# Randomized Algorithms

### Balls and Bins: Some basics

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### Outline

- The Birthday Paradox Revisited
- Balls into Bins
- Poisson Distribution

• By definition of  $e^x$ :

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$\Rightarrow e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + x + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \cdots$$

By the Binomial Theorem:

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^n \binom{n}{k} \left(\frac{1}{n}\right)^k$$

$$= \sum_{k=0}^n \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \cdot \frac{1}{n^k}$$

$$= \sum_{k=0}^n \left(\frac{1}{n}\right) \cdot \left(\frac{n-1}{n}\right) \cdots \left(\frac{n-k+1}{n}\right) \cdot \frac{1}{k!}.$$

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$$\therefore \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

Useful approaches:

$$\left(1+\frac{1}{n}\right)^{-m} = \left(\left(1+\frac{1}{n}\right)^n\right)^{-m/n} \approx e^{-m/n}.$$

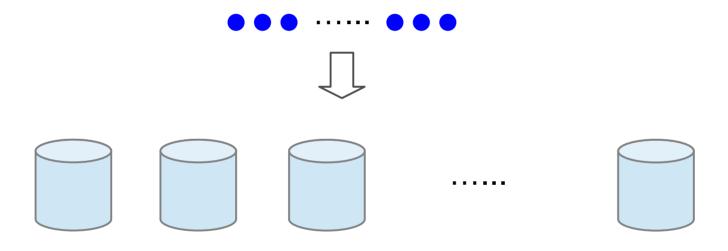
$$\frac{k^k}{k!} < \sum_{i=0}^{\infty} \frac{k^i}{i!} = e^k.$$

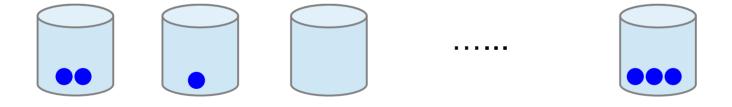
$$\therefore k! > \left(\frac{k}{e}\right)^k.$$

# Assignment 04

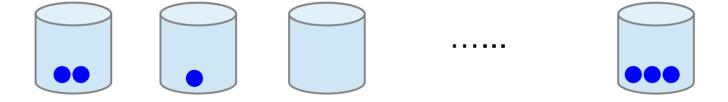
• Show that for  $|x| \le 1$ ,  $e^x(1-x^2) \le 1 + x \le e^x$ .

• Let 
$$\left(1 - \left(1 - \frac{1}{n}\right)^{km}\right)^k \approx (1 - e^{\square})^k$$
.  
Find  $\square = ?$ 

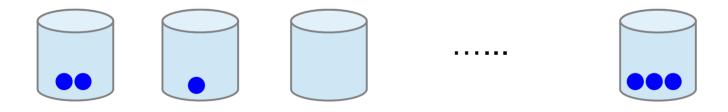




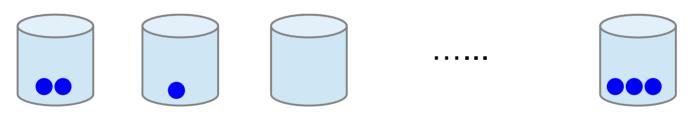
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- Let's say there are 30 people in a room. What's the probability that no two people in the room share the same birthday?

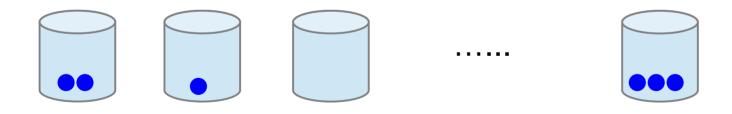


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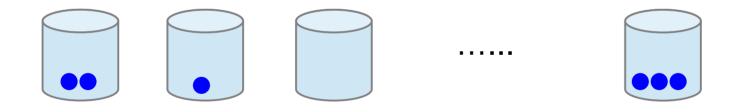
$$\frac{\binom{365}{30}30!}{365^{30}}.$$

