The Price of Anarchy in Network Creation Games

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Network creation games

- First introduced in PODC 2003.


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Network creation games [Fabrikant et al. @PODC 2003]

- n players: 1, 2, . . . , n.
- s_i: specified by a subset of $\{1, 2, \ldots, n\} \setminus \{i\} = [n] \setminus \{i\}$ as the strategy of player i.
  - The set of neighbors where player i forms a link (edge).
- G_s: the undirected graph with vertex set $[n]$ and edges corresponding to $s = \langle s_1, s_2, \ldots, s_n \rangle$.
- G_s has an edge $\{i, j\}$ if either $i \in s_j$ or $j \in s_i$.
- $d_s(i, j)$: the distance between i and j in G_s.
- G_s: an equilibrium graph (when the context is clear).

Network creation games (Two models)

The sum model

$$c_i(s) = \alpha|s_i| + \sum_{j=1}^{n} d_s(i,j).$$

The max model

$$c_i(s) = \alpha|s_i| + \max_{j=1}^{n} d_s(i,j).$$

- The total cost is $c(s) = \sum_{i=1}^{n} c_i(s)$.

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Network creation games (contd.)

Theorem [Fabrikant et al.@PODC 2003]

The PoA for the sum network creation game is $O(\sqrt{\alpha})$ for all α.

Preliminaries

Let's have a look at Fabrikant's results for $\alpha < 2$.

- $\alpha < 1$:
  - the social optimum: the complete graph.
  - ⋆ It's also a NE ($\therefore$ PoA $= 1$).

Preliminaries (contd.)

- $1 \leq \alpha < 2$:
  - The social optimum: still the complete graph (i.e., K_n).
  - Any NE must be connected and has diameter ≤ 2.
  - ⋆ K_n is NOT a NE.
  - ⋆ The worst NE: a star.
    - $\alpha \cdot |E| + |E| \cdot 2 \cdot 1 + (\binom{n}{2} - |E|) \cdot 2 \cdot 2 = (\alpha - 2) \cdot |E| + 2n(n-1)$.

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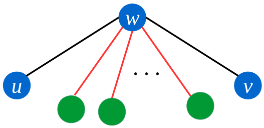

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  - $$\begin{aligned} \text{PoA} &= \frac{C(\text{star})}{C(K_n)} = \frac{(\alpha-2)\cdot(n-1) + 2n(n-1)}{\alpha\binom{n}{2} + 2 \cdot \binom{n}{2} \cdot 1} \\ &= \frac{4}{2+\alpha} - \frac{4-2\alpha}{n(2+\alpha)} \\ &< \frac{4}{3}. \end{aligned}$$

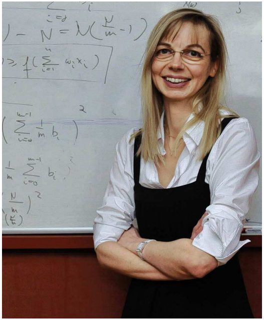
Preliminaries (contd.)

Lemma 1 [Albers et al. @SODA 2006]

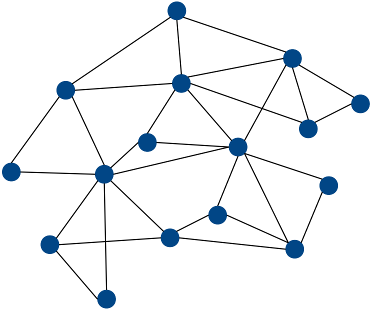
For any Nash equilibrium s and any vertex v_0 in G_s,

$$c(s) \leq 2\alpha(n-1) + n \cdot \mathrm{Dist}(v_0) + (n-1)^2.$$

- $\mathrm{Dist}(v_0) = \sum_{v \in V(G_s)} d_s(v_0, v)$.



Sketch of proving Lemma 1



- A graph G_s corresponding to a NE s.

