MATH 416: Final Exam
(Take-Home Part)

Joseph McGuire

December 11, 2018

(1.) A researcher wants to run at least one trail over a day over a period of 50 days, but no more than 75 trails in all. Show that during those 50 days, there is a period of consecutive days during which the researcher runs exactly 24 trails.

Proof. Let $a_1 = \#$ of trails ran during day 1, $a_2 = \#$ of trails ran during days 1 and 2, ..., $a_{50} = \#$ of trails ran during days 1, ..., 50. Denote $a_1, a_2, ..., a_{50}$ (*). Then since we will run at least one experiment everyday over the course of the fifty days, we have the following property:

$$1 \leq a_1 < a_2 < ... < a_{50} \leq 75$$

$$\text{iff } 25 < a_1 + 24 < a_2 + 24 < ... < a_{50} + 24 \leq 99.$$

With a_1, a_2, ..., a_{50}, $a_1 + 24$, $a_2 + 24$, ..., $a_{50} + 24$ we have 100 positive integers between 1 and 99 positive. Hence with (*) we have 100 positive integers and 99 places between 1 and 99. Thus, by the pigeonhole principle, we have at least one element in (*) that are the same. But since $a_1, a_2, ..., a_{50}$ are all distinct, that means that we have for some distinct a_i and a_j: $a_i = a_j + 24$. Thus $a_i - a_j = 24$, hence between days i and j we have exactly 24 trails are ran. □

(2.)

(a.) Set up a generating function to find the number of ways in which n identical coins can be put in five distinct envelopes (labeled A, B, C, D, E), such that envelopes A, B and C have the same number of coins in them. (No restrictions on envelopes D and E.)

Proof. Let $A, B, C, D,$ and E denote five distinct envelopes. We want to put n identical coins in A, B, C, D, E with A, B, C having the same number of coins. For D and E, if we put zero coins in the envelopes we have 1, if we put one coin in the envelope we have x, repeating this ad infinitum:

$$(1 + x + x^2 + ...)^2.$$

For A, B, C we have they must add one coin to each, anytime we add one to any of the three envelopes. Hence we have the generating function:

$$(1 + x^3 + x^6 + ...).$$

Combining these using the multiplication principle we have the following generating function:

$$(1 + x^3 + x^6 + ...)(1 + x + x^2 + ...)^2$$

$$= \left(\frac{1}{1 - x^3}\right)\left(\frac{1}{1 - x}\right)^2, \text{ by G.F(3) and G.F(4).}$$

□

(b.) Use your generating function to find the number of ways to distribute 11 coins.

Considering the generating function from part(a.) for the special case where $n = 11$, we get

$$(1 + x + x^2 + ... + x^{11})^2(1 + x^3 + x^6 + x^9).$$

Expanding this out, we get:

$$b_{11} = 30.$$

Thus there are 30 different ways to distribute 11 coins to five envelopes.

(3.) Prove that if $n \in \mathbb{Z}^+$ with $n \geq 2$, then $\binom{2n}{2} = \binom{n}{2}2^2 + n$.

(a.) Give an algebraic proof.

Proof. Let $n \in \mathbb{Z}^+$, where $n \geq 2$.
(L.H.S)

$$\binom{2n}{2} = \frac{2n!}{2!(2n-2)!} = \frac{2n(2n-1)}{2} = n(2n-1) = 2n^2 - n = n(2n-1).$$

(R.H.S)

$$\binom{n}{2}2^2 + n = \frac{n!}{2!(n-2)!}2^2 + n = \frac{n(n-1)2^2}{2} + 2 = 2n(n-1) + n = 2n^2 - 2n + n = 2n^2 - n = n(2n-1).$$

Thus $L.H.S = R.H.S$, hence our identity holds. □

(b.) Give a combonatorial proof.

