

# Counting Binary Trees and Selection Trees

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# Outline

1 Counting Binary Trees

2 Selection Trees



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# Counting Binary Trees

- Consider the following three disparate problems:
  - ① The number of distinct binary trees having  $n$  nodes.
  - ② The number of distinct permutations of the numbers from 1 to  $n$  obtainable by a [stack](#).
  - ③ The number of distinct ways of multiplying  $n + 1$  matrices.



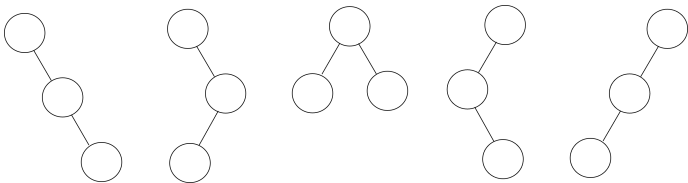
# Counting Binary Trees

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  - ② The number of distinct permutations of the numbers from 1 to  $n$  obtainable by a [stack](#).
  - ③ The number of distinct ways of multiplying  $n + 1$  matrices.
- Amazingly, **these problems have the same solution!**



# Problem One

- The number of distinct binary trees having  $n$  nodes.



★ Example of  $n = 3$ .

## Problem Two

- The number of distinct permutations of the numbers from 1 to  $n$  obtainable by a **stack**.

- ① push 1  $\rightarrow$  pop  $\rightarrow$  push 2  $\rightarrow$  pop  $\rightarrow$  push 3  $\rightarrow$  pop  $\Rightarrow$  123.
- ② push 1  $\rightarrow$  pop  $\rightarrow$  push 2  $\rightarrow$  push 3  $\rightarrow$  pop  $\rightarrow$  pop  $\Rightarrow$  132.
- ③ push 1  $\rightarrow$  push 2  $\rightarrow$  push 3  $\rightarrow$  pop  $\rightarrow$  pop  $\rightarrow$  pop  $\Rightarrow$  321.
- ④ push 1  $\rightarrow$  push 2  $\rightarrow$  pop  $\rightarrow$  pop  $\rightarrow$  push 3  $\rightarrow$  pop  $\Rightarrow$  213.
- ⑤ push 1  $\rightarrow$  push 2  $\rightarrow$  pop  $\rightarrow$  push 3  $\rightarrow$  pop  $\rightarrow$  pop  $\Rightarrow$  231.

★ Example of  $n = 3$ .



## Problem Three

- The number of distinct ways of multiplying  $n + 1$  matrices.

①  $((M_1 \times M_2) \times M_3) \times M_4.$

②  $(M_1 \times (M_2 \times M_3)) \times M_4.$

③  $M_1 \times ((M_2 \times M_3) \times M_4).$

④  $M_1 \times (M_2 \times (M_3 \times M_4)).$

⑤  $(M_1 \times M_2) \times (M_3 \times M_4).$

★ Example of  $n = 3$ .





# Stack Permutation (1/4)

- Recall: preorder, inorder and postorder traversal of a binary tree.
  - Each traversal requires a **stack**.

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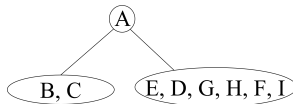
Every binary tree has a unique pair of preorder/inorder sequences.

- The number of distinct binary trees is equal to the number of **inorder permutations** obtainable from binary trees having the preorder permutation,  $1, 2, \dots, n$ .

