Randomized Algorithms

Coupon Collector’s Problem

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Review

- Expectation of discrete random variables
- Linearity of expectation.
- Bernoulli and Binomial random variable

Expectation

- The expectation of a discrete random variable X, denoted by $\mathbf{E}[X]$, is

$$\mathbf{E}[X] = \sum_i i \cdot \Pr[X = i]$$

- Example: Let X denote the sum of of dices:

$$\mathbf{E}[X] = \frac{1}{36} \cdot 2 + \frac{2}{36} \cdot 3 + \frac{3}{36} \cdot 4 + \cdots + \frac{1}{36} \cdot 12 = 7.$$

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Linearity of Expectation

- For any finite collection of discrete random variables $X_1, X_2, ..., X_n$ with finite expectations,

$$\mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{E}[X_i].$$

- For any constant c and discrete random variable X,

$$\mathbf{E}[cX] = c \cdot \mathbf{E}[X].$$

- Why is it useful?

Example

- Consider the dice-throwing example again.
  - X_1 : the outcome of die 1
  - X_2 : the outcome of die 2

$$\mathbf{E}[X_1] = \mathbf{E}[X_2] = \frac{1}{6} \cdot \sum_{j=1}^{6} j = \frac{7}{2}$$

$$\mathbf{E}[X] = \mathbf{E}[X_1 + X_2] = 7.$$

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Bernoulli random variable

- Suppose we run an experiment that succeeds with probability p and fails with probability $1-p$.

$$Y = \begin{cases} 1 & \text{if the experiment succeeds,} \\ 0 & \text{otherwise.} \end{cases}$$

- Y: Bernoulli random variable.
  - or *indicator random variable*.

$$\mathbf{E}[Y] = 1 \cdot \Pr[Y = 1] + 0 \cdot \Pr[Y = 0] = \Pr[Y = 1] = p.$$

Binomial random variable

- A binomial random variable X with parameters n and p, denoted by $B(n, p)$, is defined as

$$\Pr[X = j] = \binom{n}{j} p^j (1-p)^{n-j}.$$

for $j = 0, 1, 2, ..., n$.

- <u>Exercise</u>: Show that $\sum_{j=0}^{n} \Pr[X = j] = 1.$

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X: the number of successful trials in the n experiments.

- <u>Exercise</u>: Show that $\sum_{j=0}^{n} \Pr[X = j] = 1.$

Binomial random variable (expectation)

- $$\begin{aligned}\mathbf{E}[X] &= \sum_{j=0}^{n} j \binom{n}{j} p^j (1-p)^{n-j} \\ &= \sum_{j=0}^{n} j \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \\ &= \sum_{j=1}^{n} \frac{n!}{(j-1)!(n-j)!} p^j (1-p)^{n-j} \\ &= np \sum_{j=1}^{n} \frac{(n-1)!}{(j-1)!((n-1)-(j-1))!} p^{j-1} (1-p)^{(n-1)-(j-1)} \\ &= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k!((n-1)-k)!} p^k (1-p)^{(n-1)-k} \\ &= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{(n-1)-k} \\ &= np.\end{aligned}$$

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$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}.$$

Let's make it simpler!

- Denote a set of n Bernoulli random variables $X_1, X_2, \ldots, X_n$.
  - $X_i = 1$ if the ith trial is successful and 0 otherwise.
  - $X = X_1 + X_2 + \ldots + X_n$

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  - Compute $\mathbf{E}[X]$ using linearity of expectation:

$$\mathbf{E}[X] = \mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{E}[X_i] = np.$$

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- Imagine: *flip a coin until it lands on a head.*
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- Definition. A geometric random variable X with parameter p is

$$\Pr[X = n] = (1 - p)^{n-1} p.$$

for n = 1, 2, …

- Exercise. Show that $\sum_{n \geq 1} \Pr[X = n] = 1.$

Memoryless

- Let X be a geometric random variable X with parameter $p > 0$.
- For any $n, k > 0$, $\Pr[X = n + k \mid X > k] = \Pr[X = n]$.

