Joseph C. McGuire
Dr. Hongtao Shi

Homework #1

1

Find the roots of the equation $f(x) = x^4 - x^3 - 7x^2 + 13x - 6 = 0$.

Solution:

By inspection we see that $x = 1$ is a root: $1^4 - 1^3 - 7(1)^2 + 13(1) - 6 = 0$. So now can use the method of synthetic division to find the remaining roots, doing so we find that we end us with $f(x) = (x-1)(x^3 - 7x + 6)$. Noticing that $x = 1$ is a root for $x^3 - 7x + 6$, we use synthetic division on this polynomial and we have: $f(x) = (x-1)^2(x^2 + x - 6)$. Finally, we can use the quadratic formula to find the remaining two roots and find that:

$$x = \frac{-1 \pm \sqrt{1 - 4(1)(-6)}}{2(1)} = \frac{-1 \pm 5}{2} = -3, 2$$

So the polynomial f has the roots: $x = -3, 1, 1, 2$.

2

Determine $f(f(x))$ if $f(x) = x^2 + 2x + 1$.

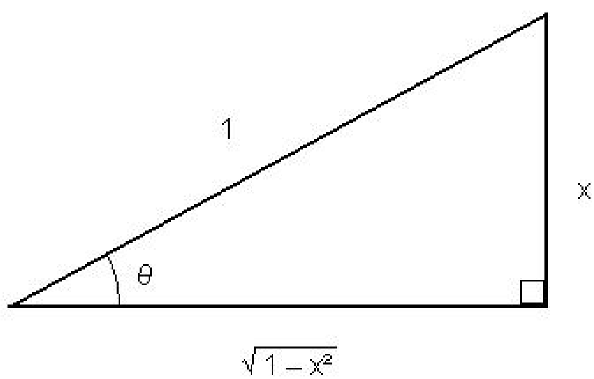
Solution:

$$\begin{aligned}(x^2+2x+1)^2 + 2(x^2+2x+1) + 1 &= x^4 + 2x^3 + x^2 + 2x^3 + 4x^2 + 2x \\ &\quad + x^2 + 2x + 1 + 2x^2 + 4x + 2 + 1 \\ &= x^4 + 4x^3 + 8x^2 + 8x + 4\end{aligned}$$

3

Show that $\sin^{-1}(x) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ if $|x| < 1$.

Solution: Since the output of sine is a ratio of magnitudes of a triangle, and arcsine is the inverse function, then we know arcsine will give back the angle that gives you $\frac{x}{1}$. So this defines a right triangle with 1 as the hypotenuse and x as the opposite side to an angle θ, this comes from the fact that x ; in other words $\sin\theta = \frac{x}{1}$, so we recover that $\arcsin x = \theta$. Using the Pythagorean theorem, we know then that the unknown side is given by: $1^2 = x^2 + a^2 \iff a = \sqrt{1-x^2}$. Hence we get the following triangle:



Notice, however, that if we take the tangent of the angle θ we will get: $\tan\theta = \frac{x}{\sqrt{1-x^2}}$. Thus $\arctan\left(\frac{x}{\sqrt{1-x^2}}\right) = \theta$.

Therefore $\arcsin(x) = \theta = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$

4

Find all real solutions of $3\sin\theta - 4\cos\theta = 2$. Make sure you verify your solutions that each of them does satisfy the given equation.

Solution:

$$3\sin\theta - 4\cos\theta = 2$$
$$\frac{3}{2}\sin\theta - 2\cos\theta = 1$$
$$\frac{3k}{2}\sin\theta - 2k\cos\theta = k$$

Then we know $\cos A = \frac{3k}{2}$ and $\sin A = 2k$, and thus we get $9k^2/4 + 4k^2 = 1 \iff 9k^2 + 16k^2 = 4 \iff k^2 = 4/25 \iff k = 2/5$. So we know $\cos A = \frac{3}{5}$ and $\sin A = \frac{4}{5}$, so $\tan A = \frac{4/5}{3/5} = \frac{4}{3} \iff A = \arctan 4/3 = 53.13^o,\ (180 + 53.13)^o = 233.13^o$; note their our two solutions since $\arcsin x$ only gives angles between $-90^o \leq \theta \leq 90^o$, however values shifted by 180^o are also valid solutions, the same will go for $\arctan x$.

