

Linked List

Equivalence Relations, Sparse Matrices & Doubly Linked Lists

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Outline

- 1 Equivalence Relations
- 2 Sparse Matrices Revisted
- 3 Doubly Linked Lists

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1 Equivalence Relations

2 Sparse Matrices Revisted

3 Doubly Linked Lists

Equivalence Relation

A relation over a set S is said to be an equivalence relation over S iff it is symmetric, reflexive, and transitive over S .

- **reflexive:** $x \equiv x$ for each $x \in S$.
- **symmetric:** for $x, y \in S$, if $x \equiv y$, then $y \equiv x$.
- **transitive:** for x, y, z , if $x \equiv y$ and $y \equiv z$, then $x \equiv z$.

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Let's see several examples...

- Fix an integer $n > 0$. Two integers $x, y \in \mathbb{Z}$, $x \sim y$ iff $x \equiv y \pmod{n}$.

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- Given matrices $A, B \in \mathbb{R}^{n \times n}$, $A \sim B$ iff there exists an invertible $P \in \mathbb{R}^{n \times n}$ such that $B = P^{-1}AP$.



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- Given matrices $A, B \in \mathbb{R}^{n \times n}$, $A \sim B$ iff there exists an invertible $P \in \mathbb{R}^{n \times n}$ such that $B = P^{-1}AP$.
- Given an undirected graph. For any two vertices x, y in the graph, $x \sim y \iff$ Two vertices x and y lie in the same connected component.



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Example

Given $0 \equiv 4, 3 \equiv 1, 6 \equiv 10, 8 \equiv 9, 7 \equiv 4, 6 \equiv 8, 3 \equiv 5, 2 \equiv 11, 11 \equiv 0$.

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Example

Given $0 \equiv 4, 3 \equiv 1, 6 \equiv 10, 8 \equiv 9, 7 \equiv 4, 6 \equiv 8, 3 \equiv 5, 2 \equiv 11, 11 \equiv 0$.

We have three equivalent classes:

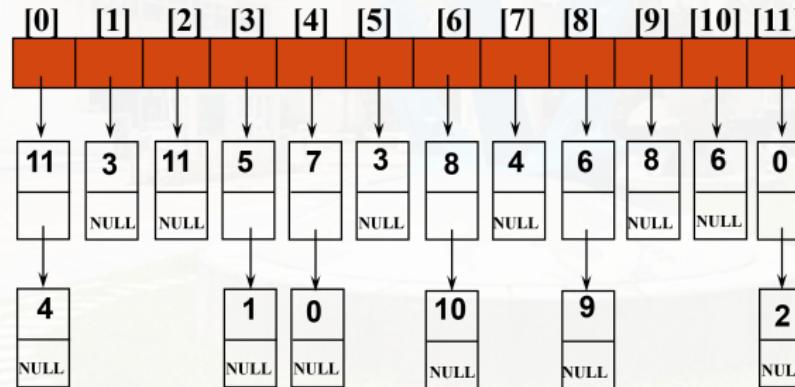
$$\{0, 2, 4, 7, 11\}, \{1, 3, 5\}, \{6, 8, 9, 10\}.$$



Lists after Giving Pairs as the Input

$0 \equiv 4, 3 \equiv 1, 6 \equiv 10, 8 \equiv 9, 7 \equiv 4,$
 $6 \equiv 8, 3 \equiv 5, 2 \equiv 11, 11 \equiv 0.$

```
typedef struct node *nodePointer;
typedef struct node {
    int data;
    nodePointer link;
};
```



The Equivalence Algorithm (Full Code [HERE](#))

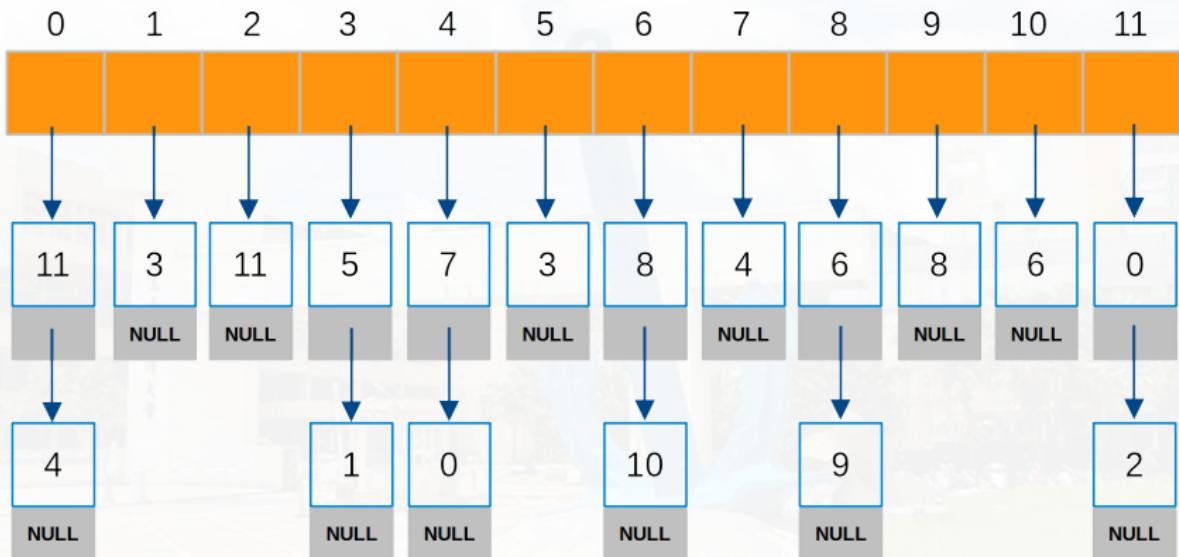
`seq[n]`: hold the head nodes of the n lists; `out[n]`: whether the object i has been printed.

```
void equivalence() {
    initialize seq[i] = NULL, out[i] to true for each i
    while (there are more pairs) {
        read the next pair <i,j>;
        put j on the seq[i] list;
        put i on the seq[j] list;
    }
    for (i=0; i<n; i++) { // the 2nd phase
        if (out[i]) {
            out[i] = false;
            output this equivalence class;
        }
    }
}
```

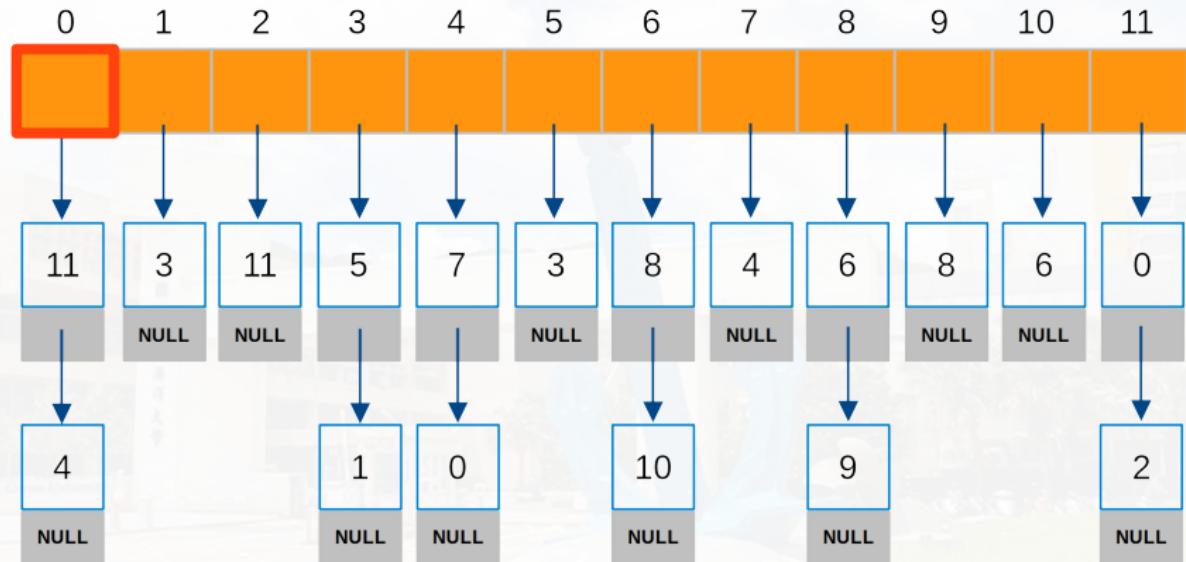


The Equivalence Algorithm (second phase in detail)

```
for (i=0; i<n; i++) { // the 2nd phase
    if (out[i]) {
        printf("\nNew Classes: %d", i);
        out[i] = false; // class involving i must be processed
        x = seq[i]; top = NULL; // initial stack
        while (1) { // find the rest of the class
            while (x) { // process the list
                j = x->data;
                if (out[j]) {
                    printf("%d", j); out[j] = false;
                    // then, push y into the stack
                    y = x->link; x->link = top; top = x; x = y;
                } else x = x->link;
            }
            if (!top) break;
            // else, the stack keeping equivalent class members is nonempty:
            x = seq[top->data]; top = top->link; // pop the stack
        }
    }
}
```

