

Randomized Algorithms

Continuous Distributions and Poisson Processes

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Outline

- Continuous Random Variables
- The Uniform Distribution
- The Exponential Distribution
- The Poisson Process

Recall: Probability function

- $\Pr(\Omega) = 1$
- For any event E , $0 \leq \Pr(E) \leq 1$.
- For any finite or enumerable collection **B** of disjoint events,

$$\Pr\left(\bigcup_{E \in \mathbf{B}} E\right) = \sum_{E \in \mathbf{B}} \Pr[E].$$

k distinct points in $[0, 1)$

- p : the probability of any given point is in $[0, 1)$.
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$$\therefore p = 0$$

Continuous distribution

- Probabilities are assigned to **intervals** rather than to individual values.
- For any $x \in \mathbf{R}$, $F(x) = \Pr[X \leq x] = \Pr[X < x]$.
 - X is continuous if $F(x)$ is a continuous function.

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$$\begin{aligned}\Pr[x < X \leq x + dx] &= F(x + dx) - F(x) \\ &\approx f(x)dx.\end{aligned}$$

Continuous distribution

$$\Pr[a \leq X \leq b] = \int_a^b f(x)dx.$$

$$\mathbf{E}[X^i] = \int_{-\infty}^{\infty} x^i f(x)dx.$$

$$\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)dx.$$

$$\begin{aligned}\mathbf{Var}[X] &= \mathbf{E}[(X - \mathbf{E}[X])^2] \\ &= \int_{-\infty}^{\infty} (x - \mathbf{E}[x])^2 f(x)dx \\ &= \mathbf{E}[X^2] - (\mathbf{E}[X])^2.\end{aligned}$$

Exercise

- **Lemma.** Let $X \geq 0$ be a continuous random variable. Then

$$\mathbf{E}[X] = \int_0^\infty \Pr[X \geq x]dx.$$

Exercise

$$\begin{aligned}\mathbf{E}[X] &= \int_0^\infty \textcolor{blue}{y} \cdot f(y) dy \\ &= \int_{y=0}^\infty f(y) \int_{x=0}^{\textcolor{blue}{y}} \textcolor{blue}{dx} dy \\ &= \int_{\textcolor{red}{x}=0}^\infty \int_{\textcolor{red}{y}=x}^\infty f(y) dy dx \quad 0 \leq x \leq y < \infty \\ &= \int_{x=0}^\infty (1 - F(x)) dx \\ &= \int_{x=0}^\infty \Pr[X \geq x] dx.\end{aligned}$$

Joint Distribution

- The joint distribution function of X and Y :

$$F(x, y) = \Pr[X \leq x, Y \leq y].$$

- X and Y have joint density function f if for all x, y ,

$$F(x, y) = \int_{-\infty}^y \int_{-\infty}^x f(u, v) dudv.$$

We denote that $f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y).$

Marginal Distribution Function

- Given a joint distribution function $F(x, y)$ over X and Y , we have the **marginal distribution functions**:

$$F_X(x) = \Pr[X \leq x], \quad F_Y(y) = \Pr[Y \leq y].$$

The corresponding **marginal density functions**:

$$f_X(x) \text{ and } f_Y(y).$$

Independence

- The random variables X and Y are independent if, for all x and y ,

$$\Pr[(X \leq x) \cap (Y \leq y)] = \Pr[X \leq x] \Pr[Y \leq y].$$



$$F(x, y) = F_X(x)F_Y(y).$$



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Example

- For $a, b > 0$, consider the joint distribution function of two random variables X and Y :

$$F(x, y) = 1 - e^{-ax} - e^{-by} + e^{-(ax+by)}, \text{ over } x, y \geq 0.$$

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$$F(x, y) = (1 - e^{-ax})(1 - e^{-by}) = F_X(x)F_Y(y).$$

$$f_X(x) = ae^{-ax}, f_Y(y) = be^{-by}, \therefore f(x, y) = f_X(x)f_Y(y).$$

Conditional Probability

$$\Pr[X \leq x \mid Y = y] = \lim_{\delta \rightarrow 0} \Pr[X \leq x \mid y \leq Y \leq y + \delta].$$

Why?

