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Portfolio #2

MATH 431
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Lesson 5 # 6

What would be the solution to problem #4 if the IC were $u(x,0) = x - x^2,\ 0 < x < 1$?

$$\begin{cases} \text{PDE} & u_t = u_{xx} \\ \text{BCs} & u(0,t) = 0 \\ & u(1,t) = 0 \\ \text{IC} & u(x,0) = 1 \end{cases}$$

Solution:

Since we have homogeneous linear BCs, we may use separation of variable. So assume that $u(x,t) = X(x)T(t)$. Then applying that to our PDE we get:

$$X''(x)T(t) = X(x)T'(t)$$

Assuming a separation constant of $k \in \mathbb{R}$, we get:

$$\frac{X''}{X} = \frac{T'}{T} = k. \tag{1}$$

Note that in this situation if we have $k \geq 0$ then we will have either $u(x,t) = 0$ for all $t \in dom(u)$ or $u(x,t) \to \infty$ as $t \to \infty$. Thus we must have $k < 0$. So let $k = -\lambda^2$ which is guaranteed to be less than 0, where $\lambda \in \mathbb{R}$. Then taking (1) and turning into a system of equations we get:

$$X'' + \lambda^2 X = 0 \tag{2}$$

$$T' + \lambda^2 T = 0 \tag{3}$$

Solving for X from (2) and T from (3) via characteristic equation we get that $T(t) = Ae^{-\lambda^2 t}$ and $X(x) = B_1 \cos \lambda x + B_2 \sin \lambda x$ for some $B_1, B_2, A \in \mathbb{C}$. So we get the general solution

$$u(x,t) = X(x)T(t) = e^{-\lambda^2 t}(A \cos \lambda x + B \sin \lambda x) \tag{4}$$

Applying our BC of $u(0,t) = 0$, we get:

$$u(0,t) = e^{-\lambda^2 t}(A \cos \lambda 0 + B \sin \lambda 0) \iff 0 = e^{-\lambda^2 t}(A) \iff A = 0$$

Thus after applying these boundary conditions we have the solution:

$$u(x,t) = e^{-\lambda^2 t}(B \sin \lambda x)$$

Applying the BC of $u(1,t) = 0$:

$$u(1,t) = e^{-\lambda^2 t}(B \sin \lambda 1) \iff 0 = e^{-\lambda^2 t} B \sin \lambda \iff 0 = \sin \lambda \iff \lambda = \pi n$$

Assuming that $B \neq 0$, and letting $n \in \mathbb{N}$. So we have the solution for our PDE and BCs:

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-(\pi n)^2 t} \sin \pi n x$$

Now we can solve for our IC $u(x,0) = x - x^2$:

$$u(x,0) = \sum_{n=0}^{\infty} A_n e^{-(\pi n)^2 0} \sin \pi n x \iff x - x^2 = \sum_{n=0}^{\infty} A_n \sin \pi n x$$

$$\iff \sin \pi m x (x - x^2) = \sum_{n=0}^{\infty} A_n \sin \pi n x \sin \pi m x$$

$$\iff \int_0^1 \sin \pi m x (x - x^2) = \int_0^1 \sum_{n=0}^{\infty} A_n \sin \pi n x \sin \pi m x$$

$$\iff \frac{2 - 2\cos \pi m - \pi m \sin \pi m}{\pi^3 m^3} = A_m \frac{1}{2}$$

By the orthogonality of the *sine* function and evaluating the left hand integral using Mathematica.

$$\iff \frac{2(2 - 2\cos \pi m)}{\pi^3 m^3} = A_m$$

Since integer multiples of π give a zero *sine* function. Then note that for all even $m's$ we have that $\cos \pi m = 1$ hence our coefficient A_m is zero, and that when m is odd, we have the coefficient is $A_m = \frac{8}{\pi^3 m^3}$. So we have the general solution as:

$$u(x,t) = \sum_{n=0}^{\infty} \frac{8}{\pi^3 (2n+1)^3} e^{-(\pi(2n+1))^2 t} \sin \pi (2n+1) x$$

Lesson 6 #2

Transform

$$\begin{cases} \text{PDE} & u_t = u_{xx} \\ \text{BCs} & u(0,t) = 0 \\ & u(1,t) = 1 \\ \text{IC} & u(x,0) = x^2 \end{cases}$$

to zero BCs and solve the new problem. What will the solution to this problem look like for different values of time? Does the solutions agree with your intuition? What is the steady-state solution? What does the transient solution look like?

