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Homework # 6

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26.

Find the average value of the inversion statistic over all permutations of n.

Proof. To show this we need to find:

$$\frac{\text{Total numbers of inversions over all permutations of n}}{\text{Total number of permutations of n}} = \frac{\sum_{\sigma \in S_n} \operatorname{inv}(\sigma)}{|S_n|}.$$

Note we can get the numerator by applying the inversion theorem for $Q-$analouges; that is use the inversion theorem to get the following:

$$\frac{d}{dq}[n]_q! = \frac{d}{dq}\sum_{\sigma \in S_n} q^{\operatorname{inv}(\sigma)} = \sum_{\sigma \in S_n} \operatorname{inv}(\sigma) q^{\operatorname{inv}(\sigma)-1}.$$

So that taking the limit of both sides as $q \to 1$ we get that:

$$\lim_{q\to 1}[n]_q! = \lim_{q\to 1}\sum_{\sigma \in S_n} \operatorname{inv}(\sigma) q^{\operatorname{inv}(\sigma)-1} = \sum_{\sigma \in S_n} \operatorname{inv}(\sigma).$$

So that using the fact that $|S_n| = n!$ we get that:

$$\frac{\lim_{q\to 1}\frac{d}{dq}[n]_q!}{|S_n|} = \frac{\sum_{\sigma \in S_n} \operatorname{inv}(\sigma)}{n!}.$$

So skipping ahead a little bit, we'll find that (look at (27.) for proof):

$$\lim_{q\to 1}\frac{d}{dq}[n]_q! = \frac{n!n(n-1)}{4}.$$

So that this gives us our result:

$$\text{Average value of inversions over all permutations of n} = \frac{\frac{n!n(n-1)}{4}}{n!} = \frac{n(n-1)}{4}.$$

$\square$

27.

Find $\lim_{q\to 1} \frac{d}{dq}[n]_q!$.

Proof. Consider the following: First I'll show by induction on n that $\frac{d}{dq}[n]_q! = \frac{d}{dq}([n]_q)[n-1]_q \dots [2]_q + [n]_q \frac{d}{dq}([n-1]_q) \dots [2]_q + \dots + [n]_q \dots \frac{d}{dq}([2]_q)$.

- (Basis)

  With $n = 1$ we get: $[1]_q! = 1$ and so that $\frac{d}{dq}[1]_q! = 0$ and with our formula above plugging in $n = 1$ also gives us 0, so this holds for $n = 1$.

- (Inductive Hypothesis)

  Suppose for some $n \in \mathbb{N}$ that our formula above holds. Now consider $n+1$:

$$\begin{aligned}
\frac{d}{dq}[n+1]_q! &= \frac{d}{dq}([n+1]_q[n]_q!) \\
&= \frac{d}{dq}([n+1]_q)[n]_q! + [n+1]_q \frac{d}{dq}[n]_q! \\
(I.H) &= \frac{d}{dq}([n+1]_q)[n]_q! \\
&\quad + [n+1]_q \left(\frac{d}{dq}([n]_q)[n-1]_q \dots [2]_q + \dots + [n]_q \dots \frac{d}{dq}[2]_q \right)
\end{aligned}$$

  Checking this with the form of the formula, this is what we want! So we have our lemma holds for any $n \in \mathbb{N}$ by the Principle of mathematical induction.

Taking the formula that we have let's take the limit as $q \to 1$. I'll do this by breaking up the limit over the sums and the products, this is a valid step as these are all polynomials in q so that their limits for finite numbers exist and are well-defined. Additionally note that for any $k \in \mathbb{N}$ with $k \geq 0$ we have that $\lim_{q\to 1}[k]_q = k$ and that

$$\begin{aligned}
\lim_{q\to 1} \frac{d}{dq}[k]_q &= \lim_{q\to 1} \frac{d}{dq}(1 + q + \dots + q^{k-1}) \\
&= \lim_{q\to 1}(1 + 2q + 3q^2 + \dots + (k-1)q^{k-2}) \\
&= 1 + 2 + 3 + \dots + (k-1) = \frac{(k-1)(k)}{2}
\end{aligned}$$

so that we get the following:

