Randomized Algorithms

Markov Chains and Random Walks

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Outline

- Markov Chains: definitions and representations
- Application: Random Walk
- Classification of States
- Stationary Distribution

Stochastic Process

- A stochastic process $\mathbf{X} = \{X(t) : t \in T\}$ is a collection of random variables.
  - t: time
  - $X(t)$: state of the process at time t.
- If T is a countably infinite set, we say $\mathbf{X}$ is a discrete time process.

Markov Chain

- A discrete time stochastic process X_0, X_1, X_2, … is a Markov chain if

states

$$\begin{aligned}\Pr[X_t = a_t \mid X_{t-1} = \boxed{a_{t-1}}, X_{t-2} = \boxed{a_{t-2}}, \ldots, X_0 = \boxed{a_0}] &= \Pr[X_t = a_t \mid X_{t-1} = a_{t-1}] \\ &= P_{a_{t-1},a_t}.\end{aligned}$$

  - Markov property.

Markov Chain

- A discrete time stochastic process X_0, X_1, X_2, … is a Markov chain if

$$\begin{aligned} \Pr[X_t = a_t \mid X_{t-1} = a_{t-1}, X_{t-2} = a_{t-2}, \ldots, X_0 = a_0] &= \Pr[X_t = a_t \mid X_{t-1} = a_{t-1}] \\ &= P_{a_{t-1}, a_t}. \end{aligned}$$

  - Markov property.
  - ✔ This does **NOT** imply that X_t is **independent** of X_0, X_1, …, X_{t-2},
    - ✔ The dependency of X_t on the past is captured in X_{t-1}.

Markov Chain

- Markov property implies:
  - ➔ The Markov chain is uniquely defined by the one-step transition matrix.

$$\mathbf{P} = \begin{pmatrix} P_{0,0} & P_{0,1} & \cdots & P_{0,j} & \cdots \\ P_{1,0} & P_{1,1} & \cdots & P_{1,j} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ P_{i,0} & P_{i,1} & \cdots & P_{i,j} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

$$\text{for all } i, \sum_{j \geq 0} P_{i,j} = 1.$$

Transition Probabilities

- $\bar{p}(t) = (p_0(t), p_1(t), p_2(t), \ldots)$.
  - $p_i(t)$: the probability that the process is at state i at time t.

$$p_i(t) = \sum_{j \geq 0} p_j(t-1) P_{j,i}.$$

$$\bar{p}(t) = \bar{p}(t-1)\mathbf{P}.$$

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- m step transition probability:

$$P^m_{i,j} = \Pr[X_{t+m} = j \mid X_t = i].$$

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$$\bar{p}(t) = \bar{p}(t-1)\mathbf{P}.$$

$$\mathbf{P}^{(m)} = \mathbf{P} \cdot \mathbf{P}^{(m-1)}$$

$$\mathbf{P}^{(m)} = \mathbf{P}^{m} \qquad \text{(by induction on } m\text{)}$$

- *m* step transition probability:

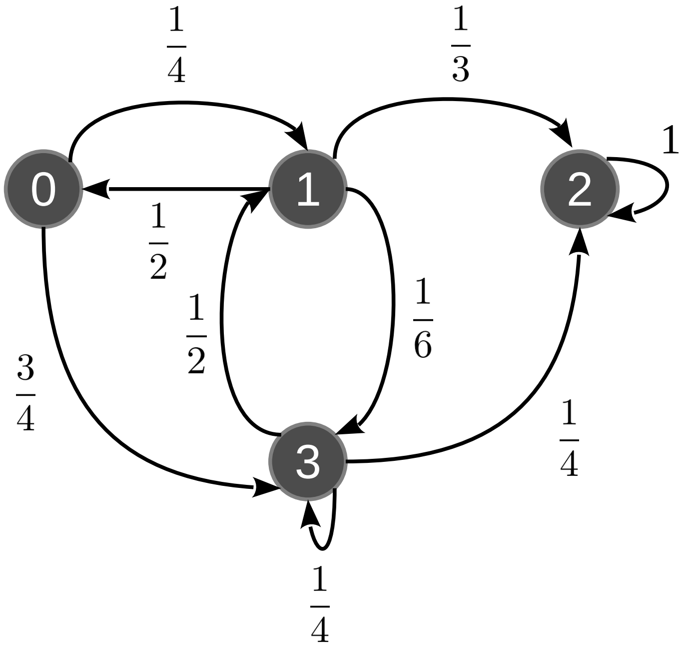
$$P^m_{i,j} = \Pr[X_{t+m} = j \mid X_t = i].$$

