Social Choice

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Outline

The Setting of Social Choice

Take voting scheme for example.

- A set O of outcomes (i.e., alternatives, candidates, etc.)
- A set A of agents s.t. each of them has a preference $\succ$ over the outcomes.
- The social choice function: a mapping from the profiles of the preferences to a particular outcome.

Outcomes & preferences

outcomes : a, b, c, d

$d \succ b \succ a \succ c$

preferences

d	***b***	***a***	***a***	***a***
b	*c*	*b*	*c*	*b*
a	*a*	*c*	*b*	*c*
c	*d*	*d*	*d*	*d*

Preferences

- A binary relation $\succ$ such that
  - for every $a, b \in O$, $a \neq b$, we have either $a \succ b$ or $b \succ a$ but NOT both.
  - for $a, b, c \in O$, if $a \succ b$ and $b \succ c$, then we have $a \succ c$.
- $\succeq$ can be defined similarly.
  - $\prec$: $\neg\succ$

Agents with preferences

- E.g., five agents (voters).
- Each agent has its preference over four candidates $\{a, b, c, d\}$.

outcomes : a, b, c, d

$d \succ b \succ a \succ c$

preferences

$\boldsymbol{d}$	$\boldsymbol{b}$	$\boldsymbol{a}$	$\boldsymbol{a}$	$\boldsymbol{a}$
b	c	b	c	b
a	a	c	b	c
c	d	d	d	d

Agents with preferences

- E.g., three agents (voters).
- Each agent has its preference over four candidates $\{a, b, c, d\}$.

v_1	v_2	v_3
d	b	a
b	c	b
a	a	c
c	d	d

Plurality rule $\Rightarrow$ a

outcomes : a, b, c, d

$d \succ b \succ a \succ c$

preferences

d	***b***	***a***	***a***	***a***
b	*c*	*b*	*c*	*b*
a	*a*	*c*	*b*	*c*
c	*d*	*d*	*d*	*d*

- Plurality rule: each agent can only give score 1 to the most preferred one and 0 to the others.

Plurality rule (contd.)

v_1	v_2	v_3
d	***b***	***a***
b	c	b
a	a	c
c	d	d

- Plurality rule: each agent can only give score 1 to the most preferred one and 0 to the others.

Plurality rule (contd.)

v_1	v_2	v_3
$\boldsymbol{d}$	$\boldsymbol{b}$	$\boldsymbol{a}$
b	c	b
a	a	c
c	d	d

- Plurality rule:

Plurality rule (contd.)

v_1	v_2	v_3
$\boldsymbol{d}$	$\boldsymbol{b}$	$\boldsymbol{a}$
b	c	b
a	a	c
c	d	d

- Plurality rule: depending on the tie-breaking rule.

Condorcet rule

v_1	v_2	v_3
d	b	a
b	c	b
a	a	c
c	d	d

- Condorcet rule:
  - a vs. b
  - a vs. c
  - a vs. d

Condorcet rule

v_1	v_2	v_3
d	b	a
b	c	b
a	a	c
c	d	d

- Condorcet rule:
  - a vs. $b \to b$
  - a vs. $c \to a$
  - a vs. $d \to a$

Condorcet rule

v_1	v_2	v_3
d	b	a
b	c	b
a	a	c
c	d	d

- Condorcet rule:
  - c vs. a
  - c vs. b
  - c vs. d

Condorcet rule

v_1	v_2	v_3
d	b	a
b	c	b
a	a	c
c	d	d

- Condorcet rule:
  - c vs. $a \to a$
  - c vs. $b \to b$
  - c vs. $d \to c$

Condorcet rule

v_1	v_2	v_3
d	b	a
b	c	b
a	a	c
c	d	d

- Condorcet rule:
  - b vs. a
  - b vs. c
  - b vs. d

Condorcet rule

v_1	v_2	v_3
d	b	a
b	c	b
a	a	c
c	d	d

- Condorcet rule:
  - b vs. $a \to b$
  - b vs. $c \to b$
  - b vs. $d \to b$

Condorcet rule

v_1	v_2	v_3
d	b	a
b	c	b
a	a	c
c	d	d

- Condorcet rule: b
  - b vs. $a \to b$
  - b vs. $c \to b$
  - b vs. $d \to b$

Borda rule

v_1		v_2		v_3	
d	**3**	b	**3**	a	**3**
b	**2**	c	**2**	b	**2**
a	**1**	a	**1**	c	**1**
c	**0**	d	**0**	d	**0**

- Borda count rule:

Borda rule

v_1		v_2		v_3	
d	**3**	b	**3**	a	**3**
b	**2**	c	**2**	b	**2**
a	**1**	a	**1**	c	**1**
c	**0**	d	**0**	d	**0**

- Borda count rule:
  - score of a: $1 + 1 + 3 = 5$.
  - score of b: $2 + 3 + 2 = 7$.
  - score of c: $0 + 2 + 1 = 3$.
  - score of d: $3 + 0 + 0 = 3$.

