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MATH 530: Discrete Mathematics
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Homework # 1

1

Using the Taylor series centered at $x = 0$, show that $(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$ where $\binom{a}{n} = \frac{a(a-1)\ldots(a-n+1)}{n!}$.

Proof. First to compute the Taylor series we'll start with taking derivatives of $f(x) = (1+x)^a$.

$$\begin{aligned} f(x) &= (1+x)^a & f(0) &= (1)^a \\ f'(x) &= a(1+x)^{a-1} & f'(0) &= a \\ f''(x) &= a(a-1)(1+x)^{a-2} & f''(0) &= a(a-1) \\ &\vdots & &\vdots \\ f^{(n)}(x) &= a(a-1)\ldots(a-(n-1))(1+x)^{a-n} & f^{(n)}(0) &= a(a-1)\ldots(a-n+1) \end{aligned}$$

So that the Taylor series of $(1+x)^a$ has coefficients: $c_n = \frac{a(a-1)\ldots(a-n+1)}{n!} = \binom{a}{n}$. Thus the Taylor series representation gives us $(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$. □

3

Verify the following identities involving the products of series:

- $\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$
- $\left(\sum_{n=0}^{\infty} a_n x^n\right)^k = \sum_{n=0}^{\infty}\left(\sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\ldots+i_k=n}} a_{i_1}\ldots a_{i_k}\right) x^n$

Solution. So we'll start with a finite sum, and then take the limit to infinity of that sum. Additionally, I'll start from right to left.

$$\begin{aligned} \sum_{n=0}^{m}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n &= \sum_{n=0}^{m} x^n (a_0 b_n + a_1 b_{n-1} + \ldots + a_{n-1} b_1 + a_n b_0) \\ &= (a_0 b_0) x^0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \ldots + (a_0 b_m + a_1 b_{m-1} + \ldots + a_m b_0) x^m \\ &= a_0 b_0 + a_0 b_1 x + a_1 b_0 x + a_0 b_2 x^2 + a_1 b_1 x^2 + a_2 b_0 x^2 + \ldots + a_0 b_m x^m + a_1 b_{m-1} x^m + \ldots + a_m b_0 x^m \\ &= a_0 (b_0 + b_1 x + b_2 x^2 + \ldots + b_{m-1} x^{m-1} + b_m x^m) \\ &+ a_1 (b_0 + b_1 x + b_2 x^2 + \ldots + b_{m-1} x^{m-1} + b_m x^m) + \ldots \\ &+ a_m (b_0 + b_1 x + b_2 x^2 + \ldots + b_{m-1} x^{m-1} + b_m x^m) \\ &= (b_0 + b_1 x + b_2 x^2 + \ldots + b_m x^m)(a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m) \\ &= \left(\sum_{n=0}^{m} b_n x^n\right)\left(\sum_{n=0}^{m} a_n x^n\right) \\ &= \left(\sum_{n=0}^{m} a_n x^n\right)\left(\sum_{n=0}^{m} b_n x^n\right) \end{aligned}$$

With the final step giving us:

$$\lim_{m\to\infty}\sum_{n=0}^{m}\left(\sum_{k=0}^{n}a_kb_{n-k}\right)x^n = \lim_{m\to\infty}\left(\sum_{n=0}^{m}a_nx^n\right)\left(\sum_{n=0}^{m}b_nx^n\right)$$

Finally

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_kb_{n-k}\right)x^n = \left(\sum_{n=0}^{\infty}a_nx^n\right)\left(\sum_{n=0}^{\infty}b_nx^n\right)$$

□

Proof. We'll show (b) through mathematical induction on k using finite sums again here up to $m \in \mathbb{N}$:

(Basis)

We'll use (a) here:

$$\left(\sum_{n=0}^{m}a_nx^n\right)^2 = \sum_{n=0}^{m}\left(\sum_{k=0}^{n}a_ka_{n-k}\right)x^n$$