$$P^m_{i,j} = \sum_{k \geq 0} P_{i,k} P^{m-1}_{k,j}.$$

for any $t \geq 0$ and $m \geq 1$,

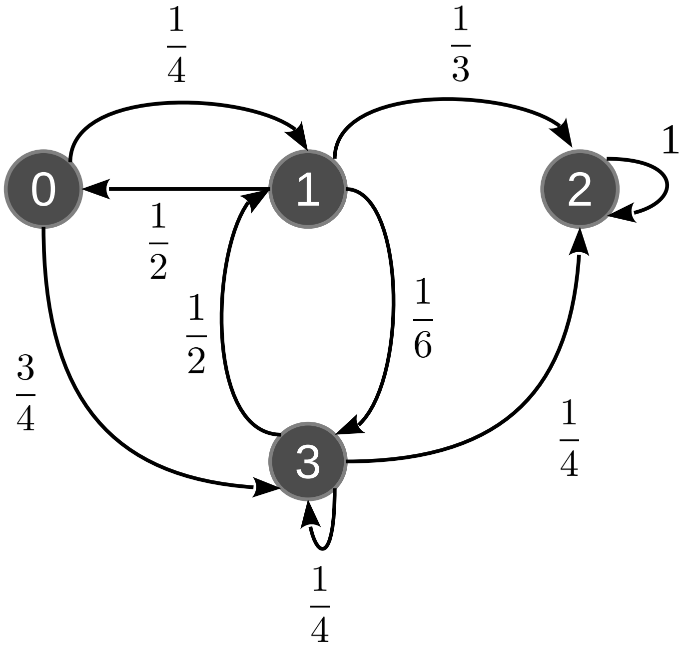
$$\bar{p}(t+m) = \bar{p}(t)\mathbf{P}^m.$$

Transition Probabilities



$$\mathbf{P} = \begin{bmatrix} 0 & 1/4 & 0 & 3/4 \\ 1/2 & 0 & 1/3 & 1/6 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 1/4 & 1/4 \end{bmatrix}$$

Transition Probabilities



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$$\mathbf{P}^3 = \begin{bmatrix} 3/16 & 7/48 & 29/64 & 41/192 \\ 5/48 & 5/24 & 79/144 & 5/36 \\ 0 & 0 & 1 & 0 \\ 1/16 & 13/96 & 107/192 & 47/192 \end{bmatrix}$$

Transition Probabilities

- If we begin in a state chosen uniformly at random: (¼, ¼, ¼, ¼), what is the probability distribution after three steps?

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$$(1/4, 1/4, 1/4, 1/4)\mathbf{P}^3 = (17/192, 47/384, 737/1152, 43/288).$$

$$\mathbf{P}^3 = \begin{bmatrix} 3/16 & 7/48 & 29/64 & 41/192 \\ 5/48 & 5/24 & 79/144 & 5/36 \\ 0 & 0 & 1 & 0 \\ 1/16 & 13/96 & 107/192 & 47/192 \end{bmatrix}$$

Exercise

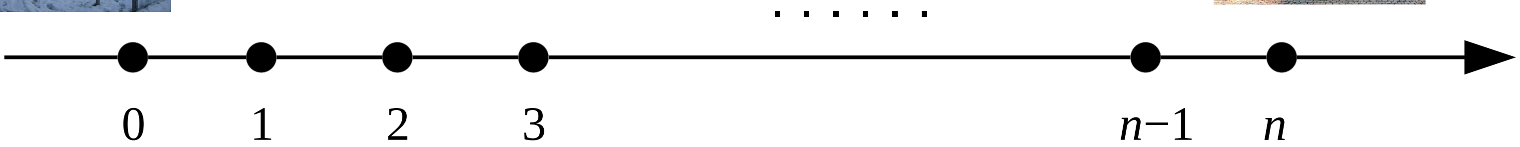
- Consider the two-state Markov chain with the following transition matrix. Find a simple expression for $P^t_{0,0}$.

$$\mathbf{P} = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

Application: Random Walks

Steppenbergallee Aachen

Aachener Dom



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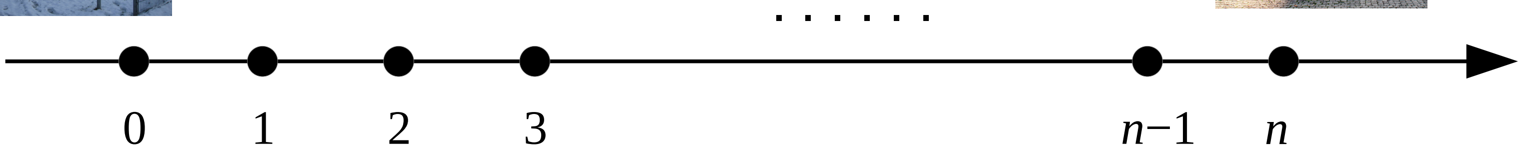

- X_i : the position after the *i*th step you've walked.

