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Homework #4

20.

Define $f_k(x) = \frac{x^2-3}{\sqrt{1+k^3x^2}}$

a.

Prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[1,3]$.

Proof. Consider the sequence $\{f_n(x)\}_{n=1}^{\infty}$ for any $k \in \mathbb{N}$, $f_k(x) = \frac{x^2-3}{\sqrt{1+k^3x^2}}$.
Then to show that $\{f_n(x)\}_{n=1}^{\infty}$ converges on the interval $x \in [1,3]$, we will use the definition of uniform convergence.
Let $\epsilon > 0$ be given and $x \in [1,3]$. Then choose and n_0 such that $\epsilon > \frac{6}{n_0^{3/2}}$. We may do this via the Archimedean Principle. And consider all $n \geq n_0$. Then we want to show that $|f_n(x) - 0| < \epsilon$.
So consider:

$$|\frac{x^2-3}{\sqrt{1+k^3x^2}}| \leq \left|\frac{3^2-3}{\sqrt{1+k^3(1)^2}}\right| \leq \frac{6}{\sqrt{1+k^3}} \leq \frac{6}{\sqrt{k^3}} \leq \frac{6}{k^{3/2}} \leq \frac{6}{n_0^{3/2}} < \epsilon$$

Thus we have that, by the definition of uniform convergence:

$$\lim_{n\to\infty} f_n(x) = 0$$

On the interval $[1,3]$. Is thus uniformly convergent on this same interval. □

b.

Prove that $\sum_{k=1}^{\infty} f_k$ converges uniformly on $[1,3]$.

Proof. Consider the series $\sum_{n=1}^{\infty}\{f_n(x)\}_{n=1}^{\infty}$ such that for any $n \in \mathbb{N}$, $f_n(x) = \frac{x^2-3}{\sqrt{1+n^3n^2}}$.
To show that this is a convergent series, we start from our in the middle of our inequality in part(a):

$$\left|\frac{x^2-3}{\sqrt{1+k^3x^2}}\right| \leq \left|\frac{x^2}{\sqrt{1+k^3x^2}}\right| \leq \frac{3^2}{\sqrt{1+k^3x^2}} \leq \frac{1}{\sqrt{1+k^3x^2}} \leq \frac{1}{\sqrt{1+k^3(1)^2}} \leq \frac{1}{\sqrt{k^3}} = \frac{1}{k^{3/2}}$$

Since $\sum \frac{1}{k^{3/2}}$ is a convergent p-series, we have that the series $\sum_{k=1}^{\infty}\left|\frac{x^2-3}{\sqrt{1+k^3x^2}}\right|$ is convergent by the Weierstrass M-Test. □

21.

a.

Find the Taylor series for $\sin x$ centered at $c = 0$, and prove that $\sin x$ is equal to this series for all $x \in \mathbb{R}$.
Solution:
First we should derive the Taylor series for this computationally, note that if $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, and $f'''(x) = -\cos x$. Evaluating this at $x = 0$: $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, and $f'''(0) = -1$. So we get the following general formula:

$$\sum_{k=0}^{\infty} \frac{(-1)^k (x)^{2k+1}}{(2k+1)!}$$

Proof. To show that this series representation is equal to $sin(x)$ we need to show that by Theorem 8.7.16 we have that there exists a $\xi \in (x, c)$ such that $R_n(x) = R_n(f, c)(x) = \frac{f^{(n+1)}(\xi)(x-c)^{n+1}}{(n+1)!}$, where c is the center of convergence.

Note that $f^{(n)}(x) = \pm \sin x$ or $\pm \cos x$, hence $|f^n(x)| \leq 1$, for all $n \in \mathbb{N}$.

So note that we have $0 \leq \left| \frac{f^{(n+1)}(\xi)(x-c)^{n+1}}{(n+1)!} \right| = \left| \frac{(x)^{n+1}}{(n+1)!} \right| \leq \frac{|x^{n+1}|}{(n+1)!}$.

Note that by Theorem 2.2.6(f) we have $\lim_{n \to \infty} \frac{|x^{n+1}|}{(n+1)!} = 0$ and $\lim_{n \to \infty} 0 = 0$.

Hence, by the Theorem 4.1.9 (Squeeze Theorem) we have that $\lim_{n \to \infty} R_n(x) = R_n(f, c)(x) = \frac{f^{(n+1)}(\xi)(x-c)^{n+1}}{(n+1)!} = 0$.

So, by definition, we have the Taylor series converges to the function $\sin x$. $\square$

b.

Evaluate $\int_0^1 \sin x^2 \, dx$ as a power series, and fully justify your answer.

Proof. First, note that by part(a), we have $\sin x^2 = \sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^{2k+1}}{(2k+1)!}$. To integrate this power series, we need to show that we have uniform convergence on $[0, 1]$.

To do this we will show that the radius of convergence $R = \infty$. So to show this we will show that $\lim_{k \to \infty} \left| \frac{\frac{1}{(2k+3)!}}{\frac{1}{(2k+1)!}} \right| = \lim_{k \to \infty} \frac{1}{(2k+2)(2k+3)} = \lim_{k \to \infty} \frac{1}{2k+2} \lim_{k \to \infty} \frac{1}{2k+3} = 0 \cdot 0 = 0$, by Theorem 2.2.6(a).

So we have that $R = \infty$. So by Theorem 8.7.3 we have the series converges uniformly for all x such that $|x| \leq 1$. Hence converges uniformly on the interval $[0, 1]$.

So know we may take the integral by Corollary 8.4.2 as follows:

$$\int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^{2k+1}}{(2k+1)!} \, dx = \sum_{k=0}^{\infty} \int_0^1 \frac{(-1)^k (x^2)^{2k+1}}{(2k+1)!} \, dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_0^1 (x^2)^{2k+1} \, dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_0^1 x^{4k+2} \, dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{1}{4k+3}$$

Thus we have

$$\int_0^1 \sin x^2 \ dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{1}{4k+3}$$

$\square$