Modeling the dynamics of a forest environment: role of the water cycle.

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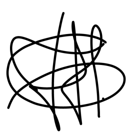
Abstract

According to the United Nations, one of the goals of sustainable development is the conservation of forests. Forests play a crucial role in our daily lives by supporting livelihoods, providing natural and medicinal plants, and much more. It is therefore obvious that the well-being of our forests should be one of our priorities. However, in the face of factors such as climate change and the risks of desertification, our forests are becoming increasingly vulnerable. In the literature, several models have been developed to study their interaction in nature, taking into account environmental parameters, and describing more or less real scenarios to assess their behavior and resilience to these factors. We present here a mathematical model based on a reaction-diffusion-advection system to describe the dynamics of forest ecosystems. First, we analyze the reaction-diffusion-advection system in our unstructured age model with the characteristic methods and transforming the initial problem into a well-known reaction-diffusion system. Then, the existence and stability of homogeneous and heterogeneous stationary solutions are examined to show that the forest ecosystem is capable of returning to its initial equilibrium state under small perturbations. In addition, we present a selection of numerical simulations for an abstract forest ecosystem to study the stability of steady states, verify the impact of disturbances such as deforestation, and to explore the effects of climate change on the dynamics of the forest ecosystem.

Keywords: Forest ecosystem, Forest model, Water resources, Reaction-diffusion-advection system, Climate change, Deforestation.

Declaration

I, the undersigned, hereby declare that the work contained in this essay is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



Joseph Junior Penlap Tamagoua, May 29, 2022.

Contents

Notations

- $\Omega \subset \mathbb{R}^2$ bounded domain, $\partial\Omega$ regular boundary of Ω,
- t time variable describing the evolution of forest ecosystem,
- $x \in \Omega$ spatial distribution variable,
- Γ border wich models the oceanic littoral,
- $\nu(x)$ outward normal vector at point $x \in \partial\Omega$,
- β rate of seed production per year,
- δ rate of seed establishment per year,
- σ average decrease rate of the water resource evaporated per year,
- α rate of maximum water production per year,
- h rate of tree mortality due to scarcity of water per year,
- h_0 maximum tree mortality rate per year,
- h_1 minimum tree mortality rate per year,
- φ_0 maximum rate of contribution to evaporation (mm $\times$ trees^{-1} $\times$ yr^{-1}),
- d rate of seed diffusion (mm $\times$ trees^{-1} $\times$ yr^{-1}),
- $m(x)$ oceanic contribution (mm $\times$ ha^{-1}),
- $a(t, x)$ advection field vector,
- $\rho(t, x)$ amount of water per surface area (mm $\times$ ha^{-1}),
- $u(t, x)$ tree density (trees $\times$ ha^{-1}),
- $\omega(t, x)$ surface density of seeds (seeds $\times$ ha^{-1}),
- $(u, v)^t = \begin{pmatrix} u \\ v \end{pmatrix}$ transpose vector.
- Spec(A) denotes the spectrum (set of eigenvalue(s)) of the matrix A.

1. Introduction

Ecosystems are parts of biomes including climatic systems of life and organisms. An ecosystem can also be defined as a place of living species (plants, animals, and bacteria) where organisms interact between themselves, and with their non-living environments (earth, sun, soil, and atmosphere) [38, 36]. Terrestrial ecosystems are one of the four ecosystem types [38] that we distinguish and which include physical environmental systems such as deserts, grasslands, coastal regions, and forests.

Forests are part of the more or less complex ecosystems of the planet because they are full of organisms of small and large scales, namely trees, animals, and bacteria [41]. This constitutes a living environment for many plant and animal species around the world. Furthermore, these forests are of great interest as they provide a multitude of goods, services, and livelihoods. To understand the functioning of forest ecosystems, it is useful to focus on the influence of certain factors in the forest environment. We refer to the impact of climate change, water resources, and deforestation.

Forests contribute to our daily lives by supporting livelihoods. In ecology, they prevent some events like erosion [4] by absorbing water running off and removing topsoil. Also affect the atmospheric temperature by transpiration process and combat global warming [21]. In the same line, forests provide a habitat for biodiversity. Concerning the health domain, it gives many important natural medicines. That is why healthy forest ecosystems are vital for human welfare and survival [8].

Therefore, forest science has evolved over the last centuries to understand how trees and forests develop, and the patterns and processes governing their evolution in time and space. According to FAO (Food and Agriculture Organization) of the United Nations, deforestation can be defined as the conversion of forests for different land uses (not necessarily due to human activity) [19]. However, over the past three decades, there has been a significant reduction in the rate of net forest loss, a parameter that takes into account both deforestation and forest expansion [39].

The concept of climate change continues to be widely debated and many researchers are working on this topic to provide more evidence. Several studies project [25] rather stressful results on climate change, namely that our forests will disappear very soon, where conditions will be more difficult; a high incidence of degradation of our forests, and death of trees due to changing climate conditions. Furthermore, the flooding in our forests will lead to competition between plant and animal species more or less adapted to the site in the wake of climate change.

Forests are undergoing many changes due to global warming [31]. On the one hand, climate change contributes to the well-being of certain categories of trees in a specific locality, and on the other hand, effects such as premature tree growth, extended seasons, and high temperatures are observed in some places. However, the susceptibility of climate change and its direct or indirect effect on the environment advocates negative consequences for forests. The observation of some changes in plant distribution [24, 29] or the increase in tree mortality due to drought and heat in forests around the world [5] may not be due to climate change accompanied by human action, but demonstrate the potential impacts of rapid climate change.

In addition to these impacts, human action through human behavior and environmental changes may also be revealed as a source activating climate change because of facts such as increased concentrations of low-level ozone, deposition of nitrogen pollutants through our daily activities (industries, vehicle use), presence of exotic insect pests and pathogens, and fires [9]. Other effects of climate change may also be important for forests. Doyle et al [15] discuss a significant impact of sea-level rise on tidal freshwater forests and tidal saltwater forests (mangroves) that extend inland from subtropical coastal areas over areas of marsh and freshwater forests [14]. However, deforestation and forest degradation observed in many countries have noted a segmentation of forests with similar effects such as the division of plots, decomposition of plant species, and many others [13].

The relationship between deforestation and climate change is mainly established at the level of chemical reactions observed in trees that absorb and store CO2 throughout their lives. For example, the world's tropical forests store more than two hundred gigatons of carbon, according to studies conducted by research centers. Worryingly, the destruction of these trees has two major negative side effects [3]. One is that forests have a significant amount of carbon that is necessarily useful to offset some of the energy emissions from deforestation [30, 10].

The effects of climate change and the distribution of water resources on the equilibrium of the forest ecosystem are not well understood. We emphasize the role that mathematical modeling plays in the ongoing process of ecology and to a better understanding of terrestrial ecosystems especially the interaction between forests and nature. Somehow, mathematics can help us explain the transitions of state parameters from one step to another. Mark Kot et al [26] stated that ecology must answer the question faced by all scientific knowledge: What do our models mean? To answer this question in our case, the modeling of density trees, their contribution to evapotranspiration, and the spatial distribution of water resources in a forest area will be carefully examined to describe the dynamics of forest ecosystems with a focus on the effects of climate change and deforestation.

1.1 Motivation and contributions

Scientists, specifically biologists, do not easily distinguish tree categories in the field based on their density. For this reason, the determination of the actual values of the model parameters becomes more or less complex, which makes the mathematical model less accurate.

In this work, we aim to formulate a mathematical model of an unstructured age model of trees which describes a dynamic forest ecosystem, by adapting an age-structured forest model based on a reaction-diffusion-advection system taking into account the effects of atmospheric activity, water resources and constraints of available data. Our contributions in this work include:

Proposing a mathematical model of forest ecosystem which includes only one category of tree based on [12]. Proposing a mathematical discretization scheme of our model and relevant simulation to assess the impact of climate change on the forest and deforestation.

1.2 Thesis outline

In Chapter 2, first presents different research works done so far in this area. Besides the introduction work, we will state the problem of our work, which is to construct an unstructured age model describing the dynamics of a forest environment of trees. To finish, useful mathematical tools and results on Caratheodory equations will be presented.

Chapter 3, presents the reaction-diffusion-advection system describing our unstructured age model. Next, we will proceed by analyzing the model mathematically, and we will end by investigating the stability of the latter system. A mathematical discretization scheme of our model will be proposed.

We conclude by presenting some perspectives for future research.

2. Background

2.1 Literature review

The authors evaluated the effects of climate change in a forest environment [43]. Northeast China is the region most affected by global warming, both in China and globally, according to the [18]. Results show that over the past decade, regional warming has increased significantly due to forest management.

In the same way, Marshet et al [20] focused on the effect of climate change on forest ecosystems also shown that Climate change alters and shifts forest ecosystems both directly and indirectly. Warming temperature directly affects rate of plant photosynthesis and respiration processes, also indirectly by increasing the risk of infestation.

Antonovsky and Korzukhin [6] first introduced a simple model of age structure dynamics of the monospecies system. The study of other factors such as the influence of insect pests upon the age structure dynamics on the equilibrium of the forest ecosystem was carried out by Antonovsky et al [32, 33]; while the seed dynamics were discussed by Kuznetsov et al [44]. In the same line, Rajasingh et al. studied the spatial effects in the forest ecosystem by considering the sensibility of parameters (see for instance [23]). Also, Bert Wuyts et al [16] in their work, presented the first analytical, numerical, and spatial heterogeneous tropical tree cover of Amazon forest versus savanna trees based on a reaction-diffusion model.

We use the mathematical modeling in this work to give precision and strategy for problem solutions and enable a systematic understanding of the dynamics of the forest ecosystem modeled. Many works and surveys were carried out to model forest ecosystems. For instance, in [37] Antonovsky et al. presented an age-structure of a forest, also in [44] and [34], age or size-structured models also have been studied but by introducing the models based on reaction-diffusion systems including cross-diffusion terms. Since we aim to build a structured forest model (which takes into account only one category of trees), we assume that the ecosystem contains a homogeneous population of trees. Also, the analysis of some factors such as the water cycle and the moisture atmospheric will be considered since extreme events (drought or rainfalls) impact the equilibrium of forests. Besides, in [12] Guillaume Cantin et al. proposed an age-structured forest model by gathering the reaction-diffusion and the advection processes. Here, the advection process is added in the original model (see for instance [34, 44]) which is explained by the moisture conservation transport. Partial differential equations (PDEs) play a crucial role in this task.

2.2 Statement of the problem

The age-structured mathematical model represented by a reaction-diffusion-advection system [12] of Guillaume Cantin et al. is used to study the behavior of forest ecosystems with respect to atmospheric activity and water resource distribution. In the same way, we suggest and construct another mathematical model that will no longer depend on the age of trees.

A forest ecosystem with ecological mechanisms was reviewed in (the book of Botkin [17]); therefore we divide the population of trees of that forest ecosystem into two categories: the young trees and the old trees. Also let us denote by $x = (x_1,\ x_2) \in \mathbb{R}^2$ the variable space, t time variable, $u(t,x)$, and $v(t,x)$ the densities of young and old trees respectively, and $\omega(t,x)$ represents the seed density. Authors in [12] proposed an innovative mathematical model for better understanding and predicting the dynamics of such a forest ecosystem. Their model is described by the following reaction-diffusion advection system of four partial differential equations.

$$\begin{cases} a.\nabla\rho &= -\sigma\rho + \varphi(\rho)v, \\ \dfrac{\partial u}{\partial t} &= \beta\delta\omega - \gamma(v,\rho)u - fu, \\ \dfrac{\partial v}{\partial t} &= fu - h(\rho)v, \\ \dfrac{\partial \omega}{\partial t} &= d\Delta\omega - \beta\omega + \alpha(\rho)v, \end{cases} \tag{2.2.1}$$

where the parameters σ, f, δ, and β are positive real coefficients. The parameter ρ represents the water quantity per unit of area, σ denotes the average decrease rate of the water resource evaporated per year; β, δ, and f represent the seed production rate, seed establishment rate, and aging rate of young trees respectively. We have the overall tree mortality rate $\gamma(v,\rho)$ defined by

$$\gamma(v,\rho) = \gamma_0(v) + \mu(\rho). \tag{2.2.2}$$

Here, the function γ_0 refers to the competition of young trees with the old ones and which is defined by a quadratic form:

$$\gamma_0(v) = r(v-b)^2 + c. \tag{2.2.3}$$

where r, b, and c are the positive constants. We can easily notice that in the second equation of the system above (2.2.1), the competition between young and old trees is highlighted by considering life resources (water, light).

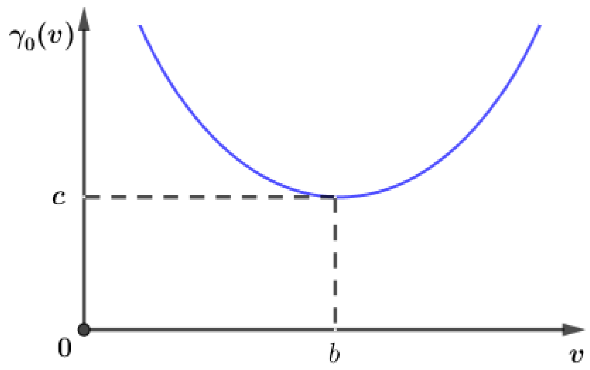


Figure 2.1: Illustration of competition trees [12].

The following interpretations lead to equation (2.2.2)

1. if there are too many old trees in the forest, the young trees will die from lack of light (they will not be able to capture enough light for their survival because of the darkness created by the deployment of the large trees), so the overall mortality of small trees will increase. On the other hand, in the absence of old trees, the overall mortality of young trees will also increase because they will be exposed to too much energy (sunlight, weather, large amount of light) (see figure 2.1).

