

Union-Find (Disjoint Set Union)

Efficient Maintenance of Disjoint Sets

Joseph Chuang-Chieh Lin (林莊傑)

Department of Computer Science & Engineering,
National Taiwan Ocean University

Fall 2025

Reference

- Lecture Notes of CS6820 2022 (Cornell University)
- Robert Endre Tarjan: Efficiency of a Good But Not Linear Set Union Algorithm. *Journal of the ACM*. Vol. 22(2) (1975) 215–225. [[DOI](#) [LINK](#)]
- R. E. Tarjan on analyzing the “union-find” data structure [[YouTube](#)]

Outline

- 1 Motivation & Abstract Data Type
- 2 Two Key Heuristics
- 3 Implementation
- 4 Why $\alpha(n)$? A Glimpse of the Analysis

Why Union-Find?

- Many graph algorithms need to maintain a **partition** of elements into **disjoint sets**.
- Example: **Kruskal's algorithm** and **Borůvka/Sollin's algorithm** for Minimum Spanning Tree (MST)
 - Cycle detection.
 - When scanning edges in nondecreasing weight:
Add edge (u, v) iff u and v are currently in **different** connected components.
- Union-Find supports this pattern in (almost) constant amortized time.

Disjoint-Set (Union-Find) ADT

Operations

- **find(v)**: return a **canonical representative** of the set containing v .
- **union(u,v)**: merge the two sets containing u and v .



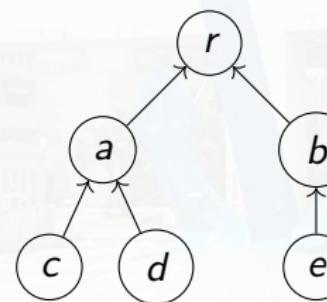
Disjoint-Set (Union-Find) ADT

Operations

- **find(v)**: return a **canonical representative** of the set containing v .
- **union(u,v)**: merge the two sets containing u and v .
- Same-set query:
 u and v are in the same set $\iff \text{find}(u) = \text{find}(v)$.
- Initially: n singleton sets.

Representing Each Set as a Rooted Tree

- Each element x stores a pointer $\text{parent}[x]$.
- The **root** is the canonical representative (for the set).



`find(x) = follow parent pointers to the root.`

Naïve Implementation: Worst-Case Can Be Bad

- If we always attach one root under the other **arbitrarily**, the tree can become a **chain**.



find(5) traverses
4 edges
Worst-case: $\Theta(n)$

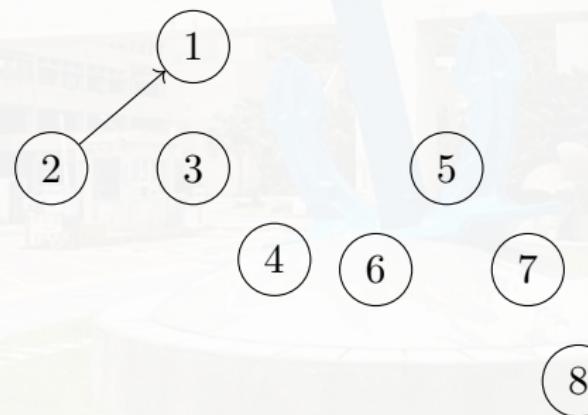
A Small Example: Unions Build a Forest

- Start with singletons $\{1\}, \{2\}, \dots, \{8\}$.
- Perform unions: $\text{union}(1,2)$, $\text{union}(3,4)$, $\text{union}(5,6)$,
 $\text{union}(7,8)$, $\text{union}(1,3)$, $\text{union}(5,7)$, $\text{union}(1,5)$.



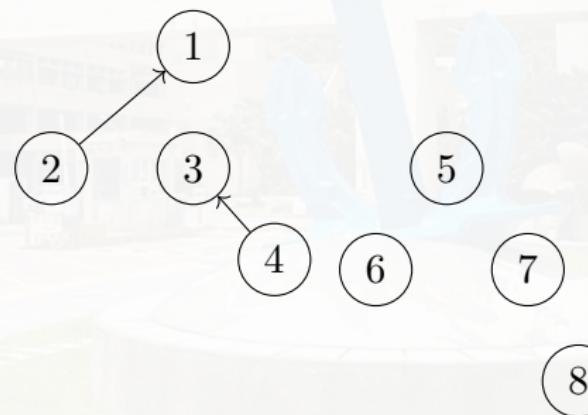
A Small Example: Unions Build a Forest

- Start with singletons $\{1\}, \{2\}, \dots, \{8\}$.
- Perform unions: $\text{union}(1,2)$, $\text{union}(3,4)$, $\text{union}(5,6)$,
 $\text{union}(7,8)$, $\text{union}(1,3)$, $\text{union}(5,7)$, $\text{union}(1,5)$.



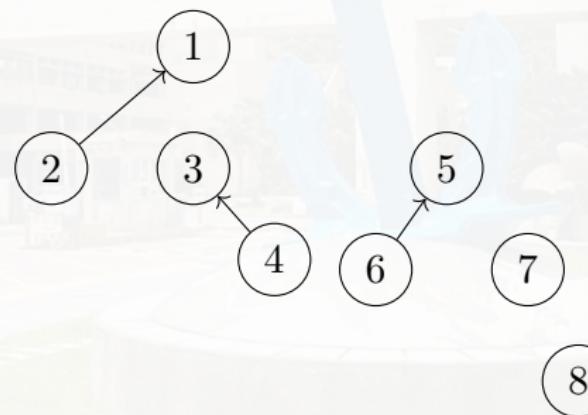
A Small Example: Unions Build a Forest

- Start with singletons $\{1\}, \{2\}, \dots, \{8\}$.
- Perform unions: $\text{union}(1,2)$, $\text{union}(3,4)$, $\text{union}(5,6)$,
 $\text{union}(7,8)$, $\text{union}(1,3)$, $\text{union}(5,7)$, $\text{union}(1,5)$.



