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Class: Math 531

Homework 1

1. What if the relax the definition of a matrix representation X to not require $X(g)$ to be invertible? This exercise gives the answer.

   (a) Let E be an $n \times n$ matrix over $\mathbb{C}$ such that $E \neq I_n, E \neq 0$, and $E^2 = E$. Show that there's an $n \times n$ matrix such that
   $$E = T^{-1} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} T$$
   for some k.

   Proof. Let E be an $n \times n$ matrix over $\mathbb{C}$ such that $E \neq I_n, E \neq 0$, and $E^2 = E$. First, note that since $E^2 = E$ and $E \neq I$, then suppose that E is invertible. Then $E^2 = E \implies E = I_n$, a contradiction. So that E isn't invertible, and hence must have an eigenvalue of 0. Setting up the eigenvalue problem with $v \neq 0$ and $\lambda \neq 0$:
   $$Ev = \lambda v \iff E^2 v = E(\lambda v) \iff Ev = \lambda Ev$$
   $$\iff Ev - \lambda Ev = 0v \iff E(1-\lambda)v = 0v \iff (1-\lambda)Ev = 0v$$
   $$\iff (1-\lambda)E = 0$$
   Where the last step is stating the two operators $(1-\lambda)E$ and the zero operator are equal. Since $E \neq 0$, this implies that $1 - \lambda = 0 \iff \lambda = 1$. Furthermore, $\lambda = 0, 1$ are the only eigenvalues of E.
   Now to complete the proof, we'll show that the associated eigenvectors of $\lambda = 0, 1$ form a basis for $\mathbb{C}^n$. To show this we can actually just use the Spectral Value Theorem states that the condition that E is normal (i.e. $E^*E = EE^*$) is equivalent to E having a basis consisting of eigenvectors. Consider the following:
   $$\begin{aligned} E^2 &= E \\ E^2E^* &= EE^* \\ EEE^* - EE^* &= 0 \\ EE^*(E - I) &= 0 \end{aligned}$$
   Since $E \neq I$ this implies $EE^* = 0$. Additionally:
   $$\begin{aligned} E^2 &= E \\ E^*E^2 &= E^*E \\ E^*EE - E^*E &= 0 \\ E^*E(E - I) &= 0 \end{aligned}$$
   Again $E^*E = 0$. Hence $E^*E = EE^*$ and so by the spectral value theorem E has a basis of eigenvectors and hence is diagonializable with a diagonal of eigenvalues. But it's eigenvalues are $1, 0$, hence E is diagonalizable and this is the result! $\square$

(b) Let G be a group and X be a function mapping G into the set of (possibly singular) $n \times n$ matrices over $\mathbb{C}$ such that $X(\epsilon) \neq 0$ and $X(gh) = X(g)X(h)$ for all $g, h \in G$. Show that there's an $n \times n$ matrix T and a matrix representation Y of G such that

$$X(g) = T^{-1} \begin{bmatrix} Y(g) & 0 \\ 0 & 0 \end{bmatrix} T$$

for all $g \in G$.

Proof. Let G be a group and X be a function mapping G into the set of (possibly singular) $n \times n$ matrices over $\mathbb{C}$ such that $X(\epsilon) \neq 0$ and $X(gh) = X(g)X(h)$ for all $g, h \in G$. This rules out that $X(g) = 0$ for all $g \in G$.

First, suppose that $X(g)$ isn't invertible. So clearly $X \neq I_n$. This implies that X has an eigenvalue of 0, and then the result follows. That is $X(g)$ has some Jordan-normal form with 0 on the bottom diagonal. That is the result for $X(g)$ being singular.

Alternatively, suppose that $X(g)$ is invertible. Then $X(g)$ doesn't have an eigenvalue of 0. Hence the result follows with the 0 blocks having size 0×0. □

2. For $\sigma = \sigma_1 \dots \sigma_n \in S_n$ written in one line notation, $\text{inv}(\sigma)$ is the number of pairs $i < j$ such that $\sigma_j < \sigma_i$. Define $\text{sign}(\sigma) = (-1)^{\text{inv}(\sigma)}$. Show that switching the position of any two integers in σ by an odd number. This fact implies

$$\text{sign}(\sigma\tau) = \text{sign}(\sigma)\text{sign}(\tau)$$

where τ is any transposition. Since any permutation can be written as a product of transpositions, this means that the sign representation $X(\sigma) = \text{sign}(\sigma)$ is indeed a matrix representation of S_n.