Solution:

Assume that we may break the solution into a steady state part and a transient part: $u(x,t) = S(x,t) + U(x,t)$, where S is the steady state and U is the transient part. Assuming that the steady state solution is the simply the linear curve between our BCs, this reasonable since we have BCs with constant coefficients we get that the steady state piece $S(x,t)$ should be of the form:

$$S(x,t) = 0 + \frac{x}{1}(1-0) = x$$

which is just the linear path between our BCs. So we have that

$$u(x,t) = x + U(x,t).$$

Taking derivatives of that we have $u_t = U_t$ and $u_{xx} = U_{xx}$. Finding our new BCs and IC we have $u(0,t) = U(0,t)$ and $u(1,t) = 1 + U(1,t)$, and $u(x,0) = x + U(x,0) \iff x^2 - x = U(x,0)$. Thus Hence we have the new IVBP:

$$\begin{cases} \text{PDE} & U_t = U_{xx} \\ \text{BCs} & U(0,t) = 0 \\ & U(1,t) = 0 \\ \text{IC} & U(x,0) = x^2 - x \end{cases}$$

Note that this is nearly identical to #6 from Lesson 5, the only difference being a sign. So we have the solution to this IVBP as:

$$U(x,t) = -\sum_{n=0}^{\infty} \frac{8}{\pi^3(2n+1)^3} e^{-(\pi(2n+1))^2 t} \sin(\pi(2n+1)x)$$

So our solution for $u(x,t)$ is as follows:

$$u(x,t) = x - \sum_{n=0}^{\infty} \frac{8}{\pi^3(2n+1)^3} e^{-(\pi(2n+1))^2 t} \sin(\pi(2n+1)x)$$

The solution will look like a line with slope 1 and a zero y-intercept with an almost imperceptibly small wave pattern to it. Note that even for small $t's$ we have the approximate solution as:

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In[54]:= expr1 = x - Sum[(8 E^(-(Pi (2 i + 1))^2*1) Sin[(Pi (2 i + 1) x)] / (Pi^3 * (2 i + 1)^3)), {i, 0, 10}]
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Out[54]= $x - \frac{8 e^{-\pi^2} \operatorname{Sin}[\pi x]}{\pi^3} - \frac{8 e^{-9\pi^2} \operatorname{Sin}[3\pi x]}{27\pi^3} - \frac{8 e^{-25\pi^2} \operatorname{Sin}[5\pi x]}{125\pi^3} - \frac{8 e^{-49\pi^2} \operatorname{Sin}[7\pi x]}{343\pi^3} - \frac{8 e^{-81\pi^2} \operatorname{Sin}[9\pi x]}{729\pi^3} - \frac{8 e^{-121\pi^2} \operatorname{Sin}[11\pi x]}{1331\pi^3} - \frac{8 e^{-169\pi^2} \operatorname{Sin}[13\pi x]}{2197\pi^3} - \frac{8 e^{-225\pi^2} \operatorname{Sin}[15\pi x]}{3375\pi^3} - \frac{8 e^{-289\pi^2} \operatorname{Sin}[17\pi x]}{4913\pi^3} - \frac{8 e^{-361\pi^2} \operatorname{Sin}[19\pi x]}{6859\pi^3} - \frac{8 e^{-441\pi^2} \operatorname{Sin}[21\pi x]}{9261\pi^3}$

