

Counting Binary Trees & Selection Trees

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Fall 2025



Outline

- 1 Counting Binary Trees
- 2 Selection Trees

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1 Counting Binary Trees

2 Selection Trees

Counting Binary Trees

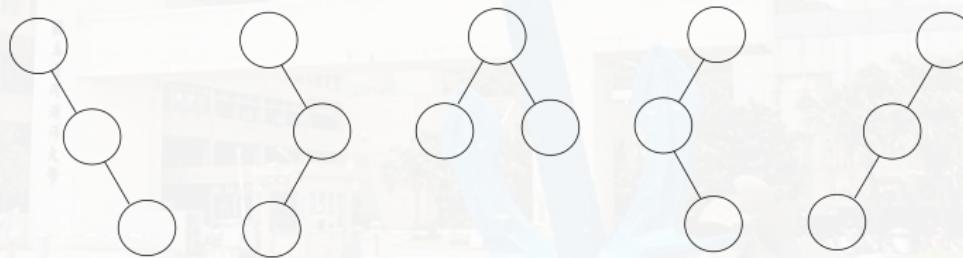
- Consider the following three disparate problems:
 - The number of distinct binary trees having n nodes.
 - The number of distinct permutations of the numbers from 1 to n obtainable by a **stack**.
 - The number of distinct ways of multiplying $n + 1$ matrices.

Counting Binary Trees

- Consider the following three disparate problems:
 - ➊ The number of distinct binary trees having n nodes.
 - ➋ The number of distinct permutations of the numbers from 1 to n obtainable by a **stack**.
 - ➌ The number of distinct ways of multiplying $n + 1$ matrices.
- Amazingly, **these problems have the same solution!**

Problem One

- The number of distinct binary trees having n nodes.



* Example of $n = 3$.

Problem Two

- The number of distinct permutations of the numbers from 1 to n obtainable by a stack.
 - ① push 1 → pop → push 2 → pop → push 3 → pop ⇒ 123.
 - ② push 1 → pop → push 2 → push 3 → pop → pop ⇒ 132.
 - ③ push 1 → push 2 → push 3 → pop → pop → pop ⇒ 321.
 - ④ push 1 → push 2 → pop → pop → push 3 → pop ⇒ 213.
 - ⑤ push 1 → push 2 → pop → push 3 → pop → pop ⇒ 231.
- ★ Example of $n = 3$.

Problem Three

- The number of distinct ways of multiplying $n + 1$ matrices.
 - ① $((M_1 \times M_2) \times M_3) \times M_4$.
 - ② $((M_1 \times (M_2 \times M_3)) \times M_4)$.
 - ③ $(M_1 \times ((M_2 \times M_3) \times M_4))$.
 - ④ $(M_1 \times (M_2 \times (M_3 \times M_4)))$.
 - ⑤ $((M_1 \times M_2) \times (M_3 \times M_4))$.

* Example of $n = 3$.

Stack Permutation (1/4)

- Recall: preorder, inorder and postorder traversal of a binary tree.
 - Each traversal requires a **stack**.

Every binary tree has a unique pair of preorder/inorder sequences.

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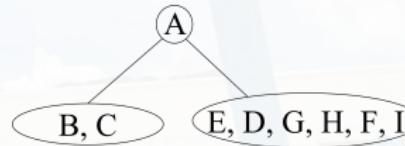
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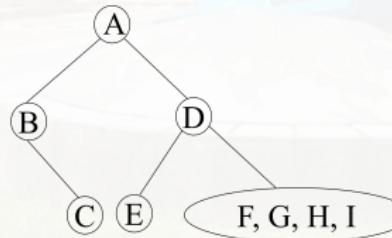
- The number of distinct binary trees is equal to the number of **inorder permutations** obtainable from binary trees having the preorder permutation, $1, 2, \dots, n$.

Stack Permutation (2/4)

- preorder: A B C E D G H F I
- inorder: B C A E D G H F I



- preorder: A B C (D E F G H I)
- inorder: B C A (E D F G H I)



Stack Permutation (3/4)

- We can show that

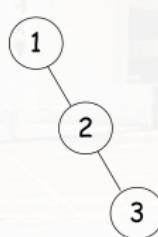
the number of distinct permutations obtainable by passing the numbers $\{1, 2, \dots, n\}$ through a stack is equal to the number of distinct binary trees with n nodes.

