

Simple Near-Optimal Auctions

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Outline

Background & Introduction

The Prophet Inequality

Simple Single-Item Auctions

Prior-Independent Mechanisms



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What we have learned

- ▶ For every single-parameter environment where agents' valuations are drawn independently from **regular** distributions, the corresponding **virtual welfare maximizer** maximizes the **expected revenue** over all **DSIC** mechanisms.
- ▶ The allocation rule:

$$\mathbf{x}(\mathbf{v}) = \arg \max_{\mathbf{x}} \sum_{i=1}^n \varphi_i(v_i) x_v(\mathbf{v})$$

for each valuation profile \mathbf{v} , where

$$\varphi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

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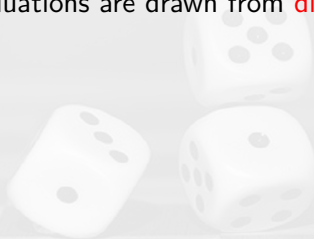
for each valuation profile \mathbf{v} , where

$$\varphi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

- ▶ When F_i 's are i.i.d. & regular, the optimal single-item auction is surprisingly simple: a second-price auction augmented with the reserved price $\varphi^{-1}(0)$.

Optimal Auctions Can Be Complex

- ▶ What if bidders' valuations are drawn from **different** regular distributions?



Optimal Auctions Can Be Complex

- ▶ What if bidders' valuations are drawn from **different** regular distributions?
- ▶ We would like to know if there is any simple and practical single-item auction formats that are at least approximately optimal.

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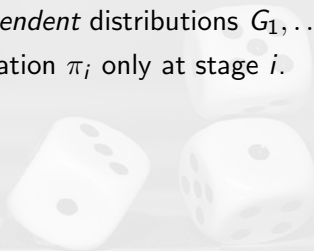
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Game with n stages (resembling the secretary problem)

- ▶ Consider the following game with n stages.
- ▶ In stage i , we are offered a nonnegative prize π_i , drawn from a distribution G_i .
- ▶ We know the *independent* distributions G_1, \dots, G_n in advance.
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 - ▶ either accept the prize and end the game, or
 - ▶ discard the prize, and then proceed to the next stage.

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 - ▶ discard the prize, and then proceed to the next stage.
- ▶ What's the risk and difficulty?

The Prophet Inequality

- ▶ It offers a simple strategy that performs almost as well as (approximately) a fully clairvoyant prophet.

Theorem (Prophet Inequality)

For every sequence G_1, \dots, G_n of n independent distributions,

- ▶ There is a strategy that guarantees expected reward $\geq \frac{1}{2} \mathbf{E}_{\pi \sim \mathbf{G}}[\max_i \pi_i]$.
 - ▶ There is such a threshold strategy, which accepts prize i if and only if $\pi_i \geq t$.
-
- ▶ $z^+ := \max\{z, 0\}$.
 - ▶ $[n] := \{1, 2, \dots, n\}$.

Proof of Prophet Inequality (1/3)

- ▶ Compare the expected payoff of a threshold strategy with that of a prophet, through **lower and upper bounds**.
- ▶ $q(t)$: the probability that the threshold strategy accepts **no prize at all**.
- ▶ First, we want to have a lower bound on

$$\psi := \mathbf{E}_{\pi \sim \mathbf{G}}[\text{payoff of the } t\text{-threshold strategy}].$$

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- ▶ The payoff: with prob. $q(t)$ we get 0 and with prob. $1 - q(t)$ we get $\geq t$.
- ▶ We can credit the baseline t with “extra credit” of $\pi_i - t$.
- ▶ We only credit the baseline t for two or more prizes $\geq t$ (\because LB).

