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Homework #8

MATH 550
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3.B.1

Give an example of a sequence $f_1, f_2, \dots$ of functions from $\mathbb{Z}^+$ to $[0, \infty)$ such that

$$\lim_{k\to\infty} f_k(m) = 0$$

for every $m \in \mathbb{Z}^+$ but $\lim\limits_{k\to\infty} \int f_k \, d\mu = 1$, where μ is counting measure on $\mathbb{Z}^+$.

Example. In the following we'll be working in the measure space of $(\mathbb{Z}^+, 2^{\mathbb{Z}^+}, \textit{counting})$, where *counting* is the counting measure.

Then note that every function is measurable in this space, since the σ-algebra is the power set of the entire space. Now define the sequence of functions $f_k(m) : \mathbb{Z}^+ \to \mathbb{Z}^+$ given by

$$f_k(m) = \chi_{\{k\}}(m)$$

for all $m \in \mathbb{Z}^+$ and $k \in \mathbb{Z}^+$. Then for any $k \in \mathbb{Z}^+$ we'll have

$$\int f_k(m) \; d\textit{counting} = \int \chi_{\{k\}}(m) \; d\textit{counting} = \textit{counting}(\{k\}) = 1,$$

for all $m \in \mathbb{Z}^+$. So that $\lim\limits_{k\to\infty} f_k(m) = 1$ for all $m \in \mathbb{Z}^+$. But taking the limit we'll get:

$$\lim_{k\to\infty} \chi_{\{k\}}(m) = \chi_{\{\infty\}}(m) = 0,$$

for all $m \in \mathbb{Z}^+$. □

3.B.2

Give an example of a sequence $f_1, f_2, \ldots$ of continuous functions from $\mathbb{R}$ to $[0,1]$ such that

$$\lim_{k\to\infty} f_k(x) = 0$$

for every $x \in \mathbb{R}$ but $\lim\limits_{k\to\infty} \int f_k \, d\mu = \infty$, where λ is Lebesgue measure on $\mathbb{R}$.

Proof. Let $(\mathbb{R}, \mathcal{L}, \lambda)$ be the measure space of Lebesgue measurable sets in $\mathbb{R}$. Consider the sequence of functions $f_k : \mathbb{R} \to [0,1]$ given by

$$f_k(x) = \frac{1}{k}\chi_{\mathbb{R}}(x)$$

for all $k \in \mathbb{Z}^+$ and $x \in \mathbb{R}$. Since $\chi_{\mathbb{R}}$ is a characteristic function of a Lebesgue measurable set, we have that f_k is Lebesgue measurable for all $k \in \mathbb{Z}^+$.

Then we have:

$$\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} \frac{1}{k}\infty = 0 \cdot \infty = 0$$

for all $x \in \mathbb{R}$. Finally, we also get:

$$\int f_k \, d\lambda = \int \frac{1}{k}\chi_{\mathbb{R}} \, d\lambda = \mu(\mathbb{R})\frac{1}{k} = \infty,$$

for all $k \in \mathbb{Z}^+$. Hence $\lim\limits_{k\to\infty} \int f_k \, d\lambda = \infty$. □

3.B.5

Let λ denote Lebesgue measure on $\mathbb{R}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function such that $\int |f| \; d\lambda < \infty$. Prove that

$$\lim_{k\to\infty} \int_{[-k,k]} f \; d\lambda = \int f \; d\lambda.$$

Proof. Let λ denote Lebesgue measure on $\mathbb{R}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function such that $\int |f| \; d\lambda < \infty$.

We'll show our conclusion by the dominated convergence theorem. We just need to show that f satisfies the hypotheses of this theorem. Then note that f is automatically Lebesgue measurable since for every Borel set B we have that $f^{-1}(B)$ is a Borel set and hence is a Lebesgue set, thus f is a Lebesgue measurable function on $\mathbb{R}$. Define the sequence of function $f_k : \mathbb{R} \to \mathbb{R}$ given by

$$f_k(x) = (\chi_{[-k,+k]}f)(x)$$

for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}^+$. By this definition we'll also get:

$$\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} (\chi_{[-k,+k]}f)(x) = (\chi_{\mathbb{R}}f)(x) = f(x),$$

for all $x \in \mathbb{R}$.

