Nash Equilibria of a Two-Party Policy Competition Game

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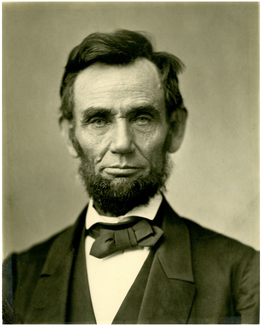
Outline

Outline

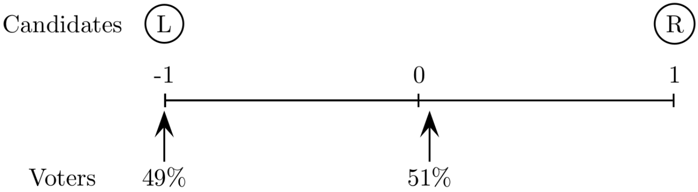
The Inspiration (an EC'17 paper)



"[. . .] and that government of the people, by the people, for the people, shall not perish from the earth."

— *Abraham Lincoln, 1863.*

Previous Work (I): Distortion of Social Choice Rules



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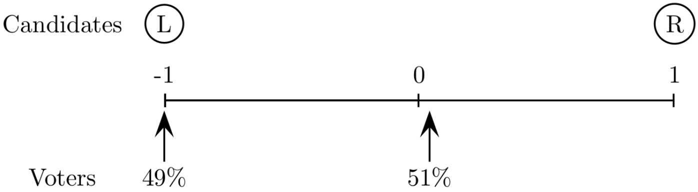
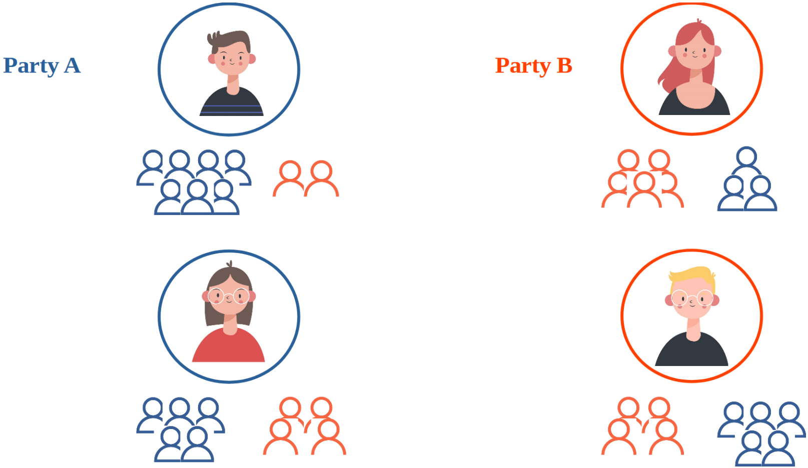


- The average distance from the population to candidate L: ≈ 0.5.
- The average distance from the population to candidate R: ≈ 1.5.
- But R will be elected as the winner in the election.

Previous Work (II): Two-Party Election Game

- Parties are players.
- Strategies: their candidates (or policies).
- A candidate beats the other candidates from other candidates of other parties with uncertainty.
- The payoff of each party: expected utility its supporters can get.


Party A
Party B

Two-Party Election Game (contd.)

- Party A: m candidates, party B: n candidates.
- Candidate A_i can bring social utility $u(A_i) = u_A(A_i) + u_B(A_i) \in [0, \beta]$ for some real $\beta \geq 0$.
- $p_{i,j}$: $\Pr[A_i \text{ wins over } B_j]$.
    - Linear: $p_{i,j} := (1 + (u(A_i) - u(B_j))/\beta)/2$
    - Natural: $p_{i,j} := u(A_i)/(u(A_i) + u(B_j))$
    - Softmax: $p_{i,j} := e^{u(A_i)/\beta}/(e^{u(A_i)/\beta} + e^{u(B_j)/\beta})$
- Payoff (reward) $r_A = p_{i,j} u_A(A_i) + (1 - p_{i,j}) u_A(B_j)$.

Two-Party Election Game (contd.)

