Instructor: J. McGuire Class: Math 118

Chapter 1: Fundamentals

Note 1 (Goals)**.**

1. *Define Numbers*
2. *Define Operations*
3. *Introduce Expressions*
4. *Equations*
5. *Inequalities*
6. *Graphing*
7. *Solving Equations with Graphs*

1 Adding and Subtracting Polynomials

Definition 1. *A* ***polynomial*** *in the variable x is an expression of the form*

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are the ***coefficients*** *of $x^0, x^1, \ldots, x^n$. a_n is the* ***leading coefficient****, n is the degree, and each piece of the sum is a* ***term*** *($a_1 x, a_2 x^2, \ldots, a_n x^n, a_0$).*

Example 1.

Polynomial	Type	Terms	Degree
$2x^2 - 3x + 4$	trinomial	$2x^2, -3x, 4$	2
$x^8 + 5x$	binomial	$x^8, 5x$	8
$8 - x + x^2 - \frac{1}{2}x^3$	four terms	$-\frac{1}{2}x^3, x^2, -x, 3$	3
$5x + 1$	binomial	$5x, 1$	1
$9x^5$	monomial	$9x^5$	5
6	monomial	6	0

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Example 2 (Adding and Subtracting Polynomials). *1. Find the sum:*

$$x^3 - 6x^2 + 2x + 4 + (x^3 + 5x^2 - 7x)$$

2. Find the difference:

$$(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x)$$

2 Multiplying Polynomials

Recall FOIL:

$$(a+b)(c+d) = \underset{\mathbf{F}}{\underset{\uparrow}{ac}} + \underset{\mathbf{O}}{\underset{\uparrow}{ad}} + \underset{\mathbf{I}}{\underset{\uparrow}{bc}} + \underset{\mathbf{L}}{\underset{\uparrow}{bd}}$$

Example 3 (Multiplying Polynomials)**.** *Expand out:*

$$(2x+1)(3x-5)$$

and

$$(2x+3)(x^2-5x+4)$$

3 Special Product Formulas

Theorem 1. *If A and B are polynomials:*

1. $(A+B)(A-B) = A^2 - B^2$ *(Difference of Squares)*
2. $(A+B)^2 = A^2 + 2AB + B^2$ *(Perfect Square)*
3. $(A-B)^2 = A^2 - 2AB + B^2$ *(Perfect Square, Also)*
4. $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$ *(Perfect Cube)*
5. $(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$ *(Perfect Cube, Also)*

Example 4. *Use the Special Product Formulas to expand:*

1. $(3x+5)^2$
2. $(x^2-2)^3$

4 Factoring

Example 5. *Factor the following:*

1. $3x^2 - 6x$
2. $8x^4y^2 + 6x^3y^3 - 2xy^4$
3. $(2x + 4)(x - 3) - 5(x - 3)$

Example 6. *Factor the following:*

1. $x^2 + 6x + 9$
2. $4x^2 - 4xy + y^2$
3. $27x^3 - 1$
4. $x^6 + 8$

5 Rational Expressions

Example 7. *Simplify*

$$\frac{x^2-1}{x^2+x-2}$$

Example 8. *Perform the following:*

$$\frac{(x-1)(x+3)}{x^2+8x}\,\frac{3x+12}{x-1}$$

Example 9 (Adding Rational Expressions)**.** *Simplify*

1. $\dfrac{3}{x-1}+\dfrac{x}{x+2}$
2. $\dfrac{1}{x^2-1}-\dfrac{2}{(x+1)^2}$

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Example 10 (Compound Fractions). *Simplify*

$$\frac{\frac{x}{y}+1}{1-\frac{y}{x}}$$

and

$$\frac{\frac{1}{a+h}-\frac{1}{a}}{h}.$$

Example 11 (Compound Fractions: Harder)**.** *Simplify:*

$$\frac{(1+x^2)^{1/2} - x^2(1+x^2)^{-1/2}}{1+x^2}$$

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Theorem 2 (Math Sins)**.** *These are very common mistakes, please try to avoid them:*

1. $(2+3)^2 \neq 2^2 + 3^2$
2. $\sqrt{2+3} \neq \sqrt{2} + \sqrt{3}$
3. $\sqrt{2^2+3^2} \neq 2+3$
4. $\frac{1}{2} + \frac{1}{3} \neq \frac{1}{2+3}$
5. $\frac{2+3}{2} \neq 3$
6. $2^{-1} + 3^{-1} \neq (2+3)^{-1}$

6 Linear Equations

Definition 2. *A **linear equation** in one variable is an equation that is equivalent to one of the form:*

$$ax + b = 0$$

where $a, b \in \mathbb{R}$ and x is a variable.

Linear equations	**Nonlinear equations**	
$4x - 5 = 3$	$x^2 + 2x = 8$	Not linear; contains the square of the variable
$2x = \frac{1}{2}x - 7$	$\sqrt{x} - 6x = 0$	Not linear; contains the square root of the variable
$x - 6 = \dfrac{x}{3}$	$\dfrac{3}{x} - 2x = 1$	Not linear; contains the reciprocal of the variable

Example 12. *Solve the equation:*

$$7x - 4 = 3x + 8$$

Example 13. *Solve for M in terms of the other variables in:*

$$F = G\frac{mM}{r^2}$$

Example 14. *The surface area A of the closed rectangle box shown in Figure 1 can be calculated from the length l, the width w, and the height h according to the formula:*

$$A = 2lw + 2wh + 2lh$$

Solve for w in terms of the other variables in the equation.

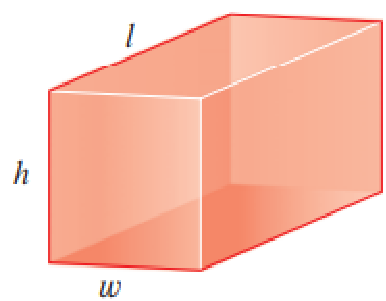

FIGURE 1 A closed rectangular box

7 Quadratic Equations

Definition 3. *A **quadratic equation** is an equation of the form*

$$ax^2 + bx + c = 0$$

where a, b, c are real numbers and $a \neq 0$.
Note that if A, B are polynomials then,

$$AB = 0 \iff A = 0 \text{ or } B = 0$$

Example 15 (Solving Simple Quadratics)**.** *Find all real solutions of each equation.*

1. $x^2 = 5$
2. $(x - 4)^2 = 5$

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Theorem 3. *To make $x^2 + bx$ into a perfect square, we add by $\left(\frac{b}{2}\right)^2$:*

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Example 16. *Find all real solutions of each equation.*

1. $x^2 - 8x + 13 = 0$
2. $3x^2 - 12x + 6 = 0$

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Theorem 4. *The solutions to the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$ is:*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Proof. Start with $ax^2 + bx + c = 0$:

$\square$

Example 17. *Using the quadratic formula solve:*

1. $3x^2 - 5x - 1 = 0$
2. $4x^2 + 12x + 9 = 0$
3. $x^2 + 2x + 2 = 0$

Example 18. *An object thrown or fired straight upward at an initial speed of v_0 ft/s will reach a height of h feet after t seconds, where h and t are related by:*

$$h = -16t^2 + v_0 t$$

Suppose that $v_0 = 800$ ft/s.

