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Homework #2

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Problem 1.17

A particle is represented (at time $t = 0$) by the wave function

$$\Psi(x,0) = \begin{cases} A(a^2 - x^2), & \text{if } -a \leq x \leq a \\ 0, & \text{otherwise} \end{cases}$$

a.)

Determine the normalization constant A.

Solution:

$$\begin{aligned}
\int_{-\infty}^{+\infty} |\Psi(x,0)|^2 \, dx &= \int_{-\infty}^{-a} 0 \, dx + \int_{-a}^{+a} A^2(a^2 - x^2)^2 \, dx + \int_{+a}^{\infty} 0 \, dx \\
&= \int_{-a}^{+a} A^2(a^4 - 2a^2x^2 + x^4) \, dx = 1 \\
& a^4x - \frac{2}{3}a^2x^3 + \frac{x^5}{5}\bigg|_{x=-a}^{+a} = \frac{1}{A^2} \\
& a^4(a + a) - \frac{2}{3}a^2(a^3 - (-a^3)) + \frac{(a^5 - (-a^5))}{5} = \frac{1}{A^2} \\
& 2a^5 - \frac{4}{3}a^5 + \frac{2}{5}a^5 = \frac{1}{A^2} \\
& \frac{2}{3}a^5 + \frac{2}{5}a^5 = \frac{1}{A^2} \\
& \frac{16}{15}a^5 = \frac{1}{A^2} \\
& A^2 = \frac{15}{16a^5} \\
& A = \sqrt{\frac{15}{16a^5}}
\end{aligned}$$

b.)

What is the expectation value of x (at time $t = 0$)?

Solution:

$$\begin{aligned}
< x > &= \int_{-\infty}^{+\infty} \Psi(x,0)\ x\Psi^*(x,0)\ dx \\
&= \int_{-a}^{+a} \frac{15}{16a^5}(a^4 - 2a^2x^2 + x^4)x\ dx \\
&= \int_{-a}^{+a} \frac{15}{16a^5}(a^4x - 2a^2x^3 + x^5)\ dx
\end{aligned}$$

Note that x, x^3, x^5 are all odd functions, so:

$$< x > = 0$$

c.)

What is the expectation value of p at time $t = 0$? (Note that you cannot get it from $p = m\frac{d<x>}{dt}$. Why not?)

Solution:

$$\begin{aligned}
< p > &= \int_{-\infty}^{+\infty} \Psi(x,0)\hat{p}\left[\Psi^*(x,0)\right]\ dx \\
&= \int_{-\infty}^{+\infty} \Psi(x,0)\left[ih\frac{d}{dx}\right]\left[\Psi^*(x,0)\right]\ dx \\
&= \int_{-\infty}^{+\infty} \frac{15}{16a^5}(a^2 - x^2)i\hbar\frac{d}{dx}\left[(a^2 - x^2)\right]\ dx \\
&= \int_{-\infty}^{+\infty} \frac{15}{16a^5}(a^2 - x^2)i\hbar\left[(0 - 2x\right]\ dx \\
&= \int_{-\infty}^{+\infty} \frac{15}{16a^5}(a^2 - x^2)i\hbar\left[(0 - 2x)\right]\ dx \\
&= \frac{-i\hbar 15}{16^5}\int_{-\infty}^{+\infty} (a^2x - x^3)\ dx
\end{aligned}$$

Since both x^3, x are both odd functions, so:

$$< p > = 0$$

$< p > \neq m\frac{d<x>}{dt}$, because $< x >$ is always a constant, so that $\frac{d<x>}{dt}$ will always be 0, which just so happens to be the answer here, but is not generally.

d.)