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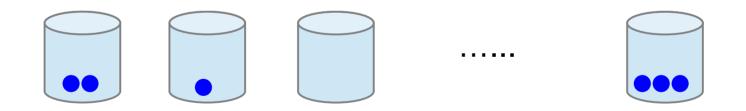
$$\left(1 - \frac{1}{365}\right) \cdot \left(1 - \frac{2}{365}\right) \cdots \left(1 - \frac{29}{365}\right).$$

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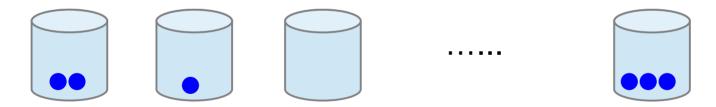
$$\left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right) = \prod_{j=1}^{m-1} \left(1 - \frac{j}{n}\right).$$

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$$\left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right) = \prod_{j=1}^{m-1} \left(1 - \frac{j}{n}\right) \approx \prod_{j=1}^{m-1} \frac{e^{-j/n}}{n} = \exp\left\{-\sum_{j=1}^{m-1} \frac{j}{n}\right\} = e^{-m(m-1)/2n} \approx e^{-m^2/2n}.$$

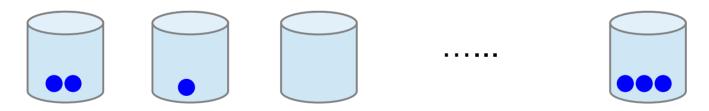
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 Matching your observation?

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    - The average load is n/n = 1?!

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$$\leq n\left(\frac{\ln\ln n}{\ln n}\right)^{3\ln n/\ln\ln n} = e^{\ln n}\left(\frac{e^{\ln\ln\ln n}}{e^{\ln\ln n}}\right)^{3\ln n/\ln\ln n}$$

$$= e^{\ln n}(e^{\ln\ln\ln n-\ln\ln n})^{3\ln n/\ln\ln n}$$

$$= e^{-2\ln n+3(\ln n)(\ln\ln n)/\ln\ln n}$$

$$\leq \frac{1}{n}.$$

• A set of  $n = 2^m$  numbers chosen uniformly at random in  $[0, 2^k]$ ,  $k \ge m$ .

#### **Bucket Sort:**

- Stage 1: place the elements into *n* buckets.
  - $j^{th}$  bucket: holds all elements whose first m binary digits corresponds to j.
    - = e.g.,  $n = 2^{10}$ , bucket 3 contains all elements whose first 10 digits are 000000011.
- Stage 2: sort each bucket using any standard sorting algorithm.

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#### **Bucket Sort:** Expected *O*(*n*) time?!

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  - The number of elements landing in a specific bucket: Binomial(n, 1/n).
- Stage 2: sort each bucket using any standard sorting algorithm.

- $X_j$ : the number of elements landing in bucket j.
- The time to sort bucket  $j: c(X_i)^2$ , for some constant c.
- The expected time for sorting in Stage 2:

$$\mathbf{E}\left[\sum_{j=1}^{n} c(X_{j})^{2}\right] = c\sum_{j=1}^{n} \mathbf{E}[X_{j}^{2}] = cn\mathbf{E}[X_{1}^{2}].$$

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 The same for all buckets.

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$$= n \cdot \frac{1}{n}\left(1 - \frac{1}{n}\right) + \left(n \cdot \frac{1}{n}\right)^{2}$$

$$= 2 - \frac{1}{n}$$

$$< 2.$$

# Assignment 05

- Suppose that *n* balls are thrown independently and uniformly at random into *n* bins.
  - Find the conditional probability that bin 1 has one ball given that exactly one ball fell into the first three bins.
  - Find the conditional expectation of the number of balls in bin 1 under the condition that bin 2 received no balls.

## Questions

- What is the probability that **a given bin is empty**?
- What is the expected number of **empty bins**?

• The probability that the *i*th bin remains empty is

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The expected "fraction"

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The probability that a given bin has r balls is

$$p_r = {m \choose r} \left(\frac{1}{n}\right)^r \left(1 - \frac{1}{n}\right)^{m-r}$$

$$= \frac{1}{r!} \frac{m(m-1)\cdots(m-r+1)}{n^r} \left(1 - \frac{1}{n}\right)^{m-r}.$$

$$\approx \frac{e^{-m/n}(m/n)^r}{r!}.$$
Expected number of bins with

Expected number of bins with exactly r balls  $\approx np_r$ .



