Simple Near-Optimal Auctions

Joseph Chuang-Chieh Lin

Department of Computer Science & Engineering,
National Taiwan Ocean University

Fall 2024

Outline

1. Background & Introduction
2. The Prophet Inequality
3. Simple Single-Item Auctions
4. Prior-Independent Mechanisms

Outline

1. Background & Introduction
2. The Prophet Inequality
3. Simple Single-Item Auctions
4. Prior-Independent Mechanisms

What we have learned

- For a single-parameter environment where agents' valuations are drawn independently from **regular** distributions, the corresponding virtual welfare maximizer maximizes the expected revenue over all **DSIC** mechanisms.
  - The allocation rule:
    $$\mathbf{x}(\mathbf{v}) = \arg\max_{X} \sum_{i=1}^{n} \varphi_i(v_i) x_v(\mathbf{v})$$
    for each valuation profile $\mathbf{v}$, where
    $$\varphi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

What we have learned

- For a single-parameter environment where agents' valuations are drawn independently from **regular** distributions, the corresponding virtual welfare maximizer maximizes the expected revenue over all **DSIC** mechanisms.
    - The allocation rule:

$$\boldsymbol{x}(\boldsymbol{v}) = \arg\max_{X} \sum_{i=1}^{n} \varphi_i(v_i) x_v(\boldsymbol{v})$$

      for each valuation profile $\boldsymbol{v}$, where

$$\varphi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

- For i.i.d. & regular F_i's, the optimal single-item auction is surprisingly simple:

What we have learned

- For a single-parameter environment where agents' valuations are drawn independently from **regular** distributions, the corresponding virtual welfare maximizer maximizes the expected revenue over all **DSIC** mechanisms.
  - The allocation rule:

$$\boldsymbol{x}(\boldsymbol{v}) = \arg\max_{X} \sum_{i=1}^{n} \varphi_i(v_i) x_v(\boldsymbol{v})$$

  for each valuation profile $\boldsymbol{v}$, where

$$\varphi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

- For i.i.d. & regular F_i's, the optimal single-item auction is surprisingly simple:
  - a second-price auction augmented with the reserve price $\varphi^{-1}(0)$.

Optimal Auctions Can Be Complex

- What if bidders' valuations are drawn from different regular distributions?

Optimal Auctions Can Be Complex

- What if bidders' valuations are drawn from different regular distributions?
- We would like to know if there is any simple and practical single-item auction formats that are at least approximately optimal.

Outline

Game with n stages (resembling the secretary problem)

- Consider the following game with n stages.
- In stage i, we are offered a nonnegative prize π_i, drawn from a distribution G_i.
- We know the *independent* distributions $G_1, \ldots, G_n$ *in advance*.
- We know the realization π_i only at stage i.

Game with n stages (resembling the secretary problem)

- Consider the following game with n stages.
- In stage i, we are offered a nonnegative prize π_i, drawn from a distribution G_i.
- We know the *independent* distributions $G_1, \ldots, G_n$ *in advance*.
- We know the realization π_i only at stage i.

⋆ After seeing π_i, we can
  - either accept the prize and end the game, or
  - discard the prize, and then proceed to the next stage.

Game with n stages (resembling the secretary problem)

- Consider the following game with n stages.
- In stage i, we are offered a nonnegative prize π_i, drawn from a distribution G_i.
- We know the *independent* distributions $G_1, \ldots, G_n$ *in advance*.
- We know the realization π_i only at stage i.

⋆ After seeing π_i, we can
  - either accept the prize and end the game, or
  - discard the prize, and then proceed to the next stage.

- What's the risk and difficulty?

The Prophet Inequality

- It offers a simple strategy that performs almost as well as (approximately) a fully clairvoyant prophet.

Theorem (Prophet Inequality)

For every sequence $G_1, \ldots, G_n$ of n independent distributions,

- There is a strategy that guarantees expected reward $\geq \frac{1}{2}\mathbf{E}_{\boldsymbol{\pi} \sim \boldsymbol{G}}[\max_i \pi_i]$.
- There is such a threshold strategy, which accepts prize i if and only if $\pi_i \geq t$.

- $z^+ := \max\{z, 0\}$.
- $[n] := \{1, 2, \ldots, n\}$.

Proof of Prophet Inequality (1/3)

- Compare the expected payoff of a threshold strategy with that of a prophet, through lower and upper bounds.
- $q(t)$: the prob. that the threshold strategy accepts no prize at all.
- First, we want to have a lower bound on

$$\psi := \mathbf{E}_{\boldsymbol{\pi} \sim \boldsymbol{G}}[\text{payoff of the } t\text{-threshold strategy}].$$

Proof of Prophet Inequality (1/3)

- Compare the expected payoff of a threshold strategy with that of a prophet, through lower and upper bounds.
- $q(t)$: the prob. that the threshold strategy accepts no prize at all.
- First, we want to have a lower bound on

$$\psi := \mathbf{E}_{\boldsymbol{\pi} \sim \boldsymbol{G}}[\text{payoff of the } t\text{-threshold strategy}].$$

- The payoff: 0 with prob. $q(t)$ and $\geq t$ with prob. $1 - q(t)$.

