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Homework #5

MATH 440
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22.

For parts (a)-(c), determine whether or not the sequences is in l^2.

a.

$$\left\{\frac{1}{3^k}\right\}_{k=1}^{\infty}$$

Solution: Yes,
consider the following:

$$\sum_{k=1}^{\infty}\left(\frac{1}{3^k}\right)^2 = \sum_{k=1}^{\infty}\frac{1}{3^{2k}} = \sum_{k=1}^{\infty}\frac{1}{9^k} = \sum_{k=1}^{\infty}\left(\frac{1}{9}\right)^k = \frac{1}{1-\frac{1}{9}} = \frac{1}{\frac{8}{9}} = \frac{9}{8} < \infty.$$

So we have that the sum of the square of this sequence is convergent, and thus

$$\left\{\frac{1}{3^k}\right\}_{k=1}^{\infty} \in l^2$$

b.

$$\left\{\frac{2}{k}\right\}_{k=1}^{\infty}$$

Solution: Yes,
consider the following:

$$\sum_{k=1}^{\infty}\left(\frac{2}{k}\right)^2 = \sum_{k=1}^{\infty}\frac{4}{k^2} = 4\sum_{k=1}^{\infty}\frac{1}{k^2}$$

Note that the sum $\sum_{k=1}^{\infty}\frac{1}{k^2} < \infty$. Thus, by Theorem 7.1.1(a) we have that the sum $4\sum_{k=1}^{\infty}\frac{1}{k^2} < \infty$. Hence we have

$$\left\{\frac{2}{k}\right\}_{k=1}^{\infty} \in l^2.$$

c.

$$\{1\}_{k=1}^{\infty}$$

Solution: No,
consider the following:

$$\sum_{k=1}^{\infty} 1^2 = \sum_{k=1}^{\infty} 1$$

Note that the we have $\lim_{k\to\infty} 1 \neq 0$, thus the sum $\sum_{k=1}^{\infty} 1 = \infty$.
Thus we have:

$$\{1\}_{k=1}^{\infty} \notin l^2$$

23.

Let X be a vector space over $\mathbb{R}$, and let $\mathbf{x}, \mathbf{y}$, and $\mathbf{z} \in X$.

i.)

If $\mathbf{x}+\mathbf{y}=\mathbf{x}+\mathbf{z}$, then $\mathbf{y}=\mathbf{x}$.

Proof. Let $\mathbf{x}+\mathbf{y}=\mathbf{x}+\mathbf{z}$ for some $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$ for some vector space X for $\mathbb{R}$. Then note that by Axiom (d) we have there exists $-\mathbf{x} \in X$ such that $\mathbf{x}+(-\mathbf{x})=\mathbf{0}$. Additionally, by Axiom (b) we have vector addition is associative. Thus this yields us the following:

$$\begin{gathered}
\mathbf{x}+\mathbf{y}=\mathbf{x}+\mathbf{z}, \text{ by Hypothesis} \\
(\mathbf{x}+-\mathbf{x})+\mathbf{y}=(\mathbf{x}+-\mathbf{x})+\mathbf{z}, \text{ by Axiom (b)} \\
\mathbf{0}+\mathbf{y}=\mathbf{0}+\mathbf{z}, \text{ by Axiom (d)} \\
\mathbf{y}=\mathbf{z}, \text{ by Axiom(c)}
\end{gathered}$$

Thus we have proven our conclusion follows from the hypothesis. $\square$

ii.)

$$0 \cdot \mathbf{x}=\mathbf{0}$$

Proof. Consider the following:

$$\begin{aligned}
\mathbf{0} &= \mathbf{x}+(-\mathbf{x}), \text{ by Axiom (c)} \\
&= 1 \cdot \mathbf{x}+(-\mathbf{x}), \text{ by Axiom (h)} \\
&= (1+0) \cdot \mathbf{x}+(-\mathbf{x}) \\
&= 1 \cdot \mathbf{x}+0 \cdot \mathbf{x}+(-\mathbf{x}), \text{ by Axiom (f)} \\
&= \mathbf{x}+(-\mathbf{x})+0 \cdot \mathbf{x}, \text{ by Axiom (h)} \\
&= \mathbf{0}+0 \cdot \mathbf{x}, \text{ by Axiom (d)} \\
&= 0 \cdot \mathbf{x}, \text{ by Axiom (c)}
\end{aligned}$$

