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Homework 1

Class: Math 520

1. **Integrating factors and existence/uniqueness theorems:** Consider the following family of first-order differential equations:

$$xy' + Ay = 1 + x^2; \qquad A \in \mathbb{R}.$$

Find all of the solutions which are bounded at $x = 0$; consider <u>all</u> the possible cases for real A.

Proof. First note the following:

$$\begin{aligned} xy' + Ay &= 1 + x^2 \\ \iff y' + \frac{A}{x}y &= \frac{1+x^2}{x}. \end{aligned}$$

We know the integrating factor of this linear ODE should then be:

$$\mu(x) = \exp\{\int \frac{A}{x}\,dx\} = \exp\{\ln(x^A)\} = x^A.$$

That is we'll look at the equation:

$$\frac{d}{dx}(yx^A) = x^{A-1}(1+x^2).$$

Note then the condition on $y(x)$ being bounded at $x = 0$ is then equivalent to the following: $\left|\lim_{x\to 0^+} y(x)\right| < \infty$ (assuming a positive domain for x). Now we'll examine the cases of A:

(a) $(A = 0)$

$$\begin{aligned} \int \frac{d}{dx}(yx^0)dx &= \int x^{-1}(1+x^2)\,dx \\ y(x) &= \int (x^{-1} + x)\,dx \\ y(x) &= \ln(x) + \frac{x^2}{2} + C \end{aligned}$$

where C is our constant of integration. For this particular $y(x)$ for it to be bounded at $x = 0$ we must have:

$$\left|\lim_{x\to 0^+} y(x)\right| = \left|\lim_{x\to 0^+} \ln(x) + \frac{x^2}{2} + C\right| = \infty,$$

for whatever choice of C. This implies when $A = 0$, there are no bounded solutions at $x = 0$.

(b) $(A > 0)$

$$\begin{aligned}
\int \frac{d}{dx}(yx^A)dx &= \int x^{A-1}(1+x^2)\ dx \\
y(x)x^A &= \int x^{A-1} + x^{A+1}\ dx \\
y(x) &= \frac{1}{x^A}\left(\frac{x^A}{A} + \frac{x^{A+2}}{A+2} + C\right) \\
&= \frac{1}{A} + \frac{x^2}{A+2} + \frac{C}{x^A} \\
&= \frac{x^A(A+2) + Ax^{A+2} + C(A+2)(A)}{A(A+2)x^A}.
\end{aligned}$$

Enforcing the boundedness condition at $x = 0$ is then bounding the following:

$$\left|\lim_{x\to 0^+} \frac{x^A(A+2) + Ax^{A+2} + C(A+2)(A)}{x^A(A+2)(A)}\right| = \infty.$$

However this is infinite at hence not bounded for whatever choice of $A > 0$ and C. Hence there are no solutions to the ODE bounded at $x = 0$ when $A > 0$.

(c) $(A < 0)$

$$\begin{aligned}
\int \frac{d}{dx}(yx^A)dx &= \int x^{A-1}(1+x^2)dx \\
y(x) &= x^{-A}\int x^{-1(1-A))}(1+x^2)dx,
\end{aligned}$$

at this point introduce the dummy variable $B = -A$, since $A < 0$ we'll have $B > 0$. So that we get:

$$\begin{aligned}
y(x) &= x^B \int x^{-1(1+B)}(1+x^2)dx \\
&= x^B \int x^{-1-B} + x^{1-B}dx \\
&= x^B\left(\frac{x^{-B}}{-B} + \frac{x^{2-B}}{2-B} + C\right) \\
&= \frac{-1}{B} + \frac{x^2}{2-B} + Cx^B \\
\lim_{x\to 0^+} y(x) &= \frac{-1}{B} = \frac{1}{A}.
\end{aligned}$$

This is for whatever choice of C, hence this is bounded at $x = 0$ always when $A < 0$ and $A \neq -2$. Above we assumed that $B \neq 2$, and then analyze that as a separate case.

