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Exam #1

MATH 440
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1.)

Show that the series diverges or converges.

$$\sum_{k=1}^{\infty} \frac{e^k}{(e^k-1)^2}$$

Proof. Consider the series:

$$\sum_{k=1}^{\infty} \frac{e^k}{(e^k-1)^2}.$$

Note that all terms of the series are positive, since $k \geq 0$ and $e^k - 1 > 0$ for all $k \in \mathbb{N}$. Then we will use the comparison test to show that this is a divergent series.

So consider the following:

$$\frac{e^k}{(e^k-1)^2} = \frac{e^k}{e^{2k} - 2e^k + 1} < \frac{e^k}{e^{2k} - 2e^k} = \frac{1}{e^k - 2}.$$

From here we will show that the series $\sum_{k=1}^{\infty} \frac{1}{e^k-2}$ is convergent by the Limit Comparison Test, again note that we have all positive series since $e^k > 2$ for all $k \in \mathbb{N}$.

Then if we take the convergent geometric series $\sum_{k=1}^{\infty} \left(\frac{1}{e}\right)^k$ let's compare this with our series $\sum_{k=1}^{\infty} \frac{1}{e^k-2}$:

$$\lim_{k\to\infty} \frac{\frac{1}{e^k}}{\frac{1}{e^k-2}} = \lim_{k\to\infty} \frac{e^k-2}{e^k} = \lim_{k\to\infty} 1 - \frac{2}{e^k} = 1 - 0 = 1$$

Hence, by the Limit Comparison Test, we have that the series $\sum_{k=1}^{\infty} \frac{1}{e^k-2} < \infty$.

Thus, by the Comparison Test and the fact that $\frac{e^k}{(e^k-1)^2} < \frac{1}{e^k-2}$,

we have the series $\sum_{k=1}^{\infty} \frac{e^k}{(e^k-1)^2} < \infty$. $\square$

2.)

Show that the series diverges or converges:

$$1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{8} + \frac{1}{4} - \frac{1}{16} + ... + \frac{1}{k} - \frac{1}{2^k} + ...$$

Proof. Consider the above series. Then note that a grouping of this series is:

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{2^k} \right)$$

We will use the contrapositive of Exercise 5 on page 25, we have that if any grouping of series is divergent then the original series is divergent. So we will use this to show that the series diverges.

Then we will show that this series diverges by the Integral Test.

First, note that we have the series in question is all positive since $2^k - 1 > 0$ for all $k \geq 1$. So to use the Integral Test, we need to show monotonic decreasing behaviour of the sequence:

$$\frac{1}{k} - \frac{1}{2^k} \geq \frac{1}{k+1} - \frac{1}{2^{k+1}}.$$

We'll reduce this using Mathematica and get:

$$2^{k+1} \geq k(k+1)$$

Since this final statement true, the whole "iff" chain is true. Hence we have the sequence is monotone decreasing.

Hence the series is monotonic decreasing and we can use the Integral Test:

$$\int_1^{\infty} = \lim_{t \to \infty} \int_1^t \frac{1}{x} - \frac{1}{2^x} = \lim_{t \to \infty} \left(Ln(t) - \frac{(\frac{1}{2})^t}{Ln\left(\frac{1}{2}\right)} \right) = \infty - 0 = \infty.$$

Thus by the Integral Test, we have that the original series diverges. □

Lemma. For all $k \in \mathbb{N}$, $k(k+1) \leq 2^{k+1}$.

Proof. Let $k \in \mathbb{N}$. Then we will prove this using The Principle of Mathematical Induction.

(Basis) Consider when k = 1:

Then $k^2 + k = 1^2 + 1 = 2$. Also $2^{k+1} = 2^2 = 4$. Hence $k^2 + k \leq 2^{k+1}$ when $k = 1$.

(Inductive Hypothesis) Assume for some $k \in \mathbb{N}$, we have $k^2 + k \leq 2^{k+1}$.

Then consider the $k+1$ case:

$$(k+1)^2 + (k+1) = k^2 + k + 2k + 2 = k^2 + k + 2(k+1)$$
$$\leq 2^{k+1} + 2(k+1), \text{ by I.H}$$
$$\leq 2^{k+1} + 2(2^k), \text{ this follows from the fact that } k+1 \leq 2^k$$
$$\leq 22^{k+1} = 2^{k+2}$$

Hence our claim holds for the $k+1$ case.

$\therefore$ By P.M.I our claim holds for all $k \geq 0$. □

3.)

