Minimum Cost Spanning Trees (MSTs)

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Outline

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1. Introduction
2. Kruskal's algorithm
3. Prim's algorithm
4. Sollin's algorithm

Cost & Minimum-Cost Spanning Tree

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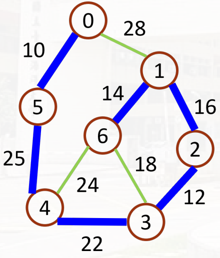
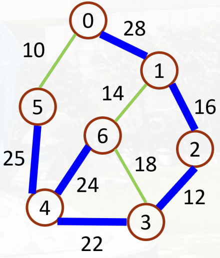


cost $= 10+25+22+12+16+14 = 99.$



cost $= 28+16+12+22+24+25 = 127.$

Greedy Methods

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  - Kruskal's algorithm
  - Prim's algorithm
  - Sollin's algorithm

⋆ Further reading: Greedy Algorithms & Matroids [link]

Greedy Method (1/2)

- Construct an optimal solution in stages.
- A feasible solution is one which works within the constrains specified by the problem.
- At each stage, we make a decision that is the best decision at that time.
- Typically, The selection of an item at each stage is based on either a least cost or a highest profit criterion.

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  - We must use only edges within the graph.
  - Exactly $n-1$ edges are used.
  - Never use edges that would produce a cycle.

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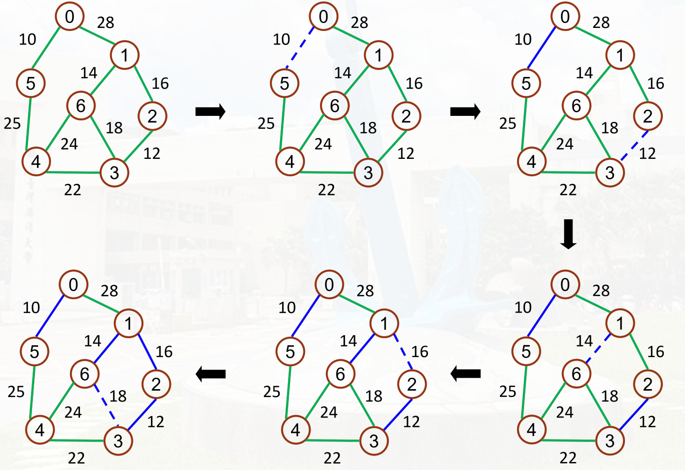
  - **How to do this? $\Rightarrow$ Union-Find Operations!**
- Exactly $n-1$ edges will be selected for inclusion in T.

- Time complexity: $O(e \log n)$.
  - Sorting the edges: $\approx e \log e < e \log n^2 = O(e \log n)$.
  - At most $2e$ find operations $\Rightarrow \approx 2e$ time.
  - At most $2n-1$ union operations $\Rightarrow \approx n \log n$ time.

Illustration of Kruskal's Algorithm



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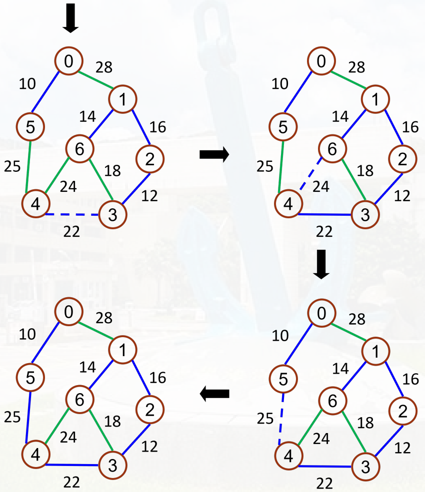

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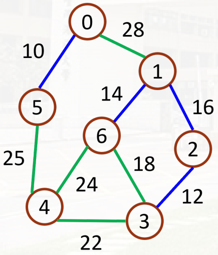

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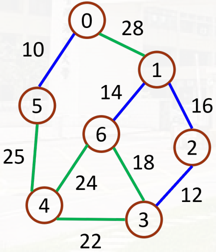

- $\{0, 5\}, \{1, 2, 3, 6\}, \{4\}$: the sets corresponding to existing subtrees.
- Vertex 3 and 6 are already in the same set $\Rightarrow$ edge $(3, 6)$ is rejected!

The Pseudo-code of Kruskal's Algorithm

```
T = { };
while (T contains fewer than n-1 edges && E is not empty) {
    choose a least cost edge (v,w) from E;
    delete (v,w) from E;
    if ((v,w) does not create a cycle in T)
        add (v,w) to T
    else
        discard (v,w);
}
if (T contains fewer than n-1 edges)
    printf("No spanning tree\n");
```

- How could "No spanning tree" happen?
- *e* union-find takes $O(e \log^* e)$ time (supplementary)

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- The main difference:
  - The set of selected edges forms a tree at all times in Prim's algorithm.
  - The set of selected edges in Kruskal's algorithm forms a *forest at each stage*.

