

Minimax Principles

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Outline

- 1 Two-Player Zero-Sum Games
- 2 Minimax Theorems
 - Yao's Minimax Principle
 - An Application: Comparison-Based Sorting



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1 Two-Player Zero-Sum Games

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- Yao's Minimax Principle
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Payoff Matrix

	Scissors	Paper	Stone
Scissors	0	1	-1
Paper	-1	0	1
Stone	1	-1	0

- Rows: Alice's choices.
- Columns: Bob's choices.
- Entry position (i, j) : state or profile.
- Entry value: the amount paid by Bob to Alice.



Payoff Matrix (the explicit form)

	Scissors	Paper	Stone
Scissors	$(0, 0)$	$(1, -1)$	$(-1, 1)$
Paper	$(-1, 1)$	$(0, 0)$	$(1, -1)$
Stone	$(1, -1)$	$(-1, 1)$	$(0, 0)$

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A Matrix M

	B1	B2	B3	B4	B5
A1	0	-1	2	-3	4
A2	-5	6	-7	8	-9
A3	10	-11	12	-13	14
A4	-15	16	-17	18	-19
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- What is $\min_j M_{1j}$? $\min_j M_{2j}$? $\min_j M_{3j}$? $\min_j M_{4j}$? $\min_j M_{5j}$?
- What is $\max_i M_{i1}$? $\max_i M_{i2}$? $\max_i M_{i3}$? $\max_i M_{i4}$? $\max_i M_{i5}$?



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• $\max_i \min_j M_{ij} =$



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- $\max_i \min_j M_{ij} = -3$.



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- $\max_i \min_j M_{ij} = -3$.
- $\min_j \max_i M_{ij} = 16$.



Exercise

Observation

For all payoff matrices M ,

$$\max_i \min_j M_{ij} \leq \min_j \max_i M_{ij}$$



Exercise

Observation

For all payoff matrices M ,

$$V_R = \max_i \min_j M_{ij} \leq \min_j \max_i M_{ij} = V_C$$

- When the equality holds, the game is said to have a solution (**saddle point**) and the value is $V = V_R = V_C$.



Hint

$$\min_j M_{ij} \leq \max_i M_{ij}?$$

Let

- $f(i) = \min_j M_{ij}, \quad j^* = \arg \min_j M_{ij}.$
- $g(j) = \max_i M_{ij}, \quad i^* = \arg \max_i M_{ij}.$



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- $\forall j, \quad M_{i,j^*} \leq M_{ij}.$

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We have

- $\forall j, M_{i,j^*} \leq M_{ij}.$
- $\forall i, M_{i,j} \leq M_{i^*,j}.$
- $\forall i \forall j, M_{i,j^*} \leq M_{i^*,j}. \text{ (since } M_{i,j^*} \leq M_{ij} \leq M_{i^*,j} \text{)}$



Example

	Scissors	Paper	Stone
Scissors	0	1	2
Paper	-1	0	1
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- Now, we have $V_R = V_C = 0$, so $V = 0$.



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- What if a game has no solution (i.e., no saddle point)?
- ★ Introduce randomization in the choice of strategies.



Example

	Scissors (33%)	Paper (33%)	Stone (33%)
Scissors (33%)	0	1	2
Paper (33%)	-1	0	1
Stone (33%)	-2	-1	0

- Now, we have $V_R = V_C = 0$, so $V = 0$.
- What if a game has no solution (i.e., no saddle point)?
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Mixed Strategies

Mixed Strategies

A mixed strategy is a probability distribution on the set of possible strategies.

- $\mathbf{p} = (p_1, \dots, p_n)$: probability distribution on the **rows** of \mathbf{M} .
- $\mathbf{q} = (q_1, \dots, q_m)$: probability distribution on the **columns** of \mathbf{M} .
- The payoff (of Alice) now becomes a **random variable**.