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- Candidate A_i can bring social utility $u(A_i) = u_A(A_i) + u_B(A_i) \in [0, \beta]$ for some real $\beta \geq 0$.
- $p_{i,j}$: $\Pr[A_i$ wins over $B_j]$. more utility for all the people, more likely to win
  - Linear: $p_{i,j} := (1 + (u(A_i) - u(B_j))/\beta)/2$
  - Natural: $p_{i,j} := u(A_i)/(u(A_i) + u(B_j))$
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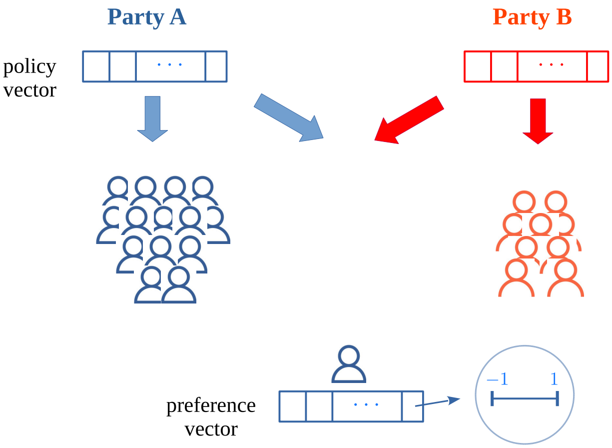
Policies and Preferences



The Setting

- Policy vectors: $\mathbf{z}_A, \mathbf{z}_B \in S \subset \mathbb{R}^k$.
  - $\|\mathbf{z}_A\| \leq 1$ and $\|\mathbf{z}_B\| \leq 1$.
  - State (or profile): $\mathbf{z} := (\mathbf{z}_A, \mathbf{z}_B)$.
- V_A and V_B: the supporters of A and B.
  - $V := V_A \dot\cup V_B$, $|V| = n$.
- Preference vector of a voter $v \in V$: $\mathbf{q}_v$.
- $Q_A := \sum_{v \in V_A} \mathbf{q}_v$, $Q_B := \sum_{v \in V_B} \mathbf{q}_v$ and $Q := Q_A + Q_B$.
- The utility

$$u_A(\mathbf{z}_A) = \sum_{v \in V_A} \mathbf{z}_A^\top \mathbf{q}_v = \mathbf{z}_A^\top Q_A,\ u_B(\mathbf{z}_A) = \sum_{v \in V_B} \mathbf{z}_A^\top \mathbf{q}_v = \mathbf{z}_A^\top Q_B.$$

$$u_A(\mathbf{z}_B) = \sum_{v \in V_A} \mathbf{z}_B^\top \mathbf{q}_v = \mathbf{z}_B^\top Q_A,\ u_B(\mathbf{z}_B) = \sum_{v \in V_B} \mathbf{z}_B^\top \mathbf{q}_v = \mathbf{z}_B^\top Q_B.$$

The Setting (Winning Prob. & Payoff)

- Winning probability:

$$\begin{aligned} p_{A \succ B} &= \frac{1}{2} + \frac{1}{4kn}(\mathbf{z}_A - \mathbf{z}_B)^\top Q, \\ p_{B \succ A} &= \frac{1}{2} + \frac{1}{4kn}(\mathbf{z}_B - \mathbf{z}_A)^\top Q. \end{aligned}$$

  - $1/4n$: a normalization factor.

- The payoffs:

$$\begin{aligned} R_A(\mathbf{z}) &= p_{A \succ B} \cdot \mathbf{z}_A^\top Q_A + p_{B \succ A} \cdot \mathbf{z}_B^\top Q_A, \\ R_B(\mathbf{z}) &= p_{B \succ A} \cdot \mathbf{z}_B^\top Q_B + p_{A \succ B} \cdot \mathbf{z}_A^\top Q_B. \end{aligned}$$

So, we can compute the gradients and Hessian...

$$\frac{\partial R_A(\mathbf{z})}{\partial \mathbf{z}_A} = \frac{1}{2}Q_A + \frac{(\mathbf{z}_A - \mathbf{z}_B)^\top Q}{4kn}Q_A + \frac{(\mathbf{z}_A - \mathbf{z}_B)^\top Q_A}{4kn}Q.$$

$$\frac{\partial R_B(\mathbf{z})}{\partial \mathbf{z}_B} = \frac{1}{2}Q_B + \frac{(\mathbf{z}_B - \mathbf{z}_A)^\top Q}{4kn}Q_B + \frac{(\mathbf{z}_B - \mathbf{z}_A)^\top Q_B}{4kn}Q.$$

$$\frac{\partial^2 R_A(\mathbf{z})}{\partial \mathbf{z}_A^2}[i,j] = \frac{1}{4kn}\left(Q[i]Q_A[j] + Q[j]Q_A[i]\right),$$

$$\frac{\partial^2 R_B(\mathbf{z})}{\partial \mathbf{z}_B^2}[i,j] = \frac{1}{4kn}\left(Q[i]Q_B[j] + Q[j]Q_B[i]\right).$$

Outline

Our Contribution

[Nash 1950]

Every FINITE game has a mixed-strategy Nash equilibrium.

