

Connected Components & Biconnected Components

Joseph Chuang-Chieh Lin (林莊傑)

Department of Computer Science & Engineering,
National Taiwan Ocean University

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Outline

1 Connected Components

- Spanning Trees
- Articulation Points & Biconnected Graph
- Finding the articulation points



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Connectivity

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Problem II

List all connected components of an (un)directed graph.

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Problem II

List all connected components of an (un)directed graph.

This can be done by making repeated calls to either $\text{dfs}(v)$ or $\text{bfs}(v)$ where v is an **unvisited vertex**.



```
void connected(void) { // dfs(0) or bfs(0)
/* determine the connected components of a graph */
    int i;
    for (i=0; i<n; i++) {
        if (!visited[i]) {
            dfs(i);
            printf("\n");
        }
    }
}
```

Analysis of connected

- If G is represented by its adjacency lists, then the total time taken by DFS is $O(e)$.



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- If G is represented by its adjacency lists, then the total time taken by DFS is $O(e)$.
- Since the for loop takes $O(n)$ time, the total time needed to generate all the connected components is $O(n + e)$.
- If G is represented by an **adjacency matrix**, then the time needed to determine the connected components is $O(n^2)$.

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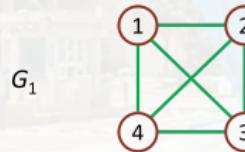
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Spanning Trees

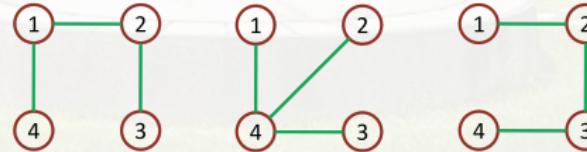
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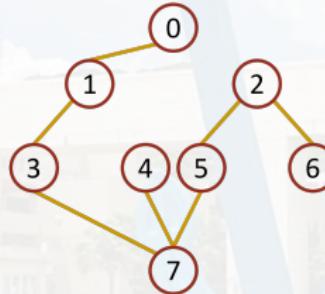
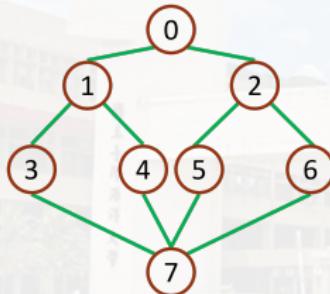
- When graph G is connected, a DFS or BFS implicitly partitions the edges in G into two sets:
 - Tree edges:** the set of edges used or traversed during the search.
 - Nontree edges:** the set of remaining edges.



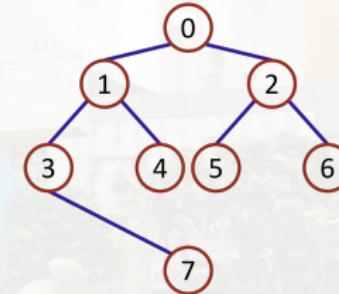
Three spanning trees of G_1 .



DFS Spanning Trees & BFS Spanning Trees



DFS (0)
spanning tree

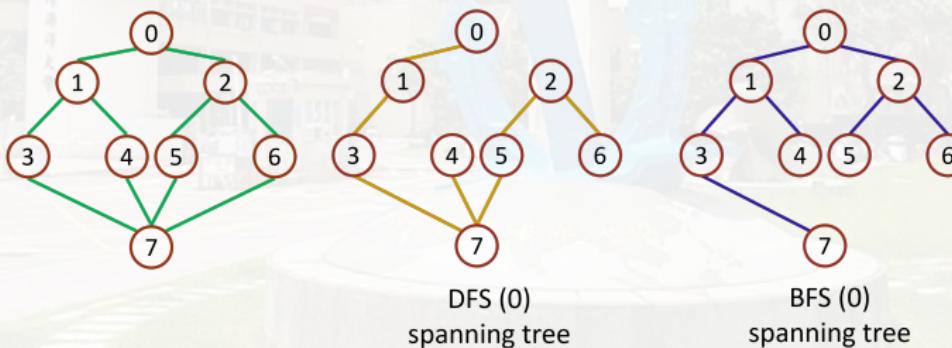


BFS (0)
spanning tree

Properties of (DFS or BFS) Spanning Trees

Property I

Suppose we add a nontree edge, (v, w) , into any spanning tree, T . The result is a cycle that consists of the edge (v, w) and all the edges on the path from w to v in T .

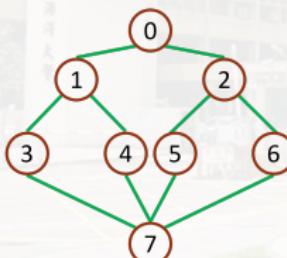


Properties of (DFS or BFS) Spanning Trees

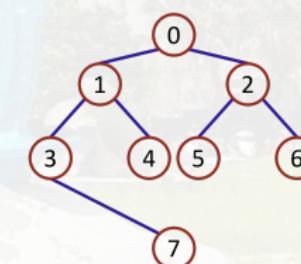
Property II

A spanning tree is a **minimal** subgraph, G' , of G such that $V(G') = V(G)$ and G' is connected.

- A spanning tree has $n - 1$ edges.



DFS (0)
spanning tree



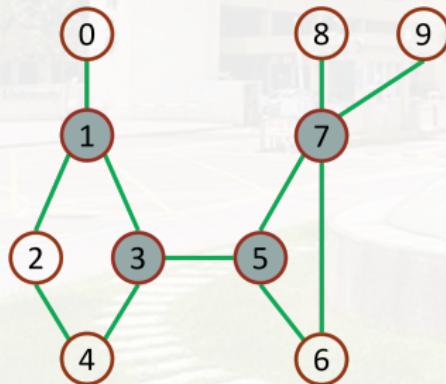
BFS (0)
spanning tree

Articulation Points

Articulation Points

An **articulation point** is a vertex v of G such that the deletion of v , together with all edges incident on v , produces a graph, G' , that has ≥ 2 connected components.

a connected graph



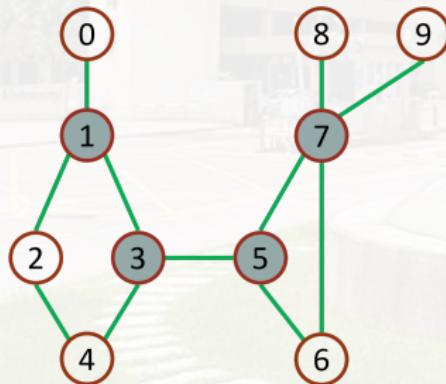
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- Four articulation points:

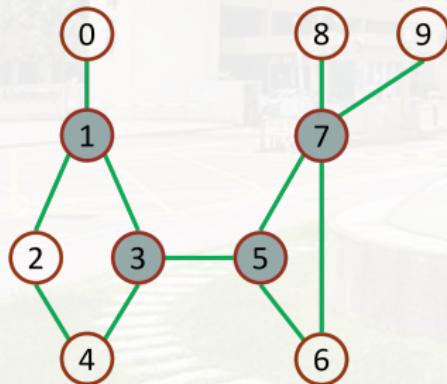


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a connected graph



- Four articulation points:
1, 3, 5, 7.

Biconnected Graph (雙連通圖)

Biconnected Graph

A biconnected graph is a connected graph that has **NO** articulation points.

Biconnected Component

A biconnected component of a connected graph G is a **maximal biconnected subgraph H of G .**

- H is “maximal”: no other subgraph that is both biconnected and properly contains H .



Biconnected Graph (雙連通圖)

Biconnected Graph

A biconnected graph is a connected graph that has **NO** articulation points.

- A connected graph G which has an articulation point $\Rightarrow G$ is NOT biconnected.

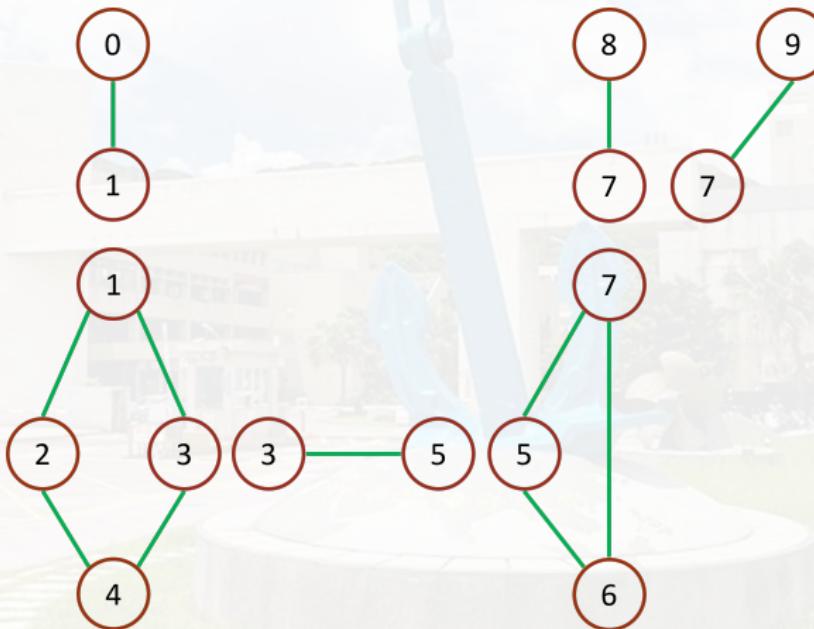
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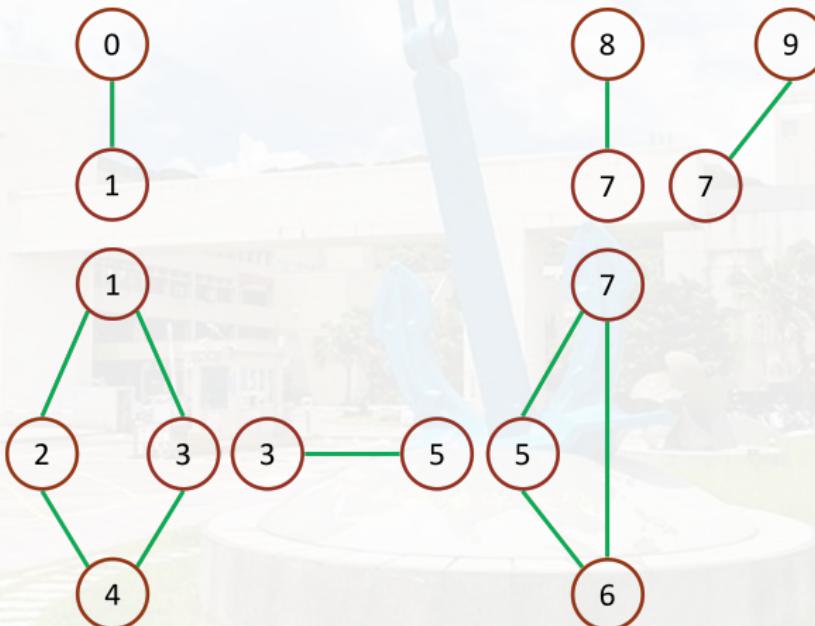
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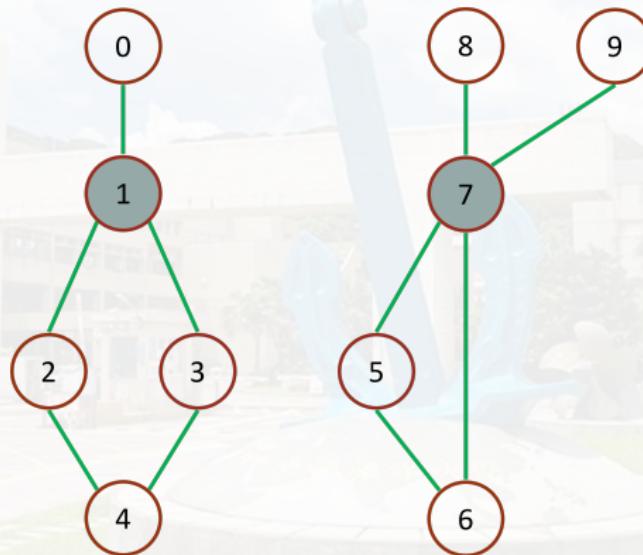
Connected Components



Biconnected Components (an Example)



Biconnected Components (NOT an Example)

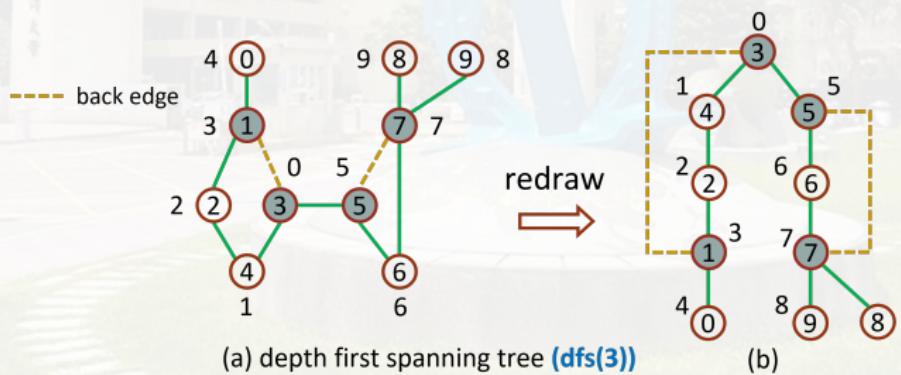


Finding articulation points (1/3)

We can find biconnected components of a graph G using any depth-first spanning of G .

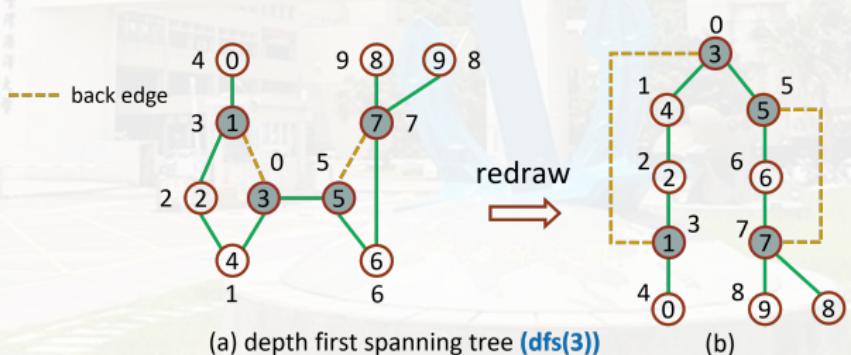
Back edges

- Tree edges: DFS
- Nontree edges: we call them **back edges**



Observations

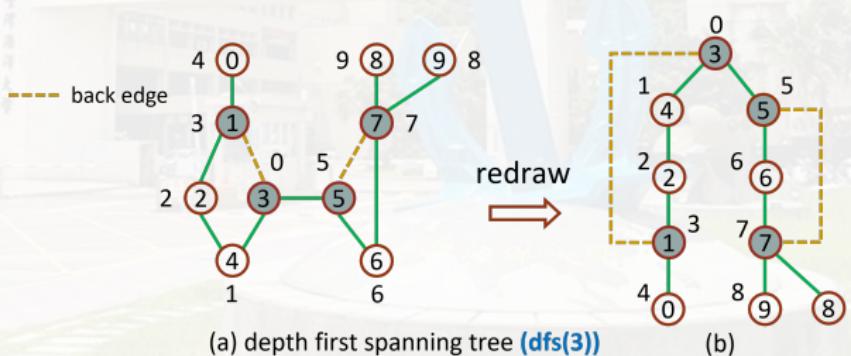
- The root of a depth first spanning tree is an articulation point if and only if it has ≥ 2 children.
- Any other vertex u is an articulation point if and only if it has ≥ 1 child w such that we cannot reach an ancestor of u using that consists of only w , descendants of w , and a single back edge.