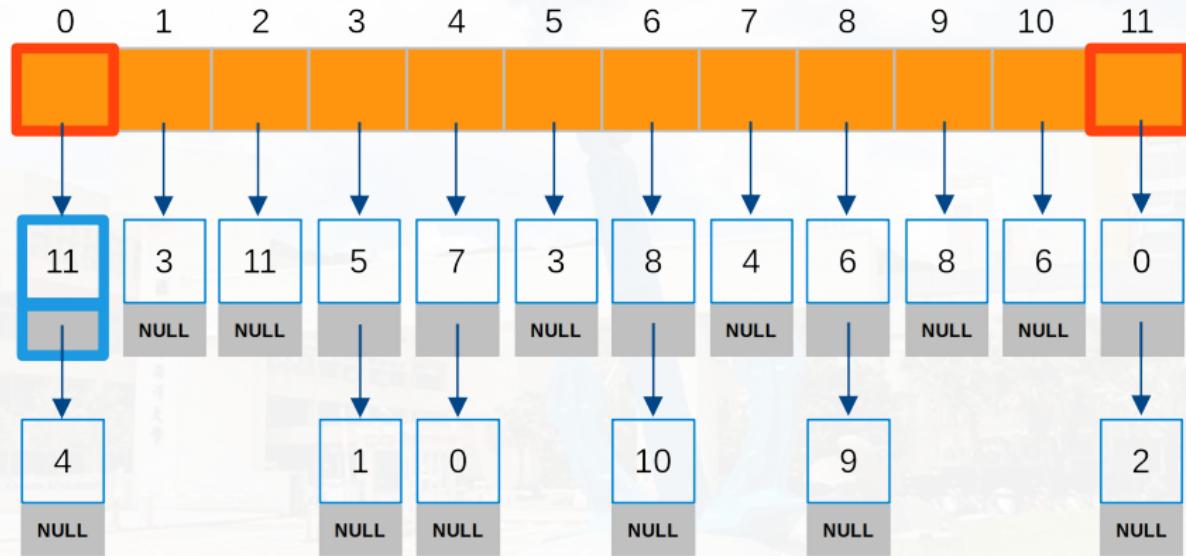


top → NULL



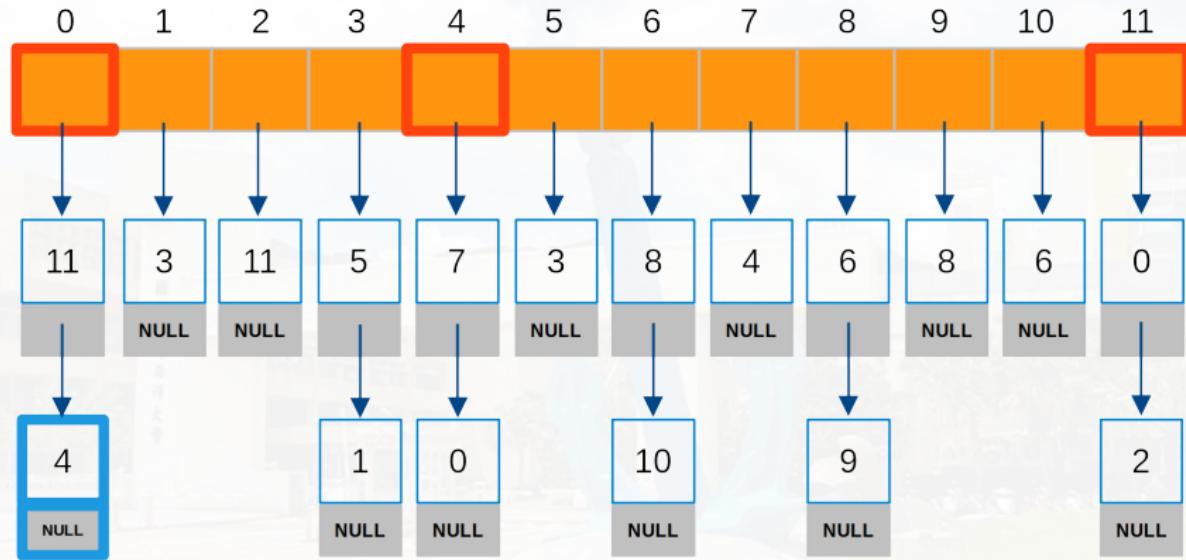
top → NULL

New Classes: 0



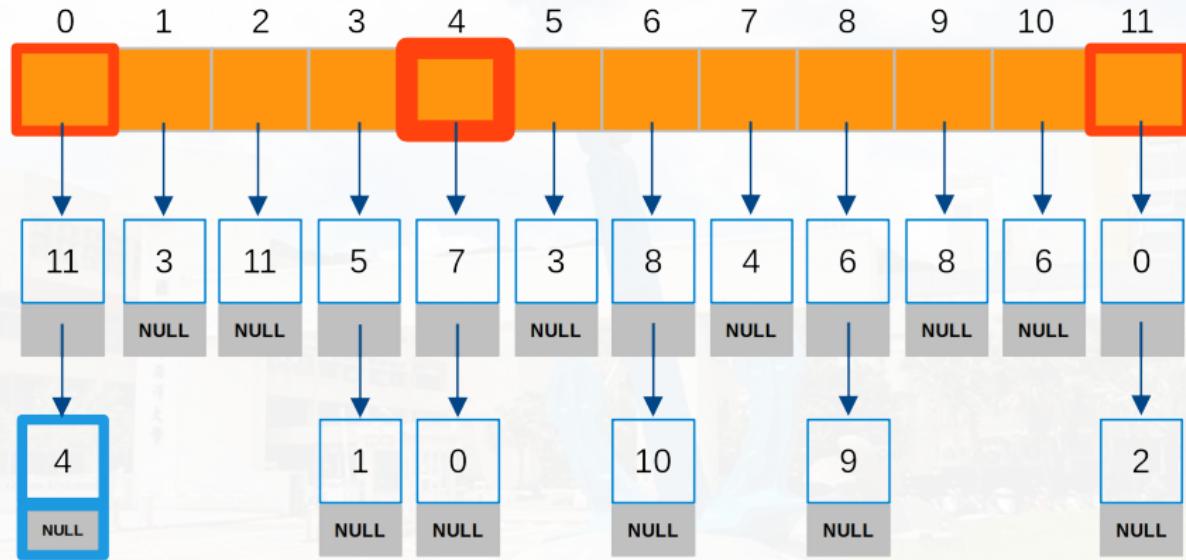
top → (11) → NULL

New Classes: 0, 11



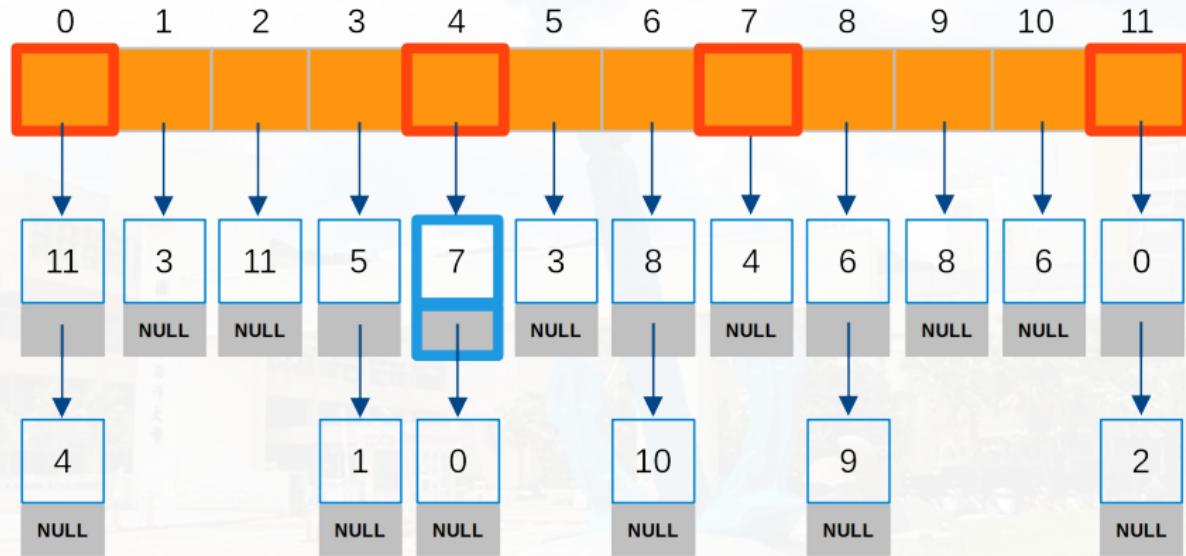
top \rightarrow (4) \rightarrow (11) \rightarrow NULL