Conditional Probability

$$\begin{aligned}\Pr[X \leq x \mid Y = y] &= \lim_{\delta \rightarrow 0} \Pr[X \leq x \mid y \leq Y \leq y + \delta] \\ &= \lim_{\delta \rightarrow 0} \frac{\Pr[(X \leq x) \cap (y \leq Y \leq y + \delta)]}{\Pr[y \leq Y \leq y + \delta]} \\ &= \lim_{\delta \rightarrow 0} \frac{F(x, y + \delta) - F(x, y)}{F_Y(y + \delta) - F_Y(y)} \\ &= \lim_{\delta \rightarrow 0} \int_{u=-\infty}^x \frac{\partial F(u, y + \delta)/\partial x - \partial F(u, y)/\partial x}{F_Y(y + \delta) - F_Y(y)} du \\ &= \int_{u=-\infty}^x \lim_{\delta \rightarrow 0} \frac{(\partial F(u, y + \delta)/\partial x - \partial F(u, y)/\partial x)/\delta}{(F_Y(y + \delta) - F_Y(y))/\delta} du \\ &= \int_{u=-\infty}^x \frac{f(u, y)}{f_Y(y)} du.\end{aligned}$$

Conditional Probability

- For example,

$$F(x, y) = 1 - e^{-ax} - e^{-by} + e^{-(ax+by)},$$

$$\Pr[X \leq 3 \mid Y = 4]$$

$$= \int_{u=0}^3 \frac{abe^{-au+4b}}{be^{-4b}} du = 1 - e^{-3a}.$$

For $a, b > 0$, consider the joint distribution function of two random variables X and Y :

$$F(x, y) = 1 - e^{-ax} - e^{-by} + e^{-(ax+by)}, \text{ over } x, y \geq 0.$$

$$F_X(x) = F(x, \infty) = 1 - e^{-ax}, F_Y(y) = F(\infty, y) = 1 - e^{-by}.$$

$$f(x, y) = abe^{-(ax+by)}.$$

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Conditional Density Function

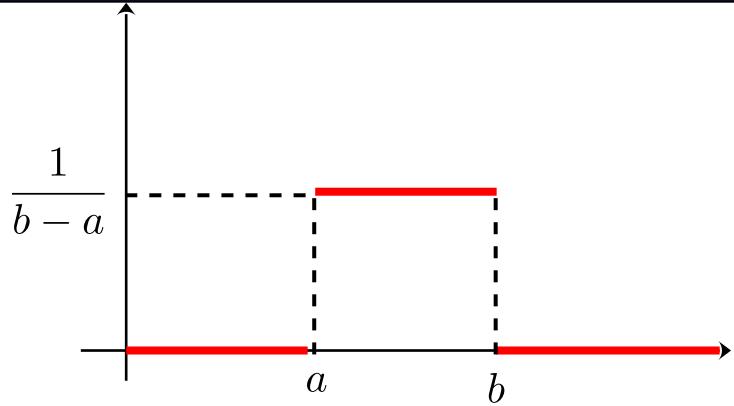
- Assume that $f_Y(y) \neq 0$ (resp., $f_X(x) \neq 0$),

$$f_{X|Y}(x, y) = \frac{f(x, y)}{f_Y(y)}$$

$$f_{Y|X}(x, y) = \frac{f(x, y)}{f_X(x)}.$$

$$\mathbf{E}[X \mid Y = y] = \int_{x=-\infty}^{\infty} x \cdot f_{X|Y}(x, y) dx.$$

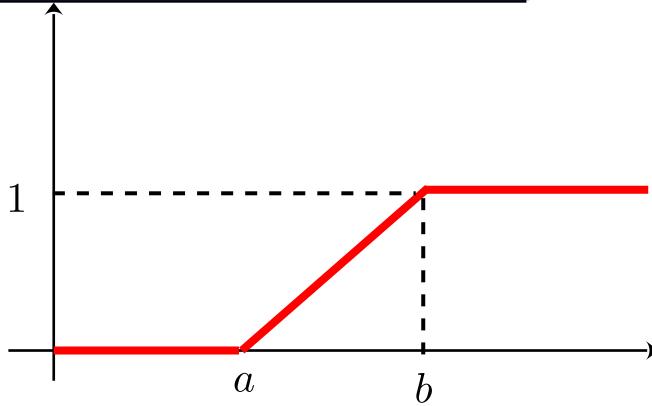
Uniform Distribution



$$f(x) = \begin{cases} 0 & \text{if } x \leq a, \\ \frac{1}{b-a} & \text{if } a \leq x \leq b, \\ 0 & \text{if } x \geq b. \end{cases}$$