Find the expectation value of $< x^2 >$.
Solution:

$$\begin{aligned}
< x^2 > &= \int_{-\infty}^{+\infty} \Psi(x,0) x^2 \Psi^*(x,0)\; dx \\
&= \int_{-a}^{+a} \frac{15}{16a^5}(x^2)(a^2-x^2)^2\; dx \\
&= \int_{-a}^{+a} \frac{15}{16a^5}(a^4 - 2a^2x^2 + x^4)(x^2)\; dx \\
&= \int_{-a}^{+a} \frac{15}{16a^5}(a^4x^2 - 2a^2x^4 + x^6)\; dx \\
&= \frac{15}{16a^5}\left[a^4\frac{x^3}{3} - 2a^2\frac{x^5}{5} + \frac{x^7}{7}\right]_{x=-a}^{+a} \\
&= \frac{15}{16a^5}\left[\frac{a^4}{3}(a^3+a^3) - \frac{2a^2}{5}(a^5+a^5) + \frac{1}{7}(a^7+a^7)\right] = \frac{15}{16a^5}\left(\frac{2a^7}{3} - \frac{4a^7}{5} + \frac{2a^7}{7}\right) \\
&= \frac{15a^2}{16}(2/3 - 4/5 + 2/7) = \frac{15a^2}{16}\frac{16}{105} = \frac{a^2}{7}
\end{aligned}$$

e.)

Find the expectation value of p^2.
Solution:
First, I'll calculate $p^2[\Psi^*(x,0]$ separately for aesthetic purposes.

$$\begin{aligned}
p^2[\Psi^*(x,0)] &= \left(i\hbar\frac{d}{dx}\right)^2\left[\sqrt{\frac{15}{16a^5}}(a^2-x^2)\right] \\
&= -\hbar^2\sqrt{\frac{15}{16a^5}}\frac{d^2}{dx^2}(a^2-x^2) = 2\hbar^2\sqrt{\frac{15}{16a^5}}
\end{aligned}$$

So that our expectation value becomes:

$$\begin{aligned}
< p^2 > &= \int_{-\infty}^{+\infty} \Psi(x,0)\hat{p}^2\ \Psi^*(x,0)\ dx \\
&= \int_{-a}^{+a} \frac{\hbar^2 15}{8a^5}(a^2 - x^2)\ dx \\
&= \frac{\hbar^2 15}{8a^5}[a^2 x - \frac{x^3}{3}]_{x=-a}^{+a} \\
&= \frac{\hbar^2 15}{8a^5}[2a^3 - \frac{2}{3}a^3] = \frac{\hbar^2 15}{8a^5}\frac{4}{3}a^3 = \frac{5\hbar^2}{2a^2}
\end{aligned}$$

f.)

Find the uncertainty in x (σ_x).
Solution:

$$\begin{aligned}
\sigma_x &= \sqrt{< x^2 > - < x >^2} \\
&= \sqrt{\frac{a^2}{7}} = \frac{a}{\sqrt{7}} = \frac{a\sqrt{7}}{7}
\end{aligned}$$

g.)

Find the uncertainty in p (σ_p).
Solution:

$$\begin{aligned}
\sigma_p &= \sqrt{< p^2 > - < p >^2} \\
&= \sqrt{\frac{5\hbar^2}{2a^3}} = \frac{\hbar}{a}\frac{5}{2}
\end{aligned}$$

h.)

Check that your results are consistent with the uncertainty principle.
Solution:

$$\sigma_x \sigma_p = \frac{a\sqrt{7}}{7}\frac{\hbar}{a}\frac{5}{2} = \frac{5\sqrt{7}}{14}\hbar \approx 0.9449\hbar \geq \frac{\hbar}{2}\checkmark$$

Problem 2.4

Calculate $< x >, < x^2 >, < p >, < p^2 >, \sigma_x$, and σ_p, for the nth stationary state of the infinite square well. Check that the uncertainty principle is satisfied. Which state comes closed to the uncertainty principle.

Solution:

$$
\begin{aligned}
< x > &= \int_{-\infty}^{+\infty} x|\psi_n(x)|^2 \; dx \\
&= \int_0^a \frac{2}{a} x \sin^2 \frac{n\pi}{a} x \; dx \\
u &= x, du = dx, dv = \sin^2 \frac{n\pi x}{a} \; dx = \frac{1}{2}(1 - \cos \frac{2\pi n x}{a}) \; dx, v = \frac{1}{2}(x - \frac{a}{2\pi n} \sin \frac{2\pi n x}{a}) \\
&= \frac{1}{2}\frac{2}{a} x(x - \frac{a}{2\pi n} \sin \frac{2\pi n x}{a})\bigg|_{x=0}^{a} - \frac{1}{a}\int_0^a (x - \frac{a}{2\pi n} \sin \frac{2\pi n x}{a}) \; dx \\
&= a - \frac{1}{a}\frac{x^2}{2}\bigg|_{x=0}^{a} + \frac{1}{2\pi n}\int_0^a \sin \frac{2\pi n x}{a} \; dx \\
&= a - \frac{a}{2} - \frac{a}{(2\pi n)^2} \cos \frac{2\pi n x}{a}\bigg|_{x=0}^{a} \\
&= \frac{a}{2} + 0 = \frac{a}{2} \\
< p > &= \int_{-\infty}^{+\infty} \psi_n(x) \hat{p}[\psi_n^*(x)] \; dx \\
&= \int_0^a \frac{2i\hbar}{a} \sin\left(\frac{\pi n}{a} x\right) \frac{\pi n}{a} \cos\left(\frac{\pi n}{a} x\right) \; dx
\end{aligned}
$$

