Mathematics for Machine Learning

— Probability & Distributions

Sum Rule, Product Rule, Bayes' Theorem & Summary Statistics

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Credits for the resource

- The slides are based on the textbooks:
  - *Marc Peter Deisenroth, A. Aldo Faisal, and Cheng Soon Ong: Mathematics for Machine Learning. Cambridge University Press. 2020.*
  - *Arnold J. Insel, Lawrence E. Spence, Stephen H. Friedberg: Linear Algebra, 4th Edition. Prentice Hall. 2013.*
  - *Howard Anton, Chris Rorres, Anton Kaul: Elementary Linear Algebra, 12th Edition. Wiley. 2019.*
- We could partially refer to the monograph:
  Francesco Orabona: A Modern Introduction to Online Learning.
  https://arxiv.org/abs/1912.13213

Outline

Outline

Sum Rule (1/2)

- $\mathbf{x}, \mathbf{y}$: random variables (vectors).
- $p(\mathbf{x}, \mathbf{y})$: joint distribution of $\mathbf{x}, \mathbf{y}$.
- $p(\mathbf{y} \mid \mathbf{x})$: conditional probability of $\mathbf{y}$ given $\mathbf{x}$.

Sum Rule

$$p(\mathbf{x}) = \begin{cases} \displaystyle\sum_{\mathbf{y} \in \mathcal{Y}} p(\mathbf{x}, \mathbf{y}) & \text{if } \mathbf{y} \text{ is discrete} \\ \displaystyle\int_{\mathcal{Y}} p(\mathbf{x}, \mathbf{y}) \mathrm{d}\mathbf{y} & \text{if } \mathbf{y} \text{ is continuous} \end{cases}$$

where $\mathcal{Y}$ stands for the states of the target space of random variable Y.

- Marginalization property.

Sum Rule (2/2)

For $\mathbf{x} = [x_1, \ldots, x_D]^\top$, the marginal

$$p(x_i) = \int p(x_1, \ldots, x_D) \mathrm{d}\mathbf{x}_{-i},$$

where "$-i$" means all except i.

Product Rule

Product Rule

$$p(\mathbf{x}, \mathbf{y}) = p(\mathbf{y} \mid \mathbf{x})p(\mathbf{x})$$

Outline

Bayes' Theorem

Bayes' Theorem

$$\underbrace{p(\mathbf{x} \mid \mathbf{y})}_{\text{posterior}} = \frac{\overbrace{p(\mathbf{y} \mid \mathbf{x})}^{\text{likelihood}} \overbrace{p(\mathbf{x})}^{\text{prior}}}{\underbrace{p(\mathbf{y})}_{\text{evidence}}}.$$

- Prior: subjective prior knowledge (before observing data).
- Likelihood $p(\mathbf{y} \mid \mathbf{x})$: the probability of $\mathbf{y}$ if we were to know the latent variable $\mathbf{x}$.
  - We call it "the likelihood of $\mathbf{x}$".
- Posterior $p(\mathbf{x} \mid \mathbf{y})$: the quantity that we know about $\mathbf{x}$ after having observed $\mathbf{y}$.

Marginal Likelihood/Evidence

$$p(\mathbf{y}) := \sum_{\mathbf{x}\in\mathcal{X}} p(\mathbf{y} \mid \mathbf{x})p(\mathbf{x}) = \mathbb{E}_X[p(\mathbf{y} \mid \mathbf{x})]$$

$$p(\mathbf{y}) := \int_{\mathbf{x}\in\mathcal{X}} p(\mathbf{y} \mid \mathbf{x})p(\mathbf{x})\mathrm{d}\mathbf{x} = \mathbb{E}_X[p(\mathbf{y} \mid \mathbf{x})].$$

Outline

Expected Value

Expected value

The expected value of a function $g : \mathbb{R} \to \mathbb{R}$ of a random variable $X \sim p(x)$ is

$$\mathbb{E}_X[g(x)] = \int_{\mathcal{X}} g(x)p(x)\mathrm{d}x,$$

or

$$\mathbb{E}_X[g(x)] = \sum_{x \in \mathcal{X}} g(x)p(x).$$

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Multivariate $X = [X_1, \ldots, X_D]^\top$

$$\mathbb{E}_X[g(\mathbf{x})] = \begin{bmatrix} \mathbb{E}_{X_1}[g(x_1)] \\ \vdots \\ \mathbb{E}_{X_D}[g(x_D)] \end{bmatrix} \in \mathbb{R}^D,$$

where $\mathbb{E}_{X_d}$: taking the expectation w.r.t. the x_d.

