

# Union-Find (Disjoint Set Union)

## Efficient Maintenance of Disjoint Sets

Joseph Chuang-Chieh Lin (林莊傑)

Department of Computer Science & Engineering,  
National Taiwan Ocean University

Fall 2025

# Reference

- Lecture Notes of [CS6820 2022 \(Cornell University\)](#)
- Robert Endre Tarjan: Efficiency of a Good But Not Linear Set Union Algorithm. *Journal of the ACM*. Vol. 22(2) (1975) 215–225. [[DOI](#) [LINK](#)]
- R. E. Tarjan on analyzing the “union-find” data structure [[YouTube](#)]

# Outline

- 1 Motivation & Abstract Data Type
- 2 Two Key Heuristics
- 3 Implementation
- 4 Why  $\alpha(n)$ ? A Glimpse of the Analysis

# Why Union-Find?

- Many graph algorithms need to maintain a **partition** of elements into **disjoint sets**.
- Example: **Kruskal's algorithm** and **Borůvka/Sollin's algorithm** for Minimum Spanning Tree (MST)
  - Cycle detection.
  - When scanning edges in nondecreasing weight:  
Add edge  $(u, v)$  iff  $u$  and  $v$  are currently in **different** connected components.
- Union-Find supports this pattern in (almost) constant amortized time.

# Disjoint-Set (Union-Find) ADT

## Operations

- **find(v)**: return a **canonical representative** of the set containing  $v$ .
- **union(u,v)**: merge the two sets containing  $u$  and  $v$ .



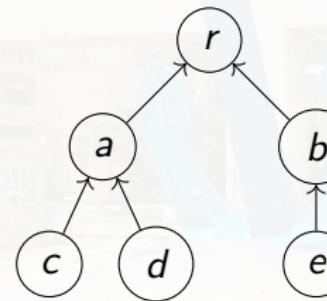
# Disjoint-Set (Union-Find) ADT

## Operations

- **find(v)**: return a **canonical representative** of the set containing  $v$ .
- **union(u,v)**: merge the two sets containing  $u$  and  $v$ .
- Same-set query:  
 $u$  and  $v$  are in the same set  $\iff \text{find}(u) = \text{find}(v)$ .
- Initially:  $n$  singleton sets.

# Representing Each Set as a Rooted Tree

- Each element  $x$  stores a pointer  $\text{parent}[x]$ .
- The **root** is the canonical representative (for the set).



`find(x) = follow parent pointers to the root.`

# Naïve Implementation: Worst-Case Can Be Bad

- If we always attach one root under the other **arbitrarily**, the tree can become a **chain**.



find(5) traverses  
4 edges  
Worst-case:  $\Theta(n)$

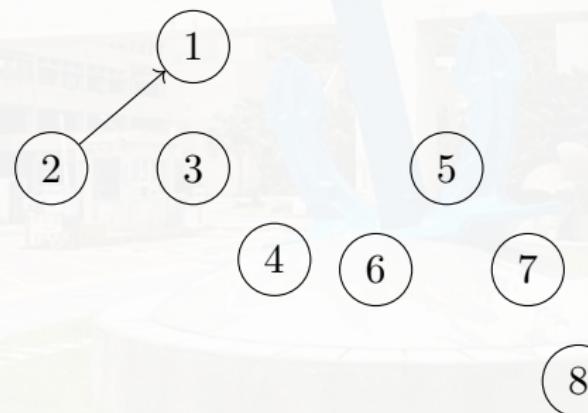
# A Small Example: Unions Build a Forest

- Start with singletons  $\{1\}, \{2\}, \dots, \{8\}$ .
- Perform unions:  $\text{union}(1,2)$ ,  $\text{union}(3,4)$ ,  $\text{union}(5,6)$ ,  
 $\text{union}(7,8)$ ,  $\text{union}(1,3)$ ,  $\text{union}(5,7)$ ,  $\text{union}(1,5)$ .



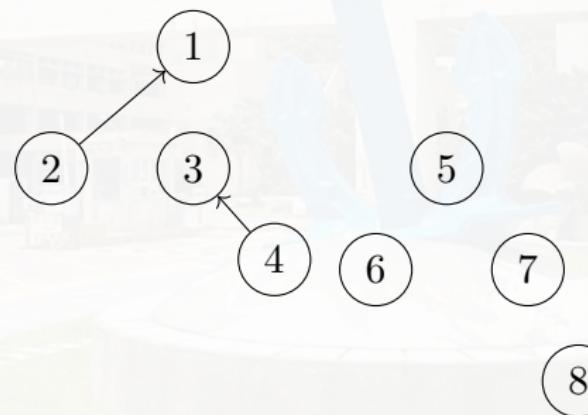
# A Small Example: Unions Build a Forest

- Start with singletons  $\{1\}, \{2\}, \dots, \{8\}$ .
- Perform unions:  $\text{union}(1,2)$ ,  $\text{union}(3,4)$ ,  $\text{union}(5,6)$ ,  
 $\text{union}(7,8)$ ,  $\text{union}(1,3)$ ,  $\text{union}(5,7)$ ,  $\text{union}(1,5)$ .



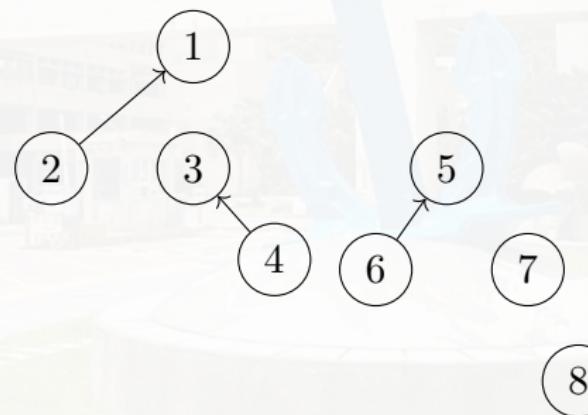
# A Small Example: Unions Build a Forest

- Start with singletons  $\{1\}, \{2\}, \dots, \{8\}$ .
- Perform unions:  $\text{union}(1,2)$ ,  $\text{union}(3,4)$ ,  $\text{union}(5,6)$ ,  
 $\text{union}(7,8)$ ,  $\text{union}(1,3)$ ,  $\text{union}(5,7)$ ,  $\text{union}(1,5)$ .



