Social Choice

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Outline

The Setting of Social Choice

Take voting scheme for example.

- A set O of outcomes (i.e., alternatives, candidates, etc.)
- A set A of agents s.t. each of them has a preference $\succ$ over the outcomes.
- The social choice function: a mapping from the profiles of the preferences to a particular outcome.

Outcomes & preferences

outcomes : a, b, c, d

$d \succ b \succ a \succ c$

preferences

d	b	a	a	a
b	c	b	c	b
a	a	c	b	c
c	d	d	d	d

Preferences

- A binary relation $\succ$ such that
  - for every $a, b \in O$, $a \neq b$, we have either $a \succ b$ or $b \succ a$ but NOT both.
  - for $a, b, c \in O$, if $a \succ b$ and $b \succ c$, then we have $a \succ c$.
- $\succeq$ can be defined similarly.
  - $\prec$: $\neg \succ$

Agents with preferences

- E.g., five agents (voters).
- Each agent has its preference over four candidates $\{a, b, c, d\}$.

outcomes : a, b, c, d $d \succ b \succ a \succ c$

preferences

d	b	a	a	a
b	c	b	c	b
a	a	c	b	c
c	d	d	d	d

Agents with preferences

- E.g., three agents (voters).
- Each agent has its preference over four candidates $\{a, b, c, d\}$.

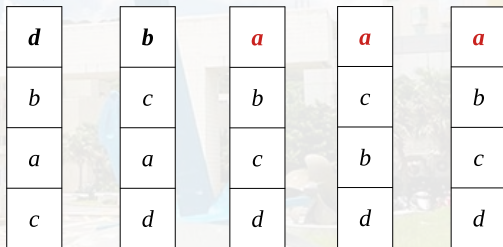| v_1	v_2	v_3
d	b	a
b	c	b
a	a	c
c	d	d

Plurality rule ⇒ a

outcomes : a, b, c, d

$d \succ b \succ a \succ c$

preferences

d	b	a	a	a
b	c	b	c	b
a	a	c	b	c
c	d	d	d	d

- Plurality rule: each agent can only give score 1 to the most preferred one and 0 to the others.

Plurality rule (contd.)

$$v_1 \qquad v_2 \qquad v_3$$

v_1	v_2	v_3
d	b	a
b	c	b
a	a	c
c	d	d

- Plurality rule: each agent can only give score 1 to the most preferred one and 0 to the others.

Plurality rule (contd.)

v_1	v_2	v_3
d	***b***	***a***
b	c	b
a	a	c
c	d	d

- Plurality rule:

Plurality rule (contd.)

v_1	v_2	v_3
d	***b***	***a***
b	*c*	*b*
a	*a*	*c*
c	*d*	*d*

- Plurality rule: depending on the tie-breaking rule.

Condorcet rule

v_1	v_2	v_3
d	b	a
b	c	b
a	a	c
c	d	d

- Condorcet rule:
  - a vs. b
  - a vs. c
  - a vs. d

Condorcet rule

- Condorcet rule:
  - a vs. $b \to b$
  - a vs. $c \to a$
  - a vs. $d \to a$

Condorcet rule

v_1	v_2	v_3
d	b	a
b	c	b
a	a	c
c	d	d

- Condorcet rule:
    - c vs. a
    - c vs. b
    - c vs. d

Condorcet rule

v_1	v_2	v_3
d	b	a
b	c	b
a	a	c
c	d	d

- Condorcet rule:
  - c vs. $a \rightarrow a$
  - c vs. $b \rightarrow b$
  - c vs. $d \rightarrow c$

Condorcet rule

v_1	v_2	v_3
d	b	a
b	c	b
a	a	c
c	d	d

- Condorcet rule:
  - b vs. a
  - b vs. c
  - b vs. d

Condorcet rule

v_1 v_2 v_3

v_1	v_2	v_3
d	b	a
b	c	b
a	a	c
c	d	d

- Condorcet rule:
  - b vs. $a \rightarrow b$
  - b vs. $c \rightarrow b$
  - b vs. $d \rightarrow b$

Condorcet rule

v_1	v_2	v_3
d	b	a
b	c	b
a	a	c
c	d	d

- Condorcet rule: b
  - b vs. $a \rightarrow b$
  - b vs. $c \rightarrow b$
  - b vs. $d \rightarrow b$

Borda rule

- Borda count rule:

Borda rule

v_1		v_2		v_3	
d	**3**	b	**3**	a	**3**
b	**2**	c	**2**	b	**2**
a	**1**	a	**1**	c	**1**
c	**0**	d	**0**	d	**0**

- Borda count rule:
  - score of a: $1 + 1 + 3 = 5$.
  - score of b: $2 + 3 + 2 = 7$.
  - score of c: $0 + 2 + 1 = 3$.
  - score of d: $3 + 0 + 0 = 3$.