$$\mathbf{E}[X] = \int_a^b \frac{x}{b-a} dx = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}.$$

$$\mathbf{Var}[X] = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \frac{(b-a)^2}{12}.$$



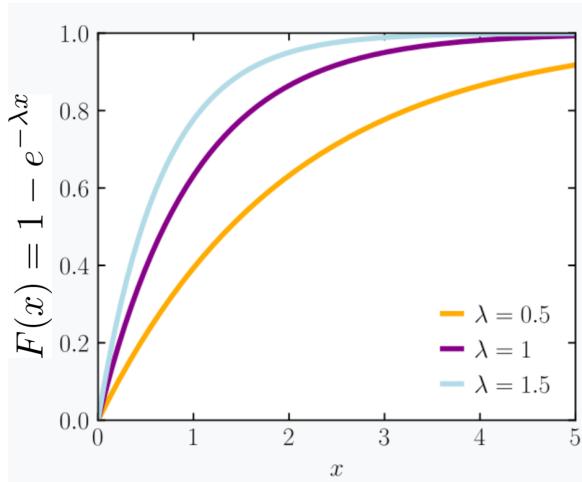
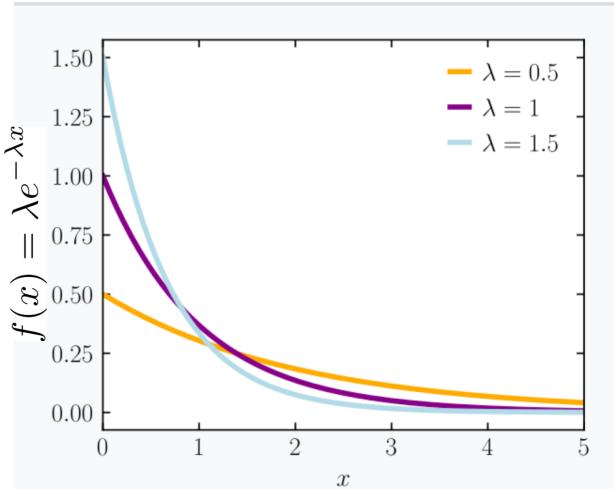
$$F(x) = \begin{cases} 0 & \text{if } x \leq a, \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b, \\ 1 & \text{if } x \geq b. \end{cases}$$

$$\mathbf{E}[X^2] = \int_a^b \frac{x^2}{b-a} dx = \frac{b^3 - a^3}{3(b-a)} = \frac{b^2 + ab + a^2}{3}.$$

Exponential Distribution

- **Definition.** An exponential distribution with parameter λ is given by the following probability distribution function:

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$



wikipedia

Exponential Distribution

Note: $\Pr[X > t] = 1 - F(t) = e^{-\lambda t}.$

$$\mathbf{E}[X] = \int_0^\infty t \lambda e^{-\lambda t} dt = \frac{1}{\lambda}. \quad (\text{Integration by parts})$$

$$\mathbf{E}[X^2] = \int_0^\infty t^2 \lambda e^{-\lambda t} dt = \frac{2}{\lambda^2}.$$

$$\mathbf{Var}[X] = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \frac{1}{\lambda^2}.$$

Exponential Distribution (memoryless)

- For an exponential random variable X with parameter λ ,

$$\Pr[X > s + t \mid X > t] = \Pr[X > s].$$

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$$\begin{aligned}\Pr[X > s + t \mid X > t] &= \frac{\Pr[X > s + t]}{\Pr[X > t]} \\ &= \frac{1 - \Pr[X \leq s + t]}{1 - \Pr[X \leq t]} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} \\ &= e^{-\lambda s} = \Pr[X > s].\end{aligned}$$

