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Homework #3

MATH 550
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2.)

Verify both bullet points in Example 2.28.

- Suppose X is a set and A is the set of subsets of X that consist of exactly one element:

  $$A = \{\{x\} : x \in X\}.$$

  Then the smallest $\sigma-$algebra on X containing A is the set of all subsets E of X such that E is countable or $X \setminus E$ is countable.

- Suppose $A = \{(0,1),(0,\infty)\}$. Then the smallest $\sigma-$algebra on $\mathbb{R}$ containing A is $\{\emptyset,(0,1),(0,\infty),(-\infty,0]\cup[1,\infty),(-\infty,0],[1,\infty),(-\infty,1),\mathbb{R}\}$

Proof. First define some general notation, we'll let T denote the set of all $\sigma-$algebras that contain our set A (whatever that is based on the context), X be the universe or ambient space we're working in, $S = \bigcap_{Y\in T} Y$, and C is the set we're trying to show S is equal to.

- So here C is the set of all countable subsets of X or subsets of X whose complements with X are countable. So then we just need to show that $S = C$ through a double inclusion proof. Note that from example (2.24) in Axler, we have that this is already a σ-algebra of X, for any set X. If X is at most countable (the same cardinality as some subset of $\mathbb{N}$), then clearly $S \subseteq C$, since C will just be the power set of X. To show $C \subseteq S$, take $E \in C$, so that E is an at most countable subset of X. Then we can write $E = \bigcup_{x\in E}\{x\}$, moreover this is an at most countable union because E is at most countable. So that since we have: $\{x\} \in A$ for all $x \in E$, $A \subseteq S$, and all at most countable unions of sets in S must be in S, it follows that $E \in S$. Thus $S = C$, when X is at most countable.

  Now assume X is uncountable. First we'll show the converse because it's easier. Take $E \in C$. If E is countable then we may write E as follows:

  $$E = \bigcup_{n=1}^{\infty}\{x_n\}.$$

  That is if E is countable, it must have an enumeration, hence we can ordered it as the set $\{x_1,\ldots\}$ so that this set is simply the union of the singletons of the elements of this enumeration. Since this collection of singletons is clearly in A and $A \subseteq S$, and S is a $\sigma-$algebra by Theorem (2.33) we must have that countable unions of sets in S are also in S. E is simply a countable union of elements of A, so that $E \in S$. If E isn't countable, then it's complement $X \setminus E$ is countable. So similarly, we can write this as a countable union of singletons:

$$X \setminus E = \bigcup_{n=1}^{\infty} \{x_n\}.$$

So that $X \setminus E \in S$, since $A \subseteq S$ and countable unions of sets from A must also be in S. But moreover, $X \setminus (X \setminus E) = E \in S$, since S is a $\sigma-$algebra. In both cases we have $E \in S$ so that $C \subseteq S$.

Conversely, by Example 2.24 we have that C is already a $\sigma-$algebra of any set X. So that since S is the smallest $\sigma-$algebra containing A, if we show that $A \subseteq C$, then since S is a subset of all σ-algebra containing A it follows that $S \subseteq C$. Suppose $\{x\} \in A$. Then clearly $\{x\}$ is finite, so that using the definition that countable sets are the same size as some subset of $\mathbb{N}$, then we have immediately that $\{x\} \in C$. Thus $A \subseteq C$ and so $S \subseteq C$. Thus $S = C$ and the smallest $\sigma-$algebra containing A for any set X is the subset of all countable subsets of X or sets with countable complements with respect to X.

- Here $C = \{\emptyset, \mathbb{R}, (0,1), (0,\infty), (-\infty,0] \cup [1,\infty), (-\infty,0], [1,\infty), (-\infty,1)\}$. So we'll show this is the smallest $\sigma-$algebra containing $A = \{(0,1), (0,\infty)\}$ simply by construction, meaning any σ-algebra containing A must contain C. Let X be an $\sigma-$algebra containing A. We automatically have both $\emptyset \in X$ and $\mathbb{R} \in X$. Then we have by closure under complimenting that $\mathbb{R} \setminus (0,1) = (-\infty,0] \cup [1,\infty) \in X$ and $\mathbb{R} \setminus (0,\infty) = (-\infty,0] \in X$. Closure under countable unions gets us: $(0,1) \cup (-\infty,0] = (-\infty,1) \in X$ and complementing that $\mathbb{R} \setminus (-\infty,1) = [1,\infty) \in X$. Hence any $\sigma-$algebra $X \supseteq C$. To check that this is a $\sigma-$algebra, note that C has $\emptyset \in C$, all complements are included in C and countable unions as well, by construction. So that C is the smallest $\sigma-$algebra on $\mathbb{R}$ containing A. Omitting anyone of the sets would break one of the axioms, meaning C is the smallest such $\sigma-$algebra.

