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Group Assignment # 3

MATH 550
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#1

Let (a_n) be a bounded sequence.

(a)

Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

Proof. First, note that the sequence is decreasing, since $y_1 = \sup\{a_1, a_2, \ldots\} \geq y_2 = \sup\{a_2, a_3, \ldots\} \geq \ldots$. Furthermore, because the set $\{a_1, a_2, \ldots\}$ is bounded, we have that, for $n \in \mathbb{N}$, the set $\{a_n, a_{n+1}, \ldots\}$ is also bounded. So employing the monotone convergence theorem for sequences gives us that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, \ldots\} = \inf_{n\in\mathbb{N}} \sup\{a_n, a_{n+1}, \ldots\}.$$

□

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(b)

The *limit superior* of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n,$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

Definition. We can define the limit inferior as

$$\liminf a_n \equiv \lim_{n\to\infty} \inf\{a_n, a_{n+1}, \ldots\} = \lim_{n\to\infty} \inf_{k\geq n} a_k.$$

This exists for any bounded sequence as the $\inf a_n$ must exist for a bounded sequence, and furthermore the sequence $\inf\limits_{k\geq n} a_k$ is increasing and bounded, meaning we can employ the monotone convergence theorem for sequences to get that:

$$\lim_{n\to\infty} \inf_{k\geq n} a_k = \sup_{n\in\mathbb{N}} \inf_{k\geq n} a_k.$$

$\square$

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(c)

Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

Proof. Note for any bounded sequence $\inf_{k\geq n} a_k \leq a_n \leq \sup_{k\geq n} a_k$ for all $n \in \mathbb{N}$. So that we can just take the limit as $n \to \infty$ of the above inequality to get:

$$\lim_{n\to\infty} \inf_{k\geq n} a_k \leq \lim_{n\to\infty} \sup_{k\geq n}.$$

□

Example. Define the sequence

$$a_n = \begin{cases} 1 & \text{if n is odd} \\ 2 & \text{if n is even} \end{cases}.$$

Notice that the sequence doesn't converge, but

$$\lim_{n\to\infty} \inf_{k\geq n} a_k = \lim_{n\to\infty} \inf\{1,2\} = 1,$$

and

$$\lim_{n\to\infty} \sup_{k\geq n} a_k = \lim_{n\to\infty} \sup\{1,2\} = 2.$$

Hence we have a strict inequality here. □

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(d)

Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Proof. Let a_n be any sequence with values in $\mathbb{R}$.

For the forwards direction, suppose that $\liminf a_n = \limsup a_n$. Then note that for any $n \in \mathbb{N}$, that $\inf_{k \geq n} a_k \leq a_n \leq \sup_{k \geq n}$. Using the squeeze theorem here, we then get that $\lim_{n\to\infty} a_n = a = \limsup a_n = \liminf a_n$.

In the backwards direction, suppose that $\lim_{n\to\infty} a_n$. Then note that $\inf\{a_1, a_2, \ldots\} \leq \inf_{k \geq n} a_k \leq a_n \leq \sup_{k \geq n} a_k \leq \sup\{a_1, a_2, \ldots\}$. So then since the sup and inf are the least upper bound and greatest lower bound, respectively, we have that there must exist a_m and a_p such that $a_m \leq \inf_{k \geq n} a_k \leq a_n \leq \sup_{k \geq n} a_k \leq a_p$. Furthermore, since (a_n) is a convergence sequence we have any subsequence shares the limit of (a_n), hence we can employ the squeeze theorem once more to get:

$$\lim_{n\to\infty} a_n = \liminf a_n = \limsup a_n.$$

Thus we have our result! $\square$

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2.

Let (x_n) be a sequence of real numbers.

Let $L = \{a \in [-\infty, +\infty] : a \text{ is a limit of some subsequence of } (x_n)\}$. Prove that $\sup L = \limsup x_n$.

Solution is attached.

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17.

Suppose that $(X, \mathcal{S}, \mu)$ is a measure space and $f_1, f_2, \ldots$ is a sequence of non-negative $\mathcal{S}$-measurable functions on X. Define a function $f : X \to [0, \infty]$ by $f(x) = \liminf_{n\to\infty} f_k(x)$.

(a)

Show that f is an $\mathcal{S}$-measurable function.

Solution attached.