1. *When does the bullet fall back to ground level?*
2. *When does it reach a height of 6400 ft?*
3. *When does it reach a height of 2 miles (10560 ft.)?*
4. *How high is the highest point the bullet reaches?*

Note: **always check your solutions whenever you're dealing with a non-quadratic or non-linear equation!**

8 Other Equations

Example 19 (Fractional Equations). *Solve the equation*

$$\frac{3}{x} - \frac{2}{x-3} = -\frac{12}{x^2-9}$$

Example 20 (Radical Equations)**.** *Solve*

$$2x = 1 - \sqrt{2 - x}.$$

Hint: Has an extraneous solution!

Example 21 (Quadratic Type). *Find all solutions of the equation:*

$$x^4 - 8x^2 + 8 = 0$$

Example 22 (Fractional Powers)**.** *Solve:*

$$x^{1/3} + x^{1/6} - 2 = 0.$$

Example 23. *Solve*

$$|2x - 5| = 3$$

9 Complex Numbers

There are some equations that do not have real solutions, for example:

$$x^2 = -4 \iff x = \pm\sqrt{-4},$$

but $\sqrt{-1}$ is not a real number! We need to introduce a new number! We'll call this number the imaginary number and define it $i = \sqrt{-1}$ or $i^2 = -1$.

Definition 4. *A* ***complex number*** *is an expression of the form*

$$a + bi$$

where a and b are real numbers and $i^2 = -1$. a is the ***real part*** *and b is the* ***imaginary part****, and $x + yi = a + bi$ if and only if $a = x$ and $y = b$.*

Example 24 (Adding and Subtracting and Multiplying Complex Numbers)**.** *Write all of the following in standard form:*

1. $(3 + 5i) + (4 - 2i)$
2. $(3 + 5i) - (4 - 2i)$
3. $(3 + 5i)(4 - 2i)$
4. i^{23}

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Definition 5 (Complex Conjugate and Magnitude)**.** *If $z = a + bi$, then it's* ***complex conjugate*** $\overline{z} = a - bi$.

And $|z|$ is the ***complex magnitude*** *and it's defined by:*

$$|z| = \sqrt{z\overline{z}} = \sqrt{a^2 + b^2} \iff |z|^2 = z\overline{z} = a^2 + b^2.$$

We will use this for division:

Example 25 (Dividing)**.** *Express the following in the form $a + bi$:*

1. $\dfrac{3 + 5i}{1 - 2i}$
2. $\dfrac{7 + 3i}{4i}$

Note 2. *An important note, we have to be careful with powers involving roots and negative numbers:*

$$\sqrt{-2}\sqrt{-3} \neq \sqrt{(-2)(-3)}.$$

The primary reason we care about complex numbers is because they give us complete solutions for quadratic equations:

Example 26. *Solve:*

1. $x^2 + 9 = 0$
2. $x^2 + 4x + 5 = 0$

Example 27. *Show that the solutions of the equation*

$$4x^2 - 24x + 37 = 0$$

are complex conjugates of each other.

10 Making and Using Models

One of the primary reasons math is as useful as it is, is it's applications to real problems. This section will cover how we apply math to problems.

Example 28. *A car rental company charge* \$30 *a day and* \$0.15 *a mile for renting a car. Helen rents a car for two days, and her bill comes to* \$108. *How many miles did she drive?*

Definition 6. *If we want to find the interest earned on some amount of money, maybe we have a bank loan or car payment that has some set interest rate, how much do we owe?*

If I is the total interest paid, P is the principal loan or amount, r is the yearly interest rate, and t is the number of years, then:

$$I = Prt.$$

Example 29. *Mary inherits* $\$100,000$ *and invest it in two certificates of deposit. One certificate pays* $\%6$ *and the other* $\%4\frac{1}{2}$ *simple interest annually. If Mary's total interest is* $\$5025$*, how much money is invested at each rate?*

Example 30. *A square garden has a walkway 3 feet wide around its outer edge and the plot is x wide and long. If the area of the entire garden, including the walkways, is $18,000\ ft^2$, what are the dimensions of the planted area?*

Example 31. *A rectangular lot is* $8ft$ *longer than it is wide and has an area of* $2900\ ft^2$. *Find the dimensions of the lot.*

Example 32. *A man who is* 6 *ft tall wishes to find the height of a certain four-story building. He measures its shadow and finds it to be 28 ft long, while his own shadow is* $3\frac{1}{2}$ *feet long. How tall is the building?*

Solution. We can use the fact that similar triangle will have the same ratio of height to base:

$$\frac{h}{28} = \frac{6}{3.5}$$

□

Theorem 5 (Mixture Problems)**.** *If we have some amount of substance x, that's dissolved in a solution of volume V, then the concentration C of the substance is given by:*

$$C = \frac{x}{V}$$

Example 33. *A manufacturer of soft drinks advertises their orange soda as 'naturally flavored,' although it contains only* %5 *orange juice. A new federal regulation stipulates that to be called 'natural,' a drink must contain at least* 10% *fruit juice. How much pure orange juice must this manufacturer add to* 900 *gal of orange juice to conform to the new regulation?*

Solution. Identify your variables:

- x = the amount in gallons of pure orange juice to be added
- 900 + x = Amount of mixture
- 0.05(900) = 45, Amount of orange juice in the first vat
- amount of orange juice in the second vat = x
- 0.10(900 + x) = Amount of orange juice in the mixture

We need first vat + second vat = mixture So that:

$$45 + x = 0.1(900 + x)$$

□

Theorem 6. *Problems involving time to do something require a rate, that is, if I do one thing in H hours, that's the fraction:* $1/H$.

