Mathematics for Machine Learning
— Probability & Distributions
Gaussian Distribution & Change of Variables/Inverse Transform

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Credits for the resource

- The slides are based on the textbooks:

  - *Marc Peter Deisenroth, A. Aldo Faisal, and Cheng Soon Ong: Mathematics for Machine Learning. Cambridge University Press. 2020.*
  - *Arnold J. Insel, Lawrence E. Spence, Stephen H. Friedberg: Linear Algebra, 4th Edition. Prentice Hall. 2013.*
  - *Howard Anton, Chris Rorres, Anton Kaul: Elementary Linear Algebra, 12th Edition. Wiley. 2019.*

- We could partially refer to the monograph:
  Francesco Orabona: A Modern Introduction to Online Learning. https://arxiv.org/abs/1912.13213

Outline

Outline

Introduction

- The Gaussian distribution (a.k.s. normal distribution) is the most well-studied probability distribution for continuous-valued random variables.

- Widely used in statistics and machine learning.

Gaussian Distributions Overlaid with Samples
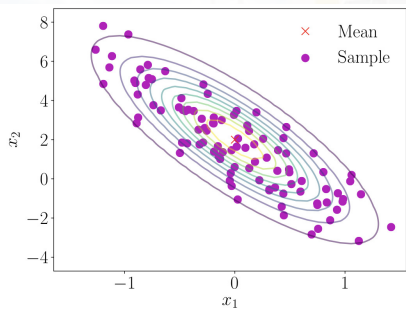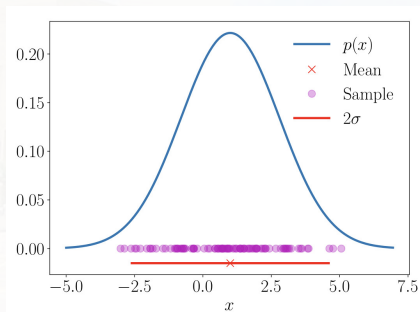
Univariate & Multivariate Gaussian

The probability density functions.

Univariate

$$p(x \mid \mu, \sigma^2) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

$\boldsymbol{\Sigma} = \mathbb{V}_X[\mathbf{x}] = \text{Cov}_X[\mathbf{x}, \mathbf{x}]$.

Multivariate

$$p(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-\frac{D}{2}} \det(\boldsymbol{\Sigma})^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right),$$

for $\mathbf{x} \in \mathbb{R}^D$.

We write $p(\mathbf{x}) = \mathcal{N}(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma})$ or $X \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

Gaussian distribution of two random variables x_1, x_2.
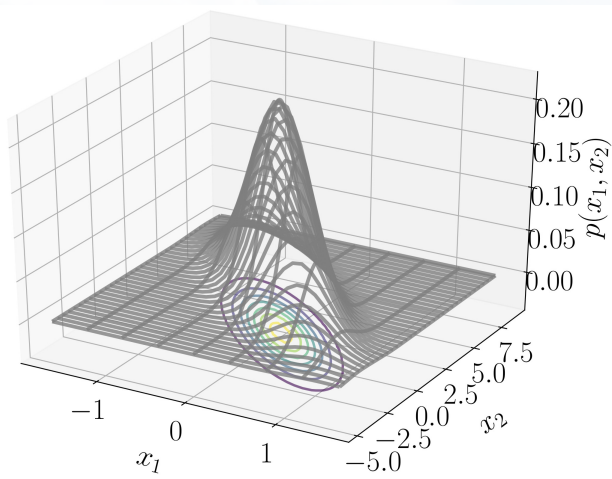
Outline

Marginals and Conditionals of Gaussians

- Let X, Y be two multivariate random variables.