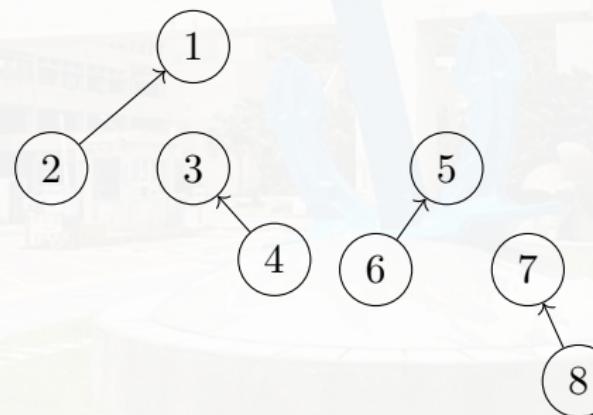
# A Small Example: Unions Build a Forest

- Start with singletons  $\{1\}, \{2\}, \dots, \{8\}$ .
- Perform unions:  $\text{union}(1,2)$ ,  $\text{union}(3,4)$ ,  $\text{union}(5,6)$ ,  
 $\text{union}(7,8)$ ,  $\text{union}(1,3)$ ,  $\text{union}(5,7)$ ,  $\text{union}(1,5)$ .



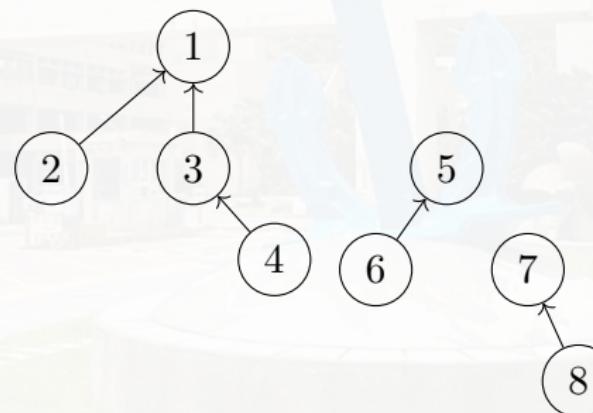
# A Small Example: Unions Build a Forest

- Start with singletons  $\{1\}, \{2\}, \dots, \{8\}$ .
- Perform unions:  $\text{union}(1,2)$ ,  $\text{union}(3,4)$ ,  $\text{union}(5,6)$ ,  
 $\text{union}(7,8)$ ,  $\text{union}(1,3)$ ,  $\text{union}(5,7)$ ,  $\text{union}(1,5)$ .



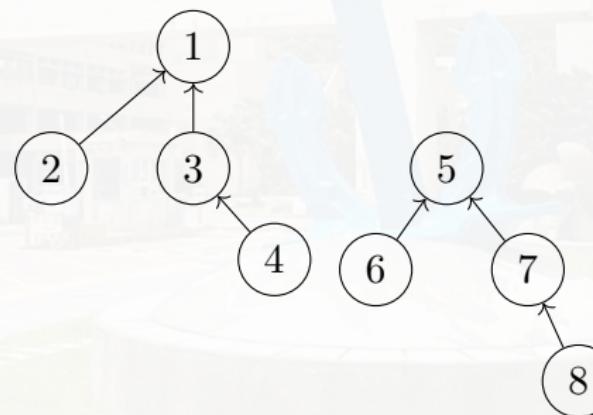
# A Small Example: Unions Build a Forest

- Start with singletons  $\{1\}, \{2\}, \dots, \{8\}$ .
- Perform unions:  $\text{union}(1,2)$ ,  $\text{union}(3,4)$ ,  $\text{union}(5,6)$ ,  
 $\text{union}(7,8)$ ,  $\text{union}(1,3)$ ,  $\text{union}(5,7)$ ,  $\text{union}(1,5)$ .



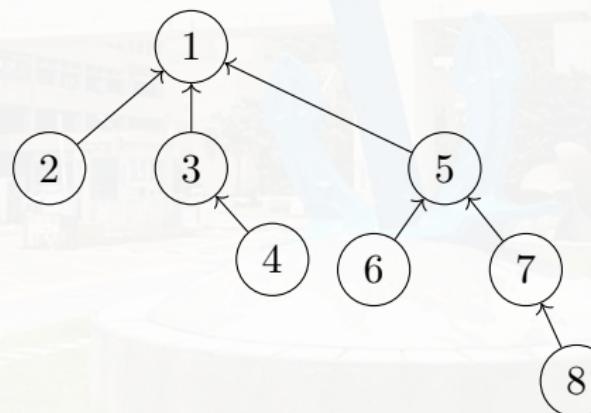
# A Small Example: Unions Build a Forest

- Start with singletons  $\{1\}, \{2\}, \dots, \{8\}$ .
- Perform unions:  $\text{union}(1,2)$ ,  $\text{union}(3,4)$ ,  $\text{union}(5,6)$ ,  
 $\text{union}(7,8)$ ,  $\text{union}(1,3)$ ,  $\text{union}(5,7)$ ,  $\text{union}(1,5)$ .



