Connected Components & Biconnected Components

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Outline

1. Connected Components
   - Spanning Trees
   - Articulation Points & Biconnected Graph
   - Finding the articulation points

Outline

Connectivity

Problem I

Determine if an (un)directed graph is connected.

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List all connected components of an (un)directed graph.

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Problem II

List all connected components of an (un)directed graph.

This can be done by making repeated calls to either `dfs(v)` or `bfs(v)` where v is an unvisited vertex.

```c
void connected(void) { // dfs(0) or bfs(0)
/* determine the connected components of a graph */
    int i;
    for (i=0; i<n; i++) {
        if (!visited[i]) {
            dfs(i);
            printf("\n");
        }
    }
}
```

Analysis of `connected`

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- Since the for loop takes $O(n)$ time, the total time needed to generate all the connected components is $O(n + e)$.
- If G is represented by an adjacency matrix, then the time needed to determine the connected components is $O(n^2)$.

Spanning Trees

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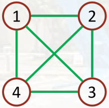
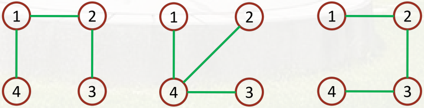
- When graph G is connected, a DFS or BFS implicitly partitions the edges in G into two sets:
  - Tree edges: the set of edges used or traversed during the search.
  - Nontree edges: the set of remaining edges.

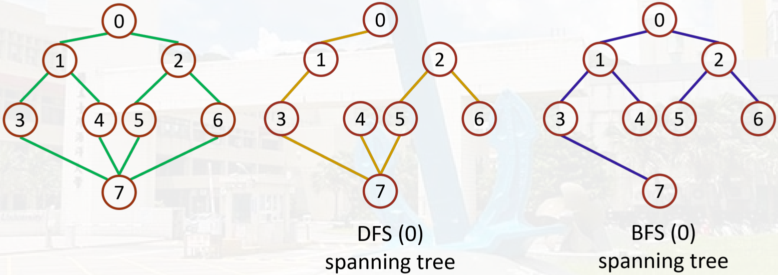
G_1



Three spanning trees of G_1.



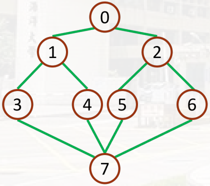
DFS Spanning Trees & BFS Spanning Trees

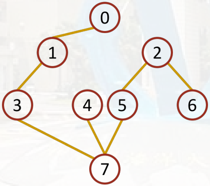


Properties of (DFS or BFS) Spanning Trees

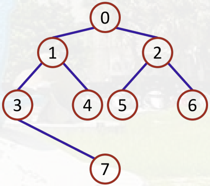
Property I

Suppose we add a nontree edge, (v, w), into any spanning tree, T. The result is a cycle that consists of the edge (v, w) and all the edges on the path from w to v in T.





DFS (0)
spanning tree



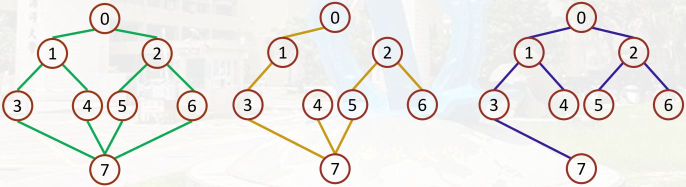
BFS (0)
spanning tree

Properties of (DFS or BFS) Spanning Trees

Property II

A spanning tree is a minimal subgraph, G', of G such that $V(G') = V(G)$ and G' is connected.

- A spanning tree has $n - 1$ edges.



DFS (0)
spanning tree

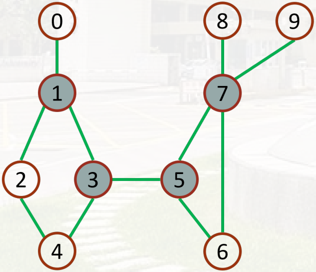
BFS (0)
spanning tree

Articulation Points

Articulation Points

An **articulation point** is a vertex v of G such that the deletion of v, together with all edges incident on v, produces a graph, G', that has ≥ 2 connected components.

a connected graph



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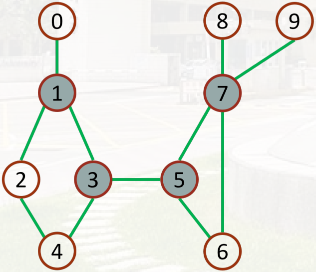

- Four articulation points:

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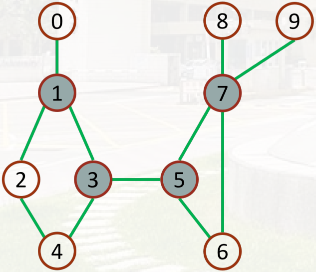

- Four articulation points: 1, 3, 5, 7.

Biconnected Graph (雙連通圖)

Biconnected Graph

A biconnected graph is a connected graph that has NO articulation points.

Biconnected Component

A biconnected component of a connected graph G is a maximal biconnected subgraph H of G.

- H is "maximal": no other subgraph that is both biconnected and properly contains H.

Biconnected Graph (雙連通圖)

Biconnected Graph

A biconnected graph is a connected graph that has NO articulation points.

- A connected graph G which has an articulation point $\Rightarrow$ G is NOT biconnected.

Biconnected Component

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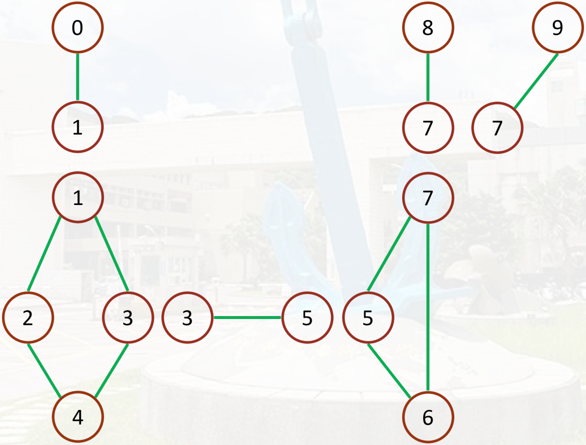
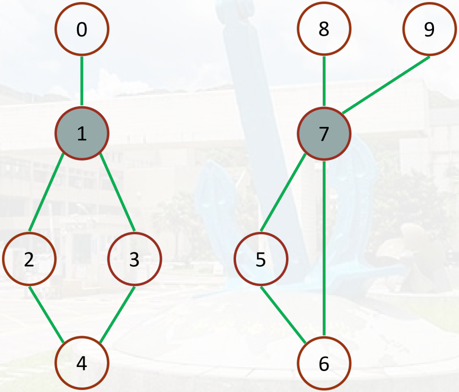
Biconnected Components (an Example)



Biconnected Components (NOT an Example)

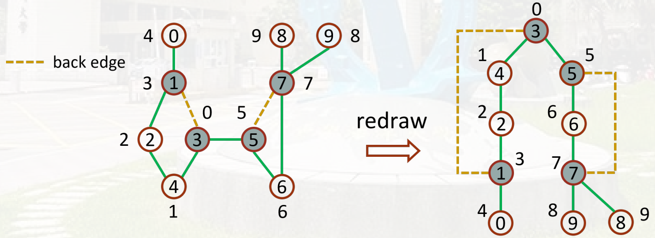


Finding articulation points (1/3)

We can find biconnected components of a graph G using any depth-first spanning of G.

Back edges

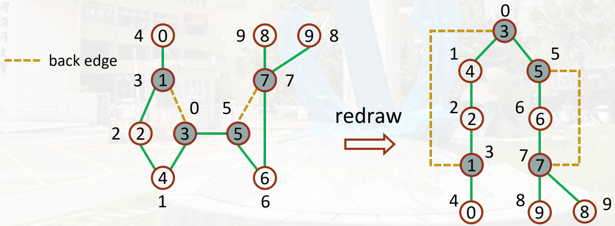
- Tree edges: DFS
- Nontree edges: we call them back edges



(a) depth first spanning tree **(dfs(3))** (b)

Observations

- The root of a depth first spanning tree is an articulation point if and only if it has ≥ 2 children.
- Any other vertex u is an articulation point if and only if it has ≥ 1 child w such that we **cannot** reach an ancestor of u using that consists of only w, descendants of w, and a single back edge.



(a) depth first spanning tree **(dfs(3))** (b)

- v_5 is an articulation point

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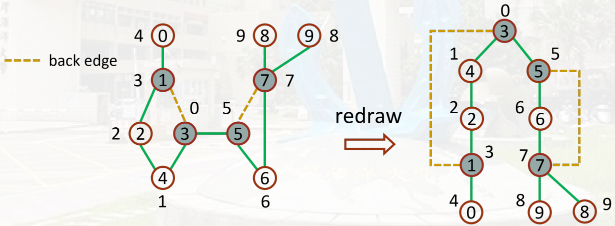
- The root of a depth first spanning tree is an articulation point if and only if it has ≥ 2 children.
- Any other vertex u is an articulation point if and only if it has $\geq$ 1 child w such that we **cannot** reach an ancestor of u using that consists of only w, descendants of w, and a single back edge.



(a) depth first spanning tree (dfs(3)) (b)

- v_5 is an articulation point, but v_6 is NOT.

