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Homework #5

MATH 550
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2.D.4

The phrase *nontrivial interval* is used to denote an interval of $\mathbb{R}$ that contains more than one element. Recall that an interval might be open, closed, or neither.

(a)

Prove that the union of each collection of nontrivial intervals of $\mathbb{R}$ is the union of a countable subset of that collection.

Proof. Let $A = \{\text{non-trivial intervals in } \mathbb{R}\}$ Consider the union over some subcollection of $\Omega \subseteq A$:

$$\bigcup_{\alpha \in \Omega} I_\alpha,$$

with each I_α an interval in $\mathbb{R}$. Now clearly for any countable subcollection we have that $\bigcup_{\alpha \in \Omega} I_\alpha \supseteq \bigcup_{n=1}^{\infty} I_{\alpha_n}$ for any choice of our α_n's. Now we'll show that there exists some subcollection of Ω such that the opposite directions follows.
Take $x \in \bigcup_{\alpha \in \Omega} I_\alpha$. Then for some $\alpha \in \Omega$ we have that $x \in I_\alpha$, for the interval I_α. Note that every interval contains rationals, because they are non-trivial. Define the subcollection:

$$B = \{I_{\alpha_n} : n \in I_{\alpha_n} \text{ for } n \in \mathbb{Q} \text{ and } I_{\alpha_n} \text{ is the maximal such interval}\}.$$

This is clearly a countable subcollection since we're indexing with respect to the rationals. Now, clearly x is close to some $r \in \mathbb{Q}$, because of the density of $\mathbb{Q} \subseteq \mathbb{R}$, so that $x \in I_r$ for some $r \in \mathbb{Q}$. Otherwise, x would not be in any interval, since every interval has a dense number of rationals contained in them. So that $x \in I_{\alpha_r}$. Hence $x \in \bigcup_{k \in \mathbb{Q}} I_{\alpha_k}$. □

(b)

Prove that the union of each collection of nontrivial intervals of $\mathbb{R}$ is a Borel set.

Proof. Consider the union of nontrivial intervals given by:

$$\bigcup_{\alpha \in \Omega} I_\alpha = \bigcup_{n=1}^{\infty} I_n.$$

The equality comes from the result of $(a.)$.

For all I_n in the above union are either half-open-closed, closed, or open intervals. The open intervals are Borel by definition, then by (2.30) the closed intervals (a closed set) and half-closed-open intervals are Borel. Thus by the axioms of the $\sigma-$algebra of Borel sets we have that $\bigcup_{n=1}^{\infty} I_n$ is a Borel set. □

(c)

Prove that there exists a collection of closed intervals of $\mathbb{R}$ whose union isn't a Borel set.

Proof. Since we have that this isn't a non-trivial interval, we can have degenerate intervals such as $[a, a]$ for $a \in \mathbb{R}$. By (2.67) we have that there exists a non-Borel subset of $\mathbb{R}$, call it B. Then:

$$B = \bigcup_{b \in B} \{b\} = \bigcup_{b \in B} [b, b].$$

Thus the collection of closed intervals given by: $\{[b, b] : b \in B\}$ has a union that isn't Borel. Showing that $(b.)$ holds only for non-trivial intervals. □

2.D.5

Prove that if $A \subset \mathbb{R}$ is Lebesgue measurable, then there exists an increasing sequence $F_1 \subset F_2 \subset \ldots$ of closed sets contained in A such that

$$\left| A \setminus \bigcup_{k=1}^{\infty} F_k \right| = 0.$$

Proof. Let $A \subset \mathbb{R}$ be Lebesque measurable. Then by theorem (2.71) we have that there exists a sequence of closed sets $F_1, F_2, \ldots$ contained in A such that $\left| A \setminus \bigcup_{k=1}^{\infty} F_k \right| = 0$. Define the sequence $C_1 = F_1$ and $C_n = C_{n-1} \cup F_n$ for all $n \in \mathbb{N}$. Then by this definition we'll get that $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} C_n$. To show this let $x \in \bigcup_{n=1}^{\infty} C_n$, then for some $n \in \mathbb{N}$ we have $x \in C_n = F_1 \cup F_2 \cup \ldots \cup F_n$. So $x \in F_i$ for some $F_i \in \{F_1, \ldots, F_n\}$ hence $x \in \bigcup_{n=1}^{\infty} F_n$ so $\bigcup_{n=1}^{\infty} C_n \subseteq \bigcup_{n=1}^{\infty} F_n$. The other direction follows from our definition of C_n. So moreover, by this definitions $C_1 \subseteq C_2 \subseteq \ldots \subseteq C_n \subseteq \ldots$ and note that the finite union of closed sets remains closed, so that each of these C_n's is closed. Thus we have the sequence $\{C_n\}_{n=1}^{\infty}$ has all of the characteristics we want. Furthermore:

$$\left|A \setminus \bigcup_{k=1}^{\infty} F_k\right| = \left|A \setminus \bigcup_{k=1}^{\infty} C_k\right| = 0.$$

□

2.D.8

Prove that the collection of Lebesgue measurable subsets of $\mathbb{R}$ is translation invariant. More precisely, prove that if $A \subset \mathbb{R}$ is Lebesgue measurable and $t \in \mathbb{R}$, then $t + A$ is Lebesgue measurable.