Borda rule

- Borda count rule: b.
    - score of a: $1 + 1 + 3 = 5$.
    - score of b: $2 + 3 + 2 = 7$.
    - score of c: $0 + 2 + 1 = 3$.
    - score of d: $3 + 0 + 0 = 3$.

Inefficiency of Borda Count

Inefficiency of Borda Count

- Who is the winner by Borda counting?

Inefficiency of Borda Count

v_1: a 2, b 1, c 0

v_2: a 2, b 1, c 0

v_3: a 2, b 1, c 0

v_4: b 2, c 1, a 0

v_5: b 2, c 1, a 0

- Who is the winner by Borda counting? a: 6, b: 7, c: 2.

Inefficiency of Borda Count

v_1

a	2
b	1
c	0

v_2

a	2
b	1
c	0

v_3

a	2
b	1
c	0

v_4

b	2
c	1
a	0

v_5

b	2
c	1
a	0

- Who is the winner by Borda counting? a: 6, b: 7, c: 2.
- Condorcet principle follows?

Inefficiency of Borda Count

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v_5

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c	1
a	0

- Who is the winner by Borda counting? a: 6, b: 7, c: 2.
- Condorcet principle follows? $a \succ b$, $a \succ c$.

Inefficiency of Borda Count

- Who is the winner by Borda counting? a: 6, b: 7, c: 2.
- Condorcet principle follows? $a \succ b$, $a \succ c$.
- Who is the winner under the plurality rule?

Inefficiency of Borda Count

v_1	v_2	v_3	v_4	v_5
a 2	a 2	a 2	b 2	b 2
b 1	b 1	b 1	c 1	c 1
c 0	c 0	c 0	a 0	a 0

- Who is the winner by Borda counting? a: 6, b: 7, c: 2.
- Condorcet principle follows? $a \succ b$, $a \succ c$.
- Who is the winner under the plurality rule? a.

Successive elimination

	v_1	v_2	v_3
	b	a	c
	d	b	a
	c	d	b
	a	c	d

- Successive elimination with ordering $a \rightarrow b \rightarrow c \rightarrow d$:

Successive elimination

v_1	v_2	v_3
b	a	c
d	b	a
c	d	b
a	c	d

- Successive elimination with ordering $a \rightarrow \not{b} \rightarrow c \rightarrow d$:

Successive elimination

v_1	v_2	v_3
b	a	c
d	b	a
c	d	b
a	c	d

- Successive elimination with ordering $\not{a} \to \not{b} \to c \to d$:

Successive elimination

v_1	v_2	v_3
b	a	c
d	b	a
c	d	b
a	c	d

- Successive elimination with ordering $\not{a} \to \not{b} \to \not{c} \to d$:

Successive elimination

- Successive elimination with ordering $\not{a} \to \not{b} \to \not{c} \to d$:   *d*

Successive elimination

	v_1	v_2	v_3
	b	a	c
	d	b	a
	c	d	b
	a	c	d

- Successive elimination with ordering $\not{a} \to \not{b} \to \not{c} \to d$: d
  - The issue: all of the agents prefer b to d!