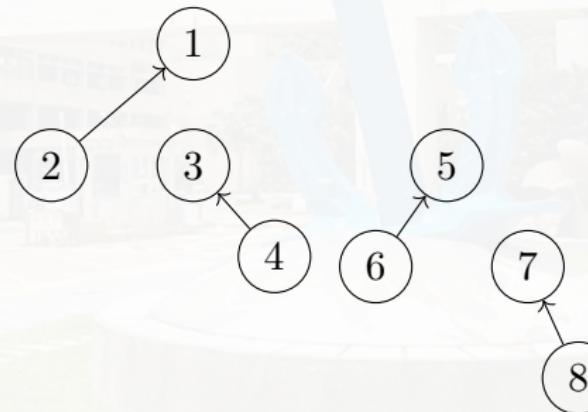
A Small Example: Unions Build a Forest

- Start with singletons $\{1\}, \{2\}, \dots, \{8\}$.
- Perform unions: $\text{union}(1,2)$, $\text{union}(3,4)$, $\text{union}(5,6)$,
 $\text{union}(7,8)$, $\text{union}(1,3)$, $\text{union}(5,7)$, $\text{union}(1,5)$.



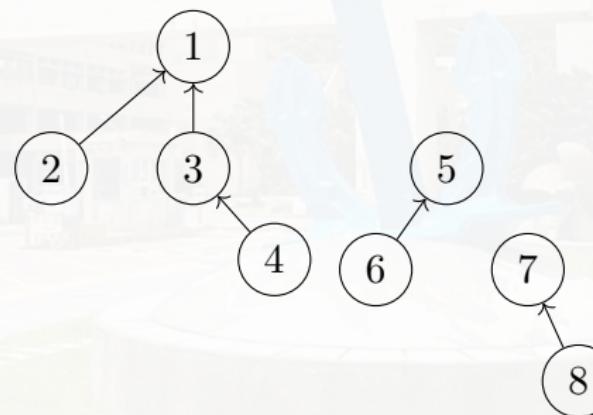
A Small Example: Unions Build a Forest

- Start with singletons $\{1\}, \{2\}, \dots, \{8\}$.
- Perform unions: $\text{union}(1,2)$, $\text{union}(3,4)$, $\text{union}(5,6)$,
 $\text{union}(7,8)$, $\text{union}(1,3)$, $\text{union}(5,7)$, $\text{union}(1,5)$.



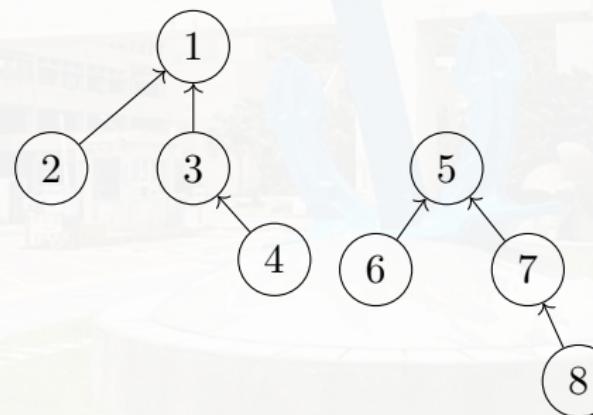
A Small Example: Unions Build a Forest

- Start with singletons $\{1\}, \{2\}, \dots, \{8\}$.
- Perform unions: $\text{union}(1,2)$, $\text{union}(3,4)$, $\text{union}(5,6)$,
 $\text{union}(7,8)$, $\text{union}(1,3)$, $\text{union}(5,7)$, $\text{union}(1,5)$.



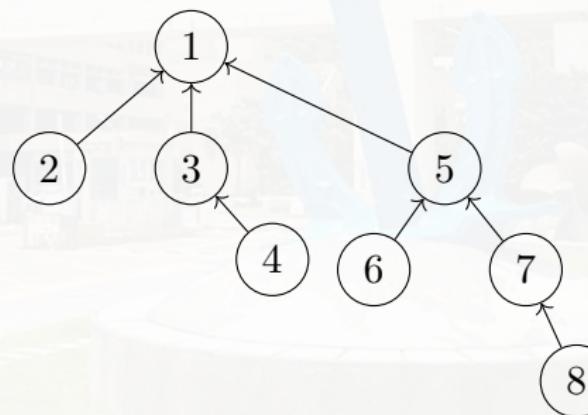
A Small Example: Unions Build a Forest

- Start with singletons $\{1\}, \{2\}, \dots, \{8\}$.
- Perform unions: $\text{union}(1,2)$, $\text{union}(3,4)$, $\text{union}(5,6)$,
 $\text{union}(7,8)$, $\text{union}(1,3)$, $\text{union}(5,7)$, $\text{union}(1,5)$.



A Small Example: Unions Build a Forest

- Start with singletons $\{1\}, \{2\}, \dots, \{8\}$.
- Perform unions: $\text{union}(1,2)$, $\text{union}(3,4)$, $\text{union}(5,6)$,
 $\text{union}(7,8)$, $\text{union}(1,3)$, $\text{union}(5,7)$, $\text{union}(1,5)$.



