Online Learning for Min-Max Discrete Problems

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Outline

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Online learning framework (1/4)

We focus on cost minimization problems.

- Decision space: $\mathcal{X}$.
- State space: $\mathcal{Y}$.
- Cost function $f : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$.

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A perspective of an iterative adversarial game with T rounds.

1. The algorithm first chooses an action $\mathbf{x}^t \in \mathcal{X}$.
2. The (adversarial) nature reveals $\mathbf{y}^t \in \mathcal{Y}$ that could depend on $\mathbf{x}^t$.
3. The algorithm observes the state $\mathbf{y}^t$ and suffers a loss $f^t(\mathbf{x}^t) = f(\mathbf{x}^t, \mathbf{y}^t)$.

Online learning framework (2/4)

The objective of the player: minimize the accumulative cost

$$\sum_{t=1}^{T} f(\mathbf{x}^t, \mathbf{y}^t).$$

Online Learning Algorithms

An algorithm that decides the actions $\mathbf{x}^t$ before observing $\mathbf{y}^t$ for each t.

- The efficiency measure: regret.

$$R_T = \sum_{t=1}^{T} f(\mathbf{x}^t, \mathbf{y}^t) - \sum_{t=1}^{T} f(\mathbf{x}^*, \mathbf{y}^t),$$

where $\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^{T} f(\mathbf{x}, \mathbf{y}^t)$ (static).

Online learning framework (3/4)

- We aim for algorithms with $R_T = O(T^c)$, for $0 \leq c < 1$.
    - Vanishing regret (or no-regret).

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- We aim for algorithms with $R_T = O(T^c)$, for $0 \leq c < 1$.
  - Vanishing regret (or no-regret).
- A computational efficiency concern:
  - It coulde be NP-hard to compute $\mathbf{x}_t$'s even for $T = 1$ and $\mathbf{y}^1$ is revealed beforehand.

A relaxed notion: α-regret

$$R_T^\alpha = \sum_{t=1}^{T} f(\mathbf{x}^t, \mathbf{y}^t) - \alpha \sum_{t=1}^{T} f(\mathbf{x}^*, \mathbf{y}^t).$$

- Goal: vanishing α-regret for some $\alpha \geq 1$.

Online learning framework (4/4)

Polynomial Time Vanishing α-Regret Algorithms

An online learning algorithm which

- computes $\mathbf{x}^t$ in poly(n, t), where n is the input instance size.
- the (expected) regret is bounded by poly$(n)T^c$, for some constant $0 \leq c < 1$.

- For the case $\alpha = 1$, we call it a polynomial time vanishing regret algorithm.

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The regret is polynomial in n and sublinear in T.

Main Contribution (1/8)

Cardinality constrained problems

Given an n-elements set $\mathcal{U}$, a set of constraints $\mathcal{C}$ on $2^{\mathcal{U}}$, and an integer k.

Goal: Determine whether there exists a feasible solution of size $\leq k$.

Min-Max-$\mathcal{P}$

Given a cardinality problem $\mathcal{P}$ where all the elements in $\mathcal{U}$ are given non-negative weights.

Goal: Compute a feasible solution such that the maximum weight of all its elements is minimized.

Main Contribution (2/8)

Online Min-Max-$\mathcal{P}$

An online learning variant of min-max-$\mathcal{P}$ such that

- the set of elements in $\mathcal{U}$ and the set of constraints $\mathcal{C}$ remain static.
- the weights on the elements of $\mathcal{U}$ change over time.

Main Contribution (2/8)

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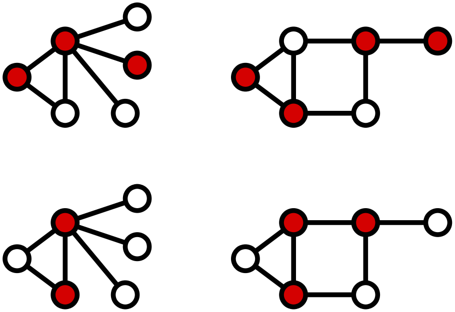
- the weights on the elements of $\mathcal{U}$ change over time.

Example: Min-Max Vertex Cover

- **Static:** Given a graph $G = (V, E)$, where each $v \in V$ has weight $w(v) \geq 0$. Find a vertex cover $V' \subseteq V$ which minimizes $w(V') = \max\{w(v) \mid v \in V'\}$.
- **Online-version:**
  - There are T rounds, a weight function w^t on the vertices for each round t.
  - An algorithm has to pick a vertex cover V'_t of G and suffers a loss $w(V'_t) = \max\{w(v) : v \in V'_t\}$.