$$
\begin{aligned}
\lim_{q\to 1}\frac{d}{dq}[n]_q! &= \lim_{q\to 1}\left((\frac{d}{dq}[n]_q)[n-1]_q\dots[2]_q + \dots + [n]_q[n-1]_q\dots(\frac{d}{dq}[2]_q)\right) \\
&= \left(\frac{(n-1)n}{2}(n-1)\dots(2)(1)\right) + \left((n)\frac{(n-2)(n-1)}{2}(n-2)\dots(2)\right) \\
&+ \dots + \left(n(n-1)\dots\frac{3(2)}{2}(2)(1)\right) + \left(n(n-1)\dots(3)\frac{2(1)}{2}\right) \\
&= \frac{n!(n-1)}{2} + \frac{n!(n-2)}{2} + \dots + \frac{n!(2)}{2} + \frac{n!(1)}{2} \\
&= \frac{n!}{2}((n-1)+(n-2)+\dots+2+1) \\
&= \frac{n!}{2}\frac{(n-1)n}{2} = \frac{n!n(n-1)}{4}
\end{aligned}
$$

An we have our solution:

$$\lim_{q\to 1}\frac{d}{dq}[n]_q = \frac{n!n(n-1)}{4}$$

□

28.

Find $\lim\limits_{q\to -1} \frac{d}{dq}[n]_q!$.

Proof. First, note that we can rewrite the lemma that we arrived to in the previous proof as:

$$\frac{d}{dq}([n]_q!) = \frac{d}{dq}([n]_q)[n-1]_q \ldots [2]_q + [n]_q\frac{d}{dq}([n-1]_q)[n-2]_q \ldots [2]_q + \ldots + [n]_q[n-1]_q \ldots \frac{d}{dq}[2]_q.$$

Then note that each of these are polynomials in q so that is we can break the limit $\lim\limits_{q\to -1}$ over the sums and products because each are continuous in q. But then note that $\lim\limits_{q\to -1}[2]_q = \lim\limits_{q\to -1}(1+q) = 0$. So every term of $\frac{d}{dq}[n]_q!$ that contains a $[2]_q$ goes to 0 in the above expression. So then $\lim\limits_{q\to -1}\frac{d}{dq}[n]_q! = \lim\limits_{q\to -1}\left[[n]_q \ldots [3]_q \frac{d}{dq}[2]_q\right] = (\lim\limits_{q\to -1}[n]_q)\ldots(\lim\limits_{q\to -1}[3]_q)(\lim\limits_{q\to -1}\frac{d}{dq}[2]_q)$. But note that $\lim\limits_{q\to -1}[4]_q = \lim\limits_{q\to -1}(1+q+q^2+q^3) = 0$. So that $\lim\limits_{q\to -1}\frac{d}{dq}[n]_q = 0$ for $n \geq 4$.

Then $[3]_q! = (1+q+q^2)(1+q) = 1+q+q^2+q+q^2+q^3 = 1+2q+2q^2+q^3$, then $\frac{d}{dq}[3]_q = 2+4q+3q^2$ taking the limit of this as $q \to -1$:

$$\lim_{q\to -1}\frac{d}{dq}[3]_q = 1.$$

Then $[2]_q = 1+q$ so that:

$$\lim_{q\to -1}\frac{d}{dq}[2]_q! = 1.$$

And finally

$$\lim_{q\to -1}\frac{d}{dq}[1]_q = 0.$$

□

29.

Let $a_{n,k}$ be the number of permutations in S_n with exactly k descents. By inserting "$n+1$" into a permutation of n, show that $a_{n+1,k} = (k+1)a_{n,k} + (n+1-k)a_{n,k-1}$.

Proof. Taking a permutation of n, there are two distinct ways we can insert $n+1$ into this: (1.) Insert $n+1$ and cause no new descents. So that to get $a_{n+1,k}$ we have k descents already and add none. (2.) Insert $n+1$ and cause a new descent. So that to get to $a_{n+1,k}$ we have $k-1$ descents so that adding one with $n+1$ will give us $a_{n+1,k}$.,

(1.) Note in this case, we can either place it after the last element, and this will cause no new descents. Placing it elsewhere, the only way we can do this without causing a new descent is by placing in between two elements that already are a descent. So that in total, there are k ways to do so, since we have that many descents, the way we arrive at this arrangement is the problem $a_{n,k}$. Adding on the $n+1$ getting added to the last space on the permutation, we have a total of: $(k+1)a_{n,k}$ possibilities.