Siméon Poisson (1781–1840) Wikipedia

• A discrete Poisson random variable X with parameter  $\mu$  is given by the following probability distribution on j = 0, 1, 2, ...:

$$\Pr[X=j] = \frac{e^{-\mu}\mu^{j}}{j!}.$$



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$$\frac{e^{-m/n}(m/n)^r}{r!}.$$



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• Try to verify if it's proper:

$$\sum_{j=0}^{\infty} \Pr[X=j] = \sum_{j=0}^{\infty} \frac{e^{-\mu} \mu^j}{j!}$$

$$= e^{-\mu} \sum_{j=0}^{\infty} \frac{\mu^j}{j!}$$

$$= 1.$$



Siméon Poisson (1781–1840) Wikipedia

• The expectation:

$$\mathbf{E}[X] = \sum_{j=0}^{\infty} j \Pr[X = j] = \sum_{j=1}^{\infty} j \frac{e^{-\mu} \mu^j}{j!}$$

$$= \mu \cdot \sum_{j=1}^{\infty} \frac{e^{-\mu} \mu^{j-1}}{(j-1)!}$$

$$= \mu \cdot \sum_{j=0}^{\infty} \frac{e^{-\mu} \mu^j}{j!}$$

$$= \mu.$$



Siméon Poisson (1781 - 1840)Wikipedia

$$\Pr[X + Y = j]$$

$$\Pr[X + Y = j] = \sum_{i=1}^{n} \Pr[(X = k) \cap (Y = j - k)]$$

- The sum of finite number of independent Poisson random variables is a Poisson random variable.
- $= \sum_{k=0}^{J} \frac{e^{-\mu_1} \mu_1^k}{k!} \frac{e^{-\mu_2} \mu_2^{(j-k)}}{(j-k)!}$

$$= \frac{e^{-(\mu_1 + \mu_2)}}{j!} \sum_{k=0}^{j} \frac{j!}{k!(j-k)!} \mu_1^k \mu_2^{(j-k)}$$

$$= \frac{e^{-(\mu_1 + \mu_2)}}{j!} \sum_{k=0}^{j} {j \choose k} \mu_1^k \mu_2^{(j-k)}$$
$$= \frac{e^{-(\mu_1 + \mu_2)} (\mu_1 + \mu_2)^j}{j!}.$$

$$= \frac{e^{-(\mu_1 + \mu_2)}(\mu_1 + \mu_2)^j)}{j!}.$$



Siméon Poisson (1781–1840) Wikipedia

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For any t,

$$\mathbf{E}[e^{tX}] = \sum_{k=0}^{\infty} \frac{e^{-\mu} \mu^k}{k!} e^{tk} = e^{\mu(e^t - 1)} \sum_{k=0}^{\infty} \frac{e^{-\mu e^t} (\mu e^t)^k}{k!}$$

$$= e^{\mu(e^t-1)}.$$



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$$= e^{\mu(e^t - 1)}.$$

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t) = e^{(\mu_1 + \mu_2)(e^t - 1)}$$

### Limit of the Binomial Distribution

- Theorem. Let  $X_n \sim \text{Binomial}(n, p)$  be a binomial random variable
  - *p*: a function of *n*
  - −  $\lim_{n\to\infty} np = \lambda$  is a constant, independent of *n*.

Then for any fixed k,

$$\lim_{n \to \infty} \Pr[X_n = k] = \frac{e^{-\lambda} \lambda^k}{k!}.$$

## Scenario

- *m* balls into *n* bins.
  - m = f(n);
  - $\lim_{m\to\infty} m/n = \lambda$ ;
  - $X_m$ : the number of balls in a specific bin.
    - Binomial(m, 1/n).
- From the theorem:

$$\lim_{m \to \infty} \Pr[X_m = r] = \frac{e^{-m/n} (m/n)^r}{r!}.$$