Memoryless

- Let X be a geometric random variable X with parameter $p > 0$.
- For any n, $k > 0$, $\Pr[X = n + k \mid X > k] = \Pr[X = n]$.
- *Proof.*

$$\begin{aligned}
\Pr[X = n + k \mid X > k] &= \frac{\Pr[(X = n + k) \cap (X > k)]}{\Pr[X > k]} \\
&= \frac{\Pr[X = n + k]}{\Pr[X > k]} \\
&= \frac{(1-p)^{n+k-1}p}{\sum_{i=k}^{\infty}(1-p)^i p} \\
&= \frac{(1-p)^{n+k-1}p}{(1-p)^k} \\
&= (1-p)^{n-1}p = \Pr[X = n].
\end{aligned}$$

The mean of a geometric r.v. $X(p)$

$$\begin{aligned}
\mathbf{E}[X] &= \sum_{j=1}^{\infty} j \Pr[X = j] \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{j} \Pr[X = j] \\
&= \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \Pr[X = j] \\
&= \sum_{i=1}^{\infty} \Pr[X \geq i].
\end{aligned}$$

$$\mathbf{E}[X] = \sum_{i=1}^{\infty} (1-p)^{i-1} = \frac{1}{1-(1-p)} = \frac{1}{p}.$$

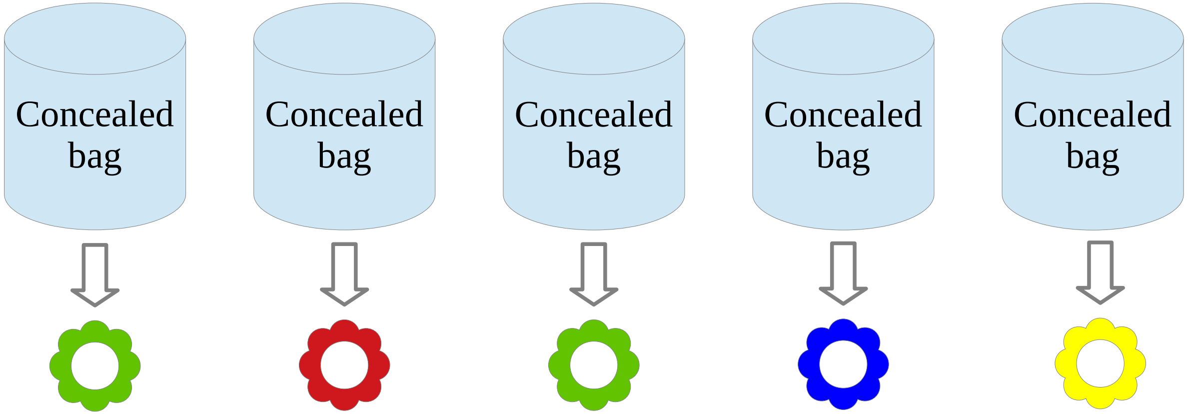
$$\Pr[X \geq i] = \sum_{k=i}^{\infty} (1-p)^{k-1} p = (1-p)^{i-1}$$

Coupon Collector’s Problem

▲一名網友在臉書社團「爆廢公社公開版」表示，多年前為了收集連鎖便利商店 7-11 的一款贈品，花了不少金額，還拿到許多重複的款式，貼文引發 3 千多名網友共鳴。（圖／翻攝自爆廢公社公開版）