- v_5 is an articulation point

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- v_5 is an articulation point, but v_6 is NOT.

Finding articulation points (2/3)

$\text{dfn}(v)$

The depth first numbers, or dfn , of the vertices give the sequence in which the vertices are visited during the depth first search.

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- If u is an ancestor of v in the depth first spanning tree, then $\text{dfn}(u) < \text{dfn}(v)$.

Finding articulation points (2/3)

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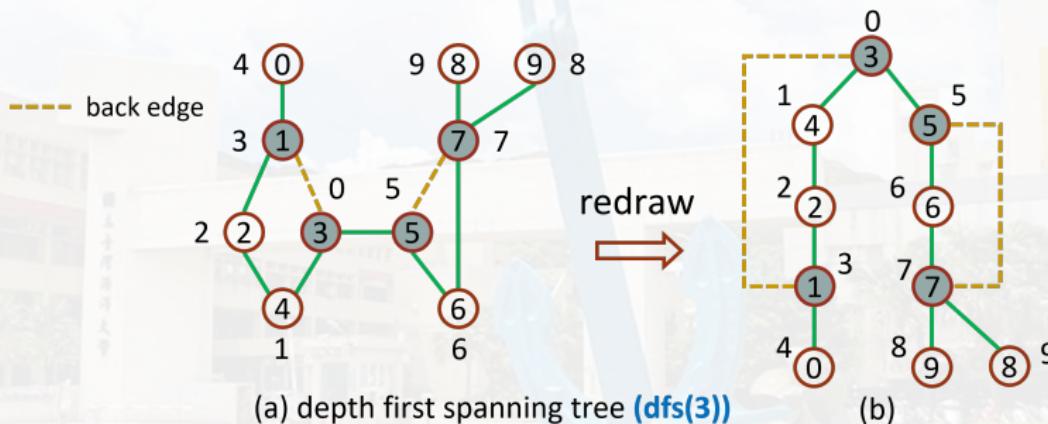
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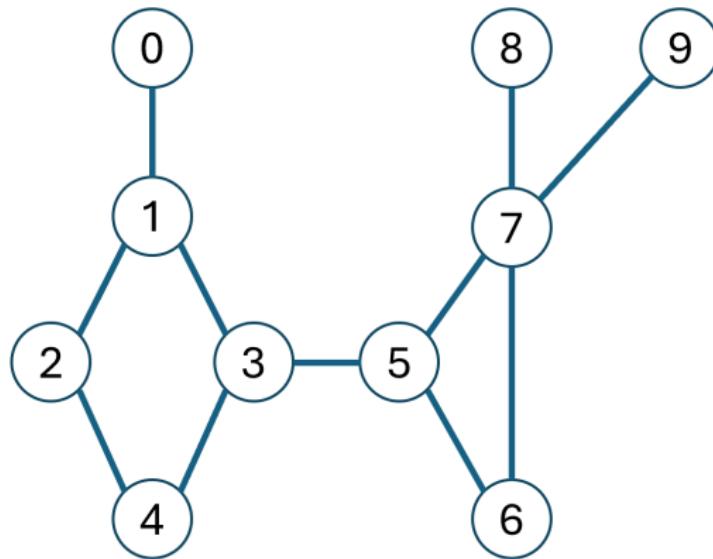
$\text{low}(v)$

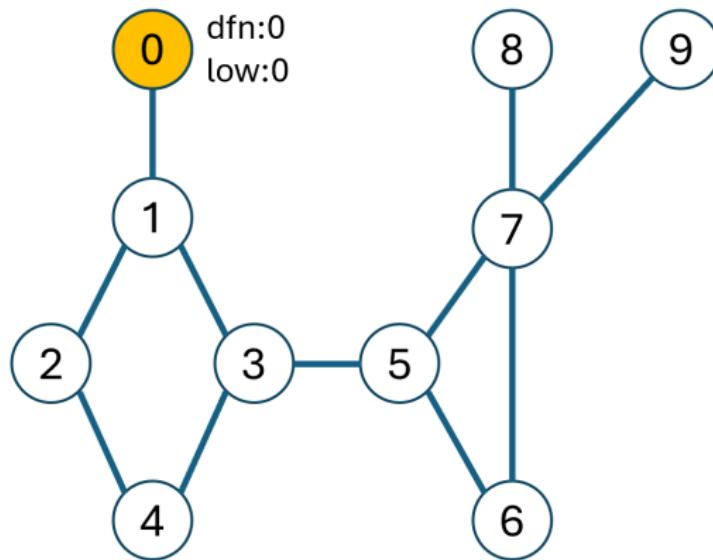
The $\text{low}(u)$ value of vertex u is the lowest depth first number that we can reach from u using a path of descendants followed by at most 1 back edge.

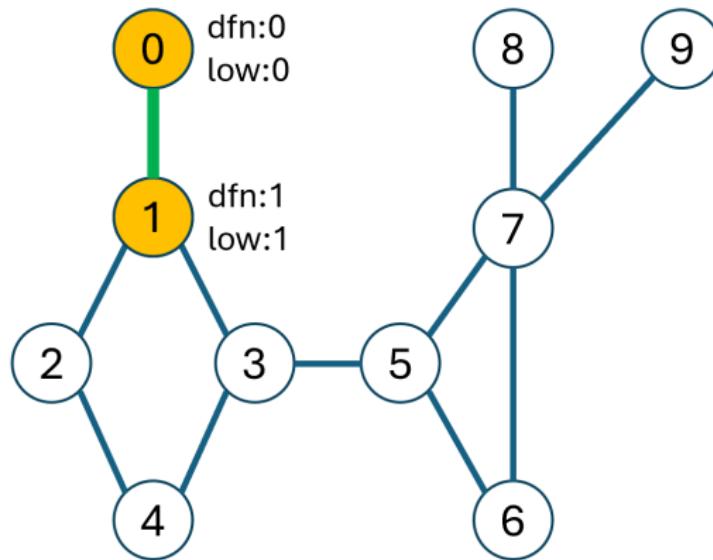
$$\text{low}(u) = \min \left\{ \begin{array}{l} \text{dfn}(u), \\ \min\{\text{low}(w) \mid w \text{ is a child of } u\}, \dots (*) \\ \min\{\text{dfn}(w) \mid (u, w) \text{ is a back edge}\} \dots (**) \end{array} \right.$$

