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Portfolio's #1 and 2

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Lesson 2

#2

Suppose the rod has a constant internal heat source, so that the basic equation describing the heat flow within the rod is

$$u_t = \alpha^2 u_{xx} + 1 \qquad 0 < x < 1$$

Suppose we fix the boundaries' temperatures by $u(0,t) = 0$ and $u(1,t) = 1$. What is the steady-state temperature of the rod? In other words, does the temperature $u(x,t)$ converge to a constant temperature $U(x)$ independent of time?

HINT Set $u_t = 0$. It would be useful to graph this temperature. Start with an initial temperature of zero and draw some temperature profiles.

Solution:

First, we assume that $u_t = 0$, by doing so we are saying that the temperature is not changing with respect to time. Hence the temperature has reached a steady-state. Then by introducing that into our equation we get:

$$0 = \alpha^2 u_{xx} + 1 \qquad 0 < x < 1$$

$$\iff -1 = \alpha^2 u_{xx} \qquad 0 < x < 1$$

Now we have a simple ODE that only is dependent on x. Hence we can solve it in a straightforward manner:

$$-1 = \alpha^2 u_{xx} \iff \int -1\ dx = \int (\alpha^2 u_{xx}) dx$$

$$\iff -x + A = \alpha^2\ u_x, \text{ where A is an arbitrary constant} \iff \int (-x + A)\ dx = \int (\alpha^2 u_x)\ dx$$

$$\iff \frac{-x^2}{2} + Ax + B = \alpha^2 u$$

$$\iff u = \frac{1}{\alpha^2}\left(\frac{-x^2}{2} + Ax + B\right), \text{ where B is some arbitrary constant}$$

Next, introducing our boundary conditions $u(0,t) = 0$ and $u(1,t) = 1$, we can solve for our arbitrary constants:

First, consider when $u(0,t) = 0$:

$$u(x,t) = \frac{1}{\alpha^2}\left(\frac{-x^2}{2} + Ax + B\right) \implies u(0,t) = \frac{1}{\alpha^2}\left(\frac{0}{2} + A(0) + B\right) \implies 0 = \frac{1}{\alpha^2}B$$

$$\implies B = 0$$

Next, consider when $u(1,t) = 1$:

$$u(x,t) = \frac{1}{\alpha^2}\left(\frac{-x^2}{2} + Ax\right) \implies u(1,t) = \frac{1}{\alpha^2}\left(\frac{-1}{2} + A\right) \implies 1 = \frac{1}{\alpha^2}\left(\frac{-1}{2} + A\right)$$

$$\alpha^2 + \frac{1}{2} = A$$

Now we have the equation $U(x) = \frac{1}{\alpha^2}\left(\frac{-x^2}{2} + (\alpha^2 + \frac{1}{2})x\right) = \frac{-x^2}{2\alpha^2} + x + \frac{x}{2\alpha^2} = \frac{-1}{2\alpha^2}(x^2 - x) + x$. Thus, we have the equation we wanted for our given IBVP.

Intuitively, this makes sense, since this ends up being roughly a straight line between our boundary conditions. Which makes sense for our steady-state.

Lesson 3

#4

Suppose a metal rod laterally insulated has an initial temperature of 20^oC but immediately thereafter has one end fixed at 50^oC. The rest of the rod is immersed in a liquid solution of temperature 30^oC. What would be the IBVP that describes this problem?

Solution: So the governing PDE here is still the basic heat equation $u_t = \alpha^2 u_{xx}$, where α^2 is some diffusivity constant determined by the material of our metal rod of length L. Then we have an initial condition of

$$u(0, x) = 20[^oC] \qquad 0 < x < L.$$

This results from the fact that the rod starts off at 20^oC. Next we have a boundary condition on one end of the rod as

$$u(L, t) = 50[^oC] \qquad 0 < t < \infty.$$

This results from one end of the rod being fixed at 50^oC, note that we arbitrarily goes the end of the rod to be the one fixed at that temperature. Finally, we note that since the rod is laterally insulated the only end feeling the effects of the liquid solution is at $x = 0$.

