Existence of Pure-Strategy Nash Equilibria in a Two-Party Policy Competition Game

Extending to the General Case

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a joint work with

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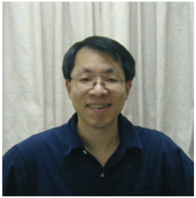


Chi-Jen Lu

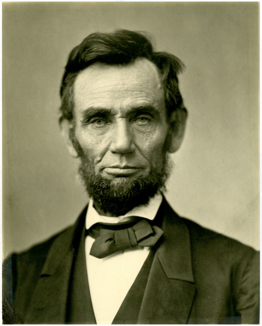
Po-An Chen

Chih-Chieh Hung

Outline

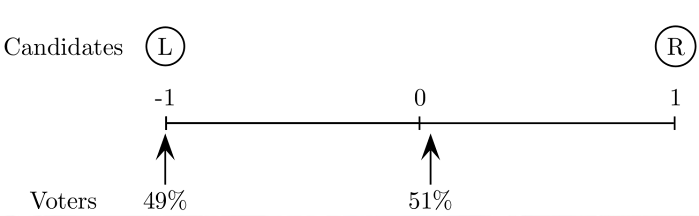
Outline

The Inspiration (an EC'17 paper)

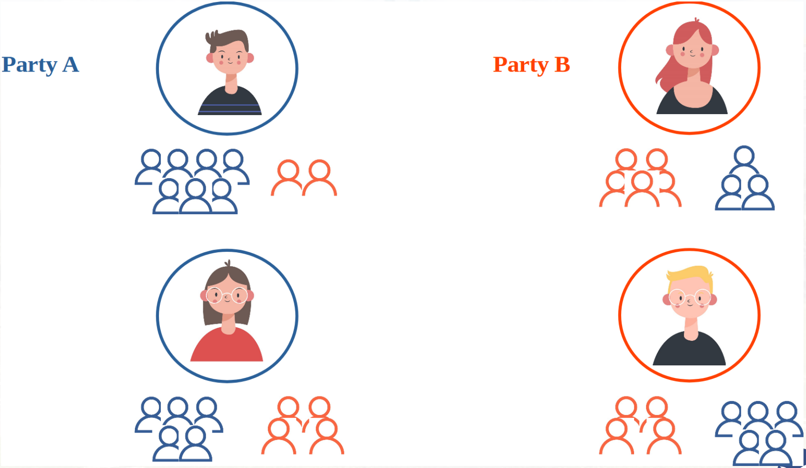


"[. . .] and that government of the people, by the people, for the people, shall not perish from the earth."
— Abraham Lincoln, 1863.

Previous Work (I): Distortion of Social Choice Rules



Previous Work (II): Two-Party Election Game



Previous Work (II): Two-Party Election Game

- Parties are players.
- Strategies: their candidates (or policies).
- A candidate beats the other candidates from other candidates of other parties with uncertainty.
- The payoff of each party: expected utility its supporters can get.

Previous Work (II): Two-Party Election Game (contd.)

- Party A: m candidates, party B: n candidates.
- Candidate A_i can bring social utility $u(A_i) = u_A(A_i) + u_B(A_i) \in [0, \beta]$ for some real $\beta \geq 0$.
- $p_{i,j}$: $\Pr[A_i \text{ wins over } B_j]$.
  - E.g., Linear: $p_{i,j} := (1 + (u(A_i) - u(B_j))/\beta)/2$
- Payoff (reward) $r_A = p_{i,j} u_A(A_i) + (1 - p_{i,j}) u_A(B_j)$.

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- Party A: m candidates, party B: n candidates.
- Candidate A_i can bring social utility $u(A_i) = u_A(A_i) + u_B(A_i) \in [0, \beta]$ for some real $\beta \geq 0$.
- $p_{i,j}$: $\Pr[A_i \text{ wins over } B_j]$. more utility for all the people, more likely to win
  - E.g., Linear: $p_{i,j} := (1 + (u(A_i) - u(B_j))/\beta)/2$
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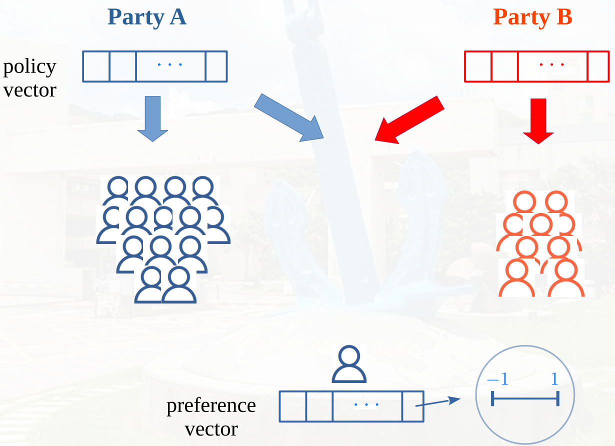
Outline

