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Portfolio #3

Lesson 8 # 2

Solve the problem:

$$\begin{array}{rll} \text{PDE} & u_t = u_{xx} - u + x & 0 < x < 1,\ 0 < t < \infty \\ \text{BCs} & u(0,t) = 0 & 0 < t < \infty \\ & u(1,t) = 1 & \\ \text{IC} & u(x,0) = 0 & 0 \le x \le 1 \end{array}$$

by

(a.) changing the non-homogeneous BCs to homogeneous ones,

(b.) transforming into a new equation without the term $-u$,

(c.) solving the resulting problem.

Solution:
(a.)
Note that we can use our technique of decomposing our solution $u(x,t)$ into transient and stead-state terms. Assume

$$u(x,t) = S(x,t) + U(x,t),$$

where S is our steady-state piece and U is our transient piece. Then note that by our BCs, our steady-state should be a straight line from the points (0,0) to (1,1), on our interval that means $S(x,t) = x$. Thus $u(x,t) = x + U(x,t)$.
Additionally, that means $u_t = U_t$, $u_{xx} = U_{xx}$, $u(0,t) = U(0,t)$, $u(1,t) = 1 + U(1,t)$, and $u(x,0) = x + U(x,0)$. Introducing this into our IVBP we get:

$$\begin{array}{rll} \text{PDE} & U_t = U_{xx} - U + (-x + x) & 0 < x < 1,\ 0 < t < \infty \\ \text{BCs} & U(0,t) = 0 & 0 < t < \infty \\ & U(1,t) = 0 & \\ \text{IC} & U(x,0) = -x & 0 \le x \le 1 \end{array}$$

(b.)
Now we want to assume that $U(x,t)$ is of the form:

$$U(x,t) = e^{-t} w(x,t),$$

where e^{-t} accounts for the heat loss across the lateral boundary, and $w(x,t)$ accounts for the temperature profile without this lateral heat loss. Then note that we get the following for our new U term:

$$\begin{aligned}
U_t &= e^{-t}w_t - e^{-t}w \\
U_{xx} &= e^{-t}w_{xx} \\
U(0,t) &= e^{-t}w(0,t) \iff w(0,t) = 0 \\
U(1,t) &= e^{-t}w(1,t) \iff w(1,t) = 0 \\
U(x,0) &= e^{-0}w(x,0) \iff w(x,0) = -x
\end{aligned}$$

So we get $U_t = U_{xx} - U \iff e^{-t}w_t - e^{-t}w = e^{-t}w_{xx} - e^{-t}w \iff w_t = w_{xx}$. So we get the following IVBP:

$$\begin{array}{rlr}
\text{PDE} & w_t = w_{xx} & 0 < x < 1,\ 0 < t < \infty \\
\text{BCs} & w(0,t) = 0 & 0 < t < \infty \\
 & w(1,t) = 0 & \\
\text{IC} & w(x,0) = -x & 0 \le x \le 1
\end{array}$$

(c.)
Thus we have the solution to this as:

$$w(x,t) = \sum_{n=1}^{\infty} a_n e^{-(\pi n)^2 t} \sin n\pi x$$

$$a_n = \frac{1}{2}\int_0^1 (-x)\sin n\pi x \; dx = \frac{2}{\pi n}(-1)^n$$

Thus we have $U(x,t) = e^{-t}\sum_{n=1}^{\infty} \frac{2(-1)^n}{\pi n} e^{-(\pi n)^2 t} \sin \pi n x$. Finally, solving $u(x,t)$:

$$u(x,t) = x - \frac{2e^{-t}}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{-(\pi n)^2 t} \sin \pi n x$$

Lesson 9 # 3

Solve the problem

$$\text{PDE} \qquad u_t = u_{xx} + \sin \pi x \qquad 0 < x < 1 \quad 0 < t < \infty$$
$$\text{BCs} \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t < \infty$$
$$\text{IC} \qquad u(x,0) = 1 \qquad 0 \le x \le 1$$

by the method of eigenfunctions expansion.
Solution:
First, assume that our solution u is of the form: $u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$, for eigenfunctions $X_n(x), T_n(t)$. And that $\sin \pi x$ can be decomposed with some eigenfunction $f_n(t)$ as follows: $\sin \pi x = \sum_{n=1}^{\infty} f_n(t) X_n(x)$.
Note that for the eigenfunction $X_n(x)$, this has the associated Strum-Louisville problem of:

$$\text{ODE } X'' + \lambda^2 X = 0$$
$$\text{BCs} \begin{cases} X(0) = 0 \\ X(1) = 0 \end{cases}$$

Having worked with this in the past, we know that the solution to this is: $X_n(x) = \sin \pi n x$. So now we have:

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \pi n x$$
$$\sin \pi x = \sum_{n=1}^{\infty} f_n(t) \sin \pi n x \ . \tag{1}$$

Looking at (1) we know can solve for $f_n(t)$ analytically by just noting that if we give $f_n(t)$ the following definition, the equality holds:

$$f_n(t) = \begin{cases} 1 & n = 1 \\ 0 & n \neq 0 \end{cases} \ .$$

Reformulating our IVBP we get:

$$\text{PDE} \quad \sum_{n=1}^{\infty} T_n'(t) \sin \pi n x - \sum_{n=1}^{\infty} -(n\pi)^2 T_n(t) \sin \pi n x - \sum_{n=1}^{\infty} f_n(t) \sin \pi n x = 0$$

$$\text{BCs} \begin{cases} \sum_{n=1}^{\infty} T_n(t) \sin 0 = 0 \iff 0 = 0 \\ \sum_{n=1}^{\infty} T_n(t) \sin \pi n = 0 \iff 0 = 0 \end{cases}$$

