

# Social Choice

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# Outline

- 1 Introduction to Social Choice
- 2 Peer-Grading in MOOCs
  - Preliminaries
  - Correctness of Recovered Pairwise Rankings
  - Proofs



# The Setting of Social Choice

Take voting scheme for example.

- A set  $O$  of **outcomes** (i.e., alternatives, candidates, etc.)
- A set  $A$  of agents s.t. each of them has a **preference**  $\succ$  over the outcomes.
- The **social choice function**: a mapping from the profiles of the preferences to a particular outcome.

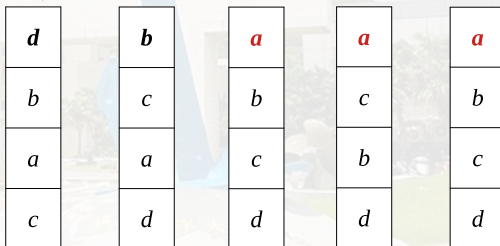


# Outcomes & preferences

outcomes :  $a, b, c, d$

preferences

$$d \succ b \succ a \succ c$$



<b><math>d</math></b>	<b><math>b</math></b>	<b><math>a</math></b>	<b><math>a</math></b>	<b><math>a</math></b>
$b$	$c$	$b$	$c$	$b$
$a$	$a$	$c$	$b$	$c$
$c$	$d$	$d$	$d$	$d$

# Preferences

- A binary relation  $\succsim$  such that
  - for every  $a, b \in O$ ,  $a \neq b$ , we have either  $a \succ b$  or  $b \succ a$  but NOT both.
  - for  $a, b, c \in O$ , if  $a \succ b$  and  $b \succ c$ , then we have  $a \succ c$ .
- $\precsim$  can be defined similarly.
  - $\precsim: \precsim$




# Agents with preferences

- E.g., five agents (voters).
- Each agent has its preference over four candidates  $\{a, b, c, d\}$ .

outcomes :  $a, b, c, d$

preferences

$d \succ b \succ a \succ c$



<b><math>d</math></b>	<b><math>b</math></b>	<b><math>a</math></b>	<b><math>a</math></b>	<b><math>a</math></b>
$b$	$c$	$b$	$c$	$b$
$a$	$a$	$c$	$b$	$c$
$c$	$d$	$d$	$d$	$d$

# Agents with preferences

- E.g., three agents (voters).
- Each agent has its preference over four candidates  $\{a, b, c, d\}$ .

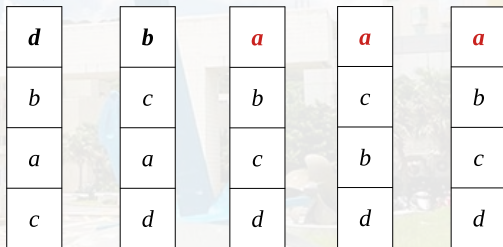
$v_1$	$v_2$	$v_3$
$d$	$b$	$a$
$b$	$c$	$b$
$a$	$a$	$c$
$c$	$d$	$d$

# Plurality rule $\Rightarrow a$

outcomes :  $a, b, c, d$

$d \succ b \succ a \succ c$

preferences



<b><math>d</math></b>	<b><math>b</math></b>	<b><math>a</math></b>	<b><math>a</math></b>	<b><math>a</math></b>
$b$	$c$	$b$	$c$	$b$
$a$	$a$	$c$	$b$	$c$
$c$	$d$	$d$	$d$	$d$

- Plurality rule: each agent can only give score 1 to the most preferred one and 0 to the others.



# Plurality rule (contd.)

$v_1$	$v_2$	$v_3$
<i>d</i>	<i>b</i>	<i>a</i>
<i>b</i>	<i>c</i>	<i>b</i>
<i>a</i>	<i>a</i>	<i>c</i>
<i>c</i>	<i>d</i>	<i>d</i>

- Plurality rule: each agent can only give score 1 to the most preferred one and 0 to the others.

# Plurality rule (contd.)

$v_1$	$v_2$	$v_3$
<i>d</i>	<i>b</i>	<i>a</i>
<i>b</i>	<i>c</i>	<i>b</i>
<i>a</i>	<i>a</i>	<i>c</i>
<i>c</i>	<i>d</i>	<i>d</i>

- Plurality rule:

# Plurality rule (contd.)

$v_1$	$v_2$	$v_3$
<i>d</i>	<i>b</i>	<i>a</i>
<i>b</i>	<i>c</i>	<i>b</i>
<i>a</i>	<i>a</i>	<i>c</i>
<i>c</i>	<i>d</i>	<i>d</i>

- Plurality rule: depending on the tie-breaking rule.

# Condorcet rule

$v_1$	$v_2$	$v_3$
$d$	$b$	$a$
$b$	$c$	$b$
$a$	$a$	$c$
$c$	$d$	$d$

- Condorcet rule:

- $a$  vs.  $b$
- $a$  vs.  $c$
- $a$  vs.  $d$

# Condorcet rule

$v_1$	$v_2$	$v_3$
$d$	$b$	$a$
$b$	$c$	$b$
$a$	$a$	$c$
$c$	$d$	$d$

- Condorcet rule:

- $a$  vs.  $b \rightarrow b$
- $a$  vs.  $c \rightarrow a$
- $a$  vs.  $d \rightarrow a$

# Condorcet rule

$v_1$	$v_2$	$v_3$
$d$	$b$	$a$
$b$	$c$	$b$
$a$	$a$	$c$
$c$	$d$	$d$

- Condorcet rule:

- $c$  vs.  $a$
- $c$  vs.  $b$
- $c$  vs.  $d$

# Condorcet rule

$v_1$	$v_2$	$v_3$
$d$	$b$	$a$
$b$	$c$	$b$
$a$	$a$	$c$
$c$	$d$	$d$

- Condorcet rule:

- $c$  vs.  $a \rightarrow a$
- $c$  vs.  $b \rightarrow b$
- $c$  vs.  $d \rightarrow c$