$\square$

4.)

Suppose S is the smallest $\sigma-$algebra on $\mathbb{R}$ containing $\{(r, n] : r \in \mathbb{Q}, n \in \mathbb{Z}\}$. Prove that S is the collection of Borel subsets of $\mathbb{R}$.

Proof. So note that by definition 2.29 of Borel set's this is to show that any open subset of $\mathbb{R}$ is contained in S, where S is the smallest $\sigma-$algebra on $\mathbb{R}$ containing $\{(r, n] : r \in \mathbb{Q} n \in \mathbb{N}\}$.

So let $O \subseteq \mathbb{R}$ be an open subset of $\mathbb{R}$. Using some topology facts, we know then that O is the union of some sequence of open intervals. So that if we can show that any open interval is in S, then this will show that O is also in S, since O would be the countable union of those open intervals which are guarenteed to be in S, since S is a $\sigma-$algebra by Theorem 2.27. That is, the smallest $\sigma-$algebra containing A is the intersection of all the $\sigma-$algebras containing A.

So then take $(a, b) \subset \mathbb{R}$. We'll then show that we can write this as a countable union of elements of S. So first, note that S must contain $\mathbb{R} \setminus (r, n] = (-\infty, r] \cup (n, \infty)$ for any $r \in \mathbb{Q}$ and $n \in \mathbb{N}$ since S is closed under complementation, so that for $r' \in \mathbb{Q}$ such that $r' < r$ we have $(r', n] \cap ((-\infty, r] \cup (n, \infty)) = (r', r] \in S$ by the fact that countable intersections of sets in $\sigma-$algebra must be in the $\sigma-$algebra. So we have $(r', r] \in S$ for all $r, r' \in \mathbb{Q}$ with $r' \leq r$, so we'll show that we can then form (a, b) as a union of such intervals. Take $\{l_1, l_2, \ldots\}$ to be a decreasing sequence in $\mathbb{Q}$ such that $\lim\limits_{n\to\infty} l_n = a$ and $l_1 > a$, this is guaranteed to exist as the rationals and the irrationals are both dense in $\mathbb{R}$. Similarly define $\{r_1, r_2, \ldots\}$ to be an increasing sequence in $\mathbb{Q}$ such that $\lim\limits_{n\to\infty} r_n = b$ and $r_1 < b$. Now we'll show by a double inclusion proof that $(a, b) = \bigcup\limits_{n=1}^{\infty} (l_n, r_n]$.

Starting off with the forward's direction, take $x \in (a, b)$. Then $a < x < b$, so since $\lim\limits_{n\to\infty} l_n = a$ and $\lim\limits_{n\to\infty} r_n = b$, a, b are cluster points of the set $\{l_n : n \in \mathbb{N}\}$ and $\{r_n : n \in \mathbb{N}\}$ respectively. That is, any neighborhood of a and b contain elements in $\{l_n : n \in \mathbb{N}\}$ and $\{r_n : n \in \mathbb{N}\}$ respectively. So that we can find a suitable r_n and l_m such that $a < r_n < x < l_m < b$. Since the r_n's are decreasing and l_n's are increasing there must exist a $N \in \mathbb{N}$ such that $a < r_N \leq r_n < x < l_m \leq l_N < b$. So that we get $x \in (r_N, l_N) \subseteq (r_N, l_N]$ so that $x \in \bigcup\limits_{n=1}^{\infty} (r_n, l_n]$. Hence $(a, b) \subseteq \bigcup\limits_{n=1}^{\infty} (r_n, l_n]$.

Conversely, take $x \in \bigcup\limits_{n=1}^{\infty} (r_n, l_n]$. Then for some $n \in \mathbb{N}$ we have $x \in (r_n, l_n]$, but since $a < r_n < x \leq l_n < b$ we have that $x \in (a, b)$. Hence we have that any open interval (a, b) can be written as $(a, b) = \bigcup\limits_{n=1}^{\infty} (r_n, l_n]$.