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- The payoff (of Alice) now becomes a **random variable**.

$$\mathbb{E}[\text{payoff}] = \mathbf{p}^\top \mathbf{M} \mathbf{q} = \sum_{i=1}^n \sum_{j=1}^m p_i M_{ij} q_j.$$



Best over distributions

$$V_R = \max_{\mathbf{p}} \min_{\mathbf{q}} \mathbf{p}^T \mathbf{M} \mathbf{q}$$

$$V_C = \min_{\mathbf{q}} \max_{\mathbf{p}} \mathbf{p}^T \mathbf{M} \mathbf{q}$$



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von Neumann's Minimax Theorem

For any two-player zero-sum game specified by a matrix \mathbf{M} ,

$$\max_{\mathbf{p}} \min_{\mathbf{q}} \mathbf{p}^T \mathbf{M} \mathbf{q} = \min_{\mathbf{q}} \max_{\mathbf{p}} \mathbf{p}^T \mathbf{M} \mathbf{q}.$$

- The saddle-point exists here and the two distributions \mathbf{p} and \mathbf{q} are called **optimal mixed-strategies**.



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For any two-player zero-sum game specified by a matrix M ,

$$\max_{\mathbf{p}} \min_{\mathbf{q}} \mathbf{p}^T M \mathbf{q} = \min_{\mathbf{q}} \max_{\mathbf{p}} \mathbf{p}^T M \mathbf{q}.$$

- Once \mathbf{p} is fixed, $\mathbf{p}^T M \mathbf{q}$ is a **linear** function of \mathbf{q} and can be minimized by setting 1 to the q_j with the smallest coefficient in the function.
- If C knows the distribution \mathbf{p} being used by R , then its optimal strategy is a pure strategy.



Loomis' Theorem

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For any two-player zero-sum game specified by a matrix M ,

$$\max_{\mathbf{p}} \min_j \mathbf{p}^\top M \mathbf{e}_j = \min_{\mathbf{q}} \max_i \mathbf{e}_i^\top M \mathbf{q}.$$

- \mathbf{e}_k : a unit vector with value 1 in the k th position and 0's elsewhere.



Example (when \mathbf{q} is fixed)

	$q_1 = \frac{1}{8}$	$q_2 = \frac{1}{2}$	$q_3 = \frac{3}{8}$
	Scissors	Paper	Stone
p_1 Scissors	0	1	-1
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- $\mathbf{p}^\top \mathbf{M} \mathbf{q} = \frac{1}{8}p_1 + \frac{1}{4}p_2 + (-\frac{3}{8})p_3.$

So we should choose $\mathbf{p} = [0 \ 1 \ 0]^\top$ for utility maximization.



Example (when \mathbf{q} is fixed; Nash equilibrium)

	$q_1 = \frac{1}{3}$	$q_2 = \frac{1}{3}$	$q_3 = \frac{1}{3}$
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- $\mathbf{p}^\top \mathbf{M} \mathbf{q} = \frac{1}{3}p_1 + \frac{1}{3}p_2 + \frac{1}{3}p_3.$

So we should choose $\mathbf{p} = [\text{? } \text{? } \text{?}]^\top$ for utility maximization.



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- $\mathbf{p}^\top \mathbf{M} \mathbf{q} = \frac{1}{3}p_1 + \frac{1}{3}p_2 + \frac{1}{3}p_3.$

So we should choose $\mathbf{p} = [\text{? } \text{? } \text{?}]^\top$ for utility maximization.

Can you find any $\mathbf{p} \neq [\frac{1}{3} \ \frac{1}{3} \ \frac{1}{3}]^\top$ which leads to better expected payoff?



Exercise (5%)

Determine the value V_R of the following 2×2 matrix game and give optimal mixed strategies for the two players.

$$\begin{pmatrix} 5 & 6 \\ 7 & 4 \end{pmatrix}$$



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The Intuitive Idea

- View the **algorithm designer** as the column player C .
 - The columns: the set of all possible algorithms.
 - Each column: a pure strategy of C ; a deterministic algorithm which is always correct.
 - ★ V_C : the **worst-case running time** of any deterministic algorithm.
- View the **adversary choosing the input** as the row player R .
 - The rows: the set of all possible inputs (of fixed size).
 - Each row: a pure strategy of R ; a specific input.
 - ★ V_R : the **non-deterministic complexity** of the problem.
- The payoff from C to R : some real-valued measure of the performance of an algorithm.
 - E.g., **running time**, solution quality, space, etc.



When considering mixed-strategies

- A mixed-strategy for C : a probability distribution over the space of always correct deterministic algorithms (Las Vegas).
- A mixed-strategy for R : a probability distribution over the space of all inputs.

Distributional Complexity

The expected running time of the **best** deterministic algorithm for the worst distribution on the inputs.



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The expected running time of the **best** deterministic algorithm for the worst distribution on the inputs.