# Condorcet rule

$v_1$	$v_2$	$v_3$
$d$	$b$	$a$
$b$	$c$	$b$
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$c$	$d$	$d$

- Condorcet rule:

- $b$  vs.  $a$
- $b$  vs.  $c$
- $b$  vs.  $d$



# Condorcet rule

$v_1$	$v_2$	$v_3$
$d$	$b$	$a$
$b$	$c$	$b$
$a$	$a$	$c$
$c$	$d$	$d$

- Condorcet rule:

- $b$  vs.  $a \rightarrow b$
- $b$  vs.  $c \rightarrow b$
- $b$  vs.  $d \rightarrow b$

# Condorcet rule

$v_1$	$v_2$	$v_3$
$d$	$b$	$a$
$b$	$c$	$b$
$a$	$a$	$c$
$c$	$d$	$d$

- Condorcet rule:  $b$

- $b$  vs.  $a \rightarrow b$
- $b$  vs.  $c \rightarrow b$
- $b$  vs.  $d \rightarrow b$

# Borda rule

$v_1$	$v_2$	$v_3$																								
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- Borda count rule:

# Borda rule

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- Borda count rule:

- score of  $a$ :  $1 + 1 + 3 = 5$ .
- score of  $b$ :  $2 + 3 + 2 = 7$ .
- score of  $c$ :  $0 + 2 + 1 = 3$ .
- score of  $d$ :  $3 + 0 + 0 = 3$ .

# Borda rule

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- Borda count rule:  $b$ .
  - score of  $a$ :  $1 + 1 + 3 = 5$ .
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# Inefficiency of Borda Count

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- Who is the winner by Borda counting?

# Inefficiency of Borda Count

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- Who is the winner by Borda counting?  $a$ : 6,  $b$ : 7,  $c$ : 2.



# Inefficiency of Borda Count

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- Who is the winner by Borda counting?  $a$ : 6,  $b$ : 7,  $c$ : 2.
- Condorcet principle follows?



# Inefficiency of Borda Count

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- Who is the winner by Borda counting?  $a$ : 6,  $b$ : 7,  $c$ : 2.
- Condorcet principle follows?  $a \succ b$ ,  $a \succ c$ .

# Inefficiency of Borda Count

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$c$	1																																	
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- Who is the winner by Borda counting?  $a$ : 6,  $b$ : 7,  $c$ : 2.
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- Who is the winner under the plurality rule?

# Inefficiency of Borda Count

$v_1$	$v_2$	$v_3$	$v_4$	$v_5$																														
<table><tr><td><math>a</math></td><td>2</td></tr><tr><td><math>b</math></td><td>1</td></tr><tr><td><math>c</math></td><td>0</td></tr></table>	$a$	2	$b$	1	$c$	0	<table><tr><td><math>a</math></td><td>2</td></tr><tr><td><math>b</math></td><td>1</td></tr><tr><td><math>c</math></td><td>0</td></tr></table>	$a$	2	$b$	1	$c$	0	<table><tr><td><math>a</math></td><td>2</td></tr><tr><td><math>b</math></td><td>1</td></tr><tr><td><math>c</math></td><td>0</td></tr></table>	$a$	2	$b$	1	$c$	0	<table><tr><td><math>b</math></td><td>2</td></tr><tr><td><math>c</math></td><td>1</td></tr><tr><td><math>a</math></td><td>0</td></tr></table>	$b$	2	$c$	1	$a$	0	<table><tr><td><math>b</math></td><td>2</td></tr><tr><td><math>c</math></td><td>1</td></tr><tr><td><math>a</math></td><td>0</td></tr></table>	$b$	2	$c$	1	$a$	0
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# Successive elimination

$v_1$	$v_2$	$v_3$
$b$	$a$	$c$
$d$	$b$	$a$
$c$	$d$	$b$
$a$	$c$	$d$

- Successive elimination with ordering  $a \rightarrow b \rightarrow c \rightarrow d$ :

# Successive elimination

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# Successive elimination

$v_1$	$v_2$	$v_3$
$b$	$a$	$c$
$d$	$b$	$a$
$c$	$d$	$b$
$a$	$c$	$d$

- Successive elimination with ordering  $\not a \rightarrow b \rightarrow c \rightarrow d$ :  $d$ 
  - The issue: all of the agents prefer  $b$  to  $d$ !

# Successive elimination

$v_1$	$v_2$	$v_3$
$b$	$a$	$c$
$d$	$b$	$a$
$c$	$d$	$b$
$a$	$c$	$d$

- Successive elimination with ordering  $a \rightarrow b \rightarrow c \rightarrow d$ :  $d$
- Successive elimination with ordering  $a \rightarrow c \rightarrow b \rightarrow d$ :

# Successive elimination

$v_1$	$v_2$	$v_3$
$b$	$a$	$c$
$d$	$b$	$a$
$c$	$d$	$b$
$a$	$c$	$d$

- Successive elimination with ordering  $a \rightarrow b \rightarrow c \rightarrow d$ :  $d$
- Successive elimination with ordering  $a \rightarrow c \rightarrow b \rightarrow d$ :  $b$

# Successive elimination (sensitive to the agenda order)

$v_1$	$v_2$	$v_3$
$b$	$a$	$c$
$d$	$b$	$a$
$c$	$d$	$b$
$a$	$c$	$d$

- Successive elimination with ordering  $a \rightarrow b \rightarrow c \rightarrow d$ :  $d$
- Successive elimination with ordering  $a \rightarrow c \rightarrow b \rightarrow d$ :  $b$
- Successive elimination with ordering  $b \rightarrow c \rightarrow a \rightarrow d$ :



# Successive elimination (sensitive to the agenda order)

$v_1$	$v_2$	$v_3$
$b$	$a$	$c$
$d$	$b$	$a$
$c$	$d$	$b$
$a$	$c$	$d$

- Successive elimination with ordering  $a \rightarrow b \rightarrow c \rightarrow d$ :  $d$
- Successive elimination with ordering  $a \rightarrow c \rightarrow b \rightarrow d$ :  $b$
- Successive elimination with ordering  $b \rightarrow c \rightarrow a \rightarrow d$ :  $a$



# Condorcet Winner vs. Plurality

- Let's say we have 1,000 agents each of which has a preference over three candidates  $A, B, C$ .
  - 499 agents for  $A \succ B \succ C$ .
  - 3 agents for  $B \succ C \succ A$ .
  - 498 agents for  $C \succ B \succ A$ .
- Who is the Condorcet winner?



