MATH 416 Mid-Term 2 Review Answers

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November 06, 2018

1 How many permutations of the 26 letters of the English alphabet contain at least one of the following words: "dog", "fish", or "bird" as consecutive letters? (Use Inclusion-Exclusion)

Proof. So first we define the sets $A_1 = \{$ The set of permutations containing the word "dog" $\}$, $A_2 = \{$ The set of permutations containing the word "fish" $\}$, and $A_3 = \{$ The set of permutations containing the word "bird"$\}$. Then by inclusion exclusion we have
$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$. Then if we note that A_2 and A_3, and A_1 and A_3 (since the pairs share at least 1 letter), are mutually exclusive, we have
$|A_2 \cap A_3| = 0$ and $|A_1 \cap A_2 \cap A_3| = 0$ and $|A_1 \cap A_3| = 0$. Hence
$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2|$. So note that if we fix the words and only choose where the first letter in the permutation is going to go we have: $|A_1| = P(24, 24)$ and the same reasoning follows for A_2 and A_3, except there we get $P(23, 23)$ because of the extra letter in the words. Hence
$|A_1| + |A_2| + |A_3| = P(24, 24) + 2 * P(23, 23)$. For $|A_1 \cap A_2|$ we treat the blocks A_1 and A_2 as two separate blocks. Hence we have 21 choices for the first block, and then 20 choices for the second block, and then 19 permutations for the remaining letters in the alphabet. Hence $|A_1 \cap A_2| = P(21, 21)$. Thus
$|A_1 \cup A_2 \cup A_3| = P(24, 24) + 2 * P(23.23) - P(21, 21)$. □

2 What is the number of non-negative integers less than 10,000 such that the sum of their digits is 21? (Use Inclusion-Exclusion)

Proof. First we want to set up a stars-and-bars problem: $x_1 + x_2 + x_3 + x_4 = 21$, where $0 \leq x_1, x_2, x_3, x_4 \leq 9$. Since we have the constraint that $x_i \leq 9$, we will solve this using the inclusion-exclusion principle. Let $A_i = \{$ Solution with the constraint that $x_i \geq 10.\}$, for $i \in \{1, 2, 3, 4\}$. Then we have the following solution to our problem $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4| = |S| - (|A_1 \cup A_2 \cup A_3 \cup A_4|)$. Note that by inclusion-exclusion we also have:
$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|) + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|$. Note that all of the three or four term intersections are zero, since the least the x_i could be is 10, hence we can't have them add up to 21. Hence
$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|)$.
So for any of the two-term intersections we have their cardinality as $\binom{1+3}{1}$, since 6 of those terms we have that cardinality of all six is $6 * \binom{4}{1}$. Similarly for the one term-cardinalites we have the equation $x_1 + x_2 + x_3 + x_4 = 21$ where one of the $x_i \geq 10$. Hence we get the equation $x_1 + x_2 + x_3 + x_4 = 11$, which has $\binom{11+3}{3}$ solutions by star-and-bars. We do this four times. Finally, to find $|S|$, we just use regular stars and bars and get $\binom{24}{3}$. Hence the final solution is $\binom{24}{3} - 4\binom{14}{3} + 6\binom{4}{1} = 592$ □

3 In each case, find the generating function and explain how it can be used to find the number of ways to insert tokens worth \$1, \$2, and \$5 into a vending machine to pay for an item that costs n dollars.

3.1 Assume that the order in which the tokens are inserted doesn't matter.

In this case for \$1 token we have $(x^0 + x^1 + x^2 + ...)$ ways to pay, in the case of \$2 token we have $(x^0 + x^2 + x^4 + ...)$, in the case of the \$5 token $(x^0 + x^5 + x^{10} + ...)$. Hence our generating function is:

$$p(x) = (1 + x + x^2 + ...)(1 + x^2 + x^4 + ...)(1 + x^5 + ...) = \frac{1}{1-x} * \frac{1}{1-x^2} * \frac{1}{1-x^5}$$

3.2 Assume that the order in which the tokens are inserted matters.

So for our first choice, we can either insert a \$1, \$2, or \$5 token. Since every step we're making a choice to either insert a \$1, \$3, or \$5 token, we get the following generating function:

$$p(x) = 1 + (x + x^2 + x^5) + (x + x^2 + x^5)^2 + ... = \frac{1}{1-(x+x^2+x^5)}.$$

4 Setup a generating function and use it to find the number of ways in which eleven identical blocks can be given to four children, if