Hence we get the final boundary condition of:

$$u_x(0, t) = \lambda[u(0, t) - 30[^oC]] \qquad 0 < t < \infty$$

Where $\lambda = \frac{h}{k}$, for the metal dependent parameters h (heat-exchange coefficient) and k (thermal conductivity).

So we get the IVBP formulated as

$$\begin{cases} \text{PDE} & u_t = \alpha^2 u_{xx} \\ \text{IC} & u(0, x) = 20 \quad 0 < x < L \\ \text{BCs} & u_x(0, t) = \lambda[u(0, t) - 30] \qquad 0 < t < \infty \\ & u(L, t) = 50 \end{cases}$$

Lesson 4

#3

Derive the heat equation

$$u_t = \frac{1}{c\rho}\frac{\partial}{\partial x}[k(x)u_x] + f(x,t)$$

for the situation where the thermal conductivity $k(x)$ depends on x.

Proof. Starting off from the heat conservation law:

Net change of heat inside$[x, x+\Delta x]$ = Net flux of heat across the boundaries + Total heat generated inside $[x, x+\Delta x]$.

We have that the heat inside of the rod segment is going to be $\int_x^{x+\Delta x} Ac\rho u(s,t)\ ds$. Next we have that the "Net flux of heat across the boundaries" is given by $A[k(x+\Delta x)u_x(x+\Delta x, t) - k(x)u_x(x,t)]$. Since the flux of the temperature profile at any point x will be the scaled by the thermal conductivity. Finally the "Total heat generated inside" of the segments is given by an arbitrary function $\int_x^{x+\Delta x} Af(s,t)\ ds$. Next we want to see how the heat inside of the rod segments is going change by time, hence we get the left side of the equation as:

$$\frac{d}{dt}\int_x^{x+\Delta x} Ac\rho u(s,t)\ ds = \int_x^{x+\Delta x} c\rho u_t(s,t)\ ds.$$

Combining all these we get:

$$\int_x^{x+\Delta x} Ac\rho u_t(s,t)\ ds = A[k(x+\Delta x)u_x(x+\Delta x,t) - k(x)u_x(x,t)] + \int_x^{x+\Delta x} Af(s,t)\ ds$$

$$\iff \int_x^{x+\Delta x} c\rho u_t(s,t)\ ds = [k(x+\Delta x)u_x(x+\Delta x,t) - k(x)u_x(x,t)] + \int_x^{x+\Delta x} f(s,t)\ ds$$

$$\iff c\rho u_t(\xi_1,t)(\Delta x) = [k(x+\Delta x)u_x(x+\Delta x,t) - k(x)u_x(x,t)] + f(\xi_2,t)(\Delta x)$$

$$\text{by the MVT and where } x < \xi_1, \xi_2 < x+\Delta x$$

Next note that by the definition of a derivative we have:

$$k'(x) \approx \frac{k(x+\Delta x) - k(x)}{\Delta x} \iff k(x+\Delta x) \approx k'(x)(\Delta x) + k(x)$$

Thus note that:

$$k(x+\Delta x)u_x(x+\Delta x,t) - k(x)u_x(x,t)$$

$$\approx (k'(x)(\Delta x) + k(x))u_x(x+\Delta x,t) - k(x)u_x(x,t)$$
$$\approx u_x(x+\Delta x,t)k'(x)(\Delta x) + k(x)u_x(x+\Delta x,t) - k(x)u_x(x,t)$$
$$\approx u_x(x+\Delta x,t)k'(x)\Delta x + k(x)(u_x(x+\Delta x,t) - u_x(x,t))$$

Plugging this back into our previous equation

$$c\rho u_t(\xi_1,t)(\Delta x) = u_x(x+\Delta x)k'(x)\Delta x + k(x)(u_x(x+\Delta x,t) - u_x(x,t)) + f(\xi_2,t)(\Delta x)$$
$$\iff c\rho u_t(\xi_1,t) = u_x(x+\Delta x,t)k'(x) + k(x)\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x} + f(\xi_2,t)$$
$$\iff \lim_{\Delta x\to 0}(c\rho u_t(\xi_1,t)) = \lim_{\Delta x\to 0}\left(u_x(x+\Delta x,t)k'(x) + k(x)\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x} + f(\xi_2,t)\right)$$

