Trees

Trees, Binary Trees & Representations

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Outline

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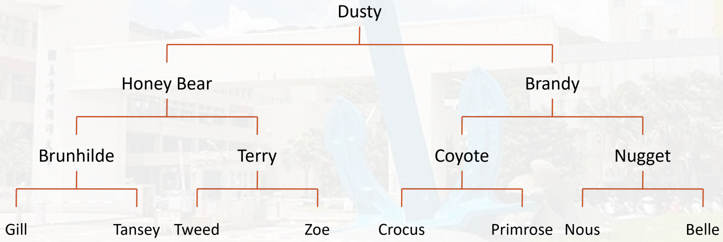
Introduction

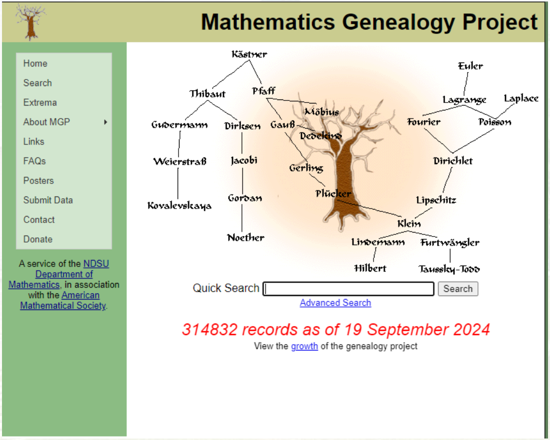
- Intuitively, a tree structure organized data in a hierarchical manner.

Example: Pedigree Chart



Example: Mathematical Genealogy Project

Figure reference: https://www.mathgenealogy.org/



Definitions

Tree

- A tree is a finite set of one or more nodes such that:
  - There is a specially designated node called root.
  - The remaining nodes are partitioned into $n \geq 0$ disjoint sets, $T_1, \ldots, T_n$, where each of these sets is a tree.
  - $T_1, \ldots, T_n$: subtrees of the root.

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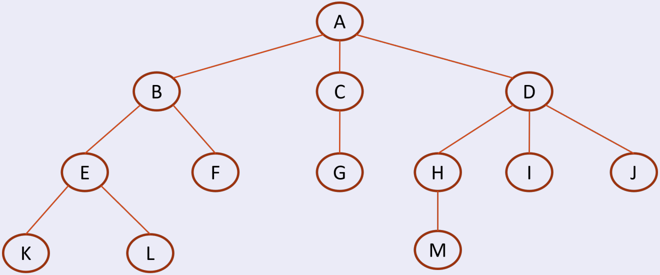
  - $T_1, \ldots, T_n$: subtrees of the root.
    - The root of a tree connects to the roots of its subtrees.

Definitions

Node

- A node stands for the item of information plus the branches to other nodes.



Definitions

Degree

- The number of subtrees of a node is called its degree.

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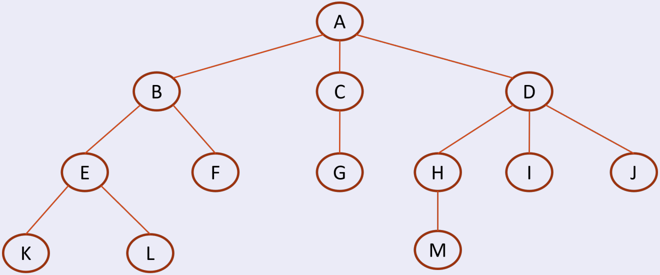
  - $\deg(A) = 3$, $\deg(C) = 1$, $\deg(F) = 0$.



Definitions

Leaf, children, parent

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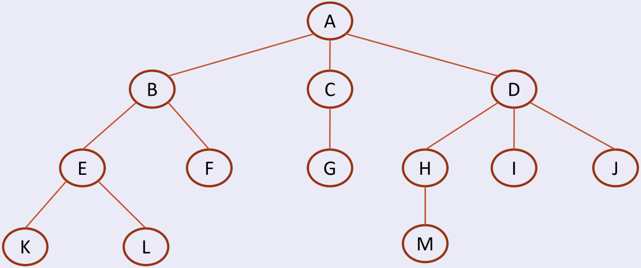
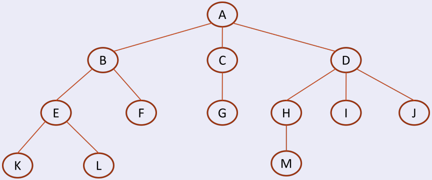
- A node that has degree 0 is called a leaf or terminal.
- The roots of the subtrees of a node X are the children of X. X is the parent of its children.



