Counting Binary Trees & Selection Trees

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Outline

1. Counting Binary Trees
2. Selection Trees

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Counting Binary Trees

- Consider the following three disparate problems:
  1. The number of distinct binary trees having n nodes.
  2. The number of distinct permutations of the numbers from 1 to n obtainable by a stack.
  3. The number of distinct ways of multiplying $n+1$ matrices.

Counting Binary Trees

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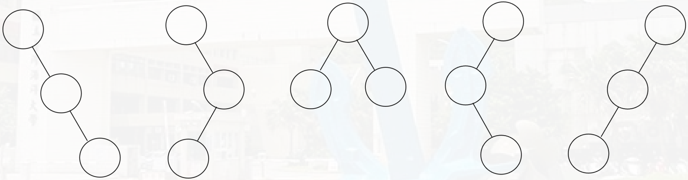
  1. The number of distinct binary trees having n nodes.
  2. The number of distinct permutations of the numbers from 1 to n obtainable by a stack.
  3. The number of distinct ways of multiplying $n+1$ matrices.
- Amazingly, these problems have the same solution!

Problem One

- The number of distinct binary trees having n nodes.



$\star$ Example of $n = 3$.

Problem Two

- The number of distinct permutations of the numbers from 1 to n obtainable by a stack.

1. push 1 $\rightarrow$ pop $\rightarrow$ push 2 $\rightarrow$ pop $\rightarrow$ push 3 $\rightarrow$ pop $\Rightarrow$ 123.
2. push 1 $\rightarrow$ pop $\rightarrow$ push 2 $\rightarrow$ push 3 $\rightarrow$ pop $\rightarrow$ pop $\Rightarrow$ 132.
3. push 1 $\rightarrow$ push 2 $\rightarrow$ push 3 $\rightarrow$ pop $\rightarrow$ pop $\rightarrow$ pop $\Rightarrow$ 321.
4. push 1 $\rightarrow$ push 2 $\rightarrow$ pop $\rightarrow$ pop $\rightarrow$ push 3 $\rightarrow$ pop $\Rightarrow$ 213.
5. push 1 $\rightarrow$ push 2 $\rightarrow$ pop $\rightarrow$ push 3 $\rightarrow$ pop $\rightarrow$ pop $\Rightarrow$ 231.

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Problem Three

- The number of distinct ways of multiplying $n+1$ matrices.

1. $(((M_1 \times M_2) \times M_3) \times M_4)$.
2. $((M_1 \times (M_2 \times M_3)) \times M_4)$.
3. $(M_1 \times ((M_2 \times M_3) \times M_4))$.
4. $(M_1 \times (M_2 \times (M_3 \times M_4)))$.
5. $((M_1 \times M_2) \times (M_3 \times M_4))$.

$\star$ Example of $n = 3$.

Stack Permutation (1/4)

- Recall: preorder, inorder and postorder traversal of a binary tree.
  - Each traversal requires a stack.

Every binary tree has a unique pair of preorder/inorder sequences.

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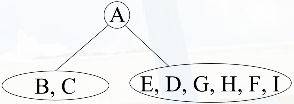
  - Each traversal requires a stack.

Every binary tree has a unique pair of preorder/inorder sequences.

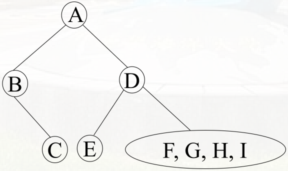
- The number of distinct binary trees is equal to the number of inorder permutations obtainable from binary trees having the preorder permutation, $1, 2, \ldots, n$.

Stack Permutation (2/4)

- preorder: A B C E D G H F I
- inorder: B C A E D G H F I



- preorder: A B C (D E F G H I)
- inorder: B C A (E D F G H I)



Stack Permutation (3/4)

- We can show that

  the number of distinct permutations obtainable by passing the numbers $\{1, 2, \ldots, n\}$ through a stack is equal to the number of distinct binary trees with n nodes.

