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Midterm 1

Class: Math 520

1

Adapt Laplace's Method to find the leading-order expansion for

$$I(x) = \int_0^\infty e^{-t^2} e^{xt}\, dt \qquad \text{as } x \to +\infty.$$

This isn't a standard case for Laplace's method. Heuristically justify each step.

Solution. Rewrite this as:

$$I(x) = \int_0^\infty e^{xt-t^2}\, dt.$$

Then note that exponentiation preserves order, so that the maximum of e^{xt-t^2} on $[0, \infty)$ occurs when:

$$\frac{d}{dt}(xt - t^2) = 0 \iff x - 2t_M = 0 \iff t_M = \frac{x}{2} \qquad \text{x held fixed },$$

where t_M is the maximum value of e^{xt-t^2} on $[0, \infty)$ for fixed x. Then define $s = \frac{t}{x}$ so that $s_M = \frac{t_M}{x} = \frac{1}{2}$. Then we'll have $dt = x\ ds$:

$$I(x) = x \int_0^\infty e^{x^2 s - x^2 s^2}\, ds = x \int_0^\infty e^{x^2(s-s^2)}\, ds.$$

Then we'll "expand" $s^2 - s$ about $c = \frac{1}{2}$ to get: $s^2 - s = \frac{-1}{4} + (s - \frac{1}{2})^2$.
Then note the asymptotic relation:

$$\int_0^\infty f(t) e^{x\phi(t)}\, dt \sim \int_{c-\epsilon}^{c+\epsilon} f(t) e^{x\phi(t)}\, dt,$$

where c is the global maximum of $\phi(t)$ on $[0, \infty)$.
So then we'll have:

$$I(x) \sim I(x;\epsilon) = x \int_{1/2-\epsilon}^{1/2+\epsilon} e^{-x^2\left(\frac{-1}{4} + \left(s - \frac{1}{2}\right)^2\right)}\, ds.$$

Let $u = x^2$ and $v = s - \frac{1}{2}$ giving us:

$$I(x;\epsilon) = \sqrt{u} \int_{-\epsilon}^{\epsilon} e^{-u(-1/4+v^2)}\, dv = \sqrt{u} e^{u/4} \int_{-\epsilon}^{\epsilon} e^{-uv^2}\, dv.$$

Substitute with $w = v^2 \iff dw = 2v\ dv \iff \dfrac{dw}{2v} = dv$:

$$\sqrt{u}e^{u/4}\frac{1}{2}\int_0^{\epsilon} w^{-1/2}e^{-uw}\ dw \sim \sqrt{u}e^{u/4}\frac{1}{2}\int_0^{\infty} w^{-1/2}e^{-uw}\ dw.$$

Then we'll have:

$$I(x) \sim xe^{x^2/4}\frac{\Gamma(\frac{1}{2})}{2} = \frac{xe^{x^2/4}\sqrt{\pi}}{2} \qquad x \to \infty.$$

□

2

Find a two-term expansion for each root of the following algebraic equation.

a $\epsilon x^3 + x^2 - 1 = 0, 0 < \epsilon << 1.$

b $\epsilon x^3 + (x-1)^2 = 0, 0 < \epsilon << 1.$

Solution. (a) This is a singular perturbation, so assume:

$$x(\epsilon) = z\delta(\epsilon),$$

where $z = O(1)$. Then this gives us the perturbation equation:

$$\epsilon\delta^3 z^3 + \delta^2 z^2 - 1 = 0.$$

This gives us terms of order:

$$\{\epsilon\delta^3, \delta^2, 1\}.$$

- ($\delta^2 \sim 1$) This gives us: $\delta \sim 1$ choose $\delta = 1$ giving us the equation:

  $$\epsilon z^3 + z^2 - 1 = 0.$$

  just the original equation. These solutions will correspond to the "finite" roots of this equation. Assume a regular expansion:

  $$z(\epsilon) = z_0 + z_1\epsilon + O(\epsilon^2).$$

  This gives us:

  $$\begin{aligned} z^2 &= z_0^2 + 2z_0z_1\epsilon + O(\epsilon^2) \\ \epsilon z^3 &= z_0^3\epsilon + O(\epsilon^2). \end{aligned}$$

Plugging these into the perturbation equation and matching coefficients:

$$\epsilon^0 : z_0^2 = 1 \iff z_0 = \pm 1$$
$$\epsilon^1 : z_0^3 + 2z_0 z_1 = 0.$$