Cost Model

- **union(u, v):**
 - Usually does two **find** operations, then one pointer update.
 - Time dominated by the two **find** calls.
- **find(u):**
 - Time proportional to the **length of the path** from u to the root.

Goal

Support a sequence of m operations on n elements in **near-linear** total time.

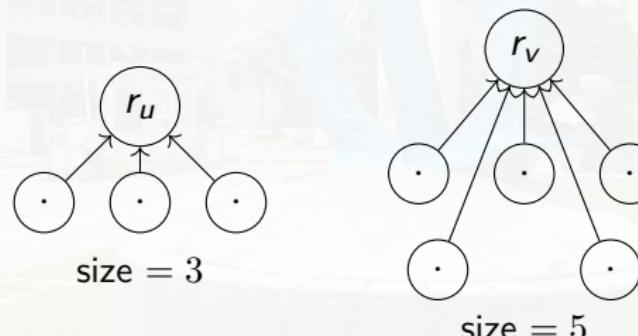


Outline

- 1 Motivation & Abstract Data Type
- 2 Two Key Heuristics
- 3 Implementation
- 4 Why $\alpha(n)$? A Glimpse of the Analysis

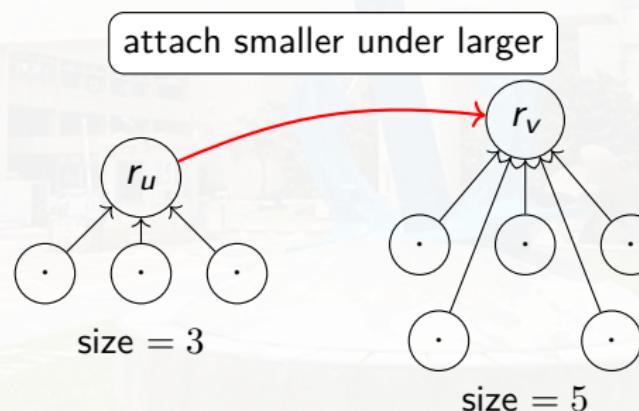
Heuristic #1: Union by Size (Smaller \rightarrow Larger)

- Maintain $\text{size}[\text{root}] = \text{number of nodes in that tree}$.
- On $\text{union}(u, v)$:
 - Find roots r_u, r_v .
 - Make the root of the **smaller** tree point to the root of the **larger** tree.



Heuristic #1: Union by Size (Smaller \rightarrow Larger)

- Maintain $\text{size}[\text{root}] = \text{number of nodes in that tree}$.
- On $\text{union}(u, v)$:
 - Find roots r_u, r_v .
 - Make the root of the **smaller** tree point to the root of the **larger** tree.



Heuristic #2: Path Compression (During find)

- After finding the root r for a query node x :
 - traverse the path again and set every visited node's parent directly to r .
- This makes **future** find operations faster.



`find(c)` follows 3 edges

Before

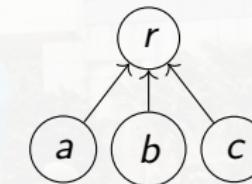
Heuristic #2: Path Compression (During find)

- After finding the root r for a query node x :
 - traverse the path again and set every visited node's parent directly to r .
- This makes **future** find operations faster.



`find(c)` follows 3 edges

Before



path compressed

After



Key Observations (w/ vs. w/o Path Compression)

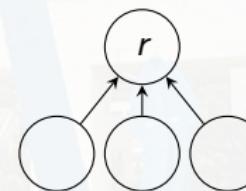
- Consider a fixed sequence σ of m operations on n elements.
- With or without path compression:
 - the partition into sets at each time is the **same**,
 - the **roots** (representatives) are the same.
- The difference: path compression can later make a node a **non-descendant** of a former ancestor.

Example

Time t : after $\text{union}(u, v)$ (and r already has a larger set)



$\text{size}(v) = 2$

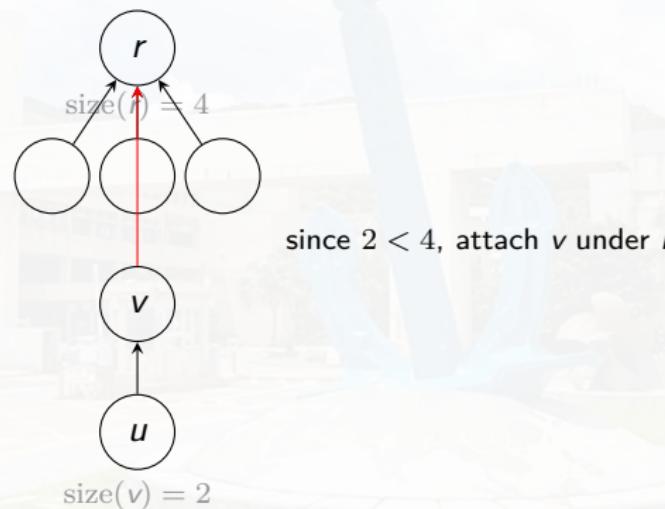


$\text{size}(r) = 4$

Now u is a descendant of v .

Example

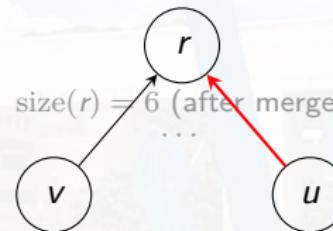
Later: after $\text{union}(v, r)$ by size (no compression yet)



Path is $u \rightarrow v \rightarrow r$, so u is still a descendant of v .

Example

After $\text{find}(u)$ with path compression



u now bypasses v

Now u is *not* a descendant of v (even though it was earlier).

