Multi-Modal Multi-Objective Evolutionary Optimization for Problems with Solutions of Variable-Length

Amiram Moshaiov, Yosef Breslav, Eliran Farhi
School of Mechanical Engineering, Tel-Aviv University, Tel-Aviv, Israel
moshaiov@tauex.tau.ac.il, yosef_breslav@mail.tau.ac.il, eliran1orit@gmail.com

***Abstract* - This paper focuses on solving a special kind of multi-modal multi-objective optimization problems (MMOPs) in which solutions are of variable length. First, problem definition and solution framework is suggested to allow using standard multi-modal multi-objective evolutionary algorithms (MMEAs) to solve the considered type of problems. Next, a real-life example of the considered type of problems is suggested concerning optimal antennas' layout-allocation design for a wireless communication network. Finally, a modification to NSGA-II is suggested and employed to solve such layout problems. When compared with other MMEAs, it is shown that the proposed algorithm provides not only better solution diversity in the decision-space, but also solutions with superior performance vectors. It is suggested here that this is attributed to the type of archive that is used here.**

Keywords— multi-concept optimization; multi-modal multi-objective optimization; wireless communication network; location optimization; variable number of dimensions; variable length problem, MMEAs

I. INTRODUCTION

Multi-Objective Evolutionary algorithms (MOEAs) have been widely used to search for Pareto-optimal solutions to multi-objective problems. Commonly, in such algorithms there is no special effort to find more than one solution per each of the optimal performance vectors. However, it is well known that decision-makers might wish to consider multiple solutions of either the same optimal performance or nearly the same (i.e., equivalent solutions). In fact, in recent years, such an understanding has motivated the rapid development of evolutionary algorithms to solve Multi-modal Multi-objective Optimization Problems (MMOPs) [1, 2]. Yet, to the best of our knowledge, such developments have been restricted to solving problems with solutions of fixed-length.

According to [3], many design problems involve a decision on the optimal number of some components to be used. Such optimization problems are often described as problems of variable number of dimensions [3], also known as variable length problems. An extensive review on evolutionary approaches to solve such problems can be found in [4], which also describes many real-life variable-length problems. As demonstrated here, MMOPs may involve solutions of variable length. Such MMOPs are hereby referred to as variable-length-MMOPs (vl-MMOPs), whereas those that are based on a decision space of a fixed dimension are hereby termed as fixed-length-MMOPs (fl-MMOPs).

When solving vl-MMOPs, special attention should be given to their definition and solution approach. This is primarily due to the fact that while comparable in a shared objective-space, solutions from different search spaces are commonly not similar in an intuitive way. This may lead to the wrong conclusion that existing Multi-modal Multi-objective Evolutionary Algorithms (MMEAs) cannot be used to solve vl-MMOPs. Here, it is suggested and demonstrated that a vl-MMOP can be solved by a simple search framework that is based on existing MMEAs. The main assumption is that the considered vl-MMOP involves a finite number of decision spaces, which is denoted in the following as n_{ds}.

According to the proposed solution approach, the considered vl-MOP is divided into a set of n_{ds} auxiliary fl-MMOPs, where the i-th auxiliary problem involves solutions strictly from the i-th decision space. While separated by the decision spaces, all solutions of the auxiliary fl-MMOPs share the objective space of the original vl-MMOP. Such an approach assumes that equivalent solutions may belong to separated spaces but their equivalency is defined by way of their performances in the mutual objective space of the considered vl-MMOP. Solving the auxiliary problems can be done in a coupled (dependent/ simultaneous) or a decoupled (independent/sequential) approach. Here, we employ a decoupled approach, which allows using existing MMEAs to solve vl-MMOPs without any modifications. The proposed procedure involves posteriori sorting of the union of solutions from all the auxiliary problems to reveal the equivalent optimal solutions of the considered vl-MMOP. As discussed in section II, the proposed solution framework has been inspired by the concept-based approach (e.g., [5]).

Another contribution of this paper is the real-life problem example that is devised to demonstrate the considered type of vl-MMOPs. The suggested example is motivated by the need to change wireless communication networks due to the rapid development of communication technology and the increasing use of sensors and actuators as edge devices. Optimizing the positions of the antennas and their number in such networks is one of the main challenges during the design or re-design process. The optimal antennas' layout-allocation problem, which is devised here, constitutes a vl-MMOP. It involves three conflicting objectives. The first is the minimization of the number of antennas, the second is the maximization of the communication connectivity, and the third is the minimization of the involved cost.

In addition to the above contributions, this study suggests a new modification to NSGA-II, [6] for solving fl-MMOPs. While, not the first attempt to adapt NSGA-II for such problems (e.g., [7-9]), it appears that it is the first to include an archive rather than just elitism. Given that it is hard to predict the number of equivalent solutions in a given MMOP, the lack of an archive in the previous attempts is considered a disadvantage. To highlight this difference, the proposed algorithm is termed Archived Multi-Modal NSGA-II, or in short amm-NSGA-II.

Finally, the applicability of the proposed algorithm and solution framework is demonstrated using the antennas' layout-allocation problem. In addition, the results as obtained by the proposed amm-NSGA-II are compared with those obtained by existing MMEAs.

The rest of this paper is organized as follows. Section II describes vl-MMOPs and the suggested solution approach. Next, Section III details the considered real-life problem. Section IV outlines the measures for selection, whereas Section V describes the proposed algorithm. In Section VI the performance indicators for algorithm comparisons are introduced. Section VII includes a numerical study with fixed-length problems, whereas Section VIII provides such a study with problems of variable-length. Finally, a discussion is given in Section IX, and the conclusions are outlined in Section X.

