

# A Sketch of Nash's Theorem from Fixed Point Theorems

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# Reference

- Lecture Notes in 6.853 Topics in Algorithmic Game Theory [[link](#)].
- *Fixed Point Theorems and Applications to Game Theory*. Allen Yuan. The University of Chicago Mathematics REU 2017. [[link](#)].
  - REU = Research Experience for Undergraduate students.



# Outline

- 1 Brouwer's Fixed Point Theorem
  - Sketch of the Proof of Nash's Theorem (from Brouwer's Theorem)
- 2 Kakutani's Fixed Point Theorem
  - Pure Nash Equilibria of Pure Strategic Games
    - Preliminaries
    - Main Theorem I & The Proof
  - Mixed Nash Equilibria of Finite Strategies Games
    - Preliminaries & Assumptions
    - Main Theorem II & the Proof



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## 1 Brouwer's Fixed Point Theorem

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- Pure Nash Equilibria of Pure Strategic Games
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# The Setting

- A set  $N$  of  $n$  players.
- Strategy set  $S_i = \{s_{i,1}, \dots, s_{i,k_i}\}$  for each player  $i \in N$ ,  $k_i$  is bounded.
- Utility function:  $u_i$  for each player  $i$ .
- $\Delta := \Delta_1 \times \Delta_2 \times \dots \times \Delta_n$ : a Cartesian product of  $(\Delta_i)_{i \in N}$ .
  - For  $x \in \Delta$ ,  $x_i(s)$  denotes the probability mass on strategy  $s \in S_i$ .
  - $\Delta_i = \{(x_i(s_{i,1}), x_i(s_{i,2}), \dots, x_i(s_{i,k_i})) \mid x_i(s_{i,j}) \geq 0 \ \forall j; \sum_j x_i(s_{i,j}) = 1\}$ .
  - $x_i \in \Delta_i$ : a **mixed strategy**.



# Nash's Theorem

Nash (1950)

Every game  $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$  has a Nash equilibrium.

- **Note:**  $u_i(\mathbf{x}) := \sum_{s \in S_i} x_i(s) \cdot u_i(s; \mathbf{x}_{-i})$ .



# Nash's Theorem

## Nash (1950)

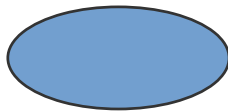
Every game  $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$  has a Nash equilibrium.

- **Note:**  $u_i(\mathbf{x}) := \sum_{s \in S_i} x_i(s) \cdot u_i(s; \mathbf{x}_{-i})$ .
- No player wants to deviate to the other strategy unilaterally.



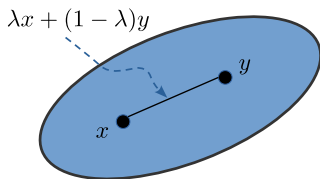


open &  
bounded

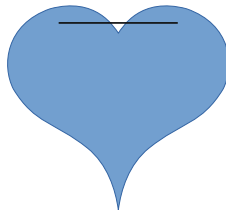


closed &  
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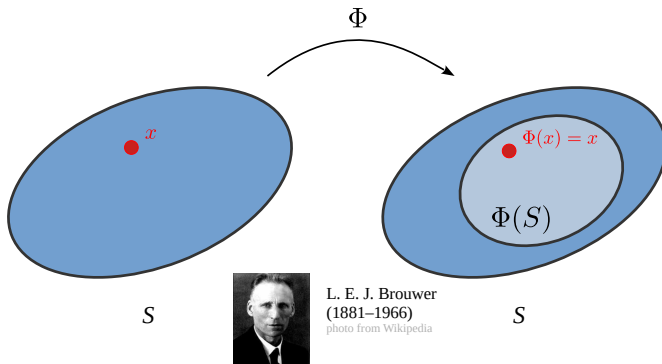


convex



not convex

# Fixed Point



# Brouwer's Fixed Point Theorem

## Brouwer's Fixed-Point Theorem

Let  $D$  be a convex, compact (closed and bounded) subset of the Euclidean space. If  $f : D \mapsto D$  is continuous, then there exists  $x \in D$  such that

$$f(x) = x.$$

- **Idea:** We want the function  $f$  to satisfy the conditions of Brouwer's fixed point theorem.



# Brouwer's Fixed Point Theorem

## Brouwer's Fixed-Point Theorem

Let  $D$  be a convex, compact (closed and bounded) subset of the Euclidean space. If  $f : D \mapsto D$  is continuous, then there exists  $x \in D$  such that

$$f(x) = x.$$

- **Idea:** We want the function  $f$  to satisfy the conditions of Brouwer's fixed point theorem.
- Try to relate utilities of players to a function  $f$  like above.



# The Gain function

## Gain

Suppose that  $\mathbf{x}' \in \Delta$  is given. For a player  $i$  and strategy  $s_i \in S_i$  (or  $s_i \in \Delta_i$ ), we define the **gain** as

$$\text{Gain}_{i,s_i}(\mathbf{x}') = \max\{u_i(s_i; \mathbf{x}'_{-i}) - u_i(\mathbf{x}), 0\},$$

which is non-negative.

