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Problem Set #2

PHYS 325
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1

If $2y + \sin(y) = x^4 + 4x^3 + (2\pi - 5)$, show that $\frac{dy}{dx} = 16$ when $x = 1$.

Solution:

Note that plugging $x = 1$ into our problem that we end up with:

$$2y(1) + \sin(y(1)) = (1) + 4(1) + 2\pi - 5$$

$$2y(1) + \sin(y(1)) = 2\pi$$

One such value that works with this problem is $y(1) = \pi$ Now we will use implicit differentiation.

$$\begin{aligned}2y + \sin(y) &= x^4 + 4x^3 + (2\pi - 5)\\ 2y' + y'\cos(y) &= 4x^3 + 12x^2 + 0\\ y'(x) &= \frac{4x^3 + 12x^2}{2 + \cos(y(x))}\end{aligned}$$

Evaluating this at $x = 1$:

$$\begin{aligned}y'(x) &= \frac{4(1) + 12(1)}{2 + \cos(y(1))}\\ &= \frac{16}{2 + \cos(\pi)}\\ &= \frac{16}{1} = 16\checkmark\end{aligned}$$

2

Show that the smallest value taken by the following function $f(x) = 3x^4 + 4x^3 - 12x^2 + 6$ is -26.

Solution:

$$\begin{aligned} f'(x) &= 12x^3 + 12x^2 - 24x \\ 0 &= 12x(x^2 + x - 2) \\ 0 &= x(x^2 + x - 2) \end{aligned}$$

Using the quadratic formula we get that: $x = -2, 0, 1$.
Finding $f''(x)$:

$$\begin{aligned} f''(x) &= 36x^2 + 24x - 24 \\ f''(0) &= -24 \\ f''(1) &= 36 \\ f''(-2) &= 72 \end{aligned}$$

So our min's our at $1, -2$:

$$\begin{aligned} f(1) &= 3 + 4 - 12 + 6 = 1 \\ f(-2) &= 3(-2)^4 + 4(-2)^3 - 12(-2)^2 + 6 = -26 \end{aligned}$$

So the minimum value of f is -26.

3

Use the first principles (definition of differentiation) to find the first derivative of $\frac{x-2}{x+2}$.
Solution:

$$\lim_{\Delta x \to 0} \frac{\frac{x+\Delta x-2}{x+\Delta x+2} - \frac{x-2}{x+2}}{\Delta x}$$
$$\lim_{\Delta x \to 0} \frac{\frac{(x+\Delta x-2)(x+2)}{(x+\Delta x+2)(x+2)} - \frac{(x-2)(x+\Delta x+2)}{(x+2)(x+\Delta x+2)}}{\Delta x}$$
$$\lim_{\Delta x \to 0} \frac{\frac{x^2+2x+x\Delta x+2\Delta x-2x-4-(x^2+x\Delta x+2x-2x-2\Delta x-4)}{(x+2)(x+\Delta x+2)}}{\Delta x}$$
$$\lim_{\Delta x \to 0} \frac{\frac{4\Delta x}{(x+2)(x+\Delta x+2)}}{\Delta x}$$
$$\lim_{\Delta x \to 0} \frac{4}{(x+2)(x+\Delta x+2)} = \frac{4}{(x+2)^2}$$

4

Use integration by parts to evaluate

$$\int_0^{\frac{\pi}{2}} x^2 \sin(x)\ dx.$$

Solution:

$u = x^2, du = 2x\ dx, dv = \sin(x)\ dx, v = -\cos(x)$

$$-x^2\cos(x)\Big|_{x=0}^{\pi} + \int_0^{\pi} 2x\cos(x)\ dx$$
$$\pi^2 + 2\int_0^{\pi} x\ \ \cos(x)\ dx$$

$u = x, du = dx, dv = cos(x)\ dx, v = \sin(x)$

$$\pi^2 + 2\left[x\sin(x)\Big|_{x=0}^{\pi} - \int_0^{\pi}\sin(x)\ dx\right]$$
$$= \pi^2 + 2\cos(x)\Big|_{x=0}^{\pi}$$
$$= \pi^2 - 4$$

5

The gamma function $\Gamma(n)$ is defined for all integers n, greater than -1, by $\Gamma(n+1) = \int_0^\infty x^n e^{-x}\ dx$ (this also works if n is not an integer). Find a recurrence relation connecting $\Gamma(n+1)$ and $\Gamma(n)$ and calculate the value of $\Gamma(\frac{5}{2})$, given $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Solution:

So first we'll attempt an IBP on this, with $u = x^n, du = nx^{n-1}\ dx, dv = e^{-x}\ dx, v = -e^{-x}$:

$$\begin{aligned}\Gamma(n+1) &= \int_0^{+\infty} x^n e^{-x}\ dx \\ &= -x^n e^{-x}\Big|_{x=0}^{+\infty} + n\int_0^{+\infty} x^{n-1}e^{-x}\ dx\end{aligned}$$

Here note that $e^{-x} \to 0$ as $x \to \infty$, quicker than $x^n \to 0$, so then notice that our new integral is just:

$$\Gamma(n+1) = n\ \Gamma(n).$$

So that given $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we have $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$ and finally that:

$$\Gamma\left(\frac{5}{2}\right) = \frac{\sqrt{\pi}}{4}$$

6

For positive integer n, prove the following (given that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$):

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\sqrt{\pi} = \frac{(2n)!}{4^n n!}\sqrt{\pi}$$

Solution:

We know that the definition of the Gamma function is given by:

$$\Gamma(n) = \int_0^{+\infty} x^{n-1}e^{-x}\ dx$$

. So to show the above, we will plug in $n+\frac{1}{2}$ into this formula:

$$\Gamma(n+\frac{1}{2}) = \int_0^{+\infty} x^{n-\frac{1}{2}}e^{-x}\ dx$$

Let $u = x^{n-\frac{1}{2}}, du = (n-\frac{1}{2})x^{n-\frac{3}{2}}\ dx, dv = e^{-x}\ dx, v = -e^{-x}$

$$\begin{aligned}
&= -x^{n-\frac{1}{2}}e^{-x}\Big|_{x=0}^{+\infty} + \int_0^{\infty} x^{n-\frac{3}{2}}e^{-x}\ dx \\
&= \frac{2n-1}{2}\int_0^{\infty} x^{n-\frac{3}{2}}e^{-x}\ dx \\
&= \frac{2n-1}{2}\left[-x^{n-\frac{3}{2}}e^{-x} + \int_0^{\infty} x^{n-\frac{5}{2}}e^{-x}\ dx\right] \\
&= \frac{2n-1}{2}\frac{2n-3}{2}\int_0^{\infty} x^{n-\frac{5}{2}}e^{-x}\ dx
\end{aligned}$$

Repeating this process n times, until we get to $\Gamma(\frac{1}{2})$:

$$\begin{aligned}
\Gamma\left(n+\frac{1}{2}\right) &= \frac{(2n-1)(2n-3)\cdots(5)(3)(1)}{2^n}\Gamma\left(\frac{1}{2}\right) = \frac{(2n-1)\cdots 3\cdot 1}{2^n}\sqrt{\pi} \\
&= \frac{(2n)(2n-1)\cdots(3)(2)(1)}{2^n(2n)(2n-2)\cdots(4)(2)}\sqrt{\pi} = \frac{(2n)!}{2^n 2^n (n)!}\sqrt{\pi} \\
&= \frac{(2n)!}{4^n(n)!}\sqrt{\pi}
\end{aligned}$$