Partially Ordered Sets and Möbius Inversion

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1 Partially Ordered Sets

In this paper we will be looking at the mathematical objects known as a partially ordered sets, some lines that can be drawn from posets to combonatorics and graph theory, and finally how they relate to the process of Möbius inversion.

Definition 1.0. A *partially ordered set*, or poset, is any set that is grouped with a binary relation " $\leq$ " that is reflexive, anti-symmetric, and transitive; that is, it has the following properties:

1. For all $t \in P$, $t \leq t$. (Reflexive)
2. If $s \leq t$ and $t \leq s$, then $s = t$. (Anti-Symmetric)
3. If $s \leq t$ and $t \leq u$, then $s \leq u$. (Transitive)

Example 1.1. The simplest of a poset is the real numbers and the $\leq$ relation. That is,

1. $x \leq x$, for all $x \in \mathbb{R}$,
2. For all $x, y \in \mathbb{R}$, $x \leq y$ and $y \leq x$ implies $x = y$,
3. If $x, y, z \in \mathbb{R}$ and $x \leq y$ and $y \leq z$, then $x \leq z$.

Hence it satisfies all the conditions of a partial ordering. So $\mathbb{R}$ paired with $\leq$ is then a called a poset, or partially ordered set. Later on, we will learn that this structure of $(\mathbb{R}, \leq)$ actually constitutes what is called a totally ordered set, or a linear ordering.

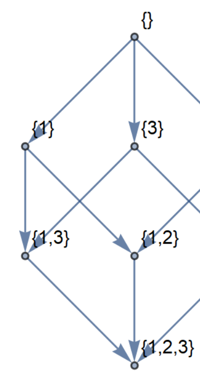
While the partial ordering on the real numbers is rather "well behaved", a general poset doesn't necessarily have to be so nice. As we get into trickier posets, it becomes useful to be able to see how all the elements of the poset are related. One way of doing this is called a Hasse diagram.

Definition 1.2. A *Hasse diagram* is a visualization of a finite poset, such that the vertices are the elements of the set, and the edges connecting the vertices describe the relation between two elements via the "$<$" ordering of the set. A Hasse diagram follows the following properties

1. If $x < y$ in the poset, then the vertex corresponding to x appears lower in the diagram than the point corresponding to y [3].

Remark 1.3. Note that here $a < b$ or $a > b$ is equivalent to $a \leq b$ with $a \neq b$ or $a \geq b$ with $a \neq b$, respectively. Notice that when we compare the $<$ and $\leq$ relations, the only property they share from the definition of a partial ordering is that of transitivity; that is $<$ is not reflexive and not anti-symmetric. We call the relation $<$ a strict partial order, or simply a strict ordering on a set P.

Example 1.4. The following is the Hasse diagram of the partially ordered set of $P(\{1, 2, 3\})$ partially ordered by containment; that is, the poset $(P(\{1, 2, 3\}), \subseteq)$.



Partially ordered sets and their associated Hasse diagrams have many applications, in areas as diverse as economics, physics, and everyday problems such as comparing ways of travel. In fact, any kind of situation where we can introduce some kind of order, posets can be used.

Example 1.5. An interesting application of posets appears in physics [2]. In the theory of relativity, it becomes useful to define the potential for an event to influence another event. This kind poset is defined as the set of all possible events occurring in the universe, paired with the causal relation; that is, we say X causally influences Y if and only if $X \leq Y$. Note that it satisfies all the properties of a poset:

1. All events X are trivially causally influenced by themselves; i.e. $X \leq X$

2. If an event X is causally influenced by an event Y, and Y was causally influenced by X, then X and Y must be the same event; i.e if $Y \leq X$ and $X \leq Y$, then $X = Y$.

3. If an event X causally influences an event Y and Y causally influences an event Z, then X causally influences Z; i.e if $X \leq Y$ and $Y \leq Z$, then $X \leq Z$.

This kind of poset is used as one way to derive relativity using a minimal set of assumptions. This is called a *causal set* in physics.

Now that we have some way to visualize posets and have seen some examples of their applications, it becomes useful to define how two posets can be related to each other. Specifically, what it means for two posets to be almost identical to each other.