Expected Value (contd.)

Mean

For $\mathbf{x} \in \mathbb{R}^D$,

$$\mathbb{E}_X[\mathbf{x}] = \begin{bmatrix} \mathbb{E}_{X_1}[x_1] \\ \vdots \\ \mathbb{E}_{X_D}[x_D] \end{bmatrix} \in \mathbb{R}^D,$$

where

- $\mathbb{E}_{X_d}[x_d] = \int_{\mathcal{X}} x_d p(x_d) \mathrm{d}x_d$ if X is continuous ;
- $\mathbb{E}_{X_d}[x_d] = \sum_{x_i \in \mathcal{X}} x_i p(x_d = x_i) \mathrm{d}x_d$ if X is discrete.

Linearity of Expectation

Let $f(\mathbf{x}) = ag(\mathbf{x}) + bh(\mathbf{x})$ for $a, b \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^D$.

$$
\begin{aligned}
\mathbb{E}_X[f(\mathbf{x})] &= \int f(\mathbf{x})p(\mathbf{x})\mathrm{d}x \\
&= \int [ag(\mathbf{x}) + bh(\mathbf{x})]p(\mathbf{x})\mathrm{d}x \\
&= a\int g(\mathbf{x})p(\mathbf{x})\mathrm{d}x + b\int h(\mathbf{x})p(\mathbf{x})\mathrm{d}x \\
&= a\mathbb{E}_X[g(\mathbf{x})] + b\mathbb{E}_X[h(\mathbf{x})].
\end{aligned}
$$

Linearity of Expectation (Discrete Case)

Let $f(\mathbf{x}) = ag(\mathbf{x}) + bh(\mathbf{x})$ for $a, b \in \mathbb{R}$ and $\mathbf{x} \in \mathcal{X}$.

$$
\begin{aligned}
\mathbb{E}_X[f(\mathbf{x})] &= \sum_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x})p(\mathbf{x}) \\
&= \sum_{\mathbf{x}\in\mathcal{X}} [ag(\mathbf{x}) + bh(\mathbf{x})]p(\mathbf{x}) \\
&= a\sum_{\mathbf{x}\in\mathcal{X}} g(\mathbf{x})p(\mathbf{x}) + b\sum_{\mathbf{x}\in\mathcal{X}} h(\mathbf{x})p(\mathbf{x}) \\
&= a\mathbb{E}_X[g(\mathbf{x})] + b\mathbb{E}_X[h(\mathbf{x})].
\end{aligned}
$$

Covariance

The (univariate) covariance between two univariate random variables $X, Y \in \mathbb{R}$ is

$$\mathrm{Cov}_{X,Y}[x, y] := \mathbb{E}_{X,Y}[(x - \mathbb{E}_X[x])(y - \mathbb{E}_Y[y])].$$

Omit the subscript.

$$\mathrm{Cov}[x, y] := \mathbb{E}[xy] - \mathbb{E}[x]\mathbb{E}[y].$$

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Note that

$$\mathrm{Cov}[x, x] := \mathbb{E}[x^2] - (\mathbb{E}[x])^2$$

is the variance and denoted by $\mathbb{V}_X[x]$ and $\sqrt{\mathrm{Cov}[x, x]}$ denoted by $\sigma(x)$ is called the standard deviation.

Covariance of Multivariate R.V.'s

Covariance (Multivariate)

Consider random variables X and Y with states $\mathbf{x} \in \mathbb{R}^D$ and $\mathbf{y} \in \mathbb{R}^E$. The covariance between X and Y:

$$\mathrm{Cov}[\mathbf{x}, \mathbf{y}] =$$

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$$\mathrm{Cov}[\mathbf{x}, \mathbf{y}] = \mathbb{E}[\mathbf{x}\mathbf{y}^\top] - \mathbb{E}[\mathbf{x}]\mathbb{E}[\mathbf{y}]^\top = \mathrm{Cov}[\mathbf{y}, \mathbf{x}]^\top \in \mathbb{R}^{D \times E}.$$

Variance (Multivariate)

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The variance of a random variables X with states $\mathbf{x} \in \mathbb{R}^D$ and mean $\boldsymbol{\mu} \in \mathbb{R}^D$ is

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- The covariance matrix of the multivariate X.