Example 34. *Because of an anticipated heavy rainstorm, the water level in a reservoir must be lowered by 1 ft. Opening spillway A lowers the level by this amount in 4 hours, whereas opening the smaller spillway B does the job in 6 hours. How long will it take to lower the water level by 1 ft if both spillways are opened?*

Solution. In one hour A lowers the water level by $1/4$ ft and B lowers it by $1/6$ ft. With both open, we'll need: $1/4 + 1/6 = 1/x$ hours to lower the water level by 1 foot. $\square$

Theorem 7. *With problems involving distance and speed, we have:*

$$\textit{Distance} = \textit{Rate} \times \textit{time}$$

Example 35. *A jet flew from New York to Los Angeles, a distance of 4200 km. The speed for the return trip was 100 km / h faster than the outbound speed. If the total trip took 13 hours of flying time, what was the jet's speed from New York to Los Angeles?*

Solution. Let

$$\text{s} = \text{Speed from New York to Los Angeles}$$

so then

$$s + 100 = \text{Speed from Los Angeles to New York.}$$

So then:

$$\frac{4200}{s} + \frac{4200}{s+100} = 13$$

is the total trip time. $\square$

Example 36. *Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours, because air generally rises over land and falls over water in the daytime, so flying over water requires more energy. A bird is released from point A on an island, 5 mi from B, the nearest point on a straight shoreline. The bird flies to a point C on the shoreline and then flies along the shoreline to its nesting area D, as shown in Figure 5. Suppose the bird has 170 kcal of energy reserves. It uses 10 kcal / mi flying over land and 14 kcal/mi flying over water.*

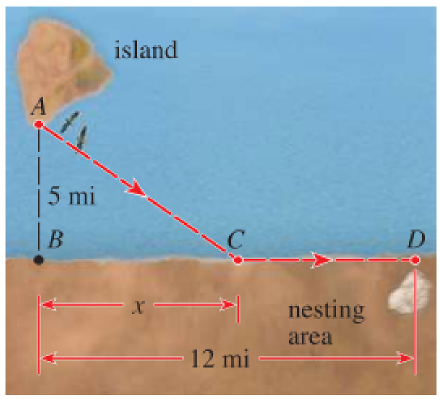


FIGURE 5

1. *Where should the point C be located so that the bird uses exactly 170 kcal of energy during its flight?*
2. *Does the bird have enough energy reserves to fly directly from A to D?*

Solution. $x =$ Distance from B to C, then the energy used over water is:

$$14\sqrt{x^2+25}$$

and over land:

$$10(12-x)$$

So that:

$$170 = 14\sqrt{x^2+25} + 10(12-x).$$

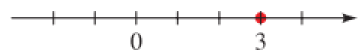
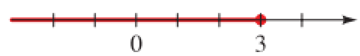
□

11 Inequalities

So far we know how to solve equations and use them model equations, but how about inequalities?

$$4x + 7 \leq 19$$

This is an example of a **linear inequality**, since this involves just one power of x. We can solve this one by manipulating the inequality, graphically these are:

		Solution	Graph
Equation:	$4x + 7 = 19$	$x = 3$	
Inequality:	$4x + 7 \leq 19$	$x \leq 3$	

In general these are the rules that we need to follow:

RULES FOR INEQUALITIES

Rule	Description
1. $A \leq B \quad \Leftrightarrow \quad A + C \leq B + C$	**Adding** the same quantity to each side of an inequality gives an equivalent inequality.
2. $A \leq B \quad \Leftrightarrow \quad A - C \leq B - C$	**Subtracting** the same quantity from each side of an inequality gives an equivalent inequality.
3. If $C > 0$, then $A \leq B \quad \Leftrightarrow \quad CA \leq CB$	**Multiplying** each side of an inequality by the same *positive* quantity gives an equivalent inequality.
4. If $C < 0$, then $A \leq B \quad \Leftrightarrow \quad CA \geq CB$	**Multiplying** each side of an inequality by the same *negative* quantity *reverses the direction* of the inequality.
5. If $A > 0$ and $B > 0$, then $A \leq B \quad \Leftrightarrow \quad \frac{1}{A} \geq \frac{1}{B}$	**Taking reciprocals** of each side of an inequality involving *positive* quantities *reverses the direction* of the inequality.
6. If $A \leq B$ and $C \leq D$, then $A + C \leq B + D$	Inequalities can be added.
7. If $A \leq B$ and $B \leq C$, then $A \leq C$	Inequality is transitive.

12 Solving Linear Inequalities

Linear inequalities are anything that looks like :

$$mx + b \leq 0, mx + b \geq 0, mx + b < 0, \text{ or } mx + b > 0$$

Example 37. *Solve the inequality $3x < 9x + 4$, and sketch the solution set.*

Example 38. *Solve the inequalities:*

$$4 \leq 3x - 2 < 13$$

13 Solving Nonlinear Inequalities

Theorem 8. *The sign of a product or quotient is determined by it's factors:*

1. *If a product or a quotient has an even number of negative factors, then its value is positive*
2. *If a product or a quotient has an odd number of negative factors, then its value is negative*

For example, $(x-2)(x-3) \leq 0$ if and only if only one of the factors $x-2$ and $x-3$ are negative.

Non linear inequalities are harder to solve, but we'll introduce a procedure that always works!

Example 39. *Solve the inequality:*

$$x^2 \leq 5x - 6$$

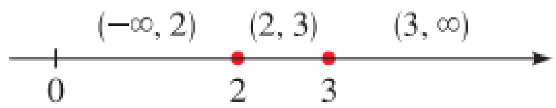


FIGURE 3

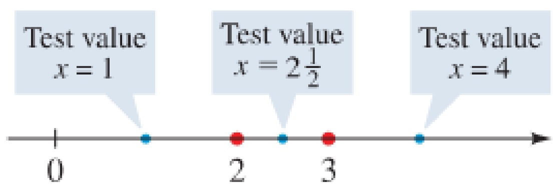


FIGURE 4

Interval	$(-\infty, 2)$	$(2, 3)$	$(3, \infty)$
Sign of $x - 2$ Sign of $x - 3$	$-$ $-$	$+$ $-$	$+$ $+$
Sign of $(x-2)(x-3)$	$+$	$-$	$+$

		2		3	
Sign of $x - 2$	$-$		$+$		$+$
Sign of $x - 3$	$-$		$-$		$+$
Sign of $(x-2)(x-3)$	$+$		$-$		$+$

GUIDELINES FOR SOLVING NONLINEAR INEQUALITIES

1. **Move All Terms to One Side.** If necessary, rewrite the inequality so that all nonzero terms appear on one side of the inequality sign. If the nonzero side of the inequality involves quotients, bring them to a common denominator.
2. **Factor.** Factor the nonzero side of the inequality.
3. **Find the Intervals.** Determine the values for which each factor is zero. These numbers will divide the real line into intervals. List the intervals that are determined by these numbers.
4. **Make a Table or Diagram.** Use **test values** to make a table or diagram of the signs of each factor on each interval. In the last row of the table determine the sign of the product (or quotient) of these factors.
5. **Solve.** Use the sign table to find the intervals on which the inequality is satisfied. Check whether the **endpoints** of these intervals satisfy the inequality. (This may happen if the inequality involves $\leq$ or $\geq$.)