New Classes: 0, 11, 4



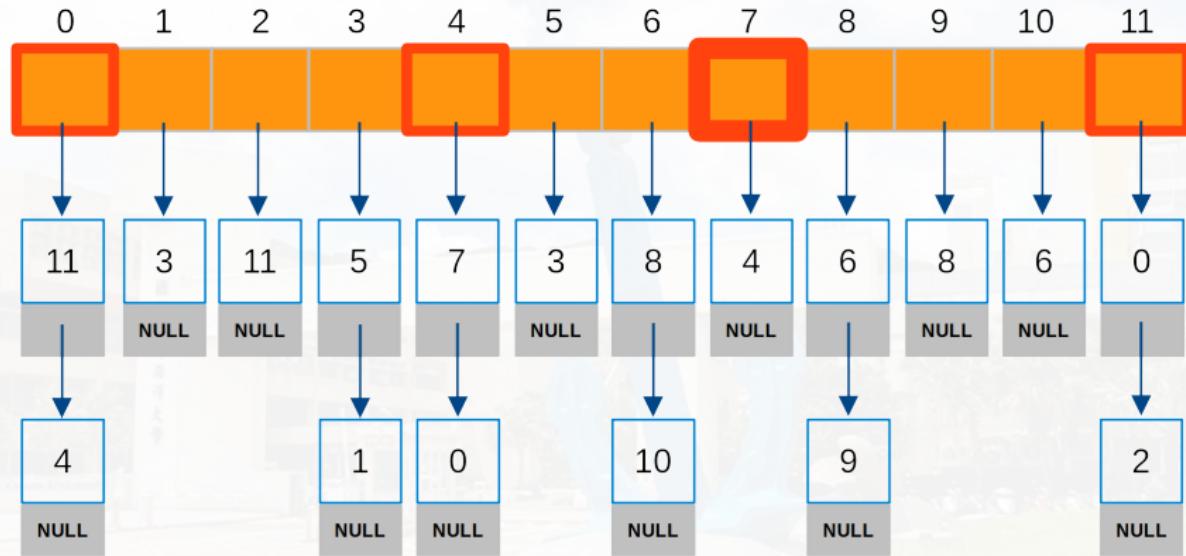
top → (11) → NULL

New Classes: 0, 11, 4



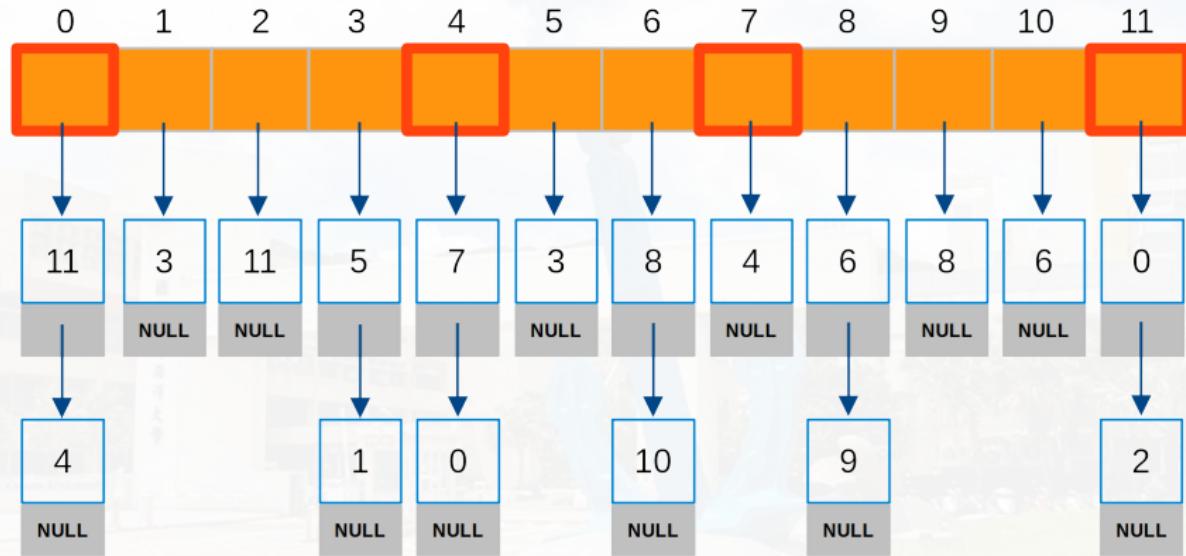
top → (7) → (4) → (11) → NULL

New Classes: 0, 11, 4, 7



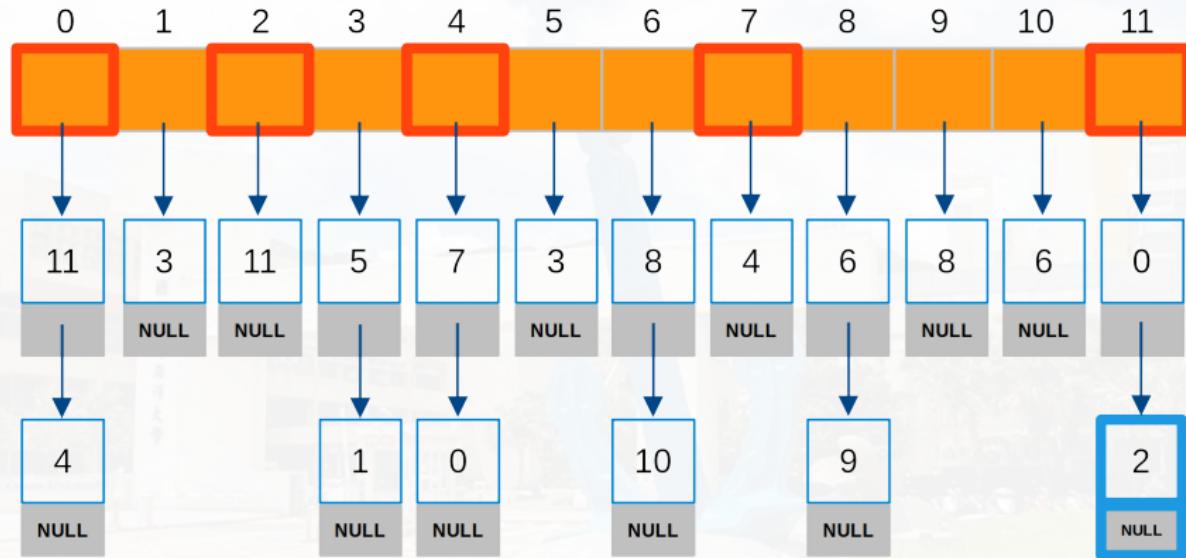
top → (4) → (11) → NULL

New Classes: 0, 11, 4, 7



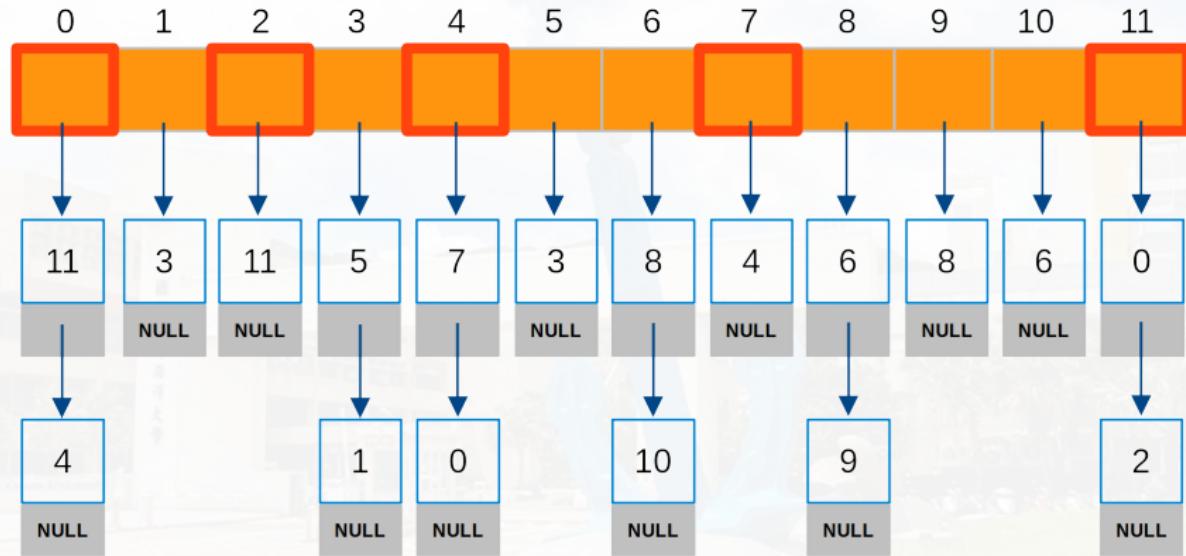
