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Homework # 4

16

Let $\alpha > 0$ and n be a nonnegative integer.

a.)

Use induction to show that $\int\limits_0^1 x^{\alpha-1}(1-x)^n \, dx = \frac{n!}{(\alpha)(\alpha+1)\ldots(\alpha+n)}$.

Proof.

- (Basis)

  For $n = 0$ we have:

$$\begin{aligned}\int_0^1 x^{\alpha-1} \, dx &= \frac{x^\alpha}{\alpha}\bigg|_{x=0}^{1} \\ &= \frac{1}{\alpha} \\ &= \frac{0!}{(\alpha)} \checkmark.\end{aligned}$$

- (Inductive Hypothesis)

  Assume for some positive integer n that we have

$$\int_0^1 x^{\alpha-1}(1-x)^n \, dx = \frac{n!}{(\alpha)\ldots(\alpha+n)}.$$

Then consider the following:

$$\begin{aligned}
\int_0^1 x^{\alpha-1}(1-x)^{n+1}\ dx &= \int_0^1 x^{\alpha-1}(1-x)(1-x)^n\ dx \\
&= \int_0^1 x^{\alpha-1}(1-x)^n\ dx + \int_0^1 -x^{\alpha}(1-x)^n\ dx \\
(I.H) &= \frac{n!}{(\alpha)(\alpha+1)\dots(\alpha+n)} + \int_0^1 x^{\alpha}(1-x)^n\ dx \\
(I.B.P) &= \frac{n!}{(\alpha)(\alpha+1)\dots(\alpha+n)} \\
&\quad - \left(x^{\alpha}(-1)\frac{(1-x)^{n.+1}}{n+1}\Big|_{x=0}^{1} - \frac{-\alpha}{n+1}\int_0^1 x^{\alpha-1}(1-x)^{n+1}\ dx \right) \\
\int_0^1 x^{\alpha-1}(1-x)^{n+1}\ dx &= \frac{n!}{(\alpha)(\alpha+1)\dots(\alpha+n)} - \frac{\alpha}{n+1}\int_0^1 x^{\alpha-1}(1-x)^{n+1}\ dx \\
\left(1+\frac{\alpha}{n+1}\right)\int_0^1 x^{\alpha-1}(1-x)^{n+1}\ dx &= \frac{n!}{(\alpha)(\alpha+1)\dots(\alpha+n)} \\
\int_0^1 x^{\alpha-1}(1-x)^{n+1}\ dx &= \frac{(n+1)n!}{(\alpha+n+1)(\alpha)(\alpha+1)\dots(\alpha+n)} \\
&= \frac{(n+1)!}{\alpha(\alpha+1)\dots(\alpha+n)(\alpha+n+1)}.
\end{aligned}$$

Therefore, by the principle of mathematical induction our result holds for all $n \in \mathbb{N}$.

$\square$

b.)

Assuming that the limit and integral can be interchanged, use $\lim\limits_{n\to\infty} \int\limits_0^n x^{\alpha-1}\left(1-\frac{x}{n}\right)^n\ dx$ to show that

$$\Gamma(\alpha) = \lim_{n\to\infty} \frac{n!n^{\alpha}}{\alpha(\alpha+1)\dots(\alpha+n)}.$$

Proof. First, note that for $\lim\limits_{n\to\infty} \int\limits_0^n x^{\alpha-1}\left(1-\frac{x}{n}\right)^n\ dx$, taking the limit of $\left(1+\frac{-x}{n}\right)^n$ as $n\to\infty$ gives us e^{-x} and since the limits of this integral are 0 to n we'll get that:

$$\lim_{n\to\infty}\int_0^n x^{\alpha-1}\left(1-\frac{x}{n}\right)^n\ dx = \int_0^\infty x^{\alpha-1}e^{-x}\ dx = \Gamma(\alpha).$$

To show the equality we'll use $(a.)$ with a change of variables to evaluate the integral. Take $u = xn, du = n\ dx$ so $x = 0$ becomes $u = 0$ and $x = 1$ becomes $u = n$ so that we get the following:

$$\begin{aligned}\int_0^1 x^{\alpha-1}(1-x)^n\ dx &= \frac{1}{n}\int_{u(0)}^{u(1)} \left(\frac{u}{n}\right)^{\alpha-1}\left(1-\frac{u}{n}\right)^n\ du \\ &= \frac{1}{n^\alpha}\int_0^n u^{\alpha-1}\left(1-\frac{u}{n}\right)^n\ du.\end{aligned}$$