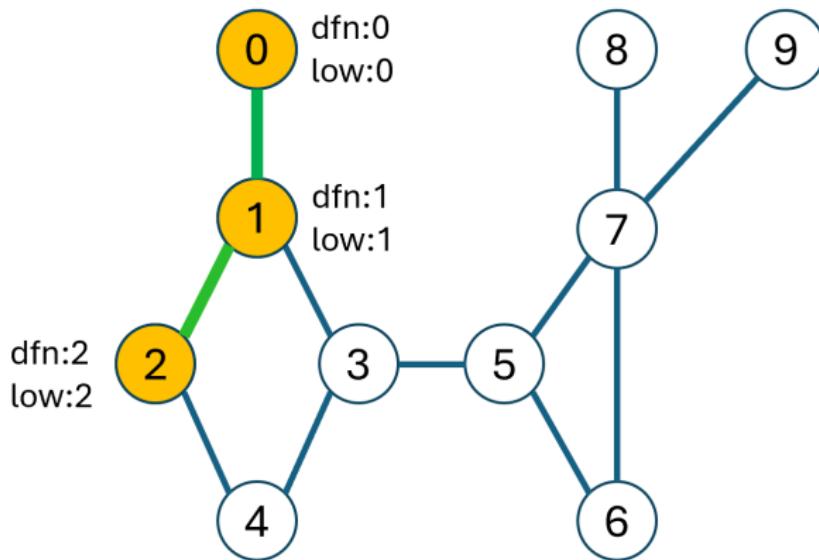
Example of Computing dfn and low values

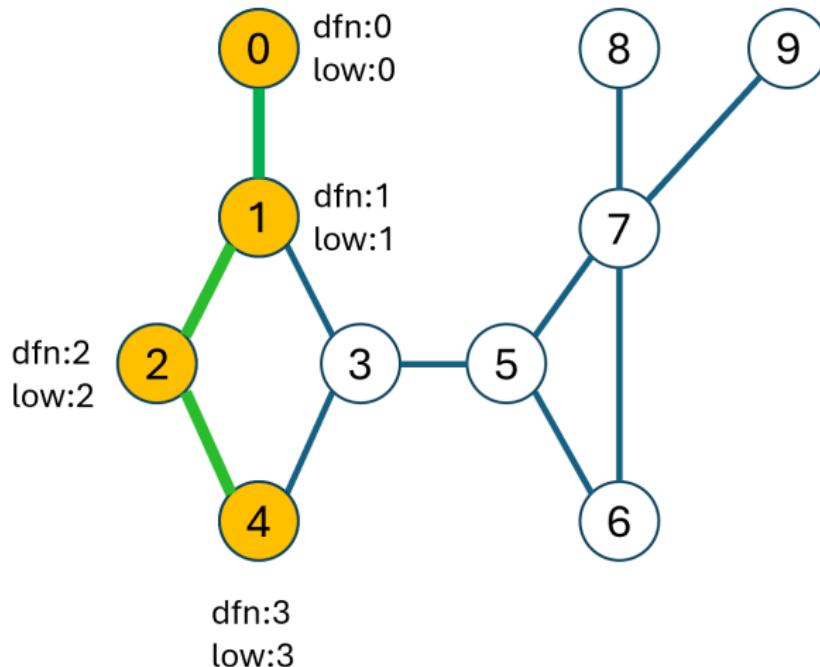


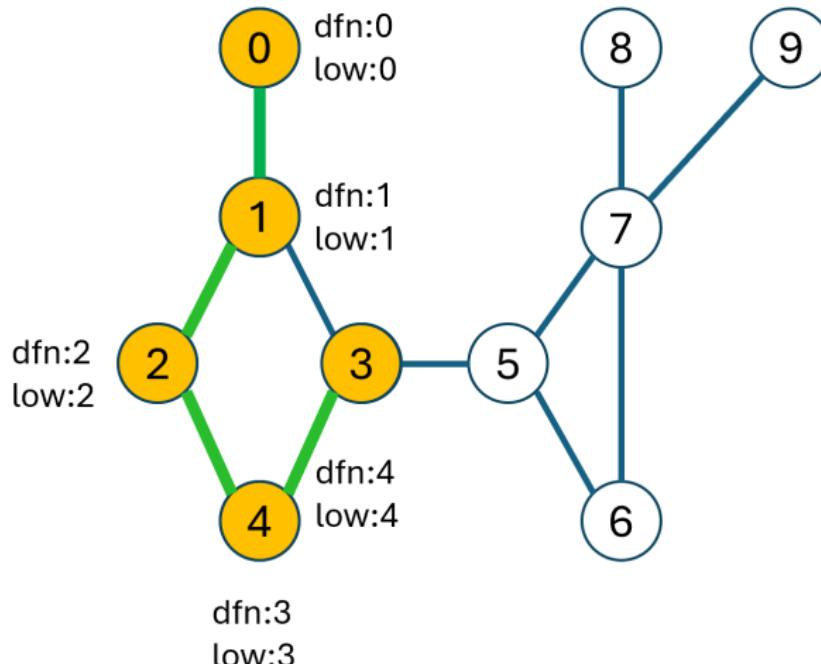


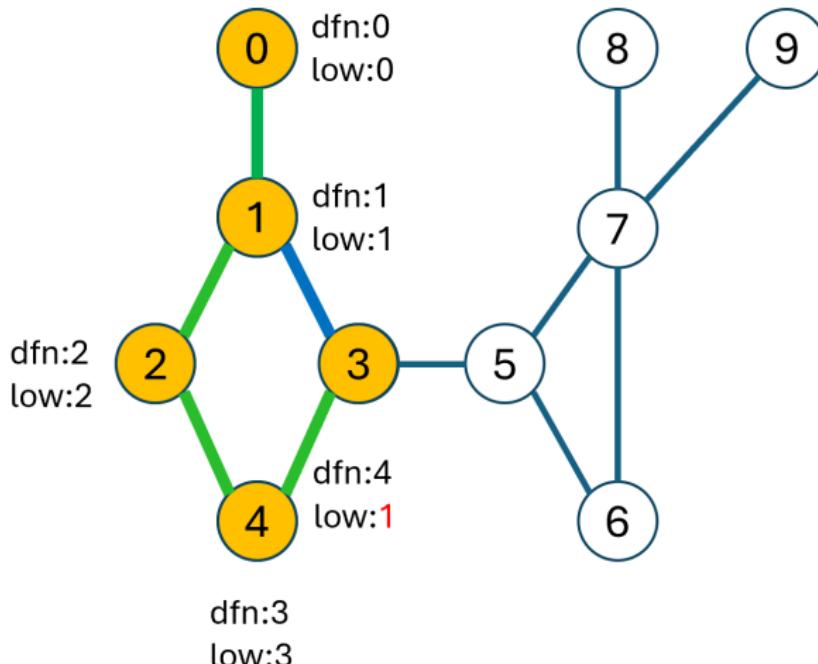


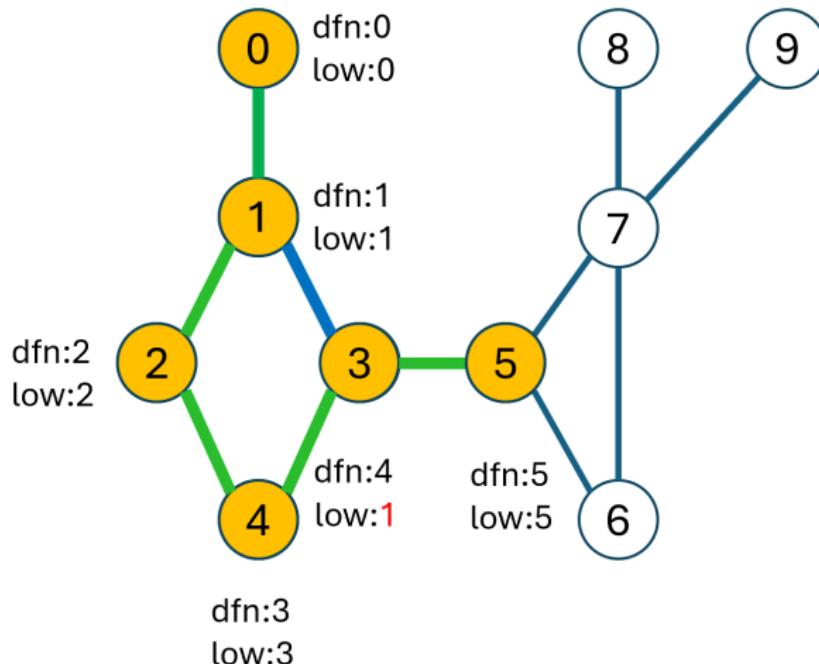


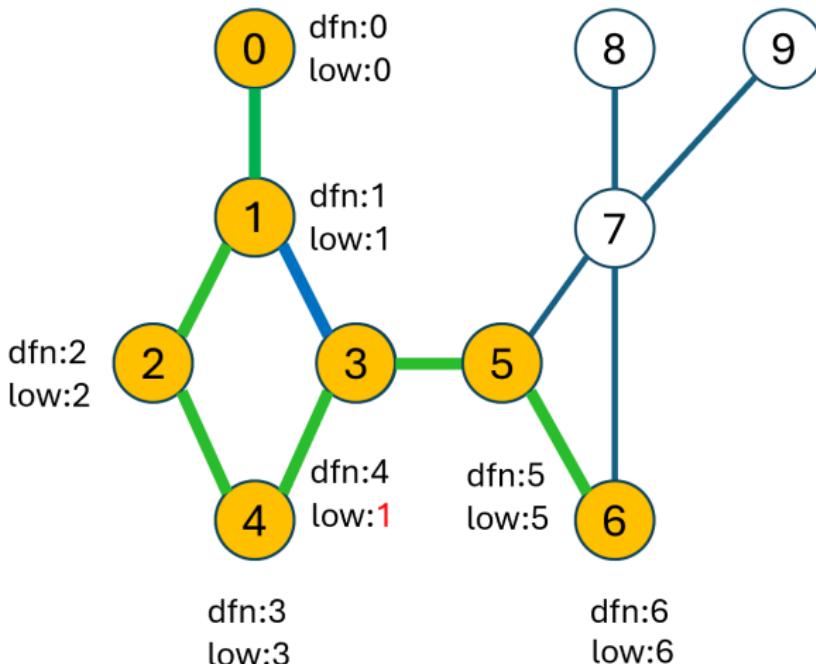


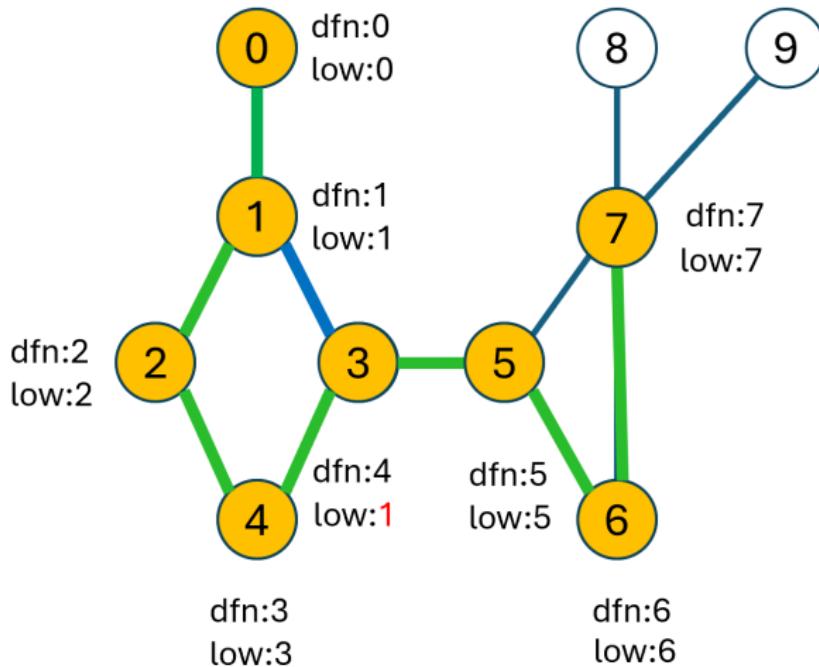


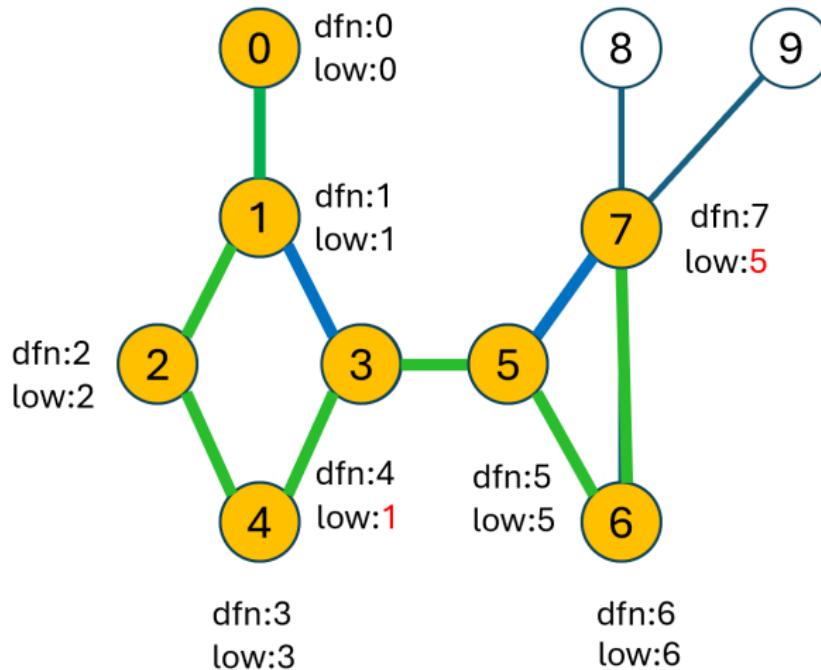


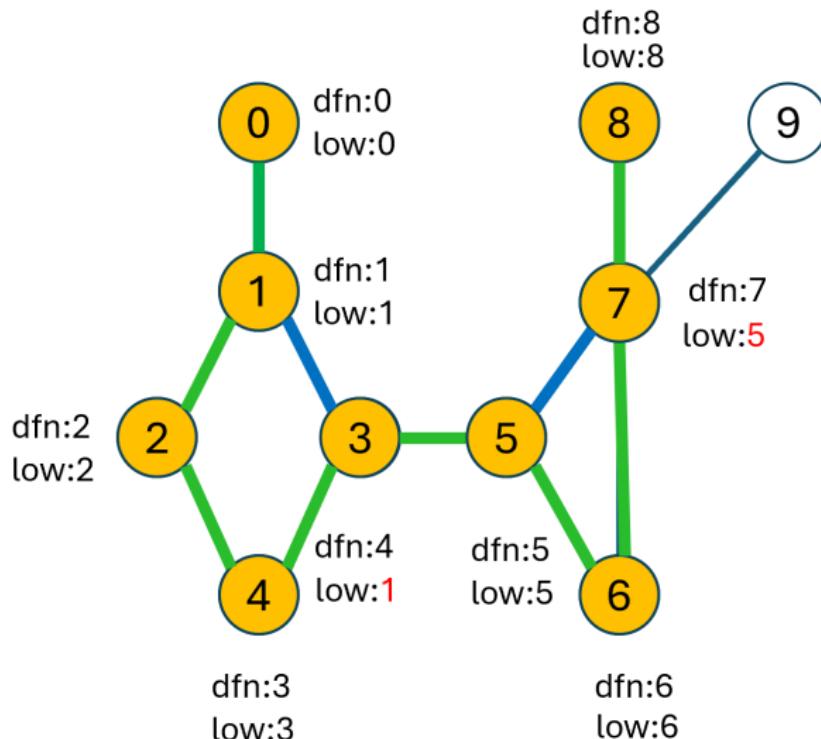


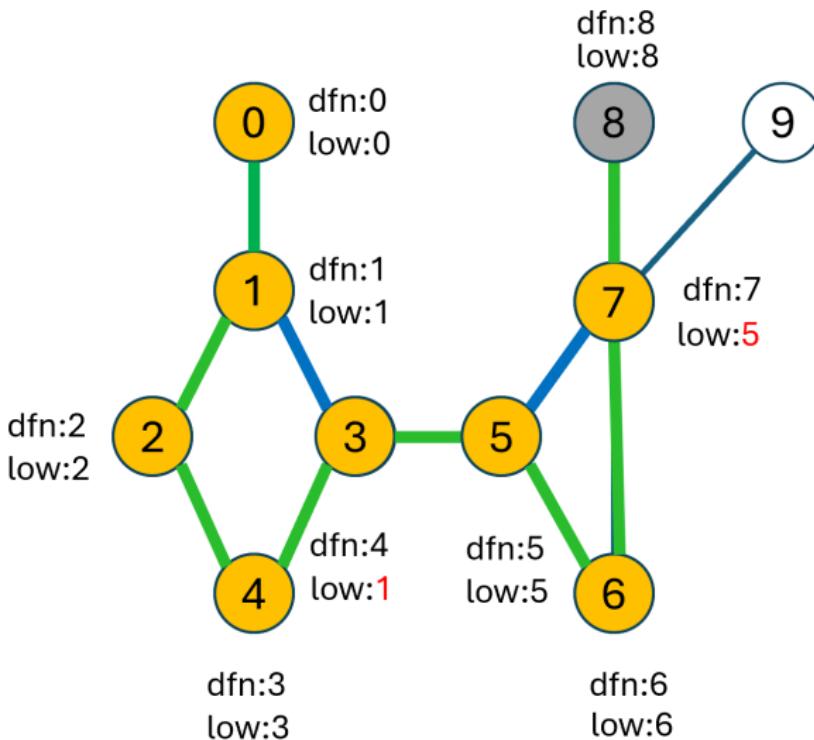


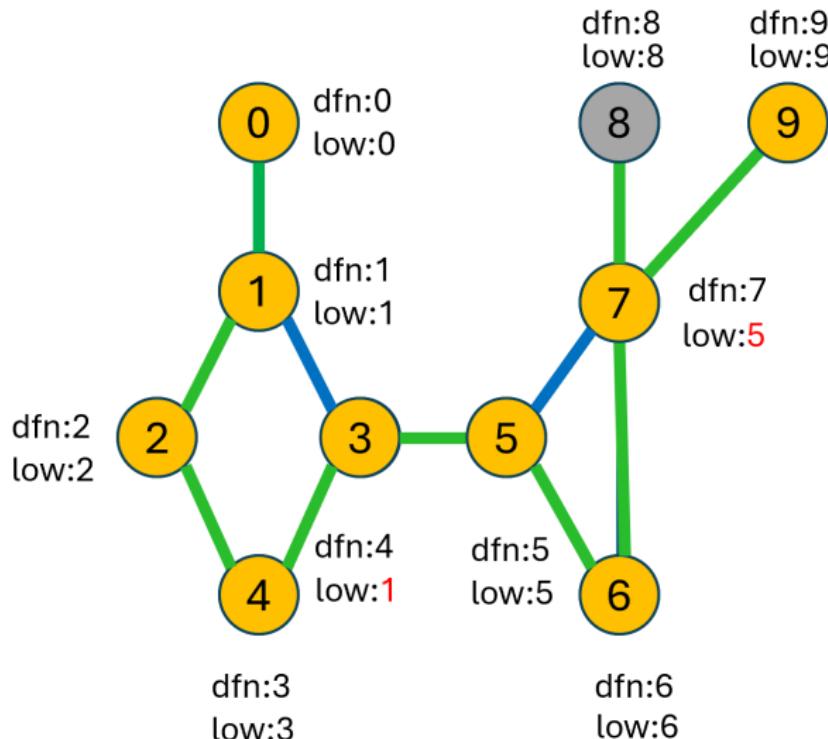


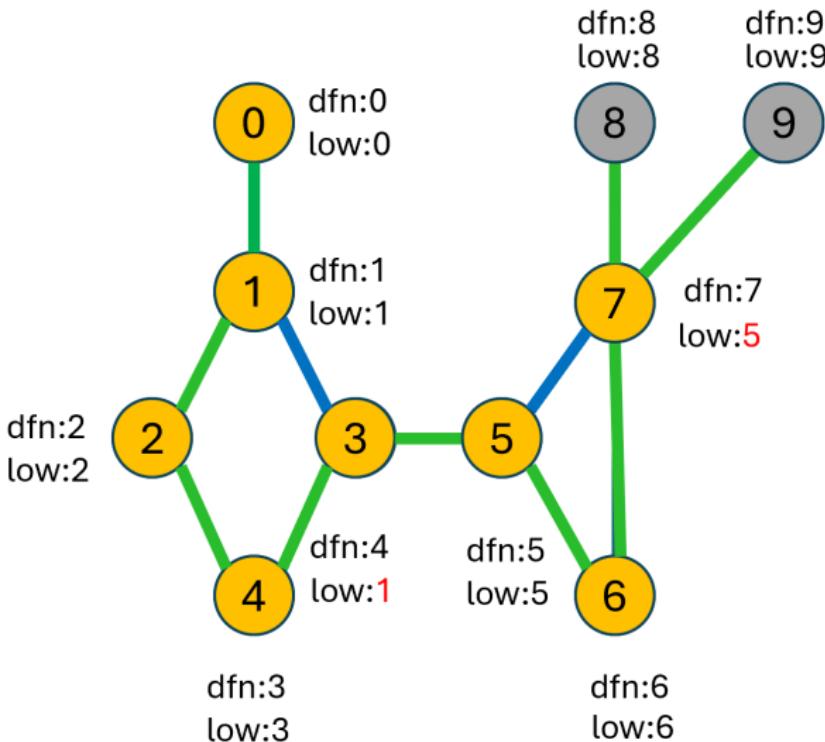


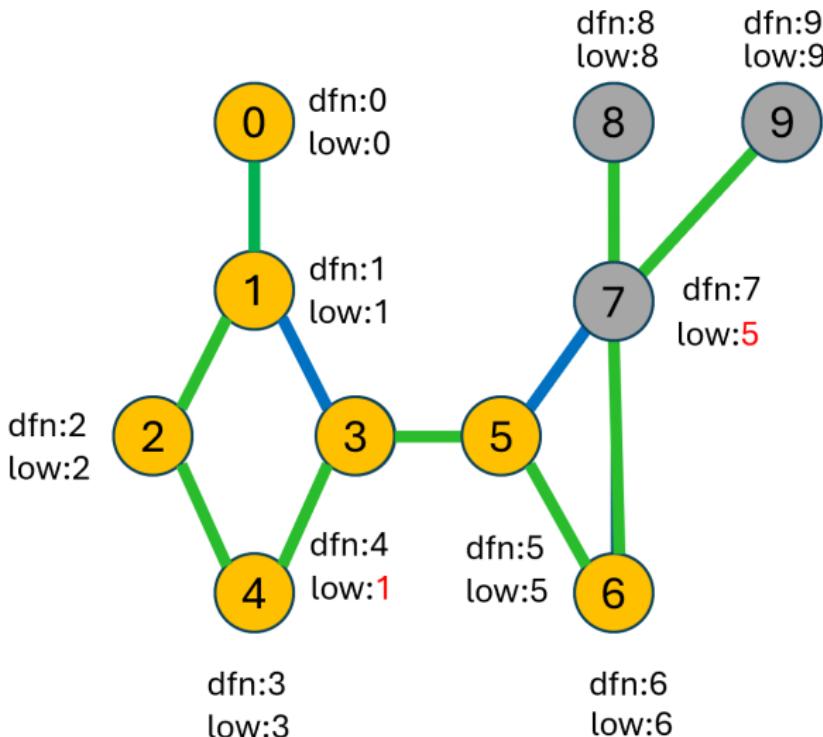


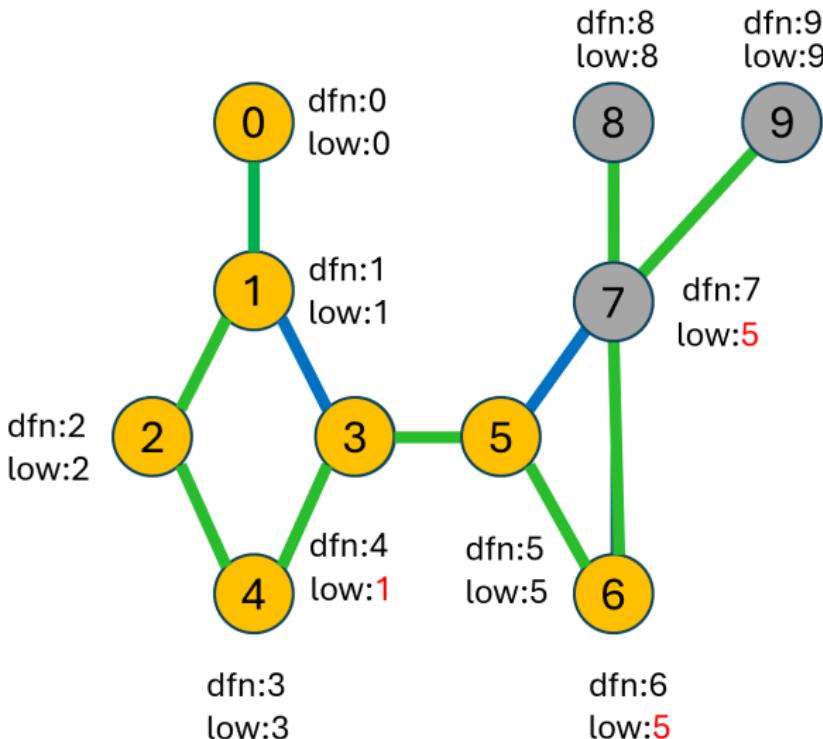


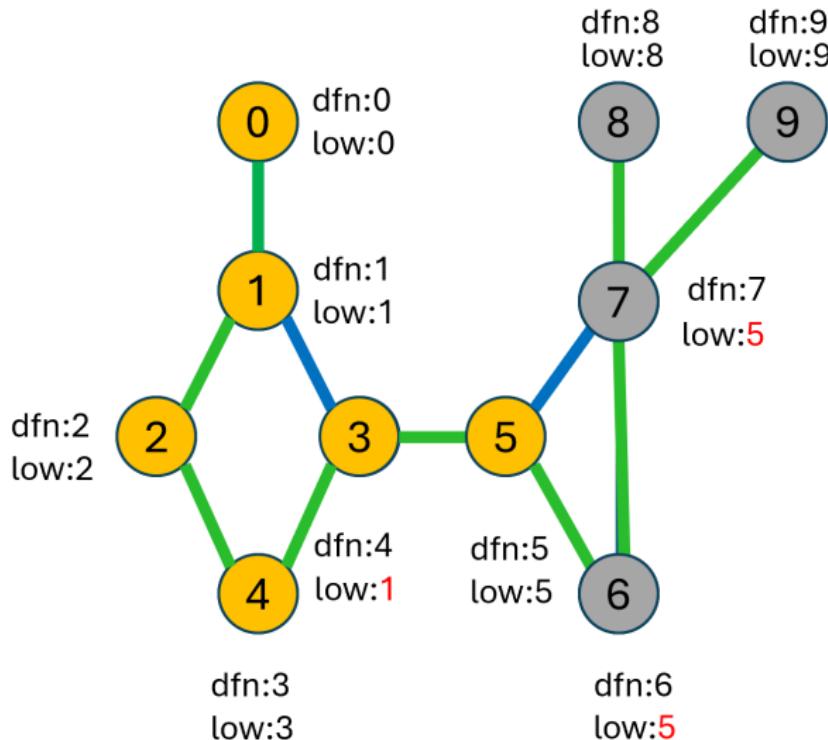


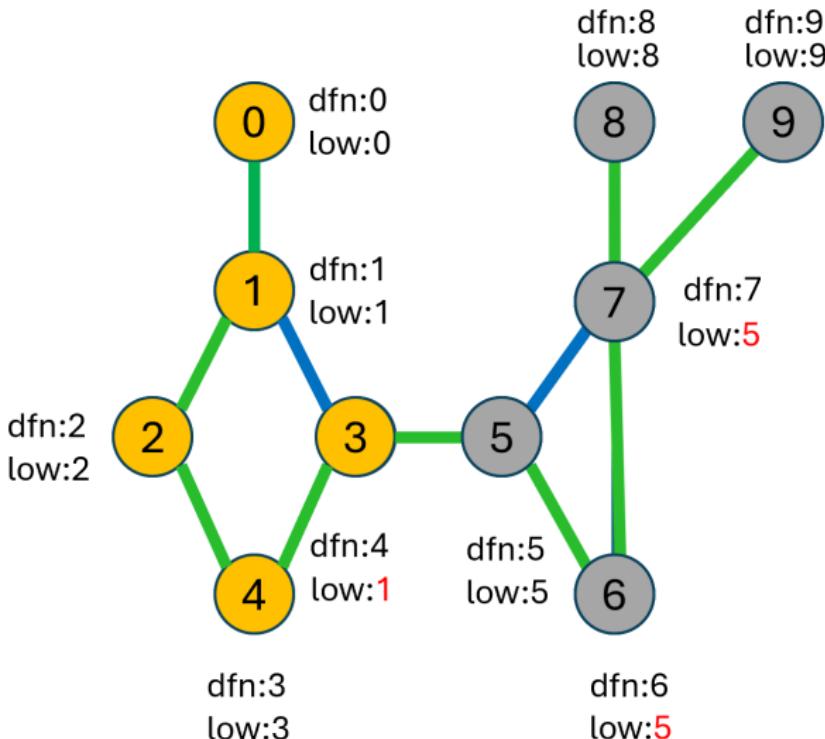


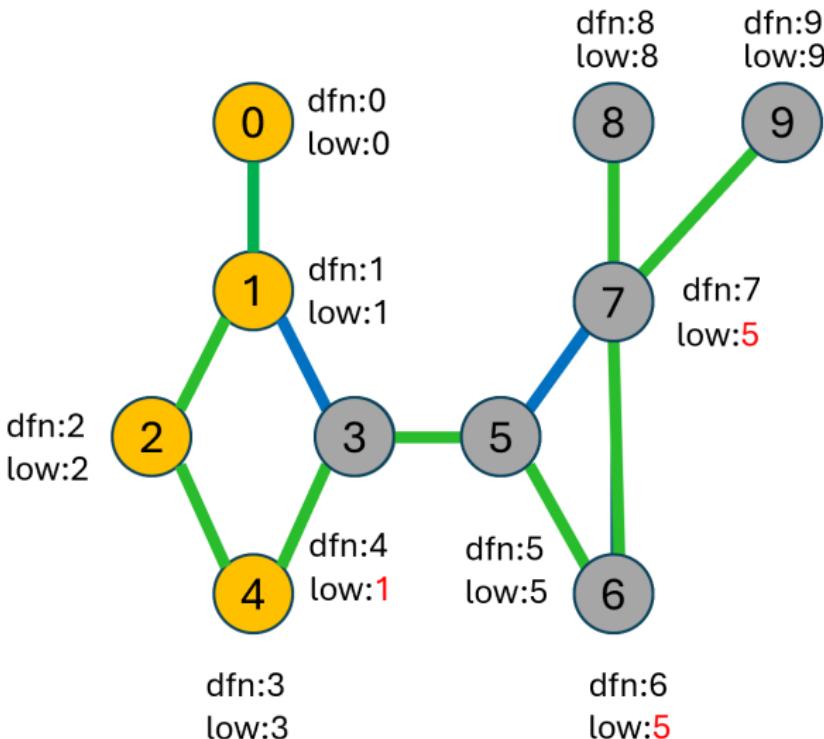


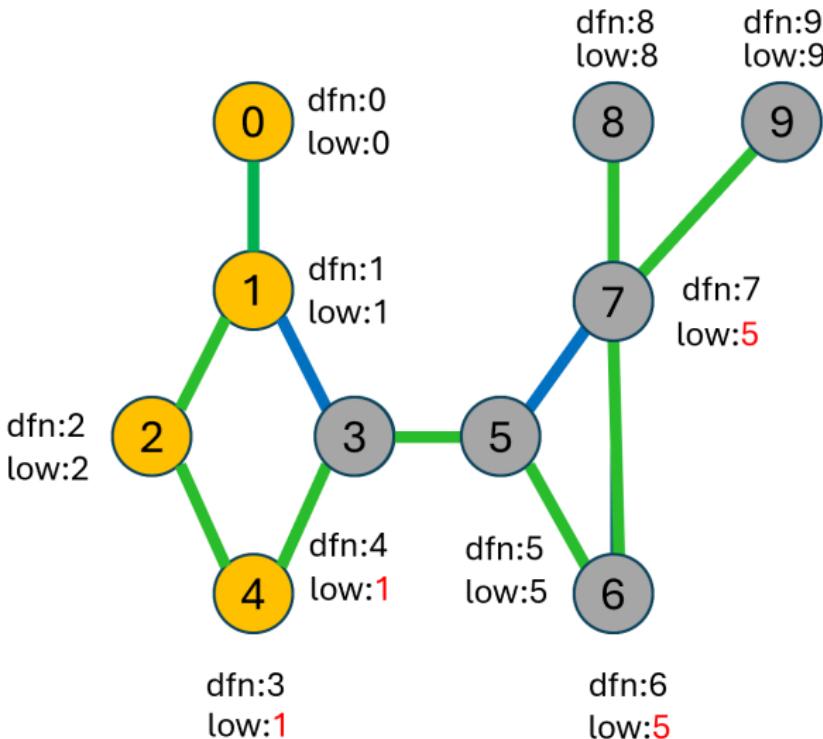


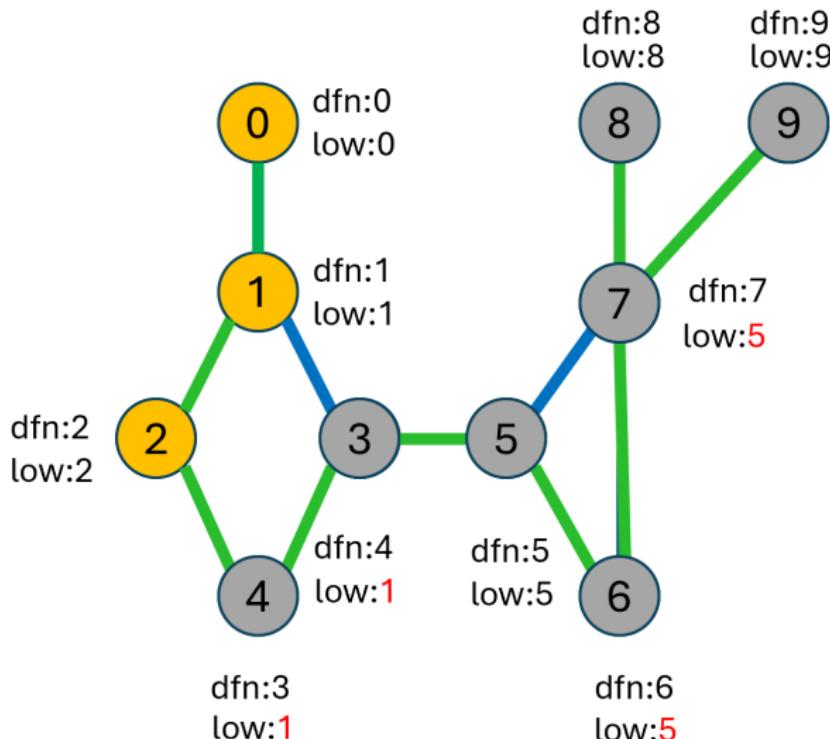


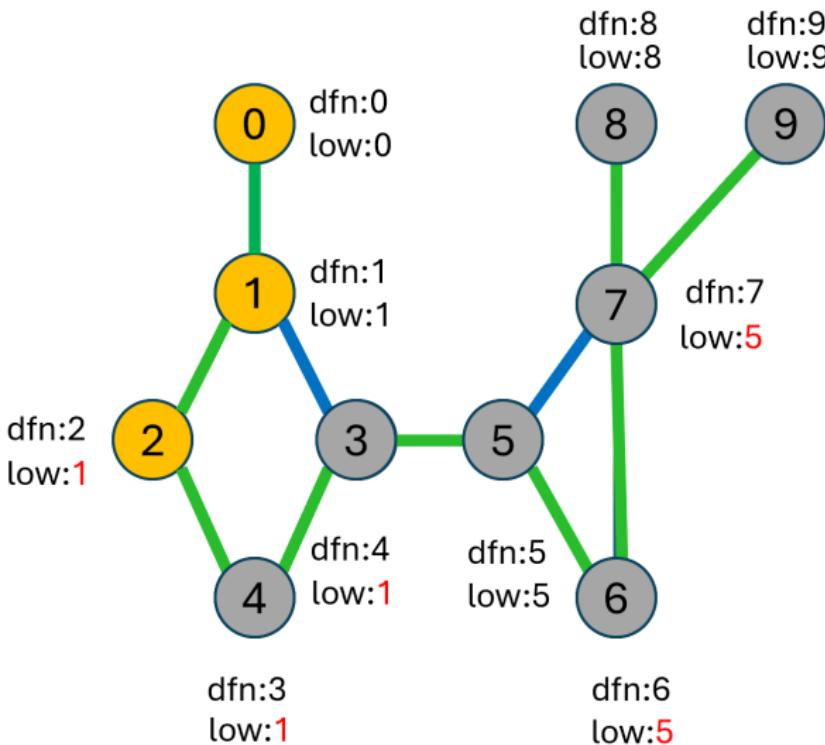


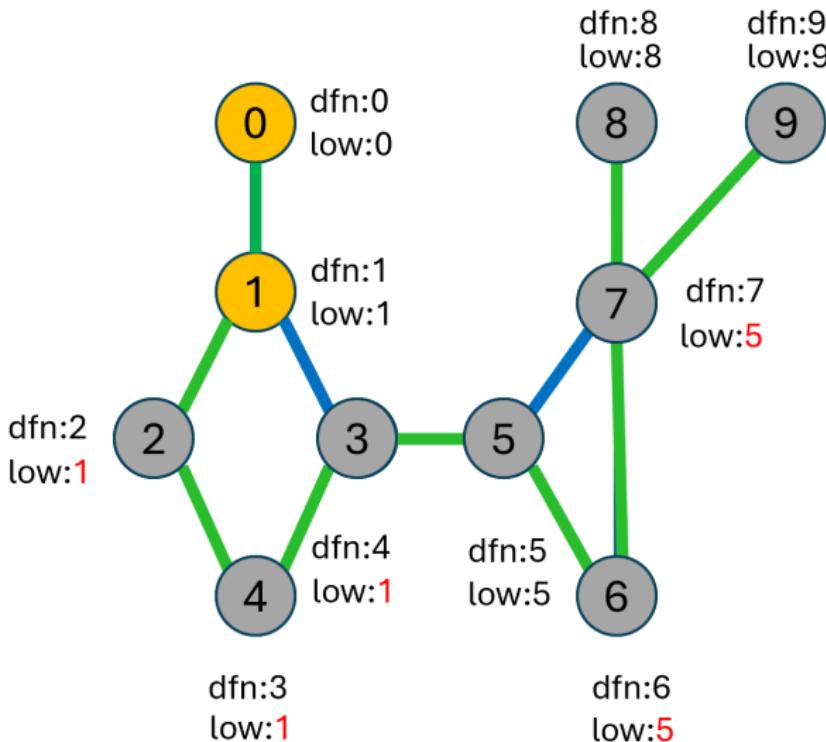


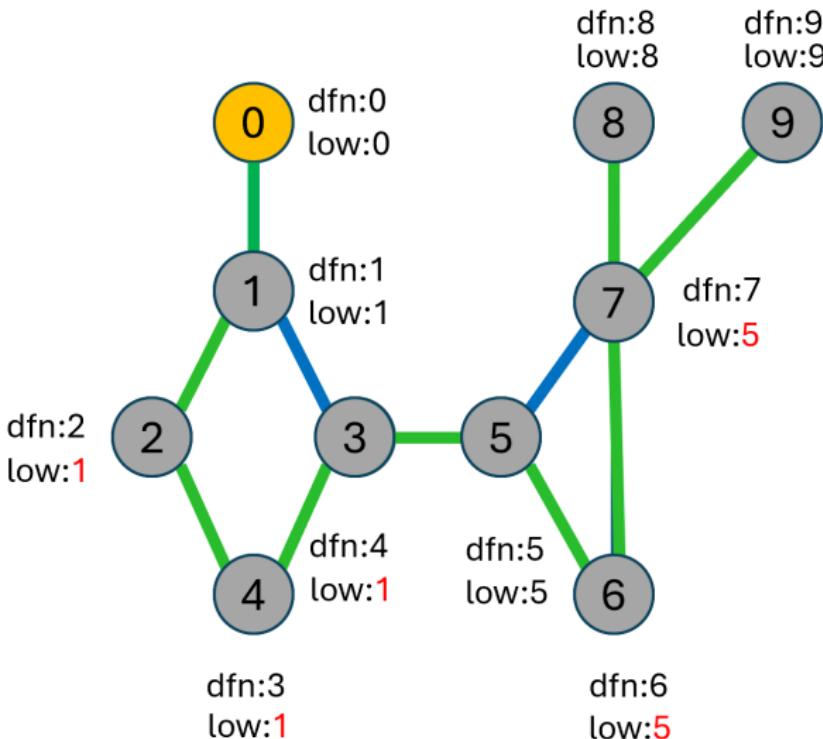


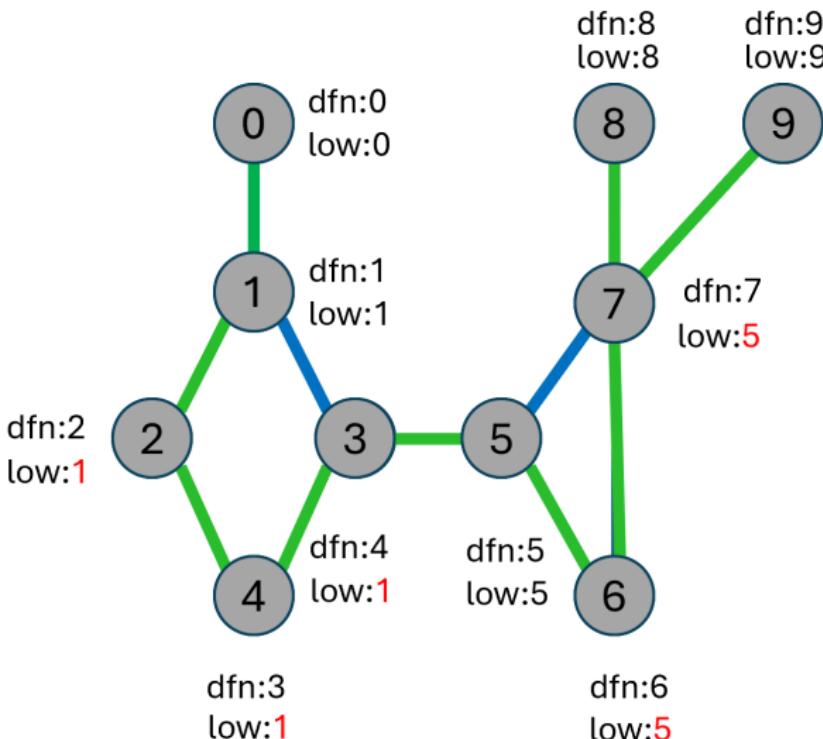












The codes for computing dfn and low

Time complexity: $O(e)$.

```
void dfn_low(int u, int v) {
    /* compute dfn and low while performing a dfs
    search beginning at vertex u, v is the parent
    of u (if any) */
    node_pointer ptr;
    int w;
    dfn[u] = low[u] = num++;
    for (ptr = graph[u]; ptr; ptr = ptr->link) {
        w = ptr->vertex;
        if (dfn[w] < 0) {
            /* w is an unvisited vertex */
