# MATH 340: Real Analysis Study Guide Mid-Term 2

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## 1 Page 59 #3

### Prove Theorem 2.2.3

**Theorem 1.** *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. If $\{b_n\}$ is bounded and $\lim_{n\to\infty}(a_n) = 0$, then $\lim_{n\to\infty}(a_n b_n) = 0$.*

*Proof.* Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers, and assume $\lim_{n\to\infty}(a_n) = 0$ and $\{b_n\}$ is bounded.

Let $\epsilon > 0$ be given. Then, by definition 2.1.9 (Bounded Sequences), there exists $M > 0$ such that $|b_n| \leq M$ for all $n \in \mathbb{N}$. Additionally, for $\frac{\epsilon}{M}$ there exists $n_0 \in \mathbb{N}$ such that $|a_n - 0| < \frac{\epsilon}{M}$ for all $n \geq n_0$ . So for all $n \geq n_0$, we have $|a_n| < \frac{\epsilon}{M}$ and since $|b_n| \leq M$, multiplying those inequalities together we get $|a_n||b_n| < \frac{\epsilon}{M} * M = \epsilon$ for all $n \geq n_0$, $|a_n b_n - 0| < \epsilon$. Thus $\lim_{n\to\infty} a_n b_n = 0$. □

## 2 Page 59 #5(a)

### If $p > 0$, prove that $\lim_{n\to\infty} \frac{1}{n^p}$.

*Proof.* Assume $p > 0$. Let $\epsilon > 0$ be given. By the Archimedean Property (Remark on page 28), there exists $n_0$ such that $\frac{1}{n_0} < \epsilon^{\frac{1}{p}}$. Then, whenever $n \geq n_0$, we have $\frac{1}{n} \leq \frac{1}{n_0} < \epsilon^{\frac{1}{p}}$. So for all $n \geq n_0$ we have $\frac{1}{n} < \epsilon^{\frac{1}{p}}$, so for all $n \geq n_0$ we have $(\frac{1}{n})^p < \epsilon$. Hence $|\frac{1}{n^p} - 0| < \epsilon$.
$\therefore\ \lim_{n\to\infty} \frac{1}{n^p} = 0$. □

## 3 Page 65 #1

**Let $I_n = [a_n, b_n]$, $n \in \mathbb{N}$, be closed and bounded intervals satisfying $I_n \supset I_{n+1}$ for all $n$. Prove that $\bigcap_{n=1}^{\infty} I_n = [a, b]$. Where $a = sup\{a_n : n \in \mathbb{N}\}$ and $b = inf\{b_n : n \in \mathbb{N}\}$**

*Proof.* ($\subseteq$) Let $x \in \bigcap_{n=1}^{\infty} I_n$ and $I_n = [a_n, b_n]$. Then, $x \in I_n$ for all $n \in \mathbb{N}$. So for all $n \in \mathbb{N}$, $a_n \leq x \leq b_n$. Hence $x \geq a_n$, and so x is an upper bound of the set $\{a_n : n \in \mathbb{N}\}$. So $x \geq a$. Similarly, $x \leq b_n$ for all $n \in \mathbb{N}$, so x is a lower bound for the set $\{b_n : n \in \mathbb{N}\}$. So $x \leq b$. Thus $x \in [a, b]$. Hence $\bigcap_{n=1}^{\infty} I_n \subseteq [a, b]$.
($\supseteq$) Let $x \in [a, b]$, where $a = sup\{a_n : n \in \mathbb{N}\}$ and $b = inf\{b_n : n \in \mathbb{N}\}$. Thus, $a \leq x \leq b$. So $sup\{a_n : n \in \mathbb{N}\} \leq x \leq inf\{b_n : n \in \mathbb{N}\}$. Then, for all $n \in \mathbb{N}$, $a_n \leq x \leq b_n$. So x is both an upper bound and a lower bound of the previously mentioned sets, so $x \in [a_n, b_n]$ for all $n \in \mathbb{N}$. Since $[a_n, b_n] = I_n$, we have $x \in \bigcap_{n=1}^{\infty} I_n$. Thus $[a, b] \subseteq \bigcap_{n=1}^{\infty} I_n$.
$\therefore\ \bigcap_{n=1}^{\infty} I_n = [a, b]$. □

