 **The Products of Measures** 

**Note 1** (The Purpose of this Subsection). *As with this entire section, we'll be dealing with product spaces in this subsection. In particular, we'll introduce the concept of integrating over cross-sections which will naturally lead us to the iterated integral. This iterated integral will then lead us to the definition of product measures, the idea being "how do I create a new measure (in $\mathbb{R}^2$) when I already have two measures (possibly the same)?" This finally lead us to a generalization of the concept of area, for example what's the measure of rectangle in $\mathbb{R}^2$? As with everything in Measure Theory this very basic question will take a while to develop an answer to.*

**Definition 1** (Finite Measure; $\sigma-$finite measure).

- *A measure $\mu$ on a measurable space $(X, \mathcal{S})$ is called finite if $\mu(X) < \infty$.*
- *A measure is called $\sigma-$finite if the whole space can be written as the countable union of sets with finite measure.*
- *More precisely, a measure $\mu$ on a measurable space $(X, \mathcal{S})$ is called $\sigma-$finite if there exists a sequence $X_1, X_2, \dots$ sets in $\mathcal{S}$ such that*

$$X = \bigcup_{k=1}^{\infty} X_k \qquad \textit{and} \qquad \mu(X_k) < \infty \textit{ for every } k \in \mathbb{Z}^+.$$

**Note 2.** *Recall, in Topology, the concept of a space being countable versus second-countable. This analogy will help us in the next example.*

**Example 1** (Finite and $\sigma-$Finite Measures).

- *Lebesgue measure on the interval $[0, 1]$ is a finite measure.*
- *Lebesgue measure on $\mathbb{R}$ is not a finite measure but is a $\sigma-$finite measure. (Similar to $\mathbb{R}$ being uncountable, but being second-countable.)*
- *Counting measure on $\mathbb{R}$ isn't a $\sigma-$finite measure (because the countable union of finite sets is a countable set).*

 **The Products of Measures** 

**Theorem 1** (Measure of Cross Section is a Measurable Function). *Suppose* $(X, \mathcal{S}, \mu)$ *and* $(Y, \mathcal{T}, \nu)$ *are* $\sigma-$*finite measure spaces. If* $E \in \mathcal{S} \otimes \mathcal{T}$, *then*

*(a)* $x \mapsto \nu([E]_x)$ *is an* $\mathcal{S}$*-measurable function on* $X$*;*

*(b)* $y \mapsto \mu([E]^y)$ *is a* $\mathcal{T}$*-measurable function on* $Y$*.*

**Note 3.** *The proof of this theorem shows that the set on which the functions defined above are* $\mathcal{S}-$*measurable or* $\mathcal{T}$*-measurable are all of* $\mathcal{S} \otimes \mathcal{T}$, *implying the conclusion of the statement. Importantly this utilizes the framework of monotone classes, how those interact nicely with the facts we have about using limits to define the unions and intersections of increasing and decreasing sets, and also the Monotone Class Theorem. This theorem will be of particular importance in developing the idea of the iterated integral, since we'll be fixing variables we aren't integrating with respect to. This will act as a cross-section-type function similar to the above theorem.*

**Definition 2** (Integration Notation). *Suppose* $(X, \mathcal{S}, \mu)$ *is a measure space and* $g : X \to [-\infty, \infty]$ *is a function. The notation*

$$\int g(x)\ d\mu(x) \qquad \textit{means} \qquad \int g\ d\mu,$$

*where* $d\mu(x)$ *indicates that variables other than* $x$ *should be treated as constants.*

**Example 2** (Example of Integrals). *If* $\lambda$ *is Lebesgue measure on* $[0, 4]$, *then*

$$\int_{[0,4]} (x^2 + y)\ d\lambda(y) = 4x^2 + 8 \qquad \textit{and} \qquad \int_{[0,4]} (x^2 + y)\ d\lambda(x) = \frac{64}{3} + 4y.$$

**Example 3** (More Examples). *Let* $\lambda$ *be the Lebesgue measure on* $[0, 1]$, *then*

$$\int_{[0,1]} x^2 + y^2 + z^2\ d\lambda(x) = \frac{x^3}{3} + y^2 x + z^2 x \Bigg|_{x=0}^{x=1} = \frac{1}{3} + y^2 + z^2$$

$$\int_{[0,1]} x^2 + y^2 - 2xy \; d\lambda(y) = x^2 y + \frac{y^3}{3} - 2xy^2 \Bigg|_{y=0}^{1} = x^2 + \frac{1}{3} - 2x$$

$$\int_{[0,1]} \cos(xyz) \; d\lambda(z) = \int_{[0,xy]} \frac{1}{xy} \cos(u) \; d\lambda(u) = \frac{1}{xy}(-\sin(u)) \Bigg|_{u=0}^{u=xy} = \frac{-1}{xy}(\sin(xy))$$

*where in the last integral we just made a* $u-$*sub of* $u = xyz$ *and* $du = xy(d\lambda(z))$ *with bounds of* $[0, xy]$.