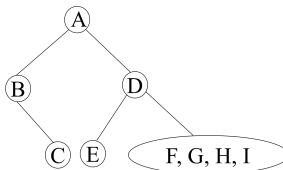


## Stack Permutation (2/4)

- preorder: A B C E D G H F I
- inorder: B C A E D G H F I



- preorder: A B C (D E F G H I)
- inorder: B C A (E D F G H I)



## Stack Permutation (3/4)

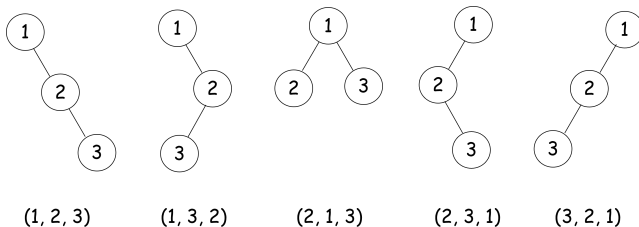
- We can show that

the number of distinct permutations obtainable by passing the numbers  $\{1, 2, \dots, n\}$  through a stack is equal to the number of distinct binary trees with  $n$  nodes.

- ① push 1  $\rightarrow$  pop  $\rightarrow$  push 2  $\rightarrow$  pop  $\rightarrow$  push 3  $\rightarrow$  pop  $\Rightarrow$  123.
- ② push 1  $\rightarrow$  pop  $\rightarrow$  push 2  $\rightarrow$  push 3  $\rightarrow$  pop  $\rightarrow$  pop  $\Rightarrow$  132.
- ③ push 1  $\rightarrow$  push 2  $\rightarrow$  push 3  $\rightarrow$  pop  $\rightarrow$  pop  $\rightarrow$  pop  $\Rightarrow$  321.
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- ⑤ push 1  $\rightarrow$  push 2  $\rightarrow$  pop  $\rightarrow$  push 3  $\rightarrow$  pop  $\rightarrow$  pop  $\Rightarrow$  231.



## Stack Permutation (4/4)



# Go Back to the Matrix Multiplication

- Computing the product of  $n$  matrices are related to the distinct binary tree problem.
- $n = 3$ :
  - ①  $(M_1 \times M_2) \times M_3$ .
  - ②  $M_1 \times (M_2 \times M_3)$ .
- $n = 4$ :
  - ①  $((M_1 \times M_2) \times M_3) \times M_4$ .
  - ②  $(M_1 \times (M_2 \times M_3)) \times M_4$ .
  - ③  $M_1 \times ((M_2 \times M_3) \times M_4)$ .
  - ④  $M_1 \times (M_2 \times (M_3 \times M_4))$ .
  - ⑤  $(M_1 \times M_2) \times (M_3 \times M_4)$ .



# Matrix Multiplication (2/2)

- $b_n$ : the number of different ways to compute the product of  $n$  matrices.
- Trivially,  $b_1 =$



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- $b_n$ : the number of different ways to compute the product of  $n$  matrices.
- Trivially,  $b_1 = 1$ ,  $b_2 = 1$ .
- We have also derived that  $b_3 = 2$  and  $b_4 = 5$ .
- We can compute that

$$b_n = \sum_{i=1}^{n-1} b_i b_{n-i}, \text{ for } n > 1.$$



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- Similarly, the number of **distinct binary trees** of  $n$  nodes is

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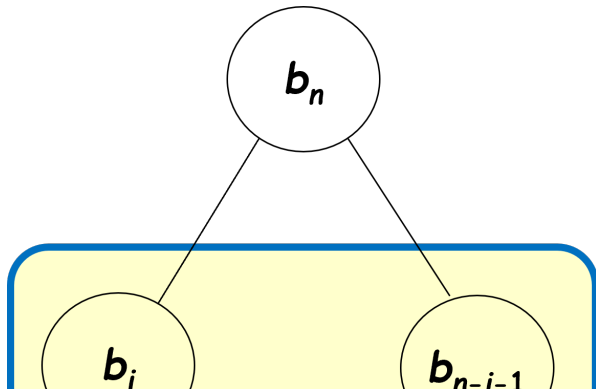
$$b_n = \sum_{i=0}^{n-1} b_i b_{n-1-i}, \text{ for } n \geq 1 \text{ and}$$



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$$b_n = \sum_{i=0}^{n-1} b_i b_{n-1-i}, \text{ for } n \geq 1 \text{ and } b_0 = 1.$$



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# Discussions