Note then that $|f_k(x)| = |(\chi_{[-k,+k]}f)(x)| \leq |f(x)|$ for all $x \in \mathbb{R}$, by our hypothesis we also have that $\int |f| \; d\lambda < \infty$. Hence employing the dominated convergence theorem gives us:

$$\begin{aligned}
\lim_{k\to\infty} \int_{[-k,k]} f \; d\lambda &= \lim_{k\to\infty} \int \chi_{[-k,+k]}f \; d\lambda \\
&= \int \lim_{k\to\infty} (\chi_{[-k,+k]}f) \; d\lambda \\
&= \int \chi_{\mathbb{R}}f \; d\lambda \\
&= \int f \; d\lambda.
\end{aligned}$$

Which is the result we wanted to prove. $\square$

3.B.6

Let λ denote Lebesgue measure on $\mathbb{R}$. Give an example of a continuous function $f : [0, \infty) \to \mathbb{R}$ such that $\lim\limits_{t\to\infty} \int_{[0,t]} f \, d\lambda$ exists (in $\mathbb{R}$) but $\int_{[0,\infty)} f \, d\lambda$ isn't defined.

Example. Let λ denote Lebesgue measure on $\mathbb{R}$.

Define the function:

$$f(x) = \begin{cases} \dfrac{\sin(x)}{\sqrt{x}} & \text{if } x > 0, \\ 0 & x = 0. \end{cases}$$

Note that we have $\lim\limits_{x\to 0} \dfrac{\sin(x)}{\sqrt{x}} = \dfrac{\lim\limits_{x\to 0} \cos(x)}{\lim\limits_{x\to 0} \frac{1}{2}\frac{1}{\sqrt{x}}} = 0$, so that since f is continuous on $x > 0$, and we have $\lim\limits_{x\to 0} f(x) = 0$, we can conclude f is continuous on $[0, \infty)$.

Now, note that since this function is continuous on its, its Riemann integral is defined on any bounded subset of its domain. So that the Riemann integral and Lebesgue integral's will agree on $[0, t]$ for any $t > 0$. Furthermore, we'll have that:

$$f^+(x) = \begin{cases} \dfrac{\sin(x)}{\sqrt{x}} & \text{if } x \in [0, \pi] \cup [2\pi, 3\pi] \cup \dots \\ 0 & \text{otherwise} \end{cases}$$

and that

$$f^-(x) = \begin{cases} -\dfrac{\sin(x)}{\sqrt{x}} & \text{if } x \in (\pi, 2\pi) \cup (3\pi, 4\pi) \cup \dots \\ 0 & \text{otherwise} \end{cases}.$$

So to show that the limit $\lim\limits_{t\to\infty} \int f \, d\lambda < \infty$, we will bound the limit with a convergence series. We'll show that separately the integrals: $\int_{[0,t]} f^+ \, d\lambda$ and $\int_{[0,t]} f^- \, d\lambda$ are bounded by a sum, and then combine these and take a limit to show that $\lim\limits_{t\to\infty} \int_{[0,t]} f \, d\lambda < \infty$.

To bound $\int_{[0,t]} f^+ \, d\lambda$ we'll start off with example and then show that it's bounded by a sum $\sum a_n$. First, note that since f^+ is positive only on intervals such as $[0, \pi], [2\pi, 3\pi], \dots$, so we just need to evaluate the integral on these intervals. Now consider the following:

$$0 \leq \int_0^\pi \frac{\sin(x)}{\sqrt{x}} dx \leq \int_0^\pi 2 = 2\pi,$$

then

$$\frac{2}{\sqrt{3\pi}} = \int_{2\pi}^{3\pi} \frac{\sin(x)}{\sqrt{3\pi}} \leq \int_{2\pi}^{3\pi} \frac{\sin(x)}{\sqrt{x}} dx \leq \int_{2\pi}^{3\pi} \frac{\sin(x)}{\sqrt{2\pi}} = \frac{2}{\sqrt{2\pi}}$$

similarly

$$\frac{2}{\sqrt{5\pi}} = \int_{4\pi}^{5\pi} \frac{\sin(x)}{\sqrt{5\pi}} \leq \int_{4\pi}^{5\pi} \frac{\sin(x)}{\sqrt{x}} dx \leq \int_{4\pi}^{5\pi} \frac{\sin(x)}{\sqrt{4\pi}} dx = \frac{2}{\sqrt{4\pi}}$$

. In fact doing this for any $k \in \mathbb{N}$, we'll find:

$$\frac{2}{\sqrt{(2k+1)\pi}} = \int_{2k\pi}^{(2k+1)\pi} \frac{\sin(x)}{\sqrt{(2k+1)\pi}} \leq \int_{2k\pi}^{(2k+1)\pi} \frac{\sin(x)}{\sqrt{x}} dx \leq \frac{2}{\sqrt{2k\pi}}.$$

