

# Shortest Paths

## Dijkstra's Algorithm & Bellman-Ford Algorithm

Joseph Chuang-Chieh Lin (林莊傑)

Department of Computer Science & Engineering,  
National Taiwan Ocean University

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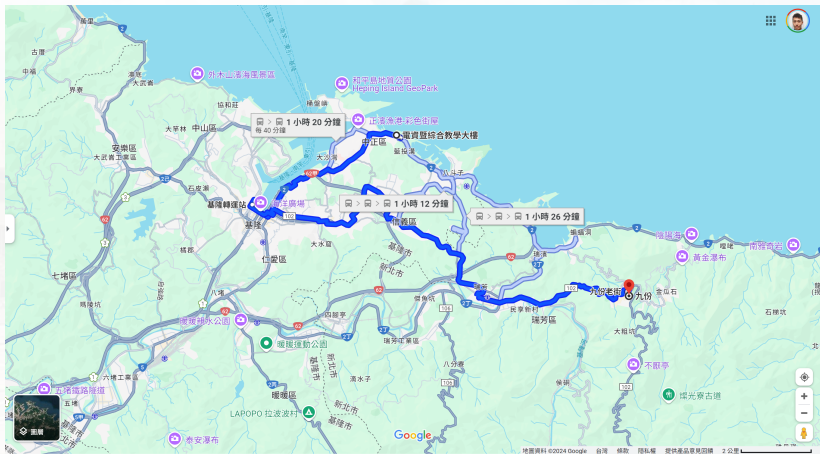


# Outline

- 1 Introduction
- 2 Dijkstra's Algorithm
- 3 Bellman-Ford Algorithm for General Weights



## Shortest path(s) from NTOU to Jiufen Old Street.



# Shortest Paths

- Model the problem via a graph.
- vertices  $\mapsto$  locations (e.g., stations, restaurants, gas stations, etc.)
  - Including the **source** and the **destination**.
- edges  $\mapsto$  highways, railways, roads, etc.
  - edge **weight**: tolls, the distance, passing-through time, etc.



# Shortest Paths

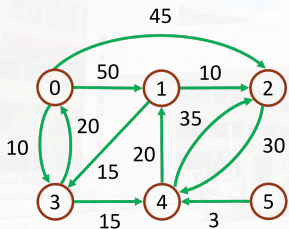
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## Questions

- Is there a path from NTOU to Jiufen?
- If it exists, which one is the **shortest**?



# Single Source/All Destinations (Nonnegative Edge Costs)



	path	length (cost)
1	0, 3	10
2	0, 3, 4	25
3	0, 3, 4, 1	45
4	0, 2	45

Notations:

- A directed graph  $G = (V, E)$ ; a weight function  $w(e)$ ,  $w(e) > 0$  for any edge  $e \in E$ .
- $v_0$ : source vertex.
- If  $(v_i, v_j) \notin E$ ,  $w(v_i, v_j) = \infty$ .



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- Let  $S$  denote the set of vertices, including  $v_0$ , whose shortest paths have been found.
- For  $v \notin S$ , let  $\text{dist}[v]$  be the length of the shortest path starting from  $v_0$ , going through vertices ONLY in  $S$ , and ending in  $v$ .

# Dijkstra's Algorithm

- At the first stage, we add  $v_0$  to  $S$ , set  $\text{dist}[v_0] = 0$  and determine  $\text{dist}[v]$  for each  $v \notin S$ .



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- Adding  $w$  to  $S$ , and updating  $\text{dist}[v]$  for  $v$ , where  $v \notin S$  currently.



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- Repeat the vertex addition process until  $S = V(G)$



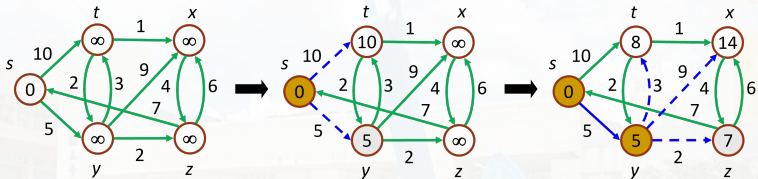
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- Adding  $w$  to  $S$ , and updating  $\text{dist}[v]$  for  $v$ , where  $v \notin S$  currently.
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Time complexity:  $O(n^2)$ .

