

Heaps

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Outline

1 Introduction

- Building a heap

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Heaps

Max Tree

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Max Heap

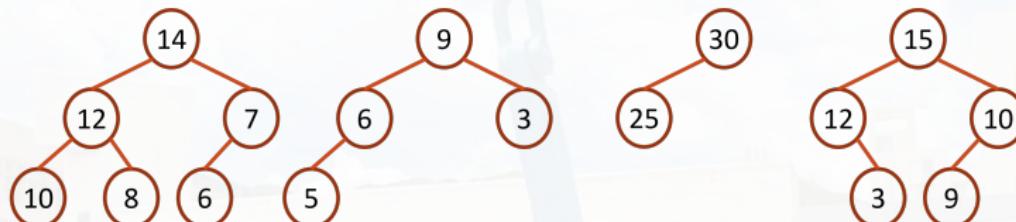
A complete binary tree that is also a max tree.

Min Heap

A complete binary tree that is also a min tree.



Examples: Max & Min Trees

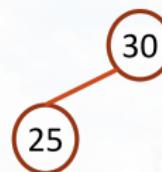
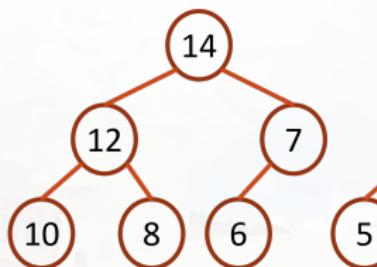


Max Trees

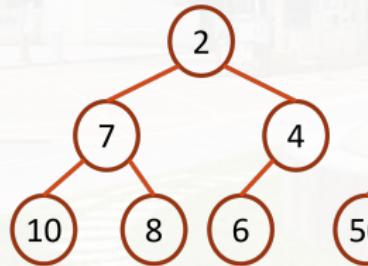


Min Trees

Examples: Max & Min Heaps



Max Heaps



Min Heaps

The Key Application: Priority Queues

- Heaps are frequently used to implement **priority queues**.
- In this kind of queue,
 - the element to be **deleted** is the one with **highest** (or **lowest**) priority.
 - at **any time**, an element with **arbitrary priority** can be **inserted** into the queue.

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- Insert the new node **next to the last element** in the array.
- A heap is a **complete** binary tree.

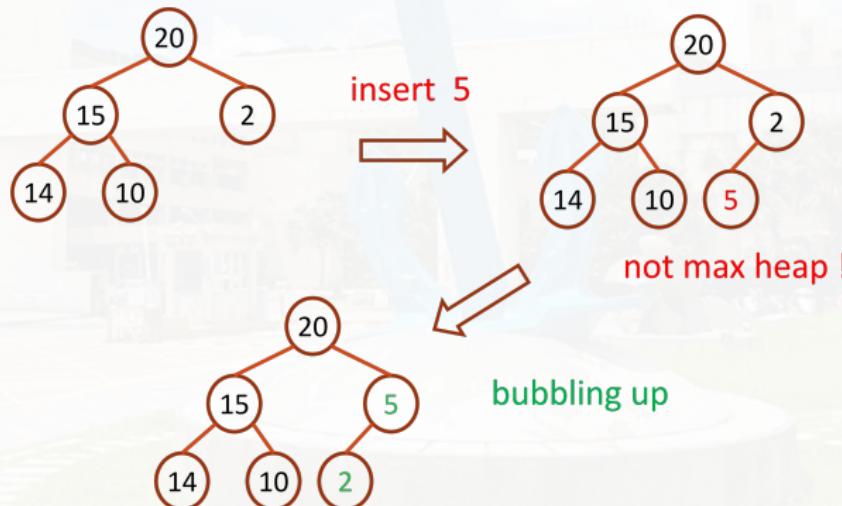
Insertion into a Max Heap

- The **bubbling process**.
 - It begins at the new node of the tree and moves toward the root.