(b)

Prove that

$$\int f\ d\mu \leq \liminf_{n\to\infty} \int f_k\ d\mu.$$

Proof. First, note that $\inf\{f_n, f_{n+1}, \ldots\}$ is the greatest lower bound of the set $\{f_n, f_{n+1}, \ldots\}$, that is for all $k \in \{n, n+1, n+2, \ldots\}$, $\inf\{f_n, f_{n+1}, \ldots\} \leq f_k$. Additionally, by (a) we have that $\inf\{f_n, f_{n+1}, \ldots\}$ is $\mathcal{S}-$measurable, so that we can employ (3.8) (integration is order preserving) to get

$$\int \inf\{f_n, f_{n+1}, \ldots\}\ d\mu \leq \int f_k\ d\mu,$$

for all $k \in \{n, n+1, \ldots\}$. That is, $\int \inf\{f_n, f_{n+1}, \ldots\}\ d\mu$ is a lower bound of the set $\{\int f_n\ d\mu, \int f_{n+1}\ d\mu, \ldots\}$, so that this must less than or equal to the greatest lower bound, $\inf\{\int f_n\ d\mu, \int f_{n+1}\ d\mu, \ldots\}$. In symbols that is

$$\int \inf\{f_n, f_{n+1}, \ldots\}\ d\mu \leq \inf\{\int f_n\ d\mu, \int f_{n+1}\ d\mu, \ldots\}$$

.

To finish the proof we'll employ our friend the monotone convergence theorem. From 2.4.7 of Abbott, we showed that $\inf\{f_n(x), f_{n+1}(x) \ldots\}$ is an increasing sequence (for fixed $x \in X$), additionally for all $n \in \mathbb{N}$, $\inf_{k\geq n} f_k(x)$ is an $\mathcal{S}$-measurable function by (a). Hence we can employ the monotone convergence theorem (for integration) after applying limits to both sides of the inequality:

$$\begin{aligned}
\lim_{n\to\infty} \int \inf\{f_n, f_{n+1}, \ldots\}\ d\mu &\leq \lim_{n\to\infty} \inf\{\int f_n\ d\mu, \int f_{n+1}\ d\mu, \ldots\} \\
\iff \int \lim_{n\to\infty} \inf\{f_n, f_{n+1}, \ldots\}\ d\mu &\leq \lim_{n\to\infty} \inf\{\int f_n\ d\mu, \int f_{n+1}\ d\mu, \ldots\} \\
\iff \int (\liminf_{n\to\infty} f_n)\ d\mu &\leq \liminf_{n\to\infty} (\int f_n\ d\mu) \\
\iff \int f\ d\mu &\leq \liminf_{n\to\infty} \int f_n\ d\mu.
\end{aligned}$$

This is our result! □

(c)

Give an example show that the inequality in (b) can be strict inequality even when $\mu(X) < \infty$ and the family of functions $\{f_k\}_{k\in\mathbb{Z}^+}$ is uniformly bounded.

Example. Note that by the paragraph following (2.94) we have that for any Lebesgue measurable set $A \subseteq \mathbb{R}$, that we can define a measure space as all the Lebesgue measurable subsets of A along with the standard outer measure $\lambda : L_A \to [-\infty, \infty]$. So we'll be looking at the finite measure space $([0,2], L_{[0,2]}, \lambda)$ where $L_{[0,2]}$ is the set of all Lebesgue measurable subsets of $[0,2]$ and λ is outer measure.

First, note that $\lambda([0,2]) = 2 < \infty$. Define the sequence of simple functions $f_n : [0,2] \to [0,\infty]$ given by

$$f_n(x) = \begin{cases} 1\chi_{[0,1]}(x) & \text{if n is odd} \\ \frac{1}{2}\chi_{[0,2]}(x) & \text{if n is even} \end{cases},$$

for all $n \in \mathbb{N}$ and $x \in [0,2]$. This sequence of functions is uniformly bounded by 1. Then note that importantly this sequence doesn't converge for any $x \in [0,2]$, since if $x \in [0,1]$ we'll have $\{f_n(x), f_{n+1}(x), \ldots\} = \{1, \frac{1}{2}\}$, when $x \in (1,2]$ we'll have $\{f_n(x), f_{n+1}(x), \ldots\} = \{\frac{1}{2}, 0\}$. But note that:

$$\inf\{\int f_n(x)\ d\lambda, \int f_{n+1}(x)\ d\lambda, \ldots\} = \inf\{1\lambda([0,1]), \frac{1}{2}\lambda([0,2])\} = 1,$$

where we used (3.4) to evaluate the above integrals. But that

$$\inf\{f_n(x), f_{n+1}(x), \ldots\} = \begin{cases} \frac{1}{2} & \text{if } x \in [0,1] \\ 0 & \text{if } x \in (0,2] \end{cases} = \frac{1}{2}\chi_{[0,1]} + 0\chi_{(1,2]}.$$

So that when we evaluate the integral of this we'll get:

$$\int (\frac{1}{2}\chi_{[0,1]} + 0\chi_{(1,2]})\ d\lambda = \frac{1}{2}\lambda([0,1]) = \frac{1}{2},$$

where we used (3.7) to evaluate the integral above. Since $\frac{1}{2} < 1$, we have a strict inequality! $\square$