Policies and Preferences



The Setting

- Policy vectors: $\mathbf{z}_A, \mathbf{z}_B \in S \subset \mathbb{R}^k$.
    - $\|\mathbf{z}_A\| \leq 1$ and $\|\mathbf{z}_B\| \leq 1$.
    - State (or profile): $\mathbf{z} := (\mathbf{z}_A, \mathbf{z}_B)$.

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  - $\|\mathbf{z}_A\| \leq 1$ and $\|\mathbf{z}_B\| \leq 1$.
  - State (or profile): $\mathbf{z} := (\mathbf{z}_A, \mathbf{z}_B)$.
- V_A and V_B: the supporters of A and B.
  - $V := V_A \dot{\cup} V_B$, $|V| = n$.
- Preference vector of a voter $v \in V$: $\mathbf{q}_v$.
- $Q_A := \sum_{v \in V_A} \mathbf{q}_v$, $Q_B := \sum_{v \in V_B} \mathbf{q}_v$, $Q := Q_A + Q_B$, $\|Q_A\|, \|Q_B\| \leq 1$.

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- The utility

$$u_A(\mathbf{z}_A) = \sum_{v \in V_A} \mathbf{z}_A^\top \mathbf{q}_v = \mathbf{z}_A^\top Q_A,\ u_B(\mathbf{z}_A) = \sum_{v \in V_B} \mathbf{z}_A^\top \mathbf{q}_v = \mathbf{z}_A^\top Q_B.$$

$$u_A(\mathbf{z}_B) = \sum_{v \in V_A} \mathbf{z}_B^\top \mathbf{q}_v = \mathbf{z}_B^\top Q_A,\ u_B(\mathbf{z}_B) = \sum_{v \in V_B} \mathbf{z}_B^\top \mathbf{q}_v = \mathbf{z}_B^\top Q_B.$$

The Setting (Winning Prob. & Payoff)

- Winning probability:

$$
\begin{aligned}
p_{A\succ B} &= \frac{1}{2} + \frac{1}{4}(\mathbf{z}_A - \mathbf{z}_B)^\top Q, \\
p_{B\succ A} &= \frac{1}{2} + \frac{1}{4}(\mathbf{z}_B - \mathbf{z}_A)^\top Q.
\end{aligned}
$$

  - 1/4: a normalization factor.

- The payoffs:

$$
\begin{aligned}
R_A(\mathbf{z}) &= p_{A\succ B} \cdot \mathbf{z}_A^\top Q_A + p_{B\succ A} \cdot \mathbf{z}_B^\top Q_A, \\
R_B(\mathbf{z}) &= p_{B\succ A} \cdot \mathbf{z}_B^\top Q_B + p_{A\succ B} \cdot \mathbf{z}_A^\top Q_B.
\end{aligned}
$$

So, we can compute the gradients and Hessian...

$$\frac{\partial R_A(\mathbf{z})}{\partial \mathbf{z}_A} = \frac{1}{2}Q_A + \frac{(\mathbf{z}_A - \mathbf{z}_B)^\top Q}{4kn}Q_A + \frac{(\mathbf{z}_A - \mathbf{z}_B)^\top Q_A}{4kn}Q.$$

$$\frac{\partial R_B(\mathbf{z})}{\partial \mathbf{z}_B} = \frac{1}{2}Q_B + \frac{(\mathbf{z}_B - \mathbf{z}_A)^\top Q}{4kn}Q_B + \frac{(\mathbf{z}_B - \mathbf{z}_A)^\top Q_B}{4kn}Q.$$

$$\frac{\partial^2 R_A(\mathbf{z})}{\partial \mathbf{z}_A^2}[i,j] = \frac{1}{4}\left(Q[i]Q_A[j] + Q[j]Q_A[i]\right),$$

$$\frac{\partial^2 R_B(\mathbf{z})}{\partial \mathbf{z}_B^2}[i,j] = \frac{1}{4}\left(Q[i]Q_B[j] + Q[j]Q_B[i]\right).$$

Outline

Previous Contributions

[Nash 1950]

Every FINITE game has a mixed-strategy Nash equilibrium.