That is, any open interval is in S, since $(r, l] \in S$ for all $r, l \in \mathbb{Q}$, so that we have O being the countable union of open intervals gives us that any open subset of $\mathbb{R}$ is in S. Hence S contains the collection of Borel subsets of $\mathbb{R}$. $\square$

5.)

Suppose S is the smallest $\sigma-$algebra on $\mathbb{R}$ containing $\{(r, r+1) : r \in \mathbb{Q}\}$. Prove that S is the collection of Borel subsets of $\mathbb{R}$.

Proof. Let S be the smallest $\sigma-$algebra on $\mathbb{R}$ containing $A = \{(r, r+1) : r \in \mathbb{Q}\}$. Similar to (4.) we'll show that any open subset $O \subseteq \mathbb{R}$, is then in S by showing that any open interval (a, b) must be in S.

Let $r \in \mathbb{Q}$, then $(r, r+1) \in S$. Similar to (4.) then for any $r < r' < r+1 < r'+1$ we have $(r', r'+1) \cap (r, r+1) = (r', r+1] \in S$ for any $r \in \mathbb{Q}$ by Theorem (2.25), implying that for any $q, p \in \mathbb{Q}$ we have $(q, p] \in S$. That is we can write any rational number as $r+1$ for $r \in \mathbb{Q}$, but it's easier to just write this as $p \in \mathbb{Q}$.

So the same proof used in (4.) can be applied here for the open interval $(a, b) \subseteq \mathbb{R}$, where we can define rational sequence $(r_n, l_n]$ such that $\lim_{n\to\infty} r_n = a$, $\lim_{n\to\infty} l_n = b$ where $\{r_n\}_{n=1}^{\infty}$ is a decreasing sequence and $\{l_n\}_{n=1}^{\infty}$ is an increasing sequence with $a < r_1 \leq l_1 < b$. So that we'll get $(a, b) = \bigcup_{n=1}^{\infty} (r_n, l_n]$ and since each $(r_n, l_n] \in S$ for all $n \in \mathbb{N}$ we have that their countable union is also in S and so that any open interval is also in S. Meaning that since any open subset of $\mathbb{R}$, O, is a countable union over some sequence of open intervals, we have that $O \in S$. Thus S contains every open subset of $\mathbb{R}$ and so that S contains all the Borel subsets of $\mathbb{R}$. $\square$

7.)

Prove that the collection of Borel subsets of $\mathbb{R}$ is translation invariant. More precisely, prove that if $B \subset \mathbb{R}$ is a Borel set and $t \in \mathbb{R}$, then $t + B$ is a Borel set.

Proof. Let $B \subseteq \mathbb{R}$ be a Borel set and $t \in \mathbb{R}$.

Now define the function $f_t : \mathbb{R} \to \mathbb{R}$ by $f_t(x) = x + t$. So since t is constant and thus continuous, as well as x being continuous. We have that f_t is Borel measurable by Theorem (2.46) for any $t \in \mathbb{R}$. Additionally, we have $f_t^{-1}(X) = X + (-t)$ for any $X \subseteq \mathbb{R}$. So since f_t is Borel measurable we have that for any $t \in \mathbb{R}$ that $f_t^{-1}(B) = B + (-t)$ is a Borel set. Swapping $-t \to t$ we get our result, that $B + t$ is a Borel set. Thus the collection of Borel sets are translation invariant. $\square$

9.)

Give an example of a measurable space (X, S) and a function $f : X \to \mathbb{R}$ such that $|f|$ is $S-$measurable but f isn't $S-$measurable.

Proof. Take the $\sigma-$algebra from 2.36 bullet 3 with $S = \{\emptyset, \mathbb{R}, (-\infty, 0), [0, \infty)\}$. Then we have by 2.36 that f is $S-$measurable if and only if f is constant on $(-\infty, 0)$ and f is constant on $[0, \infty)$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x \in (-\infty, 0) \\ -1 & x = 0 \\ 1 & x \in (0, \infty). \end{cases}$$

By the if and only if, we have that f is not $S-$measurable, but that

$$|f(x)| = \begin{cases} 0 & x \in (-\infty, 0) \\ 1 & x \in [0, \infty) \end{cases}$$

is $S-$measurable. $\square$

11.)

