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Homework #6

MATH 550
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2.E.5

Give an example to show that Egorov's Theorem can fail without the hypothesis that $\mu(X) < \infty$.

Proof. Consider the measure space of $(\mathbb{R}, 2^{\mathbb{R}}, \mu)$ we'll define μ as follows:

$$\mu(E) = \begin{cases} \infty & \text{, if } x \in 0 \\ 0 & \text{otherwise} \end{cases}.$$

To show this is a measure note that $0 \notin \emptyset$ hence $\mu(\emptyset) = 0$, next let $\{E_n\}_{n=1}^{\infty}$ be a disjoint sequence of subsets of $\mathbb{R}$, then $\mu\left(\bigcup\limits_{n=1}^{\infty} E_n\right) = \begin{cases} \infty & \text{, if } x \in \bigcup\limits_{n=1}^{\infty} \\ 0 & \text{, otherwise} \end{cases}$. Similarly $\sum\limits_{n=1}^{\infty} \mu(E_n) = \begin{cases} \infty & \text{, if } 0 \in E_n \text{ for some } n \in \mathbb{N} \\ 0 & \text{, if } 0 \notin E_n \text{ for any } n \in \mathbb{N} \end{cases} = \begin{cases} \infty & \text{, if } 0 \in \bigcup\limits_{n=1}^{\infty} E_n \\ 0 & \text{, otherwise} \end{cases}$. Hence we have that $\mu(\bigcup\limits_{n=1}^{\infty} E_n) = \sum\limits_{n=1}^{\infty} \mu(E_n)$ for any disjoint sequence of subsets of $\mathbb{R}$, hence $(\mathbb{R}, 2^{\mathbb{R}}, \mu)$ is a measure space on $\mathbb{R}$. Then note that any function is measurable on this space by (2.36). Additionally, $\mu(\mathbb{R}) = \infty$. Consider the sequence of functions $\{f_k\}_{k\in\mathbb{N}}$, such that $f_k : \mathbb{R} \to \mathbb{R}$ for all $k \in \mathbb{N}$ and they're given by

$$f_k(x) = \begin{cases} 1 & , x \in (-\frac{1}{k}, \frac{1}{k}) \\ 0 & \text{, otherwise} \end{cases}.$$

Note though that the converges pointwise to the function:

$$f(x) = \begin{cases} 1 & , x = 0 \\ 0 & \text{, otherwise} \end{cases}.$$

That is, $\{f_k\}$ converges pointwise to the characteristic function $\chi_{\{0\}}$. To show this, note that for all $x \neq 0$, that we'll get $\lim\limits_{k\to\infty} f_k(x) = \lim\limits_{k\to\infty} 0 = 0$, and for $x = 0$ we'll get $\lim\limits_{k\to\infty} f_k(0) = \lim\limits_{k\to\infty} 0 = 0$. Because this is not a continuous function on $\mathbb{R}$ we can conclude by (2.84) that the convergence must be pointwise.

So to show that Egorov's Theorem fails here, let $\epsilon_0 = 1$. Then note that for any $E \in L$ that contains 0, the sequence $\{f_k\}$ will not converge uniformly since f isn't continuous on this set (employing the fact that for all $k \in \mathbb{N}$, f_k is continuous at $x = 0$, and then applying Theorem 2.84; that is since each f_k is continuous at $x = 0$, and f isn't the convergence must be at most pointwise). Save for the singleton $\{0\}$, and on this set we have that $\mu(\mathbb{R} \setminus \{0\}) = \infty$.

Additionally, on any set where $0 \notin E$ we'll have $\mu(\mathbb{R} \setminus E) = \infty$ because $0 \in \mathbb{R} \setminus E$, and so $\mu(\mathbb{R} \setminus E) = \infty > 1 = \epsilon_0$. This covers every possible set in $2^{\mathbb{R}}$ hence we have that there exists a $\epsilon_0 > 0$ that for all $E \in 2^{\mathbb{R}}$, either $\mu(\mathbb{R} \setminus E) \geq \epsilon_0$ or $\{f_n\}_{n\in\mathbb{N}}$ doesn't converge uniformly on E. $\square$