## 4 Page 66 #13

**For each $n \in \mathbb{N}$, let $s_n = 1 + 1/2 + ... + 1/n$. Show that $\{s_n\}$ is monotone increasing but not bounded above.**

*Proof.* We have, for all $n \in \mathbb{N}$, $s_{n+1} - s_n = \frac{1}{n+1} > 0$. So $s_{n+1} > s_n$ for all $n \in \mathbb{N}$. Hence $\{s_n\}$ is monotonic increasing. Next, we will show that $s_{2^n} \geq 1 + \frac{n}{2}$ for all $n \in \mathbb{N}$, which will show $\{s_n\}$ is not bounded above.
(Basis) n = 1. Then $s_{2^n} = s_2 = 1 + 1/2 \geq 3$. Thus our conclusion holds when n = 1.
(Inductive Hypothesis) Assume $s_{2^k} \geq 1 + \frac{k}{2}$ for some $k \in \mathbb{N}$. Then consider the k + 1 case:

$$\begin{aligned} s_{2^{k+1}} &= (1 + \tfrac{1}{2} + \tfrac{1}{3} + ... + \tfrac{1}{2^k}) + \tfrac{1}{2^k+1} + ... + \tfrac{1}{2^{k+1}} \\ &\geq (1 + \tfrac{k}{2}) + \tfrac{1}{2^k+1} + ... + \tfrac{1}{2^{k+1}}. \end{aligned}$$

This is $1 + \frac{k}{2} +$ ($2^k$ terms, each bigger than or equal to $\frac{1}{2^{k+!1}}$). So
$s_{2^{k+1}} \geq 1 + \frac{k}{2} + 2^k(\frac{1}{2^{k+1}}) = 1 + \frac{k}{2} + \frac{1}{2} = 1 + \frac{k+1}{2}$. Thus if our claim holds for the $k^{th}$ case, then it will hold for the $k + 1$ case. Thus by the Principle of Mathematical Induction, our claim holds for all $n \in \mathbb{N}$. □

# 5 Page 66 #16

**Let $0 < b < 1$. For each $n \in \mathbb{N}$, let $s_n = 1 + b + b^2 + ... + b^n$. Prove that $\{s_n\}$ is monotone increasing and bounded above. Find $\lim_{n\to\infty}(s_n)$.**

*Proof.* By page 64, $s_n = \frac{1-b^{n+1}}{1-b}$, now we show $s_{n+1} \geq s_n$ for all $n \in \mathbb{N}$. Because $1 + b + b^2 + ... + b^n + b^{n+1} \geq 1 + b + b^2 + ... + b^n > 0$, since $b > 0$, $s_n$ is monotonic increasing. By Theorem 2.3.4(b), $\lim_{n\to\infty}(s_n) = \lim_{n\to\infty}(\frac{1-b^{n+1}}{1-b}) = \frac{1}{1-b}$.Additionally, to show that the sequence is bounded above. Note that $s_n = \frac{1-b^{n+1}}{1-b} = \frac{1}{1-b} - \frac{b^{n+1}}{1-b} < \frac{1}{1-b}$ since $b^{n+1}$ and $1 - b$ are positive (because $0 < b < 1$). Thus all terms in the sequence are less than $\frac{1}{1-b}$, hence the sequence is bounded above. □

# 6 Page 72 #8

**Let A be a non-empty subset of $\mathbb{R}$ that is bounded above and let $\alpha = \sup(A)$. If $\alpha \notin A$, prove that $\alpha$ is a limit point of A.**