- Smaller than the deterministic complexity since the algorithms **knows** the input distribution.



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Distributional Complexity

The expected running time of the **best** deterministic algorithm for the worst distribution on the inputs.

- Smaller than the deterministic complexity since the algorithms **knows** the input distribution.
- Loomis' Theorem implies that
the distributional complexity = the least possible expected running time achievable by any randomized algorithm.



Corollary

- Let Π be a problem with a finite set \mathcal{I} of input instances of fixed size.
- Let \mathcal{A} be a finite set of deterministic algorithms.
- Let $C(I, A)$ denote the running time of algorithm $A \in \mathcal{A}$ on input $I \in \mathcal{I}$.
- Let \mathbf{p} be a probability distribution over \mathcal{I} .
- Let \mathbf{q} be a probability distribution over \mathcal{A} .

Let $I_{\mathbf{p}}$ be a random input chosen according to \mathbf{p} and $A_{\mathbf{q}}$ be a randomized algorithm chosen according to \mathbf{q} . Then

$$\max_{\mathbf{p}} \min_{\mathbf{q}} \mathbb{E}[C(I_{\mathbf{p}}, A_{\mathbf{q}})] = \min_{\mathbf{q}} \max_{\mathbf{p}} \mathbb{E}[C(I_{\mathbf{p}}, A_{\mathbf{q}})]$$

and

$$\max_{\mathbf{p}} \min_{A \in \mathcal{A}} \mathbb{E}[C(I_{\mathbf{p}}, A)] = \min_{\mathbf{q}} \max_{I \in \mathcal{I}} \mathbb{E}[C(I, A_{\mathbf{q}})].$$

Result by Andrew C.-C. Yao

Yao's Minimax Principle

For all distributions \mathbf{p} over \mathcal{I} and \mathbf{q} over \mathcal{A} ,

$$\min_{A \in \mathcal{A}} \mathbb{E}[C(I_{\mathbf{p}}, A)] \leq \max_{I \in \mathcal{I}} \mathbb{E}[C(I, A_{\mathbf{q}})]$$



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The expected running time of the optimal deterministic algorithm for an **arbitrarily chosen input distribution** \mathbf{p} is a lower bound on the expected running time of the optimal Las Vegas randomized algorithm for problem Π .



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For all distributions \mathbf{p} over \mathcal{I} and \mathbf{q} over \mathcal{A} ,

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The expected running time of the optimal deterministic algorithm for an **arbitrarily chosen input distribution** \mathbf{p} is a lower bound on the expected running time of the optimal Las Vegas randomized algorithm for problem Π .

- Trick: choose a suitable \mathbf{p} and be aware of that the deterministic algorithm **knows** \mathbf{p} .



Extension to Monte Carlo Type Randomized Algorithms

Proposition [Yao FOCS 1977]

For

- all distributions \mathbf{p} over \mathcal{I} ,
- all distributions \mathbf{q} over \mathcal{A} ,
- any $\epsilon \in [0, 1/2]$,

we have

$$\frac{1}{2} \left(\min_{A \in \mathcal{A}} \mathbb{E}[C_{2\epsilon}(I_{\mathbf{p}}, A)] \right) \leq \max_{I \in \mathcal{I}} \mathbb{E}[C_{\epsilon}(I, A_{\mathbf{q}})]$$

- $\mathbb{E}[C_{\epsilon}(I_{\mathbf{p}}, A)]$: the expected running time of a deterministic algorithm A that errs with probability $\leq \epsilon$.



Outline

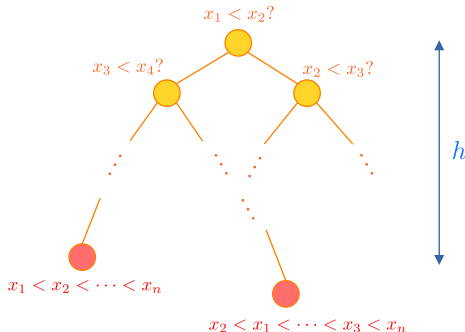
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Comparison-Based Sorting Algorithms



- Examples: MergeSort, QuickSort, BubbleSort, SelectionSort, HeapSort, etc.
- Non-examples: RadixSort, BucketSort, etc.