Definition

Siblings, degree, ancestors

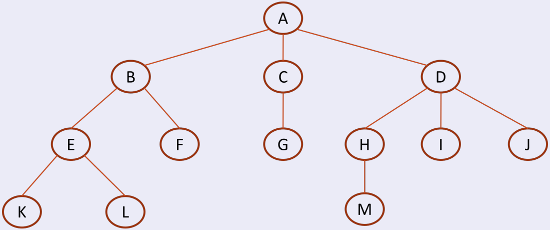
- Children of the same parent are said to be siblings.
  - Example: H, I and J are siblings; B, C and D are siblings.
- The degree of a tree is the **maximum** of the degree of the nodes in the tree.
  - The tree in this example has degree 3.
- The ancestors of a node are all the nodes along the path from the root to that node.
  - The ancestors of M are A, D, and H.



Definition

Level, height or depth

- The level of a node:
  - the root: 1.
  - if a node is at level k, then its children are at level $k+1$.
  - Example: $\text{level}(A) = 1$, $\text{level}(H) = 3$, $\text{level}(L) = 4$.
- The height or depth of a tree is defined to be the maximum level of any node in the tree.
  - The depth of the tree in this example is 4.

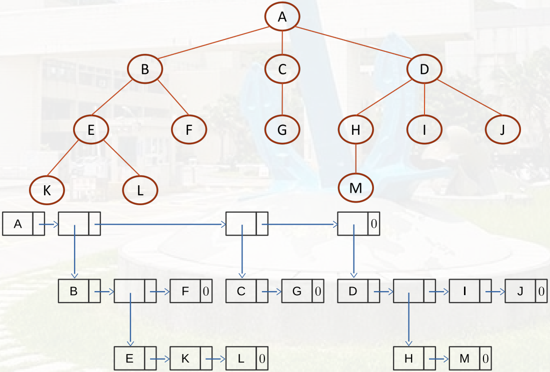


Representation of Trees

- The tree in the example can be written as

$$(A(B(E(K, L), F), C(G), D(H(M), I, J))).$$

- **Rule:** root node $\rightarrow$ list of its subtrees.



A Possible Node Structure of a Tree of Degree k

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data	child 1	child 2	$\cdots$	child k

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- However, one only uses nodes of a fixed size to represent tree nodes in practice.

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- Then, how to choose such a fixed size?

Waste of Space

Lemma 5.1

If T is a k-ary tree (i.e., a tree of degree k) with n nodes ($n \geq 1$), each having a fixed size, then $n(k-1)+1$ of the nk child fields are 0.

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Proof

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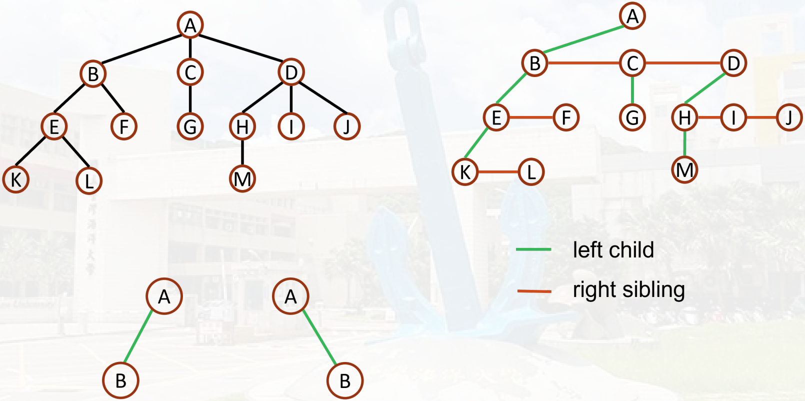
  - The total number of child fields in a k-ary tree with n nodes is nk.
  - Thus, the number of zero fields is $nk-(n-1)=n(k-1)+1$.