- Concatenate their states to be $[\mathbf{x}^\top, \mathbf{y}^\top]$.

$$p(\mathbf{x}, \mathbf{y}) = \mathcal{N}\left(\left[\begin{array}{c} \boldsymbol{\mu}_x \\ \boldsymbol{\mu}_y \end{array}\right], \left[\begin{array}{cc} \boldsymbol{\Sigma}_{xx} & \boldsymbol{\Sigma}_{xy} \\ \boldsymbol{\Sigma}_{yx} & \boldsymbol{\Sigma}_{yy} \end{array}\right]\right),$$

  where $\boldsymbol{\Sigma}_{xx} = \text{Cov}[\mathbf{x}, \mathbf{x}]$, $\boldsymbol{\Sigma}_{yy} = \text{Cov}[\mathbf{y}, \mathbf{y}]$, $\boldsymbol{\Sigma}_{xy} = \text{Cov}[\mathbf{x}, \mathbf{y}]$.

- By [Bishop 2006], the conditional distribution $p(\mathbf{x} \mid \mathbf{y})$ is also Gaussian.

$$\begin{array}{rcl} p(\mathbf{x} \mid \mathbf{y}) &=& \mathcal{N}(\boldsymbol{\mu}_{x|y}, \boldsymbol{\Sigma}_{x|y}) \\ \boldsymbol{\mu}_{x|y} &=& \boldsymbol{\mu}_x + \boldsymbol{\Sigma}_{xy}\boldsymbol{\Sigma}_{yy}^{-1}(\mathbf{y} - \boldsymbol{\mu}_y) \\ \boldsymbol{\Sigma}_{x|y} &=& \boldsymbol{\Sigma}_{xx} - \boldsymbol{\Sigma}_{xy}\boldsymbol{\Sigma}_{yy}^{-1}\boldsymbol{\Sigma}_{yx}. \end{array}$$

$$p(\mathbf{x}) = \int p(\mathbf{x}, \mathbf{y})\mathrm{d}\mathbf{y} = \mathcal{N}(\mathbf{x} \mid \boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{xx}).$$

Example

Consider
$$p(x_1, x_2) = \mathcal{N}\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0.3 & -1 \\ -1 & 5 \end{bmatrix} \right).$$
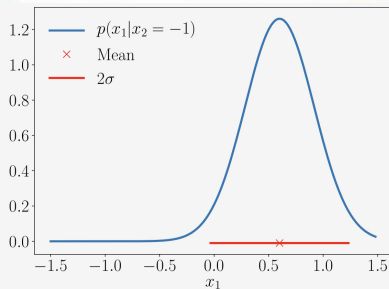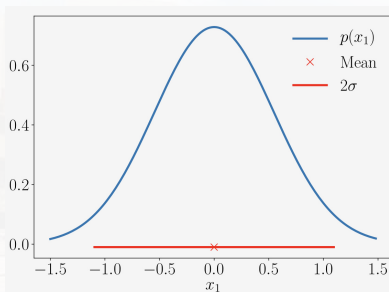
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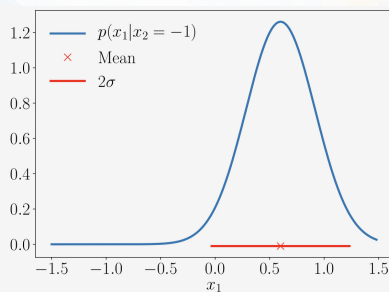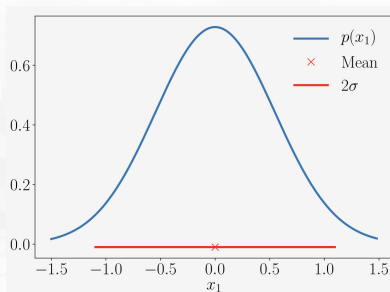
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Thus, $p(x_1 \mid x_2 = -1) =$