Vertex Cover (VC)



Static Min-Max VC is polynomial-time solvable

- VC_W: Given an integer W, determine if G has a vertex cover of maximum weight $\leq W$.
    - Pick all vertices of weight $\leq W$ and see if this is a vertex cover.

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    - The optimum solution: find the smallest W such that VC_W is affirmative.

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  - Pick all vertices of weight $\leq W$ and see if this is a vertex cover.
  - The optimum solution: find the smallest W such that VC_W is affirmative.
    - Check all values W in $\{w(v) : v \in V(G)\}$.

Main Contribution (3/8)

$[A, B]$-Gap-$\mathcal{P}$

- Given $0 \leq A < B \leq 1$.
- The decision problem where given an instance of $\mathcal{P}$ such that $|\mathbf{x}_{opt}| \leq An$ or $|\mathbf{x}_{opt}| \geq Bn$.
- **Goal:** Decide whether $|\mathbf{x}_{opt}| < Bn$.

Main Theorem I

Assume that $[A, B]$-Gap-$\mathcal{P}$ is NP-complete, for $0 \leq A < B \leq 1$. Then for every $\alpha < \frac{B}{A}$, there is no (randomized) polynomial-time vanishing α-regret algorithm for online min-max-$\mathcal{P}$ unless NP $=$ RP.

Main Contribution (4/8)

Corollary 1

- The online min-max vertex cover problem does not admit a polynomial time vanishing $(\sqrt{2}-\epsilon)$-regret algorithm unless NP $=$ RP.
- It does not admit a polynomial time vanishing $(2-\epsilon)$-regret algorithm unless Unique Game is in RP.

Corollary 2

If a cardinality problem $\mathcal{P}$ is NP-complete, then there is no polynomial time vanishing regret algorithm for online min-max-$\mathcal{P}$ unless NP $=$ RP.

- Set $\alpha = 1, A = \frac{k}{n}, B = \frac{k+1}{n} = A + \frac{1}{n}$
Deciding if $|\mathbf{x}_{opt}| \leq k \Leftrightarrow$ deciding if $|\mathbf{x}_{opt}| \leq An$ or $|\mathbf{x}_{opt}| \geq Bn$.

Main Contribution (5/8)

Algorithm 2: OGD-based algorithm for Online MinMax Vertex Cover.

1 Select an arbitrary fractional vertex cover $x^1 \in \mathcal{Q}$.
2 **for** $t = 1, 2, \ldots$ **do**
3 Round x^t to X^t: $X_i^t = 1$ if $x_i^t \geq 1/2$ and $X_i^t = 0$ otherwise.
4 Play $X^t \in \{0,1\}^n$. Observe w^t (weights of vertices) and incur the cost $f^t(X^t) = \max_i w_i^t X_i^t$.
5 Update $y^{t+1} = x^t - \frac{1}{\sqrt{t}} g^t(x^t)$.
6 Project y^{t+1} to $\mathcal{Q}$ w.r.t the ℓ_2-norm: $x^{t+1} = \mathrm{Proj}_{\mathcal{Q}}(y^{t+1}) := \arg\min_{x \in \mathcal{Q}} \|y^{t+1} - x\|_2$.

- We consider the relaxation:

$$\min_{\mathbf{x} \in \mathcal{Q}} \max_{i \in V} w_i x_i,$$

  - $\mathcal{Q} := \{\mathbf{x} : x_i + x_j \geq 1, \forall (i,j) \in E, 0 \leq x_i \leq 1, \forall i \in V\}$.
  - a sub-gradient $g^t(\mathbf{x}^t) = [0, 0, \ldots, w_i^t, 0, \ldots, 0]$ with w_i in coordinate $\arg\max_{1 \leq i \leq n} w_i^t x_i^t$ and 0 otherwise.
  - Round the solution: $X_{i+1} = 1$ if $x_i^{t+1} \geq 1/2$ and 0 otherwise.

Main Contribution (6/8)

Theorem (OGD for online Min-Max VC)

Let $W = \max_{1 \le t \le T} \max_{1 \le i \le n} w_i^t$. Then, after T steps, Algorithm 2 achieves

$$\sum_{t=1}^{T} \max_{1 \le i \le n} w_i^t X_i^t \le 2 \cdot \min_{X^* \in \mathcal{X}} \sum_{t=1}^{T} \max_{1 \le i \le n} w_i^t X_i^* + 3W\sqrt{nT}$$

Main Contribution (7/8)

- Follow-The-Regularized-Leader (FTRL): an algorithm which is less predictable and more stable:

$$\mathbf{x}^t = \arg\min_{\mathbf{x}\in\mathcal{X}} \left(\sum_{\tau=1}^{t-1} f(\mathbf{x}, \mathbf{y}^\tau) + R(\mathbf{x}) \right),$$

where $R(\mathbf{x})$ is the regularization term.