Proof. Let $t \in \mathbb{R}$. Let $A \subset \mathbb{R}$ be a Lebesgue measurable subset of $\mathbb{R}$.

We'll show a small Lemma first.

Lemma 1. *For any $t \in \mathbb{R}$ and $A, B \subseteq \mathbb{R}$ $B \setminus A + t = (B + t) \setminus (A + t)$.*

Proof. Let $x \in (B \setminus A) + t$. Then $x = a + t$ for some $a \in B \setminus A$. So that $a \in B$ and $a \notin A$. Hence $x \in B + t$ and $x \notin A + t$. Thus $x \in (B + t) \setminus (A + t)$. Conversely, let $x \in (B + t) \setminus (A + t)$. Then $x \in B + t$ and $x \notin A + t$. That is, $x - t \in B$ and $x - t \notin A$. Thus $x - t \in B \setminus A$. Hence $x \in (B \setminus A) + t$.

And we have our result! □

Now note that since $A \subseteq \mathbb{R}$ is a Lebesgue measurable subset of $\mathbb{R}$ we have that there exists a Borel set $B \subseteq A$ such that $|A \setminus B| = 0$ (2.71). Note that outer measure is translation invariant (2.7) so that $|A \setminus B + t| = 0$ By our lemma we then have $|A \setminus B + t| = |(A + t) \setminus (B + t)| = 0$. Moreover, by exercise $(2.B.7)$ if B is Borel, $B + t$ is Borel. Thus by (2.71) we have that $A + t$ is Lebesgue measurable. □

2.D.10

Prove that if A and B are disjoint subsets of $\mathbb{R}$ and B is Lebesgue measurable, then $|A \cup B| = |A| + |B|$.

Proof. Let $\epsilon > 0$ be given. Let $A, B \subseteq \mathbb{R}$ and A and B are disjoint. Suppose B is Lebesgue measurable

Then since B is Lebesgue measurable by (2.71)(b) there exists a closed set $F \subset B$ such that $|B \setminus F| < \epsilon$. Then consider the following:

$$\begin{aligned} |A \cup B| &\geq |A \cup F| \\ &= |A| + |F| \\ &= |A| + |B| - |B \setminus F| \\ &\geq |A| + |B| - \epsilon. \end{aligned}$$

The first inequality comes from $F \subset B$, the first equality comes from that since $F \subseteq B$ and B is disjoint from A, F is disjoint from A thus apply (2.63), the second equality comes from applying (2.63) with $|B| = |F \cup (B \setminus F)| = |F| + |B \setminus F|$. So this holds for any $\epsilon > 0$, thus $|A \cup | \geq |A| + |B|$. Since we still have subadditivity, we have $|A \cup B| \leq |A| + |B|$. Thus $|A \cup B| = |A| + |B|$. Our result. $\square$

2.D.11

Prove that if $A \subset \mathbb{R}$ and $|A| > 0$, then there exists a subset of A that is not Lebesgue measurable.

Proof. We'll show this by showing that every set $A \subseteq \mathbb{R}$ with $|A| > 0$ has a subset where outer measure isn't additive; that is, there exists disjoint subsets $B, C \subseteq A$ such that $|B \cup C| \neq |B| + |C|$. This will imply that B, C aren't Lebesgue measurable, sense the set of all Lebesgue measurable sets over $\mathbb{R}$ with outer measure is a measure space and thus has additivity by (2.72).

Let $A \subset \mathbb{R}$ and $|A| > 0$. Define the set $A_n = A \cap [n, n+1)$. Then $A = \bigcup\limits_{n \in \mathbb{N}} A_n$ and at least one of these A_n's must have no-zero measure; that is, for some $n \in \mathbb{N}$, $|A_n| > 0$. Otherwise for all $n \in \mathbb{N}$, $|A_n| = 0$. Hence by subadditivity of outer measure we would have $|A| = |\bigcup\limits_{n \in \mathbb{N}} A_n| \leq \sum\limits_{n \in \mathbb{N}} |A_n| = 0$, giving us $|A| = 0$, a contradiction. So at least one A_n has non-zero measure, $|A_n| > 0$. We'll show that there are subsets of this A_n that aren't additive over outer measure and hence there are some non-Lebesgue measurable sets of $A_n \subset A$. Translate $A_n = A \cap [n, n+1)$ to $[0, 1)$ via: $A_n - n = A \cap [n, n+1) - n = (A - n) \cap [0, 1)$ by our lemma 1. So then outer measure is translation invariant so $|A_n| = |A_n - n| > 0$ and furthermore, A_n is bounded in the interval $[0, 1)$.