Finding articulation points (2/3)

dfn(v)

The depth first numbers, or dfn, of the vertices give the sequence in which the vertices are visited during the depth first search.

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Finding articulation points (2/3)

dfn(v)

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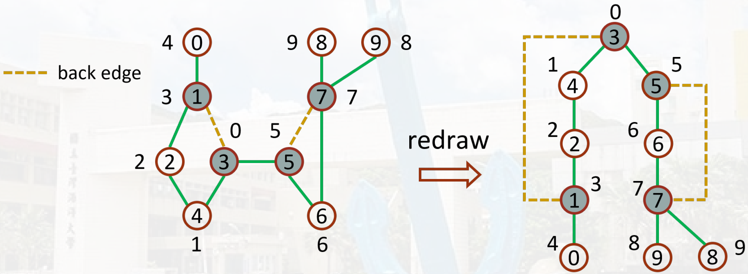
- If u is an ancestor of v in the depth first spanning tree, then $\text{dfn}(u) < \text{dfn}(v)$.

low(v)

The $\text{low}(u)$ value of vertex u is the lowest depth first number that we can reach from u using a path of descendants followed by at most 1 back edge.

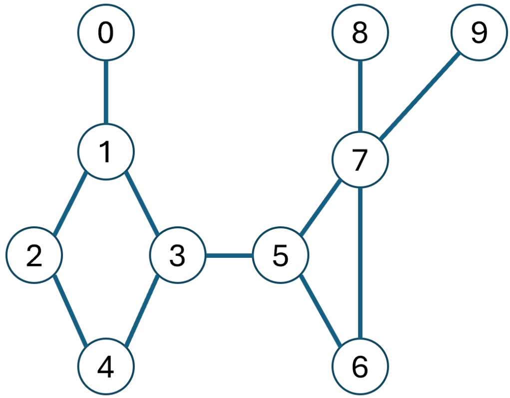
$$\text{low}(u) = \min \begin{cases} \text{dfn}(u), \\ \min\{\text{low}(w) \mid w \text{ is a child of } u\}, \cdots (*) \\ \min\{\text{dfn}(w) \mid (u, w) \text{ is a back edge}\} \cdots (**) \end{cases}$$

Example of Computing dfn and low values



(a) depth first spanning tree **(dfs(3))** (b)