1. push 1 $\to$ pop $\to$ push 2 $\to$ pop $\to$ push 3 $\to$ pop $\Rightarrow$ 123.
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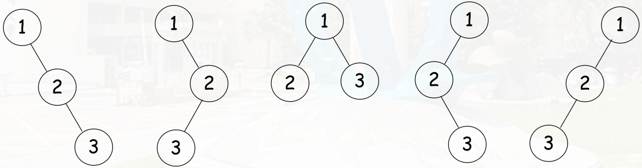
Stack Permutation (4/4)

⋆ Output: inorder traversal:

- visit node → push to stack;
  - going left → keep visiting next node
  - going right → pop the stack
- the leaf → pop the stack until empty



(1, 2, 3) (1, 3, 2) (2, 1, 3) (2, 3, 1) (3, 2, 1)

Go Back to the Matrix Multiplication

- Computing the product of *n* matrices are related to the distinct binary tree problem.
- $n = 3$:
  1. $((M_1 \times M_2) \times M_3)$.
  2. $(M_1 \times (M_2 \times M_3))$.
- $n = 4$:
  1. $(((M_1 \times M_2) \times M_3) \times M_4)$
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- $n = 4$:
  1. $(((M_1 \times M_2) \times M_3) \times M_4)$ (push, pop, push, pop, push, pop)
  2. $((M_1 \times (M_2 \times M_3)) \times M_4)$ (?)
  3. $(M_1 \times ((M_2 \times M_3) \times M_4))$ (?)
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Matrix Multiplication (2/2)

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- b_n: the number of different ways to compute the product of n matrices.
- Trivially, $b_1 = 1$, $b_2 = 1$.
- We have also derived that $b_3 = 2$ and $b_4 = 5$.
- We can compute that

$$b_n = \sum_{i=1}^{n-1} b_i b_{n-i}, \text{ for } n > 1.$$

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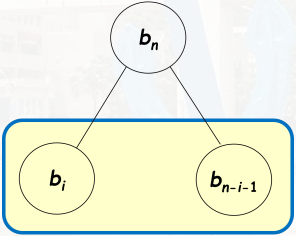

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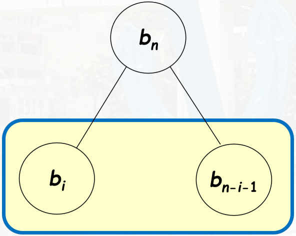

- But, how to compute b_n exactly?

The Generating Function Trick

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$$=$$

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&= \quad 1 + x \sum_{j=0}^{\infty} \sum_{k=0}^{j} b_k b_{j-k} x^j \\
&= \quad 1 + x \left(\sum_{j=0}^{\infty} b_j x^j \right)^2 = 1 + xB(x)^2.
\end{aligned}
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The Generating Function Trick

- By the recurrence relation we get:

$$xB(x)^2 = B(x) - 1.$$

- Solving the recurrence relation, we have

$$\begin{aligned} B(x) &= \frac{1-\sqrt{1-4x}}{2x} \\ &= \frac{1}{2x}\left(1 - \sum_{n\geq 0}\binom{1/2}{n}(-4x)^n\right) \\ &= \sum_{m\geq 0}\binom{1/2}{m+1}(-1)^m 2^{2m+1} x^m. \end{aligned}$$

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$$\therefore b_n = \frac{1}{n+1}\binom{2n}{n}.$$