            dfn_low(w, u); // visit w from u
            low[u] = MIN2(low[u], low[w]); // (*)
        } else if (w != v)
            low[u] = MIN2(low[u], dfn[w]); // (**)
    }
}
```

```
short int dfn [MAX_VERTICES];
short int low[MAX_VERTICES];
int num = 0;

void init(void) {
    int i;
    for(i = 0; i < n; i++) {
        visited[i] = false;
        dfn[i] = low[i] = -1;
    }
    num = 0;
}
```

bootstrapping by
`dfn_low(x, -1)`



Finding articulation points (3/3)

articulation points

u is an articulation point iff one of the following conditions are satisfied:

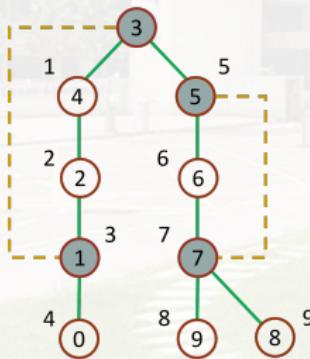
- u is the root of the spanning tree and has two or more children.
- u is not the root of the spanning tree and u has a child w such that $\text{low}(w) \geq \text{dfn}(u)$.

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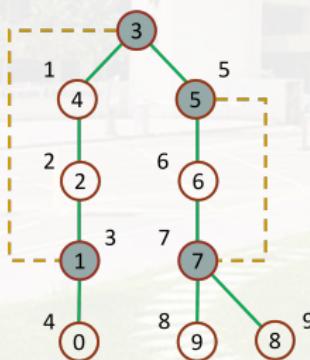
vertex	0	1	2	3	4	5	6	7	8	9
dfn	4	3	2	0	1	5	6	7	9	8
low	4	0	0	0	0	5	5	5	9	8

Finding articulation points (3/3)

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vertex	0	1	2	3	4	5	6	7	8	9
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low	4	0	0	0	0	5	5	5	9	8

- articulation points: 1, 3, 5, 7.

Connected Components

Finding the articulation points

Code for Biconnected Components ($O(n + e)$ time)

```

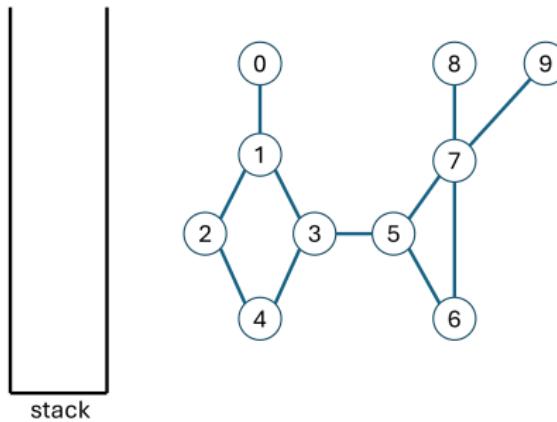
void bicon(int u, int v) { /* dfn[] = -1, num = 0, s is an empty stack initially*/
/* v is the parent node of u in the DFS spanning tree */