top → (11) → NULL

New Classes: 0, 11, 4, 7



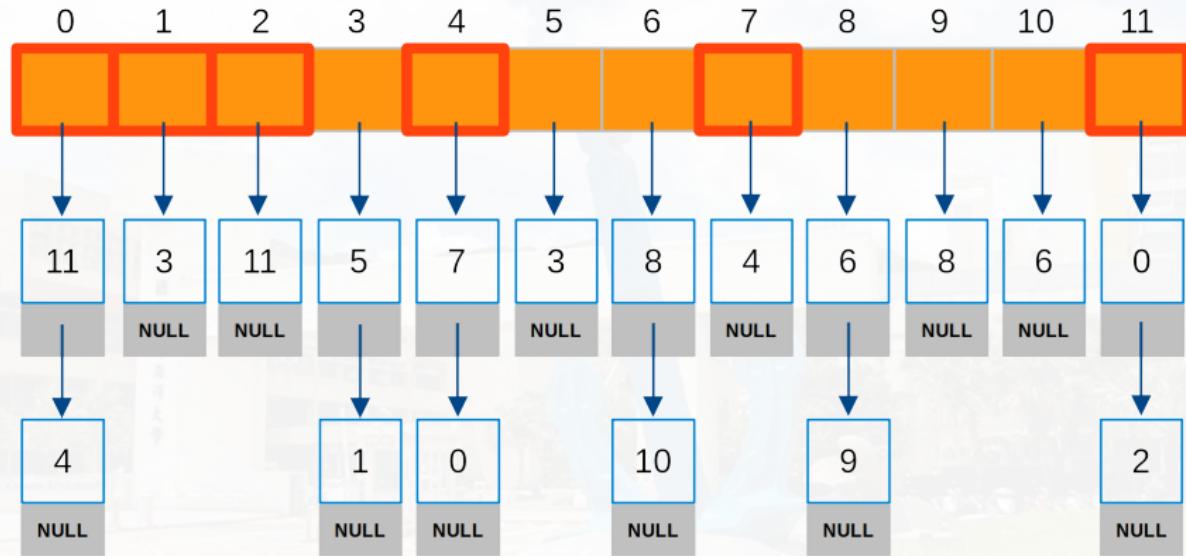
top → (2) → (11) → NULL

New Classes: 0, 11, 4, 7, 2



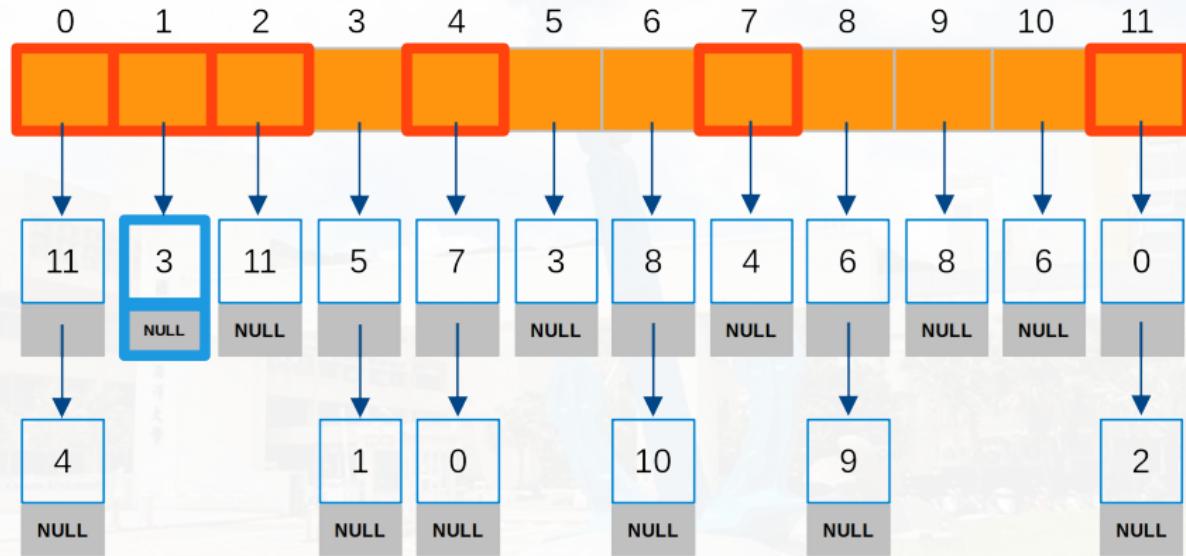
top → NULL

New Classes: 0, 11, 4, 7, 2



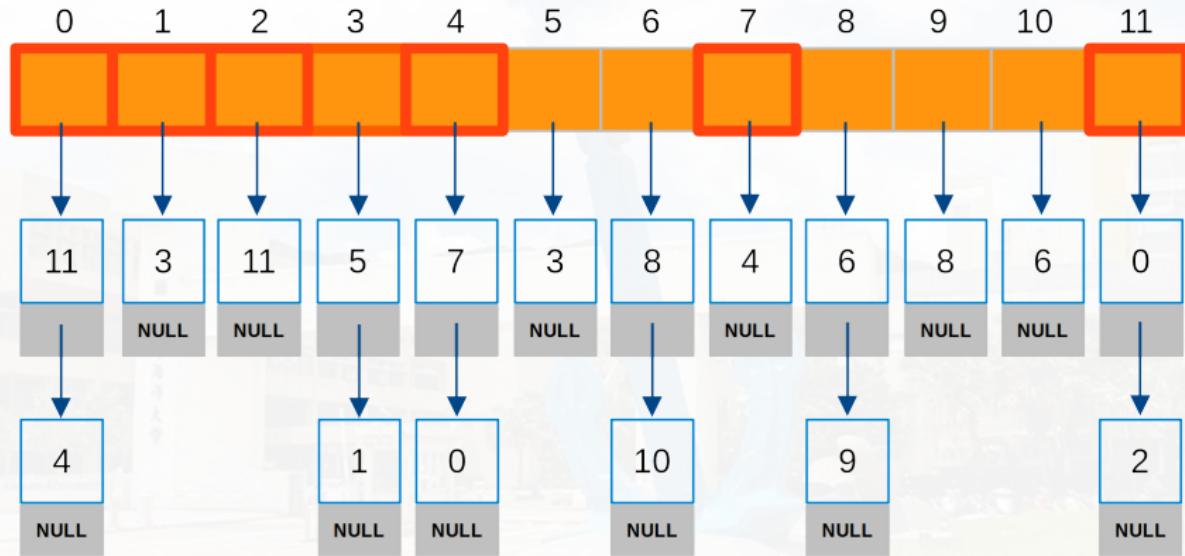
top → NULL

New Classes: 0, 11, 4, 7, 2 New Classes: 1