vertex	0	1	2	3	4	5	6	7	8	9
dfn	4	3	2	0	1	5	6	7	9	8
low	4	0	0	0	0	5	5	5	9	8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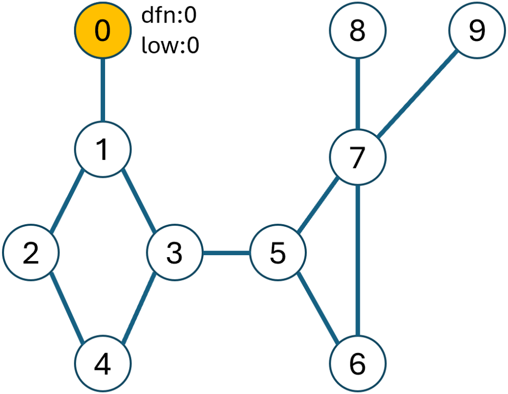
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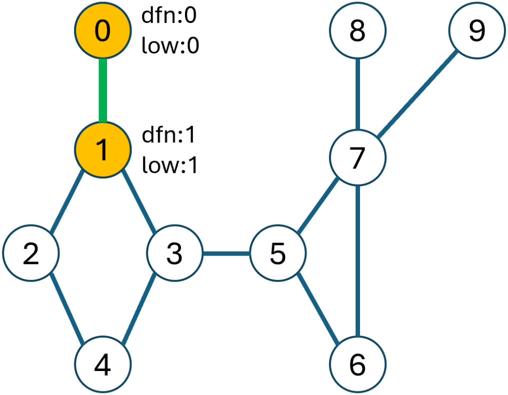
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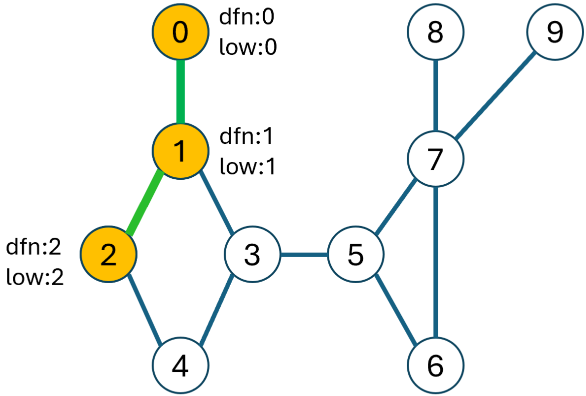
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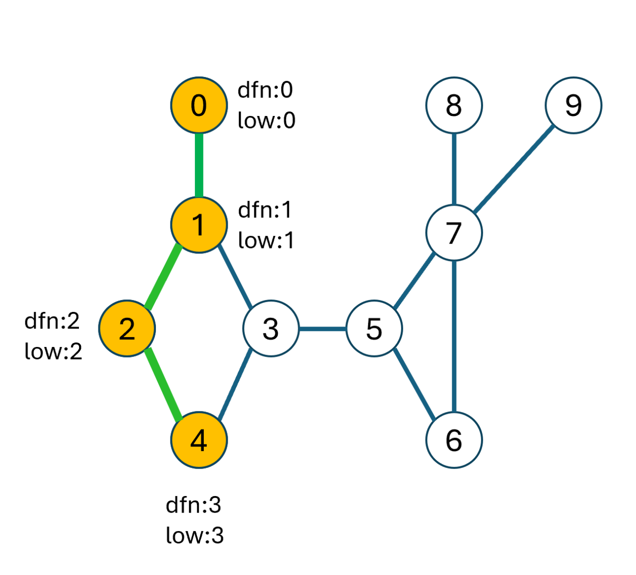
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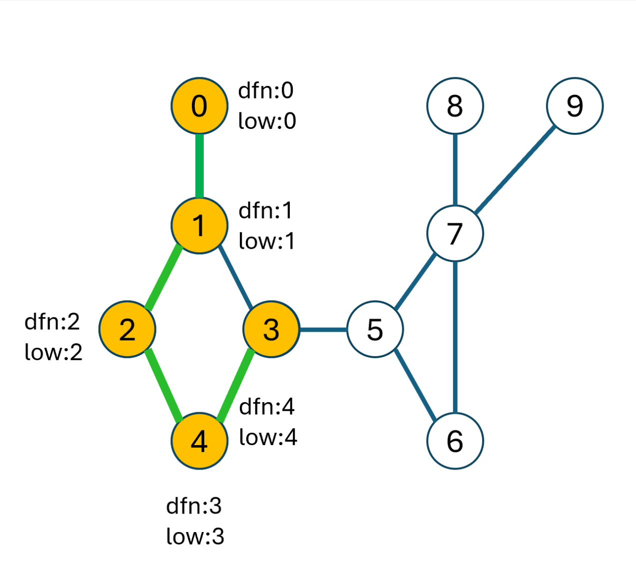
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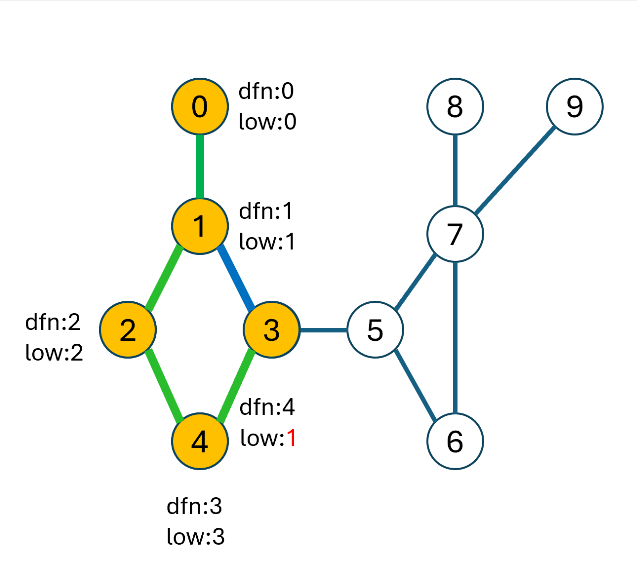
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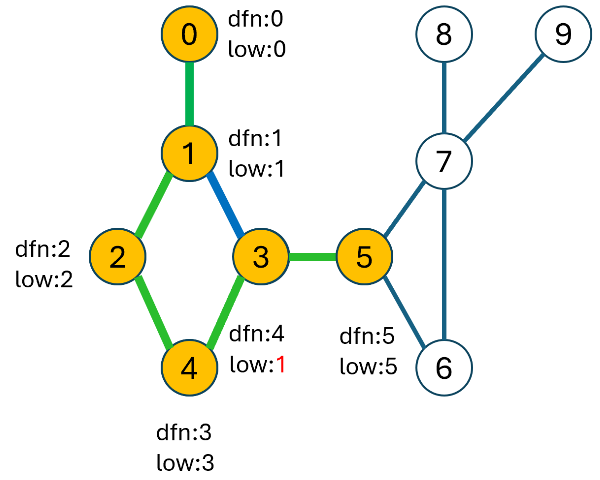
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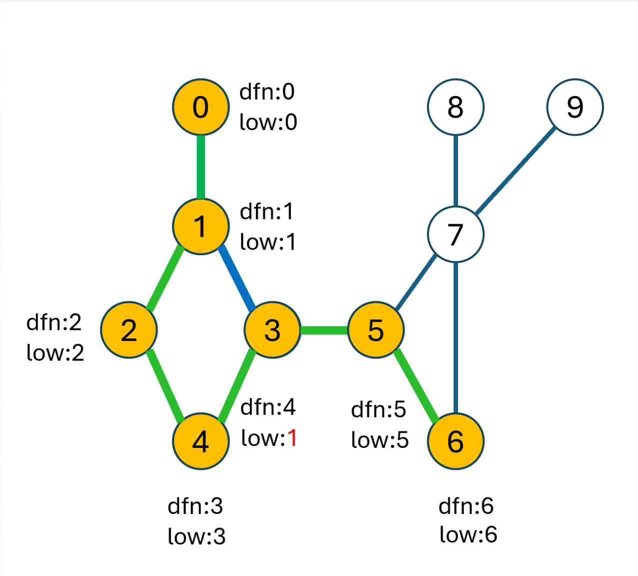
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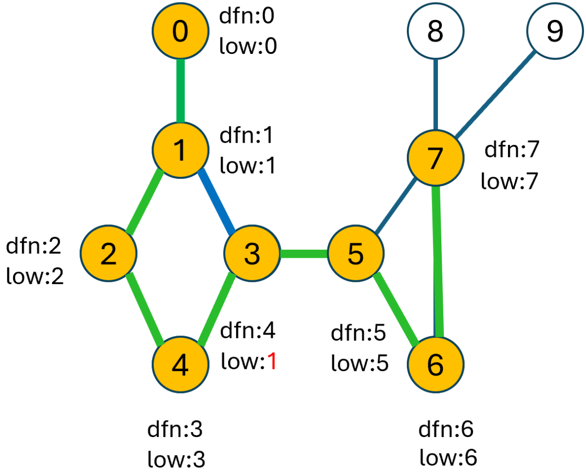
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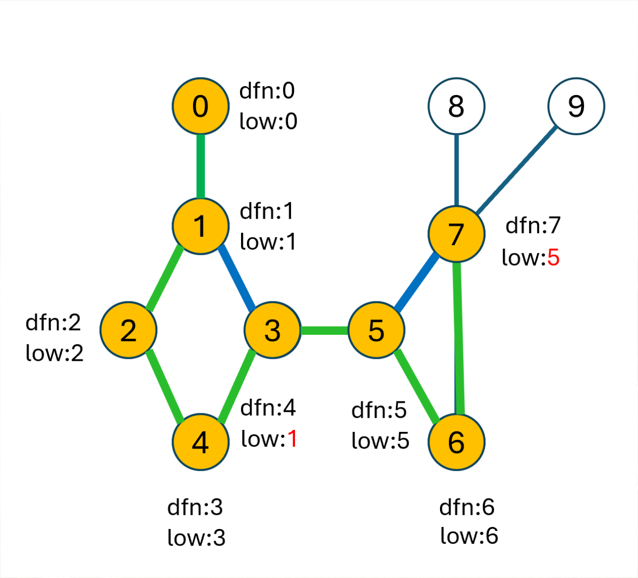