Correlation Coefficient

Correlation

The correlation between two random variables X, Y is

$$\mathrm{corr}[x, y] = \frac{\mathrm{Cov}[x, y]}{\sqrt{\mathbb{V}[x]\mathbb{V}[y]}} \in [-1, 1].$$

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- By the Cauchy–Schwarz inequality.

Inner product of two random variables

Assumption: U, V are two real random variables, $\mathbb{E}[U^2], \mathbb{E}[V^2] < \infty$.

Definition. $\langle U, V \rangle := \mathbb{E}[UV]$.

- *Well-defined:* $2|UV| \le U^2 + V^2 \Rightarrow \mathbb{E}[UV] \le |UV| < \infty$.

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- *Symmetry:* $\langle U, V\rangle = \mathbb{E}[UV] = \mathbb{E}[VU] = \langle V, U\rangle$.
- *Positive-definite:* $\langle U, U\rangle = \mathbb{E}[U^2] \geq 0$; if $\mathbb{E}[U^2] = 0$, then $U = 0$ a.s. (else $\exists\, \varepsilon > 0 :\ \Pr[|U| \geq \varepsilon] > 0 \Rightarrow \mathbb{E}[U^2] \geq \varepsilon^2 \Pr[|U| \geq \varepsilon] > 0$).

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Hence $\mathbb{E}[UV]$ as an inner product is valid; thus, by Cauchy–Schwarz

$$|\mathbb{E}[UV]| = |\langle U, V \rangle| \leq \sqrt{\mathbb{E}[U^2]}\,\sqrt{\mathbb{E}[V^2]}.$$

Correlation is in $[-1, 1]$ by Cauchy–Schwarz

- Let X, Y satisfy $0 < \sigma_X^2 = \mathbb{V}(X) < \infty$ and $0 < \sigma_Y^2 = \mathbb{V}(Y) < \infty$.
- Set $\mu_X = \mathbb{E}[X]$, $\mu_Y = \mathbb{E}[Y]$, $U = X - \mu_X$ and $V = Y - \mu_Y$.

$$\rho_{XY} = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\mathbb{E}[(X - \mu_X)(Y - \mu_Y)]}{\sqrt{\mathbb{E}[(X - \mu_X)^2]}\sqrt{\mathbb{E}[(Y - \mu_Y)^2]}} = \frac{\langle U,\ V\rangle}{\|U\|_2\,\|V\|_2}.$$

- By the Cauchy–Schwarz inequality,

$$-1 \ \leq\ \rho_{XY}\ \leq\ 1.$$

Note: If $\sigma_X = 0$ or $\sigma_Y = 0$, correlation is undefined.

Empirical Means & Covariances

In machine learning, we need to learn from empirical observations of data.

Empirical Mean & Covariance

The empirical mean vector: arithmetic average of the observations for each variable:

$$\bar{\mathbf{x}} := \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i,$$

for $\mathbf{x}_i \in \mathbb{R}^D$. The empirical covariance matrix is a $D \times D$ matrix

$$\boldsymbol{\Sigma} := \frac{1}{N}\sum_{i=1}^{N}(\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})^\top.$$

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- $\boldsymbol{\Sigma}$ is symmetric, positive semidefinite.

Computing the Empirical Variance (1D Example)

Approaches:

1. By definition $\Rightarrow \mathbb{V}_X[x] := \mathbb{E}_X[(x-\mu)^2]$.
    - Two-pass; numerically stable.
2. $\mathbb{V}_X[x] = \mathbb{E}_X[x^2] - (\mathbb{E}_X[x])^2$.
    - One-pass; more efficient but numerically unstable.
3. Averaging pairwise differences between all pairs of observations.

$$\frac{1}{N^2}\sum_{i,j=1}^{N}(x_i - x_j)^2 = {\color{red}2}\left[\frac{1}{N}\sum_{i=1}^{N}x_i^2 - \left(\frac{1}{N}\sum_{i=1}^{N}x_i\right)^2\right].$$

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   - Two-pass; numerically stable.
2. $\mathbb{V}_X[x] = \mathbb{E}_X[x^2] - (\mathbb{E}_X[x])^2$.
   - One-pass; more efficient but numerically unstable.
3. Averaging pairwise differences between all pairs of observations.

$$\frac{1}{N^2}\sum_{i,j=1}^{N}(x_i - x_j)^2 = 2\left[\frac{1}{N}\sum_{i=1}^{N}x_i^2 - \left(\frac{1}{N}\sum_{i=1}^{N}x_i\right)^2\right].$$

   - Twice of the 2nd approach (left-hand side: $O(N^2)$).
   - Interesting perspective to compute the left-hand side target.