So then, since there must exist a $N \in \mathbb{N}$ (taking advantage of f^+ being zero every other than intervals such as $[2k\pi, (2k+1)\pi]$) such that:

$$\int_{[0,t]} f^+ \, d\lambda = \int_{[0,N\pi]} f^+ \, d\lambda$$

we can bound this as follows (without loss generality, assume that N is even, switch $N-1$ and N if N is odd):

$$\sum_{k=1}^{N-1} \frac{2}{\sqrt{(2k+1)\pi}} \leq \int_{[0,N\pi]} f^+ \, d\lambda \leq 2\pi + \sum_{k=1}^{N} \frac{2}{\sqrt{2k\pi}}.$$

Applying a similar process to f^-:

$$\frac{2}{\sqrt{2\pi}} = \int_{\pi}^{2\pi} \frac{-\sin(x)}{\sqrt{2\pi}} dx \leq \int_{\pi}^{2\pi} \frac{-\sin(x)}{\sqrt{x}} dx \leq \int_{\pi}^{2\pi} \frac{-\sin(x)}{\sqrt{\pi}} dx = \frac{2}{\sqrt{\pi}}$$

repeating this process for any off $k \in \mathbb{N}$ we'll have:

$$\frac{2}{\sqrt{(2k)\pi}} \leq \int_{(2k-1)\pi}^{(2k)\pi} -\frac{\sin(x)}{\sqrt{x}} \, dx \leq \frac{2}{\sqrt{(2k-1)\pi}}.$$

So that we can deduce similar to the previous case (without loss of generality assume N is even, and switch $N-1$ to N if N is odd)

$$-\sum_{k=1}^{N-1} \frac{2}{\sqrt{(2k-1)\pi}} \leq -\int_{[0,t]} f^- \, d\lambda = -\int_{[0,N\pi]} f^- \, d\lambda \leq -\sum_{k=1}^{N} \frac{2}{\sqrt{2k\pi}}.$$

Note then taking the limit as both $N \to \infty$ with these inequalities we'll get that:

$$\infty = \sum_{k=1}^{\infty} \frac{2}{\sqrt{2k\pi}} \leq \int_{[0,\infty)} f^- \, d\lambda$$

and

$$\infty = \sum_{k=1}^{\infty} \frac{2}{\sqrt{(2k+1)\pi}} \leq \int_{[0,\infty)} f^+ \ d\lambda.$$

Both of the sums above are divergent p-series. So that $\int_{[0,\infty)} f \ d\lambda$ isn't defined because both $\int_{[0,\infty)} f^- \ d\lambda$ and $\int_{[0,\infty)} f^+ \ d\lambda$ are infinite.
But then consider the following

$$\begin{aligned}
\int_{[0,t]} f \ d\lambda &= \int_{[0,N\pi]} f \ d\lambda \\
&= \int_{[0,N\pi]} f^+ \ d\lambda - \int_{[0,N\pi]} f^- \ d\lambda \\
&\leq 2\pi + \sum_{k=1}^{N} \frac{2}{\sqrt{2k\pi}} - \sum_{k=1}^{N} \frac{2}{\sqrt{2k\pi}} = 2\pi < \infty.
\end{aligned}$$

Thus we have that taking limit as $t \to \infty$ gives us that:

$$\lim_{t\to\infty} \int_{[0,t]} f \ d\lambda \leq 2\pi,$$

so that the limit exists in $\mathbb{R}$ $\square$

3.B.9

Verify the assertion in Example 3.41.

Example. *Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $E_1, \ldots, E_n$ are disjoint subsets of X. Suppose $a_1, \ldots, a_n$ are distinct nonzero real numbers. Then*

$$a_1\chi_{E_1} + \ldots + a_n\chi_{E_n} \in \mathcal{L}^1(\mu)$$

if and only if $E_k \in \mathcal{S}$ and $\mu(E_k) < \infty$ for all $k \in \{1, \ldots, n\}$. Furthermore,

$$\|a_1\chi_{E_1} + \ldots + a_n\chi_{E_n}\|_1 = |a_1|\mu(E_1) + \ldots + |a_n|\mu(E_n).$$

Proof. Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $E_1, \ldots, E_n$ are disjoint subsets of X. Suppose $a_1, \ldots, a_n$ are distinct nonzero real numbers.

