Chow Groups and Characteristic Classes

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The purpose of this essay is to explore the use of characteristic classes in algebraic geometry. In particular, we'll mostly be discussing intersection theory of varieties and how characteristic classes come in to play. We're going to adopt the language and setting of "Intersection Theory" by Fulton. For the remainder, when we say **scheme** we mean a scheme X over an algebraically closed field K together with a finite covering by affine open sets, each of whose coordinate rings are finitely generated K-algebras. A **variety** is an integral separated scheme (in the above sense). The local ring of a scheme X along a subvariety V is denoted $\mathscr{O}_{V,X}$ with maximal ideal $m_{V,X}$. The function field of a variety X is denoted $R(X)$.

Definition A *vector bundle* of rank r on a scheme X is a scheme E and a morphism $\pi : E \to X$ together with an open covering $\{U_i\}$ of X and isomorphisms

$$\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{A}^r_K$$

such that over $U_i \cap U_j$, the composition $\varphi_i \circ \varphi_j^{-1}$ are linear, i.e. given by a morphism

$$g_{ij} : U_i \cap U_j \to GL_r(K)$$

satisfying the cocycle condition.

Next lets introduce where our Chern classes are going to live. Let X be a scheme. The group of k-cycles is the free abelian group generated by the closed subvarieties of X of dimension k, denoted $Z_k(X)$. Given such a $V \subset X$ we denote by $[V]$ the corresponding element of $Z_k(X)$. Each k-cycle can be written as a finite formal sum $\sum_i n_i[V_i]$ with coefficients in $\mathbb{Z}$, where the V_i are closed subvarieties of X of dimension k.

Definition: Let W be a variety, V a subvariety of codimension one, $f \in R(W)^*$. We define the **order of vanishing** of f along V as

$$ord_V(f) = \ell(\mathscr{O}_{V,W}/\langle r\rangle) - \ell(\mathscr{O}_{V,W}/\langle s\rangle)$$

where $f = \frac{r}{s}$ with $r, s \in \mathscr{O}_{V,W}$.

Example: When we have a discrete valuation ring A (such as when W is non-singular along V) with discrete valuation v and uniformizer π, every element $x \in A$ can be written as $x = u\pi^e$ with u a unit, $e \geq 0$. Then $\ell(A/\langle x\rangle) = v(x) = e$.

For any $(k+1)$-dimensional subvariety W of X, and $f \in R(W)^*$, we define a k-cycle on X by

$$div(f) := \sum_V ord_V(f)[V]$$

where we sum over subvarieties V of W of codimension one. We say a k-cycle, α, is *rationally equivalent to zero* if there are a finite number of $(k+1)$-dimensional subvarieties $W_i \subseteq X$ and $f_i \in R(W_i)^*$ such that $\alpha = \sum div(f_i)$. The collection of k-cycles rationally equivalent to zero form a subgroup $R_k(X) \subset Z_k(X)$ generated by the cycles of rational functions of subvarieties of X of dimension $k+1$. The quotient group

$$CH_k(X) = Z_k(X)/R_k(X)$$

is called the kth Chow group. We put

$$CH_*(X) := \bigoplus_{k\geq 0} CH_k(X)$$

and call this graded group the *Chow group* of X.

Examples: (1) The Chow groups of affine space are

$$CH_k(\mathbb{A}^n) = \begin{cases} \mathbb{Z} \ if \ \ k = n \\ 0 \ \ otherwise \end{cases}$$

(2) $CH_k(\mathbb{P}^n) \cong \mathbb{Z}$ for all $0 \leq k \leq n$ with an isomorphism $[V] \mapsto degV$ for k-dimensional subvariety $V \subset \mathbb{P}^n$.
(3) If X is a smooth projective curve, $Z_0(X)$ is the divisor class group, and $CH_0(X) = Pic(X)$.
In general, computing these groups are very hard.

With some conditions on the morphisms, we have push-forwards and pull-backs of cycles:

Theorem(s):(a) Let $f : X \to Y$ be a proper [1] morphism, then there is an induced homomorphism $f_* : CH_k(X) \to CH_k(Y)$ i.e. rational equivalence pushes forward under proper maps.
(b) Let $f : X \to Y$ be a flat morphism of relative dimension n [2], then there are induced homomorphisms $f^* : CH_k(Y) \to CH_{k+n}(X)$.

