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Homework # 8

MATH 530
October 19, 2024

39.

Prove the following identities by either proving that they have the same generating functions or by proving them with a bijection.

a.

Show that the number of integer partitions of n with no parts divisible by d is equal to the number of integer partitions of n with no part repeated d or more times.

Proof. Note that for a general integer partition $\lambda \vdash n$ we have that the generating function for this is:

$$\frac{1}{1-x}\frac{1}{1-x^2}\cdots.$$

In this case we the generating function for integer partitions of n with no parts divisible by d will be:

$$\frac{1-x^d}{1-x}\frac{1-x^{2d}}{1-x^2}\cdots = \frac{1}{1-x}\frac{1}{1-x^2}\cdots\frac{1}{1-x^{d-1}}\frac{1}{1-x^{d+1}}\cdots\frac{1}{1-x^{2d-1}}\frac{1}{1-x^{2d+1}}\cdots.$$

And that the generating function for the number of integer partitions of n with no parts repeated d or more times is going to be:

$$(1+x+\ldots+x^{d-1})(1+x^2+\ldots+(x^2)^{d-1})\cdots = \frac{1-x^d}{1-x}\frac{1-x^{2d}}{1-x^2}\cdots = \frac{1}{1-x}\frac{1}{1-x^2}\cdots\frac{1}{1-x^{d-1}}\frac{1}{1-x^{d+1}}\cdots$$

These are the same generating function! Thus the two must be equal. □

b.

Prove that the number of integer partitions of n with both odd and distinct parts is equal to the number of integer partitions of n that are equal to their conjugate.

Proof. We'll show this by bijection.

First, note that an integer partition of n, $\lambda \vdash n$, will be equal to it's conjugate if and only if $l(\lambda) = \max(\lambda)$. In terms of their Young diagrams this is they'll be equal if and only if they are as tall as they are wide. So we'll establish a reversible process to build an integer partition that's equal to it's own conjugate :

- Take an integer partition $\lambda \vdash n$ with all odd and distinct parts.
- Take the longest such row of the partition, bend it so that if it's equal to $2n+1$ we'll have a new column of height n above the corner piece, and a new row of width n to the right of the corner piece.
- Continue bending the odd and distinct pieces in descending order until you're finished with all parts of the partition.

An illustration of this process is seen in 1:

This process is well-defined since all the pieces are odd and distinct so the folding action we're doing is doable. Additionally, the first bent piece guarantees that $l(\mu) = \max(\mu)$ for this new partition μ. Finally, this is bijective because we can just unfold the bent pieces and construct the original integer partition $\lambda \vdash n$.

We have found a bijective/invertible process to pair the two sets, thus they must be they must be of the same size. □

e.

Prove that the number of integer partitions of n in which no part appears exactly once is equal to the number of integer partitions of n where no part is congruent to 1 or 5 modulo 6.

Proof. We'll show this through a generating function equivalence. Note that the number of integer partitions of n which no part appears exactly once has a generating function of:

$$(1 + x^2 + x^3 + \ldots)(1 + x^4 + x^6 + \ldots)\ldots(1 + x^{2k} + x^{3k} + \ldots).$$

Then the number of integer partitions of n with no part is congruent to 1 or 5 modulo 6 has generating function:

$$\frac{1}{1-x^2}\frac{1}{1-x^3}\cdots\frac{1}{1-x^{6k}}\frac{1}{1-x^{6k+2}}\frac{1}{1-x^{6k+3}}\frac{1}{1-x^{6k+4}}\cdots.$$

To show the equivalence of these two functions take the left hand side and we'll work our way to the right:

$$(1 + x^2 + x^3 + \ldots)(1 + x^4 + x^6 + \ldots)\ldots(1 + x^{2k} + x^{3k} + \ldots)$$
$$= \frac{(1-x^2)(1 + x^2 + x^3 + \ldots)}{1-x^2}\frac{(1-x^4)(1 + x^4 + x^6 + \ldots)}{1-x^4}\cdots\frac{(1-x^{2k})(1 + x^{2k} + x^{3k} + \ldots)}{1-x^{2k}}$$

