

# Revenue-Maximizing Auctions

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- In previous lectures, we only focus on maximizing the social welfare, while revenue is generated only as a **side effect**.
  - Though, indeed, there are real-world scenarios that the primary objective is welfare maximization (i.e., government auctions)
- In this lecture, we:
  - Study mechanisms that are designed to raise as much revenue as possible.
  - Characterize the expected revenue-maximizing mechanisms with respect to a prior distribution over agents' valuations.



# Outline

- 1 The Challenge of Revenue Maximization
  - One Bidder and One Item
  - Bayesian Analysis
  - Multiple Bidders
- 2 Characterization of Optimal DSIC Mechanisms
  - Virtual Valuations
  - Expected Revenue Equals Expected Virtual Welfare
  - Maximizing Expected Virtual Welfare
  - Regular Distributions
  - Optimal Single-Item Auctions
- 3 Proof of the Main Lemma (5.1)



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# A trivial example

- Suppose that there is one item and only one bidder, with private valuation  $v$ .
- The direct-revelation DSIC auction: **take-it-or-leave-it**.
  - With a posted price  $r \geq 0$ , the auction's revenue is either  $r$  (if  $v \geq r$ ) or 0 (if  $v < r$ ).
- Maximizing **social welfare** is trivial:
  - Set  $r := 0$ .
  - Independent of  $v$ .



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- How should we set  $r$  in order to maximize **revenue**?
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- How should we set  $r$  in order to maximize **revenue**?
  - Note the difficulty:  $v$  is private.
  - Let's consider another point of view: Bayesian analysis.



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# Bayesian Environment

## Bayesian Environment

- **A single-parameter environment.** Assume that there is a constant  $M$  such that  $x_i \leq M$  for every  $i$  and feasible solution  $(x_1, \dots, x_n) \in X$ .
- **Independent distributions  $F_1, \dots, F_n$  with positive and continuous density functions  $f_1, \dots, f_n$ .** Assume that the private valuation  $v_i$  of participant  $i$  is drawn from  $F_i$ .
  - Also, assume that the support of every distribution  $F_i$  belongs to  $[0, v_{\max}]$  for some  $v_{\max} < \infty$ .
- ★ The mechanism designer knows the distributions  $F_1, \dots, F_n$ .
- ★ The realizations  $v_1, \dots, v_n$  of agents' valuations are still private.

# The goal now

- Among all DSIC mechanisms, the optimal mechanism is the one having the highest **expected** revenue (assuming truthful bids).
  - The expectation is w.r.t.  $F_1 \times F_2 \times \cdots \times F_n$  over valuation profiles.
- The expected revenue of a posted price  $r$  is then

$$r \cdot (1 - F(r)),$$

where  $r$  represents the revenue of a sale while  $(1 - F(r))$  represents the probability of a sale.

- Solve for the best posted price  $r^* \Rightarrow$  a **monopoly price**.



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- Solve for the best posted price  $r^* \Rightarrow$  a **monopoly price**.
- For example, if  $F$  is the uniform distribution on  $[0, 1]$ , so that  $F(x) = x$  on  $[0, 1]$ , then the monopoly price is  $\frac{1}{2}$ , achieving an expected revenue of  $\frac{1}{4}$ .



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# Single-Item Auction with Two Bidders

## Exercise

Consider a single-item auction with two bidders with valuations drawn independently from the uniform distribution on  $[0, 1]$ .

- a. Prove that the expected revenue obtained by a second-price auction (with no reserve) is  $\frac{1}{3}$ .
- b. Prove that the expected revenue obtained by a second-price auction with reserve  $\frac{1}{2}$  is  $\frac{5}{12}$ .



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# Goal

- An explicit description of an optimal (i.e., **expected revenue-maximizing**) **DSIC** mechanism for every single-parameter environment and distributions  $F_1, \dots, F_n$ .



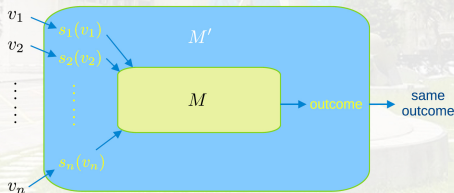
# Recall

- Every DSIC mechanism is equivalent to a direct-revelation DSIC mechanism.