Our Contribution

In this work, we show that there exists a pure-strategy Nash equilibrium (PSNE) in the two-party policy competition game for

- the degenerate case: $k = 1$
- the general case $k \geq 1$ under the consensus-reachable condition

- The two-party policy competition game is NOT a finite game.
- The above PSNE consists of dominant-strategies.

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In this work, we show that there exists a pure-strategy Nash equilibrium (PSNE) in the two-party policy competition game for

- the degenerate case: $k = 1$
- the general case $k \geq 1$ under the consensus-reachable condition
- the general case $k \geq 1$ for **non-consensus-reachable condition yet under a mild assumption.**

- The two-party policy competition game is NOT a finite game.
- The above PSNE consists of dominant-strategies.

Claim of the Egoistic Property

Claim

The egoistic property must hold in the two-party policy competition game.

- $\mathbf{z}_A^\top Q_A \geq \mathbf{z}_B^\top Q_A$ and $\mathbf{z}_B^\top Q_B \geq \mathbf{z}_A^\top Q_B$.

The General Case: $k \geq 1$ — Simplification by Polar Coordinates

- It is sufficient for party A and B to consider the space $\mathrm{span}(\{Q_A, Q_B\})$.
- Represent $\mathbf{z}_A$ (resp., $\mathbf{z}_B$) in terms of polar coordinates (r_A, θ_A) (resp., (r_A, θ_B)).
  - $r_A = \|\mathbf{z}_A\|, r_B = \|\mathbf{z}_B\|$
  - θ_A (resp., θ_B) is the angle b/w Q_A and $\mathbf{z}_A$ (resp., Q_B and $\mathbf{z}_B$).

For any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^k$, $k \geq 1$,

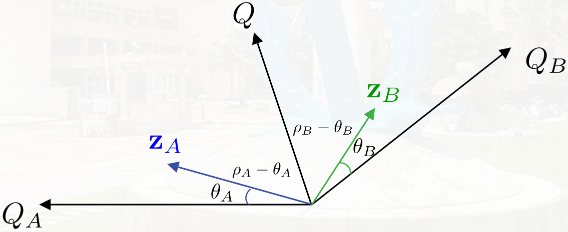
$$\langle \mathbf{u}, \mathbf{v} \rangle = \|\mathbf{u}\| \|\mathbf{v}\| \cos(\theta),$$

where θ is the angle b/w $\mathbf{u}$ and $\mathbf{v}$.

A Good Condition

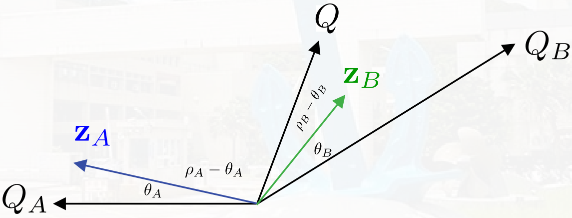
Consensus-Reachable

A two-party policy competition game is *consensus-reachable* if $Q_A^\top Q \geq 0$ and $Q_B^\top Q \geq 0$.



An Example of Not Consensus-Reachable

$\rho_A > \pi/2$.



About the Norms r_A, r_B

- A mild assumption: $\theta_A, \theta_B \leq \pi/2$.

Lemma

For the nonconsensus-reachable case where $\rho_A > \pi/2$ and $\theta_A, \theta_B \in [0, \pi/2]$, the best responses of the two players always set $r_A = r_B = 1$.

Sketch of the proof:

- Show that $\frac{\partial R_A(\mathbf{r},\boldsymbol{\theta})}{\partial r_A}\Big|_{\theta_A=\rho_A-\pi/2} \geq 0$.
- For $\theta \in [0, \rho_A - \theta_A]$, we can show that $\frac{\partial^2 R_A(\mathbf{r},\boldsymbol{\theta})}{\partial r_A^2} \leq 0$.
- Hence, it follows that $\frac{\partial R_A(\mathbf{r},\boldsymbol{\theta})}{\partial r_A} \geq 0$ for $\theta_A \in [0, \rho_A - \pi/2]$.
- Combining $\frac{\partial^2 R_A(\mathbf{r},\boldsymbol{\theta})}{\partial r_A^2} \geq 0$ for $\theta_A \in [\rho_A - \pi/2, \pi/2]$, we have $\frac{\partial R_A(\mathbf{r},\boldsymbol{\theta})}{\partial r_A} \geq 0$ for $\theta_A \in [0, \pi/2]$.

