Area Law Violations in Entanglement Measures for Spatially Inhomogeneous Quantum Spin Chains

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Abstract

In this report, we investigate the effect of slowly decaying couplings on measures of entanglement one dimensional, inhomogeneous quantum spin chains. Entanglement is often regarded as 'the' essential property of quantum systems, and one dimensional systems offer a useful (and often analytically tractable) window into how entangled systems behave. We reproduce findings from the literature about homogeneous system and systems with exponentially decaying couplings, and extend the analytical results to systems with slowly decaying couplings. We verify these results with the exact solution (in the case of quadratic models) and the Strong Disorder Renormalisation Group (SDRG) method in the case of interacting models for which an exact solution is not possible. We show that the scaling of the entanglement entropy is sub-logarithmic for the system with slowly decaying chains and that the logarithmic negativity decays more quickly than for the exponential model.

Contents

1 Introduction

Entanglement is often regarded as 'the' stylistic feature of quantum systems [1]. However, many bodied and disordered systems present analytical and computational challenges. For example, analytic solutions for higher dimensional problems are still rare, and real world implementations of useful quantum technologies such as teleportation and communication are still in their infancy.

Quantum spin chains are (in)finitely long collections of quantum spins in one dimension. The physical and mathematical definition of a spin will be giving in more detail in section A.7, but superficially these chains can be thought of as a more complicated version of a classical Ising spin chain: a sequence of abstract bodies that have one (or more) binary states. Whilst a one dimensional system may seem like a trivial object of study, one dimensional chains provide a first step towards more detailed models of higher dimensional systems (see for example [2]), and particular one dimensional quantum systems have been used to study 1+1 dimensional classical systems [3]. There are of course genuinely one dimensional systems of interest ([4]), and simple, one dimensional classical spin chains are solveable via transfer matrices [5].

The presence of disorder introduces a new richness of modelling possibilities but also creates serious problems for solutions to this models. In particular, local spatial homogeneity is violated and this loss of symmetry makes solving the models more difficult. This has prompted a wave a numerical techniques based on renormalisation group theory ([6], [7]). Further complicating the picture is the presence of interactions. The combination of disorder and interactions makes disordered quantum chains truly complex system.

However, inconsistencies have arisen in the predict scaling of certain observables and their numerically observed outcomes [7]. In particular, it is known that for gapped, spatially invariant systems, entanglement entropy must scale with an area law ([8]). An interesting research programme began in 2004 [7] to study what changes to the system Hamiltonian would result in area law violations, to exact volume scaling or logarithmic scaling. Subsequent research ([9], [1], [10]) has developed a significant pool of results mostly by enforcing a degree of spatial inhomogeneity through couplings that decay from the centre.

To motivate this research programme in area law violations, Eisert et al. [11] list four important motivations for studying measures of entanglement:

1. Black hole physics: Beckstein-Hawking radiation is proportional to the boundary of the black hole [12].

2. The theory of long range quantum correlations - is it possible to have arbitrarily long range quantum correlations away from criticality?

3. Complexity of numerical simulation: great success has been made in classical physical through mean field theory to more efficiently simulation and solve complex systems ([13]). If we understand how and when mean field theories work for quantum systems, this will help us to effectively model and solve complex quantum systems.

4. Topological entanglement entropy, a novel order parameter, cannot be described by local conditions.

In this report, we will investigate the scaling of two measures of entanglement, entanglement entropy and entanglement negativity, with varying degrees of spatial inhomogeneity. We will verify prior results for spatial homogeneous systems and systems whose couplings decay exponentially, before extending some results for exponential systems to negativity and investing a new type of inhomogeneity that decays more slowly than exponential systems.

We will investigate these systems with a few analytical results, the Strong Disorder Renormalisation Group (SDRG) technique, and some exact numerical results. The SDRG technique in particular is useful for understanding the groundstate of disordered systems and is known to be asymptotically exact ([7]). We implement algorithms for the SDRG procedure and associated analyses, as well as for the exact solutions where possible, all at arbitrary machine precision.

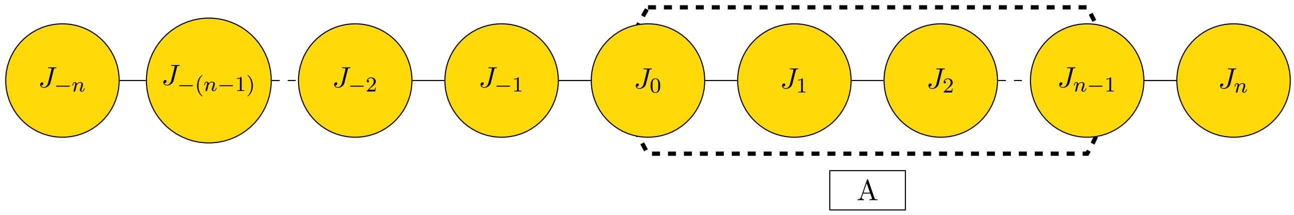

Figure 1.1: A chain of spins with the central spin labelled J_0. The subsystem A is highlighted with the dashed line. Unless otherwise stated, subsystems will always start from the centre of the chain.

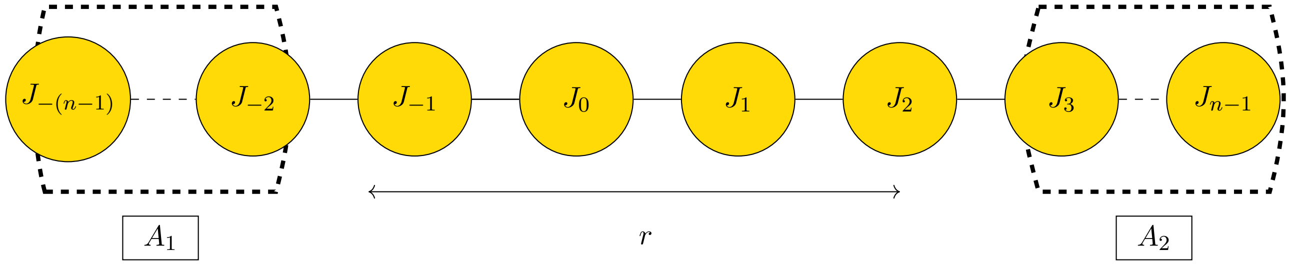

Figure 1.2: A chain of spins with a pair of disjoint subsystems, separated by a distance r. The subsystems A_1 and A_2 are highlighted with the dashed line. Unless otherwise stated, subsystems will always start from the centre of the chain.

The layout of the rest of this report is as follows: In section 2, we review current literature around area law violations in quantum systems and the SDRG procedure. In section A, we review basic quantum physics and mathematics that will be used in this report, with a focus on discrete models. In section 3, we cover different measures of quantum entanglement and the difference between entanglement entropy and negativity. In section 4, we will lay out the spin chain models to be investigated and the SDRG procedure. In section 5, we will briefly review the implementation of the SDRG procedure and demonstrate some initial verifications. In section 6 we will verify some existing results for area law violations, and in sections **??** through 8, we will demonstrate the new results. Lastly, in section 9 we will evaluate the new results and discuss areas for further research.

2 Literature Review

2.1 Area Law Violations in Quantum Systems

We will briefly survey the wide literature of area law violations in quantum systems. For a thorough review, we recommend [11]. Early work on the scaling of entropy in quantum systems focused on the study of blacks holes: in particular, Bombelli et al. [14] and Srednicki [15]. As discussed in section 1, the horizon of a black hole and the theory of Hawking radiation are of great general interest. Srednicki argued that for a field of quantum harmonic oscillators divided into a spherical 'in' region and exterior, the entropy of the 'in' region must scale with the area as opposed to the volume, as the surface area is the only region common to both subregions. In 2004 Hastings published a proof of the area law for strictly local systems [8], e.g. systems of the form:

$$H = \sum_{i=1}^{N} H_{i,i+1} \tag{2.1}$$

where $H_{i,i+1}$ is a function only of the sites i and $i+1$, and this was followed by a more general proof in 2005 by Eisert et al. [16], and by [17] it was well established that for for local, local hamiltonians, the ground state had an entropy that scaled with the area rather than the volume.

This 'area law' sets the baseline for the physics of such systems: any evidence of violations to this area law must also explain how to account for this violation. Many theoretical systems do in fact display area law violations of their quantum entropy [18], and thus a literature has sprung up to identify and explain these phenomena with a variety of tools. For example, in a very detailed survey Calabrese and Cardy [19] give a thorough treatment through $(1 + 1)d$ conformal field theory (CFT) of the entanglement entropy, and show that the entropy S_A of a subsystem A of length l at zero temperature in an infinitely long, 1D system without boundaries is:

$$S_A \sim (c/3) \log (l/a) \tag{2.2}$$

where c is the central charge and a is the lattice spacing. This is known as a *logarithmic correction* to the area law.

Page [20] shows with an information theoretic argument that does not use the harmonic oscillator framework that progressively smaller subsystems contain virtually no information on the total system, which thus explains that logarithmic scaling: as the subsystem size decreases (and as the number of subsystems increases), the entropy decreases more and more quickly, giving rise to the logarithmic scaling in l. Furthermore, area law violations do not just come in via the subsystem length. With a combination of analytical and approximate results (more on SDRG below in 2.3) Giampaola et al. show that the system size N also affects the entanglement scaling, especially under certain open conditions [21].

Most of the refernces above report on the entanglement entropy (see 3.2 for a definition), but results are not limited to this. For example, Calabrese et al. compute the scaling of the logarithmic negativity in the CFT framework [22] and

Given that the locality of the Hamiltonian and the boundary conditions affect the scaling, the working hypothesis is that the violations come from a degree of spatial inhomogeneity in certain systems. For example, in a random coupling spin chain, there will be small areas that are deeply inhomogeneous, and these contribute positively to the entropy in a way that a homogeneous model would not.

2.2 Measures of Entanglement

A proper physical and mathematical treatment of measures of entanglement will be given in section 3. Here we will review the recent literature on measures of entanglement in quantum mechanics. An excellent review is [23], section two. As a starting point, Bell [24] presents his 'game' in which Alice and Bob attempt to coordinate their observations of a potentially entangled pair of particles. A full illustration is beyond the scope of this paper, but it can be shown (ibid) that under certain conditions (see below on the CHSH inequality) this game is properly 'entangled', in that the (classical) states of both particles are not known until either one is measured.

This was developed into the CHSH inequality [25], which is now a benchmark test for entangled states. In Vedral et al. 1997, a set of axioms for entanglement to satisfy is presented [26] to provide a rigorous grounding for entanglement. This is in response to the fact that states had been found that did not violate the CHSH inequality (i.e. they did not appear to be entangled) but that were effectively entangled under local operations and classical communication (LOCC, again see section 3.1) [27]. However, there is a tradeoff between what is often called the 'operational' view of entanglement, which measures entanglement in a way that is physically intuitive, and the mathematical approach that seeks rigorous and numerically stable measures ([3], [28]). For example, the entanglement of distillation gives the number of maximally entangled pairs that can be purified from a given quantum state [29]. However, purification processes are varied and there is not yet a way of calculating this quickly. On the other hand, negativity (see section 3.2 for a definition) is well defined but does not have any obvious physical interpretation. Mathematically, perhaps the most important criteria for any measure of entanglement is that it is *monotone*[28], i.e. that for any measure E on a state ρ we have:

$$E(\rho) \geq \sum_i p_i E(p_i) \tag{2.3}$$

Which implies that any classical interference with the state (represented by the classical probability weightings p_i can only lower its entropy. This is one requirement for keeping measures of entropy focused strictly on the quantum correlations.

Single measures of entanglement have also been extended to a complete *entanglement spectrum* [30], which is in turn related to moments of the reduced density matrix (ibid, [1]). The moments of the reduced density matrix are given by its eigenvalues λ_i as follows:

$$R_\alpha = \sum_i \lambda_i^\alpha \tag{2.4}$$

Which are in turn the object of all numerical approaches the depend on matrix product states (see section 2.3 below for more information).

2.3 SDRG Procedure

Whilst in non-interacting cases, one dimensional spin chain models are often analytically solvable, this is not the case in general. This observation of course holds for more such quantum systems, and the renormalisation group (RG) approach to approximating complex systems (quantum and classical) came out of work with (Michael) Fisher and Wilson in the 1960s [31]. The renormalisation group approach involves successively integrating out the strongest factors in a model to approach a state that is accurate *even at a critical point*. Typically, order parameters will diverge in a scale free manner, i.e. according to a power law, which gives rise to fluctuations with no specific scale. This scale invariance is difficult to capture with traditional mathematical methods, and hence the RG method has proven so useful.

In this paper we will focus exclusively on *disordered* models, which puts us in the domain of the *strong disorder renormalisation group* (SDRG). For a very thorough review, see [32] and [33]. Studies of disordered systems start as early at [34] who showed that the experimental magnetisation of certain materials could be smooth even around the critical point, and that this was explainable by a degree of disorder. The key contribution was from Dasgupta, Ma, and Hu ([35], [36]) who defined a rule for the elimination of bonds that was generalised by (Daneil) Fisher [6]. Since then the SDRG procedure has been used to investigate a great variety of challenging disordered problems ([37], [38], [39], [7], [9], [1], [10]).

The SDRG procedure is often used in complement (or contrast) to matrix product states, and in particular to the density matrix renormalisation group (DMRG)[40]. For a thorough review, see [41] and [42]. A detailed discussion of the literature on DMRG is beyond the scope of this report, but it is crucial out that in [1], the DMRG technique struggled to converge for large system size L, with significant errors relative to the exact solution (where exact solutions were not available). For this reason we have not used the DMRG method in this report.

3 Measures of Entanglement

3.1 Entanglement as a Information Phenomenon

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Entanglement is a type of correlation unique to quantum mechanics, but defining the style of correlation is difficult. One approach is the 'operational approach' [3], which says that states are entangled if and only iff (iff) they are *seperable*. Seperability implies that, given a set of classical probabilities $\{p_i\}$, and a similar set of density matrices for two subsystems A and B, the final density matrix ρ can be written as:

$$\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i \tag{3.1}$$

Thus a first (and incorrect) attempt would be to define entanglement as 'the property of any statement that cannot be reached by local operations and classical communication' (LOCC). LOCC includes, for example, experiments on quantum systems to take observations (see A.1), and traditional methods of communication (Alice: 'Bob, my spin is in state $|\psi\rangle$, how about you?'). However, as Peres shows [43], LOCC is only a subset of operations that lead to separable states. Rather, density matrices must be at least *positive partial transpose preserving*. This is known as the Peres criterion. The partial transpose is defined as follows - given a density matrix ρ defined as:

$$\rho = \sum_{ijkl} p_{kl}^{ij} |i\rangle \langle j| \otimes |k\rangle \langle l| \tag{3.2}$$

Then the partial transpose with respect to B is:

$$\rho = \sum_{ijkl} p_{kl}^{ij} |i\rangle \langle j| \otimes (|k\rangle \langle l|)^T \tag{3.3}$$

The preceeding matrix is positive iff it does not have any negative eigenvalues.