# A Small Example: Unions Build a Forest

- Start with singletons  $\{1\}, \{2\}, \dots, \{8\}$ .
- Perform unions:  $\text{union}(1,2)$ ,  $\text{union}(3,4)$ ,  $\text{union}(5,6)$ ,  
 $\text{union}(7,8)$ ,  $\text{union}(1,3)$ ,  $\text{union}(5,7)$ ,  $\text{union}(1,5)$ .



# Cost Model

- **union( $u, v$ ):**
  - Usually does two **find** operations, then one pointer update.
  - Time dominated by the two **find** calls.
- **find( $u$ ):**
  - Time proportional to the **length of the path** from  $u$  to the root.

## Goal

Support a sequence of  $m$  operations on  $n$  elements in **near-linear** total time.

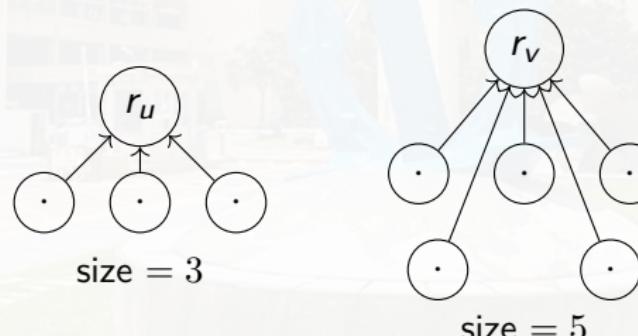


# Outline

- 1 Motivation & Abstract Data Type
- 2 Two Key Heuristics
- 3 Implementation
- 4 Why  $\alpha(n)$ ? A Glimpse of the Analysis

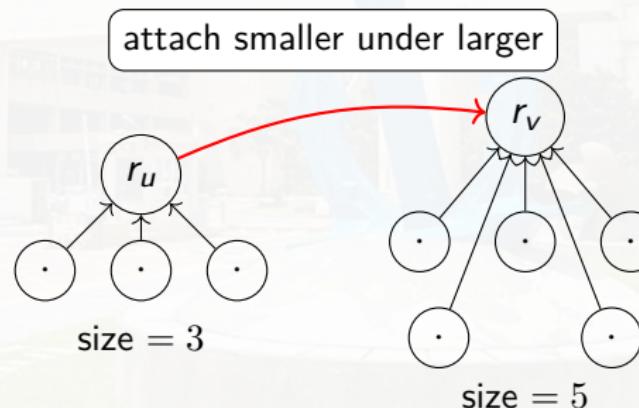
# Heuristic #1: Union by Size (Smaller $\rightarrow$ Larger)

- Maintain  $\text{size}[\text{root}] = \text{number of nodes in that tree}$ .
- On  $\text{union}(u, v)$ :
  - Find roots  $r_u, r_v$ .
  - Make the root of the **smaller** tree point to the root of the **larger** tree.



# Heuristic #1: Union by Size (Smaller $\rightarrow$ Larger)

- Maintain  $\text{size}[\text{root}] = \text{number of nodes in that tree}$ .
- On  $\text{union}(u, v)$ :
  - Find roots  $r_u, r_v$ .
  - Make the root of the **smaller** tree point to the root of the **larger** tree.



## Heuristic #2: Path Compression (During find)

- After finding the root  $r$  for a query node  $x$ :
  - traverse the path again and set every visited node's parent directly to  $r$ .
- This makes **future** find operations faster.



`find(c)` follows 3 edges

**Before**

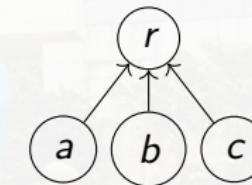
## Heuristic #2: Path Compression (During find)

- After finding the root  $r$  for a query node  $x$ :
  - traverse the path again and set every visited node's parent directly to  $r$ .
- This makes **future** find operations faster.



`find(c)` follows 3 edges

**Before**



path compressed

**After**



# Key Observations (w/ vs. w/o Path Compression)

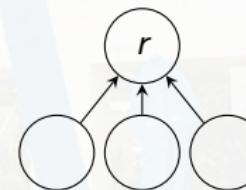
- Consider a fixed sequence  $\sigma$  of  $m$  operations on  $n$  elements.
- With or without path compression:
  - the partition into sets at each time is the **same**,
  - the **roots** (representatives) are the same.
- The difference: path compression can later make a node a **non-descendant** of a former ancestor.

# Example

Time  $t$ : after  $\text{union}(u, v)$  (and  $r$  already has a larger set)



$\text{size}(v) = 2$

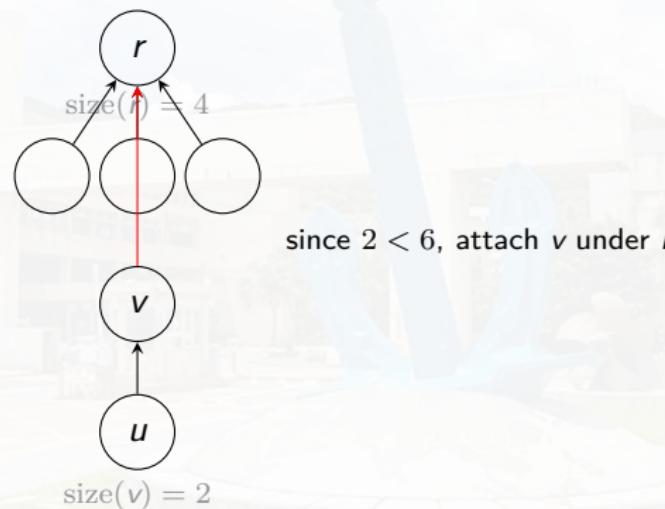


$\text{size}(r) = 6$

Now  $u$  is a descendant of  $v$ .