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In[56]:= Plot[expr1, {x, 0, 1}]
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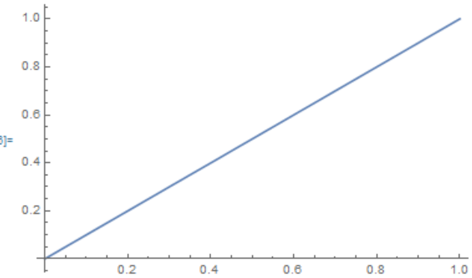
Out[56]=



This matches our intuition, since as time goes on we will just get a linear path between the two BCs. The steady-state solution is simply x on $0 \leq x \leq 1$ as $t \to \infty$. While the transient solution is:

$$U(x,t) = -\sum_{n=0}^{\infty} \frac{8}{\pi^3(2n+1)^3} e^{-(\pi(2n+1))^2 t} \sin(\pi(2n+1)x)$$

Lesson 7 #3

Solve the following problem with insulated boundaries

$$\begin{cases} \text{PDE} & u_t = u_{xx} \\ \text{BCs} & u_x(0,t) = 0 \\ & u_x(1,t) = 0 \\ \text{IC} & u(x,0) = x \end{cases}$$

Does your solution agree with your interpretation of the problem? What is the steady state solution? Does this make sense?

Solution:

Note that since we have homogeneous boundary conditions we can use separation of variables. So assume that $u(x,t)$ is of the form $u(x,t) = X(x)T(t)$. Then note that we already have the general solution of this PDE as:

$$u(x,t) = e^{-\lambda^2 t}[A \sin \lambda x + B \cos \lambda x)]$$

Where $-\lambda^2$ is the separation constant and $A, B \in \mathbb{C}$. Then to apply the boundary conditions we need to take the x-derivative of this solution:

$$u_x(x,t) = e^{-\lambda^2 t}[A\lambda \cos \lambda x - B\lambda \sin \lambda x]$$

Applying our BC of $u_x(0,t) = 0$:

$$u_x(0,t) = e^{-\lambda^2 t}[A\lambda \cos \lambda 0 - B\lambda \sin \lambda 0] \iff 0 = e^{-\lambda^2 t} A\lambda \iff 0 = A\lambda$$

Note that if $\lambda = 0$, then we would have the solution $u(x,t)$ is a constant, which wouldn't be the most interesting case. So assume that $\lambda \neq 0$ for this BC.

$$\iff A = 0$$

Hence we now have $u(x,t) = e^{-\lambda^2 t} B \cos \lambda x$. Now applying the BC $u_x(1,t) = 0$. So we have:

$$u_x(0,t) = e^{-\lambda^2 t}(-B)\lambda \sin(\lambda) \iff 0 = e^{-\lambda^2 t}(-B)\lambda \sin(\lambda) \iff 0 = \lambda \sin(\lambda)$$

Note here we will also assume that $\lambda \neq 0$ for the same reason that we assumed it for our other BC.

$$\iff 0 = \sin(\lambda) \iff \lambda = \pi n, \text{ for some } n \in \mathbb{N}.$$

Then we have the solution to the PDE and BCs:

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-(\pi n)^2 t} \cos(\pi n x)$$

Now applying IC we have:

$$u(x,0)=\sum_{n=0}^{\infty}A_n e^{-(\pi n)^2 0}\cos(\pi n) \iff x=\sum_{n=0}^{\infty}A_n\cos(\pi n x)$$

$$\iff \cos(\pi m x)x=\sum_{n=0}^{\infty}A_n\cos(\pi n x)\cos(\pi m x)$$

$$\iff \int_0^1 \cos(\pi m x)x\ dx=\int_0^1\sum_{n=0}^{\infty}A_n\cos(\pi n x)\cos(\pi m x)\ dx$$