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  - Need an optimization oracle over the observed history.

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Multi-instance version of min-max-$\mathcal{P}$

Given an integer $N > 0$, a set $\mathcal{X}$ of feasible solutions, and N objective functions $f_1, f_2, \ldots, f_N$ over $\mathcal{X}$.

Goal: Minimize $\sum_{i=1}^{N} f_i(\mathbf{x})$ over $\mathcal{X}$.

Main Contribution (8/8)

Examples:

- Min-max vertex cover
  - Weight function $w : V \mapsto \mathbb{R}^+$ on the vertices.
- Min-max perfect matching
  - Weight function $w : E \mapsto \mathbb{R}^+$ on the edges.
  - The weight of the heaviest edge on the perfect matching is minimized.
- Min-max path
  - Given a graph $G = (V, E)$ and two vertices s, t, and a weight function $w : E \mapsto \mathbb{R}^+$ on the edges.
  - The weight of the heaviest edge in the s–t path is minimized.

Main Theorem II

The multi-instance version of min-max perfect matching, min-max path and min-max vertex cover are APX-hard.

Outline

Proof of Main Theorem I

Main Theorem I

Assume that the problem $[A, B]$-Gap-$\mathcal{P}$ is NP-complete, for $0 \le A < B \le 1$. Then for every $\alpha < \frac{B}{A}$, there is no (randomized) polynomial-time vanishing α-regret algorithm for online min-max-$\mathcal{P}$ unless $\mathrm{NP} = \mathrm{RP}$.

- Assumption: a vanishing α-regret algorithm $\mathcal{O}$ as an oracle for online min-max-$\mathcal{P}$ with $\alpha = \frac{B}{A} - \epsilon = (1 - \epsilon')\frac{B}{A}$, for $\epsilon > 0$.
- Devise a polynomial time algorithm that
  - answers 'yes' with prob. $< D < 1$ if $|\mathbf{x}_{opt}| \le An$
  - answers 'no' if $|\mathbf{x}_{opt}| \ge Bn$.

⋆ **Note:** if $|\mathbf{x}_{opt}| \ge Bn$, all the solutions $\mathbf{x}_t$ computed by $\mathcal{O}$ must have size $\ge Bn$.

Algorithm for the $[A, B]$-Gap-$\mathcal{P}$

1. **for** $t = 1, 2, \ldots, T$ **do**
   - Choose $\mathbf{x}^t \in \mathcal{X}$ according to the random distribution given by $\mathcal{O}$.
   - **if** $|\mathbf{x}^t| < Bn$ **then return** 'yes' (i.e., $|\mathbf{x}_{opt}| \leq An$).
   - Fix a weight vector w^t by assigning weight 1 to an element of $\mathcal{U}$ chosen uniformly at random and weight 0 to all other elements.
   - Feed the weight vector and the cost $f^t(\mathbf{x}^t) = \max_{u \in \mathbf{x}^t} w^t(u)$ back to $\mathcal{O}$.
2. **return** 'No' (i.e., $|\mathbf{x}_{opt}| \geq Bn$).

Proof of Main Theorem I (contd.)

- Assume that $|\mathbf{x}_{opt}| \leq An$.
- Let E be the event that the algorithm returns 'No'.
  - It finds $|\mathbf{x}_t| \geq Bn$ at each step $t \in [T]$.
- We get

$$\Pr[E] \;=\; \Pr\left[\bigcap_{t=1}^{T} \left\{|\mathbf{x}^t| \geq Bn\right\}\right]$$

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- We get

$$\begin{aligned} \Pr[E] &= \Pr\left[\bigcap_{t=1}^{T} \{|\mathbf{x}^t| \geq Bn\}\right] \leq \Pr[X \geq TBn] \leq \frac{\mathbf{E}[X]}{TBn} \\ &= \frac{\sum_{t=1}^{T} \mathbf{E}[|\mathbf{x}^t|]}{TBn} = \frac{\sum_{t=1}^{T} \mathbf{E}[f^t(\mathbf{x}^t)]}{TB}. \end{aligned}$$

where $X = \sum_{t=1}^{T} |\mathbf{x}^t|$, and $\mathbf{E}[f^t(\mathbf{x}^t)] = \mathbf{E}[|\mathbf{x}^t|]/n$.

Proof of Main Theorem I (contd.)