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- Who is the winner under the plurality rule?



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- Who is the Condorcet winner? **B**.
- Who is the winner under the plurality rule? **A**.



# Exercise

## On Borda Count & Condorcet

We have five voters with the following preferences (ordering) over the outcomes  $A$ ,  $B$ ,  $C$ , and  $D$ .

- $B \succ C \succ A \succ D$ .
- $B \succ D \succ C \succ A$ .
- $D \succ C \succ A \succ B$ .
- $A \succ D \succ B \succ C$ .
- $A \succ D \succ C \succ B$ .

Who is the winner by the Borda Count rule?

Who is the Condorcet winner?



Let's consider a practical application in MOOCs.



# MOOCs

- MOOCs: Massive Online Open Courses
  - e.g., Coursera, EdX.



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  - ▷ Ask each student to grade a SMALL number of her peers' assignments.





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- Outsourcing the grading task to the students.
- They may have incentives to assign LOW scores to everybody else.
  - ▶ Ask each student to grade a SMALL number of her peers' assignments.
  - Then merge individual rankings into a global one.



# Terminologies

- $\mathcal{A}$ : universe of  $n$  elements (students).
- $(n, k)$ -grading scheme:  
a collection  $\mathcal{B}$  of size- $k$  subsets (**bundles**) of  $\mathcal{A}$ , such that each element of  $\mathcal{A}$  belongs to exactly  $k$  subsets of  $\mathcal{B}$ .
- The **bundle graph**:  
Represent the  $(n, k)$ -grading scheme with a bipartite graph.
- $\prec_b$ : a ranking of the element  $b$  contains (partial order).



# The aggregation rule

An aggregation rule:

profile of partial rankings  $\rightarrow$  complete ranking of all elements.

- Borda:

*SPRING FEAST 2016 BALLOT*

a	LE BLE D'OR	金色三季 JINSE SANJI	5
b	CRYSTAL SPOON	水晶勺 CRYSTAL SPOON	3
c	Bei Yuan Restaurant	北园餐厅 BEI YUAN	1
d	Tasty Steak	TASTY TASTY	2
e	Capricciosa	卡普里奇奥 CAPRICCIOSA	4

*SPRING FEAST 2016 BALLOT*

a	LE BLE D'OR	金色三季 JINSE SANJI	5
b	CRYSTAL SPOON	水晶勺 CRYSTAL SPOON	4
c	Bei Yuan Restaurant	北园餐厅 BEI YUAN	2
d	Tasty Steak	TASTY TASTY	1
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*SPRING FEAST 2016 BALLOT*

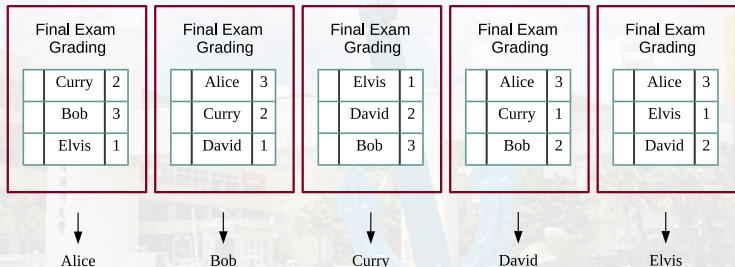
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d	Tasty Steak	TASTY TASTY	3
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- a: 14; b: 12; c: 4; d: 6; e: 9.

$a \succ b \succ e \succ d \succ c$ .

# Order-revealing grading scheme

An aggregation rule in peer grading (Borda):



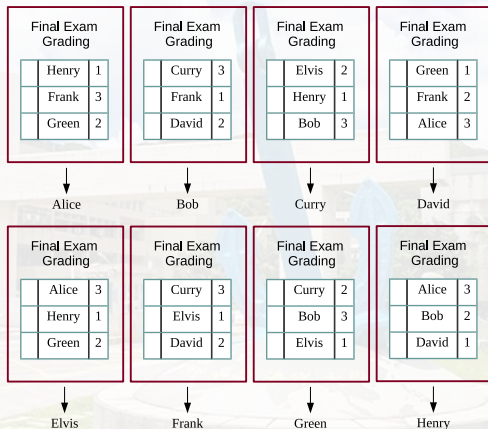
- Alice: 9; Bob: 8; Curry: 5; David: 5; Elvis: 3.

Alice  $\prec$  Bob  $\prec$  Curry  $\prec$  David  $\prec$  Elvis.

## Assumption (perfect grading)

Each student grades the assignments in her bundle **consistently** to the ground truth.

# Order-revealing grading scheme (contd.)

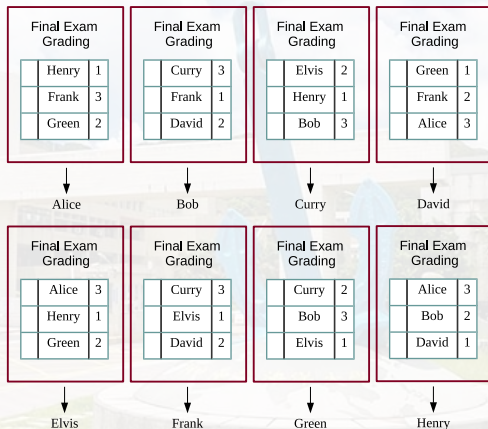


- Alice: 9; Bob: 8; Curry: 8; David: 5; Elvis: 4; Frank: 6; Green: 5; Henry: 3.

