

# Heaps

Joseph Chuang-Chieh Lin (林莊傑)

Department of Computer Science & Engineering,  
National Taiwan Ocean University

Fall 2025



# Outline

## 1 Introduction

- Building a heap

# Outline

## 1 Introduction

- Building a heap

# Heaps

## Max Tree

A **max tree** is a tree in which

- the key value in each node  $\geq$  the key values in its children.



# Heaps

## Max Tree

A **max tree** is a tree in which

- the key value in each node  $\geq$  the key values in its children.

## Min Tree

A **min tree** is a tree in which

- the key value in each node  $\leq$  the key values in its children.



# Heaps

## Max Tree

A **max tree** is a tree in which

- the key value in each node  $\geq$  the key values in its children.

## Min Tree

A **min tree** is a tree in which

- the key value in each node  $\leq$  the key values in its children.

## Max Heap

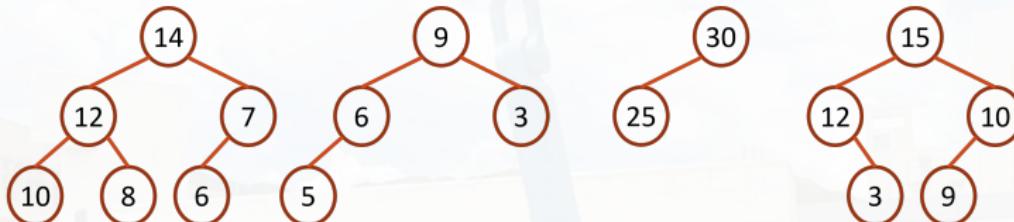
A complete binary tree that is also a max tree.

## Min Heap

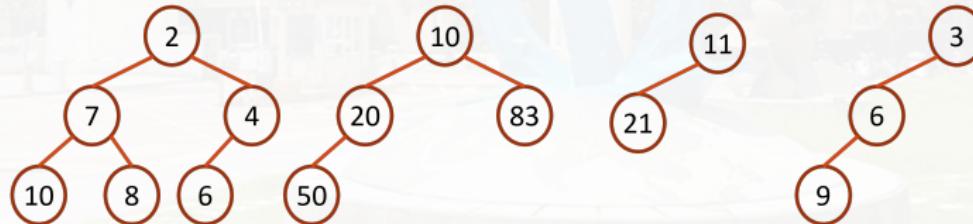
A complete binary tree that is also a min tree.



# Examples: Max & Min Trees

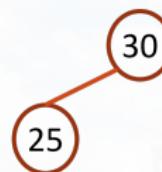
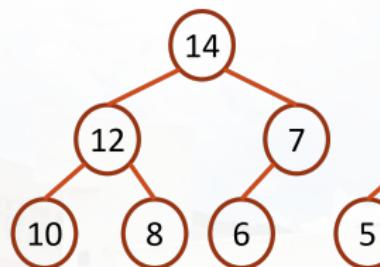


Max Trees



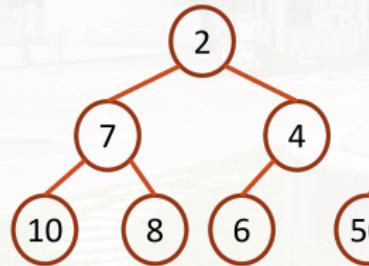
Min Trees

# Examples: Max & Min Heaps



Max Heaps

---



Min Heaps

# The Key Application: Priority Queues

- Heaps are frequently used to implement **priority queues**.
- In this kind of queue,
  - the element to be **deleted** is the one with **highest** (or **lowest**) priority.
  - at **any time**, an element with **arbitrary priority** can be **inserted** into the queue.

# Some Important Notes

- It's straightforward to implement a heap using an array (WHY?).



# Some Important Notes

- It's straightforward to implement a heap using an array (WHY?).
- Insert the new node **next to the last element** in the array.

# Some Important Notes

- It's straightforward to implement a heap using an array (WHY?).
- Insert the new node **next to the last element** in the array.
- A heap is a **complete** binary tree.