We can rewrite the index here so that $i_1 = k$ and $i_2 = n-k$ so that $i_1 + i_2 = n$ and we iterate over $i_1, i_2 \geq 0$. Giving us

$$\left(\sum_{n=0}^{m}a_nx^n\right)^2 = \sum_{n=0}^{m}\left(\sum_{\substack{i_1,i_2\geq 0\\ i_1+i_2=n}}\right)x^n$$

(Inductive Hypothesis)

Assume for some $k \in \mathbb{N}$ we have

$$\left(\sum_{n=0}^{\infty}a_nx^n\right)^k = \sum_{n=0}^{\infty}\left(\sum_{\substack{i_1,\ldots,i_k\geq 0\\ i_1+\ldots+i_k=n}}\right)x^n$$

Then we have:

$$\left(\sum_{n=0}^{m}a_nx^n\right)^k\left(\sum_{n=0}^{m}a_nx^n\right) = \left(\sum_{n=0}^{m}a_nx^n\right)\left(\sum_{n=0}^{m}\left(\sum_{\substack{i_1,\ldots,i_k\geq 0\\ i_1+\ldots+i_k=n}}a_{i_1}\ldots a_{i_k}\right)x^n\right) \qquad \text{by the inductive hypothesis}$$

$$\left(\sum_{n=0}^{m}a_nx^n\right)^{k+1} = \sum_{n=0}^{m}\left(\sum_{j=0}^{n}a_{n-j}\left(\sum_{\substack{i_1,\ldots,i_k\geq 0\\ i_1+\ldots+i_k=j}}a_{i_1}a_{i_2}\ldots a_{i_k}\right)\right)x^n$$

With a careful relabeling with $i_{k+1} = n - j$ note then that we can bring the outer sum of:

$$\sum_{j=0}^{n}a_{i_{k+1}}\sum_{\substack{i_1,\ldots,i_k\geq 0\\ i_1+\ldots+i_k=n}}a_{i_1}\ldots a_{i_k}$$

5. Let a_n be the number of ways to tile a $2 \times n$ chessboard with dominoes of sizes 2×1 and 2×2. For example, there are 11 such tilings when $n = 4$:

Find a recurrence for a_n, the generating function for a_n, and a formula for a_n.

into the inner sum since the above is equivalent to summing over $i_1, ..., i_k, i_{k+1} \geq 0$ with $i_1 + i_2 + \ldots + i_k + (i_{k+1}) = j + (n - j) = n$. So that we have our result:

$$\left(\sum_{n=0}^{m} a_n x^n\right)^{k+1} = \sum_{n=0}^{m} \left(\sum_{\substack{i_1,\ldots,i_k,i_{k+1} \geq 0 \\ i_1+\ldots+i_k+i_{k+1}=n}} \right) x^n$$

Thus by the principle of mathematical induction we have our desired result as shown above for any $k \in \mathbb{N}$. Now taking the limit as $m \to \infty$ we get that

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)^{k} = \sum_{n=0}^{\infty} \left(\sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\ldots+i_k=n}} a_{i_1} \ldots a_{i_k} \right) x^n$$

$\square$

5

Proof. Recurrence Relation. First note that $a_0 = 1$ (the null arrangement, placing no blocks), $a_1 = 1$ (Placing one vertical block), I claim then that $a_n = 2a_{n-2} + a_{n-1}$ for $n \geq 0$. First, note that we have three options when placing the first block. It's either a 2×2, a vertical 2×1, or a horizontal 2×1. Note that in the choice of a horizontal 2×1, we also lose the space directly above it as well, since we're forced to place another 2×1 on top of it. Moreover, these are independent choices. Placing a 2×2 first, we have lost 2 columns, so that we have a_{n-2} arrangements left. Placing a vertical 2×1, we have only lost one column so we have a_{n-1} arrangements left. Finally, placing one and in turn two 2×1 horizontally, we get that we have a total of a_{n-2} possible arrangements. Combining the three we get that $a_n = 2a_{n-2} + a_{n-1}$. $\square$