Whilst the Peres criterion is strictly correct, LOCC presents a useful heuristic for defining a set of axioms that any measure of entanglement must satisfy [26]. There is significant debate over what a minimal set of such axioms should be. Following [3], we use the following three necessary conditions for a good measure of entanglemnt E:

1. *Monotonicity*: $E(\rho) \geq \sum_i p_i E(p_i)$ (see section 2.2)
2. *Convexity*: $\sum_i p_i E(\rho_i) \geq E\left(\sum_i p_i \rho_i\right)$
3. *Additivity under the tensor product*: $E(\otimes_i \rho_i) = \sum_i E(\rho_i)$

The additivity requirement is exceptionally useful as it allows the calculation of product states, including the random singlet phase that will be discussed in 4.2. The convexity requirement is perhaps the only one that can be considered contentious. The right hand side of the convexity condition describes a *mixed state*, i.e. a classical combination of quantum states. The way that measures of entanglement deal with mixed states is crucial [3]. Convexity implies that a classical mixing of states cannot produce entanglement, which is prima facie a desirable property of an entanglement measure. However, Plenio shows that for entanglement negativity (see below, section 3.2) the monotonicity requirement is sufficient to guarantee its suitability as a measure of entanglement. For general measures of entanglement, however, it would be important to revisit the convexity requirement.

3.2 Entanglement Entropy and Negativity

We will now define the key measures of entanglement used in this report. Firstly, we define the general Rényi entropies:

$$S_n(\rho_A) = \frac{1}{1-n} \log \operatorname{Tr} \rho_A^n \tag{3.4}$$

where ρ_A is the reduced density matrix over the subsystem A, found by taking the partial trace over the B subsystem:

$$\rho_A = \mathrm{Tr}_B\left[\rho\right] = \sum_j \underline{\left(I_A \otimes \langle j|_B\right) \rho \left(I_A \otimes |j\rangle_B\right)} \tag{3.5}$$

Rényi entropies were introduced in [44] as a generalisation of Shannon entropies. Whilst in theory Rényi entropies capture meaningul quantum correlations, they do not satisfy the useful subadditivity condition:

$$S\left(\rho_{A\cup B}\right) \leq S\left(\rho_A\right) + S\left(\rho_B\right) \tag{3.6}$$

For bipartite systems $A \cup B$. Fortunately, taking the limit $n \to 1$ we recover the von Neumann entanglement entropy:

$$S\left(\rho\right) = -\,\mathrm{Tr}\left(\rho \log \rho\right) \tag{3.7}$$

which is subadditive [45]. Note that all Rényi entropies are symmetric with respect to the subsystem, i.e. $S_n\left(\rho_A\right) = S_n\left(\rho_B\right)$ [3], which also suggests that an area law would be reasonable to expect: if $|A| > |B|$, it would be odd for $S\left(\rho_A\right) = S\left(\rho_B\right)$ if entropy scaled with volume.

Unfortunately, the entanglement entropy is only measurable if the system is in a pure state and if it is bipartite [3]. The only other measurable form of entanglement is the *negativity*. Before defining negativity, it is useful to define the *partial trace*:

$$\left\langle \varphi_A \varphi_B \left| \rho_A^{T_2} \right| \varphi'_A \varphi'_B \right\rangle \equiv \left\langle \varphi_A \varphi'_B \left| \rho_A \right| \varphi'_A \varphi_B \right\rangle \tag{3.8}$$

where $\{\varphi_X\}$ is a basis for subsystem X [1]. We also define the *trace distance* of A:

$$||A|| = \mathrm{Tr}\,\sqrt{AA^\dagger} \tag{3.9}$$

which in turns lets us define the negativity:

$$\mathcal{N}(\rho_A) = \frac{||\rho_A^{T_2}|| - 1}{2} \tag{3.10}$$

Assuming A is Hermitian, the trace distance of A is sum of the absolute value of the eigenvalues of A[3]. Hence the negativity measures the 'negativity' of A. Given the discussion about the Peres criterion, it is clear that this quantity measures the negativity of the partially transposed density matrix. Whilst this is monotone [46], it is not additive. However, we can take the logarithmic negativity:

$$\varepsilon(\rho_A) = \ln ||\rho_A^{T_2}|| \tag{3.11}$$

In this report we will compute the entanglement entropy and the entanglement negativity for a variety of systems, both analytically and with numerical methods. In the following section we will outline the principle model that we study as well as the SDRG procedure.

4 The Random Inhomogeneous Chain and the SDRG method

4.1 Random Inhomogeneous 1D Chain

The random inhomogeneous spin-$\frac{1}{2}$ XXZ chain with L spins and open boundary conditions (OBC) has the Hamiltonian:

$$\mathcal{H} = \sum_{i=1}^{L-1} J_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right) \tag{4.1}$$

S_i^x is a spin component x operator acting on site i, J_i is a random coupling connecting site i to $i+1$, and Δ is an anisotropy parameter. For periodic boundary conditions (PBC) and additional term for $i = L$ is needed. With $\Delta = 1$ we have the XXX random chain and with $\Delta = 0$ we have the XX chain. With the XX chain the model is analytically solvable [21].

More details of how the couplings $\{J_i\}$ are calculated for the in will be given in more detail in the relevant sections. For now it is sufficient to explain the principle disordered contribution to the random couplings, namely the probability distribution:

$$P_\delta(J) \equiv \delta^{-1} J^{-1+1/\delta} \tag{4.2}$$

Clearly as $\delta \to 0$ the disordered contributions tend to 1 and we recover the clean spin chain. For $\delta \to 1$ we approach a uniform distribution on $[0, 1]$. As $\delta \to \infty$ we approach the infinite randomness fixed point (IRFP), which describes the asymptotic state of the distribution of $\{J_i\}$ under successive SDRG steps (see 4.3) [6].

4.2 SDRG Procedure

Approximation of the state missing (RSP)

We will summarise the SDRG procedure for the inhomogeneous 1D spin chain following [1]. To begin with, we identify the strongest coupling J_M and consider the energy of this interaction $\mathcal{H}_0$:

$$\mathcal{H}_0 = J_M \vec{S}_l \cdot \vec{S}_r \tag{4.3}$$

The groundstate of this microsystem is

$$|s\rangle \equiv 2^{-1/2} \left(|\uparrow_l \downarrow_r\rangle - |\downarrow_l \uparrow_r\rangle \right) \tag{4.4}$$

Treating this as a perturbation, we can calculate an effective coupling between the spins $l-1$ and $r+1$[1]:

$$J' = \frac{J_l J_r}{(1+\Delta) J_M} \tag{4.5}$$

which identifies a *flow* of couplings from one step to the next:

$$(\cdots, J_l, J_M, J_r, \cdots)_L \to \left(\cdots, \frac{J_l J_r}{(1+\Delta) J_M}, \cdots \right)_{L-2} \tag{4.6}$$

Importantly, given that $J_M > J_{(l/r)}$, the new coupling J' is smaller than either and the energy scale of the model is lowered.

4.3 SDRG Flow

The resulting distribution of couplings in terms of the SDRG step m can be given quantitatively, following [6]. We introduce logarithmic variables:

$$\beta_i^{(m)} \equiv \ln \frac{J_M^{(m)}}{J_i^{(m)}}, \quad \Gamma^{(m)} \equiv \ln \frac{J_M^{(0)}}{J_M^{(m)}} \tag{4.7}$$

[1]Details of the perturbation calculations can be found in [1] as well.

where $J_M^{(m)}$ is the strongest coupling at the SDRG step m. The flow equation to be solved is given by [6] as:

$$\frac{dP}{d\Gamma} = \frac{\partial P(\beta)}{\partial \beta} P(0) \times \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 \delta\left(\beta - \beta_1 - \beta_2\right) P\left(\beta_1\right) P\left(\beta_2\right) \tag{4.8}$$

which is solved with the ansatz:

$$P^*(\beta) = \frac{1}{\Gamma} \exp\left(-\frac{\beta}{\Gamma}\right) \tag{4.9}$$

Equation 4.8 is an attractor for *any* initial distribution of the couplings, not just the distribution described in equation 4.2.

As an initial test of the SDRG procedure we implemented, we have tracked the flow of the logarithmic couplings in figure 4.1. This is a close reproduction of a similar figure in [1] and shows that the SDRG process is indeed working correctly.

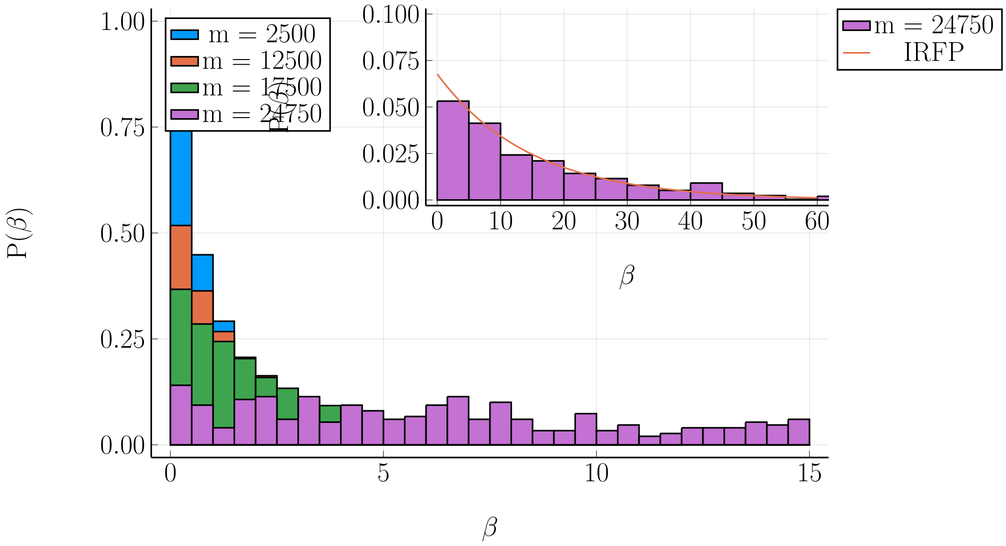

Figure 4.1: SDRG flow for $L = 50,000$ spins in the disordered XXY chain ($\Delta = 1$) where $\delta = 1$. The main plots shows the β distribution of the remaining bonds at the mth step of the SDRG process. The inset plot shows subset of the β distribution late into the SDRG process with the IRFP line overlayed. The data is a good fit to the analytical prediction.

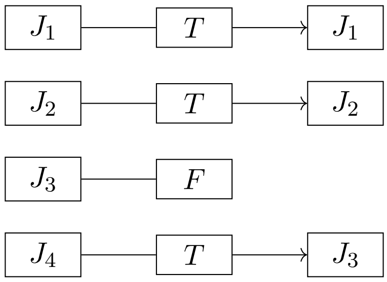

Figure 5.1: Diagram of the data masking procedure used to iteratively eliminate the bonds from data vector. The approach is size stable so minimises the need for allocations during the procedure.

5 Algorithmic Implementation and Complexity

We will briefly detail the main SDRG algorithm used and the way 'counting' was implemented, which is central to the analysis later on. Where necessary, we will add more details about how analyses were implemented as we reach them.

5.1 SDRG Algorithm

In studying disordered systems, we frequently need to take averages over disorder, i.e. many different realisations of the system in question. In many cases the number of trials must be very large for the required observables to converge - see [47] and [1] for examples of these issues. Thus an efficient and reliable algorithm is essential.

The key feature of our SDRG implementation is that it is almost totally memory static - that is, little or no extra memory is allocated in the computer every time a new disorder realisation is run. Rather, the existing memory used to hold data (e.g. bond strengths) is updated at the start of each realisation, and during the elimination process a 'mask' vector is maintained that tracks which bonds are active and should be used in calculation. The one off allocation of this mask vector is computationally very cheap compared to continually reallocating previous memory. An illustration of this approach can be seen in figure 5.1 and a psuedo-code version of our implementation can be seen in table 1.

Algorithm 1: SDRG step algorithm

Data: $\{J\}, L, \{(s_1, s_2)\}$
Result: $\{(s_1, s_2)\}$
$m \leftarrow 1$;
$active \leftarrow \{T\}^L$;
$n\_active = sum(active)$;
while $n\_active > 2$ **do**
  $J_M \leftarrow max\{(J_i\}$;
  $J_l \leftarrow J_{M-1}$;
  $J_r \leftarrow J_{M+1}$;
  $J' \leftarrow \frac{J_l J_l}{(1+\Delta)J_M}$; where do you assign new value J'?
  $active_M \leftarrow F$;
  $active_{M+1} \leftarrow F$;
  $\{(s_1, s_2)\}_m \leftarrow \{(M, M+1)\}$;
  m += 1;
  $n\_active = sum(active)$;
end
$\{(s_1, s_2)\}_{L \div 2} \leftarrow \{(J_1, J_2)\}$;
return $\{(s_1, s_2)\}$;

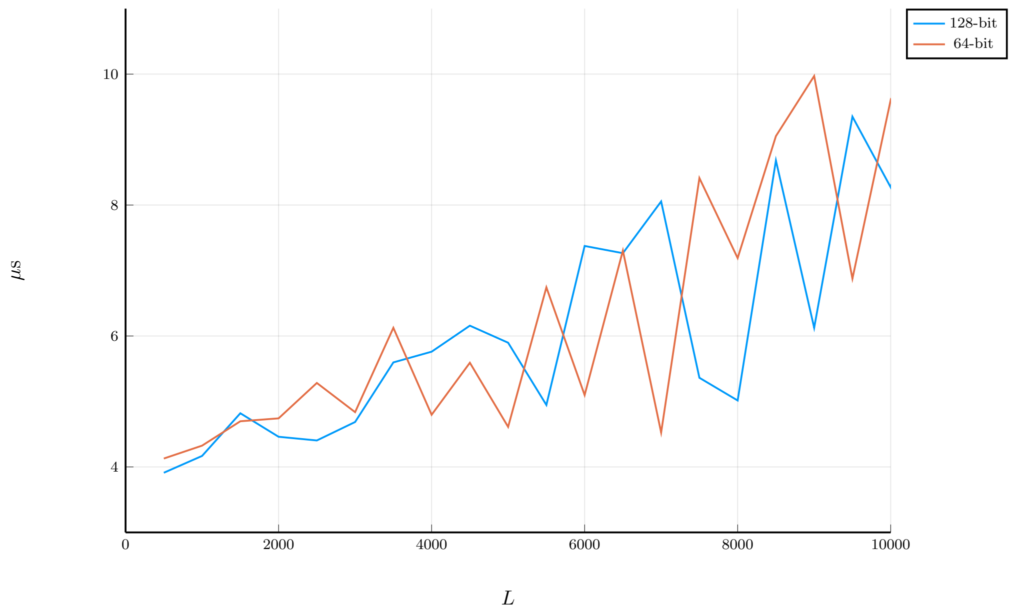

Figure 5.2: Benchmark results for our SDRG procedure. On the horizontal axis we measures the system length L, and on the vertical we report the median execution time for the complete SDRG procedure in microseconds. The execution time increases linearly in the system length for systems L in the order of thousands, and there is only a small performance penalty for using quadruple precision floating point numbers (i.e. 128 bits).