About the angles: θ_A, θ_B

- Set $x := \cos(\theta_A)$ and $y := \cos(\theta_B)$.
- Let $f(x) := R_A(r_A = 1, \theta_A)$ and $g(y) := R_B(r_B = 1, \theta_B)$, $x, y \in [0, 1]$.

$$
\begin{aligned}
f(x) = &\left(\frac{1}{2} + D_0\left(C_1\, x + \sqrt{1 - C_1^2}\,\sqrt{1 - x^2} - C_3\right)\right) D_1\, x \\
&+ \left(\frac{1}{2} - D_0\left(C_1\, x + \sqrt{1 - C_1^2}\,\sqrt{1 - x^2} - C_3\right)\right) D_1\, C_4, \\
g(y) = &\left(\frac{1}{2} + D_0\left(C_2\, y + \sqrt{1 - C_2^2}\,\sqrt{1 - y^2} - C_3'\right)\right) D_2\, y \\
&+ \left(\frac{1}{2} - D_0\left(C_2\, y + \sqrt{1 - C_2^2}\,\sqrt{1 - y^2} - C_3'\right)\right) D_2\, C_4',
\end{aligned}
$$

$$
\begin{aligned}
\text{subject to:}\quad & 0 \le D_0 \le \tfrac{1}{2},\ 0 \le D_1 \le 1,\ -1 \le C_1 < 0,\ 0 \le C_2 \le 1, \\
& 0 \le C_3 \le 1,\ -1 \le C_4 < 0,\ -1 \le C_3' \le 1,\ -1 \le C_4' \le 1, \\
& C_4 \le C_1,\ 0 \le D_2 \le 1,\ D_0 \le D_2 \text{ and } C_4' \le C_2,
\end{aligned}
$$

where $D_0 := \|Q\|/4$, $D_1 := \|Q_A\|$, $D_2 := \|Q_B\|$, $C_1 := \cos\rho_A$, $C_2 := \cos\rho_B$, $C_3 := \cos(\rho_B - \theta_B)$, $C_3' := \cos(\rho_A - \theta_A)$, $C_4 := \cos(\rho_A + \rho_B - \theta_B)$, $C_4' := \cos(\rho_A + \rho_B - \theta_A)$.

About the angles: θ_A, θ_B (contd.)

Lemma

$f(x)$ is concave and $g(y)$ is unimodal (quasi-concave).

- Therefore, Kakutani's Fixed-Point Theorem can be applied to guarantee the existence of a PSNE of the game even when it is NOT consensus-reachable.

Theorem

Under the mild condition that $\mathbf{z}_A^\top Q_A \geq 0$ and $\mathbf{z}_B^\top Q_B \geq 0$, the two-party policy competition game has at least one PSNE even when the game is not consensus-reachable.

Outline

Future and Ongoing Work (1/3)

Monotone Game

A pseudo-gradient mapping of the game

$$F : \mathcal{D} \subseteq \mathbb{R}^2 \to \mathbb{R}^2$$

is said to be *monotone* if for all $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$ in the domain one has

$$(F(\mathbf{u}) - F(\mathbf{v}))^\top (\mathbf{u} - \mathbf{v}) \geq 0.$$

Future and Ongoing Work (2/3)

Cocoercivity

A pseudo-gradient mapping of the game

$$F : \mathcal{D} \subseteq \mathbb{R}^2 \to \mathbb{R}^2$$

is said to be λ–*cocoercive* (for some $\lambda > 0$) if for all $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$ in the domain one has

$$(F(\mathbf{u}) - F(\mathbf{v}))^\top(\mathbf{u} - \mathbf{v}) \geq \lambda \left\| F(\mathbf{u}) - F(\mathbf{v}) \right\|^2.$$

Future and Ongoing Work (3/3)

Counterexample of Monotonicity

The two-party policy competition game is NOT monotone in general, and hence not cocoercive for any $\lambda \leq 1$.

Theorem

The two-party policy competition game is λ-cocoercive for $\lambda = 1/\|Q_A\|^2$ if it is voter-symmetric.

- Voter-symmetric: $Q_A = Q_B$.
- E.g., Pre-Election within-in the party.

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- Voter-symmetric: $Q_A = Q_B$.
- E.g., Pre-Election within-in the party.
- One can apply gradient-based algorithms to find a PSNE with convergence rate $O(1/T)$ in this case.

Thanks for your attention!

Q & A