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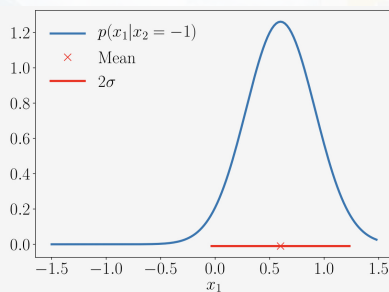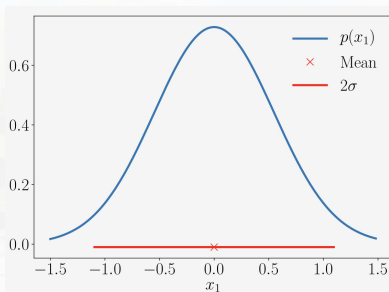
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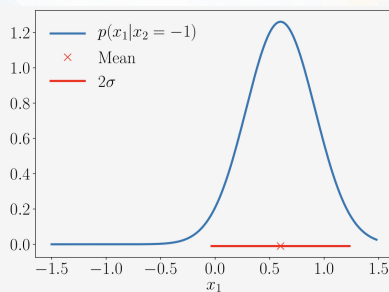
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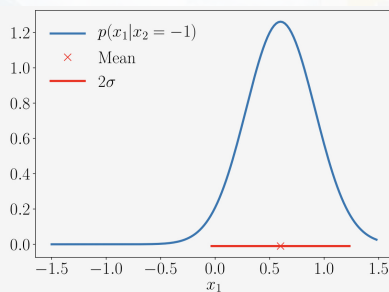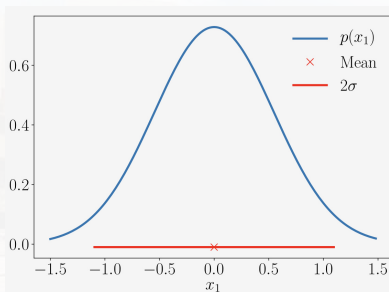
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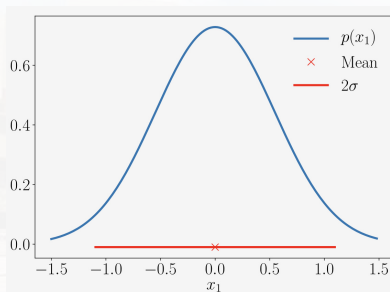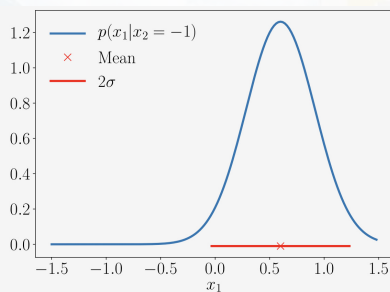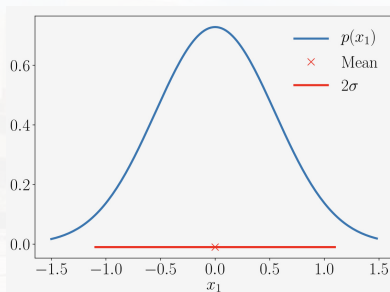
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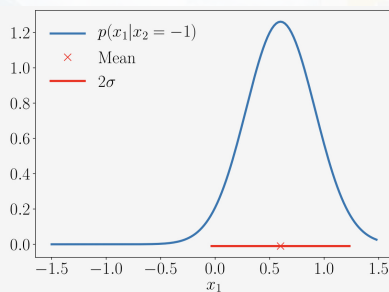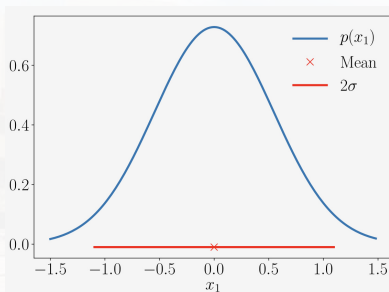
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Thus, $p(x_1 \mid x_2 = -1) = \mathcal{N}(0.6, 0.1), \qquad p(x_1) = \mathcal{N}(0, 0.3).$