4.1 Each child gets at least two blocks.

Since we're giving all four children at least 2 blocks, our powers can start at 3. Hence:

$$p(x) = (x^2 + x^3 + x^4 + x^5)^4 = x^8(\frac{1-x^4}{1-x})^4.$$

4.2 Each child gets at least two blocks, and the oldest child (Ben) gets an even number of blocks.

Again, since every kid gets at least 2 blocks, we only have 3 blocks to allocate. But that means Ben can only have 2 or 4. Hence we get:

$$p(x) = (x^2 + x^4)(x^2 + x^3 + x^4 + x^5)^3 = x^8(1 + x^2)(\frac{1-x^4}{1-x})^3.$$

5 In each case, either give an example of a graph or prove that none exists.

5.1 A simple graph with 6 vertices, whose degree sequence is: (4,4,3,3,3,3,2,1).

Not possible, by Theorem 11.1.1, since the degree sequence sums to 23. But also, for multiple reasons wrong.

5.2 A simple graph with degree sequence: (5,2,2,2,1).

Any vertex of a simple graph of order n, can have at most degree n - 1.

5.3 A simple graph with degree sequence: (3,2,2,1).

5.4 A simple graph with 6 vertices and 16 edges.

Not possible, since the largest number of edges for a order 6 graph we can have would be that of a complete graph, and that can have at most $\frac{(6-1)(6-2)}{2}$= 10 edges.

6 Draw all non-isomorphic graphs of order 5 with at most 7 edges and all of whose vertices have degree at least 2.

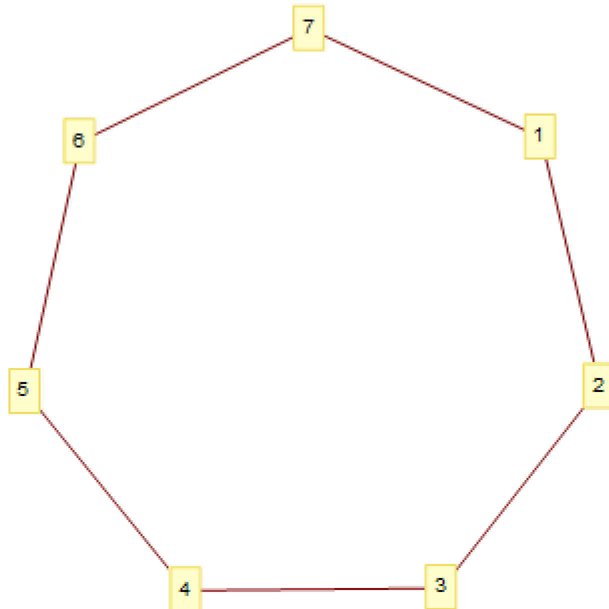

Figure 1: C_7

7 Draw all non-isomorphic graphs with degree sequence (2,2,2,2,2,2,2,2,2,2,2)

8 Which complete graphs $K_{m,n}$ have

8.1 Closed Eulerian Trails?

8.2 Open Eulerian Trails?

8.3 Hamilton Cycles?

8.4 (Open) Hamilton Paths?

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9.1 For each pair of graphs, determine whether they are isomorphic. If YES, find an isomorphism. If NO, explain why not.

9.1.1

$\phi(x_1) = y_7, \phi(x_2) = y_4, \phi(x_3) = y_3, \phi(x_4) = y_6, \phi(x_5) = y_1, \phi(x_6) = y_2, \phi(x_7) = y_5$

9.1.2

No, Graph A has degree sequence (3,3,3,3,3,3) and Graph B has degree sequence (3,3,3,3,2,2).

9.1.3

$\phi(A_1) = B_1, \phi(A_2) = B_2, \phi(A_3) = B_3, \phi(A_4) = B_6, \phi(A_5) = B_5, \phi(A_6) = B_4$

9.1.4

Not possible, since for vertex x_2, the adjacent vertices have degree 4,4,3, and 3. There is no such vertex in the second graph.

9.1.5

$\phi(x_1) = v_8, \phi(x_2) = v_1, \phi(x_3) = v_4, \phi(x_4) = v_5, \phi(x_5) = v_7, \phi(x_6) = v_2, \phi(x_7) = v_2, \phi(x_8) = v_6$

9.2 Determine which of these graphs are bipartite graphs.

10 Which of the following graphs have a closed Eulerian trail? An open Eulerian trail? Justify your answer by either finding such a trail or explaining why none exists.

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