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Homework #1

Exercise 2.

In definition of an *affine algebraic variety*, describe the set $\{F_i : i \in I\}$. In particular, what is an example of the object F_i?

Solution:

In the definition F_i is called a complex polynomial, meaning that it is a polynomial that is defined on the complex numbers. So the polynomial $p(x) = x^2 + 1$ is one such example.

Exercise 3.

For each, describe the variety in $\mathbb{C}^n$.

1. $\mathbb{V}(0)$
2. $\mathbb{V}(1)$
3. $\mathbb{V}(7)$
4. $\mathbb{V}(x_1 - 5, x_2 - 5, x_3 - 5, ..., x_n - 5)$

Solution:

1. $\mathbb{C}^n$, so this is of the whole complex numbers of dimension n.
2. $\emptyset$, this is the empty set because this the constant polynomial $p(x) = 1$, which doesn't have vanishing.
3. $\emptyset$, for the same reason as in 2.
4. $\{5\}$, since this is the common vanishing of the function $x_i - 5$, for all $1 \leq i \leq n$.

Exercise 5.

Is the closed square $(x, y) \in \mathbb{C}^2 \ : \ |x| \leq 1, |y| \leq 1$ in $\mathbb{C}^2$ an algebraic variety? Is this set closed?

Solution:

No, informally you can argue this as any nontrivial polynomial will have a vanishing that doesn't have interior points; that is, if one takes a open neighborhood of any point on the vanishing of the polynomial, there will be some points laying outside of the vanishing. This is equivalent to the zero-set of polynomials in $\mathbb{C}^n$ will be boundary points of the zero set of that polynomial. With open neighborhoods of any point in the zero set laying both inside and outside the zero set.

Exercise 6.

Provide a picture and defining equations of the following:

a.)

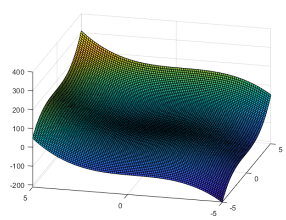
An example of the intersection of two varieties. ***Solution:***
In both examples we will consider the polynomials:

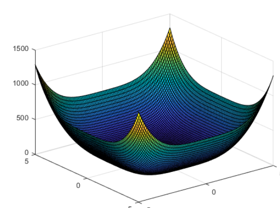
$$p_1(x,y) = x^3 + x^2 + x - 1 + y^3 + y^2 + y - 1$$

and

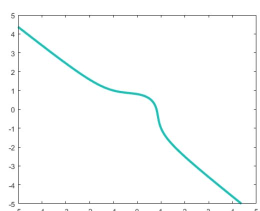
$$p_2(x,y) = x^4 + x^2 - 2 + y^4 + y^2 - 2$$

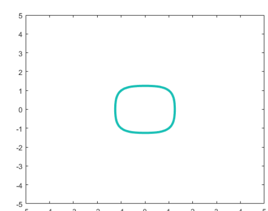
In $\mathbb{R}^3$ these functions look like, respectively:





Their roots in $\mathbb{R}^2$ look like:





Their intersection, is then the roots that these two varieties share. The intersection, then are all the points (x,y) that satisfy the equation:

$$p_1(x,y) = 0 = p_2(x,y)$$

Or, $\mathbb{V}(\{p_1, p_2\}) = \{(x,y) \in \mathbb{C}^2 \mid p_1(x,y) = 0 \text{ and } p_2(x,y) = 0\}$.

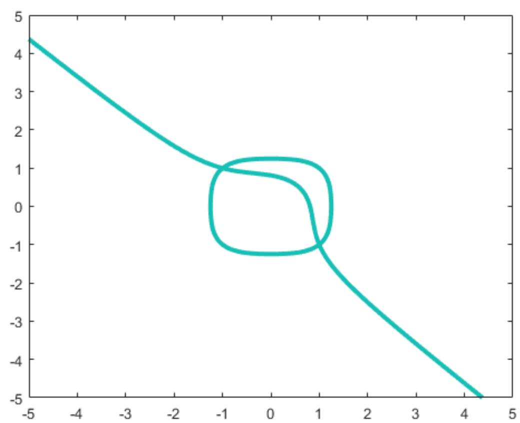
b.)

An example of the union of two varieties.

Solution:

The union of the two varieties would then be any of the roots of either p_1 or p_2.

In set notation this would be the set $\mathbb{V}(p_1) \cup \mathbb{V}(p_2)$, equivalently though, this can be written more compactly as $\mathbb{V}(p_1 p_2)$. The reason being that if $(x, y) \in \mathbb{V}(p_1) \cup \mathbb{V}(p_2)$, then either $p_1(x, y) = 0$ or $p_2(x, y) = 0$ so it follows that $p_1 \cdot p_2(x, y) = 0$. Conversely, if $p_1 \cdot p_2(x, y) = 0$, then since $\mathbb{C}$ is a field, we know either $p_1(x, y) = 0$ or $p_2(x, y) = 0$, by modern algebra results. Thus we have $\mathbb{V}(p_1) \cup \mathbb{V}(p_2) = \mathbb{V}(p_1 p_2)$. In $\mathbb{R}^2$ they look like this:



Exercise 1.1.1.

Show that every affine algebraic variety in $\mathbb{C}^n$ is closed in the Euclidean topology. (Hint: Polynomials are continuous functions from $\mathbb{C}^n$ to $\mathbb{C}$, so their zero sets are closed.)

Solution:

If we assume that all complex polynomials are continuous (not a reasonable thing to assume, should be proven at some point), then we know both open sets get sent to open sets and closed sets get sent to closed sets. So, that is, for any polynomial $p(x_1, ..., x_n) : \mathbb{C}^n \to \mathbb{C}$, that if X is a closed set and Y is an open set in the Euclidean topology on $\mathbb{C}$, then $p^{-1}(X)$ is closed in the Euclidean topology on $\mathbb{C}^n$ and $p^{-1}(Y)$ is open in the Euclidean topology on $\mathbb{C}^n$.

So then the argument we must make is that $\{0\}$ is closed in $\mathbb{C}$. Informally, if we view $\mathbb{C}$ as the complex line, then this is intuitive, since $\{0\}^{\mathrm{C}}$ would be an open interval on this line. Hence, informally, $\{0\}$ is closed since $\{0\}^{\mathrm{C}}$ is open in $\mathbb{C}$.

If we wanted to argue this formally, then we could argue that $\{0\}$ contains all of its limit points, because it is a singleton.

So since $\{0\}$ is closed in $\mathbb{C}$ and polynomials are continuous from $\mathbb{C}^n$ to $\mathbb{C}$, then their zero sets must be as well and thus their algebraic varieties are closed in $\mathbb{C}^n$.