Proof. First, note that any transposition can be written as a product of adjacent transpositions. So that to prove our proposition, we'll show the following: Let $\sigma \in S_n$ and $i \in \mathbb{N}$, then $(-1)^{\text{inv}(\sigma(i\ i+1))} = (-1)^{\text{inv}(\sigma)+1}$. That is, the product of any permutation and an adjacent transposition changes the number of inversions by an odd number.

Let $\sigma \in S_n$ and $i \in \mathbb{N}$. Then let σ have the following two-line representation:

$$\begin{pmatrix} 1 & \dots & n \\ \sigma_1 & \dots & \sigma_n \end{pmatrix}$$

where $\sigma_j = \sigma(j)$. Then the product $\sigma(i\ i{+}1)$ will be given by the function composition:

$$\begin{pmatrix} 1 & \dots & i & i+1 & \dots & n \\ \sigma_1 & \dots & \sigma_i & \sigma_{i+1} & \dots & \sigma_n \end{pmatrix} \circ \begin{pmatrix} 1 & \dots & i & i+1 & \dots & n \\ 1 & \dots & i+1 & i & \dots & n \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \dots & i & i+1 & \dots & n \\ \sigma_1 & \dots & \sigma_{i+1} & \sigma_i & \dots & \sigma_n \end{pmatrix}.$$

The only change to the permutation was that σ_i swapped with σ_{i+1}. If $\sigma_i < \sigma_{i+1}$, then this will cause one inversion with indices i and $i+1$. If $\sigma_i > \sigma_{i+1}$, then this will take away one inversion with indices i and $i+1$. So then, the change of $\mathrm{inv}(\sigma)$ to $\mathrm{inv}(\sigma(i\ i+1))$ causes a change in the parity of $(-1)^{\mathrm{inv}(\sigma)}$, that is our result:

$$(-1)^{\mathrm{inv}(\sigma(i\ \ i+1))} = (-1)^{\mathrm{inv}(\sigma)+1}.$$

$\square$

3. Let $G = \{g_1, \dots, g_n\}$ be a finite group. The right regular representation R is the representation given by the action of G on itself by right multiplication. That is,

$$R(g)_{i,j} = \begin{cases} 1 & \text{if } g_i g = g_j, \\ 0 & \text{otherwise.} \end{cases}$$

for all $g \in G$. Let L be the left regular representation. Show that $L(g)R(h) = R(h)L(g)$ for all $g, h \in G$.

Proof. To show this we'll show that for all $i = 1, \dots, n$, $j = 1, \dots, n$, $g, h \in G$ that $L(g)R(h)_{(i,j)} = R(h)L(g)_{(i,j)}$.

$$\begin{aligned} L(g)R(h)_{i,j} &= \sum_{k=1}^{n} L(g)_{i,k} R(h)_{j,k} \\ &= \sum_{\{k : k=1,2,\dots,n \text{ and } gg_i = g_k \text{ and } g_k h = g_j\}} 1 \\ &= \sum_{\{k : k=1,2,\dots,n \text{ and } gg_i h = g_j\}} 1 \\ R(h)L(g)_{i,j} &= \sum_{k=1}^{n} R(h)_{i,k} L(g)_{k,j} \\ &= \sum_{\{k : k=1,2,\dots,n \text{ and } g_i h = g_k \text{ and } gg_k = g_j\}} \\ &= \sum_{\{k : k=1,2,\dots,n \text{ and } gg_i h = g_j\}} 1 \end{aligned}$$

Notice the sums are over the same index set! Hence the sums are equivalent, and thus the matrices are the same! $\square$

4. Let X be a representation of H and $H \leq G$. Show that the induced representation $X \uparrow_H^G$ is indeed a representation.

Proof. Let X be a representation of H and $H \leq G$. To show this, we'll show the homomorphism property of the mapping $X \uparrow_H^G$. First note that $X \uparrow_H^G (\varepsilon)_{i,j} = \begin{cases} X(g_i^{-1}\varepsilon g_j) & \text{if } \varepsilon g_j \in g_i H \\ 0 & \text{otherwise} \end{cases}$ Since $\varepsilon g_j = g_j \in g_i H$ if and only if $i = j$, this tells us that $X \uparrow_H^G (\varepsilon)$ is 0 off-diagonal with block form. Additionally if $i = j$, the $X(g_i^{-1}\varepsilon g_j) = X(g_i^{-1}g_i) = X(\varepsilon) = I_n$, so that the diagonals of $X \uparrow_H^G$ will be all I_n. Thus this satisfies the first criterion of a Matrix representation.

