1 Precalculus

1.1 Functions

Definition 1.1.1 (Function). *A function is a rule that assigns to each element of one set (the* ***domain****) exactly one element of another set (the* ***range****). A function f with domain A and range B is denoted by*

$$f : A \to B$$

If f assigns to the element $x \in A$ the element $y \in B$, we say that f ***maps*** *x to y, and write either*

$$y = f(x) \text{ or } x \overset{f}{\mapsto} y$$

and call y the ***image*** *of x.*

Example 1.1. *Consider the real-valued function f, defined by the equation*

$$f(x) = -\sqrt{\frac{x+1}{|2x-1|}}.$$

1. *Determine the largest subset of $\mathbb{R}$ that can serve as a domain of f.*
2. *A point x_0 in the domain of f is called a fixed point of f if $f(x_0) = x_0$. This function f has exactly one rational fixed point. What is it?*

Solution.

1. Note that we need $x + 1 \geq 0$ and $2x - 1 \neq 0$. So that the largest domain for f in $\mathbb{R}$ is $\left[-1, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$.
2. Setting up the equation we get:

$$x_0 = -\sqrt{\frac{x_0+1}{|2x_0-1|}}$$
$$x_0^2|2x_0 - 1| = x_0 + 1.$$

We have to then seperate this into the two equations, one when $2x_0 + 1 \geq 0$ and the other when $2x_0 - 1 < 0$:

$$x_0^2(2x_0 - 1) = x_0 + 1 \qquad x_0^2(1 - 2x_0) = x_0 + 1.$$

Giving us the two cubic polynomial equations:

$$2x_0^3 - x_0^2 - x_0 - 1 = 0 \qquad -2x_0^2 + x_0^2 - x_0 - 1 = 0.$$

We can use the rational roots theorem to solve this, giving us possible roots in the first and second equations of $\pm 1, \pm\frac{1}{2}$. Rule out $x_0 = \frac{1}{2}$ as this isn't in our domain. We'll find the first equation has no rational solutions, while the second equation has a solution of $x_0 = -\frac{1}{2}$. This is our rational fixed point.

$\square$

1.1.1 Composition of Functions

Definition 1.1.2 (Composition of Functions). *Let $f : A \to B$ and $g : C \to D$ be functions, such that $B \subseteq C$ and $D \subseteq A$. Then $(g \circ f)(x) = g(f(x))$ and $(f \circ g)(x) = f(g(x))$.*

Note that in general $(g \circ f) \neq (f \circ g)$.

Example 1.2. *Consider the functions f and g such that $(f \circ g)(x) = \sqrt{x^2 + 1} - 1$. If $g(x) = x^2 + 1$, then what's the value of $f(4)$?*

Solution. Since $(f \circ g)(x) = \sqrt{x^2+1} - 1$ and $g(x) = x^2 + 1$, we can conclude $f(g(x)) = f(x^2+1) = \sqrt{x^2+1} - 1$, implying $f(x) = \sqrt{x} - 1$. So that $f(4) = \sqrt{4} - 1 = 1$. $\square$

Example 1.3. *Let f and g be real-valued functions defined on the entire real line such that $f(x) = g(x^2 - 1)$ and $g(x) = x - 1$. Find all values of x such that:*

$$(f \circ g)(x) = (g \circ f)(x).$$

Solution. $f(g(x)) = f(x-1) = g((x-1)^2 - 1) = g(x^2 - 2x + 1 - 1) = x^2 - 2x - 1$ and $g(f(x)) = g(g(x^2-1)) = g(x^2 - 1) - 1 = x^2 - 1 - 1 - 1 = x^2 - 3$. $g(f(x)) = x^2 - 3 = f(g(x)) = x^2 - 2x - 1 \iff x = 1$. $\square$

1.1.2 Inverse Functions

Definition 1.1.3 (One-to-One/Injective Function). *A function $f : A \to B$ is said to be **one-to-one** (or **injective**) if no elements in A are mapped by f to the same element in B. That is, f is one-to-one if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.*

Definition 1.1.4 (Onto/Surjective Function). *A function $f : A \to B$ is said to be **onto** (or **surjective**) if every element in B is the image of some element in A. That is if $y \in B$, then there exists a $x \in A$ such that $f(x) = y$.*

Definition 1.1.5 (Bijective Function/Inverse). *If f is both one-to-one and onto, it's said to be **bijective**, and it's guaranteed that for every element in B there is one, and only one, element in A such that $f(x) = y$. So that we may define another function $f^{-1} : B \to A$ defined by $f^{-1}(y) = x \iff f(x) = y$. The function f^{-1} is said to be the **inverse** of f. We'll have $f^{-1}(f(x)) = x$ for every $x \in A$ and $f(f^{-1}(y)) = y$ for every $y \in B$.*

Example 1.4. *Define f by the equation $f(x) = x^3 + x - 4$. Given that this function has well-defined inverse, determine the value of $f^{-1}(2)$?*

Proof. Setup the equation: $-2 = x^3 + x - 4$. Solving for x:

$$x^3 + x - 2 = 0$$

by the rational roots theorem we can conclude the possible roots will be $\pm 1, \pm 2$. $x = 1$ is the solution, since the function is bijective, this value must be unique. Hence $f^{-1}(-2) = 1$. $\square$

Example 1.5. *Let $A = (-1, 1)$ and define the bijective function f on A by the equation:*

$$f(x) = \frac{x}{1 - x^2}.$$

Find a formula for $f^{-1}(x)$ that works for every real x.

Proof.

1. Setup the equation:
$$y = \frac{x}{1 - x^2}$$
2. Solve for x:
$$(1 - x^2)y = x \iff -yx^2 - x + y = 0 \iff x = \frac{+1 \pm \sqrt{1 - 4(-y)(y)}}{-2y} = \frac{1 \pm \sqrt{1 + 4y^2}}{-2y}$$
3. The function should agree everywhere on $(-1, +1)$, note the limit as $y \to 0$, the function is undefined for the negative solution. So we can rule out the negative solution.
4. We need to rationalize the denominator so that it's defined at $y = 0$:
$$\begin{aligned}\frac{-1 + \sqrt{1 + 4y^2}}{2y} &= \frac{-1 + \sqrt{1 + 4y^2}}{2y} \frac{-1 - \sqrt{1 + 4y^2}}{-1 - \sqrt{1 + 4y^2}} \\ &= \frac{-1 - (1 + 4y^2)}{2y(-1 - \sqrt{1 + 4y^2})} \\ &= \frac{4y^2}{2y(1 + \sqrt{1 + 4y^2})} \\ &= \frac{2y}{1 + \sqrt{1 + 4y^2}}.\end{aligned}$$
This is defined at $y = 0$, hence we're happy with this.

$\square$

1.1.3 Graphs in the xy-Plane

Definition 1.1.6 (Symmetry of Graphs)**.** *The **graph** of f consists of all points (x, y) in the plane such that $y = f(x)$. A graph is said to be **symmetric with respect to the y-axis** if, whenever (x, y) is on the graph, $(-x, y)$ is also. A graph is said to be **symmetric with respect to the origin** if, whenever (x, y) is on the graph, $(-x, -y)$ is also. The graph of $y = f(x)$ and the graph of its inverse function $y = f^{-1}(x)$ are **symmetric with respect to the line** $y = x$, since if (x, y) lies on one of the graphs,* *then (y, x) lines on the other.*

Definition 1.1.7 (Graph Terminology)**.** *The **vertical-line test** says that a given graph isn't the graph of a function, if there are two (or more) points that lie on the same vertical line; this would be a violation of the definition of a function, which says that for every value of the **independent variable** (from the domain), a function assigns exactly one value to the **dependent variable** (from the range.*

*A graph is said to be **symmetric with respect to the x-axis** if, whenever (x, y) is on the graph, $(x, -y)$ is also. The x-coordinate (**abscissa**) of a point at which the graph crosses the x-axis is called an **x-intercept**, and the y-intercept (**ordinate**) of a point at which a graph crosses the y-axis is called a **y-intercept**.*

1.2 Analytical Geometry

Definition 1.2.1 (Conic Sections)**.** *They are **conic sections** and have as their universal equation the general second-degree equation in x and y:*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{1}$$

The identity of the graph depends on the values of the coefficients. If the curve isn't a straight line or circle, then its symmetry axis (or axes) will not be parallel to one of the coordinate axes if $B \neq 0$, but the coordinate axes can be rotated to simplify the equation of the curve and place the axis of the conic parallel to one of the coordinate axes.

1.2.1 Lines

Definition 1.2.2 (Point-Slope Formula)**.** *If the line isn't vertical, then the equation of the line with slope m passing through the point (x_1, y_1) can be written as:*

$$y - y_1 = m(x - x_1),$$

*this is called the **point-slope formula**. Recall that if (x_1, y_1) and (x_2, y_2) are any two points on a nonvertical line, then the **slope** is defined as:*

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

We don't define the slope of a vertical line. If two nonvertical lines have the same slope, then they're parallel (or overlapping), and if the product of the slopes of two nonvertical lines is equal to -1, then they're perpendicular.

1.2.2 Parabolas

Definition 1.2.3 (Parabolas)**.** *Let F be a given fixed point and D a given fixed line that doesn't contain F. By definition, a **parabola** is the set of points in the plane containing F and D that are equidistant from the point F (the **focus**) and the line D (the **directix**). The **axis** of a parabola is the line through* *the focus and perpendicular to the directrix. The* ***vertex*** *of a parabola is the turning point, the point on the parabola's axis that's midway between the focus and the directrix. The standard equations of the parabolas with vertex at the origin and axis either the $x-$ or $y-$axis are:*

$$y = \pm\frac{1}{4p}x^2 \text{ or } x = \pm\frac{1}{4p}y^2.$$

There are 4 catagories of parabolas:

1. *Directrix: $y = -p$, Focal point: $F = (0, p)$*

   This will be $y = \dfrac{x^2}{4p}$.

2. *Directrix: $y = p$, Focal Point: $F = (0, -p)$*

   This will be $y = -\dfrac{x^2}{4p}$.

3. *Directrix: $x = -p$, Focal Point: $F = (p, 0)$*

   This will be $x = \dfrac{y^2}{4p}$

4. *Directrix: $x = p$, Focal Point: $F = (-p, 0)$*

   This will be $x = -\dfrac{y^2}{4p}$

Example 1.8. *Find the focus of the parabola $y = x^2$ and determine the length of the latus rectum, which is the line segment with endpoints on the parabola that passes through the focus, perpendicular to the axes (or, equivalently, parallel to the directix).*

Solution. This will correspond the parabola with $p = \frac{1}{4}$, since $y = \frac{1}{4p}x^2$. Hence the Focal Point will be at $(p, 0) = \left(0, \frac{1}{4}\right)$ and the directrix is the line $y = -\frac{1}{4}$. You can then just find when $y = \frac{1}{4}$, that will be at the points $x = \pm\frac{1}{2}$. The *latus rectum* will be the distance between these two points, giving us it has length 1. □

1.2.3 Circles

Definition 1.2.4 (Circle Geometry)**.** *A* ***circle*** *is the set of points in the plane that are all at a constant, positive distance from a given fixed point. The constant distance is called the* ***radius****, and the given fixed point is the* ***center****. The standard equation of the circle with radius a, centered at the origin is*

$$x^2 + y^2 = a^2,$$

or, equivalently,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1.$$

If the center is at the point (h, k), then we can replace x by $x - h$ and y by $y - k$ and write:

$$(x-h)^2 + (y-k)^2 = a^2 \text{ or } \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} = 1.$$

Example 1.9. *A circle has $A : (-2, -3)$ and $B : (6, 1)$ as the endpoints of a diameter.*

(a) Where does this circle cross the $y-$axis?

(b) Where does the line tangent to the circle at the point B cross the $y-$axis?

Solution. (a) We'll find the defining formula of a circle first. Note that the center will be the midpoint of the line $\overline{AB}$, that is given by the midpoint formula:

$$C = \left(\frac{6 + (-2)}{2}, \frac{1 + (-3)}{2}\right) = (2, -1).$$

The distance between the center and A is then: $\sqrt{(6-2)^2 + (1 - (-1))^2} = \sqrt{20}$ The formula for the circle is then the solutions to:

$$(x-2)^2 + (y - (-1))^2 = 20.$$

That is $(x-a)^2 + (y-b)^2 = R^2$. Then just plug in $x = 0$ into this formula, giving us:

$$4 + (y+1)^2 = 20 \iff (y+1)^2 = 16 \iff y + 1 = \pm 4,$$

this gives us the two solutions of $y = 3, -5$. So the circle intersects with the $y-$axis at the points $\boxed{(0, -5), (0, 3)}$.

(b) Recall that two lines are perpendicular if and only if the product of their slopes is -1. So take the line $\overline{CB}$, this will have a slope of $\frac{1-(-1)}{6-2} = \frac{2}{4} = \frac{1}{2}$. Hence the tangent line at B must be perpendicular to $\overline{CB}$, hence the slope of the tangent line at B is exactly -2. Finally use the point-slope form of a line and plug in the point $B : (6, 1)$.

$$y - y_1 = m(x - x_1) \iff y = 1 - 2(x - 6).$$

It'll intersect the $y-$axis at $x = 0$, so that this is at $y = 1 - 2(-6) = 13$, the point $\boxed{(0, 13)}$.

□

1.2.4 Ellipses

Definition 1.2.5 (Ellipse)**.** *An **ellipse** is the set of all points in the plane such that the sum of the distances from every point on the ellipse to given two fixed points (the **foci**) is a constant. (It's a generalization of a circle). For an ellipse centered at the origin - with axes parallel to the coordinate* *axes - is:*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The long symmetry axis of the ellipse is called the ***major axis*** *(on which the foci are located), and the shorter one (on which the center lies) is called the* ***minor axis****. The end points of the major axis are called the* ***vertices****. The eccentricity of an ellipse, e, is a number between* 0 *and* 1 *that measures its "flatness". The closer e is to* 0*, the more it's closer to a perfect circle, the closer to* 1*, the ellipse flattens out.*

Definition 1.2.6 (Ellipse Formuli)**.** *For an ellipse with major axis parallel to the $x-$axis:*

1. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, *with* $a > b$
2. *foci* $= (\pm\sqrt{a^2 - b^2}, 0)$
3. *vertices* $= (\pm a, 0)$
4. *major axis length* $= 2a$
5. *minor axis length* $= 2b$.
6. *eccentricity* $= e = \dfrac{\sqrt{a^2 - b^2}}{a}$.

An ellipse with major axis parallel to the $y-$axis:

1. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, a < b$
2. *foci* $= (0, \pm\sqrt{b^2 - a^2}$
3. *vertices* $= (0, \pm b)$
4. *major axis length* $= 2b$
5. *minor axis length* $= 2a$
6. *eccentricity* $= e = \dfrac{\sqrt{b^2 - a^2}}{b}$

Example 1.10. *The sum of the distances from every point on an ellipse to the points* $(\pm 2, 0)$ *is equal to* 8*. Find the positive value of x such that the point* $(x, 3)$ *is on the ellipse.*

Solution. Start by drawing a picture. We'll have that the foci lie on the $x-$axis, implying that $a > b$. First, recall that since the foci are at ± 2, this immediately gives us that $c^2 = (\pm 2)^2 \iff c^2 = a^2 - b^2 \iff c^2 + b^2 = a^2$. That is, the triangle $\Delta(abc)$ is both a right triangle and will have hypotenuse a. So that is the magnitude of the line $b(-c)$ and bc are the same and are both just a. Hence $d(b, -c) + d(b, c) = 8 \iff$

$2a = 8 \iff a = 4$. Thus we can now solve for b:

$$\begin{aligned} b^2 + c^2 &= a^2 \\ b^2 &= a^2 - c^2 \\ b^2 &= 16 - 4 = 12. \end{aligned}$$

So $b^2 = 12$, and this gives us the equation of the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x^2}{16} + \frac{y^2}{12} = 1.$$

So then solve for the point x where $x > 0$ and $(x, 3)$ is on the ellipse is as simple as plugging in $y = 3$ and solving for x:

$$\frac{x^2}{16} + \frac{9}{12} = 1 \iff x^2 = 16\frac{1}{4} = 4 \iff x = \pm 2.$$

Thus the x we're looking for is: $\boxed{x = 2}$ $\square$

1.2.5 Hyperbolas

Definition 1.2.7 (Hyberpola)**.** *A **hyperbola** is the set of all points such that the difference between the distances from every point on the hyperbola to two fixed points (the **foci**) is a constant. The hyperbola is always two separate curves, called **branches**. A hyperbola is also different from other curves in that it has **asymptotes**, which are lines that its branches approach but never touch, as the magnitude of x and y increase.*

*The line that contains the foci is called the **focal axis**; the midpoint of the segment joining the foci is the **center** of the hyperbola; and the points at which the branches of the hyperbola intersect the focal axis are called the **vertices**. The equation for a hyperbolas centered at the origin:*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ or } \frac{x^2}{b^2} - \frac{y^2}{a^2} = 1.$$

Definition 1.2.8 (Formuli for Hyperbola)**.** *For* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$*:*

1. *Foci* $= (\pm c, 0)$ *where* $c^2 = a^2 + b^2$
2. *Vertices* $= (\pm a, 0)$
3. *Asymptotes* $= y = \frac{\pm b}{a} x$

For $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$*:*

1. *Foci* $= (0, \pm c)$ *where* $c^2 = a^2 + b^2$
2. *Vertices* $= (0, \pm b)$

3. Asymptotes $y = \pm\frac{b}{a}$

Example 1.11. *The foci of a hyperbola are at* $(\pm 3, 0)$*, and the difference between the distances from every point on the hyperbola to the foci is* 2*. What's the equation of this hyperbola?*

Solution. First, note that we can immediately identify the vertices must lie on the x-axis, meaning the equation will be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Remember that $c^2 = a^2 + b^2$, so that $9 = a^2 + b^2$ since $c = \pm 3$. Finally note that by the definition of a hyperbola the distance between a and c, $d(a, c)$, and the distance between a and $-c$, $d(a, -c)$, are respectively $c - a$ and $a - (-c) = a + c$. So that $d(a, -c) - d(a, c) = 2a$ and by the definition of this hyperbola, $2a = 2 \iff a = 1$. Hence $c^2 = 9$ and $a^2 = 1$ giving us $b^2 = 8$, hence we have the equation $\boxed{\frac{x^2}{1} - \frac{y^2}{8} = 1}$ □

1.3 Polynomial Equations

Example 1.12. *Find the monic cubic polynomial* $p(x)$ *that has real coefficients such that the equation* $p(x) = 0$ *has* 1 *and* $2 - i$ *as roots.*

Solution. So this is as simple as applying the factor theorem and the complex conjugate theorem. $(x-1)(x-(2-i))(x-(2+i)) = (x-1)(x^2 - 4x + 5) = x^3 - 5x^2 + 9x - 5$. □

Example 1.13. *For the equation* $x^2 + bx + c = 0$*, the sum of the roots is* 3 *and the sum of the squares of the roots is* 1*. Find the numerical value of* c*.*

Solution. Let x_1, x_2 be the roots of this polynomial, so that $x_1 + x_2 = 3$ and $x_1^2 + x_2^2. = 1$. The quadratic equation gives us that:

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

So that $x_1 + x_2 = -b = 3 \iff b = -3$. Finally, note that $x_1^2 + bx_1 + c = 0$ and $x_2^2 + bx_2 + c = 0$ as x_1, x_2 are both roots. Then $(x_1^2 + x_2^2) + b(x_1 + x_3) + 2c = 0 \iff 1 + 3(-3) + 2c = 0 \iff 2c = 8 \iff c = 4$. □

Example 1.14. *The equation* $3x^3 - 7x^3 + 5x^2 - 7x + 2 = 0$ *has exactly two rational roots, both of which are positive. Find the larger of these two roots.*

Solution. An application of the rational roots theorem gives us that the possible rational roots will be $\pm 1, \pm 2, \pm\frac{1}{3}, \pm\frac{2}{3}$. Notice that any negative number will be a strictly non-negative output, meaning we can eliminate all of the possible negative roots. Start off testing the with the largest and work your way down, you'll find that 2 works, so stop there. $x = 2$ is the largest rational root of the polynomial. □

1.4 Logarithms

Note: $\log(x) = \log_e(x)$ in the Math GRE

Example 1.15. *Solve for* $x : 4^x = 2^x + 3$.