To describe intersections on the level of Chow groups we need the notions of Weil divisors and Cartier divisors [3]. Let X be an n-dimensional variety, a Weil divisor on X is an $(n-1)$-cycle on X. A Cartier divisor on X is a global section of the quotient sheaf $\mathscr{K}_X^*/\mathscr{O}_X^*$ where $\mathscr{K}_X^*$ is the sheaf of invertible rational functions on X, and $\mathscr{O}_X^*$ the sheaf of nonzero regular functions on X. Whats important for us is that in this setting there is a one-to-one correspondence between Cartier divisors on X and pairs (L, s) for L a line bundle on X and s a rational section of L. So we'll talk about the line bundle $\mathscr{L}(C)$ *associated* to the Cartier divisor C. Also every Cartier divisor class determines a Weil divisor class in $CH_{n-1}(X)$. When X is smooth, Weil divisors and Cartier divisors coincide.

Definition: (First Chern class) Let L be a line bundle X. If V is a k-dimensional subvariety, then the restriction $L|_V$ is isomorphic to $\mathscr{O}_V(C)$ for some Cartier divisor C on V. Then we define the $(k-1)$-cycle $c_1(L) \cap [V] = [C]$ which is a well-defined element in $CH_{k-1}(X)$. Extending by linearity gives us a homomorphism

$$c_1(L) \cap - : Z_k(X) \to CH_{k-1}(X), \ \ \alpha = \sum n_i V_i \mapsto c_1(L) \cap \alpha := \sum n_i c_1(L) \cap V_i.$$

If L, L' are isomorphic line bundles, then we have $c_1(L) = c_1(L')$. Also if $\alpha \sim 0$ on X, then $c_1(L) \cap \alpha = 0$. Thus there is an induced homomorphism

$$c_1(L) \cap - : CH_k(X) \to CH_{k-1}(X).$$

Definition: Let C be a Cartier divisor on X and $V \subset X$ a subvariety of dimension k. We define the **intersection class** of V by C as

$$C \cdot V := c_1(\mathscr{O}_X(C)) \cap V$$

in $CH_{k-1}(V \cap |C|)$ ($|C|$ is the support of C in X). $\mathscr{O}(C)$ being in the definition can be thought of as placing C in general position with respect to V.

Examples: (1) Let $H \subset \mathbb{P}^n$ be a hyperplane, $V \subset \mathbb{P}^n$ a subvariety of dimension k. Then $H \cdot V = [H' \cap V]$ in $CH_{k-1}(V \cap H)$ where H' is the hyperplane that does not contain V.
(2) If X is complete i.e. there is a proper map $X \to Spec(K)$ the push-forward discussed above $CH_k(X) \to CH_k(Spec(K)) = \mathbb{Z}$ is denoted $\int_X$. Given zero cycle $\alpha = \sum_V n_V[V]$, $\int_X \alpha = \sum_V n_V[R(V) : K]$ is called the **degree** of α. Let C be a projective smooth curve, $X = C \times C$, and $i : C \to C \times C$ the diagonal embedding. Then we have $\int_C \left(c_1(T_C) \cap [C]\right) = 2 - 2g$ where g is the genus of C and T_C the tangent bundle of C.

We're now ready to define Characteristic classes, which will take the form of Segre classes of vector bundles, and we will invert them to get Chern classes. They will be operators of degree $-i$ acting on the Chow group $CH_*(X)$. These definitions will rely on the Chow group of the associated projective bundle in a similar way we defined Chern classes in class.

[1] A morphism of schemes (in our sense above) $f : X \to Y$ is **proper** if for any $Y' \to Y$ the projection $X \times_Y Y' \to Y'$ is a closed map.
[2] locally looks like $f : Spec(A) \to Spec(B)$ where A is a flat B-module, and for subvarieties $V \subset Y$, irreducible V' of $f^{-1}(V)$ have $dimV' = dimV + n$
[3] see Hartshorne II.6

Recall: Let $f : E \to X$ be a vector bundle of rank $e+1$, $p : \mathbb{P}(E) \to X$ the associated projective bundle. If $U \subset X$ open s.t. $E_U := f^{-1}(U)$ we define $\mathbb{P}(E)_U := p^{-1}(U) \cong U \times \mathbb{P}^e$. This map p is both proper and flat! The bundle p^*E admits a tautological line bundle $\mathscr{O}(-1)$ (the fibre of $\mathscr{O}(-1)$ a point $x \in \mathbb{P}(E)$ is the 1-dim. subspace of $E_{p(x)}$ that x represents). We denote the dual $\mathscr{O}(-1)^*$ as $\mathscr{O}(1)$.

