

Shortest Paths

Dijkstra's Algorithm & Bellman-Ford Algorithm

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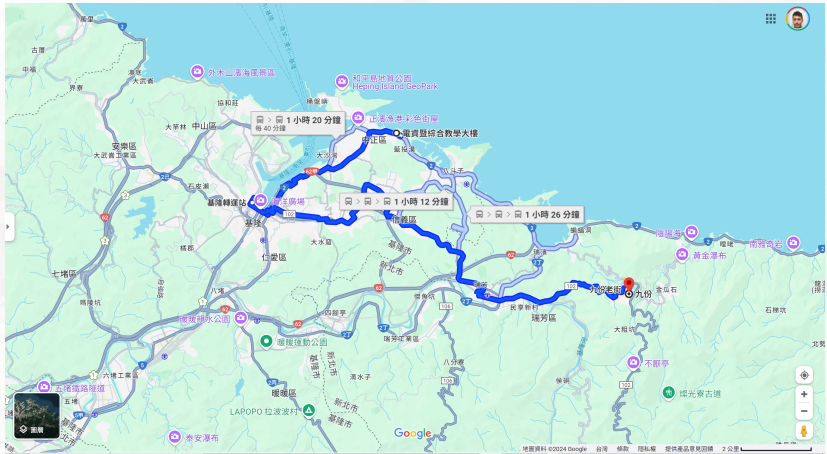


Outline

- 1 Introduction
- 2 Dijkstra's Algorithm
- 3 Bellman-Ford Algorithm for General Weights



Shortest path(s) from NTOU to Jiufen Old Street.



Shortest Paths

- Model the problem via a graph.
- vertices \mapsto locations (e.g., stations, restaurants, gas stations, etc.)
 - Including the **source** and the **destination**.
- edges \mapsto highways, railways, roads, etc.
 - edge **weight**: tolls, the distance, passing-through time, etc.



Shortest Paths

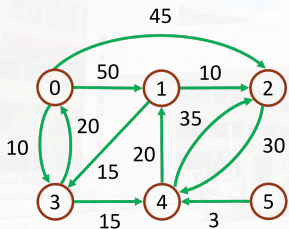
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Questions

- Is there a path from NTOU to Jiufen?
- If it exists, which one is the **shortest**?



Single Source/All Destinations (Nonnegative Edge Costs)



	path	length (cost)
1	0, 3	10
2	0, 3, 4	25
3	0, 3, 4, 1	45
4	0, 2	45

Notations:

- A directed graph $G = (V, E)$; a weight function $w(e)$, $w(e) > 0$ for any edge $e \in E$.
- v_0 : source vertex.
- If $(v_i, v_j) \notin E$, $w(v_i, v_j) = \infty$.



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- Let S denote the set of vertices, including v_0 , whose shortest paths have been found.
- For $v \notin S$, let $\text{dist}[v]$ be the length of the shortest path starting from v_0 , going through vertices ONLY in S , and ending in v .

Dijkstra's Algorithm

- At the first stage, we add v_0 to S , set $\text{dist}[v_0] = 0$ and determine $\text{dist}[v]$ for each $v \notin S$.

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- Adding w to S , and updating $\text{dist}[v]$ for v , where $v \notin S$ currently.



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- Repeat the vertex addition process until $S = V(G)$

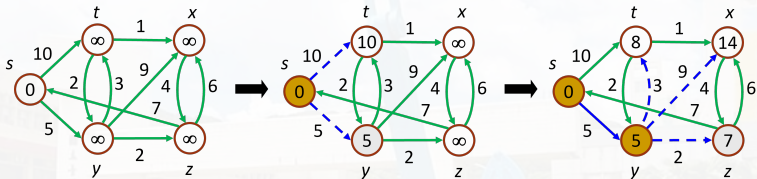


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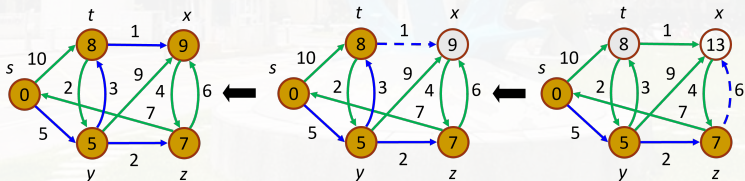
Time complexity: $O(n^2)$.

