Graph Theory

Joseph McGuire

October 19, 2024

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Notes on Graph Theory

A graph is fundamentally about relations between objects in a finite set.

Notes on Graph Theory

Example

Let $S = \mathbf{Z}$ and define the relation $R \subset \mathbf{Z} \times \mathbf{Z}$ by:

$$u \; R \; v \iff (u, v) \in R = \{(0,1), (1,0), (1,2), (2,1), (0,2), (2,0)\}.$$

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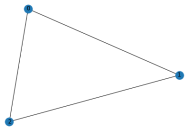

Figure: The graph of the relation

Notes on Graph Theory

Definition (Simple Graph)

Let V be a set and E be a set of ordered pairs, $E \subset V \times V$, then $G = (V, E)$ is a simple graph on V.
Where

$$\text{If } (u, v) \in E, \text{ then } (v, u) \in E.$$

Notes on Graph Theory

Example

In Figure 2 is a graph with node set $V = \{0, 1, 2\}$ and edge set

$$E = \{(0,1), (1,0), (1,2), (2,1), (0,2), (2,0)\}$$

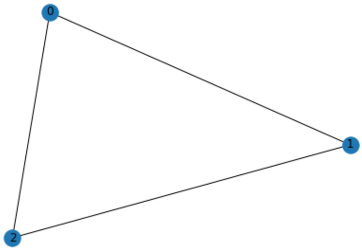

Figure: An undirected simple graph

Notes on Graph Theory

Example

A PhD student, Wayne Zachary, followed the interactions of the members of a Karate club over the course of 3 years (1970 - 1972). Over this span, there was a conflict in the club, resulting in two administrators 'Mr. Hi' and 'John A' (pseudonyms) split the group into two rival karate clubs (Think 'Karate Kid').
The graph is created by treating each individual as a node, and then every edge is a social connection between two members of the club.

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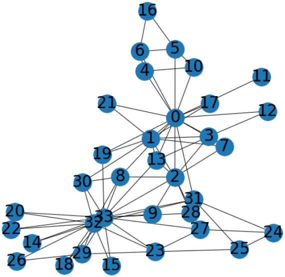

Figure: The Famous Zachary Karate Club Network

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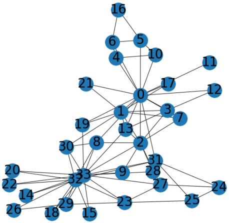

Figure: The Famous Zachary Karate Club Network

In the above, the number of connections that a node has tells us how connected that the node is, and thus how connected the person that node is representing.

Notes on Graph Theory

Definition (Degree)

Let $G = (V, E)$ be a graph. If $v \in V$, then:

$$\deg(v) = \#\text{ of edges connected to v}$$

is the **degree** of node v.

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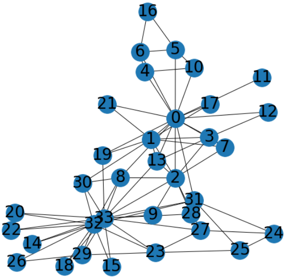
Definition (Degree Sequence)

Let $G = (V, E)$ be a graph with $|V| = n$. Then define the **degree sequence** to be the decreasing sequence:

$$(\deg(v_1), \deg(v_2), \dots, \deg(v_n)) \qquad \text{with } \deg(v_i) \leq \deg(v_{i+1}) \ \forall i.$$

Notes on Graph Theory



Example

In example 4: $\deg(0) = \deg(1) = \deg(2) = 2$. In the Zachary Karate Club example (5), $\deg(16) = \deg(15) = \deg(20) = \deg(22) = 2, \deg(11) = 1$.

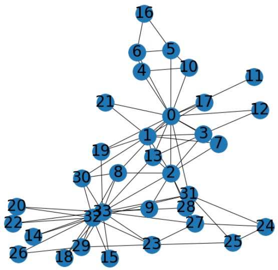
Notes on Graph Theory

Definition (**Adjacency Matrix**)

Let $G = (V, E)$ be a graph with $|V| = n \in \mathbf{Z}^+$. Then we define the $n \times n$ matrix A by:

$$A_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ share an edge} \\ 0 & \text{otherwise} \end{cases}$$

Notes on Graph Theory



Example

The adjacency matrix for Example 4 will be: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Notes on Graph Theory

Definition (Directed Graph)

Let V be a set and $E \subset V \times V$. Then $G = (V, E)$ is a **directed graph** or **di-graph**.
Usually to distinguish between simple and directed graphs, arrows are used for directed graphs. So the edge $(0, 1)$ would represented by the arrow starting at 0 and landing at 1.

Notes on Graph Theory

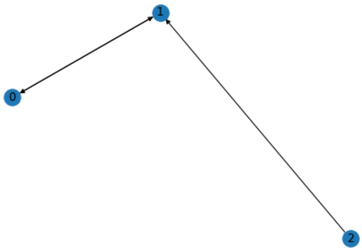


Figure: A randomly generated directed graph

Notes on Graph Theory

In many real world applications edges are not created equally, for example, the connections in a friend group might be weaker or stronger based on the quality of that friendship.

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Definition (Weighted Graph)

Let V be a set and $E \subset \{(u, v, w) : u, v \in V, w \in \mathbf{R}\}$. Then

$$G = (V, E) \text{ is a weighted graph.}$$

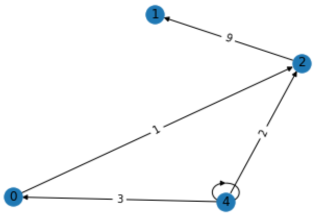
The value w for an edge $u \to v$ is called the **weight** of that edge pair.

Notes on Graph Theory

Example

Let $G = (V, E)$ be a weighted di-graph with self-loops, with adjacency matrix:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 3 & 0 & 2 & 3 \end{bmatrix}$$



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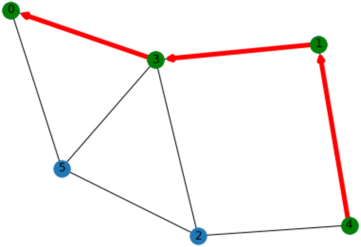
Definition (**Walk**)

On a graph $G = (V, E)$, a walk of length m on the vertices of G is a sequence:

$$(v_1, \ldots, v_m)$$

such that



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Definition (Connected)

A graph G is **connected** if

$$\text{for all } u, v \in V \text{ there exists a path between } u \text{ and } v.$$

A graph that isn't connected is **disconnected**; that is, there are two nodes $u, v \in V$ such that there is not path between them.

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Figure: A disconnected graph

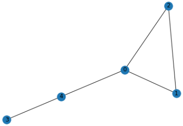

Figure: A connected graph

Notes on Graph Theory

Definition (Multi-Graph)

Let V be a set, then G is a **multi-graph** if E is a multi-set (e.g. a multi-set: $\{(1,2),(1,2),(1,3),(3,1),(3,1)\}$).

Notes on Graph Theory

Definition (**Bipartite Graph**)

Let U, V be distinct sets such that $U \cap V = \emptyset$, then G is a **bipartite graph** if $E \subset U \times V$.
That is, any edge in E must go between a member of U and V, it can't go to two members of U or two members of V.