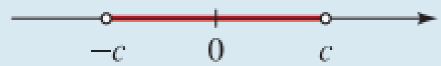
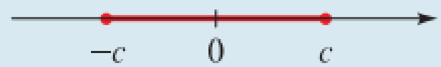
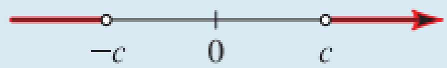
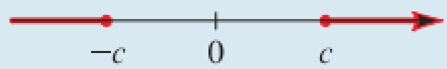
Example 40. *Solve the inequality* $x(x-1)^2(x-3) < 0$.

Example 41. *Solve the inequality*

$$\frac{1+x}{1-x} \geq 1.$$

14 Absolute Value Inequalities

PROPERTIES OF ABSOLUTE VALUE INEQUALITIES

Inequality	Equivalent form	Graph
1. $\lvert x \rvert < c$	$-c < x < c$	
2. $\lvert x \rvert \le c$	$-c \le x \le c$	
3. $\lvert x \rvert > c$	$x < -c$ or $c < x$	
4. $\lvert x \rvert \ge c$	$x \le -c$ or $c \le x$	

These properties hold when x is replaced by any algebraic expression. (In the graphs we assume that $c > 0$.)

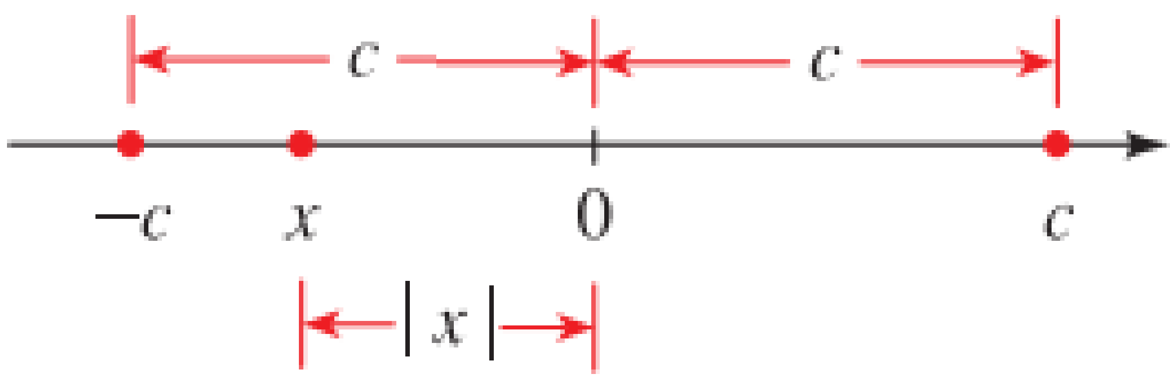


FIGURE 8

Example 42. *Solve the inequality*

$$|x - 5| < 2.$$

Example 43. *Solve the inequality*

$$|3x + 2| \geq 4.$$

15 Modeling with Inequalities

Example 44. *A carnival has two plans for tickets:*

Plan A: $5 *entrance fee and* $0.25 *each ride*

Plan B: $2 *entrance fee and* $0.5 *each ride.*

How many rides would you have to take for Plan A to be less expensive than Plan B?

Proof. The total cost for Plan A:

$$5 + 0.25x$$

and Plan B:

$$2 + 0.5x$$

□

Example 45. *The instructions on a bottle of medicine indicate that the bottle should be stored at a temperature* 5^oC *and* 30^oC*. What range of temperatures does this correspond to on the Fahrenheit scale?*

The conversion from Fahrenheit to Celsius:

$$C = \frac{5}{9}(F - 32)$$

16 The Coordinate Plane

The **Cartesian Plane or Coordinate Plane** is the plane we get by taking the real number line and extending it to two dimensions. The x**-axis** is the horizontal axis and $y-$**axis** is the vertical axis, these divide the plan into 4 **Quadrants**:

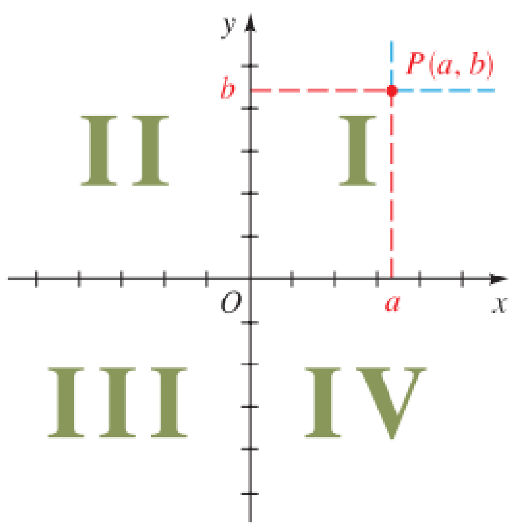


FIGURE 1

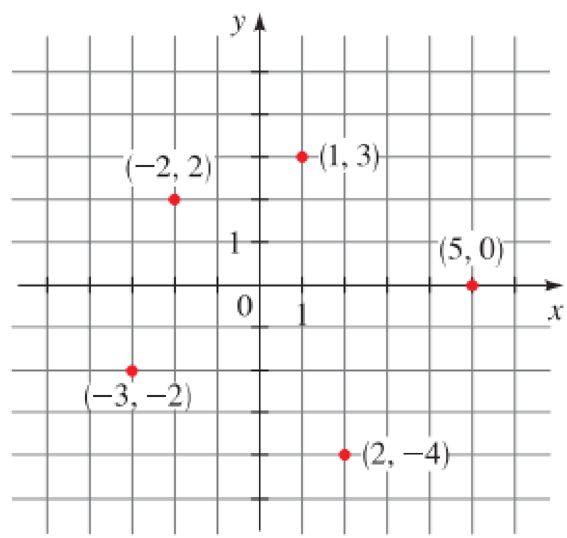


FIGURE 2

A point on the plane is called a **ordered pair** (a, b), a is the x-coordinate and b is the y-coordinate.

