

Minimum Cost Spanning Trees (MSTs)

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Outline

1 Introduction

2 Kruskal's algorithm

3 Prim's algorithm

4 Sollin's algorithm

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Cost & Minimum-Cost Spanning Tree

- The cost of a spanning tree of a weighted undirected graph is the sum of the weights of the edges in the spanning tree.



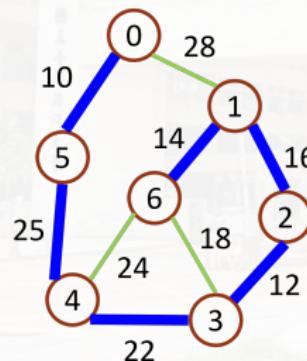
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$$\text{cost} = 10 + 25 + 22 + 12 + 16 + 14 = 99.$$

$$\text{cost} = 28 + 16 + 12 + 22 + 24 + 25 = 127.$$

Greedy Methods

- We will introduce three different **greedy algorithms** can be used to obtain a minimum cost spanning tree of a connected undirected graph.

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 - Kruskal's algorithm
 - Prim's algorithm
 - Sollin's algorithm
- ★ Further reading: Greedy Algorithms & Matroids [[link](#)]

Greedy Method (1/2)

- Construct an optimal solution in stages.
- A feasible solution is one which works within the constraints specified by the problem.
- At each stage, we make a decision that is **the best decision at that time**.
- Typically, The selection of an item at each stage is based on either **a least cost** or **a highest profit** criterion.

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 - Exactly $n - 1$ edges are used.
 - Never use edges that would produce a cycle.

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- Exactly $n - 1$ edges will be selected for inclusion in T .
- Time complexity: $O(e \log n)$ (assume using tree-based Union-Find).
 - Sorting the edges: $\approx e \log e < e \log n^2 = O(e \log n)$.
 - At most $2e$ find operations $\approx \log n$ time for each.
 - At most $2n - 1$ union operations $\approx n$ time.



Illustration of Kruskal's Algorithm

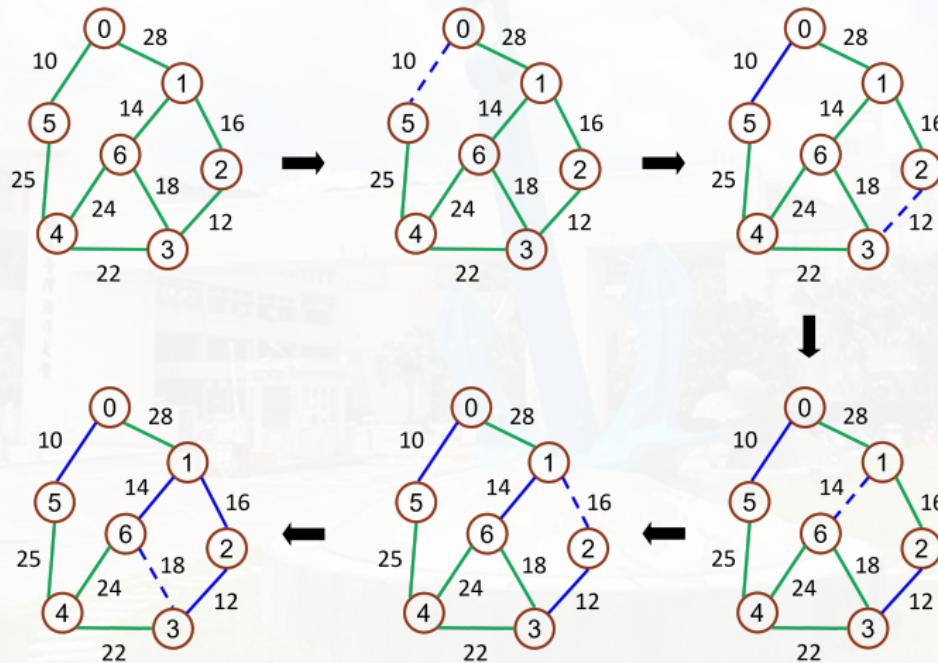
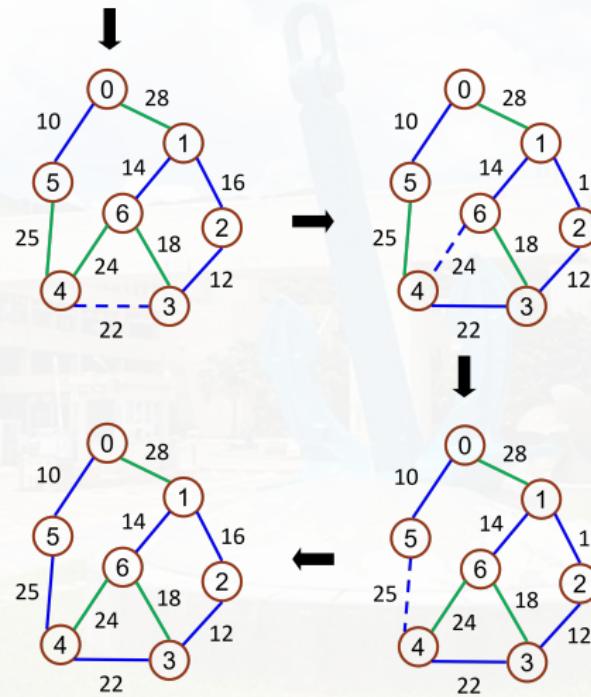


