Solutions to Past Preliminary Analysis Exams

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Spring 2020

1. *Let (f_n) be a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that f_n is bounded for each $n \in \mathbb{N}$.*

1.1. *Prove that if $f_n \to f$ uniformly on $\mathbb{R}$, then f is bounded.*

Proof. Suppose that $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of bounded functions, converging uniformly to a function $f : \mathbb{R} \to \mathbb{R}$. Then for all $n \in \mathbb{N}$ there exists a $M_n \geq 0$ such that $|f_n(x)| \leq M_n$ for all $x \in \mathbb{R}$. Additionally for all $\epsilon > 0$ there exists a $K(\epsilon) \in \mathbb{N}$ such that if $x \in \mathbb{R}$ and $n \geq K(\epsilon)$, then $|f_n(x) - f(x)| < \epsilon$. Letting $\epsilon > 0$ be fixed we have for all $n \geq K(\epsilon)$ and $x \in \mathbb{R}$, $|f_n(x) - f(x)| < \epsilon$.

$$\begin{aligned} |f_n(x) - f(x)| < \epsilon &\implies |f(x) - f_n(x)| < \epsilon \\ &\implies f(x) < \epsilon + f_n(x) \\ &\implies f(x) < \epsilon + M_n \end{aligned}$$

Both M_n and ϵ are fixed here, hence for $M = M_n + \epsilon$, for a particular $n \geq K(\epsilon)$, for a particular $\epsilon > 0$, we have $|f(x)| < M = M_n + \epsilon$ for all $x \in \mathbb{R}$. Thus f is bounded on all of $\mathbb{R}$. □

1.2. *If each f_n is continuous and $f_n \to f$ pointwise on $\mathbb{R}$, does f have to be bounded? Give a proof or a counter example.*

Solution. *Consider the sequence of functions defined by:*

$$f_n(x) = \begin{cases} |x|, & |x| \leq n \\ n, & \text{otherwise} \end{cases}$$

I'll show that f_n converges pointwise onto the function $f(x) = |x|, x \in \mathbb{R}$, that each f_n is bounded and continuous on $\mathbb{R}$, but that f is unbounded.

(Continuity of f_n) First we'll show that for any $n \in \mathbb{N}$ f_n is continuous on $(-\infty, -n) \cup (n, +\infty)$. Let $\epsilon > 0$ be given. Suppose c,x are in the interval we're considering, $(-\infty, -n) \cup (n, \infty)$ with $0 < |x - c| < \delta(\epsilon)$, then $|f_n(x) - f_n(c)| = |0| = 0 < \epsilon$. Hence $f_n(x)$ is continuous on $(-\infty, -n) \cup (n, \infty)$.

Considering $(-n, n)$ now: Let $\epsilon > 0$ be given and $\delta(\epsilon) = \epsilon$. Suppose $x, c \in (-n, n)$ and $0 < |x - c| < \delta(\epsilon) = \epsilon$. Then

$$\begin{aligned} |f_n(x) - f_n(c)| &= ||x| - |c|| \\ &\leq |x - c|, \text{by the triangle inequality} \\ &< \delta(\epsilon) = \epsilon \end{aligned}$$

Hence f_n is continuous on $(-n, n)$.

Finally to show that f_n is continuous on $x \in \{-n, n\}$ we'll use the alternative definition of continuous functions, which is f_n is continuous at a point c if a sequence in $(x_k) \to c$ in $\mathbb{R}$, then $(f_n(x_k)) \to f_n(c)$.

Let $(x_k) \to n$ in $\mathbb{R}$. Then for all $\epsilon > 0$, there exists a $K(\epsilon) \in \mathbb{N}$ such that if $k \geq K(\epsilon)$, then $|x_k - n| < \epsilon$. We'll show then that $(f_n(x_k)) \to f_n(n)$. Let $\epsilon > 0$. Then let $K_1(\epsilon) = K(\epsilon)$, where $K(\epsilon)$ is defined above. Suppose $k \geq K_1(\epsilon)$, then

$$\begin{aligned} |f_n(x_k) - f_n(n)| &= |f_n(x_k) - |n|| \\ &= |n - f_n(x_k)| \\ &\leq |n - |x_k||, \textit{since } f_n(x_k) \leq n \\ &\leq |n - x_k| \\ &< \epsilon, \textit{by our choice of } K_1(\epsilon) \end{aligned}$$

Thus the sequence $\lim_{k\to\infty}(f_n(x_k)) = f_n(n)$. *Hence* f_n *is continuous at* n, *for any* $n \in \mathbb{N}$. *Therefore,* f_n *is continuous at every point on* $\mathbb{R}$.

(Pointwise Convergence)

Let $\epsilon > 0$ *be given and* $x \in \mathbb{R}$. *Then choose* $K(\epsilon, x) = x$. *Suppose* $n \geq K(\epsilon, x)$. *Then*

$$\begin{aligned}|f_n(x) - f(x)| &= |f_n(x) - |x|| \\ &\leq |n - |x|| \\ &\leq |n - x| \\ &= n\left|1 - \frac{x}{n}\right| \\ &\leq n\left|1 - \frac{x}{K(\epsilon, x)}\right| \\ &= n|1 - 1| \\ &= 0 \\ &< \epsilon\end{aligned}$$

Thus $(f_n) \to f$, *pointwise on* $\mathbb{R}$.

(f is unbounded)

Suppose for sake of contradiction, that there exists a $M \geq 0$ *such that* $|f(x)| \leq M$, *for all* $x \in \mathbb{R}$. *Then* $||x|| \leq M \implies |x| \leq M$, *for all* $x \in \mathbb{R}$. *So that is if* $x = M + 1$, *then* $|M + 1| \leq M \implies 1 \leq 0$. *A contradiction, hence* f *must be unbounded on all of* $\mathbb{R}$.

So this is a counterexample to the claim above.

2. *Show that the function*

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

is differentiable only at $x \neq 0$.

Solution. *First, we'll use the contrapositive of the theorem stating, if* f *is differentiable on* $A \subseteq \mathbb{R}$, *then* f *is continuous on* $A \subseteq \mathbb{R}$; *that is, if* f *isn't continuous on* $A \subseteq \mathbb{R}$, *then* f *isn't differentiable on* $A \subseteq \mathbb{R}$. *So we'll show that* f *isn't continuous on* $\mathbb{R} \setminus \{0\}$. *Then we'll use the epsilon-delta definition of the derivative to show that* f *is differentiable at* 0.

First, note that irrationals are all limit points to the rationals, and vice-a-versa. This is a consequence that between any two real numbers there are irrational and rational numbers. Meaning that for any irrational number there is a sequence of rationals converging to that irrational number, since we can get arbitrarily close to any irrational with a sequence of rationals, similarly for the rationals and irrationals.

So let (a_n) *be a sequence of rational numbers such that* $(a_n) \to \zeta$, *where* $\zeta \in \mathbb{R} \setminus \mathbb{Q}$. *Then we have* $f(a_n) = a_n^2$ *for all* $n \in \mathbb{N}$, *and* $f(\zeta) = 0$. *But we know that the sequence* $(a_n) \to \zeta \neq 0$, *hence* $(a_n \cdot a_n) \to \zeta^2 \neq 0$. *Thus we have a sequence* (a_n) *converging to* ζ, *but* $f(a_n)$ *doesn't converge to* $f(\zeta)$, *and so* f *is discontinuous on* $\mathbb{R} \setminus \mathbb{Q}$.

Similarly, let (b_n) *be sequence in* $\mathbb{R} \setminus \mathbb{Q}$ *converging to* $r \in \mathbb{Q} \setminus \{0\}$. *Then* $f(b_n) = 0$, *for all* $n \in \mathbb{N}$, *and* $f(r) = r^2 \neq 0$. *Clearly* $(f(b_n)) \to 0 \neq f(r)$. *Hence* f *must be discontinuous on* $\mathbb{R} \setminus \mathbb{Q}$.