- $\mathbf{x}'_{-i} := (x_j)_{j \in N}, (x_{-i}, x_i) = \mathbf{x}$ .
- It's equal to the increase in payoff for player  $i$  if he/she were to switch to pure strategy  $s_i$ .



# Proof of Nash's Theorem (Define a response function)

- Define a function  $f : \Delta \mapsto \Delta$  that satisfies the conditions of Brouwer's fixed point theorem.
- For all  $\mathbf{x} \in \Delta$ ,  $\mathbf{y} = f(\mathbf{x})$  where for all  $i \in N$  and  $s_i \in S_i$ ,

$$y_i(s_i) := \frac{x_i(s_i) + \text{Gain}_{i;s_i}(\mathbf{x})}{1 + \sum_{s'_i \in S_i} \text{Gain}_{i;s'_i}(\mathbf{x})}.$$

- $f$  tries to boost the probability mass where strategy switching results in gains in payoff.



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- $f : \Delta \mapsto \Delta$  is continuous (verify this by yourself).
- $\Delta$  is a product of simplices so it is convex (verify this by yourself).
- $\Delta$  is closed and bounded, so it is compact.



# Proof of Nash's Theorem (Define a response function)

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- $f : \Delta \mapsto \Delta$  is continuous (verify this by yourself).
  - $\Delta$  is a product of simplices so it is convex (verify this by yourself).
  - $\Delta$  is closed and bounded, so it is compact.
- ★ Brouwer's fixed point theorem guarantees the existence of a fixed point of  $f$ .





**Claim:** Any fixed point of  $f$  is a Nash equilibrium

- It suffices to prove that a fixed point  $\mathbf{x} = f(\mathbf{x})$  satisfies:
  - $\text{Gain}_{i;s_i}(\mathbf{x}) = 0$ , for each  $i \in N$  and each  $s_i \in S_i$ .



**Claim:** Any fixed point of  $f$  is a Nash equilibrium

Prove it by contradiction.

- Assume that there is some player  $p$  who can improve his/her payoff by switching to a pure strategy, say  $s_p$ :
  - $\text{Gain}_{p;s_p}(\mathbf{x}) > 0$ .



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Prove it by contradiction.

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  - $\text{Gain}_{p;s_p}(\mathbf{x}) > 0$ .
- Note that we must have  $x_p(s_p) > 0$ , otherwise  $\mathbf{x}$  cannot be a fixed point of  $f$ .
  - From the definition of  $f$ ; the numerator would be  $> 0$ .

$$y_p(s_p) := \frac{x_p(s_p) + \text{Gain}_{p;s_p}(\mathbf{x})}{1 + \sum_{s'_p \in S_p} \text{Gain}_{p;s'_p}(\mathbf{x})}.$$



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Prove it by contradiction.

- Assume that there is some player  $p$  who can improve his/her payoff by switching to a pure strategy, say  $s_p$ :
  - $\text{Gain}_{p;s_p}(\mathbf{x}) > 0 \Rightarrow u_p(s_p; \mathbf{x}_{-p}) - u_p(\mathbf{x}) > 0.$



# Claim: Any fixed point of $f$ is a Nash equilibrium

Prove it by contradiction.

- Assume that there is some player  $p$  who can improve his/her payoff by switching to a pure strategy, say  $s_p$ :
  - $\text{Gain}_{p;s_p}(\mathbf{x}) > 0 \Rightarrow u_p(s_p; \mathbf{x}_{-p}) - u_p(\mathbf{x}) > 0$ .
- We argue that there must be some other pure strategy  $\hat{s}_p$  such that:
  - $x_p(\hat{s}_p) > 0$  and
  - $u_p(\hat{s}_p; \mathbf{x}_{-p}) - u_p(\mathbf{x}) < 0$

★ Notice that

$$u_p(\mathbf{x}) := \sum_{s \in S_p} x_p(s) \cdot u_p(s; \mathbf{x}_{-p}).$$



# Claim: Any fixed point of $f$ is a Nash equilibrium

Prove it by contradiction.

- Assume that there is some player  $p$  who can improve his/her payoff by switching to a pure strategy, say  $s_p$ :
  - $\text{Gain}_{p;s_p}(\mathbf{x}) > 0 \Rightarrow u_p(s_p; \mathbf{x}_{-p}) - u_p(\mathbf{x}) > 0.$
- We argue that there must be some other pure strategy  $\hat{s}_p$  such that:
  - $x_p(\hat{s}_p) > 0$  and
  - $u_p(\hat{s}_p; \mathbf{x}_{-p}) - u_p(\mathbf{x}) < 0 \Rightarrow \text{Gain}_{p;\hat{s}_p}(\mathbf{x}) = 0.$

★ Notice that

$$u_p(\mathbf{x}) := \sum_{s \in S_p} x_p(s) \cdot u_p(s; \mathbf{x}_{-p}).$$

- We obtain that ( $\mathbf{x}$  is not a fixed point  $\Rightarrow \nRightarrow$ )

$$y_p(\hat{s}_p) := \frac{x_p(\hat{s}_p) + \text{Gain}_{p;\hat{s}_p}(\mathbf{x})}{1 + \sum_{s'_p \in S_p} \text{Gain}_{p;s'_p}(\mathbf{x})} < x_p(\hat{s}_p).$$



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# An Extension of Brouwer's work

- Focus: **set-valued** functions.
  - Refer here for further readings.
  - Why do we consider set-valued functions?