Successive elimination

- Successive elimination with ordering $a \to b \to c \to d$:   *d*
- Successive elimination with ordering $a \to c \to b \to d$:

Successive elimination

v_1	v_2	v_3
b	a	c
d	b	a
c	d	b
a	c	d

- Successive elimination with ordering $a \to b \to c \to d$: d
- Successive elimination with ordering $a \to c \to b \to d$: b

Successive elimination (sensitive to the agenda order)

v_1	v_2	v_3
b	a	c
d	b	a
c	d	b
a	c	d

- Successive elimination with ordering $a \to b \to c \to d$: d
- Successive elimination with ordering $a \to c \to b \to d$: b
- Successive elimination with ordering $b \to c \to a \to d$:

Successive elimination (sensitive to the agenda order)

v_1	v_2	v_3
b	a	c
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c	d	b
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- Successive elimination with ordering $a \to c \to b \to d$: b
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Condorcet Winner vs. Plurality

- Let's say we have 1,000 agents each of which has a preference over three candidates A, B, C.
  - 499 agents for $A \succ B \succ C$.
  - 3 agents for $B \succ C \succ A$.
  - 498 agents for $C \succ B \succ A$.
- Who is the Condorcet winner?

Condorcet Winner vs. Plurality

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  - 3 agents for $B \succ C \succ A$.
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- Who is the Condorcet winner? B.
- Who is the winner under the plurality rule?

Condorcet Winner vs. Plurality

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  - 499 agents for $A \succ B \succ C$.
  - 3 agents for $B \succ C \succ A$.
  - 498 agents for $C \succ B \succ A$.
- Who is the Condorcet winner? B.
- Who is the winner under the plurality rule? A.

Exercise

On Borda Count & Condorcet

We have five voters with the following preferences (ordering) over the outcomes A, B, C, and D.

- $B \succ C \succ A \succ D$.
- $B \succ D \succ C \succ A$.
- $D \succ C \succ A \succ B$.
- $A \succ D \succ B \succ C$.
- $A \succ D \succ C \succ B$.

Who is the winner by the Borda Count rule?
Who is the Condorcet winner?

Let's consider a practical application in MOOCs.

MOOCs

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  - e.g., Coursera, EdX.

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  - ▷ Ask each student to grade a SMALL number of her peers' assignments.

MOOCs

- MOOCs: Massive Online Open Courses
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- Outscourcing the grading task to the students.

- They may have incentives to assign LOW scores to everybody else.
  - ▷ Ask each student to grade a SMALL number of her peers' assignments.
  - Then merge individual rankings into a global one.

Terminologies

- $\mathcal{A}$: universe of n elements (students).

- (n, k)-grading scheme:

  a collection $\mathcal{B}$ of size-k subsets (bundles) of $\mathcal{A}$, such that each element of $\mathcal{A}$ belongs to exactly k subsets of $\mathcal{B}$.

- The bundle graph:

  Represent the (n, k)-grading scheme with a bipartite graph.

- $\prec_b$: a ranking of the element b contains (partial order).

The aggregation rule

An aggregation rule:
  profile of partial rankings $\rightarrow$ complete ranking of all elements.

- Borda:

- a: 14; b: 12; c: 4; d: 6; e: 9.
    $a \prec b \prec e \prec d \prec c$.

Order-revealing grading scheme

An aggregation rule in peer grading (Borda):

Final Exam Grading		Final Exam Grading		Final Exam Grading		Final Exam Grading		Final Exam Grading	
Curry	2	Alice	3	Elvis	1	Alice	3	Alice	3
Bob	3	Curry	2	David	2	Curry	1	Elvis	1
Elvis	1	David	1	Bob	3	Bob	2	David	2

↓	↓	↓	↓	↓
Alice	Bob	Curry	David	Elvis
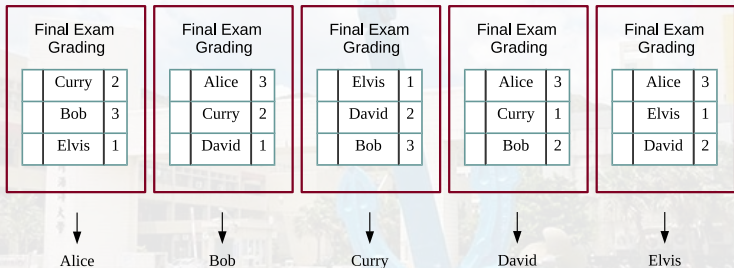
- Alice: 9; Bob: 8; Curry: 5; David: 5; Elvis: 3.
  Alice $\prec$ Bob $\prec$ Curry $\prec$ David $\prec$ Elvis.

Assumption (perfect grading)

Each student grades the assignments in her bundle consistently to the ground truth.