Application: Random Walks

- Only at the position 0 (my home) we know how to make a right step towards the destination (cathedral).

Application: Random Walks

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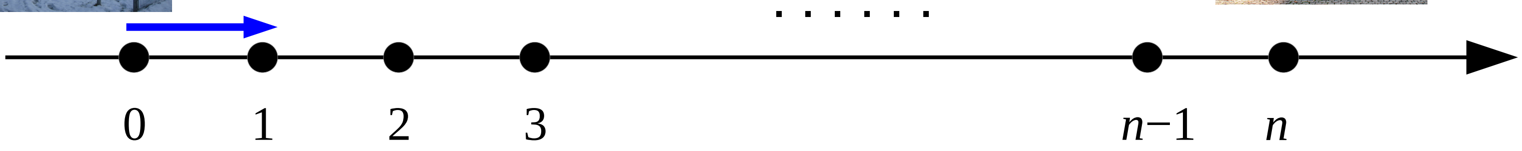

- Only at the position 0 (my home) we know how to make a right step towards the destination (cathedral).

$$\Pr[X_{i+1} = 1 \mid X_i = 0] = 1.$$

Application: Random Walks

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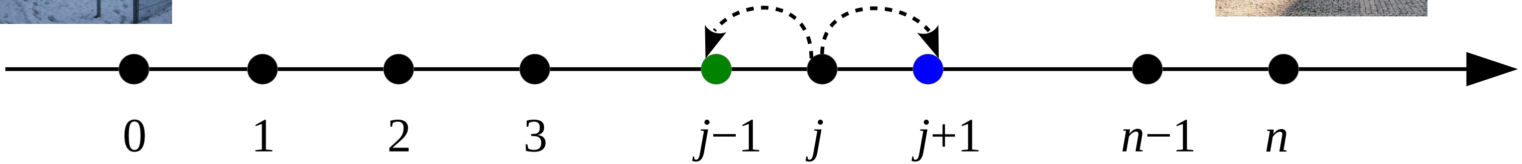

- If we are at positions 1, 2, …, $n-1$, we have no idea about the direction to go.
- Suppose then we have chance of 50% to get one step closer to the destination and 50% to get one step backward…
- How many steps we expect to walk…?

Application: Random Walks

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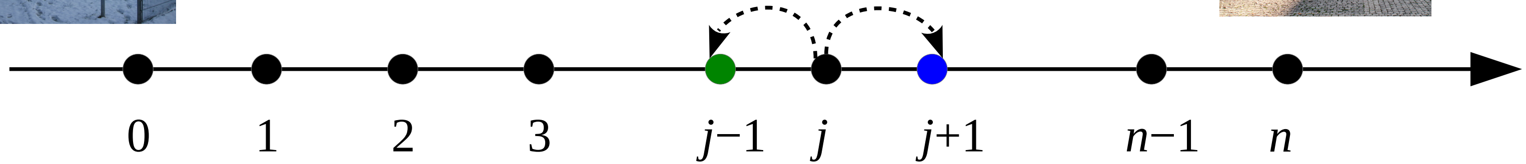

- Markov chain $X_0, X_1, X_2, \ldots$
- Z_j: random variable; the number of steps to reach n from j.
- h_j : the expected steps to reach n when starting from j.
  - $\mathbf{E}[Z_j] = h_j$.
- $h_n = 0$, $h_0 = h_1 + 1$.

Application: Random Walks

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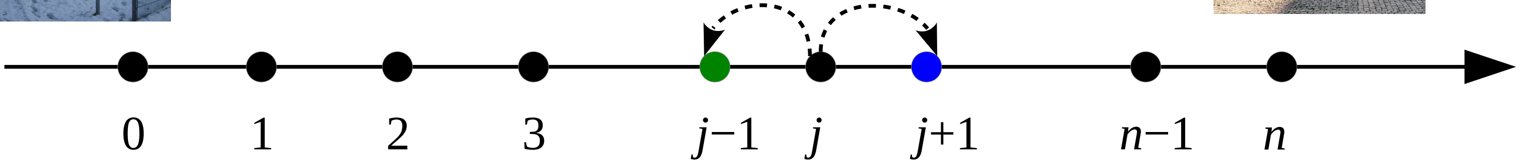

- $$\mathbf{E}[Z_j] = \mathbf{E}\left[\frac{1}{2}\cdot(1+Z_{j-1}) + \frac{1}{2}\cdot(1+Z_{j+1})\right].$$

$$h_j = \frac{h_{j-1}}{2} + \frac{h_{j+1}}{2} = \frac{h_{j-1}+h_{j+1}}{2} + 1. \quad (\text{for } 1 \le j \le n-1)$$