What We Get from These Two Heuristics

Performance Guarantee (informal)

A sequence of m union/find operations takes

$$O((m + n) \alpha(n))$$

time, where $\alpha(n)$ is the **inverse Ackermann function**.

What We Get from These Two Heuristics

Performance Guarantee (informal)

A sequence of m union/find operations takes

$$O((m + n) \alpha(n))$$

time, where $\alpha(n)$ is the **inverse Ackermann function**.

- $\alpha(n)$ grows extremely slowly that it can be effectively viewed as a constant for all realistic inputs.



What We Get from These Two Heuristics

Performance Guarantee (informal)

A sequence of m union/find operations takes

$$O((m + n) \alpha(n))$$

time, where $\alpha(n)$ is the **inverse Ackermann function**.

- $\alpha(n)$ grows extremely slowly that it can be effectively viewed as a constant for all realistic inputs.
- Why and how does this exotic function appear in the analysis?



Implementation: Two Arrays

- $\text{parent}[x]$: parent pointer; roots satisfy $\text{parent}[r] = r$.
- $\text{rank}[x]$ (a kind of “weight” of x): maintained only for roots.

Invariants

- Each set is represented by exactly one rooted tree.
- Roots are the canonical representatives.



C++ style Reference Implementation

```
struct DSU {
    vector<int> parent, rank;
    DSU(int n=0) { init(n); }
    void init(int n){
        parent.resize(n);
        rank.assign(n,1);
        for(int i=0; i<n; i++) parent[i]=i;
    }
    int find(int x) {
        if(parent[x] == x) return x;
        return parent[x] = find(parent[x]); // path compression
    }
    bool union(int a,int b){ // weighted union
        a = find(a); b = find(b);
        if(a == b) return false;
        if(rank[a] < rank[b]) swap(a,b); // union by size
        parent[b] = a;
        rank[a] += rank[b];
        return true;
    }
};
```



Practical Notes

- Union-by-**rank** is a common variant:
 - maintain an **upper bound on height** instead of exact subtree size,
 - update only when ranks tie.
- **Rule of thumb:** always combine **one** of (union by size/rank) **and** path compression.

Outline

- 1 Motivation & Abstract Data Type
- 2 Two Key Heuristics
- 3 Implementation
- 4 Why $\alpha(n)$? A Glimpse of the Analysis

Ackermann-Type Growth

- The analysis uses a hierarchy of rapidly growing functions $\{A_k\}$.
- One convenient definition:

$$A_0(x) = x + 1, \quad A_{k+1}(x) = A_k^{(x)}(x),$$

where $A_k^{(i)} := \underbrace{A_k \circ A_k \cdots \circ A_k}_{i \text{ times}}$ and A_k^0 := identity function.

Ackermann-Type Growth

- The analysis uses a hierarchy of rapidly growing functions $\{A_k\}$.
- One convenient definition:

$$A_0(x) = x + 1, \quad A_{k+1}(x) = A_k^{(x)}(x),$$

where $A_k^{(i)} := \underbrace{A_k \circ A_k \cdots \circ A_k}_{i \text{ times}}$ and A_k^0 := identity function.

Intuition

As k increases, $A_k(x)$ grows **astronomically** fast (much faster than exponentials/towers).



Concrete Examples (Small k)

For $x \geq 2$, the first few levels behave like:

$$A_0(x) = x + 1,$$

$$A_1(x) = 2x,$$

$$A_2(x) = x \cdot 2^x \quad (\text{roughly exponential}),$$

$$A_3(x) = A_2^x(x) \geq \underbrace{2^{2^{\dots^2}}}_{x} \quad (\text{a tower of } 2\text{s of height } x).$$

- $A_4(2) \geq \underbrace{2^{2^{2^{\dots^2}}}}_{2048}.$

- This is why the **inverse** function grows extremely slowly.

Ackermann Function and Its Inverse

Definitions [Ackermann and the Inverse Ackermann]

$$A(k) := A_k(2), \quad \alpha(n) = \min\{k \mid A(k) \geq n\}.$$

For example, $\alpha(2048) = 3$ ($\because A_3(2) = 2048$), $\alpha(2^{65536}) = 4$.

Ackermann Function and Its Inverse

Definitions [Ackermann and the Inverse Ackermann]

$$A(k) := A_k(2), \quad \alpha(n) = \min\{k \mid A(k) \geq n\}.$$

For example, $\alpha(2048) = 3$ ($\because A_3(2) = 2048$), $\alpha(2^{65536}) = 4$.

- $\alpha(n)$ is so small that in practice it behaves like a constant.
- The theoretical bound for Union-Find becomes **near-linear**.

High-Level Statement of the Main Result

- Let σ be a sequence of m union and find operations.

Theorem (Tarjan [JACM 1975])

Starting from n singleton sets, any sequence of m union and find operations takes

$$O((m + n) \alpha(n))$$

time when using **union-by-rank** and **path compression**.

High-Level Statement of the Main Result

- Let σ be a sequence of m union and find operations.

Theorem (Tarjan [JACM 1975])

Starting from n singleton sets, any sequence of m union and find operations takes

$$O((m + n) \alpha(n))$$

time when using **union-by-rank** and **path compression**.

- Next: the proof idea uses **ranks** and an **amortized charging** argument.

Rank of a Node

- Consider executing the same operation sequence **without** path compression.
- Let $T_m(u)$ be the subtree rooted at u in the final forest.
 - $T_t(u)$: the subtree rooted at u at time t in the execution of a sequence of m union and find instructions **without** path compressions.