Consider the series given to the right, where p is a nonzero real number. For what values of p does the series converge absolutely? Converge conditionally? Diverge? Justify your claims, and clearly state your answers.

$$\sum_{k=1}^{\infty} p^{-k} k^{p}$$

Solution:

Claim:

On the interval $(-\infty, -1) \cup (1, \infty)$ the series converges absolutely, on the interval $(-1, 0) \cup (0, 1]$ the series is divergent, and at $p = -1$ the series is conditionally convergent.

Proof. Consider the series $\sum_{k=1}^{\infty} p^{-k} k^{p}$.

Let $p \in (-\infty, -1) \cup (1, \infty)$. Then we will use the root test to show that the series is absolutely convergent on this interval, and divergent on $(-1, 0) \cup (0, 1)$.

$$\lim_{k\to\infty} \left|\frac{k^p}{p^k}\right|^{1/k} = \lim_{k\to\infty} \left|\frac{k^{\frac{p}{k}}}{p^{\frac{k}{k}}}\right| = \left|\frac{1}{p}\right| \lim_{k\to\infty} \left|k^{\frac{p}{k}}\right| = \left|\frac{1}{p}\right| \left|\lim_{k\to\infty} k^{\frac{1}{k}}\right|^{p}$$

$$= \left|\frac{1}{p}\right| \cdot 1^p, \text{ by Theorem 2.2.6(d)/2.2.1}$$

$$= \left|\frac{1}{p}\right| < 1$$

Note that this follows from Theorem 2.2.1. Additionally, note that if $p \in (-1, 0) \cup (0, 1)$ we would have $|1/p| > 1$. Thus the series is divergent on this interval by The Root Test.

Note that when $p = 1$, we have

$$\sum_{k=1}^{\infty} p^{-k} k^{p} = \sum_{k=1}^{\infty} \frac{1}{k^p}$$

This is the Harmonic Series and thus divergent.

Note that when $p = -1$, we have

$$\sum_{k=1}^{\infty} p^{-k} k^{p} = \sum_{k=1}^{\infty} (-1)^{-k} k^{-1} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^p}.$$

This is the Alternating Harmonic Series and thus is Conditionally Convergent. □

4.)

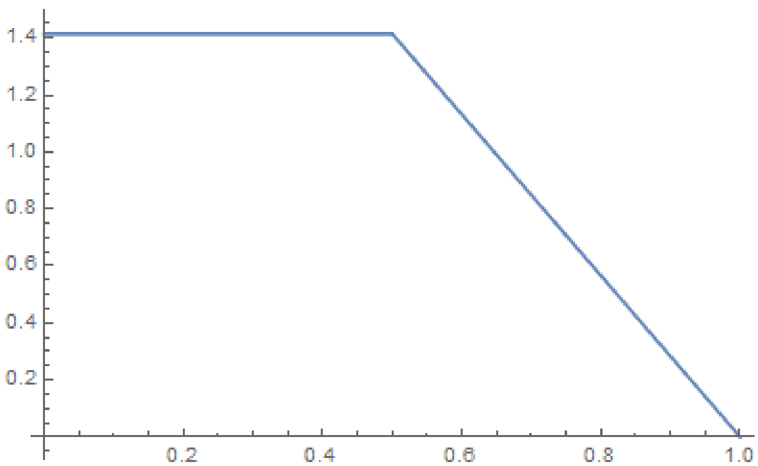
For each $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$ as shown to the right.

$$f_n(x) = \begin{cases} \sqrt{n}, & 0 < x < \frac{1}{n} \\ -n^{3/2}(x - \frac{2}{n}), & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0, & \text{otherwise} \end{cases}$$

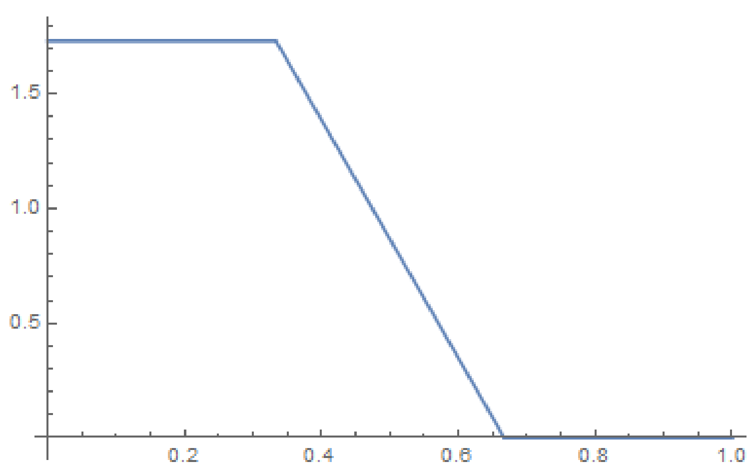
a.