Our Contribution

In this work, we show that there exists a pure-strategy Nash equilibrium (PSNE) in the two-party policy competition game for

- the degenerate case: $k = 1$
- the general case $k \geq 1$ under the consensus-reachable condition

- The two-party policy competition game is NOT a finite game.

Claim of the Egoistic Property

Egoistic property

$\langle \mathbf{z}_A, Q_A \rangle \geq \langle \mathbf{z}_B, Q_A \rangle$ and $\langle \mathbf{z}_B, Q_B \rangle \geq \langle \mathbf{z}_A, Q_B \rangle$.

Claim

The egoistic property must hold in the two-party policy competition game.

- $\mathbf{z}_A^\top Q_A \geq \mathbf{z}_B^\top Q_A$ and $\mathbf{z}_B^\top Q_B \geq \mathbf{z}_A^\top Q_B$.

The Degenerate Case: $k = 1$

$$R_A(\mathbf{z}) = \frac{1}{2}(z_A + z_B)Q_A + \frac{1}{4}QQ_A(z_A - z_B)^2,$$

$$\frac{\mathrm{d}R_A(\mathbf{z})}{\mathrm{d}z_A} = \frac{1}{2}Q_A + \frac{1}{2n}QQ_A(z_A - z_B),$$

$$\frac{\mathrm{d}^2R_A(\mathbf{z})}{\mathrm{d}z_A^2} = \frac{1}{2n}QQ_A.$$

- If $QQ_A \geq 0$ (resp., $QQ_A \leq 0$), then $R_A(\mathbf{z})$ is convex (resp., concave). $Q \geq 0$, $Q_A \geq 0$ plus the egoistic property $\Rightarrow$ $\frac{\mathrm{d}R_A(\mathbf{z})}{\mathrm{d}z_A} \geq 0$.

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⋮

- The maximizers of R_A and R_B can be solved analytically case-by-case.

The General Case: $k \geq 1$

- It is sufficient for party A and B to consider the space $\text{span}(\{Q_A, Q_B\})$.

The General Case: $k \geq 1$ — Simplification by Polar Coordinates

- It is sufficient for party A and B to consider the space $\text{span}(\{Q_A, Q_B\})$.
- Represent $\mathbf{z}_A$ (resp., $\mathbf{z}_B$) in terms of polar coordinates (r_A, θ_A) (resp., (r_A, θ_B)).
  - $r_A = \|\mathbf{z}_A\|, r_B = \|\mathbf{z}_B\|$
  - θ_A (resp., θ_B) is the angle b/w Q_A and $\mathbf{z}_A$ (resp., Q_B and $\mathbf{z}_B$).

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  - θ_A (resp., θ_B) is the angle b/w Q_A and $\mathbf{z}_A$ (resp., Q_B and $\mathbf{z}_B$).

For any two vectors $\mathbf{u}, \mathbf{v}$ in the Euclidean space $\mathbb{R}^k$ for $k \geq 1$,

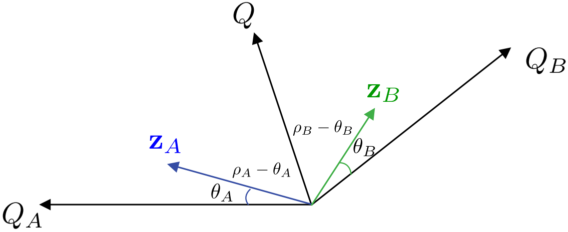
$$\langle \mathbf{u}, \mathbf{v} \rangle = \|\mathbf{u}\| \|\mathbf{v}\| \cos(\theta),$$

where θ is the angle b/w $\mathbf{u}$ and $\mathbf{v}$.