Now we will show that f *is differentiable at* 0. *Let* $\epsilon > 0$ *be given and choose* $\delta(\epsilon) = \epsilon$. *Suppose* $x \in \mathbb{R}$ *such that* $0 < |x| < \delta(\epsilon) = \epsilon$. *Then*

$$\left|\frac{f(x) - f(0)}{x - 0} - 0\right| = \left|\frac{f(x)}{x}\right|$$

If $x \in \mathbb{Q}$ *we have* $f(x) = x^2$, *hence* $\left|\frac{f(x)}{x}\right| = |x| < \delta = \epsilon$. *If* $x \in \mathbb{R}\setminus\mathbb{Q}$, *then* $f(x) = 0$, *hence* $\left|\frac{f(x)}{x}\right| = 0 < \epsilon$. *Thus we have* $f'(0) = 0$.

3. *Consider the sequence of functions*

$$f_k(x) = \frac{x^k(\sin(kx^2 + 1) + \cos(\pi - 5x))}{k!}.$$

Prove that the series $\sum_{k=0}^{\infty} f_k$ *converges uniformly on any interval of the form* $[-M, M]$ *in* $\mathbb{R}$.

Proof. First let $x \in [-M, M]$ for some $M \geq 0$. Then note that

$$\begin{aligned} f_k(x) &= \frac{x^k(\sin(kx^2+1)+\cos(\pi-5x))}{k!} \\ &\leq \frac{2x^k}{k!} \\ &\leq \frac{2|x|^k}{k!} \\ &\leq \frac{2M^k}{k!}. \end{aligned}$$

We'll employ the Weierstrass M-Test to show that $\sum\limits_{k=0}^{\infty} f_k(x)$; that is we'll show that $\sum\limits_{k=0}^{\infty} \frac{2M^k}{k!}$ is convergent. First note that the Taylor expansion of e^x is:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, x \in \mathbb{R}$$

So that $\sum\limits_{k=0}^{\infty} \frac{2M^k}{k!} = 2\sum\limits_{k=0}^{\infty} \frac{M^k}{k!} = 2e^M$. Thus, by the Weierstrass M-Test, we have $\sum\limits_{k=0}^{\infty} f_k(x)$ is converges uniformly on $[-M, M]$. □

4. *A function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz on a set $A \subseteq \mathbb{R}$ if there exists a constant $M \geq 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in A$.*

4.1. *Assume that f is a differentiable function on $\mathbb{R}$ and let f' be continuous on $[a, b]$. Prove that f is Lipschitz on $[a, b]$.*

Proof. Assume our hypothesis of f being differentiable on $\mathbb{R}$ and f' continuous on $[a, b]$. First note for any $x, y \in [a, b]$ such that $x < y$, that f is differentiable on (x, y) and f' is continuous on $[x, y]$. Then by the Mean Value Theorem for any $x, y \in [a, b]$, there's a $c \in (x, y)$ such that $f(y) - f(x) = f'(c)(y - x) \implies f'(c) = \frac{f(y)-f(x)}{y-x}$. Moreover, by the fact that f' is continuous on $[a, b]$ and that $[a, b]$ is closed and bounded hence must be compact by the Hiene-Borel-Weierstrass Theorem, $f'([a, b])$ must also be compact, because the images of compact sets under continuous functions must also be compact. Hence $f'([a, b])$ is bounded, so there exists a $M \geq 0$, such that $|f'(x)| \leq M$ for all $x \in [a, b]$. So that for any $x, y \in [a, b]$ there exists $c \in (x, y)$ such that

$$\begin{aligned} f'(c) &= \frac{f(y)-f(x)}{y-x} \\ &\leq \frac{|f(y)-f(x)|}{y-x} \\ &= |f'(c)| \\ &\leq M \end{aligned}$$

Thus there exists a $M \geq 0$, such that $|f(y) - f(x)| \leq M(y - x)$. Hence f is Lipschitz on $[a, b]$. □

4.2. *Prove that a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.*

Proof. Suppose that f satisfies the Lipschitz condition on $\mathbb{R}$. Then there exists a $M \geq 0$ such that $|f(y) - f(x)| \leq M|y - x|, x, y \in \mathbb{R}$.

We wish to show that f is uniformly continuous on $\mathbb{R}$, so let $\epsilon > 0$ and choose $\delta(\epsilon) = \frac{\epsilon}{M}$. Suppose $x, y \in \mathbb{R}$ such that $0 < |y - x| < \delta(\epsilon)$. So we have $|f(y) - f(x)| \leq M|y - x| < M\frac{\epsilon}{M} = \epsilon$. Thus f is uniformly continuous on $\mathbb{R}$. □

5.

5.1. *State the definition for $f : [a, b] \to \mathbb{R}$ to be Riemann intergratable on $[a, b]$.*

Solution. *f is Riemann integratable on $[a, b]$, if there exists a $I \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists a partition of $[a, b]$, P_ϵ, with partition points $(x_0, ..., x_n)$, with $x_0 = a$ and $x_n = b$, so that if P is a refinement of P_ϵ, then $|\sum\limits_{k=1}^{n} f(\zeta_k)(x_k - x_{k-1}) - I| < \epsilon$, with $x_{k-1} \leq \zeta_k \leq x_k$ for $k \in \{1, ..., n\}$.*

5.2. *Define $f : [0,4] \to \mathbb{R}$ to be defined by*

$$f(x) = \begin{cases} 1, & x \in [0,1) \\ 2, & x \in [1,2) \\ 3, & x \in [2,3) \\ 4, & x \in [3,4] \end{cases}.$$

Use the definition of Riemann integral to prove that f is Riemann integratable on $[0,4]$.

Proof. Let $\epsilon > 0$ be given and choose $P_\epsilon = (0,1,2,3,4)$ be a partition of $[a,b]$ defined by those partition points. Then for any refinement of P_ϵ, call it P, it must be of the form: $P = (0, x_1, ..., 1, ..., 2, ..., 3, ..., x_{n-1}, 4)$; that is, any refinement of a partition must contain the partition's partition points, in this case $0,1,2,3,4$. So that there exists indicies a,b,c,d such that $x_a = 1, x_b = 2, x_c = 3, x_d = 4$. So then

$$\begin{aligned}\sum_{k=1}^{\infty} f(\zeta_k)(x_k - x_{k-1}) &= \sum_{k=1}^{a} 1 \cdot (x_k - x_{k-1}) \\ &+ \sum_{k=a+1}^{b} 2 \cdot (x_k - x_{k-1}) \\ &+ \sum_{k=b+1}^{c} 3 \cdot (x_k - x_{k-1}) \\ &+ \sum_{k=c+1}^{\infty} 4 \cdot (x_k - x_{k-1}) \\ &= 1 + 2 + 3 + 4 \\ &= 10.\end{aligned}$$

Hence $\left| \sum_{k=1}^{n} f(\zeta_k)(x_k - x_{k-1}) - 10 \right| = 0 < \epsilon$. Thus f is Riemann integratable on $[a,b]$. □

Winter 2020

6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, period function. Prove that the set $f(\mathbb{R})$ is compact. (Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is periodic if there exists a nonzero constant P such that $f(x) = f(x+P)$ for all $x \in \mathbb{R}$.)*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous periodic with period P, so that for all $x \in \mathbb{R}$, $f(x) = f(x+P)$. To show $f(\mathbb{R})$ is compact we'll use Heine-Borel to prove the equivalent that $f(\mathbb{R})$ is closed and bounded. For any $x \in \mathbb{R}$ we have $f(x) = f(x+P)$, and since $[x, x+P]$ is compact by Heine-Borel, we have that $f(z^*) = \sup\{f(z) : z \in [x, x+P]\}$ and $f(z_*) = \inf\{f(z) : z \in [x, x+P]\}$ exist. So that $f(z_*) \leq f(z) \leq f(z^*)$ for all $z \in [x, x+P]$ and since f is periodic, we have $f(z_* + P) \leq f(z+P) \leq f(z^* + P)$, for $z + P \in [x+P, x+2P]$. Repeating this process for $[x + (n-1)P, x + nP]$ we find this is true for any $n \in \mathbb{Z}$, any $c \in \mathbb{R}$ is contained in such an interval, so that f is bounded in $\mathbb{R}$. Hence $f(\mathbb{R})$ is bounded.