Left Child-Right Sibling Representation

- Every node has ≤ 1 leftmost child and ≤ 1 closest right sibling.
- The **left child field** of each node points to its leftmost child (if any)
- The **right sibling field** points to its closest right sibling (if any).

data	
left child	right sibling


A
B
C
D
E
F
G
H
I
J
K
L
M
left child
right sibling

Outline

Binary Trees

Binary Trees

A binary tree is a finite set of nodes that

- consists of a root
- two disjoint binary trees: the left subtree and the right subtree.

Trees vs. Binary Trees

Notice

In a binary tree we distinguish between the order of the children while in a tree we do not.

- The following two binary trees are different.
  - the first binary tree has an empty right subtree
  - the second has an empty left subtree.

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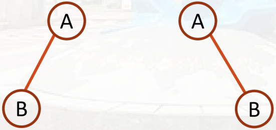
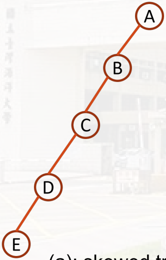
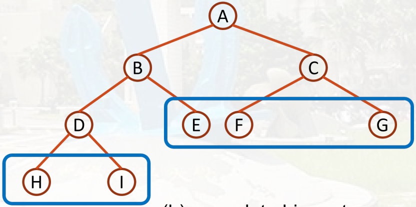


Skew Binary Trees & Complete Binary Trees

- skew: only left (or right) subtrees for each node
- complete: all leaf nodes of these trees are on two adjacent levels.



(a): skewed tree



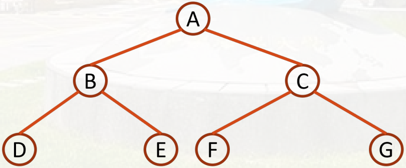
(b): complete binary tree

Properties of Binary Trees

Lemma 5.2 [Maximum Number of Nodes]

- The maximum number of nodes on level i of a binary tree is 2^{i-1}, for $i \geq 1$.
- The maximum number of nodes in a binary tree of depth k is $2^k - 1$, for $k \geq 1$.

- On level 2: 2 nodes; on level 3: 4 nodes.
- Totally $2^3 - 1 = 7$ nodes in the binary tree.



Proof of Lemma 5.2

- Induction Base:
  - The root is the only node on level 1. $2^{1-1} = 2^0 = 1$.
- Induction Hypothesis: Assume that the maximum number of nodes on level $i - 1$ is 2^{i-2}.
- Induction Step:
  - The maximum number of nodes on level $i - 1$ is 2^{i-2} by the induction hypothesis.
  - Since each node in a binary tree has a maximum degree of 2, the maximum number of nodes on level i is $2^{i-2} \cdot 2 = 2^{i-1}$.

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  - Since each node in a binary tree has a maximum degree of 2, the maximum number of nodes on level i is $2^{i-2} \cdot 2 = 2^{i-1}$.
- The maximum number of nodes in a binary tree of depth k is

$$1 + 2 + 2^2 + \cdots + 2^{k-1} = \sum_{i=1}^{k-1} 2^{i-1} = 2^k - 1.$$

Full Binary Tree

Full Binary Tree

A full binary tree of depth k is a binary tree of depth k having $2^k - 1$ nodes, for $k \geq 0$.

Remark

A binary tree with n nodes and depth k is complete iff its nodes correspond to the nodes numbered from 1 to n in the full binary tree of depth k.

- From Lemma 5.2, we know that

The height of a complete binary tree with n nodes is $\lceil \lg(n+1) \rceil$.

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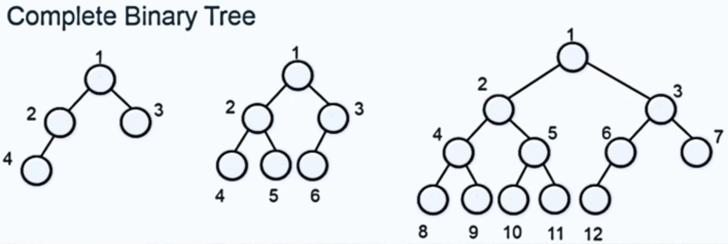
⋆ **Note:** A complete binary tree is NOT necessarily a full binary tree!

- $\lg n := \log_2 n$.