# Example

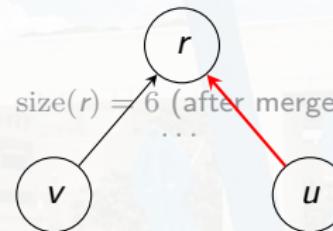
Later: after  $\text{union}(v, r)$  by size (no compression yet)



Path is  $u \rightarrow v \rightarrow r$ , so  $u$  is still a descendant of  $v$ .

# Example

After  $\text{find}(u)$  with path compression



$u$  now bypasses  $v$

Now  $u$  is *not* a descendant of  $v$  (even though it was earlier).

# What We Get from These Two Heuristics

## Performance Guarantee (informal)

A sequence of  $m$  union/find operations takes

$$O((m + n) \alpha(n))$$

time, where  $\alpha(n)$  is the inverse Ackermann function.

# What We Get from These Two Heuristics

## Performance Guarantee (informal)

A sequence of  $m$  union/find operations takes

$$O((m + n) \alpha(n))$$

time, where  $\alpha(n)$  is the **inverse Ackermann function**.

- $\alpha(n)$  grows extremely slowly that it can be effectively viewed as a constant for all realistic inputs.



# What We Get from These Two Heuristics

## Performance Guarantee (informal)

A sequence of  $m$  union/find operations takes

$$O((m + n) \alpha(n))$$

time, where  $\alpha(n)$  is the **inverse Ackermann function**.

- $\alpha(n)$  grows extremely slowly that it can be effectively viewed as a constant for all realistic inputs.
- Why and how does this exotic function appear in the analysis?



# Implementation: Two Arrays

- $\text{parent}[x]$ : parent pointer; roots satisfy  $\text{parent}[r] = r$ .
- $\text{rank}[x]$  (a kind of “weight” of  $x$ ): maintained only for roots.

## Invariants

- Each set is represented by exactly one rooted tree.
- Roots are the canonical representatives.



# C++ style Reference Implementation

```
struct DSU {
    vector<int> parent, rank;
    DSU(int n=0) { init(n); }
    void init(int n){
        parent.resize(n);
        rank.assign(n,1);
        for(int i=0; i<n; i++) parent[i]=i;
    }
    int find(int x) {
        if(parent[x] == x) return x;
        return parent[x] = find(parent[x]); // path compression
    }
    bool union(int a,int b){ // weighted union
        a = find(a); b = find(b);
        if(a == b) return false;
        if(rank[a] < rank[b]) swap(a,b); // union by size
        parent[b] = a;
        rank[a] += rank[b];
        return true;
    }
};
```



# Practical Notes

- Union-by-**rank** is a common variant:
  - maintain an **upper bound on height** instead of exact subtree size,
  - update only when ranks tie.
- **Rule of thumb:** always combine **one** of (union by size/rank) **and** path compression.

# Outline

- 1 Motivation & Abstract Data Type
- 2 Two Key Heuristics
- 3 Implementation
- 4 Why  $\alpha(n)$ ? A Glimpse of the Analysis

# Ackermann-Type Growth

- The analysis uses a hierarchy of rapidly growing functions  $\{A_k\}$ .
- One convenient definition:

$$A_0(x) = x + 1, \quad A_{k+1}(x) = A_k^{(x)}(x),$$

where  $A_k^{(i)} := \underbrace{A_k \circ A_k \cdots \circ A_k}_{i \text{ times}}$  and  $A_k^0$  := identity function.

# Ackermann-Type Growth

- The analysis uses a hierarchy of rapidly growing functions  $\{A_k\}$ .
- One convenient definition:

$$A_0(x) = x + 1, \quad A_{k+1}(x) = A_k^{(x)}(x),$$

where  $A_k^{(i)} := \underbrace{A_k \circ A_k \cdots \circ A_k}_{i \text{ times}}$  and  $A_k^0$  := identity function.

## Intuition

As  $k$  increases,  $A_k(x)$  grows **astronomically** fast (much faster than exponentials/towers).



# Concrete Examples (Small $k$ )

For  $x \geq 2$ , the first few levels behave like:

$$A_0(x) = x + 1,$$

$$A_1(x) = 2x,$$

$$A_2(x) = x \cdot 2^x \quad (\text{roughly exponential}),$$

$$A_3(x) = A_2^x(x) \geq \underbrace{2^{2^{\dots^2}}}_{x} \quad (\text{a tower of } 2\text{s of height } x).$$

$$\bullet \quad A_4(2) \geq \underbrace{2^{2^{2^{\dots^2}}}}_{2048}.$$

- This is why the **inverse** function grows extremely slowly.

# Ackermann Function and Its Inverse

Definitions [Ackermann and the Inverse Ackermann]

$$A(k) := A_k(2), \quad \alpha(n) = \min\{k \mid A(k) \geq n\}.$$

# Ackermann Function and Its Inverse

## Definitions [Ackermann and the Inverse Ackermann]

$$A(k) := A_k(2), \quad \alpha(n) = \min\{k \mid A(k) \geq n\}.$$

- $\alpha(n)$  is so small that in practice it behaves like a constant.
- The theoretical bound for Union-Find becomes **near-linear**.

# High-Level Statement of the Main Result

- Let  $\sigma$  be a sequence of  $m$  union and find operations.

## Theorem (Tarjan [JACM 1975])

Starting from  $n$  singleton sets, any sequence of  $m$  union and find operations takes

$$O((m + n) \alpha(n))$$

time when using **union-by-rank** and **path compression**.

# High-Level Statement of the Main Result

- Let  $\sigma$  be a sequence of  $m$  union and find operations.

## Theorem (Tarjan [JACM 1975])

Starting from  $n$  singleton sets, any sequence of  $m$  union and find operations takes

$$O((m + n) \alpha(n))$$

time when using **union-by-rank** and **path compression**.