(d) $(A = -2)$

$$\begin{aligned}
y(x) &= x^2 \int x^{-(1+2)}(1+x^2)\ dx \\
&= x^2 \int x^{-3} + x^{-1}\ dx \\
&= x^2 \left(\frac{x^{-2}}{-2} + \ln(x) + C \right) \\
&= \frac{-1}{2} + x^2 \ln(x) + x^2 C
\end{aligned}$$

To check to see when this is bounded at $x = 0$ is a little more complicated here, since we need to evaluate the limit:

$$\lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x^2}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{2x^3}} = \lim_{x \to 0^+} \frac{-x^2}{2} = 0.$$

So this solution is actually bounded at $x = 0$, that is the solution's:

$$y(x) = \frac{-1}{2} + x^2 \ln(x) + Cx^2.$$

In summary, the only solutions of the problem that are bounded at $x = 0$ are when $A < 0$. □

2. **General Solutions for Linear ODEs:** Find the fundamental set (described below) for initial conditions imposed at $x = 0$ for the real ODE

$$y'' + ay' + by = 0,$$

where a and b are constants.

Examine the three cases: $a^2 > 4b, a^2 = 4b, a^2 < 4b$. The fundamental set for a linear, second-order IVP are the solutions $y_1(x)$ and $y_2(x)$ where $y_1(x_0) = 1, y_1'(x_0) = 0$, and $y_2(x_0) = 0, y_2'(x_0) = 1$. Then the unique solution to the IVP for the initial conditions $y(x_0) = \alpha, y'(x_0) = \beta$ is $y(x) = \alpha y_1(x) + \beta y_2(x)$.

Proof. We'll assume a fundamental set of solution $\{y_1(x), y_2(x)\}$ such that the solution to the above linear ODE is $y(x) = Ay_1(x) + By_2(x)$. Plugging that into our ODE will give us two systems of equations and applying initial conditions

$$\begin{cases} y_1'' + ay_1' + by_1 = 0 \\ y_1(0) = 1 \\ y_1(0) = 0 \end{cases},$$

and

$$\begin{cases} y_2'' + ay_2' + by_2 = 0 \\ y_2(0) = 0 \\ y_2'(0) = 1 \end{cases}.$$

We can solve these with the method of undetermined coefficients, the domain and distinctness of these eigenvalues will determined the basis for each system.

(a) $(a^2 > 4b)$
The first IVP problem with undetermined coefficients (with $e^{\lambda x}$)will gives us the auxiliary equation:

$$\lambda^2 + a\lambda + b = 0 \qquad \iff \qquad \lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}, \lambda_2 = \frac{-a - \sqrt{a^2 - 4}}{2}.$$

With $a^2 - 4b > 0$ both are real and distinct, meaning that:

$$y_1(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}.$$

Likewise we'll end up with the solution for the other IVP problem to be:

$$y_2(x) = Ce^{\lambda_1 x} + De^{\lambda_2 x}.$$

Imposing $y_1(0) = 1, y_1'(0) = 0$ so that we have a system of equations:

$$\begin{cases} A + B = 1 \\ A\lambda_1 + B\lambda_2 = 0 \end{cases}.$$

This will give us solutions:

$$\begin{cases} A = \dfrac{\lambda_2}{\lambda_2 - \lambda_1} \\ B = \dfrac{\lambda_1}{\lambda_1 - \lambda_2} \end{cases}.$$

So that:

$$y_1(x) = \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{\lambda_1 x} + \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{\lambda_2 x}.$$

Using similar reasoning for $y_2(x) = Ce^{\lambda_1 x} + De^{\lambda_2 x}$ with it's conditions we'll get the equations:

$$\begin{cases} C + D = 0 \\ C\lambda_1 + D\lambda_2 = 1 \end{cases}.$$

Solving this we'll get:

$$\begin{cases} C = \dfrac{1}{\lambda_1 - \lambda_2} \\ D = \dfrac{1}{\lambda_2 - \lambda_1} \end{cases}.$$

So the fundamental set is then:

$$\left\{ \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{\lambda_1 x} + \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{\lambda_2 x}, \frac{1}{\lambda_1 - \lambda_2} e^{\lambda_1 x} + \frac{1}{\lambda_2 - \lambda_1} e^{\lambda_2 x} \right\}$$