Welford's Online Algorithm [1962]

- **Input:** Stream of observations $x_1, x_2, \ldots$
- **Output:** (population) variances σ^2, and unbiased variance s^2.

1. **Initialization:** $n \leftarrow 0$, $\mu \leftarrow 0$, $M_2 \leftarrow 0$;
2. for each x_i in stream, $i = 1, 2, \ldots$
   1. $n \leftarrow n + 1$;
   2. $\delta \leftarrow x - \mu, \quad \mu \leftarrow \mu + \delta/n;$ /* empirical mean update */
   3. $\delta_2 \leftarrow x - \mu, \quad M_2 \leftarrow M_2 + \delta \cdot \delta_2$ /* $M_2 = \sum_{i=1}^{n}(x_i - \mu_n)^2$ */
3. **population variance:** $\sigma^2 \leftarrow M_2/n$ (valid for $n \geq 1$);
4. **unbiased variance:** $s^2 \leftarrow M_2/(n-1)$ (valid for $n \geq 2$)

- Each increment δ and δ_2 are on on the scale of the deviation or variance, not on the scale of x and x^2.

Accuracy of Welford's Online Algorithm (Mean)

Setup. For a stream $x_1, x_2, \ldots$, maintain $\mu_n :=$ mean after n, $M_2^{(n)} = \sum_{i=1}^{n}(x_i - \mu_n)^2$. Given $(\mu_{n-1}, M_2^{(n-1)})$ and new x_n,

$$\delta = x_n - \mu_{n-1}, \quad \mu_n = \mu_{n-1} + \frac{\delta}{n}, \quad \delta_2 = x_n - \mu_n, \quad M_2^{(n)} = M_2^{(n-1)} + \delta\,\delta_2.$$

Claim (Mean exactness). $\mu_n = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_i.$

Proof

Base $n = 1$: $\mu_1 = x_1$. For the step,

$$\mu_n = \mu_{n-1} + \frac{x_n - \mu_{n-1}}{n} = \frac{(n-1)\mu_{n-1} + x_n}{n} = \frac{\sum_{i=1}^{n-1} x_i + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Accuracy of Welford's Online Algorithm (2nd Moment)

Claim. $M_2^{(n)} = \sum_{i=1}^{n}(x_i - \mu_n)^2$ is preserved by $M_2^{(n)} = M_2^{(n-1)} + \delta\,\delta_2$ with $\delta = x_n - \mu_{n-1}$ and $\delta_2 = x_n - \mu_n$.

Proof

Assume $M_2^{(n-1)} = \sum_{i=1}^{n-1}(x_i - \mu_{n-1})^2$. Then

$$\begin{aligned}\sum_{i=1}^{n}(x_i - \mu_n)^2 &= \sum_{i=1}^{n-1}\left[(x_i - \mu_{n-1}) + (\mu_{n-1} - \mu_n)\right]^2 + (x_n - \mu_n)^2 \\ &= \underbrace{\sum_{i=1}^{n-1}(x_i - \mu_{n-1})^2}_{M_2^{(n-1)}} + (n-1)(\mu_{n-1} - \mu_n)^2 + (x_n - \mu_n)^2,\end{aligned}$$

Accuracy of Welford's Online Algorithm (2nd Moment) Contd.

Since $\sum_{i=1}^{n-1}(x_i - \mu_{n-1}) = 0$. With $\mu_n - \mu_{n-1} = \delta/n$ and $\delta_2 = x_n - \mu_n = x_n - (\mu_{n-1} + \delta/n) = \delta(1 - 1/n)$,

$$\begin{aligned} (n-1)(\mu_{n-1} - \mu_n)^2 + (x_n - \mu_n)^2 &= \frac{(n-1)\delta^2}{n^2} + \frac{(n-1)^2\delta^2}{n^2} \\ &= \frac{(n-1)\delta^2}{n} \\ &= \delta\,\delta_2. \end{aligned}$$

Therefore $\sum_{i=1}^{n}(x_i - \mu_n)^2 = M_2^{(n-1)} + \delta\,\delta_2 = M_2^{(n)}$.