What follows is a word-for-word rework of the proof of (2.18) with $A_n - n$. Apologies to Axler for the plagiarism.

For $a \in A_n - n = B$, let $\tilde{a}$ be the set of numbers in B that differ from a by a rational number. If $a, b \in B$ and $\tilde{a} \cap \tilde{b} \neq \emptyset$, then $\tilde{a} = \tilde{n}$. Clearly $a \in \tilde{a}$ for each $a \in B$. Thus $B = \bigcup\limits_{a \in B} \tilde{a}$.

Let V be a set that contains exactly one element in each of the distinct sets in $\{\tilde{a} : a \in B\}$. In other words, for every $a \in B$, the set $V \cap \tilde{a}$ has exactly one element. Let $r_1, r_2, \ldots$ be an enumeration of the rationals contained within $[-2, 2]$. Then $A_n \subseteq \bigcup\limits_{k=1}^{\infty}(r_k + V)$, where the set inclusion holds because $a \in B$, then letting v be the unique element of $V \cap \tilde{a}$, we have that $a - v \in \mathbb{Q}$, which implies that $a = r_k + v \in r_k + V$ for some $k \in \mathbb{Z}^+$.

The set inclusion above, the order-preserving property of outer measure (2.5), and the countable subadditivity of outer measure (2.8) imply that

$$|B| \leq \sum_{k=1}^{\infty} |r_k + V|.$$

We know that $|A| > 0$. The translation invariance of outer measure (2.7) thus allows us to rewrite the inequality above as

$$0 < |A| \leq \sum_{k=1}^{\infty} |V|.$$

Thus $|V| > 0$.

Note that the sets $r_1 + V, r_2 + V, \ldots$ are disjoint. (As proved on page $\approx 7\pi$.) Let $n \in \mathbb{Z}^+$. Clearly

$$\bigcup_{k=1}^{n}(r_k + V) \subset [-3, 3]$$

because $V \subset B = A_n - n$ and for each $r_k \in [-2, 2]$. The set inclusion above implies that

$$\left|\bigcup_{k=1}^{n}(r_k + V)\right| \leq 6 = |[-3, 3]|(2.7). \tag{1}$$

However

$$\sum_{k=1}^{n} |r_k + V| = \sum_{k=1}^{n} |V| = n|V|. \tag{2}$$

Now 1 and 2 both suggest that we choose $n \in \mathbb{Z}^+$ such that $n|V| > 6$. Thus

$$\left|\bigcup_{k=1}^{n}(r_k + V)\right| < \sum_{k=1}^{n} |r_k + V|. \tag{3}$$

Now, note that $V \subset B$ by our construction of it with the axiom of choice. Furthermore by 3, we've seen that the collection of translations $V + r_k$'s isn't additive. Thus the $V + r_k$'s are not Lebesgue measurable sets, and since this translation of V isn't Lebesgue measurable, we can say that V isn't Lebesgue measurable by the contrapositive of translation invariance of outer measure and using (2.72). Since by our construction via the Axiom of Choice we had that $V \subset B = A_n - n = (A - n) \cap [0, 1)$. So because $V \subset A - n$, $V + n \subset A$. Since V isn't Lebesgue measurable $V + n$ is Lebesgue measurable, thus there exists a subset of A that isn't Lebesgue measurable. Which the result we wanted to prove! □

2.D.24

For $A \subseteq \mathbb{R}$, the quantity

$$\sup\{|F| : F \text{ is a closed bounded subset of } \mathbb{R} \text{ and } F \subset A\}$$

is called the *inner measure* of A.

(a)

Show that if A is a Lebesgue measurable subset of $\mathbb{R}$, then the inner measure of A equals the outer measure of A.

Proof. Denote $\text{Inn}(A)$ to be the inner measure of A.

Note the definitions $\text{Inn}(A) = \sup\{|F| : F \text{ is a compact subset of } A\}$, then $|F| \leq \text{Inn}(A)$ for all closed and bounded subsets $F \subset A$ and $|A| = \inf\{\sum_{k=1}^{\infty} l(I_k)$ for any sequence of open intervals such that $A \subset \bigcup_{k=1}^{\infty} I_k\}$.

