Section 2.7: Combining Functions

Definition 1 (Functions). *A function $f : A \to \mathbb{R}$ is a rule that works on a domain A and gives you out real numbers $\mathbb{R}$. We need 2 things to be true about this rule:*

1. *It works for every x in the domain.*
2. *For every x in the domain, $f(x)$ is unique (the vertical line test)*

Definition 2 (Algebra of Functions). *Let f and g be functions with domain A and B. Then the functions $f+g, f-g, fg, \dfrac{f}{g}$ are defined as follows.*

- $(f+g)(x) = f(x) + g(x)$ *Domain* $A \cap B$
- $(f-g)(x) = f(x) - g(x)$ *Domain*$A \cap B$
- $(fg)(x) = f(x)g(x)$ *Domain*$A \cap B$
- $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ $\{x \in A \cap B : g(x) \neq 0\}$

Remember: *that $A \cap B$ is the set of all numbers that are in both A and B, and $\{x \in A \cap B : g(x) \neq 0\}$ means every x that is in both A and B and $g(x) \neq 0$.*

Example of Definitions.

Section 2.7: Combining Functions

Example 1 (Combinations of Functions and Their Domains)**.** *Let* $f(x) = \dfrac{1}{x-2}$ *and* $g(x) = \sqrt{x}$.

(a) Find the functions $f+g, f-g, fg, \dfrac{f}{g}$ *and their domains.*

(b) Find $(f+g)(4), (f-g)(4), (fg)(4), (f/g)(4)$.

Solution

Section 2.7: Combining Functions

Example 2 (Using Graphical Addition)**.** *The graphs of f and g are shown below. Use graphical addition to graph the function $f + g$.*

Solution

Section 2.7: Combining Functions

Definition 3 (Composition of Functions)**.** *Given two functions f and g, the composite function $f \circ g$ (also called the* ***composition*** *of f and g) is defined by*

$$(f \circ g)(x) = f(g(x)).$$

The domain $f : A \to B$ is A, but notice that for g to make sense we need:

1. *Every number x in B has to work with g*
2. *Every number in x in B has to have a unique output from g; that is, $g(x)$ is a single number. (Vertical Line Test)*

So we need if $g : C \to D$ is a function, then every number in B needs to be inside of C for $f \circ g$ to be a function.

Example of Definition

Section 2.7: Combining Functions

Example 3 (Finding the Composition of Functions)**.** *Let* $f(x) = x^2$ *and* $g(x) = x - 3$.

1. *Find the functions* $f \circ g$ *and* $g \circ f$ *and their domains.*
2. *Find* $(f \circ g)(5)$ *and* $(g \circ f)(7)$.

Solution.

Section 2.7: Combining Functions

Example 4 (Finding the Composition of Functions)**.** *If* $f(x) = \sqrt{x}$ *and* $g(x) = \sqrt{2-x}$ *find the following functions and their domains.*

1. $f \circ g$
2. $g \circ f$
3. $f \circ f$
4. $g \circ g$

Solution.

Section 2.7: Combining Functions

Example 5 (Composition of Three Functions)**.** *Find* $f \circ g \circ h$ *if* $f(x) = \dfrac{x}{x+1}$, $g(x) = x^{10}, h(x) = x + 3$.

Solution.

Section 2.7: Combining Functions

Example 6 (Recognizing a Composition of Functions)**.** *Given* $F(x) = \sqrt[4]{x+9}$*, find the functions* f *and* g *such that* $F = f \circ g$*.*

Solution.

Section 2.8: One-to-One Functions and Their Inverses

Definition 4 (One-To-One Function)**.** *A function with domain A is called a* ***one-to-one function*** *if no two elements of A have the same image, that is,*

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$

Equivalently, we can say that if $x_1 = x_2$, then $f(x_1) = f(x_2)$.

Definition 5 (Horizontal Line Test)**.** *A function is one-to-one if and only if no horizontal line intersects its graph more than once.*

Solution.

Section 2.8: One-To-One Functions and Their Inverses

Example 7 (Deciding Whether a Function is One-To-One)**.** *Is the function* $f(x) = x^3$ *one-to-one?*

Solution.

Section 2.8: One-To-One Functions and Their Inverses

Example 8 (Deciding Whether a Function is One-to-One). *Is the function $g(x) = x^2$ one-to-one?*

Solution.

Section 2.8: One-To-One Functions and Their Inverses

Example 9 (Showing That a Function is One-to-One)**.** *Show that the function* $f(x) = 3x + 4$ *is one-to-one.*

Solution.

Section 2.8: One-To-One Functions and Their Inverses

Definition 6 (The Inverse of A Function). *Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by*

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

Example of Definition.

Section 2.8: One-To-One Functions and Their Inverses

Example 10 (Finding f^{-1} for Specific Values). *If* $f(1) = 5, f(3) = 7, f(8) = -10$, *find* $f^{-1}(5), f^{-1}(7), f^{-1}(-10)$.

Solution.

Section 2.8: One-To-One Functions and Their Inverses

Example 11 (Finding Values of an Inverse Function)**.** *We can find specific values of an inverse function from a table or graph of the function itself.*

(a) The table written below gives values of a function h. From the table we see that $h^{-1}(8) = 3, h^{-1}(12) = 4, h^{-1}(3) = 6$.

(b) A graph of a function f is shown below. From the graph we see that $f^{-1}(5) = 7, f^{-1}(3) = 4$.

Solution.

Section 2.8: One-To-One Functions and Their Inverses

Theorem 1 (Inverse Function Property)**.** *Let f be a one-to-one function with domain A and range B. The inverse function f^{-1} satisfies the following cancellation properties:*

- $f^{-1}(f(x)) = x$ *for every* x *in* A
- $f(f^{-1}(x)) = x$ *for every* x *in* B

Conversely, any function f^{-1} satisfying these equation is the inverse of f.

Example 12 (Verifying That Two Functions are Inverses)**.** *Show that $f(x) = x^3$ and $g(x) = x^{1/3}$ are inverses of each other.*

Solution

Section 2.8: One-To-One Functions and Their Inverses

Theorem 2 (How To Find The Inverse of a One-To-One Function)**.** *1. Write* $y = f(x)$

2. Solve this equation for x *in terms of* y *(if possible)*

3. Interchange x *and* y*. The resulting equation is* $y = f^{-1}(x)$*.*

Example 13 (Finding the Inverse of a Function)**.** *Find the inverse of the function* $f(x) = 3x - 2$.

Solution.

Section 2.8: One-To-One Functions and Their Inverses

Example 14 (Finding the Inverse of a Function)**.** *Find the inverse of the function* $f(x) = \dfrac{x^5 - 3}{2}$.

Solution.

Section 2.8: One-To-One Functions and Their Inverses

Example 15 (Finding the Inverse of a Rational Function)**.** *Find the inverse of the function* $f(x) = \dfrac{2x+3}{x-1}$.

Solution.

Section 2.8: One-To-One Functions and Their Inverses

Theorem 3 (Graphing The Inverse of a Function). *The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.*

Example 16 (Graphing the Inverse of a Function). *(a) Sketch the graph of $f(x) = \sqrt{x-2}$*

(b) Use the graph of f to sketch the graph of f^{-1}

(c) Find an equation for f^{-1}

Section 5.1: The Unit Circle

Definition 7 (The Definition of a Circle)**.** *A* ***circle*** *is all the points in the $xy-$plane, such that they are the same distance from a single point. This distance is the* ***radius*** *of a circle and the point is the* ***center****.*

If $r \geq 0$ is the radius of the circle and (x_0, y_0) is the center of the circle, then the equation that defines a circle is all of the points (x, y) such that:

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

We can also write this as:

$$\pm\sqrt{(x - x_0)^2 + (y - y_0)^2} = r.$$

Definition 8. *The* ***unit circle*** *is the circle of radius 1 centered at the origin in the $xy-$plane. Its equation is $x^2 + y^2 = 1$.*

Example 17 (A Point on The Unit Circle)**.** *Show that the point $P\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right)$ is on the unit circle.*

Solution.

Section 5.1: The Unit Circle

Example 18 (Locating a Point on the Unit Circle)**.** *The point* $P(\sqrt{3}/2, y)$ *is on the unit circle in Quadrant* IV*. Find its* $y-$*coordinate.*

Solution.

Section 5.1: The Unit Circle

Definition 9 (Arc Length). *The* ***arc length*** *is the distance of* 2 *points by traversing the circumference of the circle on which they both lie. For a circle with radius R in the xy-plane, if (x_0, y_0) and (x_1, y_1) are both on the circle, the distances between the points along the circle is then*

$$\theta R,$$

where θ is the angle between the lines from the center to the points (x_0, y_0) and (x_1, y_1).

Example of Definition.

Section 5.1: The Unit Circle

Definition 10 (Terminal Point)**.** *Suppose t is a real number. If $t \geq 0$, then move along the unit circle a distance of t in the counter clockwise direction starting off at the point* $(0, 1)$*. The point $P(x, y)$ that we end at on the unit circle, is called a* ***terminal point****. If $t < 0$, then do the same thing just in the clockwise direction.* ***Remember:*** *The circumference of a circle is 2π.*

Example of Definition.

Section 5.1: The Unit Circle

Example 19 (Finding Terminal Points). *Find the terminal point on the uni circle determined by each real number t.*

(a) $t = 3\pi$

(b) $t = -\pi$

(c) $t = -\dfrac{\pi}{2}$

Solution.

Section 5.1: The Unit Circle

Example 20 (Finding Terminal Points)**.** *Find the terminal point determined by each given real number t.*

(a) $t = -\dfrac{\pi}{4}$

(b) $t = \dfrac{3\pi}{4}$

(c) $t = -\dfrac{5\pi}{6}$

Solution.

t	Terminal Point Determined by t
0	$(0,1)$
$\pi/2$	$(\sqrt{3}/2, 1/2)$
$\pi/4$	$(\sqrt{2}/2, \sqrt{2}/2)$
$\pi/3$	$(1/2, \sqrt{2}/2)$
$\pi/2$	$(0,1)$

Section 5.1: The Unit Circle

Definition 11 (Reference Number)**.** *Let t be a real number. The* ***reference number*** $\bar{t}$ *associated with t is the shortest distance along the unit circle between the terminal point determined by t and the $x-$axis.*

Example 21 (Finding Reference Numbers)**.** *Find the reference number for each value of t.*

(a) $t = \dfrac{5\pi}{6}$

(b) $t = \dfrac{7\pi}{4}$

(c) $t = -\dfrac{2\pi}{3}$

(d) $t = 5.80$

Section 5.1: The Unit Circle

Theorem 4 (Using Reference Numbers to Find Terminal Points)**.** *To find the terminal point P determined by any value of t, we use the following steps:*

1. *Find the reference number $\bar{t}$.*
2. *Find the terminal point $Q(a, b)$ determined by $\bar{t}$.*
3. *The terminal point determined by t is $P(\pm a, \pm b)$, where the signs are chosen according to the quadrant in which this terminal point lies.*

Example 22 (Using Reference Numbers to Find Terminal Points)**.** *Find the terminal point determined by each given real number t.*

1. $t = \dfrac{5\pi}{6}$
2. $t = \dfrac{7\pi}{4}$
3. $t = -\dfrac{2\pi}{3}$

Solution.

Section 5.1: The Unit Circle

Example 23 (Finding the Terminal Point for Large t)**.** *Find the terminal point determined by* $t = \dfrac{29\pi}{6}$.

Solution.

Section 5.2: Trigonometric Functions of Real Numbers

Definition 12 (Trigonometric Functions)**.** *Let t be any real number and let $P(x, y)$ be the terminal point on the unit circle determined by t. We define*

$$\sin(t) = y \qquad \cos(t) = x \qquad \tan(t) = \frac{y}{x}\ (x \neq 0)$$

$$\csc(t) = \frac{1}{y}\ (y \neq 0) \qquad \sec(t) = \frac{1}{x}\ (x \neq 0) \qquad \cot(t) = \frac{x}{y}\ (y \neq 0)$$

*These are sometimes called the **circular functions**.*

Example 24 (Evaluating Trigonometric Functions)**.** *Find the six trigonometric functions of each given real number t.*

(a) $t = \dfrac{\pi}{3}$

(b) $t = \dfrac{\pi}{2}$

Solution.

5.2: Trigonometric Functions of Real Numbers

Theorem 5 (Special Values of The Trigonometric Functions). *The following values of the trigonometric functions are obtained from the special terminal points.*

t	$\sin(t)$	$\cos(t)$	$\tan(t)$	$\csc(t)$	$\sec(t)$	$\cot(t)$
0	0	1	0	$---$	1	$---$
$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$	2	$2\sqrt{3}/3$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$2\sqrt{3}/3$	2	$\sqrt{3}/3$
$\pi/2$	1	0	$---$	1	$---$	0

Theorem 6 (Domains of The Trigonometric Function).

Function	***Domain***
$\sin, \cos$	*All real numbers*
$\tan, \sec$	*All real numbers other than* $\frac{\pi}{2} + n\pi$ *for any integer* n
$\cot, \csc$	*All real numbers other than* $n\pi$ *for any integer,* n

Theorem 7 (Signs of The Trigonometric Functions).

Quadrant	***Positive Functions***	***Negative Functions***
I	*all*	*none*
II	$\sin, \csc$	$\cos, \sec, \tan, \cot$
III	$\tan, \cot$	$\sin, \csc, \cos, \sec$
IV	$\cos, \sec$	$\sin, \csc, \tan, \cot$

5.2: Trigonometric Functions of Real Numbers

Theorem 8 (Evaluating Trigonometric Functions For Any Real Number)**.** *To find the values of the trigonometric functions for any real number t, we carry out the following steps.*

1. ***Find the reference number.*** *Find the reference number $\bar{t}$ associated with t.*
2. ***Find the sign.*** *Determine the sign of the trigonometric function of t by noting the quadrant in which the terminal point lies.*
3. ***Find the value.*** *The value of the trigonometric function of t is the same, except possibly for sign, as the value of the trigonometric function of $\bar{t}$.*

Example 25 (Evaluating Trigonometric Functions)**.** *Find each value.*

1. $\cos \dfrac{2\pi}{3}$
2. $\tan \left(\dfrac{-\pi}{3}\right)$
3. $\sin \dfrac{19\pi}{4}$

Solution.

5.2: Trigonometric Functions of Real Numbers

Theorem 9 (Even-Odd Properties)**.** *Sine, cosecant, tangent and cotangent are odd functions; cosine and secant are even functions.*

$$\begin{array}{lll} \sin(-t) = -\sin(t) & \cos(-t) = \cos(t) & \tan(-t) = -\tan(t) \\ \csc(-t) = -\csc(t) & \sec(-t) = \sec(t) & \cot(-t) = -\cot(t) \end{array}$$

Example 26 (Even and Odd Trigonometric Functions)**.** *Use the even-odd properties of the trigonometric functions to determine each value.*

(a) $\sin\left(\frac{-\pi}{6}\right)$

(b) $\cos\left(-\frac{\pi}{4}\right)$

Solution.

5.2: Trigonometric Functions of Real Numbers

Theorem 10 (Fundamental Identities)**.** ***Reciprocal Identities***

$$\csc(t) = \frac{1}{\sin(t)} \quad \sec(t) = \frac{1}{\cos(t)} \quad \cot(t) = \frac{1}{\tan(t)} \quad \tan(t) = \frac{\sin(t)}{\cos(t)} \quad \cot(t) = \frac{\cos(t)}{\sin(t)}$$

Pythagorean Identities

$$\sin^2(t) + \cos^2(t) = 1 \quad \tan^2(t) + 1 = \sec^2(t) \quad 1 + \cot^2(t) = \csc^2(t)$$

Example 27 (Finding All Trigonometric Functions from the Value of One)**.** *If* $\cos(t) = \frac{3}{5}$ *and* t *is in Quadrant* IV, *find the values of all trigonometric functions at* t.

Solution.