Proof of Prophet Inequality (2/3)

$$\begin{aligned}
 \psi &\geq (1 - q(t))t + \\
 &\quad \sum_{i=1}^n \mathbf{E}_{\pi}[\pi_i - t \mid \pi_i \geq t, \pi_j < t \forall j \neq i] \cdot \Pr[\pi_i \geq t] \cdot \Pr[\pi_j < t \forall j \neq i] \\
 &= (1 - q(t))t + \sum_{i=1}^n \underbrace{\mathbf{E}_{\pi}[\pi_i - t \mid \pi_i \geq t] \cdot \Pr[\pi_i \geq t]}_{= \mathbf{E}_{\pi}[(\pi_i - t)^+]} \cdot \underbrace{\Pr[\pi_j < t \forall j \neq i]}_{\geq q(t) = \Pr[\pi_j < t \forall j]} \\
 &\geq (1 - q(t))t + q(t) \cdot \sum_{i=1}^n \mathbf{E}_{\pi}[(\pi_i - t)^+]
 \end{aligned}$$

Proof of Prophet Inequality (3/3)

Moreover, as to the upper bound on the prophet's expected payoff:

$$\begin{aligned}
 \psi^* &:= \mathbf{E}_\pi \left[\max_{i \in [n]} \pi_i \right] = \mathbf{E}_\pi \left[t + \max_{i \in [n]} (\pi_i - t) \right] \\
 &\leq t + \mathbf{E}_\pi \left[\max_{i \in [n]} (\pi_i - t)^+ \right] \\
 &\leq t + \sum_{i=1}^n \mathbf{E}_\pi [(\pi_i - t)^+].
 \end{aligned}$$

► Set t such that $q(t) = \frac{1}{2}$ we can complete the proof.

$$\frac{t}{2} + \frac{1}{2} \cdot \sum_{i=1}^n \mathbf{E}_\pi [(\pi_i - t)^+] \leq \psi \leq \psi^* \leq t + \sum_{i=1}^n \mathbf{E}_\pi [(\pi_i - t)^+]$$

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- ▶ Set t such that $q(t) = \frac{1}{2}$ we can complete the proof.
 - ▶ $\text{LB} := \frac{t}{2} + \frac{1}{2} \cdot \sum_{i=1}^n \mathbf{E}_{\pi} [(\pi_i - t)^+] \leq \psi \leq \psi^* \leq t + \sum_{i=1}^n \mathbf{E}_{\pi} [(\pi_i - t)^+] = 2 \cdot \text{LB}.$
 - ▶ Why $\psi \leq \psi^*$?

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Back to the motivating example

- ▶ Single-item auction with n bidders.
- ▶ Bidders' valuations are drawn independently from regular distributions F_1, \dots, F_n that might not be identical.
- ▶ Using the prophet inequality:
 - ▶ Define the i th prize as $\varphi_i(v_i)^+$ of bidder i .
 - ▶ G_i : the corresponding distribution induced by F_i (independent).
- ▶ We have (by Theorem 5.2; with maximizer $\mathbf{x} = (x_i)_{i \in [n]}$)

$$\mathbf{E}_{\mathbf{v} \sim \mathbf{F}} \left[\sum_{i=1}^n \varphi_i(v_i) x_i(\mathbf{v}) \right] = \mathbf{E}_{\mathbf{v} \sim \mathbf{F}} \left[\max_{i \in [n]} \varphi_i(v_i)^+ \right].$$

The allocation rule

Consider any allocation rule having the following form:

Virtual Threshold Allocation Rule

- ▶ Choose t such that $\Pr[\max_i \varphi_i(v_i)^+ \geq t] = \frac{1}{2}$.
- ▶ Give the item to a bidder i with $\varphi_i(v_i) \geq t$, if any, breaking ties among multiple candidate winners arbitrarily.

¹ We immediately have the following corollary:

¹What if no such t exists? An exercise!

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Corollary (Virtual Threshold Rules are Near-Optimal)

If \mathbf{x} is a virtual threshold allocation rule, then

$$\mathbf{E}_{\mathbf{v}} \left[\sum_{i=1}^n \varphi_i(v_i)^+ x_i(\mathbf{v}) \right] \geq \frac{1}{2} \mathbf{E}_{\mathbf{v}} \left[\max_{i \in [n]} \varphi_i(v_i^+) \right].$$

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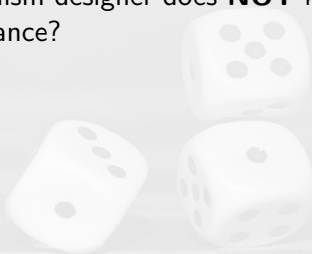
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A different critique so far

- ▶ So far, the valuation distributions are assumed to be **known to the mechanism designer in advance**.
- ▶ What if the mechanism designer does **NOT** know about the valuation distributions in advance?



