

Equilibrium Concepts

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Lecture Notes in Economics and Computation

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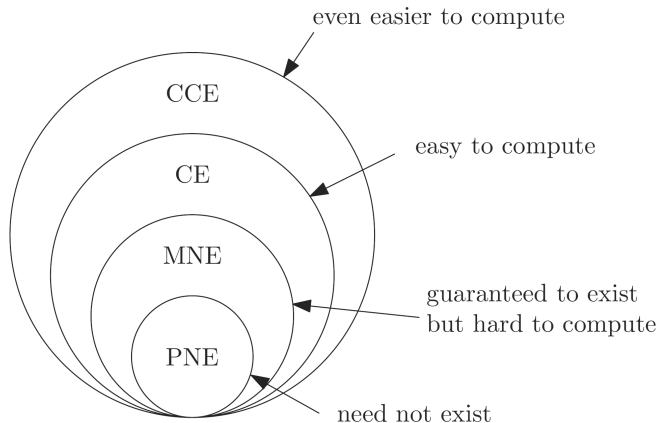


Outline

- 1 Cost Minimization and Payoff Maximization
- 2 Pure Nash Equilibria (PNE)
- 3 Mixed Nash Equilibria (MNE)
- 4 Correlated Equilibria (CE)
- 5 Coarse Correlated Equilibria (CCE)



A hierarchy of equilibrium concepts



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Cost-Minimization Games

A cost-minimization game has the following ingredients:

- a finite number of k agents;
- a finite set S_i of pure strategies for each agent i ;
- a nonnegative cost function $C_i(\mathbf{s})$ for each agent i .
 - $\mathbf{s} \in S_1 \times S_2 \times \cdots \times S_k$: a **strategy profile** or **outcome**.

For example, the network creation game.



Payoff-Maximization Games

A **payoff-maximization** game has the following ingredients:

- a finite number of k agents;
- a finite set S_i of pure strategies for each agent i ;
- a nonnegative **payoff** function $\pi_i(\mathbf{s})$ for each agent i .
 - $\mathbf{s} \in S_1 \times S_2 \times \cdots \times S_k$: a **strategy profile** or **outcome**.

For example, the Rock-Paper-Scissors game, two-party election game, etc.



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Pure Nash Equilibrium (PNE)

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A strategy profile \mathbf{s} of a cost-minimization game is a pure Nash equilibrium (PNE) if for every agent $i \in \{1, 2, \dots, k\}$ and every unilateral deviation $s'_i \in S_i$,

$$C_i(\mathbf{s}) \leq C_i(s'_i, \mathbf{s}_{-i}).$$

- \mathbf{s}_{-i} : the vector \mathbf{s} with the i th component removed.



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Mixed Nash Equilibrium (MNE)

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Distributions $\sigma_1, \dots, \sigma_k$, over strategy sets S_1, \dots, S_k respectively, of a cost-minimization game constitute a mixed Nash equilibrium (MNE) if for every agent $i \in \{1, 2, \dots, k\}$ and every unilateral deviation $s'_i \in S_i$,

$$\mathbf{E}_{\mathbf{s} \sim \sigma}[C_i(\mathbf{s})] \leq \mathbf{E}_{\mathbf{s} \sim \sigma}[C_i(s'_i, \mathbf{s}_{-i})].$$

- σ : the product distribution $\sigma_1 \times \dots \times \sigma_k$.



Product of Mixed Strategies

Player 2

		q_1 rock	q_2 scissors	q_3 paper	→ probabilities
Player 1	p_1 rock	p_1q_1 0, 0	p_1q_2 1, -1	p_1q_3 -1, 1	
	p_2 scissors	p_2q_1 -1, 1	p_2q_2 0, 0	p_2q_3 1, -1	
	p_3 paper	p_3q_1 1, -1	p_3q_2 -1, 1	p_3q_3 0, 0	

↓ probabilities

$$\begin{cases} p_1 + p_2 + p_3 = 1. \\ q_1 + q_2 + q_3 = 1. \end{cases}$$

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Correlated Equilibrium (CE)

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A **distribution σ on the set $S_1 \times \dots \times S_k$ of outcomes** of a cost-minimization game is a correlated equilibrium (CE) if for every agent $i \in \{1, 2, \dots, k\}$ and every unilateral deviation $s'_i \in S_i$,

$$\mathbf{E}_{\mathbf{s} \sim \sigma}[C_i(\mathbf{s}) \mid s_i] \leq \mathbf{E}_{\mathbf{s} \sim \sigma}[C_i(s'_i, \mathbf{s}_{-i}) \mid s_i].$$



Stop or Go?