- Next: the proof idea uses **ranks** and an **amortized charging** argument.

# Rank of a Node

- Consider executing the same operation sequence **without** path compression.
- Let  $T_m(u)$  be the subtree rooted at  $u$  in the final forest.
  - $T_t(u)$ : the subtree rooted at  $u$  at time  $t$  in the execution of a sequence of  $m$  union and find instructions **without** path compressions.

Definition (Rank): a quantity that survives path compression

$$\text{rank}(u) = 2 + \text{height} (T_m(u)).$$

$\text{height}(T)$ : the length of the longest path from leaves to the root.



# Rank of a Node

- Consider executing the same operation sequence **without** path compression.
- Let  $T_m(u)$  be the subtree rooted at  $u$  in the final forest.
  - $T_t(u)$ : the subtree rooted at  $u$  at time  $t$  in the execution of a sequence of  $m$  union and find instructions **without** path compressions.

Definition (Rank): a quantity that survives path compression

$$\text{rank}(u) = 2 + \text{height} (T_m(u)).$$

height( $T$ ): the length of the longest path from leaves to the root.

- With union-by-rank, ranks are **bounded** and **monotone** along parent pointers.



# A Key Balancing Lemma (Union-by-Rank)

## Lemma (Size vs. Height)

If we always merge the smaller tree into the larger, then for any time  $t$  and root  $u$ ,

$$|T_t(u)| \geq 2^{\text{height}(T_t(u))}.$$

# A Key Balancing Lemma (Union-by-Rank)

## Lemma (Size vs. Height)

If we always merge the smaller tree into the larger, then for any time  $t$  and root  $u$ ,

$$|T_t(u)| \geq 2^{\text{height}(T_t(u))}.$$

- Prove by induction on  $t$ .

# A Key Balancing Lemma (Union-by-Rank)

## Lemma (Size vs. Height)

If we always merge the smaller tree into the larger, then for any time  $t$  and root  $u$ ,

$$|T_t(u)| \geq 2^{\text{height}(T_t(u))}.$$

- Prove by induction on  $t$ .
- Intuition: each time the height increases by 1, the subtree size at least **doubles**.

# A Key Balancing Lemma (Union-by-Rank)

## Lemma (Size vs. Height)

If we always merge the smaller tree into the larger, then for any time  $t$  and root  $u$ ,

$$|T_t(u)| \geq 2^{\text{height}(T_t(u))}.$$

- Prove by induction on  $t$ .
- Intuition: each time the height increases by 1, the subtree size at least **doubles**.
- Consequence: the maximum rank is  $\lfloor \lg n \rfloor + 2 = O(\log n)$ .

# From Ranks to $\alpha(n)$ (Very Rough Sketch)

- Path compression changes parents, so we need to track how **fast** parent ranks can grow.
- Define **levels** using the Ackermann hierarchy:

$$\ell(u) = \max\{k \mid \text{rank}(\text{parent}(u)) \geq A_k(\text{rank}(u))\}.$$

- During path compression,  $\ell(u)$  can only **increase**.

# From Ranks to $\alpha(n)$ (Very Rough Sketch)

- Path compression changes parents, so we need to track how **fast** parent ranks can grow.
- Define **levels** using the Ackermann hierarchy:

$$\ell(u) = \max\{k \mid \text{rank}(\text{parent}(u)) \geq A_k(\text{rank}(u))\}.$$

- During path compression,  $\ell(u)$  can only **increase**.
  - Path compression changes  $\text{parent}(u)$  to a higher ancestor whose rank is larger.

# From Ranks to $\alpha(n)$ (Very Rough Sketch)

- Path compression changes parents, so we need to track how **fast** parent ranks can grow.
- Define **levels** using the Ackermann hierarchy:

$$\ell(u) = \max\{k \mid \text{rank}(\text{parent}(u)) \geq A_k(\text{rank}(u))\}.$$

- During path compression,  $\ell(u)$  can only **increase**.
  - Path compression changes  $\text{parent}(u)$  to a higher ancestor whose rank is larger.

## Charging idea

When a **find** traverses a node, either

- we can **charge** it to the **node** (and  $\ell(u)$  must eventually increase), or
- we **charge** it to the **find** itself (at most  $\alpha(n)$  times).

# From Ranks to $\alpha(n)$ (Very Rough Sketch)

- Path compression changes parents, so we need to track how **fast** parent ranks can grow.
- Define **levels** using the Ackermann hierarchy:

$$\ell(u) = \max\{k \mid \text{rank}(\text{parent}(u)) \geq A_k(\text{rank}(u))\}.$$

- During path compression,  $\ell(u)$  can only **increase**.
  - Path compression changes  $\text{parent}(u)$  to a higher ancestor whose rank is larger.