Order-revealing grading scheme (contd.)
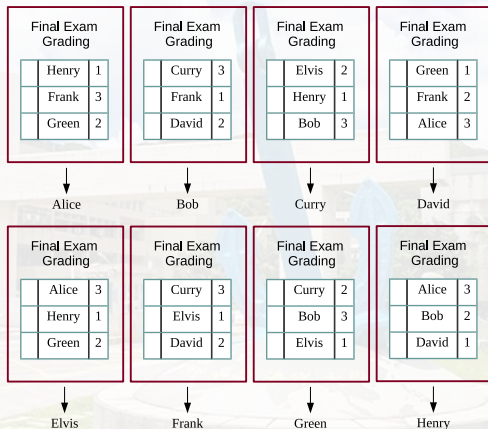
Final Exam Grading	
Henry	1
Frank	3
Green	2

Alice

Final Exam Grading	
Curry	3
Frank	1
David	2

Bob

Final Exam Grading	
Elvis	2
Henry	1
Bob	3

Curry

Final Exam Grading	
Green	1
Frank	2
Alice	3

David

Final Exam Grading	
Alice	3
Henry	1
Green	2

Elvis

Final Exam Grading	
Curry	3
Elvis	1
David	2

Frank

Final Exam Grading	
Curry	2
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Green

Final Exam Grading	
Alice	3
Bob	2
David	1

Henry

- Alice: 9; Bob: 8; Curry: 8; David: 5; Elvis: 4; Frank: 6; Green: 5; Henry: 3.

  Alice $\prec$ Bob $\prec$ Curry $\prec$ Frank $\prec$ David $\prec$ Green $\prec$ Elvis $\prec$ Henry.

Order-revealing grading scheme (contd.)
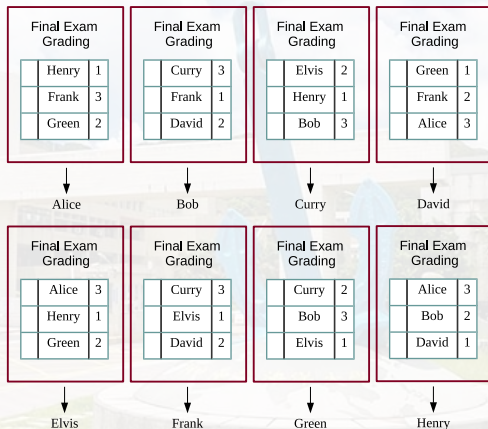
- Alice: 9; Bob: 8; Curry: 8; David: 5; Elvis: 4; Frank: 6; Green: 5; Henry: 3.

  Alice ≺ Bob ≺ Curry ≺ Frank ≺ David ≺ Green ≺ Elvis ≺ Henry.

The bundle graph

The bundle graph:
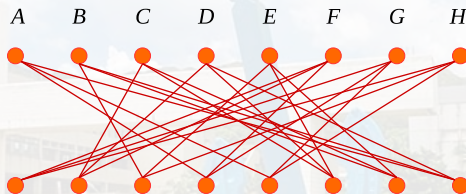
The bundle graph

The bundle graph:

$$A \quad B \quad C \quad D \quad E \quad F \quad G \quad H$$
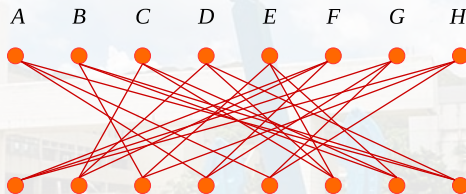
- A random k-regular graph:

  A complete bipartite $K_{n,n} \rightarrow$ removing edges $\{v, v\}$, $\forall v \rightarrow$

  repeat

  　　"draw a perfect matching uniformly at random among all perfect matchings
  　　of the remaining graph"

  for k times.

The limitation on the order revealing scheme

- The property of revealing the ground truth for certain:

  $\forall x, y \in \mathcal{A}, \exists B \in \mathcal{B}$ such that $x, y \in B$.

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  - Suppose NO bundle contains both $x, y \in \mathcal{A}$.
  - Let $\prec, \prec'$ be two complete rankings.
    - x, y are in the first two positions in $\prec, \prec'$;
    - $\prec$ and $\prec'$ differs only in the order of x and y.
  - Clearly, partial rankings within the bundles are identical in both cases.
  - No way to identify whether $\prec$ or $\prec'$ is the ground truth.