Note:

- $|\mathbf{x}_{opt}| \leq An$ (by assumption).
- Only one element of weight 1 is picked uniformly at random at each time t

Hence, $\Pr[f^t(\mathbf{x}_{opt}) = 1] \leq A$

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- Since $\mathcal{O}$ is a vanishing α-regret algorithm with $\alpha = (1-\epsilon')\frac{B}{A}$,

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Hence, $\Pr[f^t(\mathbf{x}_{opt}) = 1] \leq A \;\; \Rightarrow \;\; \sum_{t=1}^{T} \mathbf{E}[f^t(\mathbf{x}_{opt})] \leq AT$.

- Since $\mathcal{O}$ is a vanishing α-regret algorithm with $\alpha = (1-\epsilon')\frac{B}{A}$,

$$\begin{aligned}\sum_{t=1}^{T} \mathbf{E}[f^t(\mathbf{x}^t)] &\leq \alpha \sum_{t=1}^{T} \mathbf{E}[f^t(\mathbf{x}_{opt})] + \text{poly}(n)T^c \\ &\leq (1-\epsilon')BT + \text{poly}(n)T^c.\end{aligned}$$

Proof of Main Theorem I (contd.)

Hence,

$$\Pr[E] \le \frac{(1-\epsilon')BT + \mathrm{poly}(n)T^c}{BT} = (1-\epsilon') + \frac{\mathrm{poly}(n)T^{c-1}}{B}.$$

Proof of Main Theorem I (contd.)

Hence,

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We can choose $T = \left(\frac{B\epsilon'}{2\text{poly}(n)}\right)^{\frac{1}{c-1}} = \left(\frac{A\epsilon}{2\text{poly}(n)B}\right)^{\frac{1}{c-1}}$, then

$$\Pr[E] \leq 1 - \frac{\epsilon'}{2} = 1 - \frac{A\epsilon}{2B}.$$

(constant; strictly smaller than 1)

Proof of Main Theorem I (contd.)

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- We've (roughly) shown that the $[A, B]$-Gap-$\mathcal{P}$ is in RP.

The hardness result for online Min-Max VC is tight

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Select an arbitrary fractional vertex cover $x^1 \in \mathcal{Q}$.
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Theorem (OGD for online Min-Max VC)

Let $W = \max_{1 \leq t \leq T} \max_{1 \leq i \leq n} w_i^t$. Then, after T steps, Algorithm 2 achieves

$$\sum_{t=1}^{T} \max_{1 \leq i \leq n} w_i^t X_i^t \leq 2 \cdot \min_{X^* \in \mathcal{X}} \sum_{t=1}^{T} \max_{1 \leq i \leq n} w_i^t X_i^* + 3W\sqrt{nT}$$

Proof of the tightness

- The guarantee from the OGD algorithm:

$$\sum_{t=1}^{T} \max_{1\le i\le n} w_i^t x_i^t \le \min_{X^*\in\mathcal{Q}} \sum_{t=1}^{T} \max_{1\le i\le n} w_i^t x_i^* + \frac{3DG}{2}\sqrt{T}$$

  - $D \le \sqrt{n}$ (diameter of $\mathcal{Q}$).
  - $G \le W$: Lipschitz constant of g^t.
  - $\max_{1\le i\le n} X_i^t w_i^t \le 2\max_{1\le i\le n} x_i^t w_i^t$ by the rounding procedure.

Outline

Recall Main Theorem II

- Follow-The-Regularized-Leader (FTRL): an algorithm which is less predictable and more stable:

$$\mathbf{x}^t = \arg\min_{\mathbf{x}\in\mathcal{X}} \left(\sum_{\tau=1}^{t-1} f(\mathbf{x}, \mathbf{y}^\tau) + R(\mathbf{x}) \right),$$

  where $R(\mathbf{x})$ is the regularization term.
    - Need an optimization oracle over the observed history.

Multi-instance version of min-max-$\mathcal{P}$

Given an integer $N > 0$, a set $\mathcal{X}$ of feasible solutions, and N objective functions $f_1, f_2, \ldots, f_N$ over $\mathcal{X}$.

Goal: Minimize $\sum_{i=1}^{N} f_i(\mathbf{x})$ over $\mathcal{X}$.

Remark

Main Theorem II

The multi-instance version of min-max perfect matching, min-max path and min-max vertex cover are APX-hard.

- The problems $\mathcal{P}$ could be polynomially solvable when using a "sum" objective.
  - Main Theorem I cannot be applied.

Remark

Main Theorem II

The multi-instance version of min-max perfect matching, min-max path and min-max vertex cover are APX-hard.