5.2: Trigonometric Functions of Real Numbers

Example 28 (Writing One Trigonometric Function in Terms of Another)**.** *Write* $\tan(t)$ *in terms of* $\cos(t)$*, where* t *is in Quadrant III.*

Section 5.3: Trigonometric Graphs

Definition 13 (Periodic Functions). *A function f is periodic if there is a positive number p, called the **period**, such that $f(t+p) = f(t)$ for every t in the domain of f.*

Theorem 11 (Periodic Properties of Sine and Cosine). *The circumference of the circle being 2π and $\sin(t) = y$ and $\cos(t) = x$ for the terminal point $P(x, y)$ of t, so that $\sin(t+2\pi)$ and $\cos(t+2\pi)$ will be the same value for all t in your domain. That is sine and cosine have period 2π:*

$$\sin(t+2\pi) = \sin(t) \qquad \cos(t+2\pi) = \cos(t)$$

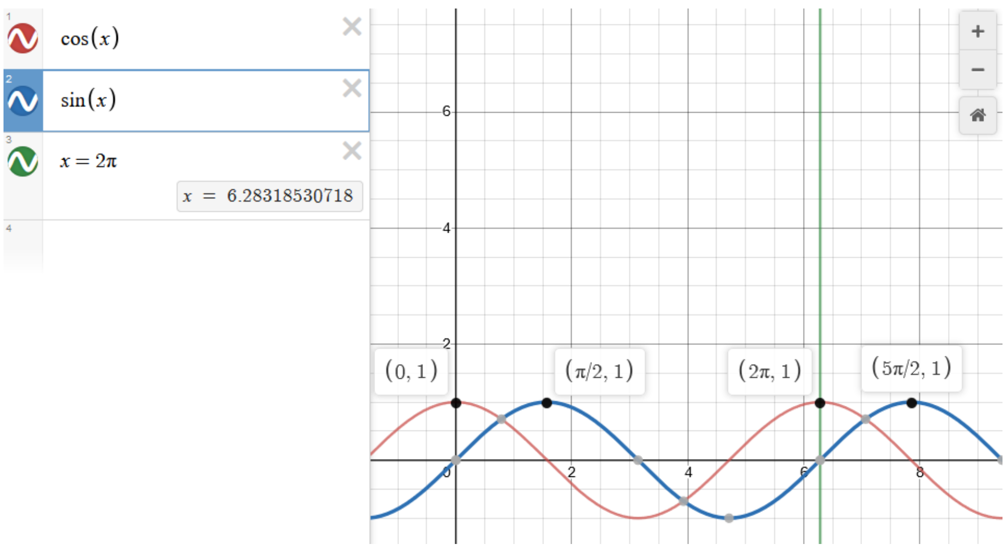

Figure 1: Notice that the graph's repeat every 2π, **Blue:** $\sin(t)$, **Red:** $\cos(t)$

5.3: Trigonometric Graphs

Example 29. *Sketch the graph of each function.*

(a) $f(x) = 2 + \cos(x)$

(b) $g(x) = -\cos(x)$

Solution.

5.3: Trigonometric Graphs

Definition 14 (Amplitude)**.** *In general, for the functions*

$$y = a\sin(x) \qquad \textit{and} \qquad y = a\cos(x)$$

the number $|a|$ *is called the amplitude and is the largest value these functions can attain.*
Note: *That if* $a < 0$*, that is* a *is negative, then the graph is inverted. For instance,* $\sin(\pi/2) = 1$ *is the maximum of the function* $\sin(x)$*, but that* $-\sin(\pi/2) = -1$ *is the minimum of the function* $-\sin(x)$ *and will have a maximum at* $x = 3\pi/2$ *since* $-\sin(3\pi/2) = -1(-1) = 1$*.*

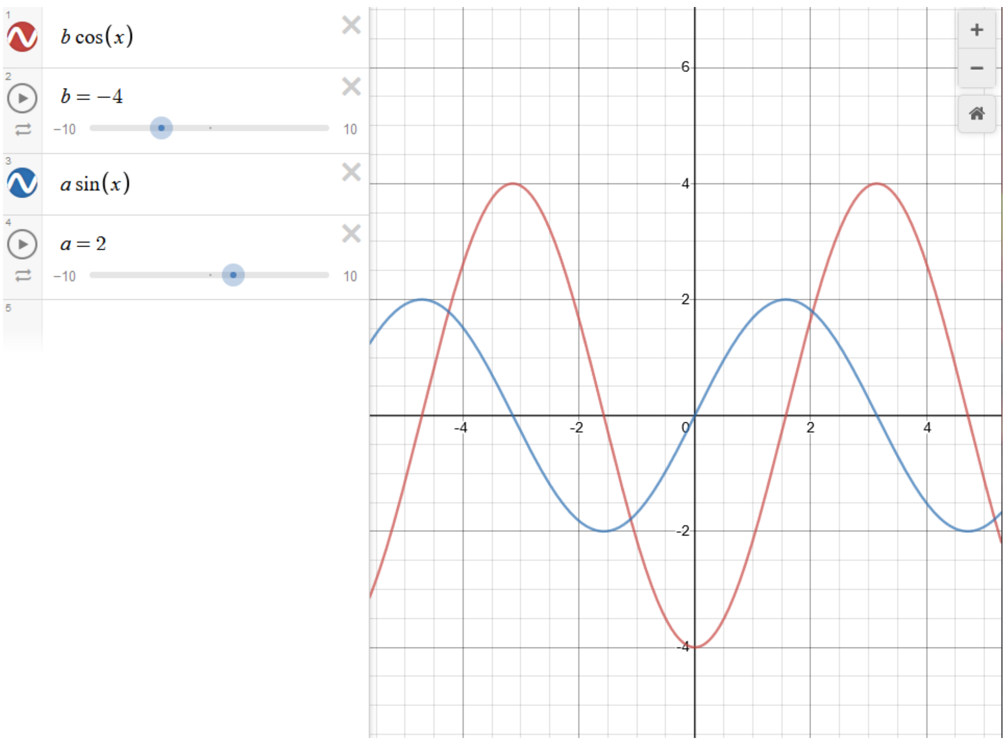

Figure 2: The **red** graph is $b\cos(x)$ where $b = -4$ and the **blue** graph is $a\sin(x)$ where $a = 2$. Notice that the maximums are these values $|a|$ and $|b|$.

5.3: Trigonometric Graphs

Example 30. *Find the amplitude of* $y = -3\cos(x)$*, and sketch its graph.*

Solution.

5.3: Trigonometric Graphs

Definition 15 (Adjusted periods)**.** *Since the sine and cosine functions have period* 2π*, the functions*

$$y = a\sin(kx) \qquad y = a\cos(kx) \qquad (k > 0)$$

complete one period as kx *varies from* 0 *to* 2π*, that is, for* $0 \leq kx \leq 2\pi$ *so adjusting this:*

$$0 \leq x \leq \frac{2\pi}{k}.$$

The sine and cosine curves

$$y = a\sin(kx) \qquad y = a.\cos(kx) \qquad (k > 0)$$

have ***amplitude*** $|a|$ *and* ***period*** $2\pi/k$*.*
An appropriate interval on which to graph one complete is $[0, 2\pi/k]$*.*

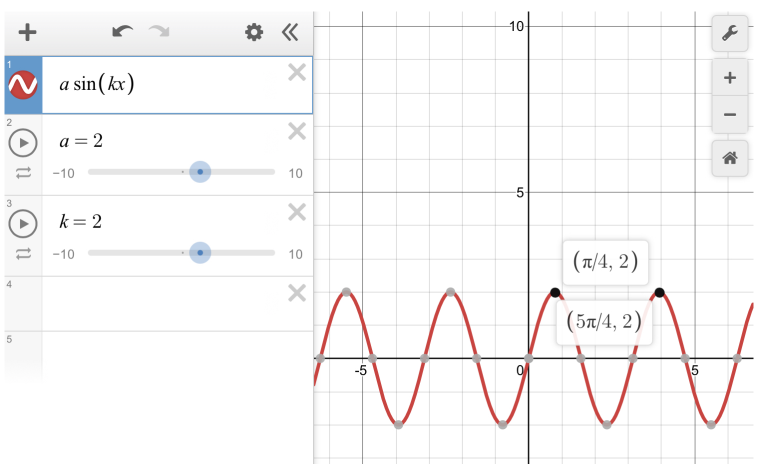

Figure 3: Notice that the function attains maximums at $\frac{\pi}{4}$ and $\frac{5\pi}{4}$, the period is π now instead of 2π

5.3: Trigonometric Graphs

Example 31. *Find the amplitude and period of each function, and sketch its graph.*

1. $y = 4\cos(3x)$
2. $y = -2\sin\left(\frac{1}{2}x\right)$

Solution.

5.3: Trigonometric Graphs

Definition 16 (Shifted Sine and Cosine Curves). *The sine and cosine curves*

$$y = a\sin(k(x-b)) \qquad y = a\cos(k(x-b)) \qquad (k>0)$$

have ***amplitude*** $|a|$, ***period*** $2\pi/k$, *and* ***horizontal shift*** b. *An appropriate interval on which to graph one complete period is* $[b, b+(2\pi/k)]$.

Example 32. *Find the amplitude, period and horizontal shift of* $y = 3\sin\left(2\left(x-\frac{\pi}{4}\right)\right)$, *and graph one complete period.*

Solution.

5.3: Trigonometric Graphs

Example 33. *Find the amplitude, period, horizontal shift of* $y = \frac{3}{4}\cos\left(2x + \frac{2\pi}{3}\right)$, *and graph one complete period.*

Solution.

5.3: Trigonometric Graphs

Example 34 (A Cosine Curve with Variable Amplitude). *Graph the functions* $y = x^2, y = -x^2$ *and* $y = x^2\cos(6\pi x)$ *on a common screen.*

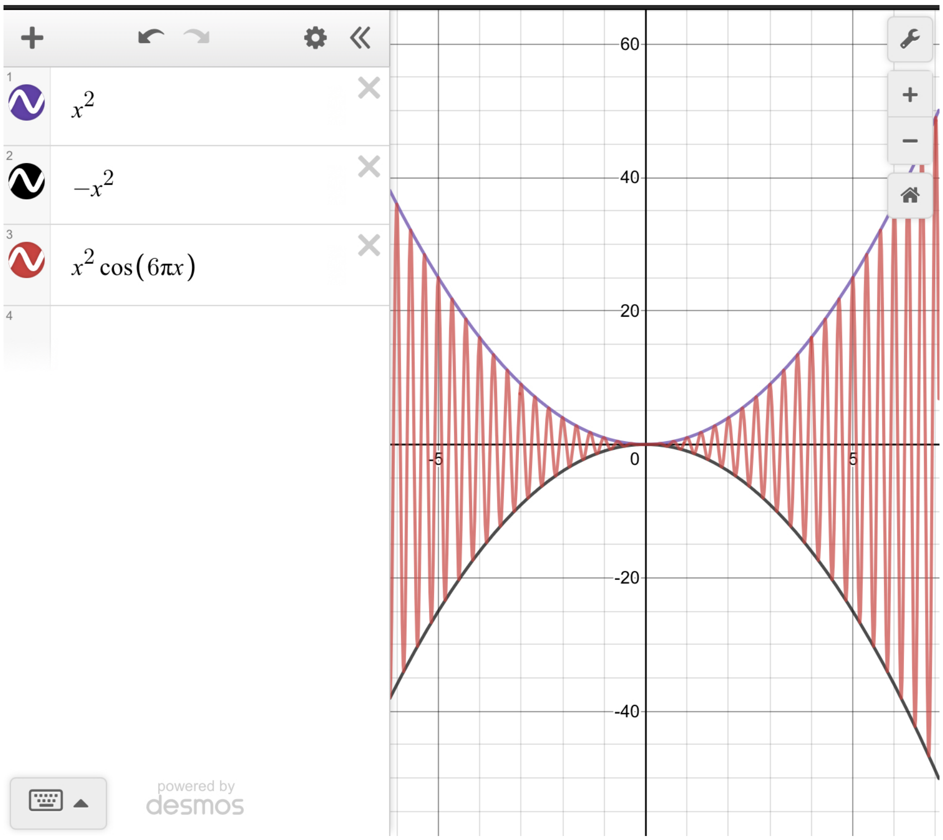

Figure 4: Notice that the function explodes as $x \to \infty$, but that the function is bounded above and below by x^2 and $-x^2$.

Note: In general, a function with a variable amplitude:

$$a(x)\sin(kx)$$

will always be bounded below by $-a(x)$ and bounded above by $a(x)$.Why??

5.3: Trigonometric Graphs

Example 35 (A Cosine Curve with Variable Amplitude)**.** *Graph the function* $f(x) = \cos(2\pi x)\cos(16\pi x)$.

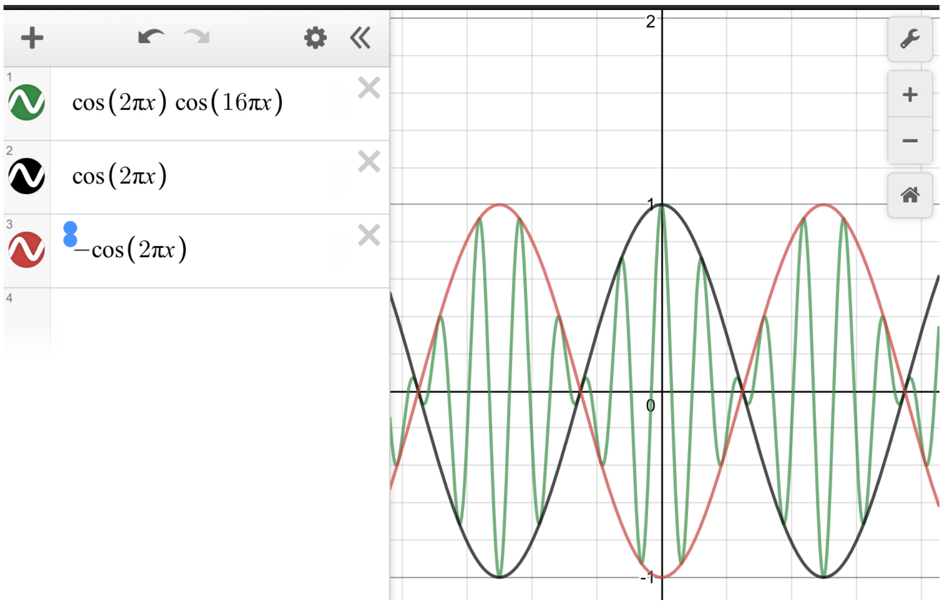

Figure 5: Notice now that the function is bounded below by the function $y = -\cos(2\pi x)$ and above by $y = \cos(2\pi x)$.

5.3: Trigonometric Graphs

Example 36 (A Sine Curve with Decaying Amplitude). *The function* $f(x) = \dfrac{\sin(x)}{x}$ *is important in calculus. Graph this function, and comment on its behavior when* x *is close to* 0.

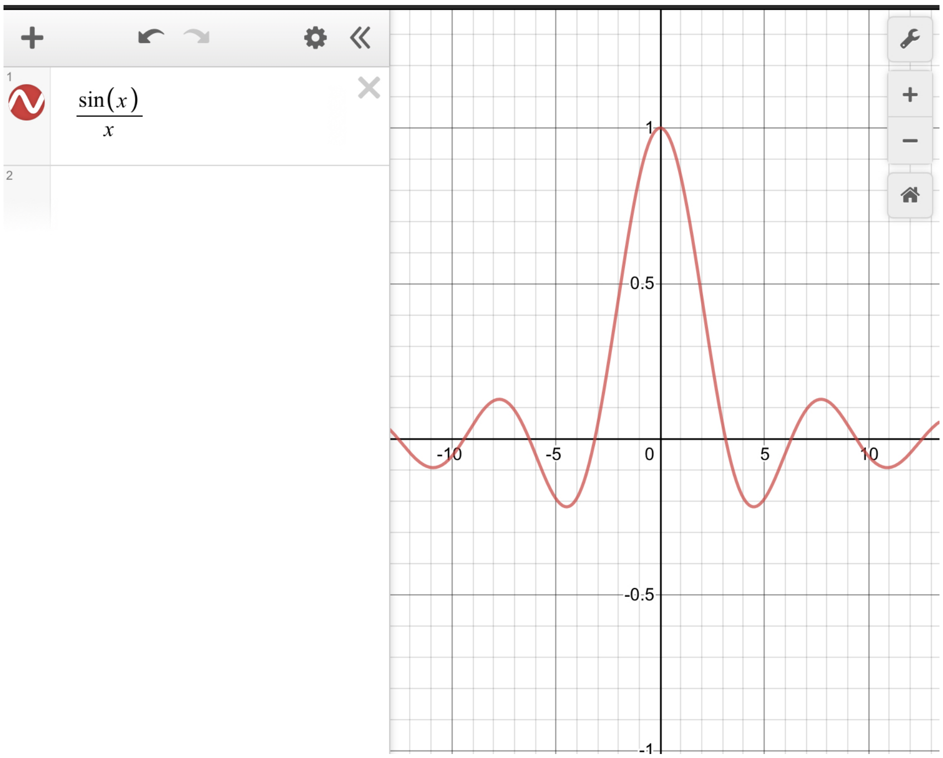

Figure 6: Notice that the behavior is as expected, but it's actually finite at $x = 0$!!!

Section 5.4: More Trigonometric Graphs

Theorem 12 (Periodic Properties)**.** *The functions tangent and cotangent have period π:*

$$\tan(x+\pi) = \tan(x) \qquad \cot(x+\pi) = \cot(x)$$

The functions cosecant and secant have period 2π:

$$\csc(x+2\pi) = \csc(x) \qquad \sec(x+2\pi) = \sec(x).$$

Because of this period, we only need to graph the functions on an interval of size π.