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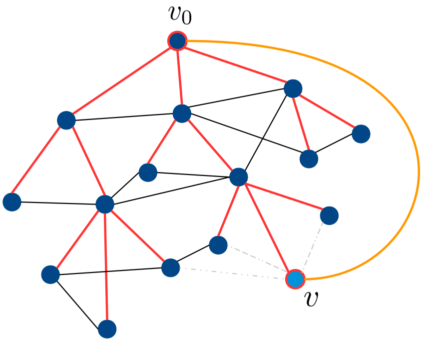

- $T(v_0)$: the shortest-path tree rooted at v_0.
- η_v: the number of tree edges built by v in $T(v_0)$.
  - ⋆ $c_v(s) \leq \alpha(\eta_v + 1) + \mathrm{Dist}(v_0) + n - 1$.
    $c_{v_0}(s) = \alpha \cdot \eta_{v_0} + \mathrm{Dist}(v_0)$.
- $c(s) = \sum_{v \in V(G_s)\setminus\{v_0\}} c_v(s) + c_{v_0}(s)$
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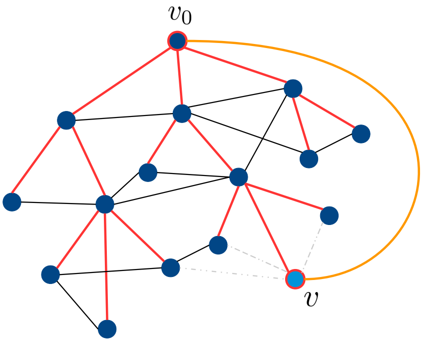

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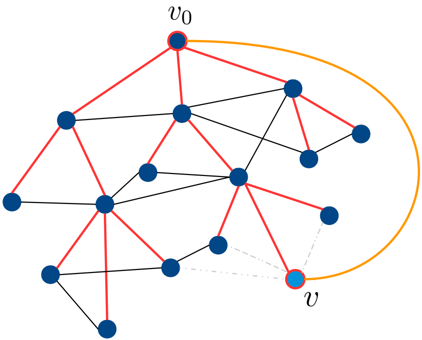

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Lemma 2

If the shortest-path tree in an equilibrium graph G_s rooted at u has depth d, then PoA $\leq d + 1$.

- For some $u \in V$,

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\text{PoA} &\leq \frac{2\alpha(n-1) + n \cdot \text{Dist}(u) + (n-1)^2}{\alpha(n-1) + n(n-1)} \\
&\leq \frac{2\alpha(n-1) + n \cdot (n-1)d + (n-1)^2}{\alpha(n-1) + n(n-1)} \\
&< \frac{2\alpha(n-1) + n(n-1)(d+1)}{\alpha(n-1) + n(n-1)} \\
&\leq \max\left\{\frac{2\alpha(n-1)}{\alpha(n-1)}, \frac{n(n-1)(d+1))}{n(n-1)}\right\} \\
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Preliminaries (contd.)

- $N_k(u)$: the set of vertices with distance $\leq k$ from u.
- $N_k := \min_{v \in V(G_s)} |N_k(v)|$.

Lemma 3

For any equilibrium graph G_s, $N_2(u) > \frac{n}{2\alpha}$ for every vertex u and $\alpha \geq 1$.

- Assume that $|\{v \in V(G_s) \mid d_s(v, u) > 2\}| \geq \frac{n}{2}$.
  - Otherwise, $|N_2(u)| \geq n/2 \geq n/(2\alpha)$.

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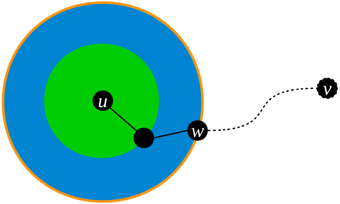

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- $|$ vertices assigned to $w \in S$ $| \leq \alpha$.
  - Otherwise, u could buy (u, w).
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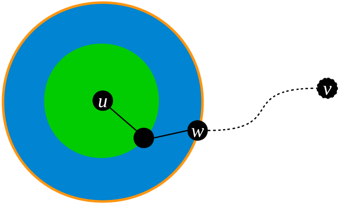

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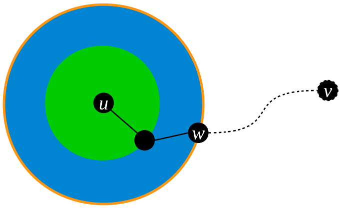

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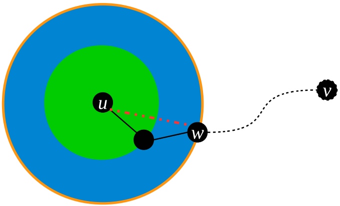

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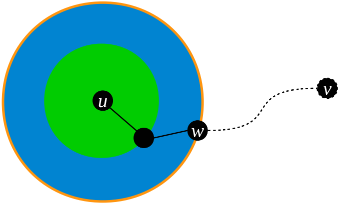

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$O(1)$ upper bounds for $\alpha = O(\sqrt{n})$

Theorem 4

For $\alpha < \sqrt{n/2}$, the PoA ≤ 6.