- The problems $\mathcal{P}$ could be polynomially solvable when using a "sum" objective.
  - Main Theorem I cannot be applied.
- Main Theorem II shows that FTRL fails to efficiently solve the online min-max-$\mathcal{P}$.

Multi-Instance Min-Max VC

- A straightforward reduction from VC (since VC is APX-hard).
- Let's say $V = \{v_1, v_2, \ldots, v_n\}$.
  Construct n weight functions $w^1, w^2, \ldots, w^n : V \mapsto \mathbb{R}$ such that
  - In w^i: we set $w^i(v_i) = 1$ and $w^i(v) = 0$ for $v \neq v_i$.

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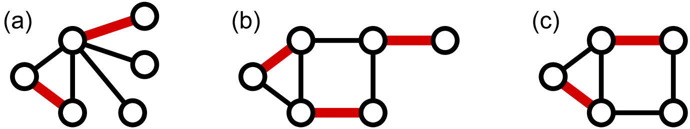
  Construct n weight functions $w^1, w^2, \ldots, w^n : V \mapsto \mathbb{R}$ such that
  - In w^i: we set $w^i(v_i) = 1$ and $w^i(v) = 0$ for $v \neq v_i$.
- Any vertex cover has total cost equal to its size.

Perfect Matching



- Maximum cardinality matchings.
- Only in (b) there is a perfect matching.

Multi-Instance Min-Max Perfect Matching (1/3)

- Reduction from the Max-3-DNF problem.
  - A 3-DNF formula: $(x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (x_1 \wedge x_3 \wedge x_4)$.
  - $(x_1 \wedge x_2 \wedge x_3)$: a clause
  - x_1 or $\neg x_2$: literals

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  - $(x_1 \wedge x_2 \wedge x_3)$: a clause
  - x_1 or $\neg x_2$: literals
- Given
  - n Boolean variables $X = \{x_1, x_2, \ldots, x_n\}$
  - m clauses $C_1, C_2, \ldots, C_m$ (conjunctions of 3 literals of X)

Goal: Determine a truth assignment $\sigma : X \mapsto \{T, F\}$ such that the number of satisfied clauses is maximized.

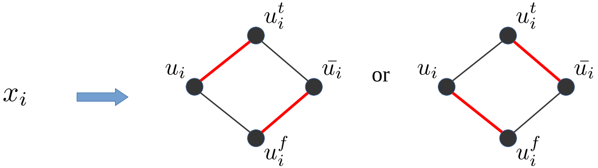
Multi-Instance Min-Max Perfect Matching (2/3)

An instance $\mathcal{I}$ of Max-3-DNF $\Rightarrow$ $G(V, E)$ and m weight functions:

Multi-Instance Min-Max Perfect Matching (2/3)

An instance $\mathcal{I}$ of Max-3-DNF $\Rightarrow G(V,E)$ and m weight functions:

- Each x_i is associated a 4-cycle on vertices $(u_i, u_i^t, \bar{u}_i, u_i^f)$.



- Weight function corresponds to clause C_j:
  - $w^j(u_i u_i^t) = 1$ if $\neg x_i \in C_i$, otherwise $w^j(u_i u_i^t) = 0$.
  - $w^j(u_i u_i^f) = 1$ if $x_i \in C_i$, otherwise $w^j(u_i u_i^f) = 0$.
  Edges incident to vertices $\bar{u}_i$ always get weight 0.

⋆ The instance $\mathcal{I}'$ of multi-instance min-max matching is constructed (in polynomial time).

Multi-Instance Min-Max Perfect Matching (3/3)

- A truth assignment σ of $\mathcal{I}$ corresponds to a matching M_σ of G.
- $\text{value}(\mathcal{I}, \sigma) = m - \text{value}(\mathcal{I}', M_\sigma)$

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- A truth assignment σ of $\mathcal{I}$ corresponds to a matching M_σ of G.
- $\text{value}(\mathcal{I}, \sigma) = m - \text{value}(\mathcal{I}', M_\sigma)$
- Assume that there exists a $(1+\epsilon)$-approximation algorithm for multi-instance min-max perfect matching, then we can get a $(1-\rho\epsilon)$ approximation algorithm for Max-3-DNF for some constant ρ.
    - PTAS-reduction.

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- A truth assignment σ of $\mathcal{I}$ corresponds to a matching M_σ of G.
- $\text{value}(\mathcal{I}, \sigma) = m - \text{value}(\mathcal{I}', M_\sigma)$
- Assume that there exists a $(1+\epsilon)$-approximation algorithm for multi-instance min-max perfect matching, then we can get a $(1-\rho\epsilon)$ approximation algorithm for Max-3-DNF for some constant ρ.
  - PTAS-reduction.
- Thus, multi-instance min-max perfect matching is APX-hard.