As expected, we get two roots with leading order behavior $z_0 = \pm 1$. For $z_0 = 1$, this then has a first-order correction that satisfies: $1 + 2z_1 = 0 \iff z_1 = \frac{-1}{2}$ and for $z_0 = -1$ we'll have:

$$-1 - 2z_1 = 0 \iff z_1 = \frac{-1}{2}.$$

So this gives us the two-term expansion for the two finite roots:

$$x_{(-1)} = -1 - \frac{1}{2}\epsilon + O(\epsilon^2)$$

$$x_{(+1)} = 1 - \frac{1}{2}\epsilon + O(\epsilon^2).$$

- ($\epsilon\delta^3 \sim 1$) This would gives us: $\delta \sim \epsilon^{-1/3}$ so that we'll get terms of order:

  $$\{1, \epsilon^{-2/3}, 1\}.$$

  This would lead to a contradiction though since $z = O(1)$ and this would give:

  $$\epsilon^{2/3} z^3 + z^2 - \epsilon^{2/3} = 0 \iff z^2 = O(\epsilon^{2/3}) \text{ as } \epsilon \to 0^+.$$

- ($\epsilon\delta^3 \sim \delta^2$) This gives us $\delta(\epsilon) \sim \frac{1}{\epsilon}$ so choose $\delta(\epsilon) = \frac{1}{\epsilon}$. The terms are then of order:

  $$\{\frac{1}{\epsilon^2}, \frac{1}{\epsilon^2}, 1\} \implies \{1, 1, \epsilon^2\}.$$

  This gives us dominant balance, with $z = O(1)$!

So we have an equation of:

$$z^3 + z^2 - \epsilon^2 = 0.$$

Assume a regular expansion of z :

$$z(\epsilon) = z_0 + z_1\epsilon^2 + O(\epsilon^4) \qquad 0 < \epsilon << 1.$$

Then this will give us:

$$\begin{aligned}
z^2 &= (z_0 + z_1\epsilon^2 + O(\epsilon^4))^2 \\
&= z_0^2 + \epsilon^2(2z_0z_1) + O(\epsilon^4) \\
z^3 &= (z_0 + z_1\epsilon^2 + O(\epsilon^4))^3 \\
&= (z_0^2 + \epsilon^2(2z_0z_1) + O(\epsilon^4))(z_0 + z_1\epsilon^2 + O(\epsilon^2)) \\
&= z_0^3 + \epsilon^2(3z_0^2z_1) + O(\epsilon^4).
\end{aligned}$$

So plugging these into the scaled perturbation equation and matching coefficients:

$$\begin{aligned}
\epsilon^0 &: z_0^2 + z_0^3 = 0 \implies z_0 = 0, 0, -1 \\
\epsilon^2 &: 2z_0z_1 + 3z_0^2z_1 = 1
\end{aligned}$$

We may rule out the roots $z_0 = 0, 0$ as these will correspond with the finite roots already accounted for. So consider $z_0 = -1$ then $-2z_1 + 3z_1 = 1 \iff z_1 = 1$. This gives us:

$$z(\epsilon) = -1 + \epsilon^2 + O(\epsilon^4).$$

So this gives us the behavior of the "infinite" root:

$$x_{(\epsilon)} = \frac{1}{\epsilon}z(\epsilon) = -\frac{1}{\epsilon} + \epsilon + O(\epsilon^3).$$

(b)

#2(b.)

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$$\varepsilon x^3 + (x-1)^2 = 0$$

Assume $x = z\delta(\varepsilon)$ where $z = O(1)$, & ~~$\delta(\varepsilon)$~~

Then the equation becomes:

$$\varepsilon \delta^3 z^3 + \delta^2 z^2 - 2z\delta + 1 = 0$$

This gives us terms of order:

$$\{\varepsilon\delta^3, \delta^2, \delta, 1\}$$

- With $\delta \sim 1$ this is our finite root case & we have:

$$\varepsilon z^3 + (z-1)^2 = 0$$

assume $z(\varepsilon) = z_0 + z_1\varepsilon^{1/2} + z_2\varepsilon + O(\varepsilon^{3/2})$ as $\varepsilon \to 0^+$.