Key: *Show that $T(u)$ has depth ≤ 5 for any $u \in V(G_s)$.*

- Suppose that $\exists v \in V(G_s)$ s.t. $d_s(u, v) \geq 6$.
- v can buy $\{u, v\}$ to decrease its distance from all vertices in $N_2(u)$ by at least 1.
  - $\because$ v has not bought it $\quad \therefore |N_2(u)| \leq \alpha$.
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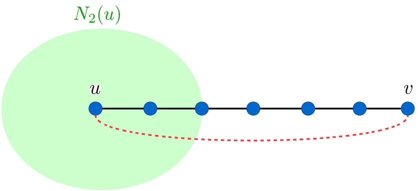

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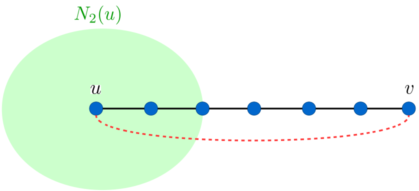

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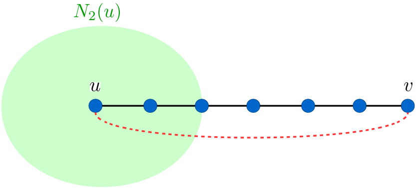

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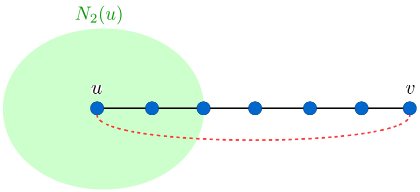

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Proof:

- Δ: maximum vertex degree of G_s.
- $N_2(u) \leq 1 + \Delta + \Delta(\Delta - 1) = 1 + \Delta^2$ for an arbitrary u.
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- Let v be a vertex with degree Δ.
- Suppose that $\exists u \in V(G_s)$ s.t. $d_s(v, u) \geq 4$.
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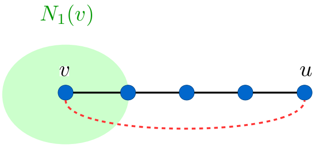

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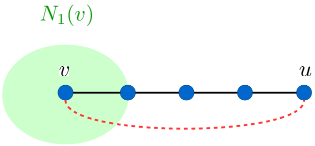

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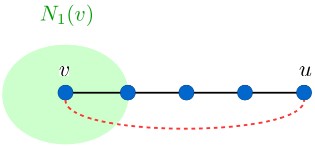

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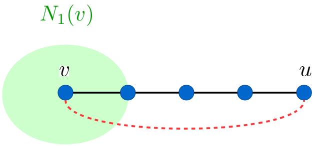

Theorem 5

For $\alpha < \sqrt[3]{n/2}$, the PoA ≤ 4.

Proof:

- Δ: maximum vertex degree of G_s.
- $N_2(u) \leq 1 + \Delta + \Delta(\Delta - 1) = 1 + \Delta^2$ for an arbitrary u.
- $1 + \Delta^2 > n/(2\alpha) > \alpha^2 \Rightarrow \Delta > \alpha - 1$.
- Let v be a vertex with degree Δ.
- Suppose that $\exists u \in V(G_s)$ s.t. $d_s(v, u) \geq 4$.
- u can buy $\{u, v\}$ to decrease its distance to all vertices in $N_1(v)$ by ≥ 1

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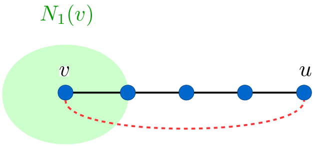


An $O(1)$ upper bound for $\alpha = O(n^{1-\epsilon})$

The main theorem

Theorem 10

For $\epsilon \geq 1/\lg n$ and $1 \leq \alpha < n^{1-\epsilon}$, PoA $\leq 4.667 \cdot 3^{\lceil 1/\epsilon \rceil} + 8$.

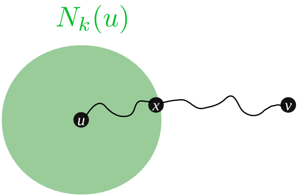
- Let's go through several lemmas and corollaries first.