- ① push 1 → pop → push 2 → pop → push 3 → pop ⇒ 123.
- ② push 1 → pop → push 2 → push 3 → pop → pop ⇒ 132.
- ③ push 1 → push 2 → push 3 → pop → pop → pop ⇒ 321.
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Stack Permutation (4/4)

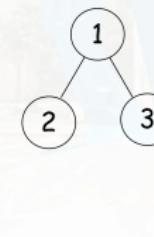
- ★ Output: inorder traversal:
- visit node → push to stack;
 - going left → keep visiting next node
 - going right → pop the stack
- the leaf → pop the stack until empty



(1, 2, 3)



(1, 3, 2)



(2, 1, 3)



(2, 3, 1)



(3, 2, 1)



Go Back to the Matrix Multiplication

- Computing the product of n matrices are related to the distinct binary tree problem.
- $n = 3$:
 - ① $((M_1 \times M_2) \times M_3)$.
 - ② $(M_1 \times (M_2 \times M_3))$.
- $n = 4$:
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- $n = 4$:
 - ① $(((M_1 \times M_2) \times M_3) \times M_4)$ (push, pop, push, pop, push, pop)
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- b_n : the number of different ways to compute the product of n matrices.
- Trivially, $b_1 = 1$, $b_2 = 1$.
- We have also derived that $b_3 = 2$ and $b_4 = 5$.
- We can compute that

$$b_n = \sum_{i=1}^{n-1} b_i b_{n-i}, \text{ for } n > 1.$$

Distinct Binary Trees

- Similarly, the number of **distinct binary trees** of n nodes is

$$b_n =$$



Distinct Binary Trees

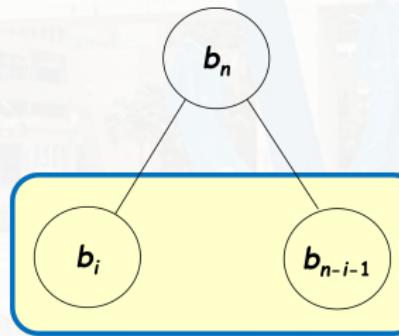
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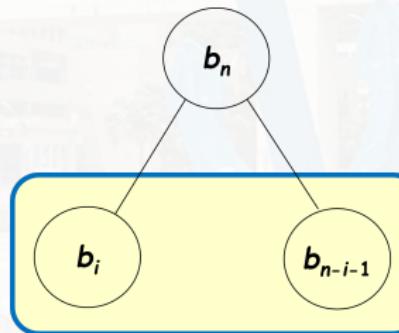
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- But, how to compute b_n exactly?

The Generating Function Trick

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 &= 1 + x \sum_{j=0}^{\infty} \sum_{k=0}^j b_k b_{j-k} x^j \\
 &= 1 + x \left(\sum_{j=0}^{\infty} b_j x^j \right)^2 = 1 + x B(x)^2.
 \end{aligned}$$



The Generating Function Trick

- By the recurrence relation we get:

$$xB(x)^2 = B(x) - 1.$$

- Solving the recurrence relation, we have

$$\begin{aligned}B(x) &= \frac{1 - \sqrt{1 - 4x}}{2x} \\&= \frac{1}{2x} \left(1 - \sum_{n \geq 0} \binom{1/2}{n} (-4x)^n \right) \\&= \sum_{m \geq 0} \binom{1/2}{m+1} (-1)^m 2^{2m+1} x^m.\end{aligned}$$

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By the Binomial Theorem...

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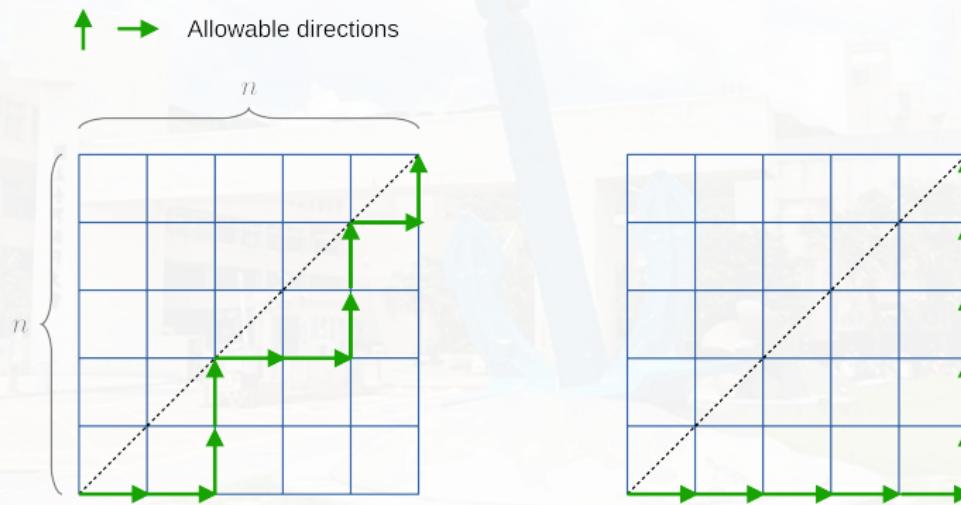
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* supplementary: Stirling's approximation