II. Problem Description and Solution Approach

A recent review in [2], defines MMOPs in a form which is restricted to solutions that share one decision-space (i.e., solutions of a fixed length). In contrast, here the problem is defined such that solutions may belong to various decision-spaces of different dimensions. Let n_o be the dimension of the objective-space $\mathbb{R}^{n_o}$ of the considered vl-MMOPs. Also, let S be the set of the considered decision-spaces, and n_{ds} be the number of such spaces (i.e., $n_{ds} = |S|$). Let X_m be the set of all the feasible solutions that are associated with the *m*-th decision-space, where n_m is the dimension of this space. In addition, let X be the set of all the feasible solutions from all the considered decision-spaces, i.e., $X = \bigcup_{m=1}^{n_{ds}} X_m$. Next, define $\boldsymbol{f}_m$: $X_m \longrightarrow \mathbb{R}^{n_o}$ as the mapping between the solutions of X_m into the objective-space. Also, let s_{mj} indicates the *j-th* solution of X_m, and let $\boldsymbol{x}_{s_{mj}}$ represents the decision-vector of s_{mj}. Now, let $\boldsymbol{y}_{s_{mj}} = \boldsymbol{f}_m\left(\boldsymbol{x}_{s_{mj}}\right)$ be the performance vector of s_{mj}, where $\boldsymbol{y}_{s_{mj}} \in \mathbb{R}^{n_o}$. In addition, let F_m be the set of all performance vectors of all the solutions of X_m, and let F be the set of all the performance vectors of all feasible solutions from all the considered decision-spaces, i.e., $F = \bigcup_{m=1}^{n_{ds}} F_m$. As argued in [2], the formal definition of MMOPs is still controversial. In [2], based on [10], Tanabe and Ishibuchi adopted the idea of equivalent Pareto-optimal solutions and discussed both a relaxed and non-relaxed versions of this term. Following [2], the considered vl-MMOP is described as follows:

Given the aforementioned sets and mappings, find $X_e \subseteq X$, which contains all the equivalent Pareto-optimal solutions of X, and $F_e \subseteq F$ that contains the associated performance-vectors of the solutions in X_e. As in [2], here equivalent solutions may include, in the relaxed case, not just the Pareto-optimal solutions but also any solution that has a performance vector that is within a prescribed distance from a Pareto-optimal solution. The above vl-MMOP description is general and can accommodate the two major types of equivalent solutions as described in [2]. In the current implementation we devised the problem such that there are multiple equivalent solutions without adhering to a relaxed definition of the equivalency. It is noted that, the above vl-MMOP description has been inspired by the multi-concept optimization problem, as in [5]. There are various ways to solve vl-MMOPs. As mentioned in the Introduction Section, in the current study we use a simple solution approach that can be implemented using existing MMEAs. It is based on the understanding that the order of non-domination sorting does not influence the final results. Hence, we propose to separately search the solutions within each X_m based on F_m, to obtain its set of equivalent Pareto-solutions $X_{me} \subseteq X_m$ and then sort the solutions within the union $X_u = \bigcup_{m=1}^{n_{ds}} X_{me}$ to obtain its set of equivalent Pareto-solutions $X_{ue} \subseteq X_u$. This procedure can be realized via separated searches within each X_m using existing MMEAs. However, in the current study we proposed our own algorithm for fl-MMOPs and implemented it according to the aforementioned approach.

III. Network Problem Example

This section describes an example of a real-life vl-MMOP. The considered problem concerns a wireless communication network. It deals with the optimal antennas' layout-allocation problem. First, the network modeling is described. To simplify the modeling, it is assumed here that all antennas are the same.

A. Network Modelling

The considered operation area, $\mathrm{A} \subseteq \mathbb{R}^2$, includes J existing buildings and K edge devices (potential customers) at fixed locations. Let a two-dimensional coordinate system X-Y be used to define a position vector of an object within the operation area A. For each of the separated auxiliary problems, the decision variables are the coordinates of the position vectors of the involved fixed number of antennas (I). The position vectors are:

$$\boldsymbol{p}_i^a \in \mathrm{A}\,,\ \ \forall \mathrm{i} \in \{1,2,\ldots,\mathrm{I}\} \qquad (1)$$

The position vectors of the j-th building and of the k-th edge device are assumed to be given. These are defined as follows:

$$\boldsymbol{p}_j^b \in \mathrm{A}, \qquad \forall \mathrm{j} \in \{1,2,\ldots,\mathrm{J}\} \qquad (2)$$
$$\boldsymbol{p}_k^{ed} \in \mathrm{A}, \qquad \forall \mathrm{k} \in \{1,2,\ldots,\mathrm{K}\} \qquad (3)$$

The above position vectors can be used to calculate the distances between each pair of objects in the operation area. Let the Euclidean distance between the i-th antenna and the k-th edge device be denoted as R_{ik}. Each such distance plays a role in the connectivity calculation of the network (see eq. 4). Similarly, let R_{ij} denotes the distance between the i-th antenna and the j-th building. These distances play a role in calculating the safety-insurance cost of the design (see eq. 6). Fig. 1 illustrates an area with such a network design. Each antenna has a danger zone, which is marked by a red curve. Buildings within such a zone are drawn in red, whereas all others are in green. The figure also shows that some edge devices have communication connectivity to the antennas, whereas some, which are marked in a red color, do not.

In the considered problem there is a tradeoff between maximizing the number of potential customers (i.e., the connectivity) and minimizing the total cost of the network change. Technically, the first objective translates into a desire to have a maximal number of edge-devices that can communicate with at least one antenna of the considered system.

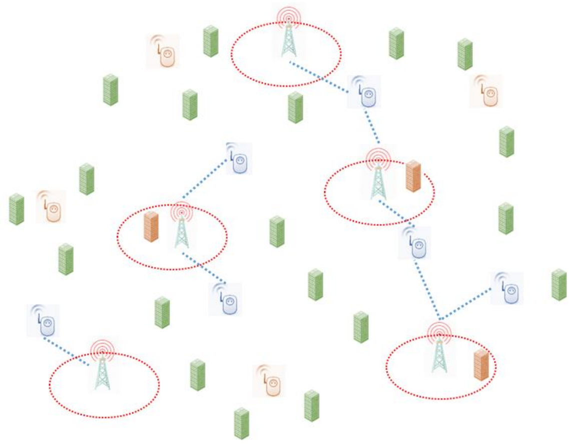

Fig. 1. A Network Example in an Operational Area

For a system with I antennas, this number is expressed as a total connectivity measure *CON*. This measure is calculated assuming that the communication signal decays with increasing ${R_{ik}}^2$. To be effective, it is assumed that the signal between the i-th antenna and the k-th edge-device should be larger than a given threshold, T. The *CON* measure is defined as follows:

$$CON = \sum_{1<k\leq K} SS_k \qquad (4)$$

where, $SS_k = 1\ if\ \exists\ i \in \{1,2,\dots,\mathrm{I}\}\ s.t. \frac{C_{ss}}{{R_{ik}}^2} > \mathrm{T}$, and C_{ss} is a given known coefficient.