Outline

Sum of Gaussians

Say X, Y are two independent Gaussian random variables with

$$X \sim \mathcal{N}(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x) \ \text{ and } \ Y \sim \mathcal{N}(\boldsymbol{\mu}_y, \boldsymbol{\Sigma}_y).$$

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Then $X + Y$ is also a Gaussian distribution with

$$X + Y \sim \mathcal{N}(\boldsymbol{\mu}_x + \boldsymbol{\mu}_y, \boldsymbol{\Sigma}_x + \boldsymbol{\Sigma}_y)$$

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Please recall $\mathbb{E}[\mathbf{x} + \mathbf{y}]$ and $\mathbb{V}[\mathbf{x} + \mathbf{y}]$.

Example

Linear Combination of Two Independent Gaussians

$$p(a\mathbf{x} + b\mathbf{y}) =$$

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$$p(a\mathbf{x} + b\mathbf{y}) = \mathcal{N}(a\boldsymbol{\mu_x} + b\boldsymbol{\mu_y}, \; a^2\boldsymbol{\Sigma_x} + b^2\boldsymbol{\Sigma_y}).$$

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Theorem [Mixture of Two Univariate Gaussian Densities]

Consider a mixture of two univariate Gaussian densities

$$p(x) = \alpha p_1(x) + (1-\alpha)p_2(x)$$

for the mixture weight $0 < \alpha < 1$ and $(\mu_1, \sigma_1^2) \neq (\mu_2, \sigma_2^2)$. Then,

$$\begin{aligned}
\mathbb{E}[x] &= \alpha\mu_1 + (1-\alpha)\mu_2 \\
\mathbb{V}[x] &= [\alpha\sigma_1^2 + (1-\alpha)\sigma_2^2] \\
&\quad + \left([\alpha\mu_1^2 + (1-\alpha)\mu_2^2] - [\alpha\mu_1 + (1-\alpha)\mu_2]^2\right).
\end{aligned}$$

Proof of the Theorem

Sketch:

1. $\mathbb{E}[x] = \int_{-\infty}^{\infty} xp(x)\mathrm{d}x = \int_{-\infty}^{\infty} (\alpha xp_1(x) + (1 - \alpha)xp_2(x))\mathrm{d}x$
   $= \alpha\mu_1 + (1 - \alpha)\mu_2.$

2. $\mathbb{E}[x^2] =$

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   - **Recall:** $\mathbb{V}_X[x] = \mathbb{E}_X[x^2] - (\mathbb{E}_X[x])^2.$

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Using ① & ② we can prove the theorem.

Linear Transformation by a Matrix (1/2)

$X \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\mathbf{y} = \boldsymbol{A}\mathbf{x}$

- The expectation: $\mathbb{E}[\mathbf{y}] = \mathbb{E}[\boldsymbol{A}\mathbf{x}] =$

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$$Y \sim \mathcal{N}(\boldsymbol{A}\boldsymbol{\mu}, \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}^\top).$$

Linear Transformation by a Matrix (2/2)

Let's consider the reverse transformation.

$Y \sim \mathcal{N}(\boldsymbol{\mu}_y, \boldsymbol{\Sigma})$, $\mathbf{y} = \boldsymbol{A}\mathbf{x}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^M$, a full rank $\boldsymbol{A} \in \mathbb{R}^{M \times N}$, $M \geq N$

- $p(\mathbf{y}) = \mathcal{N}(\mathbf{y} \mid \boldsymbol{A}\mathbf{x}, \boldsymbol{\Sigma})$.
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Linear Transformation by a Matrix (2/2)

Let's consider the reverse transformation.