Thus we have $\mathbf{0}=0 \cdot \mathbf{x}$. $\square$

iii.)

$$(-1) \cdot \mathbf{x}=-\mathbf{x}$$

Proof. By part(ii) we have:

$$\begin{gathered}
0 \cdot \mathbf{x}=\mathbf{0} \\
(1+(-1)) \cdot \mathbf{x}=\mathbf{x}+(-\mathbf{x}), \text{ by Axiom (c)} \\
1 \cdot \mathbf{x}+(-1) \cdot \mathbf{x}=\mathbf{x}+(-\mathbf{x}), \text{ by Axiom (g)} \\
\mathbf{x}+(-1) \cdot \mathbf{x}=\mathbf{x}+(-\mathbf{x}), \text{ by Axiom (h)}
\end{gathered}$$

Thus by part(i), we have $(-1)\mathbf{x}=-\mathbf{x}$. $\square$

24.

Give an example of a sequence $\mathbf{x} = \{x_k\}_{k=1}^{\infty}$ in l^2 such that $x_k > 0$ for all k, and such that $||\mathbf{x}||_2 = 5$. Clearly specify your example, and demonstrate that it works.

Solution:

Consider the sequence, $\{x_k\}_{k=1}^{\infty}$:

$$\left\{\left(\sqrt{\frac{24}{25}}\right)^{k-1}\right\}_{k=1}^{\infty}$$

Note that the square of this sequence, $\{x_k^2\}_{k=1}^{\infty}$, is:

$$\left\{\left(\frac{24}{25}\right)^{k-1}\right\}_{k=1}^{\infty}$$

Since $24/25 < 1$, we have the series $\sum_{k=1}^{\infty} x_k^2 < \infty$, by Theorem 2.7.0. Thus $\{x_k\}_{k=1}^{\infty} \in l^2$.
Additionally, all the terms of this sequence are positive. So now all we must do is show that $||\{x_k\}_{k=1}^{\infty}||_2 = 5$:

$$\left|\left|\left\{\left(\sqrt{\frac{24}{25}}\right)^{k-1}\right\}_{k=1}^{\infty}\right|\right|_2 = \sqrt{\sum_{k=1}^{\infty}\left(\sqrt{\frac{24}{25}}\right)^{2(k-1)}} = \sqrt{\sum_{k=1}^{\infty}\left(\frac{24}{25}\right)^{k-1}}$$

$$= \sqrt{\frac{1}{1-\frac{24}{25}}}, \text{ by Theorem 2.7.0}$$

$$= \sqrt{\frac{1}{\frac{1}{25}}} = \sqrt{25} = 5$$

Thus we have $||\{x_k\}_{k=1}^{\infty}||_2 = 5$.

Homework #5

25.

Let $(X, || \ ||)$ be a normed linear space.

a.

Formally state a definition of what it means for a sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ of vectors in X be Cauchy. (Suggestion: Start by rereading the definition of "convergence in the norm" that was given in the workbook. The relationship between your definition and the definition of convergence in the norm should be analogous to the relationship between Cauchy sequences of real numbers and convergent sequences of real numbers from Math 340.)

Solution:

Noting the parallels between the definition of convergence in the norm for a sequence of vectors and convergence of a sequence of real numbers, we define a sequence of vectors Cauchy in a normed linear space $(X, || \ ||)$ as follows:

A sequence of vectors $\{\mathbf{x}_n\}_{n=1}^{\infty}$ in a normed linear space $(X, || \ ||)$ is Cauchy if and only if for all $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$||\mathbf{x}_n - \mathbf{x}_m|| < \epsilon \text{ , for all } m, n \geq n_0.$$

b.

