Joseph C. McGuire
Dr. Shi

Homework #1

PHYS 460
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Problem 1.3

Consider the Gaussian distribution

$$\rho(x) = Ae^{-\lambda(x-a)^2}$$

where A, a, and λ are positive real constants. (Look up any integrals you need.)

a.)

Use Equation 1.16 to determine A.

Solution:

$$\int_{-\infty}^{+\infty} \rho(x) = 1$$

$$\int_{-\infty}^{+\infty} Ae^{-\lambda(x-a)^2} = 1$$

$$\int_{-\infty}^{+\infty} e^{-\lambda(x-a)^2} = \frac{1}{A}$$

Using a Integration Table:

$$\sqrt{\frac{\pi}{\lambda}} = \frac{1}{A}$$

$$A = \sqrt{\frac{\lambda}{\pi}}$$

b.)

Find $< x >$, $< x^2 >$, and σ.

Solution:

$$< x > = \int_{-\infty}^{+\infty} \sqrt{\frac{\lambda}{\pi}} e^{-\lambda(x-a)^2} x \, dx$$

Using an integration table:

$$= \sqrt{\frac{\pi}{\lambda}} a$$

$$< x^2 > = \int_{-\infty}^{+\infty} \sqrt{\frac{\lambda}{\pi}} e^{-\lambda(x-a)^2} x^2 \, dx$$

Using an integration table:

$$= \sqrt{\frac{\pi}{\lambda}} \frac{2a^2\lambda + 1}{\lambda}$$

$$= \frac{\sqrt{\pi}(2a^2\lambda + 1)}{\lambda^{3/2}}$$

$$\sigma_x = \sqrt{< x^2 > -(< x >)^2} = \sqrt{\frac{\sqrt{\pi}(2a^2\lambda + 1)}{\lambda^{3/2}} - \frac{a^2\pi}{\lambda}}$$

c.)

Sketch the graph of $\rho(x)$.

Solution:

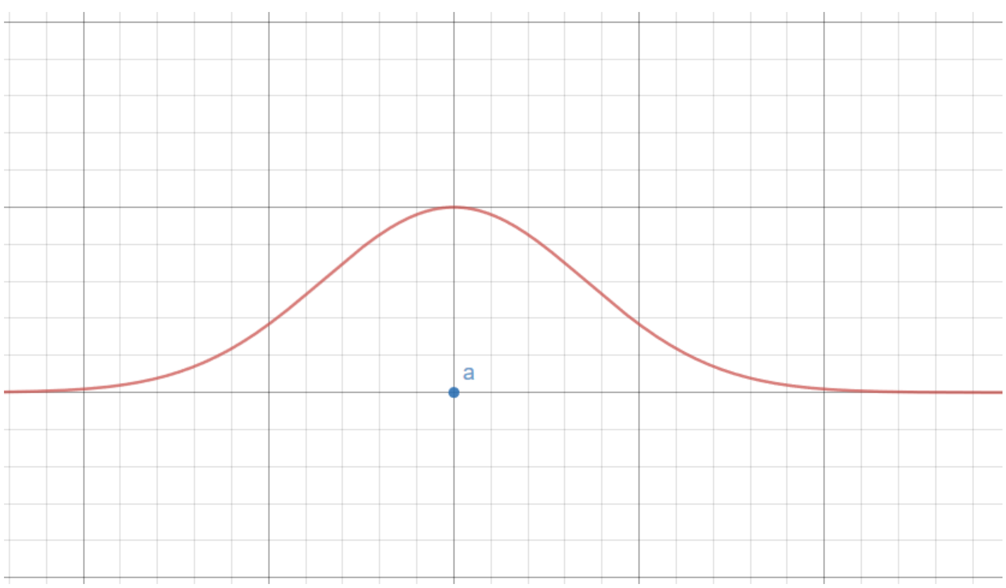

Figure 1: Gaussian Distribution centered at a

Problem 1.4

At a time $t = 0$ a particle is represented by the wave function

$$\Psi(x,0) = \begin{cases} A\frac{x}{a}, & \text{if } 0 \leq x \leq a \\ A\frac{(b-x)}{(b-a)}, & \text{if } a \leq x \leq b \\ 0, & \text{otherwise,} \end{cases}$$

where A, a, and b are constants.

a.)

