

# Shortest Path

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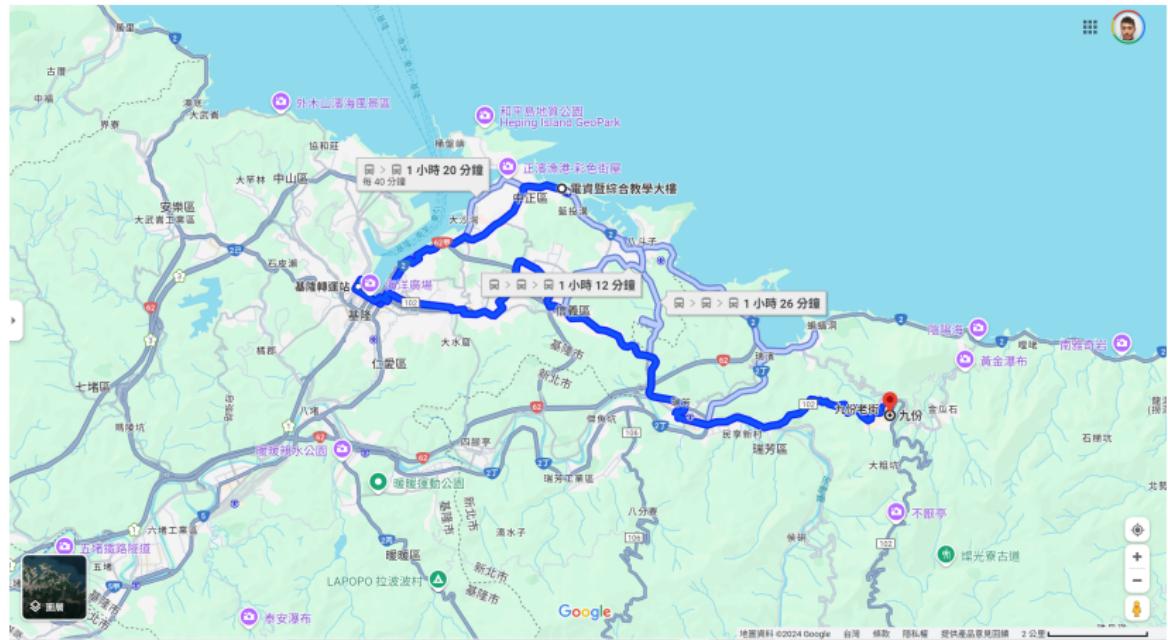
# Outline

1 Introduction

2 Dijkstra's Algorithm

3 General Weights

## Shortest path(s) from NTOU to Jiufen Old Street.



# Shortest Paths

- Model the problem via a graph.
- vertices  $\mapsto$  locations (e.g., stations, restaurants, gas stations, etc.)
  - Including the **source** and the **destination**.
- edges  $\mapsto$  highways, railways, roads, etc.
  - edge **weight**: tolls, the distance, passing-through time, etc.

# Shortest Paths

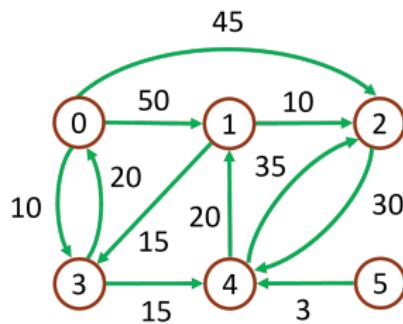
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## Questions

- Is there a path from NTOU to Jiufen?
- If it exists, which one is the **shortest**?



# Single Source/All Destinations (Nonnegative Edge Costs)



	path	length (cost)
1	0, 3	10
2	0, 3, 4	25
3	0, 3, 4, 1	45
4	0, 2	45

Notations:

- A directed graph  $G = (V, E)$ ; a weight function  $w(e)$ ,  $w(e) > 0$  for any edge  $e \in E$ .
- $v_0$ : source vertex.
- If  $(v_i, v_j) \notin E$ ,  $w(v_i, v_j) = \infty$ .



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- Let  $S$  denote the set of vertices, including  $v_0$ , whose shortest paths have been found.
- For  $v \notin S$ , let  $\text{dist}[v]$  be the length of the shortest path starting from  $v_0$ , going through vertices ONLY in  $S$ , and ending in  $v$ .