2.E.6

Suppose $(X, \mathcal{S}, \mu)$ is a measure space with $\mu(X) < \infty$. Suppose $f_1, f_2, \ldots$ is a sequence of $\mathcal{S}-$measurable functions from X to $\mathbb{R}$ such that $\lim_{k\to\infty} f_k(x) = \infty$ for each $x \in X$. Prove that for every $\epsilon > 0$, there exists a set $E \in \mathcal{S}$ such that $\mu(X \setminus E) < \epsilon$ and $f_1, f_2, \ldots$ converges uniformly to ∞ on E (measuring that for every $t > 0$, there exists $n \in \mathbb{Z}^+$ such that $f_k(x) > t$ for all integer $k \geq n$ and all $x \in E$) [The exercise above is an Egorov-type theorem for sequences of functions that converge pointwise to ∞.]

Proof. Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $\mu(X) < \infty$. Suppose $f_1, f_2, \ldots$ is a sequence of $\mathcal{S}-$measurable functions from X to $\mathbb{R}$ such that $\lim_{k\to\infty} f_k(x) = \infty$ for each $x \in X$. Let $\epsilon > 0$. The following will then be very similar to the proof of Egorov's theorem; that is, we'll be following the same proof with a caveat on the condition defining the set $A_{m,n}$.

Temporarily fix $n \in \mathbb{Z}^+$. The definition of pointwise convergence implies that

$$X = \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} \{x \in X : f_k(x) > n\} \tag{1}$$

For $m \in \mathbb{Z}^+$, let

$$A_{m,n} = \bigcap_{k=m}^{\infty} \{x \in X : f_k(x) > n\} = X.$$

The clearly $A_{1,n} \subset A_{2,n} \subset \ldots$ is an increasing sequence of sets and 1 can be rewritten as

$$\bigcup_{m=1}^{\infty} A_{m,n} = X.$$

The equation above implies (by 2.59) that $\lim_{m\to\infty} \mu(A_{m,n}) = \mu(X)$. Thus there exists a $m_n \in \mathbb{Z}^+$ such that

$$\mu(X) - \mu(A_{m_n,n}) < \frac{\epsilon}{2^n}. \tag{2}$$

Now let $E = \bigcap_{n=1}^{\infty} A_{m_n,n}$. Then

$$\begin{aligned}
\mu(X \setminus E) &= \mu\left(X \setminus \bigcap_{n=1}^{\infty} A_{m_n,n}\right) \\
&= \mu\left(\bigcup_{n=1}^{\infty} (X \setminus A_{m_n,n})\right) \\
&\leq \sum_{n=1}^{\infty} \mu(X \setminus A_{m_n,n} \\
&< \epsilon,
\end{aligned}$$

where the last inequality follows from 2.

To complete the proof, we must verify that $f_1, f_2, \ldots$ converges uniformly to f on E. To do this, suppose $\epsilon' > 0$. Let $n \in \mathbb{Z}^+$ be such that $n > \epsilon'$. Then $E \subset A_{m_n,n}$, which implies that $f_k(x) > n > \epsilon'$ for all $k \geq m_n$ and all $x \in X$. Hence $f_1, f_2, \ldots$ does indeed converge uniformly to f on E. $\square$

2.E.8

Suppose μ is the measure on $(\mathbb{Z}^+, 2^{\mathbb{Z}^+})$ defined by

$$\mu(E) = \sum_{n \in E} \frac{1}{2^n}.$$

Prove that for every $\epsilon > 0$, there exists a set $E \subset \mathbb{Z}^+$ with $\mu(\mathbb{Z}^+ \setminus E) < \epsilon$ such that $f_1, f_2, \ldots$ converges uniformly on E for every sequence of functions $f_1, f_2, \ldots$ from $\mathbb{Z}^+$ to $\mathbb{R}$ that converges pointwise on $\mathbb{Z}^+$. [This result doesn't follow from Egorov's Theorem] because here we are asking for E to depend only on ϵ. In Egorov's Theorem, E depends on ϵ and on the sequences $f_1, f_2, \ldots$.]