To given an estimate of the efficiency of this procedure, we benchmarked the code for varying levels of machine precision and vary system lengths L. The results can be seen in figure 5.2. The elimination procedure for a system of $L = 1000$ spins is on the order of four microseconds. This implies that upwards of $200,000$ SDRG eliminations can be calculated in one second.

A few numbers of system size L and number of realizations M that one can reach.

6 Existing Results and Verification

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A number of results for the scaling of entanglement measures already exist (see [1], [10]). In this section we will reproduce these results and the in the process, verify that the SDRG algorithm and related code works as expected.

6.1 Entanglement Entropy

As per [1], we measure the entanglement entropy of the disordered XX chain with and without periodic boundary conditions. The results are show in figure 6.1. In this simulation we measure the entanglement entropy of a subsystem of length l located in the left hand side of the the XXY chain (i.e. $\Delta = 1$). The position in the chain is not as relevant as it is in the later models where a strong spatial inhomogeneity is introduced, because the non-locality is much weaker and is not focused around a particular point (e.g. the centre). the number of boundaries matters

For each simulation, we draw L random couplings from the uniform distribution over $[0, 1]$ and run the SDRG algorithm (see 1). The SDRG algorithm returns a vector of tuples of sites, where each tuple represents a pair of spins. It is known from [7] that the ground state of the random spin chain (equation 4.1) is a *random singlet phase* (RSP), made up of $L \div 2$ singlets each entangled in a valence bond, e.g. a valence bond state. Each pair of singlets has the state given in equation 4.4. On this vector of singlet pairs we then run the releveant analysis. In the case of this entropy calculation, for every realisation of the the RSP we measure the entanglement entropy (equation 3.7) for all window sizes l, and maintain a running mean of the result per l. The entanglement entropy is calculated by dividing the system into the subsystem A and its complement A': the entropy is the number of singlets links between these two subsections multiplied by $\ln 2$.

We run the analysis for $L = 1000$ and $L = 2000$, and in each case calculate from $l = 10$ to $l = L \div 2$ with a gap of 10 in between. For the periodic case (figures 6.1a and 6.1c) the finite size effects are smaller, and can be easily corrected with the following two mappings:

$$\ell \to L_c \equiv \frac{L}{\pi} Y\left(\frac{\pi \ell}{L}\right) \tag{6.1}$$

$$Y(x) = \sin(x)\left(1 + \frac{4}{3} k_1 \sin^2(x)\right) \tag{6.2}$$

where $k_1 = 0.115$, given by [48]. For the open case (figures 6.1b and 6.1d) the need is especially acute.

As can be seen in all four subfigures of figure 6.1, entropy scales logarithmically with the subsystem size. This is clearly an area law violation, and perfectly follows the prediction in [7]:

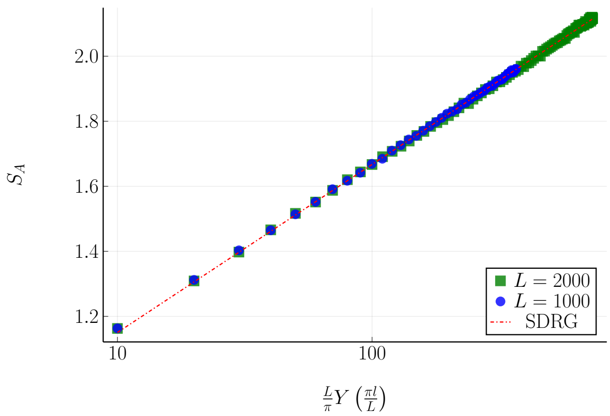
$$S_A = \frac{\ln 2}{\cancel{3}} \ln \ell + K \quad \text{prefactor depends on \# of boundaries} \tag{6.3}$$

It should be noted that figures 6.1c the degree of divergence from the log-linear trend is less than is seen in [1]. This is because we have chosen to calculate the entanglement entropy only up to $L \div 2$ as beyond half the chain, finite size effects will emerge even for the periodic implementation.

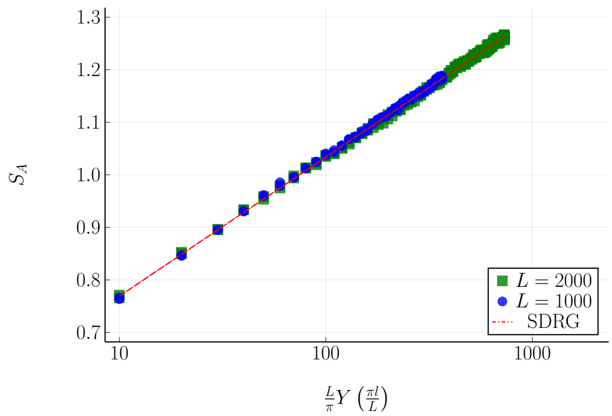
6.2 Logarithmic Negativity

In this section we recalculate the logarithmic negativity spectrum as reported in [1]. As we discussed in section 3, logarithmic negativity is in some sense a superior measure to the entanglement entropy because it is only measurable for pure states. As in [1], we calculate the logarithmic negativity of the one dimensional XXY chain (the same as in section 6.1). The pair of subsystems of length l is taken from the left hand side of the chain and extended in increments of 10 for every disorder realisation. The simulations are for the adjoint case, i.e. $r = 0$ in [1]'s notation.

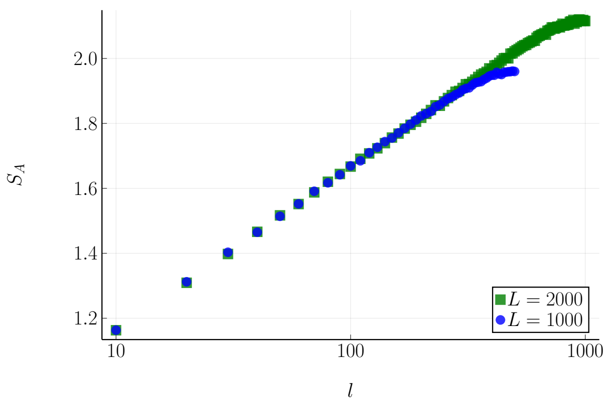
For the logarithmic negativity, we only need to measure the number of singlets shared between the two subsystems in $A = A_1 \cup A_2$, and multiply that by $\ln 2$. We run this analysis on systems of $L = 1000$ and $L = 2000$ for $50,000$ trials, as per the entanglement entropy calculations. We conducted



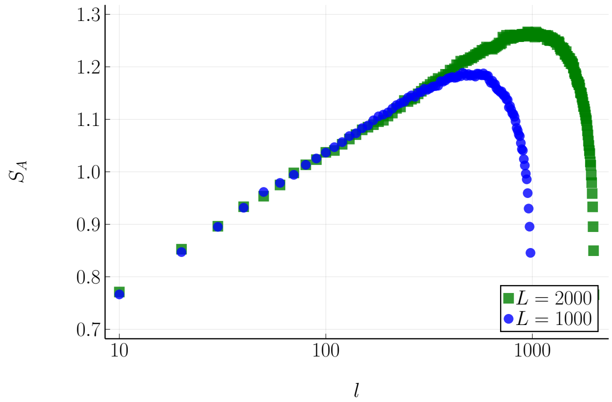
(a) Baseline entropy, PBC



(b) Baseline entropy, OBC

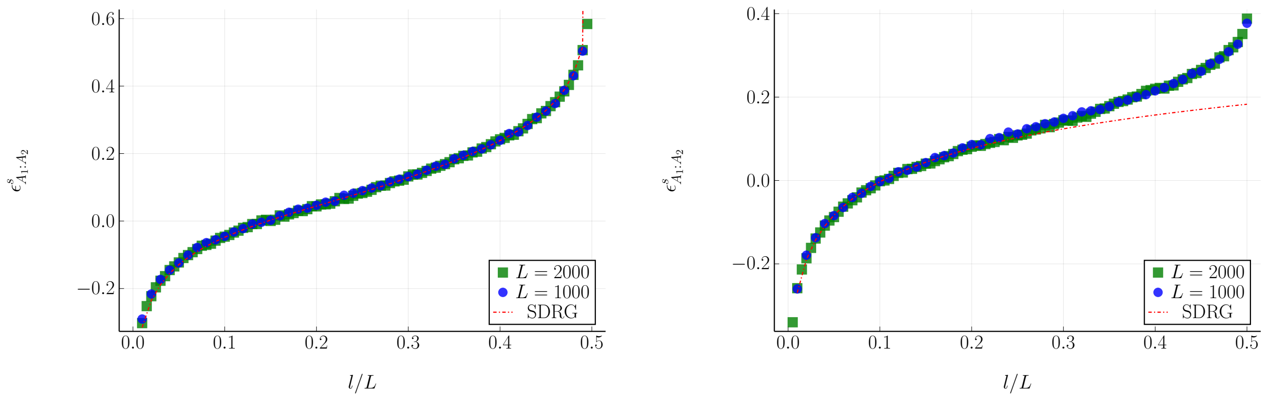


(c) Baseline entropy, no l adjustment, PBC



(d) Baseline entropy, no l adjustment, OBC

Figure 6.1: Baseline entanglement entropy, recalculated from [1]. In all figures, we measure the entanglement entropy of a subsystem of length l located in the left hand side of the the XXY chain (i.e. $\Delta = 1$). Each simulation is run for $50,000$ trials. For implementation details, see 6.1. 6.1a: entanglement entropy of the periodic chain with the adjusted subsystem length L_c. 6.1b: entanglement entropy of the open chain with the adjusted subsystem length L_c. 6.1c: entanglement entropy of the periodic chain with the unadjusted subsystem length l. 6.1d: entanglement entropy of the open chain with the unadjusted subsystem length l.



(a) Baseline negativity, PBC

(b) Baseline negativity, OBC

Figure 6.2: Shifted logarithmic negativity, recalculated from [1]. In all figures, we measure the logarithmic negativity of two adjacent subsystems of length l located in the left hand side of the the XXY chain (i.e. $\Delta = 1$). Each simulation is run for $50,000$ trials. For implementation details, see 6.2. 6.2a: shifted logarithmic negativity of the periodic chain with the subsystem length l. The fit is with equation 6.4. 6.2b: shifted logarithmic negativity of the open chain with the subsystem length l. The fit is with equation 6.5.

the analysis in the OBC and PBC cases. The results can be seen in figure 6.2. In particular we have plotted the shifted negativity as defined in [1]:

$$\mathcal{E}^s_{A_1:A_2} = \mathcal{E}_{A_1:A_2} - \frac{\ln 2}{6} \ln L \tag{6.4}$$

what is the SDRG function in the PBC case?

Which is a perfect fit for the periodic chain (figure 6.2a). For the open chain, we use the following identity for the logarithmic negativity in the case of an infinite chain:

$$\mathcal{E}_{A_1:A_2} = \frac{\ln 2}{6} \ln\left(\frac{\ell_1 \ell_2}{\ell_1 + \ell_2}\right) + k, \tag{6.5}$$

which is a good fit for $\ell/L << 1$.

In both cases (entanglement entropy and logarithmic negativity), it can be seen that the entanglement scales with a logarithmic correction to the area law.

6.3 Randbow Chain

In this section we move on to reproducing the results of [10], which involves the modifying the couplings of the random chain to include an exponentially decaying term:

$$J_i \equiv K_i \times \begin{cases} e^{-h/2}, & i = 0 \\ e^{-h|i|}, & |i| > 0 \end{cases} \tag{6.6}$$

where the K_i terms are randomly distributed coefficients as before. This is known as the 'randbow chain'. The significance of these random couplings is to enforce a *rainbow* phase if the exponential parameter h is strong enough [9] - see figure 6.3 for an illustration.

For the $h \to \infty$ limit, the entanglement entropy of a subsystem A starting in the centre of the randbow chain (properly called the rainbow chain in this phase) scales as a volume law. This is because for any subsystem l, the subsystem of size $l+1$ must contain another singlet link. For example, looking at figure 6.3, if our subsystem A starts as $A = \{J_1\}$, and then we extend it to $A = \{J_1, J_2\}$, these two subsystems have 1 and 2 singlet links in them respectively, and this holds for every other subsystem up to the maximum for $l = L \div 2$. Note that if we position the subsystem centrally in the chain, then the entanglement entropy is always zero for the strong h phase as we only have included 'complete singlets' in our subsystem (see [9] for an illustration, figure 2(b)).