## Theorem (Revelation Principle for DSIC Mechanisms)

For every mechanism  $M$  where every participant always has a dominant strategy, there is an equivalent direct-revelation DSIC mechanism  $M'$ .

- We use a simulation argument to construct  $M'$  as follows.





# Recall

- Every DSIC mechanism is equivalent to a direct-revelation DSIC mechanism.
- Hence we can pay our attention to such mechanisms.
- Assume truthful bids for the rest of our discussions.
  - $\mathbf{b} = \mathbf{v}$ .



# Expected revenue of a DSIC mechanism $(\mathbf{x}, \mathbf{p})$

- The expected revenue of a DSIC mechanism  $(\mathbf{x}, \mathbf{p})$  is

$$\mathbf{E}_{\mathbf{v} \sim \mathbf{F}} \left[ \sum_{i=1}^n p_i(\mathbf{v}) \right],$$

where the expectation is w.r.t.  $\mathbf{F} = F_1 \times \cdots \times F_n$  over agents' valuations.



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where the expectation is w.r.t.  $\mathbf{F} = F_1 \times \cdots \times F_n$  over agents' valuations.

- It's unclear how to maximize this expression...
- Later we will consider an alternative formula which only references the **allocation rule** of a mechanism.



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# Virtual Valuations

## Virtual Valuation

For an agent  $i$  with valuation distribution  $F_i$  and valuation  $v_i$  (drawn from  $F_i$ ), her virtual valuation is define as

$$\varphi(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

- For example, if  $F_i$  is the uniform distribution on  $[0, 1]$ .



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- For example, if  $F_i$  is the uniform distribution on  $[0, 1]$ .
  - $F_i(z) = z$  for  $z \in [0, 1]$ .
  - $f_i(z) = 1$ .
  - $\varphi_i(z) = z - \frac{1-z}{1} = 2z - 1$  on  $[0, 1]$ .



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  - $\varphi_i(z) = z - \frac{1-z}{1} = 2z - 1$  on  $[0, 1]$ .
- It is always at most the corresponding valuation.
- It could be *negative*.





# What do virtual valuations mean?

$$\varphi(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

- One possible interpretation:
  - $v_i$ : what you'd like to charge
  - $\frac{1 - F_i(v_i)}{f_i(v_i)}$ : inevitable revenue loss caused by not knowing  $v_i$  in advance.

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- Second interpretation:
  - $\varphi(v_i)$ : the slope of a **revenue curve** at  $v_i$ .



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# The Crucial Lemma (the proof is postponed)

## Lemma (5.1 in the Textbook)

For every single-parameter environment with valuation distributions  $F_1, \dots, F_n$ , every DSIC mechanism  $(\mathbf{x}, \mathbf{p})$ , every agent  $i$ , and every value  $\mathbf{v}_{-i}$  of the valuations of the other agents,

$$\mathbf{E}_{v_i \sim F_i}[p_i(\mathbf{v})] = \mathbf{E}_{v_i \sim F_i}[\varphi_i(v_i) \cdot x_i(\mathbf{v})].$$

- Note: the identity holds in expectation over  $v_i$ , and not pointwise.

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- Note: the identity holds in expectation over  $v_i$ , and not pointwise.
  - $\varphi_i(v_i)$  could be negative for some  $i$ .



# The Main Theorem

## Theorem (5.2 in the Textbook)

For every single-parameter environment with valuation distributions  $F_1, \dots, F_n$  and every DSIC mechanism  $(\mathbf{x}, \mathbf{p})$ ,

$$\mathbf{E}_{\mathbf{v} \sim F} \left[ \sum_{i=1}^n p_i(\mathbf{v}) \right] = \mathbf{E}_{\mathbf{v} \sim F} \left[ \sum_{i=1}^n \varphi_i(v_i) \cdot x_i(\mathbf{v}) \right].$$

- That is, the expected **revenue** equals the expected **virtual welfare**!