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Example 46. *Describe and sketch the regions given by each set:*

1. $\{(x, y) : x \geq 0\}$
2. $\{(x, y) : y = 1\}$
3. $\{(x, y) : |y| < 1\}$

17 The Distance and Midpoint Formula

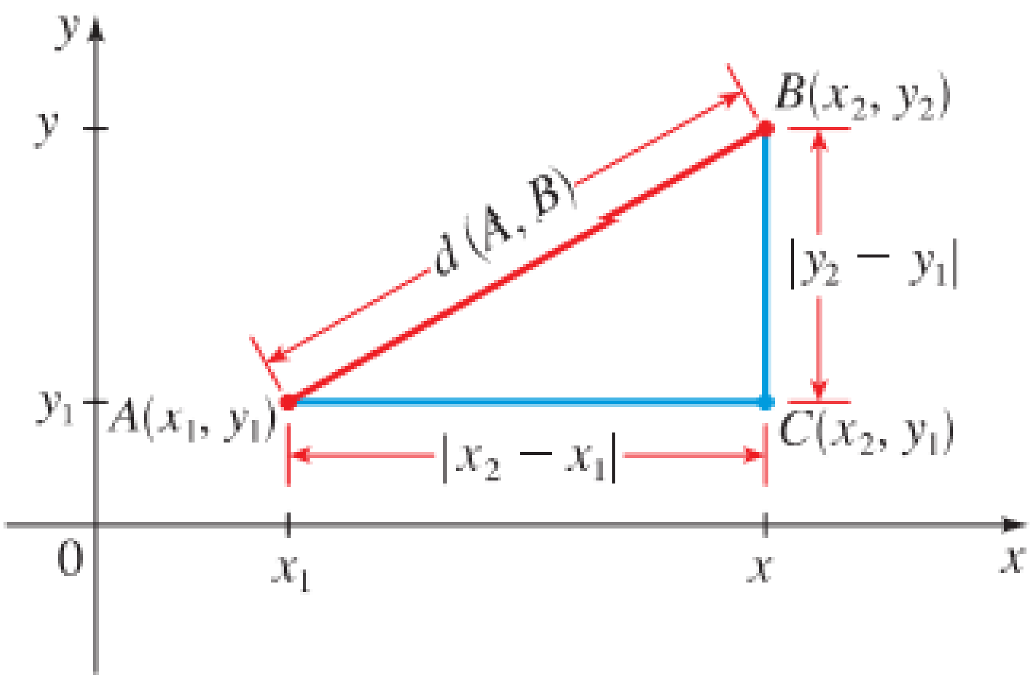


FIGURE 4

This gives us the distance formula:

Theorem 9. *The distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is:*

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 47. *Which of the points* $P(1, -2)$ *or* $Q(8, 9)$ *is closer to the point* $A(5, 3)$*?*

Now that we have distance, a natural question to ask is what is the midpoint between two points? Let's find it with a figure:

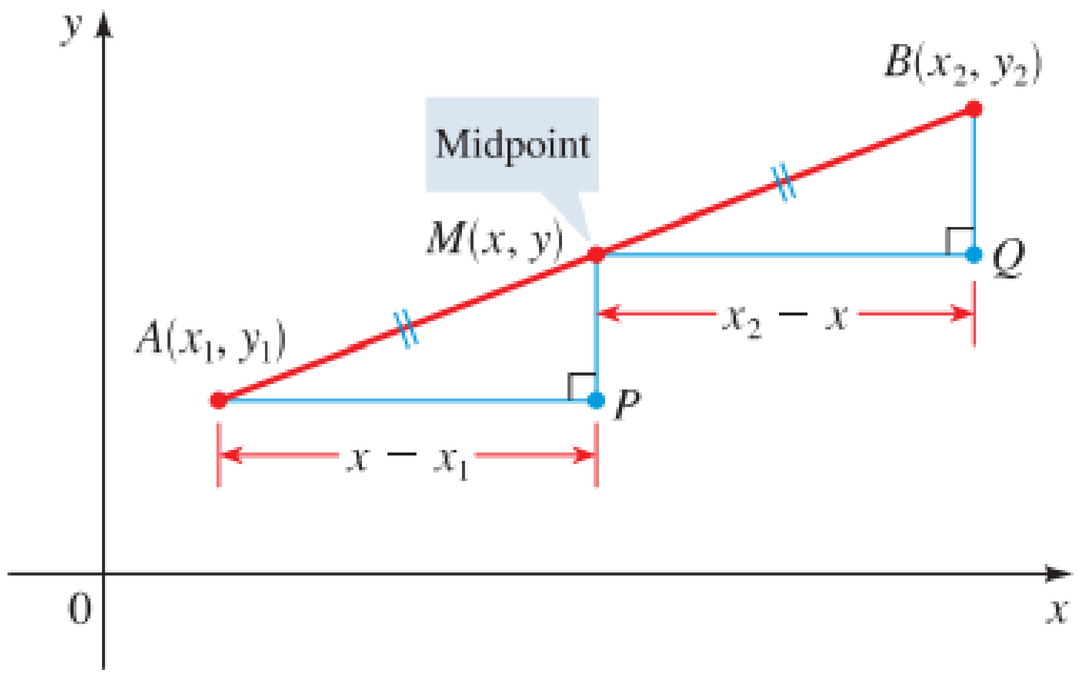


FIGURE 6

Theorem 10. *The midpoint of the line segment from $A(x_1, y_1)$ to $B(x_2, y_2)$ is:*

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example 48. *Show that the points* $S(2, 7)$ *and* $Q(4, 4)$, $P(1, 2)$ *and* $R(5, 9)$ *have the same midpoint.*

18 Graphs of Equation in Two Variables

Now to graph an equation, suppose we have:

$$y = x^2 + 1,$$

that is the vertical component is related to the horizontal component by this formula. To graph this:

Theorem 11. *The* ***graph*** *of an equation in x and y is the set of all points (x, y) in the coordinate plane that satisfy the equation.*

Example 49. *Sketch the graph of the equation*

$$2x - y = 3$$

x	$y = 2x - 3$	(x, y)
-1	-5	$(-1, -5)$
0	-3	$(0, -3)$
1	-1	$(1, -1)$
2	1	$(2, 1)$
3	3	$(3, 3)$
4	5	$(4, 5)$

Example 50. *Sketch the graph of the equation* $y = x^2 - 2$

x	$y = x^2 - 2$	(x, y)
-3	7	$(-3, 7)$
-2	2	$(-2, 2)$
-1	-1	$(-1, -1)$
0	-2	$(0, -2)$
1	-1	$(1, -1)$
2	2	$(2, 2)$
3	7	$(3, 7)$

Example 51. *Sketch the graph of the equation* $y = |x|$.