Proof. Suppose μ is the measure on $(\mathbb{Z}^+, 2^{\mathbb{Z}^+})$ defined by

$$\mu(E) = \sum_{n \in E} \frac{1}{2^n}.$$

First, we'll show a brief lemma:

Lemma 1. *Let $\{f_k\}_{k \in \mathbb{N}}$ be a sequence of functions such that for all $k \in \mathbb{N}$, $f_k : \mathbb{Z}^+ \to \mathbb{R}$. If $f_1, f_2, \ldots$ converges pointwise on $E \subseteq \mathbb{Z}^+$ to a function f, then for any finite subset $A \subseteq E$, we have that $f_1, f_2, \ldots$ converges uniformly on A.*

Proof. Let $f_1, f_2, \ldots$ converge pointwise on $E \subseteq \mathbb{Z}^+$ and $A = \{n_1, n_2, \ldots, n_m\} \subseteq E$ be a finite subset of E. Then by the definition of pointwise convergence we have that $\lim_{k \to \infty} f_k(x) = f(x)$ for all $x \in E$. That is, for all $x \in E$ and $\epsilon > 0$ we have that there exists a $N_{\epsilon,x} \in \mathbb{N}$ such that if $k \geq N_{\epsilon,x}$, then $|f_k(x) - f(x)| < \epsilon$. So then for all $x \in A \subseteq E$ this must also hold. So let $N_\epsilon = \sup_{x \in A}\{N_{\epsilon,x}\}$ where $N_{\epsilon,x} \in \mathbb{N}$ comes from the definition of pointwise convergence on E. Then we have that for all $\epsilon > 0$, if $n \geq N_\epsilon$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$. Thus $f_k \to f$ uniformly on any finite subset $A \subseteq E$. □

So the rest of the proof will follow from this lemma. Notice that for any finite subset $A \subseteq \mathbb{Z}^+$, we have that the $f_k \to f$ is uniform. Furthermore it must converge on the set $A = \{1, \ldots, N\}$ for whatever $N \in \mathbb{N}$, but notice

$$\mu(\mathbb{Z} \setminus A) = \mu(\{N+1, N+2, \ldots\}) = \sum_{n=N+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^N}$$

So choose $N \in \mathbb{N}$ large enough such that $\frac{1}{2^N} < \epsilon$.

Thus we get that $f_k \to f$ uniformly on $A \subseteq E$ and that $\mu(X \setminus A) < \epsilon$. That is our result! □

2.E.15

Suppose B is a Borel set and $f : B \to \mathbb{R}$ is a Lebesgue measurable function. Show that there exists a Borel measurable function $g : B \to \mathbb{R}$ such that

$$|\{x \in B : g(x) \neq f(x)\}| = 0$$

Proof. Suppose B is a Borel set and $f : B \to \mathbb{R}$ is a Lebesgue measurable function. Now extend f with the following:

$$\tilde{f} = \begin{cases} f(x) & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases}.$$

We'll finish off the proof by showing $\tilde{f}$ is Lebesgue measurable and employing (2.95) to get our result.

Let E be any Borel subset of $\mathbb{R}$, then

$$\tilde{f}^{-1}(E) = \begin{cases} f^{-1}(E) \cup \mathbb{R} \setminus B & \text{if } 0 \in \mathbb{R} \setminus B \\ f^{-1}(E) & \text{otherwise} \end{cases}.$$

Since B is Borel, we have that $\mathbb{R} \setminus B$ is also Borel and hence a Lebesgue set, and we get that $f^{-1}(E)$ is Lebesgue by our hypothesis that f is a Lebesgue measurable function. Thus the extension $\tilde{f}$ is Lebesgue measurable. So we can employ theorem (2.95) to get that there exists a Borel measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $|\{x \in \mathbb{R} : g(x) \neq \tilde{f}(x)\}| = 0$. Note that the restriction of g to B will also be Borel measurable by the following argument: Let E be any Borel subset of $\mathbb{R}$, then $g|_B^{-1}(E) = g^{-1}(E) \cap B$. This is a Borel subset of $\mathbb{R}$ since g is Borel measurable and B is Borel by our hypothesis. Furthermore, note that $\{x \in \mathbb{R} : g|_B(x) \neq f(x)\} \subset \{x \in \mathbb{R} : g(x) \neq \tilde{f}(x)\}$, so by order preservation of outer measure we get that $|\{x \in \mathbb{R} : g|_B(x) \neq f(x)\}| = 0$. Thus we have our result! $\square$

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Homework #6

3.A.1

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f : X \to [0, \infty]$ is an $\mathcal{S}-$measurable functions such that $\int f \; d\mu < \infty$. Explain why

$$\inf_{L} f = 0$$

for each $E \in \mathcal{S}$ with $\mu(E) = \infty$.

