Linear Algebra Problems Qualifying Exams

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1.) (2018 Spring) Let $V \subset \mathbb{R}^5$ be the subspace defined by the equation

$$x_1 - 2x_2 + 3x_3 - 4x_4 + 5x_4 = 0$$

a.) Find (with justification) a basis for V.

b.) Find (with justification) a basis of $V^\perp$, the subspace of $\mathbb{R}^5$ orthogonal to V under the usual dot product.

2.) (2018 Spring) Suppose V is a finite-dimensional real vector space and $T : V \to V$ is a linear transformation. Prove that T has at most $\dim(\text{range } T)$ distinct nonzero eigenvales.

3.) (2017 Fall) Let L be the line L parameterized by $L(t) = (2t, -3t, t)$ for $t \in \mathbb{R}$, and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that is orthogonal projection onto L.

a.) Describe $\ker(T)$ and $\text{im}(T)$, either implicitly (using equations in x, y, z) or parametrically.

b.) List the eigenvalues of T and their geometric multiplicities.

c.) Find a basis for each eigenspace of T.

d.) Let A be the matrix for T with respect to the standard basis. Find a diagonal matrix B and an invertible matrix S such that $B = S^{-1}AS$. (You do not have to compute A.)

4.) (2017 Fall) Suppose A is 5×5 matrix and v_1, v_2, v_3 are eigenvectors of A with distinct eigenvalues. Prove $\{v_1, v_2, v_3\}$ is a linearly independent set. *Hint:* Consider a minimal linear dependence relation.

5.) (2017 Spring) Let $V = \{a_0 + a_1\sqrt[3]{2} + a_2\sqrt[3]{4} | a_0, a_1, a_2 \in \mathbb{Q}\} \subseteq \mathbb{R}$. This set is a vector space over $\mathbb{Q}$.

a.) Verify V is closed under product (using the usual product operation in $\mathbb{R}$).

b.) Let $T : V \to V$ be the linear transformation defined by $T(v) = (\sqrt[3]{2} + \sqrt[3]{4})v$. Find the matrix that represents T with respect to the basis $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$ for V.

c.) Determine the characteristic polynomial for T.

6.) (2017 Spring) Suppose F is a field and A is an $n \times n$ matrix over F. Suppose further that A possesses distinct eigenvalues λ_1 and λ_2 with $\dim(\text{Null}(A - \lambda_1 I_n)) = n - 1$. Prove A is diagonalizable.

7.) (2016 Fall) Consider the following matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a.) Determine the characteristic and minimal polynomials of A.

b.) Find a basis for $\mathbb{R}^4$ consisting of generalized eigenvectors of A.

c.) Find an invertible matrix S such that $S^{-1}AS$ is in Jordan canonical form.

d.) Determine a Jordan canonical from of A.

8.) (2016 Fall) Let V be a vector space and $T : V \to V$ be a linear transformation.

a.) Prove that if T is a projection (i.e., $T^2 = T$), then V can be decomposed into the internal direct sum $V = \text{null}(t) \oplus \text{range}(t)$.

b .) Suppose V is an inner product space and $T*$ is the adjoint of T with respect to the inner product. Show that null$(T*)$ is the orthogonal complement of range(T).

c.) Suppose V is an inner product space and T is an orthogonal projection, i.e., a projection for which the null space and range are orthogonal. Show that T is self adjoint.

9.) (2016 Spring) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that expands radially by a factor of 3 around the line parameterized by $L(t) = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} t$, leaving the line itself fixed (viewed as a subspace).

a.) Find an eigenbasis for T and provide the matrix representation of T with respect to that basis.

b.) Provide the matrix representation of T with respect to the standard basis.

10.) (2015 Fall) Let $A = \begin{bmatrix} -2 & 1 & -1//5 & -2 & 2//7 & -3 & 3 \end{bmatrix}$

a.) Find the characteristic polynomial and the minimal polynomial of A.

b.) Find the Jordan canonical form of A.

11.) (2015 Spring) Suppose T is a linear transformation of a finite dimensional complex inner product space V. Let I denote the identity transformation on V. The *numerical range* of T is the subset of $\mathbf{C}$ defined by

$$W(T) := \{< T(x), x >: x \in V, ||x|| = 1\}$$

a.) Show that $W(T + cI) = W(T) + c$ for every $c \in \mathbf{C}$.

b.) Show that $W(cT) = cW(T)$ for every $c \in \mathbf{C}$.

c.) Show that the eigenvalues of T are contained in $W(T)$.

d.) Let $\mathcal{B}$ be an orthonormal basis for V. Show that the diagonal entries of $[T]_\mathcal{B}$ are contained in $W(T)$.

12.) (2014 Fall) Let V denote the real vector space of polynomials in x of degree at most 3. Let $\mathcal{B} = \{1, x, x^2, x^3\}$ be a basis for V an $T : V \to V$ be the function defined by $T(f(x)) = f(x) + f'(x)$.

a.) Prove that T is a linear transformation.

b.) Find $[T]_\mathcal{B}$, the matrix representation for T in terms of the basis $\mathcal{B}$.

c.) Is T diagonalizable? If yes, find a matrix A so that $A[T]_\mathcal{B}A^{-1}$ is diagonal, otherwise explain why T is not diagonalizable.

13.) (2014 Spring) Let $a, b \in \mathbb{R}$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation which is reflection across the plane $z = ax + by$.

a.) Find the eigenvalues of T and for each find a basis for the corresponding eigenspace.

b.) Is T diagonalizable? Justify.

c.) What is the characteristic polynomial of T?

d.) What is the minimal polynomial of T?

14.) (2014 Spring) Let $\phi : V \to W$ be a surjective linear transformation of finite dimensional linear spaces. Show that there is a $U \subseteq V$ such that $V = (\ker(\phi)) \oplus U$ and $\phi|_U : U \to W$ is an isomorphism. [Note that V is not assumed to be an inner-product space; also note that $\ker(\phi)$ is sometimes referred to as the null space of ϕ; finally $\phi|_U$ denotes the restriction of ϕ to U.]

15.) (2013 Fall) Let V be a finite dimensional vector space over $\mathbf{C}$, and let S and T be two linear transformations from V to V. Assume that $ST = TS$ and that the characteristic polynomial of S has distinct roots.

a.) Show that every eigenvector of S is an eigenvector of T.

b.) If T is nilpotent, show that $T = 0$.

16.) (2013 Fall) Let $a, b \in \mathbb{R}$ and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation which is orthogonal projection onto the plane $z = ax + by$ (with respect to the usual Euclidean inner-product on $\mathbb{R}^3$).

a.) Find the eigenvalues of T and for each find a basis for the corresponding eigenspace.

b.) Is T diagonalizable? Justify.

c.) What is the characteristic polynomial of T?

d.) What is the minimal polynomial of T?

17.) (2013 Spring) Let V be the vector space of upper triangular 2×2 matrices over $\mathbb{R}$. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$