Note that when $\Delta x \to 0$, $\xi_1, \xi_2 \to x$.

$$\iff c\rho u_t(x,t) = u_x(x,t)k'(x) + k(x)u_{xx}(x,t) + f(x,t)$$
$$\iff c\rho u_t(x,t) = \frac{\partial}{\partial x}[k(x)u_x(x,t)] + f(x,t)$$
$$\iff u_t(x,t) = \frac{1}{c\rho}\frac{\partial}{\partial x}[k(x)u_x(x,t)] + F(x,t)$$

Where $F(x,t) = \frac{1}{c\rho}f(x,t)$. $\square$

Lesson 5 # 6

What would be the solution to problem #4 if the IC were $u(x,0) = x - x^2,\ 0 < x < 1$?

$$\begin{cases} \text{PDE} & u_t = u_{xx} \\ \text{BCs} & u(0,t) = 0 \\ & u(1,t) = 0 \\ \text{IC} & u(x,0) = 1 \end{cases}$$

Solution:

Since we have homogeneous linear BCs, we may use separation of variable. So assume that $u(x,t) = X(x)T(t)$. Then applying that to our PDE we get:

$$X''(x)T(t) = X(x)T'(t)$$

Assuming a separation constant of $k \in \mathbb{R}$, we get:

$$\frac{X''}{X} = \frac{T'}{T} = k. \tag{1}$$

Note that in this situation if we have $k \geq 0$ then we will have either $u(x,t) = 0$ for all $t \in dom(u)$ or $u(x,t) \to \infty$ as $t \to \infty$. Thus we must have $k < 0$. So let $k = -\lambda^2$ which is guaranteed to be less than 0, where $\lambda \in \mathbb{R}$. Then taking (1) and turning into a system of equations we get:

$$X'' + \lambda^2 X = 0 \tag{2}$$
$$T' + \lambda^2 T = 0 \tag{3}$$

Solving for X from (2) and T from (3) via characteristic equation we get that $T(t) = Ae^{-\lambda^2 t}$ and $X(x) = B_1 \cos \lambda x + B_2 \sin \lambda x$ for some $B_1, B_2, A \in \mathbb{C}$. So we get the general solution

$$u(x,t) = X(x)T(t) = e^{-\lambda^2 t}(A \cos \lambda x + B \sin \lambda x) \tag{4}$$

Applying our BC of $u(0,t) = 0$, we get:

$$u(0,t) = e^{-\lambda^2 t}(A \cos \lambda 0 + B \sin \lambda 0) \iff 0 = e^{-\lambda^2 t}(A) \iff A = 0$$

Thus after applying these boundary conditions we have the solution:

$$u(x,t) = e^{-\lambda^2 t}(B \sin \lambda x)$$

Applying the BC of $u(1,t) = 0$:

$$u(1,t) = e^{-\lambda^2 t}(B \sin \lambda 1) \iff 0 = e^{-\lambda^2 t} B \sin \lambda \iff 0 = \sin \lambda \iff \lambda = \pi n$$

Assuming that $B \neq 0$, and letting $n \in \mathbb{N}$. So we have the solution for our PDE and BCs:

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-(\pi n)^2 t} \sin \pi n x$$

Now we can solve for our IC $u(x,0) = x - x^2$:

$$u(x,0) = \sum_{n=0}^{\infty} A_n e^{-(\pi n)^2 0} \sin \pi n x \iff x - x^2 = \sum_{n=0}^{\infty} A_n \sin \pi n x$$

$$\iff \sin \pi m x (x - x^2) = \sum_{n=0}^{\infty} A_n \sin \pi n x \sin \pi m x$$

$$\iff \int_0^1 \sin \pi m x (x - x^2) = \int_0^1 \sum_{n=0}^{\infty} A_n \sin \pi n x \sin \pi m x$$

$$\iff \frac{2 - 2\cos \pi m - \pi m \sin \pi m}{\pi^3 m^3} = A_m \frac{1}{2}$$

By the orthogonality of the *sine* function and evaluating the left hand integral using Mathematica.

