Section 10.A: Adjoints and Invertibility

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Invertible Operators Form an Open Set

Theorem

Suppose V is a Banach space. Then $\{T \in B(V) : T \text{ is invertible}\}$ is an open subset of $B(V)$.

Proof.

Suppose $T \in B(V)$ is invertible. Suppose $S \in B(V)$ and

$$\|T - S\| < \frac{1}{\|T^{-1}\|}.$$

Then

$$\begin{aligned} \|I - T^{-1}S\| &= \|T^{-1}T - T^{-1}S\| && (1) \\ &\leq \|T^{-1}\| \|T - S\| && (2) \\ &< 1. && (3) \end{aligned}$$

□

Invertible Operators Form an Open Set

Proof. (Cont.)

By Theorem 10.22, this implies that $I - (I - T^{-1}S)$ is invertible; in other words, $T^{-1}S$ is invertible. That is, we showed $\|I - T^{-1}S\| < 1$, which implies that $I - (I - T^{-1}S) = T^{-1}S$ is invertible.

Now $S = T(T^{-1}S)$ (left multiply by the identity operator), then S is the product of two invertible operators, which implies that S is invertible and has inverse $S^{-1} = (T^{-1}S)^{-1}T^{-1} = S^{-1}TT^{-1}$. We have shown that every element of the open ball of radius $\|T^{-1}\|^{-1}$ centered at T is invertible. Thus the set of invertible elements of $B(V)$ is open. $\square$

Left Invertible; Right Invertible

Definition

Suppose T is a bounded operator on a Banach space V.

- T is called *left invertible* if there exists $S \in B(V)$ such that $ST = I$.
- T is called *right invertible* if there exists $S \in B(V)$ such that $TS = I$.

(Note: These are equivalent in finite-dim. linear algebra, not the case in Hilbert spaces.)

Example

Example (Left Invertibility isn't Equivalent to Right Invertibility)

Define the right shift $T : l^2 \to l^2$ and the left shift $S : l^2 \to l^2$ by

$$T(a_1, a_2, \dots) = (0, a_1, a_2, \dots)$$

and

$$S(a_1, a_2, \dots) = (a_2, a_3, \dots).$$

Notice $ST = I$, so that T is left invertible and S is right invertible. But $TS(a_1, a_2, a_3, \dots) = (0, a_2, a_3, \dots)$, you lose information with the left shift!

Example

Example (Injective but not Left Invertible)

Define $T : l^2 \to l^2$ by

$$T(a_1, a_2, a_3, \dots) = \left(a_1, \frac{a_2}{2}, \frac{a_3}{3}, \dots\right).$$

Then T is an injective bounded operator on l^2.
Suppose S is an operator on l^2 such that $ST = I$. For $n \in \mathbb{Z}^+$, let $e_n \in l^2$ be the vector with 1 in the $n^{\text{th}}-$slot and 0 elsewhere. Then

$$Se_n = S(nTe_n) = n(ST)(e_n) = ne_n.$$

The equation above implies that S is unbounded. Thus T isn't left invertible, even though it is injective.

Left Invertibility Conditions

Theorem

Suppose V is a Hilbert space and $T \in B(V)$. Then the following are equivalent:

1. *T is left invertible.*
2. *there exists $\alpha \in (0, \infty)$ such that $\|f\| \leq \alpha \|Tf\|$ for all $f \in V$.*
3. *T is injective and has closed range.*
4. *T^*T is invertible.*

Proof. (1) $\implies$ (2).

First suppose (1) holds. Thus there exists $S \in B(V)$ such that $ST = I$. If $f \in V$, then $\|f\| = \|S(Tf)\| \leq \|S\|\|Tf\|$. Where the first equality follow by the assumption, and the inequality follows from Theorem 10.20. Thus (1) $\implies$ (2), where $\|S\| = \alpha$. $\square$

Left Invertibility Conditions

Proof. (2) $\implies$ (3).