Now to find the generating function for a_n:

Proof. Generating Function of (a_n). Let $A(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ be the generating function for the sequence (a_n) with $a_n = 2a_{n-2} + a_{n-1}$ for $n \geq 2$ and $a_0 = a_1 = 1$. Then we'll expand this out to find the generating function of (a_n):

$$\begin{aligned}
A(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \\
&= 1 + 1x + \sum_{n=2}^{\infty} (2a_{n-2} + a_{n-1}) x^n \\
&= 1 + x + 2\sum_{n=2}^{\infty} a_{n-2} x^n + \sum_{n=2}^{\infty} a_{n-1} x^n \\
&= 1 + x + 2x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} + x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} \\
&= 1 + x + 2x^2 \sum_{n=0}^{\infty} a_n x^n + x \sum_{n=1}^{\infty} a_n x^n \\
&= 1 + x + 2x^2 A(x) + x(a_0 + \sum_{n=1}^{\infty} a_n x^n) - x a_0 \\
&= 1 + x - x + 2x^2 A(x) + x \sum_{n=0}^{\infty} a_n x^n \\
A(x) &= 1 + 2x^2 A(x) + x A(x)
\end{aligned}$$

This gives us that $A(x) = \frac{-1}{2x^2+x-1}$, factoring this we find: $A(x) = \frac{-1}{(2x-1)(x+1)}$. Our generating function! □

To find the closed form of the sequence:

Closed Form of the Sequence. $A(x) = \frac{-1}{(2x-1)(x+1)}$ using partial fraction decomposition we get that $A(x) = \frac{-2}{3}\frac{1}{2x-1} + \frac{1}{3}\frac{1}{1+x}$. These are geometric series in disguise, so substituting those in:

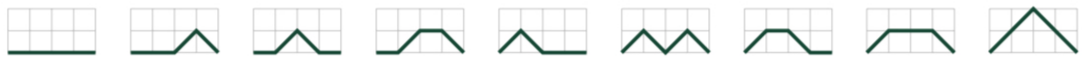
$$\begin{aligned}
A(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= \frac{2}{3}\sum_{n=0}^{\infty} (2x)^n + \frac{1}{3}\sum_{n=0}^{\infty} (-1)^n x^n \\
&= \sum_{n=0}^{\infty} \frac{(2)^{n+1} + (-1)^n}{3} x^n
\end{aligned}$$

Giving us the closed form of (a_n) :

$$a_n = \frac{2^{n+1} + (-1)^n}{3}$$

□

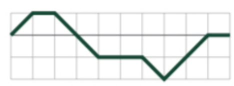
6. A Motzkin path of length n is a path in the plane which starts at $(0,0)$, ends at $(n,0)$, uses steps of the form $(1,1)$, $(1,-1)$, and $(1,0)$, and never travels below (but may touch) the x-axis. For example,



are the 9 Motzkin paths of length 4. Let m_n be the number of Motzkin paths of length n and let $M(x) = \sum_{n=0}^{\infty} m_n x^n$.

a. Show that $(M(x)-1)/x = M(x) + xM(x)^2$ and then find an explicit formula for $M(x)$.

b. Let a_n be the number of paths in the plane which start at $(0,0)$, end at $(0,n)$, and use steps of the form $(1,1)$, $(1,-1)$, and $(1,0)$. For example, one path when $n = 11$ is



By looking at the first time a path touches the x axis, show that $a_{n+2} = a_{n+1} + 2\sum_{k=0}^{n} m_k a_{n-k}$ for $n \geq 0$.

c. Show that $A(x) = 1/\sqrt{1-2x-3x^2}$.

6

a.)