Since $f(\mathbb{R})$ is bounded above by $f(z^*)$ and below $f(z_*)$, we have that the complement of $f(\mathbb{R})$ is: $(-\infty, f_*) \cup (f^*, +\infty)$, an open set. So $f(\mathbb{R})$ is closed in $\mathbb{R}$.

By Hiene-Borel, $f(\mathbb{R})$ is compact. □

7. *Consider the sequence $\{a_n\}$ given by*

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}.$$

7.1. *Prove that $\{a_n\}$ is increasing.*

Proof. We'll show the equivalent statement that $a_{n+1} - a_n \geq 0$, $n \in \mathbb{N}$.

$$\begin{aligned} a_{n+1} - a_n &= \frac{1}{n+2} + \ldots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} - \ldots - \frac{1}{2n} \\ &= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \\ &= \frac{(2n+2)(n+1) + (2n+1)(n+1) - (2n+1)(2n+2)}{(n+1)(2n+1)(2n+2)} \\ &= \frac{2n^2 + 4n + 2 + 2n^2 + 3n + 1 - 4n^2 - 4n - 2n - 2}{(n+1)(2n+1)(2n+2)} \\ &= \frac{1}{(2n+1)(2n+2)} \geq 0 \end{aligned}$$

Thus $\{a_n\}$ is increasing. □

7.2. *Prove that $\{a_n\}$ converges.*

Proof. Here we'll use the fact that for a bounded increasing sequence must converge, but won't proof the limit. Note, for $n \in \mathbb{N}$:

$$\begin{aligned} a_n &= \frac{1}{n+1} + \ldots + \frac{1}{2n} \\ &\leq \frac{n}{n+1} \\ &\leq 1 \end{aligned}$$

So $\{a_n\}$ is bounded and increasing, and hence $\lim_{n\to\infty} a_n$ exists and is less than or equal to 1. □

8. *Let $f : [-1, +1] \to \mathbb{R}$ be continuous. Suppose that f is differentiable on $(-1, 0) \cup (0, 1)$ and that $\lim\limits_{x\to 0} f'(x) = \alpha$ for some $\alpha \in \mathbb{R}$. Show that $f'(0)$ exists and that $f'(0) = \alpha$.*

Proof. We'll use the fact that f is differentiable at c if and only if the limit $\lim\limits_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists, and that the value of that limit is the derivative, if it exists.

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x}$$

This is an indeterminate form, since $\lim\limits_{x\to 0} x = 0$ and $\lim\limits_{x\to 0} f(x) - f(0) = 0$, the last following since f is continuous on $[-1, 1]$, but f and x are differentiable on $(-1, 0) \cup (0, 1)$ so that by L'Hopital's rule we have:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} f'(x) = \alpha$$

Therefore, we have $f'(0)$ exists and is exactly α. □

9. *Consider the function*

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(x^n)}{n^2 x^n}$$

9.1. *Prove that f is continuous on $[1, \infty)$.*

Proof. We will use the fact that if a sequence of functions on $D \subseteq \mathbb{R}$ converges, all of which are continuous on $D \subseteq \mathbb{R}$, and converge uniformly to a function $f : D \to \mathbb{R}$, that f will be continuous on D.

Here we'll show that the sequence of partial sums $f_n(x) = \sum\limits_{k=1}^{n} \frac{\sin(x^k)}{k^2 x^k}$ is both continuous on $[1, \infty)$ and that it converges uniformly to f, by the Weierstrass M-Test. First, we'll use the fact that $\sin(x), x$ are continuous on $\mathbb{R}$, for $n \in \mathbb{N}$, so x^n is just the multiplication of x and hence is continuous on $\mathbb{R}$, so that they're composition: $\sin(x^n)$ is continuous on $\mathbb{R}$. Similarly, n^2 is a constant for $n \in \mathbb{N}$ and x^n is continuous, so their product is continuous, and since $n^2 x^n \neq 0$ for $x \in [1, \infty)$ and $n \in \mathbb{N}$, we have each $\frac{\sin(x^n)}{x^n n^2}$ is continuous for each $n \in \mathbb{N}$. Finally, the addition of continuous functions is continuous so that $f_n(x)$ is continuous.

Now for uniform convergence, we have for $x \in [1, \infty)$:

$$\frac{\sin(x^n)}{n^2 x^n} \leq \frac{1}{n^2}$$

$\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series, so that by the Weierstrass M-Test $(f_n(x)) \to f(x)$ uniformly on $[1, \infty)$. So by the uniform convergence theorem, f is continuous on $[1, \infty)$. $\square$

9.2. *Prove that, in fact f is continuous on $(0, \infty)$.*

Proof. First, note that the McClaurin Series of $\sin(x)$ is: $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)!}$. So that $\sin(x^n) = \sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1} (x^n)^{2k-1}}{(2k-1)!} = x^n - \frac{x^3 n}{3!} + \frac{x^{5n}}{5!} - ....$ Additionally, the radius of convergence is all of $\mathbb{R}$.

This gives us

$$\sin(x^n) = x^n \left(1 - \frac{x^2 n}{3!} + \frac{x^{4n}}{5!} - ...\right) \leq x^n \left(1 + \frac{1}{3!} + \frac{1}{5!} + ...\right),$$

on $x \in (0, 1)$. Additionally note the series $\sum\limits_{k=1}^{n} \frac{1}{(2k-1)!}$ is absolutely convergent by the ratio test:

$$\lim_{n\to\infty} \frac{(2n-1)!}{(2n+1)!} = \lim_{n\to\infty} \frac{1}{(2n+1)2n} = 0$$

So that there's some positive constant c that's the sum of this series: $c = \sum\limits_{n=1}^{\infty} \frac{1}{(2n-1)!}$. So then on $x \in (0, 1)$ we have $\sin(x^n) \leq x^n c$.

So Weierstrass M-Test baby:

$$\begin{aligned} \frac{\sin(x^n)}{x^n n^2} &\leq \frac{x^n c}{x^n n^2} \\ &= \frac{c}{n^2} \end{aligned}$$

Again $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} < \infty$, so $\sum\limits_{n=1}^{\infty} \frac{c}{n^2} < \infty$. Thus by the Weierstrass M-Test, we have that $(f_n) \to f$ uniformly, and thus f is continuous on $(0, 1)$. Hence f is continuous on $(0, \infty)$. $\square$

10. *Let $f : [0, 1] \to \mathbb{R}$ be continuous. Prove that*

$$\lim_{n\to\infty} \int_0^1 f(x) x^n \, dx = 0$$

Proof. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Then since x^n is non-negative on $[0, 1]$ for any $n \in \mathbb{N}$, we have that by the first-mean value theorem there exists a $c \in (0, 1)$ such that:

$$\int_0^1 f(x) x^n \, dx = f(c) \int_0^1 x^n \, dx = f(c) \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{f(c)}{n+1}$$

So that $\lim\limits_{n\to\infty} \int_0^1 f(x) x^n \, dx = \lim\limits_{n\to\infty} \frac{f(c)}{n+1} = 0$. For the case that $f(c) = 0$, we may use L'Hopital's with n, since $x + 1$ is differentiable with respect to x and $f(c)$ is a constant and thus differentiable with respect to x. $\lim\limits_{n\to\infty} \frac{0}{n+1} = \frac{\lim_{n\to\infty} 0}{\lim_{n\to\infty} 1} = 0$. So that either way $\lim\limits_{n\to\infty} \int_0^1 f(x) x^n \, dx = 0$. $\square$

Spring 2019

11. *Show that the set $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\} \cup \{1 + \frac{1}{n} : n \in \mathbb{Z}^+\}$ is compact using the open-cover definition of compactness.*