Illustration of Dijkstra's Algorithm



During each iteration:

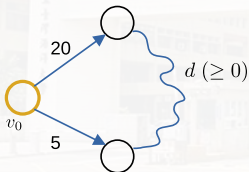
1. Update the distance of the rest vertices
2. Pick the vertex with the smallest distance value



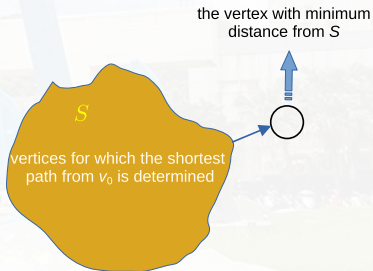
Basic Idea of Dijkstra's Algorithm

- Induction on n .

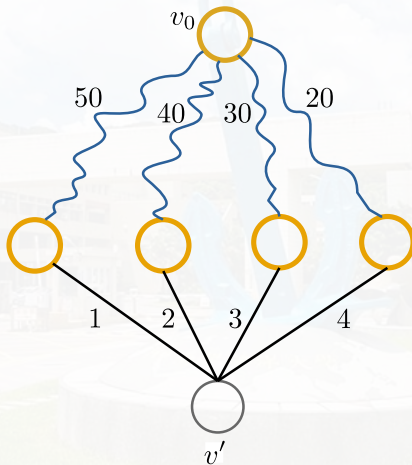
base case



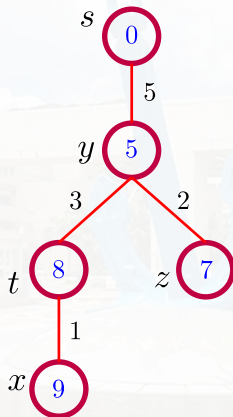
$$20 + d \geq 5 \quad (\because \text{triangular inequality})$$



Shortest path tree



Shortest path tree



The Pseudo-code of Dijkstra's Algorithm

```
S = { v0 };
dist[v0] = 0;
for each v in V - {v0} do
    dist[v] = e(v0,v); // initialization
while (S != V) do
    choose a vertex w in V - S such that dist[w] is a minimum;
    add w to S; // the other nodes in S have been utilized!
    for each v in V - S do
        dist[v] = min(dist[v], dist[w]+e(w, v));
    endfor
endwhile
```



Dijkstra's Algorithm (Functions (1/2))

```

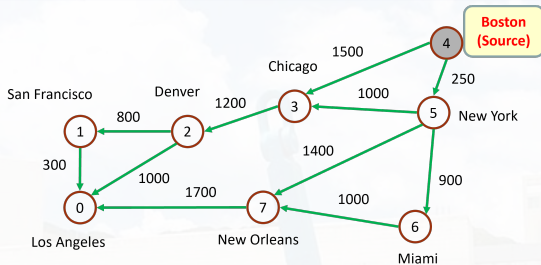
void shortestPath (int v, int cost[][MAX_VERTICES],
                  int distance[], int n, bool found[]) {
    /* distance[i]: the shortest path from vertex v to i
       found[i]: 0 if the shortest path from vertex i has not
       been found and a 1 otherwise
       cost: the adjacency matrix */
    int i, u, w;
    for (i=0; i<n; i++) {
        found[i] = false; distance[i] = cost[v][i];
    }
    found[v] = true; //initialization
    distance[v] = 0; //initialization
    for (i=0; i<n-1; i++) {
        u = choose(distance, n, found);
        found[u] = true;
        for (w=0; w<n; w++)
            if (!found[w])
                if (distance[u] + cost[u][w] < distance[w])
                    distance[w] = distance[u]+cost[u][w];
    }
}

```

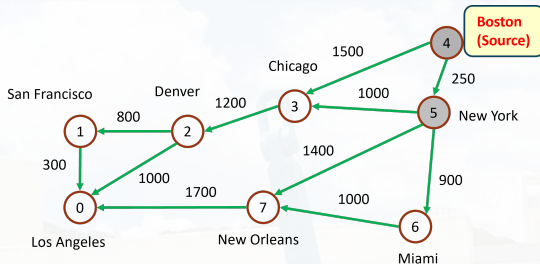


Dijkstra's Algorithm (Functions (2/2))