Suppose

$$a_1\chi_{E_1} + \ldots + a_n\chi_{E_n} \in \mathcal{L}^1(\mu)$$

. Then by $2.B.13$ we have that $E_k \in \mathcal{S}$ for all $k\{1, \ldots, n\}$. Furthermore since this is a simple function we have that:

$$\left|\sum_{k=1}^{n} a_k\mu(E_k)\right| < \infty.$$

By the hypothesis that $a_1, \ldots, a_n$ are all nonzero and by the triangle inequality we get that $\mu(E_k) < \infty$ for all $k \in \{1, \ldots, n\}$.

Conversely, suppose $E_k \in \mathcal{S}$ and $\mu(E_k) < \infty$ for all $k \in \{1, \ldots, n\}$. Then we have by $2.B.13$ that $a_1\chi_{E_1} + \ldots + a_n\chi_{E_n}$ is $\mathcal{S}$-measurable. Furthermore, since each a_k are finite and $\mu(E_k) < \infty$ for all $k \in \{1, \ldots, n\}$, giving us:

$$\sum_{k=1}^{n} |a_k|\mu(E_k) < \infty.$$

Since we have that $\sum_{k=1}^{n} |a_k\mu(E_k)| \leq \left|\sum_{k=1}^{n} a_k\mu(E_k)\right| \leq \left|\int \sum_{k=1}^{n} a_k\chi_{E_k}\right| \leq \int \left|\sum_{k=1}^{n} a_k\chi_{E_k}\right| \, d\mu$, we have that $\sum_{k=1}^{n} a_k\chi_{E_k} \in \mathcal{L}^1(\mu)$.

To finish this off, note that the above inequality gives us that $|a_1|\mu(E_1) + \ldots + |a_n|\mu(E_n) \leq \int \left|\sum_{k=1}^{n} a_k\chi_{E_k}\right| \, d\mu$, the other direction is a nice application of the triangle inequality and integration preserving inequalities:

$$\int \left|\sum_{k=1}^{n} a_k\chi_{E_k}\right| \, d\mu \leq \int \sum_{k=1}^{n} |a_k|\chi_{E_k} \, d\mu = \sum_{k=1}^{n} |a_k|\mu(E_k).$$

Giving us that

$$\|a_1\chi_{E_1} + \ldots + a_n\chi_{E_n}\|_1 = |a_1|\chi_{E_1} + \ldots + |a_n|\chi_{E_n},$$

which is what we wanted to prove. $\square$

3.B.10

(a)

Suppose $(X, \mathcal{S}, \mu)$ is a measure space such that $\mu(X) < \infty$. Suppose p, r are positive numbers with $p < r$. Prove that if $f : X \to [0, \infty)$ is an $\mathcal{S}-$measurable function such that $\int f^r \, d\mu < \infty$, then $\int f^p \, d\mu < \infty$.

Proof. Suppose $(X, \mathcal{S}, \mu)$ is a measure space such that $\mu(X) < \infty$. Suppose p, r are positive numbers with $p < r$.

Now let $f : X \to [0, \infty)$ is an $\mathcal{S}-$measurable function such that $\int f^r \, d\mu < \infty$. Then define the set $E = f^{-1}([0, 1))$, so that $X \setminus E = f^{-1}([1, \infty))$. Moreover, since both $[1, \infty)$ and $[0, 1)$ are Borel sets, we have that $X \setminus E$ and E are in $\mathcal{S}$. So that we'll have $f^p(x) < f^r(x)$ for all $x \in E$, giving us:

$$\int_E f^p \, d\mu < \int_E f^r < \infty.$$

To finish off the proof, we'll use theorem (3.25) to give us:

$$\int_{X \setminus E} f^p \, d\mu \leq \mu(X \setminus E) \sup_{X \setminus E} f \leq \mu(X \setminus E) < \infty,$$

the second inequality comes from f^p attaining at most a value of 1 on $X \setminus E$, and then we use our hypothesis that to get $\mu(X \setminus E)$
$\mu(X) < \infty$. Combining these we get our result:

$$\int_E f \, d\mu + \int_{X \setminus E} f \, d\mu = \int_X f \, d\mu = \int f \, d\mu < \infty + \infty = \infty.$$

□

(b)

Give an example to show that the result in part(a) can be false without the hypothesis that $\mu(X) < \infty$.