2. regarding the influence of water resources on the overall mortality of seedlings, this mortality will also increase when there is a lack of water resources and desertification of the soil can be observed, and it will decrease when the amount of water reaches a certain threshold. This last situation is described by the term $\mu(\rho)$ in equation (2.2.2). Most often, due to a shortage of sufficient elements such as water, sunlight, and carbon dioxide, plants may slow down the process of photosynthesis or even stop it. This will then affect the internal supply of nutrients.

2.3 Some Mathematical tools

Let Ω be a regular domain which is enclosed by a curve $\partial\Omega$.

2.3.1 Some definitions and properties. In this part, we recall some necessary and important notions from partial differential equations theory.

2.3.2 Definition. (Invariant set [27]) The set Ω is called invariant domain for (2.3.1), if any solution with initial condition in Ω stays inside Ω for all $x > 0$.

2.3.3 Lemma. ([27]) Let $\nu(x)$ denote the unit outward normal such that $\nu \in \partial\Omega$. The set Ω is an invariant domain if

$$f(x).\nu(x) < 0, \quad \forall\ \nu(x) \in \partial\Omega, \tag{2.3.1}$$

2.3.4 Lemma. (Gronwall Integral) Let $a, b \geq 0,\ \ 0 \leq \alpha, \beta < 1,$ and $T \in (0, +\infty)$. Then, there exists a constant $M = M(b, \alpha, \beta, T) > 0$ such that for each integral function $u : [0, T] \longmapsto \mathbb{R}$, which satisfies

$$0 \leq u(t) \leq \frac{a}{t^\alpha} + b \int_0^t \frac{u(s)}{(t-s)^\beta} ds, \qquad \forall\ T \in [0, +\infty], \tag{2.3.2}$$

where

$$u(t) \leq \frac{a}{t^\alpha} M, \qquad \forall\ T \in [0, +\infty]. \tag{2.3.3}$$

2.4 Caratheodory differential equations

Here, we consider a differential equation

$$\dot{x}(t) = f(t, x), \tag{2.4.1}$$

such that $f(t, x)$ is discontinuous in t and continuous in x, and let

$$x(t) = x(t_0) + \int_{t_0}^t f(s, x(s)) ds. \tag{2.4.2}$$

Then, functions satisfying (2.4.2) called solutions of (2.4.1). Let assume that the function $f(t, x)$ holds the following properties:

(a) $f(t, x)$ is measurable in t for each x;

(b) $f(t, x)$ is well defined and continuous respect to the second variable x for almost all t;

(c) $|f(t, x)| \leq m(t), \quad m(t)$ is a constant function;

(d) $|f(t,x_1) - f(t,x_2)| \leq k(t) \ |x_1 - x_2|, \quad \forall \ x_1, \ x_2 \in \mathbb{R}, \ t > 0.$ In other words, f is said to be absolutely continuous on each closed interval included in Ω.

Conditions (a)-(b)-(c)-(d) are called caratheodory conditions for the system (2.4.1).

2.4.1 Lemma. Let f a function satisfying conditions (a), (b), and (c). Then, equation (2.4.1) is called a Caratheodory equation. Furthermore, if f holds also the last condition (d), then x defined in (2.4.2) is called a solution of the Caratheodory equation (2.4.1).

2.4.2 Definition. Let $f : \Omega \longmapsto \mathbb{R}$ be a measurable function and $(\Omega, \mathbb{A}, \mu)$ a measure space. f is said to be summable if the Lebesgue integral of the absolute value of f exists and is finite, i.e.

$$f \in L^1(\Omega) \qquad \text{or} \qquad \int_\Omega |f| \, d\mu < +\infty.$$

2.4.3 Theorem. *([45]) For $t_0 \leq t \leq t_0 + a$ and $\|x - x_0\| \leq \epsilon$. Let a function $f(t,x)$ satisfying Caratheodory's conditions above. Then, there exists a solution of the problem*

$$\begin{cases} \dot{x}(t) & = & f(t, x(t)), \\ x(t_0) & = & x_0, \end{cases} \tag{2.4.3}$$

on a closed interval Γ of Ω, and the following inequalities are satisfied

$$0 < d \leq a, \qquad \varphi(t_0 + d) \leq \epsilon, \qquad \varphi(t) = \int_{t_0}^{t} m(s) \ ds,$$

where d is arbitrary chosen, and $m(t)$ is from the third caratheodory condition (c) above.

2.4.4 Remark. The existence of Caratheodory solution stated in theorem 2.4.3 is called Filippov's theorem [1].

2.4.5 Theorem. *([45]) Let $\mathcal{D}$ a domain such that $(t,x) \in \mathcal{D}$, and f a function satisfying Caratheodory conditions. There exists a summable function $k(t)$ with each point (t,x_1) and (t,x_2) of $\mathcal{D}$ such that*

$$|f(t,x_1) - f(t,x_2)| \leq k(t) \ |x_1 - x_2| \, .$$

therefore, there exists at most one solution of the problem (2.4.1)

3. Unstructured age model

Based on [12] we present another mathematical model translated by a reaction-diffusion which describes the ongoing process of forest coupled with an advection equation which models the seed dispersion.

3.1 Forest model presentation

Let $\Omega \subset \mathbb{R}^2$ a bounded domain with its regular boundary $\partial\Omega$, which sketches a geographic region settled by a forest surface.

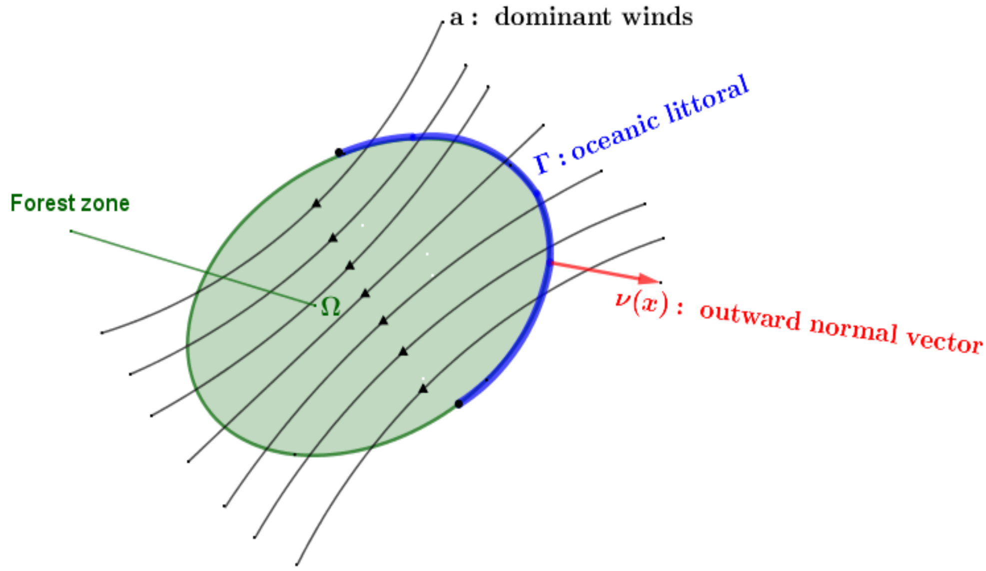


Figure 3.1: Geographical representation of a forest area [12].

As we can see, the dominant winds over the region Ω are oriented by the advection field across the boundary Γ, thus drawing the oceanic coastline. In addition, we denote the progression of the fast dominant winds by τ; besides the precipitation and vegetation timescales in the case of the Amazon region which is the world's largest rainforest illustrated in [12], we can also mention the case of the Daintree region which is one of the largest in Australia and one of the oldest rainforest in the world with a marked seasonality.

In this work, For rapidly evolving processes, we consider their average rate in the forest system study process. We now defined the oceanic littoral denoted by Γ which also is known as an invariant domain (see lemma 2.3.3) and given by:

$$\Gamma = \{x \in \partial\Omega,\ a(x).\nu(x) < 0\}\,,$$

where $\nu(x)$ represents the outward normal vector at point $x \in \partial\Omega$, $a = a(x)$ an advection field which crosses the domain Ω and representing the average dynamic of dominant winds per year, and τ is assumed invariant (does not vary anymore).

Authors in [12] mentioned the influence of climate change on the advection field a; and which most often acts at a very long time scale than the forest ecosystem time scale t, and they modelized these disturbances of the field advection a by changing the parameters instead of considering a long time scale in their model.

Let us denote by $\rho(t,x)$ the amount of water per unit of area, by $u(t,x)$ the surface density of trees and by $\omega(t,x)$ the surface density of seeds for each $t \geq 0$ and each $x \in \omega$.

Cantin et al [12] considered that the water resource has two distinct origins, the first contribution stems from climatic evaporation over maritime zones while the second contribution comes from the evapotranspiration over forests. We provide the following moisture conservation equation:

$$\frac{\partial \rho}{\partial t} + a.\nabla \rho = -\sigma \rho + \varphi(\rho) u, \tag{3.1.1}$$

where σ represents the average decrease rate of the water resource evaporated over an ocean zone per year, and $a.\nabla\rho$ represents the moisture advection.

In [17], Botkin et al considered that the per year averaged water kinematics is much more faster than the forest kinematics for a category of trees. In the case of non-linear transport equation, this fast convergence has been proved (see for instance [40]). Thus, the distribution of the water resource can be expressed by the following equation:

$$a.\nabla \rho(t,x) = -\sigma \rho(t,x) + \varphi(\rho(t,x)) u(t,x), \tag{3.1.2}$$

On the other hand, the contribution of trees to evapotranspiration process denoted by $\varphi(\rho)u$, satisfies the following system:

$$\varphi(0) = 0, \qquad \lim_{\rho \to +\infty} \varphi(\rho) = \varphi_0 > 0, \tag{3.1.3}$$

where $\varphi \geq 0$ if $u \geq 0$. Next, we consider that the forest evapotranspiration process is vanishing when the water resource decreases. We supplement the stationary advection equation (3.1.2) by:

$$\rho(t,x) = m(x), \qquad x \in \Gamma, \tag{3.1.4}$$

where m represents the average oceanic contribution to the water cycle per year.

3.1.1 Structure forest ecosystem. We start this part by showing a graphical representation of our forest model (see figure 3.2).

Next, based on the graph below, we infer the following reaction-diffusion-advection system which is the reduced form of the one presented in [12] by considering only one category of trees.

$$\begin{cases} a.\nabla \rho &=& -\sigma \rho + \varphi(\rho) u, \\ \dfrac{\partial u}{\partial t} &=& \beta \delta \omega - \gamma(\rho) u, \\ \dfrac{\partial \omega}{\partial t} &=& d\Delta \omega - \beta \omega + \alpha(\rho) u. \end{cases} \tag{3.1.5}$$

In this case, $\gamma(\rho)$ represents the overall tree mortality and it is described by:

$$\gamma(\rho) = k + h(\rho), \tag{3.1.6}$$

where k represents the average mortality rate of trees and it can be updated depending on the forest type we study and $h(\rho)$ the mortality rate of trees due to the scarcity of water resources. As mentioned

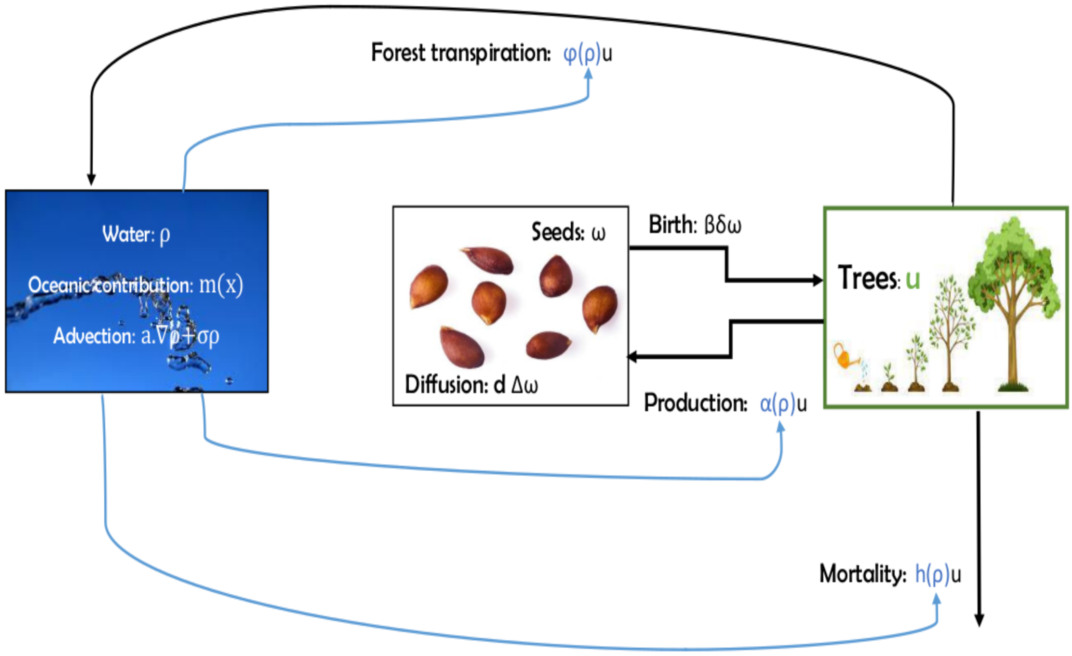


Figure 3.2: A compartmental model of our forest ecosystem is graphically represented. The blue arrows just highlight the dependence on water resources in each part of the process while the black ones describe the ongoing process in a forest ecosystem.

earlier, the fast dynamic of atmospheric concerning the slow dynamic of forest evolution confirms the assumption of a stationary advection equation for the water resource distribution. Therefore, initial conditions are given as follows:

$$u(0,x) = u_0(x), \qquad \omega(0,x) = \omega_0(x), \qquad x \in \Omega, \tag{3.1.7}$$

where u_0 and w_0 are smooth non-negative functions defined in Ω. Since the dispersion of seeds is not necessarily uniform on the whole domain $\Omega \cup \partial\Omega$, therefore conditions on ρ (3.1.4) and the Neumann boundary one on ω which specifies the value that the solution has to satisfy along the boundary domain $\partial\omega$.