Definition (Rank): a quantity that survives path compression

$$\text{rank}(u) = 2 + \text{height} (T_m(u)).$$

$\text{height}(T)$: the length of the longest path from leaves to the root.

Rank of a Node

- Consider executing the same operation sequence **without** path compression.
- Let $T_m(u)$ be the subtree rooted at u in the final forest.
 - $T_t(u)$: the subtree rooted at u at time t in the execution of a sequence of m union and find instructions **without** path compressions.

Definition (Rank): a quantity that survives path compression

$$\text{rank}(u) = 2 + \text{height} (T_m(u)).$$

height(T): the length of the longest path from leaves to the root.

- With union-by-rank, ranks are **bounded** and **monotone** along parent pointers.



A Key Balancing Lemma (Union-by-Rank)

Lemma (Size vs. Height)

If we always merge the smaller tree into the larger, then for any time t and root u ,

$$|T_t(u)| \geq 2^{\text{height}(T_t(u))}.$$

A Key Balancing Lemma (Union-by-Rank)

Lemma (Size vs. Height)

If we always merge the smaller tree into the larger, then for any time t and root u ,

$$|T_t(u)| \geq 2^{\text{height}(T_t(u))}.$$

- Prove by induction on t .

A Key Balancing Lemma (Union-by-Rank)

Lemma (Size vs. Height)

If we always merge the smaller tree into the larger, then for any time t and root u ,

$$|T_t(u)| \geq 2^{\text{height}(T_t(u))}.$$

- Prove by induction on t .
- Intuition: each time the height increases by 1, the subtree size at least **doubles**.

A Key Balancing Lemma (Union-by-Rank)

Lemma (Size vs. Height)

If we always merge the smaller tree into the larger, then for any time t and root u ,

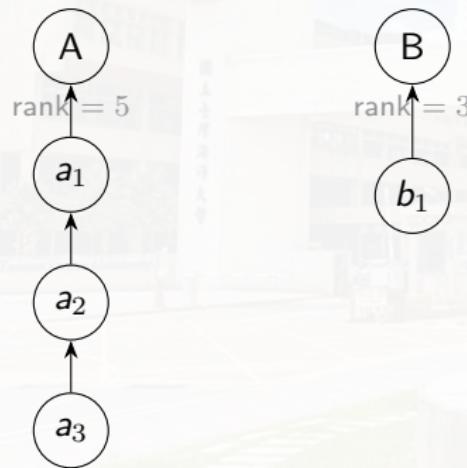
$$|T_t(u)| \geq 2^{\text{height}(T_t(u))}.$$

- Prove by induction on t .
- Intuition: each time the height increases by 1, the subtree size at least **doubles**.
- Consequence: the maximum rank is $\lfloor \lg n \rfloor + 2 = O(\log n)$.

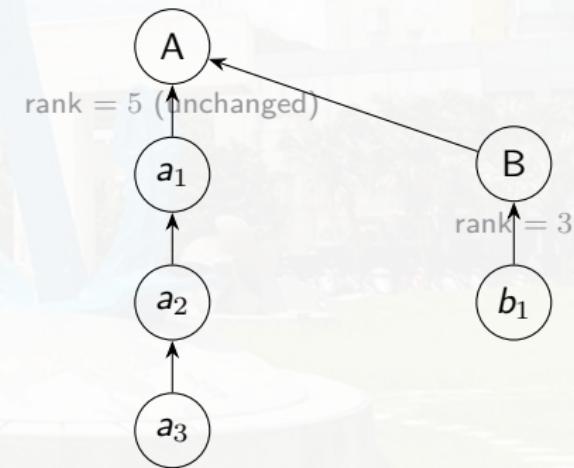
Note: Effect on the rank after union operation (1/2)

Example 1: unequal heights (no rank increase)

Before union



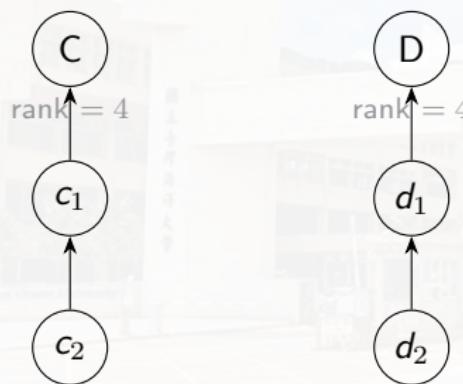
After union



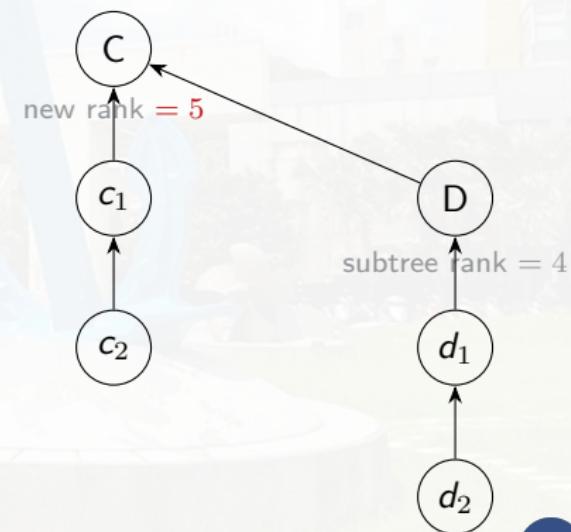
Note: Effect on the rank after union operation (2/2)

Example 2: equal heights (rank increases)

Before union



After union



From Ranks to $\alpha(n)$ (Rough Sketch (1/2))

- Path compression changes parents, so we need to track how **fast** parent ranks can grow.
- Define **levels** using the Ackermann hierarchy:
$$\ell(u) = \max \left\{ k \mid \text{rank}(\text{parent}(u)) \geq A_k(\text{rank}(u)) \right\}.$$
- During path compression, $\ell(u)$ can only **increase**.

