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Schwinger's Quantum Link Model: Restriction to physical space.

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Abstract

The Schwinger model of 1-1D electrodynamics can be approximated by a well-designed Quantum Link Model. This finite-dimensional system can be studied using classical or quantum simulations. The exact diagonalization of this quantum link model is used to illustrate the string breaking phenomenon and to confirm the potential of quantum simulations using cold atoms in optical lattice in the context of gauge theory. To facilitate performing this exact diagonalization, a small algorithm is developed to restrict calculations to only the physical part (obeying Gauss' law) of the system.

Contents

1 Introduction

A Quantum Link Model (QLM) is composed of a lattice of interacting quantum degree of freedom, making it a Quantum Many Body Problem (QMBP). A path between gauge theory and QLM [2], through lattice gauge theory, has been traced, so that a good understanding of QLM offers a perspective for the study of gauge theory.

As a QMBP, many methods already developed (Exact Diagonalization, Tensors Network,..) can be used to study properties of a QLM. Classical simulations techniques offer a first possibility, but the exponential scaling of these methods severely limits the size of the system studied. Sometimes, symmetry of the system makes it possible to push back the limit of the simulations : In the case of a gauge theory such as the Schwinger model, a physical condition greatly reduces the size of the Hilbert space describing the system and allows to simulate slightly larger configuration.

However, with advances in experimental methods for the manipulation of cold atoms, an another path opens up. By adjusting the parameters of an optical lattice, an experimenter can physically implement the dynamic of certain QLM. This method hides the exponential complexity within the physical system itself. But the topology of the designed optical lattice restricts the possibilities, and some properties of the original QLM can only be approximated (such as gauge invariance).

In this work, based on the article [1], we focus on the example of a QLM describing a discretised version of the Schwinger model. Using exact diagonalization, and an algorithm developed to focus on the physical sector, we illustrate the string breaking phenomenon. In a second part, we investigate how gauge invariance and physical conditions can be imposed on an experimental realisation of a QLM.

2 Methods

2.1 1D Quantum Link Model for the Schwinger Model

A Quantum Link Model (QLM) is composed of a chain of N fermions and N spins. Each fermionic degree of freedom is manipulated using creation-annihilation operators ψ_x, $\psi_x^\dagger$ and $n_x = \psi_x^\dagger \psi_x$ [3]. Similarly, the spin degrees of freedom are controlled by ladder operators S_x^+, S_x^- and S_x^z. The Hilbert space describing the system is the tensor product of that of each individual degree of freedom, so its dimension is :

$$\dim(\mathcal{H}_{\text{Full}}) = (2 \times (2S+1))^N \tag{1}$$

Where S is the maximum spin. This dimensionality grows exponentially has the size of the system N increases.

A basis vector for this space $\mathcal{H}_{\text{Full}}$ can be written

$$|\psi\rangle = \bigotimes_{n=0}^{N-1} \left(S_n^+\right)^{s_n+S} \left(\psi_n^\dagger\right)^{n_n^F} |0\rangle \otimes |-S\rangle \tag{2}$$

where $s_n \in \{-S, -S+1, ..., S\}$ and $n_n^F \in \{0, 1\}$.

The dynamical properties of the QLM are contained in its Hamiltonian. A 1D QLM Hamiltonian can be designed to represent a lattice version of the Schwinger model [4] (1+1D electrodynamics) which is an example of a U(1) gauge theory.

$$H_S = -t \sum_{n=0}^{N-1} \left(\psi_n^\dagger S_n^+ \psi_{n+1} + \psi_{n+1}^\dagger S_n^- \psi_n\right) + m \sum_{n=0}^{N-1} (-1)^n \psi_n^\dagger \psi_n + \frac{g^2}{2} \sum_{n=0}^{N-1} (S_n^z)^2 \tag{3}$$

This Hamiltonian has a gauge invariance reflected by its commutation properties $[H_S, \tilde{G}_n] = 0$ with the gauge generators:

$$\tilde{G}_n = \psi_n^\dagger \psi_n - S_n^z + S_{n-1}^z + \frac{1}{2}\left((-1)^n - 1\right) \tag{4}$$

In principle, H_S can be written in the basis defined by (2). In addition, to simulate the Schwinger model, a discrete version of Gauss' law must be imposed to obtain physical states [1].