Proof. Notice that $4^x = 2^{2x}$, so we can write this as a quadratic form in 2^x, $2^{2x} - 2^x - 3 = 0$. This has solutions of:

$$2^x = \frac{+1 \pm \sqrt{1 + 4(3)}}{2}.$$

Clearly $2^x \geq 0$, so reject the negative solution. Giving us:

$$2^x = \frac{1 + \sqrt{13}}{2} \iff x = \log_2\left(\frac{1 + \sqrt{13}}{2}\right).$$

□

Example 1.16. *If* $x^2 + y^2 = 14xy$, *then* $\log(k(x+y)) = \frac{1}{2}(\log(x) + \log(y))$ *for some constant* k. *Find the value of* k.

Solution.

$$\begin{aligned}
\log(k(x+y)) &= \frac{1}{2}(\log(x) + \log(y)) \\
\log(k(x+y)) &= \log((xy)^{1/2}) \\
k(x+y) &= (xy)^{1/2} \\
k^2(x+y)^2 &= xy \\
k^2(x^2 + 2xy + y^2) &= xy \\
k^2(16xy) &= xy \\
k^2 &= \frac{1}{16}.
\end{aligned}$$

Since $k(x+y) > 0$ for the logarithm to be defined, we can reject the negative solution again leaving us with $\boxed{k = \frac{1}{4}}$. □

Example 1.17. *Simplify* $(\log_{xy}(x^y))(1 + \log_x(y))$.

Proof. Let's remember our friend the change of base formula to get:

$$\log_{xy}(x^y) = \frac{\log_x(x^y)}{\log_x(xy)} = \frac{y}{1 + \log_x(y)}.$$

Hence this reduces to:

$$(\log_{xy}(x^y))(1 + \log_x(y)) = y.$$

□

1.5 Trigonometry

Definition 1.5.1 (Exact Values of Trig Functions)**.** *Remember values of* $\sin(\theta)$ *and the rest will follow from later trig identities.*

1. $\sin(0) = 0 = \cos(90^o)$
2. $\sin(30^o) = \dfrac{1}{2} = \cos(60^o)$
3. $\sin(45^o) = \dfrac{1}{\sqrt{2}} = \cos(45^o)$
4. $\sin(60^o) = \dfrac{3}{\sqrt{2}} = \cos(30^o)$
5. $\sin(90^o) = 1 = \cos(0^o)$

Definition 1.5.2 (Trig Identities)**.** 1.

$$\sin(-x) = -\sin(x) \text{ and } \cos(-x) = \cos(x)$$

2.

$$\sin(x + y) = \sin(x)\cos(y) + \sin(y)\cos(x)$$

3.

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

4. $\cos(\theta) = \sin(90^o - \theta)$ *and* $\sin(\theta) = \cos(90^o - \theta)$.

Example 1.18. *What is the exact value of* $\tan\left(\dfrac{\pi}{12}\right)$*?*

Solution. Build the table using the fact that $\sin^2(x) + \cos^2(x) = 1$ to find the values of $\cos(x)$.

Degrees	0^o	30^o	45^o	60^o	90^o
Radians	0	$2\pi/12$	$3\pi/12$	$4\pi/12$	$6\pi/12$
$\sin(x)$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos(x)$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0

Notice the symmetry, that will save you a lot of time. Finally, we can use the addition of angle identities to get:

$$
\begin{aligned}
\tan(\pi/12) &= \tan(15^o) \\
&= \tan(45^o - 30^o) \\
&= \frac{\sin(45^o - 30^o}{\cos(45^o - 30^o)} \\
&= \frac{\sin(45^o)\cos(30^o) - \sin(30^o)\cos(45^o)}{\cos(45^o)\cos(30^o) + \sin(45^o)\sin(30^o)} \\
&= \frac{\frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{2}\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2}} \\
&= \frac{\sqrt{3}-1}{\sqrt{3}+1} \\
&= 2 - \sqrt{3}
\end{aligned}
$$

□

Example 1.19. *If* $\sin(\theta) = \frac{1}{3}$, *what's the value of* $\sec(2\theta)$?

Solution. So note first that: $\cos^2(\theta) = 1 - \sin^2(\theta) = \frac{8}{9}$ and the angle addition identity $\cos(\theta + \theta) = \cos^2(\theta) - \sin^2(\theta) = \frac{8}{9} - \frac{1}{9} = \frac{7}{9}$ so that $\sec(2\theta) = \frac{1}{\cos(2\theta)} = \frac{9}{7}$. □

Example 1.20. *Express the following in simplest form, in terms of* $x : \sin(2\arctan(x))$.

Solution. First, note that $\arctan(x)$ is an angle in a right triangle with $\theta = \arctan(x) \iff \tan(\theta) = \frac{x}{1}$ so that the hypotenuse of the right triangle is $\sqrt{x^2+1}$. So that $\sin(2\theta) = \sin(\theta+\theta) = \sin(\theta)\cos(\theta) + \sin(\theta)\cos(\theta) = 2\cos(\theta)\sin(\theta)$ by the sine angle addition identity. Finally, since we know the opposite side to θ is x, adjacent is 1 and hypotenuse is $\sqrt{x^2+1}$ gives us that $\sin(2\arctan(x)) = \sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\frac{x}{\sqrt{x^2+1}}\frac{1}{\sqrt{x^2+1}} = \frac{2x}{x^2+1}$. □

Example 1.21. *Evaluate* $\arccos\left(\frac{2}{\sqrt{5}}\right) + \arccos\left(\frac{3}{\sqrt{10}}\right)$.

Solution. So again interpret the two as angles in a right triangle:

$$\theta_1 = \arccos\left(\frac{2}{\sqrt{5}}\right), \theta_2 = \arccos\left(\frac{3}{\sqrt{10}}\right) \iff \cos(\theta_1) = \frac{2}{\sqrt{5}}, \cos(\theta_2) = \frac{3}{\sqrt{10}}.$$

Notice then that for triangle 1, we'll have adjacent to angle θ_1 is 2 and the hypotenuse is $\sqrt{5}$, so that the opposite side is 1. Similarly for triangle 2, we'll have adjacent to angle θ_2 is 3 and hypotenuse $\sqrt{10}$ and opposite 1. So $\sin(\theta_1) = \dfrac{1}{\sqrt{5}}$ and $\sin(\theta_2) = \dfrac{1}{\sqrt{10}}$. So by angle-addition identity:

$$\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) = \frac{2}{\sqrt{5}}\frac{3}{\sqrt{10}} - \frac{1}{\sqrt{5}}\frac{1}{\sqrt{10}} = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}}.$$

Using our new found knowledge of exact values of trig functions, this implies $\theta_1 + \theta_2 = 45^o = \dfrac{\pi}{4}$. This is our answer! $\square$

Review Questions 1.1. *Find the sum of the roots of the equation $\sqrt{x-1} + \sqrt{2x-1} = x$.*

Solution. By inspection, note that $x \geq 1$ for this function. Then consider the following:

$$\begin{aligned}
\sqrt{x-1} + \sqrt{2x-1} &= x \\
\sqrt{2x-1} &= x - \sqrt{x-1} \\
2x - 1 &= x^2 + (x-1) - 2x\sqrt{x-1} \\
2x &= x^2 + x - 2x\sqrt{x-1} \\
2 &= x + 1 - 2\sqrt{x-1}(x \neq 0) \\
1 - x &= -2\sqrt{x-1} \\
x^2 - 2x + 1 &= 4x - 4 \\
x^2 - 6x + 5 &= 0 \\
x &= \frac{6 \pm \sqrt{36-20}}{2} \\
x &= 5, 1.
\end{aligned}$$

So that $5 + 1 = 6$. $\square$

Review Questions 1.2. *Determine the set of positive values of x that satisfy the following inequality:*

$$\frac{1}{x} - \frac{1}{x-1} > \frac{1}{x-2}.$$

Solution. Note that this is a rational inequality. Hence we want to get it into the form $\dfrac{P(x)}{D(x)} > 0$ where

$P(x)$ and $D(x)$ are both polynomials in x. So:

$$\begin{aligned}
\frac{1}{x} - \frac{1}{x-1} &> \frac{1}{x-2} \\
\frac{x-1-x}{x(x-1)} &> \frac{1}{x-2} \\
\frac{1}{x(1-x)} - \frac{1}{x-2} &> 0 \\
\frac{x-2-x(1-x)}{x(1-x)(x-2)} &> 0 \\
\frac{x-2-x+x^2}{x(1-x)(x-2)} &> 0 \\
\frac{2-x^2}{x(x-1)(x-2)} &> 0.
\end{aligned}$$

So now we need find the positive values of x such that $2 - x^2 = 0$ and $x(x-1)(x-2) = 0$. We see that these are $x = 0, 1, \sqrt{2}, 2$. So test values in between these intervals. For $x = 0.1$ we'll end up with a positive, meaning that this inequality holds for $x \in (0, 1)$. For $x = 1.1$, we'll end up with $2 - (1.1)^2 > 0$ and $x(x-1)(x-2) < 0$ so that this can't hold on $(1, \sqrt{2})$. For $x = 1.8$, we'll end up with $2 - (1.8)^2 < 0$ and $1.8(1.8-1)(1.8-2) < 0$, hence the fraction of these will be positive so this holds for $x \in (\sqrt{2}, 2)$. Finally we see $x = 3$ will give us $2 - 3^2 < 0$ and $3(3-1)(3-2) > 0$ so that this doesn't holds for $x > 2$.

Thus this inequality holds on this set: $\boxed{(0,1) \cup (\sqrt{2}, 2)}$ □

Review Questions 1.3. *Solve for* $x : |x+1| - |x| + 2|x-1| = 2x - 1$.

Proof. So notice that the absolute values of the equation require we break this into cases: $x \geq 0$ or $x < 0$, $x - 1 \geq 0$ or $x - 1 < 0$, $x + 1 \geq 0$ or $x + 1 < 0$. So these are actually intervals that x will fall into, specifically: $(-\infty, -1], (-1, 0), 0, (0, 1), [1, \infty)$.

1. $x \in (-\infty, -1]$

$$\begin{aligned}
|x+1| - |x| + 2|x-1| &= 2x - 1 \\
-x - 1 + x - 2x + 2 &= 2x - 1 \\
-2x + 1 &= 2x - 1 \\
2 &= 4x \\
x &= \frac{1}{2}
\end{aligned}$$

   Clearly $\frac{1}{2} \notin (-\infty, -1]$ so that we can rule out this solution.

2. $x \in (-1, 0)$

$-1 < x < 0$, $0 < x+1 < 1$, $-2 < x-1 < -1$

$$\begin{aligned} |x+1| - |x| + 2|x-1| &= 2x - 1 \\ x + 1 + x - 2x + 2 &= 2x - 1 \\ 4 &= 2x \\ x &= 2 \end{aligned}$$

Again not in the interval, so rule that out.

3. $x \in [0, 1)$

   $0 \leq x < 1$, $1 \leq x+1 \leq 2$, $-1 \leq x-1 < 0$

$$\begin{aligned} |x+1| - |x| + 2|x-1| &= 2x - 1 \\ x + 1 - x - 2x + 2 &= 2x - 1 \\ 4 &= 2x \\ x &= 2 \end{aligned}$$

   Same as before, no solution.

4. $x \geq 1$

   $1 \leq x$, $2 \leq x+1$, $0 \leq x-1$

$$\begin{aligned} |x+1| - |x| + 2|x-1| &= 2x - 1 \\ x + 1 - x + 2x - 2 &= 2x - 1 \\ 0 &= 0 \end{aligned}$$

   Hence any $x \geq 1$ satisfies this equation, and we're done.

□

Review Questions 1.4. *Let f be a function such that $f(n+1) = 1 - [f(n)]^2$.*

Solution.

$$\begin{aligned} f(n+2) &= 1 - (f(n+1))^2 \\ &= 1 - (1 - (f(n))^2)^2 \\ &= 1 - (1 + (f(n))^4 - 2(f(n))^2) \\ &= 2(f(n))^2 - (f(n))^4. \end{aligned}$$

□

Review Questions 1.5. *Let f be a real-valued function whose inverse is given by the equation:*

$$f^{-1}(x) = x(1 + x^2) + (1 - x^2)$$

What's the value of $f(f^{-1}(f(2)))$?

Solution. So we can immediately recognize that $f(f^{-1}(x)) = x = f^{-1}(f(x))$, so that we just need to calculate $f(2)$. To find this solve for x: $f^{-1}(x) = 2$.

$$\begin{aligned} 2 &= x(1+x^2) + (1-x^2) \\ 2 &= x + x^3 + 1 - x^2. \end{aligned}$$

By inspection, or the rational roots theorem, we see that $x = 1$ is a solution to the above. Since the inverse is well-defined, we can conclude this is the unique solution to this equation. Hence $x = 1$. So that $f(f^{-1}(f(2))) = f(2) = 1$. □

Review Questions 1.6. *Let f, g, h be real-valued functions defined for all positive x such that:*

$$(f \circ g)(x) = (g \circ h)(x)$$

If $f(x) = x + 1$ and $g(x) = \sqrt{x}$, what's $h(x)$?

Solution.

$$\begin{aligned} f(g(x)) &= g(h(x)) \\ \sqrt{x} + 1 &= \sqrt{h(x)} \\ h(x) &= \sqrt{x}^2 + 1 + 2\sqrt{x}.h(x) & &= \sqrt{x}(\sqrt{x}+2) + 1. \end{aligned}$$

□

Review Questions 1.7. *What's the equation of all points in the xy-points that are equidistant from points $(-1, 4)$ and $(5, -2)$?*

Proof. This is a trick question in some ways, don't over complicate this. The formula for these 2 points to be equidistant:

$$\sqrt{(x-(-1))^2 + (y-4)^2} = \sqrt{(x-5)^2 + (y-(-2))^2} \iff (x+1)^2 + (y-4)^2 = (x-5)^2 + (y+2)^2.$$

Solving for x or y :

$$\begin{aligned} (x+1)^2 + (y-4)^2 &= (x-5)^2 + (y+2)^2 \\ x^2 + 2x + 1 + y^2 - 8y + 16 &= x^2 - 10x + 25 + y^2 + 4y + 4 \\ 2x - 8y + 17 &= 29 + 4y - 10x \\ 12x &= 12y + 12 \\ x - y &= 1. \end{aligned}$$

This is the defining equation. □

Review Questions 1.8. *Which of the following best describes the graph of the equation $x^2+y^2-2x+4y+5=0$ in the $xy-$plane?*

Solution. You can complete the square here to get the following:

$$x^2-2x+1^2+y^2+2(2y)+2^2=(x-1)^2+(y-(-2))^2=0.$$

This is a circle centered at $(1,-2)$ with radius 0, this is just a point. □

Review Questions 1.9. *Let C be the curve in the $xy-$plane described by the equation $x^2+4y^2=16$. If every point (x,y) on C is replaced by the point $\left(\frac{1}{2}x,y\right)$, what's the area enclosed by the resulting curve?*

Solution. First, setup the change of variables $x\mapsto \frac{x}{2}=x'$ and $y\mapsto y'$. So that $x=2x'$ and $y=y'$. Hence we'll end up with the equation:

$$x^2+4y^2=16 \iff (2x')^2+4(y')^2=16 \iff (x')^2+(y')^2=4.$$

This is just a circle with radius 2, hence we'll have an area of $\pi 2^2=4\pi$. □

Review Questions 1.10. *Every point on the parabola $y=\sqrt{2x-1}$ is equidistant from the $y-$axis and which of the following points?*

Solution. First, remember that a parabola is the set of all point in $xy-$plane that are equidistant from a point, the foci, and a line, the directix. So we just need to find the foci here. So we need to put this into the standard form of a parabola:

$$\pm\frac{y^2}{4\rho}=x,$$

the foci will be $(\rho,0)$. Doing this

$$\begin{aligned} y&=\sqrt{2x-1}\\ y^2&=2x-1\\ y^2+1&=2x\\ \frac{y^2+1}{2}&=x\\ \frac{y^2}{2}&=x-\frac{1}{2}. \end{aligned}$$

We're almost done, use a transformation of variables to make this easier $x\mapsto x'=x-\frac{1}{2}$ and $y\mapsto y'$. This gives us the standard parabola:

$$\frac{(y')^2}{2}=x'.$$

This is a parabola that spans the first and fourth quadrants, now solve for the foci with:

$$\frac{(y')^2}{4\rho} = \frac{(y')^2}{2} \iff \rho = \frac{1}{2}.$$

Hence the foci in will be in $x' = \frac{1}{2} \iff x - \frac{1}{2} = \frac{1}{2}$. That is the point:

$$(1, 0)$$

□

Review Questions 1.11. *One of the foci of the hyperbola* $y^2 = \left(\frac{x}{a}\right)^2 + 1$ *is the point* $(0, \sqrt{2})$. *Find* a.

