

Simple Near-Optimal Auctions

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Outline

- 1 Background & Introduction
- 2 The Prophet Inequality
- 3 Simple Single-Item Auctions
- 4 Prior-Independent Mechanisms



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What we have learned

- For a single-parameter environment where agents' valuations are drawn independently from **regular** distributions, the corresponding **virtual welfare maximizer** maximizes the **expected revenue** over all **DSIC** mechanisms.
 - The allocation rule:

$$\mathbf{x}(\mathbf{v}) = \arg \max_{\mathbf{x}} \sum_{i=1}^n \varphi_i(v_i) x_v(\mathbf{v})$$

for each valuation profile \mathbf{v} , where

$$\varphi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$



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- For i.i.d. & regular F_i 's, the optimal single-item auction is surprisingly simple:
 - a **second-price auction augmented with the reserve price $\varphi^{-1}(0)$**



Optimal Auctions Can Be Complex

- What if bidders' valuations are drawn from **different** regular distributions?



Optimal Auctions Can Be Complex

- What if bidders' valuations are drawn from **different** regular distributions?
- We would like to know if there is any simple and practical single-item auction formats that are at least approximately optimal.



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Game with n stages (resembling the secretary problem)

- Consider the following game with n stages.
- In stage i , we are offered a nonnegative prize π_i , drawn from a distribution G_i .
- We know the *independent* distributions G_1, \dots, G_n in advance.
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 - either accept the prize and end the game, or
 - discard the prize, and then proceed to the next stage.



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 - discard the prize, and then proceed to the next stage.
- What's the risk and difficulty?



The Prophet Inequality

- It offers a simple strategy that performs almost as well as (approximately) a fully clairvoyant prophet.

Theorem (Prophet Inequality)

For every sequence G_1, \dots, G_n of n independent distributions,

- There is a strategy that guarantees expected reward $\geq \frac{1}{2} \mathbf{E}_{\pi \sim \mathbf{G}}[\max_i \pi_i]$.
 - There is such a threshold strategy, which accepts prize i if and only if $\pi_i \geq t$.
-
- $z^+ := \max\{z, 0\}$.
 - $[n] := \{1, 2, \dots, n\}$.



Proof of Prophet Inequality (1/3)

- Compare the expected payoff of a threshold strategy with that of a prophet, through **lower and upper bounds**.
- $q(t)$: the prob. that the threshold strategy accepts **no prize at all**.
- First, we want to have a **lower bound** on

$$\psi := \mathbf{E}_{\pi \sim \mathbf{G}}[\text{payoff of the } t\text{-threshold strategy}].$$



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- The payoff: **0** with prob. $q(t)$ and $\geq t$ with prob. $1 - q(t)$.
- Credit the baseline t with “extra credit” of $\pi_i - t$ if only one prize satisfies $\pi_i \geq t$.
- Only credit the baseline t for two or more prizes $\geq t$ (\because LB).



Proof of Prophet Inequality (2/3)

$$\begin{aligned}\psi &\geq (1 - q(t))t + \\ &\quad \sum_{i=1}^n \mathbf{E}_{\pi}[\pi_i - t \mid \pi_i \geq t, \pi_j < t \forall j \neq i] \cdot \Pr[\pi_i \geq t] \cdot \Pr[\pi_j < t \forall j \neq i] \\ &= (1 - q(t))t + \sum_{i=1}^n \underbrace{\mathbf{E}_{\pi}[\pi_i - t \mid \pi_i \geq t] \cdot \Pr[\pi_i \geq t]}_{= \mathbf{E}_{\pi}[(\pi_i - t)^+]} \cdot \underbrace{\Pr[\pi_j < t \forall j \neq i]}_{\geq q(t) = \Pr[\pi_j < t \forall j]} \\ &\geq (1 - q(t))t + q(t) \cdot \sum_{i=1}^n \mathbf{E}_{\pi}[(\pi_i - t)^+]\end{aligned}$$



Proof of Prophet Inequality (3/3)

Moreover, as to the **upper bound** on the **prophet's** expected payoff:

$$\begin{aligned}
 \psi^* &:= \mathbf{E}_{\pi} \left[\max_{i \in [n]} \pi_i \right] = \mathbf{E}_{\pi} \left[t + \max_{i \in [n]} (\pi_i - t) \right] \\
 &\leq t + \mathbf{E}_{\pi} \left[\max_{i \in [n]} (\pi_i - t)^+ \right] \\
 &\leq t + \sum_{i=1}^n \mathbf{E}_{\pi} [(\pi_i - t)^+].
 \end{aligned}$$

- Set t such that $q(t) = \frac{1}{2}$ we can complete the proof.

$$\frac{t}{2} + \frac{1}{2} \cdot \sum_{i=1}^n \mathbf{E}_{\pi} [(\pi_i - t)^+] \leq \psi \leq \psi^* \leq t + \sum_{i=1}^n \mathbf{E}_{\pi} [(\pi_i - t)^+]$$