Proof. Since we have $\int f \; d\mu < \infty$, this tells us that

$$\sup_{P}\{\sum_{j=1}^{n} \mu(E_j) \inf_{E_j} f\} < \infty,$$

by the definition of the integral of a non-negative function. So that since for any $E \in \mathcal{S}$, $\{E, X \setminus E\}$ is a valid $\mathcal{S}-$partition of X we have that $\mu(E) \inf_{E} f + \mu(X \setminus E) \inf_{X \setminus E} f < \infty$ for any $E \in \mathcal{S}$. Suppose that $\mu(E) = \infty$, then since all of the above values are non-negative we can conclude that $\mu(E) \inf_{E} f < \infty$ and $\mu(X \setminus E) \inf_{X \setminus E} f < \infty$. Hence if $\mu(E) = \infty$, we then must have $\inf_{E} = 0$ for this remain less than ∞. Thus we have our result! $\square$

3.A.2

Suppose X is a set, $\mathcal{S}$ is a $\sigma-$algebra on X, and $c \in X$. Define the Dirac measure δ_c on $(X, \mathcal{S})$ by

$$\delta_c(E) = \begin{cases} 1 & \text{if } c \in E \\ 0 & \text{if } c \notin E \end{cases}.$$

Prove that if $f : X \to [0, \infty]$ is $\mathcal{S}-$measurable, then $\int f d\delta_c = f(x)$. [Careful: $\{c\}$ may not be in S.]

Proof. First, we 'll show that this result holds for nonnegative simple functions. Let $f = c_1\chi_{E_1} + \ldots + c_n\chi_{E_n}$ be a simple function with $\{c_1, \ldots, c_n\} \subseteq [0, \infty]$. Then by the proof of (3.13) we have that we can change this into the standard representation without adjusting the value of the function. That is there exists a $\{b_1, \ldots, b_m\} \subset [0, \infty]$ and disjoint sets (relabeling the E's) such that $f = b_1\chi_{E_1} + b_2\chi_{E_2} + \ldots + b_m\chi_{E_m}$. Additionally, notice that only one of these sets can contain c, so that $f(c) = b_j\chi_{E_j}(c) = b_j$ for some $j \in \{1, \ldots, m\}$ such that $c \in E_j$. So that by (3.7) we have that

$$\int f d\delta_c = \int \left(\sum_{k=1}^{m} b_k \chi_{E_k} \right) d\delta_c = \sum_{k=1}^{m} b_k \delta_c(E_k) = b_j \cdot 1,$$

where the last equality comes from that all the Dirac measures for all the E_k's except for the aforementioned E_j will be 0. Hence we have that $f(c) = \int f d\delta_c$ for nonnegative simple functions f.

We'll then use the previous case in conjunction with the simple function approximation (2.89) and the monotone convergence theorem (3.11) to show this for every $\mathcal{S}$-measurable function f. Let $f : X \to [0, \infty]$ be an $\mathcal{S}$-measurable function. By the simple function approximation there exists an increasing sequence of simple functions, call these $\{f_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} f_k(x) = f(x)$ for all $x \in X$. So then consider the following:

$$\begin{aligned} f(c) &= \lim_{k\to\infty} f_k(c) \\ &= \lim_{k\to\infty} \int f_k(x) d\delta_c \\ &= \int \lim_{k\to\infty} f_k(x) d\delta_c \\ &= \int f(x) d\delta_c. \end{aligned}$$

The first step follows from pointwise convergence of $f_k \to f$, we use our previous case in the second step, the third step comes from the monotone convergence theorem, and the final step comes from the simple function approximation. Thus we have our result! □

3.A.3

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f : X \to [0, \infty]$ is an $\mathcal{S}$-measurable function. Prove that

$$\int f \, d\mu \text{ if and only if } \mu(\{x \in X : f(x) > 0\}) > 0.$$

Proof. Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f : X \to [0, \infty]$ is an $\mathcal{S}$-measurable function.