# An Extension of Brouwer's work

- Focus: **set-valued** functions.
  - Refer here for further readings.
  - Why do we consider set-valued functions?
    - Best-responses.



# Upper Semi-Continuous (having a closed graph)

## Upper semi-continuous functions

Let

- $\mathbb{P}(X)$ : all nonempty, closed, convex subsets of  $X$ .
- $S$ : a nonempty, compact, and convex set.

Then the set-valued function  $\Phi : S \mapsto \mathbb{P}(S)$  is **upper semi-continuous** if

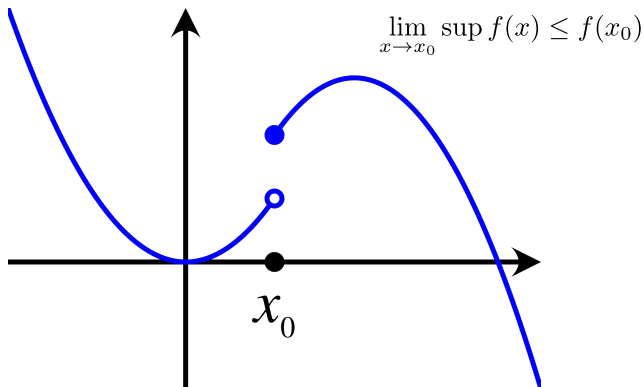
for arbitrary sequences  $(\mathbf{x}_n)_{n \in \mathbb{N}}, (\mathbf{y}_n)_{n \in \mathbb{N}}$  in  $S$ , we have

- $\lim_{n \rightarrow \infty} \mathbf{x}_n = \mathbf{x}_0$ ,
- $\lim_{n \rightarrow \infty} \mathbf{y}_n = \mathbf{y}_0$ ,
- $\mathbf{y}_n \in \Phi(\mathbf{x}_n)$  for all  $n \in \mathbb{N}$ ,

imply that  $\mathbf{y}_0 \in \Phi(\mathbf{x}_0)$ .

Removable discontinuity, Sequentially compact, Bolzano–Weierstrass theorem.





(Figure from Wikipedia)

# Fixed Point of Set-Valued Functions

## Fixed Point (Set-Valued)

A fixed point of a set-valued function  $\Phi : S \mapsto \mathbb{P}(S)$  is a point  $\mathbf{x}^* \in S$  such that  $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$ .



# Kakutani's Theorem for Simplices

## Kakutani's Theorem for Simplices (1941)

If  $S$  is an  $r$ -dimensional closed simplex in a Euclidean space and  $\Phi : S \mapsto \mathbb{P}(S)$  is upper semi-continuous, then  $\Phi$  has a fixed point.



# Kakutani's Fixed-Point Theorem

## Kakutani's Fixed-Point Theorem (1941)

If  $S$  is a **nonempty, compact, convex set** in a Euclidean space and  $\Phi : S \mapsto \mathbb{P}(S)$  is upper semi-continuous, then  $\Phi$  has a fixed point.



# Kakutani's Fixed-Point Theorem

## Kakutani's Fixed-Point Theorem (1941)

If  $S$  is a **nonempty, compact, convex set** in a Euclidean space and  $\Phi : S \mapsto \mathbb{P}(S)$  is upper semi-continuous, then  $\Phi$  has a fixed point.

- We won't go over its proof.
- Instead, we will delve into how it can be used to prove Nash's Theorem from the perspectives of set-valued functions and best-responses.





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# Cartesian product of Sets

## Cartesian Product

For a family of sets  $\{A_i\}_{i \in N}$ ,  $\prod_{i \in N} A_i = A_1 \times A_2 \times \cdots \times A_n$  denotes the Cartesian product of  $A_i$  for  $i \in N$ .

## Profile

for  $x_i \in A_i$ , then  $(x_i)_{i \in N}$  is called a (strategy) profile.



# Binary Relation

## Binary Relation

- A binary relation on a set  $A$  is a subset of  $A \times A$  consisting of all pairs of elements.
- For  $a, b \in A$ , we denote by  $R(a, b)$  if  $a$  is related to  $b$ .



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## Properties on Binary Relations

- **Completeness:** For all  $a, b \in A$ , we have  $R(a, b)$ ,  $R(b, a)$ , or both.
- **Reflexivity:** For all  $a \in A$ , we have  $R(a, a)$ .
- **Transitivity:** For  $a, b, c \in A$ , if  $R(a, b)$  and  $R(b, c)$ , then we have  $R(a, c)$ .



# Preference Relation

## Preference Relation

A preference relation is a **complete, reflexive, and transitive** binary relation.

- Denote by  $a \succsim b$  if  $a$  is related to  $b$ .
- Denote by  $a \succ b$  if  $a \succsim b$  but  $b \not\succsim a$ .
- Denote by  $a \sim b$  if  $a \succsim b$  and  $b \succsim a$ .



