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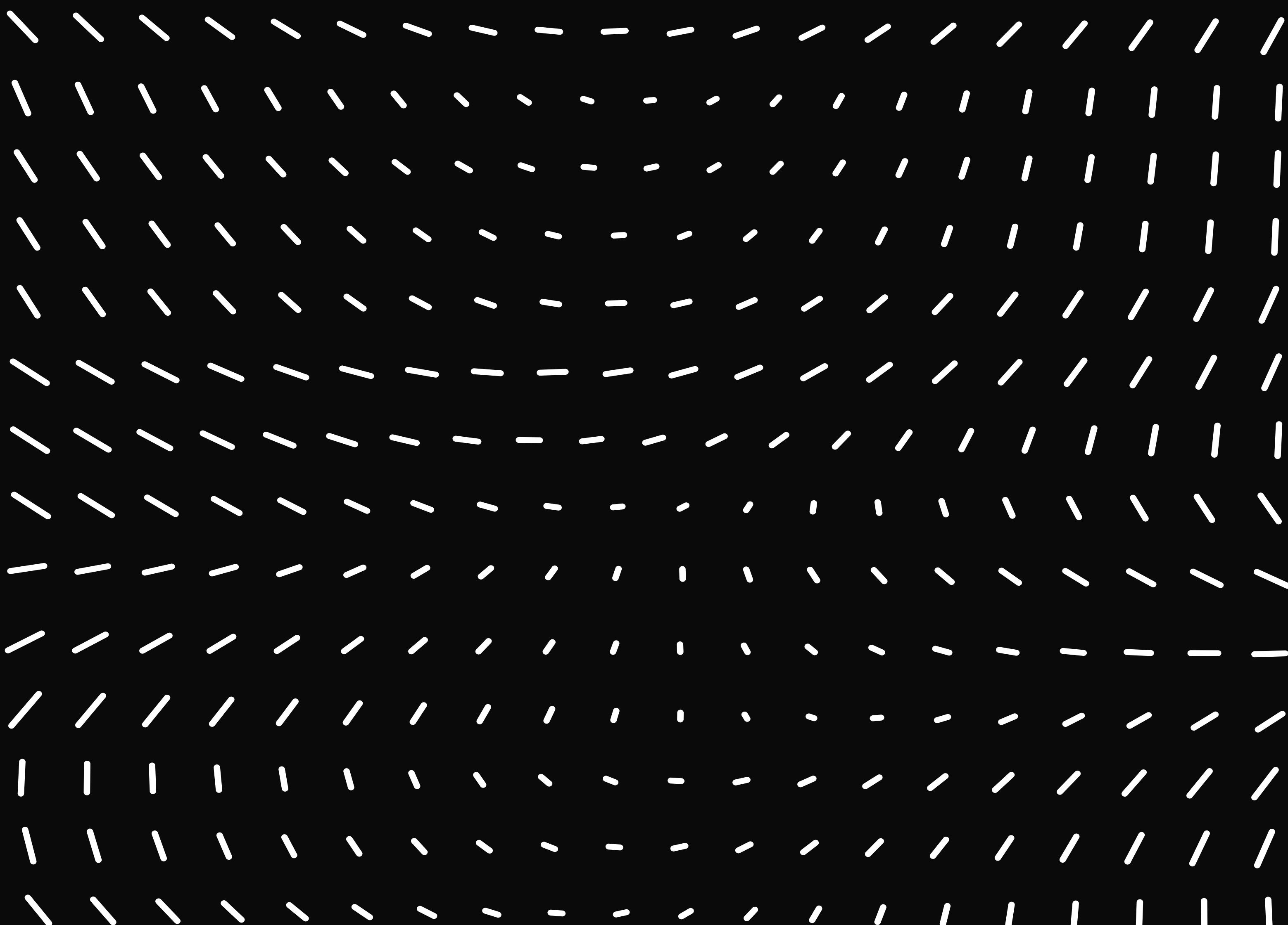
birth-death-supression Markov process
and wildfires



Questions

1. What is a birth-death process?

   How is markov different than basic coupled ODE?