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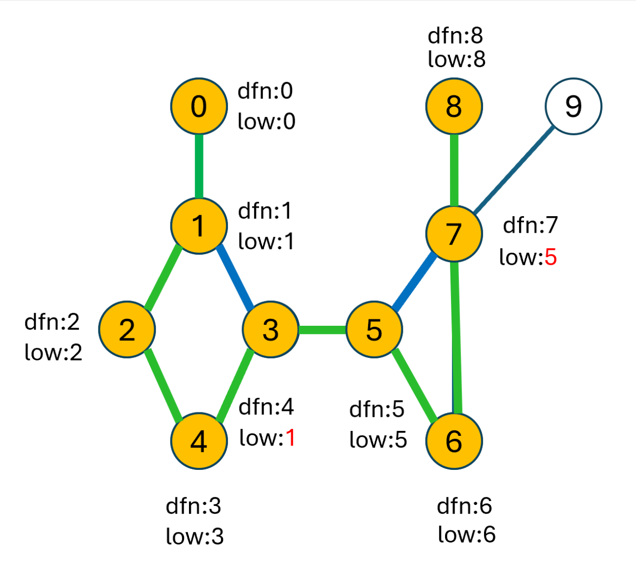
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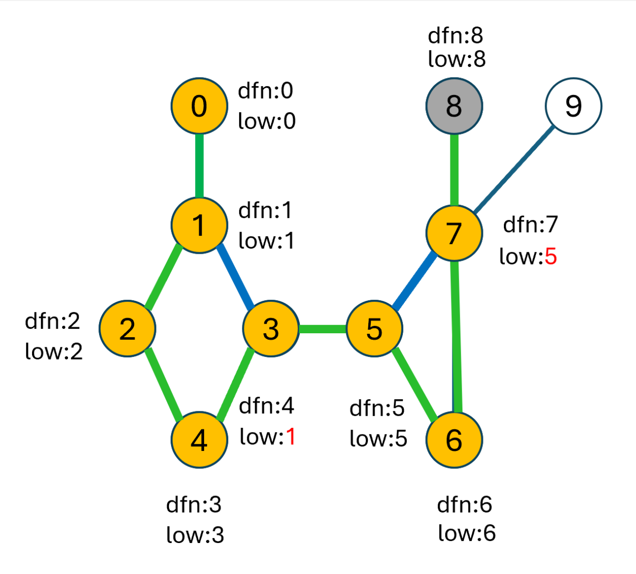
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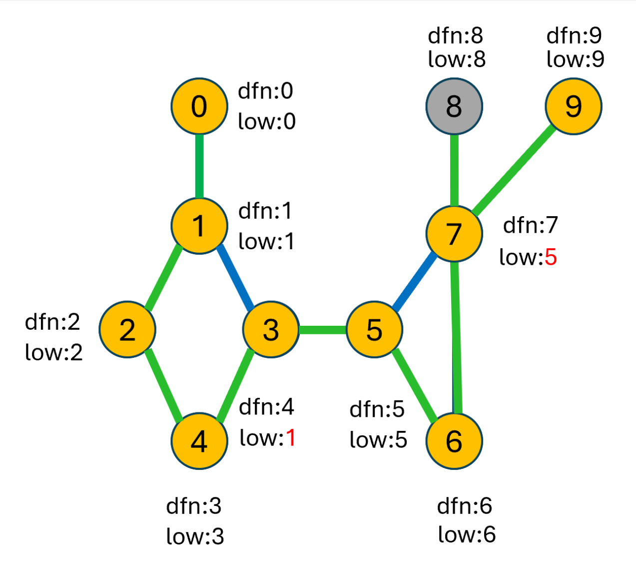
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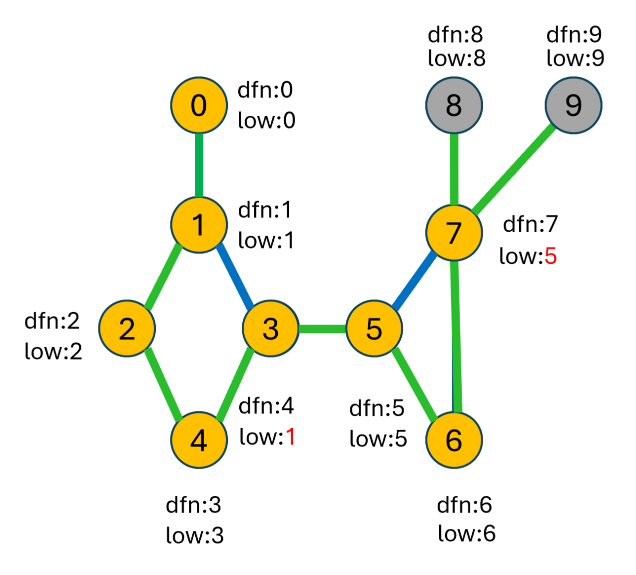
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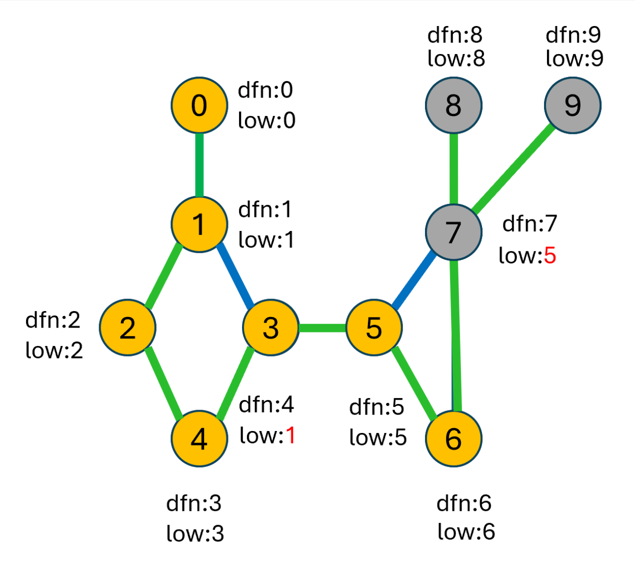
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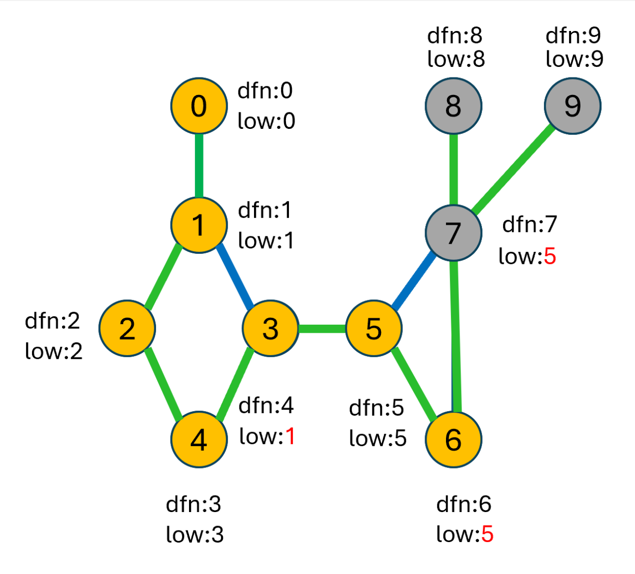
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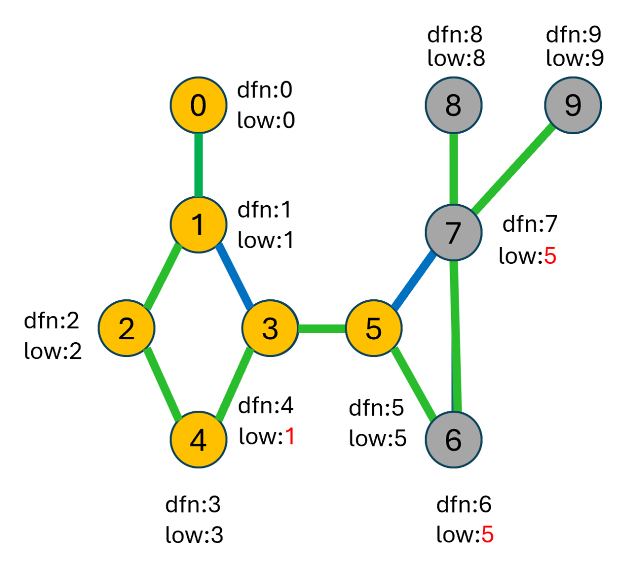
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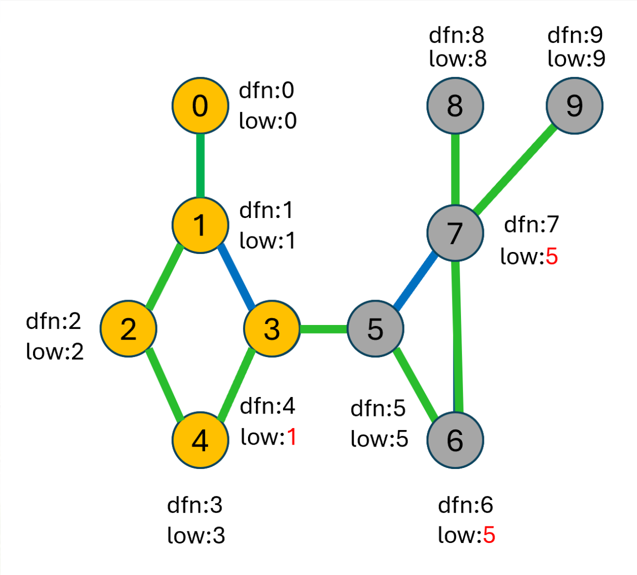
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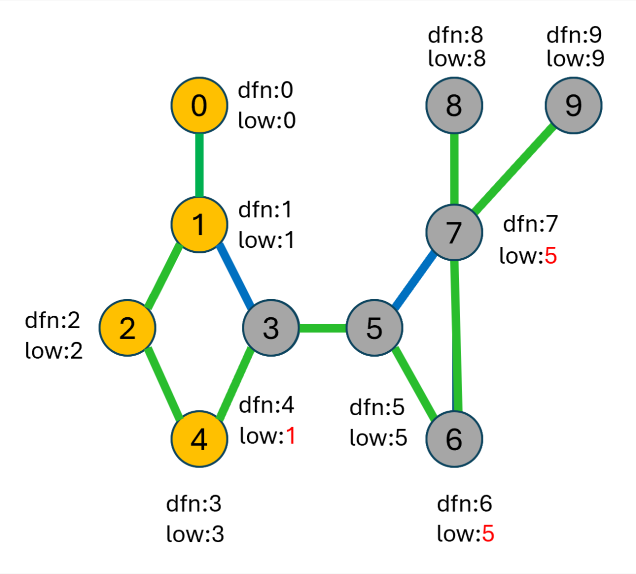
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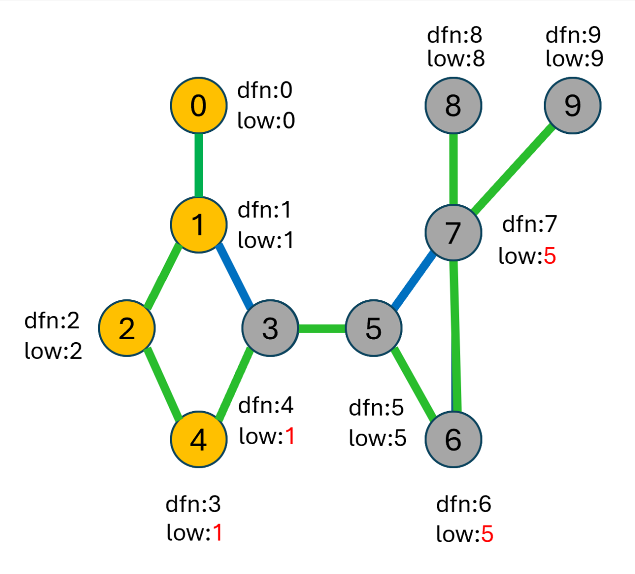
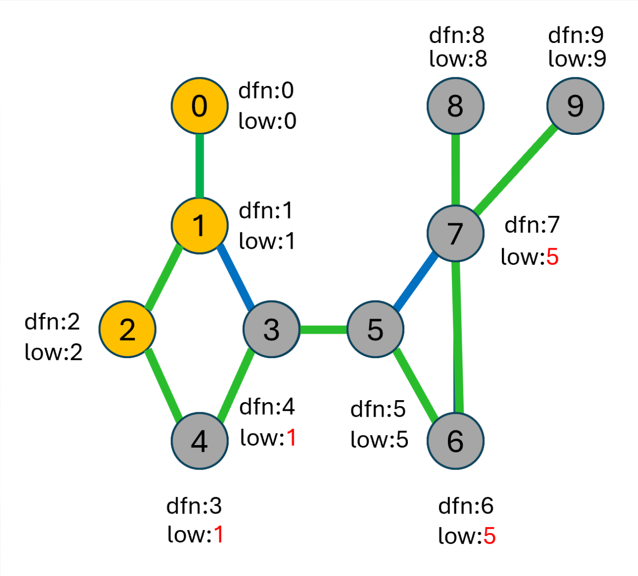
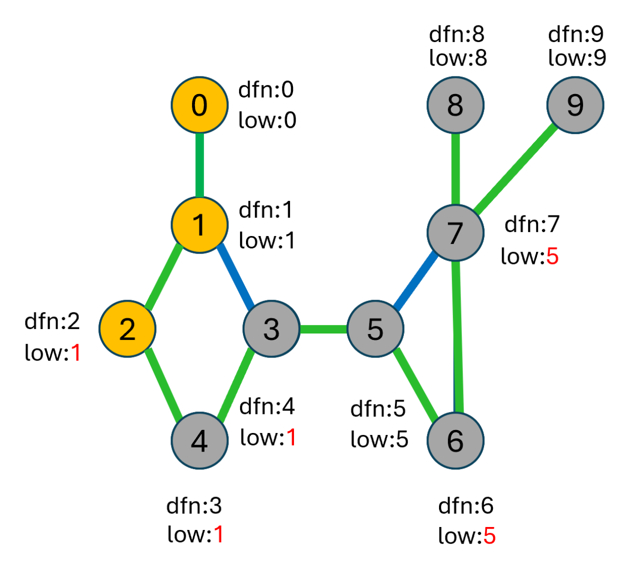
4
6
dfn:3
low:3
dfn:6
low:5