Consequences. Population variance: $\sigma^2 = M_2^{(n)}/n$ (for $n \geq 1$); unbiased sample variance: $s^2 = M_2^{(n)}/(n-1)$ (for $n \geq 2$).

Outline

Basic Rules

Simple Rules & Exercise

Consider two random variables X, Y with states $\mathbf{x}, \mathbf{y} \in \mathbb{R}^D$. Then,

$$\begin{aligned} \mathbb{E}[\mathbf{x} \pm \mathbf{y}] &= \mathbb{E}[\mathbf{x}] \pm \mathbb{E}[\mathbf{y}] \\ \mathbb{V}[\mathbf{x} \pm \mathbf{y}] &= \mathbb{V}[\mathbf{x}] + \mathbb{V}[\mathbf{y}] \pm \mathrm{Cov}[\mathbf{x}, \mathbf{y}] \pm \mathrm{Cov}[\mathbf{y}, \mathbf{x}] \quad \text{(Exercise)}. \end{aligned}$$

- **Note:** For a constant vector $\mathbf{b} \in \mathbb{R}^D$, $\mathbb{V}(\mathbf{x} \pm \mathbf{b}) = \mathbb{V}[\mathbf{x}]$ because $\mathbb{V}[\mathbf{b}] = \mathbb{E}[\mathbf{b}\mathbf{b}^\top] - \mathbb{E}[\mathbf{b}]\mathbb{E}[\mathbf{b}]^\top = \mathbf{b}\mathbf{b}^\top - \mathbf{b}\mathbf{b}^\top = \mathbf{0}$ and

$$\mathrm{Cov}(\mathbf{x}, \mathbf{b})$$

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$$\begin{aligned} \mathbb{E}[\mathbf{x} \pm \mathbf{y}] &= \mathbb{E}[\mathbf{x}] \pm \mathbb{E}[\mathbf{y}] \\ \mathbb{V}[\mathbf{x} \pm \mathbf{y}] &= \mathbb{V}[\mathbf{x}] + \mathbb{V}[\mathbf{y}] \pm \mathrm{Cov}[\mathbf{x}, \mathbf{y}] \pm \mathrm{Cov}[\mathbf{y}, \mathbf{x}] \quad \text{(Exercise)}. \end{aligned}$$

- **Note:** For a constant vector $\mathbf{b} \in \mathbb{R}^D$, $\mathbb{V}(\mathbf{x} \pm \mathbf{b}) = \mathbb{V}[\mathbf{x}]$ because $\mathbb{V}[\mathbf{b}] = \mathbb{E}[\mathbf{b}\mathbf{b}^\top] - \mathbb{E}[\mathbf{b}]\mathbb{E}[\mathbf{b}]^\top = \mathbf{b}\mathbf{b}^\top - \mathbf{b}\mathbf{b}^\top = \mathbf{0}$ and

$$\mathrm{Cov}(\mathbf{x}, \mathbf{b}) = \mathbb{E}[\mathbf{x}\mathbf{b}^\top] - \mathbb{E}[\mathbf{x}]\mathbb{E}[\mathbf{b}]^\top$$

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Simple Rules & Exercise

Consider two random variables X, Y with states $\mathbf{x}, \mathbf{y} \in \mathbb{R}^D$. Then,

$$\begin{aligned} \mathbb{E}[\mathbf{x} \pm \mathbf{y}] &= \mathbb{E}[\mathbf{x}] \pm \mathbb{E}[\mathbf{y}] \\ \mathbb{V}[\mathbf{x} \pm \mathbf{y}] &= \mathbb{V}[\mathbf{x}] + \mathbb{V}[\mathbf{y}] \pm \mathrm{Cov}[\mathbf{x}, \mathbf{y}] \pm \mathrm{Cov}[\mathbf{y}, \mathbf{x}] \quad \text{(Exercise)}. \end{aligned}$$

