# Exploring Parameter-Dependent Behaviour in the Chaos Game

## A Simple Analysis of Fractals

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**Abstract:**

The Chaos Game is a simple iterative process that makes surprisingly rich fractal structures. The purpose of this project is to examine how varying parameters in the Chaos Game, particularly the number of vertices $N$ and the contraction ratio $r$, affect the geometry and fractal dimension of the resulting patterns. Motivated by program 611 of *Experiments in Discrete Dynamics* [1], this report analyzed unexpected patterns and behaviours arising from specific parameter values, particularly $r > 1$. Using numerical simulations, the study examined self-similar fractals, transitions to chaotic behaviour, and symmetrical patterns that emerge under different parameter choices. The results reveal strong dimension sensitivity to $r$, the emergence of stable fractals at $r = \frac{e}{2}$, the transient chaotic structures with odd symmetry at $r = 2$, challenging the conventional requirement of $0 < r < 1$.

## Introduction:

The Chaos Game generates fractals (self-similarity and non-integer box-counting dimension) by repeatedly moving a point towards randomly selected vertices of an $N$-polygon. When the contraction ratio satisfies $0 < r < 1$, the mapping is sometimes a fractal. The most well-known mapping is when one chooses $(N, r) = (3,\ 0.5)$, which is Sierpiński's Triangle.

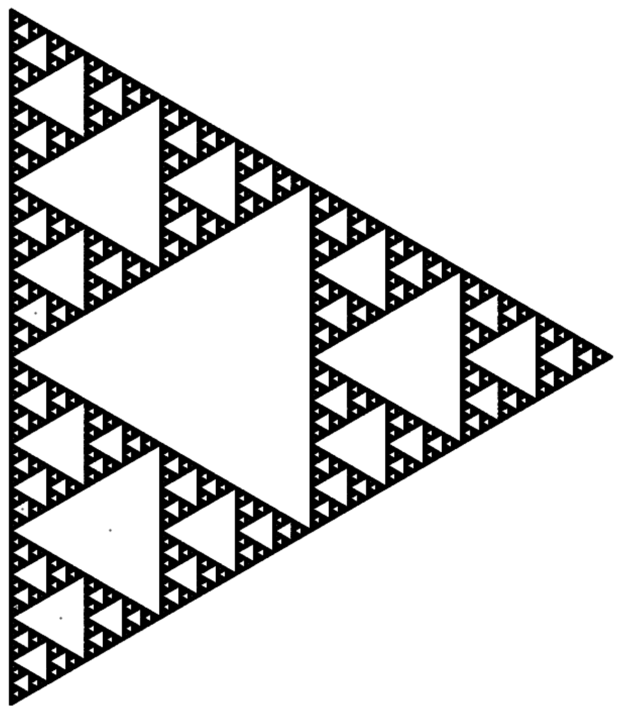


**Figure 1** - Sierpiński's Triangle Generated by the Chaos Game

However, nothing in the algorithm forces *r* to be restricted to values less than 1. This program allows users to experiment with settings where the contraction factor is allowed to exceed 1, and surprisingly, some of these settings produced bounded, symmetric, structured patterns, rather than simply diverging. This observation motivates a more systematic and mathematical investigation of the Chaos Game outside the standard contractive ratio.

This project will explore how the geometry and the fractal dimension of the Chaos Game change when one violates this contraction requirement. Thus, leading to the central question of this project:

What happens to the Chaos Game when the scaling parameter $r$ is greater than 1?

## Mathematical and Computational Foundations:

Suppose for an $N$-polygon, there are points:

$$P_1 = (a_1, b_1),\ P_2 = (a_2, b_2),\ ...,\ P_n = (a_n, b_n).$$

An initial point $z_0 = (x_o, y_o)$ will be chosen, and one of the vertices $P_i$ will randomly be selected. Then, the next point is generated from [2]:

$$z_{k+1} = w_i(z_k) = (x_{k+1}, y_{k+1})$$

where,

$$x_{k+1} = (1 - r)x_k + ra_i,$$

$$y_{k+1} = (1 - r)y_k + rb_i.$$

In these equations, $w_i$ is some similarity transformation that reduces the new point by a factor of $r$.

## Methodology:

To investigate the behaviour of the Chaos Game when the contraction ratio exceeds 1, a computational approach using numerical simulations and fractal dimension analysis was initiated.

**Simulation Parameters:** Each Chaos Game simulation was generated using 150,000 iterations to ensure sufficient density for visual analysis. The initial point $z_0$, was placed randomly for each simulation. For the vertex selection at each iteration, the program used Python's random library, giving each vertex of the $N$-polygon equal probability of being chosen.