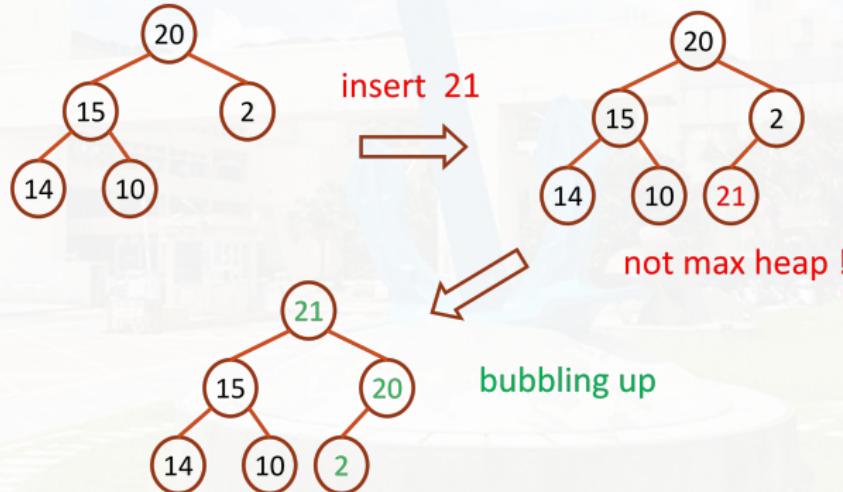
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The Code for Insertion into a Max Heap

- Consider the following declarations:

```
#define MAX_ELEMENTS 200 /* maximum heap size+1 */  
#define HEAP_FULL (n) (n == MAX_ELEMENTS -1)  
#define HEAP_EMPTY (n) (!n)  
typedef struct {  
    int key;  
    /* other fields */  
} element;  
element heap[MAX_ELEMENTS];  
int n = 0;
```

The Code for Insertion into a Max Heap

```
void push (element item, int *n) {
    /* insert item into a max heap of current size *n */
    int i;
    if (HEAP_FULL(*n)) {
        printf("The heap is full.\n");
        exit(EXIT_FAILURE);
    } // O(1) time
    i = ++(*n);
    while ((i != 1) && (item.key > heap[i/2].key)) {
        heap[i] = heap[i/2];
        i /= 2;
    } // O(lg n) time
    heap[i] = item; // O(1) time
}
```

- The time complexity of the insertion: $O(\lg n)$.

Deletion from a Max Heap

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- When an element is to be deleted from a max heap, it is **ALWAYS** taken from the root of the heap.
- The steps of deletion from a Max heap:
 - delete the root node.
 - insert the last node into the root (say r).
 - use the **bubbling up process** to ensure that the resulting heap remains a max heap (a.k.a. **heapify** at r).

Illustration of Deletion from a Max Heap

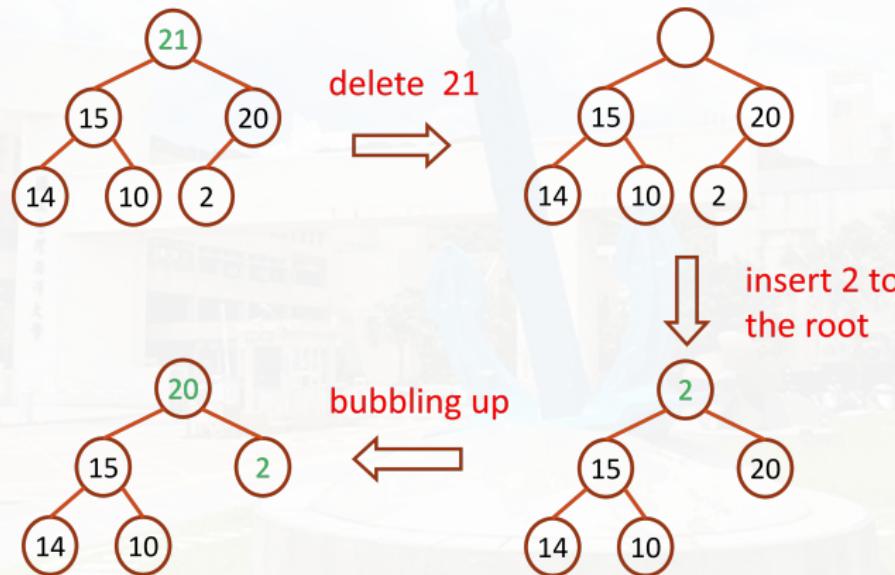
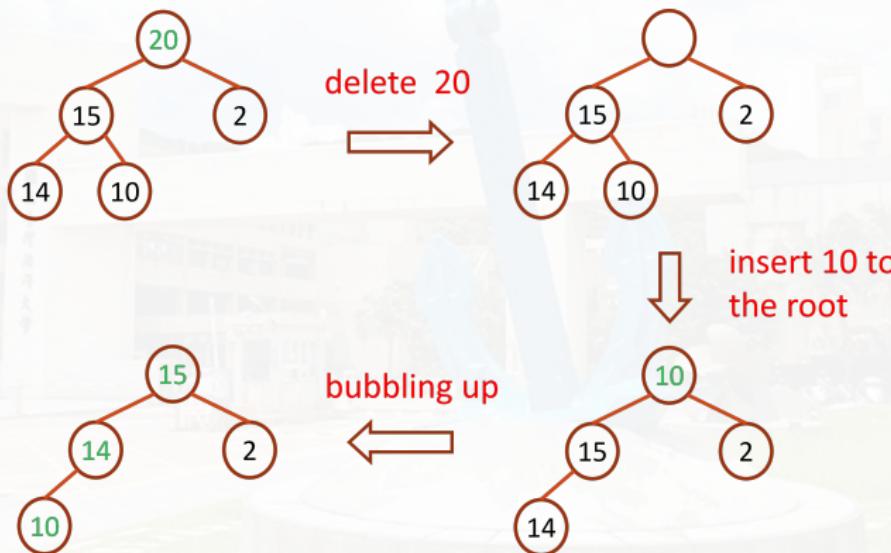