Discussion

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    - Follow The Leader (FTL) Algorithm.

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- How about just *following the one with best performance*?
  - Follow The Leader (FTL) Algorithm.

- First, we assume to make no assumptions on $\mathcal{K}$ and $\{f_t : L \mapsto \mathbb{R}\}$.
- At time t, we are given previous cost functions $f_1, \ldots, f_{t-1}$, and then give the solution

$$\mathbf{x}_t := \arg\min_{\mathbf{x} \in \mathcal{K}} \sum_{k=1}^{t-1} f_k(\mathbf{x}).$$

What is FTL and Why not FTL?

- How about just *following the one with best performance*?
  - Follow The Leader (FTL) Algorithm.

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That is, the best solution for the previous $t-1$ steps.

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  That is, the best solution for the previous $t-1$ steps.
- It seems reasonable and makes sense, doesn't it?

FTL leads to "overfitting"

t:	1
$\mathbf{x}_t$:	$(0.5, 0.5)$
ℓ_t:	$(0, 0.5)$
$f_t(\mathbf{x}_t)$:	0.25
$\arg\min_{\mathbf{x}} \sum_{k=1}^{t} f_k(\mathbf{x})$:	$(1, 0)$

FTL leads to "overfitting"

t:	1	2
$\mathbf{x}_t$:	$(0.5, 0.5)$	$(1, 0)$
ℓ_t:	$(0, 0.5)$	$(1, 0)$
$f_t(\mathbf{x}_t)$:	0.25	1
$\arg\min_{\mathbf{x}} \sum_{k=1}^{t} f_k(\mathbf{x})$:	$(1, 0)$	$(0, 1)$

FTL leads to "overfitting"

t:	1	2	3
$\mathbf{x}_t$:	$(0.5, 0.5)$	$(1, 0)$	$(0, 1)$
ℓ_t:	$(0, 0.5)$	$(1, 0)$	$(0, 1)$
$f_t(\mathbf{x}_t)$:	0.25	1	1
$\arg\min_{\mathbf{x}} \sum_{k=1}^{t} f_k(\mathbf{x})$:	$(1, 0)$	$(0, 1)$	$(1, 0)$

FTL leads to "overfitting"

t:	1	2	3	4
$\mathbf{x}_t$:	$(0.5, 0.5)$	$(1, 0)$	$(0, 1)$	$(1, 0)$
ℓ_t:	$(0, 0.5)$	$(1, 0)$	$(0, 1)$	$(1, 0)$
$f_t(\mathbf{x}_t)$:	0.25	1	1	1
$\arg\min_{\mathbf{x}} \sum_{k=1}^{t} f_k(\mathbf{x})$:	$(1, 0)$	$(0, 1)$	$(1, 0)$	$(0, 1)$

FTL leads to "overfitting"

t:	1	2	3	4	5
$\mathbf{x}_t$:	$(0.5, 0.5)$	$(1, 0)$	$(0, 1)$	$(1, 0)$	$(0, 1)$
ℓ_t:	$(0, 0.5)$	$(1, 0)$	$(0, 1)$	$(1, 0)$	$(0, 1)$
$f_t(\mathbf{x}_t)$:	0.25	1	1	1	1
$\arg\min_{\mathbf{x}} \sum_{k=1}^{t} f_k(\mathbf{x})$:	$(1, 0)$	$(0, 1)$	$(1, 0)$	$(0, 1)$	$(1, 0)$

FTL leads to "overfitting"

t:	1	2	3	4	5	...
$\mathbf{x}_t$:	$(0.5, 0.5)$	$(1, 0)$	$(0, 1)$	$(1, 0)$	$(0, 1)$	...
ℓ_t:	$(0, 0.5)$	$(1, 0)$	$(0, 1)$	$(1, 0)$	$(0, 1)$	...
$f_t(\mathbf{x}_t)$:	0.25	1	1	1	1	...
$\arg\min_{\mathbf{x}} \sum_{k=1}^{t} f_k(\mathbf{x})$:	$(1, 0)$	$(0, 1)$	$(1, 0)$	$(0, 1)$	$(1, 0)$	...