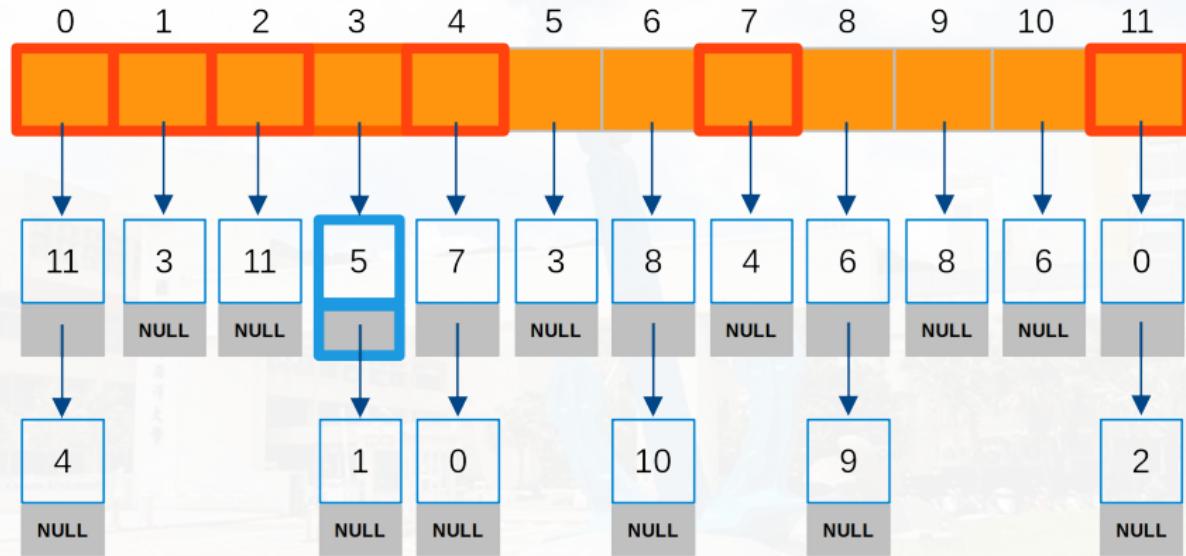
top → (3) → NULL

New Classes: 0, 11, 4, 7, 2 New Classes: 1, 3



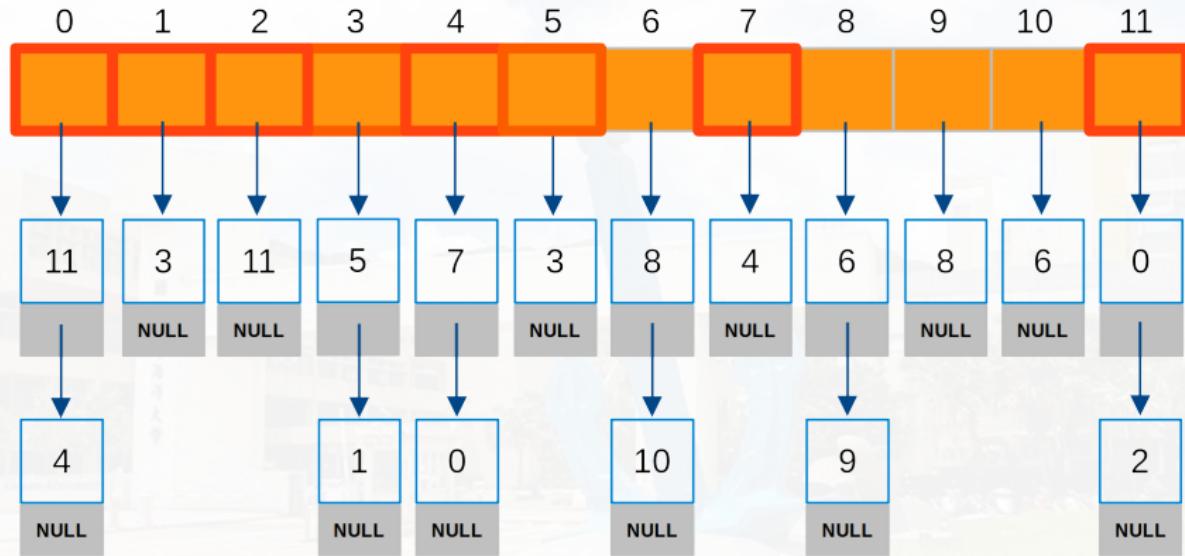
top → NULL

New Classes: 0, 11, 4, 7, 2 New Classes: 1, 3



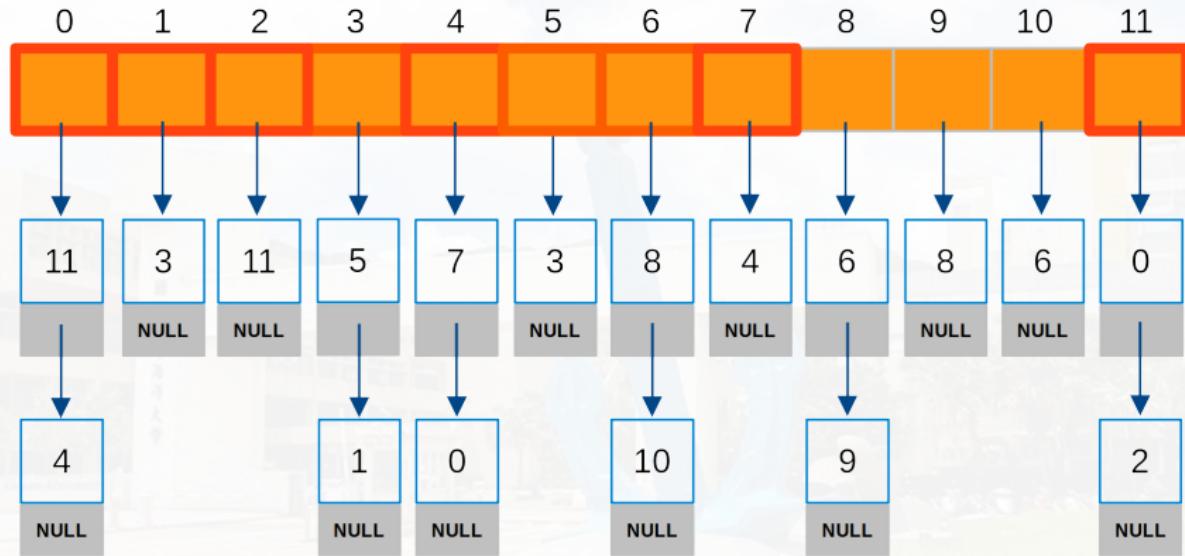
top → (5) → NULL

New Classes: 0, 11, 4, 7, 2 New Classes: 1, 3, 5