$$\frac{\partial \omega}{\partial \nu}(t,x) = 0, \qquad t \geq 0, \qquad x \in \partial\Omega, \tag{3.1.8}$$

In the figure 3.3, the function $\varphi(\rho)$ describes the water evaporation process over forests. Secondly, the function $h(\rho)$ represents the mortality rate of trees concerning a variation of the water resource, and $\alpha(\rho)$ gives the shape of the production of seeds by trees. Since the lack of water resources affects the mortality of trees, as we can see on the graph above (b), h is a positive decay function of ρ such that

$$h(0) = h_0 > 0, \qquad \lim_{\rho \to +\infty} h(\rho) = h_1 > 0. \tag{3.1.9}$$

Thus, a lack of water resources increases tree mortality, but mortality becomes constant (no longer varies) when available water resources exceed a given threshold. Water has a positive effect on seed growth by hydrating the seed coat and increasing its permeability. In addition to the evapotranspiration process that occurs in the atmosphere, forests play an important role in the water cycle in that water

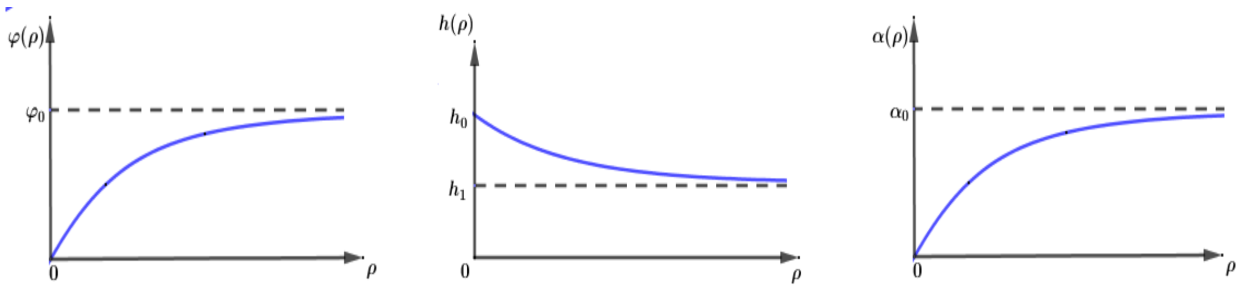


Figure 3.3: Graph of functions φ, h, and α respectively [12].

vapor is released from the leaves and stems of plants. For these reasons, the $\varphi(\rho)$ and $\alpha(\rho)$ functions are non-negative functions.

Furthermore, the evolution of seed density is described by the last equation in the system (3.1.5). This evolution admits a tree production expressed by the term $\alpha(\rho)u$ and where the function α satisfies similar properties as a function φ, given by:

$$\alpha(0) = 0, \qquad \lim_{\rho \to +\infty} \alpha(\rho) = \alpha_0 > 0. \tag{3.1.10}$$

Finally, the germination procedure of seeds is described by the term $\beta\delta\omega$ where $\beta > 0$, and $\delta > 0$ represent the rate of seed production and seed establishment respectively, then the dispersion of seeds density conducts by the effect of air diffusion, denoted by the Laplace operator weighted by diffusion coefficient $d > 0$. The following table gives us the unknown parameters of our model

Unknown	Symbol	Unit
Water resource	$\rho(t,x)$	mm $\times$ ha^{-1}
Trees density	$u(t,x)$	trees $\times$ ha^{-1}
Seeds density	$\omega(t,x)$	seeds $\times$ ha^{-1}

Table 3.1: Unknowns of the forest model (3.1.5).

Analysis of the reaction-diffusion-advection system

In order to simplify the study of our model, which is a model without age structure based on a reaction-diffusion-advection system. We use the method of characteristics to reduce the initial system to an equivalent classical reaction-diffusion system still reflecting the studied model. Subsequently, we prove the existence and uniqueness of the global time solution of our problem through the properties of the semi-groups and Caratheodory (see section 2.4).

Here, we will prove that system (3.1.5) possesses global solutions in time with non-negative and bounded components ρ, u, and ω, for any non-negative and sufficiently smooth initial conditions u_0 and ω_0 defined in Ω. Some of the solutions may possess discontinuities, see for instance [11] for the case of model without water resources.

3.2 Simplification to a reaction-diffusion system

In this section, we aim to parametrize the advection equation to reduce the system (3.1.5) into a reaction-diffusion system. From here till the end of this work, the real-valued functions α, φ, and h are assumed to be continuous and differentiable, i.e.

$$\begin{array}{rcl} 0 \leq \alpha(s) \leq \alpha_0, & |\alpha(s_1) - \alpha(s_2)| & \leq & \alpha_0 \, |s_1 - s_2| \, , \\ 0 \leq \varphi(s) \leq \varphi_0, & |\varphi(s_1) - \varphi(s_2)| & \leq & \varphi_0 \, |s_1 - s_2| \, , \\ h_1 \leq h(s) \leq h_0, & |h(s_1) - h(s_2)| & \leq & h_0 \, |s_1 - s_2| \, , \end{array} \tag{3.2.1}$$

for all s, s_1, $s_2 \in \mathbb{R}^+$, with positive coefficients α_0, φ_0, h_0, and h_1. Following the same idea, we also make hypothesis on the advection field $a = (a_1, a_2)$, by assuming that orbit $x = (x_1, x_2)$ describes the characteristic lines of a. x is defined by:

$$\begin{cases} \dfrac{\partial x_1(s)}{\partial s} = a_1 \left(x_1(s), x_2(s) \right), \\ \dfrac{\partial x_2(s)}{\partial s} = a_2 \left(x_1(s), x_2(s) \right), \end{cases} \tag{3.2.2}$$

adding the fact that the initial condition $x(0) = x_0 \in \Gamma$ (oceanic littoral) spans the domain Ω so that each point $x \, in\Omega$ belongs to at most one characteristic line. Let $\{\xi(x_0, s)\}_{0 \leq s \leq S(x_0)}$ the single orbit of system (3.2.2) originating from $x_0 \in \Gamma$. We assume that:

$$\xi(x_0, 0) = x_0, \quad \xi(x_0, s) \in \Omega \ \text{ if } 0 < s < S(x_0), \quad \xi(x_0, S(x_0)) \in \partial\Omega \backslash \Gamma, \tag{3.2.3}$$

and a is fairly regular, so that the family of orbits $\{\xi(x_0, s)\}_{0 \leq s \leq S(x_0)}$ parametrized by $x_0 \in \Gamma$ change continuously with x_0, thus if $\|\tilde{x}\|_{\mathbb{R}^2}$ converges to $\|x_0\|_{\mathbb{R}^2}$, for $\tilde{x} \in \Gamma$, then

$$\sup_{0 \leq s \leq \min(S(\tilde{x}), S(x_0))} \|\xi(\tilde{x}, s) - \xi(x_0, s)\|_{\mathbb{R}^2} \to 0. \tag{3.2.4}$$

Next, we suppose that there exists a constant $\bar{S} > 0$ so that:

$$S(x_0) \leq \bar{S}, \qquad \forall \ x_0 \in \Gamma, \tag{3.2.5}$$

the above hypothesis also holds if Γ is compact, i.e. if Γ is closed and bounded. Next, for each $x \in \Omega$, we denote the unique pair $(x_0, s) \in \Gamma \times \mathbb{R}$ by:

$$(x_0, s) = \left(\zeta_1(x), \zeta_2(x) \right), \qquad \text{with} \ \ x = \xi(x_0, s). \tag{3.2.6}$$

Based on the notation (3.2.6), the unique orbit of system (3.2.2) passing through $x \in \Omega$ is now $\{\xi(\zeta_1(x), s)\}_{0 \leq s \leq S(\zeta_1(x))}$. The continuity of a implies the one of ζ_1 and ζ_2.

We set q as a function defined almost everywhere in Ω. We have:

$$q(x) = q \left(\xi(\zeta_1(x), \xi(\zeta_2(x)) \right),$$

for x almost everywhere in Ω. We introduce the following notation

$$\tilde{q}(x_0, s) = q \circ \xi(x_0, s), \quad x_0 \in \Omega, \quad s \in [0, S(x_0)] \, . \tag{3.2.7}$$

In order to make things simple, we may use $\tilde{q}(s)$ instead of $\tilde{q}(x_0, s)$. Now, we introduce and define the following Banach space:

$$L^\infty_+(\Omega) = \{u \in L^\infty(\Omega), \quad u \geq 0 \text{ a.e on } \Omega\},$$

where

$$L^\infty(\Omega) = \left\{u, \quad \|u\|_\infty = \sup_{x\in\Omega} |u(x)| < \infty\right\}.$$

In other words, $L^\infty_+(\Omega)$ denotes the subset of L^∞ with non-negative functions almost everywhere. For each $u \in L^\infty_+(\Omega)$ and each $m \in L^\infty_+(\Omega)$, we consider the stationary advection equation

$$\begin{cases} a.\nabla\rho = -\sigma\rho + \varphi(\rho)u, & x \in \Omega, \\ \rho(x) = m(x), & x \in \Gamma, \end{cases} \tag{3.2.8}$$

Our aim is to study the following operator:

$$\begin{aligned} \psi : L^\infty_+(\Omega) &\longrightarrow L^\infty_+(\Omega), \\ u &\longmapsto \rho. \end{aligned} \tag{3.2.9}$$

where ρ is the solution of advection equation (3.2.8).

In the next theorem, we will discuss about the posedness of operator ψ defined in (3.2.9).

3.2.1 Theorem. *For $x \in \Omega$ almost everywhere (a.e) and $u \in L^\infty_+(\Omega)$, the defined operator ψ in (3.2.9) exists and it is uniquely determined along the characteristic lines of the advection field a by:*

$$\psi(u)(x) = m(\zeta_1(x))e^{-\sigma\zeta_2(x)} + \int_0^{\zeta_2} \varphi\left(\tilde{\rho}(\zeta_1(x), \tau)\right) \ \tilde{u}(\zeta_1(x), \tau)e^{-\sigma(\zeta_2(x)-\tau)}, \tag{3.2.10}$$

where $(x_0, s) = (\zeta_1(x), \zeta_2(x))$ is defined with notation (3.2.6).

Furthermore, the operator ψ is continuous in $L^\infty_+(\Omega)$ and we have:

$$\|\psi(u+h) - \psi(u)\| \leq \|h\|_\infty \times \frac{\varphi_0}{\sigma} e^{\varphi_0 \bar{S}\|u\|_\infty}, \quad \forall\, u, h \in L^\infty_+(\Omega). \tag{3.2.11}$$

Therefore, the operator ψ is differentiable in $L^\infty_+(\Omega)$ and for all $x \in \Omega$ a.e and each $u, h \in L^\infty_+(\Omega)$ we have

$$\begin{aligned} D\psi(u)h(x) = &\int_0^{\zeta_2} \varphi\left(\tilde{\rho}\left(\zeta_1(x), \tau\right)\right) \tilde{h}\left(\zeta_1(x), \tau\right) e^{-\sigma(\zeta_2(x)-\tau)} d\tau \\ + &\int_0^{\zeta_2} \varphi'\left(\tilde{\rho}\left(\zeta_1(x), \tau\right)\right) \tilde{v}\left(\zeta_1(x), \tau\right) D\psi(u)\tilde{h}\left(\zeta_1(x), \tau\right) e^{-\sigma(\zeta_2(x)-\tau)} d\tau, \end{aligned} \tag{3.2.12}$$

its norm holds

$$\|D\psi(u)\|_{\mathcal{L}(L^\infty(\Omega), L^\infty(\Omega))} \leq \frac{\varphi_0}{\sigma} e^{\bar{S}\|u\|_\infty}. \tag{3.2.13}$$

Proof. The sketch of the proof of theorem 3.2.1 can be found in the appendix. □

3.2.2 Well posedness of the reduced system. Since the advection equation has been parameterized see (3.2.8, 3.2.9) to reduce the initial system to a reaction-diffusion system, then according to theorem 3.2.1, system (3.1.5) becomes:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) &= \beta\delta\omega - \gamma(\psi(u))u, & t>0, \quad x\in\Omega, \\ \dfrac{\partial \omega}{\partial t}(t,x) &= d\Delta\omega - \beta\omega + \alpha(\psi(u))u, & t>0, \quad x\in\Omega, \\ \dfrac{\partial \omega}{\partial \nu}(t,x) &= 0, & t>0, \quad x\in\partial\Omega, \end{cases} \tag{3.2.14}$$

where $\psi(u)$ models the water resource dependence in the tree life process, and (3.2.14) is a non linear reaction diffusion system. Let pose $\Lambda = -d\Delta + \beta$ a linear operator in $L^2(\Omega)$ with the Neumann boundary condition on $\partial\Omega$; thus $U = (u,\omega) \in X = (L^\infty(\Omega) \times L^2(\Omega))$. The system (3.2.14) can be written as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) &= \beta\delta\omega - \gamma(\psi(u))u, & t>0, \quad x\in\Omega, \\ \dfrac{\partial \omega}{\partial t}(t,x) &= -\Lambda\omega + \alpha(\psi(u))u, & t>0, \quad x\in\Omega, \\ \dfrac{\partial \omega}{\partial \nu}(t,x) &= 0, & t>0, \quad x\in\partial\Omega. \end{cases} \tag{3.2.15}$$

Which is still equivalent to:

$$\begin{pmatrix} \dfrac{\partial u}{\partial t} \\ \dfrac{\partial \omega}{\partial t} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \Lambda \end{pmatrix} \begin{pmatrix} u \\ \omega \end{pmatrix} = \begin{pmatrix} \beta\delta\omega - \gamma(\psi(u))u + u \\ \alpha(\psi(u))u \end{pmatrix}, \tag{3.2.16}$$

Let us denote by $A = \text{diag}(1,\Lambda)$ the diagonal operator in X; A is a positive self-adjoint which generates an analytic semi-group and admits fractional powers in its domain $\mathcal{D}(A) = L^\infty(\Omega) \times H^2_N(\Omega)$ where

$$H^2_N(\Omega) = \left\{ v \in H^2(\Omega), \quad \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}.$$

The domains of its fractional powers A^η, $0<\eta<1$ are also well known; also

$$U = (u,\omega)^t \in \mathcal{D}(A^\eta), \qquad \frac{dU}{dt} = \left(\frac{\partial u}{\partial t}, \frac{\partial \omega}{\partial t} \right)^t$$

and the non linear operator F is defined by:

$$F(U) = \begin{bmatrix} \beta\delta\omega - \gamma(\psi(u))u + u \\ \alpha(\psi(u))u \end{bmatrix} = \begin{bmatrix} F_1(U) \\ F_2(U) \end{bmatrix}.$$

Based on the embedding properties of Sobolev spaces, we have:

$$\mathcal{D}(\Lambda^\eta) = H^{2\eta}_N(\Omega) \subset \mathcal{C}(\bar{\Omega}), \quad \text{with} \quad \frac{1}{2} < \eta < 1.$$

Hence,

$$\mathcal{D}(A^\eta) \subset (L^\infty(\Omega))^3.$$

Therefore, the system (3.2.14) is now formulated as an abstract Cauchy problem defined as follows:

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), & t>0, \\ U(0) = U_0, & U_0 \in \mathcal{Z}, \end{cases} \tag{3.2.17}$$

where $\mathcal{Z} = \{U = (u,\omega)^t \in X,\ u(x) \geq 0,\ \omega(x) \geq 0 \text{ a.e. on } \Omega\}$. In other words, $\mathcal{Z}$ is a subset of X containing functions which are non-negative (either constant or positive).