   → Stochastic instead of deterministic: DE in prob instead of pop

   $$p_n(t) = P(j(t) = n)$$

   $$\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) + \mu_{n+1} p_{n+1}(t) - (\lambda_n + \mu_n) p_n(t)$$

2. Why does $\delta = 1$ fix units of time? it seems like parcels burn at different rates and we now rely on an assumption that they burn equally fast

3. What markov property ensures eq 6 is indep. of s?

   → require $\beta(t) = \beta$ and $\delta(t) = \delta$ for homogeneity and Markov

   * still okay for piecewise-constant.

   → is the markov property just that any state transition occurs by the transition matrix, and does not depend on previous states?

4. What is the spectral measure do?

   How is it determined?

5. What are the error bars on escape probability in fig 11 B, D?

   → Lower certainty should bring the prediction closer to 50%, rather than fluctuating before a decisive last-minute turn.

Notes

Goal: Analytically determine timelines and distance scales for fires to support decision-making for optimal resource allocation

Previous work provided a model for wildfire under supression

this work provides analytical solution

Discretized model - firelets population $j(t)$
System size J
Burned area $F(t)$
initial outbreak N

Figures show two views - Population vs time and
Spatial representation at fixed time
with 3 states: burned, burning, unburned

Firelet birth rate $\beta \cdot j(t)$

firefighting rate γ

Goal: find t such that $j(t) = 0$ with Probability P_0
or find "escape probability" $P(F \geq J) \sim J^{-\alpha}$
(experimentally $\alpha \approx \frac{1}{2}$)

Suppression-free model reproduces $P(F \geq J)$ and introducing γ allows computation of decision outcomes

Known Analogue for model: BDI birth·death·immigration

BD Models are solved using orthogonal polynomials

I created a wildfire spread Process class and reproduced fig 2

"Solving" the BDS process means Characterizing $P(j(t)=0)$ "absorption probability" and $P(F(t) \geq J)$ "escape probability"

Define the model

aggregate birth rate: $\lambda_j \equiv \beta j$

'' death rate: $\mu_j \equiv \delta j + \gamma$

To first order the probs of active born increase or decrease in time Δt are

$$P(j+1 | j) \simeq \lambda_j \Delta t$$

$$P(j-1 | j) \simeq \mu_j \Delta t$$

$$P(j | j) \simeq 1 - (\lambda_j - \mu_j) \Delta t$$

Then to set the timescale, redefine in terms of $\delta = 1$

$$\beta' = \frac{\beta}{\delta} \qquad \gamma' = \gamma/\delta$$

Then add Boundary Conditions:

$\lambda_{j=0} = 0$ No new ignition after total extinguishment

$\mu_{j=0} = 0 \quad \forall \gamma$ No extinguishment after total extinguishment

$j(t)$ is not formally bounded from above, but the simulation stops for $F(t) \geq J$

Markov modeling

define transition matrix

$$P_{nm}(t) = \text{Prob}\left[\, j(t+t_0) = m \mid j(t_0) = n \,\right)$$

Determine Absorption (total extinguishment) probability

$$\boxed{P_A(t) = \mu_1 \int_0^t P_1(\tau)\, d\tau}$$

Formulate as a Kolmogorov backward eqn:

For $j(0) = N$, $p_n(t) = P_{Nn}(t)$ is a row vector $\langle p(t)|$

$$\langle p(t)| = \langle p(0)|\, P(t)$$

$$\frac{d}{dt}\langle p(t)| = \langle p(t)|\, A$$

Where A is the transition coefficients

$$A = \begin{bmatrix} -(\lambda_1+\mu_1) & \lambda_1 & 0 & 0 & \cdots \\ \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \cdots \\ 0 & \mu_3 & \ddots & & \\ \vdots & \vdots & & \ddots & \end{bmatrix}$$

Then linearity implies

$$\boxed{\frac{d}{dt} P(t) = P(t)\, A}$$

First order Hamilton eqn
"master equation"

"A generates time translations of $P(t)$"