Illustration of Deletion from a Max Heap



The Code for Deletion from a Max Heap

```
element pop(int *n) {
    /* delete element with the highest key from the heap */
    int parent, child;
    element item, temp;
    if (HEAP_EMPTY(*n)) {
        fprintf(stderr, "The heap is empty\n");
        exit(EXIT_FAILURE);
    }
    /* save value of the element with the highest key */
    item = heap[1];
    /* use last element in heap to adjust heap */
    temp = heap[(*n)--];
    parent = 1;
    child = 2; // default: the left child
    while (child <= *n) { // O(lg n) time
        /* find the larger child of the current parent */
        if ((child < *n) && (heap[child].key < heap[child+1].key))
            child++; // okay, it's the right child!
        if (temp.key >= heap[child].key) break; // the new root is the maximum!
        /* if the max-child gets larger key, move to the next lower level */
        heap[parent] = heap[child];
        parent = child;
        child *= 2;
    }
    heap[parent] = temp;
    return item;
}
```

Time Complexity of the Deletion from a Max Heap

- Delete the root node: $O(1)$.
- Insert the last node to the root: $O(1)$.
- Since the height of the heap is $\lceil \lg(n + 1) \rceil$, the while loop is iterated for $O(\lg n)$ times.
- Thus, the overall time complexity: the time complexity of the deletion: $O(\log n)$.

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- The above process is correct and requires $O(n \log n)$ time.

An $O(n)$ time algorithm for building a (max) heap

Input: n numbers: x_1, x_2, \dots, x_n .

Efficient Heap Construction

- ① For each input number x_i , insert x_i into array A at $A[i - 1]$ one by one.
- ② For $i = \lfloor n/2 \rfloor - 1$ down to 0:
 - Run $\text{heapify}(A, i)$