Sketch the graph of f_n for $n = 2, n = 3$, and $n = 4$.

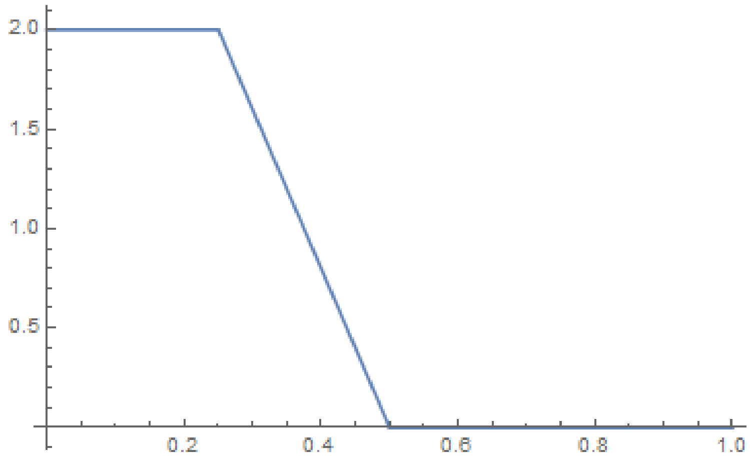
Solution: For $f_1(x)$:



$f_2(x)$:



$f_3(x)$:



b.

Prove that $\{f_n\}_{n=1}^{\infty}$ converges pointwise but <u>not</u> uniformly to a function f, and clearly state what the function is.

Solution:

I claim that $\lim_{n\to\infty} f_n(x) = f(x) = 0$ for all $x \in [0,1]$.

Proof. Note that when we have $x = 0$ we have $f_n(x) = 0$, and when $x = 1$ we have $f_n(x) = 0$ for all $n \geq 3$, so we will consider $x \in (0,1)$.

Let $x \in (0,1)$. Note that we can choose a n_0 such that $\frac{2}{n_0} < x < 1$. Then for all $n \geq n_0$, we have $\frac{2}{n} \leq \frac{2}{n_0} < x < 1$. Hence $f_n(x) = 0$ and we have $\lim_{n\to\infty} f_n(x) = 0$.

$\therefore$ In any possible case we have that

$$\lim_{n\to\infty} f_n(x) = 0.$$

□

Now we will prove that this sequence is not uniformly convergent.

Proof. To show that this sequence isn't uniformly convergent on the interval $[0,1]$, we will use Theorem 8.2.3. Note that to show this we can choose our ϵ, n, m and x, such that $x \in [0,1]$ and $n, m \geq n_0$.

Here we choose $\epsilon = 1$, $n = n_0$ and $m = n_0 + 1$, and $x = \frac{1}{n_0}$.

Note that n, m are clearly greater than n_0, and $0 \leq \frac{1}{n_0} \leq 1$ for all $n_0 \geq 1$. Additionally we have $\frac{1}{n_0} \leq \frac{1}{n_0} \leq \frac{2}{n_0}$ and $\frac{1}{n_0+1} \leq \frac{1}{n_0}$ and $\frac{1}{n_0} \leq \frac{2}{n_0+1} \iff n_0 + 1 \leq 2n_0 \iff 1 \leq n_0$ (Hence this is true for all $n_0 \geq 1$).

So by this previous fact we have the following:

$$\left| f_n\left(\frac{1}{n_0}\right) - f_m\left(\frac{1}{n_0}\right) \right| = \left| -(n_0)^{3/2}\left(\frac{1}{n_0} - \frac{2}{n_0}\right) - (-1)(n_0+1)^{3/2}\left(\frac{1}{n_0} - \frac{1}{n_0+1}\right) \right|$$

$$= \left| -(n_0)^{3/2}\left(\frac{1}{n_0} - \frac{2}{n_0}\right) + (n_0+1)^{3/2}\left(\frac{1}{n_0} - \frac{1}{n_0+1}\right) \right|$$

$$= \left|(-1)\frac{-(n_0)^{3/2}}{n_0} + \frac{(n_0+1)^{3/2}}{n_0} - \frac{(n_0+1)^{3/2}}{n_0+1}\right|$$

$$= \left|(n_0)^{1/2} + \frac{(n_0+1)^{3/2}}{n_0} - (n_0+1)^{1/2}\right| = \left|n_0^{1/2} + (n_0+1)^{1/2}\left(\frac{n_0+1}{n_0} - 1\right)\right|$$

$$\left|n_0^{1/2} + (n_0+1)^{1/2}\left(1 + \frac{1}{n_0} - 1\right)\right| = \left|n_0^{1/2} + \frac{(n_0+1)^{1/2}}{n_0}\right| \geq 1 = \epsilon.$$

Thus we have the sequence doesn't meet the Cauchy Criterion (Theorem 8.2.3.) on the interval $[0,1]$, and is thus not uniformly convergent on the interval $[0,1]$. □

c.