$$\text{IC} \quad \sum_{n=1}^{\infty} T_n(0) \sin \pi n x = 1$$

Note that the PDE as formulated about turns into:

$$\sum_{n=1}^{\infty} \sin \pi n x \, (T_n'(t) + T_n(t) - f_n(t)) = 0$$

This gives us the following ODE:

$$T_n'(t) + T_n(t) = f_n(t)$$

Solving this ODE via integrating factors we get: $T_n(t) = T_n(0)e^{-(\pi n)^2 t} + \int_0^1 f_n(\tau) e^{-(\pi n)^2(t-\tau)} d\tau$, where $T_n(0) = 2\int_0^1 \sin \pi n \xi \, d\xi = \frac{2(1-\cos \pi n)}{\pi n} = \frac{2(1-(-1)^n)}{\pi n}$, when n is even, we have $T_{2n}(0) = 0$, when n is odd, we have $T_{2n-1}(0) = \frac{4}{\pi(2n-1)}$. Combining what we know about $T_n(0)$ and $f_n(t)$, we can solve explicitly for $T_n(t)$:

$$T_1(t) = \frac{4e^{-\pi^2 t}}{\pi} + \frac{1 - e^{-\pi^2 t}}{\pi^2}$$

$$T_{2n}(t) = 0$$

$$T_{2n+1}(t) = \frac{4e^{-(\pi(2n+1))^2 t}}{(2n+1)\pi} \ .$$

So our solution is:

$$u(x,t) = \left(\frac{4e^{-\pi^2 t}}{\pi} + \frac{1 - e^{-\pi^2 t}}{\pi^2} \right) \sin \pi x + \sum_{n=1}^{\infty} \frac{4e^{-(\pi(2n-1))^2 t} \sin(\pi(2n+1)x)}{(2n+1)\pi}$$

Lesson 9 #4

Find the solution of

$$\text{PDE } u_t = u_{xx} + \sin(\lambda_1 x) \quad 0 < x < 1, \quad 0 < t < \infty$$

$$\text{BCs} \begin{cases} u(0,t) = 0 \\ u_x(1,t) + u(1,t) = 0 \end{cases} \quad 0 < t < \infty$$

$$\text{IC } u(x,0) = 0 \qquad 0 \le x \le 1$$

by the method of eigenfunctions expansion where λ_1 is the first root of the equations $\tan(\lambda) = -\lambda$. What are the eigenfunctions $X_n(x)$ in the problem?
Solution:
First, assume that we can decompose both u and our extra term into eigenfunction as follows:

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

$$\sin \lambda_1 x = \sum_{n=1}^{\infty} f_n(t) X_n(x)\ .$$

Note that this has the associated Strum-Louisville Problem:

$$\text{ODE} \qquad X'' + \lambda^2 X = 0$$

$$\text{BCs} \qquad \begin{cases} X(0) = 0 \\ X'(1) + X(1) = 0 \end{cases}$$

Note that by our PDE we already know that $X_n(x) = A\sin(\lambda_n x) + B\cos(\lambda_n x)$. Applying our BC of $X(0) = 0$ we get $B = 0$, we get $X_n(x) = A\ \sin(\lambda_n x)$. Next applying our BC of $X'(1) + X(1) = 0$, we get:

$$A\lambda_n \cos\lambda_n + A\sin\lambda_n = 0 \iff \lambda_n \cos\lambda_n = -\sin\lambda_n, \text{ since } A \neq 0$$

$$\tan\lambda_n = -\lambda_n.$$

Thus we have the form our eigenvalues with an eigenfunction of: $X_n(x) = A_n \sin(\lambda_n x)$. Assume that A_n can be absorbed in the $f_n(t)$'s. Now we can solve for our $f_n(t)$ in $\sin(\lambda_1 x) = \sum_{n=1}^{\infty} f_n(t) \sin\lambda_n x$, using orthogonality we can get this down to:

$$\int_0^1 \sin\lambda_1 x \sin\lambda_m x\, dx = \int_0^1 \sum_{n=1}^{\infty} f_n(t) \sin\lambda_n x \sin\lambda_m\, dx \iff f_m(t) = 2\int_0^1 A_n \sin\lambda_1 x \sin\lambda_m x\, dx$$

$$f_n(t) = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$$

Now we want to plug our series representations back into the PDE and we get the following:

$$\text{PDE} \qquad \sum_{n=1}^{\infty} T_n'(t) \sin \lambda_n x + (\lambda_n)^2 T_n(t) \sin \lambda_n x - f_n(t) \sin \lambda_n x = 0$$

$$\text{BC} \qquad \text{reduce to } 0 = 0$$

$$\text{IC} \qquad \sum_{n=1}^{\infty} T_n(0) \sin \lambda_n x = 0$$

We can formulate this as a ODE initial value problem as:

$$\text{ODE} \qquad T_n'(t) + T_n(t) = f_n(t)$$

Solving this using integrating factors:

$$T_n(t) = T_n(0) e^{-(\lambda_n)^2 t} + \int_0^t e^{-(\lambda_n)^2 (t-\tau)} f_n(\tau)\ d\tau$$

$$T_n(0) = \int_0^1 0 \sin \lambda_n \xi\ d\xi = 0$$

So

$$T_n(t) = \int_0^t e^{-\lambda_n^2 (t-\tau)} f_n(\tau) d\tau$$

by our solution to $f_n(\tau)$:

$$T_{n \neq 1}(t) = 0$$

$$T_1(t) = \int_0^1 e^{-\lambda_1^2 (t-\tau)} d\tau = \frac{1 - e^{-\lambda_1 t}}{\lambda_1}$$

So our general solution is:

$$u(x,t) = \left(\frac{1 - e^{\lambda_1^2 t}}{\lambda_1^2} \right) \sin(\lambda_1 x)$$