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  - Denote by  $a \sim b$  if  $a \succsim b$  and  $b \succsim a$ .
- 
- $a \succsim b$ :  $a$  is **weakly preferred to**  $b$ .
  - $a \sim b$ : agent is indifferent between  $a$  and  $b$ .



# Continuity on a Preference relation

## Continuous Preference Relation

A preference relation is **continuous** if:

whenever there exist sequences  $(a_k)_{k \in \mathbb{N}}$  and  $(b_k)_{k \in \mathbb{N}}$  in  $A$  such that

- $\lim_{k \rightarrow \infty} a_k = a$ ,
- $\lim_{k \rightarrow \infty} b_k = b$ ,
- and  $a_k \succsim b_k$  for all  $k \in \mathbb{N}$

we have  $a \succsim b$ .



# Strategic Games

## Strategic Games

A strategic game is a tuple  $\langle N, (A_i), (\succsim_i) \rangle$  consisting of

- a finite set of **players**  $N$ .
  - for each player  $i \in N$ , a nonempty set of **actions**  $A_i$ .
  - for each player  $i \in N$ , a **preference relation**  $\succsim_i$  on  $A = \prod_{j \in N} A_j$ .
- 
- A strategic game is **finite** if  $A_i$  is finite for all  $i \in N$ .





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- 
- A strategic game is **finite** if  $A_i$  is finite for all  $i \in N$ .
  - **Note:**  $\succsim_i$  is not defined on  $A_i$  only, but instead on the set of all  $(A_j)_{j \in N}$ .



# PNE w.r.t. a Preference Relation

## Pure Nash Equilibrium (PNE) with $(\succsim_i)$

A (pure) Nash equilibrium (PNE) of a strategic game  $\langle N, (A_i), (\succsim_i) \rangle$  is a profile  $\mathbf{a}^* := (a_i)_{i \in N}$  such that for all  $i \in N$ , we have

$$(\mathbf{a}_{-i}^*, a_i^*) \succsim_i (\mathbf{a}_{-i}^*, a'_i) \text{ for all } a'_i \in A_i.$$



# Best-Response Function

## Best-Response Functions

The **best-response** function of player  $i$ ,

$$BR_i : \prod_{j \in N \setminus \{i\}} A_j \mapsto \mathbb{P}(A_i),$$

is given by

$$BR_i(\mathbf{a}_{-i}) = \{a_i \in A_i \mid (\mathbf{a}_{-i}, a_i) \succeq_i (\mathbf{a}_{-i}, a'_i) \text{ for all } a'_i \in A_i\}.$$



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- $BR_i$  is set-valued.
- Recall:  $\mathbb{P}(X)$  includes all nonempty, closed, and convex subsets of  $X$ .



# PNE w.r.t. a Preference Relation

- Alternative definition of NE.

## Pure Nash Equilibrium (PNE) with $(\succsim_i)$

A Nash equilibrium of a strategic game  $\langle N, (A_i), (\succsim_i) \rangle$  is a profile  $\mathbf{a}^* := (a_i)_{i \in N}$  such that  $\mathbf{a}_i^* \in BR_i(\mathbf{a}_{-i}^*)$  for all  $i \in N$ .

- Thus, to prove the existence of a PNE for a strategic game  $\langle N, (A_i), (\succsim_i) \rangle$ , it suffices to show that:



# PNE w.r.t. a Preference Relation

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- Thus, to prove the existence of a PNE for a strategic game  $\langle N, (A_i), (\succsim_i) \rangle$ , it suffices to show that:
  - There exists a profile  $\mathbf{a}^* \in A$  such that for all  $i \in N$  we have  $\mathbf{a}_i^* \in BR_i(\mathbf{a}_{-i}^*)$ .



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# General Idea

- Let  $BR : A \mapsto \mathbb{P}(A)$  be

$$BR(\mathbf{a}) = \prod_{i \in N} BR_i(\mathbf{a}_{-i}).$$

- We seek for some  $\mathbf{a}^* \in A$  such that  $\mathbf{a}^* \in BR(\mathbf{a}^*)$ .





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- We can then use Kakutani's Fixed-Point Theorem to show that  $\mathbf{a}^*$  exists.



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- We seek for some  $\mathbf{a}^* \in A$  such that  $\mathbf{a}^* \in BR(\mathbf{a}^*)$ .
- We can then use Kakutani's Fixed-Point Theorem to show that  $\mathbf{a}^*$  exists.
- Yet, we need to verify the conditions under which Kakutani's Fixed-Point Theorem holds.