> $Y \sim \mathcal{N}(\boldsymbol{\mu}_y, \boldsymbol{\Sigma})$, $\mathbf{y} = \boldsymbol{A}\mathbf{x}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^M$, a full rank $\boldsymbol{A} \in \mathbb{R}^{M \times N}$, $M \geq N$
>
> - $p(\mathbf{y}) = \mathcal{N}(\mathbf{y} \mid \boldsymbol{A}\mathbf{x}, \boldsymbol{\Sigma})$.
>   - **Note:** $\boldsymbol{A}$ might not be invertible (not squared).
>
> - $\mathbf{y} = \boldsymbol{A}\mathbf{x} \iff \boldsymbol{A}^\top \mathbf{y} = \boldsymbol{A}^\top \boldsymbol{A} \mathbf{x} \iff (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top \mathbf{y} = \mathbf{x}$.
>   - This works even for non-invertible $\boldsymbol{A}$!
>
> - The variance: $\mathbb{V}[\mathbf{x}] = \mathbb{V}[(\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top \mathbf{y}] = (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top \boldsymbol{\Sigma} \boldsymbol{A} (\boldsymbol{A}^\top \boldsymbol{A})^{-1}$.
>
> - Thus, we have
>
> $$X \sim \mathcal{N}((\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top \boldsymbol{\mu}_y, (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top \boldsymbol{\Sigma} \boldsymbol{A} (\boldsymbol{A}^\top \boldsymbol{A})^{-1}).$$

Exercise

Another example of *reverse transformation*.

$Y \sim \mathcal{N}(\boldsymbol{\mu}_y, \boldsymbol{\Sigma})$ and $\mathbf{y} = \boldsymbol{A}\mathbf{x}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^M$, and $\boldsymbol{A}$ is invertible

- $p(\mathbf{y}) = \mathcal{N}(\mathbf{y} \mid \boldsymbol{A}\mathbf{x}, \boldsymbol{\Sigma})$.

- Compute $\mathbb{E}[\mathbf{x}]$.

- Compute $\mathbb{V}[\mathbf{x}]$.

- Derive $X \sim \mathcal{N}(?, ?)$.

A Sampling Approach

We want to obtain samples from a multivariate $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

- However, we only have a sampler of $\mathcal{N}(\mathbf{0}, \boldsymbol{I})$ at hand.

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- To derive $\boldsymbol{A}$: Use Cholesky decomposition of the covariance matrix $\boldsymbol{\Sigma}$.
    - $\boldsymbol{A}$ will be triangular and efficient for computation.

Outline

Motivation

Consider the following examples.

- Assuming that X is a random variable distributed according to some well-known distribution, then what is the distribution of X^2?

- Assuming that X_1, X_2 are two univariate standard normal distributions, then what is the distribution of $\frac{1}{2}(X_1 + X_2)$?

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Consider the following examples.

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- What if the transformation is nonlinear?
  - Closed-form expressions are not readily available.

Straightforward for Discrete Random Variables

Example: Univariate Random Variables

Given

- A discrete random variable X with pmf $\Pr[X = x]$.

- An invertible function $U(x)$.

Consider the transformed random variable $Y := U(X)$ with pmf
$\Pr[Y = y]$. Then

$$
\begin{aligned}
\Pr[Y = y] &= \Pr[U(X) = y] \quad \text{(transformation of interest)} \\
&= \Pr[X = U^{-1}(y)] \quad \text{(inverse)}
\end{aligned}
$$

where we can observe $x = U^{-1}(y)$.

Two Approaches

- So far we considered the discrete case (e.g., $\Pr[X = x]$).

- For continuous distributions, we will consider the two approaches:
  1. Cumulative distribution (Distribution Function Technique).
  2. Change-of-variable.

Outline

1. Gaussian Distribution
   - Marginals and Conditionals of Gaussians
   - Sums and Linear Transformations

2. Change of Variables
   - Distribution Function Technique
   - Change of Variables

3. Case Study: Multivariate Gaussian

Distribution Function Technique

Note: a cdf of X: $F_X(x) = \Pr[X \leq x]$.