Example. Let $(\mathbb{Z}^+, 2^{\mathbb{Z}^+}, counting)$ be the measure space on $\mathbb{Z}^+$ with the counting measure. Then we have that all sequence $a_n : \mathbb{Z}^+ \to [0, \infty)$ are $2^{\mathbb{Z}^+}-$measurable and that integration in this space is summation. Furthermore, $counting(\mathbb{Z}^+) = \infty$. Consider the sequence given by $a_n = \dfrac{1}{n}$. Then we have that $\int a_n^1 \ dcounting = \sum_{k=1}^{\infty} \dfrac{1}{k} = \infty$, as this is the harmonic series. But that $\int a_n^2 \ dcounting = \sum_{k=1}^{\infty} \dfrac{1}{k^2} < \infty$, as this is a convergence p-series. $\square$

1 3.B.11

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f \in \mathcal{L}^1(\mu)$. Prove that

$$\{x \in X : f(x) \neq 0\}$$

is the countable union of sets with finite $\mu-$measure.

Proof. Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f \in \mathcal{L}^1(\mu)$. Then we have that $\int |f| \, d\mu < \infty$ and that f is $\mathcal{S}$-measurable.

Note that $\{x \in X : f(x) \neq 0\} = |f|^{-1}((0, \infty])$, and that since f is $\mathcal{S}-$measurable then so is $|f|$. Note then that we can partition $|f|^{-1}((0, \infty])$ as follows:

$$|f|^{-1}((0,\infty]) = |f|^{-1}((0, \frac{1}{N}])\cup\bigcup_{k=1}^{N} |f|^{-1}((\frac{1}{k+1}, \frac{1}{k}])\cup\bigcup_{k=1}^{N} |f|^{-1}((k, k+1])\cup|f|^{-1}((N, \infty]). \quad (1)$$

Moreover, this is an $\mathcal{S}$-partition of X, as $|f|$ is $\mathcal{S}$-measurable. So that since $\int |f| \, d\mu < \infty$ we can conclude that for each $k \in \{1, \ldots, N\}$ that (define $A_k = (\frac{1}{k+1}, \frac{1}{k}]$ and $B_k = (k, k+1]$):

$$\inf_{A_k} \mu(A_k) \leq \frac{1}{k+1}\mu(A_k) \leq \mu(A_k) < \infty$$

and

$$\inf_{B_k} \mu(B_k) \leq (k+1)\mu(B_k) < \infty$$

both implying $\mu(A_k), \mu(B_k) < \infty$, since $k \neq -1$, for all $k \in \{1, \ldots, N\}$. Since our N was arbitrary in the above, for any $k \in \mathbb{N}$ we can choose an $N \geq k$ such that the above argument implies that $\mu(A_k) < \infty$ and that $\mu(B_k) < \infty$. So that $\mu(|f|^{-1}((\frac{1}{k+1}, \frac{1}{k}])) < \infty$ and $\mu(|f|^{-1}((k, k+1])) < \infty$ for all $k \in \mathbb{N}$.

Now we only need to show that $\mu(|f|^{-1}(\{\infty\})) < \infty$. Note that since $\{\infty\}$ is an extended Borel set, we can write an $\mathcal{S}$-partition as $(|f|^{-1}(\{\infty\}), |f|^{-1}((0, \infty)))$. Additionally, we would have $\inf_{|f|^{-1}(\{\infty\})} |f| = \infty$, so that for $\int |f| \, d\mu$ to remain finite, it follows that $\mu(|f|^{-1}(\{\infty\})) = 0$.

Because we can write $|f|^{-1}((0, \infty])$ in the form seen in 1, it follows that $\{x \in X : f(x) \neq 0\} = |f|^{-1}((0, \infty))$ is the countable union of sets with finite μ-measure. $\square$

3.B.14

Let λ denote Lebesgue measure on $\mathbb{R}$.

(a)

Let $f(x) = \dfrac{1}{\sqrt{x}}$. Prove that $\int_{[0,1]} f \; d\lambda = 2$.

Proof. Consider the following calculation where $\int_{[0,1]} f \; d\lambda$ is the Lebesgue integral of f on $[0,1]$ and $\int\limits_0^1 f \; dx$ is the Riemann integral of f on $[0,1]$.

$$\begin{aligned}\int_{[0,1]} f \; d\lambda &= \int_0^1 \frac{dx}{\sqrt{x}} \\ &= \frac{x^{1/2}}{1/2}\bigg|_0^1 = 2(1-0) = 2.\end{aligned}$$

□

(b)

Let $f(x) = \dfrac{1}{1+x^2}$. Prove that $\int_{\mathbb{R}} f \; d\lambda = \pi$.