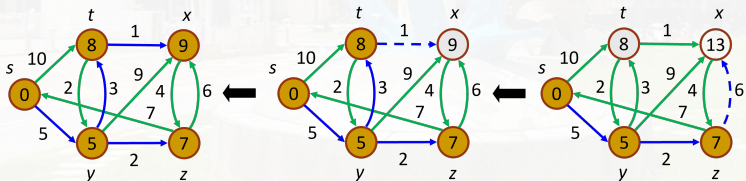


# Illustration of Dijkstra's Algorithm



During each iteration:

1. Update the distance of the rest vertices
2. Pick the vertex with the smallest distance value

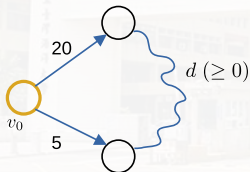




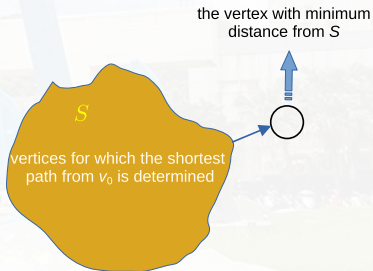
# Basic Idea of Dijkstra's Algorithm

- Induction on  $n$ .

base case



$$20 + d \geq 5 \quad (\because \text{triangular inequality})$$



# The Pseudo-code of Dijkstra's Algorithm

```
S = { v0 };  
dist[v0] = 0;  
for each v in V - {v0} do  
    dist[v] = e(v0,v); // initialization  
while (S != V) do  
    choose a vertex w in V - S such that dist[w] is a minimum;  
    add w to S; // the other nodes in S have been utilized!  
    for each v in V - S do  
        dist[v] = min(dist[v], dist[w]+e(w, v));  
    endfor  
endwhile
```



# Dijkstra's Algorithm (Functions (1/2))

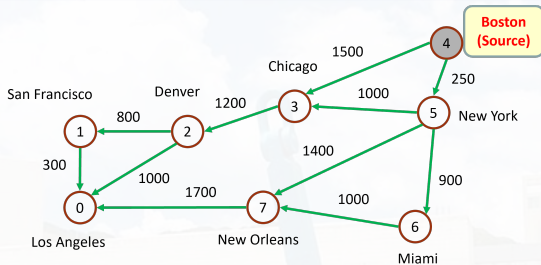
```
void shortestPath (int v, int cost[][MAX_VERTICES],
                  int distance[], int n, bool found[]) {
    /* distance[i]: the shortest path from vertex v to i
       found[i]: 0 if the shortest path from vertex i has not
       been found and a 1 otherwise
       cost: the adjacency matrix */
    int i, u, w;
    for (i=0; i<n; i++) {
        found[i] = false; distance[i] = cost[v][i];
    }
    found[v] = true; //initialization
    distance[v] = 0; //initialization
    for (i=0; i<n-1; i++) {
        u = choose(distance, n, found);
        found[u] = true;
        for (w=0; w<n; w++)
            if (!found[w])
                if (distance[u] + cost[u][w] < distance[w])
                    distance[w] = distance[u]+cost[u][w];
    }
}
```



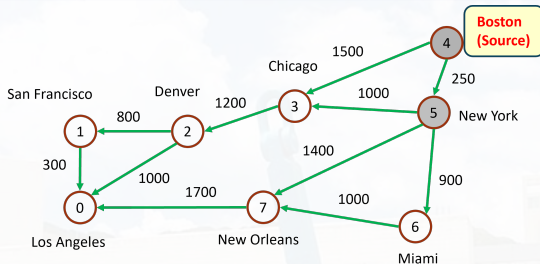
# Dijkstra's Algorithm (Functions (2/2))

```
int choose (int distance[], int n, short int found[]) {  
    /* find the smallest distance not yet checked */  
    int i, min, min_pos;  
    min = INT_MAX;  
    min_pos = -1;  
    for (i=0; i<n; i++)  
        if (distance[i] < min && !found[i]) {  
            min = distance[i];  
            min_pos = i;  
        }  
    return min_pos;  
}
```

