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Homework # 3

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The generating function for the number of permutations of n with only even sized cycles is

$$\sqrt{\frac{1}{1-x^2}}.$$

(The number of such permutations is $1^2 \cdot 3^2 \cdot 5^2 \ldots (n-1)^2$ if n is even and 0 if n is odd.)

a.)

Use the exponential formula to prove that

$$\sum_{n=0}^{\infty} (\text{the number of permutations of } n \text{ with only odd sized cycles}) \frac{x^n}{n!} = (1+x)\sqrt{\frac{1}{1-x^2}}.$$

Proof. By the exponential formula we have the generating function for this sequence is $A(x,y) = e^{yC(x)}$. Taking $y = 1$ will simply give us the number of such odd length permutations. So that our generating function of this will be $A(x) = e^{C(x)}$.

Note then that since the number of cycles of length n is simply $|C_n| = (n-1)!$, we can simply plug in an odd number say $2n+1$ we get: $|C_{2n+1}| = ((2n+1)-1)! = (2n)!$ for any $n \in \mathbb{N}$ including 0. To only get these odd-length cycles, we'll take $|C_{2n}| = 0$. So then we get that the exponential generating function of $|C_{2n+1}|$ is: $\sum_{n=0}^{\infty} |C_{2n+1}| \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (2n)! \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$.

So then we have $C(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = (1 + \frac{x^3}{3} + \frac{x^5}{5} + \ldots)$. But note that we can rewrite this as follows:

$$\begin{aligned}
C(x) &= 1 + \frac{x^3}{3} + \frac{x^5}{5} + \dots \\
&= (1 + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \dots) - \left(\frac{x^2}{2} + \frac{x^4}{4} + \dots\right) \\
&= \log\left(\frac{1}{1-x}\right) - \frac{1}{2}\left(x^2 + \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3} + \dots\right) \\
&= \log\left(\frac{1}{1-x}\right) - \frac{1}{2}\log\left(\frac{1}{1-x^2}\right) \\
&= \log\left(\frac{1}{1-x}\right) + \log(\sqrt{1-x^2}) \\
&= \log\left(\frac{\sqrt{1-x^2}}{1-x}\right) \\
&= \log\left(\frac{(1-x)^{1/2}(1+x)^{1/2}}{1-x}\right) & = \log\left(\frac{(1+x)^{1/2}}{(1-x)^{1/2}}\right).
\end{aligned}$$

So then this gives us a generating function:

$$e^{\log\left(\sqrt{\frac{1+x}{1-x}}\right)} = \sqrt{\frac{1+x}{1-x}}.$$

But note that if we take $\frac{1+x}{(1-x^2)^{1/2}} = \frac{1+x}{(1-x)^{1/2}(1+x)^{1/2}} = \frac{(1+x)^{1/2}}{(1-x)^{1/2}}$. Thus we have our result:

$$\sum_{n=0}^{\infty}(\text{the number of permutations of } n \text{ with only odd sized cycles})\frac{x^n}{n!} = (1+x)\sqrt{\frac{1}{1-x^2}}.$$

□

b.)

The coefficients of x^2 in (1) and (2) are the same. Therefore the number of permutations of $2n$ with only even sized cycles is equal to the number of permutations of $2n$ with only odd sized cycles. Find a bijection between these two sets of permutations.

Proof. Define the set of odd cycle permutations O_{2n} and the set of even cycle permutations E_{2n}. I'll impose the restriction that all permutations are ordered in such a way that we have the least element in the cycle at the front of each cycle in a given permutation. This is reasonable, as each cycle will have a least element because it's finite. Additionally, we'll order all of the cycles in such a way that they are ordered from smallest first element in the cycle to the largest first element in the cycle; e.g (1 2) (3 4) or (1) (2 4) (3 5).