x	$y = \lvert x \rvert$	(x, y)
-3	3	$(-3, 3)$
-2	2	$(-2, 2)$
1	1	$(1, 1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
2	2	$(2, 2)$
3	3	$(3, 3)$

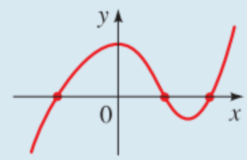
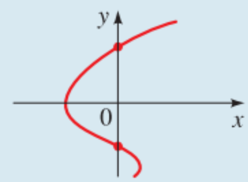
Example 52. *Use a graphing calculator (or Desmos) to graph the following equation in the viewing rectangle* $[-5, 5]$ *by* $[-1, 2]$

$$y = \frac{1}{1 + x^2}.$$

19 Intercepts

The points of an equation that intercept the axes tells us a lot about the equations and the overall shape of the graph, here's how we find them:

DEFINITION OF INTERCEPTS

Intercepts	How to find them	Where they are on the graph
x-intercepts: The x-coordinates of points where the graph of an equation intersects the x-axis	Set $y = 0$ and solve for x	
y-intercepts: The y-coordinates of points where the graph of an equation intersects the y-axis	Set $x = 0$ and solve for y	

Example 53. *Find the $x-$ and $y-$intercepts of the graphs of the equation $y = x^2 - 2$.*

20 Circles

One type of equation that pops up quite a lot and has a relatively nice form is the circle:

EQUATION OF A CIRCLE

An equation of the circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

This is called the **standard form** for the equation of the circle. If the center of the circle is the origin $(0, 0)$, then the equation is

$$x^2 + y^2 = r^2$$

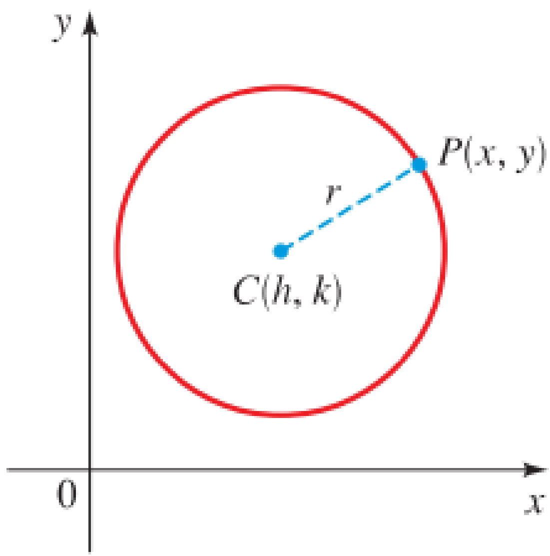


FIGURE 13

Example 54. *Graph each equation.*

1. $x^2 + y^2 = 25$
2. $(x-2)^2 + (y+1)^2 = 25$

Example 55. 1. *Find an equation of the circle with radius* 3 *and center* $(2, -5)$

2. *Find an equation of the circle that has the points* $P(1, 8)$ *and* $Q(5, -6)$ *as the endpoints of a diameter.*

Example 56. *Show that the equation*

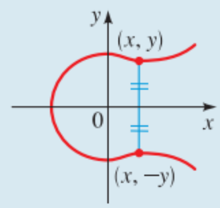
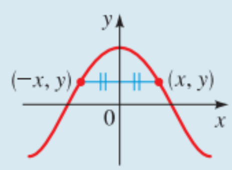
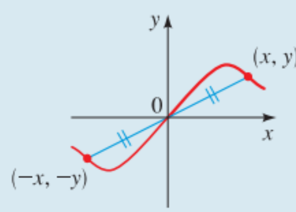
$$x^2 + y^2 + 2x - 6y + 7 = 0$$

represents a circle, and find the center and radius of the circle. (Hint: Complete the square.)

21 Symmetry

Symmetries are a way of identifying a kind of equivalence by a reflection or rotation, for example we saw the graph of $y = x^2 - 2$ was symmetric by a reflection about the $y-$axis. In general these make graphing a little easier for us, and identify important properties of functions. Here's how we identify symmetries:

TYPES OF SYMMETRY

Symmetry	Test	Graph	Property of Graph
With respect to the *x*-axis	Replace y by $-y$. The resulting equation is equivalent to the original one.		Graph is unchanged when reflected about the x-axis. See Figures 14 and 19.
With respect to the *y*-axis	Replace x by $-x$. The resulting equation is equivalent to the original one.		Graph is unchanged when reflected about the y-axis. See Figures 9, 10, 11, 12, 14, and 18.
With respect to the origin	Replace x by $-x$ and y by $-y$. The resulting equation is equivalent to the original one.		Graph is unchanged when rotated 180° about the origin. See Figures 14 and 20.

Example 57. *Test the equation $x = y^2$ for symmetry and sketch the graph.*

Example 58. *Test the equation $y = x^3 - 9x$ for symmetry.*

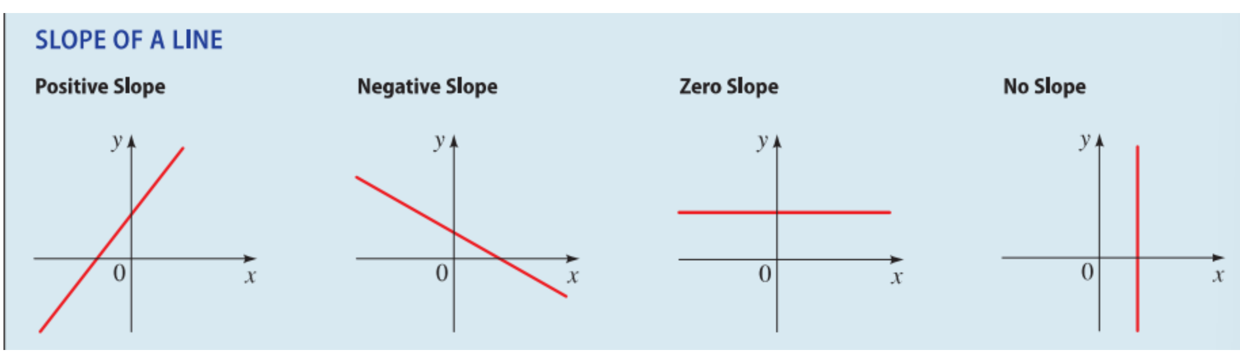
22 The Slope of a Line

SLOPE OF A LINE

The **slope** m of a nonvertical line that passes through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.