3.2.3 Theorem. *For any initial condition $U_0 \in \mathcal{Z}$, the Cauchy problem (3.2.17) possesses a unique local in time solution $U = (u, \omega)^t$ defined on $Y = [0, T_{U_0}]$ with*

$$\begin{cases} u \in \mathcal{C}\left(Y, L^\infty(\Omega)\right) \cap \mathcal{C}^1\left((0, T_{U_0}], L^\infty(\Omega)\right), \\ \omega \in \mathcal{C}\left((0, T_{U_0}], \mathcal{D}(\Lambda)\right) \cap \mathcal{C}\left(Y, L^2(\Omega)\right) \cap \mathcal{C}^1\left((0, T_{U_0}], L^2(\Omega)\right), \end{cases} \tag{3.2.18}$$

where $T_{U_0} = Cte(\|U_0\|_X) > 0$. Furthermore, the local solution U satisfies

$$t\,\|AU(t)\|_X + \|U(t)\|_X \le C_{U_0}, \quad 0 < t \le T_{U_0}, \quad C_{U_0} = Cte(\|U_0\|_X) > 0. \tag{3.2.19}$$

3.2.4 Notation. $C_{U_0} = \text{Cte}(\|U_0\|_X) > 0$ means that C_{U_0} is a positive constant which depends only on $\|U_0\|_X$.

Proof. (proof of theorem 3.2.3) Since $\mathcal{D}(A^\eta)$ is dense in X ($\overline{\mathbb{D}(A^\eta)} = X$) and $0 \le \eta < 1$, it remains to show that the non linear F is bounded in order to conclude. Let $U = (u, \omega)^t$ and $\tilde{U} = (\tilde{u}, \tilde{\omega})^t$ in $\mathcal{D}(A^\eta)$, we have to show that

$$\left\|F(U) - F(\tilde{U})\right\|_X \le k\left(\|U\|_X + \left\|\tilde{U}\right\|_X\right)\left[\left\|A^\eta(U - \tilde{U})\right\|_X + \left(\|U\|_X + \left\|\tilde{U}\right\|_X\right)\left\|U - \tilde{U}\right\|_X\right].$$

The aim is to estimate the quantities $\left\|F_1(U) - F_1(\tilde{U})\right\|_\infty$ and $\left\|F_2(U) - F_2(\tilde{U})\right\|_\infty$, we have:

$$\begin{aligned} \left\|F_1(U) - F_1(\tilde{U})\right\|_\infty &= \sup_{x\in\Omega} \left|\beta\delta(\omega - \tilde{\omega}) - \gamma(\psi(u))u + \gamma(\psi(\tilde{u}))\tilde{u} + (u - \tilde{u})\right|, \\ &\le \beta\delta\,\|\omega - \tilde{\omega}\|_\infty + \|\gamma(\psi(u))u - \gamma(\psi(\tilde{u}))\tilde{u}\|_\infty + \|u - \tilde{u}\|_\infty, \\ \left\|F_1(U) - F_1(\tilde{U})\right\|_\infty &\le \beta\delta\,\|\omega - \tilde{\omega}\|_\infty + P + \|u - \tilde{u}\|_\infty, \end{aligned} \tag{3.2.20}$$

with $P = \|\gamma(\psi(u))u - \gamma(\psi(\tilde{u}))\tilde{u}\|_\infty$. Let us expand the expression of P,

$$\begin{aligned} P &= \|\gamma(\psi(u))u - \gamma(\psi(\tilde{u}))\tilde{u}\|_\infty, \\ &= \|(k + h(\psi(u)))u - (k + h(\psi(\tilde{u})))\tilde{u}\|_\infty, \\ &\le k\,\|u - \tilde{u}\|_\infty + \|h(\psi(u))u - h(\psi(\tilde{u}))\tilde{u}\|_\infty, \\ &\le k\,\|u - \tilde{u}\|_\infty + \|h(\psi(u))u - h(\psi(u))\tilde{u} + h(\psi(u))\tilde{u} - h(\psi(\tilde{u}))\tilde{u}\|_\infty, \\ &\le k\,\|u - \tilde{u}\|_\infty + \|h(\psi(u))u - h(\psi(u))\tilde{u}\|_\infty + \|h(\psi(u))\tilde{u} - h(\psi(\tilde{u}))\tilde{u}\|_\infty, \\ &\le k\,\|u - \tilde{u}\|_\infty + h_0\,\|u - \tilde{u}\|_\infty + h_0\,\|u\|_\infty\,\|\psi(u) - \psi(\tilde{u})\|_\infty, \\ &\le k\,\|u - \tilde{u}\|_\infty + h_0\,\|u - \tilde{u}\|_\infty + h_0\,\|u\|_\infty\,\frac{\varphi_0}{\sigma}\,\|u - \tilde{u}\|_\infty\; e^{\varphi_0\bar{S}\|u\|_\infty}, \\ P &\le k\,\|u - \tilde{u}\|_X + h_0\,\|u - \tilde{u}\|_X + h_0\,\|u\|_X\,\frac{\varphi_0}{\sigma}\,\|u - \tilde{u}\|_X\; e^{\varphi_0\bar{S}\|u\|_X}. \end{aligned} \tag{3.2.21}$$

Thus, by substituting the approximation of P (3.2.21) in (3.2.20), we get:

$$\left\|F_1(U) - F_1(\tilde{U})\right\|_\infty \le g\left(\|U\|_X + \left\|\tilde{U}\right\|_X\right)\left(\|U\|_X + \left\|\tilde{U}\right\|_X\right)\left\|U - \tilde{U}\right\|_X. \tag{3.2.22}$$

with g an increasing function.

Next, we consider

$$\begin{aligned} \left\|F_2(U) - F_2(\tilde{U})\right\|_2 &= \|\alpha(\psi(u))u - \alpha(\psi(\tilde{u}))\tilde{u}\|_2, \\ &= \|\alpha(\psi(u))u - \alpha(\psi(\tilde{u}))u + \alpha(\psi(\tilde{u}))u - \alpha(\psi(\tilde{u}))\tilde{u}\|_2, \\ &\le \|\alpha(\psi(u))u - \alpha(\psi(\tilde{u}))u\|_2 + \|\alpha(\psi(\tilde{u}))u - \alpha(\psi(\tilde{u}))\tilde{u}\|_2, \\ &\le \|u\|_\infty\,\|\alpha(\psi(u)) - \alpha(\psi(\tilde{u}))\|_2 + \|\alpha(\psi(u))\|_\infty\,\|u - \tilde{u}\|_2 \\ &\le \alpha_0\,\|u\|_\infty\,\|\psi(u) - \psi(\tilde{u})\|_2 + \alpha_0\,\|u - \tilde{u}\|_2, \quad \text{because } \alpha \text{ is continue (see 3.2.1).} \end{aligned}$$

Since $L^\infty(\Omega) \subset L^2(\Omega)$ (Lebesgue properties), then

$$\|\psi(u) - \psi(\tilde{u})\|_2 \leq \|u - \tilde{u}\|_2 \, \frac{\varphi_0}{\sigma} \, e^{\varphi_0 \bar{S}\|u\|_2} \leq \|u - \tilde{u}\|_\infty \, \frac{\varphi_0}{\sigma} \, e^{\varphi_0 \bar{S}\|u\|_\infty}.$$

Therefore,

$$\begin{aligned}
\left\|F_2(U) - F_2(\tilde{U})\right\|_2 &= \|\alpha(\psi(u))u - \alpha(\psi(\tilde{u}))\tilde{u}\|_2 \,, \\
&\leq \alpha_0 \|u\|_\infty \|u - \tilde{u}\|_\infty \frac{\varphi_0}{\sigma} \, e^{\varphi_0 \bar{S}\|u\|_\infty} + \alpha_0 \|u - \tilde{u}\|_2 \,, \\
&\leq c_1 \left[\|u\|_\infty \|u - \tilde{u}\|_\infty e^{\varphi_0 \bar{S}\|u\|_\infty} + \|u - \tilde{u}\|_2\right], \qquad (3.2.23)
\end{aligned}$$

Furthermore, we know that $\|u - \tilde{u}\|_2 \leq C_2 \; \|A^\eta(u - \tilde{u})\|_X$, then

$$\begin{aligned}
\left\|F_2(U) - F_2(\tilde{U})\right\|_2 &= C_3 \left[\|u\|_X \|u - \tilde{u}\|_X e^{\varphi_0 \bar{S}\|u\|_X} + \|A^\eta(u - \tilde{u})\|_X\right], \\
&\leq C_3 \left[(\|u\|_X + \|\tilde{u}\|_X)(\|u\|_X + \|\tilde{u}\|_X)\|u - \tilde{u}\|_X + (\|u\|_X + \|\tilde{u}\|_X)\|A^\eta(u - \tilde{u})\|_X\right], \\
&\leq C_3 (\|u\|_X + \|\tilde{u}\|_X)\left[(\|u\|_X + \|\tilde{u}\|_X)\|u - \tilde{u}\|_X + \|A^\eta(u - \tilde{u})\|_X\right]. \qquad (3.2.24)
\end{aligned}$$

where C_3 is a continuous increasing function. From (3.2.24) and (3.2.22), we conclude that F is bounded.

Since the conditions of [45, theorem 4.4] are satisfied, then the abstract Cauchy problem (3.2.17) possesses a unique local solution $U = (u, \omega)^t$ in the function space

$$\begin{aligned}
u &\in \mathcal{C}(Y, L^\infty(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}], L^\infty(\Omega)) \,, \\
\omega &\in \mathcal{C}((0, T_{U_0}], \mathcal{D}(\Lambda)) \cap \mathcal{C}\left(Y, L^2(\Omega)\right) \cap \mathcal{C}^1\left((0, T_{U_0}], L^2(\Omega)\right).
\end{aligned}$$

□

Therefore, we formulate the following corollary,

3.2.5 Corollary. For each $U_0 \in \mathcal{Z}$ and $m \geq 0$, $m \in \Gamma$, the reaction diffusion advection problem (3.2.15) with conditions (3.2.14)-(3.2.17) possesses a unique local solution $U = (\rho, u, \omega)^t$ defined on $[0, T_{U_0}]$, $T_{U_0} > 0$ such that

$$\begin{aligned}
\rho, \; u &\in \mathcal{C}(Y, L^\infty(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}], L^\infty(\Omega)) \,, \\
\omega &\in \mathcal{C}((0, T_{U_0}], \mathcal{D}(\Lambda)) \cap \mathcal{C}\left(Y, L^2(\Omega)\right) \cap \mathcal{C}^1\left((0, T_{U_0}], L^2(\Omega)\right).
\end{aligned}$$

3.2.6 Theorem. *(Non-negativity of solutions) Let $0 \leq u_0 \in L^\infty(\Omega)$ and $0 \leq \omega_0 \in L^2(\Omega)$. System (3.2.14) admits a unique non-negative local solution such that:*

$$\begin{cases}
0 \leq u \in \mathcal{C}(Y, L^\infty(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}], L^\infty(\Omega)) \,, \\
0 \leq \omega \in \mathcal{C}((0, T_{U_0}], \mathcal{D}(\Lambda)) \cap \mathcal{C}\left(Y, L^2(\Omega)\right) \cap \mathcal{C}^1\left((0, T_{U_0}], L^2(\Omega)\right),
\end{cases} \qquad (3.2.25)$$

Proof. The uniqueness of local solution (u, ω) in function space (3.2.18) leads to theorem 3.2.3. Let $(\hat{u}, \hat{\omega}) \in X$, We define the following cut-off function ϑ by:

$$\vartheta(\hat{u}) = \begin{cases} \hat{u} & \text{if} \;\; \hat{u} \geq 0, \\ 0 & \text{if} \;\; \hat{u} < 0. \end{cases} \qquad \text{and} \qquad \vartheta(\hat{\omega}) = \begin{cases} \hat{\omega} & \text{if} \;\; \hat{\omega} \geq 0, \\ 0 & \text{if} \;\; \hat{\omega} < 0. \end{cases} \qquad (3.2.26)$$