Note: $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and $\cos(\pi/2) = 0$, so that we'll have a **vertical asymptote** at $x = \dfrac{\pm pi}{2}/$

5.4: More Trigonometric Graphs

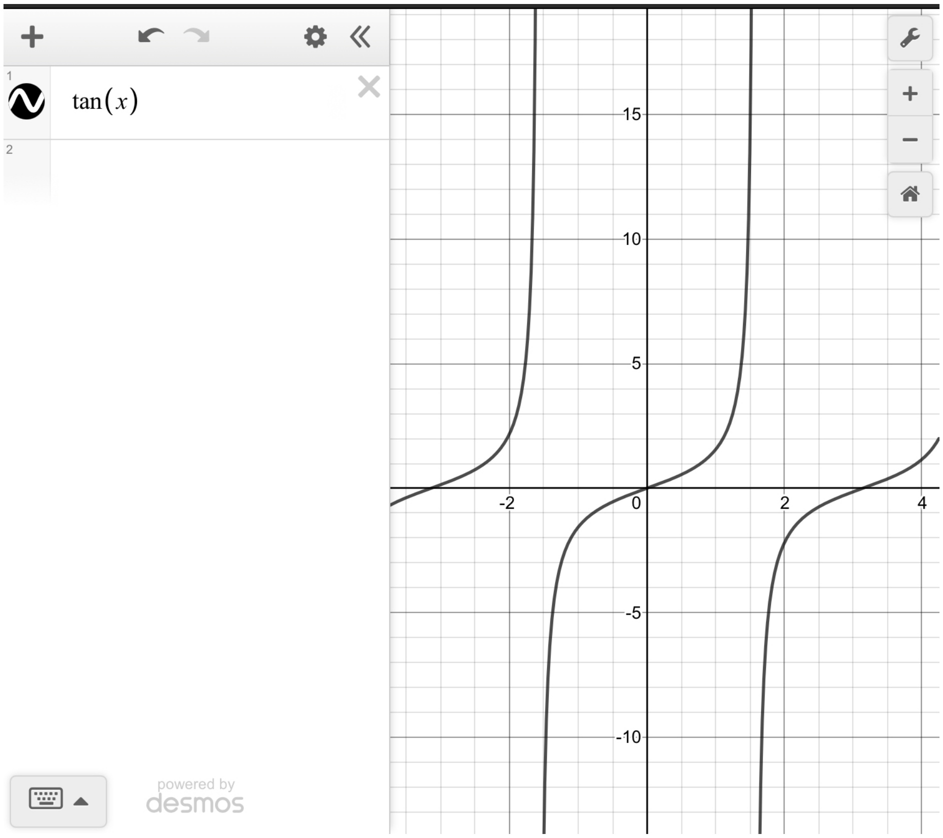

Figure 7: Notice that this is a periodic function, but has vertical asymptotes!

Example 37 (Graphing Tangent Curve)**.** *Graph each function.*

(a) $y = 2\tan(x)$

(b) $y = -\tan(x)$

5.4: More Trigonometric Graphs

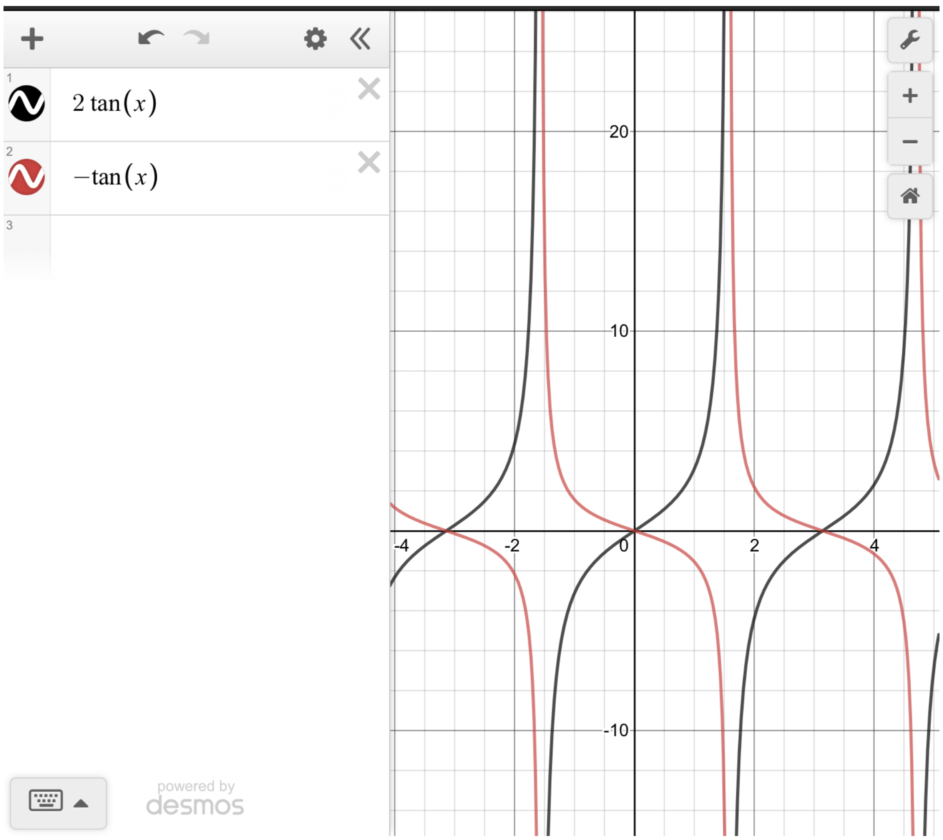

Figure 8: Notice that the vertical asymptotes are mirrored, and that $2\tan(x)$ is slightly larger than $\tan(x)$ as expected

Definition 17 (Tangent and Cotangent Curves)**.** *Take the tangent and cotangent functions have period π, the functions*

$$y = a\tan(kx) \qquad y = a\cot(kx) \qquad (k > 0)$$

complete one period as kx varies between 0 and π, that is $0 \leq kx \leq \pi$. Solve for x gives us:

$$y = a\tan(kx) \qquad y = a\cot(kx) \qquad (k > 0)$$

have period $\dfrac{\pi}{k}$.

To graph one period of $y = a\tan(kx)$, an appropriate interval is $\left(-\dfrac{\pi}{2k}, \dfrac{\pi}{2k}\right)$.

5.4: More Trigonometric Graphs

To graph one period of $y = a\cot(kx)$, *an appropriate interval is* $\left(0, \dfrac{\pi}{k}\right)$

5.4: More Trigonometric Graphs

Example 38 (Graphing Tangent Curves). *Graph each function.*

(a) $y = \tan(2x)$

(b) $y = \tan\left(2\left(x - \frac{\pi}{4}\right)\right)$

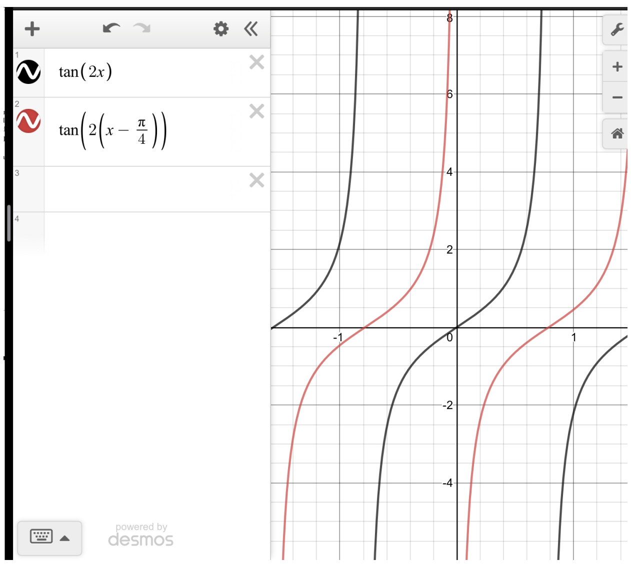

Figure 9: Notice that these are the same graph!

5.4: More Trigonometric Graphs

Example 39 (A Horizontally Shifted Cotangent Curve). *Graph the function* $y = 2\cot\left(3x - \dfrac{\pi}{4}\right)$.

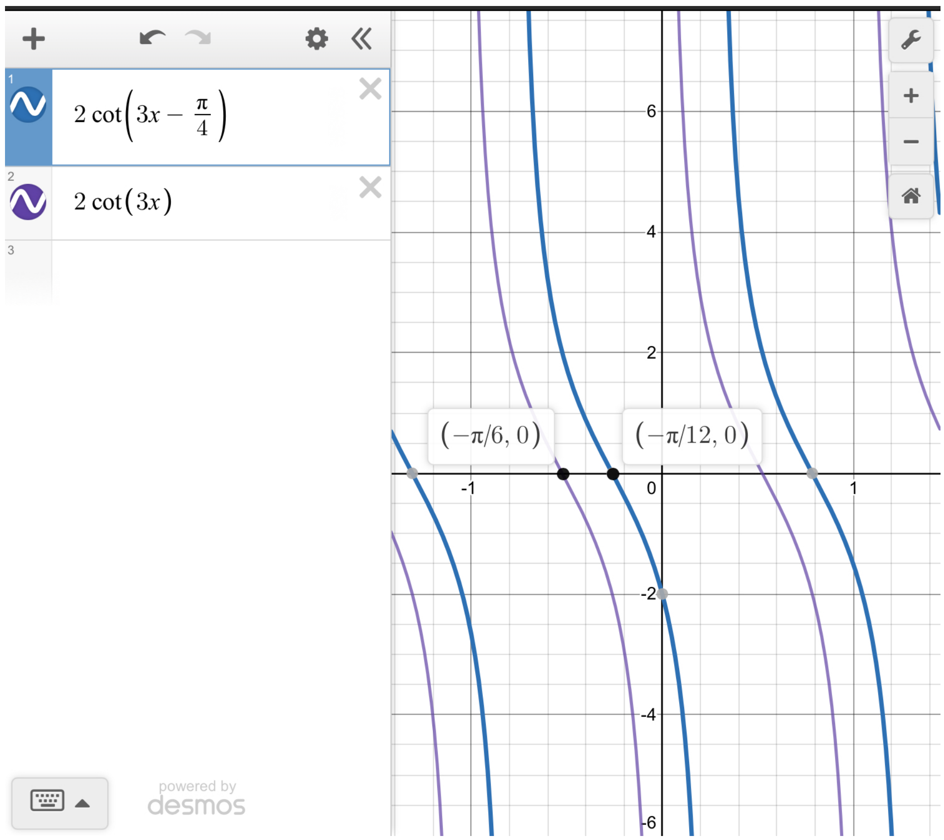

Figure 10: These are perfectly shifted graphs, to by a factor of $\pi/6$

5.4: More Trigonometric Graphs

Definition 18 (Cosecant and Secant Curves)**.** *The functions*

$$y = a\csc(kx) \qquad y = a\sec(kx) \qquad (k > 0)$$

have period $2\pi/k$.

An appropriate interval on which to graph one complete period is $(0, 2\pi/k)$

5.4: More Trigonometric Graphs

Example 40 (Graphing Cosecant Curves). *Graph each function.*

(a) $y = \dfrac{1}{2}\csc(2x)$

(b) $y = \dfrac{1}{2}\csc\left(2x + \dfrac{\pi}{2}\right)$

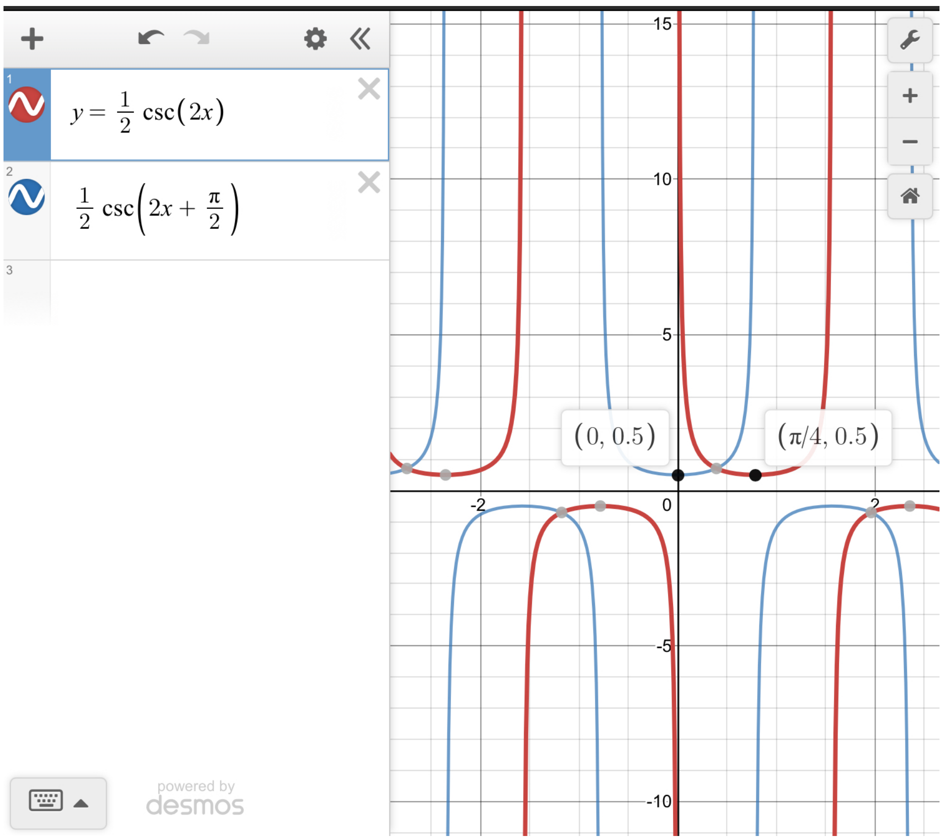

Figure 11: Notice that these are shifted by $\pi/4$!

5.4: More Trigonometric Graphs

Example 41 (Graphing a Secant Curve)**.** *Graph* $y = 3\sec\left(\frac{x}{2}\right)$.

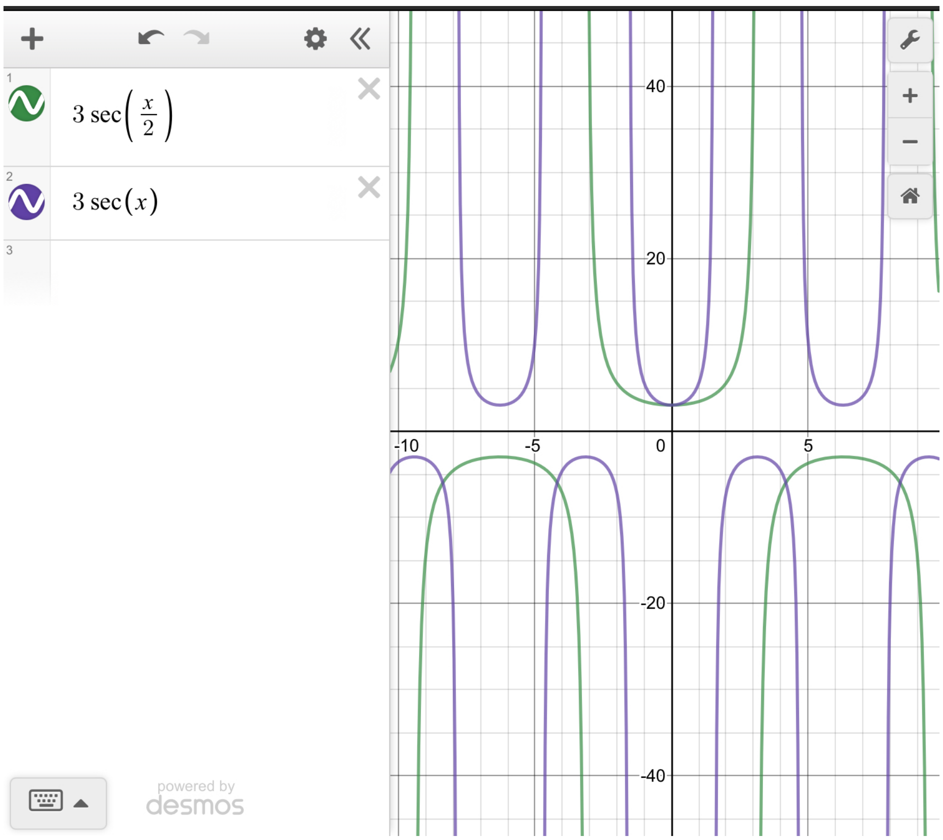

Figure 12: Notice that similar to the graphs of cosine and sine the k value will affect the width of each full wave.

Section 5.5: Inverse Trigonometric Functions and Their Graphs

Note 1. *We'll begin by remembering that for a function to have an inverse in the first place, that we need the function to be* $1-1$.

To do this with the trigonometric functions, we take the functions and just look at them for one half of a period.

Why? Let's look at the graphs of the function.