# Dijkstra's Algorithm

- At the first stage, we add  $v_0$  to  $S$ , set  $\text{dist}[v_0] = 0$  and determine  $\text{dist}[v]$  for each  $v \notin S$ .

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- Adding  $w$  to  $S$ , and updating  $\text{dist}[v]$  for  $v$ , where  $v \notin S$  currently.

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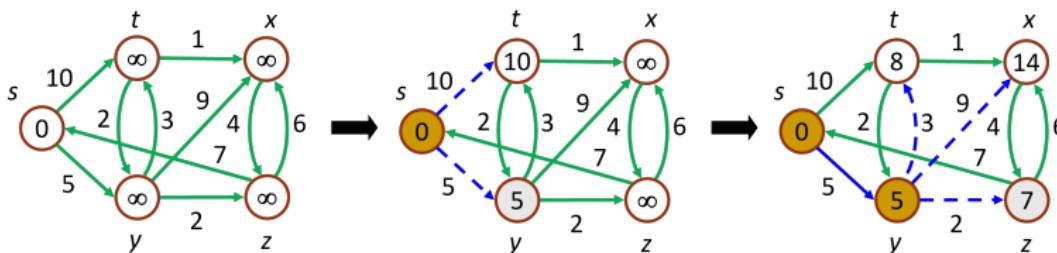
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- Repeat the vertex addition process until  $S = V(G)$

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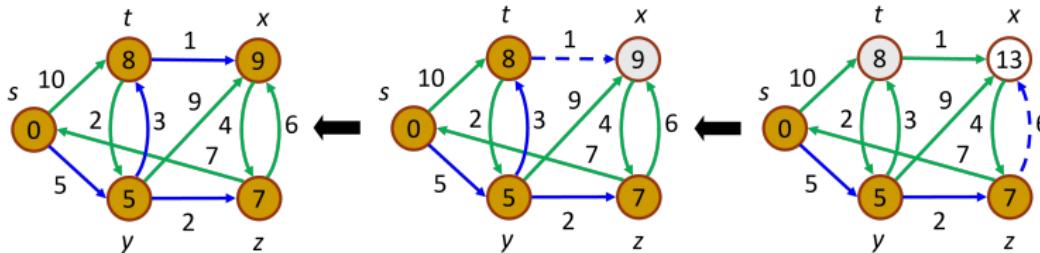
Time complexity:  $O(n^2)$ .

## Illustration of Dijkstra's Algorithm



**During each iteration:**

1. Update the distance of the rest vertices
  2. Pick the vertex with the smallest distance value



# The Pseudo-code of Dijkstra's Algorithm

```
S = { v0 };
dist[v0] = 0;
for each v in V - {v0} do
    dist[v] = e(v0,v); // initialization
while (S != V) do
    choose a vertex w in V - S such that dist[w] is a minimum;
    add w to S;
    for each v in V - S do
        dist[v] = min(dist[v], dist[w]+e(w, v));
    endfor
endwhile
```

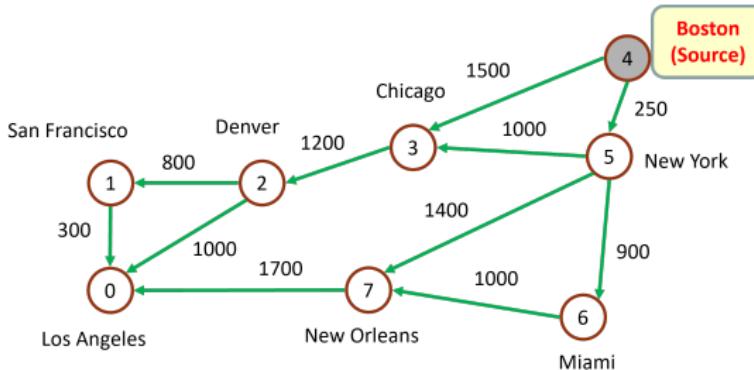
# Dijkstra's Algorithm (Functions (1/2))

```
void shortestPath (int v, int cost[][] [MAX_VERTICES],
                  int distance [], int n, short int found []) {
    /* distance[i]: the shortest path from vertex v to i
     * found[i]: 0 if the shortest path from vertex i has not
     * been found and a 1 otherwise
     * cost: the adjacency matrix */
    int i, u, w;
    for (i=0; i<n; i++) {
        found [i] =FALSE; distance[i] = cost[v][i];
    }
    found[v] = TRUE; //initialization
    distance[v] = 0; //initialization
    for (i=0; i<n-1; i++) {
        u = choose(distance, n, found);
        found[u] = TRUE;
        for (w=0; w<n; w++)
            if (!found[w])
                if (distance[u] + cost[u][w] < distance[w])
                    distance[w] = distance[u]+cost[u][w];
    }
}
```