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- ▶ Next, we consider that
 - ▶ Bidders' valuations are still drawn from such valuation distributions;
 - ▶ Yet, these distributions are still unknown to the mechanism designer.
 - ★ We use the distributions in the *analysis*, but **NOT** in the design of mechanisms.
- ▶ Goal: design a good **prior-independent** mechanism.
 - ▶ Such a mechanism makes NO reference to a valuation distribution.

A Beautiful Result from Auction Theory

- ▶ The expected revenue of an optimal single-item auction is at most that of a second-price auction (with no reserved price) with **one extra** bidder.

Theorem [Bulow-Klemperer Theorem (1989)]

We have

- ▶ F : a regular distribution;
- ▶ n : a positive integer.
- ▶ \mathbf{p} : the payment rule of the second-price auction with $n + 1$ bidders.
- ▶ \mathbf{p}^* : the payment rule of the optimal auction for F with n bidders.

Then,

$$\mathbf{E}_{\mathbf{v} \sim F^{n+1}} \left[\sum_{i=1}^{n+1} p_i(\mathbf{v}) \right] \geq \mathbf{E}_{\mathbf{v} \sim F^n} \left[\sum_{i=1}^n p_i^*(\mathbf{v}) \right]$$

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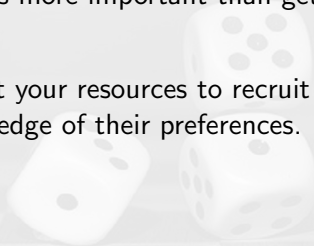
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Interpretation of the Bulow-Klemperer Theorem

- ▶ Extra competition is more important than getting the auction format just right.
- ▶ It is better to invest your resources to recruit more serious participants than sharpen your knowledge of their preferences.



Proof of the Bulow-Klemperer Theorem (1/3)

- It's tricky to compare two sides of the inequality directly.



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- ▶ Let's consider a fictitious auction \mathcal{A} below to facilitate the comparison.

The Fictitious Auction \mathcal{A}

1. Simulate an optimal n -bidder auction for F on the first n bidders.
2. If the item was not awarded in the first step, then give the item to the $(n + 1)$ th bidder for free.

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1. Simulate an optimal n -bidder auction for F on the first n bidders.
 2. If the item was not awarded in the first step, then give the item to the $(n + 1)$ th bidder for free.
- ▶ The expected revenue of \mathcal{A} equals that of an optimal auction with n bidders.
 - ▶ The right-hand side of the inequality.

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The second-price auction maximizes the expected revenue over all DSIC auctions that **always allocate the item**.

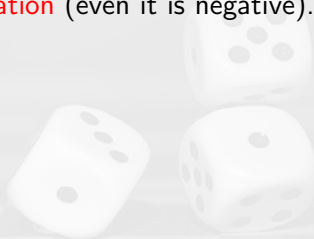
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- ▶ From previous lectures, it suffices to maximize the expected virtual welfare.



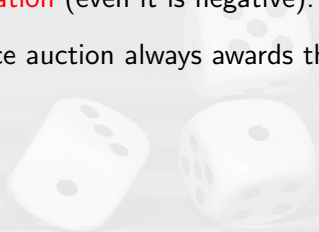
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- ▶ From previous lectures, it suffices to maximize the expected virtual welfare.
- ▶ The allocation rule with maximum possible expected virtual welfare *subject to always allocating the item* always awards the item to a bidder with the **highest virtual valuation** (even it is negative).



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- ▶ All bidders share the same nondecreasing virtual valuation function φ .
 - ▶ A bidder with highest valuation also has the highest virtual valuation.

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- ▶ All bidders share the same nondecreasing virtual valuation function φ .
 - ▶ A bidder with highest valuation also has the highest virtual valuation.
- ▶ Hence, the second-price auction maximizes expected revenue subject to always awarding the item.