Proof of Prophet Inequality (1/3)

- Compare the expected payoff of a threshold strategy with that of a prophet, through lower and upper bounds.
- $q(t)$: the prob. that the threshold strategy accepts no prize at all.
- First, we want to have a lower bound on

$$\psi := \mathbf{E}_{\boldsymbol{\pi}\sim\boldsymbol{G}}[\text{payoff of the } t\text{-threshold strategy}].$$

- The payoff: 0 with prob. $q(t)$ and $\geq t$ with prob. $1 - q(t)$.
- Credit the baseline t with "extra credit" of $\pi_i - t$ if only one prize satisfies $\pi_i \geq t$.
- Only credit the baseline t for two or more prizes $\geq t$ ($\because$ LB).

Proof of Prophet Inequality (2/3)

$$\begin{aligned}
\psi &\geq (1-q(t))t + \\
&\quad \sum_{i=1}^{n} \mathbf{E}_{\boldsymbol{\pi}}[\pi_i - t \mid \pi_i \geq t, \pi_j < t\ \forall j \neq i] \cdot \Pr[\pi_i \geq t] \cdot \Pr[\pi_j < t\ \forall j \neq i] \\
&= (1-q(t))t + \sum_{i=1}^{n} \underbrace{\mathbf{E}_{\boldsymbol{\pi}}[\pi_i - t \mid \pi_i \geq t] \cdot \Pr[\pi_i \geq t]}_{= \mathbf{E}_{\boldsymbol{\pi}}[(\pi_i - t)^+]} \cdot \underbrace{\Pr[\pi_j < t\ \forall j \neq i]}_{\geq q(t) = \Pr[\pi_j < t\, \forall j]} \\
&\geq (1-q(t))t + q(t) \cdot \sum_{i=1}^{n} \mathbf{E}_{\boldsymbol{\pi}}[(\pi_i - t)^+]
\end{aligned}$$

Proof of Prophet Inequality (3/3)

Moreover, as to the upper bound on the prophet's expected payoff:

$$\begin{aligned}\psi^* := \mathbf{E}_{\boldsymbol{\pi}}\left[\max_{i\in[n]} \pi_i\right] &= \mathbf{E}_{\boldsymbol{\pi}}\left[t + \max_{i\in[n]}(\pi_i - t)\right] \\ &\le t + \mathbf{E}_{\boldsymbol{\pi}}\left[\max_{i\in[n]}(\pi_i - t)^+\right] \\ &\le t + \sum_{i=1}^{n} \mathbf{E}_{\boldsymbol{\pi}}[(\pi_i - t)^+].\end{aligned}$$

- Set t such that $q(t) = \frac{1}{2}$ we can complete the proof.
  $\frac{t}{2} + \frac{1}{2}\cdot\sum_{i=1}^{n} \mathbf{E}_{\boldsymbol{\pi}}[(\pi_i - t)^+] \le \psi \le \psi^* \le t + \sum_{i=1}^{n} \mathbf{E}_{\boldsymbol{\pi}}[(\pi_i - t)^+]$

Proof of Prophet Inequality (3/3)

Moreover, as to the upper bound on the prophet's expected payoff:

$$
\begin{aligned}
\psi^* := \mathbf{E}_{\boldsymbol{\pi}}\left[\max_{i\in[n]} \pi_i\right] &= \mathbf{E}_{\boldsymbol{\pi}}\left[t + \max_{i\in[n]}(\pi_i - t)\right] \\
&\le t + \mathbf{E}_{\boldsymbol{\pi}}\left[\max_{i\in[n]}(\pi_i - t)^+\right] \\
&\le t + \sum_{i=1}^{n} \mathbf{E}_{\boldsymbol{\pi}}[(\pi_i - t)^+].
\end{aligned}
$$

- Set t such that $q(t) = \frac{1}{2}$ we can complete the proof.
  - LB$:= \frac{t}{2} + \frac{1}{2} \cdot \sum_{i=1}^{n} \mathbf{E}_{\boldsymbol{\pi}}[(\pi_i - t)^+] \le \psi \le \psi^* \le t + \sum_{i=1}^{n} \mathbf{E}_{\boldsymbol{\pi}}[(\pi_i - t)^+] = 2 \cdot$ LB.