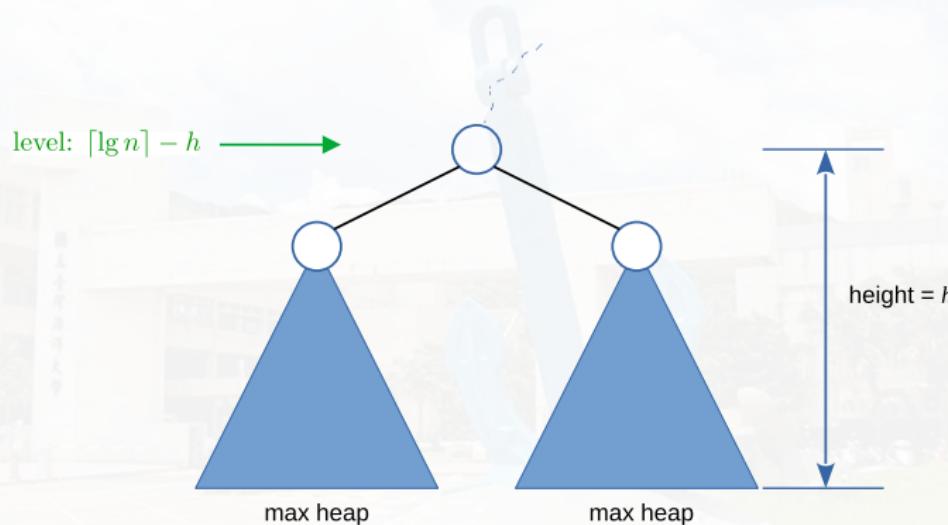
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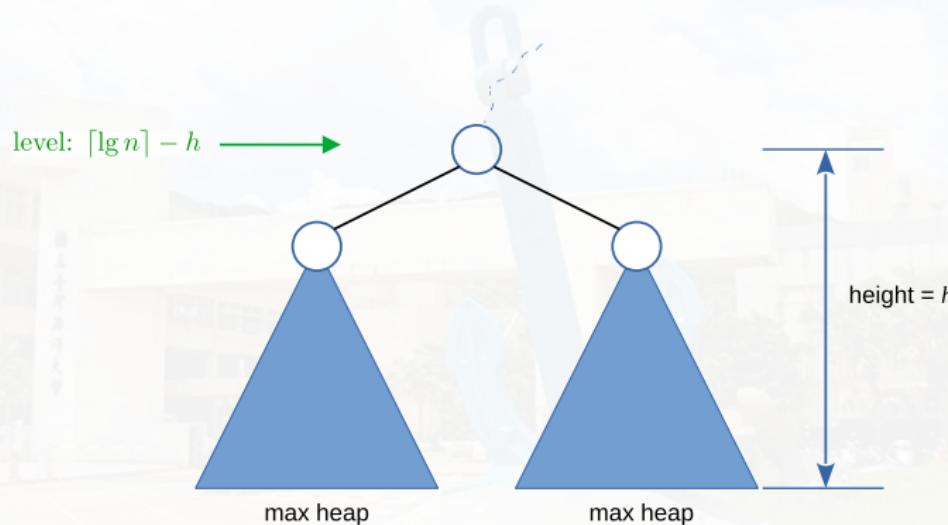
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- That is, we build a heap in a bottom-up fashion!

Heap recursive view (bottom-up)

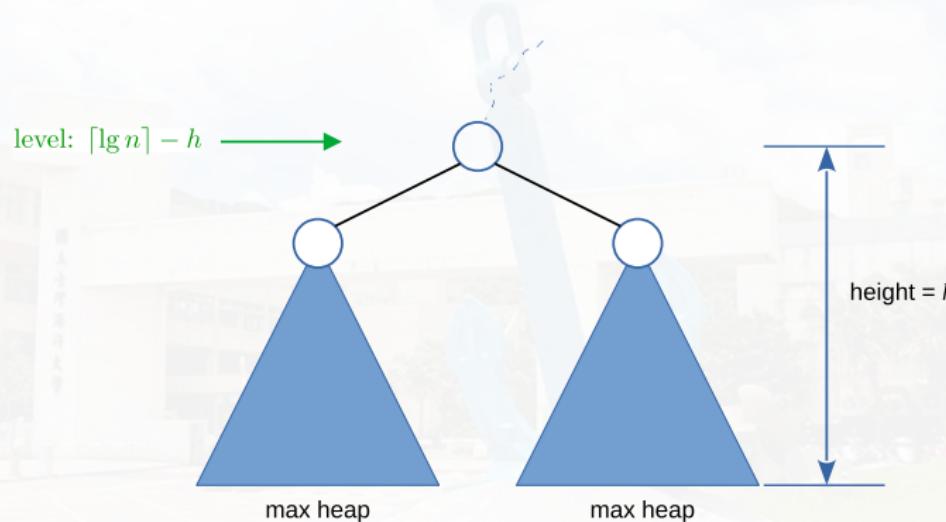


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- Take the maximum of the roots of the two subtrees.

heapify(A, r) for a Max Heap of n nodes (複習)

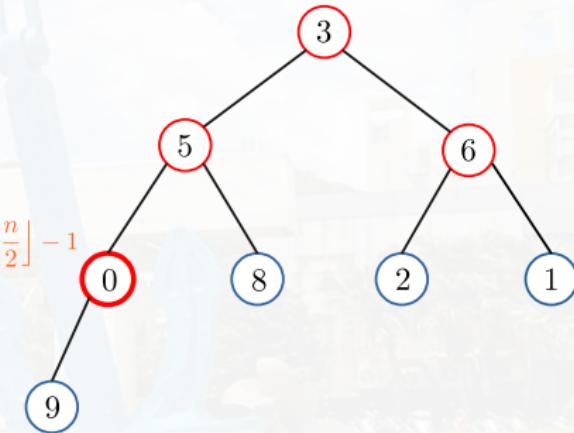
- Assumption: the left and right subtrees of r are already max heaps.

```
void heapify(element heap[], int i) {
    int parent = i; int child = 2 * parent; // default child: left child
    element temp = heap[parent];
    // each loop goes down one level, so O(log n) time in a complete binary tree.
    while (child <= n) {
        /* choose the larger child */
        if (child < n && heap[child].key < heap[child + 1].key) child++;
        /* if temp already >= max(child), we're done */
        if (temp.key >= heap[child].key) break;
        /* move the larger child up one level */
        heap[parent] = heap[child];
        parent = child;
        child = 2 * parent;
    }
    heap[parent] = temp;
}
```

