J. McGuire Class: Math 500

Chapter 5: Product Measures

5A: Product of Measure Spaces

Topic: Products of Sigma-Algebras

Definition 1 (Rectangle)**.** *Suppose X and Y are sets. A rectangle in $X \times Y$ is a set of the form $A \times B$, where $A \subset X$ and $B \subset Y$.*

Definition 2 (Product of Two σ-algebras; $S \otimes T$)**.** *Suppose (X, S) and (Y, T) are measurable spaces. Then*

- *the product $S \otimes T$ is defined to be the smallest $\sigma-$algebra on $X \times Y$ that contains*
$$\{A \times B : A \in S, B \in T\};$$
- *a measurable rectangle in $S \otimes T$ is a set of the form $A \times B$, where $A \in S$ and $B \in T$.*

Definition 3 (Cross Sections of Sets)**.** *Suppose X and Y are sets and $E \subset X \times Y$. Then for $a \in X$ and $b \in Y$, the cross sections $[E]_a$ and $[E]^b$ are defined by*

$$[E]_a = \{y \in Y : (a, y) \in E\} \qquad \text{and} \qquad [E]^b = \{x \in X : (x, b) \in E\}.$$

Theorem 1 (Cross Sections of Measurable Sets are Measurable)**.** *Suppose S is a $\sigma-$algebra on X and T is a $\sigma-$algebra on Y. If $E \in S \otimes T$, then*

$$[E]_a \in T \text{ for every } a \in X \qquad \text{and} \qquad [E]^b \in S \text{ for every } b \in Y.$$

Definition 4 (Cross Sections of Functions)**.** *Suppose X and Y are sets and $f : X \times Y \to \mathbb{R}$ is a function. Then for $a \in X$ and $b \in Y$, the cross section function $[f]_a : Y \to \mathbb{R}$ and $[f]^b : X \to \mathbb{R}$ are defined by*

$$[f]_a(y) = f(a, y) \text{ for } y \in Y \qquad \text{and} \qquad [f]^b(x) = f(x, b) \text{ for } x \in X.$$

Chapter 5: Product Measures

Theorem 2 (Cross Sections of Measurable Functions are Measurable). *Suppose S is a $\sigma-$algebra on X and T is a $\sigma-$algebra on Y. Suppose $f : X \times Y \to \mathbb{R}$ is an $S \otimes T$-measurable function. Then*

$$[f]_a \text{ is a } T\text{-measurable function on } Y \text{ for every } a \in X$$

and

$$[f]^b \text{ is an } S\text{-measurable function on } X \text{ for every } b \in Y.$$

Topic: Monotone Class Theorem

Definition 5 (Algebra). *Suppose W is a set and A is a set of subsets of W. Then A is called an algebra on W is the following three conditions are satisfied:*

- $\emptyset \in A$;
- *if $E \in A$, then $W \setminus E \in A$;*
- *if E and F are elements of A, then $E \cup F \in A$.*

Theorem 3 (The Set of Finite Unions of Measurable Rectangles is an Algebra). *Suppose (X, S) and (Y, T) are measurable spaces. Then*

(a) the set of finite unions of measurable rectangles in $S \otimes T$ is an algebra on $X \times Y$;

(b) every finite union of measurable rectangles in $S \otimes T$ can be written as a finite union of disjoint measurable rectangles in $S \otimes T$.

Definition 6 (Monotone Class). *Suppose W is a set and M is a set of subsets of W. Then M is called a monotone class on W if the following two conditions are satisfied:*

- *If $E_1 \subset E_2 \subset \ldots$ is an increasing sequence of sets in M, then $\bigcup_{k=1}^{\infty} E_k \in M$;*
- *If $E_1 \supset E_2 \supset \ldots$ is a decreasing sequence of sets in M, then $\bigcap_{k=1}^{\infty} E_k \in M$.*

Chapter 5: Product Measures

Theorem 4 (Monotone Class Theorem)**.** *Suppose A is an algebra on a set W. Then the smallest $\sigma-$algebra containing A is the smallest monotone class containing A.*

Topic: Products of Measures

Definition 7 (Finite Measure; σ-finite measure)**.**

- *A measure μ on a measurable space (X, S) is called finite if $\mu(X) < \infty$.*
- *A measure is called $\sigma-$finite if the whole space can be written as the countable union of sets with finite measure.*
- *More precisely, a measure μ on a measurable space (X, S) is called $\sigma-$finite if there exists a sequence $X_1, X_2, \ldots$ of sets in S such that*

$$X = \bigcup_{k=1}^{\infty} X_k \qquad \textit{and} \qquad \mu(X_k) < \infty \textit{ for every } k \in \mathbb{Z}^+.$$