Let $\hat{u}_1, \hat{u}_2 \in \mathbb{R}$, we have that:

$$|\vartheta(\hat{u}_1) - \vartheta(\hat{u}_2)|_{\mathbb{R}} = |\vartheta(\hat{u}_1) - \vartheta(\hat{u}_2)|_{\mathbb{R}_+} + |\vartheta(\hat{u}_1) - \vartheta(\hat{u}_2)|_{\mathbb{R}_-} = |\hat{u}_1 - \hat{u}_2|\,.$$

Let consider this similar problem

$$\left\{\begin{array}{rcl} \dfrac{\partial\vartheta(\hat{u})}{\partial t}(t,x) &=& \beta\delta\vartheta(\hat{\omega}) - \gamma(\psi(\vartheta(\hat{u})))\vartheta(\hat{u}), \qquad t>0, \quad x\in\Omega, \\ \dfrac{\partial\vartheta(\hat{\omega})}{\partial t}(t,x) &=& d\Delta\vartheta(\hat{\omega}) - \beta\vartheta(\hat{\omega}) + \alpha(\psi(\vartheta(\hat{u})))\vartheta(\hat{u}), \qquad t>0, \quad x\in\Omega, \\ \dfrac{\partial\vartheta(\hat{\omega})}{\partial \nu}(t,x) &=& 0, \qquad t>0, \quad x\in\partial\Omega, \\ \vartheta(\hat{u})(0,x) &=& u_0(x), \qquad \vartheta(\hat{\omega})(0,x) = \omega_0(x), \qquad x\in\Omega. \end{array}\right. \tag{3.2.27}$$

In particular, for $\hat{u} \geq 0$ and $\hat{\omega} \geq 0$, $\vartheta(\hat{u}) = \hat{u}$, $\vartheta(\hat{\omega}) = \hat{\omega}$, system (3.2.27) becomes:

$$\left\{\begin{array}{rcl} \dfrac{\partial\hat{u}}{\partial t}(t,x) &=& \beta\delta\hat{\omega} - \gamma(\psi(\hat{u}))\hat{u}, \qquad t>0, \quad x\in\Omega, \\ \dfrac{\partial\hat{\omega}}{\partial t}(t,x) &=& d\Delta\hat{\omega} - \beta\hat{\omega} + \alpha(\psi(\hat{u}))\hat{u}, \qquad t>0, \quad x\in\Omega, \\ \dfrac{\partial\hat{\omega}}{\partial \nu}(t,x) &=& 0, \qquad t>0, \quad x\in\partial\Omega, \\ \hat{u}(0,x) &=& u_0(x), \qquad \hat{\omega}(0,x) = \omega_0(x), \qquad x\in\Omega. \end{array}\right. \tag{3.2.28}$$

Following the same path as in section 3.2.2, we can infer that system (3.2.28) admits a unique local solution $\hat{U} = (\hat{u}, \hat{\omega})$ in the function space (3.2.18). Then, $\hat{U} = (\hat{u}, \hat{\omega})$ is also a unique local solution of system (3.2.14).

Therefore, by uniqueness of solutions, we conclude that $U = (u, \omega) = (\hat{u}, \hat{\omega}) = \hat{U}$ in $Y = [0, T_{U_0}]$. Hence, system (3.2.14) possesses a unique non-negative local solution in the function space (3.2.18). □

3.2.7 Proposition. (Global solutions) Let $0 \leq u_0 \in L^\infty(\Omega)$, and $0 \leq \omega_0 \in L^2(\Omega)$. Based on the theorem 2.4.3, since the local solution $U = (\rho, u, \omega)^t$ of (3.2.14)-(3.2.15)-(3.2.17) in the function space

$$\begin{cases} 0 \leq \rho,\ u \in \mathcal{C}\left(Y, L^\infty(\Omega)\right) \cap \mathcal{C}^1\left((0, T_{U_0}], L^\infty(\Omega)\right), \\ 0 \leq \omega \in \mathcal{C}\left((0, T_{U_0}], \mathcal{D}(\Lambda)\right) \cap \mathcal{C}\left(Y, L^2(\Omega)\right) \cap \mathcal{C}^1\left((0, T_{U_0}], L^2(\Omega)\right). \end{cases}$$

satisfies (3.2.19). More precisely, the following estimate is satisfied

$$\|u(t)\|_{L^\infty(\Omega)} + \|\omega(t)\|_{L^2(\Omega)} \leq C\left[e^{-t\eta}\left(\|u_0\|_{L^\infty(\Omega)} + \|\omega_0\|_{L^2(\Omega)}\right)\right], \quad \eta>0,\ 0<t\leq T_{U_0}, \tag{3.2.29}$$

Then, according to [45, corollary 4.3], the reaction-diffusion advection system (3.2.14)-(3.2.15)-(3.2.17) possesses a unique global solution $U = (\rho, u, \omega)^t$ in the function space

$$\begin{array}{rcl} 0 \leq \rho,\ u &\in& \mathcal{C}\left(Y, L^\infty(\Omega)\right) \cap \mathcal{C}^1\left((0, T], L^\infty(\Omega)\right), \\ 0 \leq \omega &\in& \mathcal{C}\left((0, T], \mathcal{D}(\Lambda)\right) \cap \mathcal{C}\left(Y, L^2(\Omega)\right) \cap \mathcal{C}^1\left((0, T], L^2(\Omega)\right). \end{array}$$

The proof of proposition 3.2.7 is similar to the one done in dynamical systems under Dirichlet conditions, see [42, section 5].

3.3 Stability analysis of the model

First of all, we consider that we are in the situation of a healthy forest that, despite the problems it encounters, maintains a certain balance (continuous evolution, constant supply). However, we are interested in their state of reaction to the phenomena studied so far, namely climate change, and the distribution of water resources. This is why stability analysis is important to evaluate the possible equilibrium states of the forest model taking into account these factors.

In this section, the purpose is to investigate the stationary equilibrium of the reaction-diffusion advection system. We start by checking both the existence of stationary homogeneous solutions of system (3.1.5) which are constant in time (solution does not vary on time) and uniform in space and also where solutions are constant in time but not necessarily uniform in space in case of heterogeneous solutions.

3.3.1 Proposition. [12] Let us assume that the regular function is vanishing (i.e. $\rho = m(x) = 0$), for all $x \in \Gamma$, then system (3.1.5) possesses a unique stationary homogeneous solution $\bar{U} = (\bar{\rho}, \bar{u}, \bar{\omega}) = (0, 0, 0)$.

Regarding figure 3.3, the graphs involved in the model. To make it a recall, the function $\alpha(\rho)$ models the seed production, $h(\rho)$ models the total mortality of trees while $\varphi(\rho)$ describes the tree contribution to water evaporation over the forest, all of them are a function of water resource ρ. To reflect a real scenario, these functions are explicitly chosen as a non-negative function and defined as follows:

$$\alpha(\rho) = \frac{\alpha_0 \rho}{1+\rho}, \qquad \varphi(\rho) = \frac{\varphi_0 \rho}{1+\rho}, \qquad h(\rho) = \frac{h_0 + h_1 \rho}{1+\rho}. \tag{3.3.1}$$

Let $\bar{U} = (\bar{\rho}, \bar{u}, \bar{\omega})^t$ a stationary solution of system (3.1.5). We have

$$\begin{cases} \bar{\rho} = m(x), \\ a.\nabla \bar{\rho} = -\sigma \bar{\rho} + \varphi(\bar{\rho})\bar{u}, \\ \dfrac{\partial \bar{u}}{\partial t} = \beta\delta\bar{\omega} - \gamma(\bar{\rho})\bar{u}, \\ \dfrac{\partial \bar{\omega}}{\partial t} = d\Delta\bar{\omega} - \beta\bar{\omega} + \alpha(\bar{\rho})\bar{u}. \end{cases} \tag{3.3.2}$$

3.3.2 Stationary homogeneous solution. Here, a stationary solution $\bar{U}$ satisfies the following homogeneous system:

$$\begin{cases} \bar{\rho} = m(x), \\ \sigma\bar{\rho} - \varphi(\bar{\rho})\bar{u} = 0, \\ \beta\delta\bar{\omega} - \gamma(\bar{\rho})\bar{u} = 0, \\ -\beta\bar{\omega} + \alpha(\bar{\rho})\bar{u} = 0. \end{cases} \Leftrightarrow \begin{cases} \bar{\rho} = m(x), \\ \sigma\bar{\rho} = \varphi(\bar{\rho})\bar{u}, \\ \beta\delta\bar{\omega} = \gamma(\bar{\rho})\bar{u}, \\ \beta\bar{\omega} = \alpha(\bar{\rho})\bar{u}. \end{cases} \tag{3.3.3}$$

The system (3.3.3) may not admit a stationary homogeneous solution because the first equation $\rho = m(x)$ depends on the regular function $m(x)$ which is not necessarily uniform on the whole boundary Γ. For that reason, we distinguish two cases.

1. **Oceanic contribution is vanishing** (m(x)=0). This case models a vanishing oceanic contribution. A real situation can be explored in [35] where Kristen Melon presented the seafloor morseling and shrubby algae invasion in the gulf of Maine. In this case, we consider $m(x) = 0$, for each $x \in \Gamma$.

Then, system (3.3.3) becomes:

$$\begin{cases} \bar{\rho} = 0, \\ \varphi(0)\bar{u} = 0, \\ \beta\delta\bar{\omega} = \gamma(0)\bar{u}, \\ \beta\bar{\omega} = \alpha(0)\bar{u}. \end{cases} \tag{3.3.4}$$

From (3.3.4), we get $\bar{u} = 0$; therefore, from the third equation of the system above, we also have $\bar{\omega} = 0$. Hence, the trivial solution $\bar{U} = (\bar{\rho}, \bar{u}, \bar{\omega}) = (0, 0, 0)$ is a unique stationary homogeneous solution of the system (3.1.5), and proposition 3.3.1 holds.

2. **Oceanic contribution is not vanishing**. In this case, under the condition $\bar{m} = m(x) > 0$ for each $x \in \Gamma$ system (3.3.3) becomes:

$$\begin{cases} \bar{\rho} = \bar{m}, \\ \sigma\bar{m} = \varphi(\bar{m})\bar{u}, \\ \beta\delta\bar{\omega} = \gamma(\bar{m})\bar{u}, \\ \beta\bar{\omega} = \alpha(\bar{m})\bar{u}. \end{cases} \tag{3.3.5}$$

Next, by substituting each unknown component in the latter system, we get:

$$\bar{u} = \frac{\sigma\bar{m}}{\varphi(\bar{m})}, \qquad \text{and} \qquad \bar{\omega} = \frac{\alpha(\bar{m})}{\beta}\bar{u}. \tag{3.3.6}$$

We introduce the expressions of φ and α in (3.3.5), and we obtain:

$$\begin{cases} \bar{u} = \dfrac{\sigma\bar{m}}{\varphi(\bar{m})} = \dfrac{\sigma\bar{m}}{\dfrac{\varphi_0\bar{m}}{1+\bar{m}}} = \dfrac{\sigma}{\varphi_0}(1+\bar{m}), \\ \bar{\omega} = \dfrac{\alpha_0\bar{m}}{\beta(1+\bar{m})}\dfrac{\sigma}{\varphi_0}(1+\bar{m}) = \bar{m}\dfrac{\sigma}{\beta}\dfrac{\alpha_0}{\varphi_0}, \end{cases} \tag{3.3.7}$$

and $\bar{m}$ has also to satisfy $f(m) = 0$ where f is defined by:

$$f(m) = h_0 + k + (k - \delta\alpha_0 + h_1)\bar{m}.$$

Indeed, u, and ω solve the third equation of the system (3.3.5), i.e.

$$\beta\delta\bar{\omega} = \gamma(\bar{m})\bar{u}, \qquad \Rightarrow \qquad \beta\delta\bar{\omega} = [k + h(\bar{m})]\,\bar{u},$$

therefore,

$$\beta\delta\frac{\bar{m}\alpha_0\sigma}{\beta\varphi_0} = \left[k + \frac{h_0 + h_1\bar{m}}{1+\bar{m}}\right]\frac{\sigma}{\varphi_0}(1+\bar{m}),$$

thus,

$$\delta\alpha_0\bar{m} - k(1+\bar{m}) + h_0 + h_1\bar{m} = 0,$$

So,

$$f(m) = h_0 + k + (k - \delta\alpha_0 + h_1)\bar{m}.$$

The polynomial $f(m)$ can admit at most one positive root. Furthermore, if there exists the positive solution of the latter first-degree polynomial equation corresponds to a stationary homogeneous solution to system (3.1.5) which can be parametrized by the following set of equations:

$$\bar{\rho} = \bar{m}, \qquad \bar{u} = \frac{\sigma}{\varphi_0}(1+m), \qquad \bar{\omega} = \frac{\alpha_0}{\beta}\frac{\sigma}{\varphi_0}m. \tag{3.3.8}$$

The equilibrium states of the forest ecosystem are characterized by the stationary homogeneous solutions of system (3.1.5). In other words, water resource is uniformly distributed over the forest. Next, if the oceanic contribution $m = m(x) > 0$ is slightly disturbed, then the forest system will tend to return to its original equilibrium state.

3.3.3 Stationary inhomogeneous solutions. Here, the stationary inhomogeneous solutions as opposed to the previous case are not necessarily uniform in space (depending on the space variable), and it can be defined in the form of $U(t,x) = U(x) = (\rho(x), u(x), \omega(x))$, which holds the following system:

$$\begin{cases} \rho = m(x), & x \in \Gamma, \\ a.\nabla\rho = -\sigma\rho + \varphi(\rho)u, & x \in \Omega, \\ \dfrac{\partial u}{\partial t} = \beta\delta\omega - \gamma(\rho)u, & x \in \Omega, \\ \dfrac{\partial \omega}{\partial t} = d\Delta\omega - \beta\omega + \alpha(\rho)u, & x \in \Omega, \\ \dfrac{\partial \omega}{\partial \nu}(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.3.9}$$

We distinguish two cases, the case where the density of trees u and density of seeds ω are stationary homogeneous while $\rho = \rho(x)$ is stationary inhomogeneous and vice versa. Next, we pay more attention to the first case because we would like to explore how far the non-uniform distribution of water resource ρ affects the equilibrium of the forest.