Definition 1.6. Two posets P and Q are *isomorphic*, denoted $P \approx Q$, if and only if, there exists an order-preserving bijection $\phi : P \to Q$ such that $x \leq y$ in $P \iff \phi(x) \leq \phi(y)$ in Q. The mapping ϕ is called an isomorphism between P and Q.

Note that if two posets are isomorphic, then their Hasse diagrams are also isomorphic in the graph theoretic notation; that is, they have all the properties that isomorphic graphs share. In fact the two are equivalent, two Hasse diagrams are isomorphic in the terms of graph theory if and only if the two corresponding posets are isomorphic.

However, just as in graph theory, a bijection is not always an isomorphism, meaning that it is possible for two sets to have the same number of elements, but not share the same adjacency's or "shape", as we will see in the next example.

Example 1.7. Consider the two sets $\mathbb{Z}$ and $\mathbb{N}$ each with the traditional partial ordering $a \leq b$. Then we'll define the mapping $\phi : \mathbb{N} \to \mathbb{Z}$ by

$$\phi(x) = \begin{cases} x & \text{if x is even} \\ -(x+1) & \text{if x is odd} \end{cases}$$

.

This mapping can be show to be a bijection between the two sets. However, consider the elements $2, 3 \in \mathbb{N}$. Clearly we have $2 \leq 3$ in $\mathbb{N}$, but $\phi(2) = 2$, and $\phi(3) = -4$, so $\phi(3) \leq \phi(2)$ in $\mathbb{Z}$. Hence this is a bijection between the two sets $\mathbb{N}$ and $\mathbb{Z}$, but does not preserve the partial ordering, thus ϕ is not an isomorphism between these two posets.

Definition 1.8. For any poset P, if all elements of P are *comparable*, that is, if for any $x, y \in P$, $x \leq y$ or $y \leq x$, then P is called a *chain*. If no two elements in P are comparable, then P is called an *antichain*.

Example 1.9. Consider the set of singleton sets up to n, such that $n \in \mathbb{N}$, ordered by containment; i.e the set $\{\{1\},\{2\},\{3\},...,\{n\}\}$ paired with $\subseteq$. Note that no two elements of this set are comparable, hence this set is an antichain. However, if we were to consider the set: $\{\emptyset,\{1\},\{1,2\},...,\{1,2,...,n\}\}$ ordered with containment, this poset is a chain, since any two elements are either supersets of each other or subsets.

Some convenient consequences follow from this definition of chains and antichains:

Definition 1.10. For any $S \subseteq P$ where P is a poset, S is a *subposet* of P if the following condition holds: $x, y \in S$ where $x \leq y$ in S if and only if $x \leq y$ in P.

Note that from this definition, similar to general set theory, every poset P is a subposet of itself and that the empty set $\emptyset$ paired with the partial ordering is vacuously a subposet of every poset P.

Theorem 1.11. Let P be a chain and Q be an antichain.

1. Any subposet of P is a chain.

2. Any subposet of Q is an antichain.

Proof. Let P, Q be partially ordered sets with the arbitrary partial ordering $\leq$ such that P is a chain and Q is an antichain.

1. Note that for P to be a chain, then we have for all $x, y \in P$ either $x \leq y$ or $y \leq x$. Let E be a subposet of P, then if $x, y \in E$, then $x, y \in P$ and either $x \leq y$ or $y \leq x$. Hence E is a chain.

2. Note that for Q to be an antichain, no two element in Q are comparable. Hence if P is a subposet of Q, then no two elements in P are comparable. Therefore, P is an antichain.

$\square$

This result might seem trivial, but is convenient when dealing with subposets of chains or antichains.

Definition 1.12. Given a poset P, if there exists a $x \in P$ such that no y in P is greater than x, then x is a *maximal element* of the poset P. Moreover, if for all $z \in P$, $z \leq x$, then x is called the *maximum element* of P; denoted $\hat{1}$.

If a subposet A of P

Definition 1.13. Given a poset Q, if there exists a $x \in Q$ such that no y in Q is less than x, then x is a *minimal element* of the poset Q. Moreover, if for all $y \in Q$ $x \leq y$, then x is called the *minimum element* of Q; denoted $\hat{0}$.

Note that from this definition, for any partially ordered set P, if there isn't at least one element comparable to every other element of the poset, then P has no maximum element. This is not true, however, for maximal elements. The same holds true for minimum and minimal elements.