From Ranks to $\alpha(n)$ (Rough Sketch (1/2))

- Path compression changes parents, so we need to track how **fast** parent ranks can grow.
- Define **levels** using the Ackermann hierarchy:

$$\ell(u) = \max \left\{ k \mid \text{rank}(\text{parent}(u)) \geq A_k(\text{rank}(u)) \right\}.$$

- During path compression, $\ell(u)$ can only **increase**.
 - Path compression changes $\text{parent}(u)$ to a higher ancestor (larger rank), so $\ell(u)$ is monotone.

From Ranks to $\alpha(n)$ (Rough Sketch (1/2))

- Path compression changes parents, so we need to track how **fast** parent ranks can grow.
- Define **levels** using the Ackermann hierarchy:

$$\ell(u) = \max \left\{ k \mid \text{rank}(\text{parent}(u)) \geq A_k(\text{rank}(u)) \right\}.$$

- During path compression, $\ell(u)$ can only **increase**.
 - Path compression changes $\text{parent}(u)$ to a higher ancestor (larger rank), so $\ell(u)$ is monotone.

From Ranks to $\alpha(n)$ (Rough Sketch (1/2))

- Path compression changes parents, so we need to track how **fast** parent ranks can grow.
- Define **levels** using the Ackermann hierarchy:

$$\ell(u) = \max \left\{ k \mid \text{rank}(\text{parent}(u)) \geq A_k(\text{rank}(u)) \right\}.$$

- During path compression, $\ell(u)$ can only **increase**.
 - Path compression changes $\text{parent}(u)$ to a higher ancestor (larger rank), so $\ell(u)$ is monotone.



From Ranks to $\alpha(n)$ (Rough Sketch (2/2))

$$\ell(u) = \max \left\{ k \mid \text{rank}(\text{parent}(u)) \geq A_k(\text{rank}(u)) \right\}.$$

Charging idea (just bookkeeping for the analysis, not in the code)

When a `find` traverses a node u , we charge its $O(1)$ cost as follows:

- **Node-charge (same-level step):** if the traversal is “same-level” for u (e.g., $\ell(\text{parent}(u)) = \ell(u)$), then we `charge` it to the **node u** .
- **Find-charge (level-increasing step):** if the traversal is “level-increasing” (e.g., $\ell(\text{parent}(u)) > \ell(u)$), then we `charge` it to the **find** itself.

From Ranks to $\alpha(n)$ (Rough Sketch (2/2))

$$\ell(u) = \max \left\{ k \mid \text{rank}(\text{parent}(u)) \geq A_k(\text{rank}(u)) \right\}.$$

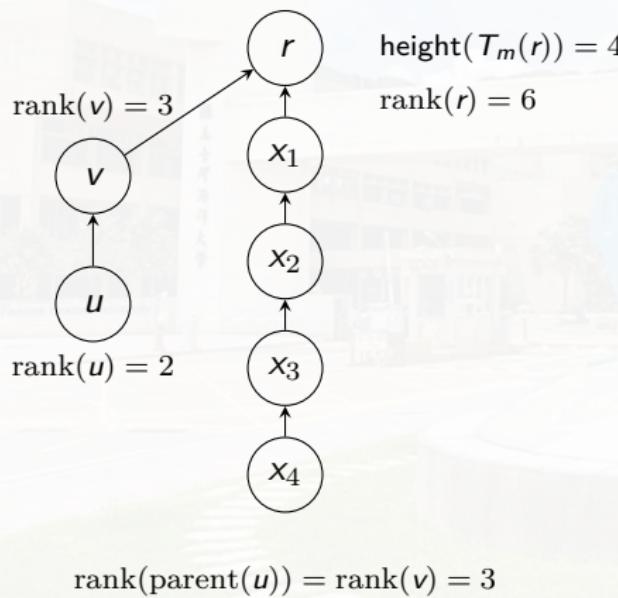
Charging idea (just bookkeeping for the analysis, not in the code)

When a `find` traverses a node u , we charge its $O(1)$ cost as follows:

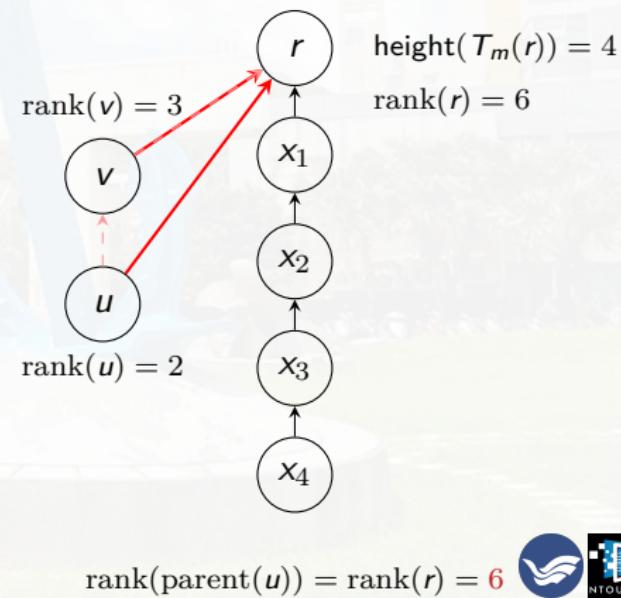
- **Node-charge (same-level step):** if the traversal is “same-level” for u (e.g., $\ell(\text{parent}(u)) = \ell(u)$), then we **charge** it to the **node u** .
 - After compression, $\text{rank}(\text{parent}(u))$ strictly increases, so u makes progress (either $\ell(u)$ increases, or it advances within the same level via the auxiliary index).
- **Find-charge (level-increasing step):** if the traversal is “level-increasing” (e.g., $\ell(\text{parent}(u)) > \ell(u)$), then we **charge** it to the **find** itself.