Min (exponential random variables)

- **Lemma.** If X_1, X_2, \dots, X_n are independent exponential random variables with parameters $\lambda_1, \lambda_2, \dots, \lambda_n$, respectively, then $\min(X_1, X_2, \dots, X_n)$ is exponential random variable with parameter $\sum_{i=1}^n \lambda_i$

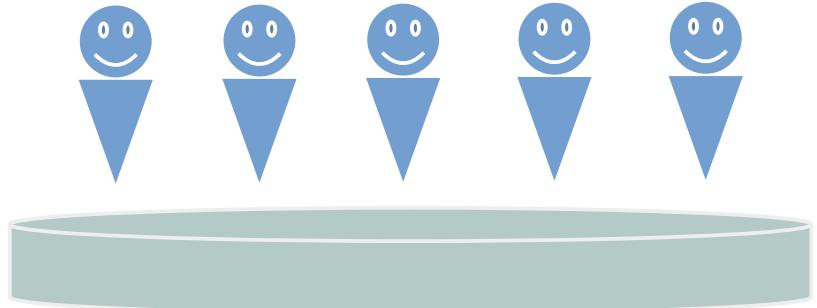
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$$\begin{aligned}\Pr[\min(X_1, X_2, \dots, X_n) > x] &= \Pr[(X_1 > x) \cap (X_2 > x) \cap \dots \cap (X_n > x)] \\ &= \Pr[X_1 > x] \cdot \Pr[X_2 > x] \cdots \Pr[X_n > x] \\ &= e^{-\lambda_1 x} \cdot e^{-\lambda_2 x} \cdots e^{-\lambda_n x} \\ &= e^{-\sum_{i=1}^n \lambda_i}.\end{aligned}$$

Scenario

- An airline ticket counter with n service agents.
- The time agent i takes per customer:
 - Exponential distribution, parameter λ_i
- You are at the head of the line and wondering how long, in average, you wait for an agent to serve you...

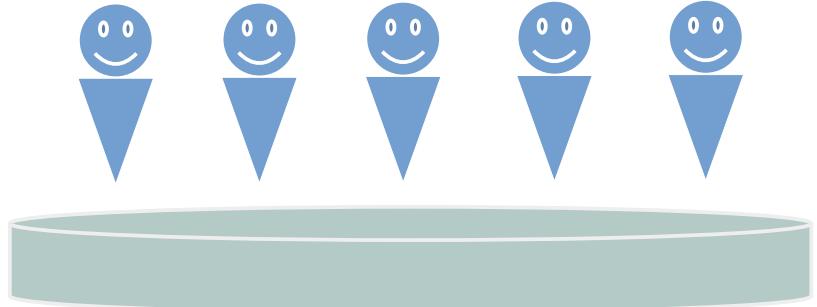


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The time until the first agent is free

- Exponential distribution with parameter $\sum_{i=1}^n \lambda_i$
- Expected waiting time: $1 / \sum_{i=1}^n \lambda_i$



The Poisson Process



Siméon Poisson
(1781–1840)
Wikipedia

- Counting process.
 - E.g., arrivals of customers to a queue, emissions of radioactive particles, price surges in the stock markets, etc.
- $N(t)$: the number of events in the interval (say $[0, t]$).
- A **stochastic counting process**: $\{N(t): t \geq 0\}$

The Poisson Process



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- A **Poisson process** $\{N(t): t \geq 0\}$ with parameter λ is a stochastic counting process such that the following conditions hold.
 1. $N(0) = 0$.
 2. (Independent & stationary increments) For any $t, s > 0$,
 - the distribution of $N(t+s) - N(s)$ is identical to the distribution of $N(t)$;
 - for disjoint intervals $[t_1, t_2]$ and $[t_3, t_4]$, the distribution of $N(t_2) - N(t_1)$ is independent of the distribution of $N(t_4) - N(t_3)$.
 3. $\lim_{t \rightarrow 0} \frac{\Pr[N(t) = 1]}{t} = \lambda$.
 4. $\lim_{t \rightarrow 0} \frac{\Pr[N(t) \geq 2]}{t} = 0$.