Theorem 1.14. Every finite poset has minimal and maximal elements.

Proof. Let P be a finite partially ordered set paired with the arbitrary partial ordering $\leq$.
(Minimal Elements)
Then for sake of contradiction suppose that P has no minimal elements. So that means for all $x \in P$, there exists $y \in P$ such that $y \leq x$. So either $y = x$ or $y < x$.

If $y = x$, then that means that x has only itself that it is $\leq$. Hence no other element is less than x, and x is a minimal element. A contradiction.

If $y < x$, then that means for all $x \in P$, there exists $y < x$, but that would mean that P is infinite. A contradiction.

Thus in any possible case, we have a contradiction. Therefore P must have at least one minimal element.
(Maximal Elements) Suppose, for sake of contradiction, that P has no maximal elements. Then for all $x \in P$ there exist $y \in P$ such that $x \leq y$. So either $y = x$ or $x < y$.

If x = y, then x has no greater element then itself, hence x is a maximal element. A contradiction.

If $x < y$, then that means for all $x \in P$, there exists a $y \in P$ such that $x < y$, continuing this on ad infinitum, we get that P is infinite. A contradiction.

Thus, in any possible case we have a contradiction. Therefore P must have at least one maximal element. □

Note that while every partially ordered set has minimal and maximal elements, but the same is not true for maximum and minimum elements.

Example 1.15. Consider the antichain of $\{\{1\},\{7\}\}$ with the partial order of $\subseteq$. Then since no two elements are comparable, there can't be any maximum or minimum elements.

Note that from the previous example, that while that poset has no maximum or minimum element, that it does have maximal and minimal elements. In fact it turns out that all elements of the poset are both maximal and minimal elements. We will formally write this out this in the next theorem.

Theorem 1.16. Every element of an antichain is both a maximal and minimal element.

Proof. Let P be a partially ordered set paired with the arbitrary partial ordering $\leq$ such that P is an antichain. Then for any $x \in P$, x isn't comparable to any other element of P. Hence there exists no element in P such that x is $\leq$ or $\geq$ to that element. Thus x is both a minimum element of P and a maximal element of P. $\square$

Example 1.17. Consider the subposet $\{\emptyset,\{1\},\{1,2\},\{1,2,3\}\}$ of the poset from Example 1.10. Then the element $\{1,2,3\}$ is a maximal element of the subposet, moreover it is the maximum element of the subposet, since every element of the poset is contained within this set.

Definition 1.18. Let P be a poset with the arbitrary partial ordering $\leq$, $x, y \in P$, and $x < y$. If there exists no $z \in P$ such that $x < z < y$, then we say y covers x. Note that this definition is equivalent to saying that in the Hasse diagram, there exists an edge between x and y.

Definition 1.19. We call a collection of disjoint chains whose union forms a poset, a *chain covering* of a partially ordered set. In symbols that is:

Let P be a poset, A be an index set, and $\{C_\alpha\}_{\alpha\in A}$ be a collection of chains such that $C_\alpha \subseteq P$ for all $\alpha \in A$. If we have,

$$\bigcup_{\alpha\in A} C_\alpha = P \text{ and } \bigcap_{\alpha\in A} C_\alpha = \emptyset,$$

we call $\{C_\alpha\}_{\alpha\in A}$ a chain covering of P.

Note that from this definition, we have the trivial result that any chain is a chain covering of itself.

Example 1.20. Consider the poset and Hasse diagram from Example 1.5. Then let us construct a chain covering of the poset $(\mathcal{P}(\{1,2,3\}), \subseteq)$. Let $A_1 = \{\},\{3\}$, $A_2 = \{\{1\},\{1,3\}\}$, $A_3 = \{\{2\},\{2,3\}\}$, and $A_4 = \{\{1,2\},\{1,2,3\}\}$. Note that for any of the A_i's, $i \in \{1,2,3,4\}$, we have A_i is a chain, and that

$$A_1 \cup A_2 \cup A_3 \cup A_4 = P(\{1,2,3\}) \text{ and } A_1 \cap A_2 \cap A_3 \cap A_4 = \emptyset.$$

Hence, by definition, we have $\{A_1, A_2, A_3, A_4\}$ is a chain covering of the poset $(\mathcal{P}(\{1,2,3\}), \subseteq)$

Definition 1.21. Let P be a poset and A be an antichain of P. If A is the largest such antichain of P, we call A the *maximum antichain* of P. If we have A cannot be extended any further and remain an antichain, then we call A a *maximal antichain*. We define a set C be a *maximum chain* or *maximal chain* in a similar manner.