    nodePointer ptr;
    int w, x, y;
    dfn[u] = low[u] = num++;
    for (ptr = graph[u]; ptr; ptr = ptr->link) {
        w = ptr->vertex;
        if (v != w && dfn[w] < dfn[u]) {
            push(u,w); /* add edge (u,w) into stack s */
            if (dfn[w] < 0) { /* w is not visited yet */
                bicon(w, u);
                low[u] = MIN2(low[u],low[w]);
                if (low[w] >= dfn[u]) {
                    printf("New biconnected component:");
                    do { /* pop an edge from stack s */
                        pop(&x, &y);
                        printf("<%d,%d>", x, y);
                    } while (!(x == u) && (y == w));
                    printf("\n");
                }
            } else if (w != v)
                low[u] = MIN2(low[u],dfn[w]);
        }
    }
}

```

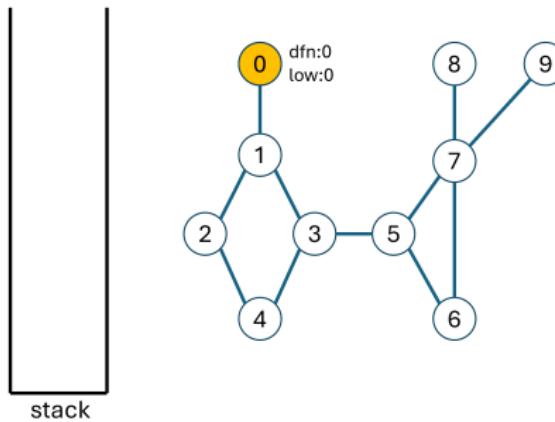
If we have $\text{low}(w) \geq \text{dfn}(v)$ whenever $\text{dfn_low}(u, w)$ returns, then a new biconnected component is found!



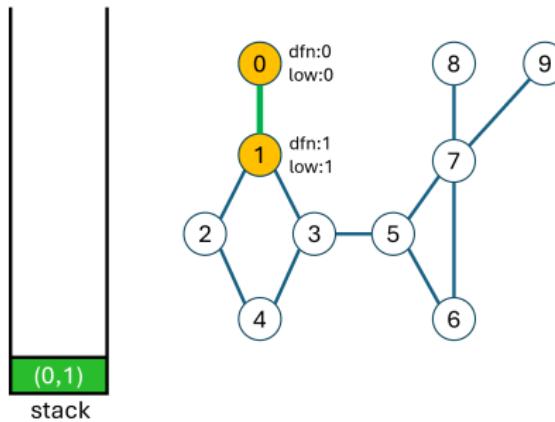
Illustration



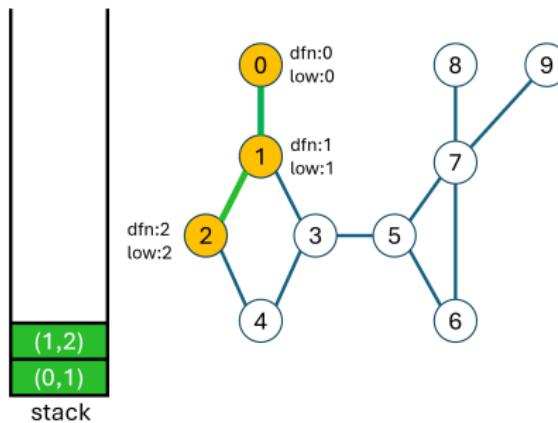
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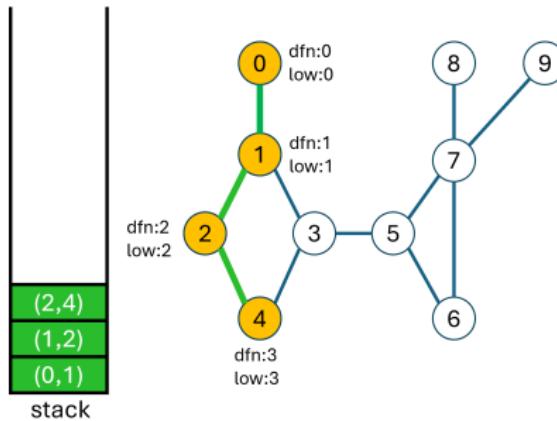
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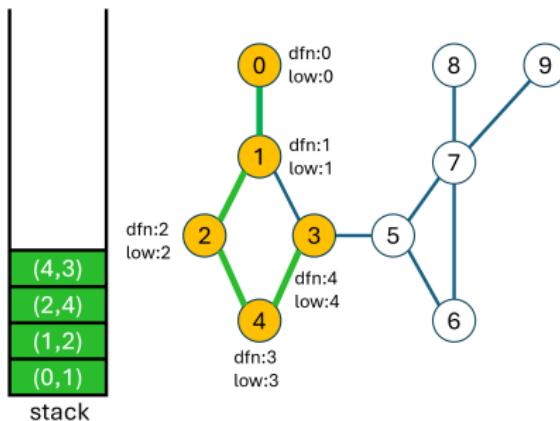
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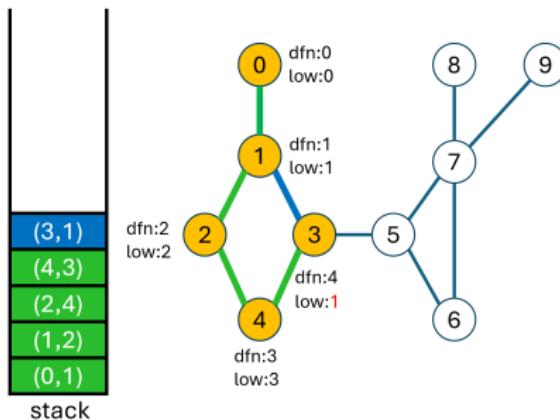
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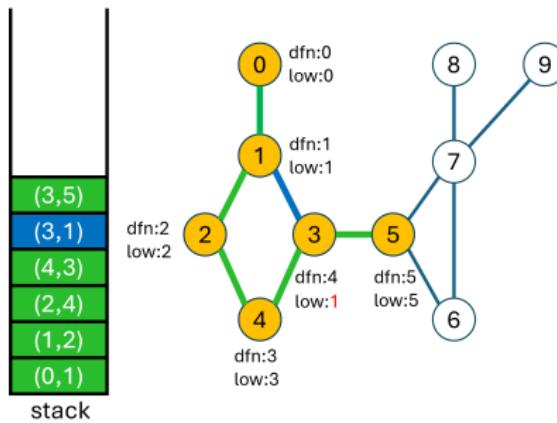
Illustration



