# A Sketch of Nash's Theorem from Fixed Point Theorems

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## Reference


- Lecture Notes in 6.853 Topics in Algorithmic Game Theory [link].
- *Fixed Point Theorems and Applications to Game Theory.* Allen Yuan. The University of Chicago Mathematics REU 2017. [link].
  - REU = Research Experience for Undergraduate students.

# Outline

# Outline

1. Brouwer's Fixed Point Theorem
   - Sketch of the Proof of Nash's Theorem (from Brouwer's Theorem)
2. Kakutani's Fixed Point Theorem
   - Pure Strategy Nash Equilibria of Pure Strategic Games
     - Preliminaries
     - Main Theorem I & The Proof
   - Mixed Nash Equilibria of Finite Strategies Games
     - Preliminaries & Assumptions
     - Main Theorem II & the Proof

# The Setting

- A set $N$ of $n$ players.
- Strategy set $S_i = \{s_{i,1}, \ldots, s_{i,k_i}\}$ for each player $i \in N$, $k_i$ is bounded.
- Utility function: $u_i$ for each player $i$.
- $\mathbf{\Delta} := \Delta_1 \times \Delta_2 \times \cdots \Delta_n$: a Cartesian product of $(\Delta_i)_{i \in N}$.
  - For $\mathbf{x} \in \mathbf{\Delta}$, $x_i(s)$ denotes the probability mass on strategy $s \in S_i$.
  - $\Delta_i = \{(x_i(s_{i,1}), x_i(s_{i,2}), \ldots, x_i(s_{i,k_i})) \mid x_i(s_{i,j}) \geq 0 \ \forall j; \sum_j x_i(s_{i,j}) = 1\}$.
  - $x_i \in \Delta_i$: a mixed strategy.

# Nash's Theorem

**Nash (1950)**

Every game $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ has a Nash equilibrium.

- **Note:** $u_i(\boldsymbol{x}) := \sum_{s \in S_i} x_i(s) \cdot u_i(s; \boldsymbol{x}_{-i})$.

# Nash's Theorem

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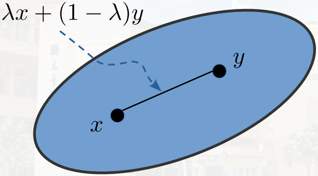
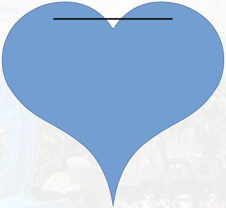
Every game $\langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$ has a Nash equilibrium.

- **Note:** $u_i(\boldsymbol{x}) := \sum_{s\in S_i} x_i(s)\cdot u_i(s;\boldsymbol{x}_{-i})$.
- No player wants to deviate to the other strategy unilaterally.


open &
bounded
closed &
bounded

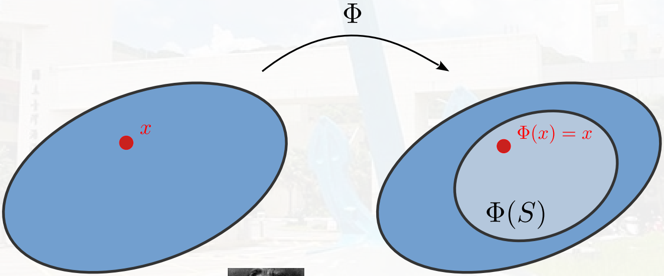



convex



not convex

# Fixed Point




L. E. J. Brouwer
(1881–1966)
photo from Wikipedia

# Brouwer's Fixed Point Theorem

## Brouwer's Fixed-Point Theorem

Let $D$ be a convex, compact (closed and bounded) subset of the Euclidean space. If $f : D \to D$ is continuous, then there exists $x \in D$ such that

$$f(x) = x.$$

- **Idea:** We want the function $f$ to satisfy the conditions of Brouwer's fixed point theorem.

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$$f(x) = x.$$

- **Idea:** We want the function $f$ to satisfy the conditions of Brouwer's fixed point theorem.
- Try to relate utilities of players to a function $f$ like above.

# The Gain function

**Gain**

Suppose that $\boldsymbol{x}' \in \boldsymbol{\Delta}$ is given. For a player $i$ and strategy $s_i \in S_i$ (or $s_i \in \Delta_i$), we define the gain as

$$\mathrm{Gain}_{i,s_i}(\boldsymbol{x}') = \max\{u_i(s_i; \boldsymbol{x}'_{-i}) - u_i(\boldsymbol{x}), 0\},$$

which is non-negative.