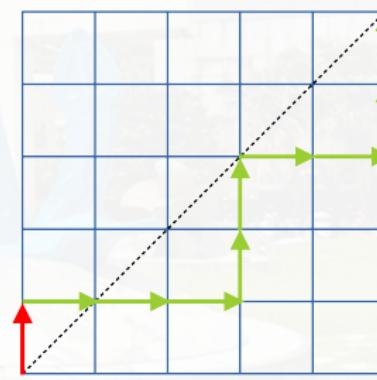
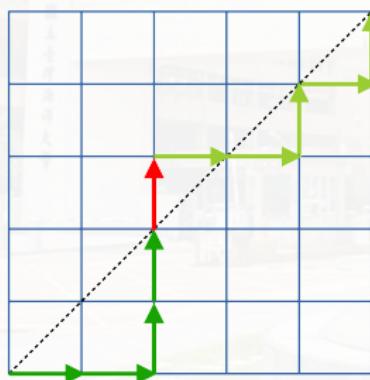
Catalan number by monotonic lattice paths (1/5)



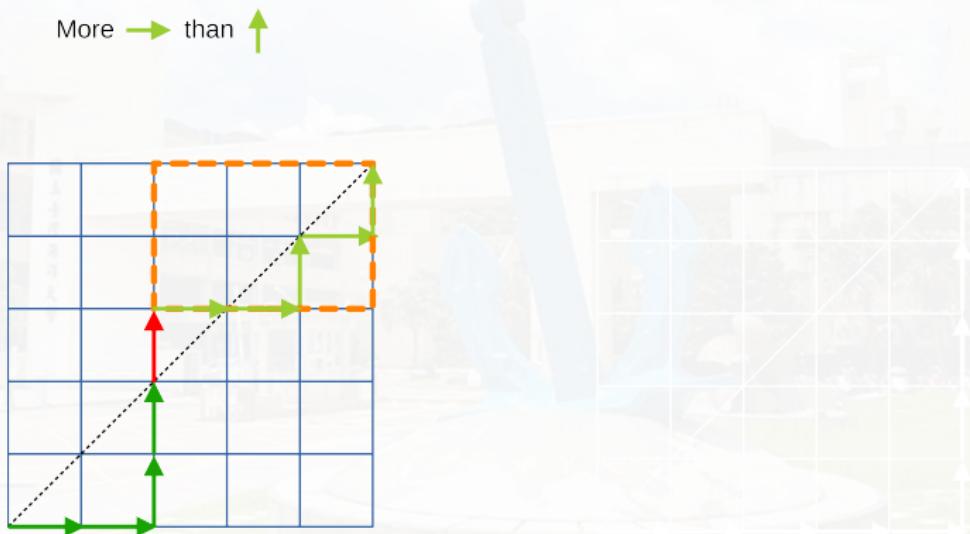
Catalan number by monotonic lattice paths (2/5)



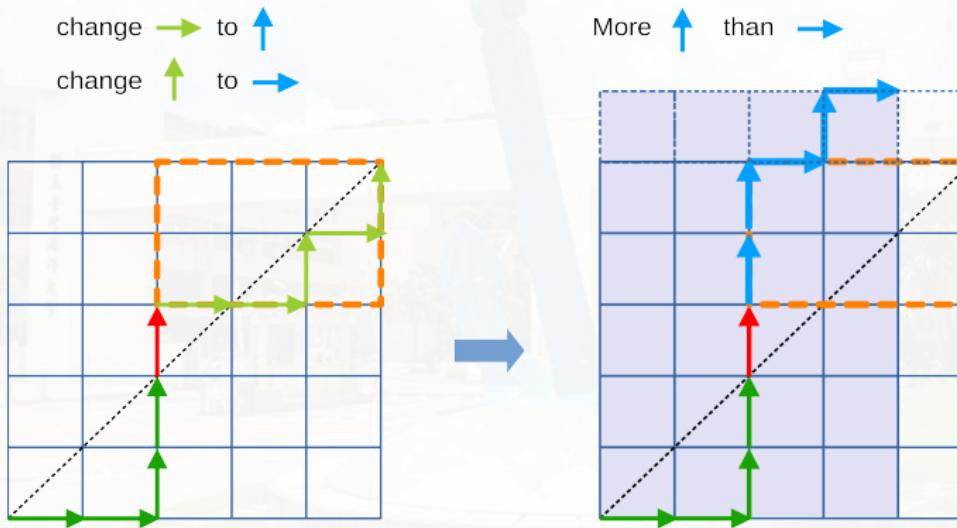
First time breaking the diagonal



Catalan number by monotonic lattice paths (3/5)