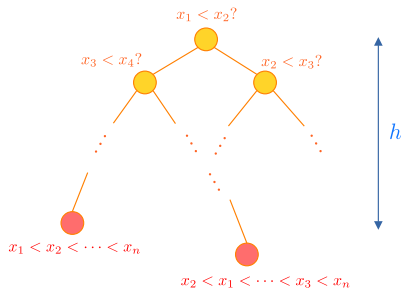
Our Goal

Theorem

Any comparison-based Las Vegas sorting algorithm requires expected $\Omega(n \log n)$ time steps.

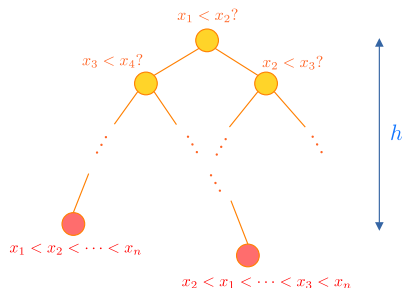


Analysis (1/3)



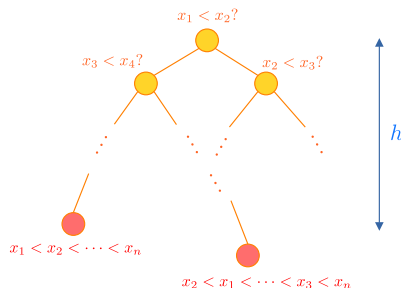
- A decision tree which models any comparison-based sorting algorithm.

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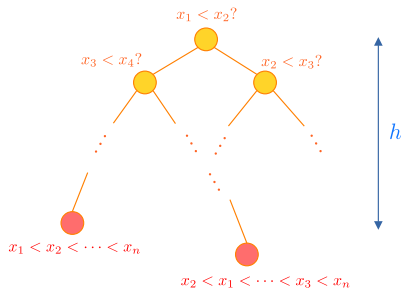
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 - Assume that the set of all permutations is **uniformly** distributed.

Analysis (1/3)



- A decision tree which models any comparison-based sorting algorithm.
- Each tree leaf corresponds to a permutation (i.e., sorted result).
 - Assume that the set of all permutations is **uniformly** distributed.
- Tree depth h : number of comparisons made by the algorithm.



Analysis (2/3)

- By the pigeonhole principle, we must have $2^h \geq n!$.

¹Note that $\lg_2(\cdot) = \log_2(\cdot)$.



Analysis (2/3)

- By the pigeonhole principle, we must have $2^h \geq n!$.
- Thus¹,

$$\begin{aligned}h \geq \lg n! &= \lg n(n-1) \cdots 2 \cdot 1 = \sum_{i=2}^n \lg i \\&\geq \sum_{i=n/2+1}^n \lg i \geq \sum_{i=n/2+1}^n \lg \left(\frac{n}{2}\right) \\&= \frac{n}{2} \lg \left(\frac{n}{2}\right) = \Omega(n \log n).\end{aligned}$$

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Analysis (2/3)

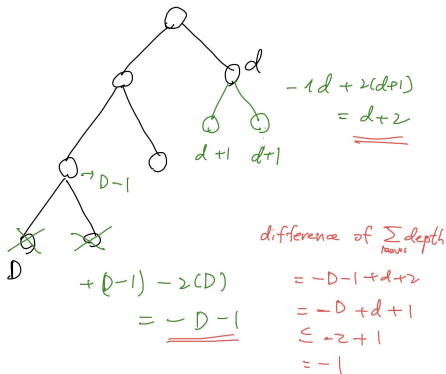
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- **Note:** This only bounds the maximum depth of a leaf in the tree.

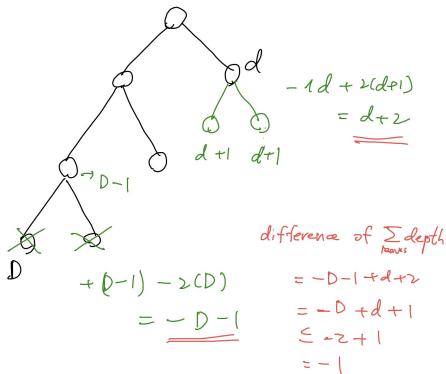
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Analysis (3/3)



The average (i.e., expected) depth of the decision tree minimized when the tree is a completely balanced.

Analysis (3/3)



The average (i.e., expected) depth of the decision tree minimized when the tree is a completely balanced. $\implies \Omega(\lg n!) = \Omega(n \log n)$ expected depth.

Discussions