Lemma 6

For any vertex u in an equilibrium graph G_s, if $|N_k(u)| > n/2$, then $|N_{2k+2\alpha/n}(u)| = n$.

Proof:

- Assume to the contrary that $|N_{2k+2\alpha/n}(u)| < n$.
  - $\exists v \in V(G_s)$ s.t. $d_s(u, v) \geq 2k + 1 + 2\alpha/n$.



$$d_s(u, v) \geq 2k + 1 + 2\alpha/n$$
$$d_s(u, x) + d_s(x, v) \geq d_s(u, v)$$
$$\Rightarrow d_s(x, v) \geq k + 1 + 2\alpha/n$$

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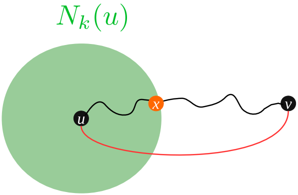

$$d_s(x, v) \geq k + 1 + 2\alpha/n \quad \to \quad \leq k + 1.$$
$$\mathrm{Dist}(v) \text{ decreases by } \geq N_k(u) \cdot 2\alpha/n.$$
$$\therefore \alpha \geq |N_k(u)| \cdot 2\alpha/n \quad \text{(Contradiction)}$$

Lemma 6

For any vertex u in an equilibrium graph G_s, if $|N_k(u)| > n/2$, then $|N_{2k+2\alpha/n}(u)| = n$.

- Setting $\alpha < n/2$ & $\alpha < 12n \lg n$:

Corollary 7

For any vertex $u \in V(G_s)$ with $\alpha < n/2$, if $|N_k(u)| > n/2$, then $|N_{2k+1}(u)| = n$.

Corollary 8

For any vertex $u \in V(G_s)$ with $\alpha < 12n \lg n$, if $|N_k(u)| > n/2$, then $|N_{2k+24 \lg n}(u)| = n$.

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Lemma 9

If $|N_k(u)| > Y$ for every vertex u in an equilibrium graph G_s, then

- either $|N_{2k+3}(u)| > n/2$ for some u
- or $|N_{3k+3}(u)| \geq N_k \cdot n/\alpha > Yn/\alpha$ for every u.

- Obviously true if $\exists u$ s.t. $|N_{2k+3}(u)| > n/2$.
- Assume that for every u, $|N_{2k+3}(u)| \leq n/2$.

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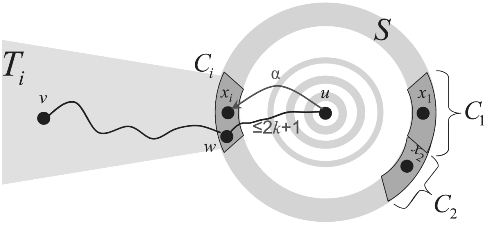
Proof of Lemma 9



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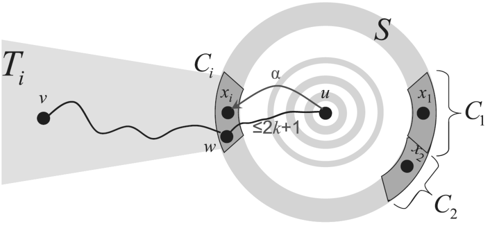
- $S = \{v \in V(G_s) \mid d_s(v, u) = 2k + 3\}$.
- Select a subset of S (*center points*) by the following algorithm:
  - Unmark all vertices in S;
  - Repeatedly select an unmarked $x \in S$ as a center point, mark all unmarked vertices in $\{x' \in S \mid d_s(x', x) \leq 2k\}$, and assign these vertices to x.

Proof of Lemma 9 (contd.)



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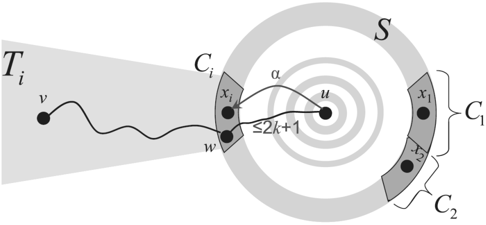
- Suppose that we select $x_1, x_2, \ldots, x_\ell$ as center points.
- $\ell \geq n/\alpha$. WHY?

  We prove it later...

Proof of Lemma 9 (contd.)