# Dijkstra's Algorithm (Functions (2/2))

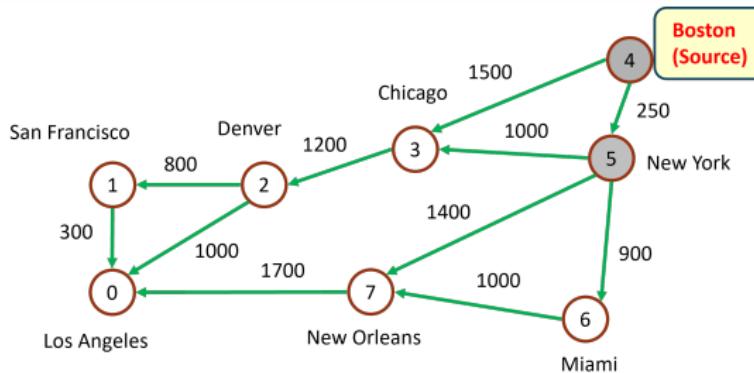
```
int choose (int distance[], int n, short int found[]) {  
    /* find the smallest distance not yet checked */  
    int i, min, min_pos;  
    min = INT_MAX;  
    min_pos = -1;  
    for (i=0; i<n; i++)  
        if (distance[i] < min && !found[i]) {  
            min = distance[i];  
            min_pos = i;  
        }  
    return min_pos;  
}
```





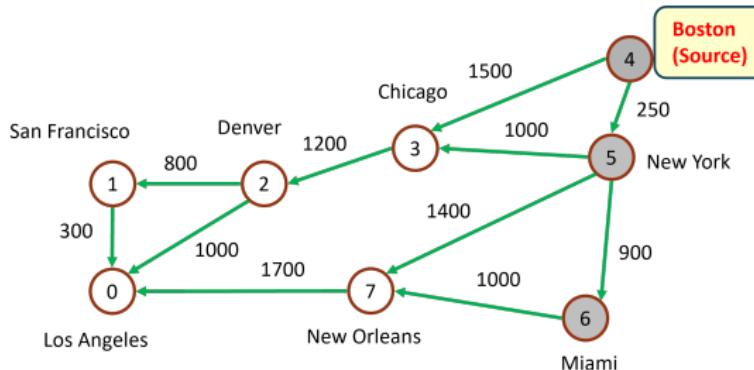
Iteration	Vertex Select.	Distance							
		LA	SF	DEN	CHI	BOS	NY	MIA	NO
		[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
initial	—	∞	∞	∞	1500	0	250	∞	∞





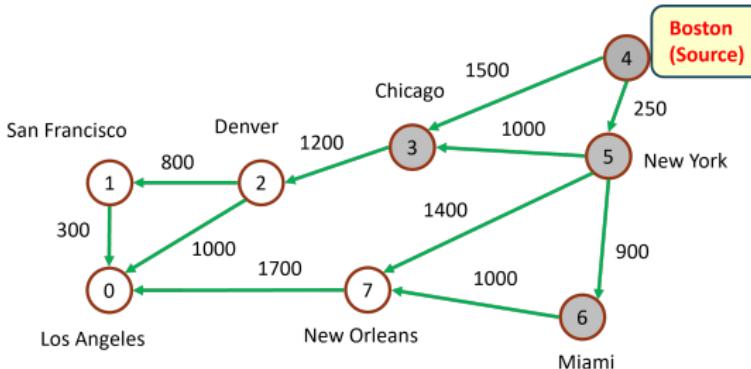
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1	5	∞	∞	∞	1250	0	250	1150	1650