First, we'll go through a proof of $m_{n+2} = m_{n+1} + \sum_{k=0}^{n} m_k m_{n-k}$ for $n \geq 0$. Then show that the equality holds and then find a generating function for $M(x)$.

Proof. Recurrence Relation. First, note that $m_0 = 1$(The null path), $m_1 = 1$. Then for $n \geq 0$, we have exactly two choices at $(0,0)$ going straight with a step of $(1,0)$ or going diagonal $(1,1)$. Moreover, these are independent, if I make one the other is not possible. So that if I go straight with step $(1,0)$, the problem of getting to $(n+2,0)$ has m_{n+1} possible paths.

If I go diagonal with step $(1,1)$, then I'll break this down into two problems that are dependent on each other. Let's note that at some point we have to get back down to the x-axis, and that we have to pass through the line $y = 1$ to do so. So that letting $k+2$ be the x-value that we hit the x-axis, immediately before that we were at $(k+1,1)$. Then $k+1$ has possible values of 1 to $n+1$. The number of paths to get to this $k+1$ is then the problem of getting from $(1,1)$ to $(k+1,1)$ so that this is just the problem of m_k.

The remaining problem of getting from $(k+2,0)$ to $(n+2,0)$ has a total of $m_{n+2-(k+2)} = m_{n-k}$. We get that the total number of paths after going up a diagonal from $(0,0)$ and getting to $(n+2,0)$ is exactly $m_k m_{n-k}$ for $k \in \{0,...,n\}$. Of course we have to sum over all possible k's so that we have in total $\sum_{k=0}^{n} m_k m_{n-k}$.

Combining this with the other choice $m_{n+2} = m_{n+1} + \sum_{k=0}^{n} m_k m_{n-k}$, for $n \geq 0$.

Alternatively, we can write this as $m_{n+2} = m_{n+1} + \sum_{k=0}^{n} m_k m_{n-k}, n \geq 0$. $\square$

Proof. Generating Function and Equation. Let $M(x) = \sum_{n=0}^{\infty} m_n x^n$ be the generating function for the sequence (m_n) where $m_0 = m_1 = 1$ and $m_n = m_{n-1} + \sum_{k=0}^{n-2} m_k m_{n-2-k}$.

Then we have the following:

$$\begin{aligned}
M(x) &= \sum_{n=0}^{\infty} m_n x^n \\
&= 1 + x + \sum_{n=2}^{\infty} m_n x^n \\
&= 1 + x + \sum_{n=2}^{\infty} (m_{n-1} + \sum_{k=0}^{n-2} m_k m_{n-2-k}) x^n \\
&= 1 + x + \sum_{n=2}^{\infty} m_{n-1} x^n + \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n-2} m_k m_{n-2-k} \right) x^n \\
&= 1 + x + x \sum_{n=2}^{\infty} m_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} m_k m_{n-2-k} x^{n-2} \\
&= 1 + x + x \sum_{n=1}^{\infty} m_n x^n + x^2 \left(\sum_{n=2} m_{n-2} x^{n-2} \right)^2 \qquad \text{by 3(a)} \\
&= 1 + x \sum_{n=0}^{\infty} m_n x^n + x^2 \left(\sum_{n=0}^{\infty} m_n x^n \right)^2 \\
&= 1 + xM(x) + x^2 M(x)^2 \\
M(x) &= 1 + xM(x) + x^2 M(x)^2 \\
\frac{M(x) - 1}{x} &= M(x) + xM(x)^2 \qquad \checkmark
\end{aligned}$$

Now to find the generating function we'll start off from the second to last line above.

$$\begin{aligned}
M(x) &= 1 + xM(x) + x^2 M(x)^2 \\
0 &= 1 + (x-1)M(x) + x^2 M(x)^2 M(x) \qquad = \frac{-(x-1) \pm \sqrt{(x-1)^2 - 4x^2}}{2x^2}
\end{aligned}$$

We want $M(x) < \infty$ at $x = 0$, but

$$\lim_{x \to 0} \frac{-(x-1) + \sqrt{(x-1)^2 - 4x^2}}{2x^2} = \frac{1+1}{0} = \infty$$

So rejecting this solution we have the generating function of $M(x)$ is:

$$M(x) = \frac{1 - x - \sqrt{(x-1)^2 - 4x^2}}{2x^2}$$

□

b.)