Goal: Find the cdf of the random variable $Y := U(X)$

1. Find the cdf

$$F_Y(y) = \Pr[Y \leq y].$$

2. Differentiating $F_Y(y)$ to get the pdf $f_Y(y)$:

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y).$$

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2 Differentiating $F_Y(y)$ to get the pdf $f_Y(y)$:

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y).$$

Note: The domain of the random variable may have changed!

Example

Example

Let X be a continuous random variable with pdf $f_X : [0,1] \to [0,1]$:

$$f_X(x) = 3x^2.$$

Goal: Find the pdf of $Y = X^2$.

$F_Y(y) = \Pr[Y \leq y]$

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\begin{aligned}
F_Y(y) &= \Pr[Y \leq y] = \Pr[X^2 \leq y] \\
&= \Pr[X \leq y^{\frac{1}{2}}] \\
&= F_X(y^{\frac{1}{2}})
\end{aligned}
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F_Y(y) &= \Pr[Y \leq y] = \Pr[X^2 \leq y] \\
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&= F_X(y^{\frac{1}{2}}) = \int_0^{y^{\frac{1}{2}}} 3t^2 \mathrm{d}t \\
&= [t^3]_0^{y^{\frac{1}{2}}} = y^{\frac{3}{2}}, \quad 0 \leq y \leq 1.
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Thus,

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \frac{3}{2} y^{\frac{1}{2}}$$

for $0 \leq y \leq 1$.

Exercise

Theorem [Casella & Berger (2002)]

Let X be a continuous random variable with a *strictly monotone* cumulative distribution function $F_X(x)$. Then, the random variable Y defined as

$$Y := F_X(X)$$

has a uniform distribution.

Exercise

Consider $f_X(x) = 3x^2$ in the previous example. Show that $Y := F_X(X)$ attains a uniform distribution.

Remark

The first approach relies on the following facts:

- We can transform the cdf of Y into an expression that is a cdf of X.

- We can differentiate the cdf to obtain the pdf.

Outline

What We have Learnt From the Calculus Course

$$\int f(g(x))g'(x)\mathrm{d}x = \int f(u)\mathrm{d}u, \text{ where } u = g(x).$$

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$$\int f(g(x))g'(x)\mathrm{d}x = \int f(u)\mathrm{d}u, \text{ where } u = g(x).$$

- Intuitively, considering $\mathrm{d}u \approx \Delta u = g'(x)\Delta x$ as the "small changes".

The Roadmap (1/2)

- Consider a univariate random variable X and an invertible function U such that $Y := U(X)$.

- Assume that X has states $x \in [a, b]$.

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If U is *strictly increasing*, then so is its inverse U^{-1}.

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$$\Pr[U(X) \leq y] = \Pr[U^{-1}(U(X)) \leq U^{-1}(y)] = \Pr[X \leq U^{-1}(y)].$$

  Then, $\quad F_Y(y) = \Pr[X \leq U^{-1}(y)] = \int_a^{U^{-1}(y)} f_X(x) \mathrm{d}x$

The Roadmap (2/2)

- To obtain the pdf, we differentiate $F_Y(y)$ w.r.t. y:

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- $\displaystyle \int f_X(U^{-1}(y)) {U^{-1}}'(y) \mathrm{d}y = \int f_X(x) \mathrm{d}x$, where $x = U^{-1}(y)$.

- Thus,
$$\begin{aligned} f_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}y} \int_a^{U^{-1}(y)} f_X(U^{-1}(y)) {U^{-1}}'(y) \mathrm{d}y \\ &= f_X(U^{-1}(y)) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}y} U^{-1}(y) \right). \end{aligned}$$

Remark

For decreasing functions,

$$f_Y(y) \;=\; -f_X(U^{-1}(y)) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}y} U^{-1}(y) \right).$$

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So for both increasing and decreasing U,

$$f_Y(y) \;=\; f_X(U^{-1}(y)) \cdot \left| \frac{\mathrm{d}}{\mathrm{d}y} U^{-1}(y) \right|.$$

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So for both increasing and decreasing U,

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- The term $\left| \frac{\mathrm{d}}{\mathrm{d}y} U^{-1}(y) \right|$ measures how much a unit volume changes when applying U.

