Homework: Week 8

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Math 100
Due: November 22, 2017
Professor Boltje
MWF 9:20a-10:25a

Chapter 10.

2.

Proposition: For every integer $n \in \mathbb{N}$, it follows that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof. Observe that if $n = 1$ the statement $1^2 = \frac{1((1)+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = 1$, so the statement is true. Now let $k \geq 1$, so $1^2 + 2^2 + 3^2 + \ldots + (k+1)^2 = 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = (k+1)\frac{k(2k+1)}{6} + (k+1) = (k+1)(\frac{k(2k+1+6k+6)}{6}) = (k+1)(\frac{2k^2+k+6k+6}{6}) = (k+1)(\frac{2k^2+7k+6}{6}) = (k+1)(\frac{2k^2+4k+3k+6}{6}) = (k+1)(\frac{2k(k+1)+3(k+2)}{6}) = (k+1)(\frac{(k+2)(2k+3)}{6}) = \frac{(k+1)(k+1+1)(2k+2+1)}{6} = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$. Therefore $1^2 + 2^2 + 3^2 + \ldots + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$. It follows by induction that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for every natural number n.

4.

Proposition: If $n \in \mathbb{N}$, then $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$.

Proof. Observe that if $n = 1$ the statement $1(1+1) = \frac{1(1+1)(1+2)}{3} \to 2 = \frac{1(2)(3)}{3} = 2$ so the statement is true for $n = 1$. Now let $k \geq 1$, so $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + (k+1)((k+1)+1) = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + k(k+1) + (k+1)((k+1)+1) = \frac{k(k+1)(k+2)}{3} + (k+1)((k+1)+1) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) = \frac{k(k+1)(k+2)+3(k+1)(k+2)}{3} = \frac{(k+1)(k+2)(k+3)}{3} = \frac{(k+1)((k+1)+1)((k+1)+2)}{3}$. It follows by induction that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ for every natural number n.

8.

Proposition: If $n \in \mathbb{N}$, then $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{n}{(n+1)!}=1-\frac{1}{(n+1)!}$.

Proof. Observe that if $n=1$, then $\frac{1}{(1+1)!}=1-\frac{1}{(1+1)!} \rightarrow \frac{1}{2}=1-\frac{1}{2}=\frac{1}{2}$, so the statement is true for $n=1$. Now let $k \geq 1$, so $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{(k+1)}{((k+1)+1)!}=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{k}{(k+1)!}+\frac{(k+1)}{((k+1)+1)!}=1-\frac{1}{(k+1)!}+\frac{(k+1)}{((k+1)+1)!}=1-\frac{1}{(k+1)!}+\frac{(k+1)}{(k+2)!}=1-(\frac{1}{(k+1)!}-\frac{(k+1)}{(k+2)!})=1-(\frac{1}{(k+1)!}-\frac{(k+1)}{(k+2)(k+1)!})=1-\frac{k+1-(k+1)}{(k+2)(k+1)!}=1-\frac{1}{(k+2)(k+1)!}=1-\frac{1}{(k+2)!}=1-\frac{1}{((k+1)+1)!}$. It follows by induction that $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{n}{(n+1)!}=1-\frac{1}{(n+1)!}$ for every natural number n.

20.

Proposition: $(1+2+3+\cdots+n)^2=1^3+2^3+3^3+\cdots+n^3$ for every $n \in \mathbb{N}$.

Proof. Observe that if $n=1$, then $1^2=1^3$, which is true. Now let $k \geq 1$, so then $(1+2+3+\cdots+(k+1))^2=(1+2+3+\cdots+k+(k+1))^2$.. If we say $a=(1+2+3+\cdots+k)$ and $b=(k+1)$ then $(a+b)^2=a^2+b^2+2ab$, then substituting back in for a and b we get $(1+2+3+\cdots+k)^2+(k+1)^2+2(1+2+3+\cdots+k)(k+1)$. Note that $(1+2+3+\cdots+k)=\frac{k(k+1)}{2}$ and $(1+2+3+\cdots+k)^2=1^3+2^3+3^3+\cdots+n^3$. So $1^3+2^3+3^3+\cdots+n^3+(k+1)^2+2\frac{k(k+1)}{2}(k+1)=1^3+2^3+3^3+\cdots+n^3+(k+1)^2+k(k+1)=1^3+2^3+3^3+\cdots+n^3+(k+1)^2(k+1)=1^3+2^3+3^3+\cdots+n^3+(k+1)^3$. It follows by induction that $(1+2+3+\cdots+n)^2=1^3+2^3+3^3+\cdots+n^3$ for every $n \in \mathbb{N}$.