The other objective deals with minimizing the total cost of changing the antennas. The total cost (denoted below as DTC) is assumed to be composed of several components including: safety-insurance cost, leasing cost, system maintenance cost, degradation cost, construction cost and the antennas' production cost. The time dependent costs, e.g., the leasing cost, are calculated based on the expected life-time of the system. The cost components can be categorized into two types. The first involves costs that are location dependent including the safety-insurance cost. $\mathrm{C}_{\mathrm{safe}}$, and the leasing cost, C_{lease}. The other type involves all the rest of the cost components which are assumed to be location independent. These cost elements are aggregated into C_{other}, which is a function of the number of antennas, I. Let the total cost of a particular solution be:

$$DTC = \mathrm{C}_{\mathrm{safe}} + C_{lease} + C_{other} \qquad (5)$$

This study assumes that the safety-insurance cost depends on the number of buildings that are at a radiation risk due to their proximity to the intended locations of the antennas. Namely, this cost is a function of the decisions on the antennas' locations, i.e., $C_{safe}(\boldsymbol{p}_1^a, \boldsymbol{p}_2^a, \dots \boldsymbol{p}_I^a)$. A safety radius R_{safe} defines the unsafe areas around the antennas' locations. Let, $C_{ij} = 1\ if\ R_{ij} < R_{safe}$ else $C_{ij} = 0$. Also, let β_j be the insurance cost associated with the j-th unsafe building, then:

$$\mathrm{C}_{\mathrm{safe}} = \beta_j \sum_{\substack{0\leq i\leq I \\ 0<j<J}} C_{ij} \qquad (6)$$

The leasing cost is assumed to vary between different sub-areas of the considered operation area, i.e., $C_{lease}(\boldsymbol{p}_1^a, \boldsymbol{p}_2^a, \dots \boldsymbol{p}_I^a)$. It is to be calculated according to the expected life-time of the system. Let the number of the operation sub-areas be P, and let the number of antennas that are located at the p-th sub-area be $\mathrm{n_p}$. Also, let the leasing cost for one antenna at the p-th sub-area be $\mathrm{C_p}$. Then, the leasing cost is:

$$\mathrm{C}_{\mathrm{lease}} = \sum_{p=1}^{P} C_p \cdot n_p \qquad (7)$$

Finally, the aggregated other costs results in:

$$\mathrm{C}_{\mathrm{other}} = \gamma \cdot \mathrm{I} \qquad (8)$$

where γ is the aggregated other costs per one antenna.

B. *Main and Auxiliary Problems*

The main network optimization problem is a vl-MMOP. This means that it aims to find <u>all the equivalent optimal locations</u> $\boldsymbol{p}_i^a \in \mathrm{A}$, $\forall \mathrm{i} \in \{1,2,\dots,\mathrm{I}^*\}$ of the minimum number of antennas I^*, with maximum connectivity and minimum cost. Namely, the main problem is:

$$Min(I), Max\ (CON), Min(DTC) \quad (9)$$

where $1\leq \mathrm{I} \leq \mathrm{M}$ and also CON=$CON(I, \boldsymbol{p}_1^a, \boldsymbol{p}_2^a, \dots \boldsymbol{p}_I^a)$ and DTC=$DTC(I, \boldsymbol{p}_1^a, \boldsymbol{p}_2^a, \dots \boldsymbol{p}_I^a)$.

The above problem is converted into a series of M two-objective auxiliary MMOPs. Each of the auxiliary problems aims to find <u>all the equivalent optimal locations</u> $\boldsymbol{p}_i^a \in \mathrm{A}$, $\forall \mathrm{i} \in \{1,2,\dots,\mathrm{I}\}$ where I is a fixed number of antennas. The n-th auxiliary problem is defined as:

$$Max\ (CON), Min(DTC) \quad (10)$$

where I=n antennas, CON=$CON(\boldsymbol{p}_1^a, \boldsymbol{p}_2^a, \dots \boldsymbol{p}_n^a)$ and DTC=$DTC(\boldsymbol{p}_1^a, \boldsymbol{p}_2^a, \dots \boldsymbol{p}_n^a)$.

Finally, it should be noted that we devised the implementation of the above design problem such that it has a discrete set of performance vectors. Moreover, it was devised such that it has an infinite number of equivalent solutions. This was achieved without adhering to the relaxed equivalency approach.

IV. SELECTION MEASURES

Each solution of each of the auxiliary problems is to be evaluated using three measures. The first is ranking, which is done using the performance vector, i.e., $(CON, DTC)^{\mathrm{T}}$. It is being calculated as in NSGA-II (Deb. et al. [6]), using non-domination sorting. The second measure is a modified crowding-distance, which is calculated based on the modified NSGA-II [11]. In [6], the definition of the crowding-distance measure is based on the distances between the performance vectors of neighboring solutions in the Pareto-front. This causes a problem when multiple equivalent solutions are associated with the same performance vector. The modified crowding-distance of [11] aims to resolve this problem by calculating the distance between neighboring performance vectors, excluding neighboring solutions of the same performance vectors. This means that all solutions with the same performance vectors are assigned with the same crowding-distance value.