# Proof of Theorem 5.2

- Taking the expectation, with respect to  $\mathbf{v}_{-i} \sim \mathbf{F}_{-i}$ , of both sides of the equation in Lemma 5.1: (i.e.,

$$\mathbf{E}_{\mathbf{v}_i \sim F_i}[p_i(\mathbf{v})] = \mathbf{E}_{\mathbf{v}_i \sim F_i}[\varphi_i(v_i) \cdot x_i(\mathbf{v})]^1$$

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<sup>1</sup>Consider  $v_i \sim F_i$  and for any  $\mathbf{v}_{-i}$  of the other agents.



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$$\mathbf{E}_{\mathbf{v} \sim F}[p_i(\mathbf{v})] = \mathbf{E}_{\mathbf{v} \sim F}[\varphi_i(v_i) \cdot x_i(\mathbf{v})].$$

- Applying the linearity of expectation twice:

$$\begin{aligned} \mathbf{E}_{\mathbf{v} \sim F} \left[ \sum_{i=1}^n p_i(\mathbf{v}) \right] &= \sum_{i=1}^n \mathbf{E}_{\mathbf{v} \sim F}[p_i(\mathbf{v})] \\ &= \sum_{i=1}^n \mathbf{E}_{\mathbf{v} \sim F}[\varphi_i(v_i) \cdot x_i(\mathbf{v})] \\ &= \mathbf{E}_{\mathbf{v} \sim F} \left[ \sum_{i=1}^n \varphi_i(v_i) \cdot x_i(\mathbf{v}) \right]. \end{aligned}$$

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# Maximization concerning only the allocation rule

- Theorem 5.2 says that: even though we only care about “payments”, we can still focus on an optimization problem concerning only the **allocation rule** of the mechanism.

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- Theorem 5.2 says that: even though we only care about “payments”, we can still focus on an optimization problem concerning only the **allocation rule** of the mechanism.
- So, how should we choose the allocation rule  $\mathbf{x}$  to maximize

$$\mathbf{E}_{\mathbf{v} \sim F} \left[ \sum_{i=1}^n \varphi_i(v_i) \cdot \mathbf{x}_i(\mathbf{v}) \right] ?$$

- An obvious approach: maximize pointwise:
  - For each  $\mathbf{v}$ , choose  $\mathbf{x}(\mathbf{v})$  to maximize the virtual welfare obtained on input  $\mathbf{v}$ , subject to feasibility of the allocation.



# Well, not so obvious...

- For example, consider a single-item auction, where the feasible constraint is  $\sum_{i=1}^n x_i(\mathbf{v}) \leq 1$  for every  $\mathbf{v}$ .



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  - ★ **Note:** virtual valuations can be negative (e.g., consider  $\varphi_i(v_i) = 2v_i - 1$  for  $v_i$  uniformly drawn from  $[0, 1]$ ).



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  - The virtual welfare is maximized by **not awarding the item to anyone**.





# An Issue/Key Question

- Such a virtual welfare-maximizing allocation rule maximizes the expected virtual welfare over **all allocation rules**.

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Is the virtual welfare-maximizing allocation rule **monotone**?

- If so, **Myerson's lemma** can be applied and the rule can be extended to a **DSIC** mechanism, hence the mechanism results in the **maximum possible expected revenue** by Theorem 5.2.



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# Regularity Comes to the Rescue

## Regular Distribution

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- For example, consider  $F$  to be the uniform distribution on  $[0, 1]$ .
- It's regular since the corresponding  $\varphi(v) = 2v - 1$  which is nondecreasing in  $v$ .



# Virtual Welfare Maximizer

Assume that  $F_i$  is **regular** for each  $i$ .

- 1 Transform the (truthfully reported) valuation  $v_i$  of agent  $i$  into  $\varphi_i(v_i)$ .
- 2 Choose the feasible allocation  $(x_1, \dots, x_n)$  that maximizes the virtual welfare  $\sum_{i=1}^n \varphi_i(v_i) x_i$ .
- 3 Charge payments according to Myerson's payment formula (refer to previous lectures).



# Virtual Welfare Maximizers Are Optimal

## Theorem 5.4

For every single-parameter environment and **regular distributions**  $F_1, \dots, F_n$ , the corresponding virtual welfare maximizer is a DSIC mechanism with the **maximum-possible expected revenue**.