Then note that we can form any even cycle permutation by "gluing" together odd cycle permutations. That is take an even cycle permutation, $\sigma_1 \ldots \sigma_m$. Take any σ_i from this permutation, so that $\sigma_i = (a_{i_1} \ldots a_{i_{2k}})$ for some $k \leq n$, so that we break this cycle into two disjoint cycles: $\sigma_{i_1}\sigma_{i_2} = (a_{i_1} \ldots a_{i_k})(a_{i_{k+1}} \ldots a_{i_{2k}})$. Cycles don't contain repeated numbers so that this is guaranteed to be disjoint.

Then if k is odd, then we're done. If k is even, break it down even further, so that continuing this until we get a chain of disjoint odd cycles: $\sigma_{i_1} \ldots \sigma_{i_m}$. This process is well-defined for any even cycle permutation, because it terminates when we have all odd length cycles and it perfectly partitions the numbers present in σ_i. This is guaranteed to be a unique output and clearly maps to the set O_{2n} because of our restriction that we have the least element of a cycle at the beginning of each cycle, so that we can treat this process as a function.

To show that this process ϕ is invertible and hence a bijection. Take a permutation from $\sigma \in O_{2n}$ so that $\sigma = (a_1 \ldots a_m)$ with $m \leq n$, where all the $a_i's$ are pairwise disjoint. Because this is a permutation of $1, \ldots, 2n$ we have that there must be an even number of cycles in σ, otherwise we would have an odd number of permutations that are all of odd length, meaning that $2n$ would be odd, a contradiction. Thus we can take any pair of odd cycles, and gluing them together will give us an even cycle. This will clearly give us an even permutation in E_{2n} and since we ordered this in such a way that we have ascending chain of starting elements in the disjoint cycles this process is well-defined, since this guarantees that output will be to be unique.

Define the process $\phi : E_{2n} \to O_{2n}$ to be as described above, since it's inverse $\phi^{-1} : O_{2n} \to E_{2n}$ is well-defined by the preceding paragraph, we have that this defines a bijection between O_{2n} and E_{2n}. $\square$

13.)

Let L_n be the set of ordered lists of the form $(C_1, \ldots, C_m)$ where $C_1, \ldots, C_m$ are cards containing disjoint sets with unions $\{1, \ldots, n\}$. This is similar to hands in the exponential formula with the difference being that hands are unordered and lists are ordered.

a.)

Let $C(x) = \sum\limits_{n=1}^{\infty} |C_n| \frac{x^n}{n!}$. Show that $\sum\limits_{n=0}^{\infty} \left(\sum\limits_{l \in L_n} y^{(\text{number of cards in } l)} \right) \frac{x^n}{n!} = \frac{1}{1 - yC(x)}$.

Proof. We can essentially mimic the proof for the exponential formula as follows:

$$\begin{aligned}
\frac{1}{1-yC(x)} &= \sum_{k=0}^{\infty}(yC(x))^k \\
&= \sum_{k=0}^{\infty} y^k (C(x))^k \\
&= \sum_{k=0}^{\infty} y^k \left(\sum_{n=1}^{\infty} |C_n| \frac{x^n}{n!}\right)^k \\
&= \sum_{k=0}^{\infty} y^k \left(\sum_{n=1}^{\infty} \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+\ldots+i_k=n}} \frac{|C_{i_1}|}{i_1!} \cdots \frac{|C_{i_k}|}{i_k!} \frac{x^n}{n!}\right) \\
&= \sum_{n=1}^{\infty} \left(\sum_{k=0}^{\infty} \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+\ldots+i_k=n}} y^k \frac{|C_{i_1}|}{i_1!} \cdots \frac{|C_{i_k}|}{i_k!}\right) \frac{x^n}{n!}.
\end{aligned}$$

Note then that the inner two sums, are sums over the cards in the list $L_n = (C_1, \ldots, C_k)$, when $k \geq n$ the remaining $C_k = 0$ since we can't partition $\{1, \ldots, n\}$ into more than C_n sets. So that we get our result:

$$\sum_{n=0}^{\infty} \left(\sum_{l \in L_n} y^{(\text{number of cards in } l)}\right) \frac{x^n}{n!} = \frac{1}{1-yC(x)}.$$

□

b.)

Use part $a.$ of this exercise to find the result in part $f.$ of Exercise 9 in Set 2.