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Time complexity: $O(n^2)$.

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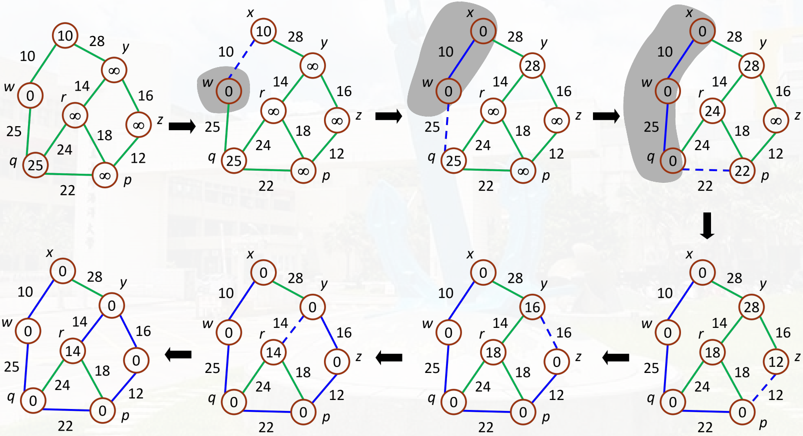

Prim's Algorithm (3/3)

```
T = {}; TV = {0};
while (T contains fewer than n-1 edges) {
    let (u,v) be a least cost edge such that u is in TV and
    v is not in TV;
        if (there is no such edge )
            break;
        add v to TV;
        add (u,v) to T;
}
if (T contains fewer than n-1 edges)
    printf("No spanning tree\n");
```

- Each vertex $v \notin TV$ has a companion vertex "near(v)" such that near(v) $\in TV$ and cost(near(v), v) is minimum over all such choices for near(v).
- Therefore, it takes $O(n)$ time to choose an edge.
- We can implement Prim's algorithm in $O(n^2)$ time.

Outline

Sollin's Algorithm (1/2)

- Sollin's algorithm selects several edges for inclusion in T at each stage.
- At the start of a stage, the selected edges, together with all the n vertices, form a spanning forest.
- During each stage, we select one edge for each tree in the forest.
  - This edge is a minimum cost edge that has exactly one vertex in the tree.

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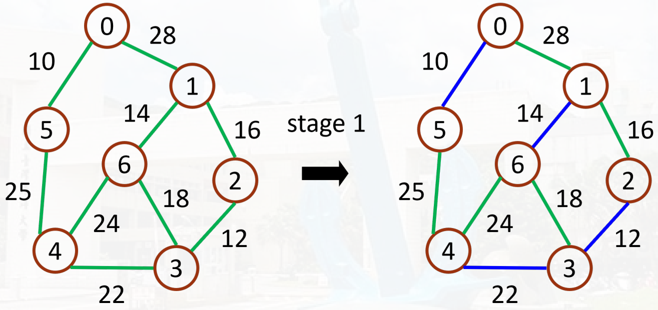
- **Note:** two trees in the forest could select the same edge.
  - We need to eliminate duplicate edges.
- At the start of the first stage the set of selected edges is empty.
- The algorithm terminates when there is only one tree at the end of a stage or no edges remain for selection.

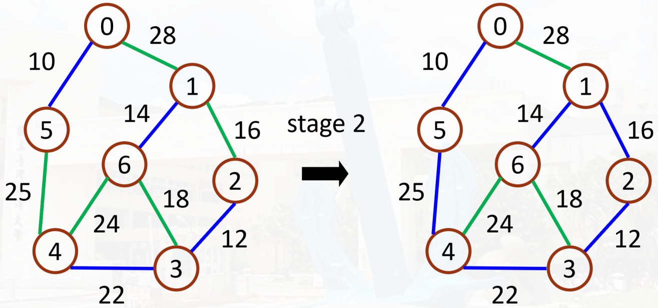
> We can implement Sollin's algorithm in $O(e \log v)$ time (WHY?).

Illustration of Sollin's Algorithm



- Stage 1: $(0,5),(1,6),(2,3),(3,2),(4,3),(5,0)$, and $(6,1)$ are selected, and then duplicates are removed.

Illustration of Sollin's Algorithm



- Stage 2: $(5,4), (1,2)$, and $(2,1)$ are selected, and then duplicates are removed.

Discussions