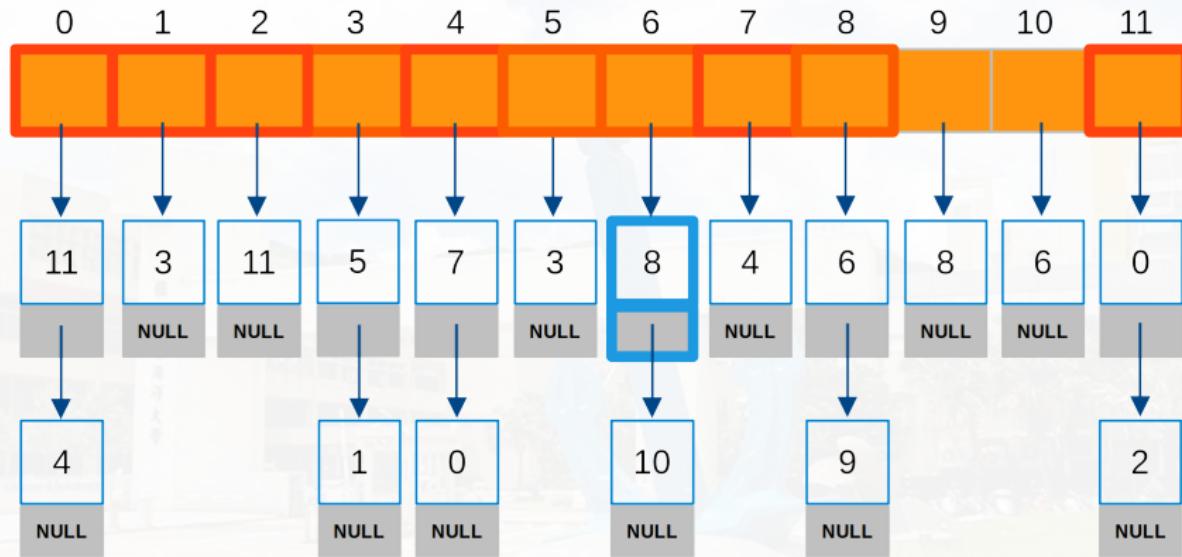
top → NULL

New Classes: 0, 11, 4, 7, 2 New Classes: 1, 3, 5



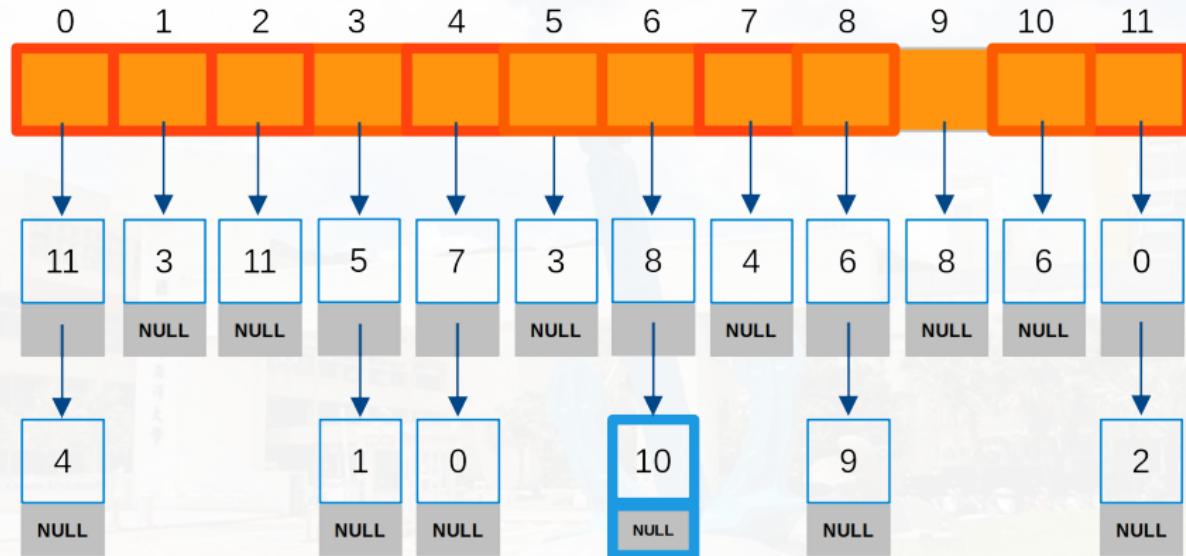
top → NULL

New Classes: 0, 11, 4, 7, 2 New Classes: 1, 3, 5 New Classes: 6



top → (8) → NULL

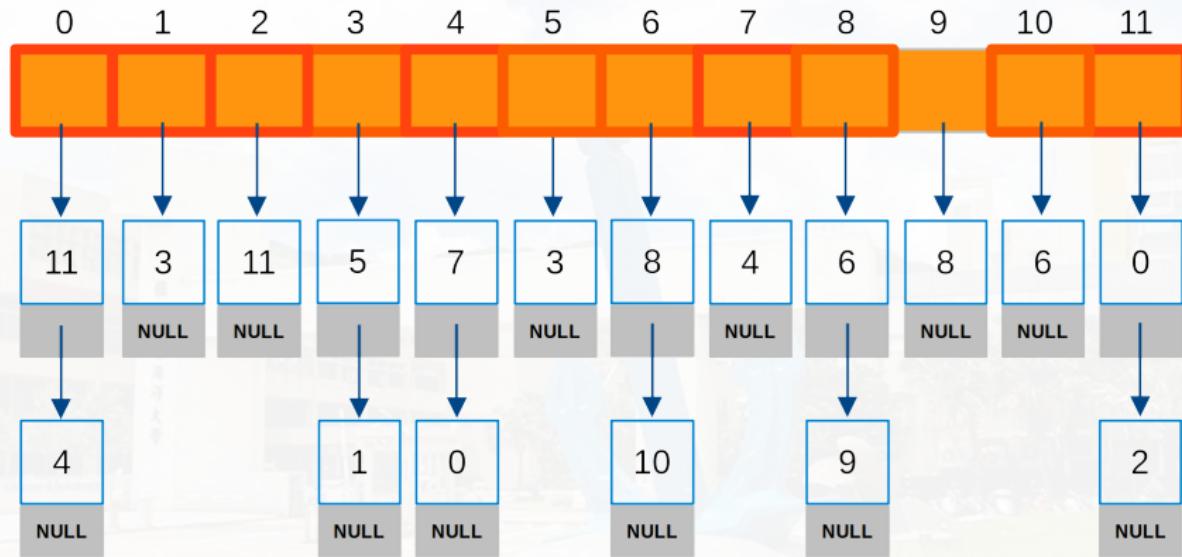
New Classes: 0, 11, 4, 7, 2 New Classes: 1, 3, 5 New Classes: 6, 8