Application: Random Walks

$$
\begin{aligned}
h_{j+1} &= 2h_j &&- h_{j-1} - 2 \\
h_j &= 2h_{j-1} &&- h_{j-2} - 2 \\
h_{j-1} &= 2h_{j-2} &&- h_{j-3} - 2 \\
h_{j-2} &= 2h_{j-3} &&- h_{j-4} - 2 \\
\cdots & \quad \cdots && \cdots \\
h_2 &= 2h_1 &&- h_0 - 2 \\
h_1 &= h_0 &&- 1
\end{aligned}
$$

Application: Random Walks

$$
\begin{array}{rcll}
h_{j+1} & = 2h_j & -\cancel{h_{j-1}} & -\ 2 \\
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\cancel{h_{j-2}} & = 2\cancel{h_{j-3}} & -\cancel{h_{j-4}} & -\ 2 \\
\cancel{\cdots} & \cdots & \cdots & \\
\cancel{h_2} & = 2h_1 & -h_0 - 2 & \\
h_1 & = h_0 & -1 &
\end{array}
$$

$$\Rightarrow h_j = h_{j+1} + 2j + 1.$$

Application: Random Walks

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\cancel{\cdots} & \quad \cdots && \cdots \\
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\cancel{h_1} &= h_0 &&-1
\end{aligned}
$$

$$\Rightarrow h_j = h_{j+1} + 2j + 1.$$

$$h_0 = h_1 + 1 = h_2 + 1 + 3 = \cdots = \sum_{i=0}^{n-1}(2i+1) = n^2.$$

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$$\Rightarrow h_j = h_{j+1} + 2j + 1.$$

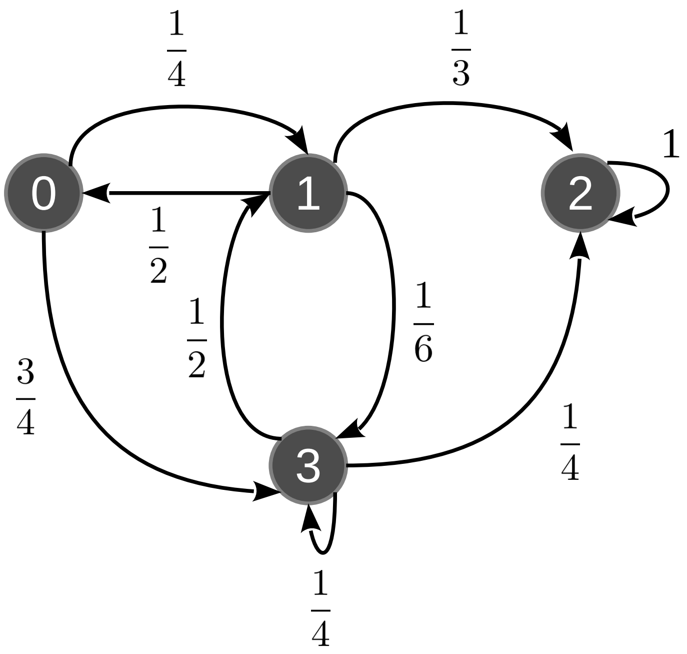
$$h_0 = h_1 + 1 = h_2 + 1 + 3 = \cdots = \sum_{i=0}^{n-1} (2i+1) = n^2.$$

$$\Pr[\text{walking steps } > 2n^2] \le \frac{n^2}{2n^2} = \frac{1}{2}.$$

Exercise

- Consider the random walk we just discussed. Now we assume that whenever position 0 is reached, with probability ½ the walk moves to position 1 and with probability ½ the walk stays at 0. What is the expected number of steps to reach *n* starting from position 0?