Now suppose (2) holds. Thus there exists $\alpha \in (0, \infty)$ such that

$$\|f\| \leq \alpha \|Tf\| \text{ for all } f \in V. \tag{4}$$

The inequality above shows that if $f \in V$ and $Tf = 0$, then $f = 0$. Thus T is injective. To show that T has closed range, suppose $f_1, f_2, \ldots$ is a sequence in V such that $Tf_1, Tf_2, \ldots$ converges in V to some $g \in V$. Thus the sequence $Tf_1, Tf_2, \ldots$ is a Cauchy sequence in V. The inequality 4 then implies that $f_1, f_2, \ldots$ is a Cauchy sequence in V. Thus $f_1, f_2, \ldots$ converges in V to some $f \in V$, which implies that $Tf = g$. Hence $g \in \operatorname{range} T$, completing the proof that T has a closed range, and completing the proof that (2) $\implies$ (3). $\square$

Left Invertibility Conditions

Proof. (3) $\implies$ (1).

Suppose now that (3) holds, so that T is injective and has closed range. We want to show that (1) holds.
Let $R : \operatorname{range} T \to V$ be the inverse of the one-to-one linear function $f \mapsto Tf$ that maps V onto $\operatorname{range} T$. Because $\operatorname{range} T$ is a closed subspace of V and thus is a Banach space [by Theorem 6.16(b)(A closed subset of a complete metric space is complete)], the Bounded Inverse Theorem (1-1 Bounded linear maps on Banach Spaces have bounded linear inverses) implies that R is a bounded linear map. Let P denote the orthogonal projection of V onto the closed subspace $\operatorname{range} T$. Define $S : V \to V$ by

$$Sg = R(Pg)$$

$\square$

Left Invertibility Conditions

Proof. (3) $\implies$ (1) Cont.

Then for each $g \in V$, we have

$$\|Sg\| = \|R(Pg)\| \leq \|R\|\|Pg\| \leq \|R\|\|g\|,$$

where the last inequality comes from (8.37(d), projections "shrink" vectors). The inequality above implies that S is a bounded operator on V. If $f \in V$, then

$$S(Tf) = R(P(Tf)) = R(Tf) = f.$$

Thus $ST = I$, which means that T is left invertible, completing the proof that (3) $\implies$ (1). $\square$

Left Invertibility Conditions

Proof. (2) $\implies$ (4).

At this stage of the proof we know that (1) $\iff$ (2) $\iff$ (3). To prove that one of these implies (4), suppose (2) holds. That is, there exists a $\alpha \in (0, \infty)$ such that $\|f\| \leq \alpha\|Tf\|$ for all $f \in V$. Squaring the inequality in, we see that if $f \in V$, then

$$\|f\|^2 \leq \alpha^2\|Tf\|^2 = \alpha^2 < T^*Tf, f > \leq \alpha^2\|T^*Tf\|\|f\|,$$

which implies that

$$\|f\| \leq \alpha^2\|T^*Tf\|.$$

In other words, (2) holds with T replaced by T^*T (and α replace with α^2). By the equivalence we already proved between (a) and (b), we conclude that T^*T is left invertible. Thus there exists $S \in B(V)$ such that $S(T^*T) = I$ taking the adjoints of both sides shows that $(T^*T)S^* = I$. Thus T^*T is also right invertible, which implies that T^*T is invertible. Thus (2) $\implies$ (4). $\square$

Left Invertibility Conditions

Proof. (4) $\implies$ (1).

Finally, suppose (d) holds, so T^*T is invertible. Hence there exists $S \in B(V)$ such that $I = S(T^*T) = (ST^*)T$. Thus T is left invertible, showing that (4) $\implies$ (1), completing the proof that (1) $\iff$ (2) $\iff$ (3) $\iff$ (4). $\square$

Right Invertibility Conditions

A note that the following theorem also is true in the finite-dimensional case, that is right invertibility is equivalent to surjectivity.

Theorem (Right Invertibility)

Suppose V is a Hilbert space and $T \in B(V)$. Then the following are equivalent:

1. *T is right invertible.*
2. *T is surjective.*
3. *TT^* is invertible.*

Right Invertibility

Proof. (1) $\iff$ (3).