top → (10) → (8) → NULL

New Classes: 0, 11, 4, 7, 2 New Classes: 1, 3, 5 New Classes: 6, 8, 10

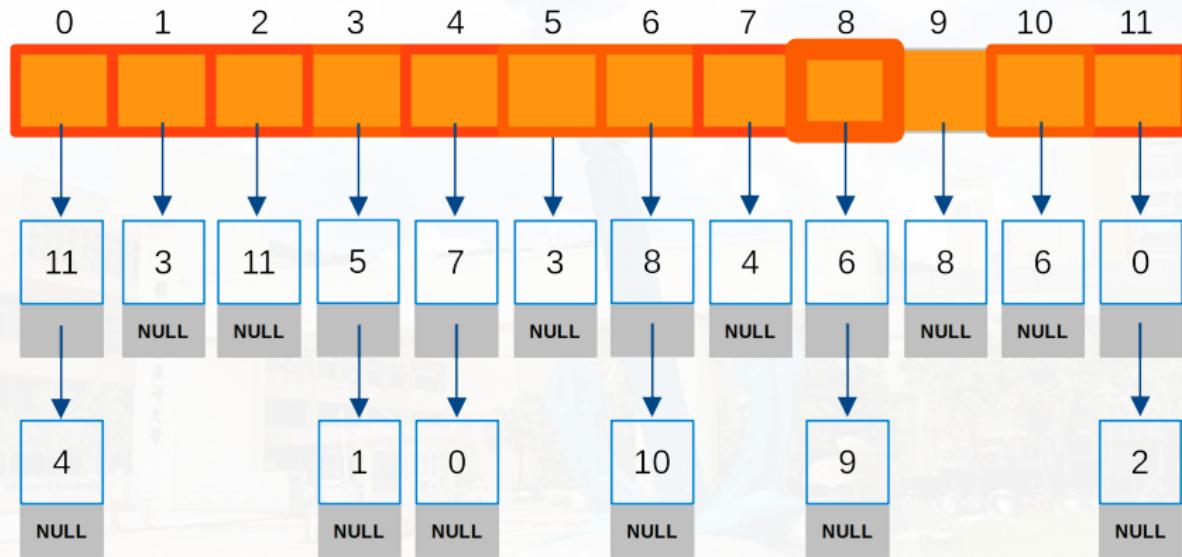




top → (8) → NULL

New Classes: 0, 11, 4, 7, 2 New Classes: 1, 3, 5 New Classes: 6, 8, 10

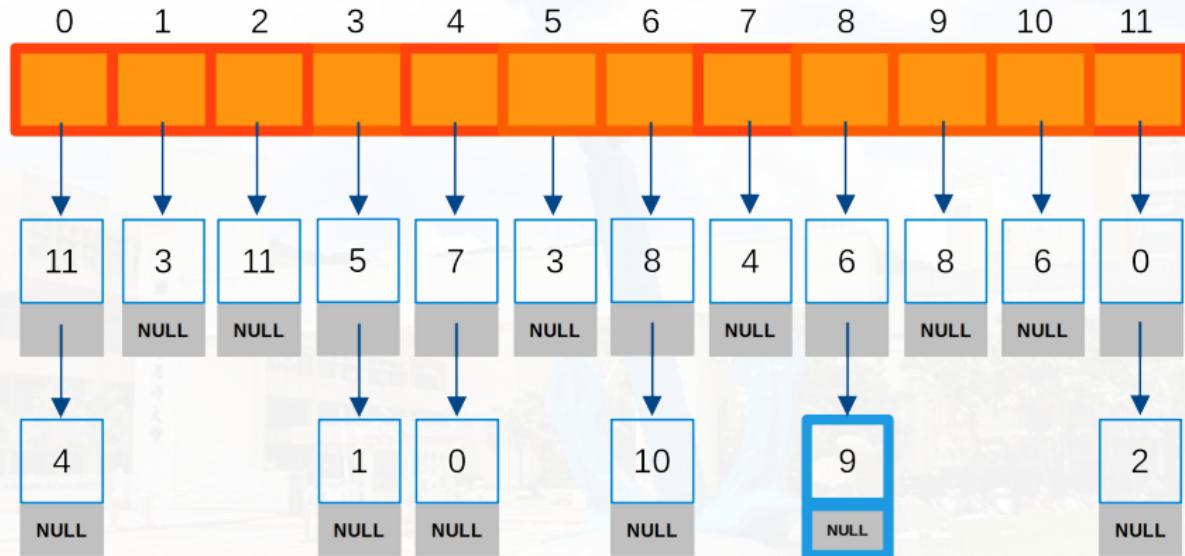




top → NULL

New Classes: 0, 11, 4, 7, 2 New Classes: 1, 3, 5 New Classes: 6, 8, 10





top → NULL

New Classes: 0, 11, 4, 7, 2 New Classes: 1, 3, 5 New Classes: 6, 8, 10, 9



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2 Sparse Matrices Revisted

3 Doubly Linked Lists

Issues for Previous Array Representation

- When we performed matrix operations such as $+$, $-$, or $*$, the number of **nonzero terms** varied.
- The sequential representation of sparse matrices suffered from the same inadequacies as the similar representation of polynomials.

Solution:

- Linked list representation for sparse matrices.

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Solution:

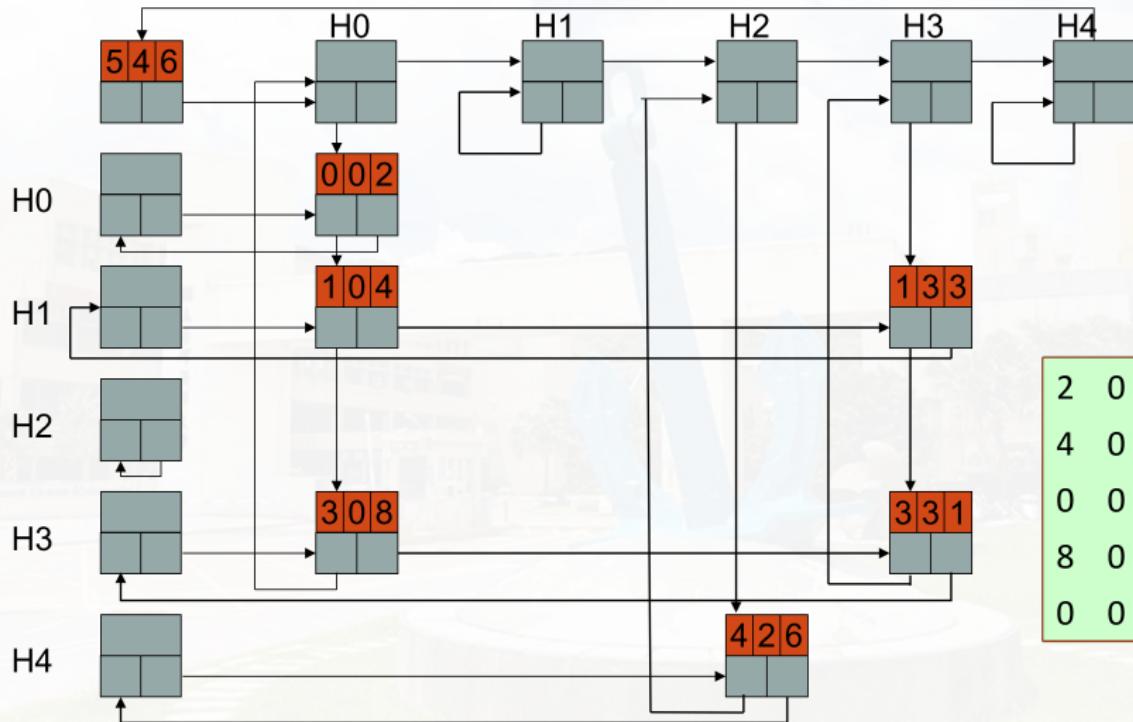
- Linked list representation for sparse matrices.
- Two types of nodes in the representation: **header nodes** and **element nodes**.

next	
down	right

header node

row	col	value
down	right	

element node



Sparse Matrix Representation

- We represent each column (row) of a sparse matrix as a circularly linked list with a header node.
- The header node for **row *i*** is also the header node for **column *i***. The number of header nodes is $\max\{\text{numRows}, \text{numCols}\}$.
- Each element node is **simultaneously** linked into two lists: a **row** list, and a **column** list.
- Each head node is belonged to three lists: a **row** list, a **column** list, and a **header node** list.

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Issues for Singly Linked Lists

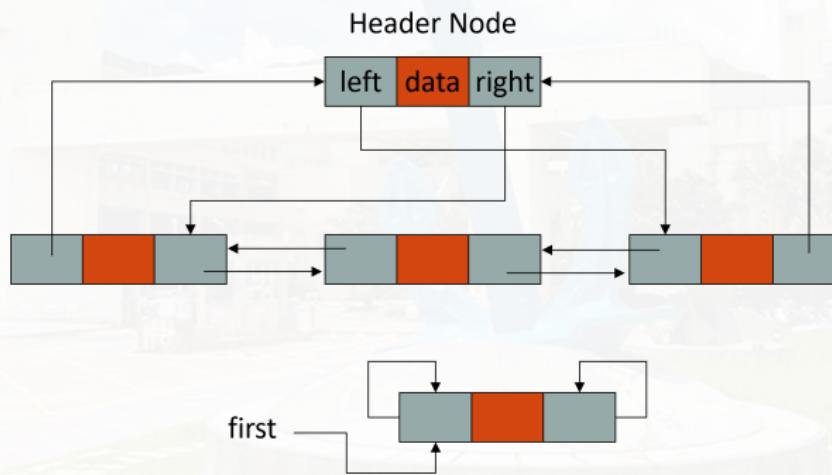
- The only way to find the node that precedes some node p is to start at the beginning of the list.
- Sometimes it is necessary to move in either direction.

Doubly linked lists:

```
typedef struct node *nodePointer;
typedef struct node {
    nodePointer llink;
    element data;
    nodePointer rlink;
};
```

- The following formula reflects the essential virtue of this structure, namely, that we can go back and forth with equal ease.