$$\iff \frac{2(2 - 2\cos \pi m)}{\pi^3 m^3} = A_m$$

Since integer multiples of π give a zero *sine* function. Then note that for all even $m's$ we have that $\cos \pi m = 1$ hence our coefficient A_m is zero, and that when m is odd, we have the coefficient is $A_m = \frac{8}{\pi^3 m^3}$. So we have the general solution as:

$$u(x,t) = \sum_{n=0}^{\infty} \frac{8}{\pi^3 (2n+1)^3} e^{-(\pi(2n+1))^2 t} \sin \pi (2n+1) x$$

Lesson 6 #2

Transform

$$\begin{cases} \text{PDE} & u_t = u_{xx} \\ \text{BCs} & u(0,t) = 0 \\ & u(1,t) = 1 \\ \text{IC} & u(x,0) = x^2 \end{cases}$$

to zero BCs and solve the new problem. What will the solution to this problem look like for different values of time? Does the solutions agree with your intuition? What is the steady-state solution? What does the transient solution look like?

Solution:

Assume that we may break the solution into a steady state part and a transient part: $u(x,t) = S(x,t) + U(x,t)$, where S is the steady state and U is the transient part. Assuming that the steady state solution is the simply the linear curve between our BCs, this reasonable since we have BCs with constant coefficients we get that the steady state piece $S(x,t)$ should be of the form:

$$S(x,t) = 0 + \frac{x}{1}(1-0) = x$$

which is just the linear path between our BCs. So we have that

$$u(x,t) = x + U(x,t).$$

Taking derivatives of that we have $u_t = U_t$ and $u_{xx} = U_{xx}$. Finding our new BCs and IC we have $u(0,t) = U(0,t)$ and $u(1,t) = 1 + U(1,t)$, and $u(x,0) = x + U(x,0) \iff x^2 - x = U(x,0)$. Thus Hence we have the new IVBP:

$$\begin{cases} \text{PDE} & U_t = U_{xx} \\ \text{BCs} & U(0,t) = 0 \\ & U(1,t) = 0 \\ \text{IC} & U(x,0) = x^2 - x \end{cases}$$

Note that this is nearly identical to #6 from Lesson 5, the only difference being a sign. So we have the solution to this IVBP as:

$$U(x,t) = -\sum_{n=0}^{\infty} \frac{8}{\pi^3(2n+1)^3} e^{-(\pi(2n+1))^2 t} \sin(\pi(2n+1)x)$$

So our solution for $u(x,t)$ is as follows:

$$u(x,t) = x - \sum_{n=0}^{\infty} \frac{8}{\pi^3(2n+1)^3} e^{-(\pi(2n+1))^2 t} \sin(\pi(2n+1)x)$$

The solution will look like a line with slope 1 and a zero y-intercept with an almost imperceptibly small wave pattern to it. Note that even for small $t's$ we have the approximate solution as:

```
In[54]:= expr1 = x - Sum[(8 E^(-(Pi (2 i + 1))^2 * 1) Sin[(Pi (2 i + 1) x)] / (Pi^3 * (2 i + 1)^3)), {i, 0, 10}]
```

Out[54]=
$$x - \frac{8 e^{-\pi^2} \text{Sin}[\pi x]}{\pi^3} - \frac{8 e^{-9\pi^2} \text{Sin}[3\pi x]}{27\pi^3} - \frac{8 e^{-25\pi^2} \text{Sin}[5\pi x]}{125\pi^3} - \frac{8 e^{-49\pi^2} \text{Sin}[7\pi x]}{343\pi^3} - \frac{8 e^{-81\pi^2} \text{Sin}[9\pi x]}{729\pi^3} - \frac{8 e^{-121\pi^2} \text{Sin}[11\pi x]}{1331\pi^3} - \frac{8 e^{-169\pi^2} \text{Sin}[13\pi x]}{2197\pi^3} - \frac{8 e^{-225\pi^2} \text{Sin}[15\pi x]}{3375\pi^3} - \frac{8 e^{-289\pi^2} \text{Sin}[17\pi x]}{4913\pi^3} - \frac{8 e^{-361\pi^2} \text{Sin}[19\pi x]}{6859\pi^3} - \frac{8 e^{-441\pi^2} \text{Sin}[21\pi x]}{9261\pi^3}$$