Now that we have some vocabulary down and some examples out of the way, we will move into the first substantive theorem of this paper.

Theorem 1.22. (Dilworth's Theorem) Let P be a finite partially ordered set. Then the size of the the largest antichain is equal to the number of chains in any smallest chain cover.

The following proof is as provided by Miklós Bóna in *A Walk Through Combonatorics*[1].

Proof. Let P be a partially ordered set such that $|P| = n$. Let A be the largest antichain in P such that $|A| = m$ and $\{C_\alpha\}_{\alpha=1}^{l}$, for some $l \in \mathbb{N}$, be the smallest chain covering of P. Then we wish to show that $l = m$, we will do this by showing $l \leq m$ and $m \leq l$. Note that throughout this proof we will only use $\leq$ in the traditional usage when referring to the cardinalities of the sets, unless otherwise stated.

$(m \leq l)$

Since we have A is the largest antichain in P, we have $A \subseteq P$. Also we have $\bigcup_{\alpha=1}^{l} C_\alpha = P$, so $A \subseteq \bigcup_{\alpha=1}^{l} C_\alpha$. Since A is an antichain and $A \subseteq \bigcup_{\alpha=1}^{l} C_\alpha$, then some sub-collection of our chain cover must encompass all of A such that the size of this sub-collection is m. Hence $|A| \leq \left|\bigcup_{\alpha=1}^{l} C_\alpha\right|$, so $m \leq l$.

$(l \leq m)$

To prove the converse, we will show that if $|A| = k$ for some $k \in \mathbb{N}$, then P can be decomposed into the union of k chains; that is, if the largest antichain in P is size k, then we can decompose P into a chain cover of size k. Hence $l = m$, whence $l \leq m$. Giving us our desired result. So to do this we will proceed by using mathematical induction on the cardinality of P; that is, induction over n. Note that when $n = 0$, our theorem holds trivially, so we will start with

$n = 1$.

($Basis$) n = 1

Then we have $|P| = 1$. Hence, we have P is both a chain and an antichain. Hence $|A| = 1$ and we may decompose P into itself, yielding us a chain covering of size 1. Hence we have a basis for our claim.

(*Inductive Hypothesis*)

Assume that for all $2 \leq j \leq n$, we have that P can be composed into a chain cover the size of its largest antichain. Then take note that we can break this down into two cases:

1. Either P has an antichain A that contains at least one element that is not minimal in P, and one element that is not maximal in P.

2. Or all maximal antichains of P consist of all maximal elements or all minimal elements of P.

(*Case 1.*)

Suppose that P has a k-element antichain, call it A, that contains at least one element that is not maximal. So there exists $a \in A$. Then we may partition P into two distinct sets, call them U and L. U will be defined as the set of all elements in P, such that $x \in U$ if and only if $u \leq x$ for some $u \in A$, and L will be defined such that $x \in L$ if and only if $x \leq l$, for some $l \in A$. Notice that these two sets partition the set A, since A contains non-minimal and non-maximal elements, U and L are non-empty and preserve the partial ordering of the poset P. Moreover, $|U|, |L| \leq n$. Thus, by our induction hypothesis, we have that both U and L can be covered by k chains separately. Note that each of the k chains in U has one of the k elements of A at its bottom, and each of the k chains in L has one of the k elements of A at its top. Thus we have these $2k$ chains can be united to cover P, thus we have a chain covering of size k.

(*Case 2*)

Assume that for all maximal antichains of P consist of maximal elements only or minimal elements only. That necessarily implies that they contain all minimal elements or all maximal elements. Let x be a minimal element of P, and let y be a maximal element of P such that $x \leq y$. Let P' be the poset obtained from P by omitting x and y. Then the largest antichain of P' has $k - 1$ elements as it cannot contain all minimal elements or maximal elements of P. Moreover, P' has less than n elements, so by the inductive hypothesis, it can be decomposed into $k - 1$ chains.