Coupon Collector's Problem

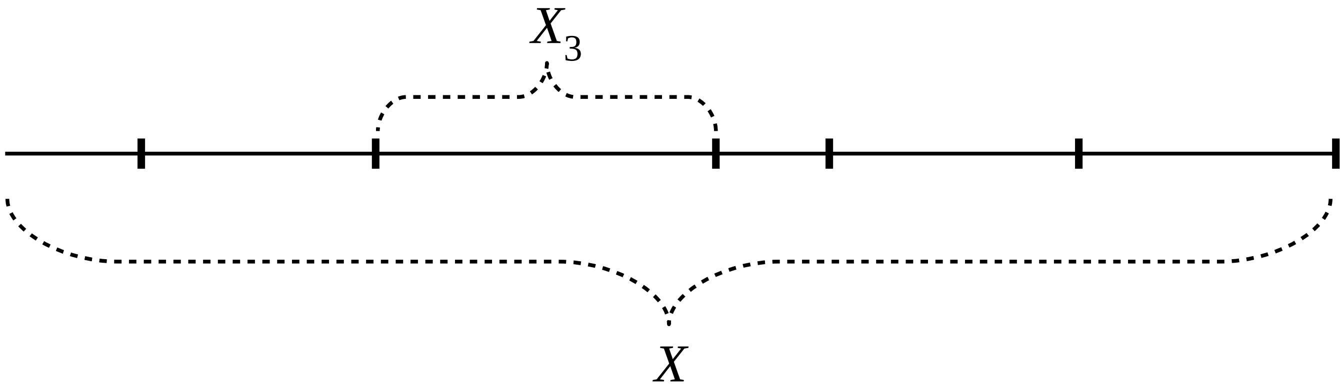
- Have you already got all of them (totally n types)?
- Have you ever thought about how much you should pay for them?



- Each bag is chosen independently and uniformly at random from the n possibilities.

Coupon Collector’s Problem

- Let X be the number of bags bought until every type of coupon is obtained.
- Let X_i be the number of bags bought while you had already got exactly $i-1$ different coupons.



Coupon Collector's Problem

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Coupon Collector's Problem

- Let X be the number of bags bought until every type of coupon is obtained.
- Let X_i be the number of bags bought while you had already got exactly $i-1$ different coupons.
  - Geometric random variables?!
  - What about $X = \sum_{i=1}^{n} X_i$?

Coupon Collector's Problem

- When exactly $i-1$ coupons have been collected, the probability of obtaining a new one is

$$p_i = 1 - \frac{i-1}{n}$$

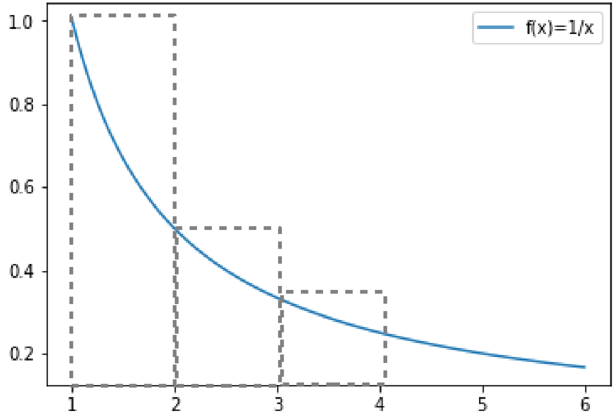
- X_i is a geometric random variable, so

$$\mathbf{E}[X_i] = \frac{1}{p_i} = \frac{n}{n-i+1}.$$

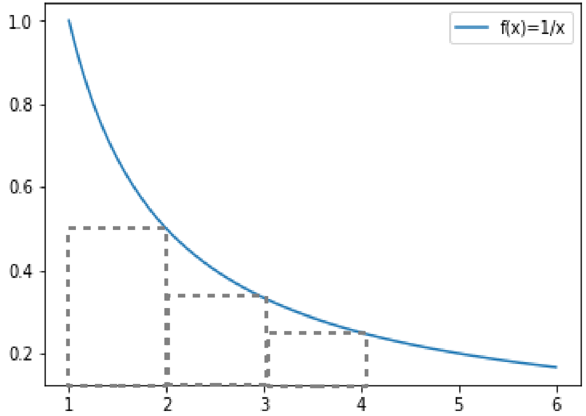
Coupon Collector's Problem (contd.)