Proof. Let $\{G_\alpha\}_{\alpha\in I}$ be an arbitrary open cover of the set described above with the arbitrary index set I, so that for all $\alpha \in I$, G_α is open in $\mathbb{R}$. Then for some $\delta > 0$, $(-\delta, \delta) \subseteq G_\alpha$ for some $\alpha \in I$. That is there's a positive constant $\delta > 0$ that we can make arbitrarily small to fit it in to some G_α, call this set G_0. This is possible because the standard topology of $\mathbb{R}$ is made up of arbitrary unions of open intervals. So then by the Archimedean principle, there exists a $N \in \mathbb{N}$ such that $0 < \frac{1}{N} < \delta$. So for all $n \geq N$ we have $\frac{1}{n} \in (-\delta, \delta)$. So that $\{0, 1/N, 1/(N+1), ...\} \subseteq (-\delta, \delta)$. Additionally there must exist a set in the collection $\{G_\alpha\}_{\alpha\in I}$ such that $\frac{1}{N-1} \in G_\alpha$, call it G_1, similarly for $\frac{1}{N-2}, ..., \frac{1}{N-(N-1)}$. Call this sequence $G_2, ..., G_{N-1}$.

Note for $G_{N-1} \supseteq (1-\epsilon, 1+\epsilon)$, for some $\epsilon > 0$. So again by the Archimedean principle, there's a $M \in \mathbb{N}$ such that $1 < 1 + \frac{1}{M} < 1 + \epsilon$. So that $\frac{1}{m} \in (1-\epsilon, 1+\epsilon)$ for all $m \geq M$, with $m \in \mathbb{N}$. Similarly as before, then there exists $G_{N+1} \in \{G_\alpha\}_{\alpha\in I}$ such that $1 + \frac{1}{M-1} \in G_{N+1}$, so on until $1 + \frac{1}{M-(M-1)} \in G_{M+N}$. So that any open cover $\{G_\alpha\}_{\alpha\in I}$ of the set $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{1 + \frac{1}{n} | n \in \mathbb{N}\} \subseteq \bigcup_{n=1}^{M+N} G_n$ admits a finite subcover. Thus the set is compact. □

12. *Show that if (a_n) is a decreasing sequence of positive numbers and $\sum_{n=1}^{\infty} a_n$ diverges, then*

$$\lim_{n\to\infty} \frac{a_1 + a_3 + ... + a_{2n-1}}{a_2 + a_4 + ... + a_{2n}} = 1$$

Proof. Assume the hypothesis. First, note that $a_{2n-1} \leq a_1, ..., a_{2n-1} \leq a_{2n-1}$ so that $na_{2n-1} \leq a_1 + ... + a_{2n-1}$. Similarly, $a_2 \leq a_1, ..., a_{2n} \leq a_1$, then $a_2 + ... + a_{2n} \leq na_1 \implies \frac{1}{na_1} \leq \frac{1}{a_2+...+a_{2n}}$. So that $\frac{na_{2n-1}}{na_1} \leq \frac{a_1+...+a_{2n-1}}{a_2+...+a_{2n}}$. Additionally, note that $a_1 \leq a_1, ..., a_{2n-1} \leq a_1 \implies a_1 + ... + a_{2n-1} \leq na_1$ and $a_{2n} \leq a_2, ..., a_{2n} \leq a_{2n} \implies na_{2n} \leq a_2 + ... + a_{2n} \implies \frac{1}{a_2+...a_{2n}} \leq \frac{1}{na_{2n}}$. So that $\frac{a_1+...+a_{2n-1}}{a_2+...+a_{2n}} \leq \frac{a_1}{a_{2n}}$. Hence $\frac{a_{2n-1}}{a_1} \leq \frac{a_1+...+a_{2n-1}}{a_2+...+a_{2n}} \leq \frac{a_1}{a_{2n}}$. But $a_{2n} \leq a_{2n-1} \leq a_1$, so that $\frac{a_{2n}}{a_1} \leq \frac{a_{2n-1}}{a_1} \implies \frac{a_{2n-1}}{a_1} \leq 1 \leq \frac{a_1}{a_{2n-1}} \leq \frac{a_1}{a_{2n}}$. Thus $\lim_{n\to\infty} \frac{a_1+...+a_{2n-1}}{a_2+...+a_{2n}} = 1$. □

13. *Show that $\sqrt{1-x} \leq 1 - \frac{x}{2} - \frac{x^2}{8}$ for all $x \in [0, 1]$.*

Proof. First note that $\sqrt{1-x}$ is a continuous function on $[0, 1]$ because it can be constructed from sums of and compositions of $1, x, x^{1/2}$, and it's for all $x \in [0, 1)$ it's derivative exists on $[0, x)$. By Taylor's theorem, for all $x \in (0, 1)$, there exists a $\gamma \in (0, x)$ for which $\sqrt{1-x} = \frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + O(x^3) = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{\gamma^{-5/2}}{3!}x^3$. Both $-x, -x^2, -x^3$ are monotonically increasing decreasing on $[0, 1]$, hence $\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{\gamma^{-5/2}}{3!}x^3 \leq 1 - \frac{x}{2} - \frac{x^2}{8}$. □

14. *Show that the following series converges uniformly on $(r, +\infty)$ for any real number $r > 1$.*

$$\sum_{n=1}^{\infty} \frac{n \ln(1 + nx)}{x^n}$$

Proof. Let $r \in (1, \infty)$. Then note that for any $x \in (r, \infty)$ we have that the derivative of $\frac{\ln(1+nx)}{x^n}$ is negative if and only if $\frac{n}{1+nx}x^n - \frac{n\ln(1+nx)}{x^{n-1}} < 0 \iff \frac{1}{(1+nx)x} < \ln(1+nx) \iff 1 < x(1+nx)\ln(1+nx)$ which is certainly true for $x \in (r, \infty), n \in \mathbb{N}$. So then $n\frac{\ln(1+nx)}{x^n} \leq \frac{n\ln(1+nr)}{r^n}$. So we will show that the series $\sum_{n=1}^{\infty} \frac{n\ln(1+nr)}{r^n} < \infty$ and employ the Weierstrass M-Test to show that this series converges uniformly on (r, ∞).

We will show convergence through the ratio test:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{(n+1)\ln(1+(n+1)r)}{r^{n+1}} \frac{r^n}{n\ln(1+nr)} \\
&= \frac{1}{r} \lim_{n\to\infty} (1 + \frac{1}{n}) \frac{\ln(1+(1+n)r)}{\ln(1+nr)}
\end{aligned}$$

We'll focus now on the last piece of the equation above:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\ln(1+(1+n)r)}{\ln(1+nr)} \\
&\le \lim_{n\to\infty} \frac{\ln((n+1)(1+r))}{\ln(1+nr)} \\
&\le \lim_{n\to\infty} \frac{\ln(n+1)}{\ln(1+nr)} + \frac{\ln(r+1)}{\ln(1+nr)} \\
&\le \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\ln\left(r+\frac{1}{n}\right)} + 0 \\
&= 0
\end{aligned}$$

Thus $\frac{1}{r}\lim_{n\to\infty}(1+\frac{1}{n})\cdot 0 = 0$ and by the Ratio test $\sum_{n=1}^{\infty} \frac{n\ln(1+nr)}{r^n} < \infty$ for any $r \in (1,\infty)$ and by the Weierstrass M-test, $\sum_{n=1}^{\infty} \frac{n\ln(1+nx)}{x^n}$ is uniformly convergent on (r,∞) for any $r \in (1,\infty)$. □

15.

15.1. *State a definition for a real valued function $f : [a,b] \to \mathbb{R}$ to be Riemann integratable.*

Solution. *f is Reimann integratable on $[a,b]$ if there exists a $I \in \mathbb{R}$ such that for all $\epsilon > 0$ there exists a partition $P_\epsilon = (x_0, x_1, ..., x_N)$ of $[a,b]$ such that any refinement of P_ϵ, call it $Q = (x_0, ..., x_1, ..., x_N, ..., x_M)$, satisfies*

$$\left|\sum_{n=1}^{M} f(\zeta_n)(x_n - x_{n-1}) - I\right| \le \epsilon$$

for any choice of $\zeta_n \in [x_{n-1}, x_n]$.