(b) ($a^2 = 4b$) In the auxiliary equation for both IVP problems will lead to a repeated real root. In this case, we'll have $y_1(x) = Ae^{\lambda x} + xBe^{\lambda x}$ where $\lambda \equiv \lambda_1 = \lambda_2$. So imposing are initial conditions of $y_1(0) = 1, y_1'(0) = 0$: $1 = A$ and $A\lambda e^{\lambda 0} + B(e^{\lambda 0} + \lambda 0 e^{\lambda 0}) = 0 \iff \lambda + B = 0 \iff B = -\lambda$. So the solution to this IVP problem is $y_1(x) = e^{\lambda x} - \lambda x e^{\lambda x}$. For y_2 we'll end up with $y_2(x) = Ce^{\lambda x} + Dxe^{\lambda x}$ then imposing $y_2(0) = 0$ and $y_2'(0) = 1$:

$$C = 0 \qquad D = 1.$$

Hence the fundamental set of solutions is then:

$$\left\{ e^{\lambda x} - \lambda x e^{\lambda x}, x e^{\lambda x} \right\}.$$

(c) ($a^2 < 4b$) In the auxiliary equation for both IVP problems will end up with a complex conjugate pair. So that: $y_1(x) = A\cos(\lambda_1 x) + B\sin(\lambda_2 x)$ and $y_2(x) = C\cos(\lambda_1 x) + D\sin(\lambda_2 x)$. Now imposing $y_1(0) = 1, y_1'(0) = 0$ gives us:

$$1 = A \qquad 0 = \lambda_2 B \implies B = 0$$

and imposing $y_2(0) = 0, y_2'(0) = 1$ gives us:

$$C = 0 \qquad 1 = \lambda_2 D \implies D = \lambda_2.$$

Notice that $\lambda_2 \neq 0$, since that would imply $\lambda_1 = 0$ and that would land us in the previous case. This gives us a fundamental set of solutions:

$$\left\{ \cos(\lambda_1 x), \frac{1}{\lambda_2} \sin(\lambda_2 x) \right\}.$$

□

Homework 1

3. **Patching:** Find two linearly independent solutions of

$$y'' + \operatorname{sgn}(x)y = 0, \qquad -\infty < x < \infty$$

where $\operatorname{sgn}(x) = \pm 1$ according to whether x is positive or negative and $\operatorname{sgn}(0) = 0$. Since the ODE is 2nd-order, the solution should be continuous and have a continuous first derivative for all x including $x = 0$. Solve for $x > 0$ and $x < 0$ separately and match at $x = 0$. This is a simpler precursor to asymptotic matching; a core concept we will develop later.

Proof. (a) ($x < 0$) This will be the problem

$$y'' - y = 0$$

using the method of undetermined coefficients with $e^{\lambda x}$ will give us the equation:

$$\lambda^2 - 1 = 0.$$

This implies that we'll have a solution for this of the form:

$$y(x) = Ae^{x} + Be^{-x},$$

when $x < 0$.

(b) ($x > 0$) Using the same method as in the previous case we'll end up with the equation:

$$\lambda^2 + 1 = 0,$$

this time $\lambda = \pm i$. That implies, through Euler's formula, solutions of the form:

$$y(x) = C\cos(x) + D\sin(x),$$

where C, D are arbitrary complex coefficients when $x > 0$.

(c) (Imposing Conditions) Since both the solution and its first derivative must be continuous at $x = 0$, we have the two conditions $Ae^0 + Be^{-0} = C\cos(0) + D\sin(0)$ and $Ae^0 - Be^{-0} = -C\sin(0) + D\cos(0)$. So this is an linear system of 2 equations with 4 unknowns, meaning we'll just be able to solve 2 unknowns in terms of the other 2:

$$\begin{cases} A + B & = C \\ A - B & = D \end{cases}.$$

Hence the solutions to this equation will be:

$$y(x) = \begin{cases} Ae^{x} + Be^{-x} & \text{if } x \leq 0 \\ (A + B)\cos(x) + (A - B)\sin(x) & \text{if } x \geq 0 \end{cases},$$

and the coefficients A, B must be determined by 2 more restrictions being imposed on the system.