Borda rule

v_1		v_2		v_3	
d	**3**	b	**3**	a	**3**
b	**2**	c	**2**	b	**2**
a	**1**	a	**1**	c	**1**
c	**0**	d	**0**	d	**0**

- Borda count rule: b.
  - score of a: $1 + 1 + 3 = 5$.
  - score of b: $2 + 3 + 2 = 7$.
  - score of c: $0 + 2 + 1 = 3$.
  - score of d: $3 + 0 + 0 = 3$.

Inefficiency of Borda Count

v_1		v_2		v_3		v_4		v_5	
a	**2**	a	**2**	a	**2**	b	**2**	b	**2**
b	**1**	b	**1**	b	**1**	c	**1**	c	**1**
c	**0**	c	**0**	c	**0**	a	**0**	a	**0**

Inefficiency of Borda Count

v_1		v_2		v_3		v_4		v_5	
a	**2**	a	**2**	a	**2**	b	**2**	b	**2**
b	**1**	b	**1**	b	**1**	c	**1**	c	**1**
c	**0**	c	**0**	c	**0**	a	**0**	a	**0**

- Who is the winner by Borda counting?

Inefficiency of Borda Count

v_1		v_2		v_3		v_4		v_5	
a	**2**	a	**2**	a	**2**	b	**2**	b	**2**
b	**1**	b	**1**	b	**1**	c	**1**	c	**1**
c	**0**	c	**0**	c	**0**	a	**0**	a	**0**

- Who is the winner by Borda counting? a: 6, b: 7, c: 2.

Inefficiency of Borda Count

v_1		v_2		v_3		v_4		v_5	
a	**2**	a	**2**	a	**2**	b	**2**	b	**2**
b	**1**	b	**1**	b	**1**	c	**1**	c	**1**
c	**0**	c	**0**	c	**0**	a	**0**	a	**0**

- Who is the winner by Borda counting? a: 6, b: 7, c: 2.
- Condorcet principle follows?

Inefficiency of Borda Count

	v_1	v_2	v_3	v_4	v_5
2	a	a	a	b	b
1	b	b	b	c	c
0	c	c	c	a	a

- Who is the winner by Borda counting? a: 6, b: 7, c: 2.
- Condorcet principle follows? $a \succ b$, $a \succ c$.

Inefficiency of Borda Count

Score	v_1	v_2	v_3	v_4	v_5
2	a	a	a	b	b
1	b	b	b	c	c
0	c	c	c	a	a

- Who is the winner by Borda counting? a: 6, b: 7, c: 2.
- Condorcet principle follows? $a \succ b$, $a \succ c$.
- Who is the winner under the plurality rule?

Inefficiency of Borda Count

v_1		v_2		v_3		v_4		v_5	
a	**2**	a	**2**	a	**2**	b	**2**	b	**2**
b	**1**	b	**1**	b	**1**	c	**1**	c	**1**
c	**0**	c	**0**	c	**0**	a	**0**	a	**0**

- Who is the winner by Borda counting? a: 6, b: 7, c: 2.
- Condorcet principle follows? $a \succ b$, $a \succ c$.
- Who is the winner under the plurality rule? a.

Successive elimination

v_1	v_2	v_3
b	a	c
d	b	a
c	d	b
a	c	d

- Successive elimination with ordering $a \to b \to c \to d$:

Successive elimination

v_1	v_2	v_3
b	a	c
d	b	a
c	d	b
a	c	d

- Successive elimination with ordering $a \to \cancel{b} \to c \to d$:

Successive elimination

v_1	v_2	v_3
b	a	c
d	b	a
c	d	b
a	c	d

- Successive elimination with ordering $\cancel{a} \to \cancel{b} \to c \to d$:

Successive elimination

v_1	v_2	v_3
b	a	c
d	b	a
c	d	b
a	c	d

- Successive elimination with ordering $\cancel{a} \to \cancel{b} \to \cancel{c} \to d$:

Successive elimination

v_1	v_2	v_3
b	a	c
d	b	a
c	d	b
a	c	d

- Successive elimination with ordering $\cancel{a} \to \cancel{b} \to \cancel{c} \to d$: d

Successive elimination

v_1	v_2	v_3
b	a	c
d	b	a
c	d	b
a	c	d

- Successive elimination with ordering $\cancel{a} \to \cancel{b} \to \cancel{c} \to d$: d
  - The issue: all of the agents prefer b to d!