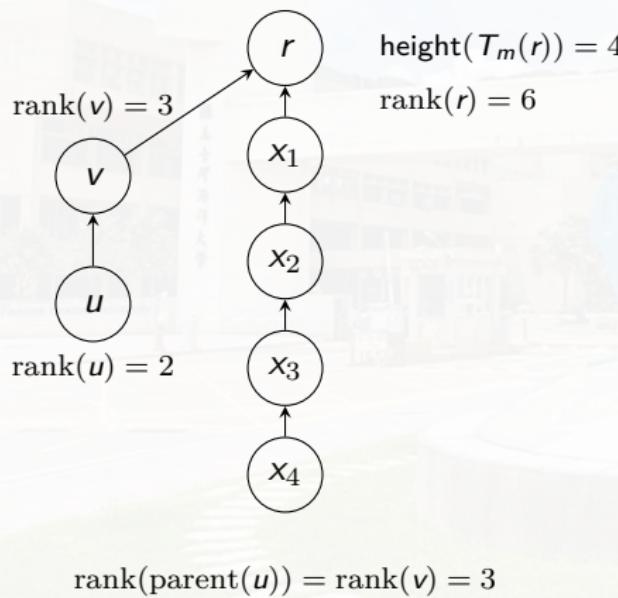
## Charging idea

When a **find** traverses a node, either

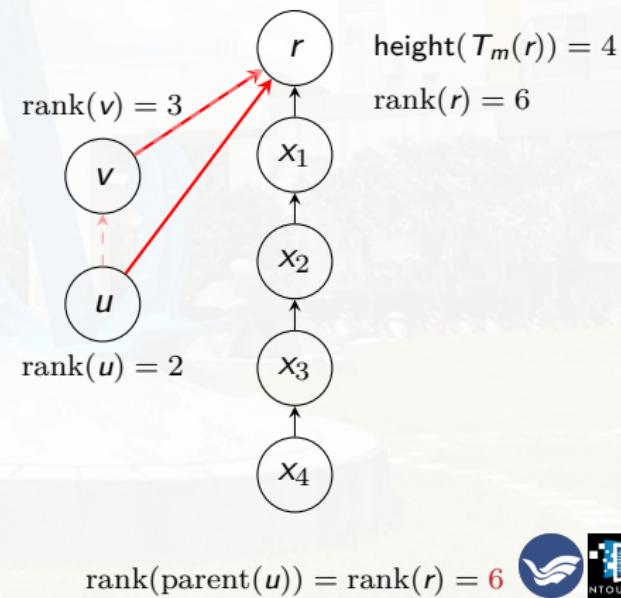
- we can **charge** it to the **node** (and  $\ell(u)$  must eventually increase), or
- we **charge** it to the **find** itself (at most  $\alpha(n)$  times).

# Example

**Before path compression**



**After  $\text{find}(u)$  with compression**



## Remark (connections to the Ackermann)

- If  $u$  has a parent, then

$$\text{rank}(\text{parent}(u)) \geq \text{rank}(u) + 1 = A_0(\text{rank}(u)).$$

- For  $n \geq 5$ , the maximum value  $\ell(u)$  can take on is  $\alpha(n) - 1$ , since if  $\ell(u) = k$ ,

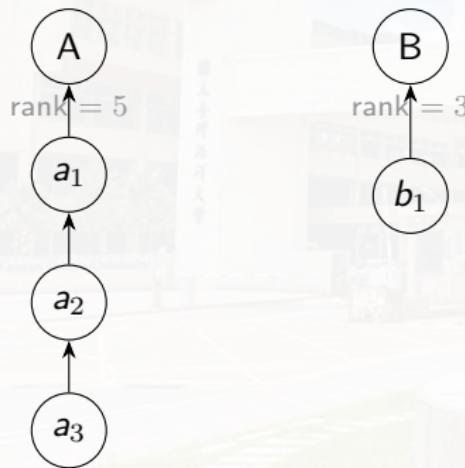
$$\begin{aligned} n &> \lfloor \lg n \rfloor + 2 \\ &\geq \text{rank}(\text{parent}(u)) \\ &\geq A_k(\text{rank}(u)) \\ &\geq A_k(2), \end{aligned}$$

therefore,  $\alpha(n) > k = \ell(u)$ .

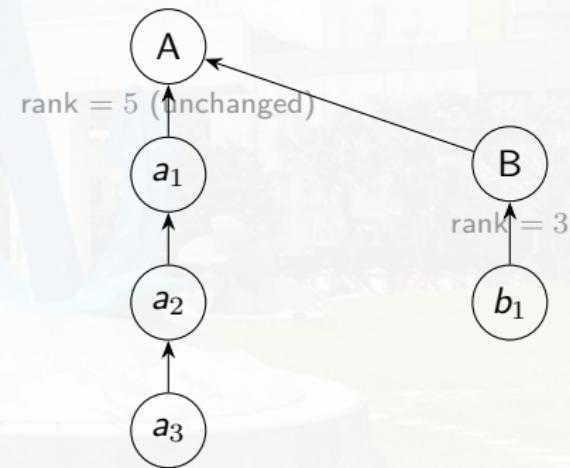
# Note: Effect on the rank after union operation

## Example 1: unequal heights (no rank increase)

Before union



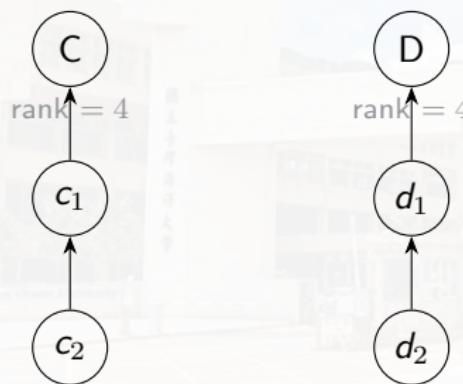
After union



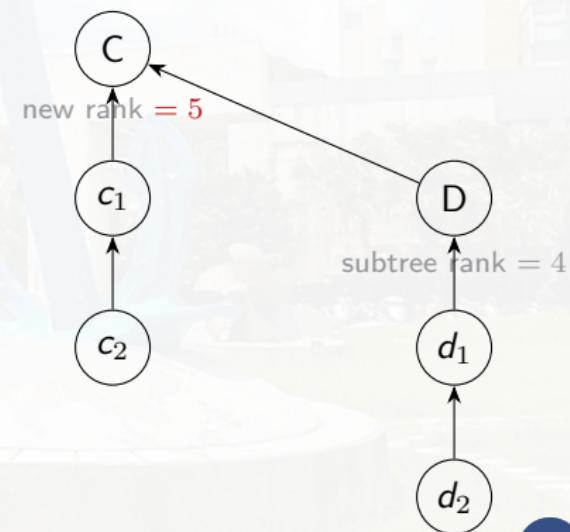
# Note: Effect on the rank after union operation

## Example 2: equal heights (rank increases)

Before union



After union



## Remark: levels $\leftrightarrow$ ranks

- Because path compression changes parents, we track **how large the parent's rank is** compared to the node's rank on an Ackermann scale.