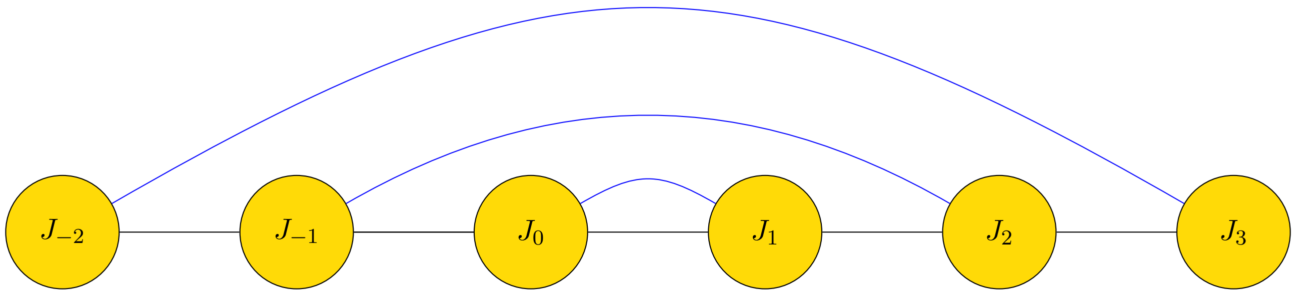


Figure 6.3: A rainbow chain as introduced in [9] and [10]. The central coupling J_0 is by definition the strongest coupling for the clean chain or for sufficiently large h, so it will always be eliminated first. The black edges represent the couplings $\{J_i\}$, whereas the blue edges represent the singlet links. Note that for the rainbow chain to make sense, we must have an odd number of links and an even number of spins in the open chain, hence the asymmetry in the diagram.

For the $h = 0$ phase we obviously recover the random chain and the results from section 6.1 hold. However, for intermediate h we observe a square root correction to the area law (originally reported in [10]). This is presented in figures 6.4 and 6.5. As h increases, the scaling of the entropy gets closer and closer volume law phase, as the effect of the 'rainbow' dominates. As we approach $h = 0$, the effect is weaker and we approach the random phase again. In 6.4 we measure the scaling of the entanglement entropy for a subsystem A of length l with its left edge on the centre of the chain. This positioning guarantees that we cpature the volume scaling as h increases. We run both experiments for $50,000$ disorder realisations for the non-interacting open XX chain.

When we measure the entropy scaling for set ratios of h/δ, we observe a data collapse as first reported in [10]. This is interestingly only the case of the SDRG procedure and does not hold in the exact solution for the XX chain (see section 6.5).

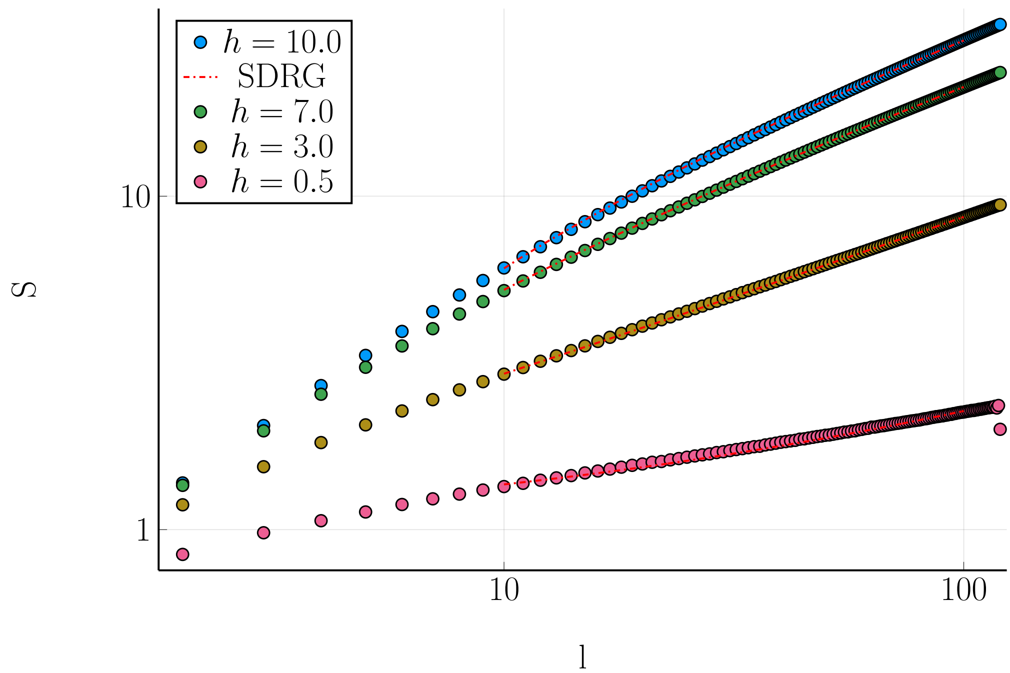


Figure 6.4: Scaling of the entanglement entropy for the open XX randbow chain whilst varying the exponential parameter h. Notice the log-log scaling on each axis. Each experiment is run for $50,000$ realisations of the disorder, and we measure the entanglement entropy of a subsystem A starting at the chain centre (offset by one to ensure the proper scaling) for each realisation. A fitted function of the form $y = a + b\sqrt{x}$ is overlayed for each value of h.

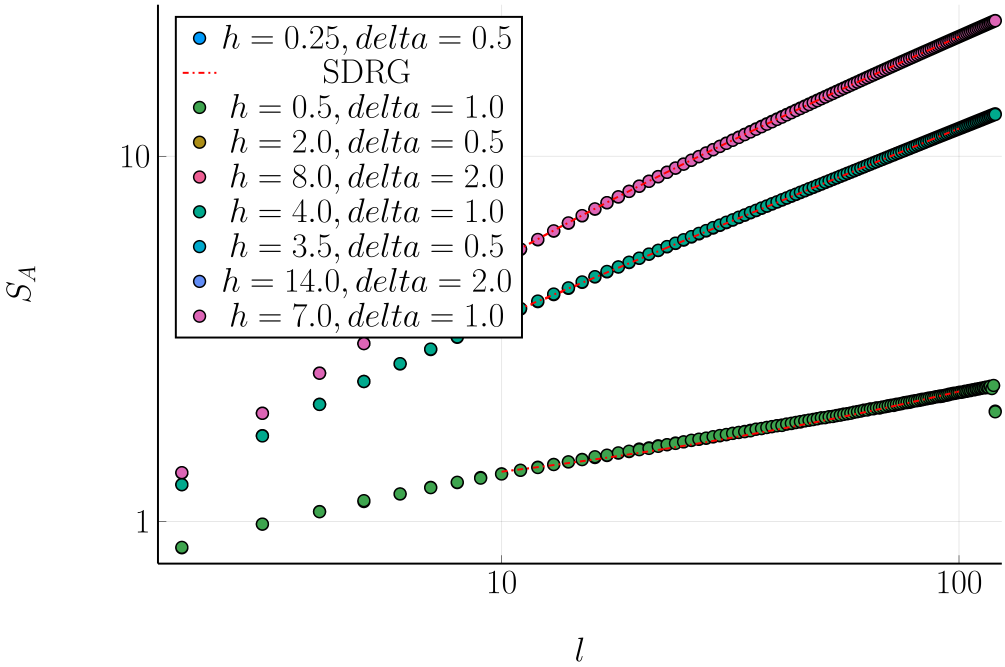


Figure 6.5: Scaling of the entanglement entropy for the open XX randbow chain whilst varying the exponential parameter h and the disorder parameter δ in fixed ratios. Again, notice the log-log scaling on each axis. Each experiment is run for $50,000$ realisations of the disorder, and we measure the entanglement entropy of a subsystem A starting at the chain centre (offset by one to ensure the proper scaling) for each realisation. A fitted function of the form $y = a + b\sqrt{x}$ is overlayed for each value of h/δ.

6.4 Randbow Subregion Analysis

To further verify the findings of [10] and to verify our own SDRG implementation, we recalculated specific elements of the contour analysis performed in that paper. Specifically, to understand the square root scaling we consider the probability densities of the sizes of the rainbow and 'bubble' regions. In any given realisation of the randbow chain, there will be subregions of continuous rainbow links and regions of continuous 'bubbles' (see figure 6.6 for an example).

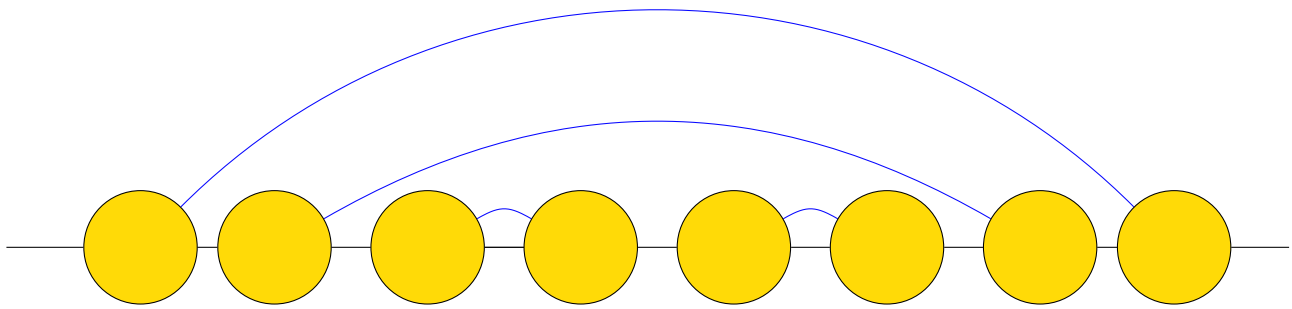


Figure 6.6: A demonstration of the bubble and rainbow subregions as named in [10]. The first two spins represent a randbow region of $l = 2$, and the second four spins represent a bubble region of $l = 4$.

We denote $P_r(l)$ the probability mass of seeing a rainbow subregion of length l, and similarly $P_b(l)$ for the probability of a bubble subregion of length l. We report these probability mass functions in figures 6.7 and 6.8 respectively, both with disorder parameter $\delta = 1$ and for a system size $L = 1000$ in the XX chain. We observe that for the rainbow distribution P_r, the probability of a region of length l decays exponentially in l, and that the rate of decay depends on h. However, for the P_b distribution, we observe a much slower power law decay, implying that there is no characteristic size of a bubble region, and furthermore that this does not depend on h at a scale visible on the plot.

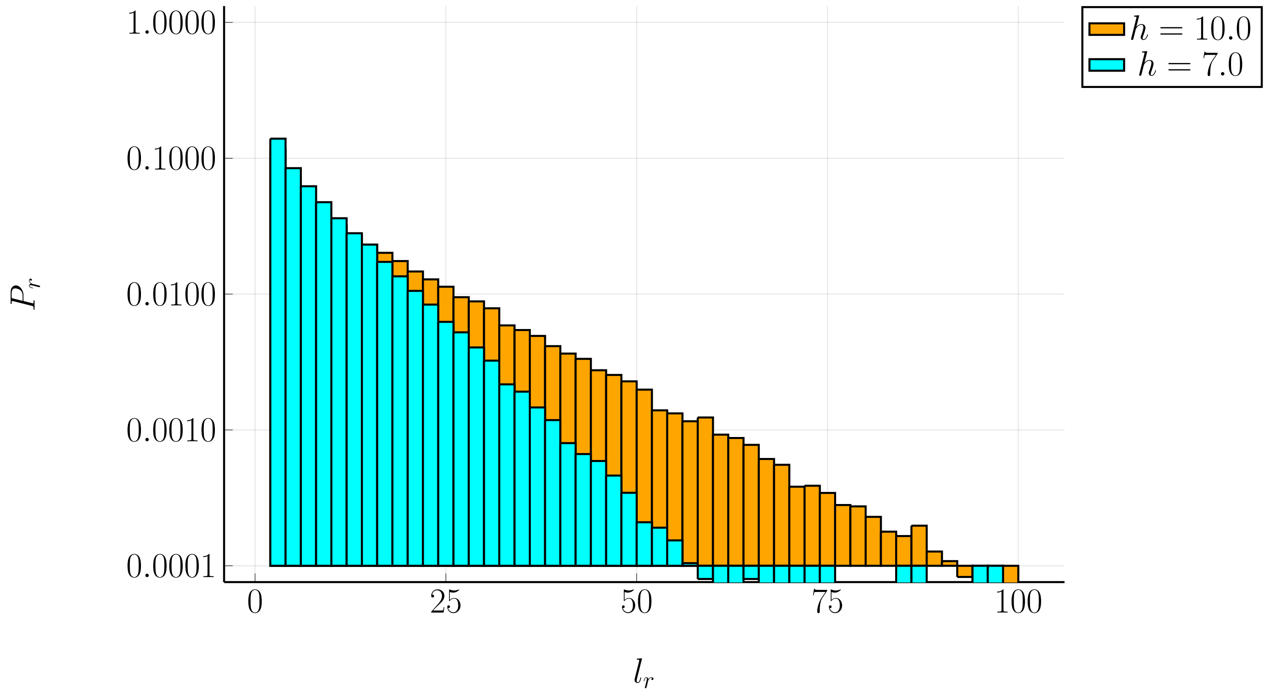


Figure 6.7: Probability density function $P_r(l)$ of the lengths of rainbow subsystems. The data were collected by averaging over 50,000 disorder realisations for $h = 10$ and $h = 7$ in the open XX chain, solved with the SDRG method. Notice the logarithmic scale on the y-axis only, suggesting an exponential decay.

This leads to an argument that the scaling of the entanglement entropy scales as a square root. We presented the following argument, adapted slightly from that given in [10] for clarity. Firstly, we observe from figure 6.8 that $P_b(l) = l^{-3/2}$, and thus:

$$\langle l_b \rangle = \int_2^l dl l P_b(l) \propto l^{1/2} \tag{6.7}$$

Secondly, calling a 'bubble region' a subsystem of consecutive bubbles, and similarly a 'rainbow region', we notice that, on average, the number of bubble regions N_b must be equal to the number of rainbow regions N_r. Furthermore, dividing the total subsystem length l by the average length of a bubble region $\langle l_b \rangle$ gives N_b, and thus:

$$\frac{l}{\langle l_b \rangle} \propto N_b = N_r \tag{6.8}$$

The entanglement entropy is equal to the number of rainbow links in the subsystem multiplied by $\ln 2$, which is equal to the number of rainbow regions multiplied by the average length of a rainbow region:

$$S_A \propto N_r \times \langle l_r \rangle \times \ln 2 \tag{6.9}$$

Bringing together equations 6.8 and 6.9, we get:

$$S_A \propto \frac{l}{\langle l_b \rangle} \times \langle l_r \rangle \times \ln 2 \propto l^{1/2} \langle l_r \rangle \ln 2 \tag{6.10}$$

which suggests the observed square root scaling.

