Homework: Week 5

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Math 100
Due: November 3, 2017
Professor Boltje
MWF 9:20a-10:25a

Chapter 5

6.

Proposition:
If $x \leq -1$ then $x^3 - x \geq 0$.
Proof. (Contrapositive) Suppose $x \in \mathbb{R}$, and it is not the case that $x > -1$, so $x \leq -1$. Then x^3 and x are less than or equal to -1. Therefore $x^3 - x \leq 0$. Thus it is not true that $x^3 - x > 0$. ■

18.

Proposition: For any $a, b \in \mathbb{Z}$, it follows that $(a+b)^3 \equiv a^3+b^3 (\text{mod } 3)$.
Proof. Consider $(a+b)^3 - a^3 + b^3$. By the process of simplification, we get the following steps: $(a+b)^3 - (a^3+b^3)$ $= (a^3 + 3a^2b + 3ab^2 + b^3) - a^3 - b^3$ $= 3a^2b + 3ab^2$. Since $a, b \in \mathbb{Z}$, we can say that $3(a^2b + ab^2)$ is an integer. Then $(a^2b + ab^2)$had the property of $(a+b)^3 - a^3 + b^3 = 3(a^2b+ab^2)$. Then it is proper to say $3|((a+b)^3 - a^3 + b^3)$. Therefore $(a+b)^3 \equiv a^3+b^3 (\text{mod} \quad n)$.
■

22.

Proposition: Let $a \in \mathbb{Z}, n \in \mathbb{N}$. If a has a remainder r when divided by n, then $a \equiv r (\text{mod } n)$.
Proof. Suppose a has a remainder r when divided by n. Then it follows that $a = qn+r$ for some integer q. We can simplify this into $\frac{a}{n} = q + \frac{r}{n} \rightarrow -q = \frac{r}{n} - \frac{r}{n}$ which becomes $-qn = r - a$ and since q is an arbitrary integer, we can say $c, c \in \mathbb{Z}, c = -q$. So then $-qn = r - a$ becomes $cn = r - a$, which clearly shows that $r - a$ is divisible by n, or just $n|r - a$. Therefore $a \equiv r (\text{mod } n)$. ■

Chapter 6

6.

Proposition: If $a, b \in \mathbb{Z}$, then $a^2 - 4b - 2 \neq 0$.

Proof. For the sake of contradiction, suppose $a^2 - 4b - 2 = 0$. Then $a^2 = 4b + 2 \to a^2 = 2(2b+1)$ which means a^2 is even, which implies a is even. Then $a = 2n$ for some integer n. Substituting back into our original equation, we get $(2n)^2 - 4b - 2 = 0$ and through we series of simplification we get $4n^2 - 4b - 2 = 0 \to 4(n^2 - b) = 2 \to 2(n^2 - b) = 1$, which implies that 1 is even because it's the result of multiplying two even numbers, but it is really not even so we've reached a contradiction. ■

16.

Proposition: If a and b are positive real numbers, then $a + b \geq 2\sqrt{ab}$.

Proof. For the sake of contradiction, suppose $a + b < 2\sqrt{ab}$. Then through a series of simplification we get

$(a+b)^2 < 4ab$

$\to a^2 + 2ab + b^2 < 4ab$

$\to a^2 + b^2 < 2ab$

$\to a^2 - 2ab - b^2 < 0$

$\to (a-b)^2 < 0$ which is a contradiction because a and b are positive real numbers and when their difference is squared, it cannot be less than 0. ■

20.

Proposition: The curve $x^2 + y^2 - 3 = 0$ has no rational points.

Proof. For the sake of contradiction, suppose $x^2 + y^2 - 3 = 0$ has a rational point. Now suppose that the positive numbers x, y have a rational point (x, y) line on the curve. Then x and y can be rewritten as $x = \frac{a}{b}, y = \frac{c}{d}, 0 < a, b, c, d \in \mathbb{N}$ where a and b are relatively prime so then $gcd(a, b) = 1$ and $gcd(c, d) = 1$. Then by substation, we get $(\frac{a}{b})^2 + (\frac{c}{d})^2 - 3 = 0 \to$ $(ad)^2 + (bc)^2 - 3(bd)^2 = 0$
$\to (ad)^2 + (bc)^2 = 3(bd)^2$. Then we must have $(ad)^2 + (bc)^2 \equiv 0(\text{mod } 3)$. But recall we must have $x \equiv 0(\text{mod } 3)$ or $x \equiv \pm 1(\text{mod } 3)$, thus $x^2 \equiv 0(\text{mod } 3)$ or $x^2 \equiv 1(\text{mod } 3)$ by properties of congruency. Then $(ad)^2 + (bc)^2 \equiv 0(\text{mod } 3) \implies (ad)^2 \equiv 0(\text{mod } 3)$ and $(bc)^2 \equiv 0(\text{mod } 3)$ again by properties of congruency. Then $(ad) \equiv 0(\text{mod } 3)$ and $(bc) \equiv 0(\text{mod } 3)$. Since 3 is prime, we must have $a \equiv 0(\text{mod } 3)$ or $d \equiv 0(\text{mod } 3)$, and $b \equiv 0(\text{mod } 3)$ or $c \equiv 0(\text{mod } 3)$. Then if $b \equiv 0(\text{mod } 3)$ then $a \not\equiv 0(\text{mod } 3)$ since $gcd(a, b) = 1$. Then since $b \equiv 0(\text{mod } 3)$ then $d \equiv 0(\text{mod } 3)$. It follows that $b = 3n, d = 3m, 0 < n, m \in \mathbb{N}$. Therefore, $(ad)^2 + (bc)^2 = 3(bd)^2 \implies (3am)^2 + (3cn)^2 = 3(9mn)^2$. Thus, we have $(am)^2 + (cn)^2 = 27(mn)^2$ so $3|(am)^2 + (cn)^2$ which means $gcd(a, b) \geq 3$ and $gcd(c, d) \geq 3$. ■

24.

Proposition: $\log_2 3$ is irrational.

Proof. For the sake of contradiction, suppose $\log_2 3$ is rational. By definition of rational numbers, that means $\log_2 3$ can be written in the form $\frac{a}{b}$ for arbitrary integers a and b. Therefore we can say $2^{\frac{a}{b}} = \frac{a}{b}$ using the definition of logarithms. Then we can simplify to $2^a = 3^b$ which we know is false because an even number raised to an integer cannot be equal to an odd number raised to an integer. ■