(2.) In this case, we want to cause a new descent. To do this, we can just place $n+1$ anywhere where there isn't a descent and not in the last place. So in total there are $n-(k-1)$ places to do this, since we only have $k-1$ descents in this case, the way that we arrange these is then the problem of $a_{n,k-1}$. In this case we have a total of $(n-(k-1))a_{n,k-1} = (n+1-k)a_{n,k-1}$ possibilities.

These are distinct cases for producing a permutation of $n+1$ with k descents, so that we can simply add these to get the total possibilities:

$$a_{n+1,k} = (k+1)a_{n,k} + (n+1-k)a_{n,k-1}.$$

□

32.

Suppose that in one line notation, the permutation $\sigma \in S_n$ has σ_i in position i. Then the inverse permutation σ^{-1} written in one line notation has i in position σ_i. Show that $\text{inv}(\sigma) = \text{inv}(\sigma^{-1})$.

Proof. Take $\sigma \in S_n$, then to show that $\text{inv}(\sigma) = \text{inv}(\sigma^{-1})$ we'll show that for every inversion in σ there is a corresponding inversion in σ^{-1}. That is $i < j$ and $\sigma_j < \sigma_i$ is an inversion in σ if and only if $\sigma_j < \sigma_i$ with $i < j$ is an inversion in σ^{-1}.

Take a permutation $\sigma \in S_n$ with $\sigma = \sigma_1 \ldots \sigma_n$ in one line notation, so that in two-line notation we can write this as

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma_1 & \sigma_2 & \ldots & \sigma_n \end{pmatrix}.$$

The process of finding the inverse is then swapping the rows of the permutation and putting them back in numerical order:

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma_1^{-1} & \sigma_2^{-1} & \ldots & \sigma_n^{-1} \end{pmatrix}.$$

One-line notation is then just the second row of each two-line representation.

Let $\sigma \in S_n$ and suppose $i < j$ and $\sigma_j < \sigma_i$, that is suppose we have an inversion in σ. In one-line notation: $\sigma = \sigma_1 \ldots \sigma_i \ldots \sigma_j \ldots \sigma_n$. We'll now show that this corresponds to an inversion in σ^{-1}. Applying the process of described previously to σ to find σ^{-1}:

$$\begin{gathered}
\sigma = \begin{pmatrix} 1 & \ldots & i & \ldots & j & \ldots & n \\ \sigma_1 & \ldots & \sigma_i & \ldots & \sigma_j & \ldots & \sigma_n \end{pmatrix} \\
\begin{pmatrix} \sigma_1 & \ldots & \sigma_i & \ldots & \sigma_j & \ldots & \sigma_n \\ 1 & \ldots & i & \ldots & j & \ldots & n \end{pmatrix} \\
\sigma^{-1} = \begin{pmatrix} 1 & \ldots & \sigma_j & \ldots & \sigma_i & \ldots & n \\ \sigma_1^{-1} & \ldots & j & \ldots & i & \ldots & \sigma_n^{-1} \end{pmatrix}.
\end{gathered}$$

Where in the last step, the first row is now in ascending numerical order. In one-line notation, $\sigma^{-1} = \sigma_1^{-1} \ldots j \ldots i \ldots \sigma_n^{-1}$. So that since $i < j$ and $\sigma_i > \sigma_j$(our labels in one-line notation for σ^{-1}) we have that this inversion in σ causes an inversion in σ^{-1}. Moreover, since σ is a bijection we have that σ^{-1} is a bijection, so that this process works in the backwards direction as well.

Thus for every inversion in σ there is a corresponding inversion in σ^{-1}, thus there number must the same i.e

$$\text{inv}(\sigma) = \text{inv}(\sigma^{-1}).$$

□

33.

Using terminology from Abstract Algebra, prove that the sign of σ (as found by writing σ as a product of transpositions) is equal to $(-1)^{\text{inv}(\sigma)}$.

Proof. First, we'll show a lemma: Let $\sigma \in S_n$ and $i \in \mathbb{N}$, then $(-1)^{\text{inv}(\sigma(i\ i+1))} = (-1)^{\text{inv}(\sigma)+1}$. That is, the product of any permutation and an adjacent transposition changes the number of inversions by an odd number.