Alice  $\prec$  Bob  $\prec$  Curry  $\prec$  Frank  $\prec$  David  $\prec$  Green  $\prec$  Elvis  $\prec$  Henry.



# Order-revealing grading scheme (contd.)



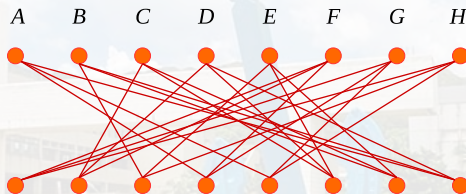
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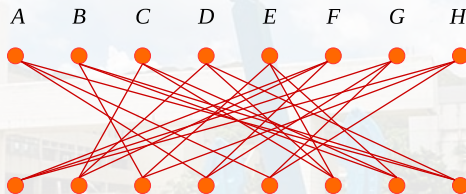
# The bundle graph

The bundle graph:



# The bundle graph

The bundle graph:



- A random  $k$ -regular graph:

A complete bipartite  $K_{n,n} \rightarrow$  removing edges  $\{v, v\}, \forall v \rightarrow$   
repeat

*“draw a perfect matching uniformly at random among all perfect matchings of the remaining graph”*

for  $k$  times.





# The limitation on the order revealing scheme

- The property of revealing the ground truth for certain:

$$\forall x, y \in \mathcal{A}, \exists B \in \mathcal{B} \text{ such that } x, y \in B.$$



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- Suppose NO bundle contains both  $x, y \in \mathcal{A}$ .
- Let  $\prec, \prec'$  be two complete rankings.
  - $x, y$  are in the first two positions in  $\prec, \prec'$ ;
  - $\prec$  and  $\prec'$  differs only in the order of  $x$  and  $y$ .
- Clearly, partial rankings within the bundles are identical in both cases.
- No way to identify whether  $\prec$  or  $\prec'$  is the ground truth.



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  - Clearly, partial rankings within the bundles are identical in both cases.
  - No way to identify whether  $\prec$  or  $\prec'$  is the ground truth.
- To reveal the ground truth with certainty:  $k = \Omega(\sqrt{n})$ .
    - $n \cdot \binom{k}{2} \geq \binom{n}{2}$ .



## Seeking for approximate order-revealing grading schemes

- Use a bundle graph with a very low degree  $k$  (independent of  $n$ ).
- Randomly permute the elements by  $\pi : U \rightarrow \mathcal{A}$  before associating them to the nodes of  $U$  of the bundle graph.
- Aiming at  $\frac{\text{\#correctly recovered pairwise relations}}{\binom{n}{2}}$ .



# The main result

## Theorem (Caragiannis, Krimpas, Voudouris@AAMAS'15)

When

- Borda is applied as the aggregation rule, and
- all the partial rankings are consistent to the ground truth,

then the expected fraction of correctly recovered pairwise relations is  $1 - O(1/\sqrt{k})$ .



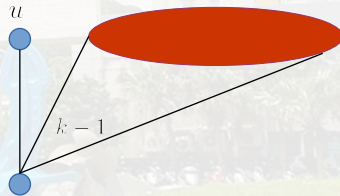
# Question

- What will happen if we assign for each student only two assignments and each assignment is graded by exactly two students?



# About who grading both $u$ and $v$

- $\lambda_{u,v} := |N(u) \cap N(v)|$ , for  $u, v \in U$ .
- $\sum_{v \in U \setminus \{u\}} \lambda_{u,v} = k(k-1)$ .



- $W_{r,q}$ : the r.v. denoting  $B(a_r) - B(a_q)$  for  $r < q$ ,  $a_r, a_q \in \mathcal{A}$ .
- $\Gamma_{u,v}^{r,q}$ : the event that  $\pi(u) = a_r$ ,  $\pi(v) = a_q$ .

$$C := \sum_{r=1}^{n-1} \sum_{q=r+1}^n \mathbb{E}[\mathbb{1}\{W_{r,q} > 0\}] = \sum_{r=1}^{n-1} \sum_{q=r+1}^n \Pr[W_{r,q} > 0]$$



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 \end{aligned}$$

- Given  $\Gamma_{u,v}^{r,q}$ ,
  - the expected Borda score of  $a_r$  is  $k + (k(k-1) - \lambda_{u,v}) \cdot \frac{n-r-1}{n-2} + \lambda_{u,v}$ .
  - the expected Borda score of  $a_q$  is  $k + (k(k-1) - \lambda_{u,v}) \cdot \frac{n-q}{n-2}$ .
- Thus

$$E[W_{r,q} \mid \Gamma_{u,v}^{r,q}] = (k(k-1) - \lambda_{u,v}) \frac{q-r-1}{n-2} + \lambda_{u,v}.$$



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$$E[W_{r,q} \mid \Gamma_{u,v}^{r,q}] = (k(k-1) - \lambda_{u,v}) \frac{q-r-1}{n-2} + \lambda_{u,v}.$$

Why?



## Calculate the Borda score from another point of view

- Element  $a_r$  gets one point for each bundle it belongs to;
  - plus one additional point for each appearance of an element with rank higher  $> r$  in the bundles  $a_r$  belongs to.



- In the bundles of containing  $a_r$ :
  - $\lambda_{u,v}$  appearances of  $a_q$  in the bundles of  $a_r$ .
  - $k(k-1) - \lambda_{u,v}$  appearances of elements different than  $a_r, a_q$ .
    - Each of them has prob.  $\frac{n-r-1}{n-2}$  to have rank higher than  $r$ .



