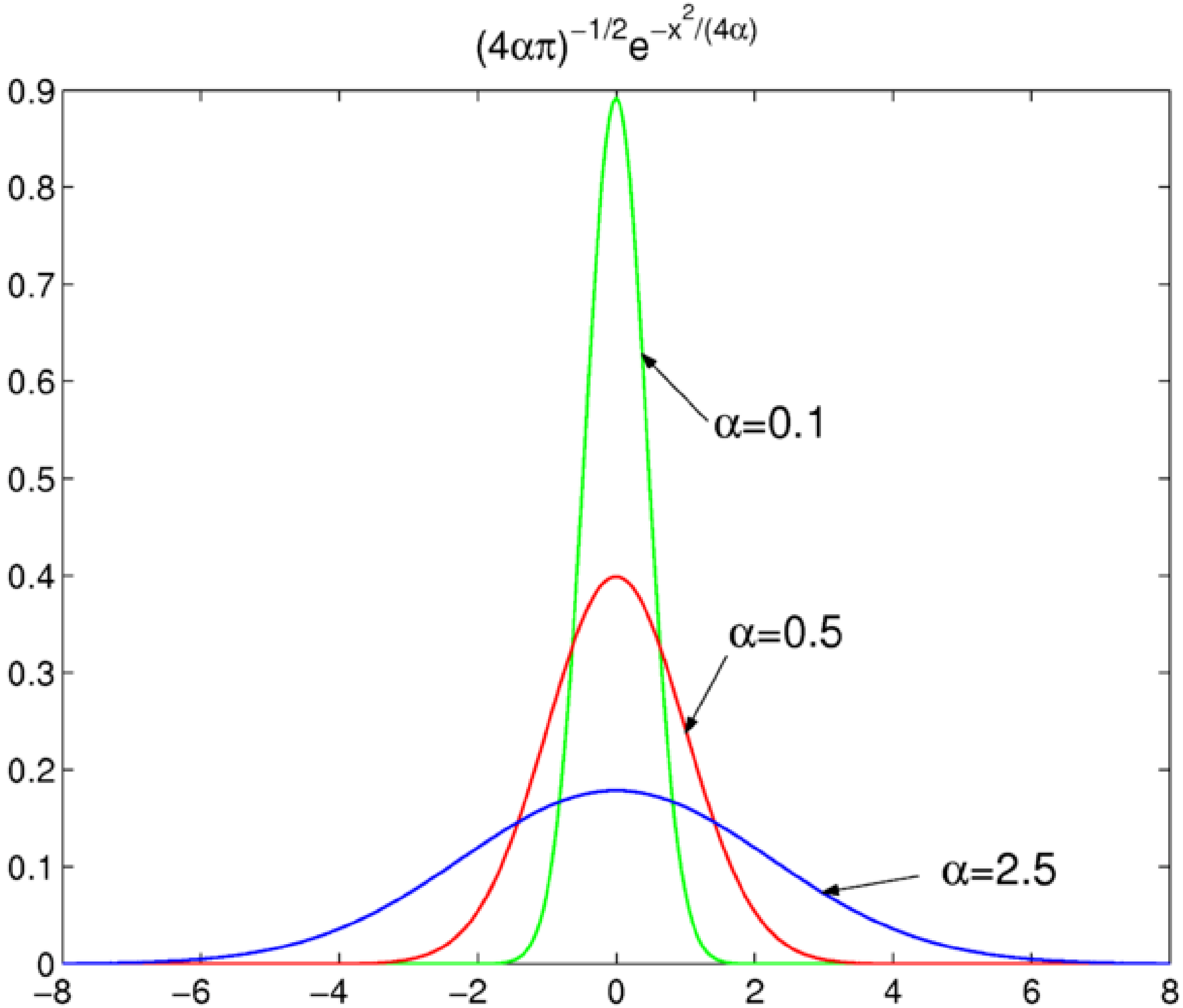

Theorem 8.2.3 (Cauchy Criterion)

A sequence $\{f_n(x)\}$ of real-valued functions defined on a set E converges uniformly on E iff for every $\epsilon > 0$, there exists an integer

$$|f_n(x) - f_m(x)| < \epsilon \text{ , for all } x \in E \text{ and all } n, m \geq n_0.$$

Proof. Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of real-valued functions over a set E (such that $E \subseteq \mathbb{R}$).
($\implies$) Suppose that $\{f_n(x)\}_{n=1}^{\infty}$ is a uniformly convergent on E.
Then let $\epsilon > 0$.

So, since $\{f_n(x)\}_{n=1}^{\infty}$ is a convergent sequence, if we choose our epsilon to be $\frac{\epsilon}{2}$, then there exists a n_0 such that:

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}$$

holds for all $n \geq n_0$.

Thus if we have a $m, n \geq n_0$, by the triangle inequality we have that:

$$|f_n(x) - f(x) + f(x) - f_m(x)| = |f_n(x) - f_m(x)|$$

$$\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| = |f_n(x) - f(x)| + |f_m(x) - f(x)|$$

Since $m \geq n_0$ we have:

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus we have shown our theorem holds in the forward direction.

$(\Longleftarrow)$

Suppose that for all $\epsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \epsilon \text{ , for all } x \in E \text{ and all } n, m \geq n_0.$$

Note that by this definition, that the sequence $\{f_n(x)\}_{n=1}^{\infty}$ forms a Cauchy sequence for all $x \in E$. Hence, by Theorem 2.6.4, we know that

$$\lim_{n \to \infty} f_n(x) = f(x)$$

for every $x \in E$.

Now we will show that the sequence converges uniformly.

Let $\epsilon > 0$ be given. Then, by our hypothesis, we have that there exists $n_0 \in \mathbb{N}$ such that for all $x \in E$ and $m, n \geq n_0$ (we fix m):

$$|f(x) - f_m(x)| = |\lim_{n \to \infty} f_n(x) - f_m(x)|$$

$$\lim_{n \to \infty} |f_n(x) - f_m(x)| \leq \epsilon \text{ for all } x \in E$$

Since this holds for all $m \geq n_0$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ converges uniformly to $f(x)$ on E.

$\square$