Then:

$$\varepsilon z^3 = z_0^3\varepsilon + O(\varepsilon^{3/2}) \quad \text{as } \varepsilon \to 0^+$$

$$z^2 = (z_0 + z_1\varepsilon^{1/2} + z_2\varepsilon + O(\varepsilon^{3/2}))^2$$

$$= z_0^2 + 2z_0z_1\varepsilon^{1/2} + (z_1^2 + 2z_0z_2)\varepsilon + O(\varepsilon^{3/2}) \quad \text{as } \varepsilon \to 0^+$$

$$-2z = -2z_0 + (-2z_1)\varepsilon^{1/2} + (-2z_2)\varepsilon + O(\varepsilon^{3/2}) \quad \text{as } \varepsilon \to 0^+$$

Then collecting ~~terms~~ coefficients of powers of ε:

ε^0: $z_0^2 - 2z_0 + 1 = 0 \to (z_0 - 1)^2 = 0 \to z_0 = 1, 1$

$\varepsilon^{1/2}$: $2z_0z_1 - 2z_1 = 0 \to$ Nothing!

ε: $z_0^3 + z_1^2 + 2z_0z_2 - 2z_2 = 0 \to z_1^2 = -1 \to z_1 = \pm i$

giving us the expansion

$$\boxed{\begin{aligned} z_{(1)}(\varepsilon) &= 1 + i\varepsilon^{1/2} + O(\varepsilon) \\ z_{(2)}(\varepsilon) &= 1 - i\varepsilon^{1/2} + O(\varepsilon) \end{aligned}}$$

(2.)(b.)

- $\varepsilon\delta^3 \sim 1$ gives orders of:

$\Rightarrow \delta^3 \sim \frac{1}{\varepsilon} \Rightarrow \{1, \varepsilon^{-2/3}, \varepsilon^{-1/3}, 1\}$

$\delta \sim \varepsilon^{-1/3} \quad \Rightarrow \{\varepsilon^{2/3}, 1, \varepsilon^{1/3}, \varepsilon^{2/3}\}$

a contradiction since

$z = O(1)$ as $\varepsilon \to 0^+$

- $\varepsilon\delta^3 \sim \delta^2 \Rightarrow \delta \sim \frac{1}{\varepsilon}$ gives orders:

$\{\varepsilon^{-2}, \varepsilon^{-2}, \varepsilon^{-1}, 1\}$

$\Rightarrow \{1, 1, \varepsilon, \varepsilon^2\}$ which is dominant balance possible with $z = O(1)$.

~~another contradiction of $z = O(1)$, $\varepsilon \to 0^+$~~

- ~~$\varepsilon\delta^3 \sim \delta \Rightarrow \delta \sim \varepsilon^{-1/2}$ gives orders:~~

~~$\{\varepsilon^{-1/2}, \varepsilon^{-1}, 1, \varepsilon^{-1/2}\}$~~

So we'll have:

$$\varepsilon\delta^3 z^3 + \delta^2 z^2 - 2\delta z + 1 = 0$$

$$\Rightarrow \varepsilon^{-2} z^3 + \varepsilon^{-2} z^2 - 2\varepsilon^{-1} z + 1 = 0$$

$$\Rightarrow z^3 + z^2 - 2\varepsilon + \varepsilon^2 = 0$$

Assume an expansion of $z(\varepsilon) \sim z_0 + z_1\varepsilon + O(\varepsilon^2)$ as $\varepsilon \to 0^+$:

$$z^2 = z_0^2 + 2z_0z_1\varepsilon + O(\varepsilon^2), \quad \varepsilon \to 0^+$$

$$z^3 = z_0^3 + (3z_0^2 z_1)\varepsilon + O(\varepsilon^2), \quad \varepsilon \to 0^+$$

so matching coefficients gives:

ε^0: $z_0^3 + z_0^2 = 0 \Rightarrow z_0^2(z_0+1) = 0 \Rightarrow z_0 = 0, 0, -1$

ε^1: $2z_0z_1 + 3z_0^2 z_1 - 2 = 0 \Rightarrow -2z_1 + 3z_1 = 2 \Rightarrow z_1 = 2 \; (z_0 = -1)$

giving us: $z(\varepsilon) = -1 + 2\varepsilon + O(\varepsilon^2) \Rightarrow \varepsilon X(\varepsilon) = -1 + 2\varepsilon + O(\varepsilon^2)$

$$\Rightarrow \boxed{X(\varepsilon) = -\varepsilon^{-1} + 2 + O(\varepsilon), \; \varepsilon \to 0^+}$$

$\square$

3

A mass attached to an aging spring subject to a damping force can be represented by the following IVP:

$$m\frac{d^2y}{d\tau^2} + a\frac{dy}{d\tau} + ke^{-r\tau}y = 0, \qquad \tau > 0; \qquad y(0) = y_0, y'(0) = 0,$$

where m is the mass of the particle; a, k, and r are positive parameters; and $y(\tau)$ is the displacement from equilibrium at time τ.