Using a u-sub with $u = \sin \frac{\pi n x}{a}$ and $du = \frac{\pi n}{a} \cos \frac{\pi n x}{a}$, however, now our limits out $[0, \sin \pi n = 0]$, thus we have $< p >= 0$.

$$
\begin{aligned}
< x^2 > &= \int_{-\infty}^{+\infty} x^2 |\psi_n(x)|^2 \; dx \\
&= \int_0^{+a} x^2 \frac{2}{a} \sin^2\left(\frac{\pi n x}{a}\right) \\
& u = x^2, du = 2x \; dx, dv = \sin^2\left(\frac{\pi n x}{a}\right) \; dx, v = \frac{1}{2}\left(x - \frac{a}{2\pi n}\sin\left(\frac{2\pi n x}{a}\right)\right) \\
&= \frac{2}{a}\frac{x^2}{2}\left(x - \frac{a}{2\pi n}\sin\left(\frac{2\pi n x}{a}\right)\right)\Bigg|_{x=0}^{+a} - \frac{2}{a}\int_0^a x(x - \frac{a}{2\pi n}\sin\left(\frac{2\pi n x}{a}\right) \; dx \\
&= a^2 - \frac{2}{a}\int_0^{+a} x^2 - \frac{a}{2\pi n} x \sin\left(\frac{2\pi n x}{a}\right) \; dx \\
&= a^2 - \frac{2}{a}\frac{x^3}{3}\Bigg|_{x=0}^{+a} + \frac{2}{a}\int_0^a \frac{a}{2\pi n} x \sin\left(\frac{2\pi n x}{a}\right) \; dx \\
& u = x, du = dx, dv = \sin\frac{2\pi n x}{a} \; dx, v = \frac{-a}{2\pi n}\cos\frac{2\pi n x}{a} \\
&= a^2 - \frac{2}{3}a^2 + \frac{2}{a}\left[x\frac{-a^2}{(2\pi n)^2}\cos\frac{2\pi n x}{a}\Bigg|_{x=0}^{a} + \frac{a^2}{(2\pi n)^2}\int_0^{+a}\cos\frac{2\pi n x}{a} \; dx\right] \\
&= a^2 - \frac{2}{3}a^2 + \frac{2}{a}\left[\frac{-a^3}{(2\pi n)^2}\cos(2\pi n) + \left(\frac{a^3}{(2\pi n)^3}\right)\sin\frac{2\pi n x}{a}\Bigg|_{x=0}^{a}\right] \\
&= a^2 - \frac{2}{3}a^2 + \frac{-a^2}{2(\pi n)^2} \\
&= \frac{a^2}{3} - \frac{a^2}{2(\pi n)^2} = \frac{2(\pi n a)^2 - 3a^2}{6(\pi n)^2}
\end{aligned}
$$

$$
\begin{aligned}
< p^2 > &= \int_{-\infty}^{+\infty} \psi_n(x)\hat{p}^2[\psi_n^*(x)] \; dx \\
&= \int_0^a \frac{-2\hbar^2}{a}\sin\left(\frac{\pi n x}{a}\right)\frac{d^2}{dx^2}\left(\sin\left(\frac{\pi n x}{a}\right)\right) \; dx \\
&= \int_0^{+a} \frac{-2(-\hbar^2)}{a}\left(\frac{\pi n}{a}\right)^2 \sin^2\left(\frac{\pi n x}{a}\right) \; dx = \frac{\hbar^2\pi^2 n^2}{a^2}\int_0^a \frac{2}{a}\sin^2(\frac{\pi n x}{a}) \; dx = \left(\frac{\hbar \pi n}{a}\right)^2
\end{aligned}
$$