Successive elimination

v_1	v_2	v_3
b	a	c
d	b	a
c	d	b
a	c	d

- Successive elimination with ordering $a \to b \to c \to d$: d
- Successive elimination with ordering $a \to c \to b \to d$:

Successive elimination

v_1	v_2	v_3
b	a	c
d	b	a
c	d	b
a	c	d

- Successive elimination with ordering $a \to b \to c \to d$: d
- Successive elimination with ordering $a \to c \to b \to d$: b

Successive elimination (sensitive to the agenda order)

v_1	v_2	v_3
b	a	c
d	b	a
c	d	b
a	c	d

- Successive elimination with ordering $a \rightarrow b \rightarrow c \rightarrow d$: d
- Successive elimination with ordering $a \rightarrow c \rightarrow b \rightarrow d$: b
- Successive elimination with ordering $b \rightarrow c \rightarrow a \rightarrow d$:

Successive elimination (sensitive to the agenda order)

v_1	v_2	v_3
b	a	c
d	b	a
c	d	b
a	c	d

- Successive elimination with ordering $a \to b \to c \to d$: d
- Successive elimination with ordering $a \to c \to b \to d$: b
- Successive elimination with ordering $b \to c \to a \to d$: a

Condorcet Winner vs. Plurality

- Let's say we have 1,000 agents each of which has a preference over three candidates A, B, C.
  - 499 agents for $A \succ B \succ C$.
  - 3 agents for $B \succ C \succ A$.
  - 498 agents for $C \succ B \succ A$.
- Who is the Condorcet winner?

Condorcet Winner vs. Plurality

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- Who is the Condorcet winner? B.

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  - 499 agents for $A \succ B \succ C$.
  - 3 agents for $B \succ C \succ A$.
  - 498 agents for $C \succ B \succ A$.
- Who is the Condorcet winner? B.
- Who is the winner under the plurality rule?

Condorcet Winner vs. Plurality

- Let's say we have 1,000 agents each of which has a preference over three candidates A, B, C.
  - 499 agents for $A \succ B \succ C$.
  - 3 agents for $B \succ C \succ A$.
  - 498 agents for $C \succ B \succ A$.
- Who is the Condorcet winner? B.
- Who is the winner under the plurality rule? A.

Exercise

On Borda Count & Condorcet

We have five voters with the following preferences (ordering) over the outcomes A, B, C, and D.

- $B \succ C \succ A \succ D$.
- $B \succ D \succ C \succ A$.
- $D \succ C \succ A \succ B$.
- $A \succ D \succ B \succ C$.
- $A \succ D \succ C \succ B$.

Who is the winner by the Borda Count rule?
Who is the Condorcet winner?

Let's consider a practical application in MOOCs.

MOOCs

- MOOCs: Massive Online Open Courses
  - e.g., Coursera, EdX.

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  - ▷ Ask each student to grade a SMALL number of her peers' assignments.

MOOCs

- MOOCs: Massive Online Open Courses
  - e.g., Coursera, EdX.
- Outscourcing the grading task to the students.
- They may have incentives to assign LOW scores to everybody else.
  - ▷ Ask each student to grade a SMALL number of her peers' assignments.
  - Then merge individual rankings into a global one.

Terminologies

- $\mathcal{A}$: universe of n elements (students).
- (n, k)-grading scheme:
  a collection $\mathcal{B}$ of size-k subsets (bundles) of $\mathcal{A}$, such that each element of $\mathcal{A}$ belongs to exactly k subsets of $\mathcal{B}$.
- The bundle graph:
  Represent the (n, k)-grading scheme with a bipartite graph.
- $\prec_b$: a ranking of the element b contains (partial order).

The aggregation rule

An aggregation rule:
profile of partial rankings $\to$ complete ranking of all elements.