Prove that the sequence of vectors $\{\mathbf{x}_n\}_{n=1}^{\infty}$ converges to $\mathbf{x} \in X$, then this sequence is Cauchy.

Proof. Let $\epsilon > 0$ be given.

Let $(X, || \ ||)$ be a normed linear space. Suppose that a sequence of vectors in X, $\{\mathbf{x_n}\}_{n=1}^{\infty}$, converges to a vector $\mathbf{x} \in X$. Then by definition we have that for all $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $||\mathbf{x}_n - \mathbf{x}|| < \epsilon$ for all $n \geq n_0$. So then there must exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$||\mathbf{x}_n - \mathbf{x}|| < \frac{\epsilon}{2}.$$

Additionally, if we take a $m \in \mathbb{N}$ such that $m \geq n_0$, then we have

$$||\mathbf{x}_m - \mathbf{x}|| < \frac{\epsilon}{2}.$$

Note that $||\mathbf{x}_m - \mathbf{x}|| = |-1| \cdot ||\mathbf{x} - \mathbf{x}_m||$, by Axiom (c) of a norm, thus we have $||\mathbf{x} - \mathbf{x}_m||$. Combining these inequalities we get

$$||\mathbf{x}_n - \mathbf{x}|| + ||\mathbf{x} - \mathbf{x}_m|| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Note that by Axiom (d) we have $||\mathbf{x}_n - \mathbf{x} + \mathbf{x} - \mathbf{x}_m|| \leq ||\mathbf{x}_n - \mathbf{x}|| + ||\mathbf{x} - \mathbf{x}_m||$. Hence we have:

$$||\mathbf{x}_n - \mathbf{x} + \mathbf{x} - \mathbf{x}_m|| < \epsilon$$

$$||\mathbf{x}_n - \mathbf{x}_m|| < \epsilon$$

Thus, by our definition we have that the sequence of vectors in X, $\{\mathbf{x}_n\}_{n=1}^{\infty}$ is Cauchy in the normed linear space $(X, || \ ||)$. $\square$

26.

For each $n \in \mathbb{N}$, let $\mathbf{e}_n$ be the sequence in l^2 defined as shown to the right. By completing the following, show that, in l^2, the analogue of the Bolzano-Weierstrass Theorem (2.4.11) is false.

a.

State a formal definition of what you think it should mean for a sequence to be bounded in l^2. Then, explain why $\{\mathbf{e}_n\}_{n-1}^{\infty}$ is a bounded sequence in l^2 according to your definition.

Solution:

Our definition is as follows:

A sequence in l^2 is bounded if there exists a positive constant such that $||\mathbf{x}_n||_2 \leq M$ for all $n \in \mathbb{N}$, where every $\mathbf{x}_n$ is a sequence in l^2.

Applying this definition to our sequence $\{\mathbf{e}_n\}_{n=1}^{\infty}$ to show it is bounded we need to show that for all $n \in \mathbb{N}$ we have $||\mathbf{e}_n||_2 \leq M$, for some positive constant M.

Note, by the definition of $||\cdot||_2$ we have for any arbitrary, but fixed $n \in \mathbb{N}$:

$$||\mathbf{e}_n||_2 = \sqrt{\sum_{k=1}^{\infty} \mathbf{e}_n^2(k)} = \sqrt{0^2 + 0^2 + ... + 1^2 + 0^2 + ...} = \sqrt{1} = 1.$$

Note that this follows by the definition of this vector in l^2. So we have that the sequence $\{\mathbf{e}_n\}_{n=1}^{\infty}$ is bounded by $M = 1$, since we have shown every vector in this sequence has a norm of 1.

b.

Prove that $\{\mathbf{e}_n\}_{n=1}^{\infty}$ has no convergent subsequence.
(Hint: First, show that $||\mathbf{e}_n - \mathbf{e}_m|| = \sqrt{2}$ if $m \neq n$.)

Proof. Let $\{\mathbf{e}_n\}_{n=1}^{\infty}$ be a sequence in l^2.