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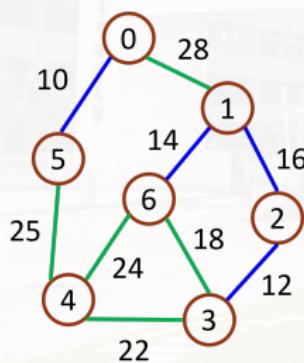


Union-Find Operations

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Union-Find Operations

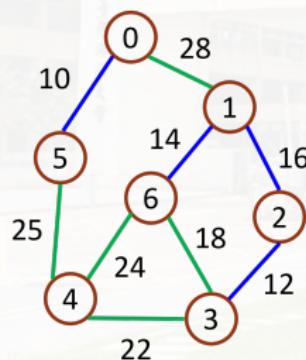
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- $\{0, 5\}, \{1, 2, 3, 6\}, \{4\}$: the sets corresponding to existing subtrees.
- Vertex 3 and 6 are already in the same set \Rightarrow edge $(3, 6)$ is rejected!

The Pseudo-code of Kruskal's Algorithm

```
T = { };
while (T contains fewer than n-1 edges && E is not empty) {
    choose a least cost edge (v,w) from E;
    delete (v,w) from E;
    if ((v,w) does not create a cycle in T)
        add (v,w) to T
    else
        discard (v,w);
}
if (T contains fewer than n-1 edges)
    printf("No spanning tree\n");
```

- How could “No spanning tree” happen?



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Prim's Algorithm (1/3)

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Prim's Algorithm (1/3)

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- The main difference:
 - The set of selected edges forms a **tree at all times** in Prim's algorithm.
 - The set of selected edges in Kruskal's algorithm forms a *forest at each stage*.

Prim's Algorithm (2/3)

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Prim's Algorithm (2/3)