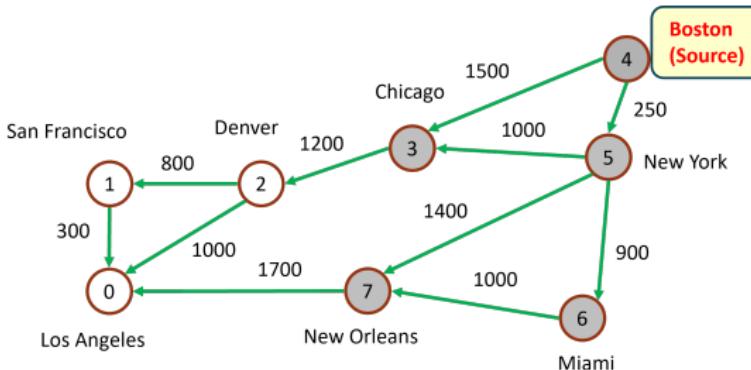
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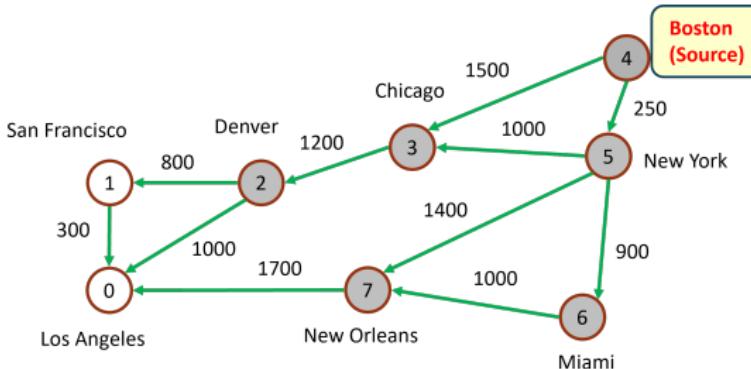
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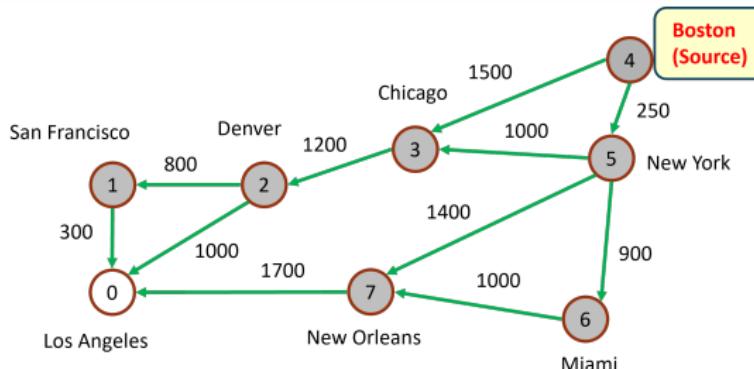
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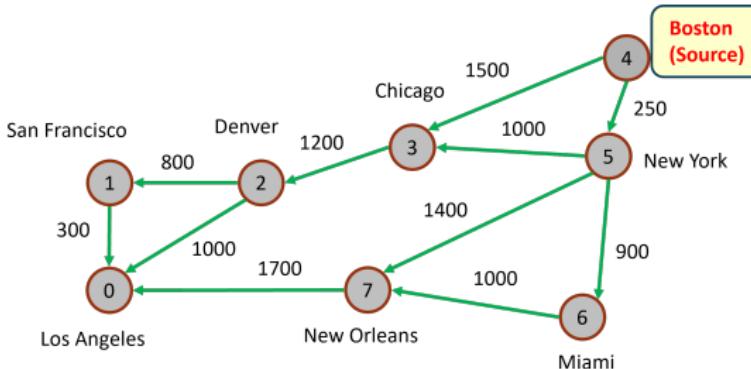
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# Single Source/All Destinations: General Weights

- **Focus:** Some edges of the directed graph  $G$  have negative length (cost).