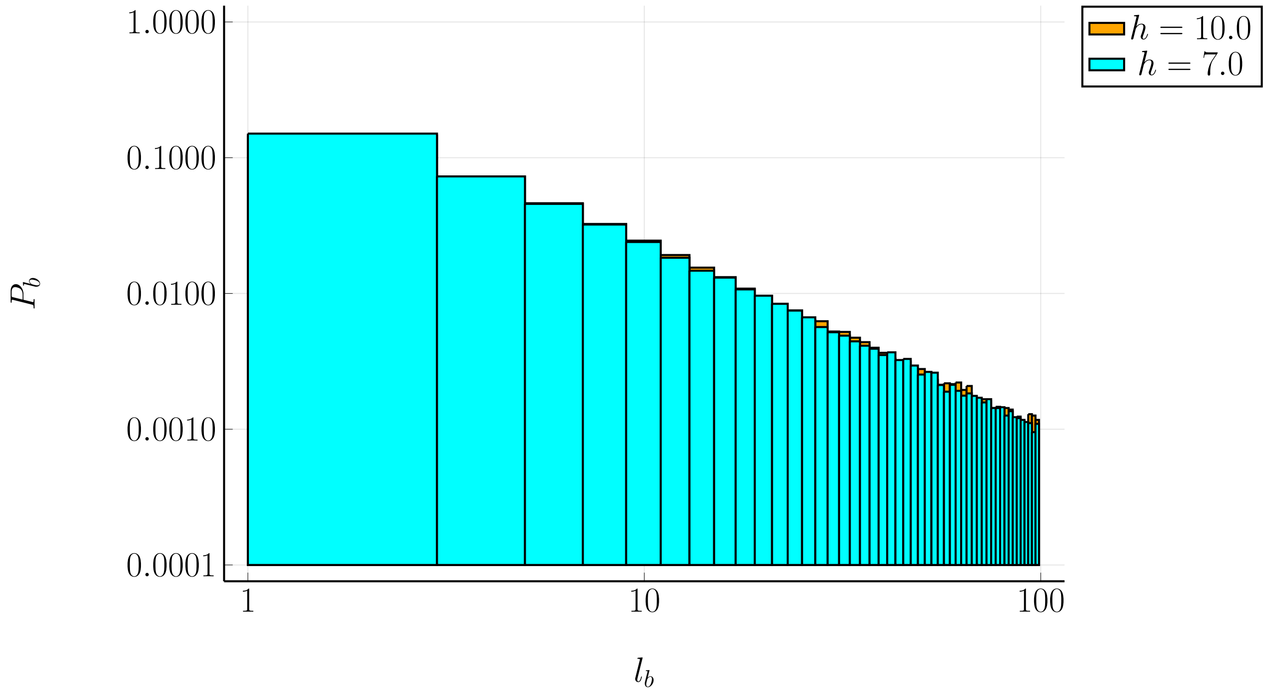


Figure 6.8: Probability density function $P_b(l)$ of the lengths of bubble subsystems. The data were collected by averaging over 50,000 disorder realisations for $h = 10$ and $h = 7$ in the open XX chain, solved with the SDRG method. Notice the logarithmic scale on both axes, suggesting a power law decay.

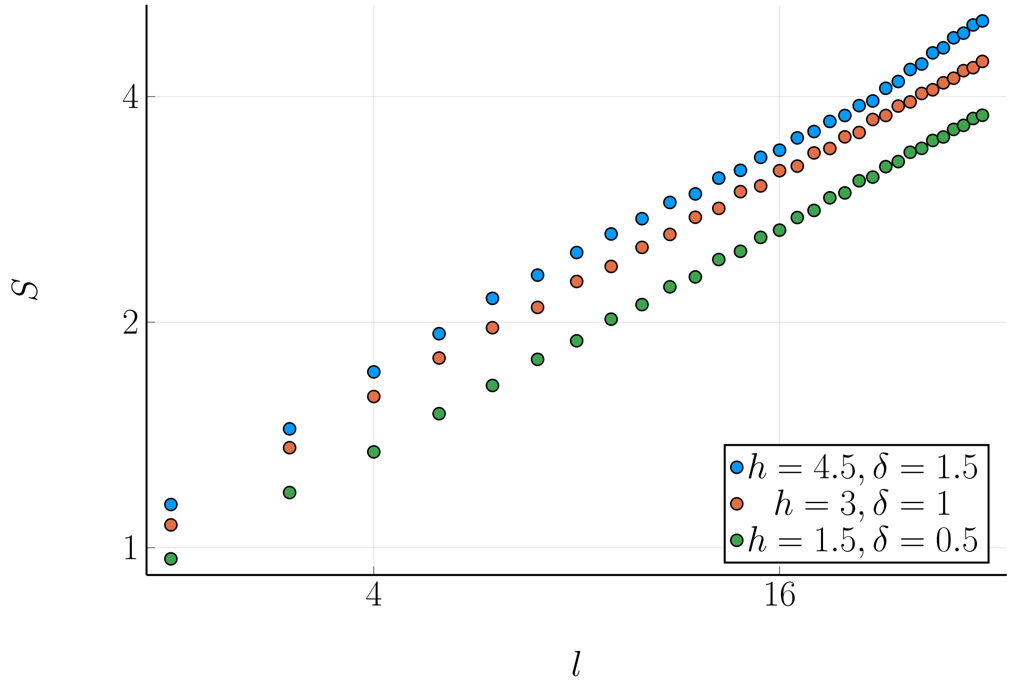
6.5 Randbow Chain Exact Solution

Finally, we analyse the exact solution of the XX chain. This is done by forming a matrix $C_{ij} = \langle c_i^\dagger c_j \rangle$ and taking a subset A of these matrix entries - this forms the partial density matrix with respect to the A subsystem, from which we can calculate the entanglement entropy trivially. For a detail derivation, see appendix B. time analysis here?

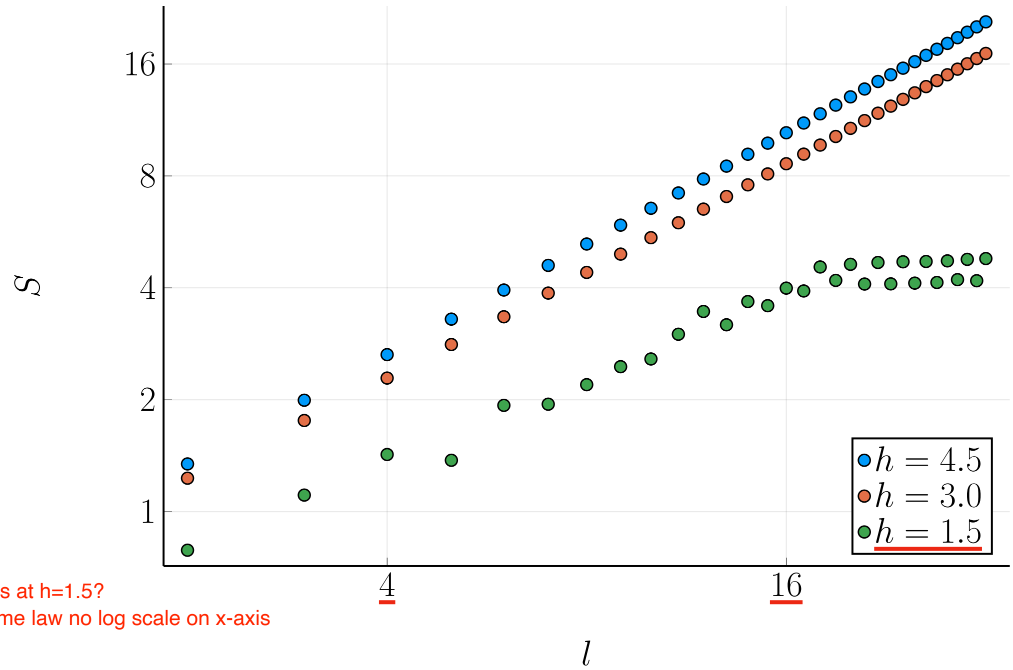
In the following experiments, we calculate the exact solution to the XX randbow chain for a system with $L = 100$ spins. We take an average over 1000 realisations of the disorder for each disordered system and calculate the entanglement entropy from $l = 1$ to $l = 32$ for each realisation. For accuracy these experiments were run with 128-bit floating point numbers, which slows down the computation significantly as most optimised linear algebra routines are designed for 64-bit (or fewer) numbers. The results for the disordered cases are shown in figure 6.9a. Note the $\log_2$ scaling on each axis. The entanglement entropy retains its square root scaling, but we observe the the data collapse onto the h/δ ratios is no longer present, but that an absolute increase in h does increase the entanglement entropy at all scales l. This suggests that the SDRG procedure is not wholly accurate in the large l limit, perhaps because in the tail region of the spin couplings the values of J_i are so small that the second order perturbation used to develop the Dasgupta-Ma rule is no longer accurate.

We also show the results for the clean XX rainbow chain, again with $L = 100$, in figure 6.9b. Here we observe the predicted volume scaling as discussed in section 6.3.

The reason why the collapse does not hold is because there is an overall additive constant which is not taken into account by SDRG
S=n +const =ln l +const



(a) Scaling of the entanglement entropy of the clean open XX chain, calculated with the exact solution to the groundstate problem with varying h and δ. We use $1,000$ trials for each value of h and δ on a system of size $L = 100$. Notice that the data collapse on the ratio h/δ is no longer present.



What happens at h=1.5?
Also, for volume law no log scale on x-axis

(b) Scaling of the entanglement entropy of the clean open XX chain, calculated with the exact solution to the groundstate problem with varying h. We use 1 trial for each value of h on a system of size $L = 100$, with $\delta = 0$. Notice the volume scaling in l.

Figure 6.9: Entanglement entropy scaling of the randbow chain with the exact solution. We calculate the solution for the $L = 100$ chain. notice the log-log plots on both figures. 6.9a: the exact solutions for the XX chain with disorder ($\delta = 1$) for $50,000$ disorder realisations. Figure 6.9b: the exact solution of the clean case. notice the much strong scaling of the entanglement entropy.

7 Entanglement negativity for the randbow chain

7.1 Analytical expectation

In the following sections, we being our extension of the existing results. We start by analysing the logarithmic negativity we expect for an open randbow chain with two adjacent intervals as we vary the subsystem length l. This experimental setup is identical to that seen in 1.2, except that we use the randbow couplings rather than the basic disordered couplings.

For the $h \to \infty$ limit, we would expect the negativity to follow the volume law. The argument is essentially the same as that made for the large h entanglement entropy: for every extra $l+1$, we introduce another singlet link between A_1 and A_2, thus we get a volume law.

For moderate values of h, we can predict a square root scaling via a very similar argument to that presented in 6.4. First, recall that the logarithmic negativity is simply the number of singlet links between the two subsystem, multiplied by $\ln 2$. Secondly, we assume that for moderate h, the groundstate of the SDRG procedure is symmetric with respect to the chain centre. This is reasonable for moderate h and is supported by further arguments with respect to the flow of elimination presented in [10].

This allows us to consider just one of the two A subsystems (as we know that the other will be identical expect for a reflection in the indices i). Any rainbow links emerging from the subsystem A_2 will be going to the subsystem A_1 and vice versa [2], and thus will contribute to the logarithmic negativity. To understand the scaling of the logarithmic negativity, then, we only need to understand the scaling of the entanglement entropy of the subsystem A_2, which we know to be a square root scaling.

It is important to point out that this argument, based on that given in [10], only holds in the non-interacting case. We will see in the numerical evidence that that is indeed the case and the interacting model shows a saturating behaviour that cannot yet be predicted analytically.

7.2 SDRG results

We measure the logarithmic negativity of two adjoint subsystems in the XX and XXY randbow chains, with $L = 1000$ and 2000 in both cases. We use the SDRG procedure for 50,000 realisations of the disorder each, and plot the regular (i.e. not shifted as per 6.2) logarithmic negativity as a function of the adjusted subsystem length l L. In both cases we use the parameters $h = 1, \delta = 1$ for the moderate inhomogeneity regime. Our results are shown in figure 7.1.

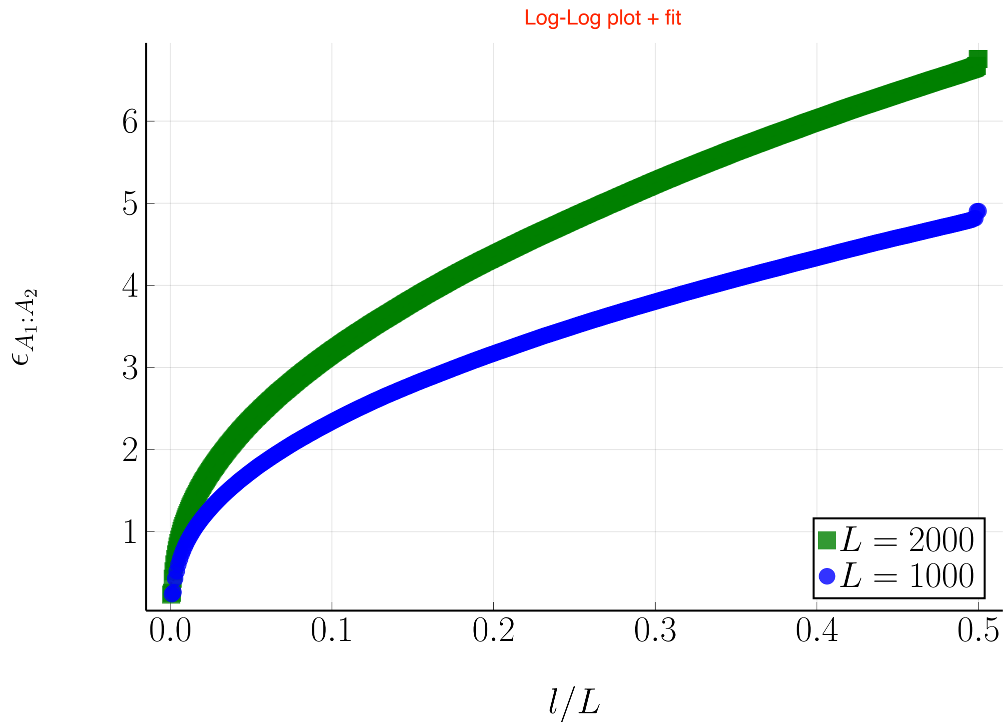
As can be seen from figure 7.1a, in the XX case the square root scaling is captured perfectly. Furthermore, for larger values of l we notice that the logarithmic negativity is higher for a given fraction of the chain. This is to be expected: for a longer chain, there will be more singlet links in a subsystem for length $0.5L$, etc.

