Partially Ordered Sets and Möbius Inversion

Joseph C. McGuire

Sonoma State

December 4, 2018

Relations

Definition: Binary Relation

Let S be a set. Then a **Relation** R is a subset of $S \times S$; $R \subseteq S \times S$, and if $(x, y) \in R$ we call them related elements of S or $x \sim y$.

Definition: Partially Ordered Set (Posets)

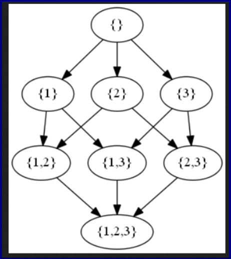
Let P be a set paired with a binary relation $\leq$; $P = (S, \leq)$ for some set S. Then P is a **Partially Ordered Set** if the relation $\leq$ has the following properties:

1. For all $x \in S$, $x \leq x$ (Reflexive)
2. If $x \leq y$ and $y \leq x$, then $x = y$ (Anti-Symmetric)
3. If $x \leq y$ and $y \leq z$, then $x \leq z$ (Transitive).

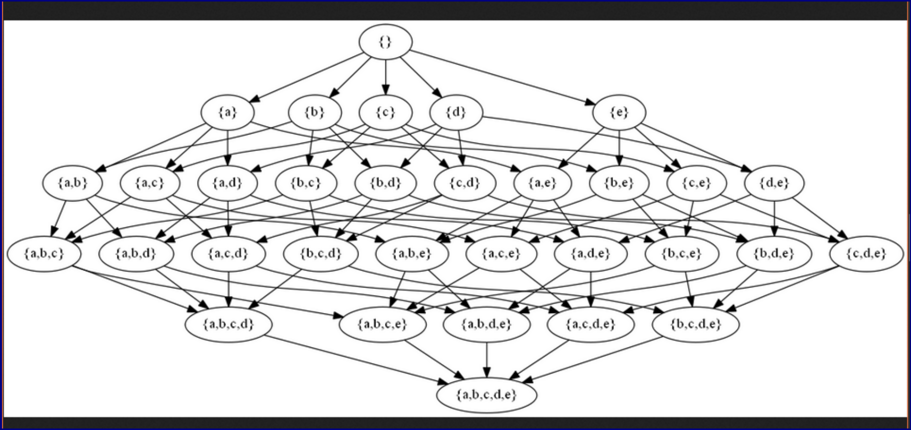
Hasse Diagram

Definition: Hasse Diagram

A visualization of a partially ordered set, where each vertex is an element of the set and the edges are how the elements are related to each other via the partial ordering.

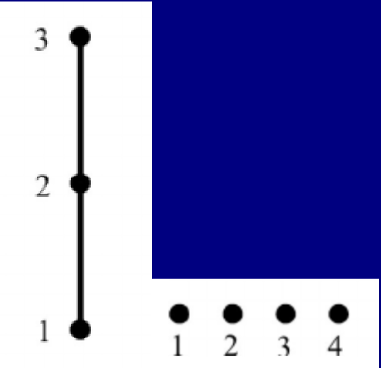


Hasse Diagram



Hasse Diagrams

1. If we follow the rules of the partial ordering and the definition of a Hasse diagram, we can actually come up with new partial orderings via Hasse diagrams.



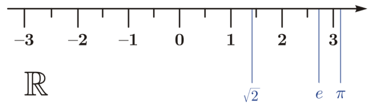
Chains and Antichains

Definition: Chain

Let C be a poset with the arbitrary partial ordering $\leq$. If $x, y \in C$ implies either $x \leq y$ or $y \leq x$. Then C is called a **Chain**.

Definition: Anti-Chain

Let A be a poset with the arbitrary partial ordering $\leq$. If $x, y \in A$ implies $x \nleq y$ and $y \nleq x$, then A is called an **Anti-Chain**.



Applications of Posets

With in Mathematics

1. Modern Algebra
2. Real Analysis
3. Poset Topology
4. Probability Theory

In The "Real" World

1. Economics
2. Physics
3. Computer Science
4. Anything Where Order Matters

Definition: Isomorphisms

Given posets P and Q. If there exists an order-preserving bijection, $\phi : P \to Q$ such that $x \leq y$ in $P \iff \phi(x) \leq \phi(y)$ in Q.

1. The Hasse diagrams of two isomorphic posets are themselves isomorphic in terms of graph theory.

Chain Covers

Definition: Chain Cover

Let P be a poset. Then if there exists a collection of chains such that

$$C_1 \cup C_2 \cup ... \cup C_n = P \text{ and } C_1 \cap C_2 \cap ... \cap C_n = \emptyset$$

. Then we call $C_1, ..., C_n$ a chain covering P.