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  - Clearly, partial rankings within the bundles are identical in both cases.
  - No way to identify whether $\prec$ or $\prec'$ is the ground truth.

- To reveal the ground truth with certainty: $k = \Omega(\sqrt{n})$.
  - $n \cdot \binom{k}{2} \geq \binom{n}{2}$.

Seeking for approximate order-revealing grading schemes

- Use a bundle graph with a very low degree k (independent of n).

- Randomly permute the elements by $\pi : U \to \mathcal{A}$ before associating them to the nodes of U of the bundle graph.

- Aiming at $\dfrac{\#\text{correctly recovered pairwise relations}}{\binom{n}{2}}$.

The main result

Theorem (Caragiannis, Krimpas, Voudouris@AAMAS'15)

When

- Borda is applied as the aggregation rule, and
- all the partial rankings are consistent to the ground truth,

then the expected fraction of correctly recovered pairwise relations is $1 - O(1/\sqrt{k})$.

Question

- What will happen if we assign for each student only two assignments and each assignment is graded by exactly two students?

About who grading both u and v

- $\lambda_{u,v} := |N(u) \cap N(v)|$, for $u, v \in U$.
  - $\sum_{v \in U \setminus \{u\}} \lambda_{u,v} = k(k-1)$.
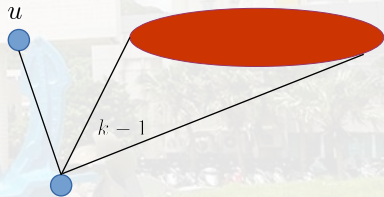
- $W_{r,q}$: the r.v. denoting $B(a_r) - B(a_q)$ for $r < q$, $a_r, a_q \in \mathcal{A}$.
- $\Gamma_{u,v}^{r,q}$: the event that $\pi(u) = a_r$, $\pi(v) = a_q$.

-
$$C := \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} \mathsf{E}[\mathbb{1}\{W_{r,q} > 0\}] = \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} \Pr[W_{r,q} > 0]$$

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& = \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} \left(1 - \sum_{u,v \in U} \Pr[W_{r,q} \le 0 \mid \Gamma_{u,v}^{r,q}] \cdot \Pr[\Gamma_{u,v}^{r,q}] \right)
\end{aligned}
$$

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&= \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} \left(1 - \frac{1}{n(n-1)} \sum_{u,v \in U} \Pr[W_{r,q} \leq 0 \mid \Gamma_{u,v}^{r,q}] \right)
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\end{aligned}
$$

- Given $\Gamma_{u,v}^{r,q}$,
  - the expected Borda score of a_r is $k + (k(k-1) - \lambda_{u,v}) \cdot \frac{n-r-1}{n-2} + \lambda_{u,v}$.
  - the expected Borda score of a_q is $k + (k(k-1) - \lambda_{u,v}) \cdot \frac{n-q}{n-2}$.
- Thus

$$E[W_{r,q} \mid \Gamma_{u,v}^{r,q}] = (k(k-1) - \lambda_{u,v})\frac{q-r-1}{n-2} + \lambda_{u,v}.$$

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$$\mathsf{E}[W_{r,q} \mid \Gamma_{u,v}^{r,q}] = (k(k-1) - \lambda_{u,v}) \frac{q-r-1}{n-2} + \lambda_{u,v}.$$

Why?

Calculate the Borda score from another point of view

- Element a_r gets one point for each bundle it belongs to;
  - plus one additional point for each appearance of an element with rank
    higher $> r$ in the bundles a_r belongs to.

- In the bundles of containing a_r:
  - $\lambda_{u,v}$ appearances of a_q in the bundles of a_r.
  - $k(k-1) - \lambda_{u,v}$ appearances of elements different than a_r, a_q.
    - Each of them has prob. $\frac{n-r-1}{n-2}$ to have rank higher than r.

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- $E[B(a_r) \mid \Gamma_{u,v}^{r,q}] = k + (k(k-1) - \lambda_{u,v})\frac{n-r-1}{n-2} + \lambda_{u,v}$.