A Good Condition

Consensus-Reachable

A two-party policy competition game is *consensus-reachable* if $Q_A^\top Q \geq 0$ and $Q_B^\top Q \geq 0$.



Gradients w.r.t. r

$$\frac{\partial}{\partial r_A} R_A(\mathbf{r}, \boldsymbol{\theta}) = \frac{1}{2}\|Q_A\|\cos(\theta_A) + \frac{1}{4kn}\Big((\mathbf{z}_A^\top Q - \mathbf{z}_B^\top Q)(\|Q_A\|\cos(\theta_A)) + (\mathbf{z}_A^\top Q_A - \mathbf{z}_B^\top Q_A)(\|Q\|\cos(\rho - \theta_A))\Big).$$

and

$$\frac{\partial^2}{\partial r_A^2} R_A(\mathbf{r}, \boldsymbol{\theta}) = \frac{1}{4kn}\|Q_A\|\|Q\|\cos(\theta_A)\cos(\rho_A - \theta_A),$$

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and

$$\frac{\partial^2}{\partial r_A^2} R_A(\mathbf{r}, \boldsymbol{\theta}) = \frac{1}{4kn}\|Q_A\|\|Q\|\cos(\theta_A)\cos(\rho_A - \theta_A),$$

- $\frac{\partial^2}{\partial r_A^2} R_A(\mathbf{r}, \boldsymbol{\theta}) \geq 0$ by the consensus-reachable condition ($\therefore$ concave).

Gradients w.r.t. $\mathbf{r}$ (contd.)

- Compute $R_A(0,\theta_A)$ and $R_A(1,\theta_A)$ we will find that

$$
\begin{aligned}
R_A(1,\theta_A) &= R_A(0,\theta_A) + \frac{1}{2}\|Q_A\|\cos(\theta_A) \\
&\quad + \frac{1}{4kn}\Big(\|Q\|\|Q_A\|\cos(\rho_A-\theta_A)\cos(\theta_A) \\
&\quad - \mathbf{z}_B^\top Q\|Q_A\|\cos(\theta_A) - \mathbf{z}_B^\top Q_A\|Q\|\cos(\rho_A-\theta_A)\Big) \\
&\geq R_A(0,\theta_A).
\end{aligned}
$$

Gradients w.r.t. $\boldsymbol{\theta}$

- Assuming $\mathbf{r} = (1, 1)$, we derive that

$$R_A(\boldsymbol{\theta}) = p(\theta_A)(\|Q_A\| \cos(\theta_A)) + (1 - p(\theta_A))(\|Q_A\| \cos(\rho_A - \theta_A)),$$

where

$$\begin{aligned} p(\theta_A) &= p_{A \succ B} = \frac{1}{2} + \frac{1}{4kn}\left(\mathbf{z}_A^\top Q - \mathbf{z}_B^\top Q\right) \\ &= \frac{1}{2} + \frac{1}{4kn}\left(\|Q\| \cos(\alpha_A - \theta_A) - \mathbf{z}_B^\top Q\right) \end{aligned}$$

and ρ_A is the angle between Q and Q_A.

Gradients w.r.t. $\boldsymbol{\theta}$ (contd.)

- It's straight-forward to derive

$$p'(\theta_A) = \frac{\|Q\|}{4kn}\sin(\alpha_A - \theta_A) \geq 0$$

and

$$p''(\theta_A) = -\frac{\|Q\|}{4kn}\cos(\alpha_A - \theta_A) \leq 0$$

- Hence,

$$\begin{aligned} R_A'(\boldsymbol{\theta}) &= p'(\theta_A)(\mathbf{z}_A^\top Q - \mathbf{z}_B^\top Q) - p(\theta_A)\sin(\theta_A)\|Q_A\| \\ &= \frac{\|Q\|}{4kn}\sin(\alpha_A - \theta_A)(\mathbf{z}_A^\top Q_A - \mathbf{z}_B^\top Q_A) \\ &\quad -\frac{1}{2}\sin(\theta_A)\|Q_A\| - \frac{1}{4kn}\left(\|Q\|\cos(\alpha_A - \theta_A) - \mathbf{z}_B^\top Q\right)\sin(\theta_A)\|Q_A\|, \end{aligned}$$

$$\begin{aligned} R_A''(\boldsymbol{\theta}_A) \quad = \quad & p''(\theta_A)(\mathbf{z}_A^\top Q_A - \mathbf{z}_B^\top Q_A) - 2p'(\theta_A)\|Q_A\|\sin(\theta_A) \\ & -p(\theta_A)\cos(\theta_A)\|Q_A\| \leq 0. \end{aligned}$$

Gradients w.r.t. $\boldsymbol{\theta}$ (contd.)