**Parameter Selection:** For choices of $r$, the investigation was catered towards specific cases where $r > 1$. Specifically, the investigation was motivated by ratios involving irrational numbers, including $r = \frac{e}{2}$, $r = \sqrt{2}$, and $r = \frac{\pi}{3}$. The number of vertices $N$ was varied across multiple values to explore how different polygons affect the resulting patterns.

**Fractal Dimension Calculation:** To quantify the fractal nature of the generated attractors, the program used the box-counting method. It overlaid grids of varying box sizes onto each attractor and counted how many boxes contains at least one point, using 10 different box sizes (from $\varepsilon = 0.1$, and reducing by a factor of 0.75 each time). It then plotted $log(N(\varepsilon))$ vs. $log(\varepsilon)$ and performed linear regression using NumPy. The fractal dimension $D_{N=n}$ was estimated as the absolute value of the slope of this fitted line.

## Software Implementation:

All the simulations used were made in Python, building from programs 607 and 611 from *Experiments in Discrete Dynamics* [1]. NumPy was utilized for numerical computations and array operations, Matplotlib for visualizations, and Python's built-in random library for random vertex selection. The modified code allowed one to explore parameter ranges beyond the standard $0 < r < 1$ constraint.

## Analysis of Evidence:

The following section examines interesting cases of $r$ values and explains the patterns that emerge.

**(a)** N = 3 **(b)** N = 5 **(c)** N = 27

**Figure 2** - Multiple $N$-polygons with $r = 2$

One can see that these figures display odd symmetry about their centroids. To show why it exhibits this behavior, consider the mathematics of the Chaos Game. By letting $r = 2$, one gets:

$$x_{k+1} = (1 - 2)x_k + 2a_i \Rightarrow - x_k + 2a_i$$

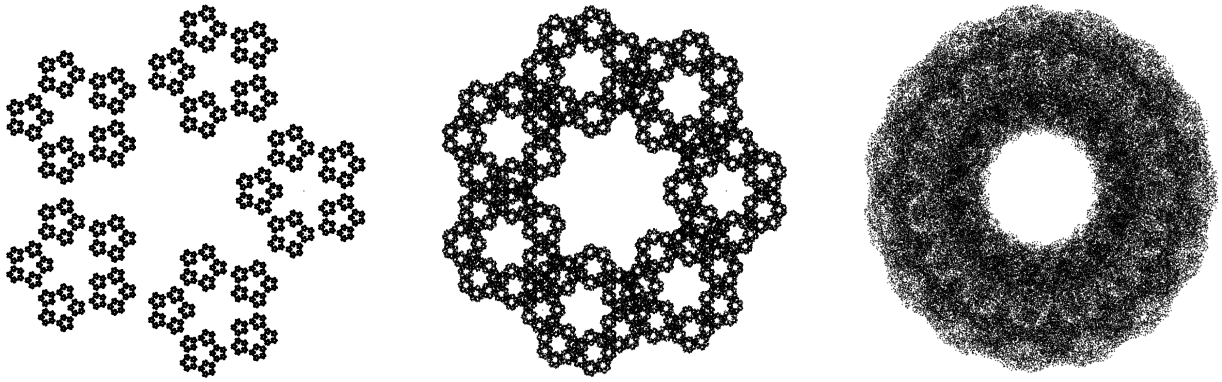
$$y_{k+1} = (1 - 2)y_k + 2b_i \Rightarrow - y_k + 2b_i$$

In vector form: $$z_{k+1} = 2P_i - z_k$$

Clearly, this is a point reflection through the vertex $P_i$. Geometrically, $P_i$ becomes the midpoint between $z_k$ and $z_{k+1}$, meaning each iteration reflects the current point through the chosen vertex. Note that different values for $N$ does not change this odd-symmetry behaviour, as it is completely dependent on the contraction ratio. Additionally, one is not able to replicate the exact images in **Figure 2**, even with the same parameters. This is because of the chaotic behaviour of $r = 2$, which implies extreme sensitivity to initial conditions, and since the program relies on iterations of random points, it is impossible to guarantee the same initial points for each program.

Despite this, $r = 2$ consistently displays odd symmetry about the polygon's centroid. This occurs because each reflection $z_{k+1} = 2P_i - z_k$ preserves distances from the center, and since vertices are symmetrically arranged, the point distribution maintains rotational symmetry. This demonstrates that chaos and symmetry can co-exist.