- Borda:

SPRING FEAST 2016 BALLOT

a	LE BLE D'OR	5
b	CRYSTAL SPOON	3
c	Bei Yuan Restaurant	1
d	Tasty Steak	2
e	Capricciosa	4

SPRING FEAST 2016 BALLOT

a	LE BLE D'OR	5
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SPRING FEAST 2016 BALLOT

a	LE BLE D'OR	4
b	CRYSTAL SPOON	5
c	Bei Yuan Restaurant	1
d	Tasty Steak	3
e	Capricciosa	2

- a: 14; b: 12; c: 4; d: 6; e: 9.
  $a \prec b \prec e \prec d \prec c$.

Order-revealing grading scheme

An aggregation rule in peer grading (Borda):

Final Exam Grading

	Curry	2
	Bob	3
	Elvis	1

$\downarrow$ Alice

Final Exam Grading

	Alice	3
	Curry	2
	David	1

$\downarrow$ Bob

Final Exam Grading

	Elvis	1
	David	2
	Bob	3

$\downarrow$ Curry

Final Exam Grading

	Alice	3
	Curry	1
	Bob	2

$\downarrow$ David

Final Exam Grading

	Alice	3
	Elvis	1
	David	2

$\downarrow$ Elvis

- Alice: 9; Bob: 8; Curry: 5; David: 5; Elvis: 3.

  Alice $\prec$ Bob $\prec$ Curry $\prec$ David $\prec$ Elvis.

Assumption (perfect grading)

Each student grades the assignments in her bundle consistently to the ground truth.

Order-revealing grading scheme (contd.)

Final Exam Grading → Alice

	Henry	1
	Frank	3
	Green	2

Final Exam Grading → Bob

	Curry	3
	Frank	1
	David	2

Final Exam Grading → Curry

	Elvis	2
	Henry	1
	Bob	3

Final Exam Grading → David

	Green	1
	Frank	2
	Alice	3

Final Exam Grading → Elvis

	Alice	3
	Henry	1
	Green	2

Final Exam Grading → Frank

	Curry	3
	Elvis	1
	David	2

Final Exam Grading → Green

	Curry	2
	Bob	3
	Elvis	1

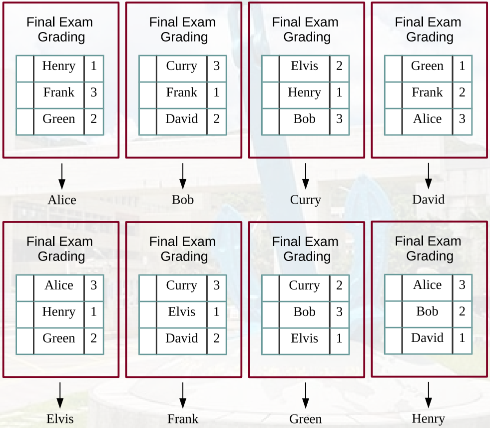
Final Exam Grading → Henry

	Alice	3
	Bob	2
	David	1

- Alice: 9; Bob: 8; Curry: 8; David: 5; Elvis: 4; Frank: 6; Green: 5; Henry: 3.

  Alice $\prec$ Bob $\prec$ Curry $\prec$ Frank $\prec$ David $\prec$ Green $\prec$ Elvis $\prec$ Henry.

Order-revealing grading scheme (contd.)

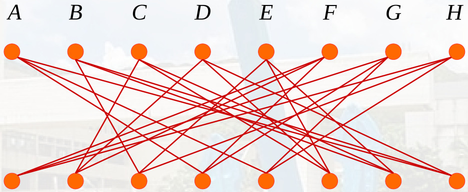


- Alice: 9; Bob: 8; Curry: 8; David: 5; Elvis: 4; Frank: 6; Green: 5; Henry: 3.

  Alice $\prec$ Bob $\prec$ Curry $\prec$ Frank $\prec$ David $\prec$ Green $\prec$ Elvis $\prec$ Henry.

The bundle graph

The bundle graph:



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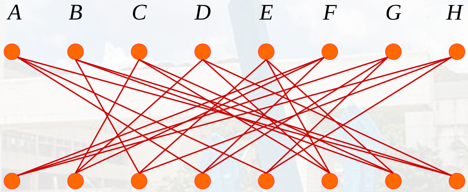

- A random k-regular graph:

  A complete bipartite $K_{n,n} \to$ removing edges $\{v, v\}$, $\forall v \to$

  repeat

  > *"draw a perfect matching uniformly at random among all perfect matchings of the remaining graph"*

  for k times.