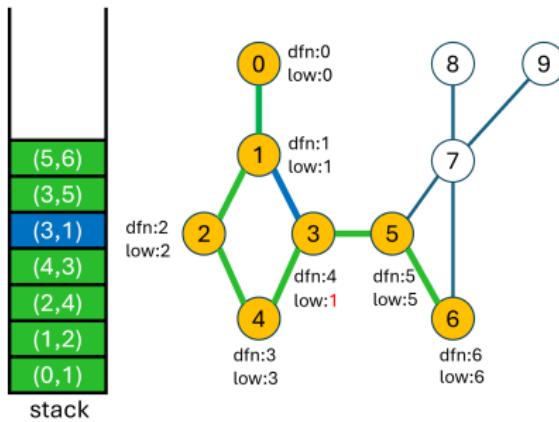
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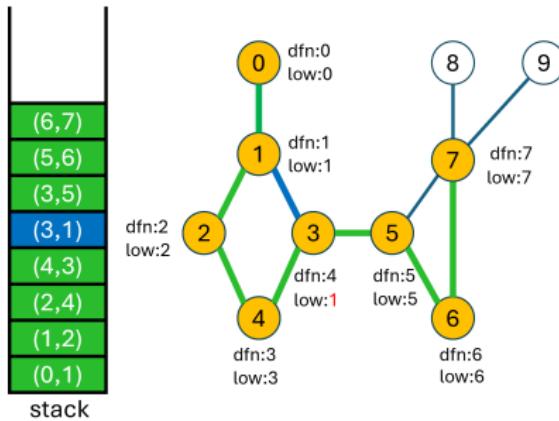
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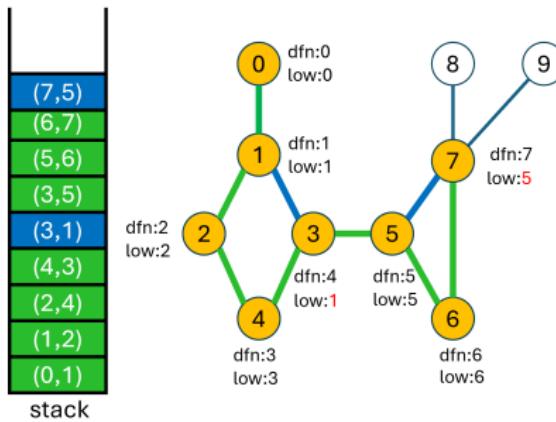
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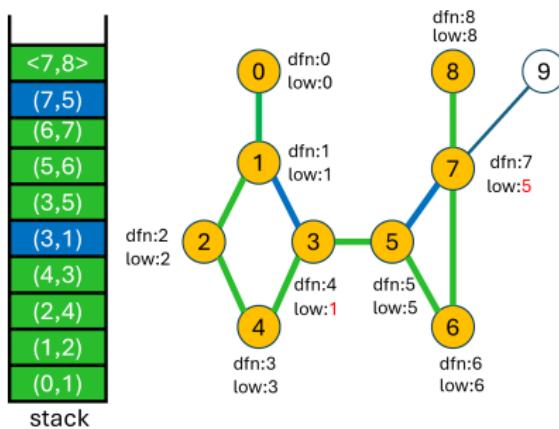
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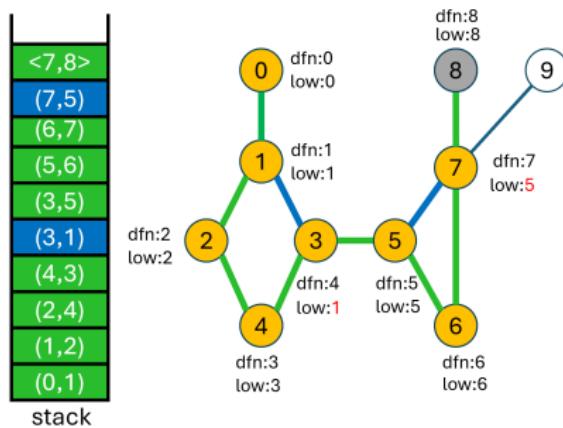
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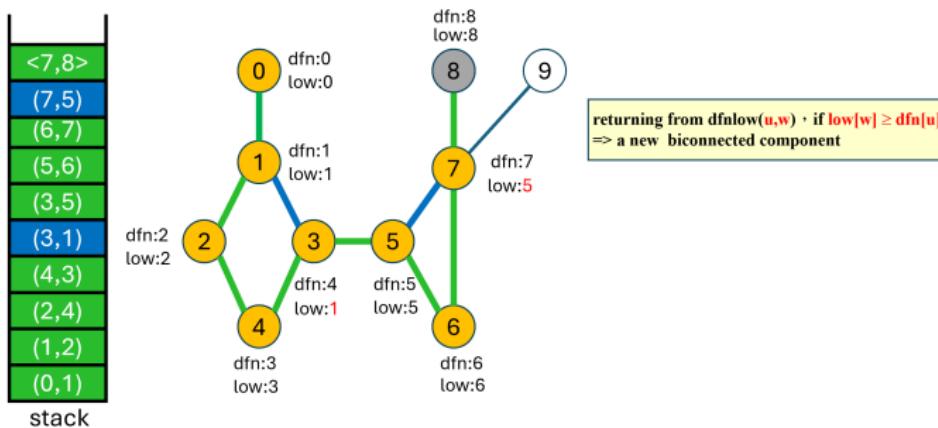
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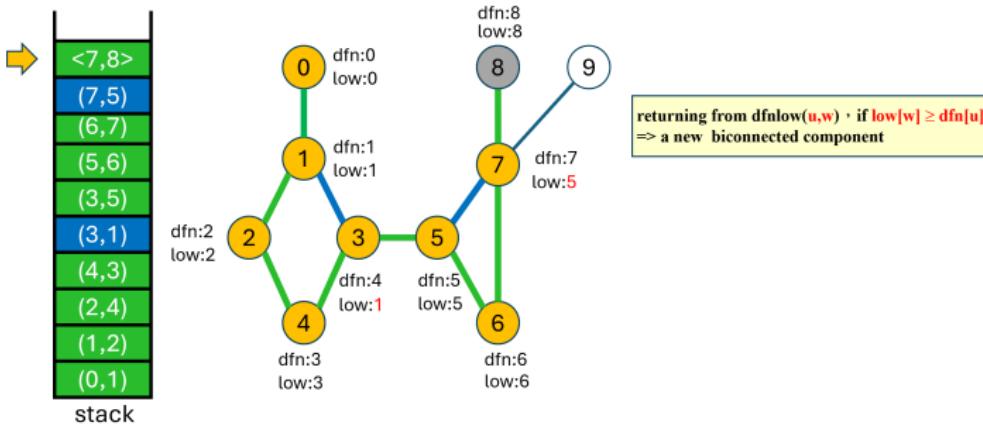
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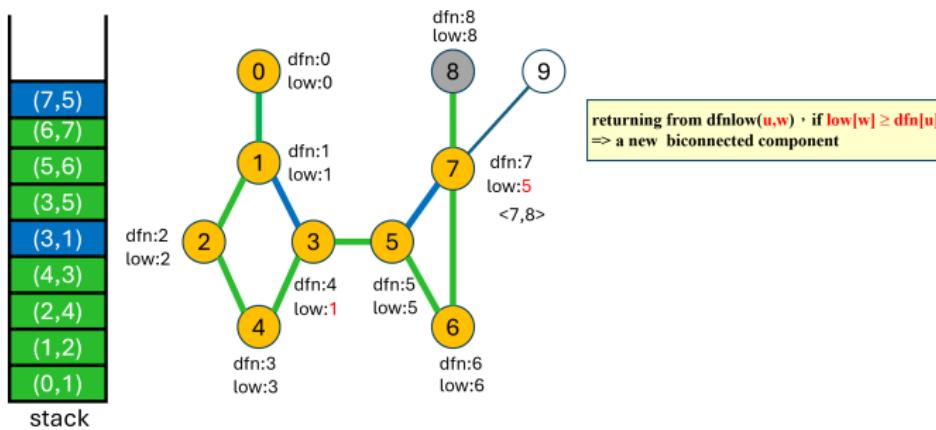
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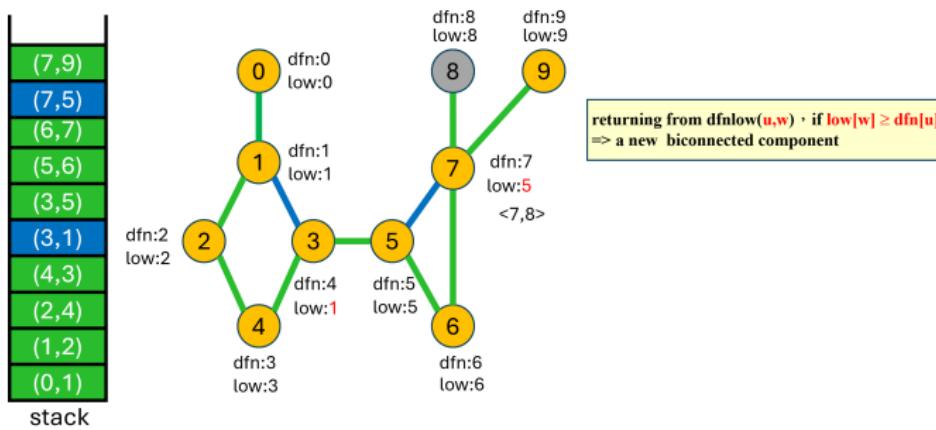
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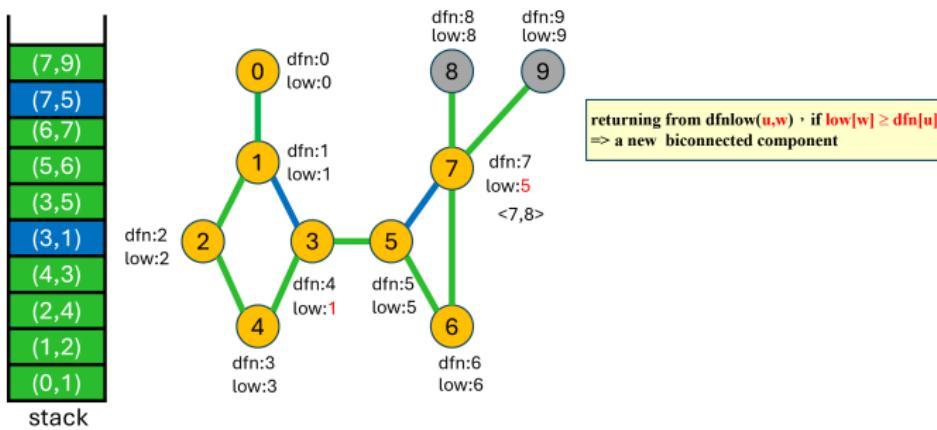
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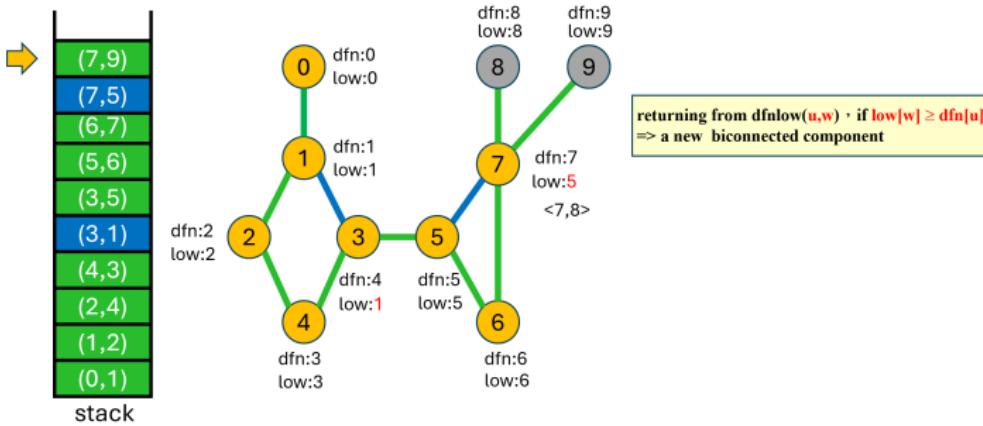
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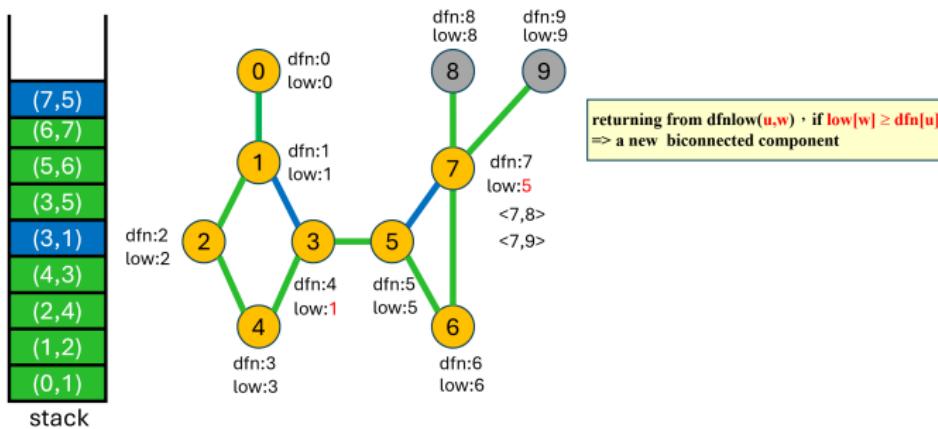