Normalize Ψ (that is, find A, in terms of a and b)

Solution:

$$\int_{-\infty}^{+\infty} |\Psi(x,0)|^2 \; dx = 1$$

$$\int_{-\infty}^{0} 0 \; dx + \int_{0}^{a} A^2 \frac{x^2}{a^2} \; dx + \int_{a}^{b} A^2 \frac{(b-x)^2}{(b-a)^2} \; dx + \int_{b}^{+\infty} 0 \; dx = 1$$

$$\int_{0}^{a} A^2 \frac{x^2}{a^2} \; dx + \int_{a}^{b} A^2 \frac{(b-x)^2}{(b-a)^2} \; dx = 1$$

$$\frac{A^2 x^3}{3a^2}\bigg|_{x=0}^{a} + \frac{-A^2}{3(b-a)^2}(b-x)^3\bigg|_{x=a}^{b} = 1$$

$$\frac{A^2 a^3}{3a^2} + \frac{A^2}{3(b-a)^2}\left(0 - -(b-a)^3\right) = 1$$

$$\frac{A^2 a}{3} + A^2 \frac{(b-a)}{3} = 1$$

$$A^2 \left(\frac{a}{3} + \frac{(b-a)}{3}\right) = 1$$

$$A^2 = \frac{3}{b}$$

$$A = \sqrt{\frac{3}{b}}$$

b.)

Sketch $\Psi(x,0)$, as a function of x.

Solution:

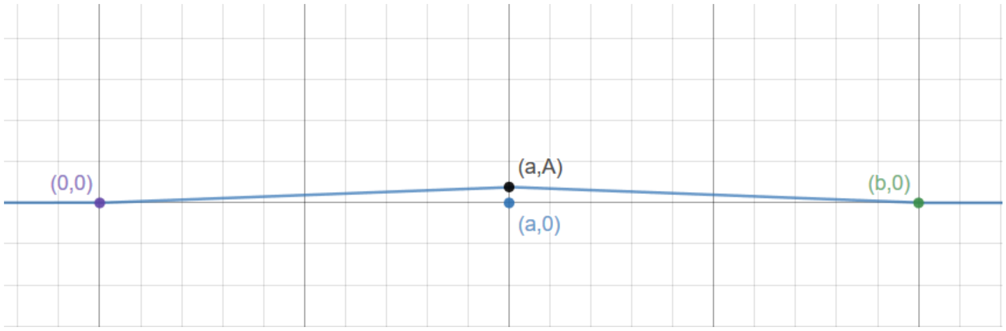

Figure 2: The distribution of $\Psi(x, 0)$

c.)

Where is the particle most likely to be found, at $t = 0$?

Solution:

Just from the sketch of the distribution we can see that the most likely position for the particle is $x = a$.

d.)

What is the probability of find the particle to the left of a? Check you result in the limiting cases $b = a$ and $b = 2a$.

Solution:

The probability is given by:

$$\int_{-\infty}^{a} |\Psi(x,0)|^2 \, dx = \int_{0}^{a} \frac{3}{b}\frac{x^2}{a^2} \, dx = \left.\frac{x^3}{a^2 b}\right|_{x=0}^{a} = \frac{a}{b}$$

So the probability is $\frac{a}{b}$. When $a = b$, then the probability is 1 and particle will certainly be found at a. This checks with our understanding, since Ψ would only be non-zero on $[0, a]$. When $b = 2a$ the probability will be $1/2$, which makes sense because that will be exactly half of our non-zero distribution.

e.)

What is the expectation value of x?

Solution:

This is:

$$\begin{aligned}
\int_{-\infty}^{\infty} x\ \Psi(x,0)\ dx &= \int_0^a \frac{3x^3}{a^2b}\ dx + \int_a^b \frac{3x}{b}\frac{(x-b)^2}{(b-a)^2}\ dx \\
&= \frac{3x^4}{4a^2b}\bigg|_{x=0}^{a} + \frac{3}{b(b-a)^2}\int_a^b x(x^2-2bx+b^2)\ dx \\
&= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2}\int_a^b x^3-2bx^2+b^2x\ dx \\
&= \frac{3a^2}{4b} + \frac{3a}{b(b-a)^2}\left(\frac{x^4}{4}-\frac{2bx^3}{3}+\frac{b^2x^2}{2}\right)\bigg|_{x=a}^{b} \\
&= \frac{3a^2}{4b} + \frac{3a}{b(b-a)^2}\left(\frac{b^4}{4}-\frac{2b^4}{3}+\frac{b^4}{2}-\left(\frac{a^4}{4}-\frac{2ba^3}{3}+\frac{a^2b^2}{2}\right)\right)
\end{aligned}$$

Problem 1.5

Consider the wave function

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t}$$

where A, λ, and ω are positive real constants. (We'll see in Chapter 2 what potential (V) actually produces such a wave function.)

a.)

Normalize Ψ.

Solution:

$$\begin{aligned}
\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 \, dx &= 1 \\
\int_{-\infty}^{+\infty} A^2 e^{-2\lambda|x|} \, dx &= 1 \\
\int_{-\infty}^{0} A^2 e^{-2(-x)} \, dx + \int_{0}^{+\infty} A^2 e^{-2x} \, dx &= 1 \\
\left.\frac{e^{2x}}{2}\right|_{-\infty}^{0} + \left.\frac{-e^{-2x}}{2}\right|_{0}^{+\infty} &= \frac{1}{A^2} \\
\frac{1}{2} + \frac{1}{2} &= \frac{1}{A^2} \\
A^2 &= 1 \\
A &= 1
\end{aligned}$$

Thus we have $\Psi(x,t) = e^{-\lambda|x|}e^{-i\omega t}$.

b.)