The limitation on the order revealing scheme

- The property of revealing the ground truth for certain:

$$\forall x, y \in \mathcal{A},\ \exists B \in \mathcal{B} \text{ such that } x, y \in B.$$

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  - Let $\prec, \prec'$ be two complete rankings.
    - x, y are in the first two positions in $\prec, \prec'$;
    - $\prec$ and $\prec'$ differs only in the order of x and y.
  - Clearly, partial rankings within the bundles are identical in both cases.
  - No way to identify whether $\prec$ or $\prec'$ is the ground truth.

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    - $\prec$ and $\prec'$ differs only in the order of x and y.
  - Clearly, partial rankings within the bundles are identical in both cases.
  - No way to identify whether $\prec$ or $\prec'$ is the ground truth.
- To reveal the ground truth with certainty: $k = \Omega(\sqrt{n})$.
  - $n \cdot \binom{k}{2} \geq \binom{n}{2}$.

Seeking for approximate order-revealing grading schemes

- Use a bundle graph with a very low degree k (independent of n).
- Randomly permute the elements by $\pi : U \to \mathcal{A}$ before associating them to the nodes of U of the bundle graph.
- Aiming at $\dfrac{\#\text{correctly recovered pairwise relations}}{\binom{n}{2}}$.

The main result

Theorem (Caragiannis, Krimpas, Voudouris@AAMAS'15)

When

- Borda is applied as the aggregation rule, and
- all the partial rankings are consistent to the ground truth,

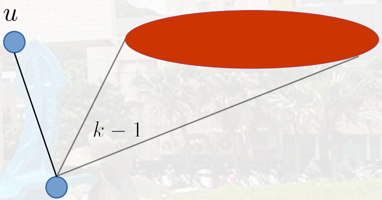
then the expected fraction of correctly recovered pairwise relations is $1 - O(1/\sqrt{k})$.

Question

- What will happen if we assign for each student only two assignments and each assignment is graded by exactly two students?

About who grading both u and v

- $\lambda_{u,v} := |N(u) \cap N(v)|$, for $u, v \in U$.
  - $\sum_{v \in U \setminus \{u\}} \lambda_{u,v} = k(k-1)$.



- $W_{r,q}$: the r.v. denoting $B(a_r) - B(a_q)$ for $r < q$, $a_r, a_q \in \mathcal{A}$.
- $\Gamma_{u,v}^{r,q}$: the event that $\pi(u) = a_r$, $\pi(v) = a_q$.

- $$C := \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} \mathrm{E}[\mathbb{1}\{W_{r,q} > 0\}] = \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} \Pr[W_{r,q} > 0]$$

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- $\Gamma^{r,q}_{u,v}$: the event that $\pi(u) = a_r$, $\pi(v) = a_q$.

- $$\begin{aligned} C &:= \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} \mathrm{E}[\mathbb{1}\{W_{r,q} > 0\}] = \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} \Pr[W_{r,q} > 0] \\ &= \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} \left(1 - \sum_{u,v \in U} \Pr[W_{r,q} \leq 0 \mid \Gamma^{r,q}_{u,v}] \cdot \Pr[\Gamma^{r,q}_{u,v}]\right) \\ &= \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} \left(1 - \frac{1}{n(n-1)} \sum_{u,v \in U} \Pr[W_{r,q} \leq 0 \mid \Gamma^{r,q}_{u,v}]\right) \end{aligned}$$

- Given $\Gamma_{u,v}^{r,q}$,
  - the expected Borda score of a_r is $k + (k(k-1) - \lambda_{u,v}) \cdot \frac{n-r-1}{n-2} + \lambda_{u,v}$.
  - the expected Borda score of a_q is $k + (k(k-1) - \lambda_{u,v}) \cdot \frac{n-q}{n-2}$.
- Thus

$$\mathrm{E}[W_{r.q} \mid \Gamma_{u,v}^{r,q}] = (k(k-1) - \lambda_{u,v})\frac{q-r-1}{n-2} + \lambda_{u,v}.$$

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- Thus

$$\mathrm{E}[W_{r.q} \mid \Gamma_{u,v}^{r,q}] = (k(k-1) - \lambda_{u,v})\frac{q-r-1}{n-2} + \lambda_{u,v}.$$

Why?