Definition: (Segre Classes) Suppose we're given a vector bundle and associated projective bundle as above. For each cycle $\alpha \in CH_k(X)$ we define homomorphisms

$$s_i(E) \cap - : CH_k(X) \to CH_{k-i}(X), \quad \alpha \mapsto s_i(E) \cap \alpha := p_*\Big(c_1(\mathscr{O}(1))^{e+1} \cap p^*\alpha\Big)$$

where $p^*\alpha$ is the flat pull-back from $CH_k(X)$ to $CH_{k+e}\mathbb{P}(E)$, and letting $c_1(\mathscr{O}(1))$ act $e+i$ times on $p^*\alpha$ will send it to $CH_{k-i}(X)$. The operator $s_i(E)$ is called the ith Segre class of the bundle E. These Segre classes satisfy many nice properties:
(a) $s_0(E) = id$ and $s_i(E) = 0$ for $i < 0$ and $i > dim(X)$.
(b) (naturality) if $f : Y \to X$ a flat morphism and $\alpha \in CH_*(X)$, then $s_i(f^*E) \cap f^*\alpha = f^*(s_i(E) \cap \alpha)$,
(c) (projection) if $f : Y \to X$ proper and $\alpha' \in CH_*(Y)$, then $f_*(s_i(f^*E) \cap \alpha') = s_i(E) \cap f_*\alpha'$,
(d) (commutativity) if E, F vector bundles over X and $\alpha \in CH_*(X)$, then $s_i(E) \cap s_j(F) \cap \alpha = s_j(F) \cap s_i(E) \cap \alpha$,
(e) if E is a line bundle, then $s_1(E) = -c_1(E)$.
For a proof see Fulton "Intersection Theory" pg. 48 (I'm running out of room).

We consider the formal power series $s_t(E) = 1 + s_1(E)t + s_2(E)t^2 + \cdots$ and define the **Chern polynomial** $c_t(E) = 1 + c_1(E)t + c_2(E)t^2 + \cdots$ to be the inverse power series $c_t(E) = s_t(E)^{-1}$ which turns out to be an actual polynomial. In particular,

$$c_0(E) = 1,$$
$$c_1(E) = -s_1(E),$$
$$c_n(E) = -s_1(E)c_{n-1}(E) - s_2(E)c_{n-2}(E) - \cdots - s_n(E).$$

The **total Chern class** is again the sum $c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_r(E)$ where $r = rank(E)$. i.e. $c(E) \cap \alpha = \sum_{i=0}^{r} c_i(E) \cap \alpha$ for all $\alpha \in CH_*(X)$ and $c_i(E) \cap CH_k(X) \subseteq CH_{k-i}(X)$. The **total Segre class** is defined analogously. The Chern classes satisfy similar properties as the Segre classes did above, however they also satisfy an additional two properties:

Proposition: (a) For all bundles $E \to X$, and all $i > rank(E)$, $c_i(E) = 0$.
(b) (Whitney sum) For any exact sequence $0 \to E' \to E \to E'' \to 0$ of vector bundles on X, $c(E) = c(E')c(E'')$ i.e. $c_k(E) = \sum_{i+j=k} c_i(E')c_j(E'')$.

Analogous to the case of vector bundles over manifolds, there is also a "splitting principle" in this setting where we consider a family of vector bundles over X and a flat map $f : Y \to X$. Then pulling back each bundle in the family has a filtration by subbundles where each quotient is a line bundle.

Example: Lets compute the Chern classes of the tangent bundle T_X of $X = \mathbb{P}^n$. We have an exact sequence of vector bundles on X

$$0 \to \mathscr{O}_X \to \mathscr{O}_X(1)^{\oplus(n+1)} \to T_X \to 0.$$

By the properties of Chern classes we have $c(\mathscr{O}_X) = 1$ and $c(\mathscr{O}_X(1)) = 1 + H$ wjere H is the divisor class of a hyperplane in X. So by the Whitney sum formula we have

$$c(T_X) = c(\mathscr{O}_X(1))^{n+1}/c(\mathscr{O}_X) = (1+H)^{n+1}$$

i.e. $c_k(T_X) = \binom{n+1}{k} \cdot H^k$ where H^k is the class of a linear subspace of X of codimension 1.

There are many more things to say but not enough space, however I want to mention a couple. In the above we defined the intersection class of a divisor and a subvariety. However we can also define the intersection class in a more general way for arbitrary codimension. For a nonsingular variety X this allows us to define a product $CH_*(X) \otimes CH_*(X) \to CH_*(X)$ giving $CH_*(X)$ a ring structure, which we call the Chow ring. Then there are Poincare duality type theorems and Riemann-Roch theorems that hold, but a lot of this requires algebraic K-theory. On the other hand, when X is defined over $\mathbb{C}$, $X(\mathbb{C})$ carries an analytic structure, then there is a cycle map $cl : CH_*(X) \to H_*(X)$ to ordinary homology with locally finite support, which is compatible with Chern classes.