Solution. Putting this in standard form of a parabola:

$$y^2 - \frac{x^2}{a^2} = 1.$$

The key here is to remember that the foci of the hyperbolas will be $\pm c = \pm\sqrt{a^2 + b^2}$. In this case $b = 1$ and so $c^2 = a^2 + 1$ will determine the foci. We're given the foci is $\pm\sqrt{2}$, hence $2 = 1 + a^2 \iff a^2 = 1 \iff \boxed{a = 1}$. □

Review Questions 1.12. *Which of the following polynomials* $p(x)$ *has the property that* $\sqrt{3} - \sqrt{2}$ *is a root of the equation* $p(x) = 0$*?*

Proof. The idea here is that if $\sqrt{3} - \sqrt{2}$ is a root, then we must have:

$$p(x) = (x - (\sqrt{3} - \sqrt{2}))(x - (\sqrt{3} + \sqrt{2}))(x - (-\sqrt{3} - \sqrt{2}))(x - (-\sqrt{3} + \sqrt{2})),$$

this is just the radical conjugate theorem. After a lot of nasty algebra we get that this polynomial is

$$x^4 - 10x^2 + 1.$$

□

Review Questions 1.13. *When the polynomial* $p(x)$ *is divided by* $x - 1$, *it leaves a remainder of* 1, *and when* $p(x)$ *is divided by* $x + 1$, *it leaves a remainder of* -1. *Find the remainder when* $p(x)$ *is divided by* $x^2 - 1$.

Solution. The key fact here is to use a version of remainder theorem for polynomials:

$$\begin{aligned} p(x) &= (x-1)q_1(x) + r_1(x) \\ p(x) &= (x+1)q_2(x) + r_2(x) \\ p(x) &= (x^2-1)q_3(x) + r_3(x) \end{aligned}$$

So then we'll have $p(1) = r_1(1) = 1$ and $p(-1) = r_2(-1) = -1$. So since $p(1) = 1$ and $x^2 - 1 = (1)^2 - 1 = 0$, this implies $r_3(1) = 1$ and similarly $r_3(-1) = -1$. So that since the remainder has to have degree less than the dividend we'll have $r_3(x)$ has degree 1 or 0. Clearly $r_3(x)$ isn't constant and is thus determined by the points $(1, 1)$ and $(-1, -1)$ this is just the line x:

$$r_3(x) = x.$$

□

Review Questions 1.14. *Given that $p(x)$ is a real polynomial of degree ≤ 4 such that one can find five distinct solutions to the equation $p(x) = 5$, what's the value of $p(5)$?*

Solution. Note that by the fundamental theorem of algebra, if $\deg(p(x)) \leq 4$, then if $p(x) \neq 0$, then $p(x) = 0$ has at most 4 unique solutions. Because we have 5 distinct solutions implies $p(x) - 5 = 0$ for all $x \in \mathbb{R}$, hence $p(x) = 5$ for all $x \in \mathbb{R}$. So that $p(5) = 5$. □

Review Questions 1.15. *If the roots of the equation $x^2 + Bx + 1 = 0$ are the squares of the roots of the equation $x^2 + bx + 1 = 0$, which of the following expresses B in terms of b?*

Solution. This is a little trickier, notice that for this to make sense we need to talk about the sums of these roots. Note that $x^2 + Bx + 1 = 0$ has solutions:

$$x = \frac{-B \pm \sqrt{B^2 - 1}}{2},$$

so that $x_1 + x_2 = -B$ with x_1 being the $+$ root and x_2 being the $-$ root. Next, since we have $x^2 + bx + 1 = 0$ has roots: $x_3, x_4 = \dfrac{-b \pm \sqrt{b^2 - 1}}{2}$. So that: $x_1 + x_2 = x_3^2 + x_4^2$ gives us the equation:

$$-B = \left(\frac{-b - \sqrt{b^2 - 1}}{2}\right)^2 + \left(\frac{b - \sqrt{b^2 - 1}}{2}\right)^2 \iff -4B = b^2 - 2b\sqrt{b^2 - 4} + b^2 - 4 + b^2 + b^2 - 4 - 2b\sqrt{b^2 - 4} = 4b^2 - 8.$$

This gives us the equation:

$$B = 2 - b^2.$$

□

Review Questions 1.16. *Find the largest value of b such that $1 + bi$ satisfies the equation*

$$x^3 - 3x^2 + 6x - 4 = 0$$

given that every root of this equation has the form $1 + bi$ (where b is real).

Solution. So by the conjugate roots theorem:

$$(x - (1 + bi))(x - (1 - bi)) = x^2 - 2x + (1 + b^2),$$

and by inspection $x = 1$ is also a root. So that $x^3 - 3x^2 + 6x - 4 = (x-1)(x^2 - 2x + (1+b^2)) = x^3 - 3x^2 + (3+b^2)x - (1+b^2)$. Finally this gives us:

$$1 + b^2 = 4 \iff b^2 = 3 \iff b = \pm\sqrt{3}.$$

So that since $-\sqrt{3}$ would give us only a negative number on the left-hand side, we can conclude this doesn't work. So that $b = \sqrt{3}$. □

Review Questions 1.17. *If a and x are positive numbers and $A = a^2$, express the following in its simplest form in terms of x.*

Solution.

$$\begin{aligned} a^{\log_a(x) + \log_A(x)} &= a^{\log_a(x)} a^{\log_a(x)/\log_a(A)} \\ &= x\left(a^{\log_a(x)}\right)^{1/\log_a(a^2)} \\ &= x x^{1/2} = x\sqrt{x}. \end{aligned}$$

The change of base formula and associativity of exponents is employed above. □

Review Questions 1.18. *What are the roots of the following equation?*

$$(\log(x))^2 = 2\log(x)$$

Solution. First, note that $x = 1$ is a solution to the above, just using the fact that $\log_a(1) = 0$ for any base a. So that we just need to find 1 more root:

$$\begin{aligned} (\log(x))^2 - 2\log(x) &= 0 \\ \log(x) &= 2 \\ \log_e(x) &= 2 \\ e^{\log_e(x)} &= e^2 \\ x &= e^2. \end{aligned}$$

□

Review Questions 1.19. *The hyperbolic sine function, denoted* sinh*, is defined by the equation:*

$$\sinh(x) = \frac{e^x - e^{-x}}{2}.$$

Find the formula for $\sinh^{-1}(x)$.

Solution. There's a typo in the book: it should read $(D)\log(x+\sqrt{x^2+1})$.

$$\begin{aligned}
y &= \frac{e^x - e^{-x}}{2}\\
2e^x y &= e^{2x} - 1\\
0 &= e^{2x} - 2ye^x - 1\\
e^x &= \frac{+2y \pm \sqrt{4y^2+4}}{2}\\
e^x &= y \pm \sqrt{y^2+1}\\
x &= \log(y \pm \sqrt{y^2+1})
\end{aligned}$$

Finally, drop the negative solution as $\log(\cdot)$ is only defined on positive numbers. Hence $\sinh^{-1}(x) = \log(x+\sqrt{x^2+1})$. □

Review Questions 1.20. *The hyperbolic cosine function, denoted* cosh, *is defined by the equation:*

$$\frac{e^x + e^{-x}}{2}$$

*If the hyperbolic tangent function,*tanh, *is defined by*

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

find a formula for $\tanh^{-1}(x)$.

Solution.

$$\begin{aligned}
y &= \frac{e^x - e^{-x}}{e^x + e^{-x}}\\
ye^x + ye^{-x} &= e^x - e^{-x}\\
ye^{2x} + y &= e^{2x} - 1\\
(y-1)e^{2x} + y + 1 &= 0\\
e^{2x} &= \frac{-(1+y)}{(y-1)}\\
e^{2x} &= \frac{1+y}{1-y}\\
2x &= \log\left(\frac{1+y}{1-y}\right)\\
x &= \frac{1}{2}\log\left(\frac{1+y}{1-y}\right)\\
\tanh^{-1}(x) &= \frac{1}{2}\log\left(\frac{1+x}{1-x}\right).
\end{aligned}$$

□

Review Questions 1.21. *Let x be the real number such that $\sin(\sin(x)) = \frac{1}{2}$ and $2 < x < 3$ What's the value of $\cos(-\sin(x))$?*

Solution. So note that $\cos(\sin(x)) = \cos(-\sin(x))$. Then remember back that $\sin(x) = \frac{1}{2} \iff x = 30^o$ or 150^o. So that either $\cos(\sin(x)) = \pm\frac{\sqrt{3}}{2}$. Since $2\frac{\pi}{3} \cong 2 < x < 3 \cong \pi$, we can conclude that $\cos(\sin(x))$ should be positive. So that $\cos(\sin(x)) = \frac{\sqrt{3}}{2}$. □

Review Questions 1.22. *Which one of the following is in the domain of the function $f(x) = \log(\sin(x))$? (You may use the fact that 1111 is just slightly greater than $353.64 \times \pi$.)*

Solution. So note the domain must be intervals of the form:

$$(2\pi n, (2n+1)\pi n)$$

since $\log(\cdot)$ is only defined for positive numbers. So that since $1111 \cong 353.64 \times \pi$, we can conclude that $1111 \in [353 \times \pi, 354 \times \pi]$ that is not in the domain of the function.

Next, note that $\frac{11}{\pi} \cong \frac{11}{3.14} \cong \frac{11}{3}$, and this number is clearly between 3 and 4, so that $\frac{11}{\pi}$ will lie between 3 and 4 and hence 11 is between 3π and 4π so not in our domain. So since $11110 \cong 11111$ within a margin of a fifth decimal place, 11110 is a good enough approximation. And $11110 \cong 3536.4 \times \pi$ and so lies the right of $3536 \times \pi$ and left of $3537 \times \pi$. Boom, 11111 is in the domain of $\log(\sin(x))$. □

Review Questions 1.23. *Simplify* $\tan\left(2\arcsin\left(\frac{1}{3}\right)\right)$.

Solution. So here we want to draw a triangle with an angle: $\sin(\theta) = \frac{1}{3}$. This will give us a triangle with hypotenuse 3 and an opposite side of 1, so that the adjacent side to θ will be $\sqrt{8}$. So now we'll have: $\tan(2\theta)$ with $\theta = \arcsin(1/3)$. So that:

$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{2\sin(\theta)\cos(\theta)}{\cos^2(\theta) - \sin^2(\theta} = \frac{2(\sqrt{8}/3)(1/3)}{(8/9 - 1/9} = \frac{2\sqrt{8}}{7} = \frac{4\sqrt{2}}{7}.$$

□

Review Questions 1.24. *Simplify* $\sqrt{\csc^2\left(\cot^{-1}\frac{\pi}{4}\right) - 1}$.

Solution. Let $\theta = \cot^{-1}\left(\frac{\pi}{4}\right)$, then $\cot(\theta) = \frac{\pi}{4}$. So that is the triangle determined by θ has an adjacent side of π with an opposite side of 4, meaning the hypotenuse is $\sqrt{16+\pi^2}$. So that $\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{\sqrt{\pi^2+16}}{4}$. Plugging this into the expression:

$$\sqrt{\csc^2(\cot^{-1}(\pi/4)) - 1} = \sqrt{\frac{16+\pi^2-16}{16}} = \frac{\pi}{4}.$$

□

Review Questions 1.25. *Determine the exact value of the sum* $\arctan(1)+\arctan(2)+\arctan(3)$.

Proof. First, note that $\arctan(1) = \frac{\pi}{4}$. So we just need to calculate $\arctan(2)+\arctan(3)$. Let $\theta_1 = \arctan(2)$ and $\theta_2 = \arctan(3)$ so that $\tan(\theta_1) = 3$ and $\tan(\theta_2 = 2)$. This will give us the first triangle with opposite side 3 and adjacent side 1 and hypotenuse of $\sqrt{10}$, the second triangle is opposite side 2, adjacent 1, and hypotenuse of $\sqrt{5}$. So this gives us:

$$\sin(\theta_1) = \frac{3}{\sqrt{10}} \qquad \cos(\theta_1) = \frac{1}{\sqrt{10}}$$
$$\sin(\theta_2) = \frac{2}{\sqrt{5}} \qquad \cos(\theta_2) = \frac{1}{\sqrt{5}}.$$

With these we'll calculate $\sin(\theta_1+\theta_2)$ with the angle addition identity:

$$\sin(\theta_1+\theta_2) = \frac{3}{\sqrt{10}}\frac{1}{5} + \frac{1}{\sqrt{10}}\frac{2}{\sqrt{5}} = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}}.$$

Now, note that sine attains this value at two angles:

$$\theta_1+\theta_2 = 45^o, 135^o.$$

We just need to determine the sign of $\cos(\theta_1+\theta_2)$ to determine which of these is correct, if positive it's 45^o, otherwise 135^o. So then

$$\cos(\theta_1+\theta_2) = \frac{1}{\sqrt{10}}\frac{1}{\sqrt{5}} - \frac{3}{\sqrt{10}}\frac{2}{\sqrt{5}} < 0.$$

So that $\arctan(1)+\arctan(2)+\arctan(3) = 135^o = \frac{3\pi}{4}+\frac{\pi}{4} = \boxed{\pi}$. □

2 Calculus I

2.1 Limits of Sequences

Example 2.1. *In each case, show that the sequence* (x_n) *is convergent:*

(a) $x_n = \frac{4n^3 - n^2 + 5n}{2n^3 + 6n^2 - 11}$

(b) $x_n = \sqrt{n+k} - \sqrt{n}$ *(k is a constant)*

(c) $x_n = \dfrac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n)}$

(d) $x_n = (-1)^n \left(\dfrac{1}{n^2}\right)$

(e) $x_n = (\cos(n^n))e^{-n}$

Solution. (a)

$$\begin{aligned}\lim_{n\to\infty} x_n &= \lim_{n\to\infty} \frac{4n^3 - n^2 + 5n}{2n^3 + 6n^2 - 11} \\ &= \lim_{n\to\infty} \frac{4 - \frac{1}{n} + \frac{5}{n^2}}{2 + 6\frac{1}{n} - 11\frac{1}{n^2}} \\ &= \frac{4}{2} = 2\end{aligned}$$

(b) Note we can start the sequence at large enough n so that $n+k$ is positive, denote N as being the first integer where $n + k \geq 0$ and that the end behavior of this subsequence $(x_n)_{n=N}^{\infty}$ will be the same as that of the sequence $(x_n)_{n=1}^{\infty}$.

$$\begin{aligned}\lim(x_n)_{n=1}^{\infty} &= \lim(x_n)_{n=N}^{\infty} \\ &= \lim_{n\to\infty} \sqrt{n+k} - \sqrt{n} \\ &= \lim_{n\to\infty} \sqrt{n+k} - \sqrt{n}\frac{\sqrt{n+k} + \sqrt{n}}{\sqrt{n+k} + \sqrt{n}} \\ &= \lim_{n\to\infty} \frac{n+k-n}{\sqrt{n+k} + \sqrt{n}} \\ &= \lim_{n\to\infty} \frac{k}{\sqrt{n+k} + \sqrt{n}} \\ &= 0.\end{aligned}$$

(c) Note that by the sandwich theorem, we can bound this by:

$$0 \leq x_n \leq \frac{(2n-1)}{2n} = 1 - \frac{1}{2n}$$

Regardless of what the limit is, we know it exists by the sandwich theorem and that the limit is between 0 and 1.

(d) Again by the sandwich theorem:

$$\frac{-1}{n^2} \leq \frac{(-1)^n}{n^2} \leq \frac{1}{n^2}$$

so that $\lim(x_n) = 0$ since both $\lim\limits_{n\to\infty} \dfrac{\pm 1}{n^2} = 0$.

(e) Sandwiches!

$$-1e^{-n} \leq (\cos(n^n))e^{-n} \leq 1e^{-n}$$

so that since $\lim_{n\to\infty} \pm e^{-n} = 0$ we'll have $\lim_{n\to\infty} (\cos(n^n))e^{-n} = 0$.

$\square$

2.2 Limits of Functions

Example 2.2. *Evaluate each of the following limits:*

(a) $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x}+1}$

(b) $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x}-1}$

(c) $\lim_{x\to 1^-} \dfrac{x-1}{|x-1|}$

(d) $\lim_{x\to 1^-} \lfloor x-1 \rfloor$

Solution. (a) Since $\lim_{x\to 1} x-1 = 0$ and $\lim_{x\to 1} \sqrt{x}+1 = 2$, we can use the algebraic limit theorems to state that $\lim(x_n) = 0$.