Calculate the Borda score from another point of view

- Element a_r gets one point for each bundle it belongs to;
  - plus one additional point for each appearance of an element with rank higher $> r$ in the bundles a_r belongs to.

- In the bundles of containing a_r:
    - $\lambda_{u,v}$ appearances of a_q in the bundles of a_r.
    - $k(k-1) - \lambda_{u,v}$ appearances of elements different than a_r, a_q.
        - Each of them has prob. $\frac{n-r-1}{n-2}$ to have rank higher than r.

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- $\mathrm{E}[B(a_r) \mid \Gamma^{r,q}_{u,v}] = k + (k(k-1) - \lambda_{u,v})\frac{n-r-1}{n-2} + \lambda_{u,v}$.

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- In the bundles of containing a_q:
  - $\lambda_{u,v}$ appearances of a_q in the bundles of a_r.
  - $k(k-1) - \lambda_{u,v}$ appearances of elements different than a_r, a_q.
    - Each of them has prob. $\frac{n-q}{n-2}$ to have rank higher than q.

- In the bundles of containing a_r:
  - $\lambda_{u,v}$ appearances of a_q in the bundles of a_r.
  - $k(k-1) - \lambda_{u,v}$ appearances of elements different than a_r, a_q.
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- In the bundles of containing a_q:
  - $\lambda_{u,v}$ appearances of a_q in the bundles of a_r.
  - $k(k-1) - \lambda_{u,v}$ appearances of elements different than a_r, a_q.
    - Each of them has prob. $\frac{n-q}{n-2}$ to have rank higher than q.
- $\mathrm{E}[B(a_q) \mid \Gamma^{r,q}_{u,v}] = k + (k(k-1) - \lambda_{u,v})\frac{n-q}{n-2}$.

Dealing with dependencies

Goal: $\Pr[W_{r,q} \leq 0 \mid \Gamma_{u,v}^{r,q}]$

- Given $\Gamma_{u,v}^{r,q}$, define $S = N(N(u) \cup N(v)) \setminus \{u, v\}$.
- $o : [|S|] \to S$ denotes an arbitrary ordering of nodes of S.
- X_i: the random variable denoting the rank of the element $\pi(o(i))$.
- Define the Doob martingale $Z_0, Z_1, \ldots, Z_{|S|}$ such that
  - $Z_0 = \mathrm{E}[W_{r,q} \mid \Gamma_{u,v}^{r,q}]$;
  - $Z_i = \mathrm{E}[W_{r,q} \mid \Gamma_{u,v}^{r,q}, X_1, \ldots, X_i]$.
- Hence, $W_{r,q} \mid \Gamma_{u,v}^{r,q} = Z_{|S|}$.

Martingale

Martingale

A sequence of random variables $Z_0, Z_1, \ldots, Z_m$ is a martingale w.r.t. a sequence of random variables $X_1, X_2, \ldots, X_m$ if $\forall i = 1, \ldots, m$,

$$\mathrm{E}[Z_i \mid X_1, \ldots, X_{i-1}] = Z_{i-1}.$$

Doob martingale (Joseph L. Doob (1910–2004))

- W: a random variable
- $X_1, \ldots, X_m$: a sequence of m random variables.

The sequence $Z_0, Z_1, \ldots, Z_m$ such that

- $Z_0 = \mathrm{E}[W]$;
- $Z_i = \mathrm{E}[W \mid X_1, \ldots, X_i]$, $\forall i = 1, \ldots, m$

is called a Doob martingale.

Azuma-Hoeffding inequality

Azuma-Hoeffding inequality

Let $Z_0, Z_1, \ldots, Z_m$ be a martingale with $Z_i - Z_{i-1} \leq c_i$ for $i = 1, \ldots, m$. Then, for all $t > 0$,

$$\Pr[Z_m - Z_0 \leq -t] \leq \exp\left(-\frac{t^2}{2\sum_{i=1}^{m} c_i^2}\right).$$

Dealing with dependencies (contd.)

- Given $\Gamma^{r,q}_{u,v}$, define $S = N(N(u) \cup N(v)) \setminus \{u, v\}$.
- $o : [|S|] \mapsto S$ denotes an arbitrary ordering of nodes of S.
- X_i: the random variable denoting the rank of the element $\pi(o(i))$.
- Define the Doob martingale $Z_0, Z_1, \ldots, Z_{|S|}$ such that
  - $Z_0 = \mathrm{E}[W_{r,q} \mid \Gamma^{r,q}_{u,v}]$;
  - $Z_i = \mathrm{E}[W_{r,q} \mid \Gamma^{r,q}_{u,v}, X_1, \ldots, X_i]$.
- Hence, $W_{r,q} \mid \Gamma^{r,q}_{u,v} = Z_{|S|}$.