Suppose T is a $\sigma-$algebra on a set Y and $X \in \mathrm{T}$. Let $S = \{E \in \mathrm{T} : E \subset X\}$.

a.

Show that $S = \{F \cap X : F \in \mathrm{T}\}$.

Proof. Let T be a $\sigma-$algebra on a set Y, $X \in \mathrm{T}$, and $S = \{E \in \mathrm{T} : E \subset X\}$. Suppose $E \in S$. Then $E \in \mathrm{T}$ and $E \subset X$. Since $E \cap X = E$ because $E \subset X$ we have that $E \in \{F \cap X : F \in \mathrm{T}\}$.

Conversely, take $E \in \{F \cap X : F \in \mathrm{T}\}$. Then for some $F \in \mathrm{T}$ $E = F \cap X$. Since both $F \in \mathrm{T}$ and $X \in \mathrm{T}$, and T is a $\sigma-$algebra we have by Theorem 2.25 that $X \cap F = E \in \mathrm{T}$. Additionally since $E = X \cap F \subset X$ we have $X \in S$. So that $S = \{F \cap X : F \in \mathrm{T}\}$. $\square$

b.

Show that S is a $\sigma-$algebra on X.

Proof. Note that since $\emptyset \subset X$ and that T is a $\sigma-$algebra so we have that $\emptyset \in S$. Denote $E^C = Y \setminus E$.

Next take $E \in S$. Then $E = F \cap X$ for some $F \in \mathrm{T}$. Additionally, because $F \in \mathrm{T}$, we have $Y \setminus F \in \mathrm{T}$. To show closure under complementation, we'll show $X \setminus E \in S$. So note $X \setminus E = X \setminus (F \cap X) = X \cap (F \cap X)^C = X \cap (F^C \cup X^C) = (X \cap F^C) \cup (X \cap X^C) = X \cap F^C$. So since $F \in \mathrm{T}$ and T is a $\sigma-$algebra, $F^C \in \mathrm{T}$. So that we have $X \setminus E \in S$.

Now take $\{E_k\}_{k=1}^{\infty}$ to be a sequence of sets E_k, where each $E_k \in X$. So we need to show that $\bigcup\limits_{k=1}^{\infty} E_k \in S$ to show closure under countable unions. So since each $E_k \in X$ we have that $E_k = F \cap X$ for some $F \in \mathrm{T}$. Label each F with $k \in \mathbb{N}$ so that $E_k = F_k \cap X$.

First we'll show that $\bigcup\limits_{k=1}^{\infty} (F_k \cap X) = X \cap \left(\bigcup\limits_{k=1}^{\infty} F_k\right)$. Take $x \in \bigcup\limits_{k=1} \infty(F_k \cap X)$. So that for some $k \in \mathbb{N}$ we have $x \in F_k \cap X$, hence $x \in X$ and $x \in \bigcup\limits_{k=1}^{\infty}$. Conversely, take x to be in the right hand side. So that $x \in X$ and $x \in F_k \cap X$ for some $k \in \mathbb{N}$. Hence $x \in X \cap F_k$ for some $k \in \mathbb{N}$ and so $x \in \bigcup\limits_{k=1} (F_k \cap X)$ and we have our lemma.

So that we have

$$\begin{aligned}
\bigcup_{k=1}^{\infty} E_k &= \bigcup_{k=1}^{\infty} F_k \cap X \\
&= \left(\bigcup_{k=1}^{\infty} F_k\right) \cap \left(\bigcup_{k=1}^{\infty} X\right) \\
&= X \cap \left(\bigcup_{k=1}^{\infty} F_k\right).
\end{aligned}$$

Since each $F_k \in \mathrm{T}$ we have that $\bigcup_{k=1}^{\infty} F_k \in \mathrm{T}$. So that $\bigcup_{k=1}^{\infty} E_k \in S$ and S is closed under countable unions.

Thus S is a $\sigma-$algebra on X. $\square$

13.)

Suppose (X, S) is a measurable space, $E_1, \ldots, E_n$ are disjoint subsets of X, and $c_1, \ldots, c_n$ are distinct nonzero real numbers. Prove that $c_1\chi_{E_1} + \ldots + c_n\chi_{E_n}$ is an $S-$measurable function if and only if $E_1, \ldots, E_n \in S$.