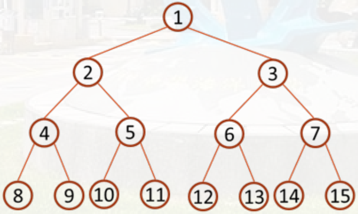

Complete Binary Tree
1
2
3
4
1
2
3
4
5
6
1
2
3
4
5
6
7
8
9
10
11
12

Array Representation for a Binary Tree

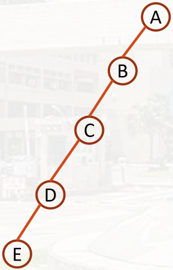
Lemma 5.4

If a complete binary tree with n nodes is represented sequentially, then for any node with index i, $1 \leq i \leq n$, we have

- parent(i) is at $\lfloor i/2 \rfloor$ if $i \neq 1$. If $i = 1$, i is at root so it has no parent.
- leftChild(i) is at $2i$ if $2i \leq n$. If $2i > n$, then i has no left child.
- rightChild(i) is at $2i + 1$ if $2i + 1 \leq n$. If $2i + 1 > n$, then i has no right child.



Binary Tree Representation: Examples



	tree
[0]	
[1]	A
[2]	B
[3]	
[4]	C
[5]	
[6]	
[7]	
[8]	D
[9]	
...	
[16]	E

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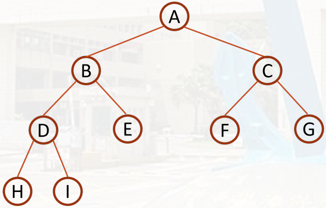


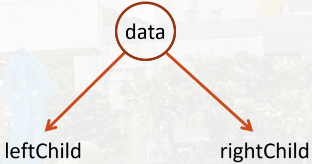
	tree
[0]	
[1]	A
[2]	B
[3]	C
[4]	D
[5]	E
[6]	F
[7]	G
[8]	H
[9]	I

Drawbacks of the Array Representation

- Waste memory space for most binary trees.
- In the worst case, a skewed tree of depth k requires $2^k - 1$ spaces.
  - Only k spaces is occupied.
- Insertion or deletion of nodes from the middle of a tree requires the movement of potentially many nodes.

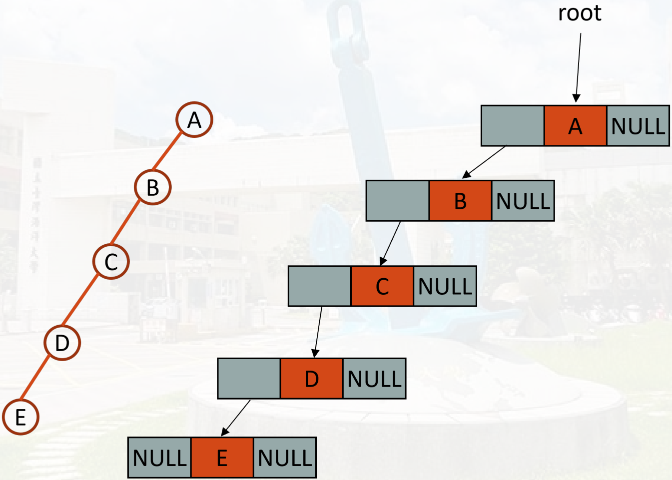
Try Linked List Representation

```
typedef struct node *treePointer;
typedef struct node {
    int data;
    treePointer leftChild, rightChild;
};
```



leftChild	data	rightChild

Example



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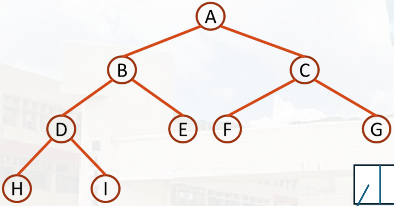

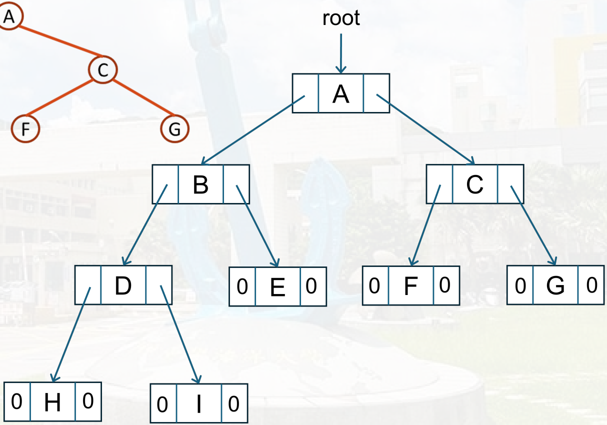

Discussions