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- Here revenue-maximizing mechanisms are almost the same as welfare-maximizing ones.
- They differ only in using *virtual* valuations in place of valuations.





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- Assume bidders are **i.i.d. with a common valuation distribution  $F$**  (hence a common virtual valuation  $\varphi$ ).
- Assume that  $F$  is strictly regular (hence  $\varphi$  is strictly increasing).
- The virtual-welfare-**maximizing** mechanism awards the item to the bidder with the highest **nonnegative** virtual valuation (if any).
  - That is, the bidder with the highest valuation.
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- The allocation rule: the same as that of a second-price auction with a **reserve price of  $\varphi^{-1}(0)$** .
- **eBay** is (roughly) the optimal auction format!



## Theorem (Myerson's Lemma)

Fix a single-parameter environment.

- (i) An allocation rule  $\mathbf{x}$  is **implementable** if and only if it is **monotone**.
- (ii) If  $\mathbf{x}$  is monotone, then there is a unique payment rule for which the direct-revelation mechanism  $(\mathbf{x}, \mathbf{p})$  is DSIC and  $p_i(\mathbf{b}) = 0$  whenever  $b_i = 0$ .
- (iii) The payment rule in (ii) is given by an explicit formula.



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- Note: the identity holds in expectation over  $v_i$ , and not pointwise.



# Sketch of the Proof (1/4)

- Assume that we have
  - a DSIC mechanism  $(\mathbf{x}, \mathbf{p})$ ;
  - the allocation rule:  $\mathbf{x}$
  - the valuation profile:  $\mathbf{v}$ .
- Recall Myerson's payment formula:

$$p_i(v_i, \mathbf{v}_{-i}) = \int_0^{v_i} z \cdot x'_i(z, \mathbf{v}_{-i}) dz.$$

for the payment made by agent  $i$ .

- Assume that  $x_i(z, \mathbf{v}_{-i})$  is differentiable.





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- Assume that  $x_i(z, \mathbf{v}_{-i})$  is differentiable.
  - The same formula holds more generally, including piecewise constant functions, for a suitable interpretation of  $x'_i(z, \mathbf{v}_{-i})$  and the corresponding integral.



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- Assume that  $x_i(z, \mathbf{v}_{-i})$  is differentiable.
  - The payments are fully dictated by the allocation rule.



# Sketch of the Proof (2/4)

- Fix an agent  $i$ . We have

$$\begin{aligned}\mathbf{E}_{v_i \sim F_i}[p_i(\mathbf{v})] &= \int_0^{v_{\max}} p_i(\mathbf{v}) f_i(v_i) dv_i \\ &= \int_0^{v_{\max}} \left[ \int_0^{v_i} z \cdot x'_i(z_i, \mathbf{v}_{-i}) dz \right] f_i(v_i) dv_i\end{aligned}$$

- 1st equality exploits the independence of agents' valuations.



## Reference

### 4.2: Expected Value and Variance of Continuous Random Variables

Last updated: Feb 28, 2020

◀ 4.1: Probability Density Functions (PDFs) and Cumulati... 4.3: Uniform Distributions ▶



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We now consider the expected value and variance for continuous random variables. Note that the interpretation of each is the same as in the discrete setting, but we now have a different method of calculating them in the continuous setting.

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#### Definition 4.2.1

If  $X$  is a continuous random variable with pdf  $f(x)$ , then the **expected value** (or **mean**) of  $X$  is given by

$$\mu = \mu_X = E[X] = \int_{-\infty}^{\infty} x \cdot f(x) dx.$$



# Sketch of the Proof (3/4)

- Reversing the order of integration in

$$\int_0^{v_{\max}} \left[ \int_0^{v_i} z \cdot x'_i(z, \mathbf{v}_{-i}) dz \right] f_i(v_i) dv_i$$

yields

$$\begin{aligned} & \int_0^{v_{\max}} \left[ \int_z^{v_{\max}} f_i(v_i) dv_i \right] z \cdot x'_i(z, \mathbf{v}_{-i}) dz \\ = & \int_0^{v_{\max}} (1 - F_i(z)) \cdot z \cdot x'_i(z, \mathbf{v}_{-i}) dz. \end{aligned}$$