To show the forwards direction of this, we'll show the contrapositive. Suppose $\mu(\{x \in X : f(x) > 0\}) = 0$. Then note that because of f's $\mathcal{S}$-measurability we have that $\{x \in X : f(x) > 0\} = f^{-1}((0, \infty]) \in \mathcal{S}$, because the set $(0, \infty]$ is an extended Borel measurable set. Because $\mathcal{S}$ is closed under complementation we have that $f^{-1}(\{0\}) \in \mathcal{S}$. For notational convenience define $A = f^{-1}((0, \infty])$ and $X \setminus A = f^{-1}(\{0\})$ (note that this notation follows from 2.33). Now, take any $\mathcal{S}$-partition of X, $\{E_1, \ldots, E_n\}$, then note that $E_i = (A \cap E_i) \cup (X \setminus A \cap E_i)$ is a disjoint union of sets in $\mathcal{S}$, since $E_i \in \mathcal{S}$ for all $i \in \{1, \ldots, n\}$ their intersections with A and $X \setminus A$ must also be in $\mathcal{S}$. Hence we can form a new $\mathcal{S}-$partition $P = \{E_1 \cap A, E_1 \cap X \setminus A, \ldots, E_n \cap A, E_n \cap X \setminus A\}$. So that

$$\begin{aligned} L(f, P) &= \sum_{k=1}^{n} \mu(A \cap E_k) \inf_{A \cap E_k} + \mu(X \setminus A \cap E_k) \inf_{X \setminus A \cap E_k} \\ &= \sum_{k=1}^{n} 0 \inf_{A \cap E_k} + \mu(E_k \cap (X \setminus A))0 \\ &= 0, \end{aligned}$$

the second equality comes from our hypothesis along with the definition of $X \setminus A = \{x \in X : f(x) = 0\}$. Since we did this for any arbitrary $\mathcal{S}-$partition of X, this implies that 0 is an upper bound of all Lebesgue lower sums, hence we have that $\int f \, d\mu = 0$.

To show the backwards direction, we'll show the contrapositive as well. Suppose $\int f \, d\mu = 0$, then $\sup_P \{L(f, P)\}$ for any $\mathcal{S}-$partition of X. Now consider the collection of sets

$$P = \{f^{-1}(\{0\}), f^{-1}([1, \infty)), f^{-1}(([\frac{1}{2}, 1)), \ldots, f^{-1}(([\frac{1}{m+1}, \frac{1}{m})), f^{-1}((0, \frac{1}{m+1}))\}$$

notice that since each of the sets above are the preimages of Borel sets in $\mathbb{R}$ and f being $\mathcal{S}-$measurable, these are all in $\mathcal{S}$. Additionally, this is a disjoint collection, because f is well-defined. So that P is an $\mathcal{S}-$partition of X. Furthermore by our hypothesis we have that $L(P, f) = 0$, so that we have the following:

$$\begin{aligned}\lim_{m\to\infty} L(f, P) &= \lim_{m\to\infty} (\mu(f^{-1}(\{0\}) \inf_{f^{-1}(\{0\}} f + \mu(f^{-1}([1,\infty)) \inf_{f^{-1}([1,\infty)} f \\ &+ \mu(f^{-1}((0, 1/m)) \inf_{f^{-1}((0,1/m))} f + \mu(f^{-1}([\frac{1}{2}, 1)) \inf_{f^{-1}[1/2,1)} f \\ &+ \ldots + \mu(f^{-1}([\frac{1}{m+1}, \frac{1}{m})) \inf_{f^{-1}([1/(m+1),1/m))} f \\ &= \mu(f^{-1}([1,\infty))) \inf_{f^{-1}([1,\infty))} f + \mu(f^{-1}(\{0\}) \inf_{f^{-1}(\{0\})} f \\ &+ \sum_{k=1}^{\infty} \mu(f^{-1}([\frac{1}{k+1}, \frac{1}{k}))) \inf_{f^{-1}([1/(k+1),1/k))} f = 0.\end{aligned}$$

