

# Knapsack Auctions

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# Outline

- 1 Knapsack Auctions
  - Welfare-Maximizing DSIC Knapsack Auctions
  - Critical Bids
  - Intractability of Welfare Maximization
- 2 Algorithmic Mechanism Design
  - The Best-Case Scenario: DSIC for Free
  - Knapsack Auctions Revisited
- 3 The Revelation Principle
  - Justifying Direct Revelation
  - Beyond Dominant-Strategy Equilibria



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Whenever there is a shared resource with *limited* capacity, you have a knapsack problem.

# Definition

- We study about another example of single-parameter environments.

## Knapsack Auctions

- Each bidder  $i$  has a publicly known size  $w_i$  and a private valuation.
- The seller has a capacity  $W$ .
- The feasible set  $X$  is defined as the 0-1 vectors  $(x_1, \dots, x_n)$  such that  $\sum_{i=1}^n w_i x_i \leq W$ .
  - $x_i = 1$ :  $i$  is a winning bidder.



# Explanations

- Each bidder's size could represent
  - the duration of a company's television ad;
  - the valuation its willingness-to-pay for its ad being shown;
  - the seller capacity the length of a commercial break.
- The situation that bidders who want
  - files stored on a *shared server*,
  - data streams sent through a shared communication channel
  - processes to be executed on a *shared* supercomputer.
  - :
  - :



# Assumptions

- We receive truthful bids and decide on our allocation rule.
- **Goal:** Devise a payment rule that extends the allocation rule to a DSIC mechanism.



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To maximize the welfare:

$$\mathbf{x}(\mathbf{b}) = \arg \max_X \sum_{i=1}^n b_i x_i.$$

The goal is to compute the **subset** of items of maximum total value that has total size bounded by  $W$ .

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- It's maximum by the assumption that bidders bid truthfully.
- ★ Check that the allocation rule  $\mathbf{x}(\cdot)$  is **monotone**.
  - Bidding higher can only get her more stuff.



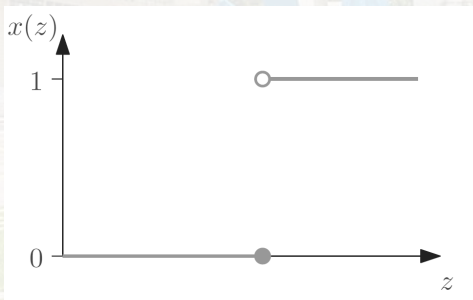
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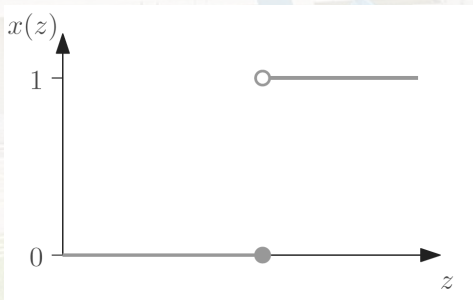
# The Guarantee from Myerson's Lemma

- Myerson's lemma guarantees the existence of a payment rule  $\mathbf{p}$  such that the mechanism  $(\mathbf{x}, \mathbf{p})$  is DSIC.
- Since the allocation rule is monotone and assigns 0 or 1 to each bidder, the allocation curve  $x_i(\cdot, \mathbf{b}_{-i})$  is 0 until some “breakpoint”  $z$ .
  - At  $z$ , the allocation jumps to 1.



## The Guarantee from Myerson's Lemma (contd.)

- If  $i$  bids less than  $z$ , she loses and pays 0.
- If  $i$  bids more than  $z$ , she pays  $\geq z \cdot (1 - 0) = z$ .
  - $z$  is the infimum bid that she could make and continue to win (holding  $\mathbf{b}_{-i}$  fixed).



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# (Recall) An ideal mechanism

## Properties of an Ideal Mechanism

- DSIC
- welfare maximizing (assuming truthful bids).
- runs in polynomial time in the input size (e.g., bids, sizes, the capacity).



# Is our mechanism for the knapsack auction ideal?

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The answer: **NO**.