Classification of States

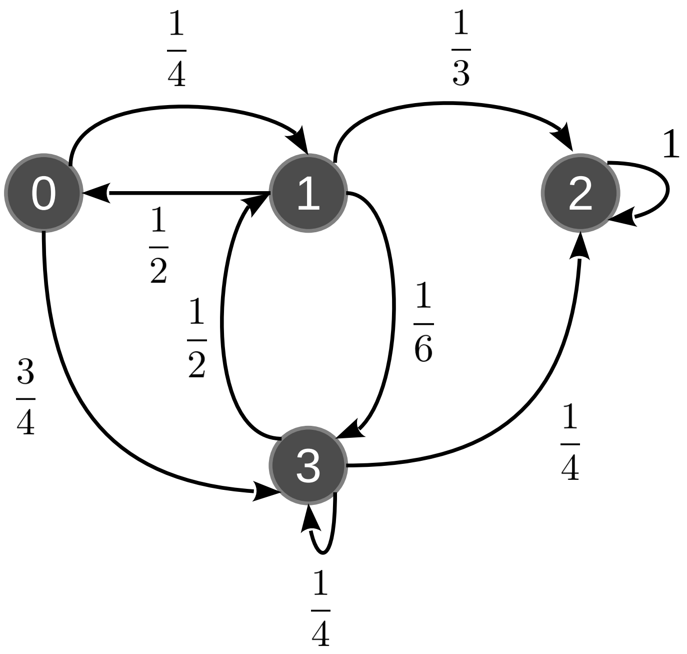


- $i \to j$ **accessible:**

  For some integer $n \geq 0, P^n_{i,j} > 0$

  - How about $2 \to 0$? $2 \to 1$?

Classification of States



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- $i \leftrightarrow j$: i and j **communicate.**

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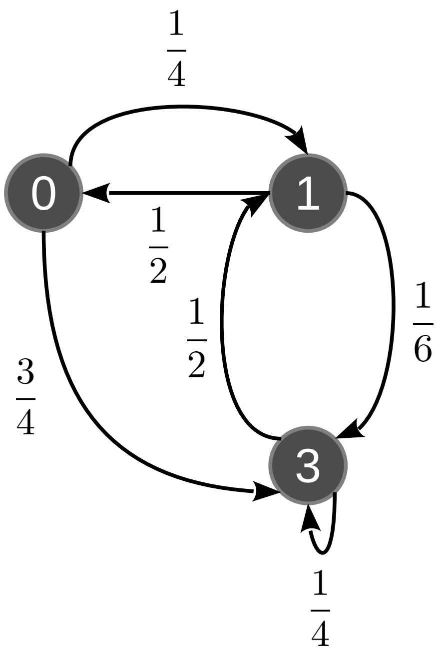


- $i \rightarrow j$ **accessible**:

  For some integer $n \geq 0, P_{i,j}^{n} > 0$

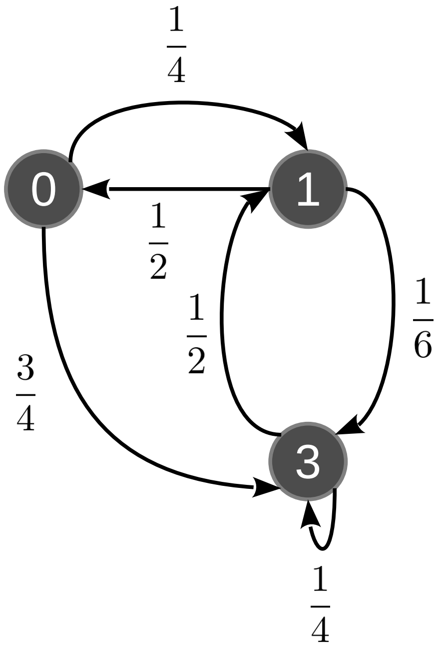
  - How about $2 \rightarrow 0$? $2 \rightarrow 1$?

- $i \leftrightarrow j$: i and j **communicate.**

The Markov chain is **irreducible.**

- Any two states communicate.

Classification of States



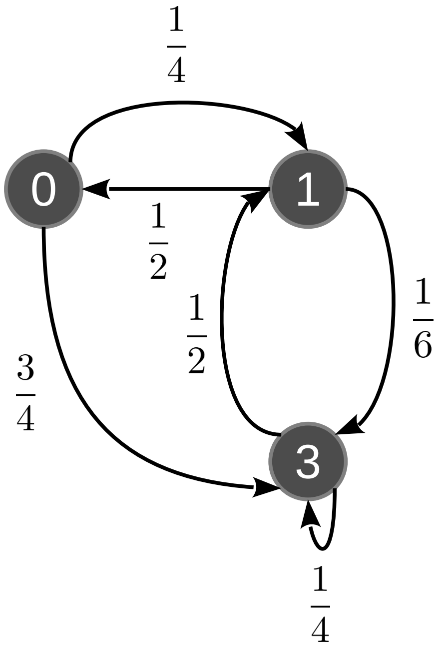
- $r^t_{i,j}$: the probability that starting at state *i*, the first transition to state *j* occurs at time *t*.

$$r^t_{i,j} = \Pr[X_t = j \text{ and, for } 1 \leq s \leq t-1, X_s \neq j \mid X_0 = i]$$

The Markov chain is **recurrent.**

- $$\sum_{t \geq 1} r^t_{i,i} = 1 \text{ for every state } i$$

Classification of States



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$$r^t_{i,j} = \Pr[X_t = j \text{ and, for } 1 \le s \le t-1, X_s \ne j \mid X_0 = i]$$