dfn:8
low:8
dfn:9
low:9
0
dfn:0
low:0
8
9
1
dfn:1
low:1
7
dfn:7
low:5
dfn:2
low:2
2
3
5
dfn:4
low:1
dfn:5
low:5
4
6
dfn:3
low:1
dfn:6
low:5


dfn:0
low:0
0
dfn:1
low:1
1
dfn:2
low:2
2
3
4
dfn:4
low:1
dfn:3
low:1
5
dfn:5
low:5
6
dfn:6
low:5
7
dfn:7
low:5
8
dfn:8
low:8
9
dfn:9
low:9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low:0
dfn:1
low:1
dfn:2
low:1
dfn:3
low:1
dfn:4
low:1
dfn:5
low:5
dfn:6
low:5
dfn:7
low:5
dfn:8
low:8
dfn:9
low:9
0
1
2
3
4
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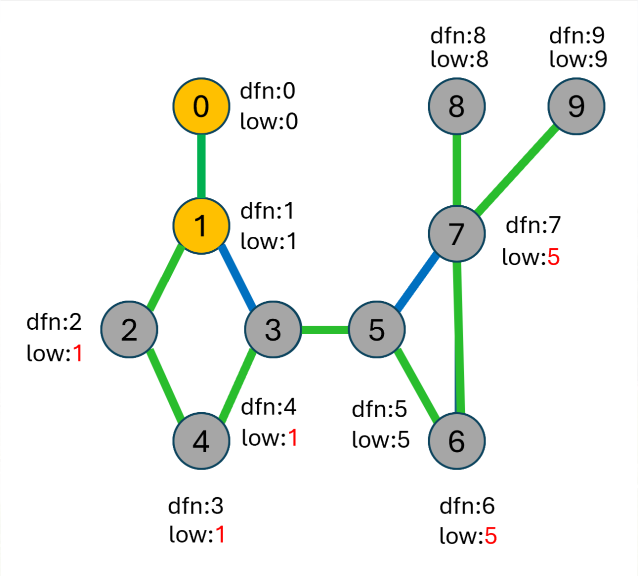
6
7
8
9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low:8
dfn:9
low:9
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dfn:0
low:0
8
9
1
dfn:1
low:1
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low:5
dfn:2
low:1
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low:1
dfn:5
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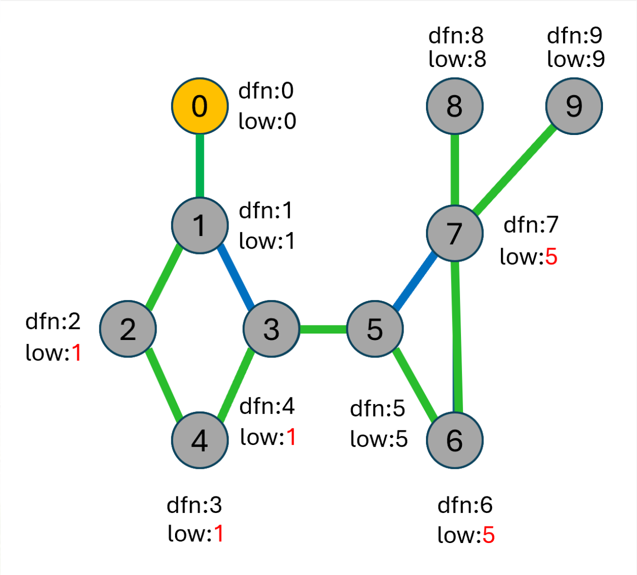
4
6
dfn:3
low:1
dfn:6
low:5


dfn:8
low:8
dfn:9
low:9
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low:0
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low:5
dfn:2
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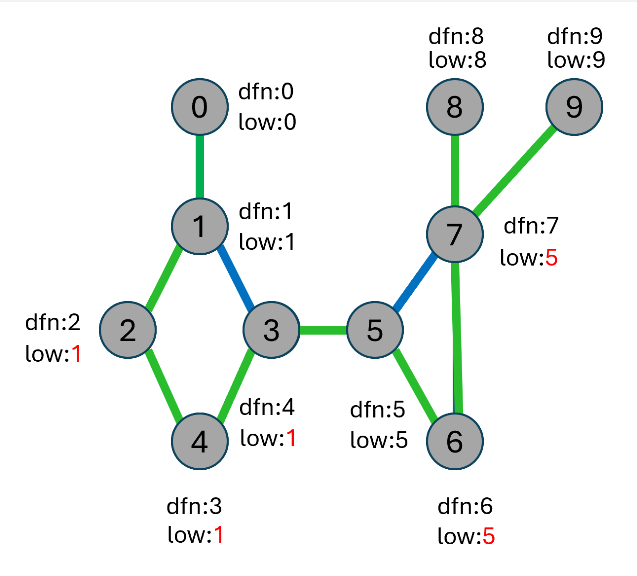
low:5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low:8
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low:5
dfn:2
low:1
2
3
5
dfn:4
low:1
dfn:5
low:5
4
6
dfn:3
low:1
dfn:6
low:5

The codes for computing dfn and low

Time complexity: $O(e)$.

```
void dfn_low(int u, int v) {
/* compute dfn and low while performing a dfs
search beginning at vertex u, v is the parent
of u (if any) */
    node_pointer ptr;
    int w;
    dfn[u] = low[u] = num++;
    for (ptr = graph[u]; ptr; ptr = ptr->link) {
        w = ptr->vertex;
        if (dfn[w] < 0) {
        /* w is an unvisited vertex */
            dfn_low(w, u); // visit w from u
            low[u] = MIN2(low[u], low[w]); // (*)
        } else if (w != v)
            low[u] = MIN2(low[u], dfn[w]); // (**)
    }
}
```

```
short int dfn [MAX_VERTICES];
short int low[MAX_VERTICES];
int num = 0;

void init(void) {
    int i;
    for(i = 0; i < n; i++) {
        visited[i] = false;
        dfn[i] = low[i] = -1;
    }
    num = 0;
}
```

bootstrapping by

`dfn_low(x, -1)`

Finding articulation points (3/3)

articulation points

u is an articulation point iff one of the following conditions are satisfied:

- u is the root of the spanning tree and has two or more children.
- u is not the root of the spanning tree and u has a child w such that $\text{low}(w) \geq \text{dfn}(u)$.

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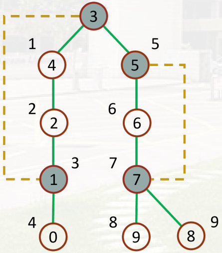

vertex	0	1	2	3	4	5	6	7	8	9
dfn	4	3	2	0	1	5	6	7	9	8
low	4	0	0	0	0	5	5	5	9	8

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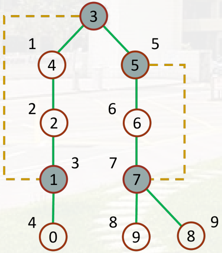


vertex	0	1	2	3	4	5	6	7	8	9
dfn	4	3	2	0	1	5	6	7	9	8
low	4	0	0	0	0	5	5	5	9	8

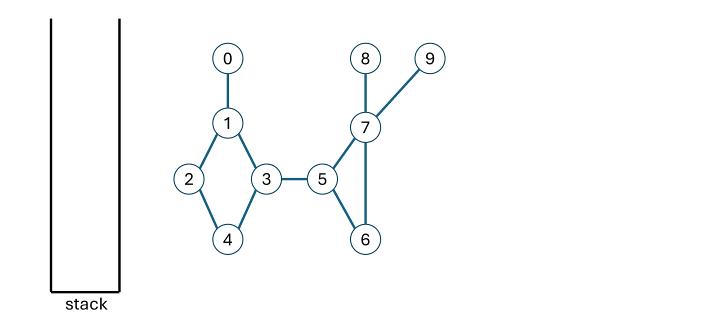
- articulation points: 1, 3, 5, 7.

Code for Biconnected Components ($O(n+e)$ time)

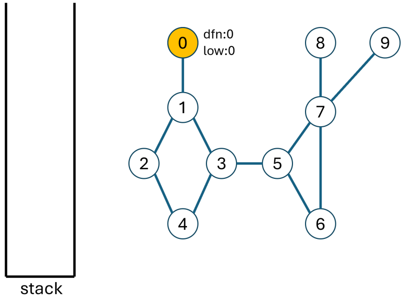
```
void bicon(int u, int v) { /* dfn[] = -1, num = 0, s is an empty stack initially*/
/* v is the parent node of u in the DFS spanning tree */
    nodePointer ptr;
    int w, x , y;
    dfn[u] = low[u] = num++;
    for (ptr = graph[u]; ptr; ptr = ptr->link) {
        w = ptr->vertex;
        if (v != w && dfn[w] < dfn[u]) {
            push(u,w); /* add edge (u,w) into stack s */
            if (dfn[w] < 0) { /* w is not visited yet */
                bicon(w, u);
                low[u] = MIN2(low[u],low[w]);
                if (low[w] >= dfn[u]) {
                    printf("New biconnected component:");
                    do { /* pop an edge from stack s */
                        pop(&x, &y);
                        printf("<%d,%d>",x, y);
                    } while (!((x == u) && (y == w)));
                    printf("\n");
                }
            } else if (w != v)
                low[u] = MIN2(low[u],dfn[w]);
        }
    }
}
```

If we have $\text{low}(w) \geq \text{dfn}(v)$ whenever $\texttt{dfn\_low}(u, w)$ returns, then a new biconnected component is found!