2.2 Physical States

The lattice version of Gauss' law is given by

$$\tilde{G}_n |\psi\rangle = 0 \tag{5}$$

Thus, any physical state should obey this condition for all n. Because of the commutation property between H_S and $\tilde{G}_n$, we know that we can concentrate calculations on a subspace $\mathcal{H}_{\text{phys}} \in \mathcal{H}_{\text{full}}$.

If we apply $\tilde{G}_n$ to a given state in the complete basis, we observe that physical states should obey the following relation between s_{n-1}, s_n and n_n^F :

	$n = \text{even}$	$n = \text{odd}$
$b_n = 0$	$s \circ s$	$s \bullet s$
$b_n = 1$	$s \bullet s+1$	$s \circ s-1$

where $n_n^F = 0 = \circ$ and $n_n^F = 1 = \bullet$.

To construct an adequate basis for $\mathcal{H}_{\text{phys}}$, we represent any basis vector by a sequence of binary numbers $\{b_k\}$ and an initial spin value $s_{\text{init}} = s_{N-1}$. Noting that:

$$s_n = \sum_{j=0}^{n} (-1)^j b_j + s_{\text{init}} \tag{6}$$

and imposing $-S \leqslant s_n \leqslant S$, the physical condition expressed in the table constraints the possible sequence of binary numbers.

Using this notation we can construct a recursive algorithm that generates all physical states for a given s_{init}:

Algorithm 1 Construct any periodic physical state for a given $s_{\text{init}} = s_{N-1}$

Require: s_{init}, j, s_{j-1}, $\{b_k\}$
  $b_j = 0$
  $s_j = s_{j-1}$
  if $j = N-1$ **and** $s_j = s_{init}$ **then**
    save $\{b_k, b_j\}$
  else
    call itself with (s_{init}, $j+1$, s_j, $\{b_k, b_j\}$)
  end if
  $b_j = 1$
  $s_j = s_{j-1} + (-1)^j$
  if $-S \leqslant s_j \leqslant S$ **then**
    if $j = N-1$ **and** $s_j = s_{init}$ **then**
      save $\{b_k, b_j\}$
    else
      call itself with (s_{init}, $j+1$, s_j, $\{b_k, b_j\}$)
    end if
  end if

The Algorithm 1 gives an ordered set of states $|s_{\text{init}}, \{b_j\}\rangle \Rightarrow |\phi_i\rangle$ that form a basis for $\mathcal{H}_{\text{phys}}$ and we can compute the Hamiltonian in that sub-basis.