Nodes to be Heapified

index

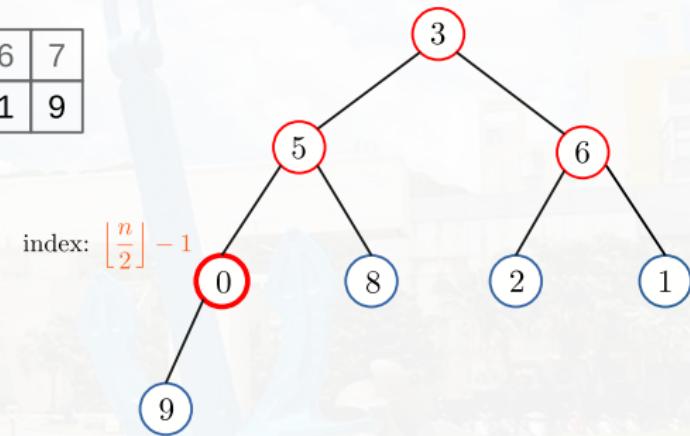
0	1	2	3	4	5	6	7
3	5	6	0	8	2	1	9

index: $\left\lfloor \frac{n}{2} \right\rfloor - 1$ 

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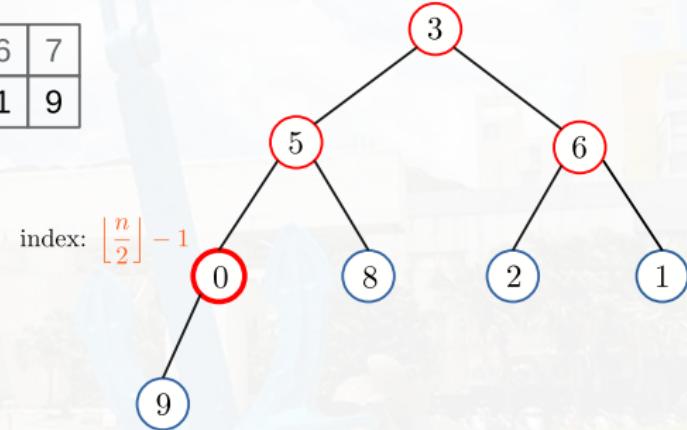
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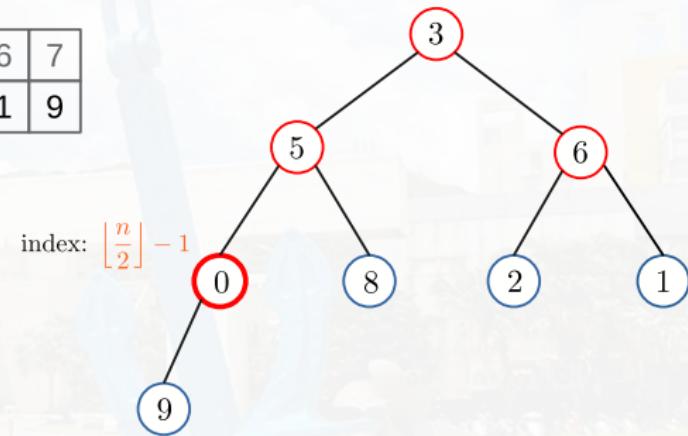


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- # Heapify steps: $\leq \sum_{h=1}^{\lg n-1} h \cdot n_h = \sum_{h=1}^{\lg n-1} h \cdot 2^{\lceil \lg n \rceil - h} \leq 2n \sum_{h=1}^{\lg n-1} \frac{h}{2^h}$.
- n_h : the number of nodes at level h .

Exercise

Show that $\sum_{h=0}^{\infty} \frac{h}{2^h} = O(1)$.



Discussions