(b)

$$\begin{aligned}
\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} &= \lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} \frac{-1-\sqrt{x}}{-1-\sqrt{x}} \\
&= \lim_{x\to 1} \frac{-x\sqrt{x}-x+\sqrt{x}+1}{1-x} \\
&= \lim_{x\to 1} \frac{(\sqrt{x}+1)(1-x)}{1-x} \\
&= \lim_{x\to 1} \sqrt{x}+1 \\
&= 2
\end{aligned}$$

(c) Note that for $x \to 1^-$ that we'll have $x < 1$ and $x - 1 < 0$ so that $|x-1| = 1-x$, hence:

$$\lim_{x\to 1^-} \frac{x-1}{|x-1|} = \lim_{x\to 1^-} \frac{x-1}{1-x} = -1.$$

(d) Note again that as $x \to 1^-$, we'll have $x - 1 < 0$, so that for all such x we'll have $\lfloor x-1 \rfloor = 0$. Hence

$$\lim_{x\to 1^-} \lfloor x-1 \rfloor = \lim_{x\to 1^-} -1 = -1.$$

$\square$

2.2.1 Limits of Functions $x \to \pm\infty$

Example 2.3. *Find the value of each of these limits (if they exist):*

(a)
$$\lim_{x\to\infty}\frac{2x^2-x+1}{x^2+4}$$

(b)
$$\lim_{x\to\infty}\frac{2x^2-x+1}{x^3+4}$$

(c)
$$\lim_{x\to-\infty}\arctan(x)$$

(d)
$$\lim_{x\to 0}\frac{1}{x}$$

Solution. 1.
$$\lim_{x\to\infty}\frac{2x^2-x+1}{x^2+4}=\lim_{x\to\infty}\frac{2-\frac{1}{x}+\frac{1}{x^2}}{1+\frac{4}{x^2}}=\frac{2}{1}=2$$

2.
$$\lim_{x\to\infty}\frac{2x^2-x+1}{x^3+4}=\lim_{x\to\infty}\frac{\frac{2}{x}-\frac{1}{x^2}+\frac{1}{x^3}}{1+\frac{4}{x^3}}=\frac{0}{1}=0$$

3. Note that arctangent is an odd function and that $\lim_{x\to\infty}\arctan(x)=\frac{\pi}{2}$
$$\lim_{x\to-\infty}\arctan(x)=-\frac{\pi}{2}.$$

4. Note that $\lim_{x\to 0^+}\frac{1}{x}=+\infty$ and $\lim_{x\to 0^-}\frac{1}{x}=-\infty$ so that the limit cannot exist, since the right and left limits do not match.

□

2.3 Continuous Functions

Example 2.4. *What value must we choose for k so that the function*
$$f(x)=\begin{cases}\dfrac{x^3-8}{x-2} & \text{if } x\neq 2\\ k & \text{if } x=2\end{cases}$$
is continuous everywhere?

Solution. Evaluate the limit as $x \to 2$, first we'll expand out this polynomial fraction. Do this with polynomial division, you should get:

$$\frac{x^3-8}{x-2} = x^2+2x+4$$

so that

$$\lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} x^2+2x+4 = 4+4+4 = 12.$$

This is the value k should be for this to be continuous: $k = 12$. $\square$

Example 2.5. *Let f be the function defined on the interval $I = (0,1)$ as follows:*

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{(in lowest terms, } m \text{ and } n \text{ are integers with } n > 0) \end{cases}$$

Show that f is continuous at every irrational point in I and discontinuous at every rational point in I.

Proof. Let $a \in I \setminus \mathbb{Q}$.

Let $\epsilon > 0$ be given. (We'll work this out not in the correct order, but, hopefully, this will be a little more clear than the explanation in the book) Let $x \in (a-\delta(\epsilon,a), a+\delta(\epsilon,a))$, then choose $\delta(\epsilon,a) > 0$ small enough such that any rational in this interval (in lowest terms $\frac{p}{q}$) has the property:

$$f\left(\frac{p}{q}\right) = \frac{1}{q} < \epsilon.$$

For instance, if a is the irrational in I such that $\frac{200}{321} \cong a$, then we can choose a small enough radius of the δ-interval such that there is no rational that has a denominator of $2, 3, \ldots, m$ for any $m \in \mathbb{Z}^+$.

This might seem a little hand-wavy, but think about approximating a with a finite decimal representation in base 10. Clearly I can restrict the interval around a such that we only omit things with a certain denominator, say for any $n \in \mathbb{N}$, I can ensure that the only rationals in the interval $(a-\delta(\epsilon), a+\epsilon)$ are of the form $\frac{p}{q}$ where $p \in \{1,\ldots,q\}$ and $q \in \{n+1, n+2, \ldots\}$. This only works because a is irrational and hence has no repeating decimal representation. Otherwise if a were rational, say $a = \frac{1}{2}$, I cannot omit any rational with a denominator of 2 centered around a, since, well, a has a denominator of 2.

Thus for any $\epsilon > 0$, we can choose a $\delta(\epsilon,a) > 0$ such that if $0 < |x-a| < |r-a| < \delta(\epsilon,a)$, then $|f(x)-f(a)| < \frac{1}{q} < \epsilon$.

Finally, take any $a \in I \cap \mathbb{Q}$. Then define the sequence $(x_n) = a - \frac{1}{n\pi + N}$ with N being a fixed constant large enough so that $0 < a - \frac{1}{\pi+N} < 1$. So that since $a - \frac{1}{n\pi+N} > a - \frac{1}{(n+1)\pi+N}$ for all $n \in \mathbb{N}$, we'll have that this is a decreasing sequence and bounded. So that it will converge to it's infinimum, which is a. So that $x_n \to a$ as $n \to \infty$, however, $f(x_n) = 0$ for all $n \in \mathbb{N}$ because $a - \frac{1}{\pi n+N}$ is an irrational number. Hence while $\lim_{n\to\infty} x_n = a$, we'll have $\lim_{n\to\infty} f(x_n) = 0$. Thus f is not continuous on any rational in $(0,1)$. $\square$

2.3.1 Theorems Concerning Continuous Functions

Theorem 2.3.1 (The Extreme Value Theorem). *If f is a function that's continuous on a closed interval $[a, b]$, then f attains an absolute minimum value, m, at some point $c \in [a, b]$, and an absolute maximum value, M, at some point $d \in [a, b]$. That is, there exists points c and d in $[a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$.*

Theorem 2.3.2 (Bolzano's Theorem). *If f is a function that's continuous on a closed interval $[a, b]$ such that $f(a)$ and $f(b)$ differ in sign, then there's a point $c \in [a, b]$ such that $f(c) = 0$.*

Theorem 2.3.3 (The Intermediate Value Theorem). *Let f be a function that's continuous on a closed interval $[a, b]$. Let m be the absolute minimum value of f on $[a, b]$, and let M be the absolute maximum value of f on $[a, b]$. Then, for every $Y \in [m, M]$, there's at least one value $c \in [a, b]$ such that $f(c) = Y$.*

Example 2.6. *Prove or give a counterexample to the following statement:*

If f is continuous on the interval $[0, 1]$ and if $0 \leq f(x) \leq 1$ for every $x \in [0, 1]$, then there exists a point $c \in [0, 1]$ such that $f(c) = c$.

Proof. Note that if $f(0) = 0$ or $f(1) = 1$ then these are fixed points, and we're done. So that assume $f(0) > 0$ and $f(1) < 1$. So that defining the function $g(x) = f(x) - x$ this function must change sign on $[0, 1]$ as $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$. Hence we can employ Bolzano's theorem to tell us that there then must exist a $c \in (0, 1)$ such that $g(c) = 0 \iff f(c) = c$. This is what we wished to prove! $\square$

2.4 The Derivative

Theorem 2.4.1 (The Inverse-Function Rule). *The* ***inverse-function rule*** *says that if f^{-1} is the inverse of f, and f has nonzero derivative at x_0, then f^{-1} has derivative at $y_0 = f(x_0)$, and $(f^{-1})'(y_0) = \dfrac{1}{f'(x_0)}$. That is the derivative of the inverse at (x_0, y_0) will be the reciprocoal of the derivative of f.*

Theorem 2.4.2 (List of Important Derivatives).

$$\begin{aligned} d(k) &= 0 & d(u^k) &= ku^{k-1}du \\ d(e^u) &= e^u & d(a^u) &= \log(a)a^u du \\ d(\log(u)) &= \frac{1}{u}du & d(\log_a(u)) &= \frac{1}{u\log(a)}du (a \neq 1) \\ d(\sin(u)) &= \cos(u)du & d(\cos(u)) &= -\sin(u)du \\ d(\tan(u)) &= \sec^2(u)du & d(\cot(u)) &= -\csc^2(u)du \\ d(\sec(u)) &= \sec(u)\tan(u)du & d(\csc(u)) &= -\csc(u)\tan(u)du \\ d(\arcsin(u)) &= \frac{du}{\sqrt{1-u^2}} & d(\arctan(u)) &= \frac{du}{1+u^2} \end{aligned}$$

Example 2.7. *Find the derivative of each of the following:*

(a) $f(x) = x^3e^{-x} - x - 3$

(b) $g(x) = \dfrac{\log(\sin^2(x))}{\cos(x)}$

(c) $h(x) = \arctan(\sqrt{x})$

Solution. (a) Using the chain and product rules:

$$\frac{d}{dx}(x^3e^{-x^3} - x - 3) = 3x^2e^{-x^3} + x^3\frac{d}{dx}(e^{-x^3}) - 1 = 3x^2e^{-x^3} + x^3(-3x^2)e^{-x^3} - 1$$

$$\boxed{3x^2e^{-x^3}(1 - x^3) - 1}$$

(b)

$$\begin{aligned} \frac{d}{dx}\frac{\log(\sin^2(x))}{\cos(x)} &= \frac{\frac{d}{dx}(\log(\sin^2(x)))}{\cos(x)} - (-\sin(x))\log(\sin^2(x))(\cos(x))^{-2} \\ &= \frac{2\sin(x)\cos(x)}{\cos(x)\sin^2(x)} + \frac{\sin(x)\log(\sin^2(x))}{\cos^2(x)} \\ &= \frac{\cos^2(x)2\sin(x)}{\sin^2(x)\cos^2(x)} + \frac{\sin^3(x)\log(\sin^2(x))}{\cos^2(x)\sin^2(x)} \\ &= \frac{2\cos^2(x) + \sin^2(x)\log(\sin^2(x))}{\sin(x)\cos^2(x)} \end{aligned}$$

(c)

$$\begin{aligned} \frac{d}{dx}\arctan(\sqrt{x}) &= \frac{1}{2}x^{-1/2}\frac{1}{1+(\sqrt{x})^2} \\ &= \frac{1}{2\sqrt{x}}\frac{1}{1+x}. \end{aligned}$$

□

Example 2.8. *What's the equation of the normal line through the origin to the curve* $y = (x^4 - 1)^3 \log(x+1)$.

Solution. Recall that two lines are perpendicular if their slopes product is -1. So to find the normal line that goes through the origin, we (1) want the $y-$intercept to be 0, and (2) the line should be perpendicular to the tangent line of the curve at the point 0. So first our answer should be $y = mx$ and we just need to determine m that is perpendicular to the curve.

So luckily for us the derivative of $f(x)$ will give you the tangent line's slop at the point x. Hence take the derivative plug in $x = 0$:

$$y'(x) = 4x^3(3)(x^4-1)^2 \log(x+1) + \frac{(x^4-1)^3}{x+1}.$$

Plugging in $x = 0$:

$$y'(0) = 0 - 1.$$

So we need $m(-1) = -1$ hence $m = 1$, thus the normal line passing through the origin is:

$$\boxed{y = x.}$$

□

2.4.1 Linear Approximations Using Differentials

Theorem 2.4.3. *To approximate the function f at the point x, we can use:*

$$\frac{df}{dx} = f'(x) \iff \frac{\Delta f}{\Delta x} \approx f'(x) \iff \frac{f(x+\Delta x) - f(x)}{\Delta x} \approx f'(x).$$

We could equivalently use $\dfrac{f(x) - f(x-\Delta x)}{\Delta x} \approx f'(x)$.

Plugging in $x = a$:

$$(\Delta x) f'(a) \approx \Delta f$$

Example 2.9. *What's an approximate value for $e^{0.1}$?*

Solution. For this, we'll use the fact that $f(x) = e^x$ then $f'(x) = e^x$ and that $f(0) = e^0 = 1$ and $f'(0) = 1$. So using our derivative approximation:

$$(\Delta x) f'(0.1) \approx f(0+0.1) - f(0) \iff 0.1 \approx f(0.1) - 1 \iff f(0.1) \approx 1.1$$

□

2.4.2 Implicit Differentiation

Theorem 2.4.4 (Implicit Function Theorem)**.** *When an equation of the form $f(x,y) = c$ actually defines y as a function of x, so that the formula we'd derive for y' using implicit differentiation gives a result that makes sense.*

If $P_0 = (x_0, y_0)$ satisfies $f(x,y) = c$ and both partial derivatives f_x and f_y are continuous in a neighborhood of P_0, then if the value of f_y is non zero at P_0, there exists a unique differentiable function, $y = g(x)$, that satisfies both $f(x,y) = c$ and $y_0 = g(x_0)$. Furthermore, the derivative of this equation is given by $y' = \frac{-f_x}{f_y}$.

2.4.3 Higher-Order Derivatives

Example 2.10. *What's the second derivative of the function $f(x) = \log(\log(x))$?*

Solution. $f' = \frac{1}{x}\frac{1}{\log(x)}$, $f'' = \frac{(-1)(\log(x)+1)}{(\log(x)x)^2}$ □

2.5 Curve Sketching

2.5.1 Properties of the First Derivative

Theorem 2.5.1 (Properties of the First Derivative)**.**

- *At a point where $f'(x) > 0$, the slope is positive, so the function is increasing.*
- *At a a point where $f'(x) < 0$, the slope is negative, so the function is decreasing.*
- *At a point where $f'(x) = 0$, the slope is zero, so the function has a horizontal tangent line. This point is called a* ***critical*** *(or* ***stationary****)* ***point*** *of f, and often signifies a turning point.*
- *At a point where $f'(x)$ doesn't exist, the function could have a vertical tangent line, or it might not be differentiable at that point. This point is also called a* ***critical*** *point of f and can sometimes signify a turning point for a function.*

2.5.2 Properties of the Second Derivative

Theorem 2.5.2 (Properties of the Second Derivative)**.**

- *At a point where $f''(x) > 0$, the curve is* ***concave up*** *(or* ***convex****), which means the curve lies above the tangent line.*
- *At a point where $f''(x) < 0$, the curve is* ***concave down*** *(or just* ***concave****), which means that the curve lies below its tangent line.*
- *An* ***inflection point*** *is a point on a curve where the second derivative changes sign; thus, the curve is concave up on one side of an inflection point and concave down on the other side. In order of the curve $y = f(x)$ to have an inflection point at a point x in the domain of f, $f''(x)$ must equal 0 or $f''(x)$ must be undefined. (This condition is not sufficient, however.)*

Example 2.11. *Sketch the curve $y = x^2(x-2)^2$.*

Solution. I won't sketch the graph here as graphing in LaTeXis a massive pain, so we'll go through analytically what this graph should be.

First, we'll have roots at $x = 0, 2$. Next taking the derivative of this function:

$$f'(x) = 2x(x-2)^2 + 2x^2(x-2).$$

Finding the extreme points: $f'(x) = 2x(x-2)((x-2)+x) = 2x(x-2)(2x-2) = 4x(x-2)(x-1) = 0$, this will give us extreme points of $x = 0, 1, 2$. Taking the second derivative, we'll see if these are minimums or maximums depending on the sign of the these functions. $f''(x) = 12(x^2 - 2x + 2/3) = 0 \iff x = 1 \pm \sqrt{\frac{1}{3}}$. So that we'll have inflection points slightly to the right and slightly to the left of $x = 1$.

So that the graph should be convex on $(-\infty, 0)$ and decreasing $(0, 1)$ it will be convex and increasing, on $(1, 2)$ it'll be decreasing and convex and $(2, \infty)$ it'll be convex and increasing. □

Theorem 2.5.3 (The n-th Derivative Test)**.** *If $f''(x_0) = 0$ and we want to determine whether it's a maximum or minimum, keep taking derivatives until $f^{(n)}(x_0) \neq 0$. If n is even and $f^{(n)}(x_0) > 0$, then f is a local minimum at x_0; if n is even and $f^{(n)}(x_0) < 0$, then x_0 is a local maximum of f.*

Alternatively, if n is odd, then x_0 is neither a minimum or maximum!

2.6 Theorems Concerning Differentiable Functions

Theorem 2.6.1 (Rolle's Theorem)**.** *Assume that f is continuous on a closed interval $[a, b]$, with $f(a) = f(b)$, and f is differentiable at every point in (a, b). If this is true, there's at least one point $c \in (a, b)$ at which $f'(c) = 0$.*

Theorem 2.6.2 (The Mean-Value Theorem (for Derivatives))**.** *Assume that f is continuous on a closed interval $[a, b]$ and differentiable at every point (a, b). Then there's at least one point $c \in (a, b)$ such that:*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Theorem 2.6.3 (Corollaries to M.V.T)**.**

1. *If f is continuous on an interval I and $f'(x) = 0$, then $f(x)$ is constant on I.*
2. *If $f'(x)$ is positive on an interval I, then f is increasing on this interval; similarly, if $f'(x)$ is negative on I, then it's decreasing.*

2.7 Max/Min Problems

Theorem 2.7.1 (Extremum's)**.**

$$\begin{array}{ll} f'(c) = 0 \text{ and } f''(c) > 0 & f(c) \text{ is a local minimum} \\ f'(c) = 0 \text{ and } f''(c) < 0 & f(c) \text{ is a local maximum} \end{array}$$

Furthermore, if f is defined on a closed interval, then the Extreme Value theorem guarantees that f will actually attain an absolute minimum and an absolute maximum on this interval. There are three possibilities for the location of the absolute extrema on a closed interval: An absolute extremum will occur at a point c such that $f'(c) = 0$, $f'(c)$ fails to exist, or an endpoint of the interval.

Example 2.12. *The sum of two nonnegative numbers, x and y, is 12. What's the largest possible product of x^2 and y¿ What's the smallest?*

Solution. That is, we want to extremize the function x^2y where $x + y = 12 \iff y = 12 - x$. We need some constraints for this solution to be unique, since we need both $x, y \geq 0$, this implies that $0 \leq x \leq 12$. So that this becomes the problem of extremizing $x^2(12 - x)$ with $x \in [0, 12]$.