30.

Proposition: F_n is the nth Fibonacci number. Show that $F_n=\frac{(\frac{1+\sqrt{5}}{2})^n-(\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$

Proof. Observe that if $n=1$, then $F_1=\frac{(\frac{1+\sqrt{5}}{2})^1-(\frac{1-\sqrt{5}}{2})^1}{\sqrt{5}}=\frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}}=1$. And if $n=2$, then $F_2=\frac{(\frac{1+\sqrt{5}}{2})^2-(\frac{1-\sqrt{5}}{2})^2}{\sqrt{5}}=\frac{(\frac{3+\sqrt{5}}{2})-(\frac{3-\sqrt{5}}{2})}{\sqrt{5}}=\frac{2\sqrt{5}}{2\sqrt{5}}=1$. Now let $k \geq 1$. Note that $F_{k+2}=F_k+F_{k+1}$. So then $\frac{(\frac{1+\sqrt{5}}{2})^k-(\frac{1-\sqrt{5}}{2})^k}{\sqrt{5}}+\frac{(\frac{1+\sqrt{5}}{2})^{k+1}-(\frac{1-\sqrt{5}}{2})^{k+1}}{\sqrt{5}}=\frac{(\frac{1+\sqrt{5}}{2})^k-(\frac{1-\sqrt{5}}{2})^k+(\frac{1+\sqrt{5}}{2})(\frac{1+\sqrt{5}}{2})^k-(\frac{1-\sqrt{5}}{2})(\frac{1-\sqrt{5}}{2})^k}{\sqrt{5}}=\frac{(\frac{1+\sqrt{5}}{2})^k+(\frac{1+\sqrt{5}}{2})(\frac{1+\sqrt{5}}{2})^k-[(\frac{1-\sqrt{5}}{2})^k+(\frac{1-\sqrt{5}}{2})(\frac{1-\sqrt{5}}{2})^k]}{\sqrt{5}}=\frac{(\frac{1+\sqrt{5}}{2})^k(1+(\frac{1+\sqrt{5}}{2}))-[(\frac{1-\sqrt{5}}{2})^k(1+(\frac{1-\sqrt{5}}{2}))]}{\sqrt{5}}=\frac{(\frac{1+\sqrt{5}}{2})^k(\frac{3+\sqrt{5}}{2})-(\frac{1-\sqrt{5}}{2})^k(\frac{3-\sqrt{5}}{2})}{\sqrt{5}}=$
Note that $\frac{3+\sqrt{5}}{2}=(\frac{1+\sqrt{5}}{2})^2$
and $\frac{3-\sqrt{5}}{2}=(\frac{1-\sqrt{5}}{2})^2$. Continuing...
$\frac{(\frac{1+\sqrt{5}}{2})^k(\frac{1+\sqrt{5}}{2})^2-(\frac{1-\sqrt{5}}{2})^k(\frac{1-\sqrt{5}}{2})^2}{\sqrt{5}}=\frac{(\frac{1+\sqrt{5}}{2})^{k+2}-(\frac{1-\sqrt{5}}{2})^{k+2}}{\sqrt{5}}$. It follows by induction that $F_n=\frac{(\frac{1+\sqrt{5}}{2})^n-(\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$ for every $n \in \mathbb{N}$.

32.

Proposition: Show that the number of n-digit binary numbers that have no consecutive 1's is the Fibonacci number F_{n+2}

Proof. Let $n=1$, then $a_1=2 \rightarrow F_{1+2}=F_3=2$. Assume that the given statement is true for $n=k$. So $a_k=F_{k+2}$. Observe that the sequence of $a_n's$ satisfy $a_{n+1}=a_n+a_{n-1}$. So $a_{n+1}=a_n+a_{n-1}=F_{n+2}+F_{(n-1)+2}=F_{n+2}+F_{n+1}=F_3=F_{(n+1)+2}$ thus the result is true for $n=k+1$. Therefore by induction, the number of n-digit binary numbers that have no consecutive 1's is the Fibonacci number F_{n+2}, where $n \in \mathbb{N}$.