Determine the expectation values of x and x^2.

Solution:

$$
\begin{aligned}
< x > &= \int_{-\infty}^{+\infty} x|\Psi(x,t)|^2 \; dx \\
&= \int_{-\infty}^{+\infty} xe^{-2\lambda|x|} \; dx \\
&= \int_{-\infty}^{+\infty} xe^{-2\lambda(-x)} \; dx + \int_{0}^{+\infty} xe^{-2\lambda x} \; dx \\
&= \frac{xe^{2\lambda x}}{2\lambda}\bigg|_{x=-\infty}^{0} - \frac{1}{2\lambda}\int_{-\infty}^{0} e^{2\lambda x} \; dx + \frac{-xe^{-2\lambda x}}{2\lambda}\bigg|_{x=0}^{\infty} + \frac{1}{2\lambda}\int_{0}^{\infty} e^{-2\lambda x} \; dx \\
&= \frac{-1}{4\lambda^2}e^{2\lambda x}\bigg|_{x=-\infty}^{0} + \frac{1}{4\lambda^2}e^{-2\lambda x}\bigg|_{x=0}^{+\infty} = \frac{-1}{4\lambda^2} + \frac{1}{4\lambda^2} = 0
\end{aligned}
$$

$$
\begin{aligned}
< x^2 > &= \int_{-\infty}^{+\infty} x^2|\Psi(x,t)|^2 \; dx \\
&= \int_{-\infty}^{0} x^2e^{2\lambda x} \; dx + \int_{0}^{+\infty} x^2e^{-2\lambda x} \; dx
\end{aligned}
$$

Evaluating this using Mathematica:

$$
= \frac{1}{4\lambda^3} + \frac{1}{4\lambda^3} = \frac{1}{2\lambda^2}
$$

Problem 1.11

The needle on a broken car speedometer is free to swing, and bounces perfectly off the pins at either end, so that if you give it a flick it is equally likely to come to a rest at any angle between 0 and π.

a.)

What is the probability density $\rho(\theta)$? Hint: $\rho(\theta)\ d\theta$ is the probability that the needle will come to rest between θ and $\theta + d\theta$. Graph $\rho(\theta)$ as a function of θ, from $-\pi/2$ to $3\pi/2$. (Of course, part of this interval is excluded, so ρ is zero there.) Make sure that the total probability is 1.

Solution:

Since it is assumed that this is a constant distribution over $[0, \pi]$, we know $\rho(\theta) = \text{Constant}$. Additionally, we know that $\int_0^\pi \rho(\theta)\ d\theta$. So that $\int_0^\pi \rho(\theta) d\theta = \int_0^\pi (\text{Constant})\ d\theta = (\text{Constant})\pi = 1$. So then $\rho(\theta) = 1/\pi$.

b.)

Compute $<\theta>$, $<\theta^2>$, and σ, for this distribution.

Solution:

$$
\begin{aligned}
<\theta> &= \int_0^\pi \theta \frac{1}{\pi}\ d\theta = \frac{\theta^2}{2\pi}\bigg|_0^\pi = \frac{\pi}{2} \\
<\theta^2> &= \int_0^\pi \theta^2 \frac{1}{\pi}\ d\theta = \frac{1}{\pi}\frac{\theta^3}{3}\bigg|_{\theta=0}^\pi = \frac{\pi^2}{3} \\
\sigma &= \sqrt{\frac{\pi^2}{3} - \frac{\pi^2}{4}} = \sqrt{\frac{\pi^2}{12}} = \frac{\pi}{\sqrt{12}}
\end{aligned}
$$

c.)

Compute $<\sin\theta>$, $\cos\theta$, and $<\cos^2\theta>$.

Solution:

$$< \sin\theta > = \int_0^\pi \sin\theta \frac{1}{\pi}\ d\theta = \frac{-\cos\theta}{\pi}\Bigg|_{\theta=0}^{\pi} = \frac{1}{\pi} + \frac{1}{\pi} = \frac{2}{\pi}$$

$$< \cos\theta > = \int_0^\pi \cos\theta \frac{1}{\pi}\ d\theta = \frac{\sin\theta}{\pi}\Bigg|_{\theta=0}^{\pi} = 0$$

$$< \cos^2\theta > = \int_0^\pi \frac{\cos^2\theta}{\pi}\ d\theta = \frac{1}{2}\int_0^\pi \frac{1}{\pi}(1+\cos 2\theta)\ d\theta = \frac{1}{2\pi}(\theta + \frac{1}{2}\sin 2\theta)\Bigg|_{\theta=0}^{\pi} = \frac{1}{2}$$