Then consider any subsequence of this sequence: $\{\mathbf{e}_{n_i}\}_{i \in I}$, where $I \subseteq \mathbb{N}$.

Then note that if we take any two distinct vectors in this sequence we have the following (W.L.O.G assume that $m > n$):

$$||\mathbf{e}_n - \mathbf{e}_m||_2 = \sqrt{\sum_{k=1}^{\infty} (\mathbf{e}_n(k) - \mathbf{e}_m(k))^2} = \sqrt{(0-0)^2 + ... + (0-1)^2 + ... + (1-0)^2 + ...} = \sqrt{1+1} = \sqrt{2}.$$

So using the definition of Cauchy established in (25.) if we choose $\epsilon = 1$, then we should have for all $n, m \geq n_0$ for some $n_0 \in \mathbb{N}$ such that $||\mathbf{e}_n - \mathbf{e}_m||_2 < 1/2$. However, we have just shown that for all $m \neq n$, we have $||\mathbf{e}_n - \mathbf{e}_m|| = \sqrt{2}$.

Thus the subsequence isn't Cauchy in l^2, and thus is not convergent in l^2. ☐

27.

Let X be a vector space over $\mathbb{R}$ with inner product $\langle\ ,\ \rangle$, and define $||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ for all $\mathbf{x} \in X$. Prove the following generalization of the Pythagorean Theorem: for any $\mathbf{x}, \mathbf{y} \in X$, we have

$$||\mathbf{x}||^2 + ||\mathbf{y}||^2 = ||\mathbf{x} + \mathbf{y}||^2 \text{ if and only if } \langle \mathbf{x}, \mathbf{y} \rangle = 0.$$

(**Important Note:** You may <u>not</u> assume that $||\ ||$ is a norm your proof, since the result you are proving was used to prove the Cauchy-Schwarz Inequality, which, in turn, is used to prove that $||\ ||$ is a norm. In particular, use only the definition of an inner product and the above definition of $||\ ||$ in your proof.)

Proof. Let X be a vector space over $\mathbb{R}$ with inner product $\langle\ ,\ \rangle$, and define $||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ for all $\mathbf{x} \in X$. Suppose $||\mathbf{x}||^2 + ||\mathbf{y}||^2 = ||\mathbf{x} + \mathbf{y}||^2$, for some $\mathbf{x}, \mathbf{y} \in X$. Then consider the following:

$$\begin{aligned}
||\mathbf{x}||^2 + ||\mathbf{y}||^2 &= ||\mathbf{x} + \mathbf{y}||^2, \text{ by hypothesis} \\
\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle &= \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle, \text{ by Def of } || \cdot || \\
\langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle &= \langle \mathbf{x}, \mathbf{x} + \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle, \text{ by Axiom (d)} \\
\langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle &= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle, \text{ by Linearity in the 2nd component} \\
\langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle &= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle, \text{ by Axiom(c)} \\
\langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle &= \langle \mathbf{x}, \mathbf{x} \rangle + 2 \cdot \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle, \text{ by scalar addition facts} \\
0 &= 2 \cdot \langle \mathbf{x}, \mathbf{y} \rangle, \text{ by scalar addition cancellation} \\
0 &= \langle \mathbf{x}, \mathbf{y} \rangle
\end{aligned}$$

Note that the above are all if-and-only-if statements, thus we have shown the theorem holds in both directions. $\square$

28.

If $\{a_k\} \in l^2$, prove that $\sum_{k=1}^{\infty} \frac{a_k}{k}$ converges absolutely.