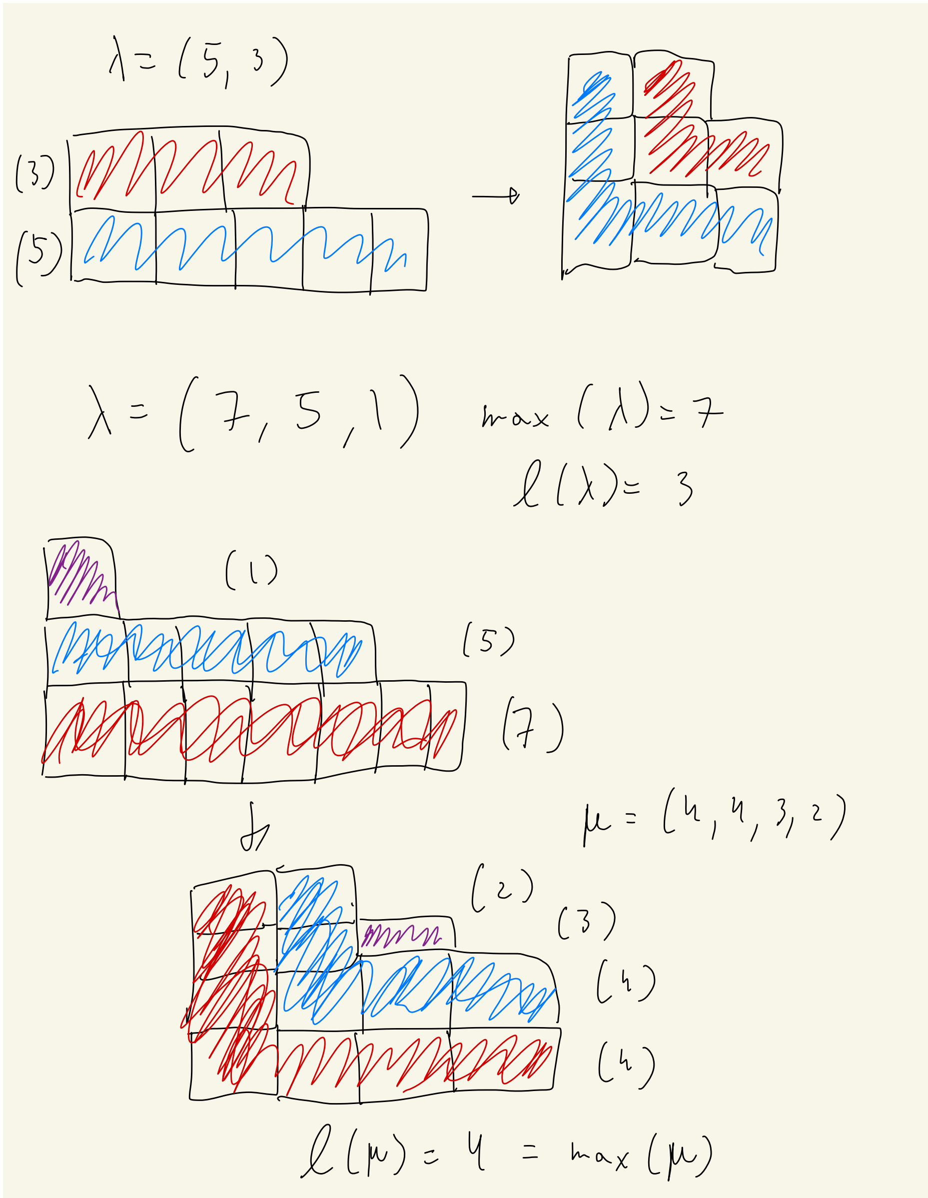

Figure 1: A poor drawing of a Young diagram

Notice that $(1+x^{2k}+x^{3k}+\dots)(1-x^{2k}) = 1+x^{2k}+x^{3k}+\dots-x^{2k}-x^{4k}-x^{5k}-\dots = 1+x^{3k}$.