- $h_{i,i}$: the expected time to return state i when starting from state i. $$h_{i,i} = \sum_{t \ge 1} t \cdot r^t_{i,i}.$$

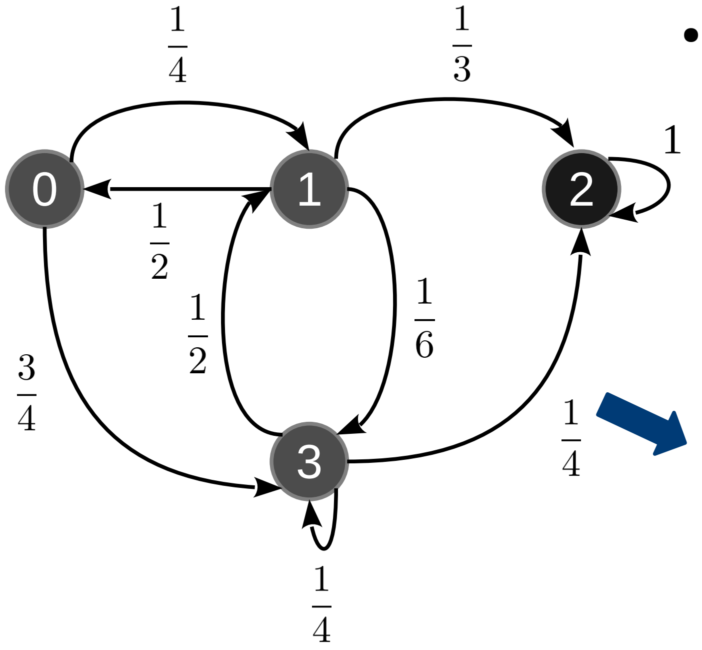
The Markov chain is **recurrent.**

- $$\sum_{t \ge 1} r^t_{i,i} = 1 \text{ for every state } i$$

- Each state i is **positive recurrent**.

$$h_{i,i} < \infty$$

Classification of States



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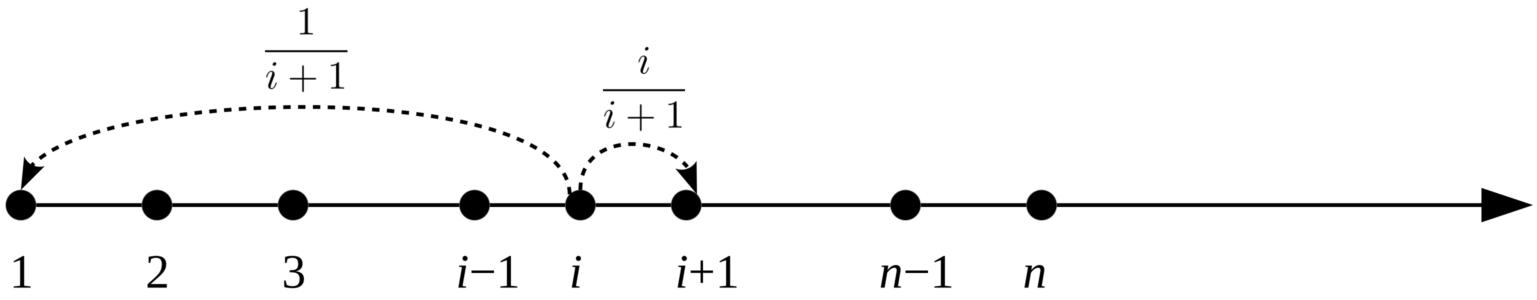
$$r^t_{i,j} = \Pr[X_t = j \text{ and, for } 1 \le s \le t-1, X_s \ne j \mid X_0 = i]$$

State 0, 1, 3 are **transient.**

- $$\sum_{t \ge 1} r^t_{1,1} < 1 \text{ for state } i \in \{0, 1, 3\}$$

Classification of States

- An example of *null* recurrent:



$$r_{1,1}^{t} = \left(\prod_{j=1}^{t-1} \frac{j}{j+1}\right) \cdot \frac{1}{t+1} = \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{t-1}{t} \cdot \frac{1}{t+1} = \frac{1}{t(t+1)}.$$