Adding the two-element chain $x \leq y$ to this chain cover of P', we get a k-element chain cover of P. $\square$

Note that this theorem is useful for any finite partially ordered set, such as the first n natural numbers ordered by divisibility, this particular poset will end up being particularly interesting in the preceding section.

Definition 1.23. Let P be a poset such that $|P| = n$ and $n \in \mathbb{N}$. Then if there exists a order-preserving bijection from P onto $\{1, 2, ..., n\}$, that is, if $x \leq y$ in P, then $f(x) \leq f(y)$ in $\{1, 2, ..., n\}$, we call this bijection a *linear extension.* Note that this is close to being the poset is isomorphic to $\{1, 2, ..., n\}$, except an isomorphism is a bi-conditional order preserving bijection, that is an "if and only if".

2 Möbius Inversion

Now that we have established what a partially ordered set is, we will begin to dive into a computational processes on partially ordered set called Möbius Inversion. Here we will begin to see the wide-ranging connections that partially ordered sets share with the world's of analysis and algebra. But first we must

Definition 2.0. Let P be a partially ordered set and $x, y \in P$ such that $x \leq y$. Then we call $\{z \in P : x \leq z \leq y\}$ the *closed interval between x and y.* If all closed intervals in P are finite, then P is called *locally finite.* Interestingly enough, if P is locally finite, that doesn't imply that P is finite.

Example 2.1. Let $P = \mathbb{N}$ paired with the partial ordering of divisibility; that is, $x \leq y$ in $\mathbb{N}$ if and only if $x|y$ in $\mathbb{N}$. Then for any $x, y \in \mathbb{N}$ we have the closed interval $[x, y]$ is finite, but the set $\mathbb{N}$ is itself infinite. Thus $\mathbb{N}$ is locally finite, but $\mathbb{N}$ isn't finite.

Definition 2.2. Let P be a partially ordered set and $I \subseteq P$. If $y \in I$ whenever $x \in I$ and $y \leq x$, then we call I an *ideal.* Moreover, if I is of the form $I = \{y : y \leq x\}$, then we call I a *principal ideal.*

Example 2.3. Let $P = \mathbb{N}$ paired with the partial ordering of divisibility as in Example 2.1. Then the ideal generated by 12 is a principal ideal such that $I = \{1, 2, 3, 4, 6, 12\}$. Note that I is also locally finite.

Definition 2.4. $Int(P)$ is defined as the set of all intervals of P.

Definition 2.5. Let P be a locally finite partially ordered set. We define the incidence algebra of $I(P)$ of P is the set of all functions $f : Int(P) \to \mathbb{R}$. Multiplication in this algebra is defined by

$$(f \cdot g)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y)$$

Note that this definition is very similar to that of matrix multiplication, and indeed can be shown to be isomorphic to the algebra of square upper triangular matrices with matrix multiplication; that is, matrices with all zero-entries below the main diagonal.

Since we have introduced the topic of algebra into our context, this begs the question: Does the indices algebra have a unity element? That is, does there exist a $e \in I(P)$ such that for all

$f \in I(P)$, we have $ef = fe = f$. The answer is yes, and we can use the previously mentioned isomorphism between square upper triangular matrices and indices algebra to find it. Recall, that for any square matrix of size $m \times m$ we have the identity matrix of I_m we give us that original matrix back when we multiply it by I_m. We can define the matrix using the *Kronecker delta function*; that is

$$I_{m_{ij}} = \delta(i,j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Conveniently enough, we can use the same function is terms of the indices algebra, i.e we have the unity element of the indices algebra is

$$\delta(x,y) = \begin{cases} 1 & x = y \\ 0 & x < y \end{cases}$$

Next we will proof this:

Theorem 2.6.

$$\delta(x,y) = \begin{cases} 1 & x = y \\ 0 & x < y \end{cases}$$

is the unity element of the indices algebra of partially ordered sets.