For $(u, \omega) = (0, 0)$, system (3.3.9) admits one solution $U = (\rho, 0, 0)$ satisfying the stationary advection system and the expression of $\rho(x)$ is explicitly given by theorem 3.2.1

$$\rho(x) = m(x_0)e^{-s\sigma}, \qquad (x_0, s) = (\zeta_1(x), \zeta_2(x)), \qquad x = \xi(x_0, s) \in \Omega.$$

3.3.4 Remark. Since we are in the case where the water resource is not everywhere uniformly distributed, then we could have a lack of water in some places of the forest and the deforestation process will lead. Cantin et al [12] stated in the case of an age structured model that the stability could correspond to reforestation if the stationary inhomogeneous solution $(\rho, 0, 0)$ is slightly perturbed and a regenerateness of the forest in case of instability.

3.4 Stability analysis of stationary solutions

The stability study of equilibrium states will be based on spectral methods; it is similar to that presented in [28]. Let $U = (\rho, 0, 0)$ the stationary inhomogeneous solution of system (3.1.5), and $\bar{U} = (0, 0)$ the stationary homogeneous solution of the reduced system (3.2.14). Now, we study its stability. For this aim, we recall the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \beta\delta\omega - \gamma(\psi(u))u, & t > 0, \quad x \in \Omega, \\ \dfrac{\partial \omega}{\partial t}(t,x) = d\Delta\omega - \beta\omega + \alpha(\psi(u))u, & t > 0, \quad x \in \Omega, \\ \dfrac{\partial \omega}{\partial \nu}(t,x) = 0, & t > 0, \quad x \in \partial\Omega, \\ u(0,x) = u_0(x), \qquad \omega(0,x) = \omega_0(x), & x \in \Omega. \end{cases} \tag{3.4.1}$$

As we did before (section 3.2.2), system above is equivalent to abstract problem which is also known as an abstract evolution equation [28],

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), & t > 0, \\ U(0) = U_0, & U_0 \in \mathcal{Z}, \end{cases} \tag{3.4.2}$$

in the function space $X = L^\infty(\Omega) \times L^2(\Omega)$, and $A = \begin{pmatrix} 1 & 0 \\ 0 & \Lambda \end{pmatrix}$ with $\mathcal{D}(A) = \{(u,\omega)^t,\ u \in L^\infty(\Omega),\ \omega \in H^2_N(\Omega)\}$. Next, we consider the non linear operator $F(U)$ given by:

$$F(U) = \begin{bmatrix} \beta\delta\omega - \gamma(\psi(u))u + u \\ \alpha(\psi(u))u \end{bmatrix}.$$

We use the spectral methods for studying the stability of our model, i.e. our stability analysis will be based on the study of set of eigenvalue of the matrix $\bar{A} = A - F^{'}(U)$, where $F^{'}(U)$ is the derivative of the non-linear operator F. The differentiability of F has been proved in [12, Appendix] and its derivative respect to each component u and ω is given by:

$$F^{'}(U) = \begin{pmatrix} 1 - \gamma(\psi(u)) - D(\psi(u))\gamma^{'}(\psi(u))u & \beta\gamma \\ \alpha(\psi(u)) + D\psi(u)\alpha^{'}(\psi(u))u & 0 \end{pmatrix} \xrightarrow{\psi(0)=\rho(x)} \begin{pmatrix} 1 - \gamma(\rho(x)) & \beta\gamma \\ \alpha(\rho(x)) & 0 \end{pmatrix} = F^{'}(U).$$

Therefore, it comes that

$$\begin{aligned} \bar{A} = A - F^{'}(U) &= \begin{pmatrix} 1 & 0 \\ 0 & \Lambda \end{pmatrix} - \begin{pmatrix} 1 - \gamma(\rho(x)) & \beta\delta \\ \alpha(\rho(x)) & 0 \end{pmatrix} = \begin{pmatrix} \gamma(\rho(x)) & -\beta\delta \\ -\alpha(\rho(x)) & -d\Delta + \beta \end{pmatrix}, \\ \bar{A} &= \begin{pmatrix} k + h(\rho(x)) & -\beta\delta \\ -\alpha(\rho(x)) & -d\Delta + \beta \end{pmatrix} = \begin{pmatrix} N & -\beta\delta \\ -\alpha(\rho(x)) & -d\Delta + \beta \end{pmatrix}, \quad \text{with} \quad N = k + h(\rho(x)). \end{aligned}$$

Based on theorem A.2 in [28], we have that Spec$(\bar{A}) \cap \{\lambda \in \mathbb{C},\ \text{Re}(\lambda) = 0\} = \emptyset$. We consider now the following proper value problem

$$(\lambda I - \bar{A}) \begin{pmatrix} u \\ \omega \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad I \text{ square identity matrix,}$$

where, $(c_1, c_2) \in X$. The latter system is equivalent to:

$$\begin{cases} (\lambda - N)u + \beta\delta\omega = & c_1, \\ \alpha(\rho)u + (\lambda - \Lambda)\omega = & c_2. \end{cases} \tag{3.4.3}$$

After substituting u in system (3.4.3), we get the following equation:

$$[(\lambda - N)(\Lambda - \lambda) + \beta\delta\alpha(\rho)]\,\omega = c_1\alpha(\rho) - (\lambda - N)c_2.$$

- if $\lambda \in \mathbb{C}$ solves the first degree polynomial equation

$$(\lambda - N) = 0, \tag{3.4.4}$$

then ω cannot belong to $H^2(\Omega)$, which implies $\lambda \in$ Spec$(\bar{A})$.

- in case of $\lambda \in \mathbb{C}$ does not satisfy equation (3.4.4), then $\lambda \in \text{Spec}(\bar{A})$ if and only if

$$\lambda + \beta\delta\frac{\alpha(\rho)}{N-\lambda} \in \text{Spec}(\Lambda).$$

  Indeed, $\lambda \in \text{Spec}(\bar{A})$ if and only if $(\lambda - N)(\Lambda - \lambda) + \beta\delta\alpha(\rho) = 0$, then

$$\begin{aligned} \Lambda - \lambda &= -\beta\delta\frac{\alpha(\rho)}{(\lambda - N)}, \\ \Lambda &= \lambda + \beta\delta\frac{\alpha(\rho)}{N-\lambda} \in \text{Spec}(\Lambda). \end{aligned}$$

Next, Atsushi et al [28] considered that $\lambda \in \text{Spec}(\bar{A})$ if and only if λ solves at least one of the quadratic equations:

$$(\lambda - N)(-d\ v_n + \beta - \lambda) + \beta\delta\alpha(\rho) = 0, \tag{3.4.5}$$

where $\{v_n\}_{n=0}^{\infty}$ are the infinite number of eigenvalues of the Laplace operator $-\Delta$ in $L^2(\Omega)$ with the Neumann initial condition given by (3.1.8). Based on the calculus done above, we state the following result.

3.4.1 Proposition. A homogeneous stationary solution $U = (u, \omega)^t$ is a hyperbolic equilibrium state if and only if $N \neq 0$ and $\beta\delta\alpha(\rho) + N(d\ v_n - \beta) \neq 0$, for all n natural number.

Proof. ($\Rightarrow$) Let assume that a homogeneous stationary solution $U = (u, \omega)^t$ is a hyperbolic equilibrium. By contraposition, we just have to prove that if $N = 0$ or $\beta\delta\alpha(\rho) + N(d\ v_n - \beta) = 0$, then U is not a hyperbolic equilibrium. From (3.4.5), we have:

$$\lambda^2 + \lambda(d\ v_n - \beta - N) = N(d\ v_n - \beta) + \beta\delta\alpha(\rho), \tag{3.4.6}$$

If $N = 0$ or $\beta\delta\alpha(\rho) + N(d\ v_n - \beta) = 0$, equation (3.4.6) will become:

$$\lambda^2 + \lambda(d\ v_n - \beta) = 0, \tag{3.4.7}$$

therefore, $\lambda = 0$ is an eigenvalue of $\bar{A}$.

($\Leftarrow$) Let $N \neq 0$ and $\beta\delta\alpha(\rho) + N(dv_n - \beta) \neq 0$, for $n \in \mathbb{N}$. Equation (3.4.4) does not admit imaginary solution. Furthermore, we set $\lambda = 0 + ib$, $b \in \mathbb{R}$ solution of equation (3.4.5).

$$\begin{aligned} \lambda^2 + \lambda(dv_n - \beta - N) &= N(dv_n - \beta) + \beta\delta\alpha(\rho), \\ -b^2 + ib(dv_n - \beta - N) &= N(dv_n - \beta) + \beta\delta\alpha(\rho). \end{aligned}$$

By identification, we obtain the following system:

$$\begin{cases} b^2 = N(\beta - dv_n) - \beta\delta\alpha(\rho) \\ b = dv_n - \beta - N. \end{cases} \quad \Rightarrow \quad \begin{cases} b^2 = N(\beta - dv_0) - \beta\delta\alpha(\rho) \\ b = dv_0 - \beta - N. \end{cases} \tag{3.4.8}$$

In particular, for $v_n = v_0 \geq 0$, the latter system does not have solution, which is absurd. $\square$

Model discretization scheme

In this section, we propose a discretization scheme of system (3.1.5) for the simulation. The idea is to do a discretization of time t by finite differences and discretization of space x by finite elements $\mathbb{P}^1$. During the process, the advection-diffusion process and the reaction process will be treated separately. The advection field $a = a(x)$ crosses the domain Ω. Let us consider the following system:

$$\left\{\begin{array}{rcll} a.\nabla\rho &=& -\sigma\rho + \varphi(\rho)u, & \\ \dfrac{\partial u}{\partial t}(t,x) &=& \beta\delta\omega - \gamma(\rho)u, & t>0, \quad x\in\Omega, \\ \dfrac{\partial \omega}{\partial t}(t,x) &=& d\Delta\omega - \beta\omega + \alpha(\rho)u, & t>0, \quad x\in\Omega, \\ \dfrac{\partial \omega}{\partial \nu}(t,x) &=& 0, \quad \rho(t,x) = m(x) & t>0, \quad x\in\partial\Omega, \\ u(0,x) &=& u_0(x), \qquad \omega(0,x)=\omega_0(x), & x\in\Omega. \end{array}\right. \tag{3.4.9}$$

Spatial Discretization: The finite element method $\mathbb{P}^1$ in dimension 2 is based on the numerical approximation of the weak solution of the problem (3.4.9). To approximate these equations in space, we will use an equivalent form, called variational. For more precision, let us introduce the following functional spaces:

$$\begin{array}{rcl} H^1(\Omega) &=& \left\{ f\in L^2(\Omega), \ \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \in L^2(\Omega) \right\}, \\ H^2(\Omega) &=& \left\{ f\in H^1(\Omega), \ \dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial x\partial y}, \dfrac{\partial^2 f}{\partial y^2} \in L^2(\Omega) \right\}, \\ V(\Omega) &=& \left\{ f\in H^2(\Omega), \ \dfrac{\partial f}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}, \end{array} \tag{3.4.10}$$

with $L^2(\Omega)$ the space of functions $f:\omega\to\mathbb{R}$ of integrable squares. If $\Delta\omega\in L^2(\Omega)$, then we can multiply the third equation of the system by a smooth test function $v(x)\in V(\Omega)$ and then apply Green's formula

$$\int_\Omega \Delta\omega\ v = -\int_\Omega \nabla\omega.\nabla v + \int_{\partial\Omega} \frac{\partial\omega}{\partial\nu}\ v. \tag{3.4.11}$$

Next, for $v\in V(\Omega)$, we have:

$$\int_\Omega \frac{\partial\omega}{\partial t}\ v \quad = \quad -\int_\Omega d\nabla\omega.\nabla v + \int_{\partial\Omega} d\underbrace{\left(\frac{\partial\omega}{\partial\nu}\right)}_{0} v - \int_\Omega \beta\ \omega\ v + \int_\Omega \alpha(\rho)\ u\ v. \tag{3.4.12}$$

Time Discretization: The domain Ω is discretized by introducing a uniformly partitioned time mesh. Mesh points are $t_n = n\Delta t$, $n = 0,1,...,N_t$, where $\Delta t = T/N_t$ is the constant length of the time steps [7]. Let ω_n be a mesh function for $n = 0,1,...,N_t$, which approximates the exact solution ω at the mesh points $t = t_n$. (Note that $n = 0$ is the known initial condition, so ω_n is identical to the mathematical ω at this point.) The derivative $\dfrac{\partial\omega}{\partial t}$ or $\dfrac{\partial\omega}{\partial x}$ is to be replaced by a finite difference approximation. A common first-order accurate approximation to the first derivative is:

$$\frac{\partial\omega}{\partial t}(t_n) = \frac{\omega_{n+1}-\omega_n}{\Delta t}. \tag{3.4.13}$$

Therefore, (3.4.12) becomes:

$$\int_\Omega \left(\frac{\omega_{n+1}-\omega_n}{\Delta t}\right) v_n = -\int_\Omega d\nabla\omega_{n+1}.\nabla v - \int_\Omega \beta\omega_n v_n + \int_\Omega \alpha(\rho_n) u_n v_n.$$

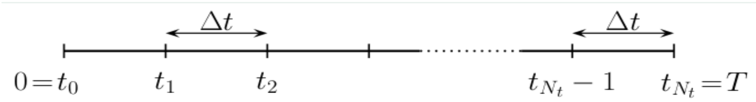


Figure 3.4: Discretized computational domain Ω.