Illustration



Illustration



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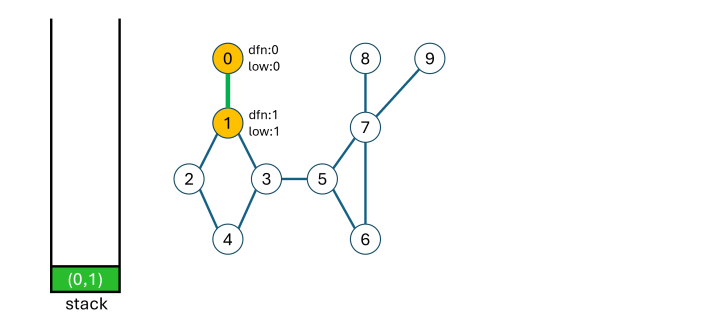

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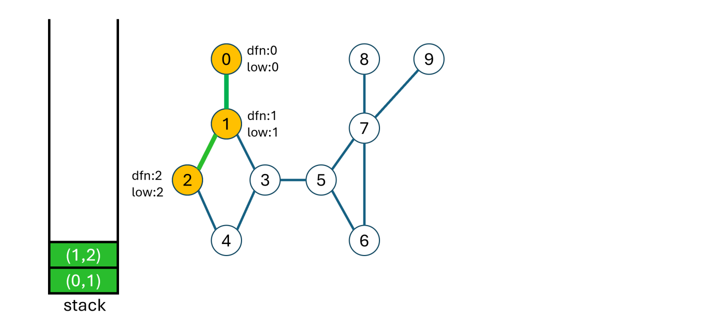

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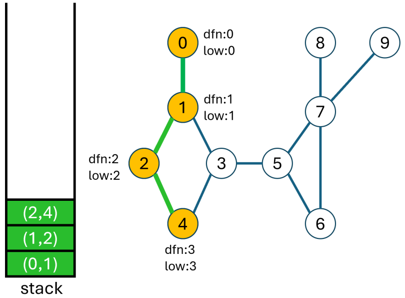

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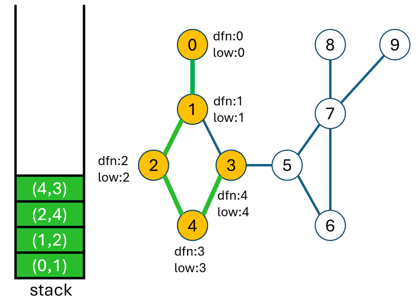

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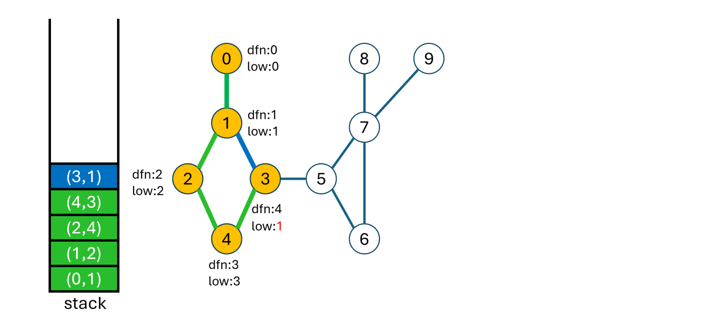

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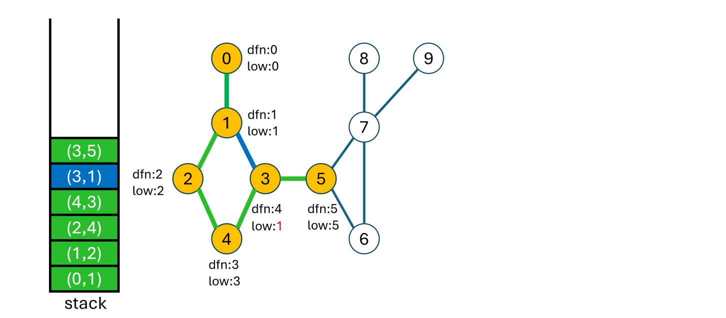

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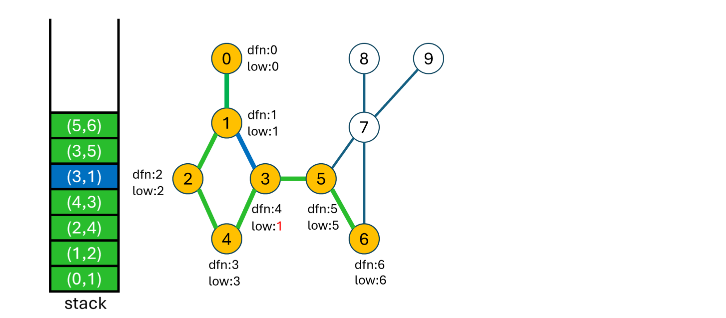

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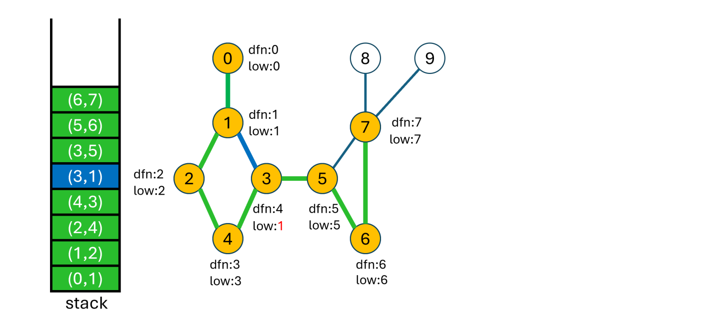

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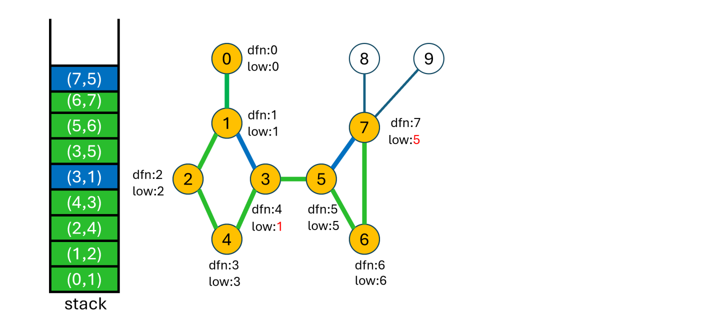

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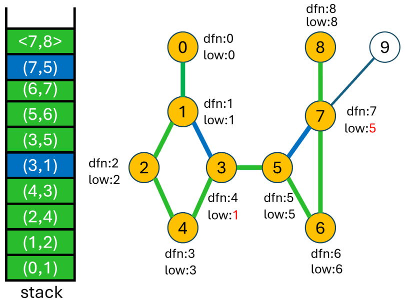

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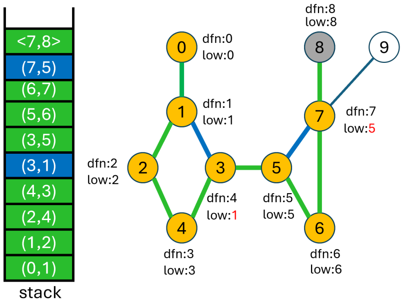

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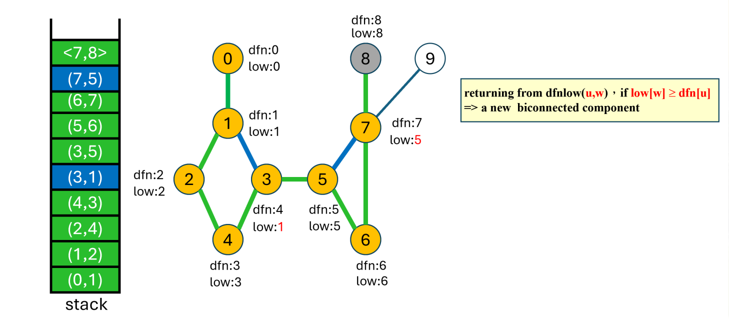

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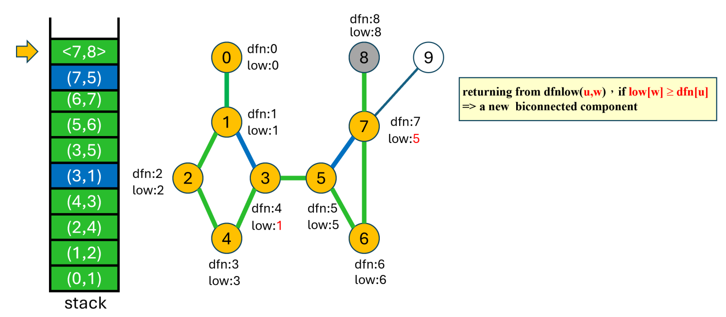