Definition 19 (Definition of Inverse Sine Function)**.** *The* ***inverse sine function*** *is the function* $\sin^{-1}$ *with domain* $[-1,1]$ *and range* $[-\pi/2, \pi/2]$ *defined by*

$$\sin^{-1}(x) = y \iff \sin(y) = x$$

The inverse since function is also called ***arcsine****, denoted by* ***arcsin****.*

Then for $y = \sin^{-1}(x)$*, this is the number in the interval* $[-\pi/2, \pi/2]$ *whose sine is* x*.*

5.5: Inverse Trigonometric Functions and Their Graphs

Theorem 13 (The Cancellation Properties)**.**

$$\sin(\sin^{-1}(x)) = x \qquad \textit{for } -1 \leq x \leq 1$$

$$\sin^{-1}(\sin(x)) = x \qquad \textit{for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

Example 42 (Evaluating the Inverse Sine Function)**.** *Find each value.*

(a) $\sin^{-1}\frac{1}{2}$

(b) $\sin^{-1}\left(\frac{-1}{2}\right)$

(c) $\sin^{-1}\frac{3}{2}$

Solution.

5.5: Inverse Trigonometric Functions and Their Graphs

Example 43 (Evaluating Expressions with Inverse Sine)**.** *Find each value.*

(a) $\sin^{-1}\left(\sin\dfrac{\pi}{3}\right)$

(b) $\sin^{-1}\left(\sin\dfrac{2\pi}{3}\right)$

Solution.

5.5: Inverse Trigonometric Functions and Their Graphs

Note 2 (Inverse Cosine)**.** *We can apply the same process as we did to define* $\sin^{-1}(x)$*, now though we need to be careful when we're cutting the graph up.*
Instead of $\left[\frac{-\pi}{2}, \frac{+\pi}{2}\right]$ *we'll have* $[0, \pi]$.

Definition 20 (The Definition of the Inverse Cosine Function)**.** *The* ***inverse cosine function*** *is the function* $\cos^{-1}$ *with domain* $[-1, 1]$ *and range* $[0, \pi]$ *defined by*

$$\cos^{-1}(x) = y \qquad \cos(y) = x$$

The inverse cosine function is also called ***arccosine****, denoted by* ***arccos****.*

5.5: Inverse Trigonometric Functions and Their Graphs

Theorem 14 (Cancellation Properties)**.**

$$\cos(\cos^{-1}(x)) = x \qquad \textbf{\textit{for}} \ -1 \leq x \leq 1$$

$$\cos^{-1}(\cos(x)) = x \qquad \textbf{\textit{for}} \ 0 \leq x \leq \pi$$

Example 44 (Evaluating The Inverse Cosine Function)**.** *Find each value.*

(a) $\cos^{-1}\dfrac{\sqrt{3}}{2}$

(b) $\cos^{-1}(0)$

(c) $\cos^{-1}\left(\dfrac{-1}{2}\right)$

Solution.

5.5: Inverse Trigonometric Functions and Their Graphs

Example 45 (Evaluating Expression with Inverse Cosine)**.** *Find each value.*

(a) $\cos^{-1}\left(\cos\dfrac{2\pi}{3}\right)$

(b) $\cos^{-1}\left(\cos\dfrac{5\pi}{3}\right)$

Solution.

5.5: Inverse Trigonometric Functions and Their Graphs

Definition 21 (Definition of The Inverse Tangent Function)**.** *The **inverse tangent function** is the function* $\tan^{-1}$ *with domain* $\mathbb{R}$ *and range* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *defined by*

$$\tan^{-1}(x) = y \iff \tan(y) = x$$

*The inverse tangent function is also called **arctangent**, denote by* arctan*.*

Theorem 15 (Cancellation Properties)**.**

$$\tan(\tan^{-1}(x)) = x \qquad \textit{for } x \in \mathbb{R}$$

$$\tan^{-1}(\tan(x)) = x \qquad \textit{for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Note 3. *Why is the the domain of* $\tan^{-1}$ *all of* $\mathbb{R}$*??*

5.5: Inverse Trigonometric Functions and Their Graphs

Example 46. *Find each value.*

(a) $\tan^{-1}(1)$

(b) $\tan^{-1}(\sqrt{3})$

(c) $\tan^{-1}(20)$

Section 5.6: Modeling Harmonic Motion

Note 4. *What does the average air temperature, animal populations, and the movement of oscillations of subatomic particles have in common?*

Repetition is everywhere in nature, with respect to time, space, and many other phenomena. When this repetition is constant, that is they do not change repetition to repetition, these are what we call simple harmonic motions. And the best model we have for these is that of periodic functions like sine and cosine!

Definition 22 (Simple Harmonic Motion)**.** *If the equation describing the displacement y of an object at a time t is:*

$$y = a\sin(\omega t) \qquad or \qquad y = a\cos(\omega t)$$

*then the object is in **simple harmonic motion**. In this case*

$$\textbf{\textit{amplitude}} = |a| \qquad \textit{Maximum displacement of the object}$$

$$\textbf{\textit{period}} = \frac{2\pi}{\omega} \qquad \textit{Time required to complete one cycle}$$

$$\textbf{\textit{frequency}} = \frac{\omega}{2\pi} \qquad \textit{Number of cycle per unit of time}$$

Section 5.6: Modeling Harmonic Behavior

Example 47. *The displacement of a mass suspended by a spring is modeled by the function*

$$y = 10\sin(4\pi t)$$

where y is measured in inches and t is seconds.

(a) Find the amplitude, period, and frequency of the motion of the mass.

(b) Sketch the graph of the displacement of the mass.

Solution.

Section 5.6: Modeling Harmonic Behavior

Example 48 (Vibrations of a Musical Note). *A sousaphone player plays the note E and sustains the sound for some time. For a pure E the variation in pressure from normal air pressure is given by*

$$V(t) = 0.2 \sin(80\pi t)$$

where V is measured in pounds per square inch and t is measured in seconds.

(a) Find the amplitude, period and frequency of V.

(b) Sketch a graph of V

(c) If the player increases the loudness of the note, how does the equation for V change?

(d) If the player is playing the note incorrectly and it's a little flat, how does the equation for V change?

Solution.

Section 5.6: Modeling Harmonic Behavior

Example 49 (Modeling a Vibrating Spring). *A mass is suspended from a spring. The spring is compressed a distance of* 4 *cm and then released. It's observed that the mass returns to the compressed position after* $\frac{1}{3}$ *s.*

(a) Find a function that models the displacement of the mass.

(b) Sketch the graph of the displacement of the mass.

Solution.

Section 5.6: Modeling Harmonic Behavior

Definition 23 (Shifted Harmonic Motion)**.** *Recall that to shift functions vertically and horizontally:*

$$y = a\sin(\omega(t-c)) + b \qquad \textit{or} \qquad y = a\cos(\omega(t-c)) + b$$

Example 50 (Modeling the Brightness of a Variable Star)**.** *A variable star is one whose brightness alternately increases and decreases. For the variable star Delta Cephei the time between periods of maximum brightness is* 5.4 *days. The average brightness (or magnitude) of the star is* 4.0*, and its brightness varies by* ± 0.35 *magnitude.*

(a) Find a function that models the brightness of Delta Cephei as a function of time.

(b) Sketch a graph of the brightness of Delta Cephei as a function of time.

Solution.

Section 5.6: Modeling Harmonic Behavior

Example 51 (Modeling the Number of Hours of Daylight)**.** *In Philadelphia ($40^o N$ latitude) the longest day of the year has* 14 *h* 50 *min of daylight, and the shortest day has* 9 *h* 10 *min of daylight.*

(a) Find a function L that models the length of daylight as a function of t, the number of days from January 1.

(b) An astronomer needs at least 11 hours of darkness for a long exposure astronomical photograph. On what days of the year are such long exposures possible?

Solution.

Section 5.6: Modeling Harmonic Behavior

Example 52 (Modeling Alternating Current)**.** *Ordinary* $110 - V$ *household alternating current varies from* $+115$ V *to* -115 V *with a frequency of* 60 *Hz (cycles per second). Find an equation that describes this variation in voltage.*

Solution.

Section 5.6: Modeling Harmonic Behavior

Definition 24 (Damped Harmonic Motion)**.** *In the situation where we have a harmonic motion, where the amplitude of the wave is decreasing over time, this is* ***Damped Harmonic Motion****.*

If the equation describing the displacement y of an object at time t is

$$y = ke^{-ct}\sin(\omega t) \qquad \text{or} \qquad y = ke^{-ct}\cos(\omega t)(c > 0)$$

then the object is in ***damped harmonic motion****. The constant c is the* ***damping constant****, k is the initial amplitude, and $\frac{2\pi}{\omega}$ is the period. (This period is really a period, that is we cannot say $f(t + p) = f(t)$ but the function will repeat that behavior with a dampened amplitude)*

Section 5.6: Modeling Harmonic Behavior

Example 53 (Modeling Damped Harmonic Motion)**.** *Two mass-spring systems are experiencing damped harmonic motion, both at* 0.5 *cycles per second and both with an initial maximum displacement of* 10 *cm. The first has a damping constant of* 0.5 *and the second has a damping constant of* 0.1*.*

(a) Find the functions of the form $g(t) = ke^{-ct}\cos(\omega t)$ *to model the motion in each case.*

(b) Graph the two functions you found in part (a). How do they differ?

Solution.

Section 5.6: Modeling Harmonic Behavior

Example 54 (A Vibrating Violin String)**.** *The G-string on a violin is pulled a distance of* 0.5 *cm above its rest position, then released and allowed to vibrate. The damping constant c for this string is determined to be* 1.4*. Suppose that the note produced is a pure G (frequency* = 200 *Hz). Find an equation that describes the motion of the point at which the string was plucked.*

Solution.

Section 5.6: Modeling Harmonic Behavior

Example 55 (Ripples on a Pond)**.** *A stone is dropped in a calm lake, causing waves to form. The up-and-down motion of a point on the surface of the water is modeled by damped harmonic motion. At some time the amplitude of the waves is measured, and* 20 *s later is found that the amplitude has dropped to* $\frac{1}{10}$ *of this value. Find the damping constant* c*.*

Solution.

Section 5.6: Modeling Harmonic Behavior

Note 5 (Same Frequency but Motion Differs)**.** *Notice that because of the periodic nature of cosine and sine, we can have functions that have the same profile, however they are shifted horizontally. We talk about this difference with phase and phase angle.*

Definition 25 (Phase or Phase Angle)**.** *Any sine curve can be expressed in the following equivalent forms:*

$$y = A\sin(kt - b) \qquad \text{The } \textbf{\textit{Phase}} \text{ is } b$$

$$y = A\sin\left(l\left(t - \frac{b}{k}\right)\right) \qquad \text{The } \textbf{\textit{horizontal shift}}\text{is } \frac{b}{k}$$

For two sine functions with the same frequency:

$$y_1 = A\sin(kt - b) \qquad y_2 = A\sin(kt - c)$$

the ***phase difference*** *between y_1 and y_2 is $b - c$. If the phase difference is a multiple of 2π, the waves are* ***in phase****; otherwise, the waves are* ***out of phase****. If two sine curves are in phase, their graphs coincide.*

Section 5.6: Modeling Harmonic Behavior

Example 56 (Finding Phase and Phase Difference)**.** *Objects are in harmonic motion modeled by the following curves:*

$$y_1 = 10\sin\left(3t - \frac{\pi}{6}\right) \qquad y_2 = 10\sin\left(3t - \frac{\pi}{2}\right) \qquad y_3 = 10\sin\left(3t + \frac{23\pi}{6}\right)$$

(a) Find the amplitude, period, phase, and horizontal shift of the curve y_1.

(b) Find the phase difference between the curves y_1 and y_2. Are the two curves in phase?

(c) Find the phase difference between the curves y_1 and y_3. Are the two curves in phase?

(d) Sketch all three curves on the same axes.

Solution.

Section 5.6: Modeling Harmonic Behavior

Example 57 (Using Phase)**.** *Ali, Brandon, Carmen are sitting in a stopped Ferris wheel as shown in the drawing below. At time $t = 0$ the Ferris wheel starts turning counterclockwise at the rate of 2 revolutions per minute.*

(a) Find sine curves that model the height of each rider above the center line of the Ferris wheel at any time $t > 0$.

(b) Find the phase difference between Brandon and Ali, between Ali and Carmen, and between Brandon and Carmen.

(c) Find the horizontal shift of Ali's equation. What's Ali's lead or lag time (relative to the red seat in the figure)?

Solution.

Focus on Modeling: Fitting Sinusoidal Curve to Data

Note 6. *We have ways of fitting data to linear, polynomial, exponential, logistic and logarithm style data trends.*

But what about periodic data?

Example 58 (Modeling the Height of a Tide)**.** *The water depth in a narrow channel varies with the tides. The table below shows the water depth over a* $12-h$ *period. A scatter plot of the data is shown in figure* 2*:*

Time	***Depth (ft)***
12:00 A.M.	*9.8*
1:00 A.M.	*11.4*
2:00 A.M.	*11.6*
3:00 A.M.	*11.2*
4:00 A.M.	*9.6*
5:00 A.M.	*8.5*
6:00 A.M.	*6.5*
7:00 A.M.	*5.7*
8:00 A.M.	*5.4*
9:00 A.M.	*6.0*
10:00 A.M.	*7.0*
11:00 A.M.	*8.6*
12:00 P.M.	*10.0*

(a) Find a function that models the water depth with respect to time.

(b) If a boat needs at least 11 *ft of water of cross the channel, during which times can is safely do so?*

Fitting Sinusodial Curves to Data

Theorem 16 (Method for Finding the function)**.** *First, note that we can use either sine or cosine since we can just shift the function horizontally. So let's assume that*

$$y = a\cos(\omega(t-c)) + b$$

and we'll have to find the b (vertical shift), a (amplitude), $\dfrac{2\pi}{\omega}$ *(period), c (horizontal shift). This will give us our model*

- *Vertical Shift*

  $b = \dfrac{1}{2}(\textit{maximum value} + \textit{minimum value})$

- *Amplitude*

  $a = \dfrac{1}{2}(\textit{maximum value} - \textit{minimum value})$

- *Period*

  $\dfrac{2\pi}{\omega} = 2 \times (\textit{time of maximum value} - \textit{time of minimum value})$

- *Horizontal Shift*

  $c = \textit{phase shift} = \textit{time of maximum value}$

- *The Model*

Fitting Sinusodial Curves to Data

The Model.

Fitting Sinusodial Curves to Data

Example 59 (Fitting a Sine Curve to Data)**.** *(a) Use a graphing device to find the sine curve that best first the depth of water data in Table from Example* 1*.*

(b) Compare your result to the model found in Example 1

Section 6.1: Angle Measure

Definition 26 (Angle)**.** *An **angle** AOB consists of two rays R_1 and R_2 with a common vertex O . We can view this as the rotation of R_1 to R_2, the ray R_1 is called the **initial side** and R_2 the **terminal side**. If the rotation is counterclockwise, the angle is considered **positive** and in the clockwise direction is **negative**.*

Definition 27 (Radian Measure)**.** *In a circle of radius 1 is drawn with the vertex of an angle at its center, then the measure of this angle is **radians** (abbreviated **rad**) is the length of the arc that subtends the angle*

Definition 28 (Relationship Between Degrees and Radians)**.**

$$180^o = \pi \ rad \qquad 1 \ rad = \left(\frac{180}{\pi}\right)^o \qquad 1^o = \frac{\pi}{180} \ rad$$

1. *To convert from degrees to radians, multiply by $\frac{\pi}{180}$.*
2. *To convert from radians to degrees, multiply by $\frac{180}{\pi}$.*

Section 6.1: Angle Measure

Example 60 (Converting Between Radians and Degrees). *(a) Express* 60^o *in radians*

(b) Express $\frac{\pi}{6}$ *rad in degrees.*

Solution.

Check your solution.

(a) $\pi/3$ rad

(b) 30^o

Section 6.1: Angle Measure

Example 61 (Coterminal Angles). *(a) Find angles that are coterminal with the angle $\theta = 30^o$ in standard position.*

(b) Find angles that are coterminal with the angle $\theta = \dfrac{\pi}{3}$ in standard position.

Solution.

Check your solution.

(a) $390^o, 750^o, -330^o$, or -690^o

(b) $\dfrac{7\pi}{3}, \dfrac{13\pi}{3}, \dfrac{-5\pi}{3}$ or $\dfrac{-11\pi}{3}$

Section 6.1: Angle Measure

Example 62 (Coterminal Angles). *Find an angle with measure between* 0^o *and* 360^o *that is coterminal with the angel of measure* 1290^o *in standard position.*

Solution.

Check your solution. 210^o

Section 6.1: Angle Measure

Definition 29 (Length of a Circular Arc). *To find the length of the arc of a circle, call this length s. Then it will be the fraction of 2π that is traveled upon $\frac{\theta}{2\pi}$, this will be times the total circumference of the circle:*

$$s = \frac{\theta}{2\pi}(2\pi r) = \theta r.$$

This will also give us:

$$\theta = \frac{s}{r}.$$

Example 63 (Arc Length and Angle Measure).

1. *Find the length of an arc of a circle with radius 10 m that subtends a central angel of 30^o.*
2. *A central angle θ in a circle of radius 4 m is subtended by an arc of length 6 m. Find the measure of θ in radians.*

Solution.

Check you solution.