- The knapsack problem is a notorious **NP**-hard problem.
  - No polynomial time implementation of the allocation rule unless **NP** = **P**.
- Hence, we would like to consider relaxing at least one of the three goals.



# An ideal mechanism

## Properties of an Ideal Mechanism

- DSIC
  - **welfare maximizing (assuming truthful bids).**
  - runs in polynomial time in the input size (e.g., bids, sizes, the capacity).
- 
- Relax the second requirement as little as possible.
  - Design a polynomial time and monotone allocation rule that comes as close as possible to the maximum possible social welfare.



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- The primary goal in approximation algorithms is to design **polynomial-time** algorithms for **NP**-hard optimization problems that are **as close to the optimal solution as possible**.





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# Approximation algorithms come to rescue?

- The primary goal in approximation algorithms is to design **polynomial-time** algorithms for **NP**-hard optimization problems that are **as close to the optimal solution as possible**.
- Algorithmic mechanism design has exactly the same goal, except that the algorithms must additionally obey a **monotonicity constraint**.
- The incentive constraints of the mechanism design goal are neatly compiled into a relatively intuitive extra constraint on the allocation rule.



# Approximation Algorithms come to rescue? (contd.)

- The design space of polynomial-time DSIC mechanisms is only **smaller than** that of polynomial-time approximation algorithms.



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- The design space of polynomial-time DSIC mechanisms is only **smaller than** that of polynomial-time approximation algorithms.
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# Approximation Algorithms come to rescue? (contd.)

- The design space of polynomial-time DSIC mechanisms is only **smaller than** that of polynomial-time approximation algorithms.
- (Imagine) The best-case scenario: DSIC constraint causes no additional welfare loss (beyond the loss from the polynomial-time requirement).
- Exact welfare maximization automatically yields a monotone allocation rule.
- Is that true for *approximate* welfare maximization?



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# Greedy approach

- Say  $S$  be a set of winners with total size  $\sum_{i \in S} w_i \leq W$ .
- We choose such a set  $S$  via a simple greedy algorithm.
- ★ We can assume that  $w_i \leq W$  for all  $i$  (why?)





# A Greedy Knapsack Heuristic

## A Greedy Algorithm

- 1 Sort and re-index the bidders so that

$$\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq \dots \geq \frac{b_n}{w_n}.$$

- 2 Pick winners in this order until one doesn't fit, and then halt.
- 3 Return either the solution from Step ② or the highest bidder:  $\arg \max_i b_i$ , whichever has larger social welfare.

## Theorem (Knapsack Approximation Guarantee)

Assuming truthful bids, the social welfare achieved by the greedy allocation is at least half of the maximum social welfare.



# Sketch of proving the theorem

- To have an upper bound on the maximum social welfare, allow bidders to be chosen **fractionally**, with the value prorated accordingly.
  - E.g., if 70% of a bidder with value 10 is chosen, then it contributes 7 to the welfare.
- Sort the bidders according to the step above, and pick winners in this order until the the capacity  $W$  is fully exhausted.
  - You can verify that this **maximizes the welfare over all feasible solutions**.



## Sketch of proving the theorem (contd.)

- In the optimal *fractional* solution, suppose that the first  $k$  bidders in the sorted order win and that the  $(k + 1)$ th bidder *fractionally* wins.
- ★ The welfare achieved by steps ① and ② in the greedy allocation rule = the total value of the first  $k$  bidders.
- ★ The welfare consisting only the highest bidder  $\geq$  the fractional value of the  $(k + 1)$ th bidder.
- The better of these two solutions  $\geq \frac{1}{2} \times$  the welfare of the optimal fractional solution.  
 $\Rightarrow$  Exercise!





30, 70

60, 60

50, 10

45, 20

40, 30



20, 50

30, 45

60, 60





80, 90

20, 10

20, 20

30, 30

# Sum up

- The greedy allocation rule is monotone (check by yourself).
- Using Myerson's lemma, we can extend it to a DSIC mechanism that runs in **polynomial time** and, assuming truthful bids, achieves social welfare at least 50% of the maximum possible.