First, it's a polynomial so that the derivative is continuous, but we'll check the endpoints first: $x = 0 \iff f(0) = 0$ and $x = 12 \iff f(12) = 0$. Now we'll use the derivative test:

$$\frac{d}{dx}(12x^2 - x^3) = 24x - 3x^2 = 0 \iff 3x(8 - x) = 0 \iff x = 0, 8$$

and $\frac{d}{dx}(24x - 3x^2) = 24 - 6x$, so that at $x = 0$, $f''(0) > 0$ and $x = 8$, $f''(8) = 24 - 48 < 0$. Hence $x = 0$ is an absolute minimum and $x = 8$ is an absolute maximum, and just $x = 0$ is a endpoint minimum but not $x = 12$. The minimum value is: $f(0) = 0$, and the maximum value is: $f(8) = 8^2(4) = 64(4) = 256$. □

Example 2.13. *A rectangle in the fourth quadrant of the $xy-$plane has adjacent sides on the coordinate axes. If the vertex opposite the origin is on the curve $y = \log(x)$, what's the maximum area this rectangle can have?*

Solution. So our constraints are given by: $x \geq 0$ and $y \leq 0$ since we are in the fourth quadrant. Now $x \geq 0$ and $y = \log(x) \leq 0$ implying that $0 < x \leq 1$. Since the rectangle has sides of the coordinate axes, this will imply that one side is x and the other $\log(x)$. Furthermore, $\log(x) \leq 0$ on this interval meaning the total area is given by:

$$A(x) = -\log(x)x.$$

Differentiate this with respect to x and set it equal to 0:

$$-\log(x) - 1 = 0 \iff \log(x) = -1 \iff x = e^{-1} = \frac{1}{e}.$$

Note that we can rule out when $x \to 0^+$ as this would give us an area $0(\infty)$ which we'll take to be 0 here although this is technically a indeterminate form.

Take the second derivative to term what this is (maximum or minimum):

$$(f'(x))' = (-\log(x) - 1)' = -\frac{1}{x}\bigg|_{x=1/e} = -e < 0$$

This is a maximum! And so the area reaches a maximum of $A(x) = -\log\left(\frac{1}{e}\right)\frac{1}{e} = \boxed{\frac{1}{e}}$. □

2.8 Related Rates

Theorem 2.8.1 (Related Rates)**.** *In general, find an expression such as:*

$$V(r) = \frac{4}{3}\pi r^3$$

and differentiate with respect to t:

$$\frac{dV}{dt} = \frac{4}{3}\pi\frac{dr}{dt}3r^2$$

where we used the chain rule. Typically, you will be given one changing quantity and you'll need to solve for some other quantity.

Example 2.14. *A ladder of length 5 m is leaning against a vertical wall. The base of the ladder is then pulled away from the wall at a rate of $\frac{1}{2}$ m/s. At the moment at which the base of the ladder is 3 m from the wall, how fast is the top of the ladder sliding down the wall?*

Solution. Note that this must be a write triangle, with base b and height h so that $b^2 + h^2 = 5^2$ where 5 is the length of the ladder and hence is the hypotenuse of this right triangle. Differentiate this with respect to time on both sides and we'll get:

$$0 = \frac{db}{dt}2b + 2h\frac{dh}{dt} = 0 \iff \frac{db}{dt}b + h\frac{dh}{dt} = 0.$$

So that

$$\frac{1}{2}3 + 4\frac{dh}{dt} = 0 \iff \frac{dh}{dt} = -\frac{3}{8}.$$

Note: that a choice of the area does not work here, as the area at any given time t will be a function of the height and base, the Pythagorean identity however must hold regardless of these values as this will always be a triangle. The implicit differentiation theorem tells us we must have an expression:

$$F(x, y) = c$$

for the solution to be unique, we do not have this for the area of the triangle., □

2.9 Indefinite Integration (Antidifferentitation)

Example 2.15. *The function $F(x)$ is the antiderivative of $f(x) = 6\sqrt{x} + \sin(x) - 1$ and satisfies $F(0) = 3$. What's $F(x)$?*

Solution. Just evaluate the integral

$$\int (6x^{1/2} + \sin(x) - 1)\ dx = 4x^{3/2} - \cos(x) - x + C.$$

Using $F(0) = 3 \iff 0 - 1 - 0 + C = 3 \iff C = 4$ gives us:

$$\boxed{F(x) = 4x^{3/2} - \cos(x) - x + 4}$$

□

2.9.1 Techniques of Integration

Theorem 2.9.1 (List of Important Integrals)**.**

$$\int k\ du = ku + C \qquad \int u^k\ du = \begin{cases} \frac{1}{k+1}u^{k+1} + C & \text{if } k \neq -1 \\ \log|u| + C & \text{if } k = -1 \end{cases}$$

$$\int e^u\ du = e^u + C \qquad \int a^u\ du = \frac{a^u}{\log(a)} + C$$

$$\int \sin(u)\ du = -\cos(u) + C \qquad \int \cos(u)\ du = \sin(u) + C$$

$$\int \sec^2(u)\ du = \tan(u) + C \qquad \int \csc^2(u)\ du = -\cot(u) + C$$

$$\int \sec(u)\tan(u)\ du = \sec(u) + C \qquad \int \csc(u)\cot(u)\ du = -\csc(u) + C$$

$$\int \frac{du}{\sqrt{1-u^2}} = \arcsin(u) + C \qquad \int \frac{du}{1+u^2} = \arctan(u) + C$$

Example 2.16. *Evaluate each of the following integrals.*

(a) $\int \tan(x)\ dx$

(b) $\int \dfrac{x\ dx}{x^4 + 2x^2 + 2}$

(c) $\int \sin^2(x)\ dx$

Solution. 1. Make a $u = \cos(x)$ and $du = -\sin(x)\ dx$ so that

$$\int \frac{\sin(x)}{\cos(x)}\ dx = \int \frac{-du}{u} = \log|\cos(x)| + C$$

2. First, note that we can breakup the denominator and let $u = x^2+1$ so that $du = 2x\ dx \iff \dfrac{du}{2} = x\ dx$

$$\int \frac{x\ dx}{x^4 + 2x^2 + 2} = \int \frac{x\ dx}{x^4 + 2x^2 + 1 + 1} = \int \frac{x\ dx}{(x^2+1)^2 + 1} = \int \frac{du}{2}\frac{1}{u^2+1} = \frac{1}{2}\arctan(u) + C = \frac{1}{2}\arctan(x^2+1) + C.$$

3. Here you can use a power-reduction formula from trigonometry to find: $\sin^2(x) = \dfrac{1}{2}(1 - \cos(2x))$.

Making our lives easier with letting $u = 2x$ and $du = 2\ dx \iff \frac{du}{2} = dx$ so that:

$$\int \sin^2(x)\ dx = \frac{1}{2}\int (1-\cos(2x))\ dx = \frac{1}{4}\int (1-\cos(u))\ du = \frac{1}{4}(2x - \sin(2x)) + C$$

□

Theorem 2.9.2 (Trig Subs)**.** *Anytime you have the following expressions use these trig subs:*

If the integrand contains	*Make this substitution*
$\sqrt{a^2-u^2}$	$u = a\sin(\theta), du = a\cos(\theta)\ d\theta$
$\sqrt{a^2+u^2}$	$u = a\tan(\theta), du = a\sec^2(\theta)\ d\theta$
$\sqrt{u^2-a^2}$	$u = a\sec(\theta), du = a\sec(\theta)\tan(\theta)\ d\theta$

Theorem 2.9.3 (Partial Fraction Decomposition)**.** *Any time we have something of a rational form, here's how we decompose it:*

$$\frac{1}{x^2(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{x-1}$$

we can then solve for the constants A, B, C, D.

2.10 Definite Integration

Definition 2.10.1 (Riemann Sums)**.**

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=0}^{n} \left(f(b) - f\left(\frac{(b-a)k}{n}\right)\right)$$

2.11 The Fundamental Theorem of Calculus

Definition 2.11.1 (The Fundamental Theorem of Calculus)**.**

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

where $F'(x) = f(x)$

Example 2.17. *Find the area of the region bounded by the $x-$axis, the line $x = 4$, and the curve $y = \sqrt{x}$.*

Solution. Draw a picture of this, this is quick and easy to setup. We should end up with $x = 0$ and $x = 4$. So that this is:

$$\int_0^4 \sqrt{x}\ dx = \frac{x^{3/2}}{3/2}\bigg|_{x=0}^4 = \frac{2}{3}4^{3/2} = \frac{2}{3}2^3 = \frac{2^4}{3} = \frac{16}{3}.$$

□

Example 2.18. *Simplify the expression*

$$\frac{d}{dx}\int_x^{x^2} \frac{t\ dt}{\log(t)}.$$

Proof. This is a classic application of the F.T.o.C:

$$\int_x^{x^2} \frac{t\ dt}{\log(t)} = F(x^2) - F(x)$$

with $F'(x) = f(x)$. And just differentiate both sides of this with respect to x:

$$\frac{d}{dx}\int_x^{x^2} \frac{t\ dt}{\log(t)} = 2xF'(x^2) - 1F'(x) = 2x\frac{x^2}{\log(x^2)} - \frac{x}{\log(x)} = \boxed{\frac{x^3 - x}{\log(x)}}.$$

□

2.11.1 The Average Values of a Function

2.11.2 Find the Area Between Two Curves

Example 2.19. *Find the area of the region in the first quadrant, bounded by the curve $y = x^3$ and $y = 4x$.*

Solution. First, find where $4x = x^3 \iff x^2 = 4 \iff x = \pm 2$ so that since we're in the first quadrant we have the interval of integration is $[0, 2]$. Now notice that $x = 1$ will give us $4x = 4$ and $x^3 = 1$ hence we'll have $4x - x^3$ is the integrand of the integral:

$$\int_0^2 4x - x^3\ dx = 2x^2 - \frac{x^4}{4}\bigg|_{x=0}^2 = 22^2 - \frac{2^4}{4} = 8 - 4 = 4$$

□

2.12 Polar Coordinates

Theorem 2.12.1 (Area of a Cardioid)**.** *For any function with radius $r(\theta) = r$ then the area of the cardioid will be*

$$\int_\alpha^\beta \frac{1}{2}(r(\theta))^2\ d\theta$$

This comes from:

$$\int_{\alpha}^{\beta}\int_{0}^{r} r\ dr\ d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2\ d\theta.$$

Example 2.20. *Find the area enclosed by the cardioid $r = 2a(1+\cos(\theta))$.*

Solution.

$$A = \frac{1}{2}\int_{0}^{2\theta} (r)^2\ d\theta = \frac{1}{2}\int_{0}^{2\pi} 4a^2(1+\cos^2(\theta)+2\cos(\theta))\ d\theta = 2a^2(2\pi+0+\frac{1}{2}\int_{0}^{2\pi}(1+\cos(2\theta))\ d\theta)$$

□

2.13 Volumes of Solids of Revolution

Theorem 2.13.1 (Disk Method)**.** *A disk whose radius is $f(x)$ and whose height is dx. The volume of this disk is $dV = \pi[f(x)]^2\ dx$, so that the total volume of the solid is:*

$$V = \int_a^b dV = \int_a^b \pi[f(x)]^2\ dx$$

If the curve is revolved around a vertical line (such as the $y-$axis), then horizontal disks are used. If the curve can be solved for x in terms of y, $x = g(y)$, the formula above becomes:

$$V = \int_a^b \pi[g(y)]^2\ dy.$$

Theorem 2.13.2 (Washer Method)**.** *If you have a volume that is rotated, but bounded by two curves:*

$$V = \int_a^b \pi[f(x)]^2 - [g(x)]^2\ dx.$$

Similarly if we can solve for $x = g(y)$ and $x = f(y)$.

Example 2.21. *What's the volume of the solid of revolution generated in each case?*

1. *The portion of the curve $y = x^2$ from $x = 1$ to $x = 2$ is revolved around the $x-$axis?*
2. *The region bounded by the curve $x = y^2 + 3$ and $x = 4y$ is revolved around the $y-$axis.*

Solution. 1.

$$\int_1^2 \pi((x^2)^2 - 0^2)\ dx = \pi\int_1^2 x^4\ dx = \pi\frac{x^5}{5}\Big|_{x=1}^{x=2} = \frac{\pi}{5}(32-1) = \frac{31\pi}{5}.$$

2. First, you need to find the bounds, so solve for y in: $y^2+3=4y \iff y^2-4y+3 \iff y = \dfrac{4\pm\sqrt{16-12}}{2} = \dfrac{4\pm 2}{2} = 3, 1$. Now, note that $y=2$ will give us: $2^2+3=7\leq 4(2)=8$. So that $4y$ is greater than y^2+3 on this interval. So that we get:

$$\begin{aligned}
\pi\int_1^3 (4y)^2-(y^2+3)^2\ dy &= \pi\int_1^3 16y^2-(y^4+6y^2+9)\ dy \\
&= \pi\int_1^3 10y^2-y^4-9\ dy \\
&= \pi\left(\frac{10y^3}{3}-\frac{y^5}{5}-9y\right)\Bigg|_{y=1}^{y=3} \\
&= \pi\left(\frac{10(27-1)}{3}-\frac{3^5-1}{5}-9(3-1)\right) \\
&= \pi\left(\frac{260}{3}-\frac{242}{5}-18\right) \\
&= \frac{\pi(260\times 5-(242\times 3)-(18\times 15))}{15} \\
&= \frac{304\pi}{15}.
\end{aligned}$$

□

2.14 Arc Length

Theorem 2.14.1 (Arc Length Formula)**.** *The arc length of a differentiable curve $y=f(x)$ or $x=g(y)$ is going to be $ds=\sqrt{(dy)^2+(dx)^2}=\sqrt{\left(\frac{dx}{dy}\right)^2+1}\ dy=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$. So that we just integrate this from a point a to b:*

$$\int_a^b\ ds = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx.$$

Similarly for dy.

Example 2.22. *What's the length of the curve $y=\sqrt{x^3}$, from $x=0$ to $x=28$?*

Solution. $y=x^{3/2}$ so that $\dfrac{dy}{dx}=\dfrac{3}{2}x^{1/2}$ and $\left(\dfrac{dy}{dx}\right)^2=\dfrac{9}{4}x$. With a u-substitution of $u=1+\dfrac{9}{4}x$ so that the limits are $[1,64]$ and $du=\dfrac{9}{4}\ dx \iff \dfrac{4}{9}\ du = dx$:

$$\int_0^{28}\sqrt{1+\frac{9}{4}x}\ dx = \frac{4}{9}\int_1^{64}u^{1/2}\ du = \frac{4}{9}\frac{2}{3}u^{3/2}\Bigg|_{u=1}^{u=64} = \frac{8}{27}(64^{3/2}-1) = \frac{8(8^3-1)}{27} = \frac{8(511)}{27} = \frac{4088}{27}$$

□

2.15 The Natural Exponential and Logarithm Functions

Theorem 2.15.1 (Exponential/Logarithmic Differentiation). *Occasionally it'll be useful to use the fact that* $e^x = y \iff \log(y) = x$ *to find derivatives of exponentials. Particularly, in the case where our base is a variable:* $f(x)^{g(x)} = y \iff \log_{f(x)}(y) = g(x) \iff \dfrac{\log(y)}{\log(f(x))} = g(x)$ *where we used the change of base formula for logarithms. So to find* $\dfrac{d}{dx}(f(x))^{g(x)} = \dfrac{dy}{dx}$ *and we can just differentiate both sides of* $\log(y) = \log(f(x))g(x)$ *to solve for* $\dfrac{dy}{dx}$.

Example 2.23. *What's the value of this limit* $a > 0$*?*

$$\lim_{x\to 0} \frac{a^x - 1}{x}$$

Solution. Note that for any $a > 0$ we'll have:

$$\frac{d}{dx}a^x = \log(a)a^x.$$

So that

$$(a^x)'(x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}$$

meaning the derivative at $x = 0$ is:

$$\log(a)a^0 = \lim_{h\to 0} \frac{a^h - 1}{h} = \log(a).$$

Hence $\boxed{\lim_{x\to 0} \frac{a^x - 1}{x} = \log(a)}$ for any $a > 0$. □

Example 2.24. *Let* A *denote the area of the region bounded by the curve* $y = \dfrac{1}{x}$*, the* $x-$*axis, and the vertical lines* $x = 1$ *and* $x = a$ *(where* $a > 1$*). In terms of* A*, what's the area of the region bounded by the curve* $y = \dfrac{1}{x}$*, the* $x-$*axis, and the vertical lines* $x = a^2$ *and* $x = a^3$*?*

Solution. Since $A = \int_1^a \frac{dx}{x} = \log(a) - \log(1) = \log(a) - 0 = \log(a)$ we can use this to get our answer:

$$\int_{a^2}^{a^3} \frac{dx}{x} = \log(a^3) - \log(a^2) = 3\log(a) - 2\log(a) = \log(a) = \boxed{A}.$$

□

Example 2.25. *What's the derivative of the function $f(x) = x^{\sqrt{x}}$?*

Solution. We can setup the equation: $x^{\sqrt{x}} = y \iff \log_x(y) = \sqrt{x} \iff \frac{\log(y)}{\log(x)} = \sqrt{x} \iff \log(y) = \log(x)\sqrt{x}$ hence

$$\frac{d}{dx}(x^{\sqrt{x}}) = \frac{dy}{dx} \iff \frac{\frac{dy}{dx}}{y} = \frac{\sqrt{x}}{x} + \log(x)\frac{1}{2\sqrt{x}}$$
$$\frac{dy}{dx} = \boxed{\frac{x^{\sqrt{x}}}{\sqrt{x}}\left(1 + \frac{\log(x)}{2}\right)}$$

where we used the fact that $y = x^{\sqrt{x}}$ above. □

2.16 L'Hopital's Rule

Theorem 2.16.1 (L'Hopital's Rule)**.** *If two functions f and g have an interdeterminate form with a limit as $x \to a$ such as:*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} = \frac{\infty}{\infty}$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

And the limit needn't even exist, so we can replace $x \to a$ with $x \to a^-$ or $x \to a^+$.

Example 2.26. *Find each of the following limits.*

1. $\lim_{x\to 0} \frac{\sin(4x)}{x}$
2. $\lim_{x\to 1} \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$
3. $\lim_{x\to 0} \frac{x^2}{1 - \cos(x)}$
4. $\lim_{x\to \infty} \frac{1}{x\left(\frac{\pi}{2} - \arctan(x)\right)}$
5. $\lim_{x\to 1^+} x^{1/(x^2-1)}$

Solution. 1.

$$\lim_{x\to 0} \frac{\sin(4x)}{x} = \lim_{x\to 0} \frac{4\cos(4x)}{1}$$
$$= \boxed{4}$$

2.

