MATH 340 Presentation Problem #13

Joseph C. McGuire
Dr. S. Brannen

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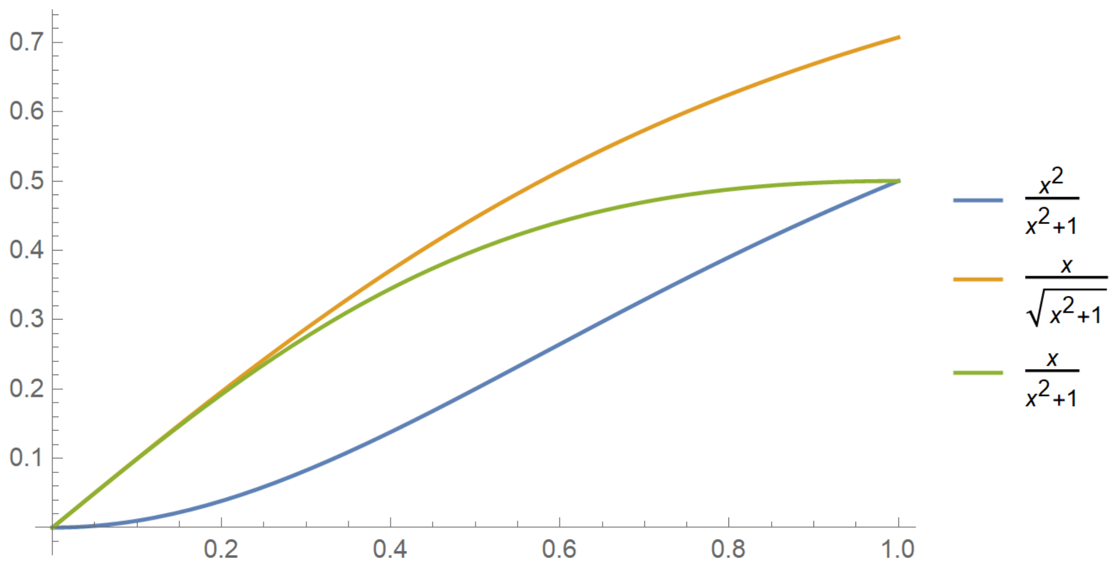

(4.) (Page 147) Show that the following function is a Lipschitz function.

(b.)

$$g(x) = \frac{x}{x^2+1}, \ \mathbf{Dom(g)} = [0, \infty)$$

Proof. Let $g : [0, \infty) \to \mathbb{R}$ be a function defined by $g(x) = \dfrac{x}{x^2+1}$ for all $x \in [0, \infty)$ and let $M = 1$. Let $y \in [0, \infty)$, then take note of the following inequalities: $\dfrac{1}{y^2+1} \leq 1$ and $\dfrac{1}{x^2+1} \leq 1$, for all $x, y \in [0, \infty)$. Also note that since

$$0 < 1$$

$$\text{iff, } x^2 < x^2 + 1$$

$$\text{iff, } \frac{x^2}{x^2+1} < 1$$

$$\text{iff, } x^2 < x^2 + 1$$

$$\text{iff, } x < \sqrt{x^2+1}$$

$$\text{iff, } \frac{x}{\sqrt{x^2+1}} < 1.$$

Furthermore, consider the following, for $x \in [0, \infty)$:

$$0 \leq x^4 + x^2$$

$$x^2 + 1 \leq x^4 + 2x^2 + 1$$

$$x^2 + 1 \leq (x^2 + 1)^2$$

$$\sqrt{x^2 + 1} \leq x^2 + 1$$

$$\frac{1}{x^2 + 1} \leq \frac{1}{\sqrt{x^2 + 1}}$$

$$\frac{x}{x^2 + 1} \leq \frac{x}{\sqrt{x^2 + 1}}$$

Thus combining $\frac{x}{x^2 + 1} \leq \frac{x}{\sqrt{x^2 + 1}}$ and $\frac{x}{\sqrt{x^2 + 1}} < 1$, we get

$$\frac{x}{x^2 + 1} < 1. \tag{1}$$

Now finally let us consider the following inequality:

$$|g(x) - g(y)| \leq M|x - y| \tag{2}$$

$$\text{iff, } \left| \frac{x}{x^2 + 1} - \frac{y}{y^2 + 1} \right| \leq |x - y|$$

$$\text{iff, } \left| \frac{x + xy^2 - yx^2 - y}{(x^2 + 1)(y^2 + 1)} \right| \leq |x - y|$$

(Then note that we can factor the following: $(x - y)(1 - xy) = x - x^2y - y + xy^2$.) Thus

$$\text{iff, } \left| \frac{(x - y)(1 - xy)}{(x^2 + 1)(y^2 + 1)} \right| \leq |x - y|$$

$$\text{iff, } \left| \frac{1 - xy}{(x^2 + 1)(y^2 + 1)} \right| \leq 1$$

Then we can get rid of the absolute values by breaking it down in to cases: (1.) $xy - 1 \leq 0$ and (2.) $xy - 1 > 0$.

1. Assume $xy - 1 \leq 0$.
   Thus we have $xy \leq 1$. Hence

   $$\left| \frac{1 - xy}{(x^2 + 1)(y^2 + 1)} \right| \leq 1 \text{ iff } \frac{xy - 1}{(x^2 + 1)(y^2 + 1)} \leq 1.$$

   Which is true since we know the following inequalities hold:

   $$\frac{xy - 1}{(x^2 + 1)(y^2 + 1)} \leq \frac{xy}{(x^2 + 1)(y^2 + 1)} \leq 1, \text{ by (1).}$$

   Thus we have (2) hold true when $xy - 1 \leq 0$.

2. Assume $xy - 1 > 0$.
   Then we have $xy > 1$.

$$\left|\frac{1-xy}{(x^2+1)(y^2+1)}\right| \le 1 \text{ iff } \frac{1-xy}{(x^2+1)(y^2+1)} \le 1$$

   . This is true since $xy > 1$ iff $-xy < -1$ iff $1 - xy < 0$. Thus (2) holds true when $xy - 1 > 0$.

Thus in any possible case we have that the inequality (2) holds true.
Thus

$$g(x) = \frac{x}{x^2+1} \text{ is Lipshitz on } x \in [0, \infty)$$

$\square$