Let $\lim_{n\to\infty} f_n(x)$ denote the pointwise limit of $\{f_n\}_{n=1}^{\infty}$. Show, by direct calculation, that

$$\int_0^1 \lim_{n\to\infty} f_n(x)\ dx = \lim_{n\to\infty} \int_0^1 f_n(x) dx.$$

Solution:

Note that by our work in part(b), we have $\lim_{n\to\infty} f_n(x) = 0$. So we have:

$$\int_0^1 \lim_{n\to\infty} f_n(x) = \int_0^1 0\ dx = C\Big|_0^1 = 0$$

Where C is some arbitrary constant.

Consider the following for the converse of what we wish to show:

$$\lim_{n\to\infty} \int_0^1 f_n(x)\ dx = \lim_{n\to\infty}\left(\int_0^{1/n} f_n(x)\ dx + \int_{1/n}^{2/n} f_n(x)\ dx + \int_{2/n}^1 f_n(x)\ dx\right)$$

$$\lim_{n\to\infty} \int_0^1 f_n(x)\ dx = \lim_{n\to\infty}\left(\int_0^{1/n} \sqrt{n}\ dx + \int_{1/n}^{2/n} -n^{3/2}\left(x - \frac{2}{n}\right)\ dx + \int_{2/n}^1 0\ dx\right)$$

$$= \lim_{n\to\infty}\left(\frac{\sqrt{n}}{n} + \frac{3\sqrt{n}}{2n} + 0\right) = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\right) + \lim_{n\to\infty}\left(\frac{3}{2}\frac{1}{\sqrt{n}}\right) + \lim_{n\to\infty} 0, \text{ by Theorem 2.2.1}$$

$$0 + \frac{3}{2}\cdot 0 + 0, \text{ by Corollary 2.2.2}$$

$$= 0$$

Hence we have

$$\lim_{n\to\infty}\int_0^1 f_n(x) = 0$$

and therefore

$$\int_0^1 \lim_{n\to\infty} f_n(x)\ dx = \lim_{n\to\infty} \int_0^1 f_n(x) dx.$$

d.

Explain why the results of parts(b) and (c) above do not contradict Theorem 8.4.1.

Solution:

Note, that in part(b) we have shown that the sequence is not uniformly convergent, so $\{f_n(x)\}$ doesn't meet the hypothesis from Theorem 8.4.1. Hence the statement is still true, but doesn't tell us anything in this situation.

5.)

Consider the series
$$\sum_{k=1}^{\infty} \frac{2^k}{3^k x + 2^k}$$

a.

Prove that the series converges uniformly on $[a, \infty)$ for each fixed $a > 0$.
Solution:

Proof. Consider the series:
$$\sum_{k=1}^{\infty} \frac{2^k}{3^k x + 2^k}$$
On the interval $[a, \infty)$ for some fixed $a > 0$.

To show that this converges uniformly on $[1, \infty)$ we will use the Weierstrass M-Test. First note that for all $x \in [a, \infty)$ we have $a \leq x \iff \frac{1}{x} \leq \frac{1}{a}$.
$$\left| \frac{2^k}{3^k x + 2^k} \right| = \frac{2^k}{3^k x + 2^k} < \frac{2^k}{3^k x} \leq \frac{1}{a} \left(\frac{2}{3} \right)^k$$
Note that the series $\sum_{k=1}^{\infty} \frac{1}{a} \left(\frac{2}{3}\right)^k = \frac{1}{a} \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k = \frac{1}{a}\left(\frac{1}{1-\frac{2}{3}}\right) = \frac{3}{a} < \infty$.

Thus by the Weierstrass M-Test, we have that the series $\sum_{k=1}^{\infty} \frac{2^k}{3^k x+2^k}$ is convergent uniformly on the interval $[a, \infty)$. □

b.

Prove that this series converges pointwise but not uniformly on the interval $(0, \infty)$.

Proof. Fix $x \in (0, \infty)$ and consider the series $\sum_{k=1}^{\infty} \frac{2^k}{3^k x+2^k}$. Note that we can treat this as a normal series since we have fixed x and can treat it as a constant. Hence we can use all of the test's we have for series we've covered so far.