Outline

Back to the motivating example

- Single-item auction with n bidders.
- Bidders' valuations are drawn independently from regular distributions $F_1, \ldots, F_n$ that might not be identical.
- Using the prophet inequality:
  - Define the ith prize as $\varphi_i(v_i)^+$ of bidder i.
  - G_i: the corresponding distribution induced by F_i (independent).
- We have (by Theorem 5.2; with maximizer $\boldsymbol{x} = (x_i)_{i\in[n]}$)

$$\mathbf{E}_{\boldsymbol{v}\sim\boldsymbol{F}}\left[\sum_{i=1}^{n}\varphi_i(v_i)x_i(\boldsymbol{v})\right] = \mathbf{E}_{\boldsymbol{v}\sim\boldsymbol{F}}\left[\max_{i\in[n]}\varphi_i(v_i)^+\right].$$

  - The expected revenue of the optimal auction.

The allocation rule

Consider any allocation rule having the following form:

Virtual Threshold Allocation Rule

- Choose t such that $\Pr[\max_i \varphi_i(v_i)^+ \geq t] = \frac{1}{2}$.
- Give the item to a bidder i with $\varphi_i(v_i) \geq t$, if any, breaking ties among multiple candidate winners arbitrarily.

[1] We immediately have the following corollary:

[1] What if no such t exists? An exercise!

The allocation rule

Consider any allocation rule having the following form:

Virtual Threshold Allocation Rule

- Choose t such that $\Pr[\max_i \varphi_i(v_i)^+ \geq t] = \frac{1}{2}$.
- Give the item to a bidder i with $\varphi_i(v_i) \geq t$, if any, breaking ties among multiple candidate winners arbitrarily.

[1] We immediately have the following corollary:

Corollary (Virtual Threshold Rules are Near-Optimal)

If $\boldsymbol{x}$ is a virtual threshold allocation rule, then

$$\mathbf{E}_{\boldsymbol{v}}\left[\sum_{i=1}^{n} \varphi_i(v_i)^+ x_i(\boldsymbol{v})\right] \geq \frac{1}{2}\mathbf{E}_{\boldsymbol{v}}\left[\max_{i\in[n]} \varphi_i(v_i^+)\right].$$

[1] What if no such t exists? An exercise!

Outline

A different critique so far

- So far, the valuation distributions are assumed to be known to the mechanism designer in advance.
- What if the mechanism designer does **NOT** know about the valuation distributions in advance?

A different critique so far

- So far, the valuation distributions are assumed to be known to the mechanism designer in advance.
- What if the mechanism designer does **NOT** know about the valuation distributions in advance?
- Next, we consider that
  - Bidders' valuations are still drawn from such valuation distributions;
  - Yet, these distributions are still unknown to the mechanism designer.

A different critique so far

- So far, the valuation distributions are assumed to be known to the mechanism designer in advance.
- What if the mechanism designer does **NOT** know about the valuation distributions in advance?
- Next, we consider that
  - Bidders' valuations are still drawn from such valuation distributions;
  - Yet, these distributions are still unknown to the mechanism designer.
  - ⋆ We use the distributions in the *analysis*, but **NOT** in the design of mechanisms.
- Goal: design a good prior-independent mechanism.
  - Such a mechanism makes NO reference to a valuation distribution.

A Beautiful Result from Auction Theory

- The expected revenue of an optimal single-item auction is at most that of a second-price auction (with no reserved price) with one extra bidder.

Theorem [Bulow-Klemperer Theorem (1989)]

We have

- F: a regular distribution;
- n: a positive integer.
- $\boldsymbol{p}$: the payment rule of the second-price auction with $n+1$ bidders.
- $\boldsymbol{p}^*$: the payment rule of the optimal auction for F with n bidders.

Then,

$$\mathbf{E}_{\boldsymbol{v}\sim F^{n+1}}\left[\sum_{i=1}^{n+1} p_i(\boldsymbol{v})\right] \geq \mathbf{E}_{\boldsymbol{v}\sim F^{n}}\left[\sum_{i=1}^{n} p_i^*(\boldsymbol{v})\right]$$

A Beautiful Result from Auction Theory

- The expected revenue of an optimal single-item auction is at most that of a second-price auction (with no reserved price) with one extra bidder.

Theorem [Bulow-Klemperer Theorem (1989)]

We have

- F: a regular distribution;
- n: a positive integer.
- $\boldsymbol{p}$: the payment rule of the second-price auction with $n+1$ bidders.
- $\boldsymbol{p}^*$: the payment rule of the second-price auction (optimal) with reserve price $\varphi^{-1}(0)$.