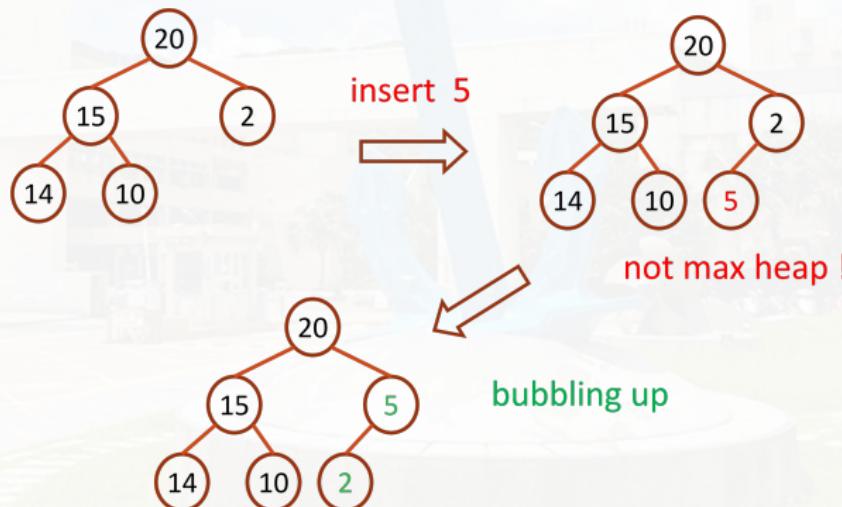
# Insertion into a Max Heap

- The **bubbling process**.
  - It begins at the new node of the tree and moves toward the root.



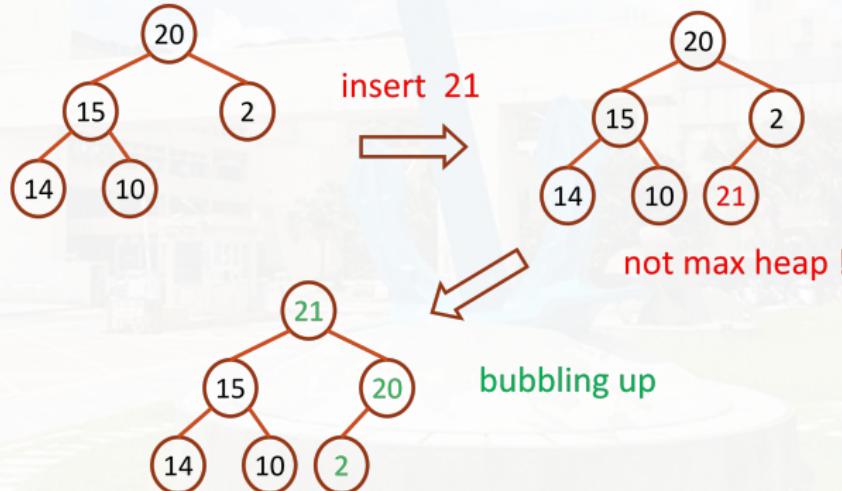
# Insertion into a Max Heap

- The **bubbling process**.
  - It begins at the new node of the tree and moves toward the root.



# Insertion into a Max Heap

- The **bubbling process**.
  - It begins at the new node of the tree and moves toward the root.



# The Code for Insertion into a Max Heap

- Consider the following declarations:

```
#define MAX_ELEMENTS 200 /* maximum heap size+1 */  
#define HEAP_FULL (n) (n == MAX_ELEMENTS -1)  
#define HEAP_EMPTY (n) (!n)  
typedef struct {  
    int key;  
    /* other fields */  
} element;  
element heap[MAX_ELEMENTS];  
int n = 0;
```

# The Code for Insertion into a Max Heap

```
void push (element item, int *n) {
    /* insert item into a max heap of current size *n */
    int i;
    if (HEAP_FULL(*n)) {
        printf("The heap is full.\n");
        exit(EXIT_FAILURE);
    } // O(1) time
    i = ++(*n); // the insertion index is n+1
    /* try bubbling up if necessary */
    while ((i != 1) && (item.key > heap[i/2].key)) {
        heap[i] = heap[i/2];
        i /= 2;
    } // O(lg n) time
    heap[i] = item; // O(1) time
}
```

- The time complexity of the insertion:  $O(\lg n)$ .