$\square$

4. **A transformation of a nonlinear ODE:**

The non-linear first-order differential equation

$$y' + p(x)y = q(x)y^n, \qquad n \neq 1$$

is called Bernoulli's equation. Show that the transformation $u = y^{1-n}$ reduces the equation to a first-order linear equation. Use this transformation to find the general solution(s) to

$$(i)\ y' - y = xy^{1/2} \qquad \text{and} \qquad (ii) y' + 2xy + y^2 = 0.$$

Proof. (i) $(y' - y = xy^{1/2})$ With $n = \dfrac{1}{2}$ this will gives us a transformation of $u = y^{1/2}$ and $y = u^2$ with $u' = y'\dfrac{1}{2y^{1/2}}$; that is $2uu' = y'$. So that we'll get:

$$\begin{aligned} 2uu' - u^2 &= xu \\ \iff 2u' - u &= x \\ \iff u' - \frac{1}{2}u &= \frac{x}{2}. \end{aligned}$$

Now this is an equation solvable with integrating factors: $\mu(x) = \exp\left\{\int -\dfrac{dx}{2}\right\} = e^{-x/2}$. So that:

$$\begin{aligned} \frac{d}{dx}\left(e^{-x/2}u\right) &= \frac{x}{2}e^{-x/2} \\ \iff e^{-x/2}u(x) + C &= \int \frac{xe^{-x/2}}{2}\ dx \end{aligned}$$

make a $u-$sub of $u = \dfrac{x}{2}$ with $du = \dfrac{dx}{2} \iff dx = 2du$ ($u(x)$ is still our dependent variable in the equation). We now have

$$e^{-x/2}u(x) + C = 2\int ue^{-u}\ du$$

integration by parts with $u = u$, $du = du$ and $dv = e^{-u}du, v = -e^{-u}$ will give us:

$$\begin{aligned}
e^{-x/2}u(x) + C &= 2\left(-ue^{-u} + \int e^{-u}du\right) \\
e^{-x/2}u(x) &= 2\left(-\frac{x}{2}e^{-x/2} - e^{-x/2} - D\right) \\
u(x) &= 2\left(-\frac{x}{2} - 1 - De^{x/2}\right) \\
y^{1/2}(x) &= \left(-x - 2 - Ee^{x/2}\right) \\
y(x) &= (-x - 2 - Ee^{x/2})^2 \\
&= (x + 2 + Ee^{x/2})^2.
\end{aligned}$$

Where C, D, E were all changes in constants of integration. That is our solution!

(ii) $(y' + (2x)y = -y^2)$ Here $n = 2$ so that we'll use the transformation $u = y1 - 2 = y^{-1}, y = u^{-1}$ with $u' = -1y'y^{-2} \iff u' = -y'u^2 \iff \frac{-u'}{u^2} = y'$:

$$\begin{aligned}
\frac{-u'}{u^2} + 2x\frac{1}{u} &= -\frac{1}{u^2} \\
\iff u' - 2xu &= 1.
\end{aligned}$$

This again is a much easier problem of integrating factors, namely $\mu(x) = \exp\{\int -2x \; dx\} = \exp\{-x^2\}$. So that we'll have the equation:

$$\frac{d}{dx}\left(ue^{-x^2}\right) = e^{-x^2}.$$

Let a be an arbitrary lower bound of integration so that:

$$\begin{aligned}
u(x)e^{-x^2} &= \int_a^x e^{-t^2} \; dt \\
u(x) &= \int_a^x e^{x^2 - t^2} \; dt \\
\frac{1}{y(x)} &= \int_a^x e^{x^2 - t^2} \; dt \\
y(x) &= \left(\int_a^x e^{x^2 - t^2} dt\right)^{-1}
\end{aligned}$$

Since this integral has no elementary antiderivative, we'll leave the solution in an un-integrated form as seen above.