Proof. Consider the following calculation for $k \in \mathbb{Z}^+$

$$\begin{aligned}\int_{[-k,+k]} f \; d\lambda &= \int_{-k}^{+k} \frac{dx}{1+x^2} \\ &= \arctan(x)\bigg|_{x=-k}^{x=+k} \\ &= \arctan(k) - \arctan(-k).\end{aligned}$$

Taking the limit as □

(c)

Let $f(x) = \dfrac{\sin(x)}{x}$. Show that the integral $\int_{(0,\infty)} f \; d\lambda$ isn't defined but $\lim\limits_{t\to\infty} \int_{(0,t)} f \; d\lambda$ exists in $\mathbb{R}$.

Proof. Let $f(x) = \dfrac{\sin(x)}{x}$.

Note that we will skip some of the work shown to get to some of the inequalities presented, but that the procedure to arrive at these is identical to the process laid out in 3.B.6. Note that:

$$f^+ = \begin{cases} \dfrac{\sin(x)}{x} & x \in \bigcup_{k=0}^{\infty}[2k\pi, (2k+1)\pi] \\ 0 & \text{otherwise} \end{cases}$$

and

$$f^- = \begin{cases} -\dfrac{\sin(x)}{x} & x \in \bigcup_{k=0}^{\infty}((2k+1)\pi, 2k\pi) \\ 0 & \text{otherwise} \end{cases}.$$

Using the identical reasoning as laid out in $3.B.6$ we will get the following inequalities: (for $N \in \mathbb{N}$, without loss of generality N being even, for $N\pi \le t$)

$$\sum_{k=1}^{N-1} \frac{2}{(2k+1)\pi} \le \int_{(0,t)} f^+ \, d\lambda \le 2\pi + \sum_{k=1}^{N} \frac{2}{2k\pi}$$

and

$$-\sum_{k=1}^{N-1} \frac{2}{(2k+1)\pi} \le -\int_{(0,t)} f^- \, d\lambda \le -\sum_{k=1}^{N} \frac{2}{2k\pi}.$$

Giving us

$$\int_{(0,t)} f \, d\lambda = \int_{(0,t)} f^+ \, d\lambda - \int_{(0,t)} f^- \, d\lambda \le 2\pi,$$

while

$$\infty = \sum_{k=1}^{\infty} \frac{2}{(2k+1)\pi} \le \int_{(0,\infty)} f^+ \, d\lambda$$

and

$$\infty = \sum_{k=1}^{\infty} \frac{2}{2k\pi} \le \int_{(0,\infty)} f^- \, d\lambda.$$

So that $\int_{(0,\infty)} f \, d\lambda$ isn't defined, while $\lim\limits_{t\to\infty} \int_{(0,t)} f \, d\lambda \le 2\pi$. $\square$

3.B.15

Prove or give a counterexample: If G is an open subset of $(0,1)$, then χ_G is Riemann integrable on $[0,1]$.

Proof. Consider the Smith-Volterra-Cantor Set. The construction is under taken by removing open middle fourths of the interval $[0,1]$. That is the first set in the construction is $C_1 = [0,3/8] \cup [5/8,1]$, the second is $C_2 = [0,5/32] \cup [7/32,3/8] \cup [5/8,25/32] \cup [27/32,1]$, we continue on in this manner for C_n. Then define $C = \bigcap_{n=1}^{\infty} C_n$. Then since C_n is closed for all $n \in \mathbb{N}$, we have that C is closed as it's the countable intersection of closed sets.

Now we'll find the measure of this set. Note that then that $[0,1] \setminus C_1 = \frac{1}{4}, [0,1] \setminus C_2 = \frac{1}{4} + \frac{1}{8}, [0,1] \setminus C_3 = \frac{1}{4} + \frac{1}{8} + \frac{1}{16}, \ldots$. Since this is an increasing sequence of sets, we can take the limit of these measures to get that: $\lambda([0,1] \setminus C) = 1 - \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 - \frac{2}{4} = \frac{1}{2}$. Since C is closed, the set $[0,1] \setminus C$ must be open and has measure $\frac{1}{2}$.