15.2. *Let $f : [0,1] \to \mathbb{R}$ be Thomae's function, defined by*

$$f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ for relatively prime natural numbers } m \text{ and } n \\ 0, & \text{otherwise} \end{cases}.$$

Show that f is Riemann integratable.

Proof. Let $\epsilon > 0$ and $I = 0$ from the above definition. Then define the set $B = \{1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, ..., \frac{1}{N}, ..., \frac{N-1}{N}\}$. Note that by the definition of Thomae's function, if $x \notin B_N$, then $f(x) \le \frac{1}{N}$ and if $x \in B_N$, then there are at most $2N$ outputs larger than $f(x)$. Define the partition $P_\epsilon = (x_0, ..., x_N)$ such that $\frac{1}{N} < \frac{\epsilon}{2}$ and that $x_n - x_{n-1} < \frac{\epsilon}{4\#B}$. So then, we'll break the set $\{\zeta_i : i \in \{1, ..., M\}$ and $\zeta_i \in [x_{i-1}, x_i]\}$ into two distinct sets, $\{\zeta_i \in B : i \in \{1, ..., M\}$ and $\zeta_i \in [x_{i-1}, x_i]\}$ and it's complement relative to the tagged set. So that we define the set of indicies as N_1 and N_2, respectively, so that $\#N_1 + \#N_2 = \#M$. That is, for $n \in N_1$, $\zeta_n \in B$ and $n \in N_2, \zeta_n \notin B$.

$$\begin{aligned}
|\sum_{n=1}^{M} f(\zeta)(x_n - x_{n-1})| &= \sum_{n\in N_1} f(\zeta_n)(x_n - x_{n-1}) + \sum_{n\in N_2} f(\zeta_n)(x_n - x_{n-1}) \\
&\le \sum_{n\in N_1} f(\zeta_n)(x_n - x_{n-1}) + \sum_{n\in N_2} \frac{(x_n - x_{n-1})}{N} \\
&\le 1(2\#B)(x_{Max} - x_{x_{Max-1}}) + \frac{1}{N} \\
&\le \frac{2\#B\epsilon}{4\#B} + \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

Therefore Thomae's function is Riemann integratable on $[0,1]$ with integral equal to 0. □

Spring 2018

16. *Prove that the series* $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ *converges by showing that the sequence of partial sums is Cauchy.*

Proof. First, playing around with geometric sums. Note that the geometric partial sums, for $|x| \in (0,1)$, are given as: $\sum_{n=1}^{n} x^{k-1} = \frac{1-x^{n+1}}{1-x}$. So that we get the following partial sums for $|x| \in (0,1)$:

$$\begin{aligned}
\sum_{k=1}^{n} x^k &= \frac{x - x^{n+2}}{1-x} \\
\sum_{k=1}^{n} kx^{k-1} &= \frac{1-(n+2)x^{n+1}}{1-x} + \frac{x-x^{n+2}}{1-x} \\
\sum_{k=1}^{n} kx^{k} &= \frac{x-(n+2)x^{n+2}}{1-x} + \frac{x^2-x^{n+3}}{1-x} \\
\sum_{k=1}^{n} k^2x^{k-1} &= \frac{1-(n+2)^2x^{n+1}}{1-x} + \frac{x-(n+2)x^{n+2}}{(1-x)^2} + \frac{2x-(n+3)x^{n+2}}{(1-x)^2} + \frac{2x^2-2x^{n+3}}{(1-x)^3} \\
\sum_{k=1}^{n} k^2x^{k} &= \frac{x-(n+2)^2x^{n+2}}{1-x} + \frac{x^2-(n+2)x^{n+3}}{(1-x)^2} + \frac{2x^2-(n+3)x^{n+2}}{(1-x)^2} + \frac{2x^3-2x^{n+4}}{(1-x)^3}.
\end{aligned}$$

Next we'll use the final piece in the above, to get:

$$\sum_{k=1}^{n} k^2x^k \leq \frac{x}{1-x} + \frac{x^2}{(1-x)^2} + \frac{2x^2}{(1-x)^2} + \frac{2x^3}{(1-x)^3}$$

Now the proof. Let $\epsilon > 0$ and choose $N(\epsilon) \in \mathbb{N}$ to be an natural number. Suppose $m \geq n \geq N(\epsilon)$, then:

$$\begin{aligned}
\left|\sum_{k=1}^{m} k^2\frac{1}{3^k} - \sum_{k=1}^{n} k^2\frac{1}{3^k}\right| &= \left|\sum_{k=1}^{m} \frac{k^2}{3^k} - \frac{3}{2} + \frac{3}{2} - \sum_{k=1}^{n} \frac{k^2}{3^k}\right| \\
&\leq \left|\sum_{k=1}^{m} \frac{k^2}{3^k} - \frac{3}{2}\right| + \left|\sum_{k=1}^{n} \frac{k^2}{3^k} - \frac{3}{2}\right| \\
&\leq |\frac{3}{2} - \frac{3}{2}| + |\frac{3}{2} - \frac{3}{2}| \\
&= 0 < \epsilon.
\end{aligned}$$

Thus the series $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ is Cauchy. □

17.

17.1. *Let* (f_n) *be a sequence of functions defined on* $A \subseteq \mathbb{R}$ *that converges uniformly on* A *to a function* f*. Prove that if each* f_n *is continuous at* $c \in A$*,then* f *is continuous at* c*.*

Proof. Let (f_n) be a sequence of functions, such that $f_n : A \subseteq \mathbb{R} \to \mathbb{R}$, converge uniformly to f on A. So that for all $\epsilon > 0$, there exists a $N(\epsilon) \in \mathbb{N}$ such that if $n \geq N(\epsilon)$ and $x \in A$, then $|f_n(x) - f(x)| < \epsilon$. Additionally, suppose that f_n is continuous at $c \in A$. So that for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f_n(x) - f_n(c)| < \epsilon$ for any $n \in \mathbb{N}$.

Now let $\epsilon > 0$. Choose $N(\epsilon)$, such that if $n \geq N(\epsilon)$ and $x \in A$, then $|f_n(x) - f(x)| < \frac{\epsilon}{3}$. Now fix n to be any $n \geq N(\epsilon)$. Choose $\delta(\epsilon) > 0$ be such that $|f_n(x) - f_n(c)| < \frac{\epsilon}{3}$. Suppose $|x - c| < \delta(\epsilon)$ Then:

$$\begin{aligned}
|f(x) - f(c)| &= |f(x) - f_n(x) + f_n(x) + f_n(c) - f_n(c) - f(c)| \\
&\leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)| \\
&\leq \frac{\epsilon}{3}(\text{uniform convergence}) + \frac{\epsilon}{3}(\text{continuity of } f_n \text{ at } c) + \frac{\epsilon}{3}(\text{uniform convergence}) \\
&= \epsilon
\end{aligned}$$

Thus f is continuous at $c \in A$. □

17.2. *Give an example to show that the result above is false if we only assume that* (f_n) *converges pointwise to* f *on* A.

Solution. *Consider the function* $f_n : [0,1] \to \mathbb{R}$ *given by* $f_n : x \mapsto x^n$, *for* $n \in \mathbb{N}$. *I'll show that this function converges pointwise to the piecewise function* $f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$, *that each* f_n *is continuous on* $[0,1]$, *but that* f *is discontinuous at* $x = 1$.