1. $\frac{5\pi}{3}$ m
2. $\frac{3}{2}$ rad

Section 6.1: Angle Measure

Definition 30 (Area of a Circular Sector)**.** *Similar to before, to find the area over an area with angle θ and radius r, let A be this total area. So the area in this region is going to be $\frac{\theta}{2\pi}$ which is the fraction of the total angle of the circle times the total area:*

$$A = \frac{\theta}{2\pi}(\pi r^2) = \frac{1}{2}r^2\theta.$$

NOTE! *This is only true for radians.*

Example 64 (Area of a Sector)**.** *Find the area of a sector of a circle with central angle 60^o if the radius of the circle is 3 m.*

Solution.

Check your solution. $\frac{3\pi}{2}\ m^2$

Section 6.1: Angle Measure

Definition 31 (Linear and Angular Speed)**.** *Suppose a point moves along a circle. There are two ways to describe the motion of the point: linear speed and angular speed.* ***Linear speed*** *is the rate at which the distance traveled is changing, so linear speed is the distance traveled divided by the time elapsed.* ***Angular speed*** *is the rate at which the central angle* θ *is changing, so angular speed is the number of radians this angle changes divided by the time elapsed.*

Suppose a point moves along a circle of radius r *and the ray from the center of the circle to the point traverses* θ *radians in time* t*. Let* $s = r\theta$ *be the distance from the point travels in time* t*. Then the speed of the object is given by*

Angular Speed $\qquad \omega = \dfrac{\theta}{t}$

Linear Speed $\qquad v = \dfrac{s}{t}$

Section 6.1: Angle Measure

Example 65 (Finding Linear and Angular Speed)**.** *A boy rotates a stone in a* 3*-ft-long sling at the rate of* 15 *revolutions every* 10 *seconds. Find the angular and linear velocities of the stone.*

Solution.

Check your solution. $\omega = 3\pi$, $\nu = 9\pi$ ft / s

Section 6.1: Angle Measure

Theorem 17 (Relationship Between Linear And Angular Speed)**.** *Note that* $v = \frac{s}{t} = \frac{r\theta}{t} = r\frac{\theta}{t} = r\omega$. *So if a point moves along a circle of radius* r *with angular speed* ω, *then its linear speed* v *is given by*

$$v = r\omega$$

Example 66 (Find Linear Speed From Angular Speed)**.** *A women is riding a bicycle whose wheels are* 26 *in. in diameter. If the wheels rotate at* 125 *revolutions per minute (rpm), find the speed (in mi / h) at which she is traveling.*

Check your answer. $\cong 9.7$ mi/h

Section 6.5: The Law of Sines

Note 7 (What Do I Need to Solve a Triangle?). *To solve any triangle, not necessarily a right triangle, we will always need at least* 3 *pieces of information. This can come in exactly* 4 *forms:*

1. *One side and two angels (ASA or SAA)*
2. *Two sides and the angle opposite one of those sides (SSA)*
3. *Two sides and the included angle (ASS)*
4. *Three sides (SSS)*

In this section, we'll learn how to solve the first 2 *with the last* 2 *we will need the Law of Cosines.*

Theorem 18 (The Law of Sines). *The Law of Sines in words is that any triangle the sides are proportional to the corresponding opposite angle. In symbols if we have a triangle with angles A, B, C and opposite sides of a, b, c, then*

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

Section 6.5: The Law of Sines

Example 67 (Tracking a Satellite (SAA)). *A satellite orbiting the earth passes directly overhead at observation stations in Phoenix and Los Angeles,* 340 *miles apart. At an instant when the satellite is between these two stations, its angle of elevation is simultaneously observed to be* 60^o *at Pheonix and* 75^o *at Los Angeles. How far is the satellite from Los Angeles?*

Check your solution. $b \approx 416$

Section 6.5: The Law of Sines

Example 68 (Solving a Triangle (SAA)). *Solve the triangle in Figure* 5*(page* 510*).*

Solution.

Check your solution. $a \approx 65.1$, $b \approx 134.5$, $B = 135^o$

Section 6.5: The Law of Sines

Definition 32 (The Ambiguous Case). *In the case of (ASA or SAA) the information that we are given will always uniquely determine a single triangle.* ***However****, in the case of (SSA) we are not guaranteed that such a triangle might exist given a side, side and an angle that* ***do not*** *give us a triangle!*

This means we have to be careful when dealing with triangles like this, as we have to ensure that such a triangle exists and be careful when applying the law of sines as problems such as $\cos(x) = \frac{1}{2}$ *do not have unique solution!*

Example 69 (SSA, the One-Solution Case). *Solve the triangle ABC, where* $A = 45^o$, $a = 7\sqrt{2}$ *and* $b = 7$.

Solution.

Check your solution. $B = 30^o$, $C = 105^o$, and $c \approx 13.5$

Section 6.5: The Law of Sines

Note 8. *In general, if* $\sin(A) < 1$*, we must check the angle and its supplementary angle as possibilities, because any angle smaller than* 180^o *can be in the triangle.*

Example 70 (SSA, the Two-Solution Case)**.** *Solve the triangle* ABC *if* $A = 43.1^o$*,* $a = 186.2$*,* $b = 248.6$

Solution.

Check your solution. First triangle, $C_1 = 71.1^o$, $B_1 \approx 65.8^o$, $c_1 \approx 257.8$
Second triangle $C_2 \approx 22.7^o$, $B_2 \approx 114.2^o$, $c_2 \approx 105.2$

Section 6.5: The Law of Sines

Example 71 (SSA, the No-Solution Case)**.** *Solve triangle ABC, where* $A = 42^o$, $a = 70$, $b = 122$.

Solution.

Check your solution. No solution.

Section 6.5: The Law of Sines

Example 72 (Calculating a Distance)**.** *A bird is perched on top of a pole on a steep hill, and an observer is located at point A on the side of the hill,* 110 *m downhill from the base of the pole, as shown in the figure (11 page 513). The angle of inclination of the hill is* 50^o*, and the angle* α *in the figure is* 9^o*. Find the distance from the observer to the bird.*

Solution.

Check your solution. $b \approx 137.3$

Section 6.6: The Law of Cosines

Definition 33 (The Law of Cosines)**.** *In any triangle ABC we have*

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$
$$b^2 = a^2 + c^2 - 2ac\cos(B)$$
$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

Proof.

Section 6.6: Law of Cosines

Example 73 (Length of a Tunnel)**.** *A tunnel is to be built through a mountain. To estimate the length of the tunnel, a surveyor makes the measurements shown in Figure 3(pg. 517). Use the surveyor's data to approximate the length of the tunnel.*

Solution.

Check your answer. ≈ 417 ft long

Section 6.6: Law of Cosines

Example 74 (SSS, the Law of Cosines)**.** *The sides of a triangle are* $a = 5, b = 8, c = 12$*. Find the angles of the triangle.*

Solution.

Check your solution. $A \approx 18^o, B \approx 29^o, C \approx 133^o$

Section 6.6: Law of Cosines

Example 75 (SAS, the Law of Cosines)**.** *Solve triangle ABC, where* $A = 46.5^o, b = 10.5, c = 18.0$.

Solution.

Check your solution. $a \approx 13.2, B \approx 35.3^o, C \approx 98.2^o$.

Section 6.6: Law of Cosines

Example 76 (Navigation). *A pilot sets out from an airport and heads in the direction N 20^o E, flying at 200 mi/h. After 1 h, they make a course correction and heads in the direction N 40^o E. Half an hour after that, engine trouble forces him to make an emergency landing.*

(a) Find the distance between the airport and his final landing point.

(b) Find the bearing from the airport to his final landing point.

Solution.

Check your solution.

(a) ≈ 295.95

(b) N $\approx 26.6^o$

Section 6.6: Law of Cosines

Theorem 19 (Heron's Formula)**.** *The area $\mathcal{A}$ of triangle ABC is given by*

$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}$$

*where $s = \dfrac{1}{2}(a+b+c)$ is the **semiperimeter** of the triangle; that is, s is half the perimeter.*

Proof.

Section 6.6: Law of Cosines

Example 77 (Area of a Lot). *A businessman wishes to buy a triangular lot in a busy downtown location (see Figure 9 (p. 520)) The lot frontages on the three adjacent streets are 125,280, and 315 ft. Find the area of the lot.*

Solution.

Check your solution. $\mathcal{A} \approx 17,451.6\ ft^2$

Focus on Modeling: Surveying

Example 78 (Mapping a Town). *A student wants to draw a map of his hometown. To construct an accurate map (or scale model), he needs to find distances between various landmarks in the town. The student makes the measurements shown in the figure below. Note that only one distance is measured: that between City Hall and the first bridge. All other measurements are angles. Find the distance from the bank to the first bridge.*

Solution.

Check your solution. ≈ 1.32 mi

Surveying

Example 79. *In the same town, find the distance from the bank to the cliff.*

Solution.

Check your solution. ≈ 1.55 mi

Section 7.1: Trigonometric Identities

Theorem 20 (Reciprocal Identities).

$$\csc(x) = \frac{1}{\sin(x)} \qquad \sec(x) = \frac{1}{\cos(x)}$$
$$\cot(x) = \frac{1}{\tan(x)} \qquad \tan(x) = \frac{\sin(x)}{\cos(x)}$$
$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

Theorem 21 (Pythagorean Identities).

$$\sin^2(x) + \cos^2(x) = 1 \qquad \tan^2(x) + 1 = \sec^2(x)$$
$$1 + \cot^2(x) = \csc^2(x)$$

Theorem 22 (Even-Odd Identities).

$$\sin(-x) = -\sin(x) \qquad \cos(-x) = \cos(x)$$
$$\tan(-x) = -\tan(x)$$

Theorem 23 (Cofunction Identities).

$$\sin\left(\frac{\pi}{2} - x\right) = \cos(x) \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot(x)$$
$$\sec\left(\frac{\pi}{2} - x\right) = \csc(x) \qquad \cos\left(\frac{\pi}{2} - x\right) = \sin(x)$$
$$\cot\left(\frac{\pi}{2} - x\right) = \tan(x) \qquad \csc\left(\frac{\pi}{2} - x\right) = \sec(x)$$

Section 7.1: Trigonometric Identities

Example 80 (Simplifying a Trigonometric Expression)**.** *Simplify the expression* $\cos(t) + \tan(t)\sin(t)$.

Solution.

Check your solution. $\sec(t)$

Section 7.1: Trigonometric Identities

Example 81 (Simplifying by Combining Fractions). *Simplify the expression* $\dfrac{\sin(\theta)}{\cos(\theta)} + \dfrac{\cos(\theta)}{1+\sin(\theta)}$.

Solution.

Check your solution. $\sec(\theta)$

Section 7.1: Trigonometric Identities

Note 9 (Guidelines for Proving Trigonometric Identities)**.**

1. ***Start with one side.*** *Pick one side of the equation, and write it down. Your goal is to transform it into the other side. It's usually easier to start with the more complicated side.*

2. ***Use known identities.*** *Use algebra and the identities you know to change the side you started with. Bring fractional expressions to a common denominator, factor, and use the fundamental identities to simplify expressions.*

3. ***Convert to sines and cosines.*** *If you are stuck, you may find it helpful to rewrite all functions in terms of sines and cosines.*

NOTE *We only want to perform steps that are reversible. An example, if* $\sin(x) = -\sin(x)$ *this is an equation not an identity since* $x = \pi/2$ *this is not true, but clearly* $\sin^2(x) = \sin^2(x)$ *is always true, so squaring both sides of an equation is not reversible and thus we* ***don't*** *use it in a proof of an identity.*

Section 7.1: Trigonometric Identities

Example 82 (Proving an Identity by Rewriting in Terms of Sine and Cosine)**.** *Consider the equation* $\cos(\theta)(\sec(\theta) - \cos(\theta)) = \sin^2(\theta)$.

(a) Verify algebraically that the equation is an identity.

(b) Confirm graphically that the equation is an identity.

Proof.

Section 7.1: Trigonometric Identities

Example 83 (Proving an Identity by Introducing Something Extra)**.** *Verify the identity* $\dfrac{\cos(u)}{1-\sin(u)} = \sec(u) + \tan(u)$.

Proof.

Section 7.1: Trigonometric Identities

Example 84 (Proving an Identity by Working with Both Sides Separately)**.** *Verify the identity* $\dfrac{1+\cos(\theta)}{\cos(\theta)} = \dfrac{\tan^2(\theta)}{\sec(\theta)-1}$.

Proof.

Section 7.1: Trigonometric Identities

Example 85 (Trigonometric Substitution). *Substitute* $\sin(\theta)$ *for* x *in the expression* $\sqrt{1-x^2}$, *and simplify. Assume that* $0 \leq \theta \leq \pi/2$.

Section 7.2: Addition and Subtraction Formulas

Note 10 (Concepts Covered)**.** *We'll be covering:*

1. *Addition and Subtraction Formulas*
2. *Evaluating Expressions Involving Inverse Trigonometric Functions*
3. *Expressions of the form* $A \sin x + B \cos x$

Theorem 24 (Addition and Subtraction Formulas)**.** ***Formulas for sine:***

$$\sin(s+t) = \sin(s)\cos(t) + \cos(s)\sin(t)$$
$$\sin(s-t) = \sin(s)\cos(t) - \cos(t)\sin(t)$$

Formulas for cosine:

$$\cos(s+t) = \cos(s)\cos(t) - \sin(s)\sin(t)$$
$$\cos(s-t) = \cos(s)\cos(t) + \sin(s)\sin(t)$$

Formulas for tangent:

$$\tan(s+t) = \frac{\tan(s) + \tan(t)}{1 - \tan(s)\tan(t)}$$
$$\tan(s-t) = \frac{\tan(s) - \tan(t)}{1 + \tan(s)\tan(t)}$$

Proof of Cosine Identity.

Proof. (Continued)

Section 7.2: Addition and Subtraction Formulas

Example 86 (Using the Addition and Subtraction Formulas). *Find the exact values of each expression.*

(a) $\cos(75^o)$

(b) $\cos\dfrac{\pi}{12}$

Solution.

Check your solution. (a) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$, (b) $\dfrac{\sqrt{6}+\sqrt{2}}{4}$

Section 7.2: Addition and Subtraction Formulas

Example 87 (Using the Addition Formula for Sine)**.** *Find the exact value of the expression* $\sin(20^o)\cos(40^o)+\cos(20^o)\sin(40^o)$.

Solution.

Check your solution. $\dfrac{\sqrt{3}}{2}$

Section 7.2: Addition and Subtraction Formulas

Example 88 (Proving an Identity)**.** *Verify the identity* $\dfrac{1+\tan(x)}{1-\tan(x)} = \tan\left(\dfrac{\pi}{4}+x\right)$.

Solution.

Section 7.2: Addition and Subtraction Formulas

Example 89 (An Identity from Calculus)**.** *If $f(x) = \sin(x)$, show that*

$$\frac{f(x+h)-f(x)}{h} = \sin(x)\left(\frac{\cos(h)-1}{h}\right) + \cos(x)\frac{\sin(h)}{h}$$

Solution.

Section 7.2: Addition and Subtraction Formulas

Example 90 (Simplifying an Expressions Involving Inverse Trigonometric Functions)**.** *Write* $\sin(\cos^{-1}(x) + \tan^{-1}(y))$ *as an algebraic expression in* x *and* y*, where* $-1 \leq x \leq 1$ *and* y *is any real number.*

Solution.

Check your solution. $\dfrac{\sqrt{1-x^2}+xy}{\sqrt{1-y^2}}$

Section 7.2: Addition and Subtraction Formulas

Example 91 (Evaluating an Expression Involving Trigonometric Functions)**.** *Evaluate* $\sin(\theta+\phi)$*, where* $\sin(\theta) = \dfrac{12}{13}$ *with* θ *in Quadrant II and* $\tan(\phi) = \dfrac{3}{4}$ *with* ϕ *in Quadrant III.*

Solution.

Check your solution. $\sin(\theta + \phi) = -\dfrac{33}{65}$

Section 7.2: Addition and Subtraction Formulas

Theorem 25 (Sums of Sines and Cosines)**.** *If A and B are real numbers, then*

$$A\sin(x) + B\cos(x) = k\sin(x+\phi)$$

where $k = \sqrt{A^2+B^2}$ and ϕ satisfies

$$\cos(\phi) = \frac{A}{\sqrt{A^2+B^2}} \qquad \textit{and} \qquad \sin(\phi) = \frac{B}{\sqrt{A^2+B^2}}$$

Proof.

Section 7.2: Addition and Subtraction Formulas

Example 92 (A Sum of Sine and Cosine Terms). *Express* $3\sin(x)+4\cos(x)$ *in the form* $k\sin(x+\phi)$*.*

Solution.

Check your solution. $5\sin(x+53.1^o)$

Section 7.2: Addition and Subtraction Formulas

Example 93 (Graphing a Trigonometric Function)**.** *Write the function* $f(x) = -\sin(2x) + \sqrt{3}\cos(2x)$ *in the form* $k\sin(2x + \phi)$*, and use the new form to graph the function.*

Solution.