# Quasi-Concave

## Quasi-Concave of $\succsim_i$

A preference relation  $\succsim_i$  over  $A$  is **quasi-concave** on  $A_i$  if for all  $\mathbf{a} \in A$ , the set

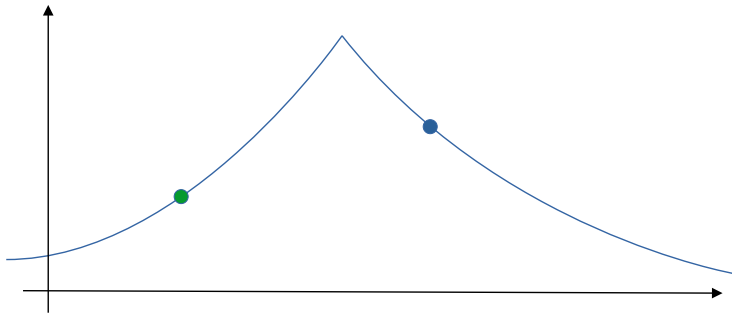
$$\{a'_i \in A_i \mid (\mathbf{a}_{-i}, a'_i) \succsim_i (\mathbf{a}_{-i}, a_i)\}$$

is **convex**.

- Then, we can consider the following theorem which guarantees the condition of a PNE.



An example of quasi-concave function.



$$f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}, \text{ for } \lambda \in [0, 1]$$



# The Main Theorem I

## Main Theorem I

The strategic game  $\langle N, (A_i), (\succsim_i) \rangle$  has a (pure) Nash equilibrium if

- $A_i$  is a nonempty, compact, and convex subset of a Euclidean space
  - $\succsim_i$  is continuous and quasi-concave on  $A_i$  for all  $i \in N$ .
- 
- We will show that  $A$  (cf.  $S$ ) and  $BR$  (cf.  $\Phi$ ) satisfy the conditions to apply Kakutani's Fixed-Point Theorem.



# Requirements for $A$ & $BR$

- $A_i$  is nonempty, compact and convex for all  $i \in N$ , so their Cartesian product (i.e.,  $A$ ) must also be nonempty, compact and convex.



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- Note that in Kakutani's Theorem,  $\Phi : S \mapsto \mathbb{P}(S)$ , where  $\mathbb{P}(S)$  denotes all **nonempty, closed, and convex** subsets of  $S$ .



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- Note that in Kakutani's Theorem,  $\Phi : S \mapsto \mathbb{P}(S)$ , where  $\mathbb{P}(S)$  denotes all **nonempty, closed, and convex** subsets of  $S$ .
- We need to show that  $BR_i(\mathbf{a}_{-i})$  is nonempty, closed, and convex for all  $\mathbf{a}_{-i} \in \prod_{j \in N \setminus \{i\}} A_j$ .





# Requirements for $A$ & $BR$

- $A_i$  is nonempty, compact and convex for all  $i \in N$ , so their Cartesian product (i.e.,  $A$ ) must also be nonempty, compact and convex.
- Note that in Kakutani's Theorem,  $\Phi : S \mapsto \mathbb{P}(S)$ , where  $\mathbb{P}(S)$  denotes all **nonempty, closed, and convex** subsets of  $S$ .
- We need to show that  $BR_i(\mathbf{a}_{-i})$  is nonempty, closed, and convex for all  $\mathbf{a}_{-i} \in \prod_{j \in N \setminus \{i\}} A_j$ .
  - Their Cartesian product  $BR(\mathbf{a})$  is then nonempty, closed and convex, too.
  - We then have  $BR : A \mapsto \mathbb{P}(A)$ .



$BR_i(\mathbf{a}_{-i})$  is nonempty

- Assume that we can construct a continuous function (utility function)  $u_i : A_i \mapsto \mathbb{R}$  such that for  $a_i, a'_i \in A_i$ ,  $(\mathbf{a}_{-i}, a_i) \succsim (\mathbf{a}_{-i}, a'_i)$  if and only if  $u_i(a_i) \geq u_i(a'_i)$ .



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- Since  $A_i$  is compact and  $u_i$  is continuous,  $u_i(A_i)$  is compact as well.
- By the [Extreme Value Theorem](#), there must exist some  $a_i^* \in A_i$  such that  $u_i(a_i^*) \geq u_i(a_i)$  for all  $a_i \in A_i$ .



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- By definition of  $u_i$ , it follows that  $(\mathbf{a}_{-i}, a_i^*) \succsim (\mathbf{a}_{-i}, a_i)$  for all  $a_i \in A_i$ , thus  $a_i^* \in BR_i(\mathbf{a}_{-i})$ .



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- So  $BR_i(\mathbf{a}_{-i})$  is nonempty.



$BR_i(\mathbf{a}_{-i})$  is closed

- Take an arbitrary  $p \in \overline{BR_i(\mathbf{a}_{-i})}$ .
- There must exist some sequence  $(p_k)_{k \in \mathbb{N}}$  such that  $p_k \in BR_i(\mathbf{a}_{-i})$  for all  $k \in \mathbb{N}$  and  $\lim_{k \rightarrow \infty} p_k = p$ .



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  - a sequence  $((\mathbf{a}_{-i}, a_i))_{k \in \mathbb{N}}$  such that  $\lim_{k \rightarrow \infty} (\mathbf{a}_{-i}, a_i) = (\mathbf{a}_{-i}, a_i)$ .





