Joseph C. McGuire
Dr. Linda Patton

Homework #7

MATH 550
October 19, 2024

3.A.11

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f_1, f_2, \ldots$ are $\mathcal{S}$-measurable functions from X to $\mathbb{R}$ such that $\sum_{k=1}^{\infty} \int |f_k| \; d\mu < \infty$. Prove that there exists $E \in \mathcal{S}$ such that $\mu(X \setminus E) = 0$ and $\lim_{k \to \infty} f_k(x)$ for every $x \in E$.

Proof. Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f_1, f_2, \ldots$ are $\mathcal{S}$-measurable functions from X to $\mathbb{R}$ such that $\sum_{k=1}^{\infty} \int |f_k| \; d\mu < \infty$. By (#10) we can interchange the above sum with integral giving us:

$$\int \sum_{k=1}^{\infty} |f_k| \; d\mu < \infty.$$

Furthermore, this gives us that the function$f : X \to [0, \infty]$ given by $f(x) = \sum_{k=1}^{\infty} |f_k(x)|$ is $\mathcal{S}$-measurable. So then we can define an $\mathcal{S}$-partition by noting that the singleton $\{\infty\}$ intersected with any Borel set in $\mathbb{R}$ is empty, hence we have that $\{\infty\}$ is an extended Borel set. So that the set $f^{-1}(\{\infty\})$ and $f^{-1}([0, \infty))$ form an $\mathcal{S}$-partition of X. But note that $\int \sum_{k=1}^{\infty} |f_k| \, d\mu = \sup_P \{L(f, P)\} < \infty$, where P is any $\mathcal{S}-$partition of X. Hence this immediately implies with the $\mathcal{S}-$partition $f^{-1}(\{\infty\}) = A$ and $f^{-1}([0, \infty)) = B$, we'll have:

$$\inf_A \mu(A) + \inf_B \mu(B) < \infty.$$

But by construction $\inf_A f = \infty$, so that for above sum to remain finite we must have $\mu(A) = 0$. So then we have that on the set $B = f^{-1}([0, \infty))$, we'll have:

$$\sum_{k=1}^{\infty} |f_k(x)| < \infty,$$

for all $x \in B$. So that is the series is absolutely convergent, which of course immediately implies normal convergence. So that we get for all $x \in B$ we'll have

$$\lim_{k \to \infty} f_k(x) = 0,$$

and that B's complement A has measure 0. This is our result! $\square$

13.

Give an example to show that the Monotone Convergence Theorem (3.11) can fail if the hypothesis that $f_1, f_2, \dots$ are nonnegative functions is dropped.

Example. Let $(\mathbb{R}, L, \lambda)$ be the measure space of all Lebesgue measurable sets with Lebesgue measure. Define the sequence of functions given by $f_k : \mathbb{R} \to \mathbb{R}$ for all $k \in \mathbb{N}$ defined as

$$f_k(x) = -\frac{1}{k}\chi_{\mathbb{R}}.$$

Then this is a Lebesgue measurable function since $\mathbb{R}$ is a Lebesgue measurable set. Furthermore, it's an increasing sequence, since $-\frac{1}{k}$ is increasing. Additionally, for all $k \in \mathbb{N}$ the integral $\int f_k \ d\mu = -\int f_k^- \ d\mu = \frac{-1}{k}\lambda(\mathbb{R}) = -\infty$. (That is, whatever $k \in \mathbb{N}$ is, it'll be absorbed by the ∞.) So that this is defined for all $k \in \mathbb{N}$. Taking the limit pointwise, we'll see that:

$$\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} -\frac{1}{k}\chi_{\mathbb{R}} = \lim_{k\to\infty} -\frac{1}{k} = 0,$$

for all $x \in \mathbb{R}$. Hence we'll have:

$$\lim_{k\to\infty} \int f_k \ d\mu = \infty \neq \int \lim_{k\to\infty} f_k \ d\mu = \int 0\chi_{\mathbb{R}} \ d\mu = 0.$$

□

14.