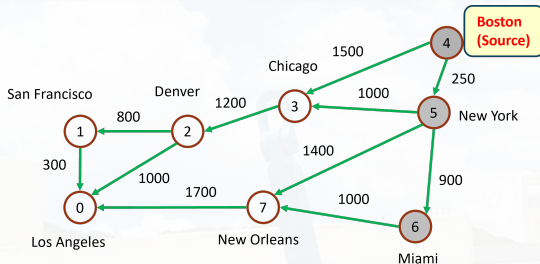
```
int choose (int distance[], int n, bool found[]) {  
    /* find the smallest distance not yet checked */  
    int i, min, min_pos;  
    min = INT_MAX;  
    min_pos = -1;  
    for (i=0; i<n; i++)  
        if (distance[i] < min && !found[i]) {  
            min = distance[i];  
            min_pos = i;  
        }  
    return min_pos;  
}
```



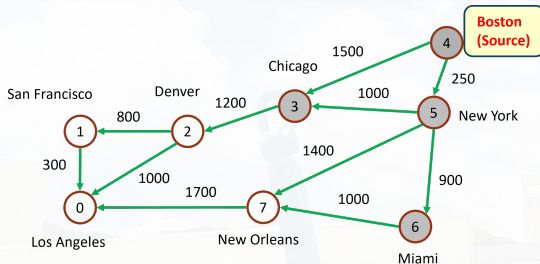
Iteration	Vertex Select.	Distance							
		LA	SF	DEN	CHI	BOS	NY	MIA	NO
		[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
initial	—	∞	∞	∞	1500	0	250	∞	∞



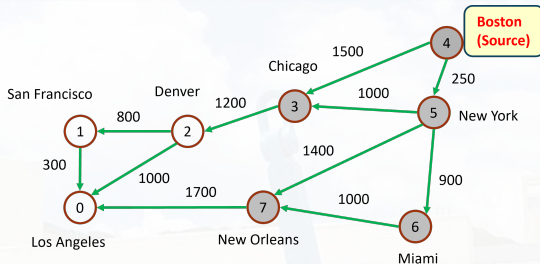
Iteration	Vertex Select.	Distance							
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		[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
initial	—	∞	∞	∞	1500	0	250	∞	∞
1	5	∞	∞	∞	1250	0	250	1150	1650



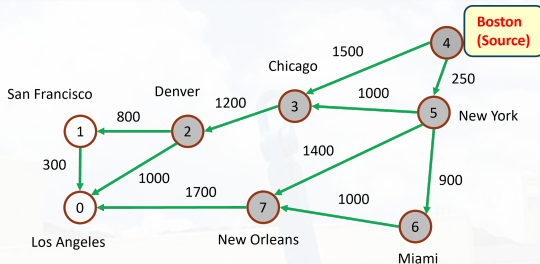
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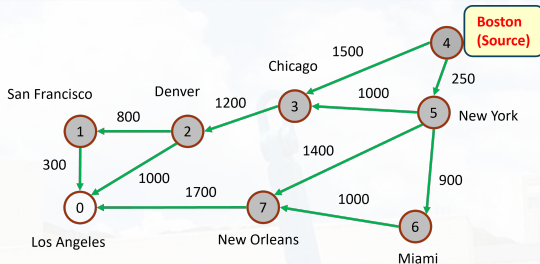
Iteration	Vertex Select.	Distance							
		LA	SF	DEN	CHI	BOS	NY	MIA	NO
		[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
initial	—	∞	∞	∞	1500	0	250	∞	∞
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3	3	∞	∞	2450	1250	0	250	1150	1650



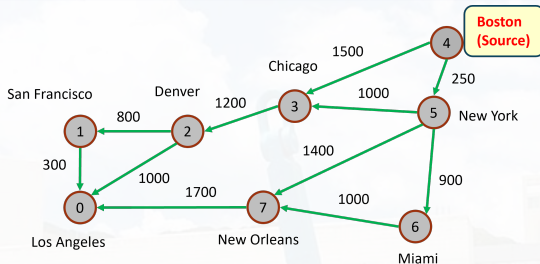
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3	3	∞	∞	2450	1250	0	250	1150	1650
4	7	3350	∞	2450	1250	0	250	1150	1650



Iteration	Vertex Select.	Distance							
		LA	SF	DEN	CHI	BOS	NY	MIA	NO
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7	0	3350	3250	2450	1250	0	250	1150	1650

Example of Using Priority Queue (MinHeap)

- Refer to the code [here](#).