Example

Before path compression



After $\text{find}(u)$ with compression



Remark (connections to the Ackermann)

- If u has a parent, then

$$\text{rank}(\text{parent}(u)) \geq \text{rank}(u) + 1 = A_0(\text{rank}(u)).$$

- For $n \geq 5$, the maximum value $\ell(u)$ can take on is $\alpha(n) - 1$, since for $\ell(u) = k$,

$$\begin{aligned} n &> \lfloor \lg n \rfloor + 2 \\ &\geq \text{rank}(\text{parent}(u)) \\ &\geq A_k(\text{rank}(u)) \\ &\geq A_k(2), \end{aligned}$$

therefore, $\alpha(n) > k = \ell(u)$.

Remark: levels \leftrightarrow ranks

- Because path compression changes parents, we track **how large the parent's rank is** compared to the node's rank on an Ackermann scale.

Level

$$\ell(u) = \max \left\{ k \mid \text{rank}(\text{parent}(u)) \geq A_k(\text{rank}(u)) \right\}.$$

- Intuition:** $\ell(u)$ is the number of “Ackermann jumps” by which the parent rank dominates the child rank.

Observation of Monotonicity: Levels only go up

- During a **find**, path compression sets $\text{parent}(u)$ to an **ancestor** (often the root), whose rank is **at least** the old parent's rank.

Observation of Monotonicity: Levels only go up

- During a **find**, path compression sets $\text{parent}(u)$ to an **ancestor** (often the root), whose rank is **at least** the old parent's rank.
- Therefore $\text{rank}(\text{parent}(u))$ is **nondecreasing over time**.

Observation of Monotonicity: Levels only go up

- During a `find`, path compression sets $\text{parent}(u)$ to an **ancestor** (often the root), whose rank is **at least** the old parent's rank.
- Therefore $\text{rank}(\text{parent}(u))$ is **nondecreasing over time**.
- Since $\ell(u)$ depends on whether $\text{rank}(\text{parent}(u))$ crosses thresholds $A_k(\text{rank}(u))$, we get:

Level monotonicity

$\ell(u)$ never decreases as operations proceed.

Observation of Monotonicity: Levels only go up

- During a `find`, path compression sets $\text{parent}(u)$ to an **ancestor** (often the root), whose rank is **at least** the old parent's rank.
- Therefore $\text{rank}(\text{parent}(u))$ is **nondecreasing over time**.
- Since $\ell(u)$ depends on whether $\text{rank}(\text{parent}(u))$ crosses thresholds $A_k(\text{rank}(u))$, we get:

Level monotonicity

$\ell(u)$ never decreases as operations proceed.

- Consequence: each node's level can increase at most $\alpha(n)$ times.



Charging scheme for the cost of find

Consider one $\text{find}(x)$ that traverses nodes on the path

$x = u_0, u_1, \dots, u_t = \text{root}$.

$$\text{Total time} \leq O(\underbrace{\#\text{node-charges}}_{\leq n\alpha(n)} + \underbrace{\#\text{operation-charges}}_{\leq m\alpha(n)}).$$

Bounding operation-charges: $\leq m\alpha(n)$

- An operation-charge happens only when $\ell(\text{parent}(u)) > \ell(u)$.
- But $\ell(u)$ is monotone and bounded:

$$0 \leq \ell(u) \leq \alpha(n).$$

- Hence each find can be charged at most $\alpha(n)$ times.

Conclusion

#operation-charges $\leq m\alpha(n)$.

Bounding node-charges

Node-charges are assigned to “same-level” traversal steps (e.g., $\ell(\text{parent}(u)) = \ell(u)$ at the time the edge is traversed).

- Fix a node u that is visited by `find`.
- After compression, $\text{rank}(\text{parent}(u))$ strictly increases, so u makes progress.
- **Important:** $\ell(u)$ may stay the same;
 - We will keep track a refined state of the progress of u .
- This state can advance at most $O(\alpha(n))$ times per node.

Claim

#node-charges in all finds = $O(n\alpha(n))$.

On the number of charges to the node

To formalize the previous slide, analyses often add an **index** within a level.

Index within level $k = \ell(u)$

Let $A_k^{(j)}$ be the j -fold iterate of A_k . Define

$$i(u) := \max \left\{ j \mid \text{rank}(\text{parent}(u)) \geq A_k^{(j)}(\text{rank}(u)) \right\}.$$

- **Note:** larger $i(u)$ means the parent's rank is not just above A_k but above many repeated applications of A_k .
- If $\ell(u)$ does not increase, path compression still tends to increase $\text{rank}(\text{parent}(u))$, so **$i(u)$ increases**.

Wait! Why do we iterate A_k ?

- By definition of ℓ , at time t we have the level- k threshold:

$$\text{rank}(\text{parent}(x)) \geq A_k(\text{rank}(x)).$$

Wait! Why do we iterate A_k ?

- By definition of ℓ , at time t we have the level- k threshold:

$$\text{rank}(\text{parent}(x)) \geq A_k(\text{rank}(x)).$$

- Path compression changes parent pointers (e.g., $x \rightarrow v$, $v = \text{parent}(x)$), so

$$\text{rank}(\text{parent}(x))$$

may *increase over time* even if $\ell(x)$ (the level) stays the same.