(a) Choose length and time scales and nondimensionalize the IVP to get:

$$u'' - u' = \epsilon e^{-\alpha t}u, t \geq 0; \qquad u(0) = 1, u'(0) = 0$$

where ϵ and $\alpha$are dimensionless parameters; t is the (dimensionless) time relative to a time scale; and $u(t)$ is the (dimensionless) displacement at time t relative to some length scale.

- What are the time and length scales used in this rescaling?
- What are ϵ and α in terms of the original parameters?

(b) Find a two-term expansion for $u(t)$ for $0 < \epsilon << 1$ using a perturbation method.

(c) Undo the scaling to find the corresponding two-term expansion for $y(\tau)$.

Solution. 1. Assume $y = Lu$ and $\tau = Tt$ where u, t are dimensionless displacement and time, respectively, relative to L and T, respectively. Note then the dimensions, where L is length, T is time, and M is mass:

$$[m] = M, [a] = \frac{ML}{T}, [k] = \frac{ML}{T^2}, [r] = T^{-1}.$$

Then the rescaled ODE:

$$m\frac{L^2}{T^2}u'' + a\frac{L}{T}u + kLe^{-rTt}u = 0, t > 0; \qquad u(0) = \frac{y_0}{L}, u'(0) = 0.$$

Choose a length-scale of $L = y_0$, giving us:

$$m\frac{y_0^2}{T^2}u'' + a\frac{y_0}{T}u + ky_0e^{-rTt}u = 0, t > 0; \qquad u(0) = 1, u'(0) = 0.$$

Now rewrite this as the following:

$$u'' + \frac{ay_0}{T}\frac{T^2}{my_0^2}u' + ky_0\frac{T^2}{my_0^2}e^{-rTt}u = 0, t > 0; \qquad u(0) = 1, u'(0) = 0,$$

equivalently:

$$u'' + \frac{aT}{my_0}u' + ky_0\frac{T^2}{my_0^2}e^{-rTt}u = 0, t > 0; \qquad u(0) = 1, u'(0) = 0.$$

Now choose the scale $T = \dfrac{my_0}{a}$ (note that since $[a^{-1}] = TM^{-1}L^{-1}$, so that $[a^{-1}my_0]$ has time scale as required. This gives us:

$$u'' + u' + k\frac{1}{my_0}\left(\frac{my_0}{a}\right)^2 e^{-r\frac{my_0}{a}t}u = 0, t > 0; \qquad u(0) = 1, u'(0) = 0$$

$$u'' + u' + \epsilon e^{-\alpha t}u = 0, t > 0; \qquad u(0) = 1, u'(0) = 0.,$$

where we've defined the dimensionless parameters:

$$\alpha \equiv \frac{rmy_0}{a} \qquad \epsilon \equiv \frac{my_0}{a^2}k.$$

And recall the time and length scales are:

$$L = y_0 \qquad T = \frac{my_0}{a}.$$

2. Now, this is a regular perturbation problem so that we will assume a regular expansion of $u(t)$:

$$u(t) = u_0 + u_1\epsilon + O(\epsilon^2) \qquad 0 < \epsilon << 1.$$

Then plugging this in a matching coefficients we end up with:

$$\epsilon^0 : u_0'' + u_0' = 0, t > 0; u_0(0) = 1, u_0'(0) = 0,$$

$$\epsilon : u_1'' + u_1' + e^{-\alpha t}u_0, t > 0; u_1(0) = 0, u_1'(0) = 0.$$

Solving the first ODE we end up with a general solution of:

$$u_0(t) = A_0 + B_0e^{-t}.$$

First impose $u_0'(0) = 0$, this gives:

$$0 = -B \implies B = 0.$$

Now impose $u_0(0) = 1$, this gives us:

$$u_0(t) = 1.$$

So that the first-order correction satisfies:

$$u_1'' + u_1' = -e^{-\alpha t}, t > 0.$$

So a homogeneous solution to this is again:

$$A_1 + B_1 e^{-t},$$

then try a particular solution $Ae^{-\alpha t}$ with the equation to gives:

$$(\alpha^2 - \alpha)Ae^{-\alpha t}.$$

From this it's clear $A = \dfrac{-1}{\alpha^2 - \alpha} = \dfrac{1}{\alpha - \alpha^2}$. So the solution to the ODE is:

$$u_1(t) = A_1 + B_1 e^{-t} - \frac{e^{-\alpha t}}{\alpha^2 - \alpha}.$$

Imposing $u_1(0) = 0$ will give us:

$$0 = A_1 + B_1 + \frac{-1}{\alpha^2 - \alpha},$$

then $u_1'(0) = 0$ gives us:

$$0 = -B_1 + \frac{\alpha}{\alpha^2 - \alpha} \iff B_1 = \frac{\alpha}{\alpha^2 - \alpha}.$$