Catalan number by monotonic lattice paths (4/5)



Catalan number by monotonic lattice paths (5/5)

- The number of monotonic lattice paths not passing the diagonal is

$$\begin{aligned}\binom{2n}{n} - \binom{2n}{n-1} &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} \\&= (2n)! \left(\frac{n+1-n}{n!(n+1)!} \right) \\&= \frac{1}{n+1} \frac{(2n)!}{n!n!} \\&= \frac{1}{n+1} \binom{2n}{n}.\end{aligned}$$

Outline

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2 Selection Trees

Scenarios of Using the Selection Trees

- External sorting.
- Data stored in each queue (run) is sorted.

Winner Selection Tree

- In the following figure, computing the first winner takes



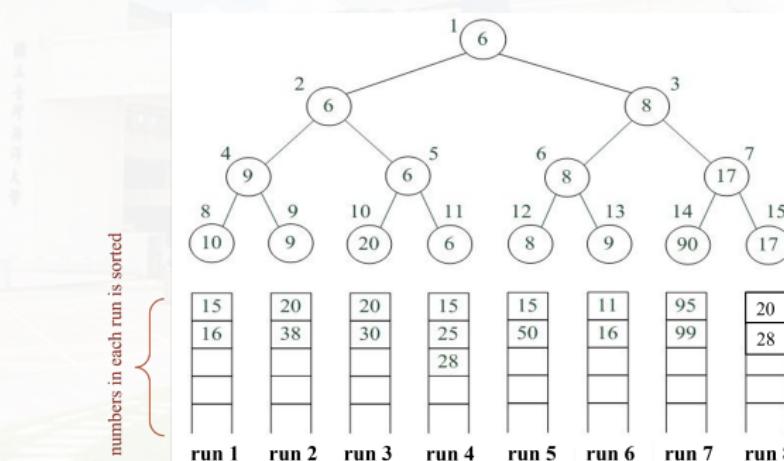
Winner Selection Tree

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Winner Selection Tree

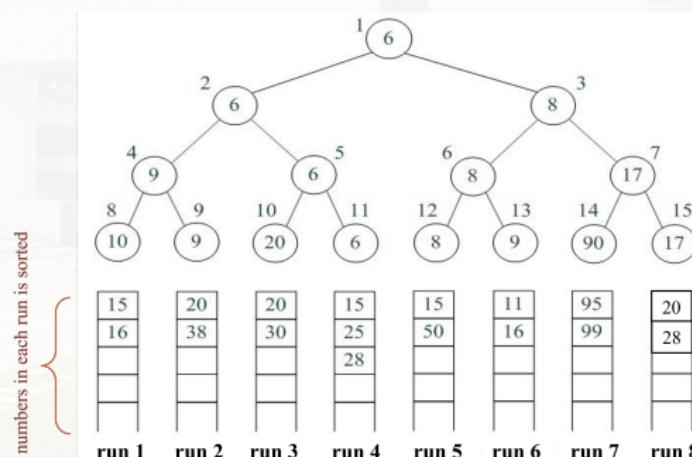
- In the following figure, computing the first winner takes $k - 1$ comparisons.
(not better?)
- But wait, how about the following iterations?



Winner Selection Tree

(Winner) Selection Tree:

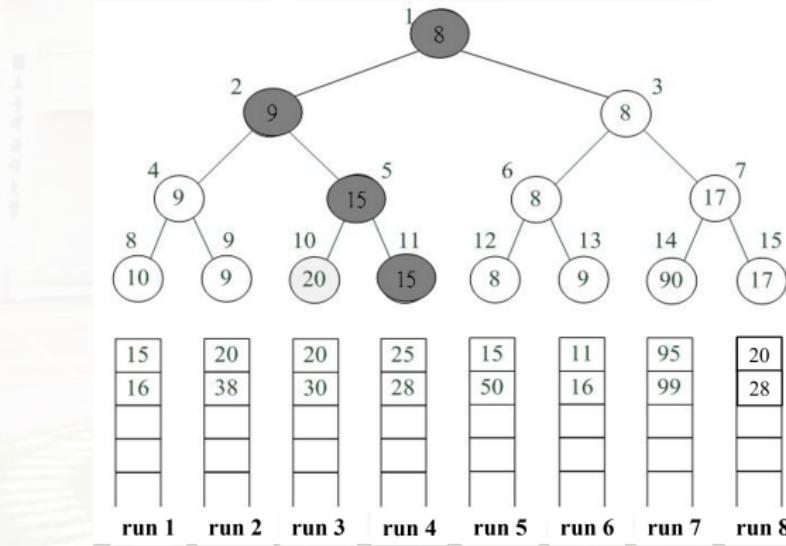
- Setting up the selection tree: $O(k)$ time.
- Restructuring: $O(\lg k)$ time.
- merging all n items: $O(n \lg k)$ time.



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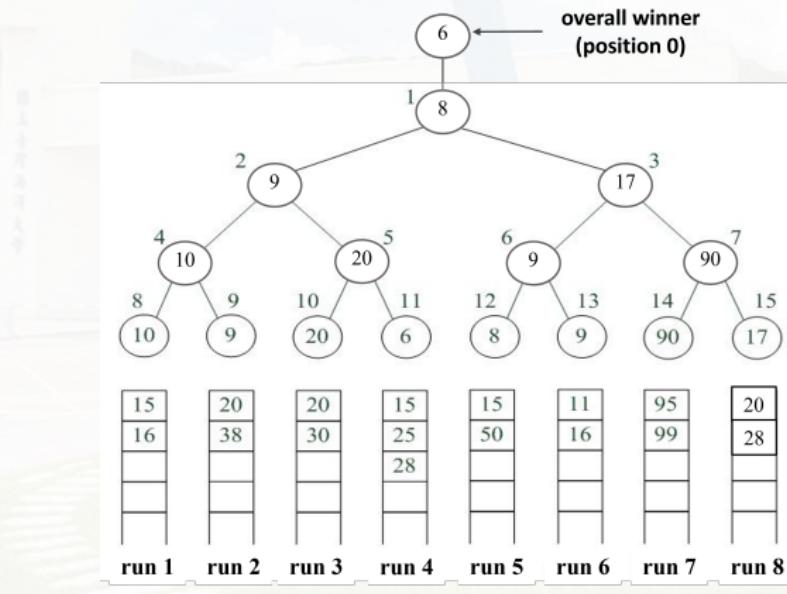
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Loser Selection Tree

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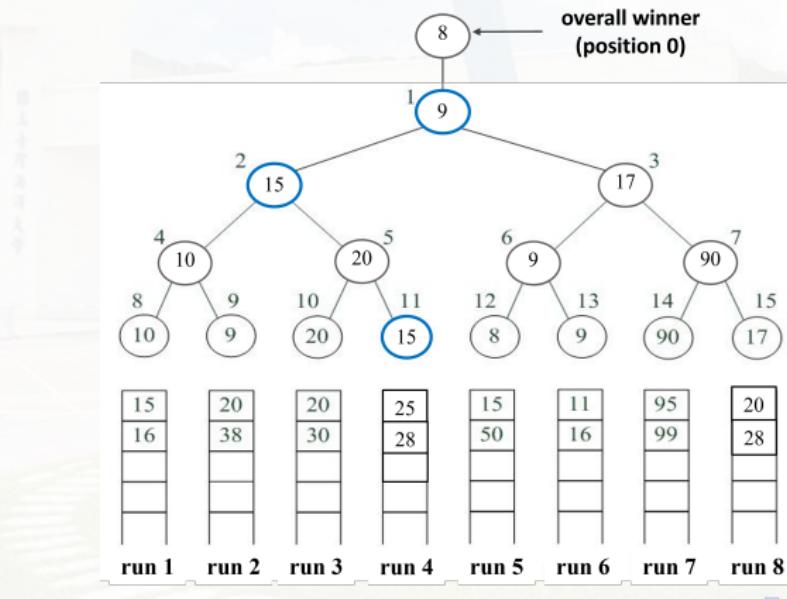
- Sibling nodes represent the losers.
- The restructuring process can be simplified.



Loser Selection Tree

Loser Tree:

- Sibling nodes represent the losers.
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Note for the Loser Selection Tree

- Comparison with the sibling is required for the first step construction.
- After the first construction, we only need to compare each node with its parent; “push” the smaller key value upward and left the “larger” key value as the **loser**.

Discussions