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- Set t such that $q(t) = \frac{1}{2}$ we can complete the proof.
- $\text{LB} := \frac{t}{2} + \frac{1}{2} \cdot \sum_{i=1}^n \mathbf{E}_{\pi} [(\pi_i - t)^+] \leq \psi \leq \psi^* \leq t + \sum_{i=1}^n \mathbf{E}_{\pi} [(\pi_i - t)^+] = 2 \cdot \text{LB}.$



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Back to the motivating example

- Single-item auction with n bidders.
- Bidders' valuations are drawn independently from regular distributions F_1, \dots, F_n that might not be identical.
- Using the prophet inequality:
 - Define the i th prize as $\varphi_i(v_i)^+$ of bidder i .
 - G_i : the corresponding distribution induced by F_i (independent).
- We have (by Theorem 5.2; with maximizer $\mathbf{x} = (x_i)_{i \in [n]}$)

$$\mathbf{E}_{\mathbf{v} \sim F} \left[\sum_{i=1}^n \varphi_i(v_i) x_i(\mathbf{v}) \right] = \mathbf{E}_{\mathbf{v} \sim F} \left[\max_{i \in [n]} \varphi_i(v_i)^+ \right].$$

- The expected revenue of the optimal auction.



The allocation rule

Consider any allocation rule having the following form:

Virtual Threshold Allocation Rule

- Choose t such that $\Pr[\max_i \varphi_i(v_i)^+ \geq t] = \frac{1}{2}$.
- Give the item to a bidder i with $\varphi_i(v_i) \geq t$, if any, breaking ties among multiple candidate winners arbitrarily.

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We immediately have the following corollary:

¹What if no such t exists? An exercise!

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Corollary (Virtual Threshold Rules are Near-Optimal)

If \mathbf{x} is a virtual threshold allocation rule, then

$$\mathbf{E}_{\mathbf{v}} \left[\sum_{i=1}^n \varphi_i(v_i)^+ x_i(\mathbf{v}) \right] \geq \frac{1}{2} \mathbf{E}_{\mathbf{v}} \left[\max_{i \in [n]} \varphi_i(v_i^+) \right].$$

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A different critique so far

- So far, the valuation distributions are assumed to be **known to the mechanism designer in advance**.
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- Next, we consider that
 - Bidders' valuations are still drawn from such valuation distributions;
 - Yet, these distributions are still unknown to the mechanism designer.



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- So far, the valuation distributions are assumed to be **known to the mechanism designer in advance**.
- What if the mechanism designer does **NOT** know about the valuation distributions in advance?
- Next, we consider that
 - Bidders' valuations are still drawn from such valuation distributions;
 - Yet, these distributions are still unknown to the mechanism designer.
 - ★ We use the distributions in the *analysis*, but **NOT** in the design of mechanisms.
- Goal: design a good **prior-independent** mechanism.
 - Such a mechanism makes NO reference to a valuation distribution.



A Beautiful Result from Auction Theory

- The expected revenue of an optimal single-item auction is at most that of a second-price auction (with no reserved price) with **one extra** bidder.

Theorem [Bulow-Klemperer Theorem (1989)]

We have

- F : a regular distribution;
- n : a positive integer.
- \mathbf{p} : the payment rule of the second-price auction with $n + 1$ bidders.
- \mathbf{p}^* : the payment rule of the optimal auction for F with n bidders.

Then,

$$\mathbf{E}_{\mathbf{v} \sim F^{n+1}} \left[\sum_{i=1}^{n+1} p_i(\mathbf{v}) \right] \geq \mathbf{E}_{\mathbf{v} \sim F^n} \left[\sum_{i=1}^n p_i^*(\mathbf{v}) \right]$$

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- \mathbf{p}^* : the payment rule of the **second-price auction (optimal)** with **reserve price $\varphi^{-1}(0)$** .

Then,

$$\mathbf{E}_{\mathbf{v} \sim F^{n+1}} \left[\sum_{i=1}^{n+1} p_i(\mathbf{v}) \right] \geq \mathbf{E}_{\mathbf{v} \sim F^n} \left[\sum_{i=1}^n p_i^*(\mathbf{v}) \right]$$

Interpretation of the Bulow-Klemperer Theorem

- Extra competition is more important than getting the auction format just right.
- It is better to invest your resources to recruit more serious participants than sharpen your knowledge of their preferences.



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The Fictitious Auction \mathcal{A}

- 1 Simulate an optimal n -bidder auction for F on the first n bidders.
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 - 2 If the item was not awarded in the first step, then give the item to the $(n + 1)$ th bidder for free.
- The expected revenue of \mathcal{A} equals that of an optimal auction with n bidders.
 - The right-hand side of the inequality.



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- Consider a stronger statement:

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- Consider a stronger statement:

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The second-price auction maximizes the expected revenue over all DSIC auctions that **always allocate the item**.



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- The allocation rule with maximum possible expected virtual welfare *subject to always allocating the item* always awards the item to a bidder with the **highest virtual valuation** (even it is negative).



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- All bidders share the same nondecreasing virtual valuation function φ .
 - A bidder with highest valuation also has the highest virtual valuation.
- Hence, the second-price auction maximizes expected revenue subject to always awarding the item.