Example 59. *Find the slope of the line that passes through the points* $P(2,1)$ *and* $Q(8,5)$

23 Point-Slope Form of the Equation of a Line

POINT-SLOPE FORM OF THE EQUATION OF A LINE

An equation of the line that passes through the point (x_1, y_1) and has slope m is

$$y - y_1 = m(x - x_1)$$

Example 60. *1. Find an equation of the line through* $(1, -3)$ *with slope* $-\dfrac{1}{2}$

2. Sketch the line

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Example 61. *Find an equation of the line through the points* $(-1, 2)$ *and* $(3, -4)$.

24 Slope-Intercept Form of the Equation of a Line

SLOPE-INTERCEPT FORM OF THE EQUATION OF A LINE

An equation of the line that has slope m and y-intercept b is

$$y = mx + b$$

Example 62. 1. *Find an equation of the line with slope* 3 *and* y*-intercept* -2

2. *Find the slope and* y*-intercept of the line* $3y - 2x = 1$.

VERTICAL AND HORIZONTAL LINES

- An equation of the vertical line through (a, b) is $x = a$.
- An equation of the horizontal line through (a, b) is $y = b$.

Example 63.

1. *An equation for the vertical line through* $(3, 5)$ *is* $x = 3$
2. *The graph of the equation* $x = 3$ *is a vertical line with* $x-$*intercept* 3
3. *An equation of the horizontal line through* $(8, -2)$ *is* $y = -2$
4. *The graph of the equation* $y = -2$ *is a horizontal line with* $y-$*intercept* -2.

25 General Equation of a Line

GENERAL EQUATION OF A LINE

The graph of every **linear equation**

$$Ax + By + C = 0 \qquad (A, B \text{ not both zero})$$

is a line. Conversely, every line is the graph of a linear equation.

Example 64. *Sketch the graph of the equation* $2x - 3y - 12 = 0$

26 Parallel and Perpendicular Lines

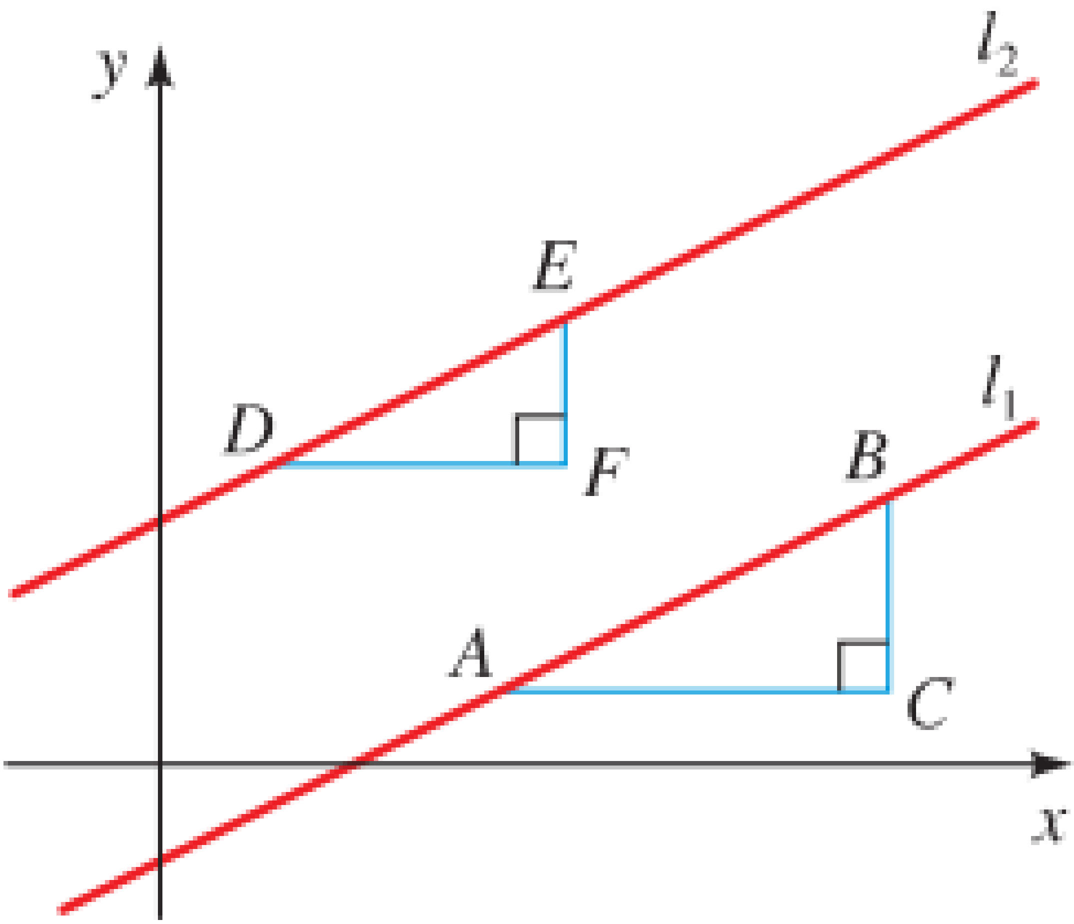

FIGURE 12

Theorem 12. *Two non-vertical lines are parallel if and only if they have the same slope.*

Proof. Let the lines l_1 and l_2 in Figure 12 have slope m_1 and m_2. If the lines are parallel, then the right triangles ABC and DEF are similar, so $m_1 = m_2$.

Conversely, if the slopes are equal then the triangles are similar, so then the lines are parallel. $\square$

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Example 65. *Find an equation of the line through the point* $(5, 2)$ *that is parallel to the line* $4x + 6y + 5 = 0$.

