Randomized Algorithms

Introduction

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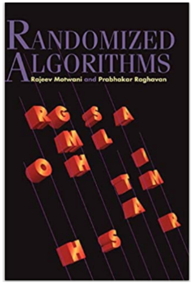
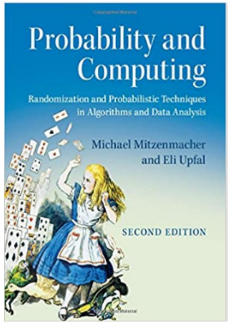
Textbooks and Materials

- Textbooks:
  - *Randomized Algorithms*. Motwani, R. and Raghavan, P., 1995. Cambridge University Press.
  - Probability and Computing: Randomization and Probabilistic Techniques in Algorithms and Data Analysis. Second Edition. M. Mitzenmacher and E. Upfal, Cambridge University Press, 2017.





- Other materials:
  - Prepared slides.

Prerequisites

- Basic undergraduate courses in
  - Algorithms
  - Data structures
  - Probability theory
  - Discrete mathematics
  - Linear Algebra
- Motivation.
- Curiosity.

Topics

- Introduction
- The Min-Cut Problem
- Discrete Random Variables & Expectations
- Randomized QuickSort
- *k*-Smallest Number Selection
- Complexities
- Minimax Principles
- Coupon Collector's Problem
- The Secretary Problem
- Moments & Deviations
- Chernoff & Hoeffding Bounds
- Balls and Bins
- Markov Chains & Random Walks
- Continuous Distributions & The Poisson Process
- The Probabilistic Method
- Lovász Local Lemma and Applications
- The Monte Carlo Method
- Other Selected Topics

Grading Policy

- Attendance (10%)
- Assignments and Quizzes (60%)
  - 4 x 15%
  - Assignments will be announced first, then we have quizzes on them.
- Final Exam/Paper Presentation (30%)
  - Depending on the teaching progress.

Grading Policy for the Presentations (If Necessary)

- Students need to group to teams
- Complete the presentation: 60 point
  - Duration for each presentation: 30~50 minutes.
- Raising questions: +4 point for each one (maximum +20 point)
- Clearly answering the teacher’s 4 questions: +5 point for each one.

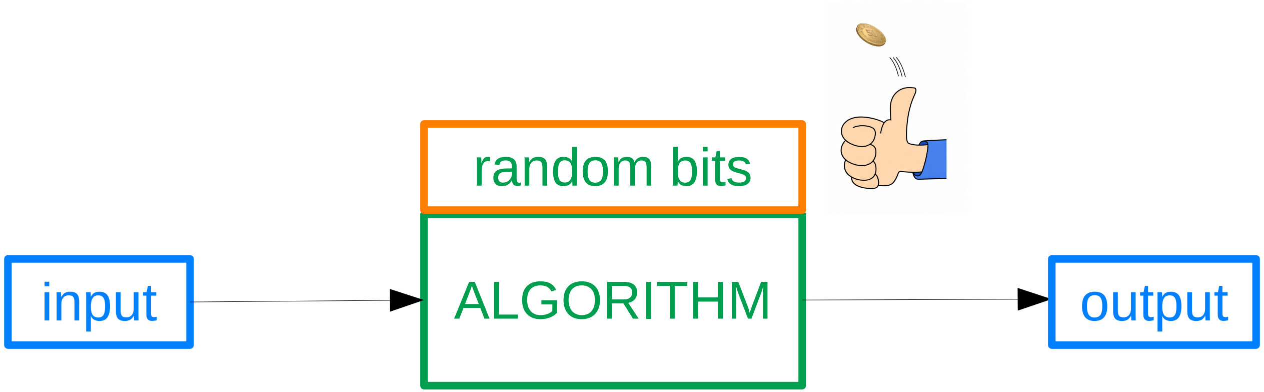
Deterministic Algorithms

vs.

Randomized Algorithms

Traditional deterministic algorithms

Randomized algorithms



Why?

- Randomized algorithms are
  - often much ***simpler*** than the best known deterministic ones.
  - often much ***more efficient*** (faster or using less space) than the best known deterministic ones.

- Sometimes ideas from the randomized algorithms lead to good deterministic algorithms.