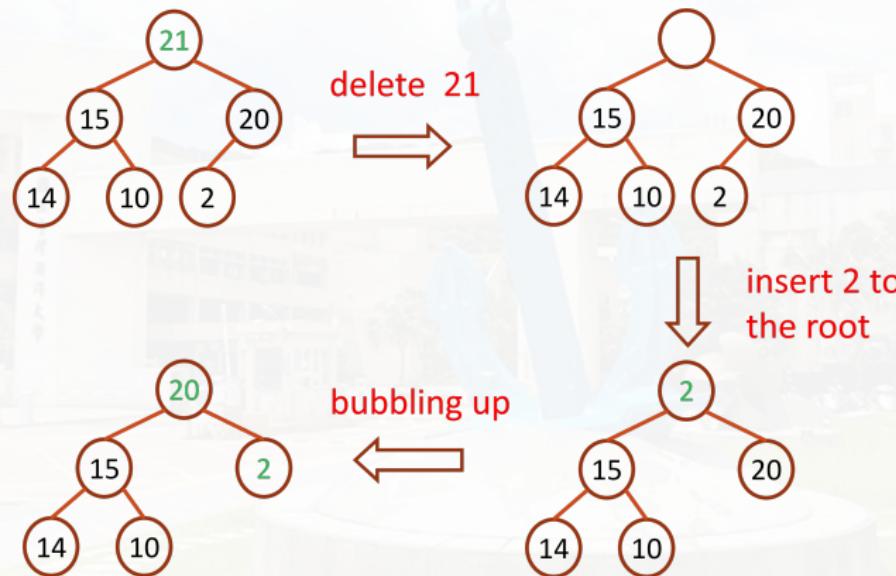
# Deletion from a Max Heap

- When an element is to be deleted from a max heap, it is **ALWAYS** taken from the root of the heap.

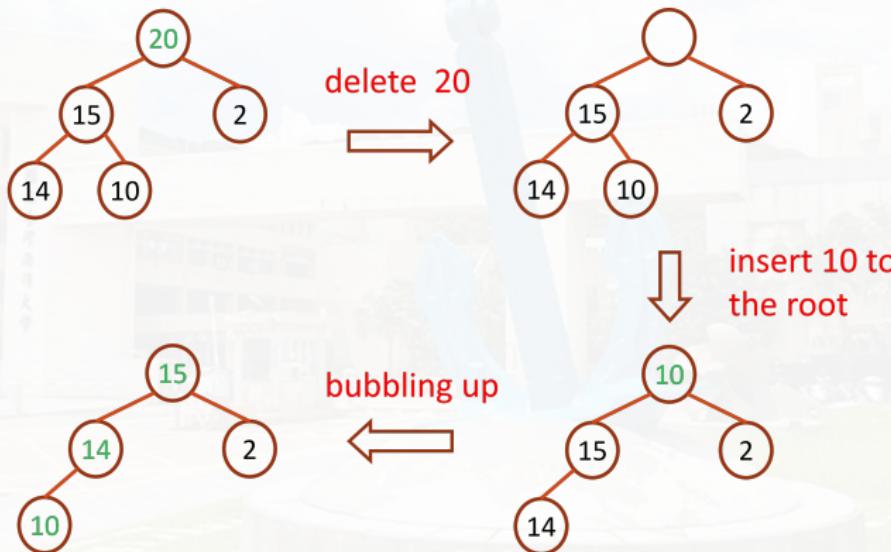
# Deletion from a Max Heap

- When an element is to be deleted from a max heap, it is **ALWAYS** taken from the root of the heap.
- The steps of deletion from a Max heap:
  - delete the root node.
  - insert the last node into the root (say  $r$ ).
  - use the **bubbling up process** to ensure that the resulting heap remains a max heap (a.k.a. **heapify** at  $r$ ).

# Illustration of Deletion from a Max Heap



# Illustration of Deletion from a Max Heap



# The Code for Deletion from a Max Heap

```
element pop(int *n) {
    /* delete element with the highest key from the heap */
    int parent, child;
    element item, temp;
    if (HEAP_EMPTY(*n)) {
        fprintf(stderr, "The heap is empty\n");
        exit(EXIT_FAILURE);
    }
    /* save value of the element with the highest key */
    item = heap[1];
    /* use last element in heap to adjust heap */
    temp = heap[(*n)--]; // start Heapify from the root
    parent = 1;
    child = 2; // default: the left child
    while (child <= *n) { // O(lg n) time
        /* find the larger child of the current parent */
        if ((child < *n) && (heap[child].key < heap[child+1].key))
            child++; // okay, it's the right child!
        if (temp.key >= heap[child].key) break; // the new root is the maximum!
        /* if the max-child gets larger key, move to the next lower level */
        heap[parent] = heap[child];
        parent = child;
        child *= 2;
    }
    heap[parent] = temp;
    return item;
}
```

# Time Complexity of the Deletion from a Max Heap

- Delete the root node:  $O(1)$ .
- Insert the last node to the root:  $O(1)$ .
- Since the height of the heap is  $\lceil \lg(n + 1) \rceil$ , the while loop is iterated for  $O(\lg n)$  times.
- Thus, the overall time complexity: the time complexity of the deletion:  $O(\log n)$ .