Proof. Let P be a partially ordered set with the arbitrary partial ordering of $\leq$. Then let $f : Int(P) \to \mathbb{R}$ be a function. Note that since f is defined on an interval we have $f = f(x,y) = f([x,y])$ for all $[x,y] \in Int(P)$. Then consider the following:

$$(f \cdot \delta)(x,y) = \sum_{x \leq z \leq y} f(x,z) \cdot \delta(z,y), \text{ from Definition 2.5.}$$

$$= f(x,x) \cdot \delta(x,y) + ... + f(x,y) \cdot \delta(y,y) = f(x,x) \cdot 0 + ... + f(x,y) \cdot 1, \text{ from the definition of } \delta$$

$$= f(x,y)$$

Thus we have retrieved our original function f, hence we have $f\delta = f$. A similar argument follows for $\delta f = f$. Thus we have δ is the unity element in the indices algebra. $\square$

Definition 2.7. Let P be a locally finite partially ordered set. Let $\zeta \in I(P)$ be defined by $\zeta(x,y) = 1$ for all $x \leq y$. We call ζ the *zeta-function* of P.

We will later relate back to each other the zeta-function and the Kronecker delta-function, but first we must get some more definitions out of the way.

Definition 2.8. Let $M = \{a_1b_1, a_2b_2, ..., a_mb_m : a_i \in \mathbb{N}$ for all i ={1,2,...,m}} be a multi-set, then if the following inequality holds we call M a *multichain*: $a_1 \leq a_2 \leq ... \leq a_m$.

An analog to Dilworth's theorem can be found by relating the zeta-function of locally finite partially ordered sets and the number of multichains of that partially ordered set.

Theorem 2.9. Let $x \leq y$ be elements of the locally finite partially ordered set P. Then the number of multichains $x = x_0 \leq x_1 \leq ... \leq x_k = y$ is equal to $\zeta^k(x,y)$; where $\zeta^k(x,y)$ refers to taking the zeta for the interval $[x_0, x_k]$.

The following proof is as provided by Miklós Bóna in *A Walk Through Combonatorics*[1].

Proof. By induction k. If $k = 1$, then we have $\zeta^1(x,y) = 1$ if and only if $x \leq y$, and the statements is true. (In fact, the statement is even true if $k = 0$. Then $\zeta^0(x,y) = \delta(x,y) = 1$ if and only if $x = y$).

Now assume that the statement is true for all positive integers less than k. Each multichain $x = x_0 \leq x_1 \leq x_2 \leq ... \leq x_k = y$ can be uniquely be decomposed to a multichain $x = x_0 \leq x_1 \leq x_2 \leq ... \leq x_{k-1} = z$, and a two-element multichain $z \leq y$, where $z \in [x,y]$. Fix such a z. Then our inductive hypothesis implies that the number of multichains $x = x_0 \leq x_1 \leq x_2 \leq ... \leq x_{k-1} = z$ is $\zeta^{k-1}(x,z)$, while the number of multichains $z \leq y$ is $\zeta(z,y)$. Summing over all z, we get that the total number of multichains $x = x_0 \leq x_1 \leq x_2 \leq ... \leq x_k = y$ is

$$\sum_{z \in [x,y]} \zeta^{k-1}(x,z)\zeta(z,y) = \zeta^k(x,y).$$

□

Since we have shown that $I(P)$ contains an identity element and that $\zeta \in I(P)$. The next natural question to ask is does ζ have an inverse in the algebra of $I(P)$? The answer is, of course, yes. Here it is again useful to put this in terms of square upper triangular matrices. The corresponding ζ-matrix is a matrix that has 1 on all its diagonals, since we have the result of

$\zeta(x,x) = 1$ for all $x \in P$. Nicely enough, this gives us that $det(\zeta - Matrix) \neq 0$. Hence ζ-Matrix has an inverse.

Definition 2.10. The corresponding inverse to the ζ function of a partially ordered set P is called the Möbius function of P, denoted $\mu = \mu_P$

However, if the reader will recall Linear Algebra results, calculating the inverse of a sufficiently large matrix is an extreme pain. Since we are dealing with upper triangular matrices, this makes it somewhat easier, and indeed allows us to use a recursive computation of the μ function of P.

Theorem 2.11. Let P be a locally finite partially ordered set and $[x,y] \in Int(P)$. Then

$$\mu(x,y) = -\sum_{x \leq z < y} \mu(x,z)$$

, if $x < y$. In other words, μ satisfies $\sum_{x \leq z \leq y} \mu(x,z) = 0$ for all $x < y$.