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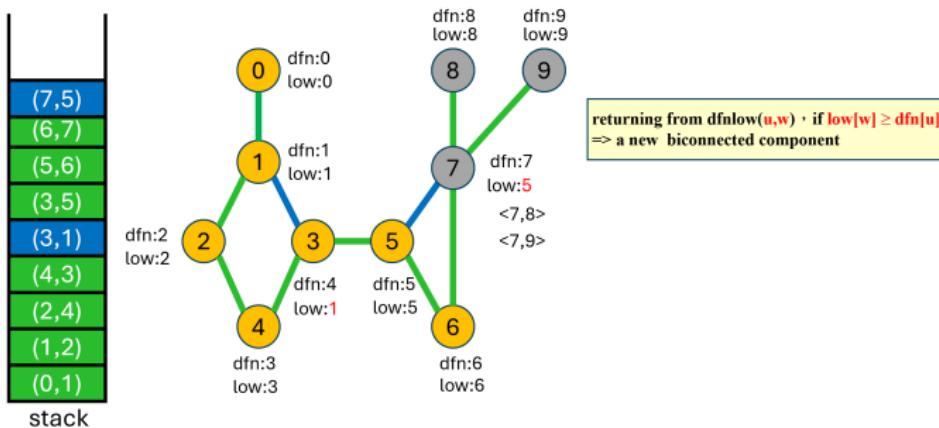
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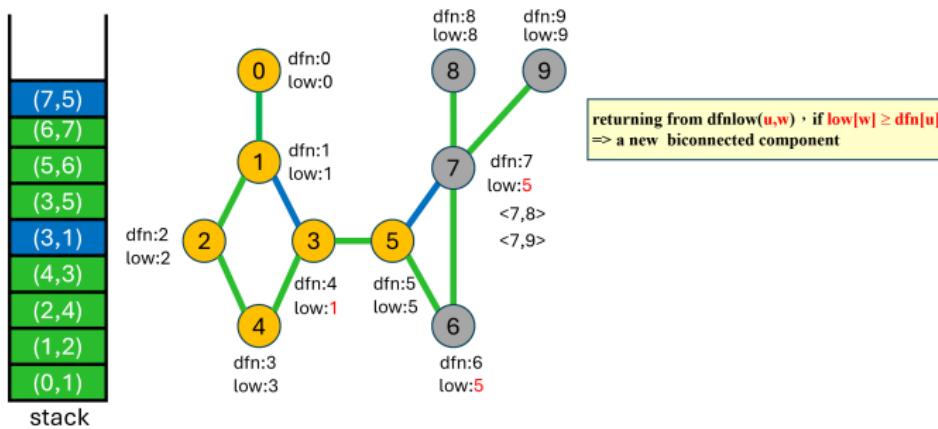
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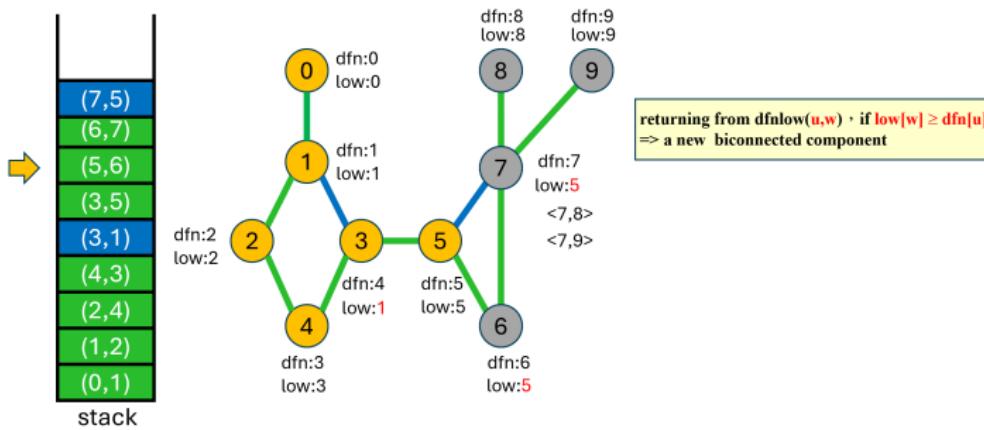
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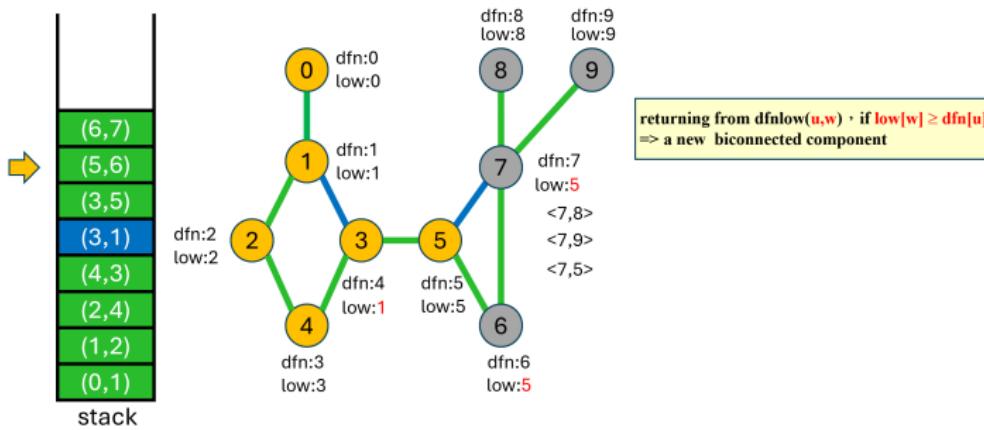
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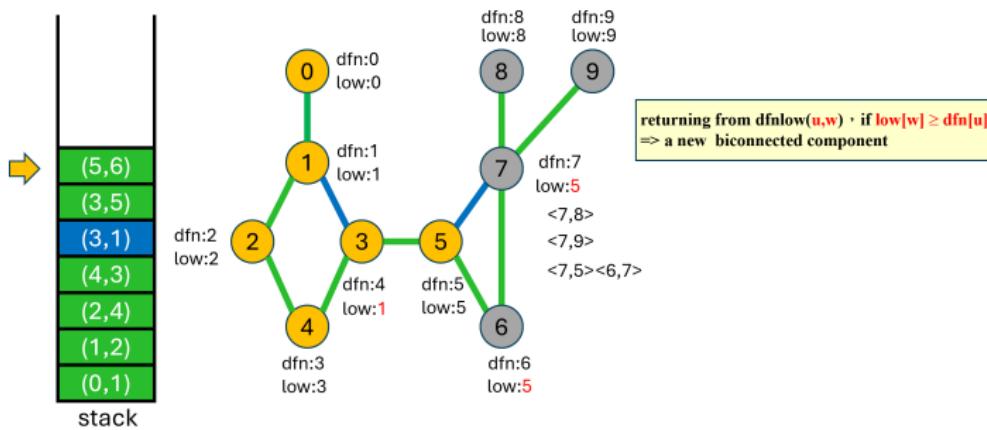
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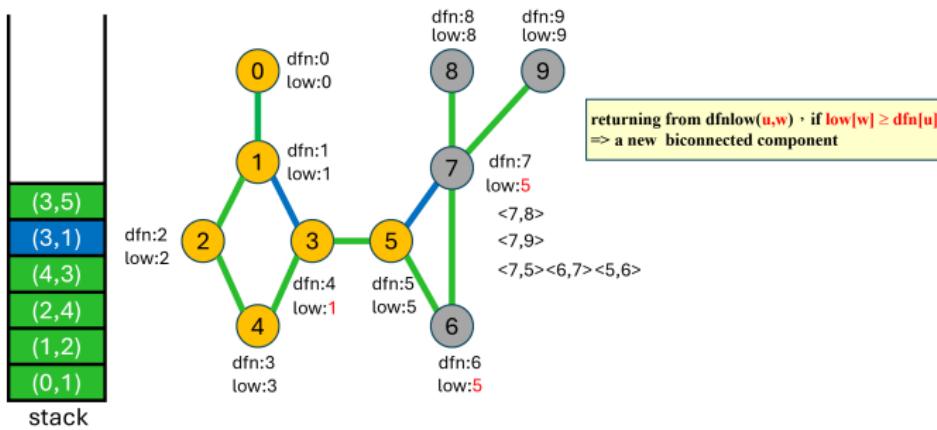
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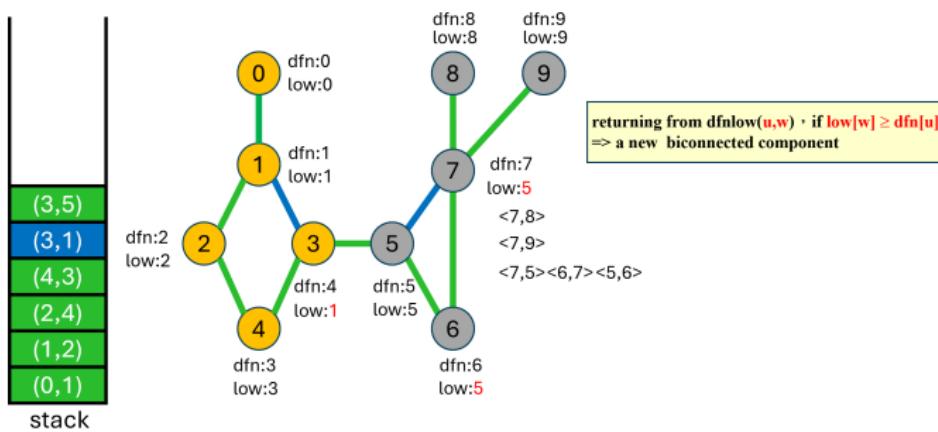
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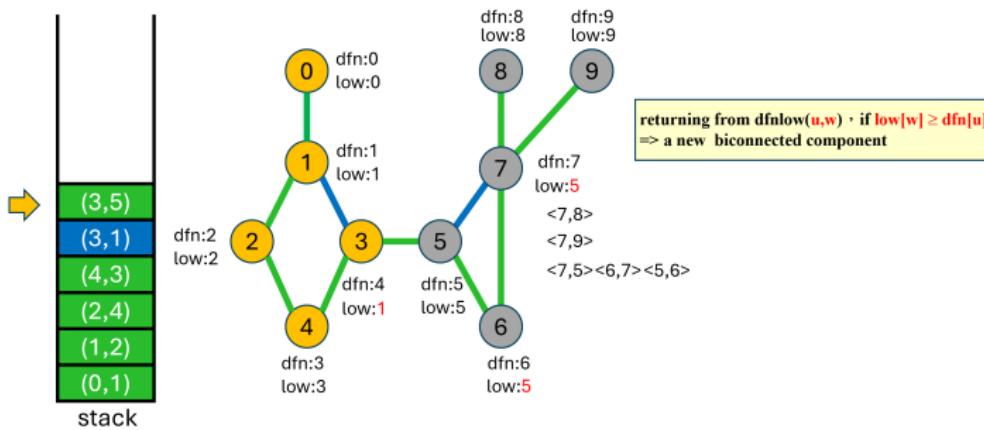
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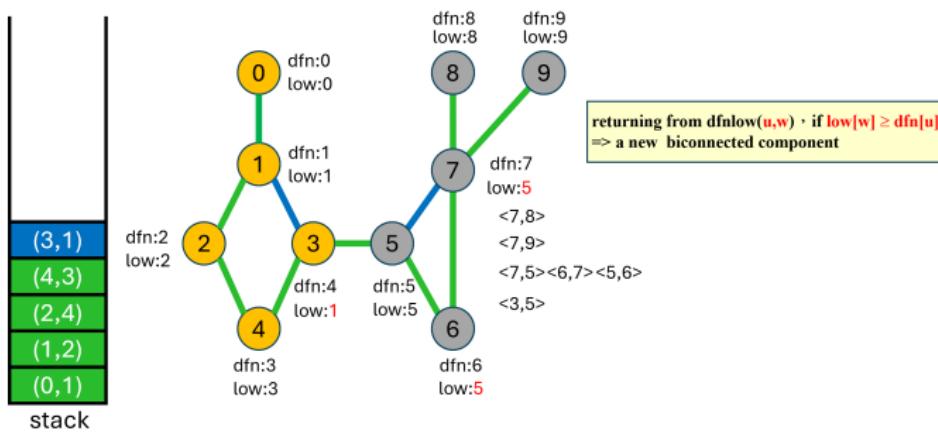
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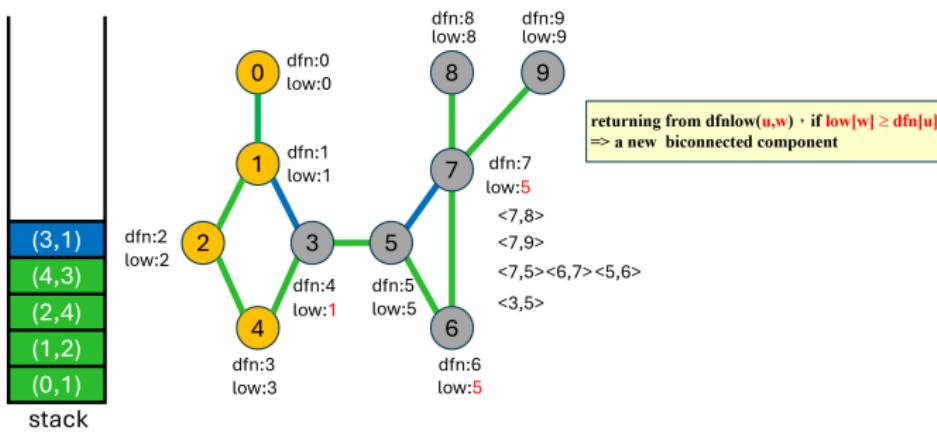
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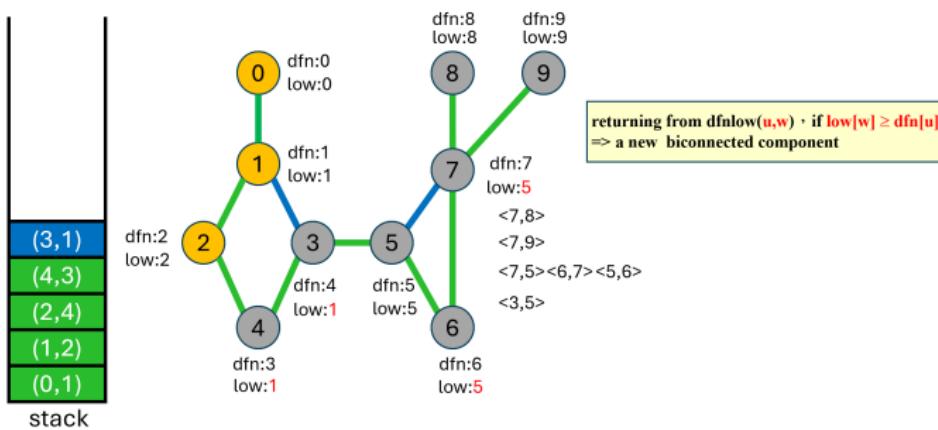