Give an example to show that the Monotone Convergence Theorem can fail if the hypothesis of an increasing sequence of functions is replaced by a hypothesis of a decreasing sequence of functions. [This exercise shows that the Monotone Convergence Theorem should be called the Increasing Convergence Theorem. However, see Exercise (20).]

Proof. Consider the measure space $(\mathbb{R}, L, \lambda)$, that is the set of Lebesgue measurable sets on $\mathbb{R}$ with Lebesgue measure. Define the sequence of functions $f_k : \mathbb{R} \to \mathbb{R}$ given by:

$$f_k(x) = \chi_{(-\infty,-k)\cup(k,\infty)}.$$

Since $(-\infty,-k) \cup (k,\infty)$ is Lebesgue measurable for all $k \in \mathbb{N}$ we have that each $f_k(x)$ is Lebesgue measurable. Furthermore:

$$\lim_{k\to\infty} \chi_{(-\infty,k)\cup(k,\infty)}(x) = \chi_{\emptyset}(x) = 0,$$

for all $x \in \mathbb{R}$. Hence we'll get that $\int \lim_{k\to\infty} f_k \, d\mu = \int 0 \, d\mu = 0$. But that $\int \chi_{(-\infty,k)\cup(k,\infty)} \, d\mu = \lambda((k,\infty)) + \lambda((-\infty,k)) = \infty$, for any $k \in \mathbb{N}$. Hence we'll have that

$$\lim_{k\to\infty} \int f_k \, d\mu = \infty \neq \int \lim_{k\to\infty} f_k \, d\mu = 0.$$

□

16.

Suppose $\mathcal{S}$ and T are $\sigma-$algebras on a set X and $\mathcal{S} \subset \mathrm{T}$. Suppose μ_1 is a measure on $(X, \mathcal{S})$, μ_2 is a measure on (X, T), and $\mu_1(E) = \mu_2(E)$ for all $E \in \mathcal{S}$. Prove that if $f : X \to [0, \infty]$ is $\mathcal{S}$-measurable, then $\int f \; d\mu_1 = \int f \; d\mu_2$.

Proof. Suppose $\mathcal{S}$ and T are $\sigma-$algebras on a set X and $\mathcal{S} \subset \mathrm{T}$. Suppose μ_1 is a measure on $(X, \mathcal{S})$, μ_2 is a measure on (X, T), and $\mu_1(E) = \mu_2(E)$ for all $E \in \mathcal{S}$.

We'll show this result holds for all simple $\mathcal{S}$-measurable functions first. Let $f = \sum\limits_{k=1}^{n} c_n \chi_{E_n}$ be a simple $\mathcal{S}$-measurable function. Then we must have that $E_1, \dots, E_n \in \mathcal{S}$. Furthermore, by (3.15) we get the following:

$$\begin{aligned} \int f \; d\mu_1 &= \int \sum_{k=1}^{n} c_n \chi_{E_n} \; d\mu_1 \\ &= \sum_{k=1}^{n} c_n \mu_1(E_n) \\ &= \sum_{k=1}^{n} c_n \mu_2(E_n) \\ &\int \sum_{k=1}^{n} c_n \chi_{E_n} \; d\mu_2. \end{aligned}$$

Hence we have our result for any $\mathcal{S}$-measurable function.

Let $f : X \to [0, \infty)$ be an $\mathcal{S}$-measurable function. So then applying (2.89) we have that there exists a sequence increasing $\mathcal{S}$-measurable simple functions $f_1, f_2, \dots$ such that $\lim\limits_{k\to\infty} f_k(x) = f(x)$ for all $x \in X$. Hence we have:

$$\begin{aligned} \int f \; d\mu_1 &= \int \lim_{k\to\infty} f_k \; d\mu_1 \\ &= \lim_{k\to\infty} \int f_k \; d\mu_1 \\ &= \lim_{k\to\infty} \int f_k \; d\mu_2 \\ &= \int \lim_{k\to\infty} f_k \; d\mu_2 \\ &= \int f \; d\mu_2. \end{aligned}$$

The second equality comes from the Monotone Convergence Theorem, the third equality is the result we showed above, and then the fourth equality is the Monotone Convergence Theorem again. This is our result! □

20.