Note that we have two cases at this point, if $m=0$, then we get:

$$\int_0^1 x\ dx=\int_0^1\sum_{n=0}^{\infty}A_n\cos(\pi n x)\cos(\pi 0 x)\ dx$$

Note that by the orthogonality of the cosine function we have that for all $m\neq n$: $\int_0^1\cos(\pi m x)\cos(\pi n x)=0$. Hence we have that the only remaining term is when $n=m=0$.

$$\left.\frac{x^2}{2}\right|_0^1=\int_0^1 A_0\cos(0)\ dx \iff \frac{1}{2}=A_0$$

This and when $n=0$ we have the solution $u_0(x,t)=\frac{1}{2}e^{-(\pi\cdot 0)^2 t}\cos(\pi x(0))=\frac{1}{2}$. Now we will look at all $n\geq 1$. So now we look at the problem:

$$\int_0^1 \cos(\pi m x)x\ dx=\int_0^1\sum_{n=1}^{\infty}A_n\cos(\pi n x)\cos(\pi m x)\ dx$$

$$\iff \frac{\pi m\sin(\pi m)+\cos(\pi m)-1}{(\pi m)^2}=A_m\frac{1}{2}$$

By the orthogonality of the cosine function and evaluating that left integral using Mathematica.

$$\iff \frac{2(\cos(\pi m)-1)}{(\pi m)^2}=A_m$$

Note that if m is odd, then we have that $\cos(\pi m)=-1$ and when m is even we have $\cos(\pi m)=1$. Thus we have that even solution will go to zero, so we have the general solution for $n\geq 1$:

$$u_{2k-1}(x,t)=\frac{2(-2)}{(\pi(2k-1))^2}e^{-(\pi(2k-1))^2 t}\cos(\pi(2k-1)x)$$

So that the general solution is:

$$u(x,t) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{-4}{(\pi(2k-1))^2} e^{-(\pi(2k-1))^2 t} \cos(\pi(2k-1)x)$$

This means that the steady-state solution is $\frac{1}{2}$ as $t \to \infty$. This makes sense, our boundary conditions tell us that the derivatives at the boundaries are zero. So as time approaches infinity we need the graph to kind of even out, to a constant function. Since the midpoint between 1 and 0 is 1/2, and by our IC we have that $u(0,0) = 0$ and $u(1,0) = 0$, it would make sense that the boundaries temperatures "meet" at 1/2.

Lesson 7 #4

What are the eigenvalues and eigenfunctions of

$$\begin{cases} ODE & X'' + \lambda X = 0 \\ BCs & X'(0) = 0 \\ & X'(1) = 0 \end{cases}$$

On the interval $0 < x < 1$.

Solution:

Note that we can solve our ODE via a characteristic equation, assuming that $X(x) = Ce^{rt}$ for some $C, r \in \mathbb{C}$. Applying this method we get the polynomial $r^2 = -\lambda \iff r = \pm\sqrt{\lambda}i$ So we get the solution:

$$X(x) = C_1 e^{\sqrt{\lambda}ix} + C_2 e^{-\sqrt{\lambda}ix} \iff X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x), \text{ where } B, A \in \mathbb{C}$$

Applying BCs:

$$X'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x) \iff 0 = B\sqrt{\lambda}$$

Note that if $\sqrt{\lambda} = 0$, then we would have $X(x) = B$ where $B \in \mathbb{C}$ which would be a trivial result, so $\sqrt{\lambda} \neq 0$. Hence $B = 0$. So now we have $X(x) = A\cos(\sqrt{\lambda}x)$. Now applying our BC $X'(1) = 0$:

$$X'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) \iff 0 = -A\sqrt{\lambda}\sin(\sqrt{\lambda}) \iff \sin(\sqrt{\lambda}) = 0$$

$$\iff \sqrt{\lambda} = \pi m$$

Hence we have the solution:

$$X(x) = \sum_{n=0}^{\infty} A\cos(\pi n x)$$

In this case our eigenvalues are:

$$\lambda_n = (\pi n)^2$$

And the eigenfunctions are:

$$X_n(x) = \cos(\sqrt{\lambda_n}x).$$