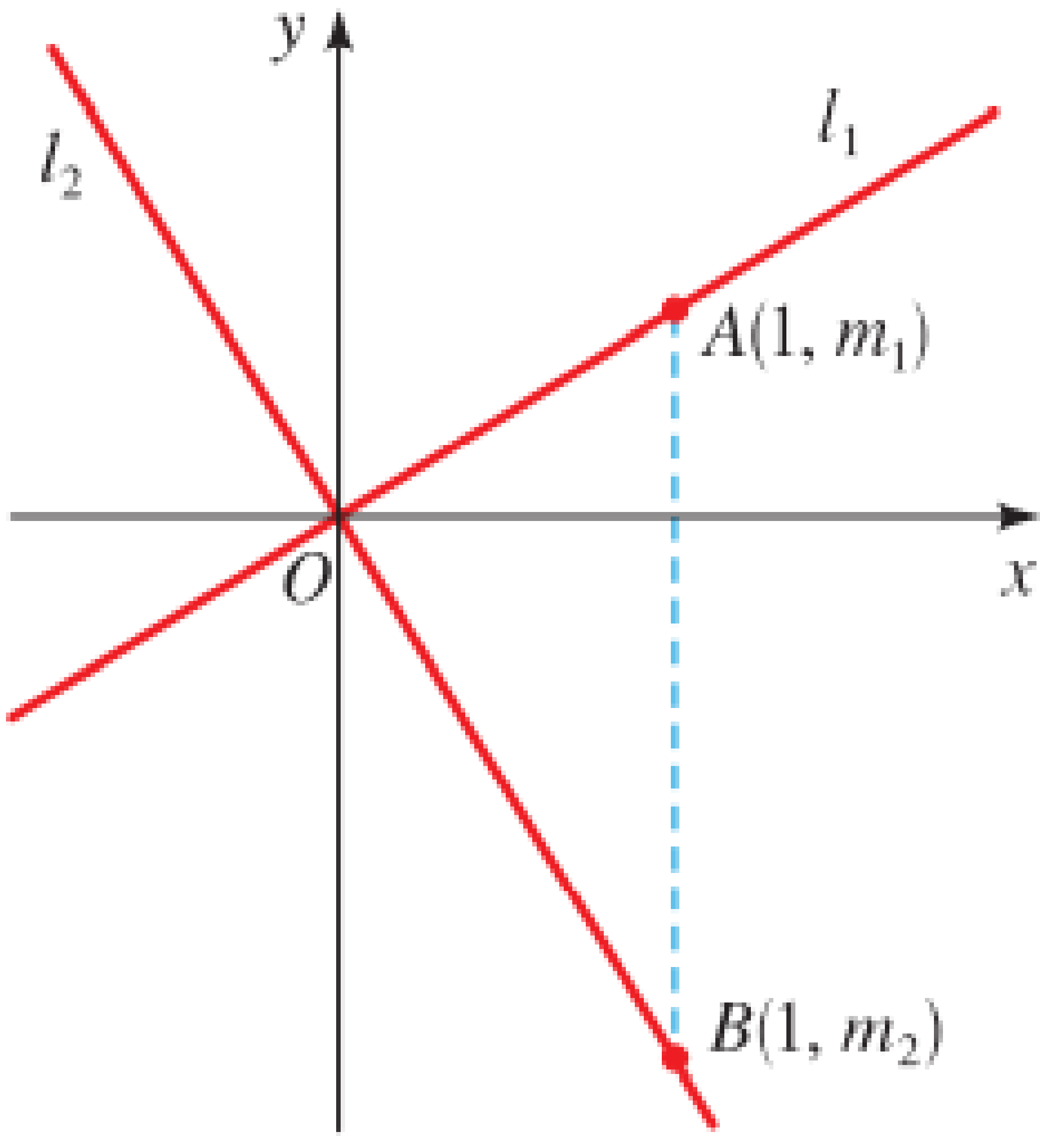

FIGURE 13

PERPENDICULAR LINES

Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$, that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$

Also, a horizontal line (slope 0) is perpendicular to a vertical line (no slope).

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Example 66. *Show that the points* $P(3,3), Q(8,17), R(11,5)$ *are the vertices of a right triangle.*

Example 67. *Find an equation of the line that is perpendicular to the line* $4x+6y+5=0$ *and passes through the origin.*

Example 68. *Use a graphing calculator (Desmos) to graph the family of lines*

$$y = 0.5x + b$$

with $b = -2, 1, 0, 1, 2$*. What property do the lines share?*

Example 69. *A swimming pool is being filled with a hose. The water's depth y (in feet) in the pool t hours after the hose is turned on is given by*

$$y = 1.5t + 2$$

1. *Find the slope and y-intercept of the graph of this equation.*
2. *What do the slope and y-intercept represent?*

27 Solving Equations Graphically

So far we have solved equation algebraically, that is we manipulated the symbols until we found a solution. This entire section we'll cover how to solve equations graphically using Desmos or a graphing calculator.

SOLVING AN EQUATION

Algebraic Method

Use the rules of algebra to isolate the unknown x on one side of the equation.

Example: $3x - 4 = 1$

$3x = 5$ Add 4

$x = \frac{5}{3}$ Divide by 3

The solution is $x = \frac{5}{3}$.

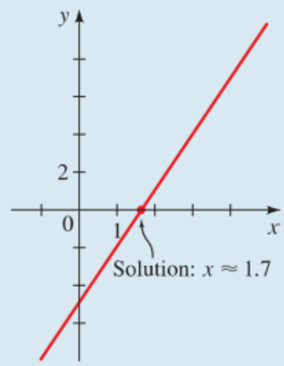
Graphical Method

Move all terms to one side, and set equal to y. Graph the resulting equation, and find the x-intercepts.

Example: $3x - 4 = 1$

$3x - 5 = 0$

Set $y = 3x - 5$ and graph. From the graph we see that the solution is $x \approx 1.7$



Example 70. *Find all real solutions of the quadratic equation. Use the algebraic method and the graphical method:*

1. $x^2 - 4x + 2 = 0$
2. $x^2 - 4x + 4 = 0$
3. $x^2 - 4x + 6 = 0$

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Example 71. *Solve the equation algebraically and graphically:*

$$5 - 3x = 8x - 20$$

Example 72. *Solve the equation* $x^3 - 6x^2 + 9x = \sqrt{x}$ *on* $[1, 6]$.

28 Solving Inequalities Graphically

SOLVING AN INEQUALITY

Algebraic Method

Use the rules of algebra to isolate the unknown x on one side of the inequality.

Example: $3x - 4 \geq 1$

$3x \geq 5$ Add 4

$x \geq \frac{5}{3}$ Divide by 3

The solution is $\left[\frac{5}{3}, \infty\right)$.

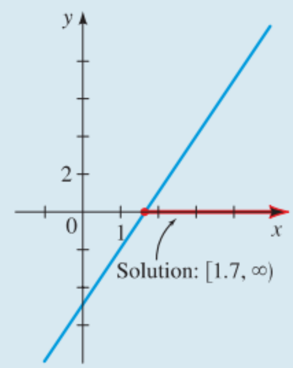
Graphical Method

Move all terms to one side, and set equal to y. Graph the resulting equation, and find the values of x where the graph is above or on the x-axis.

Example: $3x - 4 \geq 1$

$3x - 5 \geq 0$

Set $y = 3x - 5$ and graph. From the graph we see that the solution is $[1.7, \infty)$.



Example 73. *Solve the inequality* $x^2 - 5x + 6 \leq 0$ *graphically.*

Example 74. *Solve the inequality* $3.7x^2 + 1.3x - 1.9 \leq 2.0 - 1.4x$.

Example 75. *Solve the inequality* $x^3 - 5x^2 \geq -8$.