So to show that this is convergent, we will use the Comparison Test. Note that for all terms $k \in \mathbb{N}$ and $x \in (0, \infty)$ we have the terms of the series are all non-negative.
Then consider the following:
$$\frac{2^k}{3^k x + 2^k} \leq \frac{2^k}{3^k x}$$
Note that the series: $\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k \frac{1}{x} = \frac{1}{x} \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k = \frac{1}{x} \frac{1}{1-\frac{2}{3}} < \infty$.

Thus by the Comparison Test, we have the series $\sum_{k=1}^{\infty} \frac{2^k}{3^k x+2^k}$ is convergent for any $x \in (0, \infty)$. Hence it is pointwise convergent on $(0, \infty)$. □

Proof. Now we will show that this series is not uniformly convergent on this interval, $(0, \infty)$.

To do so, we will use the negation of Corollary 8.2.4. So to show this we need to show that for all $n_0 \geq 0$ there exists $\epsilon > 0$ and $x_0 \in (0, \infty)$ such that for all

$m, n \geq n_0$ we have $\left|\sum_{k=n+1}^{m} \frac{2^k}{3^k x_0 + 2^k}\right|$.

Let $n_0 > 0$ be given. Choose $\epsilon = 4$ and $x_0 = \frac{1}{4^{n_0+1}}$.
Then we have

$$\left|\sum_{k=n+1}^{m} \frac{2^k}{3^k x + 2^k}\right| \geq \frac{2^{n+1}}{3^{n+1}x_0 + 2^{n+1}} \geq \frac{2^{n+1}}{4^{n+1}x_0 + 4^{n+1}} \geq \frac{1}{2(4)^{n+1}x_0} \geq \frac{1}{2(4)^{n_0-n}}$$

$$\geq \frac{2^n}{(4)^{n_0-n}} \geq \frac{4^n}{4^{n_0}} \geq 4^{n-n_0} \geq 4 = \epsilon$$

Thus we have that $f_n(x)$ is not uniformly convergent on $(0, \infty)$. □

6.)

a.

Show that $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for all $x \in (-1,1) \setminus \{0\}$.

Proof. Let $x \in (-1,1)$ be given.
Then note that by our Lemma we have

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$

So note that if we take the limit of this as $n \to \infty$ we will have the infinite series:

$$\lim_{n\to\infty} \sum_{k=0}^{n} x^k = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{\infty} x^k$$

Then note that we break apart this middle limit such that:

$$\lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \lim_{n\to\infty} \frac{1}{1-x} - \lim_{n\to\infty} \frac{x^{n+1}}{1-x}$$

Note that since $|x| < 1$ we have $\lim_{n\to\infty} x^{n+1} = 0$. Hence $\lim_{n\to\infty} \frac{x^{n+1}}{1-x} = 0$. So we have

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} \frac{1}{1-x} = \frac{1}{1-x}$$

□

Lemma: $\sum_{k=0}^{n} x^k = \frac{1-x^{k+1}}{1-x}$ for all $x \in (-1,1) \setminus \{0\}$ and $n \in \mathbb{N}$.

Proof. Let $x \in (-1,1)$.
(Basis)
Consider when $k = 0$.
Then $x^0 = 1$ and $\frac{1-x^1}{1-x} = 1$.
(Inductive Step)
Assume for some $m \in \mathbb{N}$ we have $\sum_{k=0}^{m} x^k = \frac{1-x^{m+1}}{1-x}$. Then consider the $m + 1 case :$

$$\sum_{k=0}^{m+1} x^k = \sum_{k=0}^{m} x^k + x^{m+1} = \frac{1-x^{m+1}}{1-x} + x^{m+1}, \text{ by I.H}$$

$$= \frac{1-x^{m+1} + x^{m+1}(1-x)}{1-x} = \frac{1-x^{m+2}}{1-x}$$

Thus this formula holds for the $m + 1$.

Hence we have the m case implies the $m + 1$ case. Thus by P.M.I, we have that the claim holds for all $n \in \mathbb{N}$ □

b.

Evaluate $\int_0^{4/5} \frac{1}{1+x^8}\ dx$ as a power series, and justify your answer.

Solution:

Note that if we plug $-x^8$ into the series from part(a) we get:

$$\int_0^{4/5} \frac{1}{1+x^8}\ dx = \int_0^{4/5} \sum_{k=0}^{\infty}(-x^{8k})\ dx = \sum_{k=0}^{\infty} -\frac{x^{8k+1}}{8k+1}\Big|_0^{4/5} = -\sum_{k=0}^{\infty} \frac{(4/5)^{8k+1}}{8k+1}$$

Thus

$$\int_0^{4/5} \frac{1}{1+x^8}\ dx = -\sum_{k=0}^{\infty} \frac{(4/5)^{8k+1}}{8k+1}$$