Proof. First, note that $a_0 = 1$ (the null path) and $a_1 = 1$.

Then note that this time we have three choices $(1,0),(1,1),(1,-1)$ for our first step at $(0,0)$ since we can go below the x-axis here. These paths are independent of each other, so we just add them to find the number of all possible paths between $(0,0) \to (n,0)$.

In the case of $(1,0)$, we haven't moved from the x-axis so that this problem is equivalent to the problem a_{n-1}.

In the case of $(1,1)$ we have a similar problem presented in $(a.)$ where we have to pass through the line $y = 1$ to get to the diagonal. This problem is just m_k as presented in (a). Except this time after we get to the diagonal, our choices our now a_{n-2-k} to get to $(n,0)$ from $(k,0)$. So that in this case we have $m_k a_{n-2-k}$ paths after moving up a diagonal, for a fixed k. We then sum over k to find the total with this first move: $\sum\limits_{k=0}^{n-2} m_k a_{n-2-k}$

Nicely enough, moving down a diagonal by $(1,-1)$ is a symmetric problem to the case of moving up a diagonal $(1,1)$. So that the number of ways to get from $(0,0)$ to $(n,0)$ after moving down a diagonal initially is exactly $\sum\limits_{k=0}^{n-2} m_k a_{n-2-k}$.

Combining the three scenarios we get that $a_n = a_{n-1} + 2\sum\limits_{k=0}^{n-2} m_k a_{n-2-k}$, for $n \geq 2$. Changing the index $n \to n+2$:

$$a_{n+2} = a_{n+1} + 2\sum_{k=0}^{n} m_k a_{n-k}$$

$\square$

c.)

Proof.

$$
\begin{aligned}
A(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= 1 + x + \sum_{n=2}^{\infty} a_n x^n \\
&= 1 + x + \sum_{n=2}^{\infty} \left(a_{n-1} + 2\sum_{k=0}^{n-2} m_k a_{n-2-k} \right) x^n \\
&= 1 + x + \sum_{n=2}^{\infty} a_{n-1} x^n + 2\sum_{n=2}^{\infty}\sum_{k=0}^{n-2} m_k a_{n-2-k} x^n \\
&= 1 + x + x\sum_{n=1}^{\infty} a_n x^n + 2x^2\sum_{n=2}^{\infty}\sum_{k=0}^{n-2} m_k a_{n-2-k} x^{n-2} \\
&= 1 + x\sum_{n=0}^{\infty} a_n x^n + 2x^2\sum_{n=0}^{\infty}\sum_{k=0}^{n} m_k a_{n-k} x^n \\
&= 1 + xA(x) + 2x^2 (\sum_{n=0}^{\infty} a_n x^n)(\sum_{n=0}^{\infty} m_n x^n) \qquad \text{by 3(a)} A(x) \qquad = 1 + xA(x) + 2x^2 A(x)M(x) \\
-1 &= -A(x) + xA(x) + 2x^2 A(x)M(x) \\
1 &= A(x) - xA(x) - 2x^2 A(x)M(x) \\
1 &= A(x)(1 - x - 2x^2 M(x)) \\
1 &= A(x)(1 - x - (1 - x - \sqrt{(x-1)^2 - 4x^2})) \\
1 &= A(x)\sqrt{x^2 - 2x + 1 - 4x^2} \\
1 &= A(x)\sqrt{-3x^2 - 2x + 1} \\
A(x) &= \frac{1}{\sqrt{-3x^2 - 2x + 1}}
\end{aligned}
$$

□