- By the mean value theorem we know that there exists $\theta_A^* \in [0, \alpha_A]$ such that $R'_A(\theta_A^*) = 0$.
  - $R_A(\theta_A)$ is continuous;
  - $R'_A(0) \geq 0$;
  - $R'_A(\alpha_A) \leq 0$,

Then?

Gradients w.r.t. $\boldsymbol{\theta}$ (contd.)

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  - $R_A(\theta_A)$ is continuous;
  - $R_A'(0) \geq 0$;
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Then?

Better Responses

Define the relation $\succsim_A$ over θ_A as $(\theta', \theta_B) \succsim_A (\theta'', \theta_B)$ if $R_A(\theta', \theta_B) \geq R_A(\theta'', \theta_B)$.

Gradients w.r.t. $\boldsymbol{\theta}$ (quasi-concaveness)

Quasi-Concave

$\succsim_A$ is *quasi-concave* on $[0, \alpha_A]$ if for all $\boldsymbol{\theta} = (\theta_A, \theta_B)$, the set $\{\theta' \in [0, \alpha_A] \mid (\theta', \theta_B) \succsim_A (\theta_A, \theta_B)\}$ is convex.

Gradients w.r.t. $\boldsymbol{\theta}$ (quasi-concaveness)

Quasi-Concave

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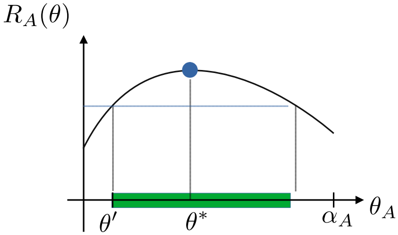

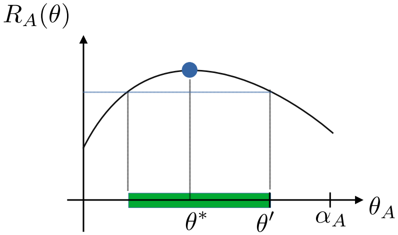

Gradients w.r.t. $\boldsymbol{\theta}$ (Kakutani's Fixed-Point Theorem)

Theorem [Concluding]

Since

- $[0, \alpha]$ is nonempty, compact and convex in $\mathbb{R}$;
- $R_A(\boldsymbol{\theta})$ is continuous w.r.t. θ_A;
- $\succsim_A$ is quasi-concave on $[0, \alpha]$

The two-party policy competition game has a pure-strategy Nash equilibrium (PSNE) under the consensus-reachable condition.

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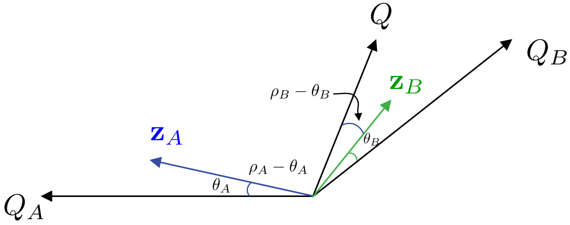
Concluding Remarks

- The unsolved case:



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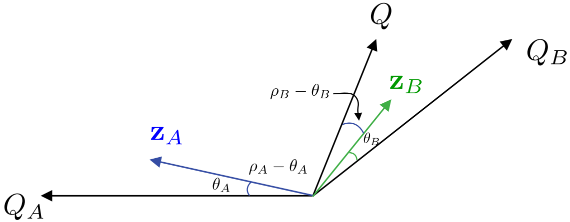

- Update the policy using gradient ascent:

$$\mathbf{z}_A \leftarrow \mathbf{z}_A + \eta \frac{\partial R_A(\mathbf{z})}{\partial \mathbf{z}_A}, \quad \mathbf{z}_B \leftarrow \mathbf{z}_B + \eta \frac{\partial R_B(\mathbf{z})}{\partial \mathbf{z}_B},$$

where η is called *learning rate*

Thanks for your attention!

Q & A