Proof. So first, note through Example (2.38) we know that χ_E is $S-$measurable if and only if $E \in S$. So let $E_1, \ldots, E_n$ be disjoint subsets of X and $c_1, \ldots, c_n$ be distinct real numbers. Suppose (X, S) is a measurable space. So note that applying the result of Example (2.38) for each χ_{E_i} for $i \in \{1, \ldots, n\}$ gets us that: $\chi_{E_1}, \ldots, \chi_{E_n}$ are $S-$measurable if and only if $E_1, \ldots, E_n \in S$. So we will show that $\chi_{E_1}, \ldots_{E_n}$ are $S-$measurable if and only if $c_1\chi_{E_1} + \ldots + c_n\chi_{E_n}$.

For the forward direction, we'll need to show that each c_i is $S-$measurable. So take c_i from the list above for $i \in \{1, \ldots, n\}$. Note then that we have:

$$c_i^{-1}((a, \infty)) = \begin{cases} \emptyset & \text{if } c_i < a \\ X & \text{if } c_i \geq a \end{cases}.$$

So that in either case both of these sets must be in S, and thus c_i is an $S-$measurable function . So by applying Theorem (2.46) we get that $c_1\chi_{E_1} + \ldots + c_n\chi_{E_n}$ is $S-$measurable. That is, each $c_i\chi_{E_i}$ is $S-$measurable by the multiplication of $S-$measurable functions is an $S-$measurable function. Then applying the fact that the addition of $S-$measurable functions is $S-$measurable inductively on n, we can get that $c_1\chi_{E_1} + \ldots + c_n\chi_{E_n}$ is $S-$measurable.

We'll show the converse of the statement by proving it's contrapositive. Suppose that $\chi_{E_1}, \ldots, \chi_{E_n}$ are all not $S-$measurable. Then we have by Example (2.38) that $E_1 \notin S, \ldots, E_n \notin S$. So to show that $c_1\chi_{E_1} + \ldots + c_n\chi_{E_n}$ is not $S - measurable$ we just need to show that there's a $B \subseteq \mathbb{R}$ that's a Borel set yet $(c_1\chi_{E_1} + \ldots + c_n\chi_{E_n})^{-1}(B) \notin S$. Note that all singletons are closed in $\mathbb{R}$ and hence are Borel sets. So then consider the following:

$$(c_1\chi_{E_1} + \ldots + c_n\chi_{E_n})^{-1}(\{c_1\}) = E_1.$$

But $E_1 \notin S$, so we have that $c_1\chi_{E_1} + \ldots + c_n\chi_{E_n}$ is not a $S-$measurable function.

So we've shown that $E_1, \ldots, E_n \in S$ if and only if $\chi_{E_1}, \ldots, \chi_{E_n}$ are $S-$measurable if and only if $c_1\chi_{E_1} + \ldots + c_n\chi_{E_n}$ is $S-$measurable. So by logical transitivity we are done! $\square$

14.)

Suppose $f_1, f_2, \ldots$ is a sequence of functions from a set X to $\mathbb{R}$. Explain why $\{x \in X :$ the sequence $f_1(x), f_2(x), \ldots$ has a limit in$\mathbb{R}\} = \bigcap\limits_{n=1}^{\infty} \bigcup\limits_{j=1}^{\infty} \bigcap\limits_{k=j}^{\infty} (f_j - f_k)^{-1}((-\frac{1}{n}, \frac{1}{n}))$.

Proof. So let $x \in X$ where $(f_n(x))$ converges to some limit point $L \in \mathbb{R}$. So that this is simply a sequence over n, since x is fixed in the above sequence of functions. Meaning that we can use the Cauchy Criterion for sequence convergence to interpret this, so that since the limit exists: for all $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that if $m \geq n \geq N$, then $|f_m(x) - f_n(x)| = |(f_m - f_n)(x)| < \epsilon$. So for all $\epsilon > 0$, we have $|(f_m - f_n)(x)| < \epsilon \implies -\epsilon < (f_m - f_n)(x) < \epsilon \implies -1\frac{(f_m - f_n)(x)}{\epsilon} < +1$. That is for all $m \geq n > N$ and $\epsilon > 0$ that we can choose a $N \in \mathbb{N}$ such that:

$$-1 < \frac{(f_m - f_n)(x)}{\epsilon} < 1$$

. That is this condition can be translated to $(f_m - f_n)^{-1}((-\frac{1}{k}, \frac{1}{k}))$ for all $k \in \mathbb{N}$. Where the k takes the place of the epsilon. So that the outer intersection accounts "for all $\epsilon > 0$", the inner union accounts for the "for some $N \in \mathbb{N}$", and the inner intersection is the condition that "for all $m \geq n \geq N$". With the "for all"'s becoming intersections and the "for some" becoming a union. □

b.)