In the XXY model, we observe the same saturation behaviour reported for the entanglement entropy in [10]. This is to be expected given the the argument made above in section 7.1: to calculate the logarithmic negativity we only need to know the entanglement entropy, and we already expect this to saturate in l.

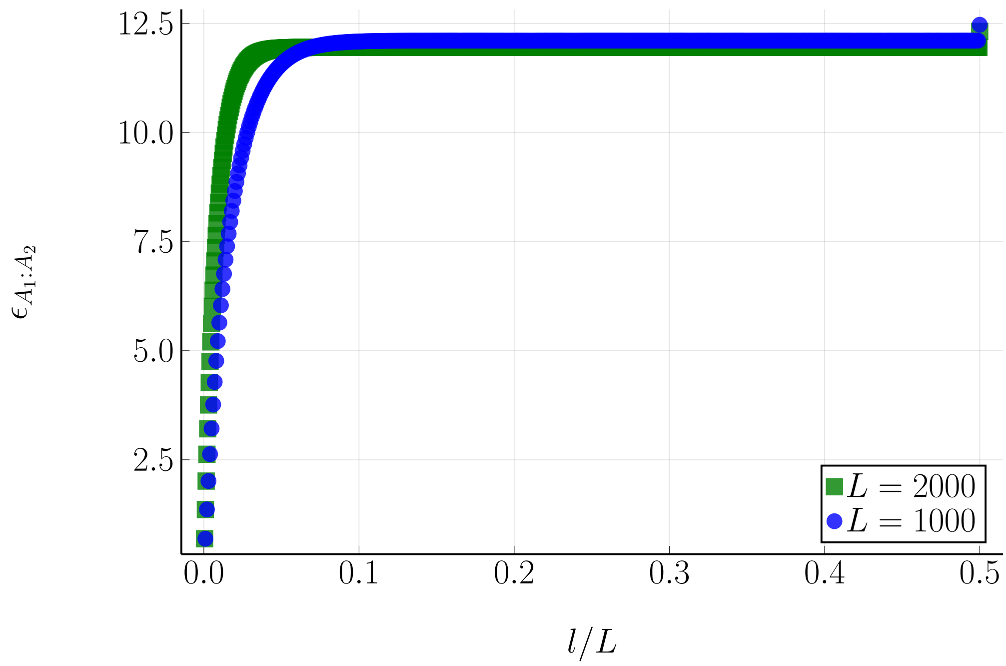
It is interesting to note that the saturation implies a return to the area law regime, and to consider why this occurs only in the interacting case. It is argued in [10] that the bubble regions in the interacting model are much more stable, which invalidates the heuristic argument presented for the square root scaling and leads to a saturation, because there is not enough space in the subsystem for rainbow links. The details of this argument are beyond the scope of this report but are covered in more detail in [10].

As an additional measure of the rate at which entanglement decays from the centre of the chain, where bonds are very strong, we measure the logarithmic negativity as we vary the interval r between two adjacent intervals in the XX and XXY chains. We consider only even r, with the interval spaced evenly over the centre of the chain (see figure 1.2 for a visualisation) with $L = 1000$. We observe that

[2] We ignore the possibility that a rainbow link could leave A_2 'to the right' and attach to the remainder of the chain. It is shown in [10] that this is a very accurate approximate from moderate h.



(a) Logarithmic negativity for the open randbow XX chain.

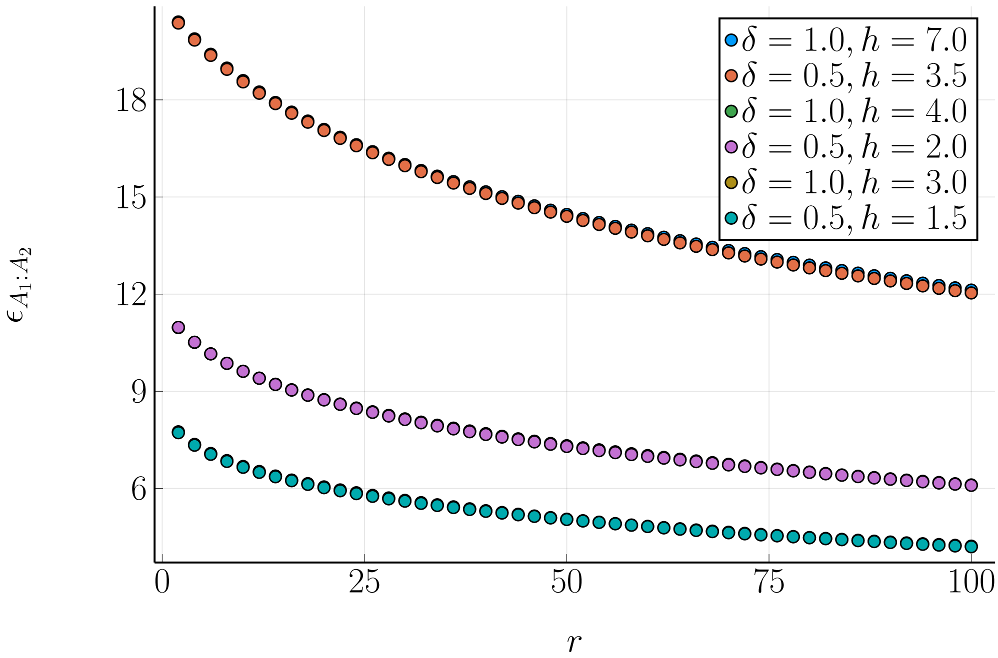


(b) Logarithmic negativity for the open randbow XXY chain.

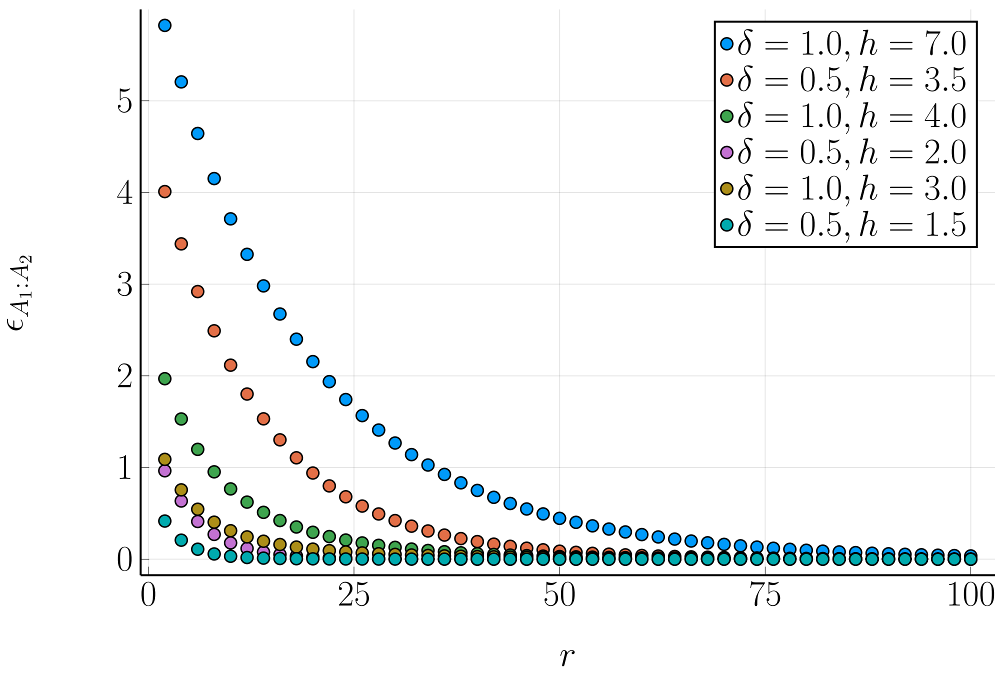
Figure 7.1: Logarithmic negativity scaling in the open randbow chain for adjoint subsystems. We set $h = 1, \delta = 1, L = 100$ in both figures and measure the logarithmic negativity over $50,000$ disorder realisations. We place the subsystems A_1 and A_2 in the centre of the chain, i.e. with the right hand edge of A_1 next to the left hand edge of A_2. In both figures we run the experiment for $L = 1000$ and $L = 2000$. 7.1a: scaling in the XX regime. A square root scaling is observed, reflecting analytical expectations and the results from 6.5. 7.1b: scaling in the XXY regime, $\Delta = 1$. We observe that the logarithmic negativity saturates quickly in both system lengths.

the in the XX case, the logarithmic negativity decays relatively slowly as r increases. This is to be expected given the previously discussed stability of the rainbow regions in the XX case relative to the bubble regions. Furthermore, we notice a strong data collapse onto the ratios h/δ, just as for the previous measures of entanglement.

However, in the interacting XXY case, the logarithmic negativity decays far more quickly, as expected from the saturating behaviour we see in figure 7.1b. This is a corollary of figure 7.1b: as we extend further into the extremes of the chain, the entanglement is generally local and we lose long distance information.



(a) Randbow central negativity with varied r, XX, OBC



(b) Randbow central negativity with varied r, XX, OBC

Figure 7.2: Logarithmic negativity scaling in the open randbow chain for disjoint subsystems as we vary r. We measure the logarithmic negativity over $50,000$ disorder realisations. We place the subsystems A_1 and A_2 in the centre of the chain separeted by an even interval r, located in the centre of the chain. $L = 1000$ and $l = 100$ in both figures. 7.2a: the XX chain. The logarithmic negativity decays relatively slowly, in line with figure 7.1a. We also observe a strong data collapse onto the ratio h/δ. 7.2b: scaling in the XXY regime, $\Delta = 1$. The logarithmic negativity decays much more quickly, in line with figure 7.1b, and there is no observable data collapse other than the $r \to 0$ limit in which all entanglement is lost.

8 Power law systems

In [9], it is mentioned that the couplings must decay relatively quickly in order for the complete rainbow chain to be enforced. In particular they suggest that:

$$J_i = \epsilon^{\alpha(i)} \tag{8.1}$$

where $\alpha(i)$ is monotonically decreasing[3]. To explore how sensitive the scaling of the rainbow and randbow phases is to the speed of this decay, we consider a new power-law system with couplings given by:

$$J_i \equiv K_i \times \begin{cases} \underline{2,} & i = 0 \\ |i|^{-\alpha}, & |i| > 0 \end{cases} \tag{8.2}$$

In the follow sections, we discuss the analyitcal expectations for the scaling of the entanglement entropy and the logarithmic negativity in the case of the power-law couplings.

8.1 Power law systems: analytical expectations

In [10], it is possible to derive analytical expectations for the randbow chain only in the strong inhomogeneous limit $h \to \infty$ (see in particular section VI.A). This is because only in the inhomogeneous limit (and with symmetric J_i) can we be sure the the central bond will always be eliminated first and that the elimination process will be symmetric with respect to the chain.

Unfortunately, for the power-law system, we cannot guarantee that the central bond will be eliminated first, nor that the process with be symmetric. The reason is that the effect of the power-law component is not enough, relative to the effect of the $\mathcal{O}(1)$ disorder factor, to rapidly reduce the consecutive couplings. Given that, as discussed in section 4.2, the elimination rule always reduces the energy scale, the couplings around the elimination site must be small enough to still be smaller than the new J' coupling. In the power-law system, this can be guaranteed to be the case. Can we check that rainbow phase is broken even for high alpha?

Therefore we cannot make any quantitative predictions about the scaling of the entanglement entropy or the logarithmic negativity for the power-law system. However, given that we do not expect the couplings to decay quickly enough to maintain a rainbow phase, we might expect that the power-law system will behave at least asymptotically as the simple disordered system. Thus we can make the following general hypotheses:

1. The entanglement entropy will scale in a manner similar to the simple disordered spin chain as seen in section 6.1.

2. The logarithmic negativity will also scale in a manner similar to the simple disordered spin chain as seen in section 6.2.

We will explore these results within the SDRG framework and with the exact solution.

8.2 Entanglement Entropy: SDRG and exact results

We start by calculating the scaling of the entanglement entropy of the open XX power-law chain with $L = 1000$ and $L = 2000$. We use $\delta = 1$ in all of our <u>experiments</u> realizations/simulations and position the subsytem A with the left hand edge on the centre of the chain, the properly capture the scale as in the randbow experiment (see 6.3). We run all of the experiments for $50,000$ disorder realisations and the results are shown in figure 8.1.

In both figures, we can see that, as expected, the entanglement entropy scales similarly to the simple disordered model. For very low l we notice that the entanglement scales very quickly, which suggests that the rainbow phase does survive for at least the first few eliminations on average. After the initial phase, the scaling becomes quasi-logarithmic, confirming the hypothesis in section 8.1.

[3] In [9] they say '$\alpha(i)$ is a function that is monotonically increasing' but this would not given the desired result, as couplings would increase exponentially from the chain centre!

Furthermore, we notice that for larger values of α, the entanglement entropy is generally larger. This accords with our intuition that for stronger inhomogeneity, the rainbow regions should be more stable and they are the regions that contribute to the entanglement entropy. Lastly, we observe some finite size effects for $l/L \approx 1/2$. We have not attempted to fit any of the analytical curves from the simple disordered model due to the very different low l region.