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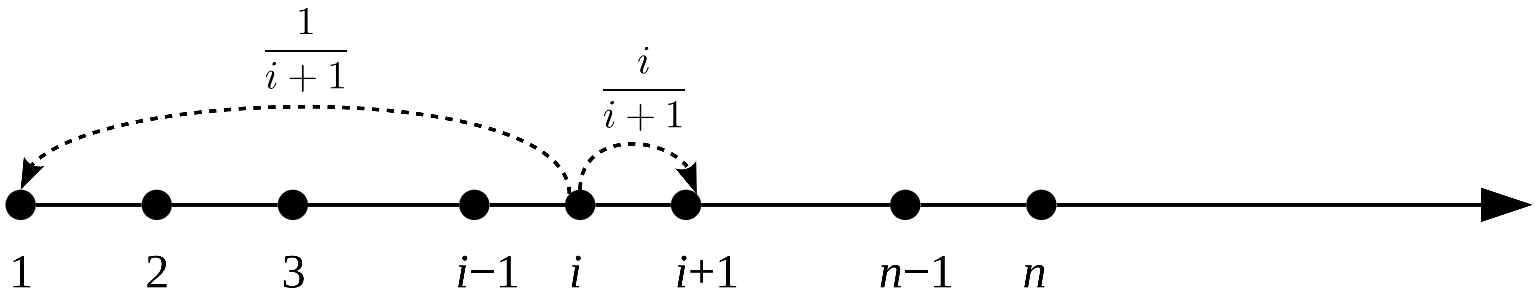

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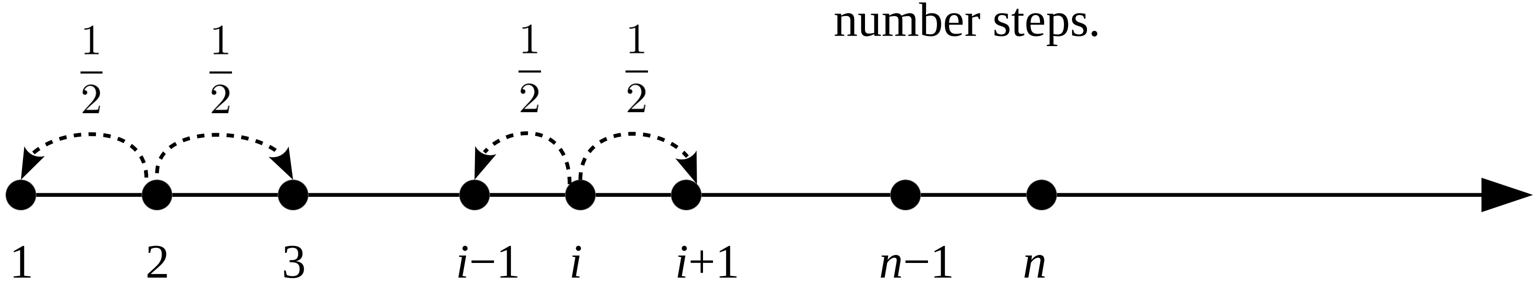
$$h_{1,1} = \sum_{t \geq 1} t \cdot r^t_{1,1} = \sum_{t \geq 1} \frac{1}{t+1} \to \infty.$$

Classification of States

- **periodic** states.

- Suppose the chain starts at 2.
- It can be at **even** number states only after **even** number steps.

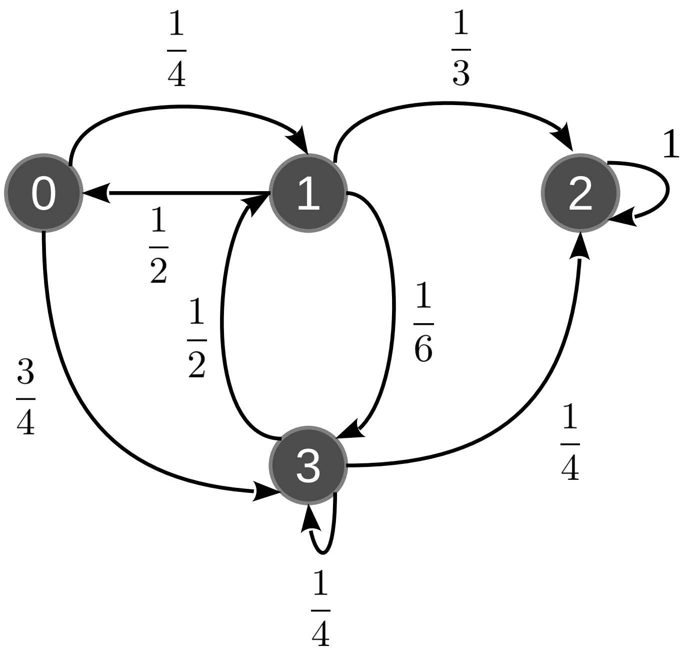


j is periodic :

$\exists \Delta > 1$ such that $\Pr[X_{t+s} = j \mid X_t = j] = 0$ unless s is divisible by Δ.

- *aperiodic* = not periodic

Classification of States



- An aperiodic, positive recurrent state is an **ergodic** state.
- **Ergodic Markov chain**: every state is ergodic.