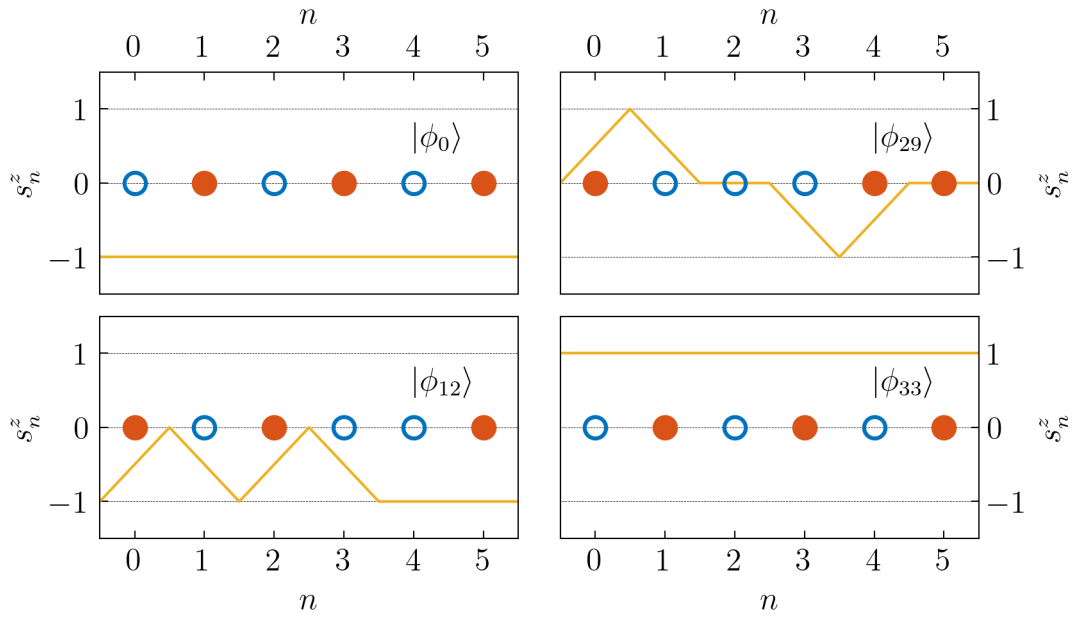

Figure 1: Few examples of physical states generated by Algorithm 1 for $S = 1$, $N = 6$.

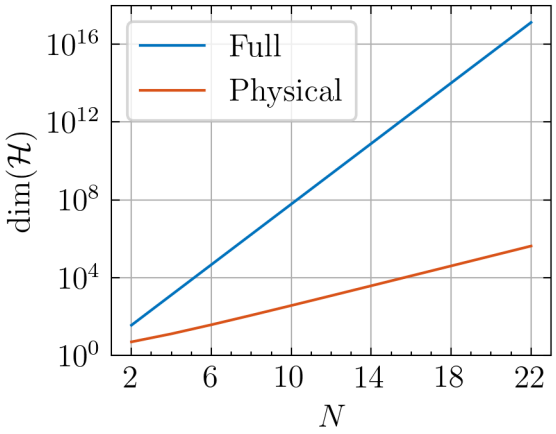

Figure 2: Comparison between dimensionality of the full and physical Hilbert space for $S = 1$ in function of N.

The algorithm presented, by focusing on the physical part of the system, considerably reduces its dimensionality. Classical simulation methods such as Exact Diagonalization (ED) are now applicable to larger system. Unfortunately, the scaling law remains exponential as shown in Figure 2. It is therefore not a fully satisfying solution.

2.3 Exact diagonalization and Time Evolution

Exact diagonalization is widely used in quantum mechanics [3] to calculate the eigenstates and energy levels of a quantum system.

$$H = PDP^\dagger \tag{7}$$

D is a diagonal matrix formed by eigenvalues of H and P contains its eigenvectors. A state $|\psi(0)\rangle$ is evolving in time according to :

$$|\psi(\tau)\rangle = e^{-i\tau H}|\psi(0)\rangle \tag{8}$$

Using ED, it is possible to rewrite this time evolution :

$$|\psi(\tau)\rangle = Pe^{-i\tau D}P^\dagger|\psi(0)\rangle \tag{9}$$

So, if the ED of H is performed, the time evolution of any state can be calculated by the mean of simple matrix multiplications. To numerically perform this ED, the `np.linalg.eigh` [5] function is used.

2.4 Experimental implementation [1]

As illustrated in Figure 2, there is an exponential scaling inherent in classical simulations that limits our ambitions.
The physical implementation of a QLM using cold atoms in an optical lattice avoids this naughty scaling. The cold atoms in the optical lattice are identified with fermions and spins of the original QLM. The experimental setup presented in [1][1](we will refer to it as the microscopic QLM) has the dynamics of a QLM with the Hamiltonian:

$$H_{micro} = -t_F \sum_{n=0}^{N-1} \left(\psi_n^\dagger \psi_{n+1} + \psi_{n+1}^\dagger \psi_n\right) - t_B \sum_{n=0}^{N-1} \left(S_n^+ + S_n^-\right) + m \sum_{n=0}^{N-1} (-1)^n \psi_n^\dagger \psi_n + U \sum_{n=0}^{N-1} (\tilde{G}_n)^2 \tag{10}$$

A state of this microscopic system belongs to $\mathcal{H}_{\text{full}}$ and can, in principle, be non-physical. The U term has a key role; by shifting upwards the energy of each non-physical state, it gives to the experimental setup, at low energy, a dynamics equal

[1]See it for detailed explanation and fixing of constants.

to the that of the Schwinger's QLM with the physical condition (5), up to the gauge invariant term ΔH.