Lemma 8

$\forall i \in \{1, 2, \ldots, |S|\}$, it holds that $|Z_i - Z_{i-1}| \leq 2(\lambda_{u,o(i)} + \lambda_{v,o(i)})$.

Lemma 3

For every k-regular bipartite graph G,
$\theta_{u,v} = 4 \sum_{z \in (N(u) \cup N(v)) \setminus \{u,v\}} (\lambda_{u,z} + \lambda_{v,z})^2 \leq 8k(k-1)(4k-3)$.

- Set $t = \mathrm{E}[W_{r,q} \mid \Gamma^{r,q}_{u,v}]$ $(= Z_0)$, by the Azuma-Hoeffding inequality:

$$\begin{aligned}
\Pr[Z_{|S|} - Z_0 \leq -t] &= \Pr[W_{r,q} \leq 0 \mid \Gamma^{r,q}_{u,v}] \\
&\leq \exp\left(-\frac{t^2}{2\sum_{i=1}^{m} c_i^2}\right) \\
&= \exp\left(-\frac{\mathrm{E}[W_{r,q} \mid \Gamma^{r,q}_{u,v}]^2}{2\theta_{u,v}}\right).
\end{aligned}$$

Back to the computation of C

$$
\begin{aligned}
C &= \sum_{r=1}^{n-1}\sum_{q=r+1}^{n}\left(1-\frac{1}{n(n-1)}\sum_{u,v\in U}\Pr[W_{r,q}\leq 0 \mid \Gamma_{u,v}^{r,q}]\right) \\
&\geq \sum_{r=1}^{n-1}\sum_{q=r+1}^{n}\left(1-\frac{1}{n(n-1)}\sum_{u,v\in U}\exp\left(-\frac{\mathrm{E}[W_{r,q}\mid\Gamma_{u,v}^{r,q}]^2}{2\theta_{u,v}}\right)\right) \\
&= \sum_{r=1}^{n-1}\sum_{q=r+1}^{n}\left(1-\frac{1}{n(n-1)}\sum_{u,v\in U}e^{-(\beta(u,v)\cdot y(q-r)+\delta(u,v))^2}\right) \\
&= \frac{n(n-1)}{2}-\frac{1}{n(n-1)}\sum_{u,v\in U}\sum_{d=1}^{n-1}(n-d)e^{-(\beta(u,v)\cdot y(d)+\delta(u,v))^2} \\
&\geq \frac{n(n-1)}{2}-\sum_{u,v\in U}\int_0^1(1-y)e^{-(\beta(u,v)\cdot y+\delta(u,v))^2}dy
\end{aligned}
$$

- $\beta(u,v)=\frac{k(k-1)-\lambda_{u,v}}{\sqrt{2\theta_{u,v}}}$;
- $\delta(u,v)=\frac{\lambda_{u,v}}{\sqrt{2\theta_{u,v}}}$;
- $y(t)=\frac{t-1}{n-2}$.

$$
\begin{aligned}
C &\geq \frac{n(n-1)}{2} - \sum_{u,v\in U} \int_0^1 (1-y)e^{-(\beta(u,v)\cdot y+\delta(u,v))^2} dy \\
&\geq \frac{n(n-1)}{2} - \sum_{u,v\in U} \frac{\beta(u,v)+\delta(u,v)}{2\beta(u,v)^2}\sqrt{\pi} \\
&\geq \frac{n(n-1)}{2} - \frac{k-1}{k(k-2)^2}\sqrt{\frac{\pi}{2}} \sum_{u,v\in U} \sqrt{\theta_{u,v}} \\
&\geq \frac{n(n-1)}{2}\left(1 - \frac{48\sqrt{2\pi}}{\sqrt{k}})\right).
\end{aligned}
$$

Claim 9

Let $\beta > 0$, $\delta \geq 0$, then $\int_0^1 (1-y)e^{-(\beta y+\delta)^2} dy \leq \frac{\beta+\delta}{2\beta^2}\sqrt{\pi}$.

- $k \geq 3$ (assumption).