Real Analysis Presentation Problem Week 12

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Use the definition to establish each of the following limits: $\lim_{x \to p} (x^3) = p^3$

Proof. Let $\epsilon > 0$ be given, $p \in \mathbb{R}$, and define $\delta = \min\{1, \frac{\epsilon}{3|p^2|+3|p|+1}\}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x^3$ for all $x \in \mathbb{R}$. Then assume $0 < |x - p| < \delta$. Then, by our definition of δ we have $|x - p| < 1$. Note, by Corollary 2.1.4(b) we have $||x| - |p|| \leq |x - p|$, and we have $||x| - |p|| < 1$. Then by Theorem 2.1.2(b), we have $\sqrt{(|x| - |p|)^2} < 1$ iff $|x| - |p| < 1$ iff $|x| < |p| + 1$.

Note from here we get $|x|^2 < (|p| + 1)^2$ iff $|x^2| < |p|^2 + 2|p| + 1$ iff $|x^2| < |p^2| + 2|p| + 1$.

Similarly: $|p||x| < |p|(|p| + 1)$ iff $|px| < |p^2| + |p|$.

Then consider the following: $|f(x) - p^3| = |x^3 - p^3| = |(x - p)(x^2 + px + p^2)| = |x - p||x^2 + px + p^2|$.

Then since we have $|x - p| < 1$ (In either case of δ this true):

$$\begin{aligned} |x - p||x^2 + px + p^2| &< |x^2 + px + p^2| \\ &\leq |x^2| + |px| + |p^2|, \text{ by the Triangle Inequality} \\ &< (|p^2| + 2|p| + 1) + (|p^2| + |p|) + |p^2| \\ &= 3|p^2| + 3|p| + 1 \end{aligned}$$

Hence $|x - p||x^2 + px + p^2| < |x - p|(3|p^2| + 3|p| + 1)$. Then by our definition of δ, we have $|x - p| < \frac{\epsilon}{3|p^2|+3|p|+1}$. Thus $|x - p||x^2 + px + p^2| < \frac{\epsilon}{3|p^2|+3|p|+1}(3|p^2| + 3|p| + 1) = \epsilon$

$\therefore$ Thus $\lim_{x \to p} (x^3) = p^3$

Q.E.D.