- $\boldsymbol{x}'_{-i} := (x_j)_{j \in N}$, $(\boldsymbol{x}_{-i}, x_i) = \boldsymbol{x}$.
- It's equal to the increase in payoff for player $i$ if he/she were to switch to pure strategy $s_i$.

# Proof of Nash's Theorem (Define a response function)

- Define a function $f : \boldsymbol{\Delta} \to \boldsymbol{\Delta}$ that satisfies the conditions of Brouwer's fixed point theorem.
- For all $\boldsymbol{x} \in \boldsymbol{\Delta}$, $\boldsymbol{y} = f(\boldsymbol{x})$ where for all $i \in N$ and $s_i \in S_i$,

$$y_i(s_i) := \frac{x_i(s_i) + \mathrm{Gain}_{i;s_i}(\boldsymbol{x})}{1 + \sum_{s_i' \in S_i} \mathrm{Gain}_{i;s_i'}(\boldsymbol{x})}.$$

  - $f$ tries to boost the probability mass where strategy switching results in gains in payoff.

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  - $f : \mathbf{\Delta} \to \mathbf{\Delta}$ is continuous (verify this by yourself).
  - $\mathbf{\Delta}$ is a product of simplicies so it is convex (verify this by yourself).
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  - $\boldsymbol{\Delta}$ is closed and bounded, so it is compact.

⋆ Brouwer's fixed point theorem guarantees the existence of a fixed point of $f$.

# **Claim:** Any fixed point of $f$ is a Nash equilibrium

- It suffices to prove that a fixed point $\boldsymbol{x} = f(\boldsymbol{x})$ satisfies:
  - $\mathrm{Gain}_{i;s_i}(\boldsymbol{x}) = 0$, for each $i \in N$ and each $s_i \in S_i$.

# **Claim:** Any fixed point of $f$ is a Nash equilibrium

Prove it by contradiction.

- Assume that there is some player $p$ who can improve his/her payoff by switching to a pure strategy, say $s_p$:
  - $\mathrm{Gain}_{p;s_p}(\boldsymbol{x}) > 0$.

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  - $\mathrm{Gain}_{p;s_p}(\boldsymbol{x}) > 0$.
- Note that we must have $x_p(s_p) > 0$, otherwise $\boldsymbol{x}$ cannot be a fixed point of $f$.
  - From the definition of $f$; the numerator would be $> 0$.

$$y_p(s_p) := \frac{x_p(s_p) + \mathrm{Gain}_{p;s_p}(\boldsymbol{x})}{1 + \sum_{s'_p \in S_p} \mathrm{Gain}_{p;s'_p}(\boldsymbol{x})}.$$

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  - $\text{Gain}_{p;s_p}(\boldsymbol{x}) > 0 \Rightarrow u_p(s_p; \boldsymbol{x}_{-p}) - u_p(\boldsymbol{x}) > 0.$

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  - $\mathrm{Gain}_{p;s_p}(\boldsymbol{x}) > 0 \Rightarrow u_p(s_p; \boldsymbol{x}_{-p}) - u_p(\boldsymbol{x}) > 0.$
- We argue that there must be some other pure strategy $\hat{s}_p$ such that:
  - $x_p(\hat{s}_p) > 0$ and
  - $u_p(\hat{s}_p; \boldsymbol{x}_{-p}) - u_p(\boldsymbol{x}) < 0$ .

⋆ Notice that

$$u_p(\boldsymbol{x}) := \sum_{s \in S_p} x_p(s) \cdot u_p(s; \boldsymbol{x}_{-p}).$$

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- We argue that there must be some other pure strategy $\hat{s}_p$ such that:
  - $x_p(\hat{s}_p) > 0$ and
  - $u_p(\hat{s}_p; \boldsymbol{x}_{-p}) - u_p(\boldsymbol{x}) < 0 \quad \Rightarrow \quad \text{Gain}_{p,\hat{s}_p}(\boldsymbol{x}) = 0.$

  ⋆ Notice that
$$u_p(\boldsymbol{x}) := \sum_{s \in S_p} x_p(s) \cdot u_p(s; \boldsymbol{x}_{-p}).$$
- We obtain that ($\boldsymbol{x}$ is not a fixed point $\Rightarrow\!\Leftarrow$)
$$y_p(\hat{s}_p) := \frac{x_p(\hat{s}_p) + \text{Gain}_{p;\hat{s}_p}(\boldsymbol{x})}{1 + \sum_{s'_p \in S_p} \text{Gain}_{p;s'_p}(\boldsymbol{x})} < x_p(\hat{s}_p).$$

# Outline

# An Extension of Brouwer's work

- Focus: set-valued functions.
  - Refer here for further readings.
  - Why do we consider set-valued functions?