$$\Delta H = \delta_F \sum_{n=0}^{N-1} \psi_n^\dagger \psi_n \left(1 - \psi_{n+1}^\dagger \psi_{n+1}\right)$$

Using ED of the microscopic QLM (for small system), we check the physical condition for low-energy states by calculating the quantity G.

$$G = \sum_{n=0}^{N-1} \langle \psi_0 | \tilde{G}_n | \psi_0 \rangle \tag{11}$$

where $|\psi_0\rangle$ is the ground state of the system. Moreover, if the physical condition is verified, we obviously have $[H_{\text{micro}}, \tilde{G}_n]|\psi\rangle \approx 0$ for all low-energy states, which therefore fixes gauge invariance.

3 Results

3.1 String Breaking

Using the Algorithm 1 with fixed boundary conditions ($s_0 = -1 = s_{N-1}$) and the ED of $H_S + \Delta H$ we can time evolve $|\phi_0\rangle$ and illustrate the string breaking phenomenon [1]. A static Quark-AntiQuark pair is separated by an electrical flux string. After time evolution, new charges emerge, destroying the electric flux, and creating two mesons at the end of the string.

The mean electric fields as function of time and position on the string is represented on Figure 3.

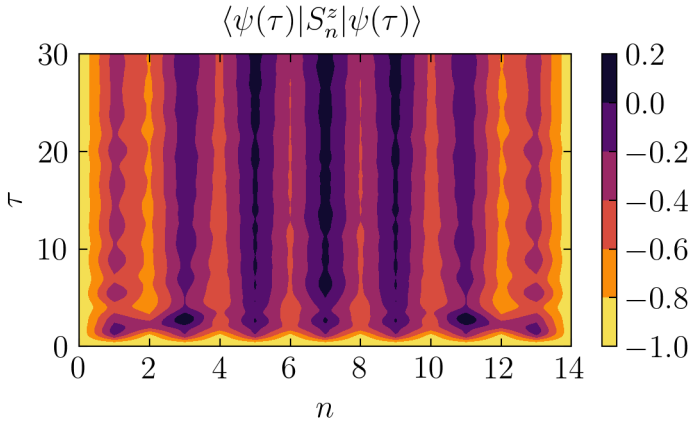

Figure 3: Time evolution of the electric field in a string breaking configuration with $N = 16$, $S = 1$, $m = 0$, $t = 1$, $g^2 = -\delta_F = \sqrt{2}t$.

The time evolution of the quantity $E = \sum_n S_n^z$ is plotted in Figure 4. We clearly see that this quantity quickly rises to approach its vacuum expectation. This is the expected behaviour for a string breaking.

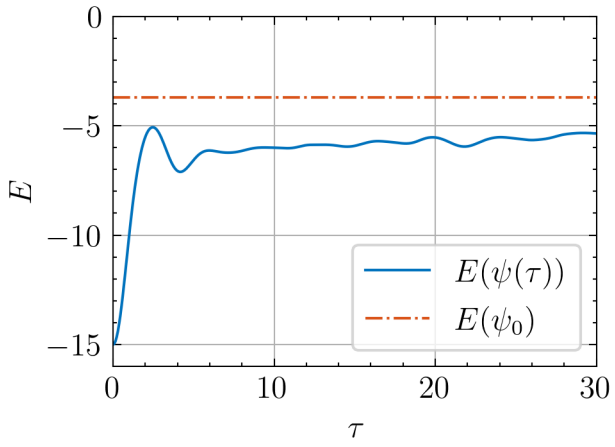

Figure 4: Time evolution of E for the simulation presented in Figure 3, E value for the ground state of the system.

3.2 Microscopic QLM convergence to Schwinger's QLM

If we are working with small $N \approx 6$, ED can be used on $\mathcal{H}_{\text{full}}$ in decent time. We therefore use ED to calculate the ground state expectation of the electric field for the microscopic and Schwinger's QLM. For $U = 20$, we observe on Figure 5 that they are in agreement.