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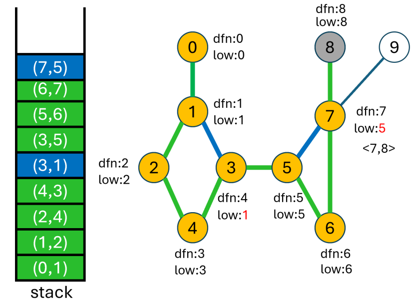
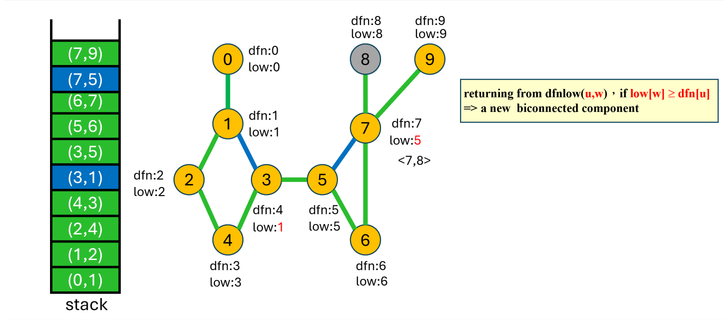


returning from dfnlow(u,w), if low[w] ≥ dfn[u]
=> a new biconnected component

Illustration



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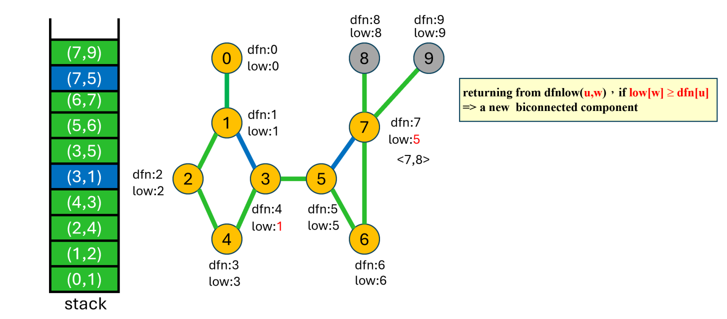

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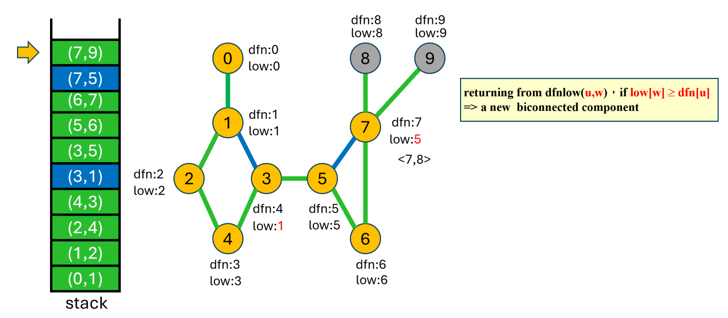

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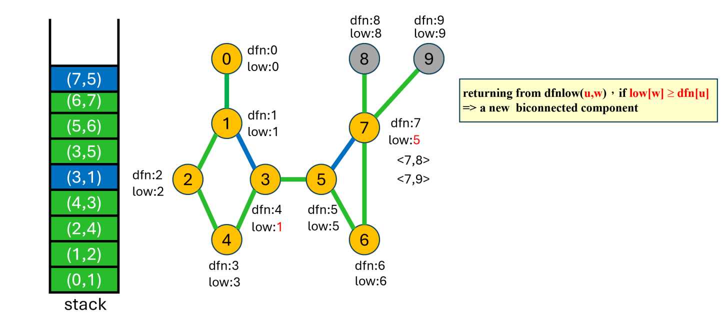

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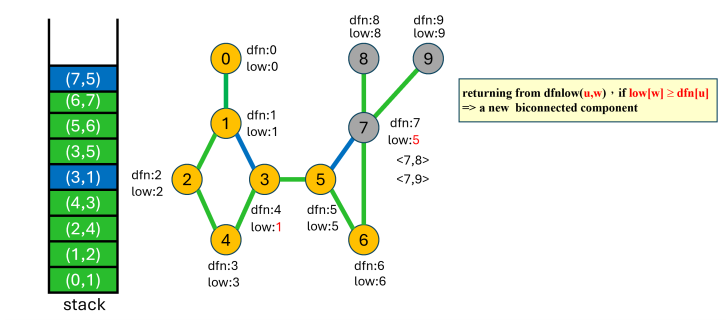

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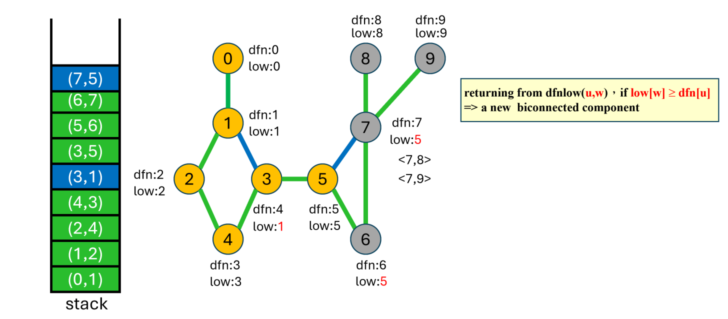

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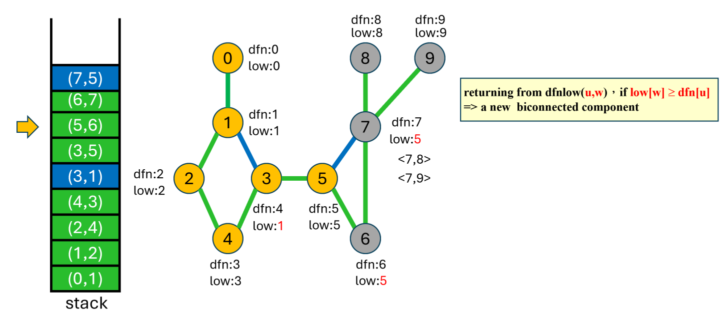

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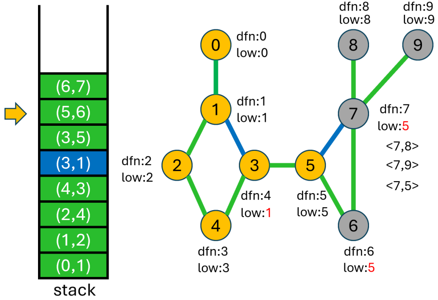

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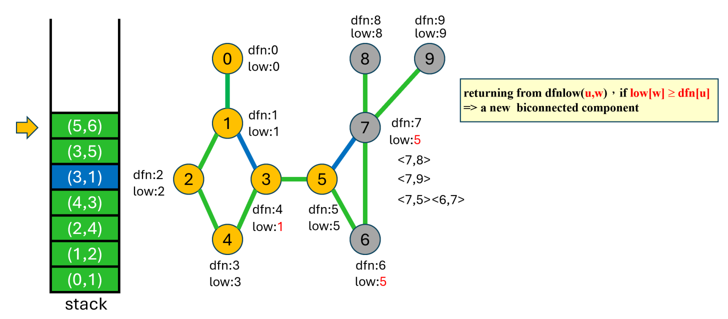

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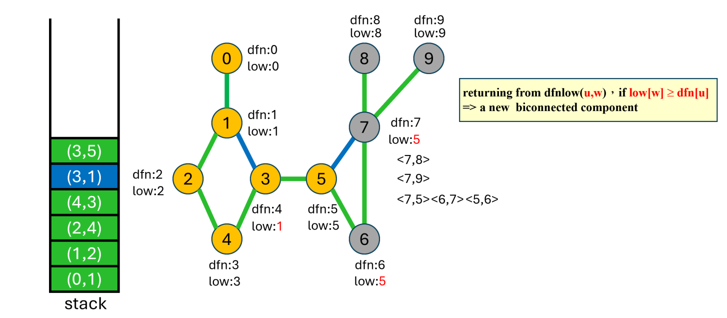

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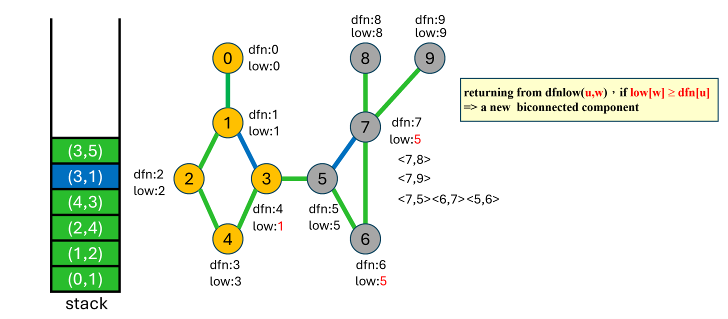

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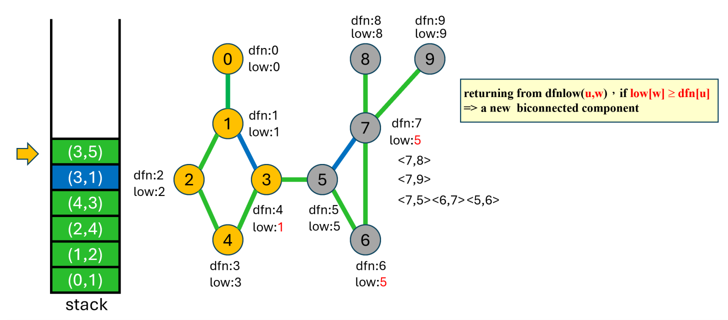

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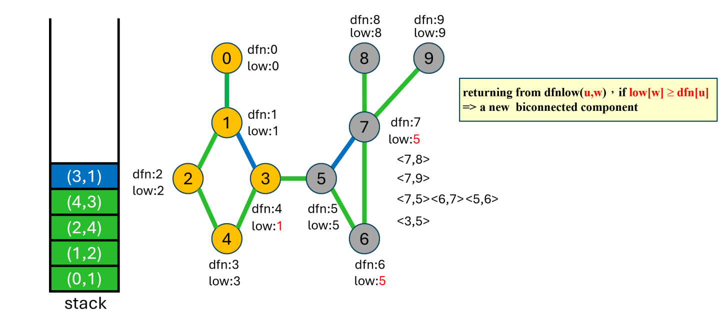