Check your solution. $2\sin\left(2\left(x + \frac{\pi}{3}\right)\right)$

Section 7.3: Double Angle, Half-Angle, and Product-to-Sum Formulas

Note 11 (Concepts Covered)**.** *1. Double-Angle Formulas*

2. Half-Angle Formulas

3. Evaluating Expressions Involving Inverse Trigonometric Functions

4. Product-Sum Formulas

Theorem 26 (Double-Angle Formulas)**.** ***Formula for sine:***

$$\sin(2x) = 2\sin(x)\cos(x)$$

Formula for cosine:

$$\begin{aligned} \cos(2x) &= \cos^2(x) - \sin^2(x) \\ &= 1 - 2\sin^2(x) \\ &= 2\cos^2(x) - 1 \end{aligned}$$

Formula for tangent:

$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$$

Proof of Double-Angle Formulas for Cosine.

Example 94 (Using the Double-Angle Formulas). *If* $\cos(x) = -\dfrac{2}{3}$ *and* x *is in Quadrant II, find* $\cos(2x)$ *and* $\sin(2x)$.

Solution.

Check your solution. $\cos(2x) = \dfrac{-1}{9}$ and $\sin(2x) = \dfrac{-4\sqrt{5}}{9}$

Example 95 (A Triple-Angle Formula)**.** *Write* $\cos(3x)$ *in terms of* $\cos(x)$.

Solution.

Check your solution. $4\cos^3(x) - 3\cos(x)$.

Example 96 (Proving an Identity)**.** *Prove the identity* $\dfrac{\sin(3x)}{\sin(x)\cos(x)} = 4\cos(x) - \sec(x)$.

Proof.

Theorem 27 (Formulas for Lowering Powers)**.**

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$
$$\tan^2(x) = \frac{1-\cos(2x)}{1+\cos(2x)}$$

Proof.

Example 97 (Lowering Powers in a Trigonometric Expression)**.** *Express* $\sin^2(x)\cos^2(x)$ *in terms of the first power of cosine.*

Solution.

Check your solution. $\dfrac{1}{8}(1 - \cos(4x))$

Theorem 28 (Half-Angle Formulas).

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos(u)}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos(u)}{2}}$$

$$\tan\frac{u}{2} = \frac{1-\cos(u)}{\sin(u)} = \frac{\sin(u)}{1+\cos(u)}$$

The choice of the + or - sign depends on the quadrant in which $u/2$ lies.

Proof.

Example 98 (Using a Half-Angle Formulas)**.** *Find the exact value of* $\sin(22.5^o)$.

Solution.

Check your solution. $\frac{1}{2}\sqrt{2-\sqrt{2}}$

Example 99 (Using a Half-Angle Formula). *Find* $\tan u/2$ *if* $\sin(u) = \frac{2}{5}$ *and* u *is in Quadrant II.*

Solution.

Check your solution. $\frac{5+\sqrt{21}}{2}$

Example 100 (Simplifying an Expression Involving an Inverse Trigonometric Function)**.**
Write $\sin(2\cos^{-1}(x))$ *as an algebraic expression in* x *only, where* $-1 \leq x \leq 1$.

Solution.

Check your solution. $2x\sqrt{1-x^2}$

Example 101 (Evaluating an Expression Involving Trigonometric Functions). *Evaluate* $\sin(2\theta)$ *where* $\cos(\theta) = -\dfrac{2}{5}$ *with* θ *in Quadrant II.*

Solution.

Check your solution. $-\dfrac{4\sqrt{21}}{25}$

Theorem 29 (Product-To-Sum Formulas)**.**

$$\begin{aligned}
\sin(u)\cos(v) &= \frac{1}{2}[\sin(u+v)+\sin(u-v)] \\
\cos(u)\sin(v) &= \frac{1}{2}[\sin(u+v)-\sin(u-v)] \\
\cos(u)\cos(v) &= \frac{1}{2}[\cos(u+v)+\cos(u-v)] \\
\sin(u)\sin(v) &= \frac{1}{2}[\cos(u-v)-\cos(u+v)]
\end{aligned}$$

Example 102 (Expressing a Trigonometric Product as a Sum)**.** *Express* $\sin(3x)\sin(5x)$ *as a sum of trigonometric functions.*

Solution.

Check your solution. $\frac{1}{2}\cos(2x) - \frac{1}{2}\cos(8x)$

Theorem 30 (Sum-to-Product Formulas).

$$\begin{aligned}
\sin(x) + \sin(y) &= 2\sin\frac{x+y}{2}\cos\frac{x-y}{2} \\
\sin(x) - \sin(y) &= 2\cos\frac{x+y}{2}\sin\frac{x-y}{2} \\
\cos(x) + \cos(y) &= 2\cos\frac{x+y}{2}\cos\frac{x-y}{2} \\
\cos(x) - \cos(y) &= -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}
\end{aligned}$$

Example 103 (Expressing a Trigonometric Sum as a Product). *Write* $\sin(7x) + \sin(3x)$ *as a product.*

Solution.

Check your solution. $2\sin(5x)\cos(2x)$

Example 104 (Proving an Identity)**.** *Verify the identity* $\dfrac{\sin(3x) - \sin(x)}{\cos(3x) + \cos(x)} = \tan(x)$.

Solution.

Section 7.4: Basic Trigonometric Equations

Note 12 (Covered Topics). *1. Basic Trigonometric Equations*

2. Solving Trigonometric Equations by Factoring

Note 13 (Method of Solving). *1. Find the solution in one period*

2. Find all solutions by adding $2k\pi$ where $k = 0, \pm 1, \pm 2, \ldots$

Example 105 (Solving a Basic Trigonometric Equation). *Solve the equation* $\sin(\theta) = \frac{1}{2}$.

Solution.

Check your solution. $\theta = \frac{\pi}{6} + 2k\pi$ and $\theta = \frac{5\pi}{6} + 2k\pi$

Section 7.4: Basic Trigonometric Equations

Example 106 (Solving a Basic Trigonometric Equation)**.** *Solve the equation* $\cos(\theta) = \dfrac{-\sqrt{2}}{2}$*, and list eight specific solutions.*

Solution.

Check your solution. $\theta = \dfrac{3\pi}{4} + 2k\pi$ and $\theta = \dfrac{5\pi}{4} + 2k\pi$

Section 7.4: Basic Trigonometric Equations

Example 107 (Solving a Basic Trigonometric Equation)**.** *Solve the equation* $\cos(\theta) = 0.65$.

Solution.

Check your solution. $\theta \approx 0.86 + 2k\pi$ and $\theta \approx 5.42 + 2k\pi$

Section 7.4: Basic Trigonometric Equations

Example 108 (Solving a Basic Trigonometric Equation). *Solve the equation* $\tan(\theta) = 2$.

Solution.

Check your solution. $\theta \approx 1.12 + k\pi$

Section 7.4: Basic Trigonometric Equations

Example 109 (Solving Trigonometric Equations)**.** *Find all solutions of the equation.*

(a) $2\sin(\theta) - 1 = 0$

(b) $\tan^2(\theta) - 3 = 0$

Solution.

Check your solution.

(a) $\theta = \frac{\pi}{6} + 2k\pi$ and $\theta = \frac{5\pi}{6} + 2k\pi$

(b) $\theta = \frac{\pi}{3} + k\pi$ and $\theta = -\frac{\pi}{3} + k\pi$

Section 7.4: Basic Trigonometric Equations

Example 110 (A Trigonometric Equation of Quadratic Type)**.** *Solve the equation* $2\cos^2(\theta) - 7\cos(\theta) + 3 = 0$.

Solution.

Check your solution. $\theta = \dfrac{\pi}{3} + 2k\pi$ and $\theta = \dfrac{5\pi}{3} + 2k\pi$

Section 7.4: Basic Trigonometric Equations

Example 111 (Solving a Trigonometric Equation by Factoring)**.** *Solve the equation* $5\sin(\theta)\cos(\theta) + 4\cos(\theta) = 0$.

Solution.

Check your solution. $\theta = \dfrac{\pi}{2} + 2k\pi$, $\theta = \dfrac{3\pi}{2} + 2k\pi$, $\theta \approx -0.93 + 2k\pi$, $\theta \approx 4.07 + 2k\pi$

Focus on Modeling: Traveling and Standing Waves

Note 14. *We have from 5.6 that the motion of a mass under going harmonic motion is just* $y = A\sin(\omega t)$. *What's the profile of the entire string??*

That would be a function of both x *and* t *with* $t = 0$ *this would be:*

$$y = A\sin(kx)$$

where y *is the height of the string above the* $x-$*axis at the point* x. *This is a kind of snapshot of the string at* $t = 0$.

Definition 34 (A Traveling Wave)**.** *The **velocity** of the wave is the rate at which it moves to the right. If the wave has velocity* v, *then it moves to the right a distance of* vt *in time* t. *So the graph is going to be shifted at time* t *to be:*

$$y(x, t) = A\sin(k(x - vt))$$

*We have **two** variables!*

1. ***If we fix x***, *then* $y(x, t)$ *is a function of* t *only, which gives the position of the fixed point* x *at time* t. *Imagine just looking at the point* x *over time.*
2. ***If we fix t***, *then* $y(x, t)$ *is a function only of* x, *whose graph is the shape of the entire string at the fixed time* t.

Traveling and Standing Waves

Example 112 (A Traveling Wave)**.** *A traveling wave is described by the function*

$$y(x,t) = 3\sin\left(2x - \frac{\pi}{2}t\right) \qquad x \geq 0$$

(a) Find the function that models the position of the point $x = \frac{\pi}{6}$ *at any time* t*. Observe that the point moves in simple harmonic motion.*

(b) Sketch the shape of the wave when $t = 0, 0.5, 1.0, 1.5,$ *and* 2.0*. Does the wave appear to be traveling to the right?*

(c) Find the velocity of the wave.

Solution.

Check your solution.

(a) $3\sin\left(\frac{\pi}{3} - \frac{\pi}{2}t\right)$

(b) It's moving to the right

(c) $\frac{\pi}{4}$

Traveling and Standing Waves

Solution. Continued

Traveling and Standing Waves

Definition 35 (Standing Wave)**.** *If two waves travel along the same string, then the movement of the string is going to be the sum of the two waves. In physics this principle is called the **superposition principle** of waves, that is when we sum to waves it gives us another wave.*

One wave is moving to the right, the other to the left, this gives us:

$$y(x, y) = A\sin(k(x - vt)) + A\sin(k(x + vt)) = 2A\sin(kx)\cos(kvt)$$

*The points $kx = 2\pi n$ for some $n = 0, \pm 1, \pm 2, \ldots$ are special as this will give us $y = 0$ for any time t. There are points that never move! Such points are **nodes**. This will give us something called a **standing wave**.*

Example 113 (A Standing Wave)**.** *Traveling Waves are generated at each end of a tank 30 ft long, with equation:*

$$y = 1.5\sin\left(\frac{\pi}{5}x - 3t\right)$$

and

$$y = 1.5\sin\left(\frac{\pi}{5}x + 3t\right)$$

(a) Find the equation of the combined wave, and find the nodes.

(b) Sketch the graph for $t = 0, 0.17, 0.34, 0.51, 0.68, 0.85$ and 1.02. Is this a standing wave?

Solution.

Check your solution. $3\sin\frac{\pi}{5}x\cos(3t)$ *and nodes* $0, 5, 10, 15, 20, 25, 30$. *This is a standing wave.*

Section 8.1: Polar Coordinates

Note 15 (Concepts Covered)**.**

1. *Definition of Polar Coordinates*
2. *Relationship Between Polar and Rectangular Coordinates*
3. *Polar Equations*

Definition 36 (Polar Coordinate System)**.** *Setting up this system, we pick a fixed point O in the plane called the **pole** (or **origin**) and draw from O a ray (half-line) called the **polar axis** Then each point P can be assigned polar coordinates $P(r, \theta)$, where*

$$r \text{ is the distance from } O \text{ to } P$$

$$\theta \text{ is the angle between the polar axis and the segment } \overline{OP}$$

We again use the convention that θ is positive if measured in a counterclockwise direction from the polar axis or negative if measured in a clockwise direction. If r is negative, then $P(r, \theta)$ is defined to be the point that lies $|r|$ units from the pole in the direction opposite to that given by θ.

Section 8.1: Polar Coordinates

Example 114 (Plotting Points in Polar Coordinates)**.** *Plot the points whose polar coordinates are given.*

(a) $(1, 3\pi/4)$

(b) $(3, -\pi/6)$

(c) $(3, 3\pi)$

(d) $(-4, \pi/4)$

Solution.

Section 8.1: Polar Coordinates

Note 16. *Similar to how we've dealt with the unit circle, the point $P(r, \theta)$ will always have an infinite number of representations, because we can represent it as either:*

$$P(r, \theta) = P(r, \theta + 2\pi n) = P(-r, \theta + (2n+1)\pi) \qquad (n = 0, \pm 1, \dots)$$

Example 115 (Different Polar Coordinates for the Same Point)**.** *(a) Graph the point with polar coordinates $P(2, \pi/3)$*

(b) Find two other polar coordinate representation of P with $r > 0$ and two with $r < 0$

Solution.

Section 8.1: Polar Coordinates

Theorem 31 (Relationship Between Polar and Rectangular Coordinates).

1. *To change from polar to rectangular coordinates, use the formulas*

$$x = r\cos(\theta) \qquad \text{and} \qquad y = r\sin(\theta)$$

2. *To change from rectangular to polar coordinates, use the formulas*

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

Example 116 (Converting Polar Coordinates to Rectangular Coordinates). *Find rectangular coordinates for the point that has polar coordinates* $(4, 2\pi/3)$.

Solution.

Check your solution. $(-2, 2\sqrt{3})$

Section 8.1: Polar Coordinates

Example 117 (Converting Rectangular Coordinates to Polar Coordinates)**.** *Find polar coordinates for the point that has rectangular coordinates* $(2, -2)$*.*

Solution.

Check your Solution. $(2\sqrt{2}, 3\pi/4)$

Section 8.1: Polar Coordinates

Note 17 (Converting Making our Lives Easier)**.** *Occasionally, as you will see, there are situations where an equation is actually easier to solve in polar coordinates and visa versa.*

Example 118 (Converting an Equation from Rectangular to Polar Coordinates)**.** *Express the equation $x^2 = 4y$ in polar coordinates.*

Solution.

Check your solution. $r = 4\sec(\theta)\tan(\theta)$

Section 8.1: Polar Coordinates

Example 119 (Converting Equations from Polar to Rectangular Coordinates)**.** *Express the polar equation in rectangular coordinates. If possible, determine the graph of the equation from its rectangular form.*

1. $r = 5\sec(\theta)$
2. $r = 2\sin(\theta)$
3. $r = 2 + 2\cos(\theta)$

Solution.

Check your solution.

1. $x = 5$
2. $x^2 + (y-1)^2 = 1$
3. $r^2 = 2r + 2r\cos(\theta)$

Section 8.2: Graphs of Polar Equations

Note 18 (Topics to be Covered).

1. *Graphing Polar Equations*
2. *Symmetry*
3. *Graphing Polar Equations with Graphing Devices*

Theorem 32 (How to Graph Polar Equations).

1. *Draw axes radially outward from origin*
2. *Label the axes with your radii,* $0, \pi/2, \pi, 3\pi/2$
3. *The* ***graph of a polar equation*** $r = f(\theta)$

Section 8.2: Graphs of Polar Equations

Example 120 (Sketching the Graph of a Polar Equation)**.** *Sketch a graph of the equation* $r = 3$*, and express the equation in rectangular coordinates.*

Solution.

Section 8.2: Graphs of Polar Equations

Note 19. *In general, the graph of the equation $r = a$ is a circle of radius $|a|$ centered at the origin. Squaring both sides of the equation will give us:*

$$x^2 + y^2 = a^2$$

this is the rectangular equation of a circle.

Example 121 (Sketching the Graph of a Polar Equation)**.** *Sketch a graph of the equation $\theta = \pi/3$, and express the equation in rectangular coordinates.*

Solution.

Check your solution. $y = \sqrt{3}x$

Section 8.2: Graphs of Polar Equations

Example 122 (Sketching the Graph of a Polar Equation)**.** *Sketch a graph of the polar equation* $r = 2\sin\theta$*.*

Solution.

Section 8.2: Graphs of Polar Equations

Theorem 33 (Circles in Polar Coordinates)**.** *The graphs of the form:*

$$r = 2a\sin\theta \qquad \textit{and} \qquad r = 2a\cos\theta$$

*are **circles** with radius $|a|$ centered at the points with polar coordinates $(a, \pi/2)$ and $(a, 0)$, respectively.*

Example 123 (Sketching the Graph of a Cardioid)**.** *Sketch a graph of* $r = 2 + 2\cos\theta$

Solution.

Section 8.2: Graphs of Polar Equations

Theorem 34 (Cardioids in Polar Coordinates)**.** *The curve in the previous example is called a* ***cardioid*** *because it's heart-shaped. In general, the graph of any equation of the form*

$$r = a(1 \pm \cos\theta) \qquad \textit{or} \qquad r = a(1 \pm \sin\theta)$$

is a cardioid.