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f_1, f_2, \ldots$ is a monotone (meaning either increasing or decreasing) sequence of $\mathcal{S}$-measurable functions. Define $f : X \to [-\infty, \infty]$ by

$$f(x) = \lim_{k \to \infty} f_k(x).$$

Prove that if $\int |f_1| \, d\mu < \infty$, then

$$\lim_{k \to \infty} \int f_k \; d\mu = \int f \; d\mu.$$

Proof. Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f_1, f_2, \ldots$ is a monotone (meaning either increasing or decreasing) sequence of $\mathcal{S}$-measurable functions. Define $f : X \to [-\infty, \infty]$ by

$$f(x) = \lim_{k \to \infty} f_k(x).$$

First, we'll show that this holds for an increasing sequence of functions $f_1, f_2, \ldots$. We'll start with a lemma:

Lemma 1. *If (f_n) is an increasing sequence of functions, then (f_n^+) is an increasing sequence of functions and (f_n^-) is a decreasing sequence of functions.*

Proof. Let (f_n) be an increasing sequence of functions, then we have for all $n \in \mathbb{N}$ that $f_n \leq f_{n+1}$. Furthermore note that $f_n^+ : R \to [0, \infty)$ is given by:

$$f_n^+(x) = \max\{f_n(x), 0\}.$$

Hence if $f_n(x) \geq 0$, we'll have that $0 \leq f_n(x) \leq f_{n+1}(x)$ implying that $f_{n+1}(x) = f_{n+1}^+(x)$. Thus $f_n^+(x) \leq f_{n+1}^+(x)$. If $f_n(x) < 0$, we'll have $f_n^+(x) = 0$. Then either $f_{n+1}^+(x) = f_{n+1}(x)$ or $f_{n+1}^+(x) = 0$. In either case we'll get that $f_n^+(x) \leq f_{n+1}^+$. So we have (f_n^+) is an increasing sequence of functions.

To show that (f_n^-) is a decreasing sequence, note the definition of f_n^- is:

$$f_n^-(x) = \max\{-f_n(x), 0\}.$$

That is, if $f_n(x) > 0$, then $f_n^-(x) = 0$. In which case, either $f_{n+1}^-(x) = -f_{n+1}(x)$ or $f_{n+1}^-(x) = 0$. In either case $f_{n+1}^- \leq f_n^-$, since $f_n \leq f_{n+1} \implies -f_{n+1} \leq -f_n$. Otherwise if $f_n(x) \leq 0$, then $f_n^-(x) = -f_n(x)$. Because we have that $f_n(x) \leq f_{n+1}(x)$, we get that $-f_{n+1}(x) \leq -f_n(x)$. Hence we have that (f_n) is a decreasing sequence of functions. That is our result! $\square$

So we have that for our increasing sequence (f_n) that (f_n^+) is increasing and that (f_n^-) is a decreasing sequence. So we'll directly apply the Monotone Convergence Theorem on (f_n^+) since $0 \leq f_1^+ \leq f_2^+ \leq \ldots$ so that:

$$\int \lim_{n\to\infty} f_n^+ \, d\mu = \lim_{n\to\infty} \int f_n^+ \, d\mu.$$

Then, note that since (f_n^-) is a decreasing sequence, we have that $f_1^- \geq f_2^- \geq \ldots$. So then the sequence $|f_1| - f_n^- = f_1^+ + f_1^- - f_n^-$ is both positive (since $f_1^- \geq f_n^-$) and increasing. As well as $\mathcal{S}$-measurable since composition and subtraction preserve $\mathcal{S}$-measurability. Hence we can apply the Monotone Convergence Theorem here as well to get:

$$\lim_{n\to\infty} \int |f_1| - f_n^- \, d\mu = \int \lim_{n\to\infty} (|f_1| - f_n^-) \, d\mu.$$