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\end{aligned}
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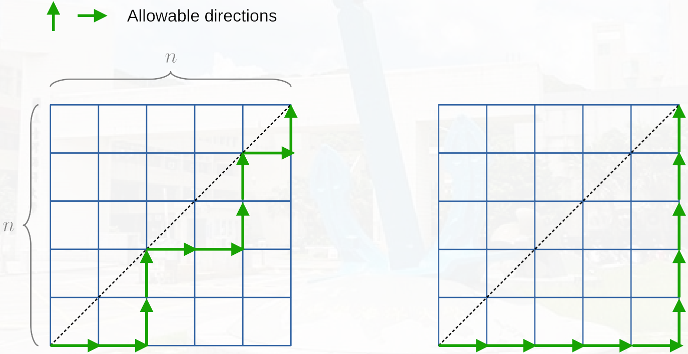
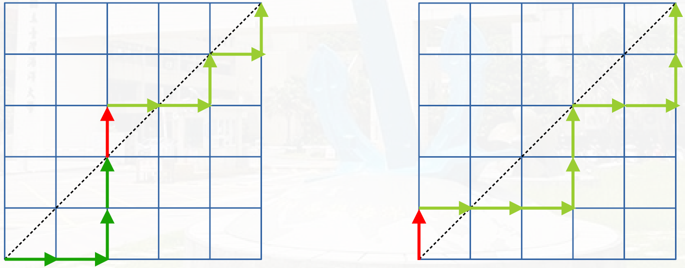
⋆ supplementary: Stirling's approximation

Catalan number by monotonic lattice paths (1/5)

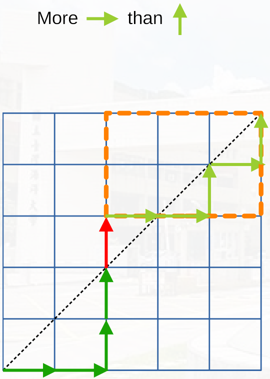


Catalan number by monotonic lattice paths (2/5)

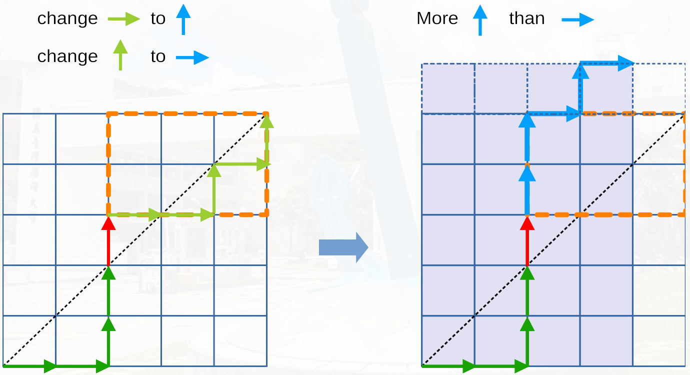
First time breaking the diagonal



Catalan number by monotonic lattice paths (3/5)



Catalan number by monotonic lattice paths (4/5)



Catalan number by monotonic lattice paths (5/5)

- The number of monotonic lattice paths not passing the diagonal is

$$\begin{aligned}
\binom{2n}{n} - \binom{2n}{n-1} &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} \\
&= (2n)!\left(\frac{n+1-n}{n!(n+1)!}\right) \\
&= \frac{1}{n+1}\frac{(2n)!}{n!n!} \\
&= \frac{1}{n+1}\binom{2n}{n}.
\end{aligned}$$

Outline

Scenarios of Using the Selection Trees

- External sorting.
- Data stored in each queue (run) is sorted.

Winner Selection Tree

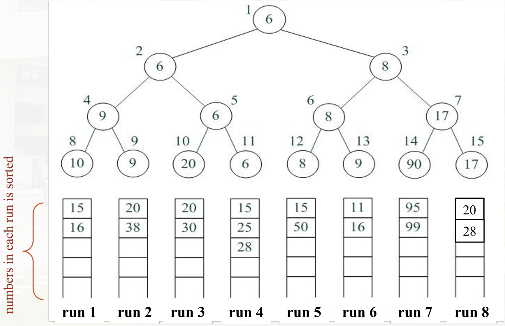
- In the following figure, computing the first winner takes

Winner Selection Tree

- In the following figure, computing the first winner takes $k-1$ comparisons.