**Note 4** (Caution While Integrating)**.** *While now we can integrate a multivariable function with respect to one of its independent variables, we still need to keep in mind the conditions that we introduced for the integral of a single-variable function. Recall that the definition of the Lebesgue integral.*

**Definition 3** (Integral of a Real-Valued Function; $\int f \; d\mu$)**.** *Suppose* $(X, \mathcal{S}, \mu)$ *is a measure space and* $f : X \to [-\infty, \infty]$ *is an* $\mathcal{S}-$*measurable function such that either* $\int f^+ \; d\mu < \infty$ *or* $\int f^- \; d\mu < \infty$*. The integral of* $f$ *with respect to* $\mu$*, denoted* $\int f \; d\mu$*, is defined by*

$$\int f \quad d\mu = \int f^+ \quad d\mu - \int f^- \; d\mu.$$

**Note 5** (For The Integral to Make Sense)**.** *Axler says "the intent of the next definition is that* $\int_X \int_Y f(x,y) \; d\nu(y) \; d\mu(x)$ *is defined only when the inner integral and then the outer integral both make sense." What that means for this new integration with respect to a single variable of a multivariable function is this:*

$$\int_Y f^+(x,y) \; d\nu(y) < \infty \qquad \textit{or} \qquad \int_Y f^-(x,y) \; d\nu(y) < \infty,$$

*again where we are treating the* $x$*'s in* $f(x,y)$ *as arbitrary fixed constants. Because of the arbitrary nature of* $x$ *though, this must be true for all* $x \in X$*.*

**Note 6** (For The Integral to Make Sense (Cont.))**.** *Now for the outer integral to "make" sense is the same idea, but notationally cumbersome, so just define the function* $g : X \to \mathbb{R}$ *by* $g(x) = \int_Y f(x,y) \; d\nu(y)$ *for all* $x \in X$*. For the outer integral to "make sense" is* *for the following to be true:*

$$\int_X g^+(x)\ d\mu(x) < \infty \qquad \text{or} \qquad \int_X g^-(x)\ d\mu(x) < \infty,$$

*this time there are no other variables other than $x$ so we don't need to use the notation of Definition 2. All of that preparation leads us to the next definition.*

**Definition 4** (Iterated Integrals)**.** *Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are measure spaces and $f : X \times Y \to \mathbb{R}$ is a function. Then*

$$\int_X \int_Y f(x,y)\ d\nu(y)\ d\mu(x) \qquad \text{means} \qquad \int_X \left( \int_Y f(x,y)\ d\nu(y) \right)\ d\mu(x).$$

*In other words, to compute $\int_X \int_Y f(x,y)\ d\nu(y)\ d\mu(x)$, first (temporarily) fix $x \in X$ and compute $\int_Y f(x,y)\ d\nu(y)$ [if this integral makes sense]. Then compute the integral with respect to $\mu$ of the function $x \mapsto \int_Y f(x,y)\ d\nu(y)$ [if this integral makes sense].*

**Example 4** (Example of Iterated Integrals)**.** *If $\lambda$ is Lebesgue measure on $[0,4]$, then*

$$\begin{aligned} \int_{[0,4]} \int_{[0,4]} (x^2 + y) d\lambda(y) d\lambda(x) &= \int_{[0,4]} (4x^2 + 8) d\lambda(x) \\ &= \frac{352}{3} \end{aligned}$$

*and*

$$\begin{aligned} \int_{[0,4]} \int_{[0,4]} (x^2 + y) d\lambda(x) d\lambda(y) &= \int_{[0,4]} \left( \frac{64}{3} + 4y \right) d\lambda(y) \\ &= \frac{352}{3}. \end{aligned}$$

*The two integrals seemingly commute! This will be proven rigorously in the next section, and will be Fubini's Theorem.*

**Note 7.** *We'll utilize this nice framework of iterated integrals to relate this calculus back to geometry in the next part of this subsection.*

**Definition 5** (Product of Two Measures)**.** *Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are $\sigma$-finite measure spaces. For $E \in \mathcal{S} \otimes \mathcal{T}$, define $(\mu \times \nu)(E)$ by*

$$(\mu \times \nu)(E) = \int_X \int_Y \chi_E(x, y) d\nu(y) d\mu(x)$$

.