Then,

$$\mathbf{E}_{\boldsymbol{v}\sim F^{n+1}}\left[\sum_{i=1}^{n+1} p_i(\boldsymbol{v})\right] \geq \mathbf{E}_{\boldsymbol{v}\sim F^{n}}\left[\sum_{i=1}^{n} p_i^*(\boldsymbol{v})\right]$$

Interpretation of the Bulow-Klemperer Theorem

- Extra competition is more important than getting the auction format just right.
- It is better to invest your resources to recruit more serious participants than sharpen your knowledge of their preferences.

Proof of the Bulow-Klemperer Theorem (1/3)

- It's tricky to compare two sides of the inequality directly.

Proof of the Bulow-Klemperer Theorem (1/3)

- It's tricky to compare two sides of the inequality directly.
- Let's consider a fictitious auction $\mathcal{A}$ below to facilitate the comparison.

The Fictitious Auction $\mathcal{A}$

1. Simulate an optimal n-bidder auction for F on the first n bidders.
2. If the item was not awarded in the first step, then give the item to the $(n+1)$th bidder for free.

Proof of the Bulow-Klemperer Theorem (1/3)

- It's tricky to compare two sides of the inequality directly.
- Let's consider a fictitious auction $\mathcal{A}$ below to facilitate the comparison.

The Fictitious Auction $\mathcal{A}$

1. Simulate an optimal n-bidder auction for F on the first n bidders.
2. If the item was not awarded in the first step, then give the item to the $(n+1)$th bidder for free.

- The expected revenue of $\mathcal{A}$ equals that of an optimal auction with n bidders.
  - The right-hand side of the inequality.

Proof of the Bulow-Klemperer Theorem (2/3)

- We argue that the expected revenue of a second-price auction with $n+1$ bidders is at least that of $\mathcal{A}$.

Proof of the Bulow-Klemperer Theorem (2/3)

- We argue that the expected revenue of a second-price auction with $n+1$ bidders is at least that of $\mathcal{A}$.
  - $\mathcal{A}$ is just a kind of auction for $n+1$ bidders subject to always allocating the item.

Proof of the Bulow-Klemperer Theorem (2/3)

- We argue that the expected revenue of a second-price auction with $n+1$ bidders is at least that of $\mathcal{A}$.
  - $\mathcal{A}$ is just a kind of auction for $n+1$ bidders subject to always allocating the item.
- Consider a stronger statement:

($n+1$) Bidders' valuations are drawn i.i.d. from a regular distribution (unknown to the designer).

Proof of the Bulow-Klemperer Theorem (2/3)

- We argue that the expected revenue of a second-price auction with $n+1$ bidders is at least that of $\mathcal{A}$.
  - $\mathcal{A}$ is just a kind of auction for $n+1$ bidders subject to always allocating the item.
- Consider a stronger statement:

($n+1$) Bidders' valuations are drawn i.i.d. from a regular distribution (unknown to the designer).

The second-price auction maximizes the expected revenue over all DSIC auctions that always allocate the item.

Proof of the Bulow-Klemperer Theorem (3/3)

- From previous lectures, it suffices to maximize the expected virtual welfare.

Proof of the Bulow-Klemperer Theorem (3/3)

- From previous lectures, it suffices to maximize the expected virtual welfare.
- The allocation rule with maximum possible expected virtual welfare *subject to always allocating the item* always awards the item to a bidder with the highest virtual valuation (even it is negative).

Proof of the Bulow-Klemperer Theorem (3/3)

- From previous lectures, it suffices to maximize the expected virtual welfare.
- The allocation rule with maximum possible expected virtual welfare *subject to always allocating the item* always awards the item to a bidder with the highest virtual valuation (even it is negative).
- Note: A second-price auction always awards the item to a bidder with the highest valuation.

Proof of the Bulow-Klemperer Theorem (3/3)

- From previous lectures, it suffices to maximize the expected virtual welfare.
- The allocation rule with maximum possible expected virtual welfare *subject to always allocating the item* always awards the item to a bidder with the highest virtual valuation (even it is negative).
- Note: A second-price auction always awards the item to a bidder with the highest valuation.
- All bidders share the same nondecreasing virtual valuation function φ.
  - A bidder with highest valuation also has the highest virtual valuation.

Proof of the Bulow-Klemperer Theorem (3/3)

- From previous lectures, it suffices to maximize the expected virtual welfare.
- The allocation rule with maximum possible expected virtual welfare *subject to always allocating the item* always awards the item to a bidder with the highest virtual valuation (even it is negative).
- Note: A second-price auction always awards the item to a bidder with the highest valuation.
- All bidders share the same nondecreasing virtual valuation function φ.
  - A bidder with highest valuation also has the highest virtual valuation.
- Hence, the second-price auction maximizes expected revenue subject to always awarding the item.