This implies that $A_1 + \dfrac{1}{\alpha} = 0 \iff A_1 = -\dfrac{1}{\alpha}$ so that:

$$u_1(t) = \frac{-1}{\alpha} + \frac{e^{-\alpha t}}{\alpha - 1} - \frac{e^{-\alpha t}}{\alpha^2 - \alpha}.$$

So that we have:

$$u(t;\epsilon) = 1 + \epsilon\left(\frac{-1}{\alpha} + \frac{e^{-\alpha t}}{\alpha - 1} - \frac{e^{-\alpha t}}{\alpha^2 - \alpha}\right) + O(\epsilon^2) \qquad \text{for } 0 < \epsilon << 1.$$

3. We have that $y = y_0 u$ and $\tau = t\dfrac{my_0}{a}$ so that:

$$y(\tau;\epsilon) = y_0 + \epsilon y_0\left(-\frac{1}{\alpha} + \frac{e^{-r\tau}}{\alpha - 1} - \frac{e^{-r\tau}}{\alpha^2 - \alpha}\right) + O(\epsilon^2) \qquad \text{for } \tau \geq 0, 0 < \epsilon << 1,$$

where $\alpha \equiv \dfrac{my_0}{a}$.

$\square$

4

Use matched asymptotics to find a leading-order uniform expansion to the solution to the boundary-values problem:

$$\epsilon y'' - y' = 1, 0 < x < 1; \qquad y(0) = \alpha, y(1) = \beta; \qquad 0 < \epsilon << 1.$$

For full credit, match using an intermediate scale; only partial credit for matching using less rigorous techniques.

Solution. By the general theory we developed for second-order perturbation equation on $[0, 1]$ we'll have a boundary layer at $x = 1$, since $-1 < 0$.

- **Outer Solution**

  Assume $0 < x < 1$ and $0 < \epsilon << 1$ with $1 - x = O(1)$ as $0 < \epsilon << 1$, as well as $y'', y', y = O(1)$. Then we'll have leading order behavior that satisfies:

  $$\Big\{ y_0' = -1 \qquad 0 < x < 1; \qquad y(0) = \alpha.$$

  Then we see that $y_0(x) = -x + A_0$, then imposing boundary conditions at $x = 0$:

  $$\alpha = A_0,$$

  so that we have:

  $$y(x; \epsilon) = \alpha - x + O(\epsilon) \qquad \text{for } x - 1 = O(1), \text{ as } 0 < \epsilon << 1.$$

- **Inner Solution**

  Assume $0 << x < 1$ and $0 < \epsilon << 1$ and rescale with $\xi = \dfrac{1-x}{\delta(\epsilon)} \iff x = 1 - \xi\delta(\epsilon)$ with $0 < \delta << 1$ with $0 < \epsilon << 1$. Then $\dfrac{d}{dx} = -\dfrac{1}{\delta}\dfrac{d}{d\xi}$ and $\dfrac{d^2}{dx^2} = \dfrac{1}{\xi^2}\dfrac{d^2}{d\xi^2}$ Then $y(x) = y(1 - \xi\delta) = Y(\xi)$ then assume $Y, Y', Y'' = O(1)$ with $\xi = O(1)$. So that the rescaled ODE becomes:

  $$\frac{\epsilon}{\delta^2}\frac{d^2Y}{d\xi^2} + \frac{1}{\delta}\frac{dY}{d\xi} = 1 \qquad \xi = O(1), 0 < \epsilon << 1.$$