Pointwise Convergence *Let* $\epsilon > 0$ *and choose* $N(x, \epsilon)$ *such that* $x^{N(\epsilon,x)} < \epsilon$. *Suppose* $x \in [0,1)$ *and* $n \geq N(\epsilon, x)$, *then*

$$\begin{aligned} |f(x) - f_n(x)| &\leq |x^n| \\ &\leq |x^{N(\epsilon)}| < \epsilon. \end{aligned}$$

Finally for $x = 1$:

$$|1 - f_n(1)| = 0 < \epsilon$$

Thus $(f_n) \to f$, *pointwise, on* $[0,1]$.

Continuity of f_n *Note that the function* $f(x) = x$ *is continuous on* $[0,1]$ *and multiplication of continuous functions preserves continuity, so that* x^n *is continuous on* $[0,1]$.

Discontinuity of f ***at*** $x = 1$ *We'll show this by giving a sequence that converges to* 1 *on* $[0,1]$, *but whose image doesn't converge to* $f(1)$. *Consider the sequence* $(a_n) = (1 - \frac{1}{n}) \to 1$, *but under* f, $f(a_n) = f\left(1 - \frac{1}{n}\right) = 0$, *for all* $n \in \mathbb{N}$. *So that* $(a_n) \to 1$ *but* $(f(a_n)) \not\to f(1)$.

18. *Consider the function*

$$f(x) = \sum_{k=1}^{\infty} \frac{x}{k(1 + kx^2)}$$

18.1. *Fix* $\epsilon > 0$. *Prove that* f *is continuous for* $|x| \geq \epsilon$.

Proof. Note that by the previous problem, that if $(f_n) \to f$ uniformly, and f_n is continuous on a set $A \subseteq \mathbb{R}$, then f is continuous on A. Here I'll employ this by showing that the series converges uniformly and that each partial sum is continuous on $|x| \geq \epsilon$, so that f is continuous on $|x| \geq \epsilon$.

First, suppose $\epsilon \leq |x|$. Note that the partial sums of the series: $\sum_{k=1}^{n} \frac{x}{k+k^2x^2}$ are continuous on $|x| \geq \epsilon$, since x is a continuous function, $k + k^2x^2$ is a continuous function for any given $k \in \{1, ..., n\}$ and never attains the value of 0. So since the sum of and division of such functions results in continuous functions each s_n is continuous on all of $\mathbb{R}$.

To show that the series converges uniformly, we'll employ the Weierstrass M-Test:

$$\begin{aligned} \frac{x}{k + k^2x^2} &\leq \frac{|x|}{k + k^2x^2} \\ &\leq \frac{|x|}{k^2x^2} \\ &\leq \frac{1}{k^2|x|} \\ &\leq \frac{1}{k^2\epsilon} \end{aligned}$$

We know $\frac{1}{\epsilon} \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges since it's a p-series with $p > 1$. Hence $\sum_{k=1} \frac{x}{k(1+kx^2)}$ converges uniformly on $|x| \geq \epsilon$. And thus by the previous problem, f is continuous on $|x| \geq \epsilon$. □

18.2. *Prove that, in fact,* f *is continuous on* $\mathbb{R}$.

Proof. First, note that for any $x \in \mathbb{R} \setminus \{0\}$ we can choose an ϵ satisfying the previous condition. The only such number we cannot do this for is $x = 0$. So we'll show that f_n is continuous on $\{0\}$, that it converges uniformly on this set, so that f is continuous at $x = 0$.

Note that $f_n(0) = 0$ for all $n \in \mathbb{N}$, so that f_n is continuous on $\{0\}$. Moreover $\frac{x}{k(1+kx^2)} \leq 0$ for all $k \in \mathbb{N}$ and $x \in \{0\}$, and since $\sum_{n=1}^{\infty} 0 = 0$, we have by the Weierstrass M-Test, the unsuprising result that f_n converges uniformly to $f = 0$ on $\{0\}$. Hence by the previous result f is continuous at $x = 0$, conjoined with the previous problem this gives us that f is continuous on $\mathbb{R}$. □

19. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $|f'(x)| < 1$ for all $x \in \mathbb{R}$.*

19.1. *Prove that f has at most one fixed point.*

Proof. Assume that $f : \mathbb{R} \to \mathbb{R}$ is differentiable with $|f'(x)| < 1$ for all $x \in \mathbb{R}$. For sake of contradiction, we'll suppose there exists $c, d \in \mathbb{R}$ such that $f(c) = c$ and $f(d) = d$, with $d \neq c$, and without loss of generality assume that $d > c$. We then have, $f(d) - f(c) = d - c$ so that $\frac{f(d)-f(c)}{d-c} = 1$. But by the mean value theorem we have that there exists a $z \in (c, d)$ such that $f(d) - f(c) = f'(z)(d - c) \implies f'(z) = \frac{f(d)-f(c)}{d-c} = 1$. A contradiction of our hypothesis that $|f'(x)| < 1$ for all $x \in \mathbb{R}$. □

19.2. *Show that the following function satisfies $|f'(x)| < 1$ for all $x \in \mathbb{R}$ but has no fixed points*

$$f(x) = \ln(1 + e^x)$$

Solution. *Taking the derivative of $f(x)$:*

$$f'(x) = \frac{e^x}{1 + e^x} = \frac{1}{1 + e^{-x}}$$

We have $e^{-x} > 0$ for all $x \in \mathbb{R}$, hence: $1 < 1 + e^{-x} \implies \frac{1}{1+e^{-x}} < 1$ for all $x \in \mathbb{R}$.

To show there are no fixed points of the function, we'll assume there is at least 1 and show this leads to a contradiction. Let $f(x) = x$ for some $x \in \mathbb{R}$, so that

$$\ln(1 + e^x) = x$$

So we have $e^x = 1 + e^x \implies 1 = 0$, a beautiful contradiction. Thus f has no fixed points on $\mathbb{R}$.

20.

20.1. *State the definition for $f : [a, b] \to \mathbb{R}$ to be Riemann integratable on $[a, b]$.*

Solution. *Define the lower (upper) Reimann sum to be defined on a partition $P = (x_0 = a, ..., x_n = b)$ such that the lower Reimann sum is: $L(P; f, x) = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$ with $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$. The upper Reimann sum: $U(P; f, x) = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$ with $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$. So we define the lower Riemann integral as $(L) \int_a^b f\ dx = \sup\{L(P; f, x) : P$ is any partition of $[a, b]\}$ and $(U) \int_a^b f\ dx = \inf\{U(P; f, x) : P$ is any partition of $[a, b]\}$.*

f is Riemann integratable if and only if $(U) \int_a^b f\ dx = (L) \int_a^b f\ dx = \int_a^b f\ dx$ with the latter being the integral of f on $[a, b]$.

20.2. *Use your definition from (a) to prove that if $f : [a, b] \to \mathbb{R}$ is continuous and $\int_a^b |f(x)|\ dx = 0$, then $f(x) = 0$ for all $x \in [a, b]$.*

Proof. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $\int_a^b |f(x)|\ dx = 0$.

First, note that f being continuous on $[a, b]$ implies $|f|$ is continuous on $[a, b]$. Moreover, by the Heine-Borel theorem and the fact that continuous functions send compact sets to compact sets, we have that $|f|$ is bounded on the interval $[a, b]$. So that for any partition $P = (x_0 = a, ..., x_N = b)$, suc that $a = x_0 < x_1 < ... < x_n = b$, we'll have for the function $|f|$:

$$\begin{aligned} 0 &\leq L(P; f, x) \\ &= \sum_{k=1} m_k(x_k - x_{k-1}) \\ &\leq U(P; f, x) \\ &= \sum_{k=1} M_k(x_k - x_{k-1}) \\ &\leq 0 \end{aligned}$$

Following from the fact that $x_k - x_{k-1} > 0$ and $|f| \geq 0$. This gives us $\sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = 0$, so that $\sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) = 0$. Since $(x_k - x_{k-1}) \neq 0$ by the definition of our partition, we have that for each $k \in \{1, ..., n\}$, $M_k - m_k = 0 \implies M_k = m_k$. And since $\sum\limits_{k=1}^{n} M_k(x_k - x_{k-1}) = 0$, with all the terms positive and $x_k \neq x_{k-1}$ for all $k \in \{1, ..., n\}$ we have each

$M_k = 0 = m_k$ for all k. So $|f(x)| = 0$ on all $x \in [x_{k-1}, x_k]$ for all $k \in \{1, ..., n\}$, implying that $|f(x)| = 0$ on $[a, b]$. This gives us:

$$\begin{aligned} |f(x)| &= 0 \\ \sqrt{(f(x))^2} &= 0 \\ (f(x))^2 &= 0 \\ f(x) &= 0 \end{aligned}$$

for all $x \in [a, b]$. □

Spring 2010

21. *Let K be a compact subset of $\mathbb{R}$ and let f be a continuous function from K into $\mathbb{R}$. Use the open cover definition of compactness to show that $f(K)$ is compact.*

Proof. Let $K \subseteq \mathbb{R}$ be compact and $f : K \to \mathbb{R}$.