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1} &= \lim_{x\to 1}\frac{1/2x^{-1/2}}{1/3x^{-2/3}}\\ &= \boxed{\frac{3}{2}}\end{aligned}$$

3.

$$\begin{aligned}\lim_{x\to 0}\frac{x^2}{1-\cos(x)} &= \lim_{x\to 0}\frac{2x}{\sin(x)}\\ &= \lim_{x\to 0}\frac{2}{\cos(x)}\\ &= \boxed{2}\end{aligned}$$

4.

$$\begin{aligned}\lim_{x\to\infty}\frac{1/x}{\frac{\pi}{2}-\arctan(x)} &= \lim_{x\to\infty}\frac{-1x^{-2}}{-\frac{1}{1+x^2}}\\ &= \lim_{x\to\infty}\frac{1+x^2}{x^2}\\ &= \lim_{x\to\infty}1+\frac{1}{x^2} = \boxed{1}\end{aligned}$$

5.

$$\lim_{x\to 1^+}x^{1/(x^2-1)} = y \iff \log(y) = \lim_{x\to 1^+}\frac{\log(x)}{x^2-1}$$

$$\begin{aligned}\lim_{x\to 1^+}\frac{\log(x)}{x^2-1} &= \lim_{x\to 1^+}\frac{\frac{1}{x}}{2x}\\ &= \lim_{x\to 1^+}\frac{1}{2x^2}\\ &= \frac{1}{2}\\ \log(y) &= \frac{1}{2}\\ y &= e^{1/2}\\ \lim_{x\to 1^+}x^{1/(x^2-1)} &= y = \boxed{e^{1/2}}\end{aligned}$$

$\square$

Example 2.27. *What's the limit of the sequence (a_n), where*

$$a_n = \frac{(\log(n))^2}{\sqrt{n}}?$$

Solution.

$$\begin{aligned}
\lim_{n\to\infty} \frac{(\log(n))^2}{\sqrt{n}} &= \lim_{x\to\infty} \frac{(\log(x))^2}{\sqrt{x}} \\
&= \lim_{x\to\infty} \frac{2\log(x)x^{-1}}{\frac{1}{2}x^{-1/2}} \\
&= \lim_{x\to\infty} \frac{4\log(x)}{x^{1/2}} \\
&= \lim_{x\to\infty} \frac{4x^{-1}}{\frac{1}{2}x^{-1/2}} \\
&= \lim_{x\to\infty} \frac{8}{x^{1/2}} \\
&= \boxed{0}
\end{aligned}$$

□

2.17 Improper Integrals

Theorem 2.17.1 (Improper Integrals First Kind)**.** *An improper integral of the first kind is defined to be an integral over an unbounded interval; that is intervals of the form:*

$$(-\infty, \infty) \quad (-\infty, a] \qquad [a, \infty)$$

where $a \in \mathbb{R}$. We define these as limits:

$$\int_{-\infty}^{\infty} f(x)\ dx = \lim_{A\to-\infty} \lim_{B\to+\infty} \int_A^B f(x)\ dx$$

and similarly for the intervals $[a, \infty)$ and $(-\infty, a]$.

Theorem 2.17.2 (Improper Integrals of the Second Kind)**.** *The difference here is that instead of integrating over an unbounded interval, we integrate over an interval that has a vertical asymptote, equivalently, integrate over a point where the function is undefined. We again define these in terms of limits, suppose that a function f is undefined at ζ in $[a, b]$ then we can define the integral as:*

$$\int_a^b f(x)\ dx = \lim_{A\to\zeta^-} \int_a^A f(x)\ dx + \lim_{B\to\zeta^+} \int_B^b f(x)\ dx$$

2.18 Infinite Series

Example 2.28. *Which of the following series converge, and which diverge?*

1. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^3+n}$
2. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2000}}{n!}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{(\log(3))^3}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{\log(n)}{n^2}$

Solution. 1. Using the comparison test and a convergent $p-$series:

$$\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^3+n} \leq \sum_{n=1}^{\infty} \frac{1}{n^3} < \infty$$

2. Using the comparison test and a divergent $p-$series:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}} \geq \sum_{n=1}^{\infty} \frac{1}{\sqrt{2}n}.$$

3. We can use the ratio test here:

$$\frac{(n+1)^{2000}}{(n+1)!} \frac{n!}{n^{2000}} = \frac{1}{n}\left(1+\frac{1}{n}\right)^{2000}$$

Since this limit exists as is 0, this series is convergent by the ratio test.

4. Use the root test here:

$$\left(\frac{n^3}{(\log(3))^n}\right)^{1/n} = \frac{(n^{1/n})^3}{\log(3)}$$

Since $\lim_{n\to\infty} n^{1/n} = 1$ we have that this limit exists and is $\frac{1}{\log(3)}$ so by the root test this series is convergent.

5. Note here we'll show that this sequence is monotonically decreasing which will enable this to use the integral test. Consider the following:

$$\frac{n^3}{(\log(3))^n} \square \frac{(n+1)^3}{(\log(3))^{n+1}}$$

$\square$

2.18.1 Alternating Series

2.19 Power Series

Example 2.29. *What's the interval of convergence for this power series?*

$$2x + x^2 + \frac{8}{9}x^3 + x^4 + \ldots = \sum_{n=1}^{\infty} \frac{2^n}{n^2} x^n$$

Solution.

$$\left|\frac{x^{n+1}}{x^n}\right| \left|\frac{2^{n+1}}{(n+1)^2} \frac{n^2}{2^n}\right| < 1$$

$$|x| 2 \left(1 + \frac{1}{n}\right)^2 < 1$$

$$|x| < \frac{1}{2} \left(1 + \frac{1}{n}\right)^{-2}$$

Taking the limit as $n \to \infty$ of both sides:

$$|x| < \frac{1}{2}$$

We have to then check the series when $x = \frac{\pm 1}{2}$. When $x = \frac{\pm 1}{2}$ we'll have:

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2} \frac{1}{(\pm 2)^n}$$

In the " $+$ "-case this will be a convergent $p-$series, so that this is an absolutely convergent series and thus the " $-$ "$-$case is also convergent. Hence the interval of convergence is: $x \in [-1/2, 1/2]$. □

2.19.1 Functions Defined by Power Series

Example 2.30. *We know that the geometric series*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

converges for all $|x| < 1$. *Use this series to find a power series expansion for*

1. $g(x) = \log(1 - x)$
2. $h(x) = \dfrac{x}{(1-x)^2}$
3. $k(x) = \arctan(x)$

Solution. This is all about playing around with derivative and integrals and change of variables of the function $\dfrac{1}{1-x}$ and using the fact that power-series expansion are preserved under those operations.

1.

$$\frac{-1}{1-x} = -\sum_{n=0}^{\infty} x^n$$
$$\int_0^x \frac{-1}{1-x} = \int_0^x \sum_{n=0}^{\infty} x^n$$
u - substitution of $u = 1 - x$and $du = -dx$
$$\int_{x=0}^{x=x} \frac{du}{u} = \sum_{n=0}^{\infty} \int_0^x x^n$$
$$\log|u|\Big|_{x=0}^{x=x} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$
$$\log|1-x| - \log(1) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$
$$\log|1-x| = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

2.

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$$
$$\frac{(-1)(-1)}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$
$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n$$

3.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$
$$\frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$$
$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$
$$\int_0^x \frac{1}{1+x^2} = \sum_{n=0}^{\infty} \int_0^x (-1)^n x^{2n}$$
$$\arctan(x) - \arctan(0) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$
$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}$$

□

2.19.2 Taylor Series

Example 2.31. *Find the power series expansions of e^x and $\sin(x)$.*

Solution. We start by taking the derivatives of the these functions and plugging in our center, $x_0 = 0$ in this case:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^x	1
1	e^x	1
2	e^x	1
$\vdots$	$\vdots$	$\vdots$

n	$f^{(n)}(x)$	$f^{(n)}(x_0)$
0	$\sin(x)$	0
1	$\cos(x)$	1
2	$-\sin(x)$	0
3	$-\cos(x)$	-1
4	$\sin(x)$	0
$\vdots$	$\vdots$	$\vdots$

These give us the Taylor coefficients: $f^{(n)}(0) = 1$ for e^x and $f^{(n)}(0) = (-1)^n$ for $\sin(x)$ (in this case, indexing $n = 1, 3, 5, \ldots$). Giving us the Taylor series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

□

2.19.3 Taylor Polynomials

Definition 2.19.1 (Taylor Polynomials)**.** *While the Taylor series itself will be absolutely convergent to the function on its interval of convergence if we terminate the series at a given $n \in \mathbb{N}$ then what we end up with an approximation, in polynomials of x, of a given smooth function.*

Taylor's theorem states that $P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$ then there exists a $c \in [a, x]$ such that:

$$f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for any given Taylor series centered at a

Review Questions 2.1. *Consider the sequence (x_n) whose terms are given by the formula*

$$x_n = \frac{(\cos(n\pi))(\sin^2(n))}{\sqrt[e]{n}}$$

for each integer $n \geq 1$. Given that the sequence converges, what's the limit?

Solution. Let's build out sandwich. Note that $\cos(\pi n) = (-1)^n$ for all $n \geq 1$ and that $\sin^2(n) \leq 1$ and that $\sqrt[2]{n} \geq \sqrt[e]{n} \geq \sqrt[3]{n}$ so that:

$$\frac{(-1)}{\sqrt[e]{n}} \leq \frac{(\cos(\pi n))(\sin^2(n))}{\sqrt[e]{n}} = \frac{(-1)^n \sin^2(n)}{\sqrt[e]{n}} \leq \frac{1}{\sqrt[e]{n}}$$

Since $\lim_{n\to\infty} \frac{\pm 1}{n^{1/e}} = 0$ we have that $\lim_{n\to\infty} x_n = 0$ by the sandwich theorem. □

Review Questions 2.2. *Let (x_n) be the sequence with $x_1 = 2$ and $x_n = \sqrt{5x_{n-1} + 6}$ for every integer $n \geq 2$. Given that this sequence converges, what's the limit?*

Proof. The key thing here is that the limit exists so that $\lim_{n\to\infty} x_n = L \in \mathbb{R}$. That since (x_{n-1}) is a subsequence of (x_n) and (x_n) sequence is convergent, we have that they share a limit of L. Since we can compose limits with the $\sqrt{\cdot}$ function we get:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt{5x_{n-1} + 6} \iff L = \sqrt{5L+6} \iff L^2 = 5L+6 \iff L = \frac{5 \pm \sqrt{25+24}}{2} = \frac{5 \pm 7}{2} = 6, -1.$$

□

Review Questions 2.3. *Let $\lfloor x \rfloor$ denote the greatest integer $\leq x$. If n is a positive integer, then*

$$\lim_{x\to -n^-} (|x| - \lfloor x \rfloor) - \lim_{x\to n^-} (|x| - \lfloor x \rfloor) =?$$

Solution. Since $x \to -n^-$ this means $x < -n^-$ for all $n \geq 1$ for the first term giving us: $|x| = -x$ and $\lfloor x \rfloor = -n - 1$. The second term is $x \to n^-$, so that $|x| = x$ and $\lfloor x \rfloor = n - 1$ so that:

$$\lim_{x\to -n^-} (|x| - \lfloor x \rfloor) - \lim_{x\to n^-} (|x| - \lfloor x \rfloor) = n - (-n-1) - (n - (n-1)) = n + 1 + n - 1 = \boxed{2n}$$

□

Review Questions 2.4. *Evaluate the following limit:*

$$\lim_{x\to 0} \frac{\arcsin(x) - x}{x^3}.$$

Solution.

$$\begin{aligned}
\lim_{x\to 0} \frac{\arcsin(x) - x}{x^3} &= \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}} - 1}{3x^2} \\
&= \lim_{x\to 0} \frac{\frac{x}{(1-x^2)^{3/2}}}{6x} \\
&= \lim_{x\to 0} \frac{1}{6} \frac{1}{(1-x^2)^{3/2}} = \frac{1}{6}.
\end{aligned}$$

□

Review Questions 2.5. *The curve whose equation is*

$$2x^2 + 3x - 2xy - y = 6$$

has two asymptotes. Identify those lines.

Solution. We can actually solve for $y = f(x)$ here:

$$\begin{aligned} 2x^2 + 3x - 6 &= y(2x+1) \\ y &= \frac{2x^2+3x-6}{2x+1} \\ &= x + 1 - \frac{7}{2x+1} \end{aligned}$$

From the second line, we get that $x = -1/2$ is a vertical asymptotes since $2x + 1 = 0$. Finally the third line gives us that we have a slant asymptote at $y = x + 1$. □

Review Questions 2.6. *If the function*

$$f(x) = \begin{cases} \dfrac{x^2 - 6x + 8}{x^3 - 2x^2 + 2x - 4} & \text{if } x \neq 2 \\ k & \text{if } x = 2 \end{cases}$$

is continuous everywhere, what's the value of k?

Solution.

$$\begin{aligned} \lim_{x\to 2} \frac{x^2 - 6x + 8}{x^3 - 2x^2 + 2x - 4} &= k \\ \lim_{x\to 2} \frac{2x - 6}{3x^2 - 4x + 2} &= k \\ \frac{-2}{12 - 8 + 2} &= k \\ k &= \frac{-1}{3} \end{aligned}$$

□

Review Questions 2.7. *Evaluate the following limit:*

$$\lim_{x\to 0} \left[\frac{1}{x^2} \int_0^x \frac{t + t^2}{1 + \sin(t)}\ dt \right]$$

Solution. Note that $\frac{d}{dx}\int_0^x \frac{t+t^2}{1+\sin(t)}\,dt = F'(x)$ where $F'(x) = \frac{x+x^2}{1+\sin(x)}$ so that:

$$\lim_{x\to 0}\left[\frac{\frac{d}{dx}\int_0^x \frac{t+t^2}{1+\sin(t)}dt}{2x}\right] = \lim_{x\to 0}\frac{\frac{x+x^2}{1+\sin(x)}}{2x}$$
$$= \frac{1}{2}\lim_{x\to 0}\frac{1+x}{1+\sin(x)} = \frac{1}{2}$$

□

Review Questions 2.8. *Determine the domain of the following function:*

$$f(x) = \arcsin(\log(\sqrt{x}))$$

Proof. Note that $\arcsin : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\log : (0,\infty) \to \mathbb{R}$, $\sqrt{\cdot} : [0,\infty) \to [0,\infty)$. So we need $x \in (0,\infty)$ and $-1 \le \log(\sqrt{x}) \le 1$. This function $1-1$ so that we can just solve for the endpoints:

$$\log(\sqrt{x}) = -1 \qquad \text{and} \qquad \log(\sqrt{x}) = 1$$

This gives us:

$$\sqrt{x} = e^{-1} \qquad \text{and} \qquad \sqrt{x} = e$$

and finally $x = e^{-2}$ and $x = e^2$ so that the domain is:

$$x \in \left[\frac{1}{e^2}, e^2\right]$$

□

Review Questions 2.9. *Evaluate the derivative of the following function at $x = e$:*

$$f(x) = \arcsin(\log(\sqrt{x}))$$

Solution.

$$\begin{aligned}
\frac{d}{dx}(\arcsin(\log(\sqrt{x}))) &= \frac{1}{\sqrt{1-(\frac{1}{2}\log(x))^2}}\frac{d}{dx}(\log(\sqrt{x})) \\
&= \frac{1}{2}\frac{1}{x}\frac{1}{\sqrt{1-\frac{\log^2(x)}{4}}} \\
&= \frac{1}{2}\frac{1}{e}\frac{1}{\sqrt{\frac{3}{4}}} \\
&= \frac{1}{2}\frac{1}{e}\frac{2}{\sqrt{3}} \\
&= \boxed{\frac{1}{e\sqrt{3}}}.
\end{aligned}$$

□

Review Questions 2.10. *For what values of m and b will the following function have a derivative for every x?*

$$f(x) = \begin{cases} x^2+x-3 & \text{if } x \leq 1 \\ mx+b & \text{if } x > 1 \end{cases}$$

Solution. First, note that we have if a function is differentiable at c then it's continuous at c. So that if f isn't continuous at c, then it isn't differentiable at c. Additionally, we need the derivative to be smooth at $x=2$ as well so continuous there. Giving us that:

$$\lim_{x\to 1^+} x^2+x-3 = \lim_{x\to 1^-} mx+b \iff -1 = m+b$$

Finally since the derivative is smooth:

$$\lim_{x\to 1^+} 2x+1 = \lim_{x\to 1^-} m \iff m = 3$$

giving us $b=-4$ so that $m=3, b=-4$ is the solution here. □

Review Questions 2.11. *If $f(x)$ is a function that's differentiable everywhere, what's the value of this limit?*

$$\lim_{h\to 0}\frac{f(x+3h^2)-f(x-h^2)}{2h^2}$$

Solution. Since f is continuous everywhere, because differentiability implies continuity, this gives us that for all $x \in \mathbb{R}$ $\lim_{h\to 0} f(x+h) = f(x)$. Now, just use L'Hoptial's rule:

$$\lim_{h\to 0}\frac{f(x+3h^2)-f(x-h^2)}{2h^2} = \lim_{h\to 0}\frac{6hf'(x+3h^2)+2hf'(x-h^2)}{4h}$$

$$= \lim_{h\to 0} \frac{1}{2} 3f'(x+3h^2) + f'(x-h^2) = \frac{4}{2} f'(x) = \boxed{2f'(x)}$$

□

Review Questions 2.12. *What's the equation of the tangent line to the curve* $y = x^3 - 3x^2 + 4x$ *at the curve's inflection point?*

Solution.

$$\begin{aligned} y &= x^3 - 3x^2 + 4x \\ y' &= 3x^2 - 6x + 4 \\ y'' &= 6x - 6 \end{aligned}$$

This gives us inflection point of $x = 1$, so just plug that into $y = f(x)$ to find the (x, y) point that that defines the line and the slope is given by $y'(1) = 1$ so that $(y(1) = 1 - 3 + 4)$:

$$y - y_0 = x - x_0 \iff y = x - 1 + 2 = x + 1 \iff \boxed{y = x + 1}.$$

□

Review Questions 2.13. *What's the slope of the tangent line to the curve* $xy(x + y) = x + y^4$ *at the point* $(1, 1)$*?*

Proof. Implicit differentiation gives us

$$2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = 1 + 4\frac{dy}{dx}y^3$$

Plugging in $x = 1$ and $y = 1$:

$$2 + \frac{dy}{dx} + 1 + 2\frac{dy}{dx} = 1 + 4\frac{dy}{dx} \iff \frac{dy}{dx} = \boxed{2}$$

□

Review Questions 2.14. *If* $f(x) = 2|x - 1| + (x - 1)^2$*, what's the value of* $f'(0)$*?*

Solution.

$$f'(x) = 2\frac{|x-1|}{x-1} + 2(x-1) \iff f'(0) = 2\frac{1}{-1} + 2(-1) = \boxed{-4}.$$

□

Review Questions 2.15. *If*

$$f(x) = \frac{e^x \arccos(x)}{\cos(x)}$$

then the slope of the line tangent to the graph of f *at its* $y-$*intercept is...*

Proof. The $y-$intercept is just the value $y(0)$, so that the $y-$intercept. The only preliminary is what is the derivative of $\arccos(x)$?