Level

$$\ell(u) = \max \left\{ k \mid \text{rank}(\text{parent}(u)) \geq A_k(\text{rank}(u)) \right\}.$$

- Intuition:**  $\ell(u)$  is the number of “Ackermann jumps” by which the parent rank dominates the child rank.
- Since  $\text{rank}(\text{parent}(u)) \leq O(\log n)$  and  $A_k(\text{rank}(u)) \geq A_k(2)$  explodes fast,

$$\ell(u) \leq \alpha(n) \quad \text{for all } u.$$



# Observation of Monotonicity: Levels only go up

- During a **find**, path compression sets  $\text{parent}(u)$  to an **ancestor** (often the root), whose rank is **at least** the old parent's rank.

# Observation of Monotonicity: Levels only go up

- During a **find**, path compression sets  $\text{parent}(u)$  to an **ancestor** (often the root), whose rank is **at least** the old parent's rank.
- Therefore  $\text{rank}(\text{parent}(u))$  is **nondecreasing over time**.

# Observation of Monotonicity: Levels only go up

- During a `find`, path compression sets  $\text{parent}(u)$  to an **ancestor** (often the root), whose rank is **at least** the old parent's rank.
- Therefore  $\text{rank}(\text{parent}(u))$  is **nondecreasing over time**.
- Since  $\ell(u)$  depends on whether  $\text{rank}(\text{parent}(u))$  crosses thresholds  $A_k(\text{rank}(u))$ , we get:

Level monotonicity

$\ell(u)$  never decreases as operations proceed.

# Observation of Monotonicity: Levels only go up

- During a `find`, path compression sets  $\text{parent}(u)$  to an **ancestor** (often the root), whose rank is **at least** the old parent's rank.
- Therefore  $\text{rank}(\text{parent}(u))$  is **nondecreasing over time**.
- Since  $\ell(u)$  depends on whether  $\text{rank}(\text{parent}(u))$  crosses thresholds  $A_k(\text{rank}(u))$ , we get:

## Level monotonicity

$\ell(u)$  never decreases as operations proceed.

- Consequence: each node's level can increase at most  $\alpha(n)$  times.



# Charging scheme for the cost of find

Consider one  $\text{find}(x)$  that traverses nodes on the path

$x = u_0, u_1, \dots, u_t = \text{root}$ .

Charge each visited internal node  $u_i$  (for  $i < t$ ) in one of two ways

- ① **Node-charge:** if after compression,  $\ell(u_i)$  increases, charge  $O(1)$  to node  $u_i$ .
- ② **Operation-charge:** otherwise, charge  $O(1)$  to this  $\text{find}$ .

Goal:

$$\text{Total time} \leq O\left(\underbrace{\#\text{node-charges}}_{\leq n\alpha(n)} + \underbrace{\#\text{operation-charges}}_{\leq m\alpha(n)}\right).$$



# Bounding node-charges: $\leq n\alpha(n)$

- A node-charge happens only when  $\ell(u)$  increases.
- But  $\ell(u)$  is monotone and bounded:

$$0 \leq \ell(u) \leq \alpha(n).$$

- Hence each node can be node-charged at most  $\alpha(n)$  times.

## Conclusion

#node-charges  $\leq n\alpha(n)$ .



## Bounding operation-charges: why only $\alpha(n)$ per find?

Operation-charges occur at nodes where compression *does not* increase  $\ell(u)$ .

- Fix a node  $u$  on the find path with level  $\ell(u) = k$ .
- “No level increase” means its new parent rank is still  $< A_{k+1}(\text{rank}(u))$ , i.e., it did *not* cross the next Ackermann threshold.
- Along a single find path, ranks are nondecreasing as you move upward.

**Claim:** The number of times a find can encounter nodes that fail to cross the next threshold is bounded by the number of levels, i.e.,  $O(\alpha(n))$ .

Claim

#operation-charges in one find =  $O(\alpha(n))$ .



# On the number of charges to the find operation

To formalize the previous slide, analyses often add an **index** within a level.

Index within level  $k = \ell(u)$

Let  $A_k^{(j)}$  be the  $j$ -fold iterate of  $A_k$ . Define

$$i(u) := \max \left\{ j \mid \text{rank}(\text{parent}(u)) \geq A_k^{(j)}(\text{rank}(u)) \right\}.$$

- **Note:** larger  $i(u)$  means the parent's rank is not just above  $A_k$  but above many repeated applications of  $A_k$ .
- If  $\ell(u)$  does not increase, path compression still tends to increase  $\text{rank}(\text{parent}(u))$ , so  **$i(u)$  increases**.

# Wait! Why do we iterate $A_k$ ?

- By definition of  $\ell$ , at time  $t$  we have the level- $k$  threshold:

$$\text{rank}(\text{parent}(x)) \geq A_k(\text{rank}(x)).$$

# Wait! Why do we iterate $A_k$ ?

- By definition of  $\ell$ , at time  $t$  we have the level- $k$  threshold:

$$\text{rank}(\text{parent}(x)) \geq A_k(\text{rank}(x)).$$

- Path compression changes parent pointers (e.g.,  $x \rightarrow v$ ,  $v = \text{parent}(x)$ ), so

$$\text{rank}(\text{parent}(x))$$

may *increase over time* even if  $\ell(x)$  (the level) stays the same.