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The number of events in a given time interval follows the **Poisson distribution**!

Poisson Process → Poisson Distribution

- **Theorem.** Let $\{N(t): t \geq 0\}$ be a Poisson process with parameter λ . For any $t, s \geq 0$ and any integer $n \geq 0$,

$$P_n(t) := \Pr[N(t + s) - N(s) = n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

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$$= \Pr[N(t) - N(0)] = \Pr[N(t) - 0]$$

- The probability that n events happen during time interval of length t .

Stochastic Process + Poisson = ?

- **Theorem.** Let $\{N(t): t \geq 0\}$ be a **stochastic process** such that
 1. $N(0) = 0$.
 2. (Independent increments) For disjoint intervals $[t_1, t_2]$ and $[t_3, t_4]$, the distribution of $N(t_2) - N(t_1)$ is independent of the distribution of $N(t_4) - N(t_3)$.
 3. For any $t, s \geq 0$, $N(t+s) - N(s)$ has a Poisson distribution with mean λt .

Then $\{N(t): t \geq 0\}$ is a Poisson process with rate λ .

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 3. For any $t, s \geq 0$, $N(\textcolor{red}{t+s}) - N(s)$ has a Poisson distribution with mean $\lambda \textcolor{red}{t}$.

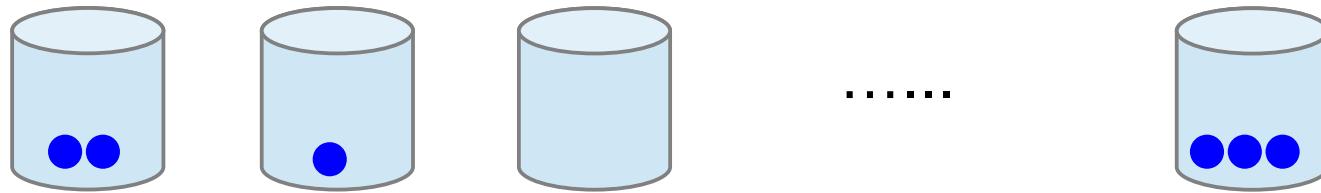
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$$\lim_{t \rightarrow 0} \frac{\Pr[N(t) = 1]}{t} = \lim_{t \rightarrow 0} \frac{e^{-\lambda t} \lambda t}{t} = \lambda.$$

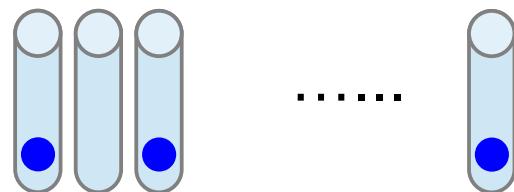
$$\lim_{t \rightarrow 0} \frac{\Pr[N(t) \geq 2]}{t} = \lim_{t \rightarrow 0} \sum_{k \geq 2} \frac{e^{-\lambda t} (\lambda \textcolor{red}{t})^k}{k! t} = 0.$$

An Intuitive Idea

- Balls: events, bins: time slots

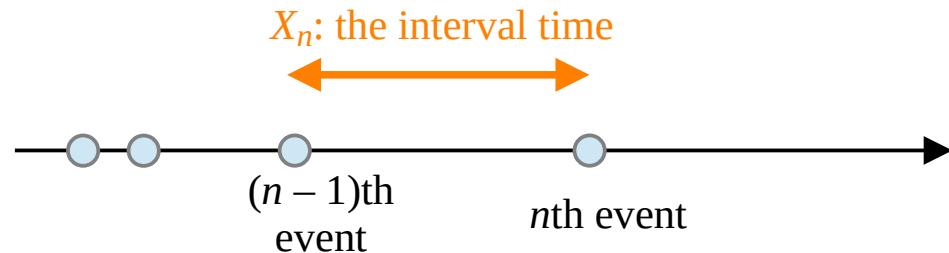


- A lot of balls into a lot of (infinitely small) bins...