The third measure is a decision-space similarity measure. It allows a comparison between solutions, within each of the auxiliary problems, based on their antennas' locations in the operation area. Let the i-th individual solution be defined as an ordered set S(i) of position vectors of the involved antennas i.e.:

$$S(i) = \{\boldsymbol{p}^a_{1(i)}, \boldsymbol{p}^a_{2(i)}, \dots \boldsymbol{p}^a_{I(i)}\} \qquad (11)$$

Given N solutions, the similarity measure for the i-th solution, in the decision-space, is defined as:

$$\mathrm{SM(i)} = \sum_{j=1}^{N} \mathrm{SimMat(j, i)} \qquad (12)$$

where SimMat is a symmetric NxN similarity matrix, such that:

$$\mathrm{SimMat(i, j)} = \sum_{n=1}^{I}\sum_{m=1}^{I} \mathrm{AS_{nm}}\ \mathrm{(i, j)} = \mathrm{SimMat(j, i)} \qquad (13)$$

and,

$$\mathrm{AS_{nm}(i, j)} = 1 \text{ if } \mathrm{i} \neq \mathrm{j}\ \&\ \left\|\mathbf{p}^{\mathrm{a}}_{\mathrm{m(i)}} - \mathbf{p}^{\mathrm{a}}_{\mathrm{n(j)}}\right\| < \mathrm{SimDist} \qquad (14)$$

$$else \quad \mathrm{SM}_{nm}(i, j) = 0$$

where, SimDist, is a predefined distance threshold.
Note that an individual has a better similarity value, SM, if its similarity value is smaller as compared with that of the opponent solution.

V. The Evolutionary Algorithm

The proposed algorithm, which is termed amm-NSGA-II, aims to find good representative sets of equivalent solutions which are associated with each of the performance vectors that represent the Pareto-front. Such a good representation aims at a spread of evenly spaced solutions in the decision space**.** The proposed algorithm follows NSGA-II with several modifications that are described in the following.

There are four main differences between amm-NSGA-II and NSGA-II. The first difference is the selection technique, which is used to create the mating pool. In contrast to NSGA-II, amm-NSGA-II uses a three level lexicographic selection that takes into considerations not only ranking and the crowding distance, but also the decision-space similarity values of the solutions (see subsection A). The second difference involves a modification to the crowding distance, as explained in Section IV. The third difference is the creation of the new population using elitism that is based on the aforementioned three measures (see subsection B). Finally, in contrast to NSGA-II, the current algorithm employs an archive that is updated at the end of each generation (see subsection C).

A. Selection

As in NSGA-II, binary tournament selection is used between individuals of the elite population. Here, a three-level lexicographic selection is employed, as compared with the two-level that are used in NSGA-II. The measures that are used for the selection are described in Section IV. The lexicographic selection involves ranking at the first level and then the modified crowding-distance (CD) in the objective-space, and finally the similarity in the decision-space.

B. Elitism

This procedure follows the one in NSGA-II with the modification of considering not only the rank and CD, but also by the similarity measure SM. The solutions in the union set of the parent and offspring, i.e., R, are first ordered by their rank. Within the rank they are ordered by their CD, solutions of the same CD, i.e., the same performance vector, are sorted by their SM.

The new population is selected using these orders until reaching the population size. First, solutions are taken from the first rank. If there is no room to all solutions of that rank, solutions are selected according to their CD and by selecting the one that has a smaller SM among solutions of the same CD. When there is still a room this procedure of selecting individuals based on their CD and SM is repeated using solutions that where not selected so far, i.e., equivalent solutions of the next smaller SM. If there is room to all solutions of the first rank, and these are less than the population size, the procedure continues to the solutions of the next rank.

C. Archive

In contrast to NSGA-II, the proposed algorithm contains an archive. First, when the archive is empty, solutions of the first rank of the elite population enter the archive. However, if the archive already contains solutions, then those of the first rank of the current elite population will be united with the archive solutions. Next, the non-dominated solutions of the union will be kept in the archive including all equivalent solutions.

VI. Comparison Approach

A. Reference Solution Approach

In general, the considered problem has no analytical solution. Moreover, as shown in the examples of Section VII, it may have an infinite set of equivalent solutions. Yet, any numerical solution is bound to produce only a finite set of solutions. Hence, testing the algorithm is somewhat problematic. To cope with this situation, we propose to create a reference solution by solving a modified problem in which the decision-space is discretized using a uniform grid, which is defined by a resolution vector $\boldsymbol{\varepsilon}$. Next, the discretized problem is solved by a full search approach to produce a reference set. In Section VII, we applied this approach to two fl-MMOPs which involve just one antenna.

B. Performance Indicators for the Decision-Space

The following two indicators are used in this study to comapre solutions. The False Discovery Rate (*FDR*) is a statistical approach used in multiple hypotheses testing to make statistical inference. Commonly, FDR is defined as the expected proportion of false discoveries, i.e., incorrectly rejected null hypothesis, among all discoveries [12]. Inspired by this approach, we propose the following *FDR* indicator to be used when comparing MMEAs:

$$FDR = \frac{FP}{TP + FP} \qquad (15)$$

where FP is the number of "False-Positive" equivalent solutions and TP is the number of "True-Positive" equivalent solutions. Since the reference solutions are based on discretization using a grid, whereas the obtained solutions are not, the definitions of FP and TP must be relaxed.

Here each of the obtained solution is transformed into the closest corner-point of the grid. Then the transformed solution is compared with the reference solutions, which are on such corners. If a reference solution exists on that corner, the solution is considered TP. If there is no reference solution at the corner of the transformed solution, then the solution is declared FP.

The second indicator is the 'Coverage' measure that calculates the ratio between the number of TP solutions and the number of solutions in the reference set.

$$Cov = \frac{TP}{|Ref|} \qquad (16)$$

VII. A Study on Fixed-length-MMOPs

This section aims to demonstrate and test the suggested algorithm on several synthetic problems. Subsection VII.A describes two initial fl-MMOPs and their results as obtained by the Full-Search approach (see Section VI.A). Next, in subsection VII.B, the aforementioned problems are used to compare the proposed algorithm with existing ones.

A. *Two fl-MMOPs and their Reference Solutions*

This subsection presents two simple initial fl-MMOPs and the results of solving them by the Full Search approach. Each of these problems involves one antenna only, and a square operation area of 2000X2000m^2, as shown in Fig. 2.