Finally, we'll show that this set is "Perfect"; that is every interval will either be disjoint from C or contain infinitely many points. So suppose that $C \cap (a,b) \neq \emptyset$. Then there exists a $x \in C$ such that $x \in (a,b)$. For any $x \in C$ we can define endpoints in the sequence (C_n) that will converge to x. Note that if $x \in C$, $x \in C_n$ for all $n \in \mathbb{N}$. Note that every time we remove the middle-fourth from every interval that x will either be in the left set or right set, every time we do this. For each cut, catalog the points that are in same side as x, call these sets A_n. Then $A_{n+1} = A_n \cap \{\text{points in the same interval as x in } C_{n+1}\}$. So that each A_{n+1} is a closed interval containing x. So in this way the sequence (A_n) is a decreasing sequence of closed intervals containing x. Now there must exists some $N \in \mathbb{N}$ such that for all $n \geq N$ we'll have $A_N \cap (a,b) \neq \emptyset$, otherwise x is an endpoint of some interval in C_n for all $n \in \mathbb{N}$. If x is an endpoint, then we can simply create a sequence of endpoints converging to x by continually removing the middle fourth of the sequences containing x. If x isn't an endpoint then $A_N \cap (a,b) \neq \emptyset$, and so that this intersection of a closed interval and open interval contains an infinite number of points. So that every new step that we make in removing the middle fourth for all $n \geq N$ will produce points in $C \cap (a,b)$, there will be a countable number of these.

So we have that the set C is perfect. Then the characteristic function of its complement $[0,1] \setminus C$ will be discontinuous on C. So that by Theorem (3.34), $\chi_{[0,1]\setminus C}$ is the characteristic function of an open set, but isn't Riemann integrable. $\square$

3.B.16

Suppose $f \in \mathcal{L}^1(\mathbb{R})$.

(a)

For $t \in \mathbb{R}$, define $f_t : \mathbb{R} \to \mathbb{R}$ by $f_t(x) = f(x-t)$. Prove that

$$\lim_{t\to 0} \|f - f_t\|_1 = 0.$$

Proof. Let $\epsilon > 0$ be given and $f \in \mathcal{L}^1(\mathbb{R})$. For $t \in \mathbb{R}$, define $f_t : \mathbb{R} \to \mathbb{R}$ by $f_t(x) = f(x-t)$.

Lemma. *Suppose λ is Lebesgue Measure on $\mathbb{R}$ and $f : \mathbb{R} \to [0,\infty]$ is a Borel measurable function such that $\int f \; d\lambda$ is defined, $f \in \mathcal{L}^1(\mathbb{R})$. Then $\int f_t \; d\lambda = \int f \; d\lambda$.*

Proof. Suppose λ is Lebesgue Measure on $\mathbb{R}$ and $f : \mathbb{R} \to [0,\infty]$ is a Borel measurable function such that $\int f \; d\lambda$ is defined. Let $E_1, \ldots, E_n$ be disjoint Lebesgue measurable sets and $a_1, \ldots, a_n \in [0,\infty]$ We'll show this for simple functions $f(x) = \sum_{n=1}^{N} a_n \chi_{E_n}(x)$. Then consider the following

$$\begin{aligned}
\int \sum_{n=1}^{N} a_n \chi_{E_n}(x) \; d\lambda &= \sum_{n=1}^{N} a_n \lambda(E_n) \\
&= \sum_{n=1}^{N} a_n \lambda(E_n + t) \\
&= \int \sum_{n=1}^{N} a_n \chi_{E_n + t}(x) \; d\lambda \\
&= \int \sum_{n=1}^{N} a_n \chi_{E_n}(x-t) \; d\lambda,
\end{aligned}$$

the first equality is the integral of a simple function, the second uses the fact that λ is translation invariant, the last equality uses the definition of

$$\chi_{E_n+t}(x) = \begin{cases} 1 & \text{if } x \in E_n + t \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } x - t \in E_n \\ 0 & \text{otherwise} \end{cases}.$$

So since this folds for simple functions, we'll employ the simple function approximation theorem to show this for any $f : \mathbb{R} \to [0,\infty)$. Let $f : \mathbb{R} \to [0,\infty)$ be a Borel measurable function. Then by the simple function approximation theorem, there exists a sequence of simple functions $f_1, f_2, \ldots$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in \mathbb{R}$. Moreover, since we can use the dominated convergence theorem ($|f_k(x)| \leq |f(x)|$ for all $x \in \mathbb{R}$ and $\int f \; d\lambda < \infty$ since $f \in \mathcal{L}^1(\mathbb{R})$) to get:

$$\begin{aligned}
\int f(x) \; d\lambda &= \int \lim_{k\to\infty} f_k(x) \; d\lambda \\
&= \lim_{k\to\infty} \int f_k(x) \; d\lambda \\
&= \lim_{k\to\infty} \int f_k(x-t) \; d\lambda \\
&= \int \lim_{k\to\infty} f_k(x-t) \; d\lambda \\
&= \int f(x-t) \; d\lambda.
\end{aligned}$$