Proof. Let P be a locally finite partially ordered set and $[x,y] \in Int(P)$.

$$\mu\zeta(x,y) = \delta(x,y) = \sum_{x \leq z \leq y} \mu(x,z)\zeta(z,y)$$

$$= \sum_{x \leq z \leq y} \mu(x,z), \text{ since for all } z \leq y \text{ we have } \zeta(z,y) = 1, \text{ by def.}$$

Thus we have $\mu\zeta(x,y) = \delta(x,y) = \sum_{x \leq z \leq y} \mu(x,z)$. Hence for all $x < y$ we have $\delta(x,y) = 0$, hence

$$\sum_{x \leq z < y} \mu(x,z) = 0.$$

□

Corollary 2.12. Let P be a locally finite partially ordered set. Let $[x,y] \in Int(P)$, and let us assume that $x \neq y$. Then

$$\mu(x,y) = \sum_{x \leq z \leq y} \mu(z,y).$$

Proof. The proof is completed in our proof of Theorem 2.11. □

These results allows us a much easier way of directly calculating the Möbius function for a given partially ordered set P. Since we can just iterate through an interval instead of deriving a general formula. Indeed since we have $\mu(x,x) = 1$, for the interval $[x,y]$ we will always start with 1 and can define the rest of the function using out recursive method. Note that this means that the terms of the series definition of the μ-function will either be 1, -1, or 0.

Example 2.13. In number theory, a specific Möbius function paired with a specific Zeta-function play an important role in the exploration of primes and the distribution of them. We'll define this function using what we know. Let $P = \mathbb{Z}^+$ paired with the partial ordering of the divides relation. Then the Möbius function of P can be shown to be

$$\mu(x,y) = \begin{cases} (-1)^k & \frac{y}{x} = p_1 p_2 ... p_k \text{ are distinct primes} \\ 0 & \frac{y}{x} = pk, \text{ for some } k \in \mathbb{Z} \text{ and p is some prime} \end{cases}$$

Taking the general idea from Theorem 2.11, we want to come up with a process on how to compute a Möbius function for any locally finite partially ordered set. This generalized process is the culmination of the paper, and is called **Möbius Inversion**.

Theorem 2.14. [Möbius Inversion Formula] Let P be a partially ordered set in which each principla ideal if finite, and let $f : P \to \mathbb{R}$ be a function. Let $g : P \to \mathbb{R}$ be defined by

$$g(y) = \sum_{x \leq y} f(x).$$

Then

$$f(y) = \sum_{x \leq y} g(x)\mu(x,y).$$

Proof. Let $x_1, x_2, ...$ be a linear extension of P. Let f be the row vector defined by $f_i = f(x_i)$, and let g be the row vector defined by $g_i = g(x_i)$. Let Z be the matrix defined by the zeta-function of P, and let M be the matrix defined by the Möbius function of P. Then the equality $g(y) = \sum_{x \leq y} f(x)$ just means

$$g = fZ$$

. Multiplying both sides by M from the right, and using the fact that $ZM = I$ we get

$$gM = f,$$

which is equivalent to our claim. [1] $\square$

3 Conclusion

The nature of partially ordered sets and applications of them are far to varied and diverse to be covered in a single paper. I hope the reader takes away this fact and the rich potential that posets offer as a mathematical object to be studied. As far as the contents of this paper goes, it was my best attempt to cover the material covered in Chapter 16 of Miklós Bóna's book *A Walk Through Combonatorics*. Somethings that might be interesting to look into from here is diving more deeply into the subject, and begin to look into lattices and begin to "peek into" the field of poset topology. Or draw even more definite parallels between the areas of modern algebra and posets and see what falls out of this. Alternatively, one road to go down is an in-depth study of the applications of posets in computer science, physics, and chemistry, in a way that a mathematican might understand. Regardless, the area of posets and Möbius Inversion offers a rich field for undergraduate research.

References

[1] Bóna, Miklós. *A Walk Through Combonatorics: An Introduction to Enumeration and Graph Theory.* New Jersey: World Scientific, 2006. Print.

[2] Carroll, Sean M. *Lecture Notes on General Relativity.* Santa Barbara: University of California, 1997. Notes.

[3] Weisstein, Eric W. "Hasse Diagram." From MathWorld–A Wolfram Web Resource.