Let $F \subset A$ be compact. Then because of the preservation of order $|F| \leq |A|$, so that $|A|$ is an upper bound of the set of these numbers. But $\text{Inn}(A)$ is the least upper bound of this set; that is:

$$\text{Inn}(A) \leq |A|.$$

For any $A \subset \mathbb{R}$

First note that if K is compact, then $|K| = \text{Inn}(K)$ since the largest compact set that's a subset of K is K.

Assume that $A \subseteq \mathbb{R}$ is a Lebesgue measurable subset of $\mathbb{R}$. Then by $(2.D.5)$ there exists an increasing sequence $F_1 \subset F_2 \subset \dots$ of closed sets contained in A such that

$$\left| A \setminus \bigcup_{k=1}^{\infty} F_k \right| = 0.$$

Either $\left|\bigcup_{k=1}^{\infty} F_k\right| = \infty$ or $\left|\bigcup_{k=1}^{\infty} F_k\right| < \infty$, in the former case we get by order preservation that $\left|\bigcup_{k=1}^{\infty} F_k\right| = |A|$. In the latter case we get through (2.A.3) that $\left|\bigcup_{k=1}^{\infty} F_k\right| = |A|$. Either way we get that they have equal measure. Now note that we can take the union $\bigcup_{k=1}^{\infty} F_k$ and rewrite this as a union of compact sets in the following manner:

$$F_1 = (F_1 \cap [-1,1]) \cup (F_1 \cap [-2,2]) \cup \dots \tag{4}$$
$$F_2 = (F_2 \cap [-1,1]) \cup (F_2 \cap [-2,2]) \cup \dots \tag{5}$$
$$\vdots \quad \vdots \tag{6}$$

Each one of these pieces is compact because it's a bounded and closed set. Define the set $K_{i,j} = F_i \cap [-j, j]$. Define the relabeled increasing sequence K_i of compact sets as the union of the boxes shown in figure 1: Each new K_i is the finite union of compact sets, hence is

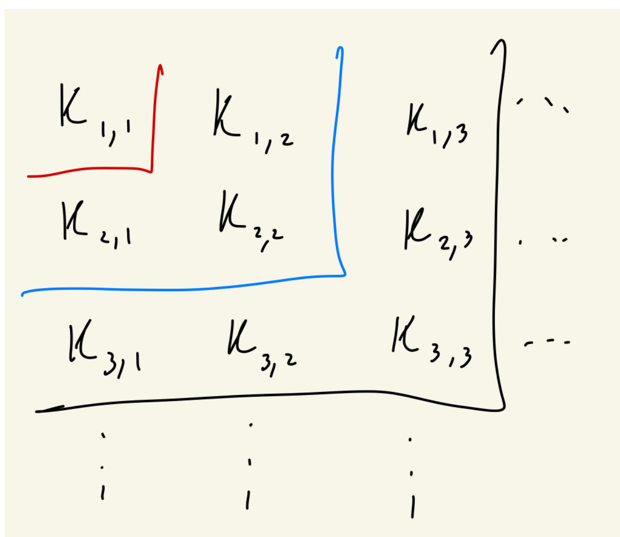

Figure 1: Union-ing the sets in the boxes

compact. Additionally, it is increasing. Hence we can use (2.59) to get the following:

$$|A| = |\bigcup_{k=1}^{\infty} F_k| = |\bigcup_{k=1}^{\infty} K_k| = \lim_{k\to\infty} |K_k|.$$

But note that $\text{Inn}(A) = \sup\{|F| : F \text{ is a compact subset of } A\}$. So that since this is an increasing sequence in the set that $\text{Inn}(A)$ is a supremum of we get that:

$$\text{Inn}(A) \geq |A|.$$

Thus $\text{Inn}(A) = |A|$. □

(b)

Show that inner measure is not a measure on the $\sigma-$algebra of all subsets of $\mathbb{R}$.

Proof. By theorem (2.22) we have that no measure can satisfy these four properties. We get that $(a.)$ is satisfied because of it's definition along with outer measure being well-defined on the power set of $\mathbb{R}$. $(b.)$ Follows from $(a.)$ because open intervals are Borel and hence Lebesgue measurable sets. Furthermore $(d.)$ can be shown: $\text{Inn}(A) = \sup\{|F| : F \text{ is a compact subset of } A\} = \sup\{|F+t| : F+t \text{ is a compact subset of } A+t\} = \text{Inn}(A+t)$. This follows since $|F| = |F+t|$ (translation invariance of outer measure) and that if $F \subset A$ is compact, then $F + t \subset A + t$ is compact. This follows since if F is closed and bounded, then $F + t$ is closed and bounded. Thus $\text{Inn}(\cdot)$ satisfies $(a), (b), (d)$ hence must not satisfy (c) hence is not a measure on all the subsets of $\mathbb{R}$. □