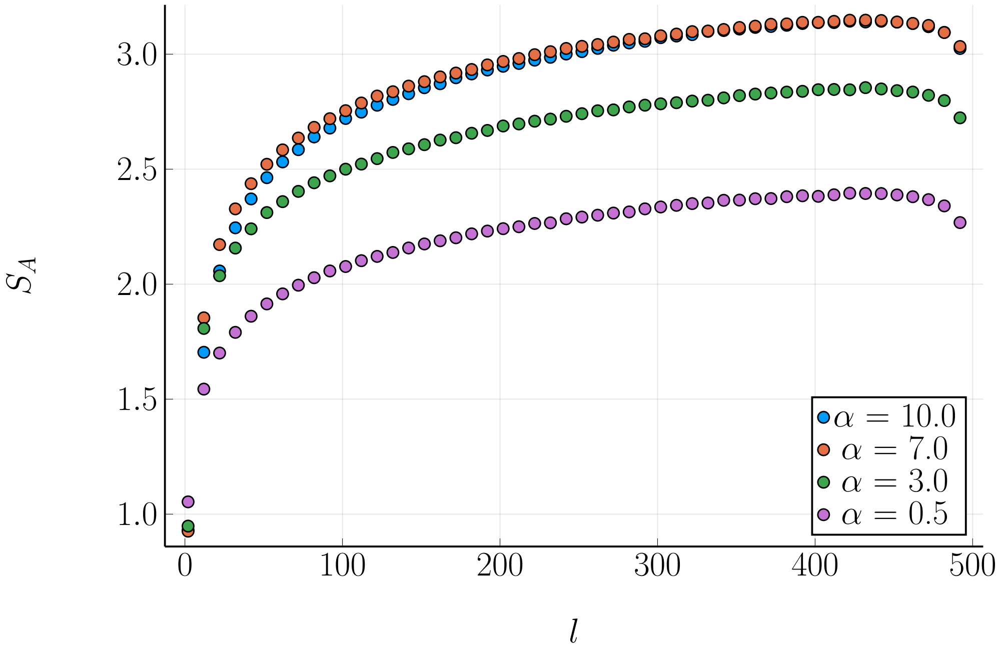
Furthermore, we measure the entanglement entropy via the exact solution, as we did for the randbow chain and as detailed in section B. Over 1000 disorder realisations, we measure the scaling of the entanglement entropy of the XX power-law chain for an open chain of length 100. Our results are shown in figure 8.2. Once again, the scaling of the entropy is logarithmic (notice the log-log scaling) and this matches the general scaling pattern observed in figures 8.1a and 8.1b. Interestingly, we observe a strong data collapse onto the ration α/δ, which accords with our intuition that the power-law system is qualitatively different to the randbow system, which did not show a data collapse in the exact solution.

8.3 Logarithmic negativity scaling

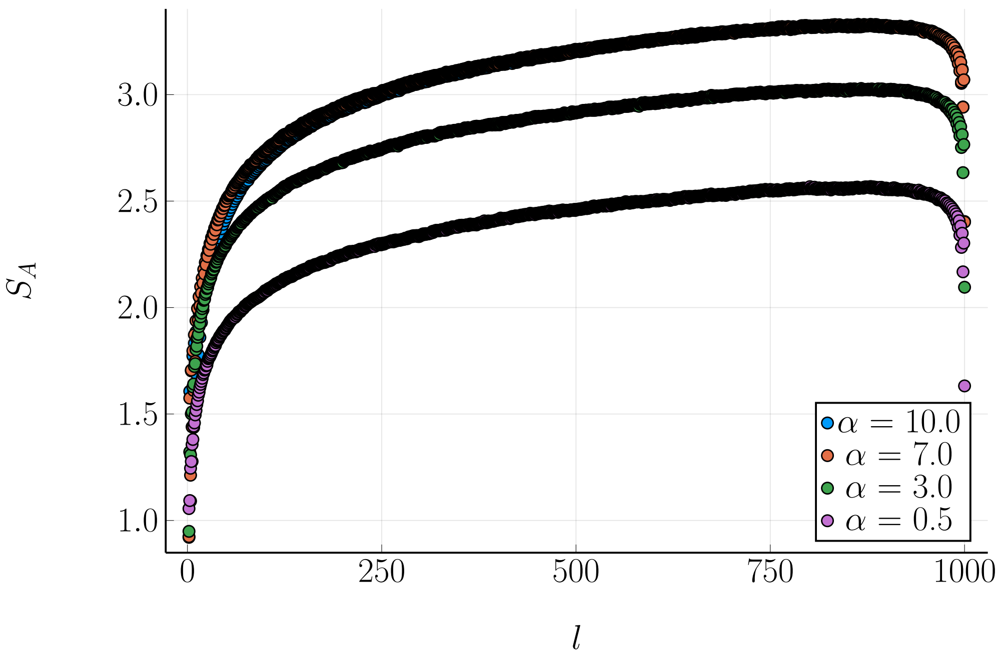
In this section, we look at the logarithmic negativity of the power-law system. Specifically we measure the logarithmic negativity of the open XX and XXY power-law chains for $L = 1000$ and $L = 2000$. We run each experiment for $50,000$ disorder realisations, and our results are shown in 8.3.

For both values of L, we observe that the scaling is identical to that of the simple disordered chain. This confirms our earlier hypotheis about the power-law systems. Interestingly, the curve for $L = 2000$ is higher in both the XX and the XXY experiments than the $L = 1000$ curve. This is best explained by the presence of some persistent rainbow regions even in the power-law model: in terms of absolute logarithmic negativity, half of the $L = 2000$ rainbow chain will contain more singlet links between the two subsystems than the equivalent subsystem of a relative size for $L = 1000$.

Finally, we calculate the scaling of the logarithmic negativity for a varying interval r between two subsystems, in a manner identical to that discussed in 7.2. Our results for the open XX and XXY chains with disorder $\delta = 1$ are reported in figure 8.4. We observe that, much like the interacting randbow chain, the logarithmic negativity decays much more quickly than in the randbow chain, which we would expected given the relative instability of the rainbow regions.



(a) Powerbow entropy via SDRG, $L = 1000$, varying α, OBC



(b) Powerbow entropy via SDRG, $L = 2000$, varying α, OBC

Figure 8.1: Entanglement entropy for the open XX power-law system, calculated via the SDRG method. We $L = 1000$ and 2000. In both figures, we calculated the entropy of a subsystem A with the left edge on the centre of the chain to capture the proper scaling over $50,000$ disorder reaslisations. For each system size we ran the experiment with a different parameter α as per equation 8.2. Figure 8.1a: $L = 1000$. Figure 8.1b: $L = 2000$.

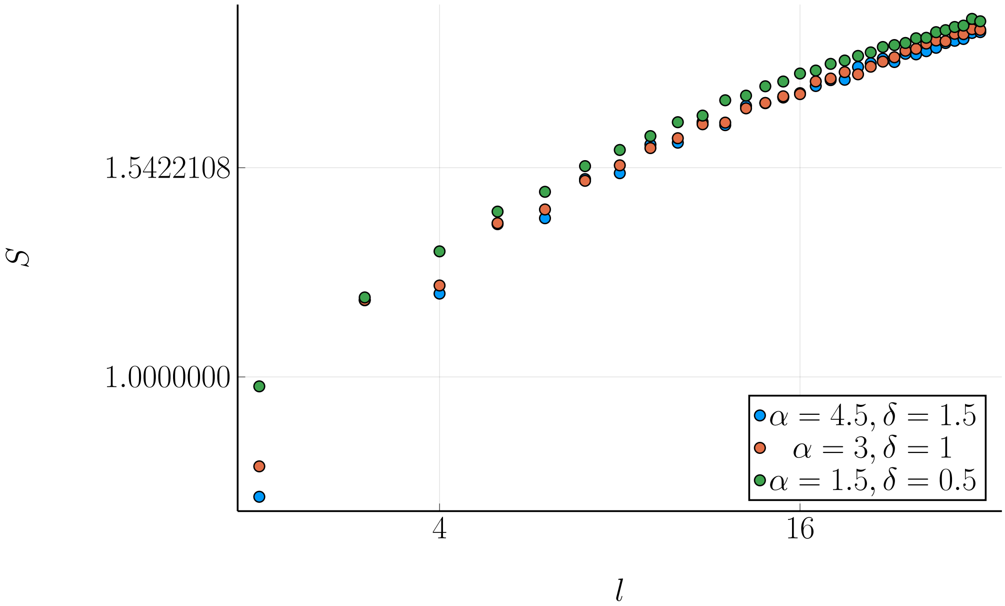
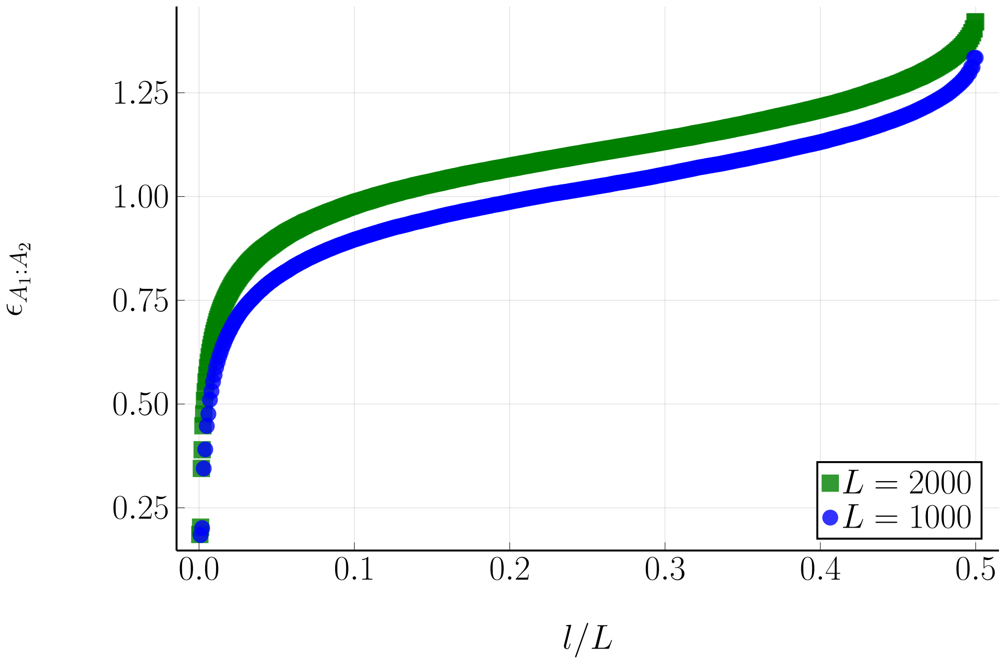
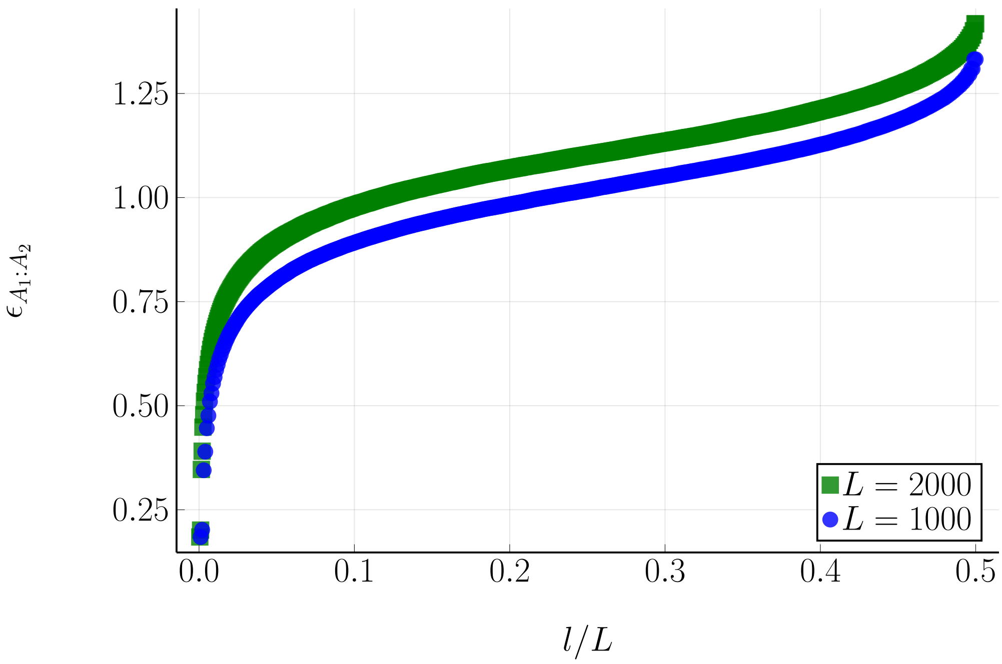


Figure 8.2: Entanglement entropy of the open XX power-law system, calculated with the exact solution over $1,000$ disorder realisations with $\delta = 1$. For each experiment we vary the α. The system length is $L = 100$. Notice the log-log scale.

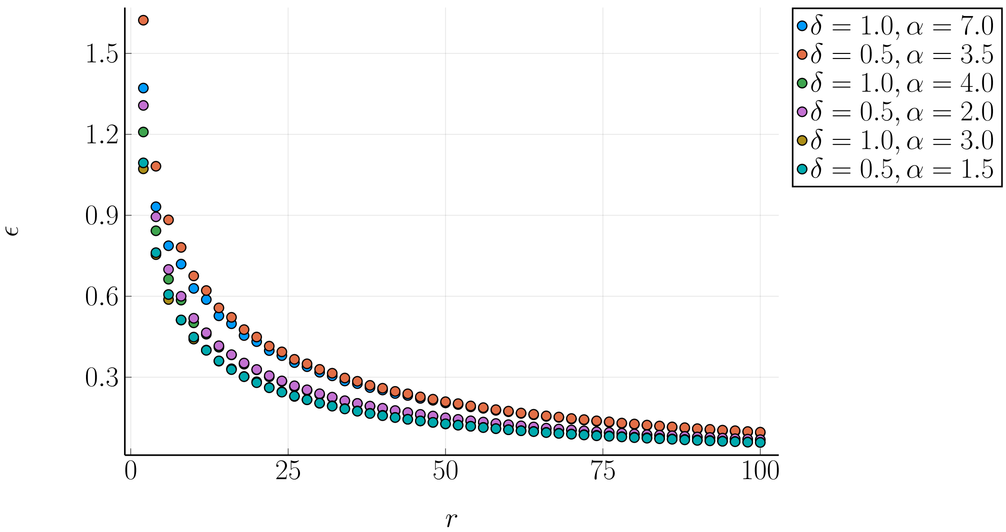


(a) Scaling of the logarithmic negativity, XX, OBC

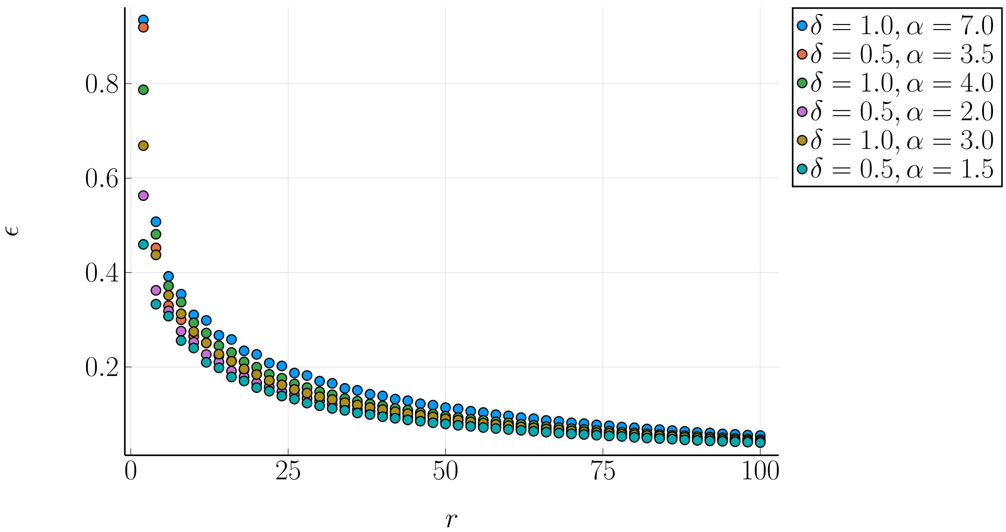


(b) Scaling of the logarithmic negativity, XXY ($\Delta = 1$), OBC

Figure 8.3: Scaling of the logarithmic negativity of the open power-law system. In each figure we calculate the negativity over $50,000$ disorder realisations whilst keeping $\alpha = 1, \delta = 1$. Figure 8.3a: logarithmic negativity for the XX chain. Figure 8.3b: logarithmic negativity for the XXY $\Delta = 1$ chain. Notice that in each subfigure, the $L = 2000$ curve is shifted upwards relative to the $L = 1000$ curve.