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- Focus: set-valued functions.
  - Refer here for further readings.
  - Why do we consider set-valued functions?
    - Best-responses.

# Upper Semi-Continuous (having a closed graph)

## Upper semi-continuous functions

Let

- $\mathbb{P}(X)$: all nonempty, closed, convex subsets of $X$.
- $S$: a nonempty, compact, and convex set.

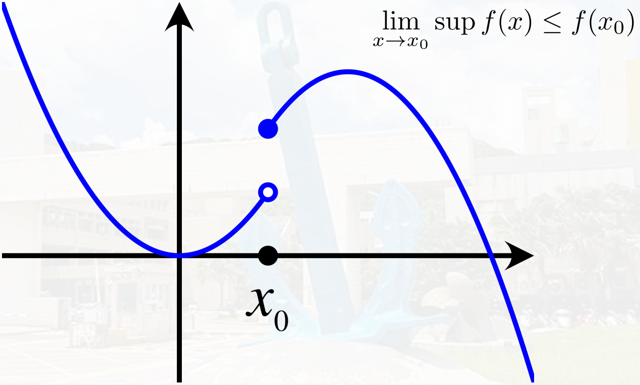
Then the set-valued function $\Phi : S \to \mathbb{P}(S)$ is upper semi-continuous if

for arbitrary sequences $(\boldsymbol{x}_n)_{n\in\mathbb{N}}, (\boldsymbol{y}_n)_{n\in\mathbb{N}}$ in $S$, we have

- $\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x}^*$,
- $\lim_{n\to\infty} \boldsymbol{y}_n = \boldsymbol{y}^*$,
- $\boldsymbol{y}_n \in \Phi(\boldsymbol{x}_n)$ for all $n \in \mathbb{N}$,

imply that $\boldsymbol{y}^* \in \Phi(\boldsymbol{x}^*)$.

Removable discontinuity, Sequentially compact, Bolzano–Weierstrass theorem.




(Figure from Wikipedia)

# Fixed Point of Set-Valued Functions

**Fixed Point (Set-Valued)**

A fixed point of a set-valued function $\Phi : S \to \mathbb{P}(S)$ is a point $\boldsymbol{x}^* \in S$ such that $\boldsymbol{x}^* \in \Phi(\boldsymbol{x}^*)$.

# Kakutani's Theorem for Simplices

**Kakutani's Theorem for Simplices (1941)**

If $S$ is an $r$-dimensional closed simplex in a Euclidean space and $\Phi : S \to \mathbb{P}(S)$ is upper semi-continuous, then $\Phi$ has a fixed point.

# Kakutani's Fixed-Point Theorem

**Kakutani's Fixed-Point Theorem (1941)**

If $S$ is a nonempty, compact, convex set in a Euclidean space and $\Phi : S \to \mathbb{P}(S)$ is upper semi-continuous, then $\Phi$ has a fixed point.

# Kakutani's Fixed-Point Theorem

**Kakutani's Fixed-Point Theorem (1941)**

If $S$ is a nonempty, compact, convex set in a Euclidean space and $\Phi : S \to \mathbb{P}(S)$ is upper semi-continuous, then $\Phi$ has a fixed point.

- We won't go over its proof.
- Instead, we will delve into how it can be used to prove Nash's Theorem from the perspectives of set-valued functions and best-responses.

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# Cartesian product of Sets

## Cartesian Product

For a family of sets $\{A_i\}_{i \in N}$, $\prod_{i \in N} A_i = A_1 \times A_2 \times \cdots \times A_n$ denotes the Cartesian product of $A_i$ for $i \in N$.

## Profile

for $x_i \in A_i$, then $(x_i)_{i \in N}$ is called a (strategy) profile.

# Binary Relation

## Binary Relation

- A binary relation on a set $A$ is a subset of $A \times A$ consisting of all pairs of elements.
- For $a, b \in A$, we denote by $R(a, b)$ if $a$ is related to $b$.

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## Properties on Binary Relations

- **Completeness**: For all $a, b \in A$, we have $R(a, b)$, $R(b, a)$, or both.
- **Reflexivity**: For all $a \in A$, we have $R(a, a)$.
- **Transitivity**: For $a, b, c \in A$, if $R(a, b)$ and $R(b, c)$, then we have $R(a, c)$.

# Preference Relation

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A preference relation is a complete, reflexive, and transitive binary relation.

- Denote by $a \succsim b$ if $a$ is related to $b$.
- Denote by $a \succ b$ if $a \succsim b$ but $b \not\succsim a$.
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- $a \succsim b$: $a$ is weakly preferred to $b$.
- $a \sim b$: agent is indifferent between $a$ and $b$.