Winner Selection Tree

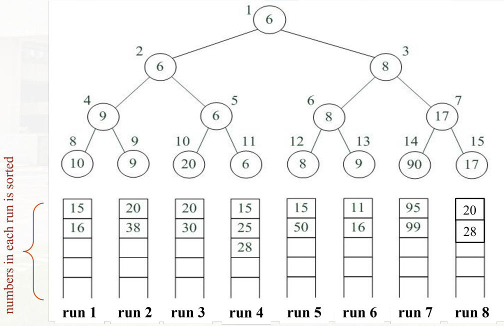
- In the following figure, computing the first winner takes $k - 1$ comparisons. (not better?)
- But wait, how about the following iterations?



Winner Selection Tree

(Winner) Selection Tree:

- Setting up the selection tree: $O(k)$ time.
- Restructuring: $O(\lg k)$ time.
- merging all n items: $O(n \lg k)$ time.



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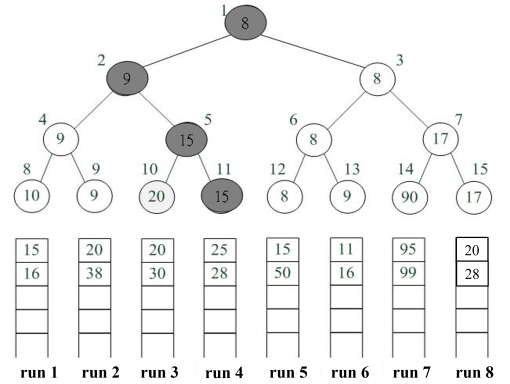
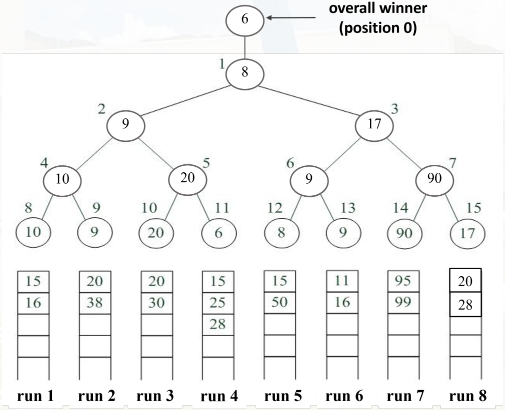


Loser Selection Tree

Loser Tree:

- Sibling nodes represent the losers.
- The restructuring process can be simplified.



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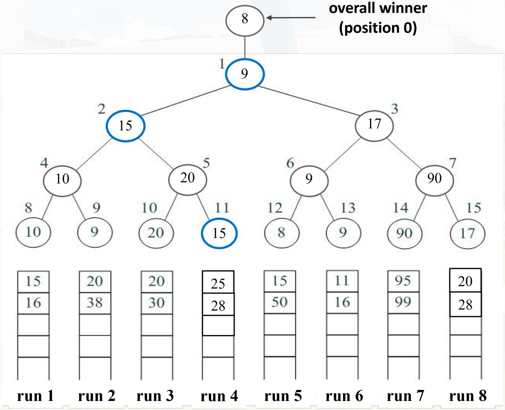
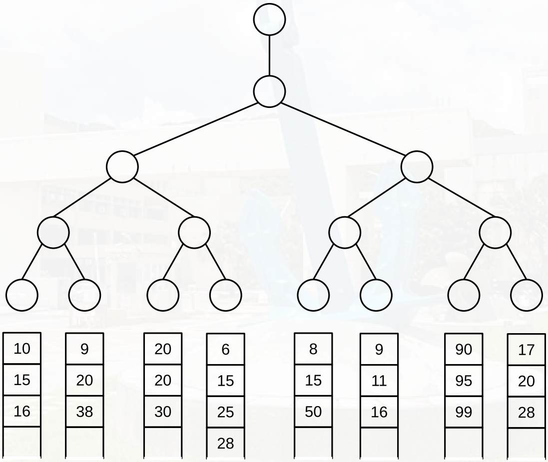