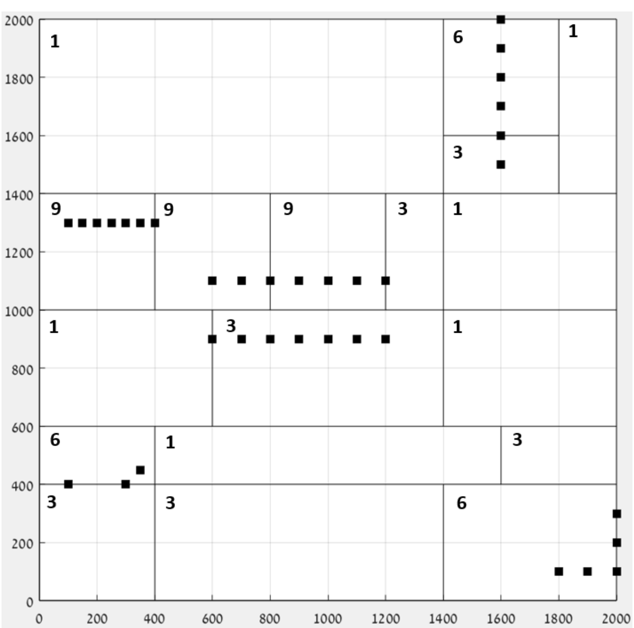

Fig. 2. Mutual Elements of the Initial fl-MMOPs

The design of the initial problems was motivated by the desire to demonstrate the obtained solutions in a clear way. Yet, the initial problems were also extended to vl-MMOPs (see Section VIII).

Fig. 2 presents all the mutual elements for the two test problems including the buildings, which are shown by black squares, and the sub-areas, which are shown as rectangles. The C_p (leasing cost per one antenna in each sub-area) is given as a number at the upper left corner of each sub-area. For simplicity, the applied costs are $\gamma = \beta_j = 1$ for both problems (see Eqs. 6 & 8). The non-mutual elements of the two fl-MMOPs are the edge devices. These devices, which differ by their amount and locations, are illustrated as blue asterisks in Fig. 3 & 4 for the 1st and 2nd problems, respectively.

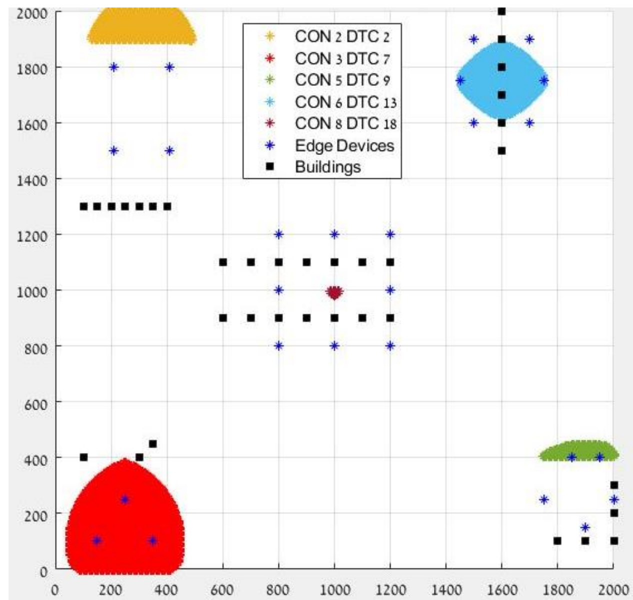

Fig. 3. Decision Space – 1st fl-MMOP

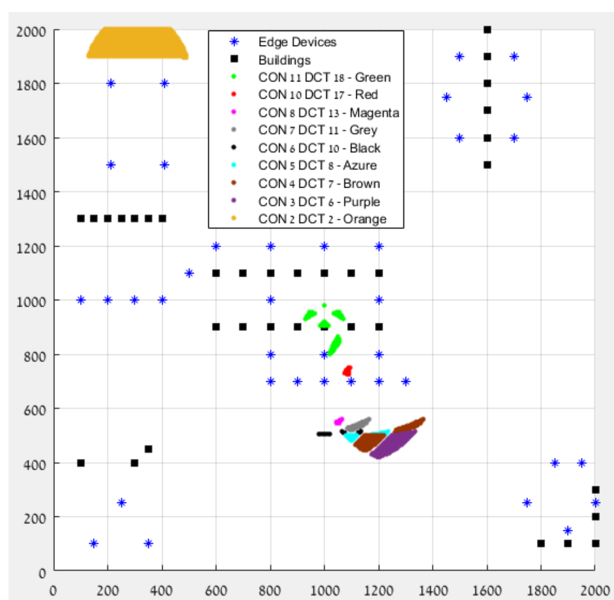

Fig. 4. Decision Space – 2nd fl-MMOP

Fig. 3 & 4 also depict the resulting optimal antenna locations, i.e., the equivalent reference solutions, as obtained by the full search procedure using the resolution vector $\boldsymbol{\varepsilon}=(5,\ 5)^T$ (see Section VI.A). The optimal antenna locations are shown as colored regions, where each color corresponds to a unique performance vector, as detailed in the legend. For example, in Fig. 3, all the equivalent Pareto optimal solution within the red region, are associated with the same performance vector in the obtained Pareto-front, i.e., *CON*=3 and *DTC*=7.

Fig. 5 & 6 show the Pareto-fronts as obtained by the Full-Search, for the 1st and 2nd problem, respectively. As can be easily observed from Fig 5 & 6, the 1st problem has five vectors in the front, whereas the second has nine. These vectors correspond to the five and nine colored regions, in Fig. 3 & 4, respectively.