$$\sigma_x = \sqrt{< x^2 > - < x >^2} = \sqrt{\frac{2(\pi n a)^2 - 3a^2}{6\pi^2 n^2} - \frac{a^2}{4}}$$
$$= \sqrt{\frac{8(\pi n a)^2 - 12a^2 - 6(\pi n a)^2}{24\pi^2 n^2}} = \frac{a}{2\pi n}\sqrt{\frac{2(\pi n)^2 - 12}{6}}$$
$$\sigma_p = \sqrt{< p^2 > - < p >^2} = \sqrt{\frac{a^2(4\pi^3 n^3 - 6\pi n)}{12\pi^3 n^3} - 0^2} = \frac{\hbar \pi n}{a}$$
$$\sigma_x \sigma_p = \frac{\hbar}{2}\sqrt{\frac{2(\pi n)^2 - 12}{6}} \geq \frac{\hbar}{2} \text{ for all } n \in \mathbb{N}$$

The minimum value of $\sigma_x\sigma_p \approx (1.1357)\frac{\hbar}{2}$ occurs at the ground state, when $n = 1$.

2.5

A particle in the infinite square well has as its initial wave function an even mixture of the first two stationary states:

$$\Psi(x,0) = A[\psi_1(x) + \psi_2(x)].$$

a.)

Normalize $\Psi(x,0)$. (That is, find A. This is very easy, if you exploit the orthnormality of ψ_1 and ψ_2. Recall that, having normalized Ψ at $t = 0$, you can rest assured that it *stays* normalized - if you doubt this, check it explicitly after doing part(b).)

Solution:

$$\int_{-\infty}^{+\infty} |\Psi(x,0)|^2 \, dx = 1$$
$$\int_0^a A^2[\psi_1(x) + \psi_2(x)]^2 \, dx = 1$$
$$\int_0^a [\psi_1^2(x) + 2\psi_1(x)\psi_2(x) + \psi_2^2(x)] \, dx = \frac{1}{A^2}$$
$$1 + 0 + 1 = \frac{1}{A^2}, \text{ by orthonormality}$$
$$A^2 = \frac{1}{2}$$
$$A = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

b.)

Find $\Psi(x,t)$ and $|\Psi(x,t)|^2$. Express the latter as a sinusoidal function of time, as in Example 2.1. To simplify the result, let $\omega \equiv \pi^2\hbar/2ma^2$.

Solution:

Now we have a normalized $\Psi(x,0)$:

$$\Psi(x,0) = \frac{\sqrt{2}}{2}\psi_1(x) + \frac{\sqrt{2}}{2}\psi_2(x)$$

So note that $A = \frac{\sqrt{2}}{2}$ is our c_1 and c_2 in our general solution, so that since $\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = 1$.

So multiplying in our time dependence we get:

$$\Psi(x,t) = \frac{\sqrt{2}}{2}\psi_1(x)\exp\left(\frac{-iE_1t}{\hbar}\right) + \frac{\sqrt{2}}{2}\psi_2(x)\exp\left(\frac{-iE_2t}{\hbar}\right)$$

So then finding $|\Psi(x,t)|^2$:

$$\begin{aligned}
|\Psi(x,t)|^2 &= \Psi(x,t)\Psi^*(x,t) \\
&= \frac{1}{4}\left(\psi_1\exp\left(\frac{-iE_1t}{\hbar}\right) + \psi_2\exp\left(\frac{-iE_2t}{\hbar}\right)\right)\left(\psi_1^*\exp\left(\frac{+iE_1t}{\hbar}\right) + \psi_2^*\exp\left(\frac{+iE_2t}{\hbar}\right)\right) \\
&= \frac{1}{4}\left(\psi_1\psi_1^* + \psi_1\psi_2^*\exp(\frac{-iE_1t}{\hbar})\exp(\frac{+iE_2t}{\hbar}) + \psi_1^*\psi_2\exp(\frac{-iE_2t}{\hbar})\exp(\frac{+iE_1t}{\hbar}) + \psi_2\psi_2^*\right) \\
&= \frac{1}{4}\left(|\psi_1|^2 + |\psi_2|^2 + \psi_1\psi_2^*\exp(\frac{-i(E_2-E_1)t}{\hbar}) + \psi_1^*\psi_2\exp(\frac{-i(E_1-E_2)t}{\hbar})\right) \\
&= \frac{1}{4}\left(|\psi_1|^2 + |\psi_2|^2 + \psi_1\psi_2^*\exp(\frac{-i(E_2-E_1)t}{\hbar}) + \left(\psi_1\psi_2^*\exp(\frac{-i(E_2-E_1)t}{\hbar})\right)^*\right)
\end{aligned}$$