  Then these terms have order:

  $$\left\{ \frac{\epsilon}{\delta^2}, \frac{1}{\delta}, 1 \right\}.$$

With $\delta \sim 1$ we end up with a contradiction because we assumed $0 < \delta(\epsilon) << 1$. With $\delta \sim \sqrt{\epsilon}$ then we'll end up with:

$$\frac{d^2Y}{d\xi^2} + \frac{1}{\sqrt{\epsilon}}\frac{dY}{d\xi} = 1 \ iff \sqrt{\epsilon}\frac{d^2Y}{d\xi^2} + \frac{dY}{d\xi} = \sqrt{\epsilon} \iff \frac{dY}{d\xi} = O(\sqrt{\epsilon}),$$

a contradiction that $\dfrac{dY}{d\xi} = O(1)$. We do end up with dominant balance with $\delta(\epsilon) = \epsilon$:

$$\frac{d^2Y}{d\xi^2} + \frac{dY}{d\xi} = \epsilon, \qquad \xi = O(1) \text{ with } 0 < \epsilon << 1.$$

At leading order, we'll then have this satisfies:

$$Y_0'' + Y_0' = 0 \qquad Y(0) = \beta,$$

since $\xi = \dfrac{1-x}{\delta}$ with $x = 1$ gives $\xi = 0$. So we have that:

$$Y_0(\xi) = C_0 + D_0 e^{-\xi},$$

imposing $Y(0) = \beta$:

$$\beta = C_0 + D_0 \implies C_0 = \beta - D_0.$$

So that the leading-order inner solution is:

$$Y(\xi; \epsilon) = \beta + D_0(e^{-\xi} - 1).$$

- **Matching**

  Now, we'll introduce an intermediate scale:

$$\nu = \frac{1-x}{\epsilon^r} \qquad 0 < r < 1.$$

  Then this will give us:

$$x = 1 - \nu\epsilon^r \qquad \xi = \frac{\nu\epsilon^r}{\epsilon} = \frac{\nu}{\epsilon^{1-r}}.$$

  So that with $\epsilon \to 0^+$ we'll end up with $\xi \to \infty$ and $x \to 1^-$. Then the limits for the outter and inner are:

$$\lim_{x \to 1^-} \alpha - x + O(\epsilon) = \alpha - 1$$

  and

$$\lim_{\xi \to \infty} \beta + D_0(e^{-\xi} - 1) = \beta - D_0.$$

  Therefore the common limit is $\beta - D_0 = \alpha - 1 \iff D_0 = \beta - \alpha + 1$.

- **Uniform Expansion**

  Now we have that at leading-order:

  $$y(x;\epsilon) = \text{Outer} + \text{Inner} - \text{Common Limit}$$

  $y(x;\epsilon) \sim \alpha - x + \beta + (\beta - \alpha + 1)(e^{-(1-x)/\epsilon} - 1) - \alpha + 1 = 1 - x + \beta + (\beta - \alpha + 1)(e^{-(1-x)/\epsilon} - 1)$, for $0 \le x \le 1$ with $0 < \epsilon << 1$.

- **Checking Answer** Checking the B.C's:

  $$y(1) = 1 - 1 + \beta + (\beta - \alpha + 1)(1 - 1) = \beta\checkmark$$

  $$y(0) = 1 - 0 + \beta + (\beta - \alpha + 1)(e^{-1/\epsilon} - 1) = 1 + \beta - \beta + \alpha - 1 + \text{exp. small} = \alpha\checkmark$$

  Checking this with the inner and outer solutions. Assume $1 - x = O(1) \implies \dfrac{1-x}{\epsilon} = O(\epsilon^{-1})$, then:

  $$1 - x + \beta + (\beta - \alpha + 1)(e^{-(1-x)/\epsilon} - 1) = 1 - x + \beta - \beta + \alpha - 1 = \alpha - x + \text{exp. small}\checkmark.$$

  Assume $1 - x = O(\epsilon)$, then $\dfrac{1-x}{\epsilon} = O(1)$ will give us:

  $$O(1) + \beta + (\beta - \alpha + 1)(e^{-(1-x)/\epsilon} - 1)\checkmark.$$

  This is a valid uniform leading-order approximation for the BVP!

$\square$

5

Use matched asymptotics to find a leading-order uniform expansion to the solution to the boundary-values problem:

$$\epsilon y'' + x^2 y' - x^3 y = 0, 0 < x < 1; \qquad y(0) = \alpha, y(1) = \beta; 0 < \epsilon << 1.$$

Your answer should involve an integral of the form $\int\limits_0^\xi \exp\{p(t)\}\ dt$ for some $p(t)$ and where ξ is the inner variable. You might find it useful to use:

$$\int_0^\xi e^{p(t)}\ dt = \int_0^\infty e^{p(t)}\ dt - \int_\xi^\infty e^{p(t)}\ dt.$$

Solution. Since $x^2 > 0$ for $x \in (0, 1]$ we have that that this cannot support a boundary layer at the right-side or the interior of the integral because of the exponential growth of a solution to the perturbation equation on this interval. However, a boundary layer might be possible where $x^2 = 0 \iff x = 0$.