Loser Tree (step-by-step)



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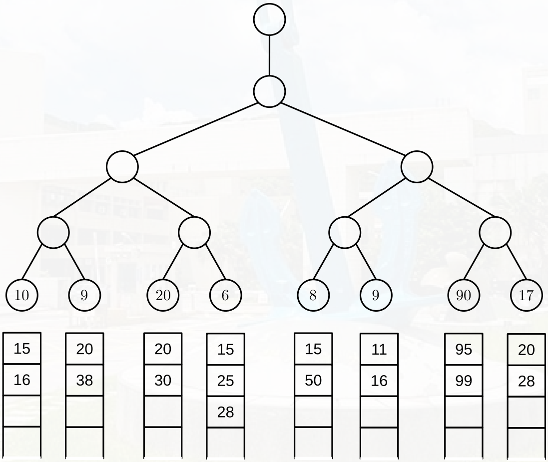

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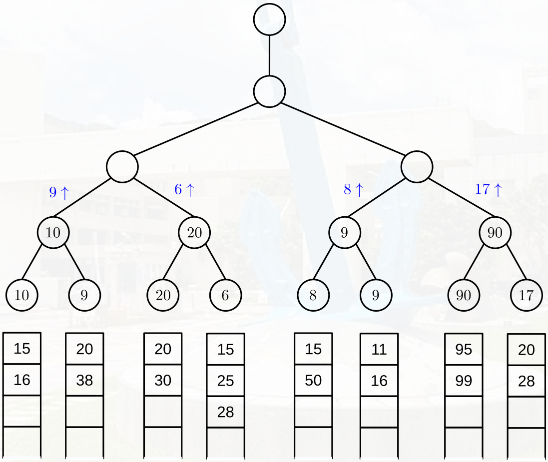

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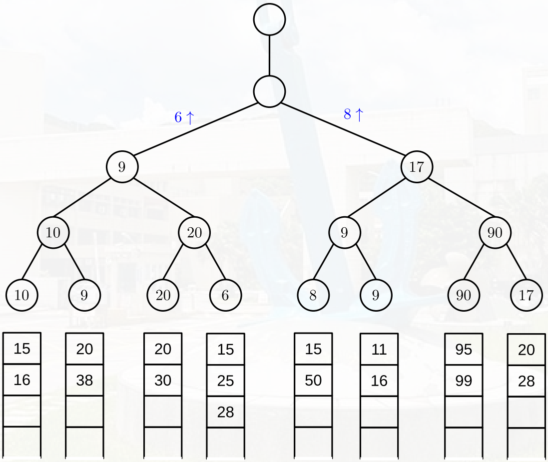

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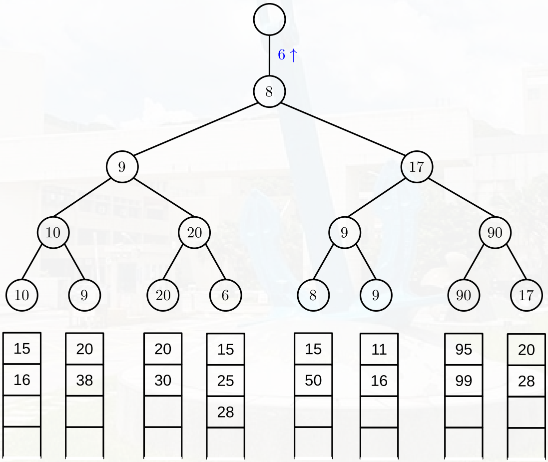

