# Randomized Algorithms

## Discrete Random Variables and Expectation

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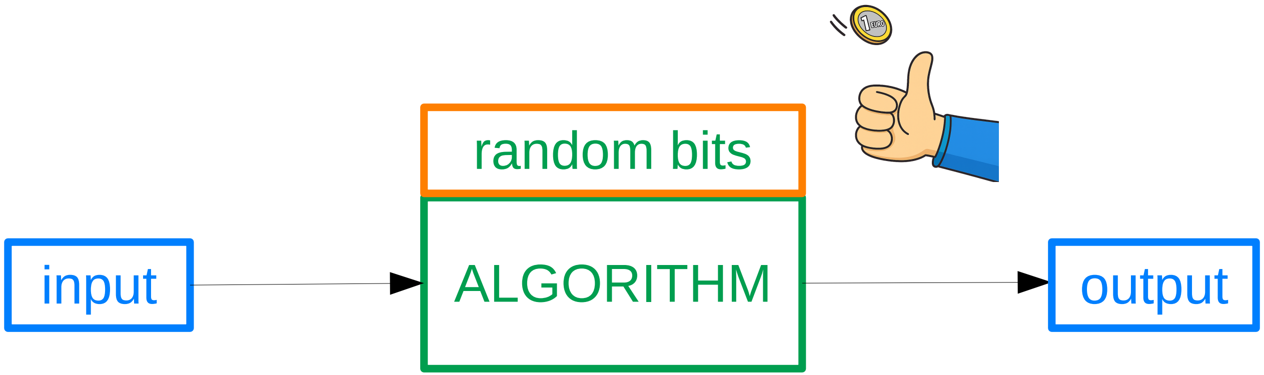
# Review

- Why randomized algorithms?
- Types of randomized algorithms.

# Randomized algorithms



# Why?

- Randomized algorithms are
  - often much ***simpler*** than the best known deterministic ones.
  - often much ***more efficient*** (faster or using less space) than the best known deterministic ones.

# Two types of randomized algorithms

- The accuracy is guaranteed.
  - Las Vegas algorithms.

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- The running time is guaranteed.
  - Monte Carlo algorithms.

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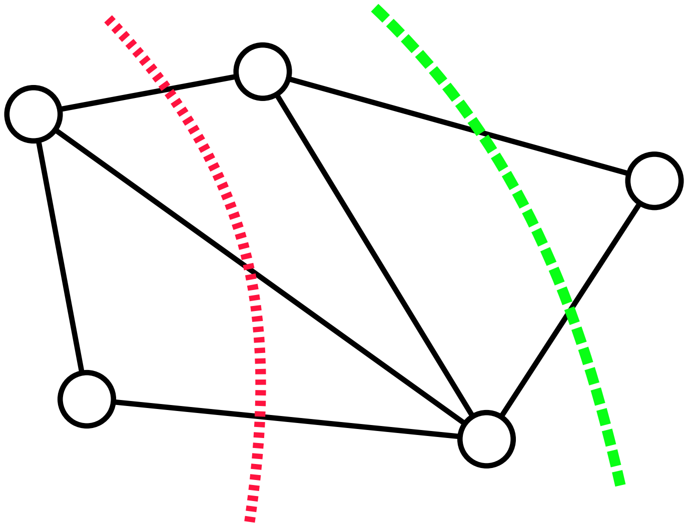
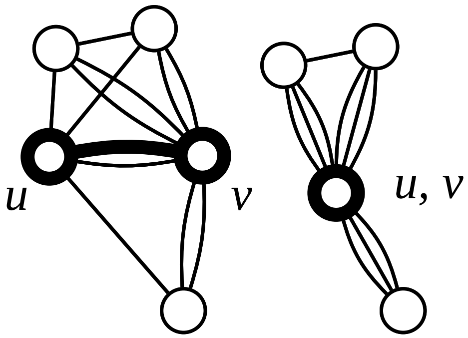
# Min-Cut



$|V| = n, |E| = m$

- A graph $G = (V, E)$ and its two “cuts”.
  - **Cut**: a partition of the vertices in $V$ into two non-empty, disjoint sets $S$ and $T$ such that
  - $S \cup T = V$
- The **cutset** of a cut:
  - $\{uv \in E \mid u \in S, v \in T\}.$
- The size of the cut:
  - the cardinality of its cutset.