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- $\Rightarrow p \in BR_i(\mathbf{a}_{-i})$



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    - By the continuity of  $\succsim_i$ , we have  $(\mathbf{a}_{-i}, p) \succsim_i (\mathbf{a}_{-i}, a_i)$  for all  $a_i \in A_i$ .
- $\Rightarrow p \in BR_i(\mathbf{a}_{-i})$  ( $\therefore BR_i(\mathbf{a}_{-i})$  is closed).



$BR_i(\mathbf{a}_{-i})$  is convex

- Consider  $a_i \in BR_i(\mathbf{a}_{-i})$ .
- $\succsim_i$  is quasi-concave on  $A_i \Rightarrow$



$BR_i(\mathbf{a}_{-i})$  is convex

- Consider  $a_i \in BR_i(\mathbf{a}_{-i})$ .
- $\succsim_i$  is quasi-concave on  $A_i \Rightarrow$

$$S = \{a'_i \in A_i \mid (\mathbf{a}_{-i}, a'_i) \succsim_i (\mathbf{a}_{-i}, a_i)\} \text{ is convex}$$

- Since  $a_i$  is a best response, the responses  $a'_i$  weakly preferable to  $a_i$  must be also best responses.



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- Since  $a_i$  is a best response, the responses  $a'_i$  weakly preferable to  $a_i$  must be also best responses.  $\Rightarrow S \subseteq BR_i(\mathbf{a}_{-i})$ .



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- $\succsim_i$  is quasi-concave on  $A_i \Rightarrow$

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- Since  $a_i$  is a best response, the responses  $a'_i$  weakly preferable to  $a_i$  must be also best responses.  $\Rightarrow S \subseteq BR_i(\mathbf{a}_{-i})$ .
- Any other best response  $a_i^* \in BR_i(\mathbf{a}_{-i})$  must be at least good as  $a_i$



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- Any other best response  $a_i^* \in BR_i(\mathbf{a}_{-i})$  must be at least good as  $a_i$   
 $\Rightarrow BR_i(\mathbf{a}_{-i}) \subseteq S$ .
- Hence, we have  $BR_i(\mathbf{a}_{-i}) = S$ , so  $BR_i(\mathbf{a}_{-i})$  is convex.





- Next, we will show that  $BR$  is upper semi-continuous.



# Recall: Upper Semi-Continuous

## Upper semi-continuous functions

Let

- $\mathbb{P}(X)$ : all nonempty, closed, convex subsets of  $X$ .
- $S$ : a nonempty, compact, and convex set.

Then the set-valued function  $\Phi : S \mapsto \mathbb{P}(S)$  is **upper semi-continuous** if

for arbitrary sequences  $(\mathbf{x}_n)_{n \in \mathbb{N}}, (\mathbf{y}_n)_{n \in \mathbb{N}}$  in  $S$ , we have

- $\lim_{n \rightarrow \infty} \mathbf{x}_n = \mathbf{x}_0$ ,
  - $\lim_{n \rightarrow \infty} \mathbf{y}_n = \mathbf{y}_0$ ,
  - $\mathbf{y}_n \in \Phi(\mathbf{x}_n)$  for all  $n \in \mathbb{N}$ ,
- imply that  $\mathbf{y}_0 \in \Phi(\mathbf{x}_0)$ .



# $BR$ is upper semi-continuous

- Consider two sequences  $(\mathbf{x}^k), (\mathbf{y}^k)$  in  $A$  such that

$$\lim_{k \rightarrow \infty} \mathbf{x}^k = \mathbf{x}^0,$$

$$\lim_{k \rightarrow \infty} \mathbf{y}^k = \mathbf{y}^0.$$

$$\mathbf{y}^k \in BR_i(\mathbf{x}^k) \text{ for all } k \in \mathbb{N}.$$

- Then we have  $\mathbf{y}_i^k \in BR_i(\mathbf{x}_{-i}^k)$  for all  $i \in N, k \in \mathbb{N}$ .



## $BR$ is upper semi-continuous

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$$\mathbf{y}^k \in BR_i(\mathbf{x}^k) \text{ for all } k \in \mathbb{N}.$$

- Then we have  $y_i^k \in BR_i(\mathbf{x}_{-i}^k)$  for all  $i \in N, k \in \mathbb{N}$ .
- For an arbitrary  $i \in N$ , we have  $(\mathbf{x}_{-i}^k, y_i^k) \succsim_i (\mathbf{x}_{-i}^k, a_i)$  for all  $a_i \in A_i$  and  $k \in \mathbb{N}$  ( $\because$  best response).



## $BR$ is upper semi-continuous (contd.)

- For each  $a_i \in A_i$ , we can construct:
  - a sequence  $((\mathbf{x}_{-i}^k, y_i^k))_{k \in \mathbb{N}}$  such that  $\lim_{k \rightarrow \infty} (\mathbf{x}_{-i}^k, y_i^k) = (\mathbf{x}_{-i}^0, y_i^0)$ .
  - a sequence  $((\mathbf{x}_{-i}^k, a_i))_{k \in \mathbb{N}}$  such that  $\lim_{k \rightarrow \infty} (\mathbf{x}_{-i}^k, a_i) = (\mathbf{x}_{-i}^0, a_i)$ .
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  - By continuity of  $\succsim_i$ , we have  $(\mathbf{x}_{-i}^0, y_i^0) \succsim_i (\mathbf{x}_{-i}^0, a_i)$  for all  $a_i \in A_i$ .