**Note 8.** *Since $[E]_x$ is $\mathcal{T}$-measurable (by theorem 5.6), the inner integral makes sense and evaluates to $\nu([E]_x)$. The outer integral makes sense only if $\nu([E]_x)$ is $\mathcal{S}$-measurable (this is true by theorem 5.20, since we've assumed $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are $\sigma$-finite measure spaces).*

**Example 5** (Measure of a Rectangle)**.** *Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are $\sigma$-finite measure spaces. Let $A \in \mathcal{S}$ and $B \in \mathcal{T}$. Then,*

$$\begin{aligned}(\mu \times \nu)(A \times B) &= \int_X \int_Y \chi_{A \times B}(x, y) d\nu(y) d\mu(x) \\ &= \int_X \nu(B) \chi_A(x, y) d\mu(x) \\ &= \nu(B) \int_X \chi_A(x) d\mu(x) \\ &= \nu(B)\mu(A)\end{aligned}$$

*Thus, we see that the product measure of a measurable rectangle is the product of the measures of the sets that make up the rectangle.*

**Theorem 2** (Theorem 5.27 Product of Two Measures is a Measure)**.** *Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are $\sigma$-finite measure spaces. Then, $\mu \times \nu$ is a measure on $(X \times Y, \mathcal{S} \otimes \mathcal{T})$.*

**Lemma 1.** *If $E_1$, $E_2$, ... are disjoint subsets of $X \times Y$, then $[\bigcup_{k=1}^{\infty} E_k]_x = \bigcup_{k=1}^{\infty} [E_k]_x$.*

*Proof of Lemma.* ($\subseteq$) Let e $\in [\bigcup_{k=1}^{\infty} E_k]_x$. Then by definition, (x,e) $\in \bigcup_{k=1}^{\infty} E_k$.

Thus, (x,e) $\in E_n$ for some $n \in \mathbb{N}$. Then by definition, e $\in [E_n]_x \subseteq \bigcup_{k=1}^{\infty} [E_k]_x$.

Thus, $[\bigcup_{k=1}^{\infty} E_k]_x \subseteq \bigcup_{k=1}^{\infty} [E_k]_x$.

$(\supseteq)$ Let $e \in \bigcup_{k=1}^{\infty} [E_k]_x$. Then by definition, $e \in [E_n]_x$ for some $n \in \mathbb{N}$. Then by definition (x,e) $\in E_n \subseteq \bigcup_{k=1}^{\infty} E_k$. Again by definition, e $\in [\bigcup_{k=1}^{\infty} E_k]_x$. Thus, $\bigcup_{k=1}^{\infty} [E_k]_x \subseteq [\bigcup_{k=1}^{\infty} E_k]_x$. Therefore, $[\bigcup_{k=1}^{\infty} E_k]_x = \bigcup_{k=1}^{\infty} [E_k]_x$, as desired. □

*Proof.* Assume $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are $\sigma$-finite measure spaces.
Note

$$\begin{aligned}(\mu \times \nu)(\emptyset) &= \int_X \int_Y \chi_{\emptyset}(x,y) d\nu(y) d\mu(x) \\ &= \int_X 0 d\mu(x) \\ &= 0\end{aligned}$$

Next, let $E_1$, $E_2$, ... be a sequence of disjoint subsets of $\mathcal{S} \otimes \mathcal{T}$.
Then,

$$\begin{aligned}(\mu \times \nu)\left(\bigcup_{k=1}^{\infty} E_k\right) &= \int_X \int_Y \chi_{\bigcup_{k=1}^{\infty} E_k}(x,y) d\nu(y) d\mu(x) \\ &= \int_X \nu\left([\bigcup_{k=1}^{\infty} E_k]_x\right) d\mu(x) \\ &= \int_X \sum_{k=1}^{\infty} \nu([E_k]_x) d\mu \\ &= \sum_{k=1}^{\infty} \int_X \nu([E_k]_x) d\mu(x) \\ &= \sum_{k=1}^{\infty} \int_X \int_Y \chi_{E_k}(x,y) d\nu(y) d\mu(x) \\ &= \sum_{k=1}^{\infty} (\mu \times \nu)(E_k)\end{aligned}$$

Therefore, we see that $\mu \times \nu$ is a measure on $(X \times Y, \mathcal{S} \otimes \mathcal{T})$. □