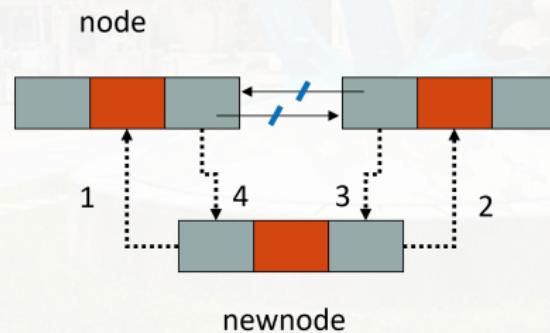
```
ptr = ptr->llink->rlink = ptr->rlink->llink
```



Empty doubly linked circular list with header node

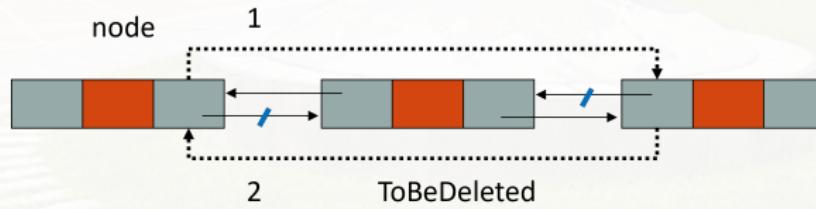
Insertion into a doubly linked circular List

```
void d_LCL_insert(nodePointer node, nodePointer newnode) {  
    /* insert newnode to the right of node */  
    newnode->llink = node;           // 1  
    newnode->rlink = node->rlink;   // 2  
    node->rlink->llink = newnode;   // 3  
    node->rlink = newnode;          // 4  
}  
  
// Question: Any other way to do this insertion?
```



Deletion from a doubly linked circular list

```
void d_LCL_delete(nodePointer node, nodePointer ToBeDeleted) {  
    /* delete from the doubly linked list */  
    if (node == ToBeDeleted)  
        printf("Deletion of preceding node is not permitted.\n");  
    else {  
        ToBeDeleted->llink->rlink = ToBeDeleted->rlink; // 1  
        ToBeDeleted->rlink->llink = ToBeDeleted->llink; // 2  
        // Question: Could we switch 1 and 2?  
        free(ToBeDeleted);  
    }  
}
```



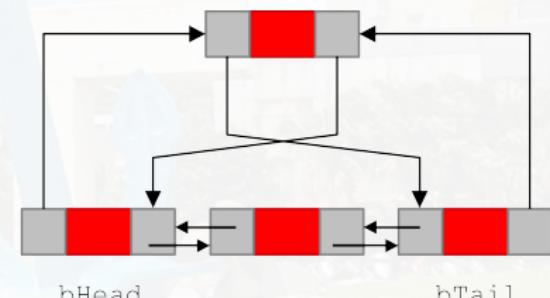
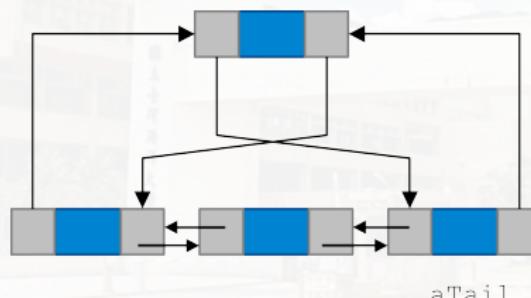
Benefit of using the node “ToBeDeleted”

- The node ToBeDeleted can be used as a guide for its right node and left node to update the links.
- No additional temporary nodePointer is required.

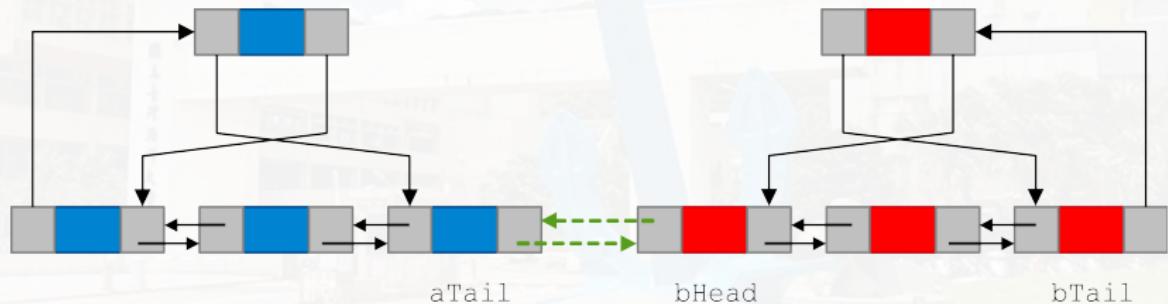
Benefit of using doubly linked circular list

- Concatenation of two lists: $O(1)$ time.
 - Using the header node.
- Inversion of a list: $O(1)$ time (consider llink as rlink and vice versa).

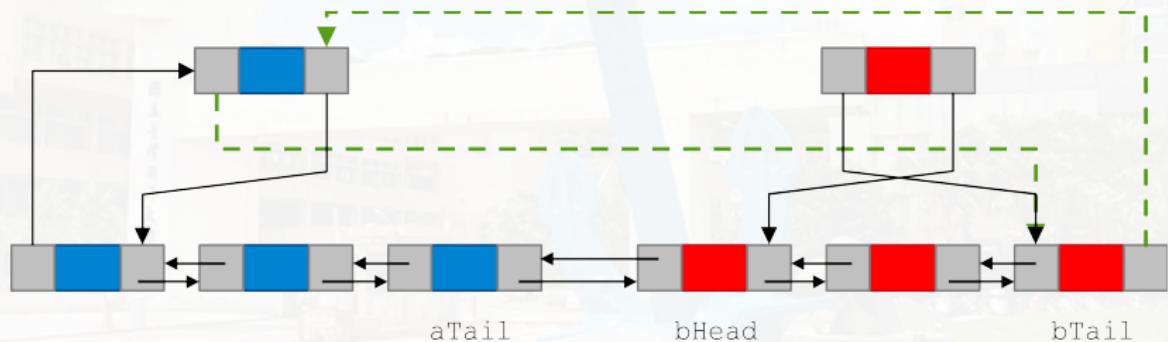
Concatenation of two doubly linked circular list



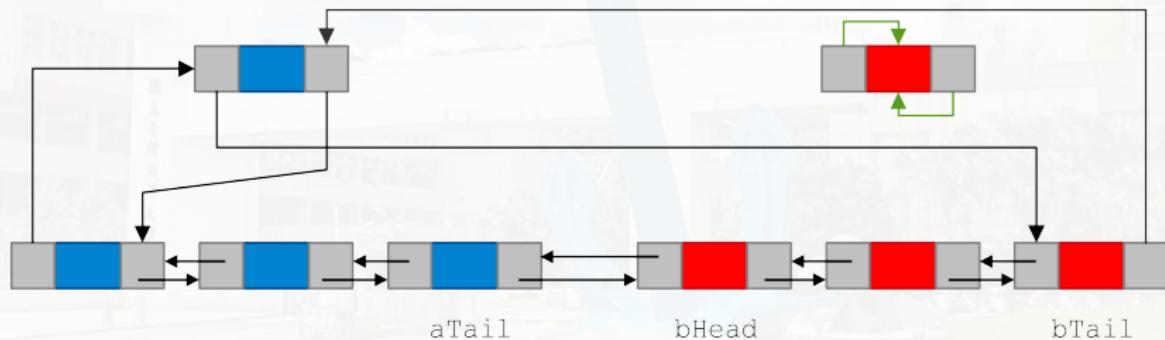
Concatenation of two doubly linked circular list



Concatenation of two doubly linked circular list



Concatenation of two doubly linked circular list



Concatenation of two doubly linked circular lists

```
void concat(Node *headerA, Node *headerB) {
    if (headerA == headerB) return; /* avoid self-append */
    if (headerB->rlink == headerB) return; /* B is empty so skipped */

    Node *aTail = headerA->llink;      // get the last node of A
    Node *bHead = headerB->rlink;      // get the first node of B
    Node *bTail = headerB->llink;      // get the last node of B

    // Stitch B after A's tail
    aTail->rlink = bHead;
    bHead->llink = aTail;

    // Close the ring back to headerA
    bTail->rlink = headerA;
    headerA->llink = bTail;

    // Make B empty
    headerB->rlink = headerB->llink = headerB;
}
```

Discussions