# Continuity on a Preference relation

## Continuous Preference Relation

A preference relation is continuous if:

whenever there exist sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ in $A$ such that

- $\lim_{k\to\infty} a_k = a$,
- $\lim_{k\to\infty} b_k = b$,
- and $a_k \succsim b_k$ for all $k \in \mathbb{N}$

we have $a \succsim b$.

# Strategic Games

### Strategic Games

A strategic game is a tuple $\langle N, (A_i), (\succsim_i) \rangle$ consisting of

- a finite set of **players** $N$.
- for each player $i \in N$, a nonempty set of **actions** $A_i$.
- for each player $i \in N$, a **preference relation** $\succsim_i$ on $A = \prod_{j \in N} A_j$.

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- A strategic game is finite if $A_i$ is finite for all $i \in N$.
- **Note**: $\succsim_i$ is not defined on $A_i$ only, but instead on the set of all $(A_j)_{j \in N}$.

# PSNE w.r.t. a Preference Relation

**Pure-Strategy Nash Equilibrium (PSNE) with $(\succsim_i)$**

A (pure-strategy) Nash equilibrium (PSNE) of a strategic game $\langle N, (A_i), (\succsim_i)\rangle$ is a profile $\boldsymbol{a}^* := (a_i)_{i\in N}$ such that for all $i \in N$, we have

$$(\boldsymbol{a}^*_{-i}, a^*_i) \succsim_i (\boldsymbol{a}^*_{-i}, a'_i) \text{ for all } a'_i \in A_i.$$

# Best-Response Function

## Best-Response Functions

The best-response function of player $i$,

$$BR_i : \prod_{j \in N \setminus \{i\}} A_j \to \mathbb{P}(A_i),$$

is given by

$$BR_i(\boldsymbol{a}_{-i}) = \{a_i \in A_i \mid (\boldsymbol{a}_{-i}, a_i) \succsim_i (\boldsymbol{a}_{-i}, a_i') \text{ for all } a_i' \in A_i\}.$$

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- $BR_i$ is set-valued.
- Recall: $\mathbb{P}(X)$ includes all nonempty, closed, and convex subsets of $X$.

# PSNE w.r.t. a Preference Relation

- Alternative definition of NE.

**Pure-Strategy Nash Equilibrium (PSNE) with $(\succsim_i)$**

A Nash equilibrium of a strategic game $\langle N, (A_i), (\succsim_i) \rangle$ is a profile $\boldsymbol{a}^* := (a_i)_{i \in N}$ such that $a_i^* \in BR_i(\boldsymbol{a}_{-i}^*)$ for all $i \in N$.

- Thus, to prove the existence of a PSNE for a strategic game $\langle N, (A_i), (\succsim_i) \rangle$, it suffices to show that:

# PSNE w.r.t. a Preference Relation

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- Thus, to prove the existence of a PSNE for a strategic game $\langle N, (A_i), (\succsim_i)\rangle$, it suffices to show that:
  - There exists a profile $\boldsymbol{a}^* \in A$ such that for all $i \in N$ we have $a_i^* \in BR_i(\boldsymbol{a}_{-i}^*)$.

# Outline

## General Idea

- Let $BR : A \to \mathbb{P}(A)$ be

$$BR(\boldsymbol{a}) = \prod_{i \in N} BR_i(\boldsymbol{a}_{-i}).$$

- We seek for some $\boldsymbol{a}^* \in A$ such that $\boldsymbol{a}^* \in BR(\boldsymbol{a}^*)$.

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- We seek for some $\boldsymbol{a}^* \in A$ such that $\boldsymbol{a}^* \in BR(\boldsymbol{a}^*)$.
- We can then use Kakutani's Fixed-Point Theorem to show that $\boldsymbol{a}^*$ exists.
- Yet, we need to verify the conditions under which Kakutani's Fixed-Point Theorem holds.

# Quasi-Concave

**Quasi-Concave of $\succsim_i$**

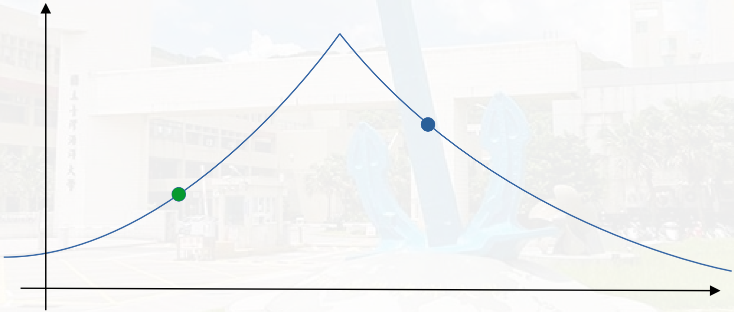
A preference relation $\succsim_i$ over $A$ is quasi-concave on $A_i$ if for all $\boldsymbol{a} \in A$, the set

$$\{a'_i \in A_i \mid (\boldsymbol{a}_{-i}, a'_i) \succsim_i (\boldsymbol{a}_{-i}, a_i)\}$$

is convex.