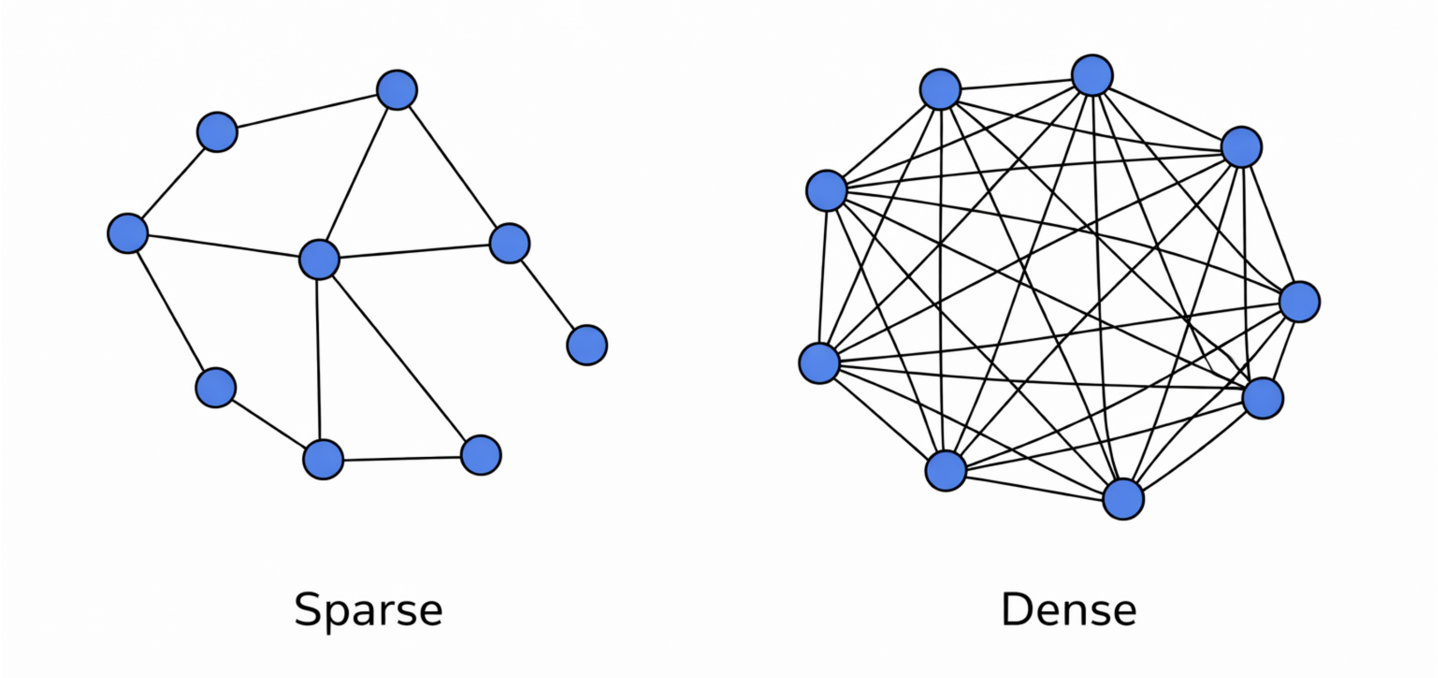
Comparisons

- It’s different from the *average-case* analysis of deterministic algorithms.
  - e.g., expected running time of a deterministic algorithm on input sampled from a distribution.

- In most scenarios, it’s NOT a heuristic algorithm.
  - The accuracy is guaranteed, or
  - The running time is guaranteed.

An illustrating example

- Determining if a graph is very dense (or sparse)



Another illustrating example

- **Problem:** find a grade-'*A*' student in a class of n students where half of them get 'A'.

- What is the time complexity for the best deterministic algorithm?
  - I mean, in the "worst case".

A randomized algorithm (from Wikipedia)

```
findingA_LV(array L, n) // Las Vegas
begin
    repeat
        Randomly select one element out of n elements.
    until 'A' is found
end
```

Assignment: Prove that the expected number of iterations is $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{2^i} \leq 2.$

A randomized algorithm (from Wikipedia)

```
findingA_MC(array L, n, k) // Monte Carlo
  begin
     i ← 0
     repeat
        Randomly select one element out of n elements.
        i ← i + 1
     until i = k or 'A' is found
  end
```

After k iterations, $\Pr[\text{find } A] = 1 - (1/2)^k$.

Randomization Helps in Making Decisions

Birthday problem (paradox)

- There are n randomly chosen people in a room.
- How ***large*** should n be such that there is at least one pair of them having the same birthday (mm/dd)?
- By the pigeonhole principle, n = 367? or 366?

Birthday problem (paradox)

- There are n randomly chosen people in a room.
- How ***large*** should n be such that there is at least one pair of them having the same birthday (mm/dd)?
- By the pigeonhole principle, n = 367? or 366?

- Let us consider this problem in the other way around.

  How ***large*** should n be such that there is at least one pair of them having the same birthday (mm/dd) with probability ≥ 0.5?