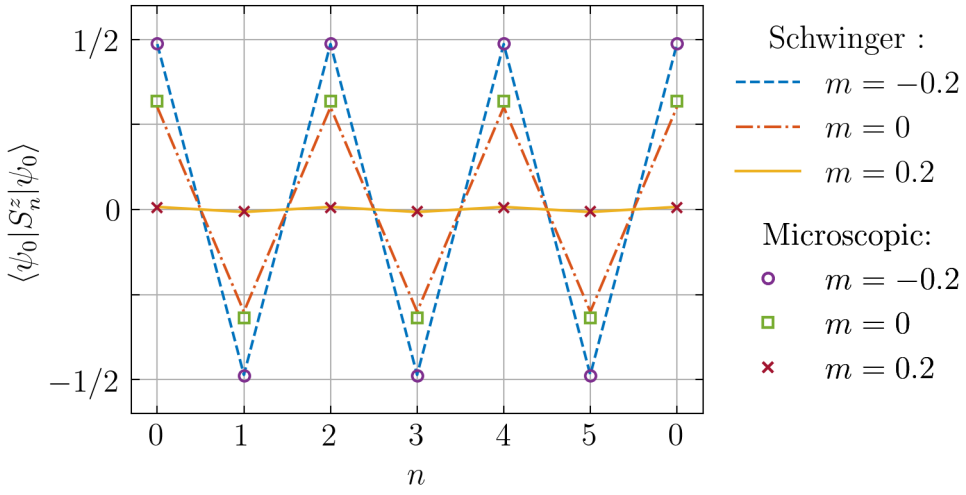

Figure 5: Ground states electric field of microscopic and Schwinger's QLM with $N = 6$, $S = \frac{1}{2}$, see [1] for the other constants.

To illustrate the role of U in the convergence from the microscopic to the Schwinger's QLM, Figure 6 draws the dependence of G on U. Thereby, this confirms that U controls how much the microscopic QLM will restricts itself to the physical space and hence to gauge invariant states.

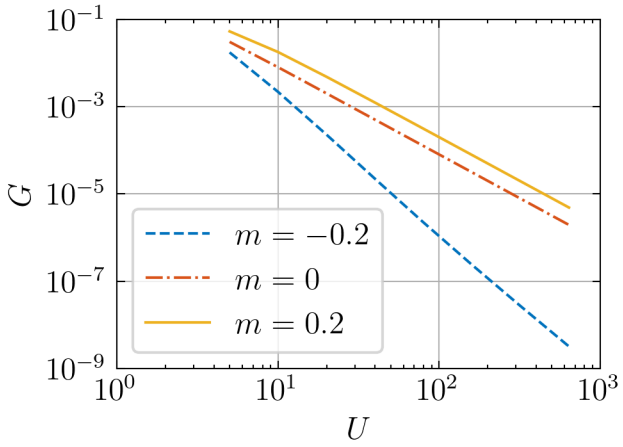

Figure 6: G in function of U with $N = 4$, $S = \frac{1}{2}$, see [1] for the other constants.

4 Discussions

In the case of Schwinger's QLM with $N = 16$, without the use of the Algorithm 1, naive storage of H requires several Gigabit. As a result, the string breaking simulation presented could not have been computed on a common post computer. However, even with this method, the exponential scaling is still present and is a clear limiting factor for more ambitious simulations. But approximation methods can be used to get rid of this scaling, for example, Matrix Product State and Tensor Network representation [6] sometimes allow the scaling to be reduced.

Despite the scaling problem, classical simulations of small systems have clearly shown there usefulness. Indeed, by using the ED of the microscopic QLM, we were able confirm that its experimental implementation can be used to simulate a lattice gauge theory.

5 Conclusion and Outlook

In this work we have illustrated how a simple gauge theory such as the Schwinger model can be simulated using classical techniques or quantum experiments. The limitation of classical simulations comes from the difficulty of working with the exponentially large Hilbert space describing the system, whereas for quantum simulations, gauge invariance must be approximated by a subtle design of the Hamiltonian. Despite its limitations, classical simulation remains and interesting way of illustrating some phenomena or verifying the behaviour of quantum simulations. Clearly, in higher dimensions, exact diagonalization will be even less effective and we will have to use other techniques or approximations in order to be able to simulate the system [6].

6 Programs

All the scripts used can be found in this GitHub repository. `https://github.com/joseph-elf/QLM`

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