Suppose (X, S) is a measurable space and $f_1, f_2, \ldots$ is a sequence of $S-$measurable functions from X to $\mathbb{R}$. Prove that $\{x \in X :$ the sequence $f_1(x), f_2(x), \ldots$ has a limit in $\mathbb{R}\}$ is an $S-$measurable subset of X.

Proof. Let (X, S) be a measurable space and $f_1, f_2, \ldots$ be a sequence of $S-$measurable functions from $X \to \mathbb{R}$ define $A = \{x \in X :$ the sequence $f_1(x), f_2(x), \ldots$ has a limit in $\mathbb{R}\}$. Then note that $f_j - f_k$ for any $k, j \in \mathbb{N}$ must then be $S-$measurable by Theorem (2.46). Note then that for any $n \in \mathbb{N}$ we have that $\left(-\frac{1}{n}, \frac{1}{n}\right)$ is open and hence a Borel set. So that $(f_j - f_k)^{-1}\left(-\frac{1}{n}, \frac{1}{n}\right)$ is a $S-$measurable subset of X by definition. Since $A = \bigcap\limits_{n=1}^{\infty} \bigcup\limits_{j=1}^{\infty} \bigcap\limits_{k=j}^{\infty} (f_j - f_k)^{-1}((-\frac{1}{n}, \frac{1}{n}))$ and S is a $\sigma - algebra$, applying Theorem (2.25(c)) to the inner intersection gives us an $S-$measurable subset. Additionally, applying the fact that countable unions of S-measurable subsets are S-measurable in a $\sigma-$algebra S on the big union, and reapplying

Theorem (2.25(c)) on the outer intersection gives us our result. The set $A = \{x \in X :$ the sequence $f_1(x), f_2(x), \ldots$ has a limit in $\mathbb{R}\}$ is S-measurable. $\square$

15.)

Suppose X is a set and $E_1, E_2, \ldots$ is a disjoint sequence of subsets of X such that $\bigcup\limits_{k=1}^{\infty} E_k = X$. Let $S = \{\bigcup_{k\in K} E_k : K \subset \mathbb{Z}^+\}$.

a.

Show that S is a $\sigma-$algebra on X.

Proof. Let X be a set and $E_1, E_2, \ldots$ be a disjoint sequence of subsets of X such that $\bigcup\limits_{k=1}^{\infty} E_k = X$. Let $S = \{\bigcup_{k\in K} E_k : K \subset \mathbb{Z}^+\}$.
To show this is a $\sigma-$algebra we'll show the three features of a $\sigma-$algebra.

- Note that taking $K = \emptyset \subseteq \mathbb{Z}^+$ we get that the $\bigcup\limits_{\emptyset} E_k = \emptyset$ so that $\emptyset \in S$.

- Now take $E \in S$. Then $E = \bigcup\limits_{k\in K} E_k$ for some $K \subseteq \mathbb{Z}^+$. So then $X \setminus E = \bigcup\limits_{k=1}^{\infty} E_k \setminus \bigcup\limits_{j\in K} E_j$. Because each E-set is disjoint and the union from $k = 1$ to ∞ we can write:

$$X \setminus E = \bigcup_{k\in Z^+} E_k \setminus \bigcup_{j\in K} E_j = \bigcup_{k\in Z^+\setminus K} E_k.$$

  And thus $X \setminus E \in S$.

- Take $\{E_k\}_{k=1}^{\infty}$ to be a sequence with each $E_k \in S$. So that for each $k \in \mathbb{Z}^+$ we have $E_k = \bigcup\limits_{j\in K_k} E_j$. So that we get the following:

$$\begin{aligned} \bigcup_{k=1}^{\infty} E_k &= \bigcup_{k=1}^{\infty} \bigcup_{j\in K_k} E_j \\ &= \bigcup_{j\in K_1\cup K_2\cup \ldots} E_j. \end{aligned}$$

  Since $\bigcup\limits_{k=1}^{\infty} K_k \subseteq \mathbb{Z}^+$ for $K_k \subseteq \mathbb{Z}^+$ we have that $\bigcup\limits_{k=1}^{\infty} E_k \in S$.