So using $(a.)$ we see that the right-hand side of the equation doesn't depend on x so that we get, changing u back to x for neatness we get that:

$$\begin{aligned}\frac{1}{n^\alpha}\int_0^n x^{\alpha-1}\left(1-\frac{x}{n}\right)^n\ dx &= \frac{n!}{\alpha(\alpha+1)\dots(\alpha+n)} \\ \int_0^n x^{\alpha-1}\left(1-\frac{x}{n}\right)^n\ dx &= \frac{n^\alpha n!}{\alpha(\alpha+1)\dots(\alpha+n)}.\end{aligned}$$

Hence our result with:

$$\Gamma(\alpha) = \lim_{n\to\infty}\int_0^n x^{\alpha-1}\left(1-\frac{x}{n}\right)^n\ dx = \lim_{n\to\infty}\frac{n^\alpha n!}{\alpha(\alpha+1)\dots(\alpha+n)}.$$

□

c.

Justify why the limit and integral can be interchanged in part b.

Proof. □

17.)

Let a_n be the number of ordered set partitions of n. Set 2 Exercise $9c$ gives $a_n = \frac{1}{2}\sum_{k=0}^{\infty} k^n 2^{-k} \approx \frac{1}{2}\int_0^\infty x^n 2^{-x}\ dx$. Use a substitution in the above intergral and then use Stirling's approximation to find $a_n \approx \frac{\sqrt{2\pi n}}{\ln 4}\left(\frac{n}{e\ln 2}\right)^n$.

Proof. Consider the following with the u-substitution of $e^{-u} = 2^{-x} \implies u = x\ln 2$ with $dx = \frac{du}{\ln 2}$

$$\begin{aligned}
a_n &= \frac{1}{2}\sum_{k=0}^{\infty} k^n 2^{-k} \\
&\approx \frac{1}{2}\int_0^{\infty} x^n 2^{-x}\ dx \\
&= \frac{1}{2\ln 2}\int_{u(0)}^{u(\infty)} \left(\frac{u}{\ln 2}\right)^n e^{-u}\ du \\
&= \frac{1}{\ln 4(\ln 2)^n}\int_0^{\infty} u^n e^{-u}\ du \\
&\approx \frac{1}{\ln 4(\ln 2)^n}\frac{\sqrt{2\pi n}n^n}{e^n} \\
&= \frac{\sqrt{2\pi n}}{\ln 4}\left(\frac{n}{e\ln 2}\right)^n.
\end{aligned}$$

And we have our result! □

18.)

Let A_n be the set of paths in $\mathbb{R}^2$ which start at $(0,0)$, end at (n,n) , and only use steps of the form $(1,0)$ and $(0,1)$. Denote the number of times the path $p \in A_n$ touches the line $y = x$ by touch(p). Let

$$A(x,t) = \sum_{n=0}^{\infty}\left(\sum_{p\in A_n} t^{\text{touch}(p)}\right)x^n.$$

a.

Let c_n be the n^{th} Catalan number. Show that

$$\sum_{p\in A_{n+1}} t^{\text{touch}(p)} = 2t\sum_{k=0}^{n} c_k\left(\sum_{p\in A_{n-k}} t^{\text{touch}(p)}\right).$$

Proof. Note that for a given $n+1 \in \mathbb{N}$ then $\sum\limits_{p\in A_{n+1}} t^{\text{touch}(p)}$ can be broken down in the following way: the number of paths that first touch the diagonal at (k,k) after starting off at $(0,0)$ is exactly c_k, then from there the remaining paths that can touch the diagonal will be $\sum_{p\in A_{n-k}} t^{\text{touch}(p)+1}$. To account for all paths touching at $(0,0)$ and $(n+1,n+1)$ there will 2 additional points to each path so that we get:

$$\sum_{p\in A_{n+1}} t^{\text{touch}(p)} = 2t\sum_{k=0}^{n} c_k \left(\sum_{p\in A_{n-k}} t^{\text{touch}(p)}\right).$$

$\square$

b.

Show that $A(x,t) = \frac{t}{1-t+2t\sqrt{\frac{1}{4}-x}}$.

c.

With the help of the corollary to the first asymptotic result in video 17, find an asymptotic formula for the average number of times a path in A_n touches the line $y=x$.