- $$\begin{aligned}\mathbf{E}[X] &= \mathbf{E}\left[\sum_{i=1}^{n} X_i\right] \\ &= \sum_{i=1}^{n} \mathbf{E}[X_i] \\ &= \sum_{i=1}^{n} \frac{n}{n-i+1} \\ &= n \cdot \boxed{\sum_{i=1}^{n} \frac{1}{i}}.\end{aligned}$$

$$H(n) = \sum_{i=1}^{n} \frac{1}{i} = \ln n + \Theta(1).$$



$$\sum_{k=1}^{n} \frac{1}{k} \geq \int_{x=1}^{n} \frac{1}{x} dx = \ln n$$



$$\sum_{k=2}^{n} \frac{1}{k} \leq \int_{x=1}^{n} \frac{1}{x} dx = \ln n$$

Coupon Collector's Problem (contd.)

- So, you are about to buy $n \ln n + \Theta(n)$ bags for collecting all the coupons (stickers)!

On conditional expectation

- There are terminologies which may confusing you.

Conditional Expectation

- Definition.

$$\mathbf{E}[Y \mid Z = z] = \sum_{y} y \Pr[Y = y \mid Z = z].$$

- Example of two dices.
  - X_1: the number showing on the first die
  - X_2: the number showing on the second die
  - $X = X_1 + X_2$

$$\mathbf{E}[X \mid X_1 = 2] = \sum_{x=3}^{8} x \cdot \frac{1}{6} = \frac{11}{2}.$$

Conditional Expectation (contd.)

- <u>Lemma</u>. For any random variables *X* and *Y*,

$$\mathbf{E}[X] = \sum_y \Pr[Y = y]\mathbf{E}[X \mid Y = y].$$

- *Proof.*

$$\begin{aligned}
\sum_y \Pr[Y = y] \cdot \mathbf{E}[X \mid Y = y] &= \sum_y \Pr[Y = y] \cdot \sum_x x \Pr[X = x \mid Y = y] \\
&= \sum_x \sum_y x \Pr[X = x \mid Y = y] \cdot \Pr[Y = y] \\
&= \sum_x \sum_y x \Pr[X = x \cap Y = y] \\
&= \sum_x x \Pr[X = x] \\
&= \mathbf{E}[X].
\end{aligned}$$

Conditional Expectation

- Lemma. For any finite collection of discrete random variables X_1, X_2, ..., X_n with finite expectations and for any random variable Y,

$$\mathbf{E}\left[\sum_{i=1}^{n} X_i \mid Y = y\right] = \sum_{i=1}^{n} \mathbf{E}[X_i \mid Y = y].$$

Conditional Expectation (contd.)

- A weird definition.
- $\mathbf{E}[Y \mid Z]$: regarded as a **random variable** *f*(*Z*).
  - It takes on the value $\mathbf{E}[Y \mid Z = z]$ when *Z* = z.

- In the previous example,

$$\mathbf{E}[X \mid X_1] = \sum_x x \cdot \Pr[X = x \mid X_1] = \sum_{X_1+1}^{X_1+6} x \cdot \frac{1}{6} = X_1 + \frac{7}{2}.$$

- So it makes sense that

$$\mathbf{E}[\mathbf{E}[X \mid X_1]] = \mathbf{E}\left[X_1 + \frac{7}{2}\right] = \frac{7}{2} + \frac{7}{2} = 7.$$

Conditional Expectation (contd.)

- <u>Theorem</u>. $\mathbf{E}[Y] = \mathbf{E}[\mathbf{E}[Y \mid Z]]$.

- *Proof.*

$$\mathbf{E}[\mathbf{E}[Y \mid Z]] = \sum_z \mathbf{E}[Y \mid Z = z] \cdot \Pr[Z = z] = \mathbf{E}[Y].$$