Wait! Why do we iterate A_k ?

- By definition of ℓ , at time t we have the level- k threshold:

$$\text{rank}(\text{parent}(x)) \geq A_k(\text{rank}(x)).$$

- Path compression changes parent pointers (e.g., $x \rightarrow v$, $v = \text{parent}(x)$), so

$$\text{rank}(\text{parent}(x))$$

may *increase over time* even if $\ell(x)$ (the level) stays the same.

- Therefore, we need a *finer progress measure within level k*.

Index within a fixed level: Using iterates A_k^i

- Consider an *iterate index* $i \geq 1$: $\text{rank}(\text{parent}(x)) \geq A_k^i(\text{rank}(x))$, where A_k^i denotes the i -fold iterate of A_k .
- It is an *analytic ruler* that counts how far $\text{rank}(\text{parent}(x))$ has advanced *within the same level* k .
- If a charge to x occurs at time t , then after compression (time $t+1$) the new parent becomes a later vertex v on the find path, and

$$\text{rank}(v) \geq A_k(\text{rank}(\text{parent}(x))) \geq A_k(A_k^i(\text{rank}(x))) = A_k^{i+1}(\text{rank}(x)).$$

- Since v is the new parent of x , this implies

$$\text{rank}(\text{parent}(x)) \geq A_k^{i+1}(\text{rank}(x)).$$

- Thus, each such event increases the *index i* by at least 1.

i is bounded before the level must increase

- While $\ell(x) = k$ remains fixed, the index i can increase only finitely many times.
- After at most $\text{rank}(x)$ such increments, the notes obtain

$$\text{rank}(\text{parent}(x)) \geq A_k^{\text{rank}(x)}(\text{rank}(x)) = A_{k+1}(\text{rank}(x)).$$

i is bounded before the level must increase

- While $\ell(x) = k$ remains fixed, the index i can increase only finitely many times.
- After at most $\text{rank}(x)$ such increments, the notes obtain

$$\text{rank}(\text{parent}(x)) \geq A_k^{\text{rank}(x)}(\text{rank}(x)) = A_{k+1}(\text{rank}(x)).$$

- Crossing the next threshold forces the level to increase:

$$\ell(x) \geq k + 1 \quad (\text{equivalently, } \ell(x) \text{ must increase}).$$

i is bounded before the level must increase

- While $\ell(x) = k$ remains fixed, the index i can increase only finitely many times.
- After at most $\text{rank}(x)$ such increments, the notes obtain

$$\text{rank}(\text{parent}(x)) \geq A_k^{\text{rank}(x)}(\text{rank}(x)) = A_{k+1}(\text{rank}(x)).$$

- Crossing the next threshold forces the level to increase:
$$\ell(x) \geq k + 1 \quad (\text{equivalently, } \ell(x) \text{ must increase}).$$
- Therefore, at most $\text{rank}(x)\alpha(n)$ such charges against x .

On the number of nodes of the same rank

Lemma

For any integer r ,

$$|\{u \mid \text{rank}(u) = r\}| \leq \frac{n}{2^{r-2}}.$$

On the number of nodes of the same rank

Lemma

For any integer r ,

$$|\{u \mid \text{rank}(u) = r\}| \leq \frac{n}{2^{r-2}}.$$

Proof sketch

- If $\text{rank}(u) = \text{rank}(v) = r$, then the subtrees $T_m(u)$ and $T_m(v)$ are disjoint.
- Hence, we have the union of these subtrees has size
$$\left| \bigcup_{\text{rank}(u)=r} T_m(u) \right| = \sum_{\text{rank}(u)=r} |T_m(u)| \leq n.$$
- Also, we have known that every node of rank r satisfies $|T_m(u)| \geq 2^{r-2}$, so

$$\sum_{\text{rank}(u)=r} |T_m(u)| \geq \sum_{\text{rank}(u)=r} 2^{r-2} = |\{u : \text{rank}(u) = r\}| \cdot 2^{r-2}.$$

- Rearranging yields $|\{u : \text{rank}(u) = r\}| \leq \frac{n}{2^{r-2}}.$

Sum up the charges against nodes

- At most $\text{rank}(x)\alpha(n)$ charges against a node x . So, there are at most

$$r\alpha(n) \frac{n}{2^{r-2}} = n\alpha(n) \frac{r}{2^{r-2}}$$

charges against nodes of rank r .

- Summing over all values of r we obtain the following bound on all charges to all vertices

$$\sum_{r=0}^{\infty} n\alpha(n) \frac{r}{2^{r-2}} = n\alpha(n) \cdot \sum_{r=0}^{\infty} \frac{r}{2^{r-2}} = 8n\alpha(n).$$

Putting it together: near-linear time

- Node-charges: at most $n\alpha(n)$ total.
- Operation-charges: at most $m\alpha(n)$ total.

Conclusion (Tarjan)

Starting from n singletons, any sequence of m union/find operations with union-by-rank (or size) plus path compression runs in

$$O((m+n)\alpha(n)) \text{ time.}$$

- Practically, $\alpha(n) \leq 4$ for any realistic n , so it behaves like “almost constant amortized time.”

Conclusion of the Amortized Analysis

- Total charge to find operations: $O(m\alpha(n))$.
- Total charge to nodes over the entire computation: $O(n\alpha(n))$.

Therefore

Total time for m operations is $O((m + n)\alpha(n))$.

- This is why Union-Find is considered almost linear time.