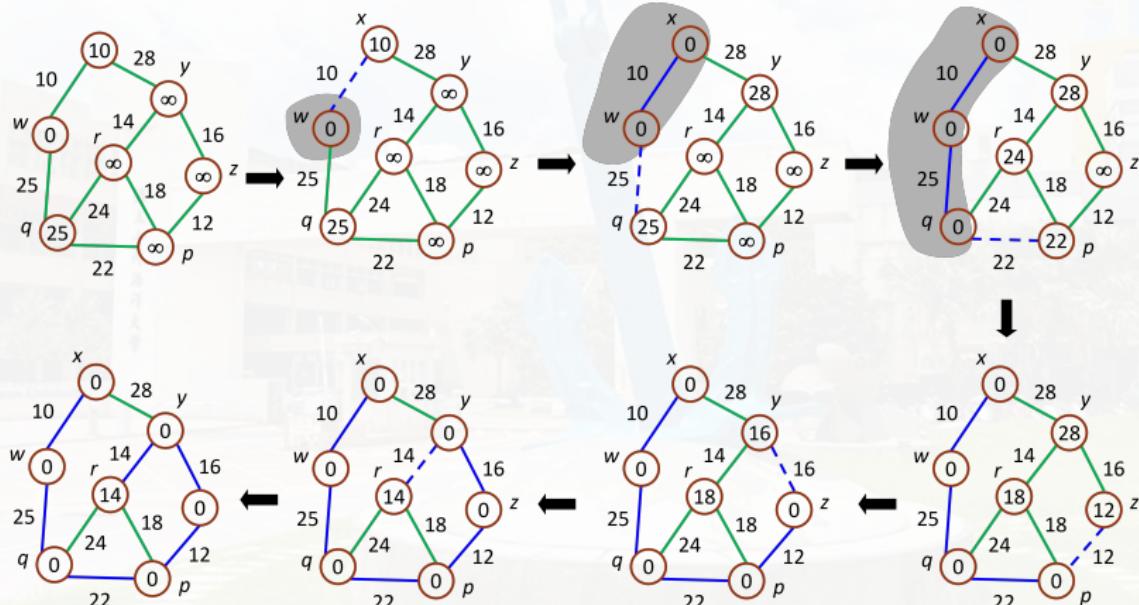
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- Repeat this edge addition until T contains $n - 1$ edges.

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Time complexity: $O(n^2)$.

Illustration of Prim's Algorithm

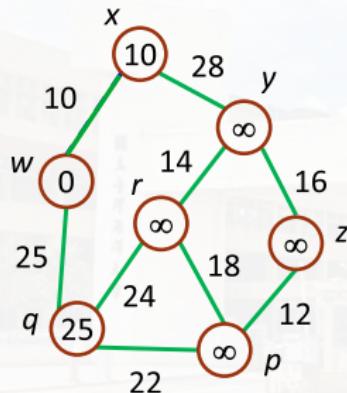


Prim's Algorithm (3/3)

```
T = {};
TV = {0};

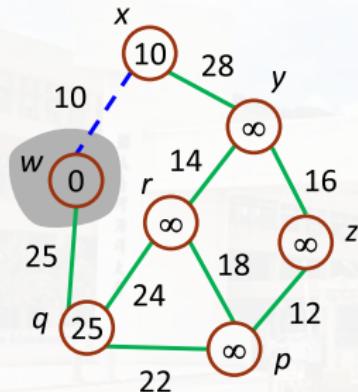
while (T contains fewer than n-1 edges) {
    let (u,v) be a least cost edge such that u is in TV and
    v is not in TV;
    if (there is no such edge )
        break;
    add v to TV;
    add (u,v) to T;
}
if (T contains fewer than n-1 edges)
    printf("No spanning tree\n");
```

- Each vertex $v \notin TV$ has a companion vertex “ $\text{near}(v)$ ” such that $\text{near}(v) \in TV$ and $\text{cost}(\text{near}(v), v)$ is minimum over all such choices for $\text{near}(v)$.
- Therefore, it takes $O(n)$ time to choose an edge.
- We can implement Prim's algorithm in $O(n^2)$ time.



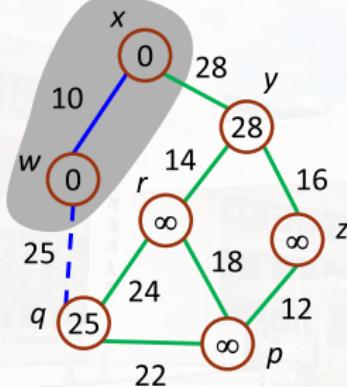
	inMST	cost
w	F	∞
x	F	∞
y	F	∞
z	F	∞
p	F	∞
q	F	∞
r	F	∞

TV = { }



	inMST	cost
w	T	∞
x	F	10
y	F	∞
z	F	∞
p	F	∞
q	F	25
r	F	∞

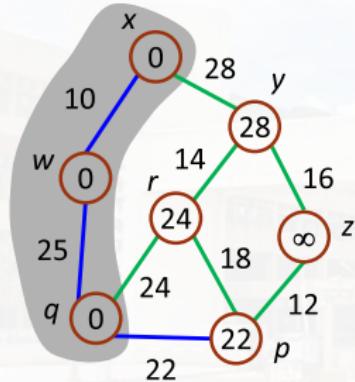
$TV = \{ w \}$



	inMST	cost
w	T	∞
x	T	10
y	F	28
z	F	∞
p	F	∞
q	F	25
r	F	∞



TV = { w, x }



	inMST	cost
w	T	∞
x	T	10
y	F	28
z	F	∞
p	F	22
q	T	25
r	F	∞



$TV = \{ w, x, q \}$

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Sollin's Algorithm (1/2)