Thus we have our result and S is a $\sigma-$algebra on X. $\square$

b.

Prove that a function from X to $\mathbb{R}$ is $S-$measurable if and only if the function on E_k for every $k \in \mathbb{Z}^+$.

Proof. Let (X, S) be a measurable space as described in $(a.)$.

For the converse, take $f : X \to \mathbb{R}$ be constant on each E_k for all $k \in \mathbb{Z}^+$. Label each for each E_k, call the constant c_k. Then we have for any Borel set B:

$$f^{-1}(B) = \bigcup_{k \in K} E_k \text{ such that for all } k \in K, c_k \in B.$$

That is K is the set of indices where $c_k \in B$. With the case that B contains none of the constant meaning that $K = \emptyset$ so that we get $f^{-1}(B) = \emptyset$. In any case we get that $f^{-1}(B) \in S$ so that f is $S-$measurable.

For the forwards direction, let $f : X \to \mathbb{R}$ be a S-measurable function. Then for some $k \in \mathbb{Z}^+$ take $a_k \in f(E_k)$. So that $\{a_k\} \subseteq f(E_k) \implies f^{-1}(\{a_k\}) \subseteq f^{-1}(f(E_k))$. By set theory facts we have that $f^{-1}(f(E_k)) = E_k$ and since f is $S-$measurable and $\{a_k\}$ is closed set and thus Borel, we have that there exists some $K \subseteq \mathbb{Z}^+$ such that $f^{-1}(\{a_k\}) = \bigcup_{j \in K} E_j \subseteq E_k$. Since each E_j is distinct and $a_k \in f(E_k)$ so that $f^{-1}(\{a_k\}) \neq \emptyset$, this gives us $f^{-1}(\{a_k\}) = E_k$. Applying set theory facts: $f(f^{-1}(\{a_k\})) = f(E_k) \implies \{a_k\} \cap f(X) = f(E_k)$, but of course $\{a_k\} \cap f(X) = \emptyset$ (not possible since $a_k \in f(E_k) \subseteq f(X)$) or $\{a_k\} \cap f(X) = \{a_k\}$. So that we have our result $f(E_k) = \{a_k\}$ and thus f is constant on each E_k for all $k \in \mathbb{Z}^+$. $\square$

17.)

Suppose X is a Borel subset of $\mathbb{R}$ and $f : X \to \mathbb{R}$ is a function such that $\{x \in X : f \text{ is not continuous at } x\}$ is a countable set. Prove that f is a Borel measurable function.

Proof. Let $X \subseteq \mathbb{R}$ be a Borel set of $\mathbb{R}$ and $f : X \to \mathbb{R}$ be a function such that $A = \{x \in X : f \text{ is not continuous at } x\}$ is a countable set. Since A is countable there exists an enumeration of it, call it $\{x_n\}_{n=1}^{\infty}$.

Note that this will mirror the proof of Theorem (2.41) with one difference. Then since f is continuous on $X \setminus A$ we have if $x \in X$, $f(x) > a$, and f is continuous at x, then there exists a δ_x such that $\delta_x > 0$ such that $f(y) > a$ for all $y \in (x - \delta_x, x + \delta_x) \cap X$. Thus

$$f^{-1}((a, \infty)) = \left(\bigcup_{x \in f^{-1}((a,\infty)) \setminus A} (x - \delta_x, x + \delta_x) \cup \bigcup_{n=}^{\infty} \{x_n\} \right) \cap X.$$

Since singletons are Borel sets by Example (2.30) we have that any countable set is also a Borel set. That is, the collection of Borel sets is a $\sigma-$algebra so that it's closed under countable unions. Because $\bigcup\limits_{n=1}^{\infty} \{x_n\} = A$ is countable and thus Borel, $\bigcup\limits_{x \in f^{-1}((a,\infty)) \setminus A} (x - \delta_x, x + \delta_x)$ is an open set and thus Borel (arbitrary unions of open intervals are always open), and by our hypothesis X is a Borel subset of $\mathbb{R}$, we have that $f^{-1}((a, \infty))$ is a Borel set. Thus by Theorem (2.39) we have that f is a Borel-measurable function. $\square$

18.)