Another Viewpoint: Interarrival

- Surprising fact: All of the X_n have the same distribution and this distribution is **exponential!**



○ : events of Poisson process

Interarrival times

- **Theorem.** X_1 has an exponential distribution with parameter λ .

$$\Pr[X_1 > t] = \Pr[N(t) = 0] = e^{\lambda t}.$$

$$F(X_1) = 1 - \Pr[X_1 > t] = 1 - e^{\lambda t}.$$

Interarrival times

- **Theorem.** X_i , $i = 1, 2, \dots$, are i.i.d exponential random variables with parameter parameter λ .

$$\begin{aligned} & \Pr[X_i > t_i \mid (X_0 = t_0) \cap (X_1 = t_1) \cap \cdots \cap (X_{i-1} = t_{i-1})] \\ &= \Pr \left[N \left(\sum_{k=0}^i t_k \right) - N \left(\sum_{k=0}^{i-1} t_k \right) = 0 \right] \\ &= e^{-\lambda t_i}. \end{aligned}$$

Interarrival times

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\therefore Poisson process & \sim distribution of $N(t_i) - N(0) = N(t_i)$

So, why the Poisson process makes $P_n(t)$ Poisson distributed?

The Poisson Process



Siméon Poisson
(1781–1840)
Wikipedia

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$$4. \lim_{t \rightarrow 0} \frac{\Pr[N(t) \geq 2]}{t} = 0.$$

Randomized Algorithms, CSIE, TKU, Taiwan

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$P_0(t)$

- The number of events in $[0, t]$ and $(t, t+h]$ are independent.

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$$\begin{aligned} & \therefore \frac{P_0(t + h) - P_0(t)}{h} \\ &= \frac{P_0(t)P_0(h) - P_0(t)}{h} \\ &= P_0(t) \frac{P_0(h) - 1}{h} \\ &= P_0(t) \frac{1 - \Pr[N(h) = 1] - \Pr[N(h) \geq 2] - 1}{h} \\ &= P_0(t) \frac{-\Pr[N(h) = 1] - \Pr[N(h) \geq 2]}{h}. \end{aligned}$$

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Solve $P'_0(t) = -\lambda P_0(t)$

$P_0(t)$

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Solve $P'_0(t) = -\lambda P_0(t)$

$$\frac{P'_0(t)}{P_0(t)} = -\lambda \Rightarrow \int \frac{P'_0(t)}{P_0(t)} dt = \int -\lambda dt$$

$$\ln P_0(t) = -\lambda t + C \quad \therefore P_0(t) = e^{-\lambda t} \quad (\because P_0(0) = 1)$$

$$P_n(t), n \geq 1$$

$$\begin{aligned} P_n(t+h) &= \sum_{k=0}^n P_{n-k}(t)P_k(h) \\ &= P_n(t)P_0(h) + P_{n-1}(t)P_1(h) + \sum_{k=2}^n P_{n-k}(t) \cdot \Pr[N(h) = k] \end{aligned}$$

$$\begin{aligned} P'_n(t) &= \lim_{h \rightarrow 0} \frac{P_n(t+h) - P_n(t)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(P_n(t)(P_0(h) - 1) + P_{n-1}(t)P_1(h) + \sum_{k=2}^n P_{n-k}(t)\Pr[N(h) = k] \right) \\ &= -\lambda P_n(t) + \lambda P_{n-1}(t). \end{aligned}$$

$$P_n(t), n \geq 1$$

$$\begin{aligned} P_n(t+h) &= \sum_{k=0}^n P_{n-k}(t)P_k(h) \\ &= P_n(t)P_0(h) + P_{n-1}(t)P_1(h) + \sum_{k=2}^n P_{n-k}(t) \cdot \Pr[N(h) = k] \end{aligned}$$

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$$\text{Solve } P'_n(t) = -\lambda P_n(t) + \lambda P_{n-1}(t). \Rightarrow e^{\lambda t}(P'_n(t) + \lambda P_n(t)) = e^{\lambda t}(\lambda P_{n-1}(t).)$$

$P_n(t), n \geq 1$

Solve $P'_n(t) = -\lambda P_n(t) + \lambda P_{n-1}(t)$.