Matrix of costs

	Stop	Go
Stop	1, 1	1, 0
Go	0, 1	5, 5

- Two PNEs.



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Stop or Go?

Matrix of costs

	Stop	Go
Stop	<small>prob. = 0</small> 1, 1	<small>prob. = 1/2</small> 1, 0
Go	<small>prob. = 1/2</small> 0, 1	<small>prob. = 0</small> 5, 5

- A CE for example.
- Cannot correspond to a MNE.



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Game of Chicken

- A.k.a. Hawk-Dove Game.
 - A model of conflict for two players.

	Dare	Chicken
Dare	0, 0	7, 2
Chicken	2, 7	6, 6

- Two PNE & One MNE.
- The expected utility of each player in the MNE:



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$$\frac{1}{3} \cdot \frac{2}{3} \cdot 7 + \frac{2}{3} \cdot \frac{1}{3} \cdot 2 + \frac{2}{3} \cdot \frac{2}{3} \cdot 6 = \frac{14}{3}$$



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Game of Chicken

- A correlated equilibrium.
 - Check that it is an equilibrium if a player is assigned “Dare”.
 - Check that it is an equilibrium if a player is assigned “Chicken Out”.

	Dare	Chicken
Dare	prob. = 0 0, 0	prob. = 1/3 7, 2
Chicken	prob. = 1/3 2, 7	prob. = 1/3 6, 6

- The expected utility for each player:
 $7 \cdot (1/3) + 2 \cdot (1/3) + 6 \cdot (1/3) = 5.$



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$$\mathbf{E}_{\mathbf{s} \sim \sigma}[C_i(\mathbf{s})] \leq \mathbf{E}_{\mathbf{s} \sim \sigma}[C_i(s'_i, \mathbf{s}_{-i})].$$

CE \subseteq CCE?

$$\begin{aligned} \mathbf{E}_{\mathbf{s} \sim \sigma}[C_i(\mathbf{s})] &= \sum_{a \in S_i} \mathbf{E}_{\mathbf{s} \sim \sigma}[C_i(\mathbf{s}) \mid s_i = a] \Pr[s_i = a] \\ &\leq \sum_{a \in S_i} \mathbf{E}_{\mathbf{s} \sim \sigma}[C_i(s'_i, \mathbf{s}) \mid s_i = a] \Pr[s_i = a] \\ &= \mathbf{E}_{\mathbf{s} \sim \sigma}[C_i(s'_i, \mathbf{s}_{-i})] \end{aligned}$$



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CCE Example

	<i>A</i>	<i>B</i>	<i>C</i>
<i>A</i>	$\text{prob.} = 1/3$ $1, 1$	$-1, -1$	$0, 0$
<i>B</i>	$-1, -1$	$\text{prob.} = 1/3$ $1, 1$	$0, 0$
<i>C</i>	$0, 0$	$0, 0$	$\text{prob.} = 1/3$ $-1.1, -1.1$

- The payoff for each player (playing according to this distribution):
 $\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 1 - \frac{1}{3} \cdot 1.1 = 0.3.$
- A player playing fixed *A* or *B* while the opponent randomized according to this distribution: $\frac{1}{3} \cdot 1 - \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 = 0.$
- A player playing fixed *C* while the opponent randomized according to this distribution: $\frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot (-1.1) < 0.$



CCE Example

	A	B	C
A	$\text{prob.} = 1/3$ $1, 1$	$-1, -1$	$0, 0$
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A	prob. = 1/3 1, 1	-1, -1	0, 0
B	-1, -1	prob. = 1/3 1, 1	0, 0
C	0, 0	0, 0	prob. = 1/3 -1.1, -1.1

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CCE Example

	A	B	C
A	$\overset{\text{prob.} = 1/3}{1, 1}$	$-1, -1$	$0, 0$
B	$-1, -1$	$\overset{\text{prob.} = 1/3}{1, 1}$	$0, 0$
C	$0, 0$	$0, 0$	$\overset{\text{prob.} = 1/3}{-1.1, -1.1}$

- A player playing fixed C and the strategy profile follows this distribution:
 \Rightarrow deviation is possible.
 - Not a CE.



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