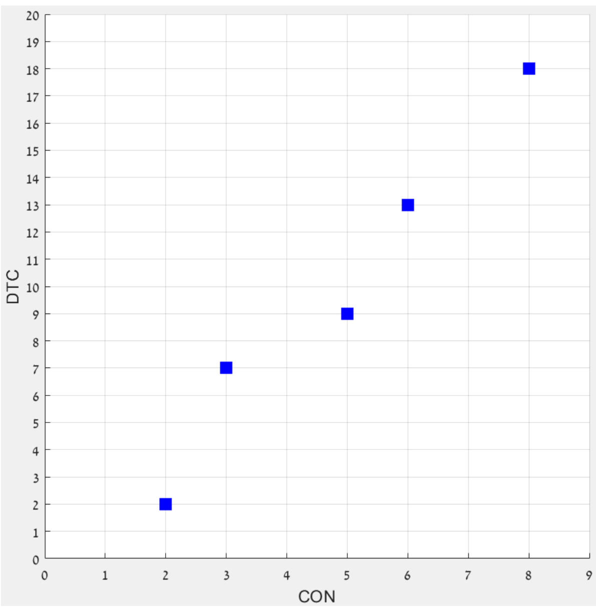

Fig. 5. Pareto Front– 1st fl-MMOP

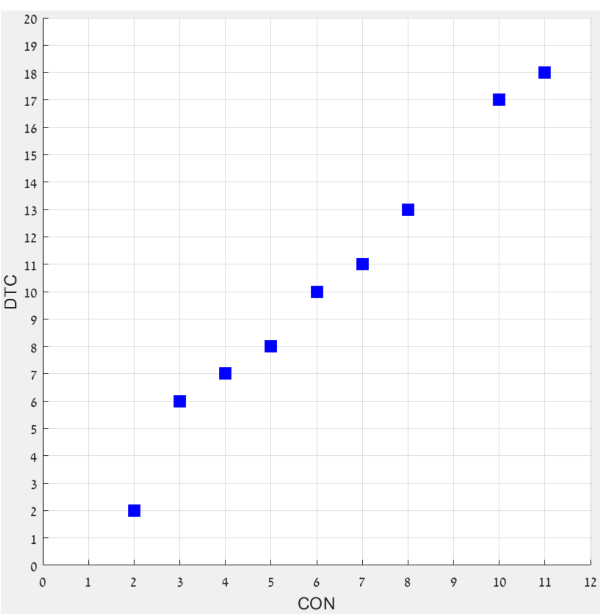

Fig. 6. Pareto Front– 2nd fl-MMOP

B. *Comparing amm-NSGA-II using the fl-MMOPs*

This subsection involves a comparison of the results of solving the fl-MMOPs of the previous subsection, as obtained by the proposed algorithm and by existing MMEAs including: CPDEA [13], MO-Ring-PSO-SCD [14] and TriMOEA-TA&R [15]. The reference solution was obtained as described in Section VI.A.

The Pareto-set and Pareto-front are shown in Fig. 5-8. For both problems, the parameters are R_{safe}=600 [m], $\beta_j = \gamma = 1$, C_{ss} = 9000, T = 0.1 and C_p (the leasing cost per area) as shown in Fig. 2. To make statistical inference, each algorithm was applied 30 times with a population size of 10 and for 200 generations.

In addition to using the decision-space indicators FDR and Coverage, which are described in Section VI.B, the IGD indicator was used for the objective-space. All the indicators were applied based on rounding the solutions to the nearest grid point. The statistical inference is based on the Wilcoxon test.

Table 1, shows the results of the comparison. It can be seen that amm-NSGA-II is superior in all comparisons ('-' sign) except for one where there is no conclusion ('=' sign). This superiority can be explained by the applied small population size and the fact that, unlike the other algorithms, the archive of the proposed algorithm has no size limitation. Illustrations of the coverage results are provided in the following figures.

Fig. 7 & 8 show the union of the equivalent solutions of the 2nd problem, for amm-NSGA-II and CPDEA, respectively.

Table 1: Algorithm Comparisons problems

			amm-NSGA-II	CPDEA	TriMOEA-TA&R	MO-Ring-PSO-SCD
Problem 1	**IGD**	mean	**0.2711**	1.6352 -	2.1408 -	0.2995 =
		variance	0.1903	0.4997	0.7017	0.1904
		median	**0.2000**	1.5185	1.9744	**0.2000**
	FDR	mean	**0.1277**	0.42 -	0.6567 -	0.4833 -
		variance	0.029	0.061	0.0572	0.0587
		median	**0.0991**	0.4	0.6	0.5
	Cov.	mean	**0.0702**	0.00074 -	0.00081 -	0.00092 -
		variance	2.8676e-04	2.5924e-04	3.49375e-04	2.29385e-04
		median	**0.0748**	0.00069	0.00079	0.00083
Problem 2	**IGD**	mean	**0.4886**	1.4493 -	2.3941 -	0.8527 -
		variance	0.2371	0.1231	0.6082	0.0193
		median	**0.4024**	1.4507	2.2384	0.8504
	FDR	mean	**0.1722**	0.5367 -	0.6564 -	0.3633 -
		variance	0.088	0.0327	0.0198	0.0155
		median	**0.2084**	0.5	0.7	0.4
	Cov.	mean	**0.031**	0.0029 -	0.0041 -	0.0043 -
		variance	3.2526e-04	4.2593e-04	2.39783e-05	3.24837e-04
		median	**0.0264**	0.0029	0.00372	0.00411
	+/=/-			**0/0/6**	**0/0/6**	**0/1/5**

VIII. A Study on variable-length-MMOPs

Here, the algorithms, as used in Section VII, are compared on two vl-MMOPs versions of the two fl-MMOPs of Section VII. In the variable-length versions of the problems, all the parameters are equal to those in the fl-MMOPs. The difference from the fl-MMOPs is the requirement to minimize the number of antennas (within the constraint value). In the first vl-MMOP, the number of antennas is constrained to 8, whereas in the 2nd problem it is constrained to 14. Here, the procedure that is suggested in Section II, is used. Namely, each vl-MMOP is converted to a set of auxiliary fl-MMOPs that differ by the number of antennas. Each of the algorithms was run 30 times for each of the auxiliary problems.