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at every element on $\mathbb{R}$. Prove that f' is a Borel measurable function from $\mathbb{R} \to \mathbb{R}$.

Proof. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at every element of $\mathbb{R}$.

Then take $x \in \mathbb{R}$ and $f'(x) > a$, so that by f being differentiable at all $x \in \mathbb{R}$, we have that $f'(x) = \lim_{y \to x} \frac{f(y)-f(x)}{y-x}$ exists for all $x \in \mathbb{R}$. We'll show that $f'(x) = \lim_{n \to \infty} \frac{f(x+\frac{1}{n)}-f(x)}{\frac{1}{n}}$ and that each $f_n(x) = \frac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}}$ is Borel measurable.

Note that an alternative definition of the derivative is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{\frac{1}{n} \to 0} \frac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}} = \lim_{n \to \infty} \frac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}}.$$

Thus the convergence of $(f_n) \to f$ is at least pointwise.

Additionally, note that from undergraduate real analysis facts that if f is differentiable on a set, it's continuous on that same set. So that f is then Borel-measurable by Theorem (2.41). Additionally with $f_n(x) = \frac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}}$, note that constant functions such as $\frac{1}{n}$ are always continuous for $n \in \mathbb{N}$ not including 0, so that by Theorem (2.46) we have that each f_n is Borel-measurable.

Thus by Theorem (2.48) we have that f' is a Borel measurable function. □

28.)

Suppose $f : B \to \mathbb{R}$ is a Borel measurable function. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in B, \\ 0 & \text{if } x \in \mathbb{R} \setminus B. \end{cases}$$

Prove that g is a Borel measurable function.

Proof. Suppose $f : B \to \mathbb{R}$ is a Borel measurable function. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in B, \\ 0 & \text{if } x \in \mathbb{R} \setminus B. \end{cases}$$

Note then also that by the paragraph just after Definition (2.40), we have that B is a Borel set, since f is Borel-measurable.

So that I claim that for any Borel set X we have:

$$g^{-1}(X) = \begin{cases} f^{-1}(X) \cup (R \setminus B) & \text{if } 0 \in X \\ f^{-1}(X) & \text{otherwise.} \end{cases}$$

Now to show this.

Take $x \in g^{-1}(X)$ so that for some $g(x) \in X$. So either $g(x) = f(x)$ if $x \in B$ or $g(x) = 0$ if $x = 0$. If $g(x) = f(x)$, then $x \in B$ and $f(x) \in X$. Hence $x \in B \cap f^{-1}(X) = f^{-1}(X)$. If $g(x) = 0$, then $0 \in X$ and so $x \in \mathbb{R} \setminus B$ by the definition of g. In the case that $0 \notin X$, then $g(x) = 0$ is a contradiction and hence we only have $g(x) = f(x)$. Thus we have $g^{-1}(X) \subseteq R.H.S$ of the equation.

Conversely, assume first $0 \in X$ and $x \in R.H.S$ of the equation. Then $x \in f^{-1}(X) \cup (R \setminus B)$, so either $x \in f^{-1}(X)$ or $x \in \mathbb{R} \setminus B$. If $x \in f^{-1}(X)$ and so $x \in B$, since $f^{-1}(X) \subseteq B$, then $f(x) \in X$ so that $g(x) \in X$ and hence $x \in g^{-1}(X)$. If $x \in \mathbb{R} \setminus B$, then $g(x) = 0$ and so $g(x) \in X$ and $x \in g^{-1}(X)$. Now assume $0 \notin X$ and $x \in f^{-1}(X)$, then $x \in B$ since $f^{-1}(X) \subseteq B$ and so $g(x) = f(x)$. Thus $x \in g^{-1}(X)$. In either case we have that $R.H.S \subseteq g^{-1}(X)$.

Thus

$$g^{-1}(X) = \begin{cases} f^{-1}(X) \cup \mathbb{R} \setminus B & \text{if } 0 \in X \\ f^{-1}(X) & \text{otherwise.} \end{cases}$$

Thus since $f^{-1}(X)$ is Borel because f is a Borel measurable function and B is Borel, we have that $\mathbb{R} \setminus B$ is Borel (closure under complementation). So that in either case $g^{-1}(X)$ is a Borel subset and hence Borel measurable. $\square$