$\square$

5. **Another transformation of a nonlinear ODE:**

   The non-linear first-order differential equation

$$y' + p(x)y = q(x)y^2 + r(x)$$

   is called Riccatti's equation (note: this is a slightly different definition of the terms from our definition in class.) Show that the transformations

$$y = \frac{v(x)}{q} \qquad \text{and} \qquad v(x) = -\frac{u'(x)}{u(x)}$$

   transform it into a second-order linear ODE with variable coefficients

$$\frac{d^2u}{dx^2} + P(x)\frac{du}{dx} + Q(x)u = 0.$$

   Proof. With the transformation $y = \frac{v}{q}$ and $v = -\frac{u'}{u}$; that is $y = \frac{-u'}{qu}$we'll end up with $y' = \frac{-u''(qu) + u'(q'u + u'q)}{(qu)^2} = \frac{-u''(qu) + u'q'u + (u')^2 q}{(qu)^2}$ so that the transformed equation will be:

$$\begin{aligned}
\frac{-u''(qu) + u'q'u + (u')^2q}{(qu)^2} + p\frac{-u'}{qu} &= q\frac{(u')^2}{q^2u^2} + r \\
-u''(qu) + u'q'u + (u')^2q + -pqu(u') &= q(u')^2 + r(q^2u^2) \\
-(q)u'' + u'q' - (pq)u' &= r(q^2)u \\
u'' + u'\left(p - \frac{q'}{q}\right) + (-rq)u &= 0.
\end{aligned}$$

   With $r = r(x), q(x) = q, p(x) = p$, this is our second-order linear ODE! □

6. **Variation of Parameters and Green's Functions:**

   Show, using the method variation-of-parameters, that the solution to the initial value problem

$$y'' + y = g(t), \qquad y(0) = 0, y'(0) = 0$$

   is

$$y(t) = \int_0^t g(\tau)\sin(t-\tau)d\tau.$$

   Note: we can define the "kernel" of this integral as the Green's function for the IVP:

$$G(t|\tau) = \begin{cases} 0 & \text{for } t < \tau \\ \sin(t-\tau) & \text{for } t > \tau; \end{cases}$$

in which case we can write the solution as the product of this Green's function and the non-homogeneous forcing term integrated over the entire domain:

$$y(t) = \int_0^\infty g(\tau)G(t|\tau)\ d\tau.$$

The Green's function is the response of the system to an idealized impulse at $t = \tau$. We'll discuss Green's functions more later.

Proof. First, we'll find the solution to the associated homogeneous problem:

$$y'' + y = 0.$$

Using the method of undetermined coefficients, with $e^{\lambda t}$, gives us $\lambda^2 + 1 = 0$. This implies $\lambda = \pm i$ and gives us the general solution for the homogeneous problem is:

$$y_H(t) = A\cos(t) + B\sin(t).$$

The variation of parameters method then tells us a particular solution to inhomogeneous problem is then

$$y_P(t) = -\cos(t)\int_0^t \frac{\sin(\tau)}{\cos^2(\tau)+\sin^2(\tau)}d\tau + \sin(t)\int_0^t \frac{\cos(\tau)}{\cos^2(\tau)+\sin^2(\tau)}\ d\tau.$$

Where the denominator comes from the fact that $(\cos(t))' = -\sin(t)$ and $(\sin(t))' = \cos(t)$. Hence we'll have:

$$y_P(t) = \int_0^t \sin(t)\cos(\tau) - \sin(\tau)\cos(t)\ d\tau = \int_0^t \sin(t-\tau)\ d\tau,$$

using the sine angle subtraction identity. Now finally imposing our initial conditions to the solution:

$$y(t) = y_H(t) + y_P(t) = A\cos(t) + B\sin(t) + y_P(t)$$

gives us:

$$0 = A + 0 + 0 \iff A = 0$$

and

$$0 = -A\sin(0) + B\cos(0) + y_P'(0) = B.$$

That is the solution to this initial value problem is just:

$$y(x) = y_P(x) = \int_0^t \sin(t-\tau)\ d\tau.$$

$\square$