Giving us:

$$\frac{(1-x^2)(1+x^2+x^3+\dots)}{1-x^2}\frac{(1-x^4)(1+x^4+x^6+\dots)}{1-x^4}\cdots\frac{(1-x^{2k})(1+x^{2k}+x^{3k}+\dots)}{1-x^{2k}}\cdots$$
$$=\frac{1+x^3}{1-x^2}\frac{1+x^6}{1-x^4}\cdots\frac{1+x^{3k}}{1-x^{2k}}\cdots$$
$$=\frac{(1-x^3)(1+x^3)}{(1-x^2)(1-x^3)}\frac{(1-x^6)(1+x^6)}{(1-x^4)(1-x^6)}\cdots\frac{(1-x^{3k})(1+x^{3k})}{(1-x^{3k})(1-x^{2k}}\cdots$$
$$=\frac{1-x^6}{(1-x^2)(1-x^3)}\frac{1-x^{12}}{(1-x^4)(1-x^6)}\frac{1-x^{18}}{(1-x^6)(1-x^9)}\cdots\frac{1-x^{6k}}{1-x^{2k}}$$
$$=\frac{1}{(1-x^2)(1-x^3)(1-x^4)(1-x^6)(1-x^8)\dots}$$

In fact the remaining terms in the denominator will be things not congruent to 1 or 5 modulo 6. Thus we've shown the equivalence of the generation functions hence the two are the same. ☐

41.

Let $p_k(n)$ be the number of integer partitions with $l(\lambda)=k$.

a.

Show there are $\binom{n-1}{k-1}$ solutions to $x_1+\ldots+x_k=n$ where $x_1,\ldots,x_k$ are positive integers. (One way is to use a "balls and bars" or "stars and bars" argument from an introductory combinatorics course.)

Proof. Let $x_1,\ldots x_k$ be positive integers and $n\in\mathbb{N}$. To count how many solutions the equation:

$$x_1+\ldots+x_k=n$$

has we'll consider this with balls-and-bars.

Let $n=1+1+\ldots+1$, where there are n-copies of 1. Then to determine the x_i's, we'll place $k-1$ bar's between the 1's. So for instance when $n=4$ and $k=2$, $1+1\Big|+1+1=4$ gives us $x_1=2, x_2=2$ with $x_1+x_2=4$. Since there are $n-1$ gaps to which a bar can be place, and there are exactly $k-1$ bars, we have that there are $\binom{n-1}{k-1}$ total solutions to the equation $x_1+\ldots+x_k=n$. ☐

b.

By considering rearrangements of the parts of partitions, show that $\binom{n-1}{k-1} \leq k!p_k(n)$.

Proof. Let $A = \{\text{all positive integer solutions to the equation } x_1 + x_2 + \ldots + x_k = n\}$ and $B = \{\text{the set of "unordered" integer partitions } \lambda \vdash n \text{ with } l(\lambda) = k\}$, where "unordered" means we drop the restriction on integer partitions being a decreasing sequence of integers. Define the map $\varphi : A \to B$ by $\varphi((x_1, x_2, \ldots, x_k)) = (x_1, x_2, \ldots, x_k)$, for all $x_1 + x_2 + \ldots + x_k$. Note then that this is a kind of embedding of A into B, since B has possibly larger cardinality than A. So the things we need to prove are: $|B| = p_k(n)k!$, φ is well-defined and one-to-one. From which, since $|A| = \binom{n-1}{k-1}$ from $(a.)$, we'll then have that $\binom{n-1}{k-1} = |A| \leq |B| = p_k(n)k!$.

To show φ is well-defined, note that $x_1, x_2, \ldots, x_k$ are all distinguishable and integers in $\mathbb{Z}^+$ since we give them a labeling from 1 to k. Moreover, $(x_1, x_2, \ldots, x_k)$ would be well-defined for every solution to $x_1 + \ldots + x_k = n$ because of the x's being distinguishable; that is associate the entry x_i in the k-tuple to the i^{th} integer in the solution. Since the integers are distinguishable by our labeling this is a well-defined association. Finally the outputs of φ will be unique for a given solution $x_1 + x_2 + \ldots + x_k = n$ since the $k-$tuple $(x_1, x_2, \ldots, x_k)$ is well-defined and $x_1 + x_2 + \ldots + x_k = n$ so that $(x_1, \ldots, x_k) \vdash n$ and $l((x_1, \ldots, x_k)) = k$ so this falls within our codomain.