- **Outer Solution** Assume $0 << x < 1$ and $0 < \epsilon << 1$, then the leading-order behavior will follow:
$$\begin{cases} x^2 y_0' - x^3 y_0 = 0 & 0 << x < 1 \\ y_0(1) = \beta & 0 < \epsilon << 1. \end{cases}$$
Since $0 << x < 1$ we have the coefficient functions are analytic for these $x-$values and this is a regular ODE. Then we may solve it as follows:
$$x^2 y_0' = x^3 y_0 \iff \frac{y_0'}{y_0} = x \iff \ln(y_0(x)) = \frac{x^2}{2} + A_0 \iff y_0(x) = A_0 \exp\left\{\frac{x^2}{2}\right\}.$$
Imposing the right-BC:
$$\beta = A_0 e^{1/2} \iff A_0 = \beta e^{-1/2} \implies y_0(x) = \beta \exp\left\{\frac{x^2 - 1}{2}\right\},$$
for $0 << x \leq 1$ and $0 < \epsilon << 1$.

- **Inner Solution** Rescale with the following $x = \xi\delta(\epsilon)$ where $\xi = O(1)$ and $0 < \delta << 1$ for $0 < \epsilon << 1$. So that $y(x) = y(\xi\delta) = Y(\xi)$, then assume $Y, Y', Y'' = O(1)$ with $\xi = O(1)$. This gives us
$$\frac{d}{dx} = \frac{1}{\delta}\frac{d}{d\xi} \qquad \frac{d^2}{dx^2} = \frac{1}{\delta^2}\frac{d^2}{d\xi^2}.$$
So that the ODE becomes:
$$\frac{\epsilon}{\delta^2} Y'' + \frac{\xi^2\delta^2}{\delta} Y' - \xi^3\delta^3 Y = 0$$
or equivalently:
$$Y'' + \frac{\delta^3\xi^2}{\epsilon} Y' - \frac{\delta^5}{\epsilon} Y = 0,$$
for $\xi = O(1)$ with $0 < \delta << 1$. These coefficients have order:
$$\left\{1, \frac{\delta^3}{\epsilon}, \frac{\delta^5}{\epsilon}\right\}.$$

If $\dfrac{\delta^3}{\epsilon} \sim \dfrac{\delta^5}{\epsilon} \implies \delta \sim 1$ a contradiction since we assumed $0 < \delta << 1$. If $\dfrac{\delta^5}{\epsilon} \sim 1$ we'll have $\delta \sim \epsilon^{1/5}$ so that the ordering becomes:

$$\left\{1, \frac{\epsilon^{3/5}}{\epsilon}, 1\right\} \implies \left\{\epsilon^{2/5}, 1, \epsilon^{2/5}\right\},$$

a contradiction since this would imply $Y'(\xi) = O(\epsilon^{2/5})$ as $\epsilon \to 0^+$. If $\dfrac{\delta^3}{\epsilon} \sim 1 \implies \delta \sim \epsilon^{1/3}$, giving us an ordering of:

$$\left\{1, 1, \epsilon^{2/3}\right\}$$

and that gives us:

$$Y'' + \xi^2 Y' - \xi\epsilon^{2/3} Y = 0.$$

We have dominant balance!

Now note that if $x = 0$, then $0 = \delta\xi \implies \xi = 0$, so that the left B.C. becomes: $Y(0) = \alpha$. Now at leading order the inner-solution satisfies:

$$\begin{cases} Y_0'' + \xi^2 Y_0' = 0 & \text{if } \xi = O(1) \\ Y_0(0) = \alpha \end{cases}.$$

Then looking at the ODE, we may solve this with a substitution of $U = Y_0'$ and using integrating factors:

$$\begin{aligned} U' + \xi^2 U &= 0 \\ \frac{d}{d\xi}\left(U(\xi)\exp\left\{\frac{\xi^3}{3}\right\}\right) &= 0 \\ U(\xi) &= B_0 \exp\left\{-\frac{\xi^3}{3}\right\} \\ Y_0'(\xi) &= B_0 \exp\left\{\frac{-\xi^3}{3}\right\} \\ Y_0(\xi) &= B_0 \int_0^\xi \exp\left\{-\frac{t^3}{3}\right\}\, dt + C_0. \end{aligned}$$

