Group Assignment #4

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Up to this point, we have discussed at length the idea of integration in the Lebesgue sense. Replacing the deficient Riemann integral is this new theory that invokes the measure of a set to determine the integral; colloquially we can partition the range of a function to approximate the integral rather than partitioning the domain.

In the typical treatment of calculus, the student learns about (Riemann) integration after much practice with the derivative. By smoothing over rough edges and equating integrals with antiderivatives, they learn to use the derivative in the evaluation of certain antiderivatives, and thus integrals. This stands in stark contrast to measure theory, which introduces the integral first, completely independent of the derivative.

So where does the derivative fit in with all this?

Derivative

Let $f : A \to \mathbb{R}$ be a function defined on an open interval $A \subseteq \mathbb{R}$, and let $b \in \mathbb{R}$. The *derivative* of f at b is defined as

$$f'(b) := \lim_{t\to 0} \frac{f(b+t) - f(b)}{t}.$$

In measure theory, this definition remains valid.

Considering calculus in its usual sense, the derivative and integral are related via the fundamental theorem of calculus. Ideally we'd like this result to pass to Lebesgue integration for Lebesgue measurable functions. As it turns out, we are in fact able to use the fundamental theorem of calculus for this very situation, with the obvious requirement that the function be continuous at the point in question; the build-up to this result has its own set of consequences that are farther-reaching than one initially anticipates.

Hardy–Littlewood maximal function

For Lebesgue measurable $h : \mathbb{R} \to \mathbb{R}$, the *Hardy–Littlewood maximal function of* h is the function $h^* : \mathbb{R} \to [0, \infty]$ given as

$$h^*(b) = \sup_{t>0} \frac{1}{2t} \int_{b-t}^{b+t} |h|.$$

Building toward the main theorems involving derivatives are a pair of inequalities named in honor of Andrei Markov and Pafnuty Chebyshev, as well as another due to Godfrey Hardy and John Littlewood. All three compare the measure of a set where a function is greater than a value to the integral of the function. The last of these inequalities requires construction of a function fittingly called the Hardy–Littlewood maximal function, as shown at the left. This function provides the inequality necessary to approximate a function by its average on small intervals.

Alongside the fundamental theorem of calculus, the notable result here is the Lebesgue differentiation theorem, which can be stated in two different ways. Interestingly, one version does not include any derivatives at all. Broadly, this first version asserts that on small intervals, a Lebesgue measurable function can be approximated arbitrarily well by its average almost everywhere. The second version invokes the derivative and combines with the fundamental theorem of calculus to allow us to think of differentiation and integration as inverse operations.

Lebesgue differentiation theorem (V1)

For $f \in \mathcal{L}^1(\mathbb{R})$,

$$\lim_{t \downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)| = 0$$

for almost every $b \in \mathbb{R}$.

The Lebesgue differentiation theorem has a particular consequence that, despite how pathological Lebesgue measurable sets can be, there does not exist a Lebesgue measurable set that contains exactly half of every interval.

The density of a subset of $\mathbb{R}$ is disjoint from the topological definition of the same word, and this measure-theoretic version is seemingly simple: it is roughly the proportion of the set that lies inside small intervals centered at a number (which may or may not be in the set). We can then prove the Lebesgue density theorem, which asserts that Lebesgue measurable $E \subseteq \mathbb{R}$ has density 1 at almost every element of E and density 0 almost everywhere not in E.

Density

For $E \subseteq \mathbb{R}$, the *density* of E at a number $b \in \mathbb{R}$ is

$$\lim_{t \downarrow 0} \frac{|E \cap (b-t, b+t)|}{2t}$$

provided the limit exists (otherwise density of is undefined).

As a final note of intrigue, the machinery developed here allows us to construct a “bad” Borel set B, one that has a (Borel measurable) characteristic function χ_B that is not continuous anywhere, and does not accept any appeal to repair this issue; that is, it cannot be modified on any set of measure 0 to become continuous. But despite this, if we define a function g in the spirit of the fundamental theorem of calculus (that is, $g(b) = \int_0^b \chi_B$), then it is differentiable almost everywhere.