Well setup the equation $x = \cos(y)$ with $y = \arccos(x)$, so that differentiating both sides: $1 = \frac{dy}{dx}(-\sin(y))$. Note the equation $x = \cos(y)$ defines a triangle with an adjacent side of x and the hypotenuse of 1, the opposite side is then $\sqrt{1-x^2}$. So that $-\frac{1}{\sin(y)} = \frac{-1}{\sqrt{1-x^2}}$ hence: $\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$.

Finally, we can use the product rule to find the slope of the tangent line at $x = 0$:

$$\begin{aligned}\frac{d}{dx}\frac{\arccos(x)e^x}{\cos(x)} &= \frac{e^x\arccos(x)}{\cos(x)} + e^x\left(-\frac{\cos(x)}{\sqrt{1-x^2}} + \frac{\sin(x)\arccos(x)}{(\cos(x))^2}\right) &= \frac{1\arccos(0)}{\cos(0)} + \frac{e^0(-1)}{\cos(0)} \\ &= \frac{\pi}{2} - 1.\end{aligned}$$

This is our answer. ☐

Review Questions 2.16. *Let $y = \frac{1}{\sqrt{x^3+1}}$. If x increases from 2 to 2.09, which of the following closely approximates the change in y.*

Solution. $\Delta y \approx f'(x)\Delta x$, here $\Delta x = 0.09$, so we just need the value of $f'(x)$ at $x = 2$.

$$\frac{d}{dx}\frac{1}{\sqrt{x^3+1}} = \frac{-1}{2}\frac{3x^2+1}{(x^3+1)^{3/2}} = \frac{-1}{2}\frac{13}{27}.$$

So that $\Delta y \approx 9\times 10^{-2}\frac{-13}{2(27)} = \frac{-13}{6}\times 10^{-2} \approx -2\times 10^{-2} = -0.02$. ☐

Review Questions 2.17. *If $f(1) = 1$ and $f'(1) = -1$, then the value of $\frac{d}{dx}\left[\frac{f(x^3)}{xf(x^2)}\right]$ at $x = 1$ is equal to...*

Solution. Using the product and chain rule of differentiation:

$$\frac{3x^2f'(x^2)}{xf(x^2)} + f(x^3)\frac{d}{dx}(xf(x^2))^{-1}$$

and $\frac{d}{dx}(xf(x^2))^{-1} = \frac{(f(x^2)+2x^2f'(x^2))(-1)}{(xf(x^2))^2}$ so that plugging in $x = 1$ we'll get:

$$-3 - 1(1-2) = -3 + 1 = \boxed{-2}.$$

☐

Review Questions 2.18. *If n is a positive integer, what's the value of the n^{th} derivative of $f(x) = \dfrac{1}{1-2x}$ at $x = \dfrac{-1}{2}$?*

Solution. Let's make a table:

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{-1}{2}\right)$
1	$\frac{2}{(1-2x)^2}$	$\frac{1}{2}$
2	$\frac{(-2)^2(-1)(-2)}{(1-2x)^3}$	$\frac{2(1)}{2}$
3	$\frac{(-2)^3(-1)(-2)(-3)}{(1-2x)^3}$	$\frac{(1)(2)(3)}{2}$
$\vdots$	$\vdots$	$\vdots$

□

Review Questions 2.19. *Let $f(x)$ be continuous on a bounded interval, $[a,b]$, where $a \neq b$, such that $f(a) = 1$ and $f(b) = 3$, and $f'(x)$ exists for every $x \in (a,b)$. What does the Mean-Value theorem say about f?*

Solution. This would give us that there exists a $c \in (a,b)$ such that $(b-a)f'(c) = f(b) - f(a)$ so that $(b-a)f'(c) = 3 - 1 = 2$. □

Review Questions 2.20. *What's the maximum area of rectangle inscribed in a semicircle of radius a?*

Solution. We can fix the origin in such a way that this semicircle is above the x-axis and that the full circle is described by the solutions to the equation

$$x^2 + y^2 = a^2.$$

So that our constraints here are that $-a \leq x \leq a$ and $0 \leq y \leq a$, that is the upper semicircle of this circle. Now the equation of y is:

$$y = \sqrt{a^2 - x^2}.$$

Additionally, any rectangle inscribed on the semi-circle has vertices of $(-x, y), (x, y), (-x, 0), (x, 0)$. So that the area of the rectangle is:

$$A(x) = 2x\sqrt{a^2 - x^2}.$$

We now differentiate this and solve for its extreme points:

$$A'(x) = \frac{2(a^2 - x^2) - 2x^2}{\sqrt{a^2 - x^2}} = 0.$$

We can assume that $y = 0$ isn't an extremum of the area since this would give us an area of 0. So that

$$0 = \frac{2(a^2 - x^2) - 2x^2}{\sqrt{a^2 - x^2}} \iff 2x^2 + x - 2a^2 = 0 \iff x = \frac{\pm a}{\sqrt{2}}.$$

Plugging this value into the function $A(x)$:

$$A\left(\frac{a}{\sqrt{2}}\right) = \frac{2a}{\sqrt{2}}\left(a^2 - \frac{a^2}{2}\right)^{1/2} = \boxed{a^2}.$$

□

Review Questions 2.21. *The following function is defined for all positive x:*

$$f(x) = \int_x^{2x} \frac{\sin(t)}{t}\, dt$$

At what value of $x \in \left(0, \frac{3\pi}{2}\right)$ does this function attain a local maximum?

Solution. We'll use the extreme value theorem here and the fundamental theorem of calculus. First let $F'(x) = \frac{\sin(x)}{x}$ so that:

$$\frac{d}{dx}\int_x^{2x} \frac{\sin(t)}{t}\, dt = 2F'(2x) - F'(x) = 0$$

$$\iff 2\frac{\sin(2x)}{2x} - \frac{\sin(x)}{x} = 0 \iff \frac{2\sin(x)\cos(x) - \sin(x)}{x} = 0$$

We can't immediately rule out $x \to 0$, but since we want to find $x \in \left(0, \frac{3\pi}{2}\right)$ that attains a local maximum. So $2\sin(x)\cos(x) = \sin(x)$, note that $\sin(x) = n\pi$ for $n = 0, \pm1, \pm2, \ldots$ so that again since $x \in \left(0, \frac{3\pi}{2}\right)$ we'll rule out $\sin(x) = 0$ so that:

$$\cos(x) = \frac{1}{2}$$

this occurs at $x = 60^o \times \frac{\pi}{180^o} = \frac{\pi}{3}$. So that this is our local maximum, $\boxed{\frac{\pi}{3}}$. □

Review Questions 2.22. *Let $f(x) = x^k e^{-x}$, where k is a positive constant. For $x > 0$, what's the maximum value attained by f?*

Solution. Again an extreme value problem:

$$kx^{k-1}e^{-x} - x^k e^{-x} = 0 \iff x^{k-1}e^{-x} = x^k e^{-x}$$

so since $e^{-x} \neq 0$ and $x^{k-1} \neq 0$ since $x > 0$ and any positive power of a positive number is positive and non-zero, we get that a solution is $x = k$. To verify that this is a maximum, we can take the second derivative:

$$k((k-1)e^{-x}x^{k-2} - xkx^{k-1}e^{-x}) - (kx^{k-1}e^{-x} - x^k e^{-x})\Bigg|_{x=k} < 0.$$

Hence is maximum. So then the maximum value of f is:

$$f(k) = \boxed{\frac{k^k}{e^k}}.$$

□

Review Questions 2.23. *The radius of a circle is decreasing at a rate of* 0.5 *cm per second. At what rate, in cm*^{2}*/sec, is the circle's area decreasing when the radius is* 4 *cm?*

Solution. This is a related rates problem, recall the area of a circle is: $A = \pi r^2$ so that:

$$\frac{dA}{dt} = \pi \frac{dr}{dt} 2r.$$

Plugging in the values of $\frac{dr}{dt} = 0.5$ and $r = 4$ will give us:

$$\frac{dA}{dt} = \pi \frac{1}{2} 24 = \boxed{4\pi}.$$

□

Review Questions 2.24. *The function* $f(x) = \int_{e^x}^{e^{2x}} t\log(t)\ dt$ *has an absolute minimum at* $x = 0$*, and a local maximum at* $x = \ldots$.

Solution. Extreme Value Theorem and Fundamental Theorem of Calculus we'll get (setup $F'(x) = x\log(x)$):

$$\frac{d}{dx}f(x) = e^{2x}2x - e^x x = 0 \iff x(2e^{2x} - e^x) = 0.$$

Since $x = 0$ is the minimum, we'll conclude that the solution to $2e^{2x} = e^x$ is the maximum:

$$2e^x = 1 \iff \boxed{x = -\log(2)}.$$

□

Review Questions 2.25. *Evaluate the following integral:*

$$\int_{-1}^{0} x^2(x+1)^3\ dx$$

Proof. Setup a u sub of $u = x + 1$ giving us bounds $[0, 1]$ and $x = u - 1$ so that:

$$\int_0^1 (u-1)^2 u^3 \, du = \int_0^1 u^3(u^2-2u+1) \, du = \frac{u^6}{6} - \frac{2}{5}u^5 + \frac{1}{4}u^4 \Bigg|_{u=0}^{u=1} = \frac{1}{6} - \frac{2}{5} + \frac{1}{4} = \frac{5-12}{30} + \frac{1}{4} = \frac{-7}{30} + \frac{1}{4} = \frac{-28+30}{120} = \boxed{-\frac{1}{60}}.$$

□

Review Questions 2.26. *If $\lfloor x \rfloor$ denotes the greatest integer $\leq x$, then $\int\limits_0^{7/2} \lfloor x \rfloor \, dx =$...*

Solution. Note that $\lfloor x \rfloor = 3$ when $x = \frac{7}{2}$. Note that $\lfloor x \rfloor = 0$ for all $0 \leq x < 1$ and $\lfloor x \rfloor = 1$ for all $1 \leq x < 2$ and $\lfloor x \rfloor = 2$ for all $2 \leq x < 3$ and $\lfloor x \rfloor = 3$ for all $3 \leq x \leq \frac{7}{2}$. Then we can just calculate the sum:

$$(1-0)0 + (2-1)1 + (3-2)2 + \frac{3}{2} = 4 + \frac{1}{2} = \boxed{\frac{9}{2}}.$$

□

Review Questions 2.27. *If*

$$f(x) = \begin{cases} -2(x+1) & \text{if } x \leq 0 \\ k(1-x^2) & \text{if } x > 0 \end{cases},$$

then the values of k for which $\int\limits_{-1}^{1} f(x) \, dx = 1$.

Solution.

$$\begin{aligned} \int_{-1}^0 -2(x+1) \, dx + \int_0^1 k(1-x^2) \, dx &= 1 \\ -2\left(\frac{x^2}{2} + x\right)\Bigg|_{x=-1}^{0} + k\left(x - \frac{x^3}{3}\right)\Bigg|_{x=0}^{1} &= 1 \\ -2(0 - (1/2 - 1)) + k(1 - 1/3) &= 1 \\ -1 + \frac{2k}{3} &= 1 \\ k &= \boxed{3}. \end{aligned}$$

□

Review Questions 2.28. *Integrate*

$$\int \frac{x^2}{\sqrt{1-x^2}}\ dx$$

Solution. Make a substitution of $\sin(\theta) = x$ with $dx = \cos(\theta)\ d\theta$, so that:

$$\begin{aligned}\int \frac{x^2}{\sqrt{1-x^2}}\ dx &= \int \frac{\sin^2(\theta)\cos(\theta)\ d\theta}{\cos(\theta)} \\ &= \int \sin^2\theta\ d\theta \\ &= \frac{1}{2}\int 1-\cos(2\theta)\ d\theta \\ &= \frac{1}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C\end{aligned}$$

To change back to x we can note that $\sin(\theta) = x$ defines a triangle with an opposite side of x and hypotenuse of 1 and adjacent of $\sqrt{1-x^2}$ so that:

$$\frac{1}{2}\left(\arcsin(x) - \sin(\theta)\cos(\theta)\right) = \boxed{\frac{1}{2}(\arcsin(x) - x\sqrt{1-x^2}) + C}.$$

□

Review Questions 2.29. *What's the area of the region in the first quadrant bounded by the curve $y = x\arctan(x)$ and the line $x = 1$?*

Solution. Since this is the first quadrant $x \geq 0$ and $y \geq 0$, so that this is the integral:

$$\int_0^1 x\arctan(x)\ dx.$$

Make a trig substitution of $\theta = \arctan(x) \iff x = \tan(\theta)$ with $dx = \sec^2(\theta)\ d\theta$ our bounds also become $\left[0, \frac{\pi}{4}\right]$:

$$\int_0^{\pi/4} \theta\tan(\theta)\sec^2(\theta)\ d\theta$$

Now integration by parts with $u = \theta$ and $du = d\theta$ and $dv = \frac{\sin(\theta)}{\cos^3(\theta)}d\theta$. We now have to evaluate:

$$\int \frac{\sin(\theta)}{\cos^3(\theta)}d\theta$$

with $u = \cos(\theta)$ and $du = -\sin(\theta)\ d\theta$ we have:

$$\int -u^{-3}\ du = \frac{\cos^{-2}(\theta)}{2} = v.$$

So using the integration by parts formula:

$$\left.\frac{\theta}{2\cos^2(\theta)}\right|_{\theta=0}^{\theta=\pi/4} - \int_0^{\pi/4} \frac{1}{2\cos^2(\theta)}\ d\theta$$

Now we can use the triangle that θ defines, with opposite side x, adjacent 1, and hypotenuse $\sqrt{x^2+1}$ to make this integral easier and $d\theta = \frac{dx}{x^2+1}$, additionally the bounds will become $[0,1]$ since $\tan(0) = 0$ and $\tan(1) = \frac{\pi}{4}$ so that:

$$\frac{1}{2}2\frac{\pi}{4} - \frac{1}{2}\int_0^1 1dx = \frac{\pi}{4} - \frac{1}{2} = \boxed{\frac{\pi-2}{4}}.$$

□

Review Questions 2.30. *Simplify the following:*

$$\exp\int_3^5 \frac{dx}{x^2-3x+2}$$

[Note: Recall that $\exp(x)$ *is a standard, alternative notation for* e^x*.]*

Solution. First, we'll evaluate the integral $\int\limits_3^5 \frac{dx}{x^2-3x+2}$ with partial fraction decomposition. By the quadratic formula:

$$x = \frac{+3 \pm \sqrt{9-8}}{2} = 1, 2$$

so that:

$$\frac{1}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2} \iff 1 = A(x-2) + B(x-1) \iff 1 = -2A - B\&0 = A + B$$

solving this system of equations will give us $A = -1$ and $B = 1$ so that we have:

$$\int_3^5 \frac{dx}{x^2-3x+2} = \int_3^5 \frac{-dx}{x-1} + \int_3^5 \frac{dx}{x-2}$$

with a simple $u = x-1$ and $u = x-2$ sub we'll get that these evaluate to:

$$-\log(4) + \log(2) + \log(3) + 0 = \log\frac{3}{2}$$

so that exponentiating that integral will just give us:

$$\boxed{\frac{3}{2}}.$$

□

Review Questions 2.31. *Calculate the area of the region in the first quadrant bounded by the graphs of $y = 8x, y = x^3$ and $y = 8$.*

Solution. With everyone of these problems, it's best to draw a picture first. If you do you'll see that we can just integrate this as follows:

$$\int_0^1 (8x - x^3)\ dx + \int_1^2 (8 - x^3)\ dx = 4x^2 - \frac{x^4}{4}\Bigg|_{x=0}^1 + \left(8x - \frac{x^4}{4}\right)\Bigg|_{x=1}^2 = \boxed{8}.$$

□

Review Questions 2.32. *Which of the following expressions give the area of the region bounded by the two circles pictured below? On circle is above the $x-$axis with $r = \sqrt{3}\sin(\theta)$ as its polar representation, and $r = 3\cos(\theta)$ is the other circle that is to the right of the $y-$axis one point at the origin.*

Solution. This is a tricky problem. So note, that for the second horizontal circle, that the origin will actually occur at $\theta = \dfrac{\pi}{2}$, while the upright circle this will occur at $\theta = 0$.

So we need to find their second intersection point. This will occur when $(x, y) = (x', y')$ of the two circles, and remembering the polar-to-rectangular and vice-versa formulas:

$$\sqrt{3}\sin(\theta)\cos(\theta) = 3\cos(\theta)\cos(\theta) \iff \tan(\theta) = \sqrt{3}$$

The acute solution to this, i.e $0 \le \theta \le 90^o$, will occur at $\theta = \dfrac{\pi}{3}$.