Now I will show that the fact that for any $z \in \mathbb{C}$, $z + \bar{z} = 2\Re(z)$.

$$z + \bar{z} = a + bi + a - bi = 2a = 2\Re(z)$$

So now we have:

$$|\Psi(x,t)|^2 = \frac{1}{4}|\psi_1|^2 + \frac{1}{4}|\psi_2|^2 + \frac{1}{2}\psi_1\psi_2^*\cos\left(\frac{(E_2-E_1)t}{\hbar}\right)$$

For the infinite quantum square well, we know:

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

And that the standing waves are real, so that:

$$|\Psi(x,t)|^2 = \frac{1}{4}|\psi_1|^2 + \frac{1}{4}|\psi_2|^2 + \frac{1}{2}\psi_1\psi_2 \cos\left(\frac{\left(\frac{4\pi^2\hbar^2}{2ma^2} - \frac{\pi^2\hbar^2}{2ma^2}\right)t}{\hbar}\right)$$

$$= \frac{1}{4}|\psi_1(x)|^2 + \frac{1}{4}|\psi_2(x)|^2 + \frac{1}{2}\psi_1(x)\psi_2(x)\cos(3\omega t)$$

c.)

Compute $< x >$. Notice that it oscillates in time. What is the angular frequency of the oscillation? What is the amplitude of the oscillation?

Solution:

Assuming that we have the solutions for standing waves in the infinite square well as:

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

Then note that:

$$|\psi_n|^2 = \frac{2}{a}\sin^2\left(\frac{n\pi x}{a}\right) = \frac{1}{a}\left(1 - \cos\left(\frac{2\pi n x}{a}\right)\right)$$

$$\begin{aligned} < x > &= \int_0^a \frac{x}{4}|\psi_1|^2 + \frac{x}{4}|\psi_2|^2 + \frac{1}{2}\psi_1\psi_2\cos(3\omega t) \\ &= \int_0^a \frac{x}{4a}\left(1 - \cos(\frac{2\pi x}{a})\right) + \frac{x}{4a}\left(1 - \cos(\frac{4\pi x}{a})\right) + \frac{x}{2}\sin^2(\frac{\pi x}{a})\sin^2(\frac{2\pi x}{a})\cos(3\omega t)\, dx \end{aligned}$$

To preserve our sanity, we'll evaluate these three integrands separately:

$$I_1 = \int_0^a \frac{x}{4a}(1 - \cos\left(\frac{2\pi x}{a}\right))\, dx \tag{1}$$

$$u = \frac{x}{4a}, du = \frac{1}{4a}, dv = 1 - \cos\frac{2\pi x}{a}, v = x - \frac{a}{2\pi}\sin(\frac{2\pi x}{a}) \tag{2}$$

$$= \frac{x}{4a}(x - \frac{a}{2\pi}\sin(\frac{2\pi x}{a}))\Big|_{x=0}^{a} - \frac{1}{4a}\int_0^a x - \frac{a}{2\pi}\sin(\frac{2\pi x}{a})\, dx \tag{3}$$

$$= \frac{a}{4} + \frac{-1}{4a}\left(\frac{x^2}{2} - (\frac{a}{2\pi})^2\cos(\frac{2\pi x}{a})\right)\Big|_{x=0}^{a} \tag{4}$$

$$= \frac{a}{4} + \frac{-1}{4a}(\frac{a^2}{2} - \frac{a^2}{4\pi^2} + \frac{a^2}{4\pi^2}) = \frac{a}{4} + \frac{-a}{8} = \frac{a}{8} \tag{5}$$