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In[56]:= Plot[expr1, {x, 0, 1}]
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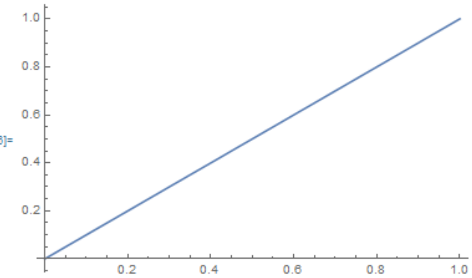
Out[56]=


This matches our intuition, since as time goes on we will just get a linear path between the two BCs. The steady-state solution is simply x on $0 \leq x \leq 1$ as $t \to \infty$. While the transient solution is:

$$U(x,t) = -\sum_{n=0}^{\infty} \frac{8}{\pi^3(2n+1)^3} e^{-(\pi(2n+1))^2 t} \sin(\pi(2n+1)x)$$

Lesson 7 #3

Solve the following problem with insulated boundaries

$$\begin{cases} \text{PDE} & u_t = u_{xx} \\ \text{BCs} & u_x(0,t) = 0 \\ & u_x(1,t) = 0 \\ \text{IC} & u(x,0) = x \end{cases}$$

Does your solution agree with your interpretation of the problem? What is the steady state solution? Does this make sense?

Solution:

Note that since we have homogeneous boundary conditions we can use separation of variables. So assume that $u(x,t)$ is of the form $u(x,t) = X(x)T(t)$. Then note that we already have the general solution of this PDE as:

$$u(x,t) = e^{-\lambda^2 t}[A \sin \lambda x + B \cos \lambda x)]$$

Where $-\lambda^2$ is the separation constant and $A, B \in \mathbb{C}$. Then to apply the boundary conditions we need to take the x-derivative of this solution:

$$u_x(x,t) = e^{-\lambda^2 t}[A\lambda \cos \lambda x - B\lambda \sin \lambda x]$$

Applying our BC of $u_x(0,t) = 0$:

$$u_x(0,t) = e^{-\lambda^2 t}[A\lambda \cos \lambda 0 - B\lambda \sin \lambda 0] \iff 0 = e^{-\lambda^2 t} A\lambda \iff 0 = A\lambda$$

Note that if $\lambda = 0$, then we would have the solution $u(x,t)$ is a constant, which wouldn't be the most interesting case. So assume that $\lambda \neq 0$ for this BC.

$$\iff A = 0$$

Hence we now have $u(x,t) = e^{-\lambda^2 t} B \cos \lambda x$. Now applying the BC $u_x(1,t) = 0$. So we have:

$$u_x(0,t) = e^{-\lambda^2 t}(-B)\lambda \sin(\lambda) \iff 0 = e^{-\lambda^2 t}(-B)\lambda \sin(\lambda) \iff 0 = \lambda \sin(\lambda)$$

Note here we will also assume that $\lambda \neq 0$ for the same reason that we assumed it for our other BC.

$$\iff 0 = \sin(\lambda) \iff \lambda = \pi n, \text{ for some } n \in \mathbb{N}.$$

Then we have the solution to the PDE and BCs:

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-(\pi n)^2 t} \cos(\pi n x)$$

Now applying IC we have:

$$u(x,0)=\sum_{n=0}^{\infty}A_ne^{-(\pi n)^2 0}\cos(\pi n) \iff x=\sum_{n=0}^{\infty}A_n\cos(\pi nx)$$

$$\iff \cos(\pi mx)x=\sum_{n=0}^{\infty}A_n\cos(\pi nx)\cos(\pi mx)$$

$$\iff \int_0^1\cos(\pi mx)x\ dx=\int_0^1\sum_{n=0}^{\infty}A_n\cos(\pi nx)\cos(\pi mx)\ dx$$