Furthermore, $r = 2$ has a counterintuitive pattern. One might expect that because $r > 1$, points would immediately diverge to infinity. However, the point reflection $z_{k+1} = 2P_i - z_k$ creates an initial “trapping” effect: when $z_k$ is near the polygon, reflections keep points within a bounded region, producing the symmetric patterns in **Figure 2**. This creates the appearance of stability at first. However, this boundedness is only temporary; as iterations continue, points gradually drift further from the origin and eventually escape to infinity. The “quasi-bounded” patterns one observes are merely transient structures rather than true attractors, though they maintain odd symmetry through this slow divergence process.



**(a)** N = 5 **(b)** N = 7 **(c)** N = 15

**Figure 3** - Multiple $N$-polygons with $r = \frac{e}{2}$

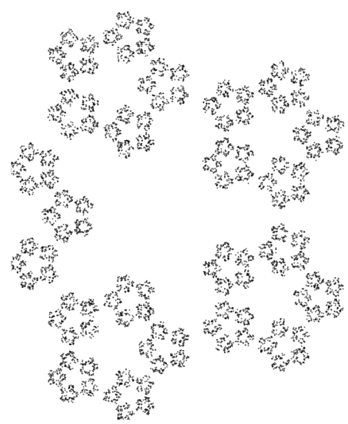


**Figure 4** - Zoomed detail of **Figure 3 (a)**, showing self-similar structure at multiple scales

When $r = \frac{e}{2}$, the Chaos Game produces stable, self-similar fractal patterns with a beautiful floral appearance. The similarity is clearly visible in **Figure 4**, which shows a magnified portion of the $N = 5$ attractor, revealing smaller copies of the petal-like structures within each larger petal. Unlike $r = 2$, these patterns are reproducible across multiple simulations, indicating stable, rather than chaotic, behaviour. The calculated fractal dimensions reveal an interesting trend: as $N$ increases, the dimension approaches 2. Specifically, $D_{N=5} = 1.54$, $D_{N=7} = 1.76$, and $D_{N=15} = 1.90$. This suggests that attractors with more vertices fill more of the 2-D plane while maintaining fractal structure. For comparison, the classical Sierpiński Triangle has dimension $D = 1.585$, so the $N = 5$ case exhibits similar geometric complexity despite operating in a completely different parameter interval ($r > 1$ vs.

$r < 1$). The stability and self-similarity observed demonstrates that the Chaos Game can still produce well-defined fractals even when the contraction ratio exceeds 1, challenging the conventional requirement.

## Conclusion:

This investigation successfully demonstrates that the Chaos Game exhibits diverse and unexpected behaviours when the contraction ratio $r$ exceeds 1, directly answering the central question. The evidence presented is convincing within the scope of computational exploration: $r = 2$ produces temporarily bounded, chaotic patterns with odd symmetry that eventually diverge to infinity, while values like $r = \frac{e}{2}$ generate stable, reproducible fractals with well-defined dimensions.

However, these results also reveal the limitations and complexities of the $r > 1$ investigation. For $r > 2$, simulations show that points diverge to infinity rapidly, producing no bounded patterns whatsoever. In the intermediate range $1 < r < 2$, the behaviour becomes increasingly complex and difficult to analyze, that is, many parameter combinations produce dense, overlapping point clouds where self-similarity is obscured, making it challenging to identify patterns as fractal or non-fractal without more sophisticated analysis.

Several improvements could strengthen these results. For example, higher iteration counts would reveal how quickly $r = 2$ patterns diverge. More systematic parameter sweeps across $1 < r < 2$ could reveal transitions between fractal, chaotic, and divergent behaviours more precisely. Most significantly, analytical mathematical proofs would provide rigorous explanations for the observed phenomena, though such proofs lie beyond the scope of this project and course.

This project demonstrated that the Chaos Game is far richer than its traditional $0 < r < 1$ formulation. This research report shows that chaos and symmetry can coexist, that fractals can emerge from non-standard parameter regimes, and that simple iterative algorithms can produce remarkably complex behaviours, including transient patterns that appear stable but eventually diverge. The computational approach, combining simulations, visualization, and dimensional analysis, proved to be effective in the investigation. Working with parameter-dependent systems required careful experimental design and revealed the value of both systematic testing and targeted investigation of special cases.

## References:

[1] Fukś, H. (2025). *Experiments in Discrete Dynamics: Low Dimensional Systems*. Programs 607 and 611.

[2] Peitgen, H-O., Jürgens, H., & Saupe, D. (2004). *Chaos and Fractals: New Frontiers of Science* (2nd ed.) Springer-Verlag. Section 6.1.