Birthday problem (paradox)

- n people: $x_1, x_2, \ldots, x_n$
- Event i: some pair of $x_1, x_2, \ldots, x_i$ have the same birthday.
- $\Pr[\text{Event2}] = 1 - \frac{364}{365}$
- $\Pr[\text{Event3}] = 1 - \frac{364}{365} \cdot \frac{363}{365}$

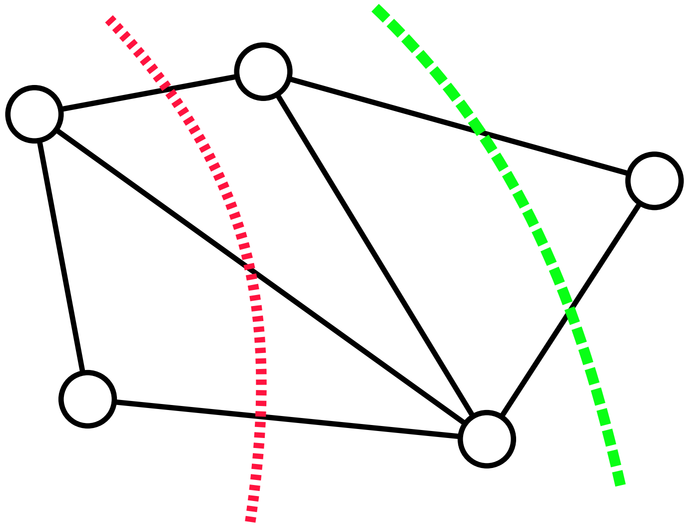
  …
- $\Pr[\text{Event23}] = 1 - \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{343}{365} \approx 0.507297.$
- 23 is much less than 366 or 367.

Birthday problem (paradox)

- **Assignment:** Compute *n* such that there is at least one pair of them having the same birthday with probability ≥ 0.9.

The Min-Cut Problem

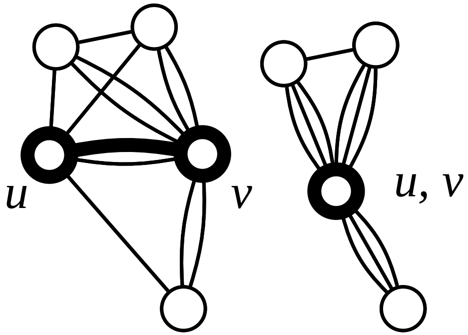
Min-Cut



$$|V| = n, |E| = m$$

- A graph $G = (V, E)$ and its two "cuts".
  - **Cut**: a partition of the vertices in V into two non-empty, disjoint sets S and T such that
  - $S \cup T = V$
- The **cutset** of a cut:
  - $\{uv \in E \mid u \in S, v \in T\}.$
- The size of the cut:
  - the cardinality of its cutset.

Edge contraction



$e = (u, v)$

$G \rightarrow \; G/e$

Karger's edge-contraction algorithm (1993)

Procedure contract ($G = (V, E)$):

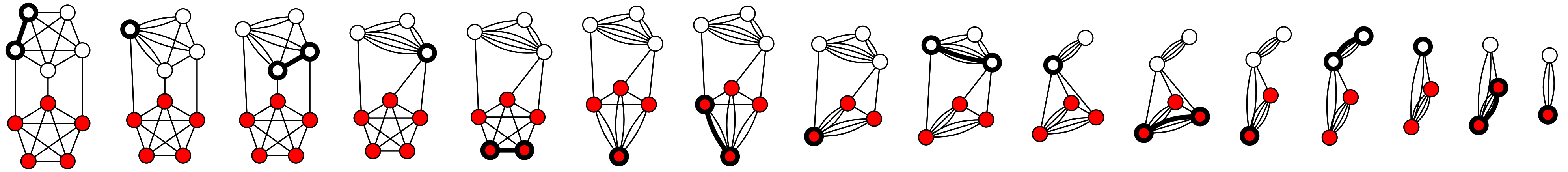
while $|V| > 2$:

  choose $e \in E$ uniformly at random

  $G \leftarrow G/e$

return the only cut in G

Time complexity: $O(m)$ or $O(n^2)$.



Analysis

- C: a specific cut of G.
- k: the size of the cut C.
- The minimum degree of G must be $\geq k$. (WHY?)
  - So, $|E| \geq nk/2$.
- The probability that the algorithm picks an edge from C to contract is

$$\frac{k}{|E|} \leq \frac{k}{nk/2} = \frac{2}{n}.$$

Analysis (contd.)

- Let p_n be the probability that the algorithm on an n-vertex graph avoids C.
- Then,

$$p_n \geq \left(1 - \frac{2}{n}\right) \cdot p_{n-1}$$

- The recurrence can be expanded as

$$p_n \geq \prod_{i=0}^{n-3} \left(1 - \frac{2}{n-i}\right) = \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdots \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{\binom{n}{2}}.$$

Analysis (contd.)

- The probability of "success" is $\binom{n}{2}^{-1}$
  - A bit too low, isn't it?
- How about repeating it for $T = \binom{n}{2} \ln n$ times, and then choose the *minimum* of them?
- The probability of NOT finding a min-cut is $\left[1 - \binom{n}{2}^{-1}\right]^T \leq \frac{1}{e^{\ln n}} = \frac{1}{n}.$
- Total running time: $O(Tm)$ or $O(Tn^2)$.

An Coding Example

- The demo code: https://onlinegdb.com/6byIpav5uJ