Supression-free model

Eqn 16: $\langle j(t) \rangle = j(0) e^{(\beta-\delta)t}$

I checked this with a plot of the mean pop for $\beta > \delta$ and $\beta < \delta$

also I checked $\Delta^2 j(t) \propto t$

Check the calculation for eqn 17

$$P_A(t) = \left(\frac{1 - e^{(\beta-1)t}}{1 - \beta e^{(\beta-1)t}} \right)^N$$

$$\lim_{\beta\to 1} P_A(t) = \lim_{\beta\to 1} \left(\frac{1 - (1 + (\beta-1)t + \dots)}{1 - \beta(1 + (\beta-1)t + \dots)} \right)^N$$

$$= \lim_{\beta\to 1} \left(\frac{-(\beta-1)t + \dots}{1 - \beta + \beta t - \beta^2 t + \dots} \right)^N$$

$$= \lim_{\beta\to 1} \left(\frac{(\beta-1)\, t + \dots}{(\beta-1) + (\beta-1)\beta t + \dots} \right)^N$$

$$= \lim_{\beta\to 1} \left(\frac{t}{1 + \beta t} \right)^N$$

$$= \left(\frac{t}{t+1} \right)^N$$

Similarly, compute the average footprint

$$\langle F(t)\rangle = \frac{N}{\beta-1}\left(\beta e^{(\beta-1)t} - 1\right)$$

using $\sum r^k = \frac{1}{1-r}$

$$\lim_{\beta\to 1} \langle F(t)\rangle = \lim_{\beta\to 1} -N\,(1+\beta+\ldots)\left(\beta\left[1+(\beta-1)t+\ldots\right]-1\right)$$

$$= \lim_{\beta\to 1} -N\,(1+\beta+\ldots)\left(\beta-\beta t+\beta^2 t+\ldots-1\right)$$

$$= \lim_{\beta\to 1} N(1+\beta t)$$

$$= N(t+1)$$

Brth-Death-Suppression model

Change of variables - $k \equiv j-1$ → absorption for $k=-1$

$$\lambda_k = \beta(k+1)$$
$$\mu_k = \delta(k+1)+\gamma$$

The supression term introduces inhomogeneity so the solution requires eigenvector decomposition.

Note: A and $P(t)$ are symmetric, noting the time-reversal of the process for $\lambda \leftrightarrow \mu$

and therefore

$$\frac{d}{dt} P(t) = A\,P(t)$$ is the Kolmogorov forward eqn

Now we diagonalize A:

assume eigenvector $|Q(x)\rangle$ and eigenvalue $-x$

$$A|Q(x)\rangle = -x|Q(x)\rangle$$

and in component form as a recurrence relation:

$$Q_0 = 1$$
$$-xQ_0 = -(\lambda_0+\mu_0)Q_0 + \lambda_0 Q_1$$
$$-xQ_k = -(\lambda_k+\mu_k)Q_k + \lambda_k Q_{k+1} + \mu_k Q_{k-1}$$

where $\deg Q_k = k$ with orthogonal polynomials Q_k

Getting the spectral form of the transition matrix $P_{nm}(t)$

The recurrence relations for Q's implies the orthogonality relation

$$\int_0^\infty Q_n(x)\, Q_m(x)\, d\sigma(x) = \frac{\delta_{nm}}{\pi_m}$$

Kronecker

where

$$\pi_0 = 1$$

$$\pi_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_0 \mu_1 \cdots \mu_{n-1}}$$

Using spectral measure $d\sigma(x)$

Most importantly, we can now compute explicitly the matrix elements:

$$P_{nm}(t) = \pi_m \int_0^\infty e^{-xt}\, Q_n(x)\, Q_m(x)\, d\sigma(x)$$

with Q's being different polynomials for $\beta < 1, \beta = 1, \beta > 1$

Skipped some heavy math...

firewalk polynomials, semigroup relation instead of mastereq,
discretized contributions to spectral measure, ...

But I've found some "Future Work"

Combine the dynamics of transitions
could shed light on $F(t)$ and not just $F(\infty)$

Thoughts & Reflections

Aside from some techical details and math I skimmed, all seems nicely accessible. I'm glad to have learned more about BD markov models with suppression, and I'm eager to implement the rest in python too. It seems like this project has wide application. I hope to work on this and improve risk-informed decision making. I think reinforcement learning would be nice to complement the analytical work but must not replace it, for accountability purposes; I would not use Deep Learning to decide the outcome when human Lives are at risk. That said, I am interested to see how a language model would handle this task. inputgrid fed to transformer, output grid from transformer. I'm not sure how different that would be from $P_{nm}(t)$, but it's something to explore. Eh, given how well next-token prediction works for numerical sequences, it probably will need serious work. but, the data is out there!