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# Dealing with dependencies

Goal:  $\Pr[W_{r,q} \leq 0 \mid \Gamma_{u,v}^{r,q}]$

- Given  $\Gamma_{u,v}^{r,q}$ , define  $S = N(N(u) \cup N(v)) \setminus \{u, v\}$ .
- $o : [|S|] \rightarrow S$  denotes an arbitrary ordering of nodes of  $S$ .
- $X_i$ : the random variable denoting the rank of the element  $\pi(o(i))$ .
- Define the **Doob martingale**  $Z_0, Z_1, \dots, Z_{|S|}$  such that
  - $Z_0 = E[W_{r,q} \mid \Gamma_{u,v}^{r,q}]$ ;
  - $Z_i = E[W_{r,q} \mid \Gamma_{u,v}^{r,q}, X_1, \dots, X_i]$ .
- Hence,  $W_{r,q} \mid \Gamma_{u,v}^{r,q} = Z_{|S|}$ .



# Martingale

## Martingale

A sequence of random variables  $Z_0, Z_1, \dots, Z_m$  is a **martingale** w.r.t. a sequence of random variables  $X_1, X_2, \dots, X_m$  if  $\forall i = 1, \dots, m$ ,

$$E[Z_i \mid X_1, \dots, X_{i-1}] = Z_{i-1}.$$

## Doob martingale (Joseph L. Doob (1910–2004))

- $W$ : a random variable
- $X_1, \dots, X_m$ : a sequence of  $m$  random variables.

The sequence  $Z_0, Z_1, \dots, Z_m$  such that

- $Z_0 = E[W]$ ;
- $Z_i = E[W \mid X_1, \dots, X_i], \forall i = 1, \dots, m$

is called a **Doob martingale**.

# Azuma-Hoeffding inequality

## Azuma-Hoeffding inequality

Let  $Z_0, Z_1, \dots, Z_m$  be a martingale with  $Z_i - Z_{i-1} \leq c_i$  for  $i = 1, \dots, m$ . Then, for all  $t > 0$ ,

$$\Pr[Z_m - Z_0 \leq -t] \leq \exp\left(-\frac{t^2}{2 \sum_{i=1}^m c_i^2}\right).$$



# Dealing with dependencies (contd.)

- Given  $\Gamma_{u,v}^{r,q}$ , define  $S = N(N(u) \cup N(v)) \setminus \{u, v\}$ .
- $o : [|S|] \mapsto S$  denotes an arbitrary ordering of nodes of  $S$ .
- $X_i$ : the random variable denoting the rank of the element  $\pi(o(i))$ .
- Define the Doob martingale  $Z_0, Z_1, \dots, Z_{|S|}$  such that
  - $Z_0 = E[W_{r,q} \mid \Gamma_{u,v}^{r,q}]$ ;
  - $Z_i = E[W_{r,q} \mid \Gamma_{u,v}^{r,q}, X_1, \dots, X_i]$ .
- Hence,  $W_{r,q} \mid \Gamma_{u,v}^{r,q} = Z_{|S|}$ .

## Lemma 8

$\forall i \in \{1, 2, \dots, |S|\}$ , it holds that  $|Z_i - Z_{i-1}| \leq 2(\lambda_{u,o(i)} + \lambda_{v,o(i)})$ .

## Lemma 3

For every  $k$ -regular bipartite graph  $G$ ,

$$\theta_{u,v} = 4 \sum_{z \in N(N(u) \cup N(v)) \setminus \{u,v\}} (\lambda_{u,z} + \lambda_{v,z})^2 \leq 8k(k-1)(4k-3).$$

- Set  $t = E[W_{r,q} \mid \Gamma_{u,v}^{r,q}] (= Z_0)$ , by the **Azuma-Hoeffding inequality**:

$$\begin{aligned} \Pr[Z_{|S|} - Z_0 \leq -t] &= \Pr[W_{r,q} \leq 0 \mid \Gamma_{u,v}^{r,q}] \\ &\leq \exp\left(-\frac{t^2}{2 \sum_{i=1}^m c_i^2}\right) \\ &= \exp\left(-\frac{E[W_{r,q} \mid \Gamma_{u,v}^{r,q}]^2}{2\theta_{u,v}}\right). \end{aligned}$$



# Back to the computation of $C$

$$\begin{aligned}
 C &= \sum_{r=1}^{n-1} \sum_{q=r+1}^n \left( 1 - \frac{1}{n(n-1)} \sum_{u,v \in U} \Pr[W_{r,q} \leq 0 \mid \Gamma_{u,v}^{r,q}] \right) \\
 &\geq \sum_{r=1}^{n-1} \sum_{q=r+1}^n \left( 1 - \frac{1}{n(n-1)} \sum_{u,v \in U} \exp \left( -\frac{E[W_{r,q} \mid \Gamma_{u,v}^{r,q}]^2}{2\theta_{u,v}} \right) \right) \\
 &= \sum_{r=1}^{n-1} \sum_{q=r+1}^n \left( 1 - \frac{1}{n(n-1)} \sum_{u,v \in U} e^{-(\beta(u,v) \cdot y(q-r) + \delta(u,v))^2} \right) \\
 &= \frac{n(n-1)}{2} - \frac{1}{n(n-1)} \sum_{u,v \in U} \sum_{d=1}^{n-1} (n-d) e^{-(\beta(u,v) \cdot y(d) + \delta(u,v))^2} \\
 &\geq \frac{n(n-1)}{2} - \sum_{u,v \in U} \int_0^1 (1-y) e^{-(\beta(u,v) \cdot y + \delta(u,v))^2} dy
 \end{aligned}$$

- $\beta(u, v) = \frac{k(k-1) - \lambda_{u,v}}{\sqrt{2\theta_{u,v}}}$ ;
- $\delta(u, v) = \frac{\lambda_{u,v}}{\sqrt{2\theta_{u,v}}}$ ;
- $y(t) = \frac{t-1}{n-2}$ .





$$\begin{aligned}
C &\geq \frac{n(n-1)}{2} - \sum_{u,v \in U} \int_0^1 (1-y) e^{-(\beta(u,v) \cdot y + \delta(u,v))^2} dy \\
&\geq \frac{n(n-1)}{2} - \sum_{u,v \in U} \frac{\beta(u,v) + \delta(u,v)}{2\beta(u,v)^2} \sqrt{\pi} \\
&\geq \frac{n(n-1)}{2} - \frac{k-1}{k(k-2)^2} \sqrt{\frac{\pi}{2}} \sum_{u,v \in U} \sqrt{\theta_{u,v}} \\
&\geq \frac{n(n-1)}{2} \left( 1 - \frac{48\sqrt{2\pi}}{\sqrt{k}} \right).
\end{aligned}$$

### Claim 9

Let  $\beta > 0$ ,  $\delta \geq 0$ , then

$$\int_0^1 (1-y) e^{-(\beta y + \delta)^2} dy \leq \frac{\beta + \delta}{2\beta^2} \sqrt{\pi}.$$

- $k \geq 3$  (assumption).