- In the bundles of containing a_r:
    - $\lambda_{u,v}$ appearances of a_q in the bundles of a_r.
    - $k(k-1) - \lambda_{u,v}$ appearances of elements different than a_r, a_q.
        - Each of them has prob. $\frac{n-r-1}{n-2}$ to have rank higher than r.
- $E[B(a_r) \mid \Gamma_{u,v}^{r,q}] = k + (k(k-1) - \lambda_{u,v})\frac{n-r-1}{n-2} + \lambda_{u,v}.$

- In the bundles of containing a_q:
    - $\lambda_{u,v}$ appearances of a_q in the bundles of a_r.
    - $k(k-1) - \lambda_{u,v}$ appearances of elements different than a_r, a_q.
        - Each of them has prob. $\frac{n-q}{n-2}$ to have rank higher than q.

- In the bundles of containing a_r:
  - $\lambda_{u,v}$ appearances of a_q in the bundles of a_r.
  - $k(k-1) - \lambda_{u,v}$ appearances of elements different than a_r, a_q.
    - Each of them has prob. $\frac{n-r-1}{n-2}$ to have rank higher than r.
- $E[B(a_r) \mid \Gamma_{u,v}^{r,q}] = k + (k(k-1) - \lambda_{u,v})\frac{n-r-1}{n-2} + \lambda_{u,v}$.

- In the bundles of containing a_q:
  - $\lambda_{u,v}$ appearances of a_q in the bundles of a_r.
  - $k(k-1) - \lambda_{u,v}$ appearances of elements different than a_r, a_q.
    - Each of them has prob. $\frac{n-q}{n-2}$ to have rank higher than q.
- $E[B(a_q) \mid \Gamma_{u,v}^{r,q}] = k + (k(k-1) - \lambda_{u,v})\frac{n-q}{n-2}$.

$$a_1 \qquad\qquad a_r \qquad\qquad a_q \qquad\qquad a_n$$

● ● ●   ● **...** ● **...** ○

Dealing with dependencies

<u>Goal:</u> $\Pr[W_{r,q} \leq 0 \mid \Gamma_{u,v}^{r,q}]$

- Given $\Gamma_{u,v}^{r,q}$, define $S = N(N(u) \cup N(v)) \setminus \{u, v\}$.

- $o : [|S|] \mapsto S$ denotes an arbitrary ordering of nodes of S.

- X_i: the random variable denoting the rank of the element $\pi(o(i))$.

- Define the Doob martingale $Z_0, Z_1, \ldots, Z_{|S|}$ such that
  - $Z_0 = \mathsf{E}[W_{r,q} \mid \Gamma_{u,v}^{r,q}]$;
  - $Z_i = \mathsf{E}[W_{r,q} \mid \Gamma_{u,v}^{r,q}, X_1, \ldots, X_i]$.

- Hence, $W_{r,q} \mid \Gamma_{u,v}^{r,q} = Z_{|S|}$.

Martingale

Martingale

A sequence of random variables $Z_0, Z_1, \ldots, Z_m$ is a martingale w.r.t. a sequence of random variables $X_1, X_2, \ldots, X_m$ if $\forall i = 1, \ldots, m$,

$$\mathsf{E}[Z_i \mid X_1, \ldots, X_{i-1}] = Z_{i-1}.$$

Doob martingale (Joseph L. Doob (1910–2004))

- W: a random variable
- $X_1, \ldots, X_m$: a sequence of m random variables.

The sequence $Z_0, Z_1, \ldots, Z_m$ such that

- $Z_0 = \mathsf{E}[W]$;
- $Z_i = \mathsf{E}[W \mid X_1, \ldots, X_i], \ \forall i = 1, \ldots, m$

is called a Doob martingale.

Azuma-Hoeffding inequality

Azuma-Hoeffding inequality

Let $Z_0, Z_1, \ldots, Z_m$ be a martingale with $Z_i - Z_{i-1} \leq c_i$ for $i = 1, \ldots, m$. Then, for all $t > 0$,

$$\Pr[Z_m - Z_0 \leq -t] \leq \exp\left(-\frac{t^2}{2\sum_{i=1}^m c_i^2}\right).$$

Dealing with dependencies (contd.)

