = \mu(X) - \mu(X \setminus E)$ with $\mu(X) < \infty$ since both $\mu(X \setminus E) < \infty$ and $\mu(E) < \infty$.

That is, I believe, the best characterization of measures on the space $(X, \mathcal{S})$ as we can do. $\square$