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Homework #3

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14.)

Test the series for convergence

$$\sum_{k=1}^{\infty} \left(\frac{1}{3k+1} \right)^k$$

Proof. Let $a_k = \left(\frac{1}{3k+1}\right)^k$. Then consider the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(\frac{1}{3k+1}\right)^k$. We will show that this series converges using the root test.

Consider the following:

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{\left(\frac{1}{3k+1}\right)^k} = \lim_{k\to\infty} \frac{1}{3k+1} = 0$$

Note that this converges by the Limit comparison test, when we compare it to the convergent sequence $\frac{1}{3k}$.

Since this limit is less than 1, then by Lite Root Test we have that this series is convergent absolutely. $\square$

15.)

Prove or disprove: If $\left|\frac{a_{k+1}}{a_k}\right| < 1$ for all $k \in \mathbb{N}$, then the series $\sum_{k=1}^{\infty} a_k$ converges.

Counterexample:

Consider the sequence defined by $a_k = \frac{1}{k}$ for all $k \in \mathbb{Z}^+$. Then note that we have $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\frac{1}{k+1}}{\frac{1}{k}}\right| = \left|\frac{k}{k+1}\right| < 1$. But not that the series $\sum_{k=1}^{\infty} \frac{1}{k}$ is a divergent series. Hence this is a counterexample to the claim.

16.)

Consider the series $\sum_{k=1}^{\infty} a_k = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + ...$, and let $\{s_n\}_{n=1}^{\infty}$ denote the sequence of partial sums of this series.

a.

Define a new series $\sum_{k=1}^{\infty} b_k$ as follows: $b_1 = a_1 + a_2 + a_3$, $b_2 = a_4 + a_5 + a_6$, $b_3 = a_7 + a_8 + a_9$, etc., so that, in general, $b_k = a_{3k-2} + a_{3k-1} + a_{3k}$ for each $k \in \mathbb{N}$. Find a general formula for b_k, and simplify so that your answer is a single fraction.

Solution: Note that for the terms of b_k are given as: $b_k = a_{3k-2} + a_{3k-1} - a_{3k}$.

Since $a_k = \frac{1}{k}$, we have $b_k = \frac{1}{3k-2} + \frac{1}{3k-1} - \frac{1}{3k}$, expanding this out using Mathematica, we get $b_k = \frac{9k^2-2}{3k(3k-1)(3k-2)}$.

b.

Prove that $\sum_{k=1}^{\infty} b_k$ diverges.

Proof. Let $b_k = \frac{9k^2-2}{3k(3k-1)(3k-2)}$. Then consider the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{9k^2-2}{3k(3k-1)(3k-2)}$. We will use the Comparison Test to show that this series diverges. Note that we may do this because we have all positive terms for all $k > 0$.

Now consider the following for $k \geq 2$:

$$\frac{9k^2-2}{3k(3k-1)(3k-2)} = \frac{9k^2-2}{27k^3-27k^2+6k} >$$

$$\frac{9k^2-9}{27k^3-27k^2+9k} = \frac{k^2-1}{3k^3-3k^2+k} = \frac{k^2-1}{k(3k^2-3k+1)} >$$

$$\frac{k^2-k}{k(3k^2-3k+1)} = \frac{k-1}{3k^2-3k+1} >$$

$$\frac{k-1}{3k^2+3} = \frac{k-1}{3(k^2+1)} > \frac{k-1}{3(k^2-1)} = \frac{k-1}{3(k-1)(k+1)} = \frac{1}{3(k+1)}$$

Note that $\sum_{k=0}^{\infty} \frac{1}{3(k+1)} < \sum_{k=0}^{\infty} \frac{1}{3k} = \frac{1}{3} \sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{3}\infty$, informally, this shows that the series is less then the Harmonic series, and hence divergent, by the Comparison Test. Hence we have, by the Comparison Test, that the series $\sum_{k=1}^{\infty} \frac{9k^2-2}{3k(3k-1)(3k-2)}$ is divergent. $\square$

c.

Use the result of part (b) to help you prove that $\sum_{k=1}^{\infty} a_k$ diverges.

Proof. Note that we have $\sum_{k=1}^{\infty} b_k$ diverges, where b_k is some grouping of our original sequence a_k.
Thus, we have by Exercise 5 on page 24 that the series $\sum_{k=1}^{\infty} b_k$ diverges. $\square$

17.)

If $\sum_{k=1}^{\infty} a_k$ converges absolutely, prove that $\sum_{k=1}^{\infty} a_k^2$ converges. Is the converse true? Is the statement true if $\sum_{k=1}^{\infty} a_k$ converges conditionally? Justify your answers.

Proof. Let $\sum_{k=1}^{\infty} a_k$ be absolutely convergent. Then note that we have $\sum_{k=1}^{\infty} |a_k| < \infty$. Hence by Problem 12 from Homework #2, we have the series $\sum_{k=1}^{\infty} |a_k|^2 = \sum_{k=1}^{\infty} a_k^2$ is convergent. $\square$

Counter-Example: The converse of this statement is false. Consider the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ and let $a_k^2 = \frac{1}{k^2}$. Then the series $\sum_{k=1}^{\infty} a_k^2$ is a convergent p-series. But the series $\sum_{k=1}^{\infty} \frac{1}{k}$ is the divergent Harmonic Series.