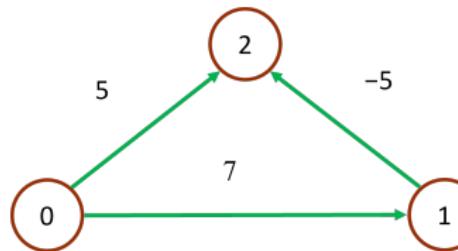
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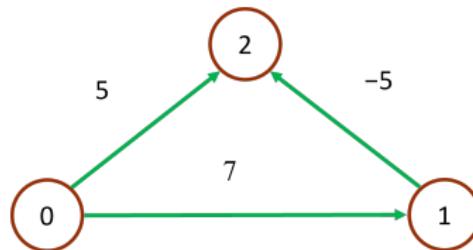
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- $\text{dist}[1] = 7, \text{dist}[2] = 5$ .

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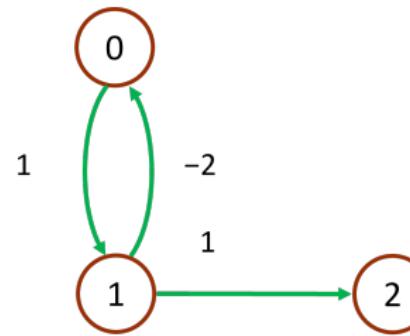
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- For example,



- $\text{dist}[1] = 7, \text{dist}[2] = 5$ .
- The shortest path from 0 to 2 is:  
 $0 \rightarrow 1 \rightarrow 2$  (length = 2).

# Workaround Solution: NO negative cycle is permitted!

- When negative edge lengths are permitted, we require that the graph have no cycles of negative length.
- This is necessary so as to ensure that shortest paths consist of a finite number of edges.



# Observations

- When there are NO cycles of negative length, there is a shortest path between any two vertices of an  $n$ -vertex graph that has  $\leq n - 1$  edges on it.

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  - Otherwise, the path must repeat at least one vertex and hence must contain a cycle.
- So, eliminating the cycles from the path results in another path with the same source and destination.
  - The length of the new path is no more than that of the original.

# Dynamic Programming Approach

$\text{dist}^k[u]$ : the length of a shortest path from the source  $v$  to  $u$  under the constraint that **the shortest path contains  $\leq k$  edges**.



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- Hence,  $\text{dist}^k[u] = \text{length}[v][u]$ , for  $0 \leq u < n$ .
  - The goal: Compute  $\text{dist}^{n-1}[u]$  for all  $u$ .
- ▷ Using Dynamic Programming.

# Sketch of Bellman-Ford Algorithm

- If the shortest path from  $v$  to  $u$  with  $\leq k$  edges has no more than  $k - 1$  edges, then  $\text{dist}^k[u] = \text{dist}^{k-1}[u]$ .

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- The recurrence relation:

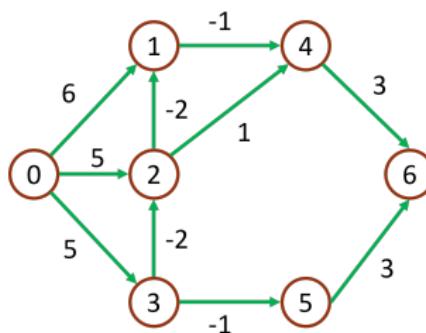
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- The recurrence relation:

$$\text{dist}^k[u] = \min\{\text{dist}^{k-1}[u], \min_i \{\text{dist}^{k-1}[i] + \text{length}[i][u]\}\}.$$

# Shortest paths with negative edge lengths (cost)

$$\text{dist}^k[u] = \min\{\text{dist}^{k-1}[u], \min_i \{\text{dist}^{k-1}[i] + \text{length}[i][u]\}\}.$$



(a) A directed graph

k	$\text{dist}^k[u]$						
	0	1	2	3	4	5	6
1	0	6	5	5	$\infty$	$\infty$	$\infty$
2	0	3	3	5	5	4	$\infty$
3	0	1	3	5	2	4	7
4	0	1	3	5	0	4	5
5	0	1	3	5	0	4	3
6	0	1	3	5	0	4	3

(b)  $\text{dist}^k$

# Discussions