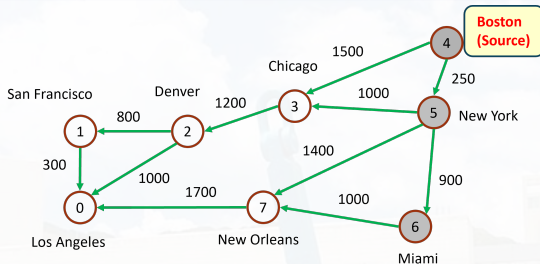




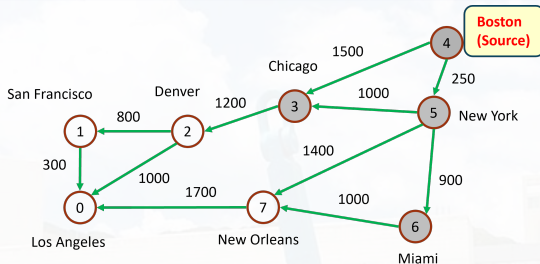
Iteration	Vertex Select.	Distance							
		LA	SF	DEN	CHI	BOS	NY	MIA	NO
		[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
initial	—	$\infty$	$\infty$	$\infty$	1500	0	250	$\infty$	$\infty$



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1	5	$\infty$	$\infty$	$\infty$	1250	0	250	1150	1650

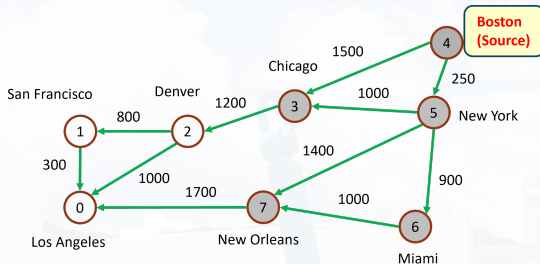


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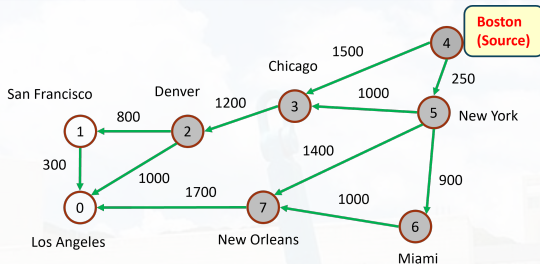


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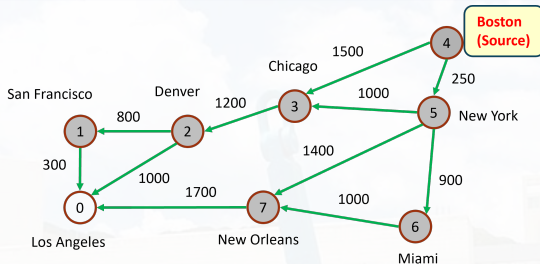




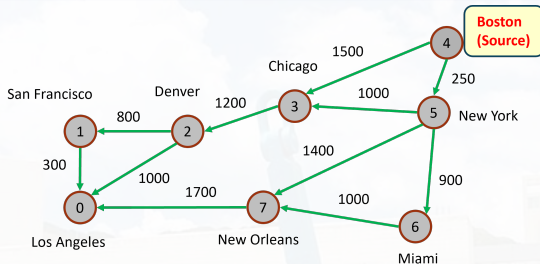
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# Single Source/All Destinations: General Weights

- **Focus:** Some edges of the directed graph  $G$  have negative length.