- Then, we can consider the following theorem which guarantees the condition of a PNE.

An example of quasi-concave function.



$$f(\lambda x + (1 - \lambda y)) \geq \min\{f(x), f(y)\}, \quad \text{for } \lambda \in [0, 1]$$

# The Main Theorem I

**Main Theorem I**

The strategic game $\langle N, (A_i), (\succsim_i)\rangle$ has a (pure-strategy) Nash equilibrium if

- $A_i$ is a nonempty, compact, and convex subset of a Euclidean space
- $\succsim_i$ is continuous and quasi-concave on $A_i$ for all $i \in N$.

- We will show that $A$ (cf. $S$) and $BR$ (cf. $\Phi$) satisfy the conditions to apply Kakutani's Fixed-Point Theorem.

# Requirements for $A$ & $BR$

- $A_i$ is nonempty, compact and convex for all $i \in N$, so their Cartesian product (i.e., $A$) must also be nonempty, compact and convex.

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- We need to show that $BR_i(\boldsymbol{a}_{-i})$ is nonempty, closed, and convex for all $\boldsymbol{a}_{-i} \in \prod_{j \in N \setminus \{i\}} A_j$.
    - Their Cartesian product $BR(\boldsymbol{a})$ is then nonempty, closed and convex, too.
    - We then have $BR : A \to \mathbb{P}(A)$.

# $BR_i(\boldsymbol{a}_{-i})$ is nonempty

- Let $u_i : A_i \to \mathbb{R}$ be a continuous function (utility function) such that for $a_i, a_i' \in A_i$, $(\boldsymbol{a}_{-i}, a_i) \succsim (\boldsymbol{a}_{-i}, a_i')$ if and only if $u_i(a_i) \geq u_i(a_i')$.

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- Since $A_i$ is compact and $u_i$ is continuous, $u_i(A_i)$ is compact as well.
- By the Extreme Value Theorem, there must exist some $a_i^* \in A_i$ such that $u_i(a_i^*) \geq u_i(a_i)$ for all $a_i \in A_i$.

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- Since $A_i$ is compact and $u_i$ is continuous, $u_i(A_i)$ is compact as well.
- By the Extreme Value Theorem, there must exist some $a_i^* \in A_i$ such that $u_i(a_i^*) \geq u_i(a_i)$ for all $a_i \in A_i$.
- By definition of $u_i$, it follows that $(\boldsymbol{a}_{-i}, a_i^*) \succsim (\boldsymbol{a}_{-i}, a_i)$ for all $a_i \in A_i$, thus $a_i^* \in BR_i(\boldsymbol{a}_{-i})$.

# $BR_i(\boldsymbol{a}_{-i})$ is nonempty

- Let $u_i : A_i \to \mathbb{R}$ be a continuous function (utility function) such that for $a_i, a_i' \in A_i$, $(\boldsymbol{a}_{-i}, a_i) \succsim (\boldsymbol{a}_{-i}, a_i')$ if and only if $u_i(a_i) \geq u_i(a_i')$.
- Since $A_i$ is compact and $u_i$ is continuous, $u_i(A_i)$ is compact as well.
- By the Extreme Value Theorem, there must exist some $a_i^* \in A_i$ such that $u_i(a_i^*) \geq u_i(a_i)$ for all $a_i \in A_i$.
- By definition of $u_i$, it follows that $(\boldsymbol{a}_{-i}, a_i^*) \succsim (\boldsymbol{a}_{-i}, a_i)$ for all $a_i \in A_i$, thus $a_i^* \in BR_i(\boldsymbol{a}_{-i})$.
- So $BR_i(\boldsymbol{a}_{-i})$ is nonempty.

# $BR_i(\boldsymbol{a}_{-i})$ is closed

- Take an arbitrary $p \in \overline{BR_i(\boldsymbol{a}_{-i})}$.
- There must exist some sequence $(p_k)_{k\in\mathbb{N}}$ such that $p_k \in BR_i(\boldsymbol{a}_{-i})$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} p_k = p$.