# Edge contraction




$e = (u, v)$

$G \rightarrow G/e$

# Karger's edge-contraction algorithm (1993)

Procedure contract ($G = (V, E)$):

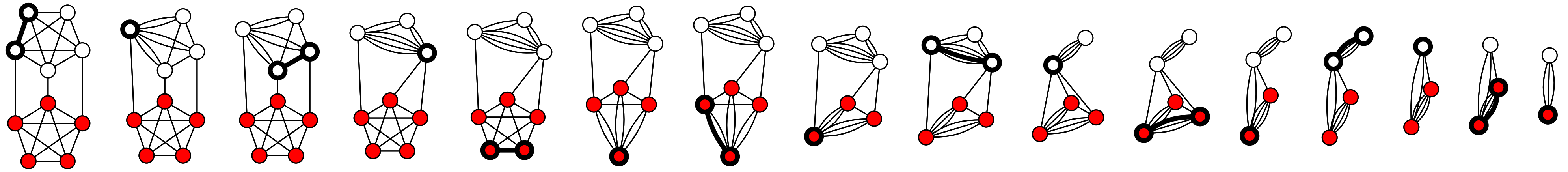
while $|V| > 2$:

choose $e \in E$ uniformly at random

$G \leftarrow G/e$

return the only cut in $G$

Time complexity: $O(m)$ or $O(n^2)$.



By Thore Husfeldt - Created in python using the networkx library for graph manipulation, neato for layout, and TikZ for drawing., 

# Analysis

- $C$: a specific cut of $G$.
- $k$: the size of the cut $C$.
- The minimum degree of $G$ must be $\geq k$. (WHY?)
  - So, $|E| \geq nk/2$.
- The probability that the algorithm picks an edge from $C$ to contract is

$$\frac{k}{|E|} \leq \frac{k}{nk/2} = \frac{2}{n}.$$

# Analysis (contd.)

- Let $p_n$ be the probability that the algorithm on an $n$-vertex graph avoids $C$.
- Then,

$$p_n \geq \left(1 - \frac{2}{n}\right) \cdot p_{n-1}$$

- The recurrence can be expanded as

$$p_n \geq \prod_{i=0}^{n-3} \left(1 - \frac{2}{n-i}\right) = \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdots \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{\binom{n}{2}}.$$

# Analysis (contd.)

- Repeat the contract algorithm for $T = \binom{n}{2} \ln n$ times, and then choose the minimum of them.

- The probability of NOT finding a min-cut is $\left[1 - \binom{n}{2}^{-1}\right]^T \leq \frac{1}{e^{\ln n}} = \frac{1}{n}.$

- Let's take a look at the exponential function $e^x$.

# Facts on $e^x$

$$e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

$$\therefore e = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\left(1 + \frac{1}{n}\right)^n = \binom{n}{0} 1 + \binom{n}{1} \frac{1}{n} + \binom{n}{2} \frac{1}{n^2} + \binom{n}{3} \frac{1}{n^3} + \cdots + \binom{n}{n} \frac{1}{n^n}$$

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

# Probability basics

A *random variable* $X$ on a sample space $\Omega$ is a real-valued function. That is,

$$X : \Omega \mapsto \mathbf{R}$$

A *discrete random variable* is a random variable that takes on only a finite or countably infinite number of values.

# Kolmogorov axioms

- 1. For any event $A \subset S$, $P(A) \geq 0$.
- 2. $\Pr[S] = 1$.
- 3. If $A_1$, $A_2$, … are mutually exclusive events, then

  $$\Pr[A_1 \cup A_2 \cup \ldots] = \Pr[A_1] + \Pr[A_2] + \cdots$$

# Useful theorems

- $\Pr[\emptyset] = 0$ for any experiment.
- For any event $A \subseteq S$, $\Pr[A] = 1 - \Pr[\overline{A}]$.
- If $A \subseteq S$, $B \subseteq S$ are any two events, then
  $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$.
- If $A \subset B$, then $\Pr[A] \leq \Pr[B]$.

# Conditional probability

- In an experiment with sample space $S$, let $B$ be any event such that $\Pr[B] > 0$.
- Then the conditional probability of $A$ occurring, given that $B$ has occurred, is

$$\Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

for any $A \subset S$.

# Example

- Suppose you are going to buy milk in a supermarket.
  - There are a total of 40 boxes for you to choose from.
  - 10 of them are corrupted (not visible on the outside).
  - ➢ Then, you are asked to buy two boxes of milk.

    **What is the probability that both boxes are good?**

# Example (contd.)

- *A*: the event that the first box you choose is good.

  *B*: the event that the second box you choose is good.