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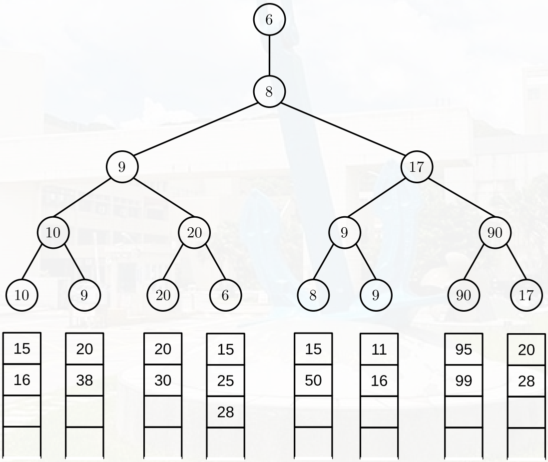

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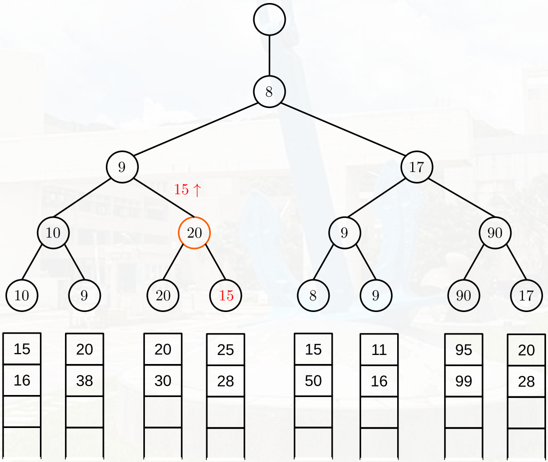

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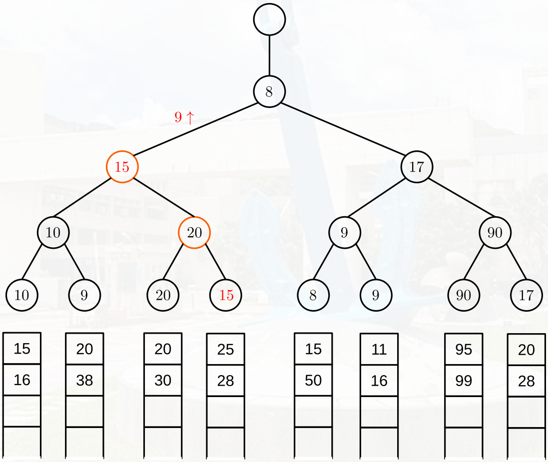

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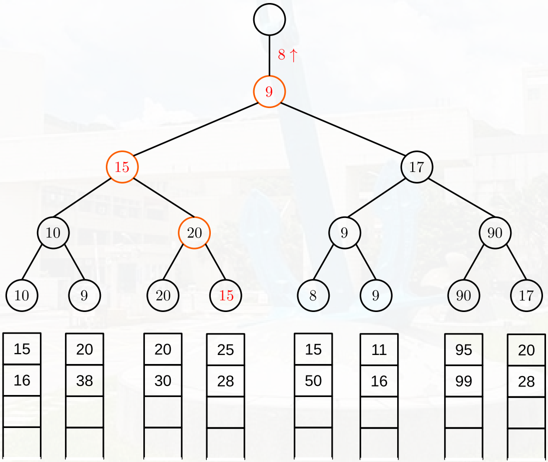

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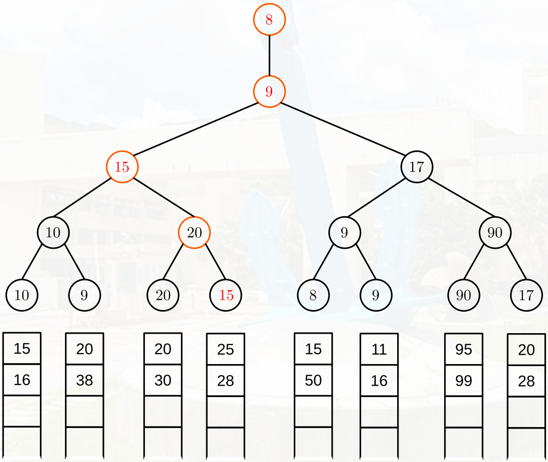

Note for the Loser Selection Tree

- Comparison with the sibling is required for the first step construction.
- After the first construction, we only need to compare each node with its parent; “push” the smaller key value upward and left the “larger” key value as the loser.