Thus, the approximated sequence of seed density is given by:

$$\int_\Omega \omega_{n+1}\ v_n = \int_\Omega \omega_n v_n - \int_\Omega d\Delta t\ \nabla\omega_{n+1}.\nabla v - \int_\Omega \beta\Delta t\ \omega_n v_n + \int_\Omega \alpha(\rho_n)\Delta t\ u_n v_n. \tag{3.4.14}$$

Let consider now the first equation of (3.4.9) given as follows:

$$\frac{\partial\rho}{\partial t} + a.\nabla\rho = -\sigma\rho + \varphi(\rho)u, \qquad \rho(t,x) = m(x). \tag{3.4.15}$$

Since the regular function $m(x)$ is not necessarily uniform, then the discontinuous Galerkin finite element methods is most adapted for approximating (3.4.15). There are may formulations possible. For instance, based on [2], equation (3.4.15) is approached by:

$$\int_\Omega \left(\frac{\rho_{n+1}-\rho_n}{\Delta t} + a.\nabla\rho\right) v_n + \int_E \left(\alpha|\nu.a| - \frac{1}{2}\nu.a\right)[\rho]v - \int_\Gamma |\nu.a|\rho v = -\int_\Omega \sigma\rho v + \int_\Omega \varphi(\rho)u, \tag{3.4.16}$$

where $\alpha \in \mathbb{R}$, E is the set of inner edges and Γ is the set of boundary edges where $\nu.a < 0$, $a = a(x)$ the advection field which crosses the domain Ω, ν represents the outward normal vector, and $[\rho]$ denotes the jump of ρ across an edge with convention that ρ^+ refers to the value on the right of oriented edge. In the simulation part, we will consider the adapted algorithmic form of the pure convection equation, which is calculated numerically using the expression "convect([a], dt, ρ)". Finally, we consider the dynamic of tree density equation

$$\frac{\partial u}{\partial t} = \beta\delta\omega - \gamma(\rho)u,$$

which is equivalent to

$$\frac{\partial u}{\partial t} = \beta\delta\omega - (k + h(\rho))\,u, \tag{3.4.17}$$

where $\gamma(\rho) = k+h(\rho)$ represents the global mortality rate of trees, k represents the average mortality rate of trees, and $h(\rho)$ the mortality rate of trees due to the scarcity of water resources. The corresponding discretization of (3.4.17) is detailed as follows:

$$\int_\Omega \left(\frac{u_{n+1}-u_n}{\Delta t}\right) v_n = \int_\Omega \beta\delta\omega_n v_n - \int_\Omega k u_n v_n - \int_\Omega h(\rho_n) u_n v_n,$$

Hence,

$$\int_\Omega u_{n+1} v_n = \int_\Omega u_n v_n + \int_\Omega \Delta t\ \beta\delta\omega_n v_n - \int_\Omega k\Delta t\ u_n v_n - \int_\Omega \Delta t\ h(\rho_n) u_n v_n. \tag{3.4.18}$$

To formulate a recursive computational algorithm, we must assume that u_n, ρ_n and ω_n have already calculated such that u_{n+1}, ω_{n+1}, and ρ_{n+1} are the unknown values to be solved by considering the discretizations (3.4.14), (3.4.16), and (3.4.18). The computational algorithm will just consist in applying apply (3.4.14), (3.4.16), and (3.4.18) successively for $n = 0, 1, ..., N_{t-1}$.

Discussions

These simulations were carried out in Debian (a GNU/Linux operating system) using FreeFem++. It is a partial differential equation solver for non-linear multi-physics systems in 1D, 2D, 3D, and 3D border domains (surface and curve), according to [22]. Table 3.2 shows the various parameter values used to represent our abstract model. As previously stated, the numerical scheme's computational method is based on discretization by finite differences in time and by finite elements in space $\mathbb{P}^1$.

Parameters	α_0	β	δ	h_0	h_1	φ_0	σ	d	k
Values	0.7	0.8	0.9	0.5	0.1	0.5	0.9	8	1

Table 3.2: Parameter values used for the numerical simulations of the forestry model (3.1.5).

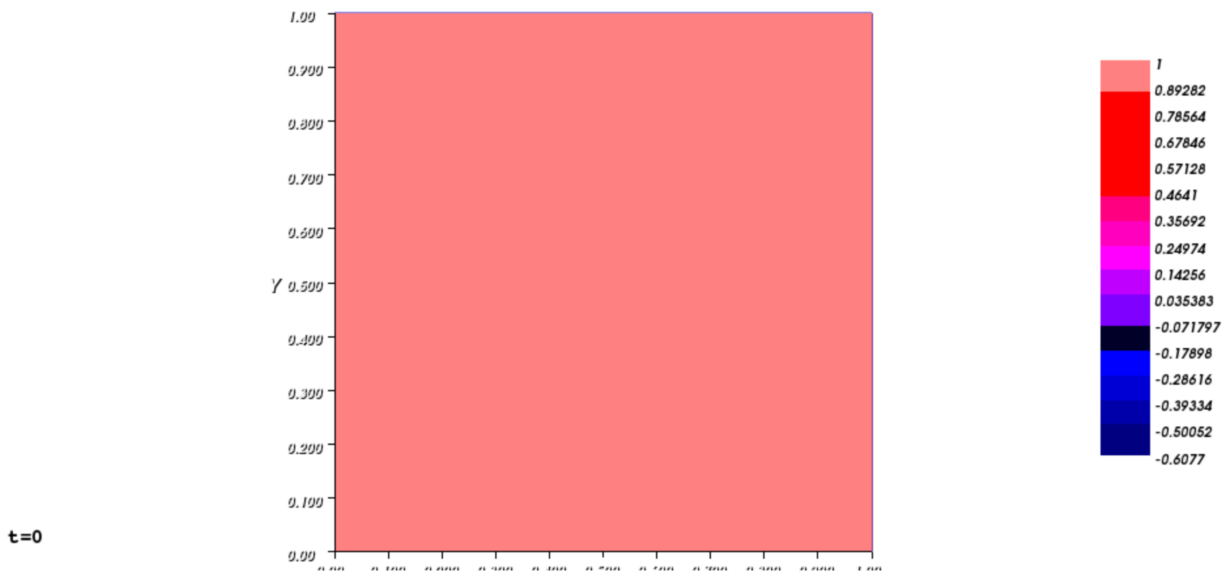


Figure 3.5: Case of the non-existence of forest. Here, everything is vanishing such as tree density $u = 0$, seed density $\omega = 0$, and oceanic contribution $m = 0$, which refers to the stationary homogeneous state of system (3.1.5). According to the forest model, just by considering the uniform vanishing of oceanic water on the whole domain Γ, it leads that the trees density and seeds density will disappear progressively.

Figures 3.6, 3.7 and 3.8 reflect a deforested area situation because of the vanishing trees, which also implies the absence of seeds. This also confirms that the inhomogeneous stationary solution $(\rho(x), 0, 0)$ describes a real deforested area scenario. We will still see a deforested area even if the oceanic contribution $m(x)$ to the water cycle is only slightly perturbed. Of course, the vanishing of seeds will lead.

Another case of non-regeneration of the forest can also occur if the trees do not contribute enough to the water cycle during transpiration. This significantly slows down the seed germination process. As a result, the mortality rate of old trees will be too high to reproduce or contribute to a healthy forest environment.

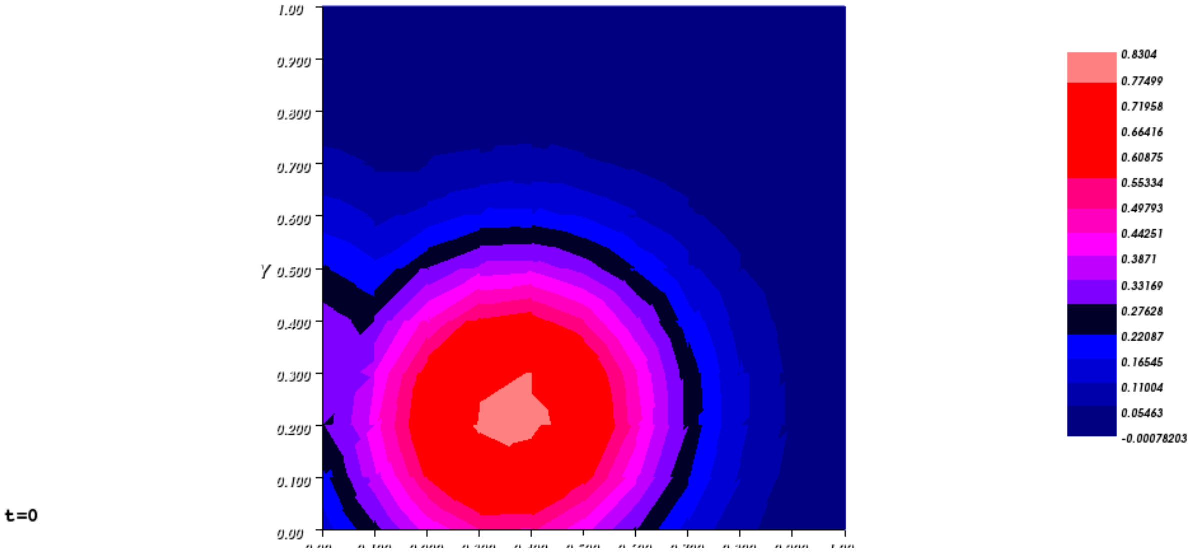


Figure 3.6: $\rho(x)$ after $t = 0$.

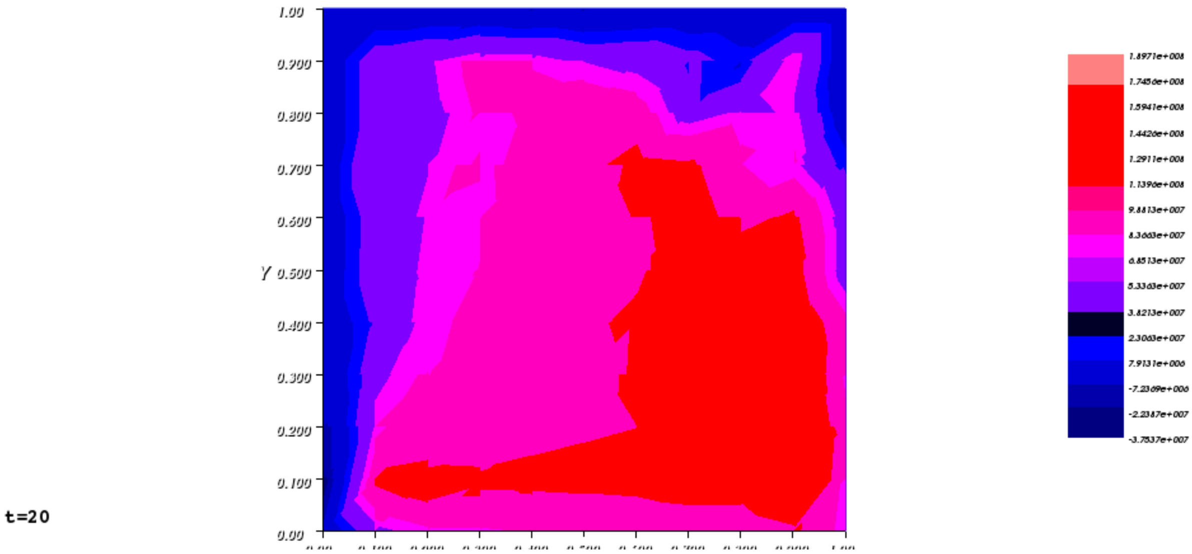


Figure 3.7: $\rho(x)$ after $t = 20$ years.

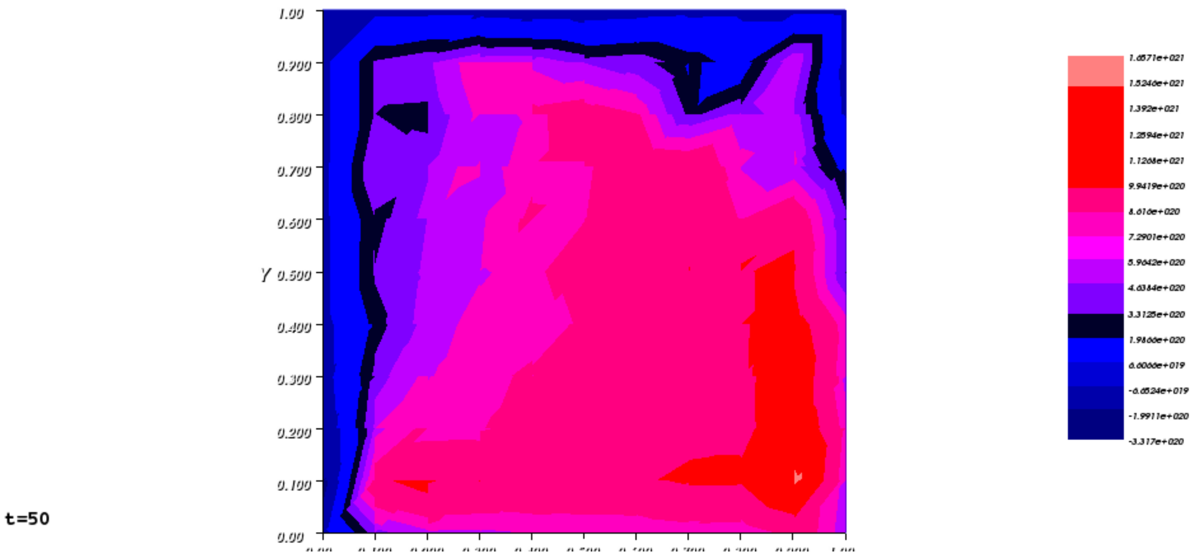


Figure 3.8: $\rho(x)$ after $t = 50$ years.