# Proof of Lemma 8

Recall:

- Given  $\Gamma_{u,v}^{r,q}$ , define  $S = N(N(u) \cup N(v)) \setminus \{u, v\}$ .
- $o : [|S|] \rightarrow S$  denotes an arbitrary ordering of nodes of  $S$ .
- $X_i$ : the random variable denoting the rank of the element  $\pi(o(i))$ .
- Define the Doob martingale  $Z_0, Z_1, \dots, Z_{|S|}$  such that
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  - $Z_i = E[W_{r,q} \mid \Gamma_{u,v}^{r,q}, X_1, \dots, X_i]$ .
- Hence,  $W_{r,q} \mid \Gamma_{u,v}^{r,q} = Z_{|S|}$ .

## Lemma 8

$\forall i \in \{1, 2, \dots, |S|\}$ , it holds that  $|Z_i - Z_{i-1}| \leq 2(\lambda_{u,o(i)} + \lambda_{v,o(i)})$ .

- $\mu_{u,v,w} = |N(u) \cap N(v) \cap N(w)|$ .



# Key to the proof of Lemma 8

- The Borda score difference  $W_{r,q}$  (conditioned on  $\Gamma_{u,v}^{r,q}$ ):



# Key to the proof of Lemma 8

- The Borda score difference  $W_{r,q}$  (conditioned on  $\Gamma_{u,v}^{r,q}$ ):
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  - **increases** for each appearance of  $\pi(o(j))$  in a bundle containing  $a_r$  but NOT  $a_q$  provided that  $r < \text{rank}(\pi(o(j))) < q$ ;
  - **increases** for each appearance of  $\pi(o(j))$  in a bundle containing BOTH  $a_r, a_q$  provided that  $r < \text{rank}(\pi(o(j))) < q$ ;
  - **decreases** for each appearance of  $\pi(o(j))$  in a bundle containing  $a_q$  but NOT  $a_r$  provided that  $\text{rank}(\pi(o(j))) > q$ .

$$W_{r,q} =$$



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$$W_{r,q} = \lambda_{u,v}$$



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$$W_{r,q} = \lambda_{u,v} + \sum_{j=1}^{|S|} (\lambda_{u,o(j)} - \mu_{u,v,o(j)}) \cdot \mathbb{1}\{X_j > r\}$$



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- The Borda score difference  $W_{r,q}$  (conditioned on  $\Gamma_{u,v}^{r,q}$ ):
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$$W_{r,q} = \lambda_{u,v} + \sum_{j=1}^{|S|} (\lambda_{u,o(j)} - \mu_{u,v,o(j)}) \cdot \mathbb{1}\{X_j > r\} + \sum_{j=1}^{|S|} \mu_{u,v,o(j)} \cdot \mathbb{1}\{r < X_j < q\}$$



# Key to the proof of Lemma 8

- The Borda score difference  $W_{r,q}$  (conditioned on  $\Gamma_{u,v}^{r,q}$ ):
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$$\begin{aligned}
 W_{r,q} &= \lambda_{u,v} + \sum_{j=1}^{|S|} (\lambda_{u,o(j)} - \mu_{u,v,o(j)}) \cdot \mathbb{1}\{X_j > r\} + \sum_{j=1}^{|S|} \mu_{u,v,o(j)} \cdot \mathbb{1}\{r < X_j < q\} \\
 &\quad - \sum_{j=1}^{|S|} (\lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > r\} - \lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > q\})
 \end{aligned}$$





# Key to the proof of Lemma 8

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  - **increases** for each appearance of  $a_q$  in the same bundle with  $a_r$ ;
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  - **increases** for each appearance of  $\pi(o(j))$  in a bundle containing BOTH  $a_r, a_q$  provided that  $r < \text{rank}(\pi(o(j))) < q$ ;
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$$\begin{aligned}
 W_{r,q} &= \lambda_{u,v} + \sum_{j=1}^{|S|} (\lambda_{u,o(j)} - \mu_{u,v,o(j)}) \cdot \mathbb{1}\{X_j > r\} + \sum_{j=1}^{|S|} \mu_{u,v,o(j)} \cdot \mathbb{1}\{r < X_j < q\} \\
 &\quad - \sum_{j=1}^{|S|} (\lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > r\} - \lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > q\}) \\
 &= \lambda_{u,v} + \sum_{j=1}^{|S|} (\lambda_{u,o(j)} \cdot \mathbb{1}\{X_j > r\} - \lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > q\}).
 \end{aligned}$$



# Key to the proof of Lemma 8 (contd.)

$$W_{r,q} = \lambda_{u,v} + \sum_{j=1}^{|S|} (\lambda_{u,o(j)} \cdot \mathbb{1}\{X_j > r\} - \lambda_{v,o(j)} \cdot \mathbb{1}\{X_j > q\})$$

$$\begin{aligned} Z_i - Z_{i-1} &= E[W_{r,q} \mid \Gamma_{u,v}^{r,q}, X_1, \dots, X_i] - E[W_{r,q} \mid \Gamma_{u,v}^{r,q}, X_1, \dots, X_{i-1}] \\ &= \sum_{j=i}^{|S|} \lambda_{u,o(j)} (\Pr[X_j > r \mid X_1, \dots, X_i] - \Pr[X_j > r \mid X_1, \dots, X_{i-1}]) - \\ &\quad \sum_{j=i}^{|S|} \lambda_{v,o(j)} (\Pr[X_j > q \mid X_1, \dots, X_i] - \Pr[X_j > q \mid X_1, \dots, X_{i-1}]) . \end{aligned}$$