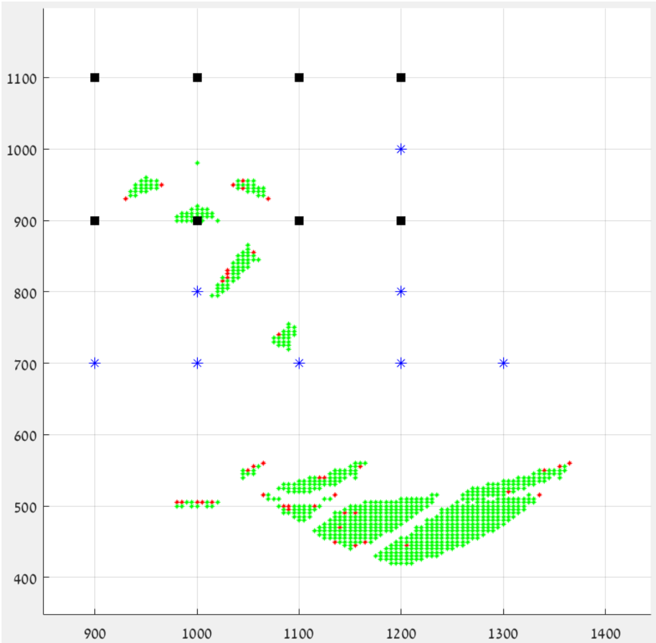

Fig. 7: amm-NSGA-II Coverage After 30 Runs – 2nd fl-MMOP

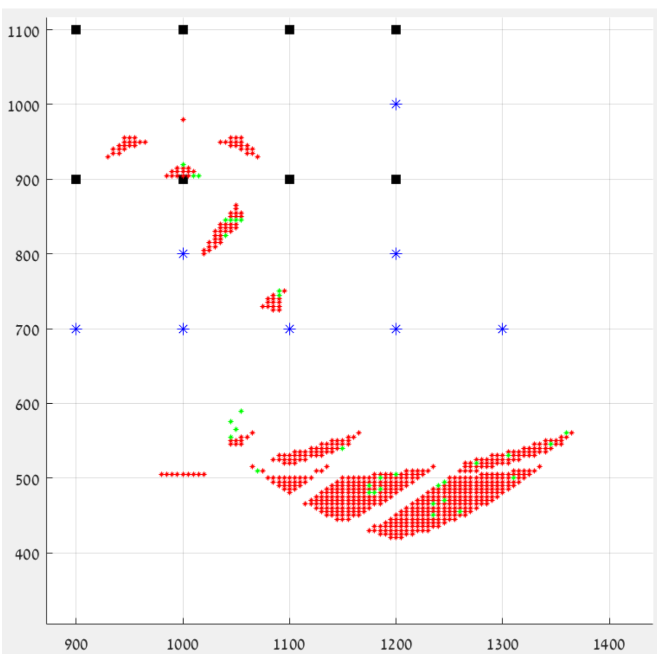

Fig. 8: CPDEA Coverage After 30 Runs – 2nd fl-MMOP

Fig. 9 & 10 show the resulting fronts for the 1st and 2nd problems respectively. It is noted that each of the shown fronts presents the performance vectors of the non-dominated solutions of the union-set of solutions from the 30 runs (per each algorithm). In Fig. 9 & 10, each of the performance vectors is associated with a number that indicates the obtained number of antennas. With increasing numbers of antennas, the results of amm-NSGA-II appear superior as compared with those of the other algorithms. However, to reach a statistical conclusion, a statistical study should be performed (as done in Section VII). Here, however, such a study is avoided for the reason which is given in the following discussion.

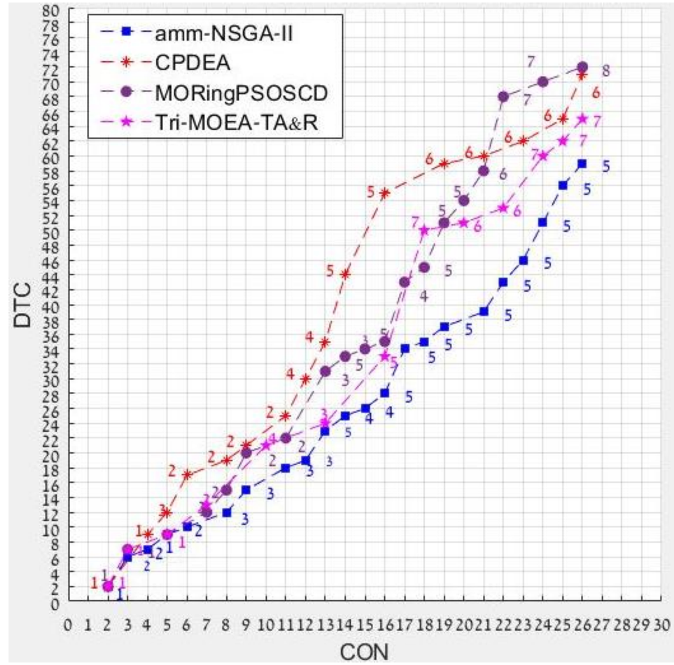

Fig. 9. Fronts Obtained by the Compared Algorithms - 1st vl-MMOP

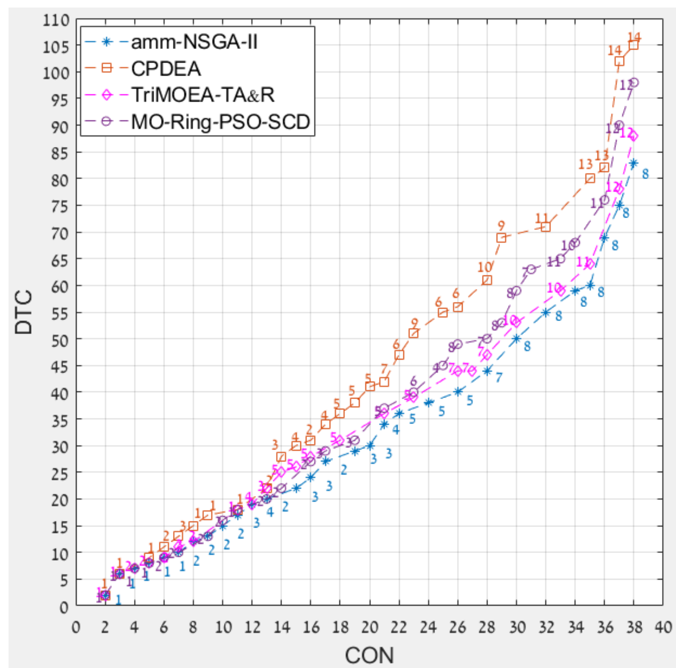

Fig. 10: Fronts Obtained by the Compared Algorithms - 2nd vl-MMOP

IX. DISCUSSION

As suggested in Section VII, it is suspected that the superiority of amm-NSGA-II can be explained by the small population size and the fact that, unlike the other algorithms, the proposed algorithm does not have a limitation on the archive size. Given this understanding, it can be suggested that statistical inference about superiority should be done only under more fair conditions. Namely, amm-NSGA-II should be compared with MMEAs that include a similar unbounded archive. At present, however, most MMEAs do not have an unbounded archive [2]. Moreover, it appears that there is no study which is dedicated to the comparison of MMEAs with unbounded archives. This situation is reflected in the work presented in [16]. The MMEA, which has been suggested in [16], is considered in [2] to be one of the few MMEAs to have an unbounded archive. Nevertheless, while having an unbounded population, the actual procedure that is used in [16] is that of a fixed-size archive. Namely, as noted in [16], for a fair comparison of the proposed algorithm, a selection procedure was applied to narrow down the population solutions into a fixed number of solutions that are actually archived.

