Minimum Cost Spanning Trees (MSTs)

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Outline

1. Introduction
2. Kruskal's algorithm
3. Prim's algorithm
4. Sollin's algorithm

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3 Prim's algorithm

4 Sollin's algorithm

Cost & Minimum-Cost Spanning Tree

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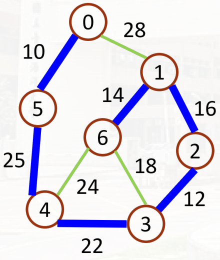
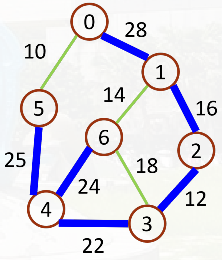


cost $= 10 + 25 + 22 + 12 + 16 + 14 = 99.$



cost $= 28 + 16 + 12 + 22 + 24 + 25 = 127.$

Greedy Methods

- We will introduce three different greedy algorithms can be used to obtain a minimum cost spanning tree of a connected undirected graph.

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  - Kruskal's algorithm
  - Prim's algorithm
  - Sollin's algorithm

⋆ Further reading: Greedy Algorithms & Matroids [link]

Greedy Method (1/2)

- Construct an optimal solution in stages.
- A feasible solution is one which works within the constrains specified by the problem.
- At each stage, we make a decision that is the best decision at that time.
- Typically, The selection of an item at each stage is based on either a least cost or a highest profit criterion.

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  - Exactly $n - 1$ edges are used.
  - Never use edges that would produce a cycle.

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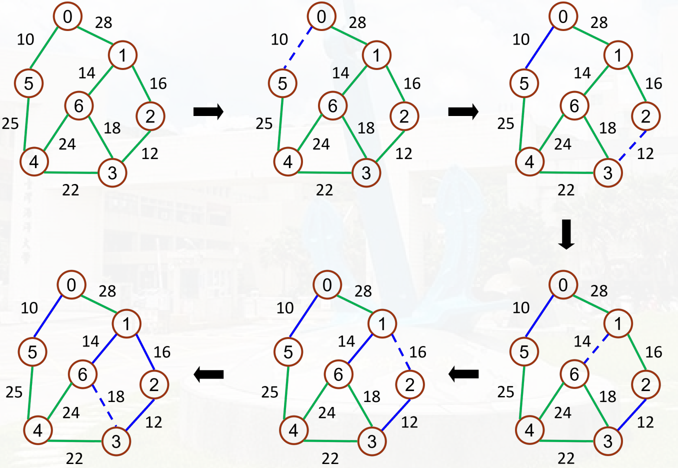
  - **How to do this? $\Rightarrow$ Union-Find Operations!**
- Exactly $n - 1$ edges will be selected for inclusion in T.

- Time complexity: $O(e \log n)$ (assume using tree-based Union-Find).
  - Sorting the edges: $\approx e \log e < e \log n^2 = O(e \log n)$.
  - At most $2e$ find operations $\approx \log n$ time for each.
  - At most $2n - 1$ union operations $\approx n$ time.

Illustration of Kruskal's Algorithm



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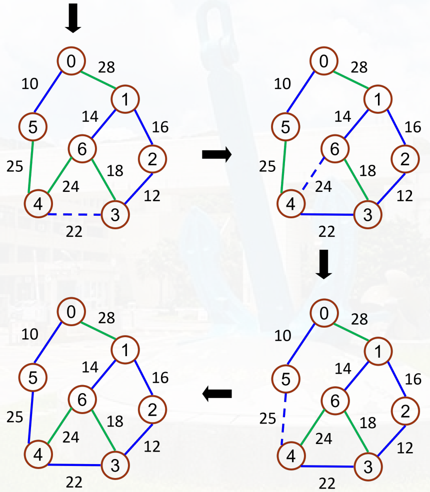

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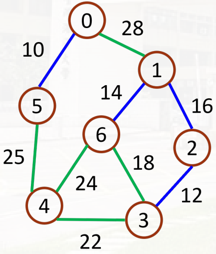

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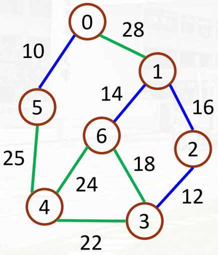

- $\{0,5\},\{1,2,3,6\},\{4\}$: the sets corresponding to existing subtrees.
- Vertex 3 and 6 are already in the same set $\Rightarrow$ edge $(3,6)$ is rejected!

The Pseudo-code of Kruskal's Algorithm

```
T = { };
while (T contains fewer than n-1 edges && E is not empty) {
    choose a least cost edge (v,w) from E;
    delete (v,w) from E;
    if ((v,w) does not create a cycle in T)
        add (v,w) to T
    else
        discard (v,w);
}
if (T contains fewer than n-1 edges)
    printf("No spanning tree\n");
```

- How could "No spanning tree" happen?

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- Another greedy MST algorithm.
- The main difference:
  - The set of selected edges forms a tree at all times in Prim's algorithm.
  - The set of selected edges in Kruskal's algorithm forms a *forest at each stage*.