The tricky part of this problem then comes in noting that the bisected section in the picture can be partitioned such that one part of the area is just $\int_{\pi/3}^{\pi/2} \dfrac{1}{2}(9\cos^2(\theta)\ d\theta$ and the other is $\int_0^{\pi/3} \dfrac{1}{2}(\sqrt{3}\sin(\theta))^2\ d\theta$. Adding these two up will give you the area. This would be:

$$\boxed{\int_0^{\pi/3} \frac{1}{2}(\sqrt{3}\sin(\theta))^2\ d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(3\cos(\theta))^2\ d\theta}$$

□

Review Questions 2.33. *Let a and b be positive numbers. The region in the second quadrant bounded by the graphs of $y = ax^2$ and $y = -bx$ is revolved around the $x-$axis. Which of the following relationships between a and b would imply that the volume of this solid of revolution is a constant, independent of a and b?*

Solution. This is in the second quadrant so that $y \ge 0$ and $x \le 0$. Additionally the two lines will have an intercept at $x = 0$, so finding the other one i.e $x \neq 0$, we'll get $ax^2 = -bx \iff ax = -b \iff x = \dfrac{-b}{a}$.
Additionally, note that $-bx \ge ax^2 \iff -b \le ax \iff \dfrac{-b}{a} \le x$, which is true for the bounds of integration $-b/a \to 0$. So this gives us the integral:

$$\pi\int_{-b/a}^0 (-ax)^2 - (bx^2)^2\ dx = -\pi\left(\frac{-b^5}{3a^2} + \frac{a^2b^5}{5a^5}\right) = \pi\frac{+2b^5}{15a^3}$$

So that this independent of a, b when $\boxed{2b^5 = a^3}$, or even when $b^5 = a^3$. Actually, any $k \neq 0$ such that $kb^5 = a^3$ would make this independent of both a and b. □

Review Questions 2.34. *The region bounded by the graphs of $y = x^2$ and $y = 6 - |x|$ is revolved around the $y-$axis. What's the volume of the generated solid?*

Solution. Note, here we have to find the function $g(y) = x$ for both curves and then find the revolved solids volume for those curves because of the shape of the solid this is simpler than just integrating these two's squared difference. We'll get the curves $x = 6 - y$ and $x = \sqrt{y}$ for $x \geq 0$, we need to integrate these from $y = 0$ to $y = 4$ for $\sqrt{y}$, and then $y = 4$ to $y = 6$ for $x = 6 - y$:

$$\pi \int_0^4 y\, dy + \pi \int_4^6 36 - 12y + y^2\, dy = 8\pi + \pi \left(36y - 6y^2 + \frac{y^3}{3}\right)\Bigg|_{y=4}^{6} = 8\pi + \pi \left(36(6-4) - 6(36-16) + \frac{1}{3}(216-64)\right)$$

$$= 8\pi + \pi\left(72 - 120 + \frac{152}{3}\right) = 8\pi + \pi(-48 + 50 + 2/3) = 10\pi + \frac{2\pi}{3} = \boxed{\frac{32\pi}{3}}.$$

□

Review Questions 2.35. *Calculate the length of the portion of the hypercycloid $x^{2/3} + y^{2/3} = 1$.*

Solution.

$$\begin{aligned}
y^{2/3} &= 1 - x^{2/3} \\
y &= (1 - x^{2/3})^{3/2} \\
\frac{dy}{dx} &= \frac{2}{3}x^{-1/3}\frac{3}{2}(1 - x^{2/3})^{1/2} \\
\frac{dy}{dx} &= x^{-1/3}(1 - x^{2/3})^{1/2} \\
\sqrt{1 + ()^2} &= \sqrt{1 + x^{-2/3} - 1} = \frac{1}{x^3} \\
\int_{1/8}^{1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx &= \int_{1/8}^{1} x^{-1/3}\, dx \\
&= \frac{x^{2/3}}{2/3}\Bigg|_{x=1/8}^{1} = \frac{3}{2}\left(1 - \frac{1}{4}\right) = \boxed{\frac{9}{8}}
\end{aligned}$$

□

Review Questions 2.36. *What positive value of a satisfies the following equation?*

$$\int_e^{a^e} \frac{dx}{x \int_a^{ax} \frac{dy}{y}} \, dx$$

Solution. First:

$$\int_a^{ax} \frac{dy}{y} = \log(ax) - \log(a) = \log(x)$$

Then $\int\limits_e^{a^e} \frac{dx}{x\log(x)}$ is a $u-$sub with $u = \log(x)$ with bounds $[1, e\log(a)]$ and $du = \frac{dx}{x}$ so that:

$$\int_e^{a^e} \frac{dx}{x\log(x)} = \int_1^{e\log(a)} \frac{du}{u} = \log(e\log(a)).$$

So we get the equation:

$$\log(e\log(a)) = 1 \iff e\log(a) = e \iff \log(a) = 1 \iff \boxed{a = e}.$$

□

Review Questions 2.37. *Evaluate the following limit:*

$$\lim_{x\to 0} (\cos(x))^{\cot^2(x)}$$

Solution. Notice, this is an indeterminant form of the form 1^∞ since $\cot(x) \to \infty$ as $x \to 0$. So L'Hoptial!

$$\begin{aligned}
y = \lim_{x\to 0} (\cos(x))^{\cot^2(x)} &\iff \log(y) = \lim_{x\to 0} \cot^2(x)\log(\cos(x)) \\
&\iff \log(y) = \lim_{x\to 0} \frac{\log(\cos(x))\cos^2(x)}{\sin^2(x)} \\
&\iff \log(y) = \lim_{x\to 0} \frac{-2\cos(x)\sin(x)\log(\cos(x)) + \cos^2(x)\frac{-\sin(x)}{\cos(x)}}{2\cos(x)\sin(x)} \\
&\iff \log(y) = \lim_{x\to 0} -\frac{2\cos(x)\sin(x)\log(\cos(x)) + \sin(x)\cos(x)}{2\cos(x)\sin(x)} \\
&\iff \log(y) = \lim_{x\to 0} -\frac{2\log(\cos(x)) + 1}{2} \\
&\iff \log(y) = \lim_{x\to 0} -\frac{1}{2} \\
&\iff \log(y) = -\frac{1}{2} \\
&\iff y = e^{-1/2} \\
&\iff \boxed{y = \frac{1}{\sqrt{e}}}
\end{aligned}$$

□

Review Questions 2.38. *Let n be a number for which the improper integral*

$$\int_e^\infty \frac{dx}{x(\log(x))^n}$$

converges. Determine the value of the integral.

Solution. $u-$sub of $u = \log(x)$ with $[1, \infty)$ and $du = \dfrac{dx}{x}$:

$$\begin{aligned}\int_e^\infty \frac{dx}{x(\log(x))^n} &= \int_1^\infty \frac{du}{u^n} \\ &= \frac{u^{-n+1}}{-n+1}\Bigg|_{u=1}^{\infty} \\ &= \frac{1}{(1-n)u^{n-1}}\Bigg|_{u=1}^{\infty} \\ &= 0 + \boxed{\frac{1}{n-1}}\end{aligned}$$

□

Review Questions 2.39. *Find the positive value of a that satisfies the equation:*

$$\int_0^a \frac{dx}{\sqrt{a^2-x^2}} = \int_0^a \frac{x\ dx}{\sqrt{a^2-x^2}}$$

Solution. Making a trig-sub of $\sin(\theta) = \dfrac{x}{a}$ so that $dx = a\cos(\theta)\ d\theta$ and $x = a\sin(\theta)$, the bounds will be $\theta = [0, \pi/2]$ so that:

$$\int_0^{\pi/2} \frac{a\cos(\theta)}{a\cos(\theta)}\ d\theta = \int_0^{\pi/2} \frac{a\sin(\theta)a\cos(\theta)}{a\cos(\theta)}\ d\theta$$

So that:

$$\frac{\pi}{2} = a\int_0^{\pi/2} \sin(\theta)\ d\theta \iff \frac{\pi}{2a} = (-\cos\frac{\pi}{2} + 1 \iff \boxed{a = \frac{\pi}{2}}$$

□

Review Questions 2.40. *Which of the following improper integrals converge?*

I $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$

II $\displaystyle\int_1^{\infty} xe^{-x}\ dx$

III $\int\limits_0^2 \frac{dx}{(2-x)^2}$

Solution. I First, you'll want to evaluate the integrals:

$$\int_0^\infty \frac{dx}{(x^2+1)^2} \qquad \int_{-\infty}^0 \frac{dx}{(x^2+1)^2}$$

seperately as limits. You could identify the fact that the transformation: $x \mapsto -x$ doesn't change the integrand so that the two integrals will be the same. That is:

$$\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2} = 2\int_0^\infty \frac{dx}{(x^2+1)^2}.$$

Now to actually integrate this thing make a trig. substitution of $x = \tan(\theta)$ the bounds of $[0,\infty)$ informally become $\left[0, \frac{\pi}{2}\right)$ and $dx = \sec^2(\theta)d\theta$:

$$\int_0^{\pi/2} \frac{\sec^2(\theta)}{(1+\tan^2(\theta))^2}\ d\theta = \int_0^{\pi/2} \frac{d\theta}{\sec^2(\theta)} = \int_0^{\pi/2} \cos^2(\theta)\ d\theta = \frac{1}{2}\int_0^{\pi/2} 1-\cos(2\theta)\ d\theta$$

Make a $u-$substitution of $u = 2\theta$ with the bounds $[0,\pi)$ now and $du = 2d\theta$ so that we now have:

$$\frac{1}{4}\int_0^\pi 1-\cos(u)\ du = \frac{1}{4}\pi$$

So that

$$\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2} = 2\frac{\pi}{4}$$

don't forget about the symmetry that accounts for $2\times$ part above. This integral converges!

II This integral isn't that bad, just use integration by parts with $u = x, du = dx$ and $dv = e^{-x}\ dx, v = -e^{-x}$ so that:

$$\int_1^\infty xe^{-x}\ dx = -xe^{-x}\Big|_{x=1}^\infty + \int_1^\infty e^{-x}\ dx = e^{-1} + (-e^{-x})\Big|_{x=1}^\infty = 2e^{-1} < \infty$$

Hence this integral converges!

III This is just a $u-$sub, you can go about this carefully by taking a limit, but we'll play a little fast and loose here so, just let $u = 2-x$ with the bounds becoming $[2,0]$ and $du = -dx$ so that the integral will be:

$$\int_0^2 \frac{dx}{x^2} = \frac{x^{-1}}{-1}\Big|_0^2 = \frac{-1}{2} + \lim_{x\to 0^+} \frac{1}{x} = \infty$$

This last integral doesn't converge!

$\square$

Review Questions 2.41. *Which of the following infinite series converge?*

I

$$\sum_{n=1} \frac{\cos^4(\arctan(n))}{n\sqrt[4]{n}}$$

II

$$\sum_{n=2}^{\infty} \frac{1}{n\log(n)}$$

III

$$\sum_{n=0}^{\infty} \frac{(n+1)^3}{5(n+3)(n+2)(n+4)}$$

Solution. I Note that $-1 \leq \cos^4(\arctan(n)) \leq 1$ for all n and that $\frac{-1}{n^2} \leq \frac{1}{n^{5/4}} \leq \frac{+1}{n^2}$ so that using the comparison test we may say that this series converges since the series $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

II For this one, since both n and $\log(n)$ are monotonically increasing functions we can use the integral test with $u = \log(x)$ and $[0, \infty)$ as the bounds and $du = \frac{dx}{x}$

$$\int_1^{\infty} \frac{dx}{x\log(x)} = \int_0^{\infty} \frac{du}{u} = \log|u|\Big|_{u\to 0}^{\infty} = \infty - (-\infty) = \infty$$

So that this sum doesn't converge.

III For this last one, just the divergence test:

$$\lim_{n\to\infty} \frac{(n+1)^3}{5(n+2)(n+3)(n+4)} = \lim_{n\to\infty} \frac{n^3}{5n^3} = \frac{1}{5} \neq 0$$

So that this series is divergent by the divergence test!

□

Review Questions 2.42. *Find the smallest value of b that makes the following statement true:*

$$\textit{If } 0 \leq a < b, \textit{ then the series } \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} a^n \textit{ converges.}$$

Solution. This is actually just an integral of convergence test:

$$|a| \frac{((n+1)!)^2}{(2n+2)!} \frac{(2n)!}{(n!)^2} < 1$$

$$|a| \frac{n^2}{(2n+2)(2n+1)} < 1 \iff |a| < \frac{(2n+1)(2n+2)}{n^2} \iff |a| < \lim_{n\to\infty} \frac{4n^2}{n^2} = 4$$

So that this is $\boxed{a = 4}$.

□

Review Questions 2.43. *Evaluate the following limit:*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{k}{n^2}-\frac{k^2}{n^3}\right]$$

Solution.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[\frac{k}{n}-\left(\frac{k}{n}\right)^2\right]$$

Note that the Riemann integral can be defined as:

$$\lim_{N\to\infty}\frac{b-a}{N}\sum_{k=1}^{N}f(x_k) \qquad \text{With } x_k=\frac{k(b-a)}{N}$$

so that this limit is actually an integral with $b=1$ and $a=0$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{k}{n^2}-\frac{k^2}{n^3}\right]=\int_0^1 x-x^2\,dx=\frac{x^2}{2}-\frac{x^3}{3}=\frac{1}{2}-\frac{1}{3}=\boxed{\frac{1}{6}}$$

□

Review Questions 2.44. *Which of the following statements are true?*

I. If $a_n \geq 0$ for every n, then:

$$\sum_{n=1}^{\infty}a_n<\infty \implies \sum_{n=1}^{\infty}\sqrt{a_n}<\infty$$

II. If $a_n \geq 0$ for every n, then:

$$\sum_{n=1}^{\infty}na_n<\infty \implies \sum_{n=1}^{\infty}a_n<\infty$$

III. If $a_n \geq 0$ and $a_{n+1}\leq a_n$ for every n, then:

$$\sum_{n=1}^{\infty}a_n^2<\infty \implies \sum_{n=1}^{\infty}(-1)^n a_n<\infty$$

Solution. I. This is false. A counter-example is the p-series with $p=2$:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty \qquad \text{but} \qquad \sum_{n=1}^{\infty}\frac{1}{n}=\infty$$

II. This is true, by the comparison test. Since $a_n \leq na_n$ for all n and $\sum_{n=1}^{\infty}na_n<\infty$ we can conclude $\sum_{n=1}^{\infty}a_n<\infty$.

III. This is true by the alternating series test: Since $a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n^2 = 0$. From the latter we can conclude that $\lim_{n\to\infty} a_n = 0$ so that by the alternating series test $\sum_{n=1}^{\infty}(-1)^n a_n < \infty$.

□

Review Questions 2.45. *If $-1 < x < 1$, then $\sum_{n=1}^{\infty} x^{2n}$ =...*

Solution. First, we have

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \iff \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} \iff \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n \iff \boxed{\frac{x^2}{(1-x^2)^2}} = \sum_{n=1}^{\infty} nx^{2n}$$

□

Review Questions 2.46. *The smallest positive integer x for which the power series $\sum_{n=1}^{\infty} \frac{n!(2n)!}{(3n)!}x^n$ doesn't converge is...*

Solution. Again this is a radius of convergence problem:

$$|x|\frac{(n+1)!(2n+2)!}{(3n+3)!}\frac{(3n)!}{(n)!(2n)!} < 1 \iff |x|\frac{(n+1)(2n+1)(2n+2)}{(3n+1)(3n+2)(3n+3)} < 1$$

$$\iff |x| < \lim_{n\to\infty} \frac{(3n+1)(3n+2)(3n+3)}{(n+1)(2n+1)(2n+2)} \iff |x| < \frac{27}{4} = 6 + \frac{3}{4} < 7$$

So $\boxed{7}$ is the smallest positive integer where the series doesn't converge. □

Review Questions 2.47. *In the Taylor series expansion (in powers of x) of the function $f(x) = e^{x^2-x}$, what's the coefficient of x^3?*

Solution. Here we just have to find:

$$\frac{f'''(0)}{3!}$$

$$\begin{aligned}
f(x) &= e^{x^2-x} \\
f'(x) &= (2x-1)e^{x^2-x} \\
f''(x) &= 2e^{x^2-x} + (2x-1)^2 e^{x^2-x} \\
&= e^{x^2-x}(2 + 4x^2 - 4x + 1) \\
&= e^{x^2-x}(4x^2 - 4x + 3) \\
f'''(x) &= (2x-1)e^{x^2-x}(4x^2-4x+3) + (8x-4)e^{x^2-x} \\
f'''(0) &= (-1)(1)(3) + (-4)(1) \\
&= -7
\end{aligned}$$

So that the Taylor series coefficient of x^3 is:

$$\boxed{\frac{-7}{3!} = \frac{-7}{6}}$$

□

Review Questions 2.48. *If k_i $(i = 0, 1, 2, 3, 4)$ are constants such that $x^4 = k_0 + k_1(x+1) + k_2(x+1)^2 + k_3(x+1)^3 + k_4(x+1)^4$ is an identity in x, what's the value of k_3?*

Solution. You could expand this out and find the x^3 coefficient that isn't very efficient though, instead we can do this:

$$x^4 = k_0 \sum_{k=0}^{0} \binom{0}{k} x^k + k_1 \sum_{k=0}^{1} \binom{1}{k} x^k + k_2 \sum_{k=0}^{2} \binom{2}{k} x^k + k_3 \sum_{k=0}^{3} \binom{3}{k} x^k + k_4 \sum_{k=0}^{4} \binom{4}{k} x^k.$$

Setting the coefficients of the identity equal to each other gives us the system of equations:

$$\begin{cases} 0 &= k_0 + k_1 + k_2 + k_3 + k_4 \\ 0 &= k_1 + \binom{2}{1}k_2 + \binom{3}{1}k_3 + \binom{4}{1}k_4 \\ 0 &= k_2 + \binom{3}{2}k_3 + \binom{4}{2}k_4 \\ 0 &= k_3 + \binom{4}{3}k_4 \\ 1 &= k_4 \end{cases}$$

Immediately we have $k_4 = 1$, so just solve for k_3 in the second to last equation:

$$0 = k_3 + 4 \iff \boxed{k_3 = -4}$$

□

Review Questions 2.49. *If the function $f(x) = e^x$ is expanded in powers of x, what's the minimum number of terms of the Taylor series that must be used to ensure the resulting polynomial will approximate $\sqrt[5]{e}$ to within 10^{-6}?*

Solution. □