$$\frac{3}{5}\sin\theta - \frac{4}{5}\cos\theta = \cos A\sin\theta - \cos\theta\sin A = \frac{2}{5}$$

Using the identity $\sin A - B = \cos A\sin B - \cos B\sin A$:

$$\sin(A-\theta) = \frac{2}{5} \iff A - \theta = \arcsin(2/5) = 23.58^o,\ (180 + 23.58)^o = 156.42^o$$

So then the possible solutions are: $\theta = A - \arcsin(2/5)$. There are four possible cases that follow:

1. $53.13^o - 23.58^o = 29.55^o$
2. $53.13^o - 203.58^o = -150.45^o = 209.55^o$
3. $233.13^o - 23.58^o = 209.55^o$
4. $233.13^o - 203.58^o = 29.55^o$

Checking these solutions we find the one that work is:

$$\theta = 209.55^o$$

5

Resolve the following function into partial fractions: $f(x) = \frac{2x+1}{x^2+3x-10}$.

Solution: Using the quadratic formula we get the roots of the denominator are: $x = -5, 2$, so $f(x) = \frac{2x+1}{(x+5)(x-2)}$. Assume $f(x) = \frac{A}{(x+5)} + \frac{B}{(x-2)}$, for some constants A, B. Then:

$$2x + 1 = A(x-2) + B(x+5) \iff 2x + 1 = Ax - 2A + Bx + 5B$$

So we get that: $A+B = 2$ and $-2A+5B = 1$. Substituting in $A = 2-B$: $-2(2-B)+5B = 1 \iff -4 + 2B + 5B = 1 \iff B = 5/7$. So $A = 2 - 5/7 = 9/7$. Thus:

$$f(x) = \frac{9/7}{(x+5)} + \frac{5/7}{(x-2)}$$

6

Suppose the parametric equations for the motion of a charged particle released from rest in electric and magnetic fields at right angles to each other take the following forms:

$$x = a(\theta - \sin(\theta)),\ y = a(1 - \cos(\theta))$$

where a is constant. Show that the tangent to the curve has a slope of $\cot(\frac{\theta}{2})$. Use a computer program, such as Mathematica, to plot the particle's trajectory $\frac{y}{a}$ as a function of $\frac{x}{a}$.

Solution: To find the slope of the tangent line we first note that $\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta}$, so we have:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

by the chain rule.

Doing the computation we find:

$$\frac{dx}{d\theta} = a(1 - \cos\theta)$$

$$\frac{dy}{d\theta} = a(\sin\theta)$$

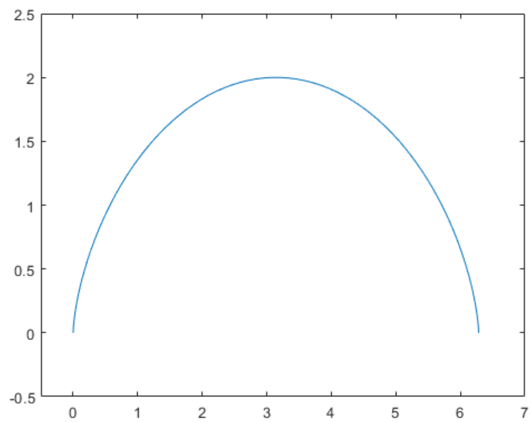
Then the slope of the tangent is the derivative dy/dx:

$$\frac{dy}{dx} = \frac{a \cdot \sin\theta}{a(1 - \cos\theta)} = \frac{\sin\theta}{1 - \cos\theta} = \frac{\frac{1}{2} * \sin\theta}{\frac{1}{2} * (1 - \cos\frac{2\theta}{2})} = \frac{\frac{1}{2} * \sin\theta}{\sin^2\frac{\theta}{2}}$$

Then note from the identity $\sin 2x = 2\sin x \cos x \iff \frac{1}{2}\sin 2x = \sin x \cos x$:

$$\frac{1/2 * \sin\theta}{\sin^2\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2} * \cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

Then I will use matlab to graph this parametric equation:


2.5
2
1.5
1
0.5
0
-0.5
0
1
2
3
4
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6
7