# Key to the proof of Lemma 8 (contd.)

$$\begin{aligned}
 Z_i - Z_{i-1} &= E[W_{r,q} \mid \Gamma_{u,v}^{r,q}, X_1, \dots, X_i] - E[W_{r,q} \mid \Gamma_{u,v}^{r,q}, X_1, \dots, X_{i-1}] \\
 &= \sum_{j=i}^{|S|} \lambda_{u,o(j)} (\Pr[X_j > r \mid X_1, \dots, X_i] - \Pr[X_j > r \mid X_1, \dots, X_{i-1}]) - \\
 &\quad \sum_{j=i}^{|S|} \lambda_{v,o(j)} (\Pr[X_j > q \mid X_1, \dots, X_i] - \Pr[X_j > q \mid X_1, \dots, X_{i-1}]) .
 \end{aligned}$$

Note that:

$$\Pr[X_j > r \mid X_1, \dots, X_{i-1}] = \frac{x+y}{n-i-1}, \quad \Pr[X_j > q \mid X_1, \dots, X_{i-1}] = \frac{y}{n-i-1} \quad (i \leq j \leq |S|)$$

$$\Pr[X_j > r \mid X_1, \dots, X_i] = \frac{x+y-\mathbb{1}\{X_i > r\}}{n-i-2}, \quad \Pr[X_j > q \mid X_1, \dots, X_i] = \frac{y-\mathbb{1}\{X_i > q\}}{n-i-2} \quad (i+1 \leq j \leq |S|)$$

- $x$ : # available ranks from  $[n] \setminus \{r, q, X_1, \dots, X_{i-1}\}$  that are between  $r$  and  $q$ ;
- $y$ : # available ranks from  $[n] \setminus \{r, q, X_1, \dots, X_{i-1}\}$  that are  $> q$ .



# Key to the proof of Lemma 8 (contd.)

$$\begin{aligned}
 Z_i - Z_{i-1} &= \lambda_{u,o(i)} \left( \mathbb{1}\{X_i > r\} - \frac{x+y}{n-i-1} \right) - \lambda_{v,o(j)} \left( \mathbb{1}\{X_i > q\} - \frac{y}{n-i-1} \right) \\
 &+ \sum_{j=i+1}^{|S|} \lambda_{u,o(j)} \left( \frac{x+y - \mathbb{1}\{X_i > r\}}{n-i-2} - \frac{x+y}{n-i-1} \right) \\
 &+ \sum_{j=i+1}^{|S|} \lambda_{v,o(j)} \left( \frac{y - \mathbb{1}\{X_i > q\}}{n-i-2} - \frac{y}{n-i-1} \right)
 \end{aligned}$$

Note that:

$$\Pr[X_j > r \mid X_1, \dots, X_{i-1}] = \frac{x+y}{n-i-1}, \quad \Pr[X_j > q \mid X_1, \dots, X_{i-1}] = \frac{y}{n-i-1} \quad (i \leq j \leq |S|)$$

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- $y$ : # available ranks from  $[n] \setminus \{r, q, X_1, \dots, X_{i-1}\}$  that are  $> q$ .



# Key to the proof of Lemma 8 (contd.)

$$\begin{aligned}
 Z_i - Z_{i-1} &= \lambda_{u,o(i)} \left( \mathbb{1}\{X_i > r\} - \frac{x+y}{n-i-1} \right) - \lambda_{v,o(j)} \left( \mathbb{1}\{X_i > q\} - \frac{y}{n-i-1} \right) \\
 &+ \sum_{j=i+1}^{|S|} \lambda_{u,o(j)} \left( \frac{x+y - \mathbb{1}\{X_i > r\}}{n-i-2} - \frac{x+y}{n-i-1} \right) \\
 &+ \sum_{j=i+1}^{|S|} \lambda_{v,o(j)} \left( \frac{y - \mathbb{1}\{X_i > q\}}{n-i-2} - \frac{y}{n-i-1} \right)
 \end{aligned}$$

Note that:

$$\Pr[X_j > r \mid X_1, \dots, X_{i-1}] = \frac{x+y}{n-i-1}, \quad \Pr[X_j > q \mid X_1, \dots, X_{i-1}] = \frac{y}{n-i-1} \quad (i \leq j \leq |S|)$$

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- $x$ : # available ranks from  $[n] \setminus \{r, q, X_1, \dots, X_{i-1}\}$  that are between  $r$  and  $q$ ;
- $y$ : # available ranks from  $[n] \setminus \{r, q, X_1, \dots, X_{i-1}\}$  that are  $> q$ .



# Key to the proof of Lemma 8 (contd.)

(Assume  $n \geq 3k(k-1) + 2$ ; Note:  $|S| \leq 2k(k-1)$ )

$$\begin{aligned}
 Z_i - Z_{i-1} &= \lambda_{u,o(i)} \left( \mathbb{1}\{X_i > r\} - \frac{x+y}{n-i-1} \right) - \lambda_{v,o(j)} \left( \mathbb{1}\{X_i > q\} - \frac{y}{n-i-1} \right) \\
 &+ \sum_{j=i+1}^{|S|} \lambda_{u,o(j)} \left( \frac{x+y - \mathbb{1}\{X_i > r\}}{n-i-2} - \frac{x+y}{n-i-1} \right) \\
 &+ \sum_{j=i+1}^{|S|} \lambda_{v,o(j)} \left( \frac{y - \mathbb{1}\{X_i > q\}}{n-i-2} - \frac{y}{n-i-1} \right) \\
 &= \left( \lambda_{u,o(i)} - \frac{\sum_{j=i+1}^{|S|} \lambda_{u,o(j)}}{n-i-2} \right) \cdot \left( \mathbb{1}\{X_i > r\} - \frac{x+y}{n-i-1} \right) \\
 &+ \left( \lambda_{v,o(i)} - \frac{\sum_{j=i+1}^{|S|} \lambda_{v,o(j)}}{n-i-2} \right) \cdot \left( \frac{y}{n-i-1} - \mathbb{1}\{X_i > q\} \right).
 \end{aligned}$$