*Proof.* Let $\alpha \notin A$, A be a non-empty subset of $\mathbb{R}$ and $\alpha = \sup(A)$. Given $\epsilon > 0$, let $\beta = \alpha - \epsilon$. Then there exists a $x \in A$ such that $\beta < x \leq \alpha$ by Theorem 1.4.4. Then $\alpha - \epsilon < x \leq \alpha < \alpha + \epsilon$, so $\alpha - \epsilon < x < \alpha + \epsilon$ hence $x \in N_\epsilon(\alpha)$ since $\alpha \notin A$ and $x \in A$ and $\alpha \neq x$. Therefore for all $\epsilon > 0$, there exists $x \in N_\epsilon(\alpha)$ and $\alpha \neq x$, so $\alpha$ is a limit point of A. □

# 7 Page 83 #1

**If $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences in $\mathbb{R}$. Prove that $\{a_n + b_n\}$ and $\{a_n b_n\}$ is Cauchy.**

## 7.1 Prove: $\{a_n + b_n\}$ is Cauchy.

*Proof.* Suppose $a_n$ and $b_n$ are Cauchy. Then for all $\epsilon > 0$, there exists $n_a, n_b \in \mathbb{N}$ such that $|a_n - a_m| < \frac{\epsilon}{2}$ and $|b_n - b_m| < \frac{\epsilon}{2}$, for all $n > n_a$ and for all $m > n_b$. Let $n_0 = \max\{n_a, n_b\}$. Then $|a_m + b_n - (a_m + b_m)| = |a_n - a_m + b_n - b_m| \leq |a_n - a_m| + |b_n - b_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. □

## 7.2 $\{a_n b_n\}$ is Cauchy

*Proof.* Let $M_1 > a_n$ for all $n \in \mathbb{N}$ and $M_2 > b_n$ for all $n \in \mathbb{N}$ (Theorem 2.6.2(b))...
$|a_n b_n - a_m b_m| = |a_n b_n - a_m b_n + a_m b_n - a_m b_m| = |b_n(a_n - a_m) + a_m(b_n - b_m)| \leq |b_n||a_n - a_m| + |a_m||b_n - b_m|$.

Since $a_n$ and $b_n$ are Cauchy, we have for all $\epsilon > 0$, there exists $n_1, n_2 \in \mathbb{N}$ such that $|a_n - a_m| < \frac{\epsilon}{2M_2}$ for all $n > m$ and $|b_n - b_m| < \frac{\epsilon}{2M_1}$ for all $n > n_2$. Let N $= maxn_1, n_2$, then

$|b_n||a_n - a_m| + |a_m||b_n - b_m| < M_2|a_n - a_m| + M_1|b_n - b_m| < M_2\frac{\epsilon}{2M_2} + M_1\frac{\epsilon}{2M_1} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$ □

# 8 Page 85 #5

## Use Mathematical Induction to prove the identity (b)

*Proof.* We will show that $a_{n+1} - a_n = (\frac{-1}{2})^{n-1}(a_2 - a_1)$, where $n \geq 3$. From identity (5) on Page 82:

$a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$, similarly $a_{n+1} = \frac{1}{2}(a_n + a_{n-1})$. So

$a_{n+1} - a_n = \frac{1}{2}(a_n + a_{n-1}) - \frac{1}{2}(a_{n-1} + a_{n-2}) = \frac{1}{2}(a_n - a_{n-2})$. Then we proceed by mathematical induction.

(Basis) n = 3.

(L.H.S) $= a_4 - a_3 = \frac{1}{2}(a_3 - a_1) = \frac{1}{2}(\frac{1}{2}(a_2 + a_1) - a_1) = \frac{1}{2}(\frac{1}{2}(a_2 - a_1)) = (\frac{1}{2})^2(a_2 - a_1)$.

(R.H.S) $= (\frac{-1}{2})^{3-1}(a_2 - a_1) = (\frac{-1}{2})^2(a_2 - a_1) = (\frac{1}{2})^2(a_2 - a_1)$.

Hence our claim holds for n = 3.

(Inductive Hypothesis)

Assume, for some $k \geq 3$ where $k \in \mathbb{N}$, $a_{k+1} - a_k = (\frac{-1}{2})^{k-1}(a_2 - a_1)$.