- Sollin's algorithm selects several edges for inclusion in T at each stage.
- At the start of a stage, the selected edges, together with all the n vertices, form a spanning forest.
- During each stage, we select one edge for each tree in the forest.
 - This edge is a minimum cost edge that has exactly one vertex in the tree.

Sollin's Algorithm (2/2)

- **Note:** two trees in the forest could select the same edge.

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 - We need to **eliminate duplicate edges**.

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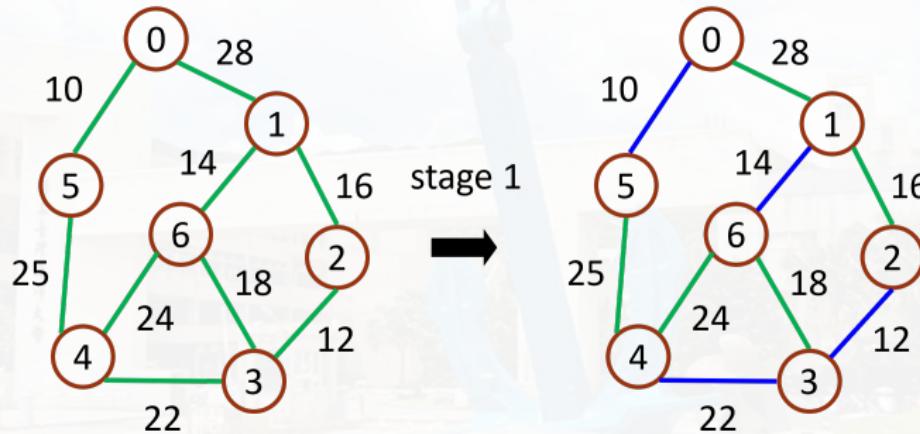
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- **Note:** two trees in the forest could select the same edge.
 - We need to **eliminate duplicate edges**.
- At the start of the first stage the set of selected edges is empty.
- The algorithm terminates when there is **only one tree at the end of a stage** or **no edges remain for selection**.

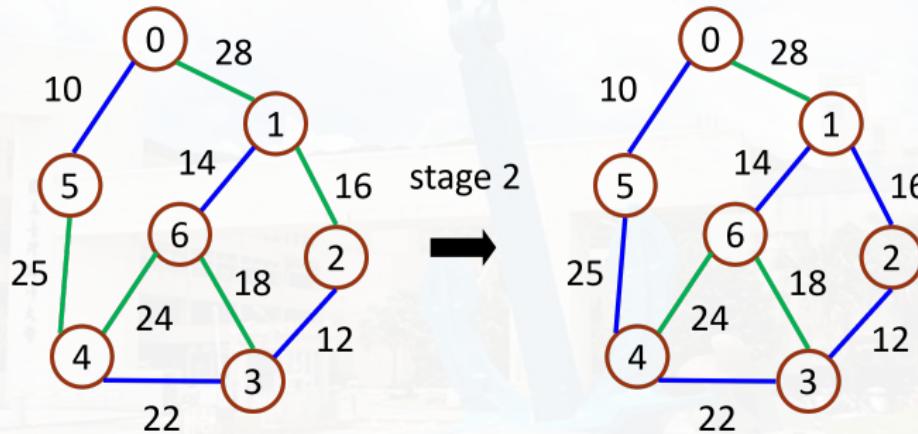
We can implement Sollin's algorithm in $O(e \log v)$ time (WHY?).

Illustration of Sollin's Algorithm



- Stage 1: $(0,5), (1,6), (2,3), (3,2), (4,3), (5,0)$, and $(6,1)$ are selected, and then duplicates are removed.

Illustration of Sollin's Algorithm



- Stage 2: $(5, 4)$, $(1, 2)$, and $(2, 1)$ are selected, and then duplicates are removed.

Discussions