Example 124 (Sketching the Graph of a Four-Leaved Rose)**.** *Sketch the curve* $r = \cos 2\theta$

Section 8.2: Graphs of Polar Equations

Theorem 35. *In general, the graph of an equation of the form*

$$r = a\cos n\theta \qquad or \qquad r = a\sin n\theta$$

*is an **n-leaved rose** if n is odd or a $2n-$leaved rose if n is even.*

Theorem 36 (Test For Symmetry)**.** *Just like in the $xy-$plane, we can use symmetry to graph polar functions.*

1. *If a polar equation is unchanged when re replace θ by $-\theta$, then the graphs is symmetric about the polar axis.*
2. *If the equation is unchanged when we replace r by $-r$, or θ by $\theta+\pi$, then the graph is symmetric about the pole.*
3. *If the equation is unchanged when re replace θ by $\pi-\theta$, then the graph is symmetric about the vertical line $\theta = \dfrac{\pi}{2}$(the $y-$axis)*

Section 8.2: Graphs of Polar Equations

Example 125 (Using Symmetry to Sketch a Limacon)**.** *Sketch a graph of the equation* $r = 1 + 2\cos(\theta)$*.*

Solution.

Section 8.2: Graphs of Polar Equations

Definition 37 (Limacon)**.** *Named after the Middle French word for snail. The graph of an equation of the form*

$$r = a \pm b\cos(\theta) \qquad \textit{or} \qquad r = a \pm b\sin(\theta)$$

is a limacon. The shape of the limacon depnds on the relative size of a and b.

Example 126 (Drawing the Graph of a Polar Equation)**.** *Graph the equation* $r = \cos\left(\frac{2\theta}{3}\right)$.

Solution.

Section 8.2: Graphs of Polar Equations

Example 127 (A Family of Polar Equations)**.** *Graph the family of polar equations* $r = 1 + c\sin(\theta)$ *for* $c = 3, 2.5, 2, 1.5, 1$. *How does the shape of the graph change as* c *changes?*

Solution.

Section 8.3: Polar Form of Complex Numbers; De Moivre's Theorem

Note 20 (Topics Covered)**.** *1. Graphing Complex Numbers*

2. Polar Form of Complex Numbers

3. De Moivre's Theorem

4. nth Roots of Complex Numbers

Definition 38 (Graphing Complex Numbers)**.** *We can graph complex numbers in rectangular coordinates, using the $x-$axis as the **real axis** and the $y-$axis as the **imaginary axis**.*

If we have a complex number $z = a + bi$ we can then plot that as the point (a, b) in the real-imaginary plane.

Example 128 (Graphing Complex Numbers)**.** *Graph the complex numbers $z_1 = 2 + 3$, $z_2 = 3 - 2i$ and $z_1 + z_2$.*

Solution.

Example 129 (Graphing Sets of Complex Numbers). *Graph each set of complex numbers.*

1. $S = \{a + bi \mid a \geq 0\}$
2. $T = \{a + bi \mid a < 1, b \geq 0\}$

Solution.

Definition 39 (The Modulus of a Complex Number)**.** *The **modulus** (or **absolute value**) of the complex number $z = a + bi$ is*

$$|z| = \sqrt{a^2 + b^2}$$

Example 130 (Calculating the Modulus)**.** *Find the moduli of the complex numbers $3+4i$ and $8 - 5i$.*

Solution.

Check your solution. $|3 + 4i| = 5$ and $|8 - 5i| = \sqrt{89}$

Example 131 (Absolute Value of Complex Numbers). *Graph each set of complex numbers.*

1. $C = \{z \mid |z| = 1\}$
2. $D = \{z \mid |z| \leq 1\}$

Solution.

Definition 40 (Polar Form of Complex Numbers)**.** *A complex number $z = a + bi$ has the **polar form** (or **trigonometric form**)*

$$z = r(\cos(\theta) + i\sin(\theta))$$

where $r = |z| = \sqrt{a^2 + b^2}$ *and* $\tan(\theta) = \frac{b}{a}$*. The number r is the **modulus** of z, and θ is an **argument** of z.*
NOTE: *The argument is not unique, and can differ by an multiple of 2π.*

Example 132 (Writing Complex Numbers in Polar Form)**.** *Write each complex number in polar form.*

1. $1 + i$
2. $-1 + \sqrt{3}i$
3. $-4\sqrt{3} - 4i$
4. $3 + 4i$

Solution.

Theorem 37 (Multiplication and Division of Complex Numbers)**.** *If the two complex numbers z_1 and z_2 have the polar forms*

$$z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1)) \qquad \textit{and} \qquad z_2 = r_2(\cos(\theta)2) + i\sin(\theta_2))$$

then

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad \textit{Multiplication}$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \quad (z \neq 0) \quad \textit{Division}$$

Proof.

Example 133 (Multiplying and Diving Complex Numbers)**.** *Let*

$$z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad \textit{and} \quad z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Find (a) $z_1 z_2$ *and* (b) $\dfrac{z_1}{z_2}$.

Solution.

Check your solution. $(a) - 2.588 + 9.659i$ and $(b)0.3864 - 0.1035i$

Theorem 38 (De Moivre's Theorem). *If $z = r(\cos\theta + i\sin\theta)$, then for any integer n*

$$z^n = r^n(\cos(\theta n) + i\sin(n\theta)).$$

Proof.

Example 134 (Finding a Power Using De Moivre's Theorem). *Find* $\left(\frac{1}{2}+\frac{1}{2}i\right)^{10}$.

Solution.

Check your solution. $\dfrac{i}{32}$

Theorem 39 (nth Roots of Complex Numbers)**.** *If $z = r(\cos\theta + i\sin\theta)$ and n is any positive integer, then z has the n distinct nth roots*

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, n-1$.

Proof.

Theorem 40 (Find the nth Roots of $z = r(\cos(\theta) + i\sin(\theta))$). *1. The modulus of each nth root is $r^{1/n}$.*

2. The argument of the first root is θ/n.

3. We repeatedly add $2\pi/n$ to get the argument of each successive root.

Example 135 (Finding Roots of a Complex Number). *Find the six sixth roots of $z = -64$, and graph these roots in the complex plane.*

Solution.

Check your solution. $w_0 = \sqrt{3} + i$, $w_1 = 2i$, $w_2 = -\sqrt{3} + i$, $w_3 = -\sqrt{3} - i$, $w_4 = -2i$, $w_5 = \sqrt{3} - i$

Example 136 (Finding Cube Roots of a Complex Number). *Find the three cube roots of $z = 2 + 2i$ and graph these roots in the complex plane.*

Solution.

Check your solution. $w_0 \approx 1.366 + 0.366i$, $w_1 = -1 + i$, $w_2 \approx -0.366 - 1.366i$

Example 137 (Solving an Equation Using the nth Roots Formula)**.** *Solve the equation* $z^6 + 64 = 0$.

Solution.

Section 8.4: Plane Curves and Parametric Equations

Note 21 (Topic Covered)**.** *1. Plane Curves and Parametric Equations*

2. Eliminating the Parameter

3. Finding Parametric Equations for a Curve

4. Using Graphing Devices to Graph Parameters

Definition 41 (Plane Curves and Parametric Equations)**.** *Think about taking a point $(x(t), y(t))$ that will move across the $xy-$plane as a function of t time.*

*If f and g are functions defined on an interval I, then the set of points $(f(t), g(t))$ is a **plane curve**. The equations*

$$x = f(t) \qquad y = g(t)$$

*where $t \in I$, are **parametric equations** for the curve, with **parameter** t.*

Example 138. *Sketch the curve defined by the parametric equations*

$$x = t^2 - 3t \qquad y = t - 1$$

Solution.

Section 8.4: Plane Curves and Parametric Equations

Theorem 41 (Eliminating the Parameter)**.** *Often a curve given by a parametric equation can be reduced down to something that's only dependent on x, y. Finding this equations called **eliminating the parameter**.*

Example 139 (Eliminating the Parameter)**.** *Eliminate the parameter in the parametric equations of Example* 1*.*

Solution.

Check your solution. $x = y^2 - y - 2$.

Section 8.4: Plane Curves and Parametric Equations

Example 140 (Modeling Circular Motion)**.** *The following parametric equations model the position of a moving object at time t (in seconds):*

$$x = \cos(t) \qquad y = \sin(t) \quad t \geq 0$$

Describe and graph the path of the object.

Solution.

Check your solution. It's a circle with radius 1!

Section 8.4: Plane Curves and Parametric Equations

Example 141 (Sketching a Parameteric Curve)**.** *Eliminate the parameter, and sketch the graph of the parametric equations*

$$x = \sin(t) \qquad y = 2 - \cos^2(t).$$

Solution.

Check your solution. $y = 1 + x^2$, $-1 \leq x \leq 1$.

Section 8.4: Plane Curves and Parametric Equations

Example 142 (Finding Parametric Equations for a Graph)**.** *Find parametric equation for the line of slope* 3 *that passes through the point* $(2, 6)$*.*

Solution.

Check your solution. $x = 2 + t$ and $y = 6 + 3t$

Section 8.4: Plane Curves and Parametric Equations

Example 143 (Parametric Equations for a Cycloid)**.** *As a circle rolls along a straight line, the curve traced out by a fixed point P on the circumference of the circle is called a* ***cycliod****. If the circle has radius a and rolls along the $x-$axis, with one position of the point P being at the origin, find parametric equations for the cycloid.*

Solution.

Check your solution. $x = a(\theta - \sin(\theta)), y = a(1 - \cos(\theta))$

Section 8.4: Plane Curves and Parametric Equations

Example 144 (Graphing Parametric Curves). *Using a graphing device to draw the following parametric curves. Discuss their similarities and differences.*

1. $x = \sin(2t), y = 2\cos(t)$
2. $x = \sin(3t), y = 2\cos(t)$

Solution.

Section 8.4: Plane Curves and Parametric Equations

Definition 42 (Polar Equations in Parametric Form)**.** *The graph of the polar equation $r = f(\theta)$ is the same as the graph of the parametric equations*

$$x = f(t)\cos(t) \qquad y = f(t)\sin(t)$$

Example 145 (Parametric Form of a Polar Equation)**.** *Consider the polar equation $r = \theta, 1 \leq \theta \leq 10\pi$.*

1. *Express the equation in parametric form.*
2. *Draw a graph of the parametric equation from part (a)*

Solution.

Check your solution. $x = t\cos(t)$ and $y = t\cos(t)$

Focus on Modeling: The Path of a Projectile

Note 22 (Projectile Motion with Gravity and Without). *Imagine firing a projectile into the air from ground level, with an initial speed v_0 and at an angle θ upward from the ground. If we were on the moon then gravity is negligible and the projectile just keeps moving indefinitely so at any time t the distance covered is $v_0 t$ so its position at time t:*

$$x = (v_0 \cos(\theta))t \qquad y = (v_0 \sin(\theta))t \quad \textit{No gravity}$$

*Reintroducing gravity, say if you were playing with your cat at a beach (no judgement from me), then tossing a toy will experience gravity. The gravity factor will **only** affect the $y-$coordinate, giving us:*

$$x = (v_0 \cos(\theta))t \qquad y = (v_0 \sin(\theta))t - \frac{gt^2}{2} \quad \textit{With gravity}$$

here $g \approx 9.8m/s^2$ accounts for the gravitational affect.

Example 146 (The Path of a Cannonball). *Find parametric equations that model the path of a cannonball fired into the air with an initial speed of 150.0 m/s at a 30^o angle of elevation. Sketch the path of the cannonball.*

Solution.

Check your solution. $x = 129.9t$ and $y = 75.0 - 4.9t^2$

The Path of a Projectile

Note 23 (Solving for Impact)**.** *We can solve for the time of impact using these equations by setting $y = 0$ and solving for t, notice that $t = 0$ is always a solution!*

Let's try to optimize this angle to get the shortest time to impact!

Theorem 42 (Time of Impact)**.**

$$\begin{aligned} 0 &= v_0 \sin(\theta) t - \frac{1}{2} g t^2 && \textit{Substitute } y = 0 \\ 0 &= t(v_0 \sin(\theta) - \frac{1}{2} g t) && \textit{Factor} \\ 0 &= v_0 \sin(\theta) - \frac{1}{2} g t && \textit{Assuming } t \neq 0 \\ t &= \frac{2 v_0 \sin(\theta)}{g} && \textit{Solving for } t \end{aligned}$$

Example 147. *Find the angle that will result in the least time to reach the ground? That's non-zero.*

The Path of a Projectile

Example 148. *What's the farthest distance we can reach?*

Section 11.1: Parabolas

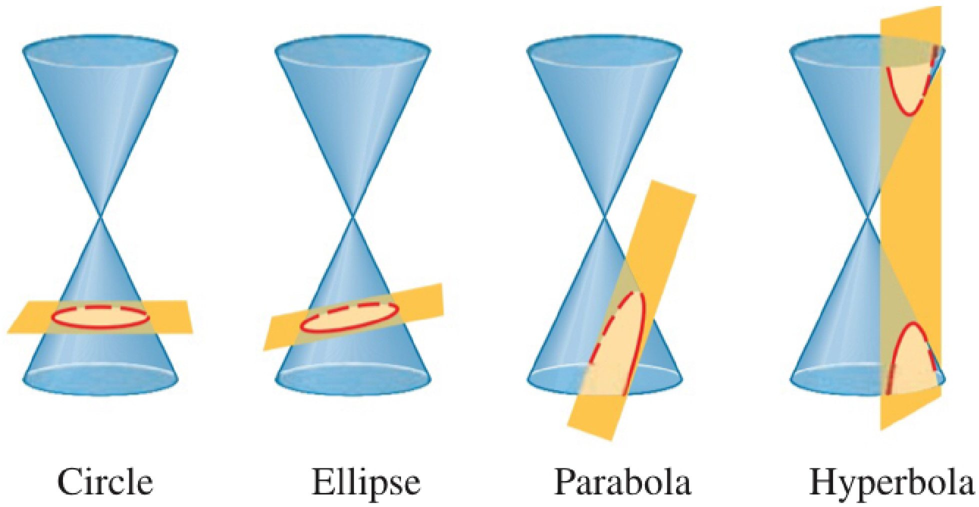

Figure 13: The Cross Section of a Cone, or Conic Section

Definition 43 (The Geometric Definition of a Parabola)**.** *A* ***parabola*** *is the set of all points in the plane that are equidistant from a fixed point F (called the* ***focus****) and a fixed line l (called the* ***directrix****).*

Drawing of Parabola.

Section 11.1: Parabolas

Deriving the Equation of a Parabola.

Check your solution. $x^2 = 4py$

Section 11.1: Parabolas

Definition 44 (Parabola with Vertical Axis). *The graph of the equation*

$$x^2 = 4py$$

is a parabola with following properties:

$$\begin{array}{rl} \textbf{\textit{VERTEX}} & V(0,0) \\ \textbf{\textit{FOCUS}} & F(0,p) \\ \textbf{\textit{DIRECTRIX}} & y=-p \end{array}$$

The parabola opens upward if $p > 0$ or downward if $p < 0$.

Drawing.

Section 11.1: Parabolas

Example 149 (Find the Equation of a Parabola)**.** *Find an equation for the parabola with vertex $V(0,0)$ and focus $F(0,2)$, and sketch its graph.*

Solution.

Check your solution. $x^2 = 8y$

Section 11.1: Parabolas

Example 150 (Finding the Focus and Directrix of a Parabola from its equation)**.** *Find the focus and directrix of the parabola $y = -x^2$, and sketch the graph.*

Solution.

Check your solution. $p = -1/4$ and $F(0, -\frac{1}{4})$ and directrix $y = \frac{1}{4}$

Section 11.1: Parabolas

Definition 45 (Parabola with Horizontal Axis). *The graph of the equation*

$$y^2 = 4px$$

is a parabola with the following properties

$$\boldsymbol{VERTEX} V(0,0)$$
$$\boldsymbol{FOCUS} F(p,0)$$
$$\boldsymbol{DIRECTRIX} x = -p$$

The parabola opens to the right if $p > 0$ *or to the left if* $p < 0$.

Drawing of a Parabola.

Section 11.1: Parabolas

Example 151 (A Parabola with Horizontal Axis)**.** *A parabola has the equation* $6x+y^2 = 0$.

1. *Find the focus and directrix of the parabola, and sketch the graph.*
2. *Use a graphing calculator to draw the graph.*

Solution.

Check your solution $F(-3/2, 0)$ and $x = 3/2$

Section 11.1: Parabolas

Definition 46 (Latus Rectum/Focal Diameter)**.** *The line segment that runs through the focus perpendicular to the axis, with endpoints of the parabola is the* ***latus rectum****. It's length is the* ***focal diameter*** *of the parabola.*

The length of the latus rectum is always $|2p|$ *and the focal diameter* $|4p|$

Example 152 (The Focal Diameter of a Parabola)**.** *Find the focus, directrix, and focal diameter of the parabola* $y = \frac{1}{2}x^2$*, and sketch its graph.*

Solution.