- **Note:** For a constant vector $\mathbf{b} \in \mathbb{R}^D$, $\mathbb{V}(\mathbf{x} \pm \mathbf{b}) = \mathbb{V}[\mathbf{x}]$ because $\mathbb{V}[\mathbf{b}] = \mathbb{E}[\mathbf{b}\mathbf{b}^\top] - \mathbb{E}[\mathbf{b}]\mathbb{E}[\mathbf{b}]^\top = \mathbf{b}\mathbf{b}^\top - \mathbf{b}\mathbf{b}^\top = \mathbf{0}$ and

$$\mathrm{Cov}(\mathbf{x}, \mathbf{b}) = \mathbb{E}[\mathbf{x}\mathbf{b}^\top] - \mathbb{E}[\mathbf{x}]\mathbb{E}[\mathbf{b}]^\top = \mathbb{E}[\mathbf{x}]\mathbf{b}^\top - \mathbb{E}[\mathbf{x}]\mathbf{b}^\top = \mathbf{0}.$$

Affine Transformation of r.v.'s (1/2)

Consider $\mathbf{y} = \boldsymbol{A}\mathbf{x} + \mathbf{b}$ and let $\boldsymbol{\Sigma} := \mathbb{V}_X[\mathbf{x}]$.

$$
\begin{aligned}
\mathbb{E}_Y[\mathbf{y}] &= \mathbb{E}_X[\boldsymbol{A}\mathbf{x} + \mathbf{b}] = \boldsymbol{A}\mathbb{E}_X[\mathbf{x}] + \mathbf{b} \\
\mathbb{V}_Y[\mathbf{y}] &= \mathbb{V}_X[\boldsymbol{A}\mathbf{x} + \mathbf{b}] = \mathbb{V}_X[\boldsymbol{A}\mathbf{x}] = \boldsymbol{A}\mathbb{V}_X[\mathbf{x}]\boldsymbol{A}^\top = \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}^\top.
\end{aligned}
$$

$$\mathbb{V}_X[\boldsymbol{A}\mathbf{x}] = \mathbb{E}_X[(\boldsymbol{A}\mathbf{x})(\boldsymbol{A}\mathbf{x})^\top] - \mathbb{E}_X[\boldsymbol{A}\mathbf{x}](\mathbb{E}_X[\boldsymbol{A}\mathbf{x}])^\top$$

$$
\begin{aligned}
\mathbb{V}_X[\boldsymbol{A}\mathbf{x}] &= \mathbb{E}_X[(\boldsymbol{A}\mathbf{x})(\boldsymbol{A}\mathbf{x})^\top] - \mathbb{E}_X[\boldsymbol{A}\mathbf{x}](\mathbb{E}_X[\boldsymbol{A}\mathbf{x}])^\top \\
&= \mathbb{E}_X[\boldsymbol{A}\mathbf{x}\mathbf{x}^\top\boldsymbol{A}^\top] - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top\boldsymbol{A}^\top
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{V}_X[\boldsymbol{A}\mathbf{x}] &= \mathbb{E}_X[(\boldsymbol{A}\mathbf{x})(\boldsymbol{A}\mathbf{x})^\top] - \mathbb{E}_X[\boldsymbol{A}\mathbf{x}](\mathbb{E}_X[\boldsymbol{A}\mathbf{x}])^\top \\
&= \mathbb{E}_X[\boldsymbol{A}\mathbf{x}\mathbf{x}^\top\boldsymbol{A}^\top] - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top\boldsymbol{A}^\top \\
&= \boldsymbol{A}\mathbb{E}_X[\mathbf{x}\mathbf{x}^\top\boldsymbol{A}^\top] - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top\boldsymbol{A}^\top
\end{aligned}
$$

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\mathbb{V}_X[\boldsymbol{A}\mathbf{x}] &= \mathbb{E}_X[(\boldsymbol{A}\mathbf{x})(\boldsymbol{A}\mathbf{x})^\top] - \mathbb{E}_X[\boldsymbol{A}\mathbf{x}](\mathbb{E}_X[\boldsymbol{A}\mathbf{x}])^\top \\
&= \mathbb{E}_X[\boldsymbol{A}\mathbf{x}\mathbf{x}^\top\boldsymbol{A}^\top] - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top\boldsymbol{A}^\top \\
&= \boldsymbol{A}\mathbb{E}_X[\mathbf{x}\mathbf{x}^\top\boldsymbol{A}^\top] - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top\boldsymbol{A}^\top \\
&= \boldsymbol{A}(\mathbb{E}_X[\boldsymbol{A}\mathbf{x}\mathbf{x}^\top])^\top - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top\boldsymbol{A}^\top
\end{aligned}
$$