(a) Logarithmic negativity with varying r, XX, OBC. Notice in this case that there is a data collapse for small r onto the ratio
$alpha/\delta$.



(b) Logarithmic negativity with varying r, XXY ($\Delta = 1$), OBC. Notice in this case that there is no obvious data collapse for small r.

Figure 8.4: Scaling of the logarithmic negativity of the open XX and XXY power-law systems as r is varied. We restrict r to even intervals and the position is the same as per figure 7.2. $L = 1000$ and $l = 100$ in both figures. We notice that the logarithmic negativity decays very quickly compared to the non-interacting randbow case, figure 7.2a.

9 Conclusion

In this report we have confirmed some key findings in the literature on entanglement scaling in random systems, as well as exploring new measures and new systems (e.g. the power-law system). In this section, we will summarise our results and consider some directions for future work.

9.1 Logarithmic negativity for the Randbow system

In section 7 we should that the scaling of the logarithmic negativity of the randbow chain was almost identical to the scaling of the entanglement entropy. In the non-interacting, XX case, we observe a square root scaling that correspondes to the XX case for the entanglement entropy. For the interacting XXY case, we see that the entanglement entropy quickly saturates, which reflects the stability of the bubble regions in this regime. We similarly note that for fixed subsytem sizes but increase intervals r, the negativity of the interacting model scales far more quickly than the non-interacting model, which is is corollary of the previous result.

9.2 Entanglement scaling for the power-law system

For a system with couplings dsitributed according to a noisy power-law, the scaling of the entanglement entropy and the logarithmic negativity changes significantly. The entanglement entropy at first scales quickly with l, as the short rainbow phase dominates and effects a volume law. However, this rainbow phase is very unstable due to the relatively slow decaying of the couplings, and from that point onwards a logarithmic scaling is observed, more in line with the simple disordered model. This was corroborated with the scaling of the exact case for a smaller system size.

For the logarithmic negativity, we observed that the power-law system scales as expected in a very similar way to the simple disordered system. This still displays an area law violation.

9.3 Limitations and Future Work

We have confirmed that area law violations can be achieved even in systems with less spatial inhomogeneity that than exponentially decaying randbow model. This contributes another step in our understanding of one dimensional quantum complex system. However, we were not able to establish any analytical results for the power-law system. This is not necessarily intractable given more time, but the power-law expression is not as analytically useful as the exponential expression, so any solution will probably not take the same form as the details of [10]. One could also argue that a better, more rigorous argument for the square root scaling of the randbow entanglement entropy is needed, but the data match very well to the heuristic used here and in [10] that we do need see this as being as important as any sort of quantitative theory of power-law systems in general.

A more exhaustive treatment could have been made of the different ways of measuring the entanglement entropy and logarithmic negativity of the power-law system - for example, calculating the exact solution for the entanglement entropy calculations for a different value of α/δ. However, we have managed to present the main results and we plan no publishing the tools used for these experiments so that they can be verified later if necessary.

In our opinion, the biggest lacuna in this scheme of work is a physical explanation for the breakdown of the area law violation when we consider the interacting case of the randbow and power-law systems. Whilst in [10] there is a good quantitative argument given in terms of projecting the SDRG procedure onto a random walk, it remains the case that we do not have a clear understanding of what happens locally in the $\Delta = 1$ case (furthermore, we have not consider values of Δ outside of 0 and 1, which could also be the subject of future work). Given that the interactions take us back into an area law phase, which we know is the result of strictly local Hamiltonians, it suggests that the interacting systems could be mapped into a strictly local system for large l and that that could give some further insight.

A Quantum Mechanics: A Brief Summary

Quantum mechanics is built on *state vectors* that have a different notion of state to classical state vectors. In a classical system, a state $s(t)$ at time t describes how the system would be observed if an experiment were conducted at time t. The mapping of moments in time to experimental outcomes is one to one: this grounds the idea of *information* in classical mechanics. To have information about the state of a system means that you know what result you will get if you observe the system.

However, in quantum mechanics, the mapping of states to experiment outcomes is now one to many. A *quantum state* does not, in general, tell us what state we will observe the system in at t - rather, it encodes a range of possible states that the system could be in when observations are drawn from experiment. The result of the experiment will not be known until the experiment has taken place.

A.1 States and Amplitudes

To demonstrate, we will start with a quantum state ψ that can be 'observed' in two possible outcomes: yes and no. ψ is a vector in $\mathbb{C}^2$, and $\psi(x)$ is the *probability amplitude* of finding ψ in state x. The probability amplitude is the working data of a quantum system; the probability that ψ is in state x is $|\psi(x)|^2$. We will represent yes with the vector y and no with the vector n. The 'overlap' of ψ on the state yes is the inner product (y, ψ), and the probability that ψ is in state y when observed is $|(y, \psi)|^2$.

Because the inner product form occurs so frequently, and because it is helpful to distinguish between column vectors and complex conjugated row vectors, we will use bra and ket form:

$$\boldsymbol{\psi} = |\psi\rangle \tag{A.1}$$

$$\boldsymbol{\psi}' = \langle\psi| \tag{A.2}$$

When $|\psi\rangle$ is describing a single particle in space, and especially when it is time dependent, it is generally called a *wave function*. This is because the state vector $|\psi\rangle$ describes a matter wave (of probability amplitude) in space.

The space of quantum states is a *vector space*. For example, a state could be equal proportions yes and no. This is the idea of *superposition*, the quantum phenomenon of objects being in two states (e.g. places) at once. It is often helpful for this space to be given an orthogonal basis, where 'states' in the classical sense are represented by orthogonal vectors. For example, in $\mathbb{C}^2$, we could have the basis $\{|y\rangle, |n\rangle\}$, and a vector could be $\frac{1}{\sqrt{2}}(|y\rangle + |n\rangle)$, equally yes and no.

We will also use *density matrices* throughout this work, where the density matrix ρ of a state is the matrix:

$$\rho = |\psi\rangle\langle\psi| \tag{A.3}$$

A.2 Normalisation and Unitary Operators

However, for the system to be meaningfully probabilistic, the probabilities associated with each state must be normalised, i.e.:

$$\sum_x \left|\,|\psi(x)\rangle\,\right|^2 = 1 \tag{A.4}$$

This in turn implies that that any function that updates the state of our system from time 0 to time t must maintain the normalisation condition. This suggests that such an map U must be norm preserving and thus orthogonal, which in $mathbbC^n$ means:

$$UU^* = U^*U = I \tag{A.5}$$

That is, the map is unitary.

A.3 Time Evolution

We will give a brief, intuitive derivation of the Schrödinger equation[4]. For a time dependent system, we have:

$$U(t)\,|\psi(0)\rangle = |\psi(t)\rangle \tag{A.6}$$

Now assuming that U can be expanded to first order, we can for small ϵ say:

$$U(\epsilon) = I + O(\epsilon) = I - i\epsilon H \tag{A.7}$$

where we have introduced the prefactor $-i\epsilon$ in front of H for convenience. H is thus the first order expansion of $U(t)$ without the prefactors.

Bringing this all together with the definition of U in equation A.6, we have:

$$|\psi(\epsilon)\rangle = U(\epsilon)\,|\psi(0)\rangle = |\psi(0)\rangle - i\epsilon H\,|\psi(0)\rangle \tag{A.8}$$

Rearranging and dividing by ϵ, and relaxing the assumption that our state started at time 0 we have

$$\lim \epsilon \to 0 \frac{|\psi(\epsilon)\rangle - |\psi(0)\rangle}{\epsilon} = \frac{\delta\,|\psi\rangle}{\delta t} = -iH\,|\psi\rangle \tag{A.9}$$

This is the time independent Schödinger equation, which will be useful in section **??**. H is the Hamiltonian of the system, which will be discussed further in section A.5.

A.4 Observables

Given a state $|\psi\rangle$, we generally want to know something about it - for example, its position, momentum, spin, energy, etc.. In quantum mechanics we use *observables* to extract these from states. Observables are represented by Hermitian linear operators[5], and their eigenvalues are the values that we can observe through experiment.

The combination of linear operators and vector spaces of states gives rise to the following useful summary[6]:

Properties of a Hermitian Operator	**Properties of an Observable A**
All the eigenvalues of the operator are real.	The values of the observable are real.
The eigenvectors of the operator are orthogonal.	The states of an observable are distinct.
The eigenvectors form a basis for the state space.	The possible values of the observable cover all of the possible values that could be observed for this system.

A.5 Hamiltonians and Groundstates

In section A.3, we defined the Hamiltonian H as part of our approximation of the time evolution operator U. The Hamiltonian is the oberservable for the total energy of the system. In dynamical systems, this will normally have a kinetic and potential term, but in this report it will focus mostly on the states of spins (see A.7 and 4.1).

The 'solution' to most physical problems is the find the *groundstate* $|\psi\rangle$ that minimises the energy. That is equivalent to solving the following minimisation problem:

$$\begin{aligned} \min_{|\psi\rangle} \quad & E \\ \text{s.t.} \quad & H\,|\psi\rangle = E\,|\psi\rangle \end{aligned} \tag{A.10}$$

[4]The following derivation of the Schrödinger equation is heavily indebted to Susskind [49]. For a more thorough review, see [50].

[5]It is often asserted that such operators must be Hermitian, i.e. $H = H^\dagger$, but this is not strictly true: see [51]. Superficially, the requirement is that the operator H on a space of dimension N have N orthogonal eigenvectors and N real (possibly degenerate) eigenvalues. Hermiticity guarantees this, hence it is a useful requirement to impose.

[6]This summary is taken almost directly from Cresser [52].

A.6 Commutators

Most operators do not commute, and the same is true within quantum mechanics. We define the *commutator*:

$$[A, B] = AB - BA \tag{A.11}$$

The *anti-commutator* is defined as:

$$\{A, B\} = AB + BA \tag{A.12}$$

Commutation relations are important in defining the relationships between operators. For example, given the position operators $\hat{x}$ and the momentum operator $\hat{p}$:

$$[\hat{x}, \hat{p}] = i\hbar\mathbb{I} \tag{A.13}$$

This relation between two operators that are Fourier transforms of one another is often referred to as the canonical commutation relation.

A.7 Spin

Spin is a form of angular momentum inherent to quantum particles. Spin can be measured in the three different dimensions of normal space, and particles each posses a *spin quantum number* s. Restrictions on this spin quantum number imply important properties about different quantum particles. Spin is quantized and takes the form:

$$S = \hbar\sqrt{s(s+1)} \tag{A.14}$$

Where s can be any half-integer. Particles with half-integer spin are *fermions* and particles with integer spins are called *bosons*.

An important property of fermions is that they obey the Pauli Exclusion Principle[53]. Consider a *creation operator* $c_i^\dagger$ that acts on a vacuum state $|0\rangle$ to create a particle at position i[7]. Adding a particle at position i and another at position j must give us the same state, up to a prefactor:

$$c_i^\dagger c_j^\dagger = \lambda c_j^\dagger c_i^\dagger \tag{A.15}$$

Restricting ourselves without loss of generality to the cases ± 1, we consider the bosonic case of $+1$ first, which implies that *the state vector is symmetric under particle exchange.* This also implies that:

$$c_i^\dagger c_j^\dagger - c_j^\dagger c_i^\dagger = [c_i^\dagger, c_j^\dagger] = 0 \tag{A.16}$$

However, for the *fermionic* case, we have the prefactor -1 and instead the anti-commutator $\{a_i^\dagger, a_j^\dagger\} = 0$, where $a_i^\dagger$ is the fermionic creation operator at i. Mostly importantly, if we set $i = j$ then:

$$a_i^\dagger a_i^\dagger + a_i^\dagger a_i^\dagger = 0 \tag{A.17}$$

$$a_i^\dagger a_i^\dagger = 0 \tag{A.18}$$

Which is exactly the Pauli Exclusion Principle: if we try to create to fermions at the same position, they annihilate and we get nothing at all. This will be relevant when we discuss the Jordan-Wigner transformation.

[7]This explanation of the exclusion principle is due to Blundell and Lancaster [50].

B Exact Solution to Non-Interacting Case

B.1 The Jordan-Wigner Transformation

The Jordan-Wigner (JW) transformation maps a system of spins into a system of (free) fermions. This is useful as it opens up a wider variety of technqiues for dealing with disordered, many body problems. We recommend [54] for a thorough overview.

Recall from A.7 that a fermion obeys the Pauli exclusion principle and that one can define an operator BLAH BLAH with commutator XYZ. Using this, we can say that the JW transformation maps spins to fermions according to the following transformations:

n is the fermion number operator

Roughly one can interpret the transformation for each spin operator as being something like 'rather than measure the spin in such a direction at site i, measure the combined effect of adding and also of annihilating a fermion at i, adjusted for the parity condition $\hat{K}_j$'. The details are beyond the scope of this report but we use them in the calculations below for the exact solution to the XX chain.

B.2 Using the JW transformation in the exact solution

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