To show φ is $1-1$, assume that $\varphi(x_1, x_2, \ldots, x_k) = \varphi(y_1, y_2, \ldots, y_k)$. Then interpreting these as unordered integer partitions we get that $(x_1, x_2, \ldots, x_k) = (y_1, y_2, \ldots, y_k)$. So then $x_1 = y_1, x_2 = y_2, \ldots, x_k = y_k$. Thus φ is a $1 - 1$ mapping into B.

Finally, we'll build the set B by take an integer partition $\lambda \vdash n$ such that $l(\lambda) = k$, call it $\lambda = (x_1, \ldots, x_k)$ where this is a traditional integer partition so that $x_1 \geq x_2 \geq \ldots \geq x_k$ and $x_1 + x_2 + \ldots + x_k = n$. So to build an unordered partition, we'll build a $k-$tuple that isn't in descending order using λ and count the ways we can do so. Take x_1, place it any where in an empty $k-$tuple, you have k ways to do so. Repeat this for x_2, you have $k - 1$ places to choose from. Repeating this for all x_i, $i \in \{1, \ldots, k\}$ we get that there are $k!$ ways to build this "unordered" integer partition. So our initial choice of λ had a total number of possibilities of $p_k(n)$. Our choice of λ dictates the resulting "unordered" integer partition, so that these are successive dependent choices giving us a total of $k!p_k(n)$. Thus $|B| = k!p_k(n)$.

Therefore, since $\varphi : A \to B$ is a $1 - 1$ map, we have that $|A| = \binom{n-1}{k-1} \leq |B| = k!p_k(n)$. $\square$

c.

By making the parts of a partition distinct, show that $k!p_k(n) \leq \binom{n+\binom{k}{2}-1}{k-1}$.

Proof. We'll define a surjective map from {solutions to the equation $x_1+x_2+x_3+\ldots+x_k = n+\binom{k}{2}$ in $(\mathbb{Z}^+)^k$} $= A$ onto {unordered integer partitions of n} $= B$. Note that in $(c.)$ we showed that $|B| = p_k(n)k!$ and that from $(a.)$, $|A| = \binom{n+\binom{k}{2}-1}{k-1}$. So all that needs to be shown is that there's a surjective map $f : A \to B$. We'll define this map in the following way: take an unordered integer partition of $n+\binom{k}{2}$, $(x_1, x_2, \ldots, x_k)$ so that $x_1+x_2+\ldots+x_k = n+\binom{k}{2} = n+1+2+\ldots+(k-1)$. If $k = n+\binom{n}{2}$, then there's one partition avalaible namely $(1, 1, \ldots, 1)$ where the length of this is k, and this is not in our set A as it's not distinct. We have that at least one of the x_i's must be greater than 2, so that subtract the least such x_i by 1 and the right side of the equation $(n+1+\ldots+(k-1))$ by 1, giving the new element $x_i - 1$. Take the next integer greater than x_i from the previous step, this is well-defined as the integer partition is distinct and must be at least greater than 3 because x_i was the least such integer greater than 2, subtract 2 from this and the right-hand side. We define the process in this manner, by taking the next larger integer in the unordered partition (note that since we're subtracting we won't have new integers introduced into this process, this will only act on the original elements of the unordered distinct partition) we subtract from that j where $j \in \{1, \ldots, k-1\}$ giving us a new element of the unordered partition. We terminate when we only have n on the right-hand side of the equation, meaning we do this process $k-1$ times. This process is well-defined as the partition starting in A is distinct and unordered. Moreover, this is a surjective process. Take an unordered integer partition $(x_1, \ldots, x_k)$ so that $x_1+x_2+\ldots+x_k = n$. We can pair this an element of A, by the following process: Take an unordered integer partition of n: $(x_1, \ldots, x_k)$ so that $x_1+\ldots+x_k = n$. Keep one element fixed, call if x_j. To all of the others add 1 through $k-1$ to each. Because no positive integer remains the same after adding different integers to it, and we have $k-1$ such integers, these will all be distinct positive integers and solutions to the equation:

$$y_1 + y_2 + \ldots + y_k = n + \binom{k}{2}.$$

Thus the distinct unordered integer partition $(y_1, \ldots, y_k)$ maps to $(x_1, \ldots, x_k)$. Hence this process is surjective and we get that $|A| = \binom{n+\binom{k}{2}-1}{k-1} \leq |B| = k!p_k(n)$. □

d.