This immediately implies that $\mu(f^{-1}([1,\infty)) = 0$ and since each term of the series must be 0 but that $\inf_{f^{-1}[1/(k+1),1/k))} f \neq 0$ we have that for all $k \in \mathbb{N}$ that $\mu(f^{-1}([\frac{1}{k+1}, \frac{1}{k}) = 0$. Thus because these sets are disjoint and μ has countable additivity we get

$$\mu(f^{-1}([1,\infty)) + \sum_{k=1}^{\infty} \mu(f^{-1}([\frac{1}{k+1}, \frac{1}{k}))) = \mu(f^{-1}((0,\infty))) = 0,$$

where we brought the sum into μ as a union (using countable additivity of measure) and rewrote the union over all these intervals as $f^{-1}((0,\infty])$. This is our result! $\square$

3.A.4

Give an example of a Borel measurable function $f : [0,1] \to (0,\infty)$ such that $L(f,[0,1]) = 0$. [Recall that $L(f,[0,1])$ denotes the lower Riemann integral, which was defined in Section 1A. If λ is Lebesgue measure on $[0,1]$, then the previous exercise states that $\int f \; d\mu > 0$ for this function f, which is what we expect for a positive function. Thus even though both $L(f,[0,1])$ and $\int f \; d\lambda$ are defined by taking the supremum of approximations from below, Lebesgue measure captures right behavior for this function f and the lower Riemann integral doesn't.]

Example. Define the function, that we'll call the modified Thomae's function $f : [0,1] \to (0,\infty)$ by

$$f(x) = \begin{cases} 1 & \text{if x is irrational or x} = 0 \text{ or x} = 1 \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ is the irreducible rational form of a rational x.} \end{cases}$$

This is well-defined on $[0,1]$ since we have that the cases partition the domain, and the function will never equal ∞ nor equal 0, since no rational numbers have denominators of that form. So we'll show that this function is Borel measurable first, let $a \in \mathbb{R}$, then

$$f^{-1}((a,\infty)) = \begin{cases} [0,1] & \text{if } a \leq 0 \\ \emptyset & \text{if } 1 \leq a \\ ([0,1] \setminus \mathbb{Q}) \cup \{0,1\} \cup \{\frac{1}{q} > a : q \in \mathbb{Z}^+\} & \text{if } 0 < a < 1. \end{cases}$$

The first two sets above are clearly Borel sets, and the third set, while strange, is a Borel set since $[0,1] \setminus \mathbb{Q}$ is Borel since both the irrationals and the interval $[0,1]$ are Borel sets, and $\{0,1\} \cup \{\frac{1}{q} > a : q \in \mathbb{Z}^+\}$ is Borel by the fact that any at most countable sets are Borel. Thus $f^{-1}((a,\infty))$ is a Borel set for any $a \in \mathbb{R}$ and thus f is a Borel measurable function.

To finish off the example, we'll note that the definition of the lower Riemann integral is given by:

$$L(f,[0,1]) = \sup_P \{L(f,[0,1],P)\}$$

where P is any partition of $[0,1]$. Take any partition $P = \{x_0, \ldots, x_N\}$ of $[0,1]$ then

$$\begin{aligned} L(f,[0,1],P) &= \sum_{n=1}^{N} \inf_{[x_{n-1},x_n]} f(x_n - x_{n-1}) \\ &= \sum_{n=1}^{N} 0(x_n - x_{n-1}) = 0. \end{aligned}$$

This requires a little proof, take any interval (x_{n-1}, x_n) from the above partition, then let $\{r_n\}_{n\in\mathbb{N}}$ be an enumeration of rationals in the interval. This interval must contain some rational r since the rationals are dense in $\mathbb{R}$ and this is a non-trivial interval. Furthermore, I can approach this rational with the subsequence $\{r - \frac{1}{n}\}_{n\in E}$ where $E \subset \mathbb{N}$. But this would imply that $\inf\limits_{[x_{k-1},x_k]} f \leq \lim f\left(r - \frac{1}{n}\right) = \lim\limits_{n\to\infty} \frac{1}{n} = 0$, where $r \in [x_{k-1}, x_k]$. Since we can do this for any interval in the above partition, implying the 0 that we get in the calculation of the Lower Riemann sum.