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\begin{aligned}
\mathbb{V}_X[\boldsymbol{A}\mathbf{x}] &= \mathbb{E}_X[(\boldsymbol{A}\mathbf{x})(\boldsymbol{A}\mathbf{x})^\top] - \mathbb{E}_X[\boldsymbol{A}\mathbf{x}](\mathbb{E}_X[\boldsymbol{A}\mathbf{x}])^\top \\
&= \mathbb{E}_X[\boldsymbol{A}\mathbf{x}\mathbf{x}^\top\boldsymbol{A}^\top] - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top\boldsymbol{A}^\top \\
&= \boldsymbol{A}\mathbb{E}_X[\mathbf{x}\mathbf{x}^\top\boldsymbol{A}^\top] - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top\boldsymbol{A}^\top \\
&= \boldsymbol{A}(\mathbb{E}_X[\boldsymbol{A}\mathbf{x}\mathbf{x}^\top])^\top - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top\boldsymbol{A}^\top \\
&= \boldsymbol{A}(\boldsymbol{A}\mathbb{E}_X[\mathbf{x}\mathbf{x}^\top])^\top - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top\boldsymbol{A}^\top
\end{aligned}
$$

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\begin{aligned}
\mathbb{V}_X[\boldsymbol{A}\mathbf{x}] &= \mathbb{E}_X[(\boldsymbol{A}\mathbf{x})(\boldsymbol{A}\mathbf{x})^\top] - \mathbb{E}_X[\boldsymbol{A}\mathbf{x}](\mathbb{E}_X[\boldsymbol{A}\mathbf{x}])^\top \\
&= \mathbb{E}_X[\boldsymbol{A}\mathbf{x}\mathbf{x}^\top\boldsymbol{A}^\top] - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top\boldsymbol{A}^\top \\
&= \boldsymbol{A}\mathbb{E}_X[\mathbf{x}\mathbf{x}^\top\boldsymbol{A}^\top] - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top\boldsymbol{A}^\top \\
&= \boldsymbol{A}(\mathbb{E}_X[\boldsymbol{A}\mathbf{x}\mathbf{x}^\top])^\top - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top\boldsymbol{A}^\top \\
&= \boldsymbol{A}(\boldsymbol{A}\mathbb{E}_X[\mathbf{x}\mathbf{x}^\top])^\top - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top\boldsymbol{A}^\top \\
&= \boldsymbol{A}\mathbb{E}_X[\mathbf{x}\mathbf{x}^\top]\boldsymbol{A}^\top - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top\boldsymbol{A}^\top
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{V}_X[\boldsymbol{A}\mathbf{x}] &= \mathbb{E}_X[(\boldsymbol{A}\mathbf{x})(\boldsymbol{A}\mathbf{x})^\top] - \mathbb{E}_X[\boldsymbol{A}\mathbf{x}](\mathbb{E}_X[\boldsymbol{A}\mathbf{x}])^\top \\
&= \mathbb{E}_X[\boldsymbol{A}\mathbf{x}\mathbf{x}^\top \boldsymbol{A}^\top] - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top \boldsymbol{A}^\top \\
&= \boldsymbol{A}\mathbb{E}_X[\mathbf{x}\mathbf{x}^\top \boldsymbol{A}^\top] - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top \boldsymbol{A}^\top \\
&= \boldsymbol{A}(\mathbb{E}_X[\boldsymbol{A}\mathbf{x}\mathbf{x}^\top])^\top - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top \boldsymbol{A}^\top \\
&= \boldsymbol{A}(\boldsymbol{A}\mathbb{E}_X[\mathbf{x}\mathbf{x}^\top])^\top - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top \boldsymbol{A}^\top \\
&= \boldsymbol{A}\mathbb{E}_X[\mathbf{x}\mathbf{x}^\top]\boldsymbol{A}^\top - \boldsymbol{A}\mathbb{E}_X[\mathbf{x}]\mathbb{E}_X[\mathbf{x}]^\top \boldsymbol{A}^\top \\
&= \boldsymbol{A}\mathbb{V}_X[\mathbf{x}]\boldsymbol{A}^\top .
\end{aligned}
$$

Affine Transformation of r.v.'s (2/2)