Show that $\binom{n+a-1}{k-1}$ $\frac{n^{k-1}}{(k-1)!}$ for any nonnegative integer a and then show that $p_k(n)$ $\frac{n^{k-1}}{k!(k-1)!}$.

Proof. We'll show the first statement showing the equivalent statement:

$$\binom{n+a-1}{k-1} \sim \frac{n^{k-1}}{(k-1)!} \iff \frac{(n+a-1)!}{(n+a-1-k+1)!} \sim n^{k-1}.$$

Before we do this recall from our asymptotic work that for any positive integer n:

$$n! \sim \sqrt{2\pi n}\frac{n^n}{e^n}.$$

Additionally, note as $n \to \infty$

$$\frac{(n+a-1)!}{(n+a-k)!} \sim \frac{\sqrt{2\pi(n+a-1)}\frac{(n+a-1)^{n+a-1}}{e^{n+a-1}}}{\sqrt{2\pi(n+a-k)}\frac{(n+a-k)^{n+a-k}}{e^{n+a-k}}}$$

As $n \to \infty$ the effects of 1 and k and a become negligible giving us:

$$\begin{aligned}\frac{(n+a-1)^{n+a-1}e^{a+n-k}}{(n+a-k)^{n+a-k}e^{a+n-1}} &\sim n^{n+a-1-n-a+k}e^{a+n-k-a-n-1} \\ &\sim n^{k-1}e^{-k-1} \\ &\sim n^{k-1}.\end{aligned}$$

Giving us our first result.

To show the final piece, first note that

$$\binom{n-1}{k-1} \leq k!p_k(n) \leq \binom{n+\binom{k}{2}-1}{k-1}.$$

This gives us:

$$\frac{\binom{n-1}{k-1}}{\binom{n+\binom{k}{2}-1}{k-1}} \leq \frac{k!p_k(n)}{\binom{n+\binom{k}{2}-1}{k-1}} \leq 1.$$

We'll show that $\lim_{n\to\infty} \frac{\binom{n-1}{k-1}}{\frac{n+\binom{k}{2}-1}{k-1}} = 1$ allowing us to use the squeeze theorem to get that $k!p_k(n) \sim$

$\binom{n+\binom{k}{2}-1}{k-1} \sim \frac{n^{k-1}}{(k-1)!}$, which is our result. So consider the following:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\binom{n-1}{k-1}}{\binom{n+\binom{k}{2}-1}{k-1}} &= \lim_{n\to\infty} \frac{\frac{(n-1)!}{(k-1)!(n-k)!}}{\frac{(n+\binom{k}{2}-1)!}{(k-1)!(n+\binom{k}{2}-1-k+1)!}} \\
&= \lim_{n\to\infty} \frac{\frac{(n-1)!}{(n-k)!}}{\frac{(n+\binom{k}{2}-1)!}{(n+\binom{k}{2}-k)!}} \\
&= \lim_{n\to\infty} \frac{(n-1)!}{(n-k)!} \frac{(n+\binom{k}{2}-k)!}{(n+\binom{k}{2}-1)!} \\
&= \lim_{n\to\infty} \frac{(n-1)!}{(n-k)!} \lim_{n\to\infty} \frac{(n+\binom{k}{2}-k)!}{(n+\binom{k}{2}-1)!} \\
&= 1(1) = 1.
\end{aligned}$$

Thus we have our result:

$$p_k(n)k! \sim \frac{n^{k-1}}{(k-1)!} \implies p_k(n) \sim \frac{n^{k-1}}{k!(k-1)!}.$$

□