Randomized Algorithms

— Randomized QuickSort

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Motivation

- We are concerned with prediction of future events and decision making.
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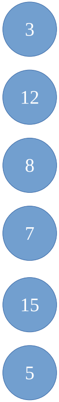
- **Note:** μ is unknown in advance and can never be observed directly.
- We need mechanisms to learn something about μ given observed outcomes of coin-flip.

Illustration (a binary tree T demonstrating RandQS)



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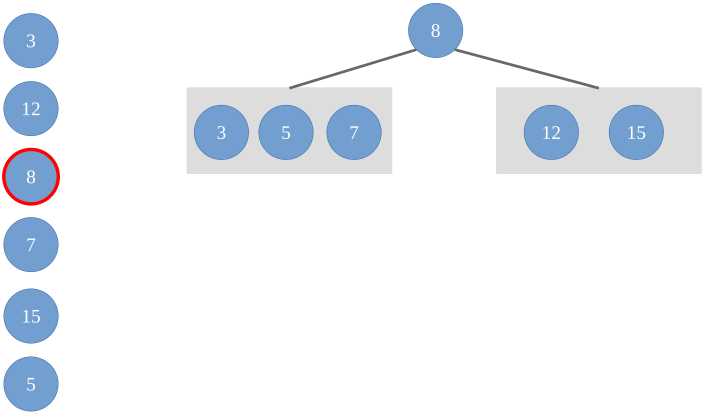

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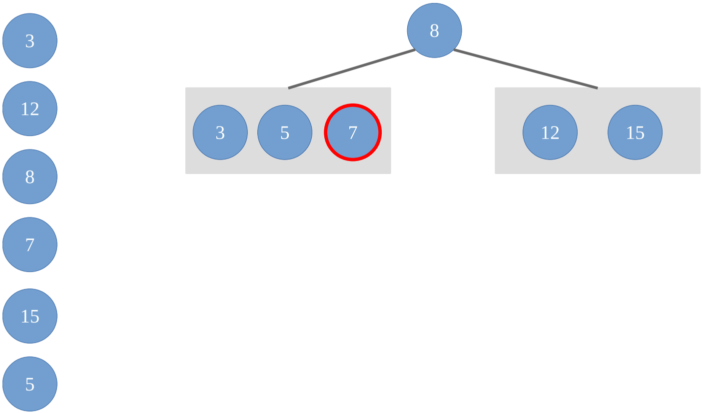

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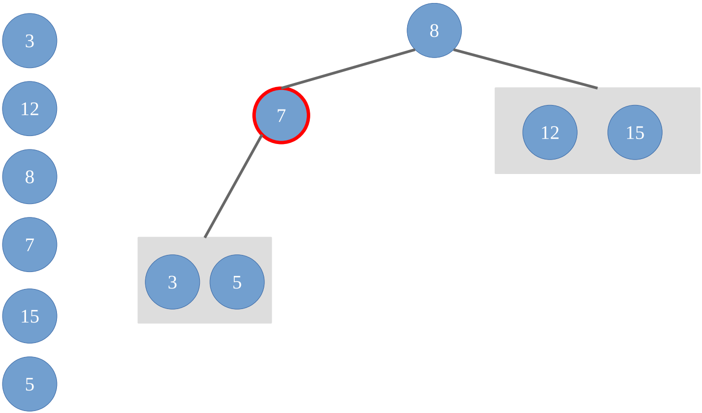

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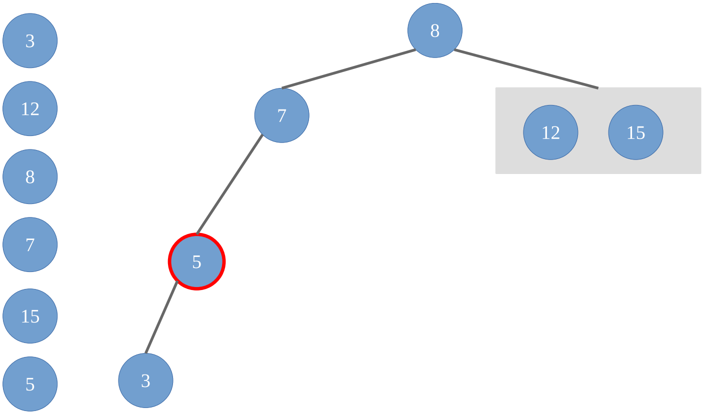

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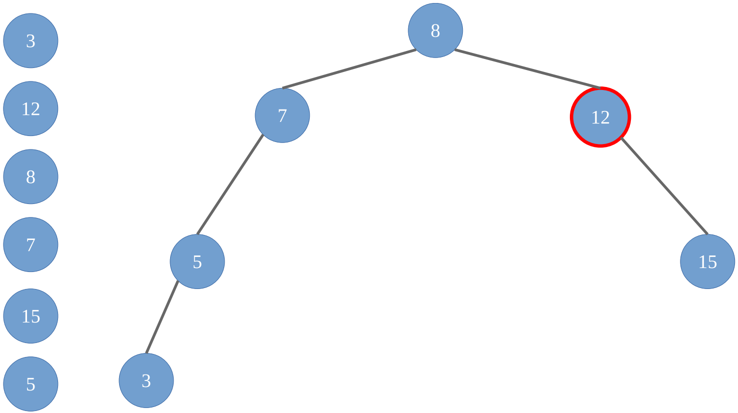

Algorithm RandQS

Input: A set of (distinct) numbers S
Output: The elements of S sorted in increasing order.

1. Choose an element $y \in S$ uniformly at random;
2. By comparing each element of S with y, compute
   - $S_1 := \{x \in S : x < y\}$;
   - $S_2 := \{x \in S : x > y\}$;
3. Recursively sort S_1 (i.e., run RandQS(S_1)) and S_2 (i.e., run RandQS(S_2)), and output the sorted version of S_1, followed by y, and then the sorted version of S_2.

Analysis (Expected Number of Comparisons)

- Comparisons are performed in Step 2.
- Let $S_{(i)}$ denote the element of rank i (i.e., the ith smallest in S).
- Define X_{ij}:
  - $X_{ij} = 1$ if $S_{(i)}$ and $S_{(j)}$ are compared in an execution.
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$$\mathbb{E}\left[\sum_{i=1}^{n} \sum_{j>i} X_{ij}\right] = \sum_{i=1}^{n} \sum_{j>i} \mathbb{E}[X_{ij}].$$

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- Note: $S_{(i)}$ and $S_{(j)}$ are compared in an execution only when one of them is an ancestor of the other in the binary tree T.

Analysis (contd.)

$$\sum_{i=1}^{n}\sum_{j>i} p_{ij} \;=\; \sum_{i=1}^{n}\sum_{j>i} \frac{2}{j-i+1}$$

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- Note that $H_n = \sum_{k=1}^{n} 1/k$

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&\le 2\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{1}{k} = O(n\log n).
\end{aligned}$$

- Note that $H_n = \sum_{k=1}^{n} 1/k \approx \Theta(\ln n)$.

Discussions