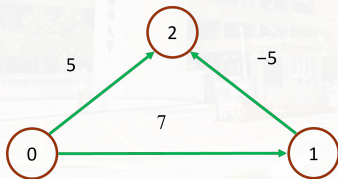
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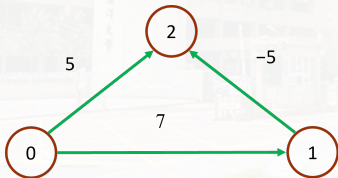


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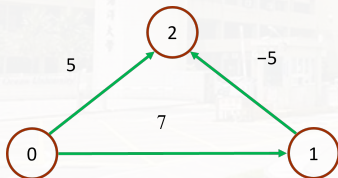
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- The shortest path from 0 to 2 is:  
 $0 \rightarrow 1 \rightarrow 2$  (length = 2).

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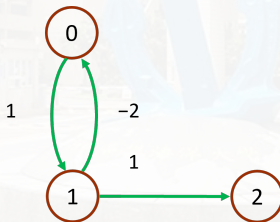
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- The shortest path from 0 to 2 is:  
 $0 \rightarrow 1 \rightarrow 2$  (length = 2).
- Dijkstra's “greedy” approach does not work here.

## Workaround Solution: NO negative cycle is permitted!

- When negative edge lengths are permitted, we require that the graph have no cycles of negative length.
- This is necessary so as to ensure that shortest paths consist of a **finite** number of edges.



# Observations

- When there are NO cycles of negative length, there is a shortest path between any two vertices of an  $n$ -vertex graph that has  $\leq n - 1$  edges on it.



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  - Otherwise, the path must repeat at least one vertex, and hence must contain a cycle.
- So, eliminating the cycles from the path results in another path with **the same source and destination**.
  - The length of the new path should be no more than that of the original.



# Dynamic Programming Approach

$\text{dist}^k[u]$ : the length of a shortest path from the source  $v$  to  $u$  under the constraint that **the shortest path contains  $\leq k$  edges**.

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  - The goal: **Compute  $\text{dist}^{n-1}[u]$**  for all  $u$ .
- ▷ Using **Dynamic Programming**.

# Bellman-Ford Algorithm (Sketch)

- If the shortest path from  $v$  to  $u$  with  $\leq k$  edges ( $k > 1$ ) has no more than  $k - 1$  edges, then  $\text{dist}^k[u] = \text{dist}^{k-1}[u]$ .

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- If the shortest path from  $v$  to  $u$  with  $\leq k$  edges ( $k > 1$ ) has exactly  $k$  edges, there exists a vertex  $i$  such that  $\text{dist}^{k-1}[i] + \text{length}[i][u]$  is minimum.
- The recurrence relation:

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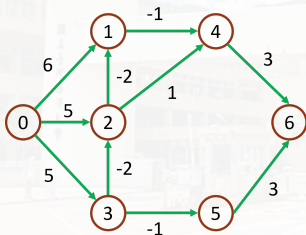
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- The recurrence relation:

$$\text{dist}^k[u] = \min\{\text{dist}^{k-1}[u], \min_i\{\text{dist}^{k-1}[i] + \text{length}[i][u]\}\}.$$



# Shortest paths with negative edge lengths (cost)

$$\text{dist}^k[u] = \min\{\text{dist}^{k-1}[u], \min_i\{\text{dist}^{k-1}[i] + \text{length}[i][u]\}\}.$$



(a) A directed graph

k	$\text{dist}^k[u]$						
	0	1	2	3	4	5	6
1	0	6	5	5	$\infty$	$\infty$	$\infty$
2	0	3	3	5	5	4	$\infty$
3	0	1	3	5	2	<b>4</b>	<b>7</b>
4	<b>0</b>	<b>1</b>	<b>3</b>	<b>5</b>	<b>0</b>	<b>4</b>	<b>5</b>
5	<b>0</b>	<b>1</b>	<b>3</b>	<b>5</b>	<b>0</b>	<b>4</b>	<b>3</b>
6	<b>0</b>	<b>1</b>	<b>3</b>	<b>5</b>	<b>0</b>	<b>4</b>	<b>3</b>