Check your solution. $F(0, 1/2)$ and $y = -1/2$.

Section 11.1: Parabolas

Example 153 (A Family of Parabolas)**.**

1. *Find equations for the parabolas with vertex at the origin and foci* $F_1(0, 1/8), F_2(0, 1/2), F_3(0, 1), F_4(0, 4)$.
2. *Draw the graphs of the parabolas in part (a). What do you conclude?*

Solution.

Section 11.1: Parabolas

Example 154 (Find the Focal Point of a Searchlight Reflector)**.** *A searchlight has a parabolic reflector that forms a "bowl", which is* 12 *in. wide from rim to rim and* 8 *in. deep, shown below. If the filament of the light bulb is located at the focus, how far from the vertex of the reflector is it?*

Solution.

Check your solution. $F(0, 9/8)$ and $x^2 = 18y$

Section 11.2: Ellipses

Note 24 (Topics Covered).

1. *Geometric Definition of an Ellipse*
2. *Equations and Graphs of Ellipses*
3. *Eccentricity of an Ellipse*

Definition 47 (Geometric Definition of an Ellipse). *An **ellipse** is the set of all points in the plane the sum of whose distances from two fixed points F_1 and F_2 is a constant. These two fixed points are the **foci** (plural of **focus**) of the ellipse.*

Deriving the Equation of an Ellipse.

Section 11.2: Ellipse

Definition 48 (Ellipse with Center at the Origin)**.** *The graph of each of the following equations is an ellipse with center at the origin and having the given properties.*

EQUATION	$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ $a>b>0$	$\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=1$ $a>b>0$
VERTICES	$(\pm a, 0)$	$(0, \pm a)$
MAJOR AXIS	*Horizontal, length* $2a$	*Vertical, length* $2a$
MINOR AXIS	*Verical, length* $2b$	*Horizontal, length* $2b$
FOCI	$(\pm c), c^2 = a^2 - b^2$	$(0, \pm c), c^2 = a^2 - b^2$

Example 155 (Sketching an Ellipse)**.** *An ellipse has the equation*

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

1. *Find the foci, the vertices, and the lengths of the major and minor axes, and sketch the graph.*

2. *Draw the graph using a graphing calculator.*

Solution.

Section 11.2: Ellipse

Example 156 (Finding the Foci of an Ellipse)**.** *Find the foci of the ellipse* $16x^2 + 9y^2 = 144$*, and sketch its graph.*

Solution.

Section 11.2: Ellipse

Example 157 (Finding the Equation of an Ellipse)**.** *The vertices of an ellipse are* $(\pm 4, 0)$*, and the foci are* $(\pm 2, 0)$*. Find its equation, and sketch the graph.*

Solution.

Section 11.2: Ellipse

Definition 49 (Definition of Eccentricty)**.** *For the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *or* $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ *(with* $a > b > 0$*), the* ***eccentricity*** e *is the number*

$$e = \frac{c}{a}$$

where $c = \sqrt{a^2 - b^2}$*. The eccentricity of every ellipse satisfies* $0 < e < 1$.

Example 158 (Finding the Equation of an Ellipse from its Eccentrity and Foci)**.** *Find the equation of the ellipse with foci* $(0, \pm 8)$ *and eccentricity* $e = \frac{4}{5}$*, and sketch its graph.*

Section 11.3: Hyperbolas

Note 25 (Topics Covered)**.** *1. Geometric Definition of a Hyperbola*

2. Equations and Graphs of Hyperbolas

Definition 50 (Geometric Definition of a Hyperbola)**.** *A **hyperbola** is the set of all points in the plane, the difference of whose distances from two fixed points F_1 and F_2 is a constant. These two fixed points are the **foci** of the hyperbola.*

Deriving the Equation of a Hyperbola.

Section 11.3: Hyperbolas

Definition 51 (Hyperbola with Center at the Origin)**.** *The anatomy of a hyperbola, there will always be two separated parts of the hyperbola, the* ***branches*** *of the hyperbola. The segment joining the two vertices on the separate branches is the* ***transverse axis*** *of the hyperbola, and the origin is called its* ***center****.*

EQUATION	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad a > 0, b > 0$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad a > 0, b > 0$
VERTICES	$(\pm a, 0)$	$(0, \pm a)$
TRANSVERSE AXIS	*Horizontal, length* $2a$	*Vertical, length* $2a$
ASYMPTOTES	$y = \pm \frac{b}{a} x$	$y = \pm \frac{a}{b} x$
FOCI	$(\pm c, 0)$, $c^2 = a^2 + b^2$	$(0, \pm c), c^2 = a^2 + b^2$

Note 26. *We'll end up with certain "slant" asymptotes with these hyperbolas. The box that we can form with these asymptotes and the vertices is the* ***central box****.*

Section 11.3: Hyperbolas

Theorem 43 (How to Sketch A Hyperbola)**.**

1. ***Sketch the Central Box.*** *This is the rectangle centered at the origin, with sides parallel to the axes, that crosses one axis at $\pm a$ and the other at $\pm b$.*
2. ***Sketch the Asymptotes.*** *These are the lines obtained by extending the diagonals of the central box.*
3. ***Plot Vertices.*** *These are the two $x-$intercepts or two $y-$intercepts.*
4. ***Sketch the Hyperbola.*** *Start at a vertex, and sketch a branch of the hyperbola, approaching the asymptotes. Sketch the other branch in the same way.*

Example 159 (A Hyperbola with Horizontal Transverse Axis)**.** *A hyperbola has the equation*

$$9x^2 - 16y^2 = 144$$

1. *Find the vertices, foci, length of the transverse axis, and asmyptotes, and sketch the graph.*
2. *Draw the graph using a graphing calculator.*

Solution.

Section 11.3: Hyperbolas

Example 160 (A Hyperbola with Vertical Transverse Axis)**.** *Find the vertices, loci, length of the transverse axis, and asmyptotes of the hyperbola, and sketch its graph.*

$$x^2 - 9y^2 + 9 = 0$$

Solution.

Section 11.3: Hyperbolas

Example 161 (Finding the Equation of a Hyperbola from its Vertices and Foci)**.** *Find the equation of the hyperbola with vertices* $(\pm 3, 0)$ *and foci* $(\pm 4, 0)$. *Sketch the graph.*

Solution.

Section 11.3: Hyperbolas

Example 162 (Finding the Equation of a Hyperbola from Its Vertices and Asmyptotes)**.** *Find the equation and foci of the hyperbola with vertices* $(0, \pm 2)$ *and asymptotes* $y = \pm 2x$. *Sketch the graph.*

Solution.

Section 11.4: Shifted Conics

Theorem 44 (Shifting Graphs of Equations). *If h, k are positive real numbers, then replacing x by $x-h$ or by $x+h$ and replacing y by $y-k$ or by $y+k$ has the following effect(s) on the graph of any equation in x and y.*

REPLACEMENT	***How the Graph Is Shifted***
1. *x replaced by $x-h$*	*Right h units*
2. *x replaced by $x+h$*	*Left h units*
3. *y replaced by $y-k$*	*Upward k units*
4. *y replaced by $y+k$*	*Downward k units*

Theorem 45 (Shifted Ellipses). *We can move the center of any ellipse to the point (h, k) with the replacement $x \to x-h$ and $y \to y-k$ so that the equation of a shifted ellipse:*

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

Section 11.4: Shifted Conics

Example 163 (Sketching the Graph of a Shifted Ellipse)**.** *Sketch a graph of the ellipse*

$$\frac{(x+1)^2}{4} + \frac{(y-2)^2}{9} = 1$$

and determine the coordinates of the foci.

Solution.

Section 11.4: Shifted Conics

Example 164 (Finding the Equation of a Shifted Ellipse). *The vertices of an ellipse are* $(-7, 3)$ *and* $(3, 3)$*, and the foci are* $(-6, 3)$ *and* $(2, 3)$*. Find the equation for the ellipse, and sketch its graph.*

Solution.

Section 11.4: Shifted Conics

Note 27. *The formula of the standard parabolas are:*

$$x^2 = 4py$$

and the shape is dependent on $p > 0$ *or* $p < 0$ *similarly for* $y^2 = 4px$.

Example 165 (Graphing a Shifted Parabola)**.** *Determine the vertex, focus and directrix, and sketch a graph of the parabola.*

$$x^2 - 4x = 8y - 28$$

Solution.

Section 11.4: Shifted Conics

Theorem 46. *The formulas for shifted hyperbolas:*

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

or

$$-\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

Example 166 (Graphing a Shifted Hyperbola)**.** *A shifted conic has the equation*

$$9x^2 - 72x - 16y^2 - 32y = 16$$

1. *Complete the square in x and y to show that the equation represents a hyperbola.*
2. *Find the center, vertices, foci, and asymptotes of the hyperbolas, and sketch its graph.*
3. *Draw the graph using a graphing calculator.*

Solution.

Section 11.4: Shifted Conics

Note 28 (Degenerate Conic). *Such a conic section is one where we end up with only a point or a pair of lines.*

Definition 52 (General Equation of a Shifted Conic). *The graph of the equation*

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where A and C are not both 0, is a conic or a degenerate conic. In the nondegenerate cases the graph is

1. *a parabola if A or C is 0,*
2. *an ellipse if A and C have the same sign (or a circle if $A = C$),*
3. *a hyperbola if A and C has opposite signs.*

Section 11.4: Shifted Conics

Example 167 (An Equation That Leads to a Degenerate Conic)**.** *Sketch the graph of the equation*

$$9x^2 - y^2 + 18x + 6y = 0$$

Section 11.5: Rotation of Axes

Note 29. *Even more general than the shifted conic sections that we covered in the previous section is the equation:*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

We'll end up showing that this too is a conic section, one that is obtained by shifting the coordinate axes.

Theorem 47 (Rotataion of Axes Formulas)**.** *Suppose that $x-$ and $y-$axes in a coordinate plane are rotated through the acute angle ϕ to produce the $X-$ and $Y-$axes, as shown in Figure 1. Then the coordinates (x, y) and (X, Y) of a point in the $xy-$ and the $XY-$plane are related as follow*

$$x = X\cos(\phi) - Y\sin(\phi) \qquad X = x\cos(\phi) + y\sin(\phi)$$

$$y = X\sin(\phi) + Y\cos(\phi) \qquad Y = -x\sin(\phi) + y\cos(\phi)$$

Proof.

Section 11.5: Rotation of Axes

Example 168 (Rotation of Axes)**.** *If the coordinate axes are rotated through* 30^o*, find the* $XY-$*coordinates of the point with* $xy-$*coordinates* $(2, -4)$*.*

Solution.

Check your solution. $(-2 + \sqrt{3}, -1 - 2\sqrt{3})$

Section 11.5: Rotation of Axes

Example 169 (Rotating a Hyperbola)**.** *Rotate the coordinate axes through* 45^o *to show that the graph of the equation* $xy = 2$ *is a hyperbola.*

Solution.

Check your solution. $\dfrac{X^2}{4} - \dfrac{Y^2}{4} = 1$

Section 11.5: Rotation of Axes

Note 30 (Eliminating the $xy-$term)**.** *Anytime we have the most general equation of a conic section:*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

we can eliminate the $xy-$term with the rotation of axes formuli. In particular, we can always rotate $xy-$axes to $XY-$axes that we get:

$$AX^2 + CY^2 + DX + EY + F = 0.$$

Theorem 48 (Simplifying the General Conic Equation)**.** *To eliminate the $xy-$term in the general conic equation*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

rotate the axes through the acute angle ϕ that satisfies:

$$\cot(2\phi) = \frac{A - C}{B}$$

Proof.

Section 11.5: Rotation of Axes

Proof. (Cont.)

Section 11.5: Rotation of Axes

Example 170 (Eliminating the $xy-$Term). *Use a rotation of axes to eliminate the $xy-$term in the equation*

$$6\sqrt{3}x^2 + 6xy + 4\sqrt{3}y^2 = 21\sqrt{3}$$

Identify and sketch the curve.

Solution.

Check your solution. $\phi = 30^o$ and $\dfrac{X^2}{3} + \dfrac{Y^2}{7} = 1$

Section 11.5: Rotation of Axes

Example 171 (Graphing a Rotated Conic). *A conic has the equation*

$$64x^2 + 96xy + 36y^2 - 15x + 20y - 25 = 0$$

1. *Use a rotation of axes to eliminate the xy−term.*
2. *Identify and sketch the graph.*
3. *Draw the graph using a graphing calculator.*

Solution.

Check your solution. $X^2 - \frac{1}{4}(Y-1)$, $\phi \approx 37^o$

Section 11.5: Rotation of Axes

Definition 53 (Identifying the Conics by the Discriminant)**.** *The graph of the equation*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is either a conic or a degenerate conic. In the nondegenerate cases the graph is

1. *a parabola if* $B^2 - 4AC = 0$
2. *an ellipse if* $B^2 - 4AC > 0$
3. *a hyperbola if* $B^2 - 4AC < 0$

The quantity $B^2 - 4AC$ *is called the* ***discriminant*** *of the equation.*

Proof.

Section 11.5: Rotation of Axes

Note 31 (Invariant)**.** *Notice that the discriminant is unchanged in the process of rotation, this property means that it is **invariant** under rotation.*

Example 172 (Identifying a Conic by the Discriminant)**.** *A conic has the equation*

$$3x^2 + 5xy - 2y^2 + x - y + 4 = 0$$

1. *Use the discriminant to identify the conic.*
2. *Confirm your answer to part (a) by graphing the conic with a graphing calculator.*

Section 11.6: Polar Equations of Conics

Note 32 (Topic Covered). *1. A Unified Geometric Description of Conics*

2. Polar Equations of Conics

Theorem 49 (Equivalent Description of Conics). *Let F be a fixed point (the* ***focus****), l a fixed line (the* ***directrix****), and let e be a fixed positive number (the* ***eccentricity****). The set of all points P such that the ratio of the distance from P to F to the distance from P to l is the constant e is a conic. That is the set of all points P such that*

$$\frac{d(P,F)}{d(P,l)} = e$$

is a conic. The conic is a parabola if $e = 1$, an ellipse if $e < 1$, or a hyperbola if $e > 1$.

Proof.

Section 11.6: Polar Equations of Conics

Proof. (Cont.)

Section 11.6: Polar Equations of Conics

Note 33. *In the previous proof, we ended up with the term* $r = e(d - r\cos(\theta))$.

Theorem 50 (Polar Equations of Conics)**.** *Solving the above for* r *above:*

$$r = \frac{ed}{1 + e\cos(\theta)}$$

Now if the directrix is chosen to be the left of the focus $(x = -d)$*, then the equation is* $r = \dfrac{ed}{1 - e\cos(\theta)}$. *If the directrix is parallel to the polar axis* $(y = d,$ *or* $y = -d)$*, then we get* $\sin(\theta)$ *instead of* $\cos(\theta)$.
So that the polar equation for a conic section is:

$$r = \frac{ed}{1 \pm e\cos(\theta)} \qquad \text{or} \qquad r = \frac{ed}{1 \pm e\sin(\theta)}$$

represents a conic with one focus at the origin and with eccentricity e*. The conic is*

1. *a parabola if* $e = 1$,
2. *an ellipse if* $0 < e < 1$,
3. *a hyperbola if* $e > 1$.

Note that we can find the directrix with:

1. *For a parabola the axis of symmetry is a perpendicular to the directrix.*
2. *For an ellipse the major axis is perpendicular to the directrix.*
3. *For a hyperbola the transverse axis perpendicular to the directrix.*

Section 11.6: Polar Equations of Conics

Example 173 (Finding a Polar Equation for a Conic). *Find a polar equation for the parabola that has its focus at the origin and whose directrix is the line $y = -6$.*

Solution.

Check your solution. $r = \dfrac{6}{1 - \sin(\theta)}$

Section 11.6: Polar Equations of Conics

Example 174 (Identifying and Sketching a Conic). *A conic is given by the polar equation*

$$r = \frac{10}{3 - 2\cos(\theta)}$$

1. *Show that the conic is an ellipse, and sketch its graph.*
2. *Find the center of the ellipse and the lengths of the major and minor axes.*

Solution.

Check your solution. $r = \dfrac{\frac{10}{3}}{1 - \frac{2}{3}\cos(\theta)}$

Section 11.6: Polar Equations of Conics

Example 175 (Identifying and Sketching a Conic)**.** *A conic is given by the polar equation*

$$r = \frac{12}{2 + 4\sin(\theta)}$$

1. *Show that the conic is a hyperbola, and sketch its graph.*
2. *Find the center of the hyperbola, and sketch the asymptotes.*

Solution.

Check your solution. $e = 2$

Section 11.6: Polar Equations of Conics

Example 176 (Rotating an Ellipse). *Suppose the ellipse of Example 2 is rotated through an angle* $\frac{\pi}{4}$ *about the origin. Find a polar equation for the resulting ellipse, and draw its graph.*