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- Complexity: $O((n + e) \lg n)$.



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Single Source/All Destinations: General Weights

- **Focus:** Some edges of the directed graph G have negative length.

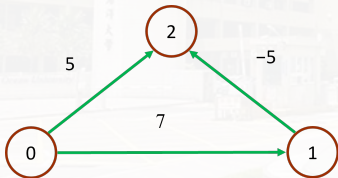


Single Source/All Destinations: General Weights

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- The function `shortestPath` may NOT work!

Single Source/All Destinations: General Weights

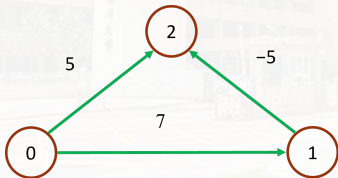
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- For example,



- $\text{dist}[1] = 7, \text{dist}[2] = 5.$

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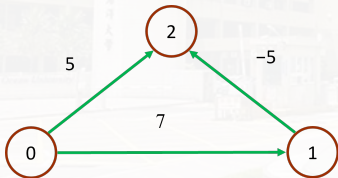
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- The shortest path from 0 to 2 is:
 $0 \rightarrow 1 \rightarrow 2$ (length = 2).

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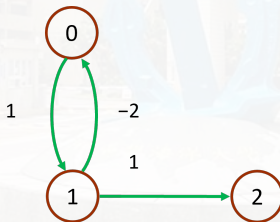
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- $\text{dist}[1] = 7, \text{dist}[2] = 5$.
- The shortest path from 0 to 2 is:
 $0 \rightarrow 1 \rightarrow 2$ (length = 2).
- Dijkstra's “greedy” approach does not work here.

Workaround Solution: NO negative cycle is permitted!

- When negative edge lengths are permitted, we require that the graph have no cycles of negative length.
- This is necessary so as to ensure that shortest paths consist of a **finite** number of edges.



Observations

- When there are NO cycles of negative length, there is a shortest path between any two vertices of an n -vertex graph that has $\leq n - 1$ edges on it.



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 - Otherwise, the path must repeat at least one vertex, and hence must contain a cycle.
- So, eliminating the cycles from the path results in another path with **the same source and destination**.
 - The length of the new path should be no more than that of the original.



Dynamic Programming Approach

$\text{dist}^k[u]$: the length of a shortest path from the source v to u under the constraint that **the shortest path contains $\leq k$ edges**.

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- Hence, $\text{dist}^1[u] = \text{length}[v][u]$, for $0 \leq u < n$.
 - The goal: **Compute $\text{dist}^{n-1}[u]$** for all u .
- ▷ Using **Dynamic Programming**.

Bellman-Ford Algorithm (Sketch)

- If the shortest path from v to u with $\leq k$ edges ($k > 1$) has no more than $k - 1$ edges, then $\text{dist}^k[u] = \text{dist}^{k-1}[u]$.

Bellman-Ford Algorithm (Sketch)

- If the shortest path from v to u with $\leq k$ edges ($k > 1$) has no more than $k - 1$ edges, then $\text{dist}^k[u] = \text{dist}^{k-1}[u]$.
- If the shortest path from v to u with $\leq k$ edges ($k > 1$) has exactly k edges, there exists a vertex i such that $\text{dist}^{k-1}[i] + \text{length}[i][u]$ is minimum.
- The recurrence relation:

Bellman-Ford Algorithm (Sketch)

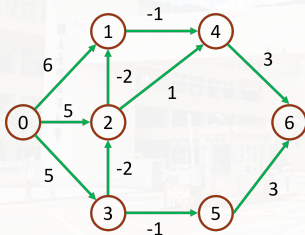
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- The recurrence relation:

$$\text{dist}^k[u] = \min\{\text{dist}^{k-1}[u], \min_i\{\text{dist}^{k-1}[i] + \text{length}[i][u]\}\}.$$