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- By the definition of $BR_i$, we know that $(\boldsymbol{a}_{-i}, p_k) \succsim_i (\boldsymbol{a}_{-i}, a_i)$ for all $a_i \in A_i$.
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$\Rightarrow$ $p \in BR_i(\boldsymbol{a}_{-i})$ ($\because$ $BR_i(\boldsymbol{a}_{-i})$ is closed).

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- Consider $a_i \in BR_i(\boldsymbol{a}_{-i})$.
- $\succsim_i$ is quasi-concave on $A_i$ $\Rightarrow$

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- Since $a_i$ is a best response, the responses $a_i'$ weakly preferable to $a_i$ must be also best responses.

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# $BR_i(\boldsymbol{a}_{-i})$ is convex

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- Since $a_i$ is a best response, the responses $a_i'$ weakly preferable to $a_i$ must be also best responses. $\Rightarrow S \subseteq BR_i(\boldsymbol{a}_{-i})$.
- Any other best response $a_i^* \in BR_i(\boldsymbol{a}_{-i})$ must be at least good as $a_i$ $\Rightarrow BR_i(\boldsymbol{a}_{-i}) \subseteq S$.
- Hence, we have $BR_i(\boldsymbol{a}_{-i}) = S$, so $BR_i(\boldsymbol{a}_{-i})$ is convex.

- Next, we will show that $BR$ is upper semi-continuous.

# Recall: Upper Semi-Continuous

## Upper semi-continuous functions

Let

- $\mathbb{P}(X)$: all nonempty, closed, convex subsets of $X$.
- $S$: a nonempty, compact, and convex set.

Then the set-valued function $\Phi : S \to \mathbb{P}(S)$ is upper semi-continuous if

for arbitrary sequences $(\boldsymbol{x}_n)_{n\in\mathbb{N}}, (\boldsymbol{y}_n)_{n\in\mathbb{N}}$ in $S$, we have

- $\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x}^*$,
- $\lim_{n\to\infty} \boldsymbol{y}_n = \boldsymbol{y}^*$,
- $\boldsymbol{y}_n \in \Phi(\boldsymbol{x}_n)$ for all $n \in \mathbb{N}$,

imply that $\boldsymbol{y}^* \in \Phi(\boldsymbol{x}^*)$.

# $BR$ is upper semi-continuous

- Consider two sequences $(\boldsymbol{x}^k), (\boldsymbol{y}^k)$ in $A$ such that
  $\lim_{k\to\infty} \boldsymbol{x}^k = \boldsymbol{x}^*$,
  $\lim_{k\to\infty} \boldsymbol{y}^k = \boldsymbol{y}^*$.
  $\boldsymbol{y}^k \in BR_i(\boldsymbol{x}^k)$ for all $k \in \mathbb{N}$.
- Then we have $y_i^k \in BR_i(\boldsymbol{x}_{-i}^k)$ for all $i \in N, k \in \mathbb{N}$.

# $BR$ is upper semi-continuous

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  $\boldsymbol{y}^k \in BR_i(\boldsymbol{x}^k)$ for all $k \in \mathbb{N}$.
- Then we have $y_i^k \in BR_i(\boldsymbol{x}_{-i}^k)$ for all $i \in N, k \in \mathbb{N}$.
- For an arbitrary $i \in N$, we have $(\boldsymbol{x}_{-i}^k, y_i^k) \succsim_i (\boldsymbol{x}_{-i}^k, a_i)$ for all $a_i \in A_i$ and $k \in \mathbb{N}$ ($\because$ best response).

# $BR$ is upper semi-continuous (contd.)

- For each $a_i \in A_i$, we can construct:
  - a sequence $((\boldsymbol{x}_{-i}^k, y_i^k))_{k \in \mathbb{N}}$ such that $\lim_{k \to \infty}(\boldsymbol{x}_{-i}^k, y_i^k) = (\boldsymbol{x}_{-i}^*, y_i^*)$.
  - a sequence $((\boldsymbol{x}_{-i}^k, a_i))_{k \in \mathbb{N}}$ such that $\lim_{k \to \infty}(\boldsymbol{x}_{-i}^k, a_i) = (\boldsymbol{x}_{-i}^*, a_i)$.
- Note that we have $(\boldsymbol{x}_{-i}^k, y_i^k) \succsim_i (\boldsymbol{x}_{-i}^k, a_i)$ for all $k \in \mathbb{N}$.
  - By continuity of $\succsim_i$, we have $(\boldsymbol{x}_{-i}^*, y_i^*) \succsim_i (\boldsymbol{x}_{-i}^*, a_i)$ for all $a_i \in A_i$.