Notice that in steps (3) and (4), that the cosine and sine terms are actually non-consequential, since they cancel out on $[0, a]$, the same will happen for I_2, so that:

$$I_2 = \frac{a}{8}$$

$$\begin{aligned}
I_3 &= \int_0^a \frac{x}{2} \sin^2\left(\frac{\pi x}{a}\right) \sin^2\left(\frac{2\pi x}{a}\right) \cos(3\omega t)\ dx \\
&= \frac{\cos(3\omega t)}{2} \int_0^a x \sin^2\left(\frac{\pi x}{a}\right) \sin^2\left(\frac{2\pi x}{a}\right)\ dx \\
&= \frac{\cos(3\omega t)}{8} \int_0^a x \left(1 - \cos\frac{2\pi x}{a}\right)\left(1 - \cos\frac{4\pi x}{a}\right) \\
&= \frac{\cos(3\omega t)}{8} \int_0^a \left(x - \cos\frac{2\pi x}{a}\right)\left(1 - \cos\frac{4\pi x}{a}\right) \\
&= \frac{\cos(3\omega t)}{8} \int_0^a \left(x - x\cos\frac{4\pi x}{a} - x\cos\frac{2\pi x}{a} + \cos\frac{2\pi x}{a}\cos\frac{4\pi x}{a}\right) \\
&= \frac{\cos(3\omega t)}{8} \int_0^a \left(x(1 - \cos(\frac{4\pi x}{a})) - x\cos(\frac{2\pi x}{a}) + \cos(\frac{2\pi x}{a})\cos(\frac{4\pi x}{a})\right) \\
&= \frac{\cos(3\omega t)}{8} \left(\int_0^a x(1 - \cos\frac{4\pi x}{a})\ dx - \int_0^a x\cos\frac{2\pi x}{a}\ dx + \int_0^a \cos\frac{2\pi x}{a}\cos\frac{4\pi x}{a}\ dx\right) \\
&= \frac{\cos(3\omega t)}{8} \left(\frac{a^2}{2} - \int_0^a x\cos\frac{2\pi x}{a}\ dx + 0\right) \\
&= \frac{\cos(3\omega t)}{8} \left(\frac{a^2}{2} - \left(\frac{a}{2\pi}\sin\frac{2\pi x}{a}\Big|_{x=0}^{a} - \int_0^a \frac{a}{2\pi}\sin\frac{2\pi x}{a}\ dx\right)\right) \\
&= \frac{\cos(3\omega t)}{8}\left(\frac{a^2}{2}\right) = \frac{\cos(3\omega t)a^2}{16}
\end{aligned}$$

This last step follows because $\sin\frac{2\pi x}{a}$ has a period of $[0, a]$ and cosine and sine are both self-orthogonal on their periods.

So that:

$$< x >= \frac{2a}{8} + \frac{\cos(3\omega t)a^2}{16} = \frac{a(a\cos(3\omega t) + 4)}{16}$$

So the amplitude of this oscillation is $\frac{a^2}{16}$ with an angular frequency of: 3ω

d.)

Compute $< p >$.

Solution:

$$< p > = \int_0^a \frac{-i\hbar\pi}{4a}\left(\sin(\frac{\pi x}{a})\exp(\frac{-iE_1t}{\hbar}) + \sin(\frac{2\pi x}{a})\exp(\frac{-iE_2t}{\hbar})\right)$$
$$* \left(\cos(\frac{\pi x}{a})\exp(\frac{-iE_1t}{\hbar}) + 2\cos(\frac{2\pi x}{a})\exp(\frac{-iE_2t}{\hbar})\right)\, dx$$

Since sine and cosine are always orthogonal on their period we have:

$$< p >= 0$$

e.)

If you measured the energy of this particle, what values might you get, and what is the probability of getting each of them? Find the expectation value of H. How does it compare with E_1 and E_2?

Solution:

The probability of each energy level occurring is the coefficient $|c_n|^2$, so that E_1's, E_2's probability of finding the particle in those levels is $\frac{1}{2}$. E_1, E_2 are the only possible energy levels for this particle. $< H >= \frac{E_1}{2} + \frac{E_2}{2} = \frac{5\pi^2\hbar^2}{4ma^2}$.