# Mathematics for Machine Learning
## — Probabilistic Modeling & Inference

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## Credits for the resource

- The slides are based on the textbooks:

  - *Marc Peter Deisenroth, A. Aldo Faisal, and Cheng Soon Ong: Mathematics for Machine Learning. Cambridge University Press. 2020.*
  - *Howard Anton, Chris Rorres, Anton Kaul: Elementary Linear Algebra. Wiley. 2019.*

- We could partially refer to the monograph: *Francesco Orabona: A Modern Introduction to Online Learning. https://arxiv.org/abs/1912.13213*

# Outline

1. Probabilistic Models & Bayesian Inference

2. Latent-Variable Models

## Motivation

- We are concerned with prediction of future events and decision making.

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- We need mechanisms to learn something about $\mu$ given observed outcomes of coin-flip.

## Probabilistic Models

- The benefit of using probabilistic models:
  - A unified and consistent set of tools from probability theory for modeling, inference, prediction, and model selection.

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- Therefore, a probabilistic model is specified by the joint distribution of all its random variables.

## Bayesian Inference (1/3)

- We have already learnt two ways of estimating model parameters $\boldsymbol{\theta}$:
    - Maximum likelihood estimation (MLE)
    - Maximum a posteriori estimation (MAP)

- We can then obtain a *single-best* value of $\boldsymbol{\theta}$ (solving an optimization problem), then we can use them to make predictions.

- Having the full posterior distribution around can be useful and leads to more robust decisions.

## Bayesian Inference (2/3)

- Bayesian inference: finding such a posterior distribution.

- For a dataset $\mathcal{X}$, a parameter prior $p(\boldsymbol{\theta})$, and a likelihood function, the posterior

$$p(\boldsymbol{\theta} \mid \mathcal{X}) = \frac{p(\mathcal{X} \mid \boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\mathcal{X})},$$

then by applying Bayes' theorem,

$$p(\mathcal{X}) = \int p(\mathcal{X} \mid \boldsymbol{\theta})p(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta}.$$

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  - It has been marginalized/integrated out.

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- The prediction becomes an average over all plausible values of $\boldsymbol{\theta}$.
    - The plausibility is encapsulated by the distribution $p(\boldsymbol{\theta})$.

## Computational Issues

- MLE or MAP yields a consistent point estimate $\theta^*$ of the parameters.
  - Key computational problem: optimization.
  - Prediction: straightforward.

- Bayesian inference yields a distribution.
  - Key computational problem: integration.
  - Prediction: solving another integration problem.

# Outline

## Latent Variables

- Sometimes it is useful to have additional variable (besides $\boldsymbol{\theta}$) as part of the model.
  - We call them latent variables.
  - They do not parametrize the model explicitly.
  - E.g., mixture of $K$ Gaussians (further reading link).

- Latent variables can
  - Describe the data-generation process.
  - Increase the interpretability of the model.
  - Simplify the structure of the model.

## In the Data Generation Process

Denote data by $\mathbf{x}$, the model parameter by $\boldsymbol{\theta}$ and the latent variables by $\mathbf{z}$, we obtain the conditional distribution:

$$p(\mathbf{x} \mid \mathbf{z}, \boldsymbol{\theta}).$$

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### A Two-Step Procedure for Parameter Learning & Inference

1. Compute the likelihood $p(\mathbf{x} \mid \boldsymbol{\theta})$ (not depending on $\mathbf{z}$).
2. Use the likelihood for parameter estimation or Bayesian inference.

## Likelihood in Terms of Marginal Distribution

What we already have: a conditional distribution

$$p(\mathbf{x} \mid \mathbf{z}, \boldsymbol{\theta}).$$

We need to marginalize out the latent variables to have the predictive distribution of the data given the model parameters $\boldsymbol{\theta}$:

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  - Resort to approximation.

## A Posterior on the Latent Variables

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# Discussions