Problem 2.8

A particle of mass m in the infinite square well (of width a) starts out in the left half of the well, and is ($t = 0$) equally likely to be found at any point in the region.

a.)

What is the initial wave function $\Psi(x, 0)$? (Assume it is real. Don't forget to normalize it.)
Solution:
So since we can assume that our probability density is constant everywhere at $t = 0$, so that:

$$\int_0^{a/2} C \ dx = 1$$

where $C \in \mathbb{R}$. So then our probability density is: $\frac{2}{a}$.
Since $|\Psi(x, 0)|^2 = \frac{2}{a}$ and Ψ is real:

$$\Psi(x, 0) = \sqrt{\frac{2}{a}} = \begin{cases} \frac{\sqrt{2a}}{a} & 0 \leq x \leq \frac{a}{2} \\ 0 & \frac{a}{2} \leq x \leq a \end{cases}$$

b.)

What is the probability that a measurement of the energy would yield the value $\frac{\pi^2 \hbar^2}{2ma^2}$?
Solution:
This is equivalent to finding $|c_1|^2$ in our general solution:

$$\begin{aligned} c_1 &= \sqrt{\frac{2}{a}} \left[\int_0^{a/2} \sin \frac{\pi x}{a} \ dx \frac{\sqrt{2a}}{a} \ dx + \int_{a/2}^{a} 0 \ dx \right] \\ &= \frac{2}{a} \int_0^{a/2} \sin \frac{\pi x}{a} \ dx \\ &= \frac{2}{\pi} \int_0^{\pi/2} \sin(u) \ du \\ &= \frac{2}{\pi} [-\cos(\pi/2) + \cos(0)] = \frac{2}{\pi} \end{aligned}$$

So that the likelihood of finding the particle at ground state E_1 is: $\frac{4}{\pi^2} \approx 0.405$.

Problem 2.10

(a.)

Construct $\psi_2(x)$.

Solution:

$$\psi_1(x) = a^+\psi_0(x)$$

Where: $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(\frac{-m\omega}{2\hbar}x^2\right)$. So then

$$\begin{aligned}
\psi_1(x) &= a^+\psi_0(x) \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left(-ip + m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(\frac{-m\omega}{2\hbar}x^2\right) \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left(-i\frac{\hbar}{i}\frac{d}{dx} + m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(\frac{-m\omega}{2\hbar}x^2\right) \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx}\exp\left(\frac{-m\omega}{2\hbar}x^2\right) + m\omega x\exp\left(\frac{-m\omega}{2\hbar}x^2\right)\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\left(\frac{m\omega x}{\hbar}\right)\exp\left(\frac{-m\omega}{2\hbar}x^2\right) + m\omega x\exp\left(\frac{-m\omega}{2\hbar}x^2\right)\right)\left(\frac{m\omega}{\pi\hbar}\right. \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left(-\left(-m\omega x\right)\exp\left(\frac{-m\omega}{2\hbar}x^2\right) + m\omega x\exp\left(\frac{-m\omega}{2\hbar}x^2\right)\right)\left(\frac{m\omega}{\pi\hbar}\right) \\
&= \frac{2}{\sqrt{2\hbar m\omega}}\left(m\omega x\exp\left(\frac{-m\omega}{2\hbar}x^2\right)\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \\
\psi_2(x) &= a^+\psi_1(x) \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\frac{2}{\sqrt{2\hbar m\omega}}\left(m\omega x\exp\left(\frac{-m\omega}{2\hbar}x^2\right)\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \\
&= \frac{1}{\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-\hbar\frac{d}{dx}x\exp(\frac{-m\omega x^2}{2\hbar}) + m\omega x^2\exp(\frac{-m\omega x^2}{2\hbar})\right] \\
&= \frac{1}{\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp(\frac{-m\omega x^2}{2\hbar})(2m\omega x^2 - \hbar)
\end{aligned}$$

adding in normalization constant:

$$\psi_2(x) = \frac{1}{\sqrt{2}}\frac{1}{\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp(\frac{-m\omega x^2}{2\hbar})(2m\omega x^2 - \hbar)$$

(b.)

Sketch ψ_0, ψ_1, and ψ_2.

Solution:

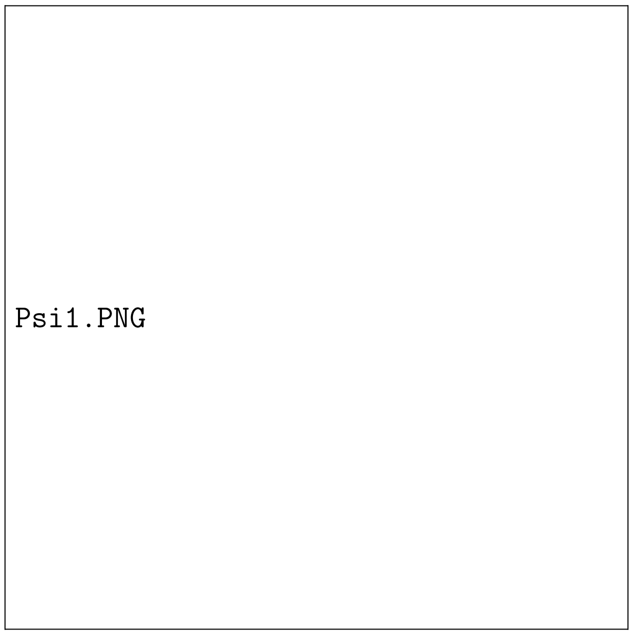

Figure 1: $\psi_0(x)$

Psi2.PNG

Figure 2: $\psi_1(x)$

Psi3.PNG

Figure 3: $\psi_2(x)$

(c.)

Check the orthogonality of ψ_0, ψ_1 and ψ_2, by explicit integration. *Hint:* If you exploit the even-ness and odd-ness of the functions, there is really only one integral left to do.

Solution:

First, note that by our graph, clearly ψ_0, ψ_2 are even, and ψ_1 is odd. So then:

$$\int_{-\infty}^{+\infty} \psi_0\psi_1 \, dx = 0 \text{ by even/odd orthogonality}$$
$$\int_{-\infty}^{+\infty} \psi_1\psi_2 \, dx = 0 \text{ by even/odd orthogonality}$$
$$\int_{-\infty}^{+\infty} \psi_0\psi_2 \, dx =$$