Proof. Let $\sigma \in S_n$ and $i \in \mathbb{N}$. Then let σ have the following two-line representation:

$$\begin{pmatrix} 1 & \dots & n \\ \sigma_1 & \dots & \sigma_n \end{pmatrix}$$

where $\sigma_j = \sigma(j)$. Then the product $\sigma(i\ i+1)$ will be given by the function composition:

$$\begin{aligned} &\begin{pmatrix} 1 & \dots & i & i+1 & \dots & n \\ \sigma_1 & \dots & \sigma_i & \sigma_{i+1} & \dots & \sigma_n \end{pmatrix} \circ \begin{pmatrix} 1 & \dots & i & i+1 & \dots & n \\ 1 & \dots & i+1 & i & \dots & n \end{pmatrix} \\ &= \begin{pmatrix} 1 & \dots & i & i+1 & \dots & n \\ \sigma_1 & \dots & \sigma_{i+1} & \sigma_i & \dots & \sigma_n \end{pmatrix}. \end{aligned}$$

The only change to the permutation was that σ_i swapped with σ_{i+1}. If $\sigma_i < \sigma_{i+1}$, then this will cause one inversion with indices i and $i+1$. If $\sigma_i > \sigma_{i+1}$, then this will take away one inversion with indices i and $i+1$. So then, the change of $\text{inv}(\sigma)$ to $\text{inv}(\sigma(i\ i+1))$ causes a change in the parity of $(-1)^{\text{inv}(\sigma)}$, that is our result:

$$(-1)^{\text{inv}(\sigma(i\ i+1))} = (-1)^{\text{inv}(\sigma)+1}.$$

$\square$

Now to the proof at hand. Take $\sigma \in S_n$, then we can decompose σ into any chain of two cycles (transpositions) as follows:

$$\sigma = \alpha_1\alpha_2 \dots \alpha_m.$$

Whatever the two-cycle decomposition is though, the parity must be the same.

So then note that we can decompose any two-cycle: $(i\ j)$ with $i < j$, into adjacent two-cycles (transpositions) (using the shorthand $(k\ m) = (k, m)$):

$$(i, j) = (i, i{+}1)\ (i{+}1, i{+}2)\ (i{+}2, i{+}3)\ \dots\ (i{+}(m{-}1), i{+}m)\ (i{+}m, j)\ (i{+}m, i{+}(m{-}1))\ \dots\ (i{+}1, i).$$

To show this note that taking the product $(i, i+1)\ (i+1, i+2)\ (i+2, i+3)\ \dots\ (i+(m-1), i+m)\ (i+m, j) = (i, i+1, i+2, \dots, i+m, j)$, so that to "delete" the extra term note that:

$$(i, i+1, i+2, \dots, i+j, m)(i\ i+1) = (i, i+1, i+2, \dots, i+j, m)(i+1\ i)$$

$$\begin{aligned}\begin{pmatrix} 1 & \dots & i & i+1 & \dots & n \\ 1 & \dots & i+1 & i+2 & \dots & n \end{pmatrix} \circ \begin{pmatrix} 1 & \dots & i & i+1 & \dots & n \\ 1 & \dots & i+1 & i & \dots & n \end{pmatrix} \\ = \begin{pmatrix} 1 & \dots & i & i+1 & i+2 & \dots & n \\ 1 & \dots & i+2 & i+1 & i+3 & \dots & n \end{pmatrix} \\ = (i, i+2, i+3, \dots, i+m, j)\end{aligned}$$

. $\square$

So we can use this process to "delete" every extra piece of the cycle $(i, i+1, i+2, \dots, i+m, j)$ so that we'll get:

$$(i\ j) = (i, i+1)(i+1, i+2)\dots(i+(m-1), i+m)(i+m, j)(i+m, i+(m-1))\dots(i+1, i).$$

Now define a new 2-cycle decomposition of σ where each 2-cycle is adjacent, we can do this by the previous work, call these β :

$$\sigma = \beta_1 \dots \beta_k.$$

Note though that the parity of k is the parity of m from $\sigma = \alpha_1 \dots \alpha_m$, because this is well-defined and unique for any permutation in S_n. So that since all the β's are adjacent transpositions we we have our result:

$$(-1)^{\mathrm{inv}(\sigma)} = (-1)^k = (-1)^m = \text{the sign of } \sigma$$