$$\begin{aligned} & \Rightarrow e^{\lambda t} (P'_n(t) + \lambda P_n(t)) = e^{\lambda t} (\lambda P_{n-1}(t)) \\ & \Rightarrow \frac{d}{dt} (e^{\lambda t} P_n(t)) = \lambda e^{\lambda t} P_{n-1}(t). \end{aligned}$$

$P_n(t), n \geq 1$

$$\begin{aligned} \text{Solve } P'_n(t) = -\lambda P_n(t) + \lambda P_{n-1}(t). & \Rightarrow e^{\lambda t} (P'_n(t) + \lambda P_n(t)) = e^{\lambda t} (\lambda P_{n-1}(t)). \\ & \Rightarrow \frac{d}{dt} (e^{\lambda t} P_n(t)) = \lambda e^{\lambda t} P_{n-1}(t). \end{aligned}$$

$$\therefore \frac{d}{dt} (e^{\lambda t} P_1(t)) = \lambda e^{\lambda t} P_0(t) = \lambda. \quad (P_0(t) = e^{-\lambda t})$$

$$\Rightarrow \int d(e^{\lambda t} P_1(t)) = \int \lambda dt.$$

$$\Rightarrow e^{\lambda t} P_1(t) = \lambda t + C$$

$$\Rightarrow P_1(t) = e^{-\lambda t} (\lambda t + C) = t \lambda e^{-\lambda t}.$$

Note: $P_1(0) = 0$
1 event occurs in zero time period

$P_n(t)$, $n \geq 1$

- Look back at what we want to have:

$$P_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

- We already have $P_0(t)$ and $P_1(t)$. $(P_0(t) = e^{-\lambda t})$
- Induction hypothesis: $(P_1(t) = t\lambda e^{-\lambda t})$

$$P_{n-1}(t) = e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

$P_n(t)$, $n \geq 1$

- Look back at what we want to have:

$$P_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

- We already have $P_0(t)$ and $P_1(t)$.
- Induction hypothesis:

$$P_{n-1}(t) = e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

Use $\frac{d}{dt}(e^{\lambda t} P_n(t)) = \lambda e^{\lambda t} P_n(t)$.

$P_n(t)$, $n \geq 1$

- Look back at what we want to have:

$$P_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

- We already have $P_0(t)$ and $P_1(t)$.
- Induction hypothesis:

$$P_{n-1}(t) = e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

Use $\frac{d}{dt}(e^{\lambda t} P_n(t)) = \lambda e^{\lambda t} P_n(t) + e^{\lambda t} \frac{d}{dt}(P_n(t)) = \lambda e^{\lambda t} P_n(t) + e^{\lambda t} \frac{\lambda^n t^{n-1}}{(n-1)!} = \lambda e^{\lambda t} P_{n-1}(t) + \frac{\lambda^n t^{n-1}}{(n-1)!}.$

$P_n(t)$, $n \geq 1$

- Look back at what we want to have:

$$P_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

- We already have $P_0(t)$ and $P_1(t)$.
- Induction hypothesis:

$$P_{n-1}(t) = e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

Use $\frac{d}{dt}(e^{\lambda t} P_n(t)) = \lambda e^{\lambda t} P_n(t) + e^{\lambda t} \frac{d}{dt}(P_n(t)) = \lambda e^{\lambda t} P_n(t) + e^{\lambda t} \frac{\lambda^n t^{n-1}}{(n-1)!} = \lambda e^{\lambda t} P_n(t) + \frac{\lambda^n t^{n-1}}{(n-1)!}.$

$$\begin{aligned}\Rightarrow \quad & \int d(e^{\lambda t} P_n(t)) = \int \frac{\lambda^n t^{n-1}}{(n-1)!} dt \\ \Rightarrow \quad & e^{\lambda t} P_n(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} \cdot \frac{t^n}{n} + C \\ \Rightarrow \quad & P_n(t) = e^{-\lambda t} \frac{\lambda^n t^n}{n!}. \quad (\text{use } P_n(0) = 0)\end{aligned}$$

Further related topics

- Stochastic counting process: **point processes**
- Hawkes (self-exciting) processes.
 - Earthquake modeling, financial analysis.
 - A reference: <https://arxiv.org/pdf/1507.02822.pdf>