This is our result! □

We're almost done, note that by Theorem (3.48) we have that these exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $\|f - g\|_1 < \dfrac{\epsilon}{3}$ and by our lemma we can bound the following $\|(f-g)_t\|_1 = \|f_t - g_t\|_1 < \dfrac{\epsilon}{3}$. So using the fact that g is continuous we get the following: Choose $\delta > 0$ such that $\|g(x) - g(x-t)\|_1 < \dfrac{\epsilon}{3}$ (we can do so since g is continuous and integration preserves inequalities). Suppose $|t| < \delta$, then consider the following:

$$\begin{aligned}
\|f(x) - f(x-t)\|_1 &= \|f(x) - g(x) + g(x) - g(x-t) + f(x-t) - g(x-t)\|_1 \\
&\leq \|f(x) - g(x)\|_1 + \|g(x) - g(x-t)\|_1 + \|f(x-t) - g(x-t)\|_1 \\
&< \frac{3\epsilon}{3} = \epsilon.
\end{aligned}$$

This is our result! □

(b)

For $t > 0$, define $f_t : \mathbb{R} \to \mathbb{R}$ by $f_t(x) = f(tx)$. Prove that

$$\lim_{t \to 1} \|f - f_t\| = 0.$$

Proof. Let $\epsilon > 0$ and $f \in \mathcal{L}^1(\mathbb{R})$. For $t > 0$, define $f_t : \mathbb{R} \to \mathbb{R}$ by $f_t(x) = f(tx)$.

Lemma. *Suppose λ is Lebesgue measure on $\mathbb{R}$ and $f : \mathbb{R} \to [0, \infty)$ and $f \in \mathcal{L}^1(\mathbb{R})$. Then $\int f_t \, d\lambda = \frac{1}{|t|} \int f \, d\lambda$ for all $t \in \mathbb{R} \setminus \{0\}$.*

Proof. Let $t \in \mathbb{R} \setminus \{0\}$. We'll show this for simple functions as we did in (a). Let $E_1, \ldots, E_n$ be Lebesgue measurable disjoint sets and $a_1, \ldots, a_n \in [0, \infty)$ such that $f(x) = \sum\limits_{k=1}^{n} a_k \chi_{E_k}(x)$ for all $x \in \mathbb{R}$. Then consider the following:

$$\begin{aligned}
\int \sum_{k=1}^{n} a_k \chi_{E_k}(x) \, d\lambda &= \sum_{k=1}^{n} a_k \lambda(E_k) \\
&= \sum_{k=1}^{n} a_k |t| \lambda(\frac{1}{t} E_k) \\
&= |t| \sum_{k=1}^{n} a_k \lambda(\frac{1}{t} E_k) \\
&= |t| \int \sum_{k=1}^{n} a_k \chi_{\frac{1}{|t|} E_k}(x) \, d\lambda \\
&= |t| \int \sum_{k=1}^{n} a_k \chi_{E_k}(xt) \, d\lambda,
\end{aligned}$$

the first equality is the integral of a simple function, the second equality is dialation property proved in $2.A$, the last equality uses the definition of the characteristic function. Dividing by $|t|$ this is our result for simple functions. Since there exists a sequence of simple functions $(f_k(x))$ converging pointwise to $(f(x))$ for all $x \in \mathbb{R}$, and increasing to $f \in \mathcal{L}^1(\mathbb{R})$, we get by the same process as in (a) we get:

$$\frac{1}{|t|} \int f(x) \, d\lambda = \int f(xt) \, d\lambda.$$

□

So with this lemma, let $f \in \mathcal{L}^1(\mathbb{R})$ then by (3.48) there exists a continuous function such that $\|f - g\|_1 < \dfrac{\epsilon}{3}$ and by our lemma $\dfrac{1}{|t|}\|f_t - h_t\|_1 < \dfrac{\epsilon}{3|t|}$. Let $\epsilon > 0$. Then choose $\delta > 0$ such that $\|g(x) - g(xt)\| < \dfrac{\epsilon}{3}$. Suppose $|x| < \delta$, then

$$\begin{aligned}
\|f(x) - f(tx)\|_1 &= \|f(x) - g(x) + g(x) - g(tx) + g(tx) - f(tx)\|_1 \\
&\leq \|f(x) - g(x)\|_1 + \|g(x) - g(tx)\|_1 + \|g(tx) - f(tx)\|_1 \\
&< \epsilon.
\end{aligned}$$

This is our result! $\square$