  Then

$$\Pr[A] = \frac{30}{40}$$

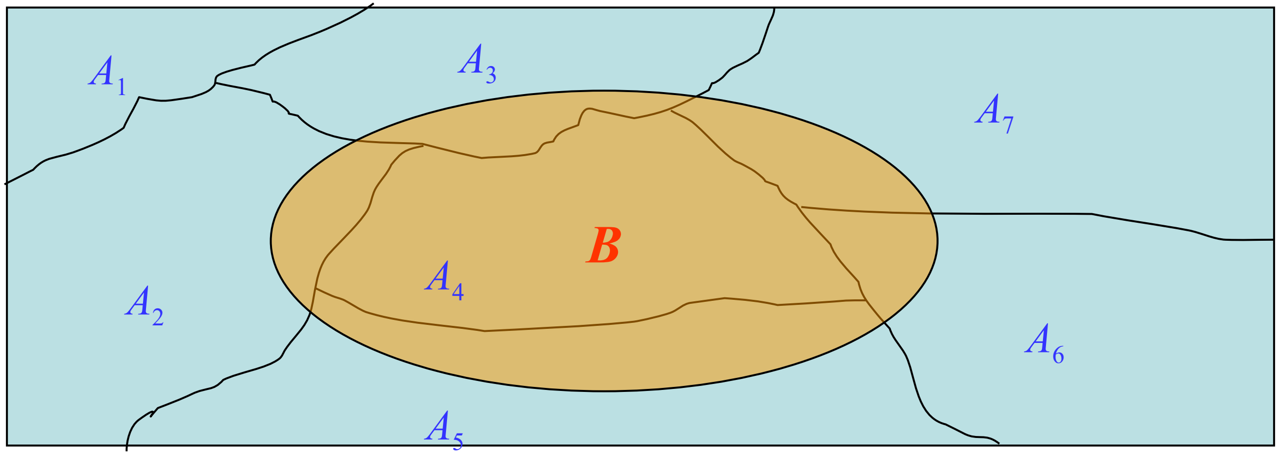
$$\Pr[B \mid A] = \frac{29}{39}$$

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B \mid A] = \frac{30}{40} \cdot \frac{29}{39} = \frac{87}{156}.$$

# Theorem of total probability

- If $A_1, A_2, \ldots, A_n$ is a partition of $S$, and $B$ is any event, then

$$\Pr[B] = \sum_{i=1}^{n} \Pr[B \mid A_i] \Pr[A_i].$$



# Bayes' theorem

- From the theorem of total probability, and granted that $A_1, A_2, \ldots, A_n$ is a partition of $S$, we have

$$\begin{aligned}\Pr[A_i \mid B] &= \frac{\Pr[A_i \cap B]}{\Pr[B]} \\ &= \frac{\Pr[A_i] \cdot \Pr[B \mid A_i]}{\sum_{i=1}^{n} \Pr[A_i] \cdot \Pr[B \mid A_i]}\end{aligned}$$

- This result is known as *Bayes' theorem*.

# Example

- Assuming a jury selected to participate in a criminal trial.
- Whether the defendant is guilty or not, there is a 95% chance of making the correct verdict.
- It is also assumed that the local police law enforcement is very strict, such that 99% of the people being tried are actually guilty.
- **If a jury is known to sentence a defendant not guilty**, **what is the probability that the defendant is really not guilty?**

# Example (contd.)

- $A_1$: the defendant is guilty
- $A_2 = \bar{A}_1$ : the defendant is not guilty.

- Let $B$ be the event that the defendant is sentenced to unguilty.

- We want to know **$\Pr[A_2 \mid B]$.**

# Example (contd.)

$$\Pr[A_2|B] = \frac{\Pr[A_2]\Pr[B|A_2]}{\Pr[A_1]\Pr[B|A_1] + \Pr[A_2]\Pr[B|A_2]}$$

$$= \frac{(0.01)(0.95)}{(0.99)(0.05) + (0.01)(0.95)}$$

$$= 0.161$$

- Before the sentence, this defendant is supposed to be unguilty with probability 1%.
- After the sentence of unguilty, the probability is increased to be **16.1%**.

# Independent events

If $A \subset S$ and $B \subset S$ are any two events with nonzero probabilities,

$A$ and $B$ are called independent if and only if $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$

That is, $\Pr[A] = \Pr[A \mid B]$ and $\Pr[B] = \Pr[B \mid A]$.