Shortest paths with negative edge lengths (cost)

$$\text{dist}^k[u] = \min\{\text{dist}^{k-1}[u], \min_i\{\text{dist}^{k-1}[i] + \text{length}[i][u]\}\}.$$



(a) A directed graph

k	$\text{dist}^k[u]$						
	0	1	2	3	4	5	6
1	0	6	5	5	∞	∞	∞
2	0	3	3	5	5	4	∞
3	0	1	3	5	2	4	7
4	0	1	3	5	0	4	5
5	0	1	3	5	0	4	3
6	0	1	3	5	0	4	3

(b) dist^k

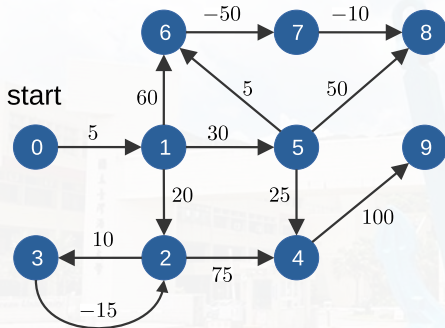
Bellman-Ford Algorithm (Pseudo-Code)

```

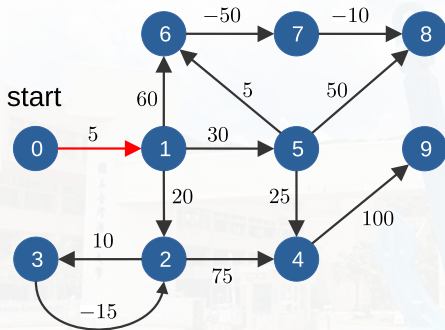
BF(int u) { // assume that the source is vertex u
    for each vertex w in V - {u}, set dist[w] = INT_MAX
    set dist[u] = 0
    for (i=0; i<n-1; i++) { // n: the number of vertices (k)
        for each edge (p,q) in the graph {
            // we can choose p with dist[p] < INT_MAX
            if (dist[p] + length[p][q] < dist[q])
                dist[q] = dist[p] + length[p][q]
        }
    }
    // Now the distances from u to every other vertex is found.
    // Repeat the following to find nodes in a negative cycle
    for (i=0; i<n-1; i++) {
        for each edge (p,q) in the graph {
            if (dist[p] + length[p][q] < dist[q])
                dist[q] = -INT_MAX
        }
    }
}