Stationary Distributions

- Recall that

$$\bar{p}(t) = \bar{p}(t-1)\mathbf{P}.$$

- Consider $\bar{p}(t) = \bar{p}(t-1)$

$$\text{That is, } \bar{\pi} = \bar{\pi}\mathbf{P}.$$

$\bar{\pi}$: a probability distribution over the states.

Stationary Distributions

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- Consider $\bar{p}(t) = \bar{p}(t-1)$

$$\text{That is, } \bar{\pi} = \bar{\pi}\mathbf{P}.$$

$\bar{\pi}$: a probability distribution over the states.

- We call it a **stationary distribution** of the Markov chain.

Stationary Distributions

- **Theorem**. Any finite, irreducible, and ergodic Markov chain has the following properties:
  1. The chain has a unique stationary distribution
  2. for all j and i,

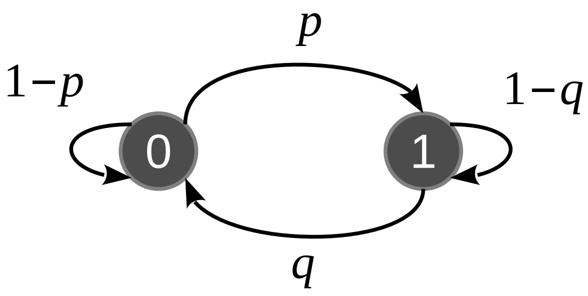
  $$\lim_{t\to\infty} P_{j,i}^t \text{ exists and it's independent of } j$$

  3. $$\pi_i = \lim_{t\to\infty} P_{j,i}^t = \frac{1}{h_{i,i}}.$$

Computing the Stationary Distribution

- Method 1: Solve the system of linear equations.
- <u>Example</u>:

$$\mathbf{P} = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}.$$



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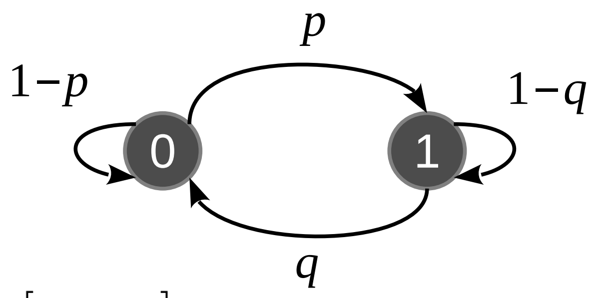


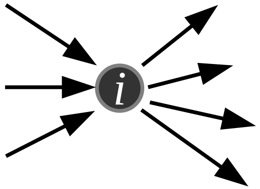
$$\bar{\pi}\mathbf{P} = \bar{\pi} \quad \Leftrightarrow \quad [\pi_0, \pi_1] \cdot \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} = [\pi_0, \pi_1].$$

$$\begin{aligned} \pi_0(1-p) + \pi_1 q &= \pi_0; \\ \pi_0 p + \pi_1(1-q) &= \pi_1; \\ \pi_0 + \pi_1 &= 1 \end{aligned} \qquad \begin{aligned} \pi_0 &= \frac{q}{p+q}. \\ \pi_1 &= \frac{p}{p+q}. \end{aligned}$$

Computing the Stationary Distribution

- Method 2: Cut-sets of the Markov chain.
- The idea:
  - For any state *i* of the chain,

$$\sum_{j=0}^{n} \pi_j P_{j,i} = \pi_i = \pi_i \cdot \sum_{j=0}^{n} P_{i,j}$$



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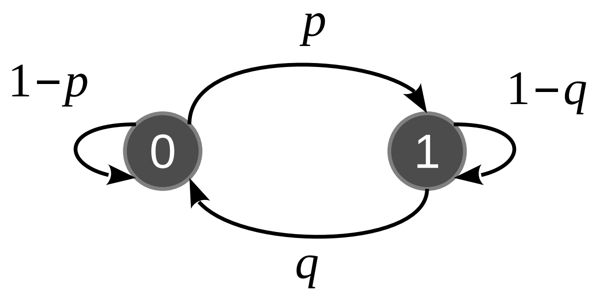

The probability of leaving state 0 must equal the probability of entering state 0

$$\pi_0 p = \pi_1 q$$

$$\begin{aligned} \pi_0 &= \frac{q}{p+q}. \\ \pi_1 &= \frac{p}{p+q}. \end{aligned}$$

Exercise

- Consider a Markov chain with state space {0, 1, 2, 3} and a transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & \frac{3}{10} & \frac{1}{10} & \frac{3}{5} \\ \frac{1}{10} & \frac{1}{10} & \frac{7}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{7}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{9}{10} & \frac{1}{10} & 0 & 0 \end{bmatrix}$$

Find the stationary distribution of the Markov chain.