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- Thus, we have $y_i^* \in BR_i(\boldsymbol{x}_{-i}^*)$ for all $i \in N$.
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- Thus, we have $y_i^* \in BR_i(\boldsymbol{x}_{-i}^*)$ for all $i \in N$.
  - $\boldsymbol{y}^* \in BR_i(\boldsymbol{x}^*)$.
- Therefore, $BR$ is upper semi-continuous.

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  By Kakutani's Fixed-Point Theorem, there exists some $\boldsymbol{a}^* \in A$ such that $\boldsymbol{a}^* \in BR(\boldsymbol{a}^*)$

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  - $\boldsymbol{y}^* \in BR_i(\boldsymbol{x}^*)$.
- Therefore, $BR$ is upper semi-continuous.

  By Kakutani's Fixed-Point Theorem, there exists some $\boldsymbol{a}^* \in A$ such that $\boldsymbol{a}^* \in BR(\boldsymbol{a}^*) \Rightarrow \boldsymbol{a}^*$ is a PNE of the strategic game.

# Outline

# Limitations of the Previous PNE Result

- Any finite game cannot satisfy the conditions.

# Limitations of the Previous PNE Result

- Any finite game cannot satisfy the conditions.
  - Each $A_i$ cannot be convex if it is finite and nonempty.

⋆ Next, we consider extending finite games into non-deterministic (randomized) strategies.

# Assumptions

- For a strategic game $\langle N, (A_i), (\succsim_i) \rangle$, we assume that we can construct a utility function $u_i : A \to \mathbb{R}$, where $A = \prod_{i \in N} A_i$.
- Each player's *expected utility* is coupled with the set of probability distributions over $A$.
- $\Delta(X)$: the set of probability distributions over $X$.
- If $X$ is finite and $\delta \in \Delta(X)$, then
  - $\delta(x)$: the probability that $\delta$ assigns to $x \in X$.
  - The support of $\delta$: $\chi(\delta) = \{x \in X \mid \delta(x) > 0\}$.

# Mixed Strategy

## Mixed Strategy

Given a strategic game $\langle N, (A_i), (u_i) \rangle$, we call

- $\alpha_i \in \Delta(A_i)$ a mixed strategy.
- $a_i \in A_i$ a pure strategy.

A profile of mixed strategies $\boldsymbol{\alpha} = (\alpha_j)_{j \in N}$ induces a probability distribution over $A$.

- The probability of $\boldsymbol{a} = (a_j)_{j \in N}$ under $\boldsymbol{\alpha}$:

$$\boldsymbol{\alpha}(\boldsymbol{a}) = \prod_{j \in N} \alpha_j(a_j). \quad \text{(a normal product)}$$

($A_i$ is finite $\forall i \in N$ and each player's strategy is resolved independently.)

prob. $= \alpha_1(t_1) \cdot \alpha_2(s_1)$

| | | $\alpha_2(s_1)$ | $\alpha_2(s_2)$ |
|---|---|---|---|
| | | $s_1$ | $s_2$ |
| $\alpha_1(t_1)$ | $t_1$ | $u_1(t_1, s_1), u_2(t_1, s_1)$ | $u_1(t_1, s_2), u_2(t_1, s_2)$ |
| $\alpha_1(t_2)$ | $t_2$ | $u_1(t_2, s_1), u_2(t_2, s_1)$ | $u_1(t_2, s_2), u_2(t_2, s_2)$ |

# Mixed Extension of $\langle N, (A_i), (u_i) \rangle$

## Mixed Extension of the Strategic Games

$\langle N, (\Delta(A_i)), (U_i) \rangle$:

- $U_i : \prod_{i \in N} \Delta(A_i) \to \mathbb{R}$; expected utility over $A$ induced by $\boldsymbol{\alpha} \in \prod_{i \in N} \Delta(A_i)$.
- If $A_j$ is finite for all $j \in N$, then

$$\begin{aligned} U_i(\boldsymbol{\alpha}) &= \sum_{\boldsymbol{a} \in A} (\boldsymbol{\alpha}(\boldsymbol{a}) \cdot u_i(\boldsymbol{a})) \\ &= \sum_{\boldsymbol{a} \in A} \left( \left( \prod_{j \in N} \alpha_j(a_j) \right) \cdot u_i(\boldsymbol{a}) \right). \end{aligned}$$

# Main Theorem II

**Main Theorem II**

Every finite strategies game has a mixed strategy Nash equilibrium.