The Main Theorem

Theorem [Billingsley (1995)]

Let $f_X(\mathbf{x})$ be the pdf of the multivariate continuous random variable X. If the vector-valued function $\mathbf{y} = U(\mathbf{x})$ is differentiable and invertible for all values within the domain of $\mathbf{x}$, then for corresponding values of $\mathbf{y}$, the pdf of $Y = U(X)$ is given by

$$f(\mathbf{y}) = f_{\mathbf{x}}(U^{-1}(\mathbf{y})) \cdot \left| \det \left(\frac{\partial}{\partial \mathbf{y}} U^{-1}(\mathbf{y}) \right) \right|.$$

Example

Example

Consider a bivariate random variable X with states $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and pdf

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \frac{1}{2\pi} \exp\left(-\frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right).$$

Then, consider a matrix $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$ defined as

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Goal: Find the pdf of the random variable Y with states $\mathbf{y} = \boldsymbol{A}\mathbf{x}$.

- $\mathbf{y} = A\mathbf{x}$

- $\mathbf{y} = \boldsymbol{A}\mathbf{x} \implies \mathbf{x} = \boldsymbol{A}^{-1}\mathbf{y}$.

$$\left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] = \boldsymbol{A}^{-1} \left[\begin{array}{c} y_1 \\ y_2 \end{array} \right]$$

- $\mathbf{y} = \boldsymbol{A}\mathbf{x} \implies \mathbf{x} = \boldsymbol{A}^{-1}\mathbf{y}$.

$$\left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] = \boldsymbol{A}^{-1} \left[\begin{array}{c} y_1 \\ y_2 \end{array} \right] = \frac{1}{ad - bc} \left[\begin{array}{cc} d & -b \\ -c & a \end{array} \right] \left[\begin{array}{c} y_1 \\ y_2 \end{array} \right].$$

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- The corresponding pdf is given by

$$f(\mathbf{x}) = f(\boldsymbol{A}^{-1}\mathbf{y}) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}\mathbf{y}^{\top}(\boldsymbol{A}^{-1})^{\top}\boldsymbol{A}^{-1}\mathbf{y}\right)$$

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- $\dfrac{\partial}{\partial \mathbf{y}} \boldsymbol{A}^{-1}\mathbf{y} = \boldsymbol{A}^{-1}$. So, $\det\left(\dfrac{\partial}{\partial \mathbf{y}} \boldsymbol{A}^{-1}\mathbf{y} \right) = \det(\boldsymbol{A}^{-1}) =$

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- Thus, $f(\mathbf{y}) = \dfrac{1}{2\pi} \exp\left(-\dfrac{1}{2}\mathbf{y}^\top (\boldsymbol{A}^{-1})^\top \boldsymbol{A}^{-1}\mathbf{y} \right) \cdot \left| \dfrac{1}{ad - bc} \right|$.

Outline

1. Gaussian Distribution
   - Marginals and Conditionals of Gaussians
   - Sums and Linear Transformations

2. Change of Variables
   - Distribution Function Technique
   - Change of Variables

3. **Case Study: Multivariate Gaussian**

Standard Multivariate Gaussian

- Let $Z = (Z_1, \ldots, Z_D)^\top$ with independent coordinates

  $$Z_i \sim \mathcal{N}(0, 1), \quad i = 1, 2, \ldots, D.$$

- The 1D standard Gaussian pdf is

  $$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

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- By independence, the joint density of Z is

$$p_Z(z_1, \ldots, z_D) = \prod_{i=1}^{D} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_i^2}{2}\right) = (2\pi)^{-D/2} \exp\left(-\frac{1}{2}\sum_{i=1}^{D} z_i^2\right).$$