This is a bad proof, I know, I'm sorry :(

Proof. Consider the case where we have $2n$ cats and we want to choose 2 cats, there are two groups one of female cats (A) and one of male cats (B), both of size n. Then we have $\binom{2n}{2}$ ways to choose 2 cats out of this group. Alternatively, if we were to only choose from the male cats or female cats, we have $\binom{n}{2}$ ways of doing so. Additionally, if we were to partition the groups further, such that we have the sets C and D

where $C \cap D = \emptyset$, where C consists of n total cats from both sets A and B while not compromising the entire set, and D is the rest of the cats. Then here we have again $2 \cdot \binom{n}{2}$ ways to do so. Finally, n ways of permuting the cats within the sets; i.e $\binom{n}{1}$. Thus we have

$$\binom{2n}{2} = 2^2 \binom{n}{2} + n$$

. $\square$

(4.) Determine which pairs of graphs from the following are isomorphic. If isomorphic, find as isomorphism. If not, explain why not.

$A \approx B$, since we have the following isomorphism:

$$\phi(A_1) = B_8, \phi(A_2) = B_3, \phi(A_3) = B_9, \phi(A_4) = B_6, \phi(A_5) = B_1, \phi(A_6) = B_4, \phi(A_7) = B_7, \phi(A_8) = B_5, \phi(A_9) = B_2.$$

But, $A \not\approx C$ since for C_9 (degree 3) we have that it is adjacent to exactly one element that is adjacent to the vertex of degree 5 (C_5), while the one vertex of degree 3 in A ($A5$), $A5$ is adjacent to no element that is adjacent to the vertex of degree 5 ($A5$). Hence $B \not\approx C$.

(5.) You have cans of paint in eight different colors. You want to paint the four 1-by-1 unit squares of a 2-by-2 board in such a way that neighboring unit squares that share an edge are painted in different colors. Two coloring schemes are considered the same if one can be obtained from the other by rotation. How many different coloring schemes like that are possible?

Placing our first color in the top-left box of the 2×2. We have 8 options for this first choice. Then moving on to the adjacent blocks, they can be one of 7 possible choices. Hence we have

$$(\binom{7}{1})^2.$$

Then finally our last square can be any of the eight colors. We have a total of $8^2(\binom{7}{1})^2$.

(6.) Each 1-by-1 unit square of a 3-by-3 square grid is to be colored either blue or red. How many coloring of a 3-by-3 grid are possible that do not have a 2-by-2 red square?

We have a total of 2^9 possibilities. Then for red-squares we can only place 4 such squares. I.e, $2^9 - 2^2$,

(7.) Can a symmetric BIBD have the following set of parameters? If it is not possible, explain why not. Otherwise, list the parameters as "potential".

(a.) (64,34,8)

Since we have a symmetric BIBD, we have $b = v$ and $k = r$, and $(b = 64, k = 34, \lambda = 3)$. So for such a design to exist, we need $\lambda(v-1) = r(k-1)$, but this isn't the case with these parameters, since $\lambda(v-1) = 3 \cdot 63 = 504$, but $r(k-1) = 34 \cdot 33 = 1122$. However, $504 \neq 1122$, thus this design isn't possible.

(b.) (15,7,3)

Again, we have $b = v$, $k = r$, and $(b = 15, k = 7, \lambda = 3)$. Then by the $BRCTheorem$, we have if there exists $x, y, z \in \mathbb{Z}$ such that

$$z^2 = (k-\lambda)x^2 + (-1)^{\frac{v-1}{2}}\lambda y^2$$

$$\iff z^2 = (7-3)x^2 + (-1)^7 3y^2$$

$$\iff z^2 = 4x^2 - 3y^2.$$

We have the ordered pair $(x, y, z) = (1, 1, 1)$ satisfies this equation. Thus, by the BRC Theorem, we have there is a potential design with these parameters.

(8.) BONUS PROBLEM. Construct a symmetric BIBD with the parameters you listed as "potential" in the previous problem. Use a "started block" method; i.e. find a difference set of size k in $\mathbb{Z}_v$ and use the difference set as a started block to construct a symmetric BIBD.