Lemma

For all $a, b \in \mathbb{R}$,

$$|ab| \leq \frac{1}{2}(a^2 + b^2)$$

Proof. Let $a, b \in \mathbb{R}$. Then we'll show that the inequality holds by showing that it reduces to a true statement via if and only if's:

$$|ab| \leq \frac{1}{2}(a^2 + b^2)\sqrt{(ab)^2} \leq \frac{1}{2}a^2 + \frac{1}{2}b^2, \text{ by Theorem 2.1.2(c)}$$

$$(ab)^2 \leq \frac{1}{4}(a^2 + b^2)^2$$

$$(ab)^2 \leq \frac{1}{4}(a^4 + 2(ab)^2 + b^4)$$

$$4(ab)^2 \leq a^4 + 2(ab)^2 + b^4$$

$$2(ab)^2 \leq a^4 + b^4$$

$$0 \leq a^4 - 2(ab)^2 + b^4$$

$$0 \leq (a^2 - b^2)^2$$

A square of any real number will always be positive, hence this last statement is true. Thus, we have our lemma holds for all $a, b \in \mathbb{R}$. $\square$

Here is the proof for our original theorem.

Proof. Let $\{a_k\}_{k=1}^{\infty} \in l^2$. Then consider the following series:

$$\sum_{k=1}^{\infty} \left| \frac{a_k}{k} \right|.$$

To show the above series is convergent, we will use the Comparison Test. By our Lemma, we have $|a_k \cdot \frac{1}{k}| \leq \frac{1}{2}(a_k^2 + \frac{1}{k^2})$ for all $k \in \mathbb{Z}^+$. Furthermore, since $\{a_k\}_{k=1}^{\infty} \in l^2$, we have $\sum_{k=1}^{\infty} a_k^2 < \infty$. Similarly, the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series.
Note that by the definition of series convergence we have:

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} a_k, \text{ where } S_n \text{ is the sequence of partial sums}$$

So if for two series $\sum b_k = b < \infty$ and $\sum a_k = a < \infty$, then we can use Theorem 2.2.1, to say that $\sum_{k=1}^{\infty} a_k + b_k = a + b < \infty$.
So by this 340 fact, we have $\sum_{k=1}^{\infty} a_k^2 + \frac{1}{k^2} < \infty$. And by Theorem 7.1.1, we have $\frac{1}{2}\sum_{k=1}^{\infty} a_k^2 + \frac{1}{k^2} < \infty$. Finally, note that all terms in the series $\left|\frac{a_k}{k}\right| \geq 0$ and $a_k^2 + \frac{1}{k^2} \geq 0$, for all $k \in \mathbb{Z}^+$.
Thus, by the Comparison Test, we have $\sum_{k=1}^{\infty} \left|\frac{a_k}{k}\right| < \infty$. $\square$

29.

a.

Let $\phi_0(x) = 1$, $\phi_1(x) = x - a_1$, and $\phi_2(x) = x^2 - a_2 x - a_3$. Determine the constants a_1, a_2, and a_3 so that the set $\{\phi_0, \phi_1, \phi_2\}$ are orthogonal on $[0, 1]$.

Solution:

Evaluating the integral using Mathematica $\int_0^1 \phi_0(x)\phi_1(x)\ dx = 0$ and solving for a_1, we get $a_1 = 1/2$. Evaluating both the integral's $\int_0^1 \phi_0(x)\phi_2(x)\ dx = 0$ and $\int_0^1 \phi_1(x)\phi_2(x)\ dx = 0$. We have $a_1 = 1/2, a_2 = 1, a_3 = -1/6$.

b.

Find the polynomial of degree less than or equal to 2 that best approximates $f(x) = \sin(\pi x)$ in the mean on $[0, 1]$.

Solution:

Using Theorem 9.1.4 and using the family of orthogonal functions introduced in part(a),
$\{\phi_0(x), \phi_1(x), \phi_2(x)\}$, we have the best approximation is found by the series:

$$S_3(x) = \sum_{k=1}^{3} c_k \phi_k(x).$$

To minimize the error we need to solve for the coefficients as follows:

$$c_k = \frac{\int_0^1 sin(\pi x)\phi_k(x)\ dx}{\int_0^1 \phi_k^2(x)\ dx}.$$

Doing this successively, for each $\phi_k(x)$ gives us the best fit approximation of:

$$\frac{\pi}{2} + \frac{60}{\pi^3}\left(\pi^2 - 12\right)\left(x^2 - x + \frac{1}{6}\right) \approx \sin(\pi x), \text{ on the interval } [0, 1].$$