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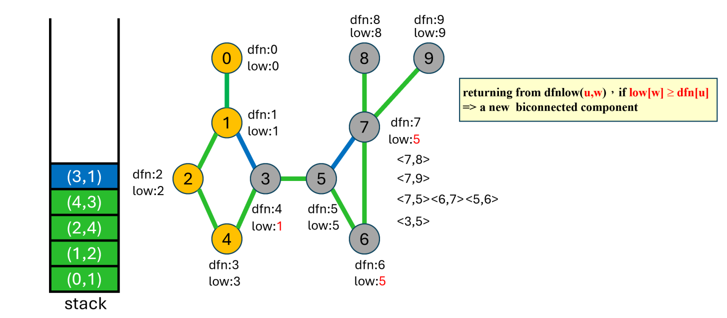

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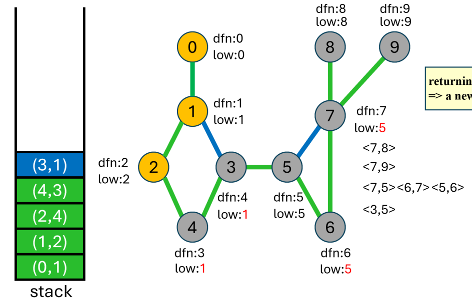

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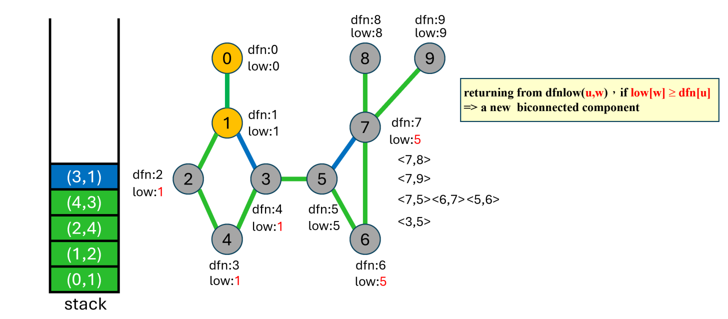

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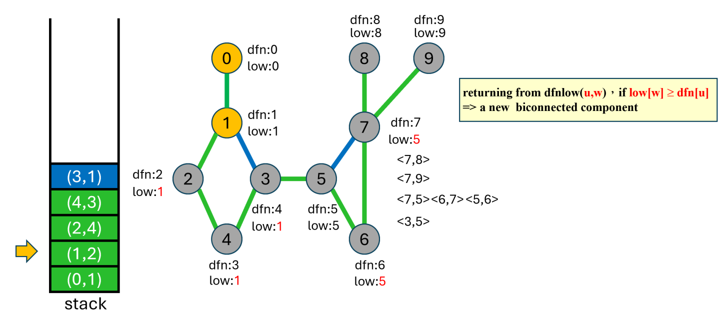

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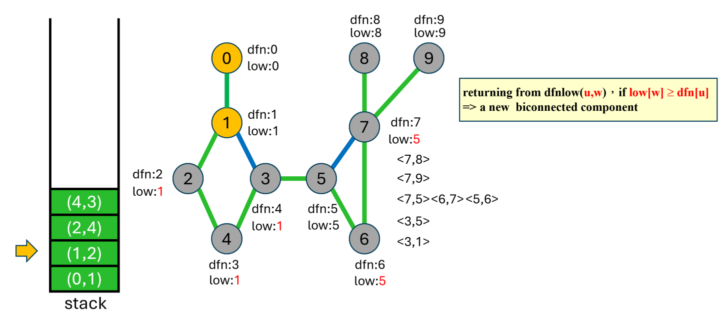

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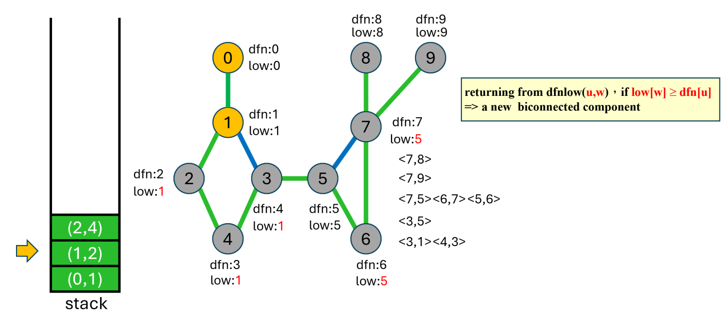

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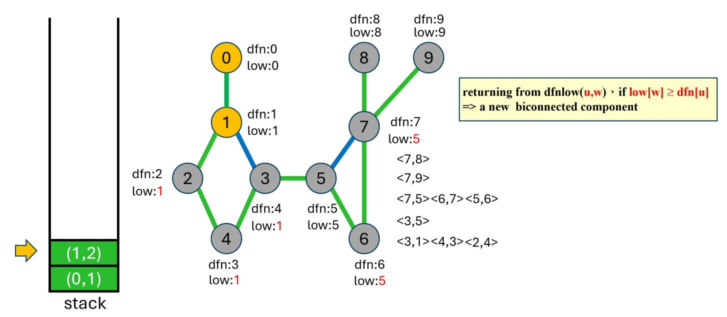

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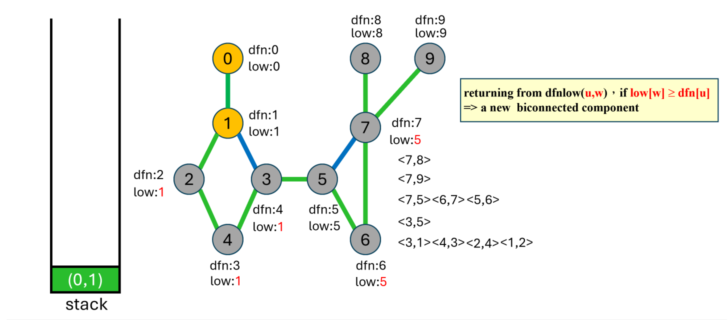

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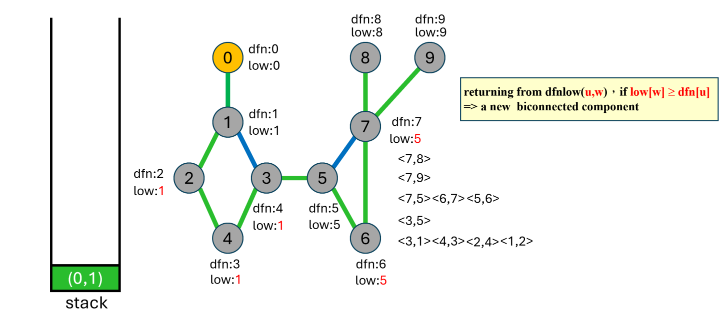

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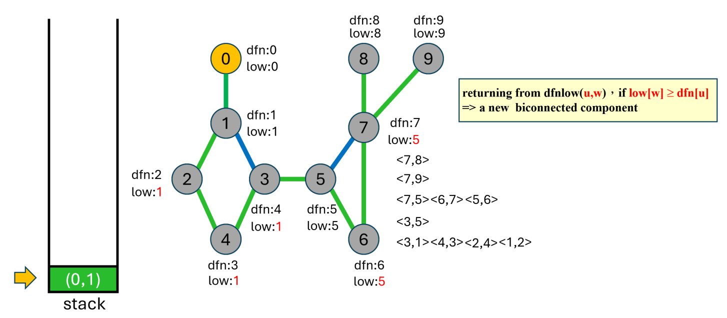

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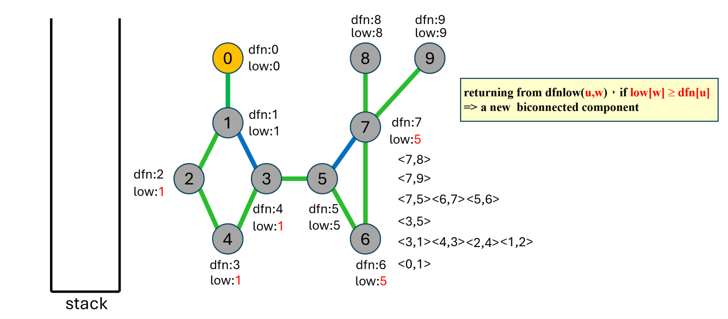

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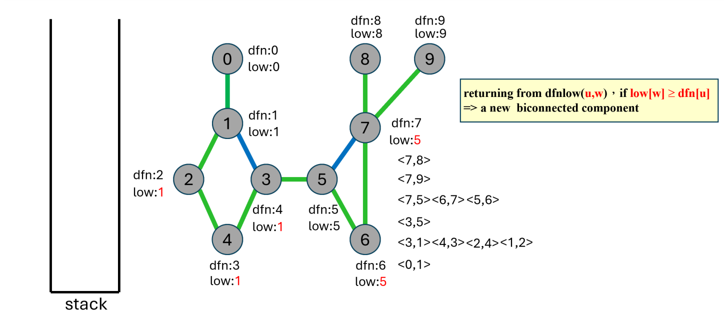

Discussions