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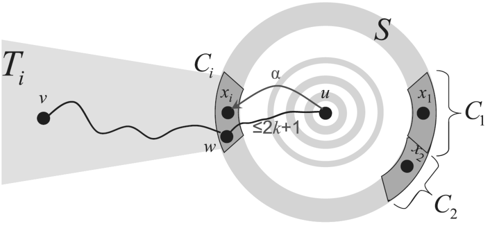

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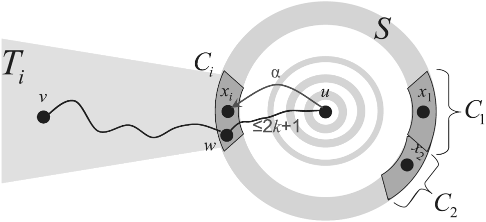

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- According to the algorithm, $d_s(x_i, x_j) > 2k$ for any pair of center points x_i, x_j.
  - $N_k(x_i) \cap N_k(x_j) = \emptyset$ for $i \neq j$.
  - By the hypothesis, $|N_k(x_i)| > Y$ for every x_i.
- Hence, $|\bigcup_{i=1}^{\ell} N_k(x_i)| = \sum_{i=1}^{\ell} |N_k(x_i)| \geq \ell \cdot N_k > \ell Y$.
- u has a path of length $\leq 3k+3$ to every vertex in $N_k(x_i)$.
- Therefore,

  $|N_{3k+3}(u)| \geq |\bigcup_{i=1}^{\ell} N_k(x_i)|$.

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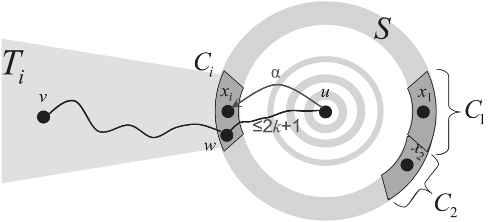

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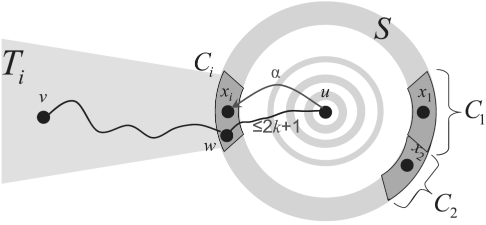

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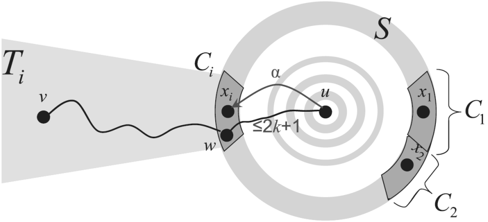

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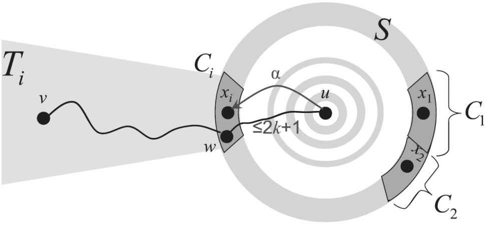

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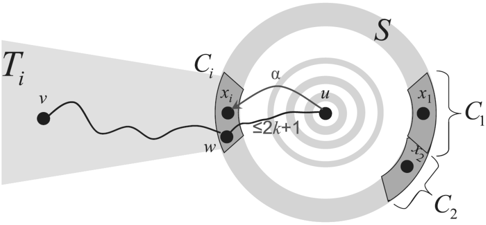
Proving $\ell \geq n/\alpha$.

- Let C_i be the vertices in S assigned to x_i.
    - $S = \bigcup_{i=1}^{\ell} C_i$.
- For each vertex v at distance $\geq 2k+3$ from u:
    - Pick any one shortest path from v to u (via exactly one vertex $w \in S$);
    - Assign v to the same center point as w.
- $T_i :=$ the set of vertices assigned to x_i and whose distance from u is $> 2k+3$.

Proof of Lemma 9 (contd.)



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- $T_i :=$ the set of vertices assigned to x_i and whose distance from u is $> 2k+3$.
- If u bought $\{u, x_i\}$, distance between u and w becomes $\leq 2k+1$.
  - u's distance to v would decrease by $\geq 2k+3-(2k+1) = 2$.
  - $\therefore \alpha \geq 2|T_i|$.
- Moreover, $|N_{2k+3}(u)| \leq n/2$ (the earliest assumption)
  $\Rightarrow \sum_{i=1}^{\ell} |T_i| \geq n/2$.
- Therefore, $\ell\alpha \geq 2\sum_{i=1}^{\ell} |T_i| \geq n$.