# Sketch of the Proof (4/4)

- Using **integration by parts**:

$$\int_0^{v_{\max}} \underbrace{(1 - F_i(z)) \cdot z}_{g(z)} \cdot \underbrace{x'_i(z, \mathbf{v}_{-i})}_{h'(z)} dz.$$

# Sketch of the Proof (4/4)

- Using **integration by parts**:

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 & \int_0^{v_{\max}} \underbrace{(1 - F_i(z)) \cdot z}_{g(z)} \cdot \underbrace{x'_i(z, \mathbf{v}_{-i})}_{h'(z)} dz. \\
 &= (1 - F_i(z)) \cdot z \cdot x_i(z, \mathbf{v}_{-i}) \Big|_0^{v_{\max}} \\
 &\quad - \int_0^{v_{\max}} x_i(z, \mathbf{v}_{-i}) \cdot (1 - F_i(z) - z f_i(z)) dz
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 &= \int_0^{v_{\max}} \underbrace{\left( z - \frac{1 - F_i(z)}{f_i(z)} \right)}_{\varphi_i(z)} x_i(z, \mathbf{v}_{-i}) f_i(z) dz
 \end{aligned}$$





# Sketch of the Proof (4/4)

- Using **integration by parts**:

$$\begin{aligned}
 & \int_0^{v_{\max}} \underbrace{(1 - F_i(z)) \cdot z}_{g(z)} \cdot \underbrace{x'_i(z, \mathbf{v}_{-i})}_{h'(z)} dz. \\
 = & \left( (1 - F_i(z)) \cdot z \cdot x_i(z, \mathbf{v}_{-i}) \right) \Big|_0^{v_{\max}} \\
 & - \int_0^{v_{\max}} x_i(z, \mathbf{v}_{-i}) \cdot (1 - F_i(z) - z f_i(z)) dz \\
 = & \int_0^{v_{\max}} \underbrace{\left( z - \frac{1 - F_i(z)}{f_i(z)} \right)}_{\varphi_i(z)} x_i(z, \mathbf{v}_{-i}) f_i(z) dz \\
 = & \mathbf{E}_{v_i \sim F_i} [\varphi_i(v_i) \cdot x_i(\mathbf{v})].
 \end{aligned}$$



# Exercise

- Consider a virtual valuation  $\varphi(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$  where  $F$  is a strictly increasing distribution function with a strictly positive density function  $f$  on the interval  $[0, v_{\max}]$ , with  $v_{\max} < \infty$ .
- For a single bidder with valuation drawn from  $F$ , for  $q \in [0, 1]$ , define  $V(q) = F^{-1}(1 - q)$  as the posted price that yields a probability  $q$  of a sale.
- Define  $R(q) = q \cdot V(q)$  as the expected revenue obtained from a single bidder when the probability of a sale is  $q$ .
- The function  $R(q)$ , for  $q \in [0, 1]$ , is the **revenue curve** of  $F$ . Note that  $R(0) = R(1) = 0$ .
- ★ Please prove that the slope of the revenue curve at  $q$  (i.e.,  $R'(q)$ ) is precisely  $\varphi(v_i)$ .



# Hint

## Theorem [Derivative of an Inverse Function]

Given an invertible function  $f(x)$ , the derivative of its inverse function  $f^{-1}(x)$  evaluated at  $x = a$  is

$$[f^{-1}]'(a) = \frac{1}{f'[f^{-1}(a)]}.$$



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- Differentiate both sides w.r.t.  $x$ :

$$1 = f'(y) \cdot \frac{dy}{dx}.$$



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- Let  $y = f^{-1}(x)$  so  $x = f(y)$ .
- Differentiate both sides w.r.t.  $x$ :

$$1 = f'(y) \cdot \frac{dy}{dx}.$$

- Thus,  $\frac{dy}{dx} = \frac{1}{f'(y)} \Rightarrow [f^{-1}]'(x) = \frac{1}{f'[f^{-1}(x)]}.$