First, consider the case where $\lim\limits_{n\to\infty} \int f_n^+ \, d\mu = \infty$. Then we'll have that since (f_n^-) is a decreasing sequence, we have that $\infty > \int f_1^- \, d\mu \geq \int f_2^- \, d\mu \geq \ldots 0$ so that $-\infty < -\int f_1^- \, d\mu \leq -\int f_2^- \, d\mu \leq \ldots \leq 0$. The important part being that $\int (-f_n^-) \, d\mu > -\infty$, hence we'll have that $\lim\limits_{n\to\infty} \int f_n^+ \, d\mu - \lim\limits_{n\to\infty} \int f_n^- \, d\mu = \infty$. Furthermore, if $\lim\limits_{n\to\infty} \int f_n^+ \, d\mu = \infty$ we have by the Monotone Convergence Theorem that $\int \lim\limits_{n\to\infty} f_n^+ \, d\mu = \infty$. Additionally $-f_n^-$ is an increasing sequence of non-positive functions, meaning that we can conclude that $\int \lim_{n\to\infty} -f_n^- \, d\mu > -\infty$. Hence we'll have that $\lim\limits_{n\to\infty} \int f_n^+ \, d\mu - \lim\limits_{n\to\infty} \int f_n^- \, d\mu = \infty$. Thus the theorem holds for this case, and increasing sequence (f_n).

Hence we can assume that $\lim\limits_{n\to\infty} \int f_n \, d\mu = \int \lim\limits_{n\to\infty} f_n \, d\mu < \infty$. So then consider the following:

$$\begin{aligned}
\lim_{n\to\infty} \int (|f_1| - f_n^-) \, d\mu + \lim_{n\to\infty} \int f_n^+ \, d\mu &= \lim_{n\to\infty} \left(\int (|f_1| - f_n^-) \, d\mu + \int f_n^+ \, d\mu \right) \\
&= \lim_{n\to\infty} \int (|f_1| - f_n^- + f_n^+) \, d\mu \\
&= \int \lim_{n\to\infty} (|f_1| - f_n^- + f_n^+) \, d\mu \\
&= \int \lim_{n\to\infty} (|f_1| + f_n) \, d\mu,
\end{aligned}$$

the first equality came from the fact that both limits exist by the definition of f, the second equality comes from the fact that both $|f_1| - f_n^-$ and f_n^+ are non-negative $\mathcal{S}$-measurable functions, the third equality comes from noting that since both sequences are increasing non-negative we can employ the monotone convergence theorem, the final equality comes from noting that $f = f^+ - f^-$ for any function. Hence we have

$$\lim_{n\to\infty} \int (|f_1| + f_n) \, d\mu = \int \lim_{n\to\infty} (|f_1| + f_n) \, d\mu.$$

And because $\lim_{n\to\infty} \int f_n \; d\mu = \int \lim_{n\to\infty} f_n \; d\mu < \infty$ and that $\int_{n\to\infty} -f_n^- \; d\mu > -\infty$ and that $\lim_{n\to\infty} \int -f_n^- \; d\mu > -\infty$, we can remove the $|f_1|$ by applying theorem (3.21) giving us our result:

$$\lim_{n\to\infty} \int f_n \; d\mu = \int \lim_{n\to\infty} f_n \; d\mu.$$

For the case where (f_n) is decreasing, take the lengthy argument above and replace f_n with $-f_n$, so that $(-f_n)$ is an increasing sequence, hence we get the same result and can employ the homogeneity of integration to remove the negative. Hence we have our result! $\square$