(Then we wish to show that $a_{k+2} - a_{k+1} = (\frac{-1}{2})^k(a_2 - a_1)$.)

$$\begin{aligned}
\text{L.H.S} = a_{k+2} - a_{k+1} &= \frac{1}{2}(a_{k+1} + a_k) - a_{k+1} \\
&= \frac{1}{2}a_{k+1} + \frac{1}{2}a_k - a_{k+1} \\
&= \frac{1}{2}a_k - \frac{1}{2}a_{k+1} \\
&= \frac{1}{2}(a_k - a_{k+1}) \\
&= \frac{-1}{2}(a_{k+1} - a_k) \\
&= \frac{-1}{2}((\frac{-1}{2})^{k-1}(a_2 - a_1)) \text{ , by our Inductive Hypothesis} \\
&= \frac{-1}{2}(\frac{-1}{2})^{k-1}(a_2 - a_1) \\
&= (\frac{-1}{2})^k(a_2 - a_1) \\
&= \text{R.H.S.}
\end{aligned}$$

Thus, by the Principle of Mathematical Induction, our claim holds for all $n \in \mathbb{N}$. □

# 9 Page 100 #2

### Show that every finite subset of $\mathbb{R}$ is closed.

*Proof.* Suppose we have some finite subset of $\mathbb{R}$, call it $A$. Then A has no limit points (Corollary 2.4.8). Thus, by Theorem 3.19, A is closed in $\mathbb{R}$ since it vacuously contains all of its limit points. $\square$

# 10 Page 100 #6(a.),#6(b.)

## 10.1 For any collection $\{O_\alpha\}_{\alpha\in A}$ of open subsets of $\mathbb{R}$, $\bigcup_{\alpha\in A} O_\alpha$ is open.

*Proof.* Assume $\{O_\alpha\}_{\alpha\in A}$ is a collection of open subsets of $\mathbb{R}$. [Show that $\bigcup_{\alpha\in A} O_\alpha$ is open.]

Let $x \in \bigcup_{\alpha\in A} O_\alpha$. Then there exists an $\alpha_0 \in A$ such that $x \in O_{\alpha_0}$. Since $O_{\alpha_0}$ is open, there exists $\epsilon > 0$ such that $N_\epsilon(x) \subseteq O_{\alpha_0}$. Then $N_{\alpha_0}(x) \subseteq \bigcup_{\alpha\in A} O_\alpha$, so x is an interior point of $\bigcup_{\alpha\in A} O_\alpha$. Thus every point in $\bigcup_{\alpha\in A} O_\alpha$ is an interior point, so $\bigcup_{\alpha\in A} O_\alpha$ is open. $\square$

## 10.2 Given an example of a countable collection $\{F_n\}_{n=1}^\infty$ of closed subsets of $\mathbb{R}$ such that $\bigcup_{n=1}^\infty F_n$ is not closed.

Let $F_n = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then each $F_n$ is closed and $\bigcup_{n=1}^\infty F_n = \bigcup_{n=1}^\infty \{\frac{1}{n}\} = \{\frac{1}{n} : n \in \mathbb{N}\}$, which has 0 as a limit point but $0 \notin \{\frac{1}{n} : n \in \mathbb{N}\}$, so it isn't closed.

# 11 Page 107 #1(a)

### Show that the set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not compact by constructing an open cover of A that doesn't have a finite subcover.

First, consider the collection of sets $\{(\frac{1}{n}, \frac{n+1}{n}) : n \in \mathbb{N}\}$. Then we will show $a \subseteq \bigcup_{n=1}^\infty((\frac{1}{n}, \frac{n+1}{n}))$.