# Wait! Why do we iterate $A_k$ ?

- By definition of  $\ell$ , at time  $t$  we have the level- $k$  threshold:

$$\text{rank}(\text{parent}(x)) \geq A_k(\text{rank}(x)).$$

- Path compression changes parent pointers (e.g.,  $x \rightarrow v$ ,  $v = \text{parent}(x)$ ), so

$$\text{rank}(\text{parent}(x))$$

may *increase over time* even if  $\ell(x)$  (the level) stays the same.

- Therefore, we need a *finer progress measure within level k*.

# Index within a fixed level: Using iterates $A_k^i$

- Consider an *iterate index*  $i \geq 1$ :  $\text{rank}(\text{parent}(x)) \geq A_k^i(\text{rank}(x))$ , where  $A_k^i$  denotes the  $i$ -fold iterate of  $A_k$ .
- It is an *analytic ruler* that counts how far  $\text{rank}(\text{parent}(x))$  has advanced *within the same level*  $k$ .
- If a charge to  $x$  occurs at time  $t$ , then after compression (time  $t+1$ ) the new parent becomes a later vertex  $v$  on the find path, and

$$\text{rank}(v) \geq A_k(\text{rank}(\text{parent}(x))) \geq A_k(A_k^i(\text{rank}(x))) = A_k^{i+1}(\text{rank}(x)).$$

- Since  $v$  is the new parent of  $x$ , this implies

$$\text{rank}(\text{parent}(x)) \geq A_k^{i+1}(\text{rank}(x)).$$

- Thus, each such event increases the *index i* by at least 1.

$i$  is bounded before the level must increase

- While  $\ell(x) = k$  remains fixed, the index  $i$  can increase only finitely many times.
- After at most  $\text{rank}(x)$  such increments, the notes obtain

$$\text{rank}(\text{parent}(x)) \geq A_k^{\text{rank}(x)}(\text{rank}(x)) = A_{k+1}(\text{rank}(x)).$$

# $i$ is bounded before the level must increase

- While  $\ell(x) = k$  remains fixed, the index  $i$  can increase only finitely many times.
- After at most  $\text{rank}(x)$  such increments, the notes obtain

$$\text{rank}(\text{parent}(x)) \geq A_k^{\text{rank}(x)}(\text{rank}(x)) = A_{k+1}(\text{rank}(x)).$$

- Crossing the next threshold forces the level to increase:

$$\ell(x) \geq k + 1 \quad (\text{equivalently, } \ell(x) \text{ must increase}).$$

# $i$ is bounded before the level must increase

- While  $\ell(x) = k$  remains fixed, the index  $i$  can increase only finitely many times.
- After at most  $\text{rank}(x)$  such increments, the notes obtain

$$\text{rank}(\text{parent}(x)) \geq A_k^{\text{rank}(x)}(\text{rank}(x)) = A_{k+1}(\text{rank}(x)).$$

- Crossing the next threshold forces the level to increase:  
$$\ell(x) \geq k + 1 \quad (\text{equivalently, } \ell(x) \text{ must increase}).$$
- Therefore, at most  $\text{rank}(x)\alpha(n)$  such charges against  $x$ .

# On the number of nodes of the same rank

## Lemma

For any integer  $r$ ,

$$|\{u \mid \text{rank}(u) = r\}| \leq \frac{n}{2^{r-2}}.$$

# On the number of nodes of the same rank

## Lemma

For any integer  $r$ ,

$$|\{u \mid \text{rank}(u) = r\}| \leq \frac{n}{2^{r-2}}.$$

## Proof sketch

- If  $\text{rank}(u) = \text{rank}(v) = r$ , then the subtrees  $T_m(u)$  and  $T_m(v)$  are disjoint.
- Hence, we have the union of these subtrees has size  
$$\left| \bigcup_{\text{rank}(u)=r} T_m(u) \right| = \sum_{\text{rank}(u)=r} |T_m(u)| \leq n.$$
- Also, we have known that every node of rank  $r$  satisfies  $|T_m(u)| \geq 2^{r-2}$ , so

$$\sum_{\text{rank}(u)=r} |T_m(u)| \geq \sum_{\text{rank}(u)=r} 2^{r-2} = |\{u : \text{rank}(u) = r\}| \cdot 2^{r-2}.$$

- Rearranging yields  $|\{u : \text{rank}(u) = r\}| \leq \frac{n}{2^{r-2}}.$

# Sum up the charges against nodes

- At most  $\text{rank}(x)\alpha(n)$  charges against a node  $x$ . So, there are at most

$$r\alpha(n) \frac{n}{2^{r-2}} = n\alpha(n) \frac{r}{2^{r-2}}$$

charges against nodes of rank  $r$ .

- Summing over all values of  $r$  we obtain the following bound on all charges to all vertices

$$\sum_{r=0}^{\infty} n\alpha(n) \frac{r}{2^{r-2}} = n\alpha(n) \cdot \sum_{r=0}^{\infty} \frac{r}{2^{r-2}} = 8n\alpha(n).$$

# Putting it together: near-linear time

- Node-charges: at most  $n\alpha(n)$  total.
- Operation-charges: at most  $m\alpha(n)$  total (each find has  $O(\alpha(n))$ ).

## Conclusion (Tarjan)

Starting from  $n$  singletons, any sequence of  $m$  union/find operations with union-by-rank (or size) plus path compression runs in

$$O((m+n)\alpha(n)) \text{ time.}$$

- Practically,  $\alpha(n) \leq 4$  for any realistic  $n$ , so it behaves like “almost constant amortized time.”

# Conclusion of the Amortized Analysis

- Total charge to find operations:  $O(m\alpha(n))$ .
- Total charge to nodes over the entire computation:  $O(n\alpha(n))$ .

Therefore

Total time for  $m$  operations is  $O((m + n)\alpha(n))$ .

- This is why Union-Find is considered **almost linear time**.