FTL leads to "overfitting"

t:	1	2	3	4	5	...
$\mathbf{x}_t$:	$(0.5, 0.5)$	$(1, 0)$	$(0, 1)$	$(1, 0)$	$(0, 1)$	...
ℓ_t:	$(0, 0.5)$	$(1, 0)$	$(0, 1)$	$(1, 0)$	$(0, 1)$	...
$f_t(\mathbf{x}_t)$:	0.25	1	1	1	1	...
$\arg\min_{\mathbf{x}} \sum_{k=1}^{t} f_k(\mathbf{x})$:	$(1, 0)$	$(0, 1)$	$(1, 0)$	$(0, 1)$	$(1, 0)$	...

optimum loss: $\approx T/2$.

FTL's loss: $\approx T$.

regret: $\approx T/2$ (linear).

Remark

- Note that the first example of no-regret analysis in this course uses a special kind of loss function.
    - Squared difference: $\|\mathbf{x}_t - \mathbf{y}_t\|_2^2$.

Outline

Online Gradient Descent (GD)

1. **Input:** convex set $\mathcal{K}$, T, $\mathbf{x}_1 \in \mathcal{K}$, learning rate $\{\eta_t\}$.
2. **for** $t \leftarrow 1$ to T **do**:
    1. Play $\mathbf{x}_t$ and observe cost $f_t(\mathbf{x}_t)$.
    2. Update and Project:

$$\begin{aligned} \mathbf{y}_{t+1} &= \mathbf{x}_t - \eta_t \nabla f_t(\mathbf{x}_t) \\ \mathbf{x}_{t+1} &= \Pi_{\mathcal{K}}(\mathbf{y}_{t+1}) \end{aligned}$$

3. **end for**

GD for online convex optimization is of no-regret

Theorem A

Online gradient descent with learning rate $\{\eta_t = \frac{D}{G\sqrt{t}}, t \in [T]\}$ guarantees the following for all $T \geq 1$:

$$\text{regret}_T = \sum_{t=1}^{T} f_t(\mathbf{x}_t) - \min_{\mathbf{x}^* \in \mathcal{K}} \sum_{t=1}^{T} f_t(\mathbf{x}^*) \leq \frac{3}{2} GD\sqrt{T}.$$

- D: the diameter of $\mathcal{K}$.
- Assume that $\nabla f_t(\mathbf{x}) \leq G$ for each $\mathbf{x} \in \mathcal{K}$.

Proof of Theorem A (1/4)

- Let $\mathbf{x}^* \in \arg\min_{\mathbf{x}\in\mathcal{K}} \sum_{t=1}^{T} f_t(\mathbf{x})$.
- Since f_t is convex, we have

$$f_t(\mathbf{x}_t) - f_t(\mathbf{x}^*) \leq (\nabla f_t(\mathbf{x}_t))^\top (\mathbf{x}_t - \mathbf{x}^*).$$

- By the updating rule for $\mathbf{x}_{t+1}$ and the Pythagorean theorem, we have

$$\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2 = \|\Pi_{\mathcal{K}}(\mathbf{x}_t - \eta_t \nabla f_t(\mathbf{x}_t)) - \mathbf{x}^*\|^2 \leq \|\mathbf{x}_t - \eta_t \nabla f_t(\mathbf{x}_t) - \mathbf{x}^*\|^2.$$

Proof of Theorem A (2/4)

- Hence

$$\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2 \le \|\mathbf{x}_t - \mathbf{x}^*\|^2 + \eta_t^2 \|\nabla f_t(\mathbf{x}_t)\|^2 - 2\eta_t (\nabla f_t(\mathbf{x}_t))^\top (\mathbf{x}_t - \mathbf{x}^*)$$

$$2(\nabla f_t(\mathbf{x}_t))^\top (\mathbf{x}_t - \mathbf{x}^*) \le \frac{\|\mathbf{x}_t - \mathbf{x}^*\|^2 - \|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2}{\eta_t} + \eta_t G^2.$$

- Summing above inequality from $t = 1$ to T and setting $\eta_t = \frac{D}{G\sqrt{t}}$ and $\frac{1}{\eta_0} := 0$ we have :

Proof of Theorem A (3/4)

$$
\begin{aligned}
2\left(\sum_{t=1}^{T} f_t(\mathbf{x}_t) - f_t(\mathbf{x}^*)\right) &\leq 2\sum_{t=1}^{T} (\nabla f_t(\mathbf{x}_t))^\top (\mathbf{x}_t - \mathbf{x}^*) \\
&\leq \sum_{t=1}^{T} \frac{\|\mathbf{x}_t - \mathbf{x}^*\|^2 - \|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2}{\eta_t} + G^2 \sum_{t=1}^{T} \eta_t \\
&\leq \sum_{t=1}^{T} \|\mathbf{x}_t - \mathbf{x}^*\|^2 \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right) + G^2 \sum_{t=1}^{T} \eta_t \\
&\leq D^2 \sum_{t=1}^{T} \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right) + G^2 \sum_{t=1}^{T} \eta_t \\
&\leq D^2 \frac{1}{\eta_T} + G^2 \sum_{t=1}^{T} \eta_t \\
&\leq 3DG\sqrt{T}.
\end{aligned}
$$