Then for any open cover of K, $\{O_\alpha\}_{\alpha \in I}$ for an arbitrary index set I, there exists a finite subcover of K.

So let $\bigcup_{\alpha \in I} O_\alpha$ be an open cover of $f(K)$. Then $f(K) \subseteq \bigcup_{\alpha \in I} O_{\alpha \in I}$. Then we'll show $K \subseteq f^{-1}\left(\bigcup_{\alpha \in I} O_\alpha\right)$. Let $x \in K$ so that $f(x) \in f(K)$, hence $f(x) \in \bigcup_{\alpha \in I} O_\alpha$. So that there exists a O_α such that $f(x) \in O_\alpha$. Thus $x \in f^{-1}(O_\alpha) \subseteq f^{-1}(\bigcup_{\alpha \in I} O_\alpha)$. Hence $K \subseteq f^{-1}(\bigcup_{\alpha \in I} O_\alpha)$.

Note $\bigcup_{\alpha \in I} O_\alpha$ is an open set in $\mathbb{R}$, because it's the union of open sets O_α. So that since f is continuous $f^{-1}(\bigcup_{\alpha \in I} O_\alpha)$ is open in $\mathbb{R}$. Furthermore, we'll show that $f^{-1}(\bigcup_{\alpha \in I} O_\alpha) \subseteq \bigcup_{\alpha \in I} f^{-1}(O_\alpha)$. Take $y \in f^{-1}(\bigcup_{\alpha \in I} O_\alpha)$. Then there exists a $x \in \bigcup_{\alpha \in I} O_\alpha$ such that $y = f(x)$. But there must exists a O_α for some $\alpha \in I$ such that $x \in O_\alpha$. So that $y \in f^{-1}(O_\alpha)$ for some $\alpha \in I$, so we have our result that $y \in \bigcup_{\alpha \in I} f^{-1}(O_\alpha)$ and that $f^{-1}(\bigcup_{\alpha \in I} O_\alpha) \subseteq \bigcup_{\alpha \in I} f^{-1}(O_\alpha)$. So that $K \subseteq \bigcup_{\alpha \in I} f^{-1}(O_\alpha)$ is an open cover of K. So that there must exist a finite subcover of K, call it $\{O_{\alpha_k}\}_{k=1}^n$, such that $K \subseteq \bigcup\limits_{k=1}^n O_{\alpha_k}$. We'll show that $f(K) \subseteq f(\bigcup\limits_{k=1}^n O_{\alpha_n}) \subseteq \bigcup\limits_{k=1}^n f(O_{\alpha_n})$. Take $y \in f(K)$, then there exists a $x \in K \subseteq \bigcup_{k=1}^n O_{\alpha_k}$, hence $y \in f(\bigcup\limits_{k=1}^n O_{\alpha_k})$. So $x \in \bigcup\limits_{k=1}^n O_{\alpha_k}$, so that there exists a O_{α_k} such that $x \in O_{\alpha_k}$. Hence $y = f(x) \in \bigcup\limits_{k=1}^n O_{\alpha_k}$. And we have $f(K) \subseteq \bigcup\limits_{k=1}^n O_{\alpha_k}$ and we have $f(K)$ is compact in $\mathbb{R}$. □

22. *Let $\{a_n\}$ be a sequence of real numbers which converges to a. Prove*

$$\lim_{n\to\infty} \frac{a_1 + ... + a_n}{n} = a.$$

Hint: *If $1 \le j \le n$ and $b_1, b_2, ..., b_n \in \mathbb{R}$, then $b_1 + b_2 + ... + b_n = (b_1 + ... + b_j) + (b_{j+1} + ... + b_n)$.*

Proof. Let $\{a\}$ be a sequence of real numbers converging to a.

Let $\epsilon > 0$ be given. Choose $N(\epsilon) \in N$ such that $|a_n - a| < \epsilon/2$ whenever $n \ge N(\epsilon)$.

Suppose $n \ge N(\epsilon)$, and let $b = \sup\{|a_n - a| : 1 \le n \le N(\epsilon)\}$, then

$$\begin{aligned} \left|\frac{a_1 + ... + a_n}{n} - a\right| &\le \frac{1}{n}|(a_1 - a) + ... + (a_n - a)| \\ &\le \frac{1}{n}|(a_1 - a) + ... + (a_{N(\epsilon)} - a) + ... + (a_n - a)| \\ &\le \frac{|a_1 - a| + ... + |a_{N(\epsilon)} - a|}{n} + \frac{|a_{N(\epsilon)+1} - a| + ... + |a_n - a|}{n} \\ &\le \frac{N(\epsilon)b}{n} + \frac{(n - N(\epsilon))\frac{\epsilon}{2}}{n}. \end{aligned}$$

For large values of $n \ge N(\epsilon)$ we can impose $\frac{N(\epsilon)}{n} < \frac{\epsilon}{2b}$, so that $|\frac{a_1+...+a_n}{n} - a| < \epsilon$. Hence $\lim\limits_{n\to\infty} \frac{a_1+...+a_n}{n} = a$, where $\lim\limits_{n\to\infty} a_n = a$. □

23. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function and differentiable on (a, b), with $f'(x) > 0$ for all $x \in (a, b)$.*

23.1. *Show f has an inverse defined on $[f(a), f(b)]$.*

Proof. We'll show that f is invertible on $[a, b]$ so that the inverse has domain $[f(a), f(b)]$. Let $f(x) = f(y)$ for some $x, y \in [a, b]$. Without loss of generality, assume $x \leq y$. Then since f is continuous on $[x, y] \subseteq [a, b]$ and differentiable on (x, y), so by Rolle's theorem there exists a $c \in (x, y)$ such that $f'(c) = 0$, a contradiction since $f' > 0$ on (a, b). So that $f(x) \neq f(y)$ for all $x, y \in [a, b]$ hence f is 1-1 on $[a, b]$. Moreover, f will always be onto $f([a, b])$. So that f is invertible on $[a, b] \to [f(a), f(b)]$. □

23.2. *Prove that f^{-1} is differentiable on $(f(a), f(b))$ and $(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$.*

Proof. Let $f : [a, b] \to \mathbb{R}$ be continuous, and differentiable on (a, b), with $f'(x) > 0$ for all $x \in (a, b)$. Then we have $f^{-1}(y)$ exists on $[f(a), f(b)]$. So show it's differentiable on all $c \in (f(a), f(b))$ we'll show that $\lim\limits_{y \to c} \frac{f^{-1}(y) - f^{-1}(c)}{y - c}$ exists for all $c \in (f(a), f(b))$ and is equal to $\frac{1}{f'(f^{-1}(y))}$. First note that

$$\frac{1}{f'(f^{-1}(c))} = \frac{1}{\lim_{f^{-1}(y) \to f^{-1}(c)} \frac{f(f^{-1}(y)) - f(f^{-1}(c))}{f^{-1}(y) - f^{-1}(c)}} = \frac{1}{\lim_{x \to d} \frac{f(x) - f(d)}{x - d}}$$

This is guranteed to exist since $f'(x) > 0$ for all $x \in (a, b)$ and $f'(x)$ is differentiable on (a, b). Then:

$$\begin{aligned}
(f^{-1})'(c) &= \lim_{f^{-1}(y) \to f^{-1}(c)} \frac{f^{-1}(y) - f^{-1}(c))}{y - c} \\
&= \lim_{x \to d} \frac{x - d}{f(x) - f(d)} \\
&= \frac{1}{\lim_{x \to d} \frac{f(x) - f(d)}{x - d}} \\
&= \frac{1}{f'(f^{-1}(y))}
\end{aligned}$$

The result that we desired. □

24.