We did this for an arbitrary partition P of $[0, 1]$ so that this immediately implies that $L(f, [0, 1]) = 0$. We have our example! $\square$

3.A.5

Verify the assertion that integration with respect to counting measure is summation (Example 3.6)

Example 3.6: Suppose μ is counting measure on $\mathbb{Z}^+$ and $b_1, b_2, \ldots$ is a sequence of nonnegative numbers. Think of b as the function from $\mathbb{Z}^+$ to $[0, \infty)$ defined by $b(k) = b_k$. Then

$$\int b \; d\mu = \sum_{k=1}^{\infty} b_k,$$

as you should verify.

Proof. Let $(\mathbb{Z}^+, 2^{\mathbb{Z}^+}, \mu)$ be the measure space over the power set of $\mathbb{Z}^+$ where $\mu(E)$ is $|E|$ if E is finite, and ∞ otherwise. Let $b_1, b_2, \ldots$ be a sequence of non-negative real valued numbers. Then by the definition of the nonnegative integrals:

$$\int b \; d\mu = \sup_P \{\sum_{j=1}^{n} \mu(E_j) \inf_{E_j} b\}$$

where $P = \{E_1, \ldots, E_n\}$ is any $\mathcal{S}$-partition of $\mathbb{Z}^+$ where $\mathcal{S} = 2^{\mathbb{Z}^+}$. Note though that we can write b as:

$$b(x) = \sum_{k=1}^{\infty} \chi_{\{k\}} b_k.$$

Where this follows from b being a sequence and hence countable. So then consider the following:

$$\begin{aligned}
\int b(x)\ d\mu &= \int \sum_{k=1}^{\infty} \chi_{\{k\}} b_k\ d\mu \\
&= \int \lim_{n\to\infty} \sum_{k=1}^{n} \chi_{\{k\}} b_k\ d\mu \\
&= \lim_{n\to\infty} \int \sum_{k=1}^{n} \chi_{\{k\}} b_k\ d\mu \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(\{k\}) b_k \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} b_k \\
&= \sum_{k=1}^{\infty} b_k \checkmark.
\end{aligned}$$

The third line comes from noting that $\{\chi_{\{k\}} b_k\}$ is an increasing sequence of functions, because the singletons are disjoint and then applying the Monotone Convergence Theorem, the fourth line comes from (3.7), the fifth comes from noting that the cardinality of $\{k\}$ is 1 for any $k \in \mathbb{N}$. We have our result! $\square$

3.A.8

Suppose λ denotes Lebesgue measure on $\mathbb{R}$. Give an example of a sequence $f_1, f_2, \ldots$ of simple Borel measurable functions from $\mathbb{R}$ to $[0, \infty)$ such that $\lim_{k\to\infty} f_k = 0$ for every $x \in \mathbb{R}$ but $\lim_{k\to\infty} \int f_k d\lambda = 1$.

Example. Let $f_k : \mathbb{R} \to [0, \infty)$ be the function defined by $f_k = \chi_{(k,k+1)}$ for all $k \in \mathbb{N}$. Since $(k, k+1)$ is Borel set for all $k \in \mathbb{N}$ we have that each f_k is a simple Borel measurable function. Furthermore

$$\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} \chi_{(k,k+1)} = \chi_\emptyset = 0,$$

for all $x \in \mathbb{R}$. But integrating this with respect to the Lebesgue measure we get that:

$$\lim_{k\to\infty} \int \chi_{(k,k+1)}\ d\lambda = \lim_{k\to\infty} \lambda((k, k+1)) = 1,$$

where the value of the integral comes from (3.4). We have our example! $\square$

3.A.9

Suppose μ is a measure on a measurable space $(X, \mathcal{S})$ and $f : X \to [0, \infty]$ is an $\mathcal{S}-$measurable function. Define $\nu : \mathcal{S} \to [0, \infty]$ by

$$\nu(A) = \int \chi_A f \; d\mu$$

for $A \in \mathcal{S}$. Prove that ν is a measure on $(X, \mathcal{S})$.