Figure 3.9: Spatial distribution of water resources $\rho(x)$ for the stationary heterogeneous solution $(\rho(x), 0, 0)$ describing a deforested area over 0, 20 and 50 years, respectively.

As inferred earlier, under mild disturbance, the forest will tend to revert to its original state. In this case, even if reforestation is practiced while water resources vary, we will likely return to the deforested state. Therefore, stability can be viewed as the inability of a deforested area to regenerate despite the reforestation process. Furthermore, instability can be considered as the ability to reproduce a healthy forest ecosystem despite disturbances such as reforestation, long climatic seasons, or short seasons.

Conclusion

In this work, we proposed an adapted mathematical model of a tree forest environment for investigating the dynamics of forest ecosystems, the role of the water cycle, and the atmospheric activity to contribute to solving a crucial problem of distinction of tree densities faced by biologists. By taking into account the effect of atmospheric activity by adding a moisture conservation equation, and the role of the water resource by taking into account oceanic input and forest transpiration, the idea was to reduce the reaction-diffusion-advection system to a classical reaction-diffusion problem that reflects the same scenario. For this purpose, the well-posedness of the model was proven by a theoretical mathematical method. Next, we proved the reduced reaction-diffusion system possesses relevant stationary solutions; among these stationary solutions, the existence of homogeneous solutions shows that our unstructured age model can return to its initial state by small perturbations. This dynamic model evolves with a mathematical method approach based on the variable coefficient of the seed production α, establishment rate δ, and tree mortality h and their major impacts on the forest ecosystem could be confirmed by numerical simulations.

Human activity and reforestation processes can be addressed to test the effectiveness of the model and study forest dynamics in these scenarios. Parameter estimation and sensitivity analysis could be examined with forest data in future work. In the near future, parameter estimation and sensitivity analysis could be performed with borehole data.

Appendix A. Some additional data

Proof. (proof of theorem 3.2.1) It will be done in three steps.

(1) We aim to solve the stationary equation (3.2.8) using the method of characteristics.

Let $\{\xi(x_0, s)\}_{0\leq s\leq S(x_0)}$ be a characteristic line of the field advection a delimited by the interval $\Big]x_0,\ \xi(x_0, S(x_0))\Big[$, with $x_0 \in \Gamma$, $\xi(x_0, S(x_0)) \in \partial\Omega\backslash\Gamma$. We defined the function $\tilde{\rho}$ along the latter characteristic line by $\tilde{\rho}(s) = \rho \circ \xi(x_0, s)$, for $0 \leq s \leq S(x_0)$; it is obvious that $\tilde{\rho}$ solves the ODE below:

$$\begin{cases} \dfrac{d\tilde{\rho}}{ds} + \sigma\tilde{\rho} = \varphi(\tilde{\rho})\tilde{u}, \quad 0 < s \leq S(x_0), \\ \qquad \tilde{\rho}(0) = m(x_0). \end{cases} \tag{A.0.1}$$

Indeed, from identity (3.2.7) $\tilde{\rho}(x_0, s) = \rho \circ \xi(x_0, s)$, it leads that

$$\begin{aligned} \frac{d\tilde{\rho}}{ds} + \sigma\tilde{\rho} = \varphi(\tilde{\rho})\tilde{u} \quad &\Rightarrow \quad \frac{d}{ds}(\rho \circ \xi(x_0, s)) + \sigma\left[\rho \circ \xi(x_0, s)\right] = \varphi(\rho \circ \xi(x_0, s))\tilde{u}, \\ &\Rightarrow \quad \frac{d}{ds}\tilde{\rho}(x_0, s) + \sigma\tilde{\rho}(x_0, s) = \varphi(\tilde{\rho}(x_0, s))\tilde{u}, \quad 0 < s \leq S(x_0). \end{aligned}$$

Furthermore, from (3.2.3) we have

$$\tilde{\rho}(0) = \rho \circ \xi(x_0, s) = \rho(x_0) = m(x_0).$$

Let $\theta(s) = \tilde{\rho}(s)e^{s\sigma}$ such that $\theta \in [0, S(x_0)]$. The function θ solves the ODE below

$$\begin{cases} \dfrac{d\theta}{ds} = g(s, \theta), \quad 0 < s \leq S(x_0), \\ \theta(0) = m(x_0). \end{cases} \tag{A.0.2}$$

Indeed,

$$\frac{d\theta}{ds} = \frac{d\tilde{\rho}}{ds}\, e^{s\sigma} + \sigma\tilde{\rho}(s)e^{s\sigma} = \frac{d\tilde{\rho}}{ds}\, e^{s\sigma} + \sigma\theta(s),$$

thus, by substituting $\frac{d\tilde{\rho}}{ds}$ above, we get the following system

$$\begin{cases} \dfrac{d\tilde{\rho}}{ds} + \sigma\tilde{\rho} = \dfrac{d\theta}{ds}\, e^{-s\sigma}, \quad 0 < s \leq S(x_0), \\ \qquad \theta(0) = m(x_0) = \tilde{\rho}(0). \end{cases} \tag{A.0.3}$$

By comparing (A.0.1) and (A.0.3), we get

$$g(s, \theta) = \varphi(\theta e^{-s\sigma})\, e^{s\sigma}\tilde{u}(s).$$

Since $u \in L^\infty(\Omega)$, then the function g is said to be continuous respect to the variable θ but discontinuous in s variable. That is why we exploit the properties of Caratheodory differential equations to solve equation (A.0.2) (see section 2.4). i.e. the following conditions should be satisfied:

- the function $g(s, \theta)$ is defined in $[0, S(x_0)] \times \mathbb{R}$, continuous in θ for almost every s;

- the function $g(.,\beta)$ is mesurable, for all $\beta \in [0, S(x_0)]$
- g is bounded i.e. $|g(s,\theta)| \leq M(s)$, with $M(s) = \varphi_0 e^{s\sigma} |\tilde{u}(s)| \in L^\infty([0, S(x_0)])$,
- Furthermore,

$$\begin{aligned}
|g(s,\theta_1) - g(s,\theta_2)| &= \left|(\varphi(\theta_1 e^{-s\sigma}) - \varphi(\theta_2 e^{-s\sigma})\ )\ e^{s\sigma}\tilde{u}(s)\right|, \\
&\leq \varphi_0 \left|\theta_1 - \theta_2\right| \left|e^{-s\sigma}\right| \left|e^{s\sigma}\right| \tilde{u}(s), \\
&\leq \varphi_0 \tilde{u}(s) \left|\theta_1 - \theta_2\right|, \\
&\leq L(s)\ \ \left|\theta_1 - \theta_2\right|.
\end{aligned}$$

Based on theorems 2.4.3 and 2.4.5, equation (A.0.2) possesses a unique local solution on $T = [0, \bar{S}]$ with $\bar{S} > 0$ and is given by

$$\theta(s) = m(x_0) + \int_0^s g(\tau, \theta(\tau)) d\tau, \quad 0 \leq s \leq \bar{S}. \tag{A.0.4}$$

The expression (A.0.4) is absolutely continuous on each closed and bounded interval T and it's non-negative.

Next, we notice that $\dfrac{d\theta}{ds} \leq \theta\ \varphi_0 \left\|u\right\|_\infty$, $s > 0$ which leads to

$$\theta(s) \leq m(x_0)\ e^{\varphi_0 S(x_0) \|u\|_\infty}, \quad \forall\ s \in T.$$

Then, we have $\bar{S} = S(x_0)$, hence the solution θ is global in $[0, S(x_0)]$. From the assumption at the beginning of the proof, we have $\tilde{\rho}(s) = \theta(s)e^{-s\sigma}$, $s \in [0, S(x_0)]$; $\tilde{\rho}$ is absolutely continuous on each compact in $[0, S(x_0)]$, and uniquely determined by

$$\tilde{\rho}(s) = m(x_0)e^{-s\sigma} + \int_0^s \varphi(\tilde{\rho}(\tau))\tilde{u}(\tau)\ e^{-\sigma(s-\tau)} d\tau, \quad s \in [0, S(x_0)],$$

So, by setting $\rho(x) = \tilde{\rho}(\xi_1(x), \xi_2(x))$, $x \in \Omega$ we can solve equation (3.2.8) since (A.0.1) has been done along each characteristic line of the advection field a. Furthermore, the principle of continuity of the solution of equation (A.0.1) is exploited to show that the operator ψ is well defined ([12], appendix).

(2) In this part, the objective is to show the continuity of ψ.

Let h, $u \in L^\infty_+(\Omega)$, $\rho_u = \psi(u)$, and $\rho_{u+h} = \psi(u+h)$. From (3.2.10), we have

$$\begin{aligned}
|\psi(u+h)(x) - \psi(u)(x)| &= \left|\int_0^s \varphi(\tilde{\rho}_{u+h}(\tau))(\tilde{u}+\tilde{h})(\tau)\ e^{-\sigma(s-\tau)} d\tau - \int_0^s \varphi(\tilde{\rho}_u(\tau))\tilde{u}(\tau)\ e^{-\sigma(s-\tau)} d\tau\right|, \\
&= \bigg| \int_0^s \left[\varphi(\tilde{\rho}_{u+h}(\tau)) - \varphi(\tilde{\rho}_u(\tau))\right] \tilde{u}(\tau)\ e^{-\sigma(s-\tau)} \\
&+ \int_0^s \varphi(\tilde{\rho}_{u+h}(\tau))\tilde{h}(\tau)\ e^{-\sigma(s-\tau)} d\tau \bigg|, \\
&\leq \left\|u\right\|_\infty \int_0^s \left|\varphi(\tilde{\rho}_{u+h}(\tau)) - \varphi(\tilde{\rho}_u(\tau))\right| d\tau + \left\|h\right\|_\infty \varphi_0 \int_0^s e^{-\sigma(s-\tau)} d\tau, \\
&\leq \left\|u\right\|_\infty \varphi_0 \int_0^s \left|\tilde{\rho}_{u+h}(\tau) - \tilde{\rho}_u(\tau)\right| d\tau + \frac{\varphi_0}{\sigma} \left\|h\right\|_\infty.
\end{aligned} \tag{A.0.5}$$

Next, we set

$$f(t) = \left|\tilde{\rho}_{u+h}(\tau) - \tilde{\rho}_u(\tau)\right|, \quad t \geq 0. \tag{A.0.6}$$

It follows from (A.0.5) that

$$f(t) \leq \|u\|_{\infty} \varphi_0 \int_0^s f(\tau) d\tau + \frac{\varphi_0}{\sigma} \|h\|_{\infty}. \tag{A.0.7}$$

By applying the lemma of Gronwall integral (2.3.4), we obtain

$$f(t) \leq \|h\|_{\infty} \frac{\varphi_0}{\sigma} e^{\varphi_0 s \|u\|_{\infty}} \leq \|h\|_{\infty} \frac{\varphi_0}{\sigma} e^{\varphi_0 \bar{S} \|u\|_{\infty}},$$

since $S(x_0) \leq \bar{S}, \ \forall \, x_0 \in \Gamma$ from (3.2.5). Hence,

$$|\rho_{u+h} - \rho_u| = |\psi(u+h) - \psi(u)| \leq \|h\|_{\infty} \frac{\varphi_0}{\sigma} \ e^{\varphi_0 \bar{S} \|u\|_{\infty}}. \tag{A.0.8}$$

In conclusion, by using the integral Gronwall lemma, we have shown that the operator ψ defined in (3.2.9) is continuous.

(3) The differentiability of operator ψ has been proven by Cantin Guillaume et al [12] by introducing a novel operator namely L_u in (A.0.6).

$\square$

Source code obtained on FreeFem++

We present in this section, the source code of computational algorithm of our abstract forest model.

Algorithm 1 Proposed simulation algorithm for our unstructured age forest model (3.1.5)

Mesh of the domain Ω
Require: int $n = 0$;
mesh Ths=square(n,n);
Parameters
Require: real $\alpha = 0.9$, $\beta = 0.8$, $\delta = 0.9$, $h0 = 0.5$, $h1 = 0.1$, $\sigma = 0.7$, $\varphi_0 = 0.8$, $d = 10$, $k = 1.$;
real $t = 10$, dtt=0.2, i;
Finite element spaces
fespace Vhs(Ths, P1);
Vhs ρ, m, ω, ω_0, u, u_0, v;
fespace Vh(Ths, P1dc);
Vh m, vh, $a_1 = x$, $a_2 = -t$, ρ;
Initial conditions
$w = x^2 + 1$, $u = 2x^2$, $\rho = \exp(-10((t-0.5)^2 + (x-0.5)^2))$;
Variational problem
problem seeds(w,v) = int2d(Ths)(ω*v)+int2d(Ths)(dtt*d*(dx(ω)*dx(v)+dy(ω)*dy(v))) +int2d(Ths)(ω_0*v)-int2d(Ths)(dtt*β*ω_0*v)+int2d(Ths)((dtt*u_0*v*α_0*m)/(1+m)) +on(1,2,3,4,ω=1);

problem trees(u,v) = int2d(Ths)(u*v)-int2d(Ths)(u_0*v)-int2d(Ths)(dtt*β*δ*ω_0*v) - int2d(Ths)(dtt*k*u_0*v)-int2d(Ths)((dtt*u_0*v)*(h0+(h1*m))/(1+m))+on(1,2,3,4,u=1);

problem advection(ρ,vh) = int2d(Ths)((ρ/dtt)*vh)-int2d(Ths)((convect([a1,a2],-dtt,m)*vh)/dtt)+int2d(Ths)(σ*m*vh) +int2d(Ths)((φ_0*m*vh*u_0)/(1+m))+on(1,2,3,4,ρ=m);

Time loop
for (i=0.; $i < t$; i+=dtt) **do**
m = ρ, $\omega_0 = \omega$, $u_0 = u$;
seeds, trees, advection;
plot(ρ, fill=1, value=1, cmm="t"=+i);
end for

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