Note that we have two cases at this point, if $m=0$, then we get:

$$\int_0^1 x\ dx=\int_0^1\sum_{n=0}^{\infty}A_n\cos(\pi nx)\cos(\pi 0x)\ dx$$

Note that by the orthogonality of the cosine function we have that for all $m\neq n$: $\int_0^1\cos(\pi mx)\cos(\pi nx)=0$. Hence we have that the only remaining term is when $n=m=0$.

$$\left.\frac{x^2}{2}\right|_0^1=\int_0^1 A_0\cos(0)\ dx \iff \frac{1}{2}=A_0$$

This and when $n=0$ we have the solution $u_0(x,t)=\frac{1}{2}e^{-(\pi\cdot 0)^2t}\cos(\pi x(0))=\frac{1}{2}$. Now we will look at all $n\geq 1$. So now we look at the problem:

$$\int_0^1\cos(\pi mx)x\ dx=\int_0^1\sum_{n=1}^{\infty}A_n\cos(\pi nx)\cos(\pi mx)\ dx$$

$$\iff \frac{\pi m\sin(\pi m)+\cos(\pi m)-1}{(\pi m)^2}=A_m\frac{1}{2}$$

By the orthogonality of the cosine function and evaluating that left integral using Mathematica.

$$\iff \frac{2(\cos(\pi m)-1)}{(\pi m)^2}=A_m$$

Note that if m is odd, then we have that $\cos(\pi m)=-1$ and when m is even we have $\cos(\pi m)=1$. Thus we have that even solution will go to zero, so we have the general solution for $n\geq 1$:

$$u_{2k-1}(x,t)=\frac{2(-2)}{(\pi(2k-1))^2}e^{-(\pi(2k-1))^2t}\cos(\pi(2k-1)x)$$

So that the general solution is:

$$u(x,t) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{-4}{(\pi(2k-1))^2} e^{-(\pi(2k-1))^2 t} \cos(\pi(2k-1)x)$$

This means that the steady-state solution is $\frac{1}{2}$ as $t \to \infty$. This makes sense, our boundary conditions tell us that the derivatives at the boundaries are zero. So as time approaches infinity we need the graph to kind of even out, to a constant function. Since the midpoint between 1 and 0 is 1/2, and by our IC we have that $u(0,0) = 0$ and $u(1,0) = 0$, it would make sense that the boundaries temperatures "meet" at 1/2.

Lesson 7 #4

What are the eigenvalues and eigenfunctions of

$$\begin{cases} ODE & X'' + \lambda X = 0 \\ BCs & X'(0) = 0 \\ & X'(1) = 0 \end{cases}$$

On the interval $0 < x < 1$.

Solution:

Note that we can solve our ODE via a characteristic equation, assuming that $X(x) = Ce^{rt}$ for some $C, r \in \mathbb{C}$. Applying this method we get the polynomial $r^2 = -\lambda \iff r = \pm\sqrt{\lambda}i$ So we get the solution:

$$X(x) = C_1 e^{\sqrt{\lambda}ix} + C_2 e^{-\sqrt{\lambda}ix} \iff X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x), \text{ where } B, A \in \mathbb{C}$$

Applying BCs:

$$X'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x) \iff 0 = B\sqrt{\lambda}$$

Note that if $\sqrt{\lambda} = 0$, then we would have $X(x) = B$ where $B \in \mathbb{C}$ which would be a trivial result, so $\sqrt{\lambda} \neq 0$. Hence $B = 0$. So now we have $X(x) = A\cos(\sqrt{\lambda}x)$. Now applying our BC $X'(1) = 0$:

$$X'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) \iff 0 = -A\sqrt{\lambda}\sin(\sqrt{\lambda}) \iff \sin(\sqrt{\lambda}) = 0$$
$$\iff \sqrt{\lambda} = \pi m$$

Hence we have the solution:

$$X(x) = \sum_{n=0}^{\infty} A\cos(\pi n x)$$

In this case our eigenvalues are:

$$\lambda_n = (\pi n)^2$$

And the eigenfunctions are:

$$X_n(x) = \cos(\sqrt{\lambda_n}x).$$