Proof. Suppose μ is a measure on a measurable space $(X, \mathcal{S})$ and $f : X \to [0, \infty]$ is an $\mathcal{S}-$measurable function. Define $\nu : \mathcal{S} \to [0, \infty]$ by

$$\nu(A) = \int \chi_A f \; d\mu$$

for $A \in \mathcal{S}$.
Then we get that $\nu(\emptyset) = \int \chi_\emptyset f \; d\mu = \int 0 \; d\mu = 0$, where the value of the integral follows from the definition of $\int \cdot \; d\mu = \sup\limits_P \{\sum\limits_{k=1}^{n} \mu(\cdot) \inf_{E_k} \cdot\}$ for any $\mathcal{S}-$partition of X.
Next a brief lemma:

Lemma 2. *For any disjoint sequence* $\{E_k\}_{k\in\mathbb{N}}$, $\chi_{\bigcup\limits_{k=1}^{\infty} E_k} = \sum\limits_{k=1}^{\infty} \chi_{E_k}$.

Proof. Let $\{E_k\}_{k\in\mathbb{N}}$ be a sequence of sets in $\mathcal{S}$. Then

$$\sum_{k=1}^{\infty} \chi_{E_k} = \begin{cases} 1 & \text{if } x \in E_k \text{ for any } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } x \in \bigcup\limits_{k=1}^{\infty} E_k \\ 0 & \text{otherwise} \end{cases} = \chi_{\bigcup\limits_{k=1}^{\infty} E_k}.$$

□

The rest of the proof follows from this lemma. Next take any disjoint sequence of sets in $\mathcal{S}$,

$\{E_k\}_{k=1}^{\infty}$, then we have

$$
\begin{aligned}
\nu\left(\bigcup_{k=1}^{\infty} E_k\right) &= \int \chi_{\bigcup\limits_{k=1}^{\infty} E_k} f \ d\mu \\
&= \int \left(\sum_{k=1}^{\infty} \chi_{E_k}\right) f \ d\mu \\
&= \int \lim_{n\to\infty} \left(\sum_{k=1}^{n} \chi_{E_k}\right) f \ d\mu \\
&= \int \lim_{n\to\infty} \left(f \sum_{k=1}^{n} \chi_{E_k}\right) \ d\mu \\
&= \int \lim_{n\to\infty} \left(\sum_{k=1}^{n} f\chi_{E_k}\right) \ d\mu \\
&= \lim_{n\to\infty} \int \left(\sum_{k=1}^{n} f\chi_{E_k}\right) \ d\mu \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \int f\chi_{E_k} \ d\mu \\
&= \sum_{k=1}^{\infty} \int f\chi_{E_k} \ d\mu \\
&= \sum_{k=1}^{\infty} \nu(E_k).
\end{aligned}
$$

The first step we used our lemma, third step we used the fact that f doesn't depend on n to bring it into the limit, the fourth step is finite distribution of multiplication of functions, the sixth step uses the Monotone Convergence Theorem (we can use this because the sequence we're considering is a non-negative sum so that it is an increasing sequence of functions, hence satisfies the hypothesis of the theorem), the seventh uses the additivity of integration over finite sums, finally the last step uses the definition of $\nu(E_k)$. Thus we have that ν has countable additivity, and maps $\emptyset \mapsto 0$, therefore ν is a measure on the measurable space $(X, \mathcal{S})$. We have our result! $\square$

3.A.10

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f_1, f_2, \ldots$ is a sequence of nonnegative $\mathcal{S}$-measurable functions. Define $f : X \to [0, \infty]$ by $f(x) = \sum_{k=1}^{\infty} f_k(x)$. Prove that

$$\int f \, d\mu = \sum_{k=1}^{\infty} \int f_k \, d\mu.$$

Proof. Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f_1, f_2, \ldots$ is a sequence of nonnegative $\mathcal{S}$-measurable functions. Define $f : X \to [0, \infty]$ by $f(x) = \sum_{k=1}^{\infty} f_k(x)$.

Consider the following

$$\begin{aligned}
\int f \, d\mu &= \int \left(\sum_{k=1}^{\infty} f_k \right) \, d\mu \\
&= \int \left(\lim_{n\to\infty} \sum_{k=1}^{n} f_k \right) \, d\mu \\
&= \lim_{n\to\infty} \int \sum_{k=1}^{n} f_k \, d\mu \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \int f_k \, d\mu \\
&= \sum_{k=1}^{\infty} \int f_k \, d\mu.
\end{aligned}$$

This is similar to $(3.A.9)$ where we used monotone convergence on step three, (3.16) on step four. We have our result! $\square$