```

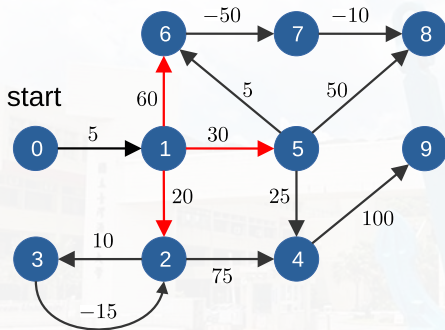




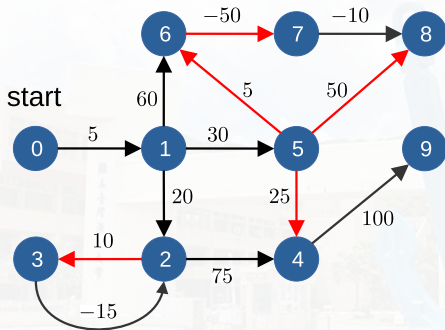
0	0
1	∞
2	∞
3	∞
4	∞
5	∞
6	∞
7	∞
8	∞
9	∞


 $i = 0$

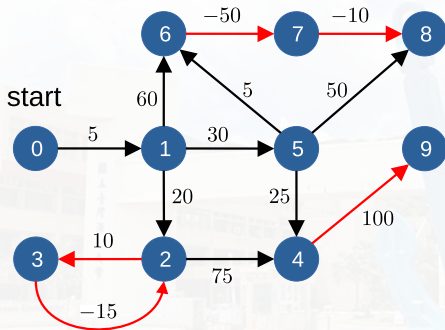
0	0
1	5
2	∞
3	∞
4	∞
5	∞
6	∞
7	∞
8	∞
9	∞


 $i = 1$

0	0
1	5
2	25
3	∞
4	∞
5	35
6	65
7	∞
8	∞
9	∞

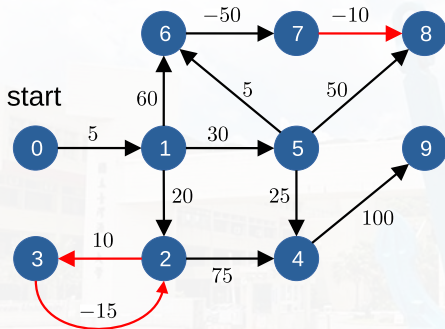
 $i = 2$

0	0
1	5
2	25
3	35
4	60
5	35
6	40
7	15
8	85
9	∞

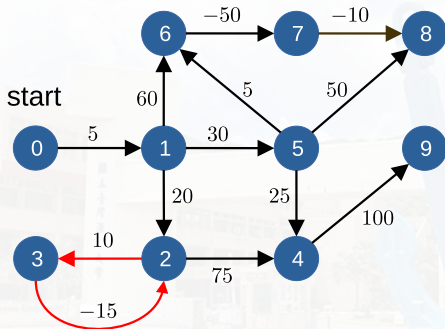


$$i = 3$$

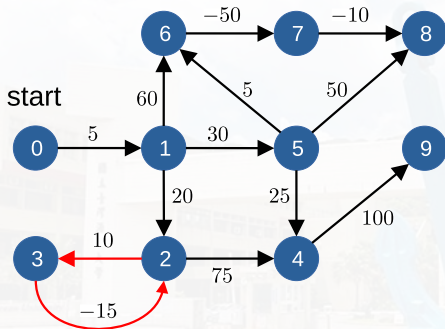
0	0
1	5
2	15
3	30
4	60
5	35
6	40
7	-10
8	5
9	160


 $i = 4$

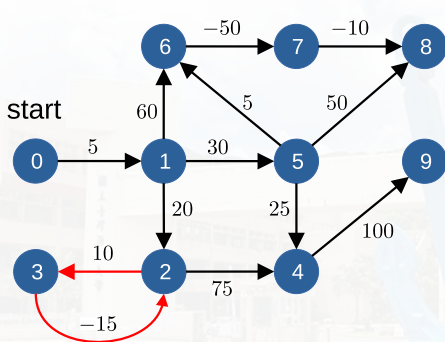
0	0
1	5
2	10
3	25
4	60
5	35
6	40
7	-10
8	-20
9	160


 $i = 5$

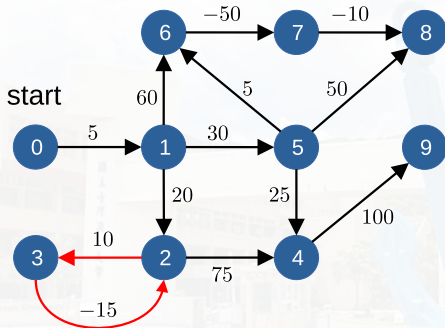
0	0
1	5
2	5
3	20
4	60
5	35
6	40
7	-10
8	-20
9	160

 $i = 6$

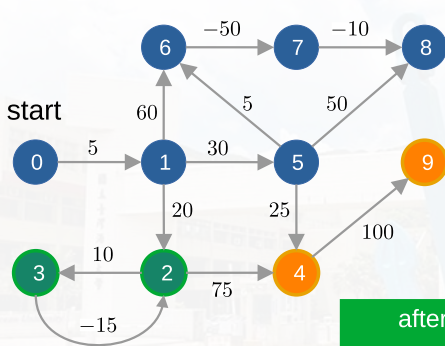
0	0
1	5
2	0
3	15
4	60
5	35
6	40
7	-10
8	-20
9	160


 $i = 7$

0	0
1	5
2	-5
3	10
4	60
5	35
6	40
7	-10
8	-20
9	160


 $i = 8$

0	0
1	5
2	-10
3	5
4	60
5	35
6	40
7	-10
8	-20
9	160



after negative cycle
detection

0	0
1	5
2	$-\infty$
3	$-\infty$
4	$-\infty$
5	35
6	40
7	-10
8	-20
9	$-\infty$

Discussions