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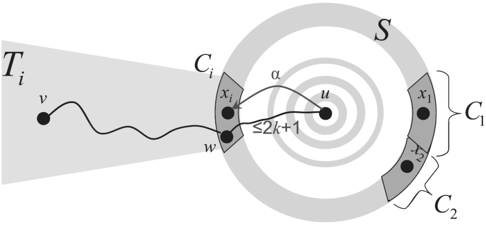

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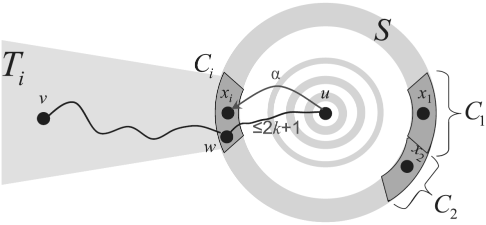

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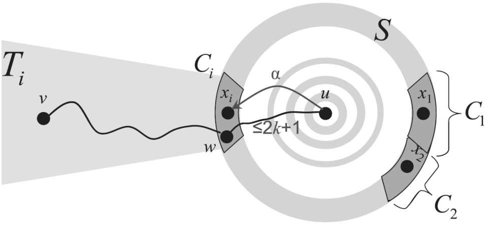

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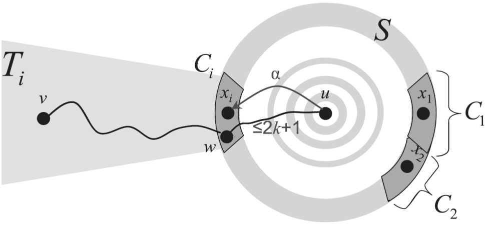

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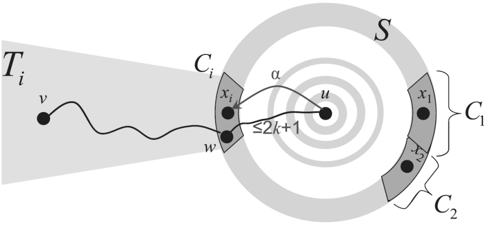

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The main theorem (proof)

Theorem 10

For $\epsilon \geq 1/\lg n$ and $1 \leq \alpha < n^{1-\epsilon}$, PoA $\leq 4.667 \cdot 3^{\lceil 1/\epsilon \rceil} + 8$.

Proof:

- Let $X = n/\alpha > n^{\epsilon}$.
- Define $a_1 = 2$ and $a_i = 3a_{i-1} + 3$ for all $i > 1$.

⋆ $N_2(u) > n/(2\alpha) = X/2$ for all u. (Lemma 3)

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Proof:

- Moreover, $X^j/2 < N_{a_j} \leq n$, so $X^j < 2n$.

  $\therefore\ j < \frac{1}{\epsilon}(1 + 1/\lg n) \quad (n^{\epsilon j} < (n/\alpha)^j = X^j < 2n)$.

  $\Rightarrow\ j < \frac{1}{\epsilon} + 1$ and hence $j \leq \lceil 1/\epsilon \rceil \quad (\because \epsilon \geq 1/\lg n)$.

- Hence, $|N_{2a_{\lceil 1/\epsilon \rceil}+3}(v)| \geq |N_{2a_j+3}(v)| > n/2$.

The main theorem (proof)

Theorem 10

For $\epsilon \geq 1/\lg n$ and $1 \leq \alpha \leq n^{1-\epsilon}$, PoA $\leq 4.667 \cdot 3^{\lceil 1/\epsilon \rceil} + 8$.

Proof:

$\because$ $\alpha < n^{1-\epsilon}$ and $\epsilon \geq 1/\lg n$, we have $\alpha < \frac{n}{n^{1/\lg n}} = n/2$.

$\therefore$ $|N_{4a_{\lceil 1/\epsilon \rceil}+7}(v)| = n$ (Corollary 7).

- Hence, the shortest path tree rooted at v has depth $\leq 4a_{\lceil 1/\epsilon \rceil} + 7$.
- PoA $\leq 4a_{\lceil 1/\epsilon \rceil} + 8$ (Lemma 2).