- Given $\Gamma_{u,v}^{r,q}$, define $S = N(N(u) \cup N(v)) \setminus \{u,v\}$.
- $o : [|S|] \mapsto S$ denotes an arbitrary ordering of nodes of S.
- X_i: the random variable denoting the rank of the element $\pi(o(i))$.
- Define the Doob martingale $Z_0, Z_1, \ldots, Z_{|S|}$ such that
    - $Z_0 = E[W_{r,q} \mid \Gamma_{u,v}^{r,q}]$;
    - $Z_i = E[W_{r,q} \mid \Gamma_{u,v}^{r,q}, X_1, \ldots, X_i]$.
- Hence, $W_{r,q} \mid \Gamma_{u,v}^{r,q} = Z_{|S|}$.

Lemma 8

$\forall i \in \{1, 2, \ldots, |S|\}$, it holds that $|Z_i - Z_{i-1}| \leq 2(\lambda_{u,o(i)} + \lambda_{v,o(i)})$.

Lemma 3

For every k-regular bipartite graph G,
$\theta_{u,v} = 4 \sum_{z \in (N(u) \cup N(v)) \setminus \{u,v\}} (\lambda_{u,z} + \lambda_{v,z})^2 \leq 8k(k-1)(4k-3)$.

- Set $t = E[W_{r,q} \mid \Gamma_{u,v}^{r,q}]\ (= Z_0)$, by the Azuma-Hoeffding inequality:

$$
\begin{aligned}
\Pr[Z_{|S|} - Z_0 \leq -t] &= \Pr[W_{r,q} \leq 0 \mid \Gamma_{u,v}^{r,q}] \\
&\leq \exp\left(-\frac{t^2}{2\sum_{i=1}^m c_i^2}\right) \\
&= \exp\left(-\frac{E[W_{r,q} \mid \Gamma_{u,v}^{r,q}]^2}{2\theta_{u,v}}\right).
\end{aligned}
$$

Back to the computation of C

$$
\begin{aligned}
C &= \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} \left(1 - \frac{1}{n(n-1)} \sum_{u,v \in U} \Pr[W_{r,q} \leq 0 \mid \Gamma_{u,v}^{r,q}] \right) \\
&\geq \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} \left(1 - \frac{1}{n(n-1)} \sum_{u,v \in U} \exp\left(-\frac{E[W_{r,q} \mid \Gamma_{u,v}^{r,q}]^2}{2\theta_{u,v}} \right) \right) \\
&= \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} \left(1 - \frac{1}{n(n-1)} \sum_{u,v \in U} e^{-(\beta(u,v) \cdot y(q-r) + \delta(u,v))^2} \right) \\
&= \frac{n(n-1)}{2} - \frac{1}{n(n-1)} \sum_{u,v \in U} \sum_{d=1}^{n-1} (n-d) e^{-(\beta(u,v) \cdot y(d) + \delta(u,v))^2} \\
&\geq \frac{n(n-1)}{2} - \sum_{u,v \in U} \int_0^1 (1-y) e^{-(\beta(u,v) \cdot y + \delta(u,v))^2} dy
\end{aligned}
$$

- $\beta(u,v) = \frac{k(k-1) - \lambda_{u,v}}{\sqrt{2\theta_{u,v}}}$;
- $\delta(u,v) = \frac{\lambda_{u,v}}{\sqrt{2\theta_{u,v}}}$;
- $y(t) = \frac{t-1}{n-2}$.

$$C \geq \frac{n(n-1)}{2} - \sum_{u,v \in U} \int_0^1 (1-y) e^{-(\beta(u,v) \cdot y + \delta(u,v))^2} dy$$

$$\geq \frac{n(n-1)}{2} - \sum_{u,v \in U} \frac{\beta(u,v) + \delta(u,v)}{2\beta(u,v)^2} \sqrt{\pi}$$

$$\geq \frac{n(n-1)}{2} - \frac{k-1}{k(k-2)^2} \sqrt{\frac{\pi}{2}} \sum_{u,v \in U} \sqrt{\theta_{u,v}}$$

$$\geq \frac{n(n-1)}{2} \left(1 - \frac{48\sqrt{2\pi}}{\sqrt{k}}\right).$$

Claim 9

Let $\beta > 0$, $\delta \geq 0$, then
$\int_0^1 (1-y) e^{-(\beta y + \delta)^2} dy \leq \frac{\beta + \delta}{2\beta^2} \sqrt{\pi}$.

- $k \geq 3$ (assumption).