Proof. Note that we can reformulate the question of (9) into a question of ordered lists of cards, since we can treat the sets in a single ordered partition as the cards, and the ordered partition itself as an ordered. Additionally, because $|C_k| = 0$ for $k > n$ we can rewrite the infinite sum over k from $(a.)$ into a finite one from $k = 1$ to n, so this just validates that $A(x, y)$ is of the form described in (9.)

So that by $(a.)$ we have that $A(x, y) = \frac{1}{1-yC(x)}$ where $C(x) = \sum_{n=1}^{\infty} |C_n| \frac{x^n}{n!}$. So that verifying that $A(x, y) = \frac{1}{1-y(e^x-1)}$ is equivalent to show that $C(x) = e^x - 1$. Well if we ask what is

$|C_n|$ for a given $\{1, \ldots, n\}$, where $|C_n|$ here is a set in the ordered partition of weight n, then there is exactly 1 way to do so. Similar to the case of unordered partitions. So that we get $|C_n| = 1$ for all $n \geq 1$ in $\mathbb{N}$. So that we have $C(x) = \sum_{n=1}^{\infty} |C_n| \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1$. Thus by $(a.)$ we have our result:

$$A(x, y) = \frac{1}{1 - yC(x)} = \frac{1}{1 - y(e^x - 1)}.$$

□

c.)

A permutation of n with ordered cycles is a list $(\sigma_1, \ldots, \sigma_m)$ where $\sigma_1, \ldots, \sigma_m$ are the cycles in a permutation of n. Let A_n be the set of permutations of n with ordered cycles and find

$$\sum_{n=0}^{\infty} \left(\sum_{l \in A_n} y^{(\text{number of cycles in } l)} \right) \frac{x^n}{n!}.$$

Proof. To show this, we will recontextualize this in terms of ordered lists and cards. So note that the disjoint cycles of a permutation form cards in a hand. Typically, we can just rearrange them however we want, and we'll get the same permutation. Clearly their unions of the cycles form the permutation of $\{1, \ldots, n\}$, so that we can put this in terms of an ordered list. With the ordered list being $(\sigma_1, \ldots, \sigma_m)$. Then our L_n is just A_n and so that applying $(a.)$ we get our result:

$$\sum_{n=0}^{\infty} \left(\sum_{l \in A_n} y^{(\text{number of cycles in } l)} \right) \frac{x^n}{n!}.$$

□

d.)

Let t_n be the total number of cards in all elements in L_n. Find a generating function involving $C(x)$ for $\sum_{n=0}^{\infty} t_n \frac{x^n}{n!}$.

Proof. Notice that our generating function from $(a.)$ we have $A(x, y) = \sum_{n=0}^{\infty} \sum_{l \in L_n} y^{(\text{number of cards in } l)} \frac{x^n}{n!}$. If we take the partial of this function with respect to y we get

$$A_y(x, y) = \sum_{n=0}^{\infty} \sum_{l \in L_n} (\#\text{ of cards in } l) y^{(\#\text{ of cards in } l) - 1} \frac{x^n}{n!}.$$

Plugging in $y = 1$ this gives us:

$$A_y(x,1) = \sum_{n=0}^{\infty} \sum_{l \in L_n} (\#\text{cards in } l) \frac{x^n}{n!} = \sum_{n=0}^{\infty} t_n \frac{x^n}{n!} = T(x).$$

Where $T(x) = \sum_{n=0}^{\infty} t_n \frac{x^n}{n!}$. Then taking the fact that $A(x,y) = \frac{1}{1-yC(x)}$, we can then say $A_y(x,y) = \frac{\partial}{\partial y}\left(\frac{1}{1-yC(x)}\right) = \frac{\partial}{\partial y}(1-yC(x))^{-1} = \frac{-(-C(x))}{(1-yC(x))^2} = \frac{C(x)}{(1-yC(x))^2}$.

Plugging in $y = 1$ we get the generating function for $A_y(x,1) = T(x) = \frac{C(x)}{(1-C(x))^2}$. $\square$