# Independent trials

- An experiment is said to consist of $n$ ***independent*** trials if and only if
  - $S = T_1 \times T_2 \times \cdots \times T_n$.
  - For every $(x_1, x_2, \ldots, x_n) \in S$,

    $\Pr[\{(x_1, x_2, \ldots, x_n)\}] = \Pr[\{x_1\}] \cdot \Pr[\{x_2\}] \cdots \Pr[\{x_n\}]$,

    where $\Pr[\{x_i\}]$ is the probability of $x_i \in T_i$ occurring on trial $i$.

# Expectation

- The expectation of a discrete random variable *X*, denoted by **E**[*X*], is

$$\mathbf{E}[X] = \sum_i i \cdot \Pr[X = i]$$

- Example: Let *X* denote the sum of of dices:

$$\mathbf{E}[X] = \frac{1}{36} \cdot 2 + \frac{2}{36} \cdot 3 + \frac{3}{36} \cdot 4 + \cdots + \frac{1}{36} \cdot 12 = 7.$$

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# Linearity of Expectation

- For any finite collection of discrete random variables $X_1$, $X_2$, ..., $X_n$ with finite expectations,

$$\mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{E}[X_i].$$

- For any constant c and discrete random variable $X$,

$$\mathbf{E}[cX] = c \cdot \mathbf{E}[X].$$

- Why is it useful?

# Example

- Consider the dice-throwing example again.
  - $X_1$ : the outcome of die 1
  - $X_2$ : the outcome of die 2

$$\mathbf{E}[X_1] = \mathbf{E}[X_2] = \frac{1}{6} \cdot \sum_{j=1}^{6} j = \frac{7}{2}$$

$$\mathbf{E}[X] = \mathbf{E}[X_1 + X_2] = 7.$$

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# Bernoulli random variable

- Suppose we run an experiment that succeeds with probability $p$ and fails with probability $1-p$.

$$Y = \begin{cases} 1 & \text{if the experiment succeeds,} \\ 0 & \text{otherwise.} \end{cases}$$


- $Y$: Bernoulli random variable.
  - or *indicator random variable*.

# Binomial random variable

- A binomial random variable $X$ with parameters $n$ and $p$, denoted by $B(n, p)$, is defined as

$$\Pr[X = j] = \binom{n}{j} p^j (1-p)^{n-j}.$$

for $j = 0, 1, 2, \ldots, n$.

- <u>Exercise</u>: Show that $\sum_{j=0}^{n} \Pr[X = j] = 1.$

# Binomial random variable (contd.)

- $$\begin{aligned}
\mathbf{E}[X] &= \sum_{j=0}^{n} j \binom{n}{j} p^j (1-p)^{n-j} \\
&= \sum_{j=0}^{n} j \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \\
&= \sum_{j=1}^{n} \frac{n!}{(j-1)!(n-j)!} p^j (1-p)^{n-j} \\
&= np \sum_{j=1}^{n} \frac{(n-1)!}{(j-1)!((n-1)-(j-1))!} p^{j-1} (1-p)^{(n-1)-(j-1)} \\
&= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k!((n-1)-k)!} p^k (1-p)^{(n-1)-k} \\
&= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{(n-1)-k} \\
&= np.
\end{aligned}$$

# Binomial random variable (contd.)

- $$\begin{aligned}
\mathbf{E}[X] &= \sum_{j=0}^{n} j\binom{n}{j}p^j(1-p)^{n-j} \\
&= \sum_{j=0}^{n} j\frac{n!}{j!(n-j)!}p^j(1-p)^{n-j} \\
&= \sum_{j=1}^{n} \frac{n!}{(j-1)!(n-j)!}p^j(1-p)^{n-j} \\
&= np\sum_{j=1}^{n} \frac{(n-1)!}{(j-1)!((n-1)-(j-1))!}p^{j-1}(1-p)^{(n-1)-(j-1)} \\
&= np\sum_{k=0}^{n-1} \frac{(n-1)!}{k!((n-1)-k)!}p^k(1-p)^{(n-1)-k} \\
&= np\sum_{k=0}^{n-1} \binom{n-1}{k}p^k(1-p)^{(n-1)-k} \\
&= np.
\end{aligned}$$

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}.$$

# Let's make it simpler!

- Denote a set of $n$ Bernoulli random variables $X_1, X_2, \ldots, X_n$.
  - $X_i = 1$ if the $i$th trial is successful and 0 otherwise.

$$\mathbf{E}[X] = \mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{E}[X_i] = np.$$