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  Writing $\sum_{i=1}^{D} z_i^2 = \|\mathbf{z}\|^2 = \mathbf{z}^\top \mathbf{z}$, we get

$$p_Z(\mathbf{z}) = (2\pi)^{-D/2} \exp\left(-\frac{1}{2} \mathbf{z}^\top \mathbf{z}\right), \quad \mathbf{z} \in \mathbb{R}^D.$$

Introducing Mean and Covariance

- Let Σ be a symmetric positive definite $D \times D$ matrix. Then there exists an invertible L such that

  $$\Sigma = LL^\top \quad \text{(e.g. Cholesky factorization)}.$$

- Define $X = \mu + LZ$. Then

  $$\mathbb{E}[X] = \mu + L\,\mathbb{E}[Z] = \mu, \quad \text{and}$$

  $$\begin{aligned}
  \text{Cov}(X) &= \mathbb{E}[(X - \mu)(X - \mu)^\top] = \mathbb{E}[LZZ^\top L^\top] \\
  &= L\,\mathbb{E}[ZZ^\top]\,L^\top = LI_D L^\top = \Sigma.
  \end{aligned}$$

- Hence X has mean μ and covariance Σ; we write $X \sim \mathcal{N}(\mu, \Sigma)$.

Change of Variables (1/2)

- The map from Z to X is affine:

$$T(\mathbf{z}) = \mu + L\mathbf{z}, \qquad X = T(Z).$$

  Its inverse is

$$z = T^{-1}(\mathbf{x}) = L^{-1}(\mathbf{x} - \mu).$$

- The Jacobian of T^{-1} (i.e., $\frac{\partial}{\partial \mathbf{x}} T^{-1}(\mathbf{x})$) is $J = L^{-1}$, so

$$\left|\det(J)\right| = \left|\det(L^{-1})\right| = (\det(L))^{-1}.$$

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$$|\det(J)| = |\det(L^{-1})| = (\det(L))^{-1}.$$

- By the change-of-variables formula,

$$p_X(\mathbf{x}) = p_Z(T^{-1}(\mathbf{x})) \, |\det(J)|.$$

Change of Variables (2/2)

- From

$$p_X(\mathbf{x}) = p_Z(T^{-1}(\mathbf{x})) \left| \det(J) \right|,$$

Plugging in p_Z and $\mathbf{z} = L^{-1}(\mathbf{x} - \mu)$, we obtain

$$
\begin{aligned}
p_X(\mathbf{x}) &= (2\pi)^{-D/2} \exp\left(-\frac{1}{2}\mathbf{z}^\top \mathbf{z} \right) (\det(L))^{-1} \\
&= (2\pi)^{-D/2} \exp\left(-\frac{1}{2}(\mathbf{x} - \mu)^\top (L^{-1})^\top L^{-1} (\mathbf{x} - \mu) \right) (\det(L))^{-1}.
\end{aligned}
$$

Final Form of the Multivariate Gaussian

- Recall that $(L^{-1})^{\top} L^{-1} = (LL^{\top})^{-1} = \Sigma^{-1}$, and

  $$\det(\Sigma) = \det(LL^{\top}) = (\det(L))^2 \implies (\det(L))^{-1} = (\det(\Sigma))^{-1/2}.$$

- Substituting into the previous expression gives

  $$p_X(\mathbf{x}) = (2\pi)^{-D/2}(\det(\Sigma))^{-1/2} \exp\left(-\frac{1}{2}(\mathbf{x} - \mu)^{\top} \Sigma^{-1}(\mathbf{x} - \mu)\right).$$

- Thus the pdf of the multivariate Gaussian $X \sim \mathcal{N}(\mu, \Sigma)$ is

  $$p(\mathbf{x} \mid \mu, \Sigma) = (2\pi)^{-D/2} \det(\Sigma)^{-1/2} \exp\left(-\frac{1}{2}(\mathbf{x} - \mu)^{\top} \Sigma^{-1}(\mathbf{x} - \mu)\right).$$

Discussions