# Outline

## 1 Introduction

- Building a heap



# How to build a heap for a set of $n$ input numbers?

- For each input number  $x$ , execute  $\text{push}(x)$ .



# How to build a heap for a set of $n$ input numbers?

- For each input number  $x$ , execute  $\text{push}(x)$ .
- The above process is correct and requires

# How to build a heap for a set of $n$ input numbers?

- For each input number  $x$ , execute  $\text{push}(x)$ .
- The above process is correct and requires  $O(n \log n)$  time.

# An $O(n)$ time algorithm for building a (max) heap

Input:  $n$  numbers:  $x_1, x_2, \dots, x_n$ .

## Efficient Heap Construction

- ① For each input number  $x_i$ , insert  $x_i$  into array  $A$  at  $A[i - 1]$  one by one.
- ② For  $i = \lfloor n/2 \rfloor - 1$  down to 0:
  - Run  $\text{heapify}(A, n, i)$

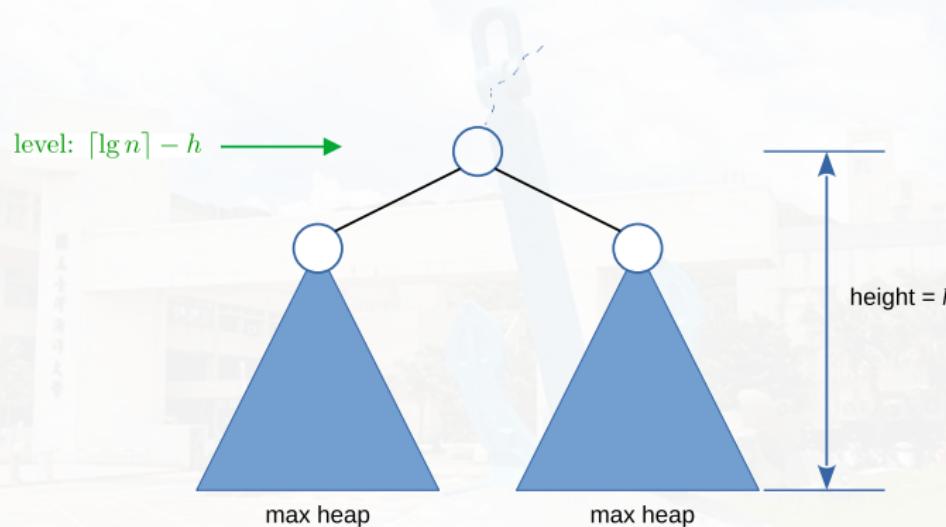
# An $O(n)$ time algorithm for building a (max) heap

Input:  $n$  numbers:  $x_1, x_2, \dots, x_n$ .

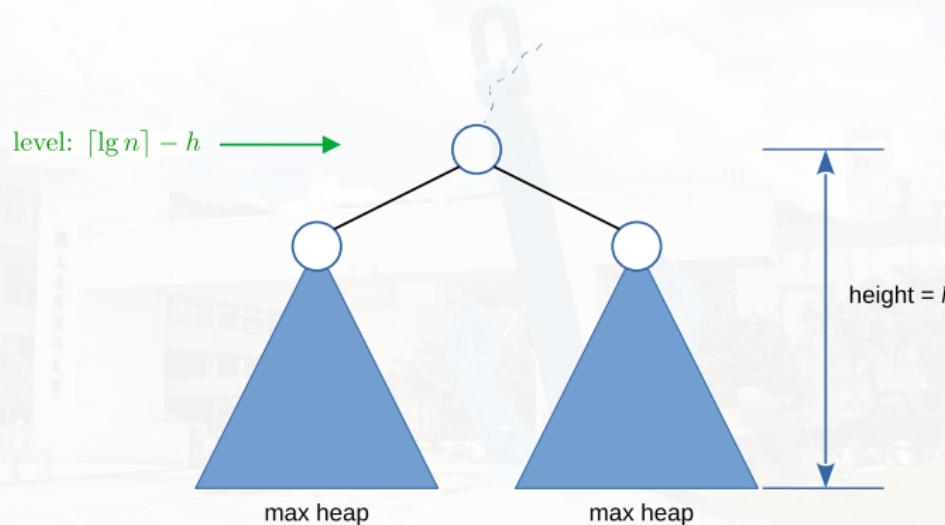
## Efficient Heap Construction

- ① For each input number  $x_i$ , insert  $x_i$  into array  $A$  at  $A[i - 1]$  one by one.
  - ② For  $i = \lfloor n/2 \rfloor - 1$  down to 0:
    - Run  $\text{heapify}(A, n, i)$
- That is, we build a heap in a bottom-up fashion!