Proof of Theorem A (4/4)

Note that we can also deduce in this way

$$\sum_{t=1}^{T} \frac{\|\mathbf{x}_t - \mathbf{x}^*\|^2 - \|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2}{\eta_t} + G^2 \sum_{t=1}^{T} \eta_t$$

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\begin{aligned}
&\sum_{t=1}^{T} \frac{\|\mathbf{x}_t - \mathbf{x}^*\|^2 - \|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2}{\eta_t} + G^2 \sum_{t=1}^{T} \eta_t \\
= \quad & \frac{\|\mathbf{x}_1 - \mathbf{x}^*\|^2}{\eta_1} + \|\mathbf{x}_2 - \mathbf{x}^*\| \left(\frac{1}{\eta_2} - \frac{1}{\eta_1} \right) + \|\mathbf{x}_3 - \mathbf{x}^*\| \left(\frac{1}{\eta_3} - \frac{1}{\eta_2} \right) + \cdots
\end{aligned}
$$

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& + \|\mathbf{x}_T - \mathbf{x}^*\| \left(\frac{1}{\eta_T} - \frac{1}{\eta_{T-1}} \right) - \frac{\|\mathbf{x}_T - \mathbf{x}^*\|}{\eta_T} + G^2 \sum_{t=1}^{T} \eta_t
\end{aligned}
$$

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\begin{aligned}
& \sum_{t=1}^{T} \frac{\|\mathbf{x}_t - \mathbf{x}^*\|^2 - \|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2}{\eta_t} + G^2 \sum_{t=1}^{T} \eta_t \\
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\leq \; & \frac{D^2}{\eta_1} + D^2 \left(\frac{1}{\eta_T} - \frac{1}{\eta_1} \right) + G^2 \sum_{t=1}^{T} \eta_t \\
= \; & \frac{D^2}{\eta_T} + G^2 \sum_{t=1}^{T} \frac{D}{G\sqrt{t}} \leq
\end{aligned}
$$

Proof of Theorem A (4/4)

Note that we can also deduce in this way

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\begin{aligned}
& \sum_{t=1}^{T} \frac{\|\mathbf{x}_t - \mathbf{x}^*\|^2 - \|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2}{\eta_t} + G^2 \sum_{t=1}^{T} \eta_t \\
= \quad & \frac{\|\mathbf{x}_1 - \mathbf{x}^*\|^2}{\eta_1} + \|\mathbf{x}_2 - \mathbf{x}^*\| \left(\frac{1}{\eta_2} - \frac{1}{\eta_1} \right) + \|\mathbf{x}_3 - \mathbf{x}^*\| \left(\frac{1}{\eta_3} - \frac{1}{\eta_2} \right) + \cdots \\
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\leq \quad & \frac{\|\mathbf{x}_1 - \mathbf{x}^*\|^2}{\eta_1} + D^2 \sum_{t=2}^{T} \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) + G^2 \sum_{t=1}^{T} \eta_t \\
\leq \quad & \frac{D^2}{\eta_1} + D^2 \left(\frac{1}{\eta_T} - \frac{1}{\eta_1} \right) + G^2 \sum_{t=1}^{T} \eta_t \\
= \quad & \frac{D^2}{\eta_T} + G^2 \sum_{t=1}^{T} \frac{D}{G\sqrt{t}} \leq \frac{D^2}{D/(G\sqrt{T})} + DG(2\sqrt{T} - 1)
\end{aligned}
$$

Proof of Theorem A (4/4)

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&\sum_{t=1}^{T} \frac{\|\mathbf{x}_t - \mathbf{x}^*\|^2 - \|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2}{\eta_t} + G^2 \sum_{t=1}^{T} \eta_t \\
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\leq \quad & \frac{D^2}{\eta_1} + D^2 \left(\frac{1}{\eta_T} - \frac{1}{\eta_1} \right) + G^2 \sum_{t=1}^{T} \eta_t \\
= \quad & \frac{D^2}{\eta_T} + G^2 \sum_{t=1}^{T} \frac{D}{G\sqrt{t}} \leq \frac{D^2}{D/(G\sqrt{T})} + DG(2\sqrt{T} - 1) \leq 3DG\sqrt{T}.
\end{aligned}
$$