It is a well-known assertion that decision-makers should not be overloaded with alternative solutions. In fact, some studies in psychology suggest that the number of alternatives that human may handle is seven plus minus two [17]. Based on such understanding, it can be argued that when solving a MMOP, special care should be taken in order to not expose the decision-makers to an excessive number of solution alternatives. This seems to be in contradiction with the use of an unbounded archive.

Yet, it should be noted that while figures such as Fig. 3 & 4, as well as Fig. 7 & 8, show a very large number of solution alternatives, such a large number of solutions can be grasped by humans. The reason is that the solution representation involves a limited number of clusters (areas). In other words, when observing the figures, humans can cognitively distinguish between each of the clusters of solutions and could eventually investigate the involved areas to select a preferred position for the antenna.

X. SUMMARY AND CONCLUSIONS

This paper suggests a new approach to deal with Multi-Modal Multi-objective Problems (MMOPs) with solutions of variable-length, i.e., vl-MMOPs. It is argued and demonstrated that such problems can be handle by existing Multi-Modal Evolutionary Algorithms (MMEAs), using a special procedure. In addition, a modification to NSGA-II is suggested to develop a new MMEA with unbounded archive. The proposed algorithm, which is termed amm-NSGA-II, is demonstrated and compared to existing algorithms using two optimal antennas layout-allocation design problems for a wireless communication network. A discussion is also provided with respect to the use of an unbounded archive for MMOPs. It is concluded that for the studied problem, the proposed algorithm appears superior to the compared MMEAs.

For a truly fair comparison, there is a need for MMEAs with unbounded archives. Yet, at present most existing MMEAs might not be suited for such a fair comparison with the proposed algorithm. This is left for future investigation. Future research may also include the development of algorithms for a simultaneous coupled search within the various decision-spaces of vl-MMOPs, as well as the development of benchmark vl-MMOPs and possibly also dedicated performance indicators.

XI. REFERENCES

[1] Li, X., Epitropakis, M.G., Deb, K. and Engelbrecht, A. Seeking multiple solutions: an updated survey on niching methods and their applications. *IEEE Transactions on Evolutionary Computation*, *21*(4), pp.518-538, 2017.

[2] Tanabe, R. and Ishibuchi, H. A review of evolutionary multimodal multiobjective optimization. *IEEE Transactions on Evolutionary Computation*, *24*(1), pp.193-200, 2020.

[3] Kadlec, P., Čapek, M., Rýmus, J., Marek, M., Štumpf, M., Jelínek, L., Mašek, M. and Kotalík, P. Design of a Linear Antenna Array: Variable Number of Dimensions Approach. In *2020 30th International Conference Radioelektronika* (pp. 1-6). IEEE, 2020.

[4] Ryerkerk, M., Averill, R., Deb, K. and Goodman, E. A survey of evolutionary algorithms using metameric representations. *Genetic Programming and Evolvable Machines*, *20*(4), pp.441-478, 2019.

[5] Avigad, G. and Moshaiov, A. Simultaneous concept-based evolutionary multi-objective optimization. *Applied Soft Computing*, *11*(1), pp.193-207, 2011.

[6] Deb, K., Pratap, A., Agarwal, S., & Meyarivan, T. A. M. T. A fast and elitist multiobjective genetic algorithm: NSGA-II. IEEE transactions on evolutionary computation, 6(2), 182-197. 2002.

[7] Deb, K., & Tiwari, S. Omni-optimizer: A generic evolutionary algorithm for single and multi-objective optimization. European Journal of Operational Research, 185(3), 1062-1087, 2008.

[8] Liang, J. J., Yue, C. T., & Qu, B. Y. Multimodal multi-objective optimization: A preliminary study. In 2016 IEEE Congress on Evolutionary Computation (CEC) (pp. 2454-2461). IEEE, July 2016.

[9] Javadi, M., Zille, H., & Mostaghim, S. Modified crowding distance and mutation for multimodal multi-objective optimization. In Proceedings of the Genetic and Evolutionary Computation Conference Companion (pp. 211-212), July 2019.

[10] Rudolph, G., & Preuss, M. A multiobjective approach for finding equivalent inverse images of pareto-optimal objective vectors. In 2009 IEEE Symposium on Computational Intelligence in Multi-Criteria Decision-Making (MCDM) (pp. 74-79). IEEE, March 2009.

[11] Fortin, F. A., & Parizeau, M. Revisiting the NSGA-II crowding-distance computation. In Proceedings of the 15th annual conference on Genetic and evolutionary computation (pp. 623-630), July 2013.

[12] Benjamini, Y., & Hochberg, Y. Controlling the false discovery rate: a practical and powerful approach to multiple testing. Journal of the Royal statistical society: series B (Methodological), 57(1), 289-300, 1995.

[13] Liu, Y., Ishibuchi, H., Yen, G. G., Nojima, Y., & Masuyama, N. Handling imbalance between convergence and diversity in the decision space in evolutionary multimodal multiobjective optimization. IEEE Transactions on Evolutionary Computation, 24(3), 551-565, 2019.

[14] Yue, C., Qu, B., & Liang, J. A multiobjective particle swarm optimizer using ring topology for solving multimodal multiobjective problems. IEEE Transactions on Evolutionary Computation, 22(5), 805-817, 2017.

[15] Liu, Y., Yen, G. G., & Gong, D. A multimodal multiobjective evolutionary algorithm using two-archive and recombination strategies. IEEE Transactions on Evolutionary Computation, 23(4), 660-674, 2018.

[16] Tanabe, R., & Ishibuchi, H. A decomposition-based evolutionary algorithm for multi-modal multi-objective optimization," in Proc. PPSN, pp. 249–261, 2018.

[17] Miller, G. A. The magical number seven, plus or minus two: Some limits on our capacity for processing information. Psychological review, 63(2), 81, 1956.