# Heap recursive view (bottom-up)

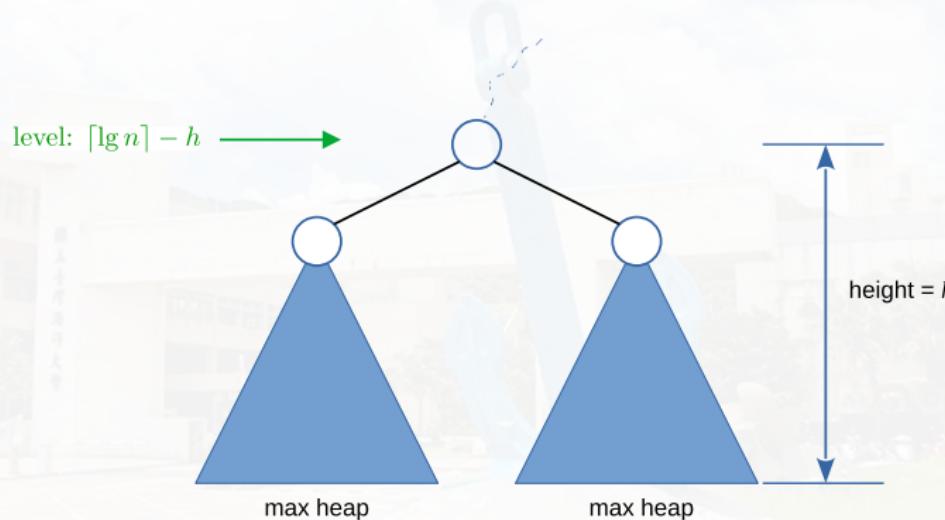


# Heap recursive view (bottom-up)



- Assumption: the left subtree & the right subtree are max heaps.

# Heap recursive view (bottom-up)



- Assumption: the left subtree & the right subtree are max heaps.
- Take the maximum of the roots of the two subtrees.

# heapify( $A, n, i$ ) for a Max Heap (nonrecursive)

- Assumption: the left and right subtrees of  $i$  are already max heaps.

```
void heapify(element A[], int n, int i) {
    int parent = i;
    int child = 2 * parent; // default child: left child
    element temp = A[parent];
    // each loop goes down one level, so O(log n) time in a complete binary tree.
    while (child <= n) {
        /* choose the larger child */
        if (child < n && A[child].key < A[child + 1].key) child++;

        /* if temp already >= max(child), we're done */
        if (temp.key >= A[child].key) break;

        /* move the larger child up one level */
        A[parent] = A[child];
        parent = child;
        child = 2 * parent;
    }
    A[parent] = temp;
}
```

# heapify( $A, n, i$ ) for a Max Heap (recursive version)

```
void heapify(element A[], int n, int i) {
    int left = 2*i + 1, right = 2*i + 2; // a heap is a complete binary tree
    int largest = i;

    if (left < n && A[left].key > A[largest].key)
        largest = left;
    if (right < n && A[right].key > A[largest].key)
        largest = right;

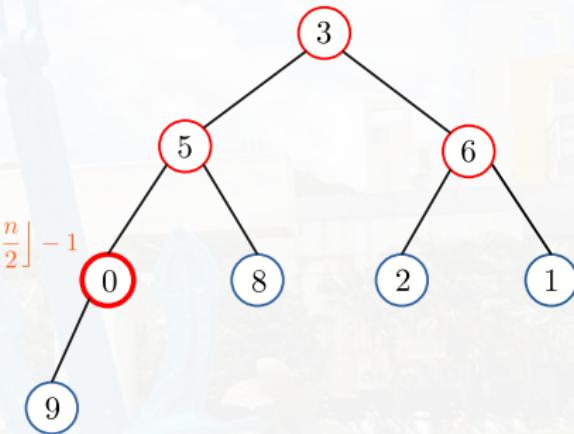
    if (largest != i) { // swap A[i] and A[largest]
        element tmp = A[i]; A[i] = A[largest]; A[largest] = tmp;
        heapify(A, n, largest);
    }
}

// IMPORTANT: We MUST heapify from the next level from the bottom
void buildMaxHeap(element A[], int n) {
    for (int i = n/2 - 1; i >= 0; --i)
        heapify(A, n, i);
}
```