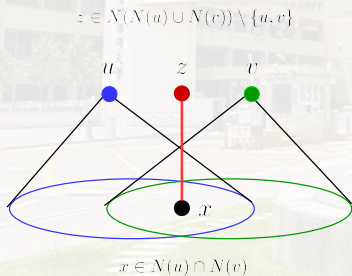


# Key to the proof of Lemma 3

## Lemma 3

For every  $k$ -regular bipartite graph  $G$ ,

$$\theta_{u,v} = 4 \sum_{z \in N(N(u) \cup N(v)) \setminus \{u,v\}} (\lambda_{u,z} + \lambda_{v,z})^2 \leq 8k(k-1)(4k-3).$$



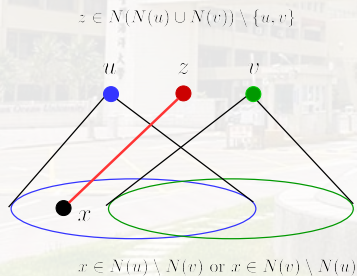
- The edge contributes 2 to the quantity  $\lambda_{u,z} + \lambda_{v,z}$ .
  - $\lambda_{u,z} + \lambda_{v,z} \leq 2k$ .
- The edge contributes  $\leq (2k)^2 - (2k-2)^2 = 8k - 4$  to the quantity  $(\lambda_{u,z} + \lambda_{v,z})^2$ .
- There are  $|N(u) \cap N(v)|(k-2)$  such edges.

# Key to the proof of Lemma 3 (contd.)

## Lemma 3

For every  $k$ -regular bipartite graph  $G$ ,

$$\theta_{u,v} = 4 \sum_{z \in N(N(u) \cup N(v)) \setminus \{u,v\}} (\lambda_{u,z} + \lambda_{v,z})^2 \leq 8k(k-1)(4k-3).$$



- The edge contributes 1 to the quantity  $\lambda_{u,z} + \lambda_{v,z}$ .
  - $\lambda_{u,z} + \lambda_{v,z} \leq 2k - 1$ .
- The edge contributes  $\leq (2k - 1)^2 - (2k - 2)^2 = 4k - 3$  to the quantity  $(\lambda_{u,z} + \lambda_{v,z})^2$ .
- There are  $2(k - |N(u) \cap N(v)|)(k - 1)$  such edges.



# Proof of the Gaussian integral (Claim 9)



$$\int_0^1 (1-y)e^{-(\beta y+\delta)^2} dy = \underbrace{\int_0^1 e^{-(\beta y+\delta)^2} dy}_A - \underbrace{\int_0^1 ye^{-(\beta y+\delta)^2} dy}_B$$

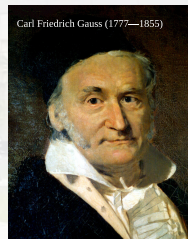
- The **error function**:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

- The **Gaussian integral**:

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}.$$

$$\therefore \int_0^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2} \quad \text{and} \quad \operatorname{erf}(x) \leq \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1.$$



- $B = \int_0^1 ye^{-(\beta y + \delta)^2} dy.$

$$\text{Let } v = \beta y + \delta \quad \therefore \begin{cases} dv = \beta dy \\ y = \frac{v - \delta}{\beta} \end{cases}.$$

$$\begin{aligned} \therefore B &= \frac{1}{\beta} \int_{\delta}^{\beta + \delta} \frac{v - \delta}{\beta} \cdot e^{-v^2} dv \\ &= \frac{1}{\beta^2} \left( \int_{\delta}^{\beta + \delta} v \cdot e^{-v^2} dv - \delta \cdot \int_{\delta}^{\beta + \delta} e^{-v^2} dv \right) \\ &= \frac{1}{2\beta^2} \int_{\delta}^{\beta + \delta} e^{-v^2} dv^2 - \frac{\delta}{\beta^2} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{2}{\sqrt{\pi}} \int_{\delta}^{\beta + \delta} e^{-v^2} dv \\ &= \frac{1}{2\beta^2} \left( -e^{-(\beta + \delta)^2} + e^{-\delta^2} \right) - \frac{\delta\sqrt{\pi}}{2\beta^2} (\text{erf}(\beta + \delta) - \text{erf}(\delta)). \end{aligned}$$

- $A = \int_0^1 e^{-(\beta y + \delta)^2} dy.$

Let  $u = \beta y + \delta \quad \therefore du = \beta dy.$

$$\begin{aligned} \therefore A &= \frac{1}{\beta} \int_{\delta}^{\beta + \delta} e^{-u^2} du \\ &= \frac{\sqrt{\pi}}{2\beta} \cdot \frac{2}{\sqrt{\pi}} \left( \int_0^{\beta + \delta} e^{-u^2} du - \int_0^{\delta} e^{-u^2} du \right) \\ &= \frac{\sqrt{\pi}}{2\beta} (\text{erf}(\beta + \delta) - \text{erf}(\delta)). \end{aligned}$$

- Thus

$$\begin{aligned} A - B &= \frac{\beta + \delta}{2\beta^2} \sqrt{\pi} (\text{erf}(\beta + \delta) - \text{erf}(\delta)) + \frac{1}{2\beta^2} (e^{-(\beta + \delta)^2} - e^{-\delta^2}) \\ &\leq \frac{\beta + \delta}{2\beta^2} \sqrt{\pi}. \end{aligned}$$