(b)  $\text{dist}^k$

# Bellman-Ford Algorithm (Pseudo-Code)

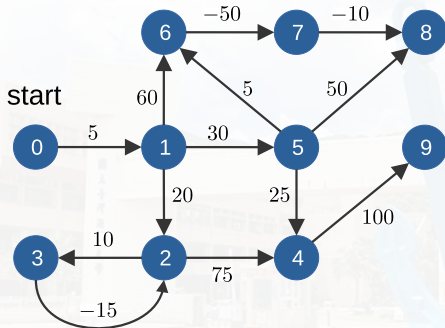
```

BF(int u) { // assume that the source is vertex u
    for each vertex w in V - {u}, set dist[w] = INT_MAX
    set dist[u] = 0
    for (i=0; i<n-1; i++) { // n: the number of vertices (k)
        for each edge (p,q) in the graph {
            // we can choose p with dist[p] < INT_MAX
            if (dist[p] + length[p][q] < dist[q])
                dist[q] = dist[p] + length[p][q]
        }
    }
    // Now the distances from u to every other vertex is found.
    // Repeat the following to find nodes in a negative cycle
    for (i=0; i<n-1; i++) {
        for each edge (p,q) in the graph {
            if (dist[p] + length[p][q] < dist[q])
                dist[q] = -INT_MAX
        }
    }
}

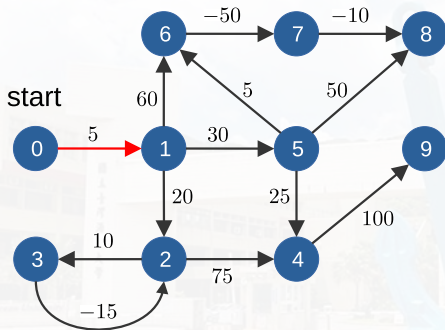
```



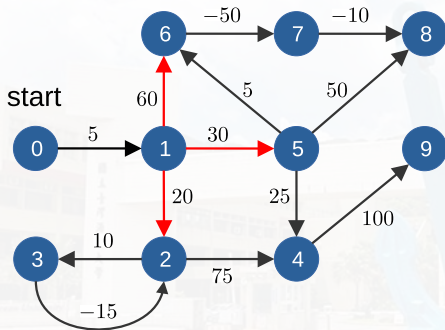




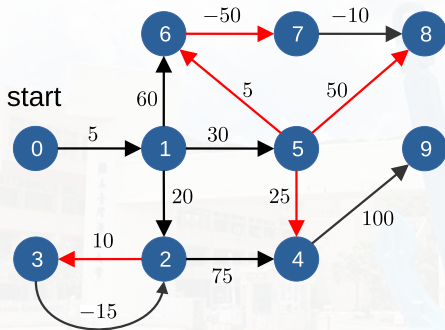
0	0
1	$\infty$
2	$\infty$
3	$\infty$
4	$\infty$
5	$\infty$
6	$\infty$
7	$\infty$
8	$\infty$
9	$\infty$


 $i = 0$ 

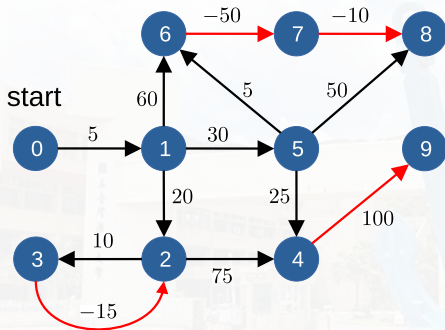
0	0
1	5
2	$\infty$
3	$\infty$
4	$\infty$
5	$\infty$
6	$\infty$
7	$\infty$
8	$\infty$
9	$\infty$