# Nodes to be Heapified

index

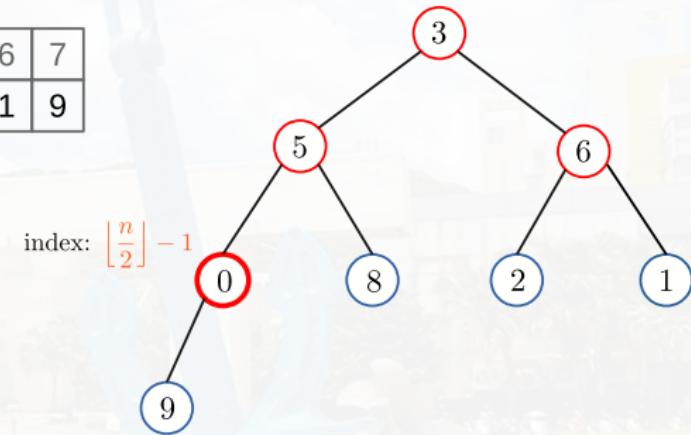
0	1	2	3	4	5	6	7
3	5	6	0	8	2	1	9

index:  $\left\lfloor \frac{n}{2} \right\rfloor - 1$ 

# Nodes to be Heapified

index

0	1	2	3	4	5	6	7
3	5	6	0	8	2	1	9

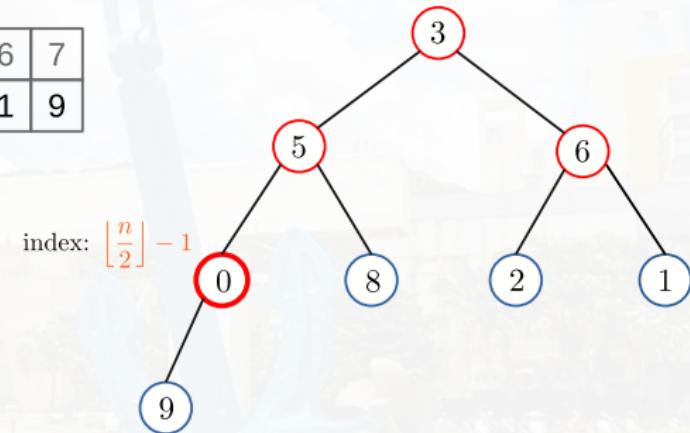


- # Heapify steps:  $\leq \sum_{h=1}^{\lg n-1} h \cdot n_h =$

# Nodes to be Heapified

index

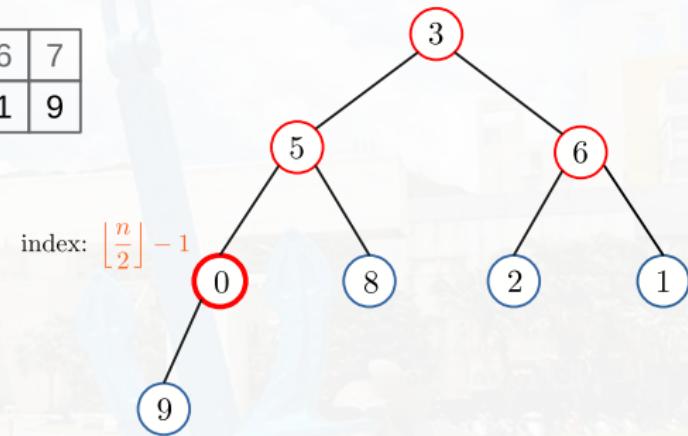
0	1	2	3	4	5	6	7
3	5	6	0	8	2	1	9



- # Heapify steps:  $\leq \sum_{h=1}^{\lg n-1} h \cdot n_h = \sum_{h=1}^{\lg n-1} h \cdot 2^{\lceil \lg n \rceil - h}$

# Nodes to be Heapified

index	0	1	2	3	4	5	6	7
	3	5	6	0	8	2	1	9



- # Heapify steps:  $\leq \sum_{h=1}^{\lg n-1} h \cdot n_h = \sum_{h=1}^{\lg n-1} h \cdot 2^{\lceil \lg n \rceil - h} \leq 2n \sum_{h=1}^{\lg n-1} \frac{h}{2^h}$ .
- $n_h$ : the number of nodes at level  $h$ .

## Exercise

Show that  $\sum_{h=0}^{\infty} \frac{h}{2^h} = O(1)$ .



# Discussions