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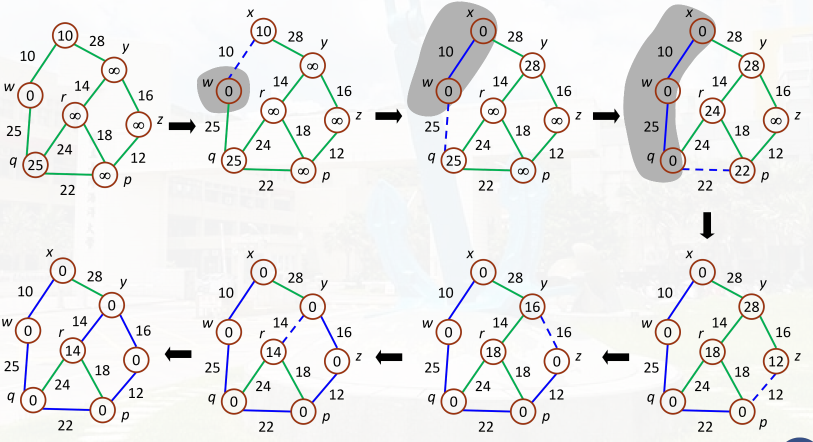
Time complexity: $O(n^2)$.

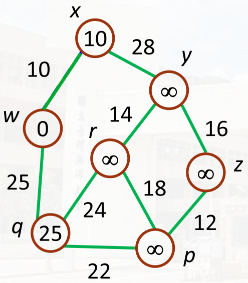
Illustration of Prim's Algorithm



Prim's Algorithm (3/3)

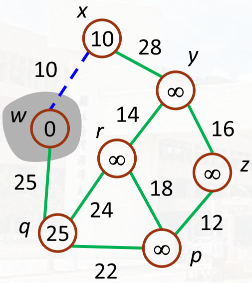
```
T = {}; TV = {0};
while (T contains fewer than n-1 edges) {
    let (u,v) be a least cost edge such that u is in TV and
    v is not in TV;
        if (there is no such edge )
            break;
        add v to TV;
        add (u,v) to T;
}
if (T contains fewer than n-1 edges)
    printf("No spanning tree\n");
```

- Each vertex $v \notin TV$ has a companion vertex "near(v)" such that near(v) $\in TV$ and cost(near(v), v) is minimum over all such choices for near(v).
- Therefore, it takes $O(n)$ time to choose an edge.
- We can implement Prim's algorithm in $O(n^2)$ time.



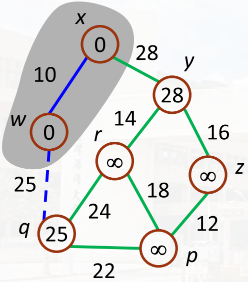
	inMST		cost
w	F	w	∞
x	F	x	∞
y	F	y	∞
z	F	z	∞
p	F	p	∞
q	F	q	∞
r	F	r	∞

TV = { }



	inMST		cost
w	T	*w*	∞
x	F	*x*	10
y	F	*y*	∞
z	F	*z*	∞
p	F	*p*	∞
q	F	*q*	25
r	F	*r*	∞

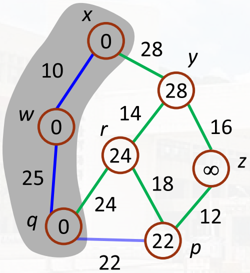
TV = { w }



	inMST	cost
w	T	∞
x	T	10
y	F	28
z	F	∞
p	F	∞
q	F	25
r	F	∞

☺

TV = { w, x }



	inMST		cost
w	T	*w*	∞
x	T	*x*	10
y	F	*y*	28
z	F	*z*	∞
p	F	*p*	22
q	T	*q*	25
r	F	*r*	∞

☺

TV = { *w*, *x*, *q* }

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Sollin's Algorithm (1/2)

- Sollin's algorithm selects several edges for inclusion in T at each stage.
- At the start of a stage, the selected edges, together with all the n vertices, form a spanning forest.
- During each stage, we select one edge for each tree in the forest.
    - This edge is a minimum cost edge that has exactly one vertex in the tree.

Sollin's Algorithm (2/2)

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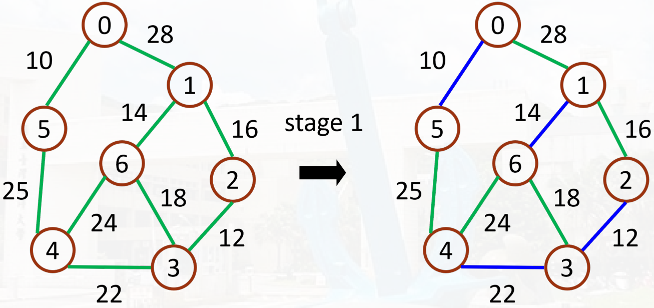
- **Note:** two trees in the forest could select the same edge.
    - We need to eliminate duplicate edges.
- At the start of the first stage the set of selected edges is empty.
- The algorithm terminates when there is only one tree at the end of a stage or no edges remain for selection.

> We can implement Sollin's algorithm in $O(e \log v)$ time (WHY?).

Illustration of Sollin's Algorithm



- Stage 1: $(0,5),(1,6),(2,3),(3,2),(4,3),(5,0)$, and $(6,1)$ are selected, and then duplicates are removed.

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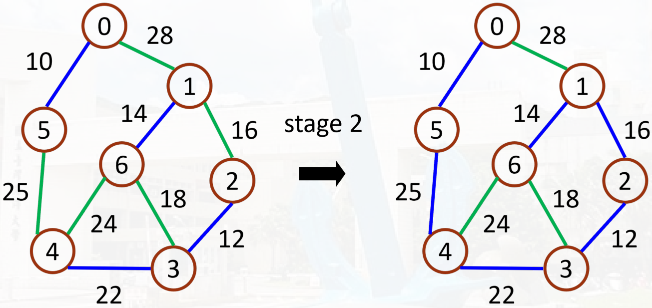

- Stage 2: $(5, 4)$, $(1, 2)$, and $(2, 1)$ are selected, and then duplicates are removed.

Discussions