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- Thus, we have  $y_i^0 \in BR_i(\mathbf{x}_{-i}^0)$  for all  $i \in N$ .
  - $\mathbf{y}^0 \in BR(\mathbf{x}^0)$ .



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- Thus, we have  $y_i^0 \in BR_i(\mathbf{x}_{-i}^0)$  for all  $i \in N$ .
  - $\mathbf{y}^0 \in BR_i(\mathbf{x}^0)$ .
- Therefore,  $BR$  is upper semi-continuous.



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- Thus, we have  $y_i^0 \in BR_i(\mathbf{x}_{-i}^0)$  for all  $i \in N$ .
  - $\mathbf{y}^0 \in BR(\mathbf{x}^0)$ .
- Therefore,  $BR$  is upper semi-continuous.

By Kakutani's Fixed-Point Theorem, there exists some  $\mathbf{a}^* \in A$  such that  $\mathbf{a}^* \in BR(\mathbf{a}^*)$





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- Thus, we have  $y_i^0 \in BR_i(\mathbf{x}_{-i}^0)$  for all  $i \in N$ .
  - $\mathbf{y}^0 \in BR(\mathbf{x}^0)$ .
- Therefore,  $BR$  is upper semi-continuous.

By Kakutani's Fixed-Point Theorem, there exists some  $\mathbf{a}^* \in A$  such that  $\mathbf{a}^* \in BR(\mathbf{a}^*) \Rightarrow \mathbf{a}^*$  is a PNE of the strategic game.



# Outline

- 1 Brouwer's Fixed Point Theorem
  - Sketch of the Proof of Nash's Theorem (from Brouwer's Theorem)
- 2 Kakutani's Fixed Point Theorem
  - Pure Nash Equilibria of Pure Strategic Games
    - Preliminaries
    - Main Theorem I & The Proof
  - Mixed Nash Equilibria of Finite Strategies Games
    - Preliminaries & Assumptions
    - Main Theorem II & the Proof



# Limitations of the Previous PNE Result

- Any **finite** game cannot satisfy the conditions.



# Limitations of the Previous PNE Result

- Any **finite** game cannot satisfy the conditions.
  - Each  $A_i$  cannot be convex if it is finite and nonempty.
- ★ Next, we consider extending finite games into **non-deterministic (randomized)** strategies.



# Assumptions

- For a strategic game  $\langle N, (A_i), (\succsim_i) \rangle$ , we assume that we can construct a utility function  $u_i : A \mapsto \mathbb{R}$ , where  $A = \prod_{i \in N} A_i$ .
- Each player's *expected utility* is coupled with the set of probability distributions over  $A$ .
- $\Delta(X)$ : the set of probability distributions over  $X$ .
- If  $X$  is finite and  $\delta \in \Delta(X)$ , then
  - $\delta(x)$ : the probability that  $\delta$  assigns to  $x \in X$ .
  - The support of  $\delta$ :  $\chi(\delta) = \{x \in X \mid \delta(x) > 0\}$ .



# Mixed Strategy

## Mixed Strategy

Given a strategic game  $\langle N, (A_i), (u_i) \rangle$ , we call

- $\alpha_i \in \Delta(A_i)$  a **mixed strategy**.
- $a_i \in A_i$  a **pure strategy**.

A profile of mixed strategies  $\alpha = (\alpha_j)_{j \in N}$  induces a probability distribution over  $A$ .

- The probability of  $\mathbf{a} = (a_j)_{j \in N}$  under  $\alpha$ :

$$\alpha(\mathbf{a}) = \prod_{j \in N} \alpha_j(a_j). \quad (\text{a normal product})$$

( $A_i$  is finite  $\forall i \in N$  and each player's strategy is resolved independently.)



prob. =  $\alpha_1(t_1) \cdot \alpha_2(s_1)$

		$\alpha_2(s_1)$	$\alpha_2(s_2)$
		$s_1$	$s_2$
$\alpha_1(t_1)$	$t_1$	$u_1(t_1, s_1), u_2(t_1, s_1)$	$u_1(t_1, s_2), u_2(t_1, s_2)$
$\alpha_1(t_2)$	$t_2$	$u_1(t_2, s_1), u_2(t_2, s_1)$	$u_1(t_2, s_2), u_2(t_2, s_2)$



## Mixed Extension of $\langle N, (A_i), (u_i) \rangle$

### Mixed Extension of the Strategic Games

$\langle N, (\Delta(A_i)), (U_i) \rangle$ :

- $U_i : \prod_{i \in N} \Delta(A_i) \mapsto \mathbb{R}$ ; expected utility over  $A$  induced by  $\alpha \in \prod_{i \in N} \Delta(A_i)$ .
- If  $A_j$  is finite for all  $j \in N$ , then

$$\begin{aligned} U_i(\alpha) &= \sum_{\mathbf{a} \in A} (\alpha(\mathbf{a}) \cdot u_i(\mathbf{a})) \\ &= \sum_{\mathbf{a} \in A} \left( \left( \prod_{j \in N} \alpha_j(a_j) \right) \cdot u_i(\mathbf{a}) \right). \end{aligned}$$





# Main Theorem II

## Main Theorem II

Every finite strategies game has a mixed strategy Nash equilibrium.