1. This is equivalent to being able to break down the Hasse diagrams into linear orderings that are all disjoint and induced subgraphs.

Dilworth's Theorem

Dilworth's Theorem

The size of the largest antichain is equal to the number of chains in any smallest chain covering.

Intervals

Closed Interval

Let P be a poset with the partial ordering $\leq$. Then the set $[x, y] = \{z : x \leq z \leq y\}$ is called the **Closed Interval** between x and y. The set of all closed intervals in P is denoted $Int(P)$.

Multiplication with Domain Int(P)

We define $I(P)$ as the set of all functions $f : Int(P) \to \mathbb{R}$ and multiplication on $I(P)$:

$$(f \cdot g)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y).$$

Definition: Locally Finite

Let P be a poset. Then if every interval $[x, y] \in Int(P)$ is finite, we call P locally finite.

Indices Algebra

Kronecker Delta Function

$\delta : Int(P) \to \mathbb{R}$ given by $\delta(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$

Zeta Function

$\zeta : Int(P) \to \mathbb{R}$ given by $\zeta(x, y) = \begin{cases} 1 & x \leq y \\ 0 & x \nleq y \end{cases}$

Indices Algebra

Theorem

δ is the identity element of $I(P)$; i.e, for all $f \in I(P)$, we have $f\delta = \delta f = f$

Proof.

Let $f \in I(P)$, then $f : Int(P) \to \mathbb{R}$. Consider the following:

$$f\delta(x,y) = \sum_{x \leq z \leq y} f(x,z)\delta(z,y)$$

By definition, we have for all $z < y$: $\delta(z,y) = 0$. Thus

$$\sum_{x \leq z \leq y} f(x,z)\delta(z,y) = f(x,x) \cdot 0 + ... + f(x,y) \cdot 1 = f(x,y).$$

Thus $\delta f = f\delta = f$. Hence δ is the identity element of $I(P)$. $\square$

Indices Algebra

1. Does ζ have an inverse in $I(P)$?

Question

Does there exist a $f \in I(P)$ such that

$$f\zeta(x, y) = \sum_{x \leq z \leq y} f(x, z)\zeta(z, y) = \delta(x, y)?$$

Definition: Möbius Function

Let P be a poset. Then it has a zeta-function such that $\zeta : Int(P) \rightarrow \mathbb{R}$ where $\zeta \in I(P)$. We call the inverse of this element, μ, such that $\zeta\mu = \mu\zeta = \delta$.

1. Note that the ζ-function is always defined as we defined it previously.
2. μ isn't that nice.

Example

Consider the poset $P = (\mathbb{Z}^+, |)$; that is, the positive integers with the partial ordering of divisibility.

$$\zeta = \begin{cases} 1 & x|y \\ 0 & x \nmid y \end{cases}$$

$$\delta = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

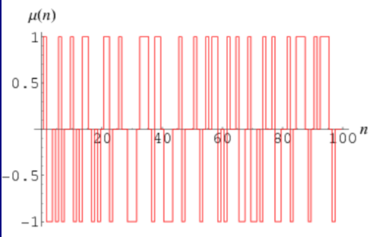
We want to find the $\mu \in I(P)$ such that:

$$\zeta\mu(x, y) = \sum_{x|z|y} \zeta(x, z)\mu(z, y) = \delta(x, y).$$

The answer is yes, and it is:

Möbius Functions

$$\mu(x,y) = \begin{cases} (-1)^k & \frac{x}{y} \nmid n^2 \text{ for any } n \in \mathbb{N} \text{ and if } \frac{x}{y} = p_1 p_2 ... p_k \\ 0 & \textit{otherwise} \end{cases}$$



Möbius Inversion

Actually calculating the μ function of a poset P can be laborious. The following Theorem helps with that:

Theorem

Let P be a locally finite poset. Let $[x, y] \in Int(P)$. Then

$$\mu(x,x) = 1 \text{ and } \mu(x,y) = -\sum_{x \leq z \leq y} \mu(x,z)$$

Möbius Inversion

Now that we have a recursive formula for the Möbius function for any given locally finite poset, we can take advantage of this to learn more about the structure of the poset. The process we can use is called **Möbius Inversion**.

Let P be a locally finite poset and let $f : P \to \mathbb{R}$ be a function. Let $g : P \to \mathbb{R}$ be defined by

$$g(y) = \sum_{x \leq y} f(x).$$

Then

$$f(y) = \sum_{x \leq y} g(x)\mu(x, y)$$

Möbius Inversion

Applications

1. Analysis of recursive sequences
2. Functional Analysis

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

References

[1] Bóna, Miklós. *A Walk Through Combonatorics: An Introduction to Enumeration and Graph Theory*. New Jersey: World Scientific, 2006. Print.