- Consider an arbitrary finite strategic game $\langle N, (A_i), (u_i) \rangle$, let $m_i := |A_i|$ for all $i \in N$.
- Represent each $\Delta(A_i)$ as a collection of vectors $\boldsymbol{p}^i = (p_1, p_2, \ldots, p_{m_i})$.
  - $p_k \geq 0$ for all $k \in [m_i]$ and $\sum_{k=1}^{m_i} p_k = 1$.
  - $\Delta(A_i)$ is a standard $m_i - 1$ simplex for all $i \in N$.

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- Represent each $\Delta(A_i)$ as a collection of vectors $\boldsymbol{p}^i = (p_1, p_2, \ldots, p_{m_i})$.
  - $p_k \geq 0$ for all $k \in [m_i]$ and $\sum_{k=1}^{m_i} p_k = 1$.
  - $\Delta(A_i)$ is a standard $m_i - 1$ simplex for all $i \in N$.
  - ⋆ $\Delta(A_i)$: nonempty, compact, and convex for each $i \in N$.
- $U_i$: continuous ($\because$ multilinear).
- Next, we show that $U_i$ is quasi-concave in $\Delta(A_i)$.

# Proof of Main Theorem II (contd.)

- Consider $\boldsymbol{\alpha} \in \prod_{i \in N} \Delta(A_i)$.
- **Goal:** Show that $S = \{\alpha'_i \in \Delta(A_i) \mid U_i(\boldsymbol{\alpha}_{-i}, \alpha'_i) \geq U_i(\boldsymbol{\alpha}_{-i}, \alpha_i)\}$ is convex.

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- **Goal:** Show that $S = \{\alpha'_i \in \Delta(A_i) \mid U_i(\boldsymbol{\alpha}_{-i}, \alpha'_i) \geq U_i(\boldsymbol{\alpha}_{-i}, \alpha_i)\}$ is convex.
- Take $\beta_i, \gamma_i \in S$, $\lambda \in [0, 1]$.
- By definition of $S$, we have
    - $U_i(\boldsymbol{\alpha}_{-i}, \beta_i) \geq U_i(\boldsymbol{\alpha}_{-i}, \alpha_i)$, and
    - $U_i(\boldsymbol{\alpha}_{-i}, \gamma_i) \geq U_i(\boldsymbol{\alpha}_{-i}, \alpha_i)$.

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- By definition of $S$, we have
  - $U_i(\boldsymbol{\alpha}_{-i}, \beta_i) \geq U_i(\boldsymbol{\alpha}_{-i}, \alpha_i)$, and
  - $U_i(\boldsymbol{\alpha}_{-i}, \gamma_i) \geq U_i(\boldsymbol{\alpha}_{-i}, \alpha_i)$.
- $\lambda U_i(\boldsymbol{\alpha}_{-i}, \beta_i) + (1 - \lambda) U_i(\boldsymbol{\alpha}_{-i}, \gamma_i) \geq$
  $\lambda U_i(\boldsymbol{\alpha}_{-i}, \alpha_i) + (1 - \lambda) U_i(\boldsymbol{\alpha}_{-i}, \alpha_i) = U_i(\boldsymbol{\alpha}_{-i}, \alpha_i)$.

# Proof of Main Theorem II (contd.)

- By the multilinearity of $U_i$, we have

$$\lambda U_i(\boldsymbol{\alpha}_{-i}, \beta_i) + (1-\lambda) U_i(\boldsymbol{\alpha}_{-i}, \gamma_i) = U_i(\boldsymbol{\alpha}_{-i}, \lambda\beta_i + (1-\lambda)\gamma_i).$$

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- So,

$$U_i(\boldsymbol{\alpha}_{-i}, \lambda\beta_i + (1-\lambda)\gamma_i) \geq U_i(\boldsymbol{\alpha}_{-i}, \alpha_i).$$

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$$\lambda\beta_i + (1-\lambda)\gamma_i \in S$$

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- So,

$$U_i(\boldsymbol{\alpha}_{-i}, \lambda\beta_i + (1-\lambda)\gamma_i) \geq U_i(\boldsymbol{\alpha}_{-i}, \alpha_i).$$

$$\lambda\beta_i + (1-\lambda)\gamma_i \in S \Rightarrow \ U_i \text{ is convex.}$$

- Thus, $U_i$ is quasi-concave in $\Delta(A_i)$.

  We are done.

# A Question

## Matching Pennies of Infinite Actions

We have two players $A$ and $B$ having utility functions $f(x,y) = (x-y)^2$ and $g(x,y) = -(x-y)^2$ respectively. $x, y \in [-1,1]$.

- Does this game has a pure Nash equilibrium?
- Why can't we use Kakutani's fixed point theorem?

# Discussions.