- Consider an arbitrary finite strategic game  $\langle N, (A_i), (u_i) \rangle$ , let  $m_i := |A_i|$  for all  $i \in N$ .
- Represent each  $\Delta(A_i)$  as a collection of vectors  $\mathbf{p}^i = (p_1, p_2, \dots, p_{m_i})$ .
  - $p_k \geq 0$  for all  $k \in [m_i]$  and  $\sum_{k=1}^{m_i} p_k = 1$ .
  - $\Delta(A_i)$  is a standard  $m_i - 1$  simplex for all  $i \in N$ .



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  - $p_k \geq 0$  for all  $k \in [m_i]$  and  $\sum_{k=1}^{m_i} p_k = 1$ .
  - $\Delta(A_i)$  is a standard  $m_i - 1$  simplex for all  $i \in N$ .
  - ★  $\Delta(A_i)$ : nonempty, compact, and convex for each  $i \in N$ .
- $U_i$ : continuous ( $\because$  multilinear).
- Next, we show that  $U_i$  is **quasi-concave** in  $\Delta(A_i)$ .



## Proof of Main Theorem II (contd.)

- Consider  $\alpha \in \prod_{i \in N} \Delta(A_i)$ .
- **Goal:** Show that  $S = \{\alpha'_i \in \Delta(A_i) \mid U_i(\alpha_{-i}, \alpha'_i) \geq U_i(\alpha_{-i}, \alpha_i)\}$  is convex.



# Proof of Main Theorem II (contd.)

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- **Goal:** Show that  $S = \{\alpha'_i \in \Delta(A_i) \mid U_i(\alpha_{-i}, \alpha'_i) \geq U_i(\alpha_{-i}, \alpha_i)\}$  is convex.
- Take  $\beta_i, \gamma_i \in S$ ,  $\lambda \in [0, 1]$ .
- By definition of  $S$ , we have
  - $U_i(\alpha_{-i}, \beta_i) \geq U_i(\alpha_{-i}, \alpha_i)$ , and
  - $U_i(\alpha_{-i}, \gamma_i) \geq U_i(\alpha_{-i}, \alpha_i)$ .



# Proof of Main Theorem II (contd.)

- Consider  $\alpha \in \prod_{i \in N} \Delta(A_i)$ .
- **Goal:** Show that  $S = \{\alpha'_i \in \Delta(A_i) \mid U_i(\alpha_{-i}, \alpha'_i) \geq U_i(\alpha_{-i}, \alpha_i)\}$  is convex.
- Take  $\beta_i, \gamma_i \in S, \lambda \in [0, 1]$ .
- By definition of  $S$ , we have
  - $U_i(\alpha_{-i}, \beta_i) \geq U_i(\alpha_{-i}, \alpha_i)$ , and
  - $U_i(\alpha_{-i}, \gamma_i) \geq U_i(\alpha_{-i}, \alpha_i)$ .
- $\lambda U_i(\alpha_{-i}, \beta_i) + (1 - \lambda) U_i(\alpha_{-i}, \gamma_i) \geq \lambda U_i(\alpha_{-i}, \alpha_i) + (1 - \lambda) U_i(\alpha_{-i}, \alpha_i) = U_i(\alpha_{-i}, \alpha_i)$ .



## Proof of Main Theorem II (contd.)

- By the multilinearity of  $U_i$ , we have

$$\lambda U_i(\alpha_{-i}, \beta_i) + (1 - \lambda) U_i(\alpha_{-i}, \gamma_i) = U_i(\alpha_{-i}, \lambda \beta_i + (1 - \lambda) \gamma_i).$$



## Proof of Main Theorem II (contd.)

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- So,

$$U_i(\alpha_{-i}, \lambda \beta_i + (1 - \lambda) \gamma_i) \geq U_i(\alpha_{-i}, \alpha_i).$$



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$$\lambda \beta_i + (1 - \lambda) \gamma_i \in S$$





## Proof of Main Theorem II (contd.)

- By the multilinearity of  $U_i$ , we have

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- So,

$$U_i(\alpha_{-i}, \lambda \beta_i + (1 - \lambda) \gamma_i) \geq U_i(\alpha_{-i}, \alpha_i).$$

$$\lambda \beta_i + (1 - \lambda) \gamma_i \in S \Rightarrow U_i \text{ is convex.}$$

- Thus,  $U_i$  is quasi-concave in  $\Delta(A_i)$ .

We are done.



# A Question

## Matching Pennies of Infinite Actions

We have two players  $A$  and  $B$  having utility functions  $f(x, y) = (x - y)^2$  and  $g(x, y) = -(x - y)^2$  respectively.  $x, y \in [-1, 1]$ .

- Does this game has a pure Nash equilibrium?
- Why can't we use Kakutani's fixed point theorem?



Thank You.