*Proof.* Let $x \in A$. Then for some $n_0 \in \mathbb{N}, x = \frac{1}{n_0}$. Then for $n = n_0 + 1$, we have the set $(\frac{1}{n_0+1}, 1 + \frac{1}{n_0+1})$. Note that $\frac{1}{n_0+1} < \frac{1}{n_0}$ and $\frac{1}{n_0} < 1 + \frac{1}{n_0+1}$. Hence $x \in (\frac{1}{n_0+1}, 1 + \frac{1}{n_0+1})$. Since $n_0 + 1 \in \mathbb{N}$ we have $x \in \bigcup_{n=1}^\infty((\frac{1}{n}, 1 + \frac{1}{n}))$. Thus $A \subseteq \bigcup_{n=1}^\infty((\frac{1}{n}, 1 + \frac{1}{n}))$. $\square$

So we have $A \subseteq \bigcup_{n=1}^\infty((\frac{1}{n}, 1 + \frac{1}{n}))$, but we will show that no finite subcover covers A. For sake of contradiction suppose we have a finite subcover of A:

$\bigcup_{k=1}^n((\frac{1}{k}, \frac{k+1}{k})) \supseteq A$. Then consider the element $\frac{1}{n+1} \in A$, but $\frac{1}{n+1} \notin \bigcup_{k=1}^n((\frac{1}{k}, \frac{k+1}{k}))$. Hence we have a open cover of A, that doesn't admit a finite subcover. Thus A isn't compact.

## 12 Page 107 #2

**Suppose $\{p_n\}$ is a convergent sequence in $\mathbb{R}$ with $\lim_{n\to\infty} p_n = p$. Prove, using the definition, that the set $A = \{p\} \cup \{p_n : n \in \mathbb{N}\}$ is a compact subset of $\mathbb{R}$.**

*Proof.* Assume $\{p_n\}$ is a convergent sequence in $\mathbb{R}$ with $\lim_{n\to\infty} p_n = p$. Assume $A = \{p\} \cup \{p_n : n \in \mathbb{N}\}$. Let $\{O_\alpha\}_{\alpha\in A}$ be an open cover of A, since $A \subseteq \bigcup_{\alpha\in A} O_\alpha$ and $p \in A$ we know $p \in \bigcup_{\alpha\in A} O_\alpha$ so $p \in O_{\alpha_0}$ for some $\alpha_0 \in A$. So there exists $\epsilon > 0$ such that $N_{\epsilon_0}(p) \subseteq O_{\alpha_0}$. Since $p_n \to p$, we know for all $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $|p_n - p| < \epsilon$. So for $\epsilon_0$, there exists $m_0 \in \mathbb{N}$ such that $n \geq m_0$ implies $|p_n - p| < \epsilon_0$. Hence when $n \geq m_0$, $p - \epsilon_0 < p_n < p + \epsilon_0$. So for all $n \geq m_0$, $p_n \in N_{\epsilon_0}(p) \subseteq O_{\alpha_0}$, so for all $n \geq m_0$, $p_n \in O_\alpha$, so there exists finitely many elements in that are possible not in $O_{\alpha_0}$. Let $E = \{p_1, p_2, ..., p_n\}$ be the finite subset, then $E \subseteq \bigcup_{\alpha\in A} O_\alpha$ and for all $\alpha \in A$, $O_\alpha$ is open. Then for all $p_i \in E$, where $i = \{1, 2, ..., n\}$, there exists $\alpha_i \in A$ such that $p_i = O_{\alpha_i}$. Now $\{O_{\alpha_i}\}_{i=0}^{n}$ is a finite open cover of E.

Thus A is compact. □

## 13 Page 107 #3

**Show that (0,1] is not compact by constructing an open cover of (0,1] that does not have a finite subcover.**

*Proof.* Let $U_n = (\frac{1}{n+1}, 2)$, for all $n \in \mathbb{N}$. Then $U_1 \subseteq U_2 \subseteq ...$ [Without Loss of Generality $U_n$ is an open cover of (0,1].]

Let $x \in (0, 1]$. Then by the Archimedean Property , there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < x$. Then $x \in (\frac{1}{n_0}, 2)$. Since $(\frac{1}{n_0}, 2) = (\frac{1}{(n_0-1)+1}), x \in_{n_0+1}$ . Also, since $U$ □