24.1. *State the definition for a function f to be Riemann integratable on $[0, 1]$.*

Solution. *Let $P = (x_0 = a, ..., x_n = b)$ be a partition of $[a, b]$, the upper/lower Riemann sums are defined as $U(P; f, x) = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$ and $L(P; f, x) = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$, respectively, where $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$ and $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$. Then we define the Riemann integral in terms of the upper/lower Riemann integrals:*

$$(U) \int_a^b f \ dx = \inf\{U(P; f, x) : P \text{ is any partition of } [a, b]\}$$

and

$$(L) \int_a^b f \ dx = \sup\{L(P; f, x) : P \text{ is any partition of } [a, b]\}$$

So that f is Riemann integratable if and only if $(U) \int\limits_a^b f \ dx = (L) \int\limits_a^b f \ dx$ with the integral value being that shared value.

24.2. *Let*

$$f(x) = \begin{cases} -1, & \text{if } x \in \mathbb{Q} \\ 1, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Use the definition stated in part (a) to prove f isn't Riemann integratable on $[0, 1]$.

Proof. Let $P = (x_0 = 0, ..., x_N = 1)$ be any partition of $[0, 1]$. Then since $\mathbb{Q}$ is dense in $\mathbb{R}$ as well as $\mathbb{R} \setminus \mathbb{Q}$ we have that $M_k = 1$ and $m_k = 0$ for all $k \in \{1, ..., N\}$, so that $L(P; f, x) = \sum_{k=1}^{N} m_k(x_k - x_{k-1}) = 0$ and $U(P; f, x) \sum_{k=1}^{N} M_k(x_k - x_{k-1}) = 1$, for any partition. Thus we have $(L) \int_0^1 f \ dx = 0$ and $(U) \int_0^1 f \ dx = 1$, so that f is not Riemann integratable on $[0, 1]$. □

24.3. *Let*

$$g(x) = \begin{cases} x, & \text{if } 0 \le x < 1 \\ x+1, & \text{if } 1 \le x \le 2 \end{cases}$$

Use the definition in part (a) to prove that g is Riemann integratable on $[0,2]$.

Proof. Let g be the function defined above.

Consider the partition of $[a,b]$, $(0, \frac{1}{N}, ..., 1, \frac{N+1}{N}, ..., 2)$. So that $L(P;f,x) = \sum\limits_{k=1}^{2N} m_k(x_k - x_{k-1}) = \frac{1}{N}\sum\limits_{k=1}^{2N} m_k = \frac{1}{N}\sum\limits_{k=1}^{N} \frac{k-1}{N} + \frac{1}{N}\sum\limits_{k=N+1}^{2N}(1+\frac{k-1}{N}) = \frac{N-1}{N} + \frac{1}{N}\sum_{k=1}^{2N}\frac{k-1}{N} = 1 - \frac{1}{N} + \frac{2N(2N-1)}{2N^2} = 1 - \frac{1}{N} + \frac{2N-1}{N} = 1 - \frac{1}{N} + 2 - \frac{1}{N}$. Taking the limit as $N \to \infty$ we have that $3 \le (L)\int_0^2 f\ dx$. The upper sum:

$$\begin{aligned} U(P;f,x) &= \frac{1}{N}\sum_{k=1}^{N-1}\frac{k}{N} + \frac{1}{N}\sum_{k=N}^{2N}(1+\frac{k}{N}) \\ &= \frac{N}{N} + \frac{1}{N}\sum_{k=1}^{2N}\frac{k}{N} \\ &= 1 + \frac{2N(2N+1)}{2N^2} \\ &= 1 + 2 + \frac{1}{N} \end{aligned}$$

Taking $N \to \infty$ we have that $(U)\int_0^2 f\ dx \le 3$. Thus

$$(U)\int_0^2 f\ dx = (L)\int_0^2 f\ dx$$

and f is Riemann integratable on $[0,2]$. □

25.

25.1. *Let $\{f_k\}$ be a sequence of real-valued functions defined on a set $E \subseteq \mathbb{R}$. State the definition for the series $\sum\limits_{k=0}^{\infty} f_k(x)$ to converges uniformly on E.*

Solution. $(\sum_{k=1}^{n} f_k) \to f$ *uniformly on D if and only if for all $\epsilon > 0$, there exists a $N(\epsilon) \in \mathbb{N}$ such that if $x \in D$ and $n \ge N(\epsilon)$, then $|\sum_{k=1}^{n} f_k(x) - f(x)| < \epsilon$.*

25.2. *Use the definition stated in (a) to prove that $\sum\limits_{k=0}^{\infty} x^k$ converges uniformly on any closed interval $[a,b]$ such that $[a,b] \subset (-1,1)$.*

Proof. Let $\epsilon > 0$ be given. Let $[a,b] \subset (-1,1)$ Choose $N(\epsilon) \in \mathbb{N}$ such that $|b^{N(\epsilon)+1}| < (1-a)\epsilon$. Suppose $n \ge N(\epsilon)$ and $x \in [a,b]$, then

$$\begin{aligned} |f_n(x) - f(x)| &= |\sum_{k=0}^{n} x^k - \frac{1}{1-x}| \\ &\le |\frac{1-x^{n+1}}{1-x} - \frac{1}{x}| \\ &\le |\frac{-x^{n+1}}{1-x}| \\ &\le |\frac{b^{n+1}}{1-x}| \\ &\le |\frac{b^{n+1}}{1-a}| \\ &\le |\frac{(1-a)\epsilon}{1-a}| < \epsilon \end{aligned}$$

Thus $\sum\limits_{k=0}^{\infty} x^k = \frac{1}{1-x}$ uniformly on $[a,b] \subset (-1,1)$. □

25.3. *Use the definition stated in (a) to prove that* $\sum_{k=0}^{\infty} x^k$ *doesn't converge uniformly on* $(-1, 1)$.

Proof. Let $0 < \epsilon_0 < 1$ be fixed, then we'll show that the series doesn't converge uniformly on $(\frac{1}{1-\epsilon_0}, 1)$. Let $N(\epsilon) \in \mathbb{N}$, $x \in (\frac{1}{1-\epsilon_0}, 1)$, and $n \geq N(\epsilon)$. Then:

$$
\begin{aligned}
|\sum_{k=0}^{n} x^k - \frac{1}{1-x}| &= |\frac{1-x^{n+1}}{1-x} - \frac{1}{1-x}| \\
&= \frac{x^{n+1}}{1-x} \\
&\geq \frac{\frac{1}{(1-\epsilon_0)^{n+1}}}{1 - \frac{1}{1-\epsilon_0}} \\
&\geq \frac{1}{(1-\epsilon_0)^{n+1}} \frac{1}{\epsilon_0}
\end{aligned}
$$

Note $1 \geq \epsilon_0(1-\epsilon_0)^{n+1}$, for $n \in \mathbb{N}$ and so $\frac{1}{(1-\epsilon_0)^{n+1}} \geq \epsilon_0$. So that:

$$
|\frac{x^{n+1}}{1-x}| \geq 1
$$

for $x \in (\frac{1}{1-\epsilon_0}, 1)$.

Hence $\sum_{k=1}^{\infty} x^k$ doesn't converge uniformly on $(\frac{1}{1-\epsilon_0}, 1)$ for a fixed $0 < \epsilon_0 < 1$, and thus doesn't converge uniformly on $(-1, 1)$. $\square$