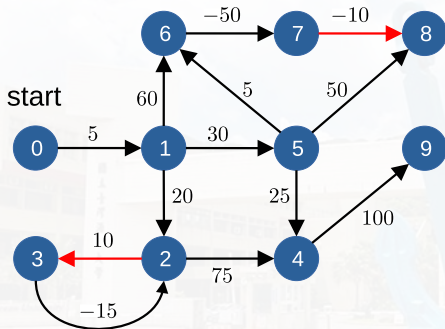
0	0
1	5
2	25
3	$\infty$
4	$\infty$
5	35
6	65
7	$\infty$
8	$\infty$
9	$\infty$

 $i = 2$ 

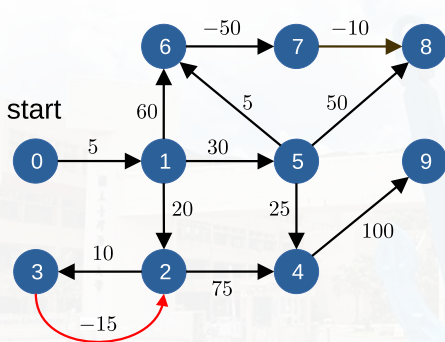
0	0
1	5
2	25
3	35
4	60
5	35
6	40
7	15
8	85
9	$\infty$

 $i = 3$ 

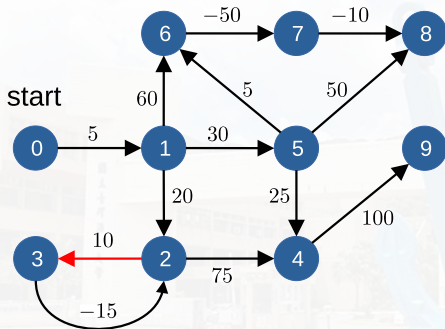
0	0
1	5
2	20
3	35
4	60
5	35
6	40
7	-10
8	5
9	160


 $i = 4$ 

0	0
1	5
2	20
3	30
4	60
5	35
6	40
7	-10
8	-20
9	160

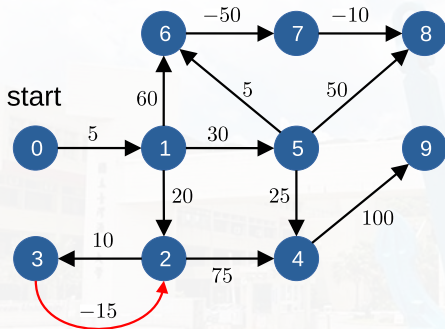

 $i = 5$ 

0	0
1	5
2	5
3	30
4	60
5	35
6	40
7	-10
8	-20
9	160

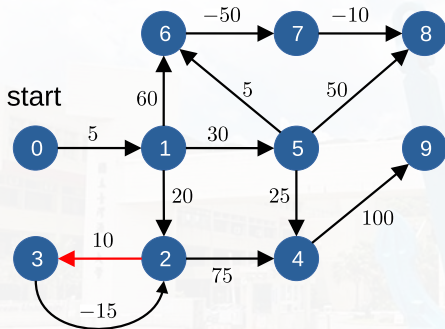

 $i = 6$ 

0	0
1	5
2	5
3	15
4	60
5	35
6	40
7	-10
8	-20
9	160

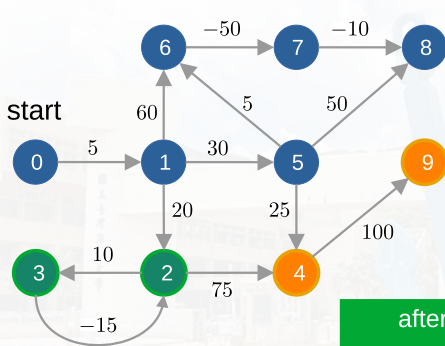



 $i = 7$ 

0	0
1	5
2	-10
3	15
4	60
5	35
6	40
7	-10
8	-20
9	160


 $i = 8$ 

0	0
1	5
2	-10
3	0
4	60
5	35
6	40
7	-10
8	-20
9	160



after negative cycle  
detection

0	0
1	5
2	$-\infty$
3	$-\infty$
4	$-\infty$
5	35
6	40
7	-10
8	-20
9	$-\infty$

# Discussions