Suppose that T is right invertible. That is, there exists some $S \in B(V)$ such that $TS = I \iff S^*T^* = I$. That is, the adjoint of T is left-invertible. So by Theorem 10.29, we have that TT^* is invertible. Conversely, we apply Theorem 10.29 and we get the result that T^* is left invertible and hence T is right invertible. This shows (1) $\iff$ (3) □

Right Invertibility

Proof. (1) $\implies$ (2).

Suppose (1) holds, that is T is right invertible. Then there exists $S \in B(V)$ such that $TS = I$. Thus $T(Sf) = f$ for every $f \in V$, which implies that T is surjective. That is, take a $f \in V$, then $Sf \mapsto f$ through the map T. This shows (1) $\implies$ (2). $\square$

Right Invertibility

Proof. (2) $\implies$ (1).

Suppose T is surjective. Define $R : (\operatorname{null} T)^{\perp} \to V$ by $R = T\big|_{(\operatorname{null} T)^{\perp}}$; that is R is the restriction of T on the orthogonal complement of the null-space of T, or the set of all things perpendicular to things that get sent to the zero-vector through T. Clearly R is injective because

$$\operatorname{null} R = (\operatorname{null} T)^{\perp} \cap (\operatorname{null} T) = \{0\}.$$

If $f \in V$, then $f = g + h$ for some $g \in \operatorname{null} T$ and some $h \in (\text{null T})^{\perp}$ (by 8.43, direct sum decomposition); thus $Tf = Th = Rh$, which implies that $\operatorname{range} T = \operatorname{range} R$. But recall that T is surjective, hence $\operatorname{range} R = V$. In other words, R is a continuous injective linear map of $(\text{null } T)^{\perp}$ onto V. The Bounded Inverse Theorem (6.83) implies that $R^{-1} : V \to (\operatorname{null} T)^{\perp}$ is a bounded linear map on V. We have $TR^{-1} = I$. Thus T is right invertible, completing the proof that (2) $\implies$ (1). $\square$

Summary of Section 10.A

- Adjoints in Hilbert spaces still preserve the norm of the map are bounded, linear, and act like their finite dimensional counterparts.
- 10.14 gives us a relatively easy way of determine whether the range of a map is a dense subset in the co-domain.
- $\|ST\| \leq \|S\|\|T\|$
- $(1-T)^{-1} = \sum_{k=0}^{\infty} T^k$ for bounded operators and $\|T\| < 1$.
- Right and Left Invertibility are not equivalent for operators!
- Left Invertibility $\iff$ Injective and has closed range
- Right Invertibility $\iff$ Surjective

Exercises

Example (Exercise 1.)

Define $T : l^2 \to l^2$ by $T(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots)$. Find a formula for T^*.

Proof.

Let $(b_1, b_2, \ldots) \in l^2$, then:

$$< T(a_1, a_2, \ldots), (b_1, b_2, \ldots) > =< (0, a_1, a_2, \ldots), (b_1, b_2, \ldots) >$$
$$= \sum_{k=1}^{\infty} a_k \overline{b_{k+1}}.$$

$\square$

Exercises

Cont.

Then we need an operator T^* such that:

$$< (a_1, a_2, \ldots), T^*(b_1, b_2, \ldots) >= \sum_{k=1}^{\infty} a_k \overline{T(b_k)} = \sum_{k=1}^{\infty} a_k \overline{b_{k+1}}.$$

Where $T(b_k)$ is a slight abuse of notation. So it's clear that the operator that matches this is such that $b_k \mapsto b_{k+1}$ for all $k \in \mathbb{Z}^+$. Or $(b_1, b_2, \ldots) \mapsto (b_2, b_3, \ldots)$, so T^* is the left-shift operator! $\square$

Exercises

Example (Exercise 2.)

Suppose V is a Hilbert space, U is a closed subspace of V, and $T : U \to V$ is defined by $Tf = f$. Describe the linear operator $T^* : V \to U$.