- Solving the recurrence relation $a_i = 3a_{i-1} + 3$ with $a_1 = 2 \Rightarrow a_i = \frac{7}{6}3^i - \frac{3}{2} < \frac{7}{6}3^i$.

$\therefore$ PoA $\leq 4 \cdot \frac{7}{6}3^{\lceil 1/\epsilon \rceil} + 8 \leq 4.667 \cdot 3^{\lceil 1/\epsilon \rceil} + 8$.

An $o(n^\epsilon)$ upper bound for $\alpha < 12n \lg n$

Lemma 11 (similar to Lemma 9)

If $N_k > Y$ in an equilibrium graph G_s, then

- either $|N_{4k+1}(u)| > n/2$ for some u
- or $|N_{5k+1}(u)| \geq N_k \cdot kn/\alpha > Ykn/\alpha$ for every u.

Theorem 12 (similar to Theorem 10)

For $1 \leq \alpha < 12n \lg n$, the PoA is $O(5^{\sqrt{\lg n}} \lg n)$.

Proof of Theorem 12

Theorem 12

For $1 \le \alpha < 12n \lg n$, the PoA is $O(5^{\sqrt{\lg n}} \lg n)$.

- Let $Z = 12 \lg n$. $\Rightarrow$ $\alpha/n < 12 \lg n = Z,\ \ n/\alpha > 1/Z$.
- $|N_Z| > Z$ ($\because$ any equilibrium graph is connected).
- Either $|N_{4k+1}(v)| > n/2$ sor some v, or $|N_{5k+1}| \ge N_k \cdot (kn/\alpha)$. (Lemma 11)
- The recurrence relation: $a_i = 5a_{i-1} + 1$, $a_0 = Z$.
  - ▷ $a_i > Z \cdot 5^i$.
  - ▷ $a_j = O(5^j \lg n)$.
- Let j be the least number s.t. $|N_{4a_j+1}(v)| > n/2$.
- $N_{a_{i+1}} \ge N_{a_i} \cdot (a_i \cdot n/\alpha) > 5^i N_{a_i}$ for each $i < j$.
- Hence $n \ge N_{a_j} > 5^{\sum_{i=1}^{j-1} i}$.
  - $\sum_{i=1}^{j-1} i = j(j-1)/2 \le \log_5 n$.
  - ▷ $j < 1 + \sqrt{2 \log_5 n} < 1 + \sqrt{\lg n}$.

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For $1 \le \alpha < 12n\lg n$, the PoA is $O(5^{\sqrt{\lg n}} \lg n)$.

- Let $Z = 12\lg n$. $\Rightarrow \alpha/n < 12\lg n = Z,\ n/\alpha > 1/Z$.
- $|N_Z| > Z$ ($\because$ any equilibrium graph is connected).
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Proof of Theorem 12 (contd.)

Theorem 12

For $1 \leq \alpha < 12n \lg n$, the PoA is $O(5^{\sqrt{\lg n}} \lg n)$.

Recall that:

Corollary 8

For any vertex $u \in V(G_s)$ with $\alpha < 12n \lg n$, if $|N_k(u)| > n/2$, then $|N_{2k+24 \lg n}(u)| = n$.

- The depth of the shortest-path tree rooted at v:
  $\leq 2(4a_j + 1) + 24 \lg n < 2(4a_{1+\sqrt{\lg n}} + 1) + 24 \lg n$ (Corollary 8).
- Therefore, PoA is $O(5^{\sqrt{\lg n}} \lg n)$. (Lemma 2 & $a_j = O(5^j \lg n)$)

A brief summary

$\alpha =$	0 – 1	1 – 2	2 – $\sqrt[3]{n/2}$	$\sqrt[3]{n/2}$ – $\sqrt{n/2}$	$\sqrt{n/2}$ – $O(n^{1-\epsilon})$	$O(n^{1-\epsilon})$ – n – $12n\lceil \lg n \rceil$	$12n\lceil \lg n \rceil$ – n^2
	1	$\frac{4}{3}$	≤ 4	≤ 6	$O(1)$	$2^{O(\sqrt{\lg n})}$	$O(1)$

— : Fabrikant *et al.* 2003

— : This paper (Demaine *et al.* 2007)

— : Albers *et al.* 2006


Thanks for your attention!