Furthermore, let $\boldsymbol{\mu} := \mathbb{E}_X[\mathbf{x}]$ and $\boldsymbol{\Sigma} := \mathbb{V}_X[\mathbf{x}]$.

$$\begin{aligned}
\mathrm{Cov}[\mathbf{x}, \mathbf{y}] &= \mathbb{E}[\mathbf{x}(\boldsymbol{A}\mathbf{x} + \mathbf{b})^\top] - \mathbb{E}[\mathbf{x}]\mathbb{E}[\boldsymbol{A}\mathbf{x} + \mathbf{b}]^\top \\
&= \boldsymbol{\mu}\mathbf{b}^\top + \mathbb{E}[\mathbf{x}\mathbf{x}^\top]\boldsymbol{A}^\top - \boldsymbol{\mu}\mathbf{b}^\top - \boldsymbol{\mu}\boldsymbol{\mu}^\top\boldsymbol{A}^\top \\
&= (\mathbb{E}[\mathbf{x}\mathbf{x}^\top] - \boldsymbol{\mu}\boldsymbol{\mu}^\top)\boldsymbol{A}^\top \\
&= \boldsymbol{\Sigma}\boldsymbol{A}^\top.
\end{aligned}$$

Outline

(Statistically) Independent

Two random variables X, Y are statistically independent if and only if

$$p(\mathbf{x}, \mathbf{y}) = p(\mathbf{x})p(\mathbf{y}).$$

If X, Y are independent, then

- $p(\mathbf{y} \mid \mathbf{x}) = p(\mathbf{y})$.
- $p(\mathbf{x} \mid \mathbf{y}) = p(\mathbf{x})$.
- $\mathbb{V}_{X,Y}[\mathbf{x} + \mathbf{y}] = \mathbb{V}_X[\mathbf{x}] + \mathbb{V}_Y[\mathbf{y}]$.
- $\mathrm{Cov}_{X,Y}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$.

Remark

Note that $\text{Cov}_{X,Y}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ does NOT necessarily imply that X and Y are independent.

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Note that $\mathrm{Cov}_{X,Y}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ does NOT necessarily imply that X and Y are independent.

- Consider a random variable X with $\mathbb{E}_X[x] = 0$ and also $\mathbb{E}_X[x^3] = 0$.

Remark

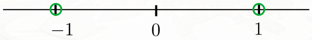
Note that $\mathrm{Cov}_{X,Y}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ does NOT necessarily imply that X and Y are independent.

- Consider a random variable X with $\mathbb{E}_X[x] = 0$ and also $\mathbb{E}_X[x^3] = 0$.
- Let $y = x^2$. Hence, Y is dependent on X.

Remark

Note that $\mathrm{Cov}_{X,Y}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ does NOT necessarily imply that X and Y are independent.

- Consider a random variable X with $\mathbb{E}_X[x] = 0$ and also $\mathbb{E}_X[x^3] = 0$.
- Let $y = x^2$. Hence, Y is dependent on X.
- $\mathrm{Cov}[x, y] = \mathbb{E}[xy] - \mathbb{E}[x]\mathbb{E}[y] = \mathbb{E}[x^3] = 0$.



Conditional Independence

Two random variables X, Y are conditionally independent given Z if and only if

$$p(\mathbf{x}, \mathbf{y} \mid \mathbf{z}) = p(\mathbf{x} \mid \mathbf{z}) p(\mathbf{y} \mid \mathbf{z}).$$

for all $\mathbf{z} \in \mathcal{Z}$.

By the product rule, we can have

$$p(\mathbf{x}, \mathbf{y} \mid \mathbf{z}) = p(\mathbf{x} \mid \mathbf{y}, \mathbf{z}) p(\mathbf{y} \mid \mathbf{z}).$$

Thus,

$$p(\mathbf{x} \mid \mathbf{y}, \mathbf{z}) = p(\mathbf{x} \mid \mathbf{z}).$$

Heads Up

If X, Y are independent, then $\mathbb{V}_{X,Y}[\mathbf{x}+\mathbf{y}] = \mathbb{V}_X[\mathbf{x}] + \mathbb{V}_Y[\mathbf{y}]$.

$$\because \quad \mathrm{Cov}_{X,Y}(\mathbf{x},\mathbf{y}) = \mathbf{0}$$

Discussions