Now we can impose the left B.C to get:

$$\alpha = C_0$$

so that we have:

$$Y_0(\xi) = B_0 \int_0^\xi e^{-t^3/3}\, dt + \alpha.$$

So note that making a $u-$sub of $u = -\dfrac{t^3}{3}, du = -t^2 dt, t = (-3u)^{1/3} = -(3u)^{1/3}$ assuming both u, t are real, so that

$$\int_0^{\xi} e^{-t^3/3}\ dt = -3^{2/3}\int_0^{\xi} u^{2/3}e^{-u}\ du.$$

So

$$Y_0(\xi) = \alpha - B_0 3^{2/3}\int_0^{\xi} u^{2/3}e^{-u}\ du,$$

for $\xi = O(1)$ as $\epsilon \to 0^+$.

- **Matching** Assume an intermediate scale of:

$$\nu = \frac{x}{\epsilon^r} \qquad 0 < r < \frac{1}{3}$$

this gives: $x = \epsilon^r \nu$ and $\xi = \dfrac{x\epsilon^r}{\epsilon^{1/3}} = \dfrac{x}{\epsilon^{1/3-r}}$, so that as $\epsilon \to 0^+$ we have $x \to 0^+$ and $\xi \to +\infty$. Now these limits are:

$$\lim_{x\to 0^+} y_{\text{out},0} = \lim_{x\to 0^+} \beta e^{(x^2-1)/2} = \beta e^{-1/2}$$

and

$$\lim_{\xi\to+\infty} Y_{\text{In},0} = \lim_{\xi\to\infty} \alpha - 3^{2/3}B_0\int_0^{\xi} u^{2/3}e^{-u}\ du = \alpha - 3^{2/3}B_0\Gamma\left(\frac{1}{3}\right).$$

So that the matching require:

$$\beta e^{-1/2} = \alpha - 3^{2/3}B_0\Gamma(1/3) \iff B_0 = \frac{\alpha - \beta e^{-1/2}}{\Gamma(1/3)3^{2/3}}.$$

- **Uniform Approximation** We then have the uniform approximation on $[0,1]$ for $0 < \epsilon << 1$ is:

$$y(x;\epsilon) = \alpha - \left(\frac{\alpha - \beta e^{-1/2}}{\Gamma(1/3)}\right)\int_0^{x/\epsilon} u^{2/3}e^{-u}\ du + \beta\exp\left\{\frac{x^2-1}{2}\right\} - \beta\exp\left\{\frac{-1}{2}\right\}.$$

- **Checking Solution** Now with $x = 0$ this satisfies:

$$y(0;\epsilon) = \alpha - 0 + \beta e^{-1/2} - \beta e^{-1/2} = \alpha\checkmark.$$

$x = 1$

$$y(1;\epsilon) = \alpha - \left(\frac{\alpha - \beta e^{-1/2}}{\Gamma(1/3)}\right)\int_0^{1/\epsilon} u^{2/3}e^{-u}\ du + \beta - \beta e^{-1/2} \approx \alpha - \alpha + \beta e^{-1/2} - \beta e^{-1/2} + \beta = \beta\checkmark,$$

where we used the fact that as $\epsilon \to 0^+$ we have:

$$\int_0^{1/\epsilon} u^{2/3} e^{-u}\ du = \Gamma\left(\frac{1}{3}\right).$$

Now checking to see if we recover the outer solution, assume $x = O(1)$ and $\frac{x}{\epsilon} = O\left(\frac{1}{\epsilon}\right)$ as $\epsilon \to +\infty$,

$$y(x;\epsilon) = \alpha - \alpha + \beta e^{-1/2} + \beta \exp\left\{\frac{x^2-1}{2}\right\} - \beta e^{-1/2} = \beta e^{(x^2-1)/2} \checkmark.$$

Assume that $\frac{x}{\epsilon} = O(1)$ and that $x = O(\epsilon)$ so that:

$$\begin{aligned} y(x;\epsilon) &= \alpha - \left(\frac{\alpha - \beta e^{-1/2}}{\Gamma(1/3)}\right)\int_0^{x/\epsilon} u^{2/3} e^{-u}\ du + \beta e^{(x^2-1)/2} - \beta e^{(x^2-1)/2} \\ &\approx \alpha - \left(\frac{\alpha - \beta e^{-1/2}}{\Gamma(1/3)}\right)\int_0^{x/\epsilon} u^{2/3} e^{-u}\ du \checkmark. \end{aligned}$$

This is a uniform leading-order approximation for y on $[0, 1]$

$\square$