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# Reiteration

- There are good reasons to strive for a DSIC guarantee.
  - Easy for a participant to **figure out what to do** in a DSIC mechanism.
  - The designer can **predict the mechanism's outcome**.



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# The DSIC Condition

## The DSIC Condition

- (a) For every valuation profile, the mechanism has a **dominant-strategy equilibrium**.
    - ★ An outcome that results from every participant playing a *dominant strategy*.
  - (b) In this dominant-strategy equilibrium, every participant **truthfully reports her private information** to the mechanism.
- The **revelation principle** asserts that:  
given (a), then (b) comes for free!

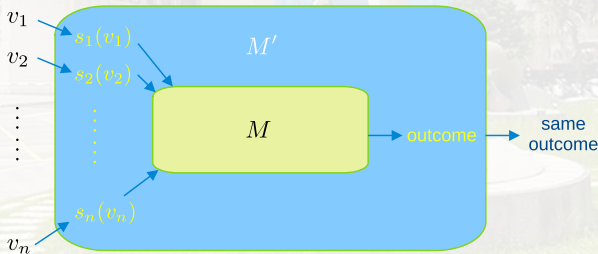


# The Revelation Principle

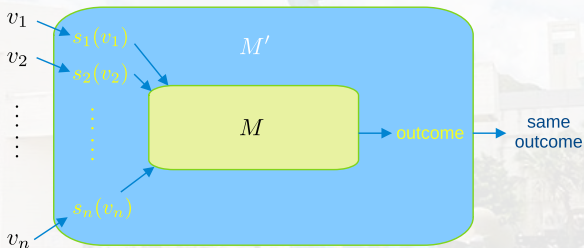
## Theorem (Revelation Principle for DSIC Mechanisms)

For every mechanism  $M$  where every participant always has a dominant strategy, there is an equivalent direct-revelation DSIC mechanism  $M'$ .

- We use a simulation argument to construct  $M'$  as follows.

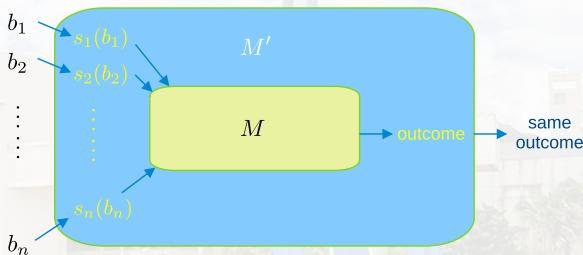


# Proof



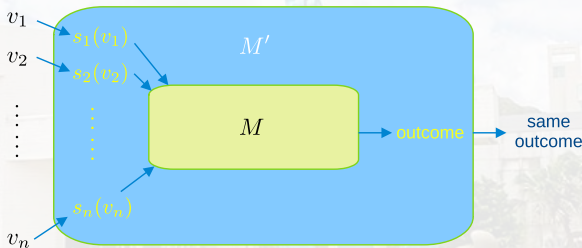
- For every participant  $i$  and its private information  $v_i$ , she has a dominant strategy  $s_i(v_i)$  in mechanism  $M$  (by assumption).

# Proof



- Construct  $M'$ , such that participants delegate the responsibility of playing the appropriate dominant strategy to  $M'$ .
  - $M'$  accepts bids  $b_1, \dots, b_n$ .
  - Then  $M'$ , which is of direct-revelation, submits the bids  $s_1(b_1), \dots, s_n(b_n)$  to the mechanism  $M$  and choose the same outcome that  $M$  does.

# Proof



- Mechanism  $M'$  is DSIC:
  - If a participant  $i$  has private information  $v_i$ , then submitting a bid other than  $v_i$  can only result in  $M'$  playing a strategy other than  $s_i(v_i)$  in  $M$ , which can only decrease  $i$ 's utility.

# What we have learned from the theorem?

- Truthfulness per se is not important.
- The difficult part is the requirement to have a dominant-strategy equilibrium.





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# Heads up

- DSIC and non-DSIC mechanisms are *incomparable*.
  - The former enjoys stronger incentive guarantees
  - The latter may enjoy better performance guarantees.