Discussions

Supplementary

- Some stuff for your reference.

Ordinary vs. Generalized Binomial Theorem

- For an integer $n \geq 0$,

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k.$$

- For an arbitrary exponent $\alpha \in \mathbb{R}$ or $\mathbb{C}$, we define

$$\binom{\alpha}{0} := 1, \qquad \binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \quad (n \geq 1).$$

- **Generalized binomial theorem:**

$$(1+z)^\alpha = \sum_{n \geq 0} \binom{\alpha}{n} z^n.$$

Analytically this converges for $|z| < 1$.

The Generalized Series Comes From Taylor expansion

- Consider $f(z) = (1 + z)^\alpha$ with arbitrary α.
- f is analytic near $z = 0$, so it has a Taylor expansion:

$$f(z) = \sum_{n \geq 0} \frac{f^{(n)}(0)}{n!} z^n.$$

- Compute derivatives:

$$f'(z) = \alpha(1 + z)^{\alpha-1}, \quad f''(z) = \alpha(\alpha - 1)(1 + z)^{\alpha-2}, \ \ldots$$

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In general, $f^{(n)}(z) = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+z)^{\alpha-n}$.
- Hence $f^{(n)}(0) = \alpha(\alpha-1)\cdots(\alpha-n+1)$, so

$$(1+z)^\alpha = \sum_{n\geq 0} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} z^n = \sum_{n\geq 0} \binom{\alpha}{n} z^n.$$

Applying to $\sqrt{1-4x}$

- Take $\alpha = \frac{1}{2}$ and $z = -4x$ in the generalized binomial theorem:

$$\sqrt{1-4x} = (1-4x)^{1/2} = \sum_{n\geq 0} \binom{1/2}{n} (-4x)^n.$$

- First few terms:

$$\sqrt{1-4x} = 1 - 2x - 2x^2 - 4x^3 - \cdots$$

  (obtained by plugging $n = 0, 1, 2, 3, \ldots$).
- This is the expansion used when deriving the generating function for the Catalan numbers.

Why not $B(x) = (1 + \sqrt{1-4x})/2x$? (1/2)

- By definition, the generating function should be

$$B(x) = \sum_{n \geq 0} b_n x^n,$$

i.e., a formal power series with only nonnegative powers of x and $b_0 = B(0) = 1$.

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- From the functional equation $xB(x)^2 = B(x) - 1$, comparing constant terms gives $B(0) - 1 = 0$, so $B(0) = 1$ is required.

Why not $B(x) = (1 + \sqrt{1-4x})/2x$? (2/2)

- Expand $\sqrt{1-4x}$ as a formal power series:

$$\sqrt{1-4x} = 1 - 2x - 2x^2 - 4x^3 - \cdots$$

  hence

$$\frac{1+\sqrt{1-4x}}{2x} = \frac{2-2x-2x^2-4x^3-\cdots}{2x} = \frac{1}{x} - 1 - x - 2x^2 - \cdots.$$

- This has a $1/x$ term and, in particular, $B(0)$ is not finite and cannot equal 1.

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- This has a $1/x$ term and, in particular, $B(0)$ is not finite and cannot equal 1.
- The other branch,

$$B(x) = \frac{1-\sqrt{1-4x}}{2x} = 1 + x + 2x^2 + 5x^3 + \cdots ,$$

  is a genuine formal power series with $B(0) = 1$, so it is the unique valid generating function for the Catalan numbers.