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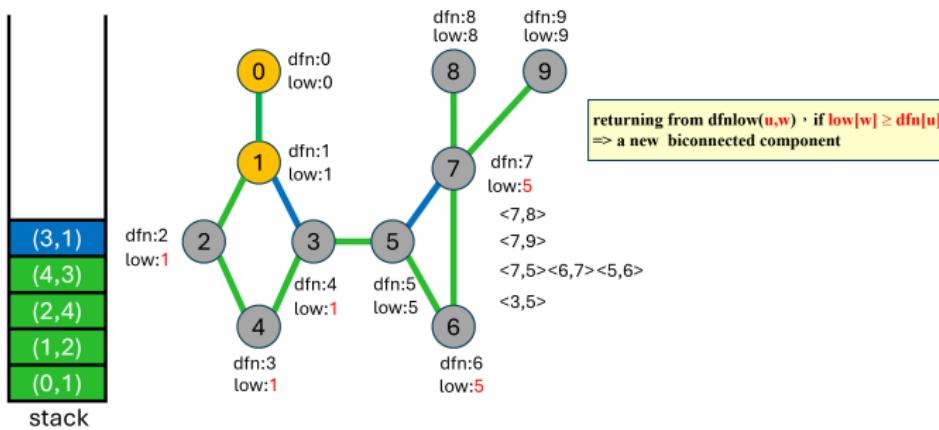
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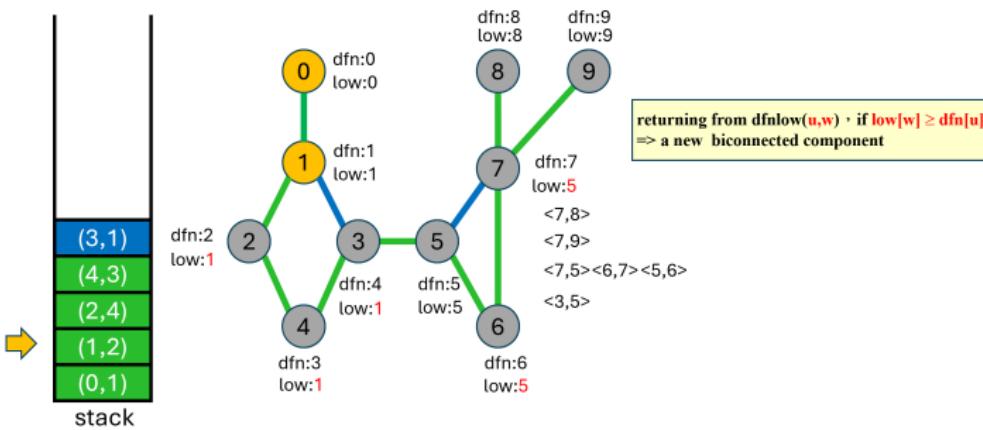
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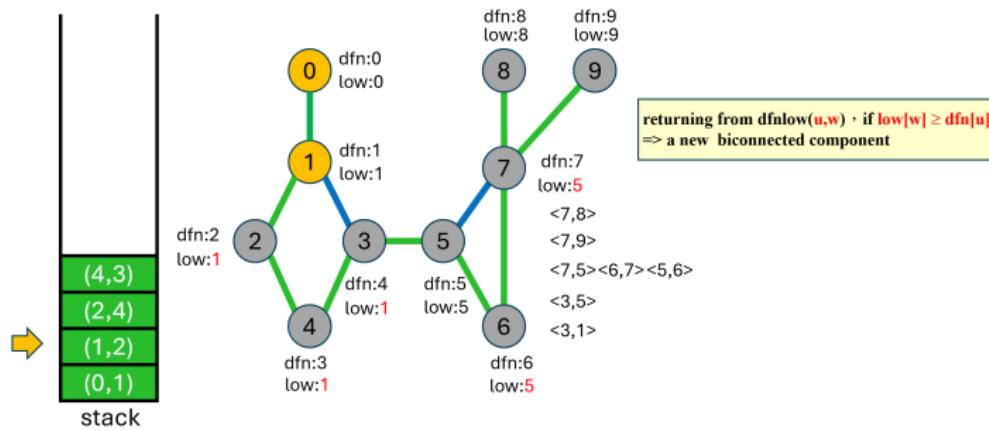
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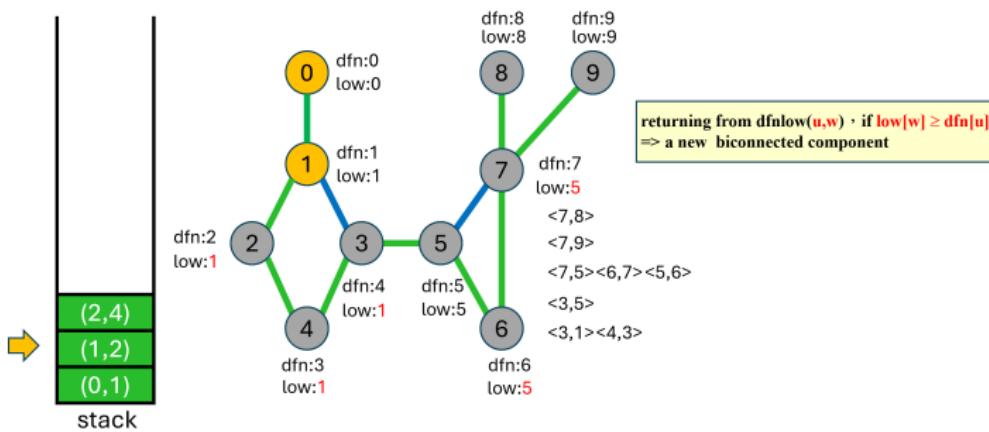
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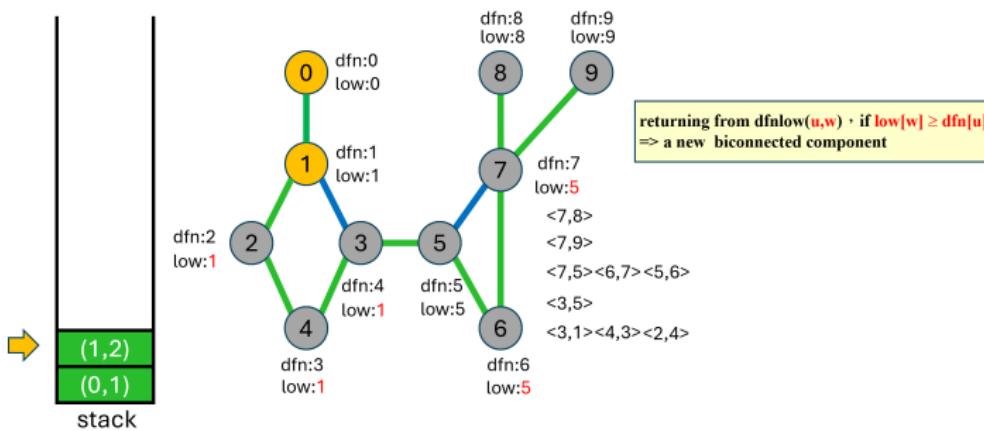
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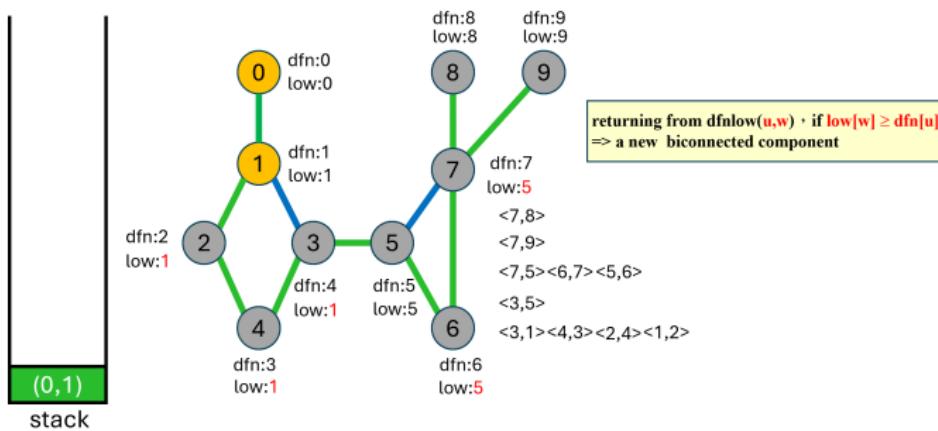
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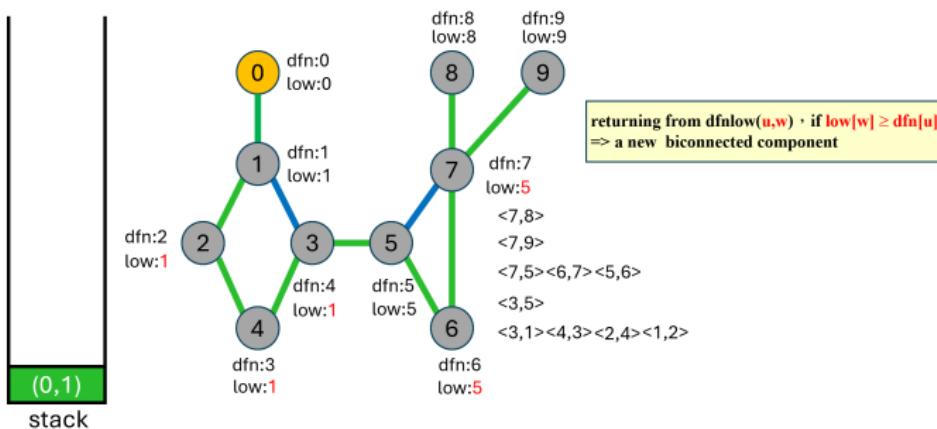
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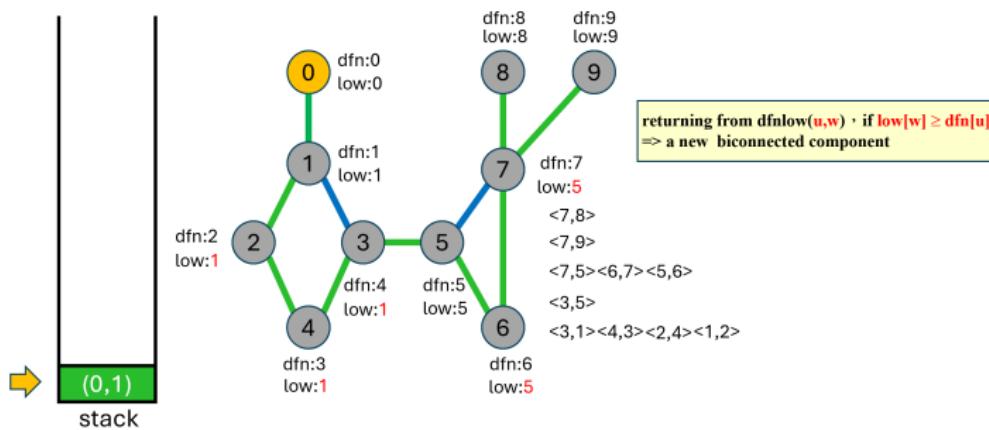
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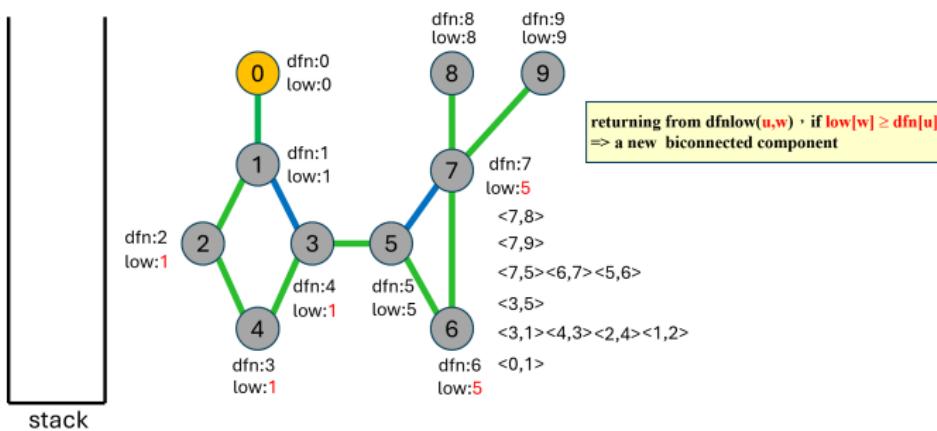
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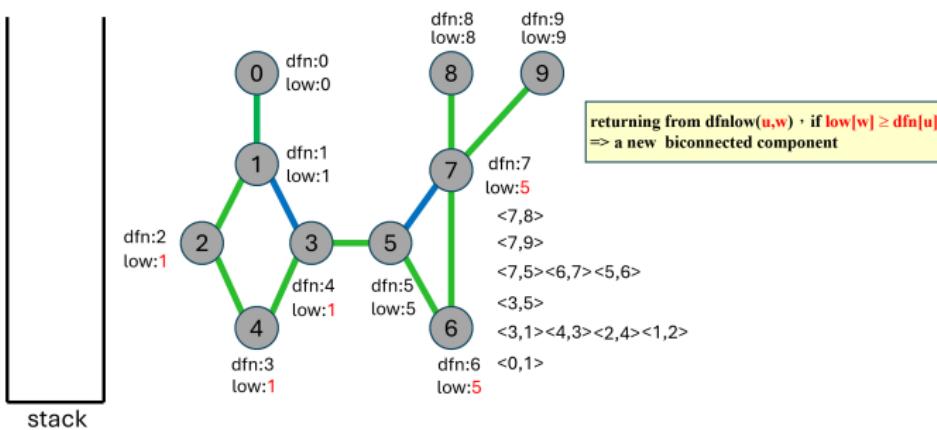
Illustration



Connected Components

Finding the articulation points

Illustration





Discussions