No, in-fact we have a more powerful claim:

If $|a_k| \geq 1$ for all $k \in \mathbb{Z}^+$ and $\sum_{k=1}^{\infty} a_k$ is conditionally convergent, then $\sum_{k=1}^{\infty} a_k^2$ is a divergent series.

Proof. Suppose we have some sequence over the reals $\{a_k\}$.

Let $|a_k| \geq 1$ for all $k \in \mathbb{Z}^+$ and the series $\sum_{k=1}^{\infty} a_k$ be conditionally convergent. Then, note that we have $a_k^2 \geq 1$ for all. More over note the following inequality:

$$|a_k| = \sqrt{a_k^2} \leq a_k^2$$

Note that since $\sum_{k=1}^{\infty} a_k$ is conditionally convergent, we have $\sum_{k=1}^{\infty} |a_k|$ is a divergent series. Since we have $|a_k|, a_k^2 \geq 0$ for all $k \in \mathbb{Z}^+$, by the Comparison Test, we have that the series $\sum_{k=1}^{\infty} a_k^2$ is divergent. $\square$

However for the case where $a_k \in (0, 1)$ for all $k \in \mathbb{Z}^+$, the claim doesn't hold for conditionally convergent series.

Counter-Example Consider the sequence $a_k = \frac{1}{k^{\frac{1}{2}}}$. Note that we have $a_k \geq a_{k+1}$ for all $k \in \mathbb{Z}^+$ and $\lim_{k\to\infty} \frac{1}{k^{\frac{1}{2}}} = 0$, hence the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{\frac{1}{2}}}$ is convergent. But note that the series $\sum_{k=1}^{\infty} \left| \frac{(-1)^k+1}{k^{\frac{1}{2}}} \right| = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}}$ is a divergent p-series. Thus the series is conditionally convergent. But then $\sum_{k=1}^{\infty} a_k^2 = \sum_{k=1}^{\infty} \left(\frac{(-1)^k+1}{k^{\frac{1}{2}}} \right)^2 = \sum_{k=1}^{\infty} \frac{1}{k}$ which the is divergent Harmonic series. Hence the claim doesn't hold when the series is conditionally convergent.

18.)

Let $\sum_{k=1}^{\infty} a_k$ be any convergent series. Prove that every rearrangement $\sum_{k=1}^{\infty} a'_k$ that leaves the position of all but finitely many terms of the series $\sum_{k=1}^{\infty} a_k$ fixed converges to the same sum that $\sum_{k=1}^{\infty} a_k$ does.

Proof. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Then suppose we have a finite rearrangement of our series called $\sum_{n=1}^{\infty} a'_n$.
Note that since we're dealing with a finite subsequence of a_n there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have that the following $\sum_{n=n_0}^{\infty} a_n = \sum_{n=n_0}^{\infty} a'_n$.
Also for

$$\sum_{n=0}^{n_0-1} a_n = \sum_{n=0}^{n_0-1} a'_n.$$

This comes from the fact that finite addition is always commutative for real numbers.
Combining these two sums we get:

$$\sum_{n=0}^{n_0-1} a_n + \sum_{n=n_0}^{\infty} a_n = \sum_{n=0}^{\infty} a'_n$$

Similarly,

$$\sum_{n=0}^{n_0-1} a'_k + \sum_{n=n_0}^{\infty} a'_n = \sum_{n=0}^{\infty} a'_n$$

By our previous work we have $\sum_{n=1}^{\infty} a'_n = \sum_{n=1}^{\infty} a_n < \infty$.
Hence our rearranged series $\sum_{n=1}^{\infty} a'_n$ is convergent to the same value as $\sum_{n=1}^{\infty} a'_n$. $\square$

19.)

For each $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$ as shown to the right, and consider the sequence $\{f_n(x)\}_{n=1}^{\infty}$

$$f_n(x) = \begin{cases} 2n^2(x-1)+2n, & \frac{n-1}{n} \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

a.

Sketch the graph of $f_n(x)$ for $n = 1, 2,$ and 3.

Attached separately.

b.

Prove that $lim_{n\to\infty} f_n(x) = 0$ for each $x \in [0,1]$.

Proof. Note that for all $n \in \mathbb{N}$ we have $0 < \frac{n-1}{n}$.

So consider the case where $x = 0$ or $x = 1$. Then we have $f_n(x) = 0$. Hence

$$\lim_{n\to\infty} f_n(x) = 0$$

Since in the limit of a constant function is the constant, by Real I results.

So we will now consider the case where $0 < x < 1$. Note that by the Archimedian Property, we can choose an $n_0 \in \mathbb{N}$ such that $0 < x < \frac{n_0-1}{n_0} < 1$.

Thus, if $n \ge n_0$, we have $x < \frac{n_0-1}{n_0} \le \frac{n-1}{n} < 1$. Hence by definition of $f_n(x)$, we have $f_n(x) = 0$. Thus

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 0 = 0.$$

Therefore in any possible case we have $\lim_{n\to\infty} f_n(x) = 0$. □

c.

Show that $\lim_{n\to\infty} \int_0^1 f_n(x)\, dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\, dx$.

Solution: Note that by part(b):

$$\lim_{n\to\infty} f_n(x) = 0$$

Then

$$\int_0^1 \lim_{n\to\infty} f_n(x)\ dx = 0.$$

Conversely:

$$\lim_{n\to\infty} \int_0^1 f_n(x)\ dx = \lim_{n\to\infty} -n^2 + 2n = \infty$$

Hence $\lim_{n\to\infty} \int_0^1 f_n(x)\ dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\ dx$