To show the second criteria, we'll show that $X \uparrow_H^G (g) X \uparrow_H^G (h) = X \uparrow_H^G (gh)$ for all $g, h \in G$. Let $i = 1, \ldots, n$, $j = 1, \ldots, n$, $g, h \in G$, then consider the following:

$$\begin{aligned}
(X \uparrow_H^G (g) X \uparrow_H^G (h))_{i,j} &= \sum_{k=1}^{n} X \uparrow_H^G (g)_{i,k} X \uparrow_H^G (h)_{k,j} \\
&= \sum_{\{k : k=1,2,\ldots,n \text{ and } gg_k \in g_i H \text{ and } hg_j \in g_k H\}} X(g_i^{-1} g g_k) X(g_k^{-1} h g_j) \\
&= \sum_{\{k : k=1,2,\ldots,n \text{ and } g_i^{-1} g g_k \in H \text{ and } g_k^{-1} h g_j \in H\}} X(g_i^{-1}(gh)g_j) \\
&= \sum_{\{k : k=1,2,\ldots,n \text{ and } g_i^{-1} g g_k g_k^{-1} h g_j \in H\}} X(g_i^{-1} g h g_j)
\end{aligned}$$

Finally, note that the sum is independent of k! So that since $g_i^{-1} g h g_j \in H \iff (gh) \in g_i H g_j^{-1}$ will only occurs once for i, j, because of the fact cosets partition a group, we'll have the sum collapses to $X(g_i^{-1}(gh)g_j)$ if $(gh)g_j \in g_i H$ and 0 (the sum has an index set of the null set) otherwise. This shows both criteria for $X \uparrow_H^G$ to be a matrix representation! $\square$

5. Show that if X is a representation of K and $K \leq H \leq G$, then $X \uparrow_K^G$ and $(X \uparrow_K^H) \uparrow_H^G$ are similar.

Proof. Suppose X is a representation of K and $K \leq H \leq G$. We'll show a lemma. Let $\{g_1, \ldots, g_m\}$ be a transversal of K as a subgroup of H and let $\{h_1, \ldots, h_n\}$ be a transversal for H as a subgroup of G. Then $\{g_i h_k : i = 1, \ldots, m, k = 1, \ldots, n\}$ is a transveral for K as a subgroup of G with $K \leq H \leq G$.

Assuming this hypothesis, we'll have that $g_i K$ is a coset within H and $h_k H$ a coset of G. Then $g_i K \subseteq h_k H$ for some k, for all $i = 1, \ldots, m$ and $k = 1, \ldots, n$. Since each $h_k H$ is pairwise disjoint and partitions G, this implies our result.

To finish the proof, note then that for any $g \in G$ that $X \uparrow_K^G (g)_{i,j} = X(g_i^{-1} g g_j)$ if $g g_j \in g_i K$ and 0 otherwise. While $X \uparrow_K^H \uparrow_H^G (g)_{i,j} = X \uparrow_K^H (h_i^{-1} g h_j)$ if $g h_j \in h_i H$ a picture illustrates the rest: That is that the matrices will be equivalent after a change of a transversal, which is guarenteed to work because of our lemma. $\square$

Homework 1

$$
\begin{array}{c|c}
 & h_1 \quad \cdots \quad h_k \\
\begin{matrix} h_1 \\ \vdots \\ h_k \end{matrix} & \begin{bmatrix} & & \\ & X\uparrow_K^H (h_i^{-1} g h_j) & \\ & & \end{bmatrix}
\end{array}
$$

$$= (X\uparrow_K^H)\uparrow_H^G (g)$$

$$
X\uparrow_K^H (h_i^{-1} g h_j) = \begin{array}{c|c}
\begin{matrix} g_1 \\ \vdots \\ g_n \end{matrix} & \begin{bmatrix} X(g_i^{-1} h_j^{-1} g g_i h_j) \end{bmatrix} \\
 & g_1 \cdots g_n
\end{array}
$$

$$
X\uparrow_K^G = \begin{array}{c|c}
 & g_i h_j \\
g_i h_j & \begin{bmatrix} & & \\ & X(g_i^{-1} h_j^{-1} g g_i h_j) & \\ & & \end{bmatrix}
\end{array}
$$

Figure 1: Caption