

Disjoint Sets Representation

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Outline

- 1 Set Representation
- 2 Set Operations

Introduction

- In this class, we study the use of trees in the representation of sets.
- For simplicity, we assume that the elements of the sets are $0, 1, 2, \dots, n - 1$.
- We also assume that the sets being represented are pairwise disjoint.

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- For simplicity, we assume that the elements of the sets are $0, 1, 2, \dots, n - 1$.
- We also assume that the sets being represented are pairwise disjoint.
 - If S_i and S_j are two disjoint sets, then $S_i \cap S_j = \emptyset$, that is, no element that is in both S_i and S_j .

Set Representation (1/2)

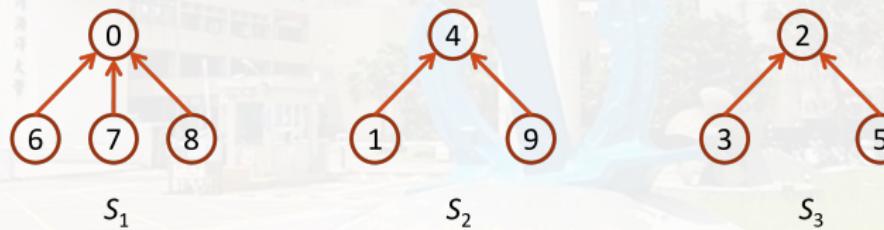
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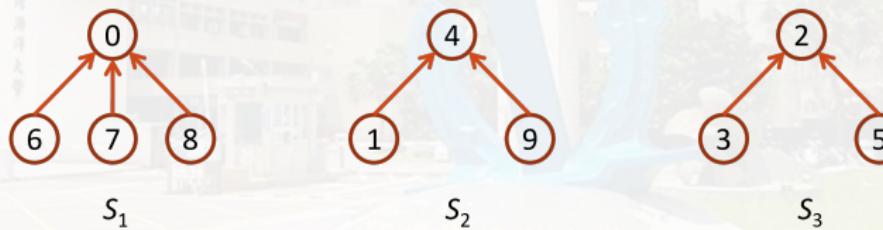
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- **Note:** for each set, we have linked the nodes from the children to the parent.

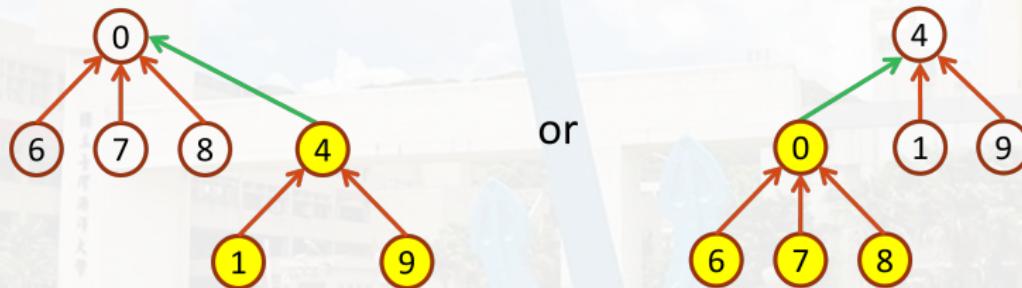
Set Representation (2/2)

- The operations that we wish to perform on these sets are:

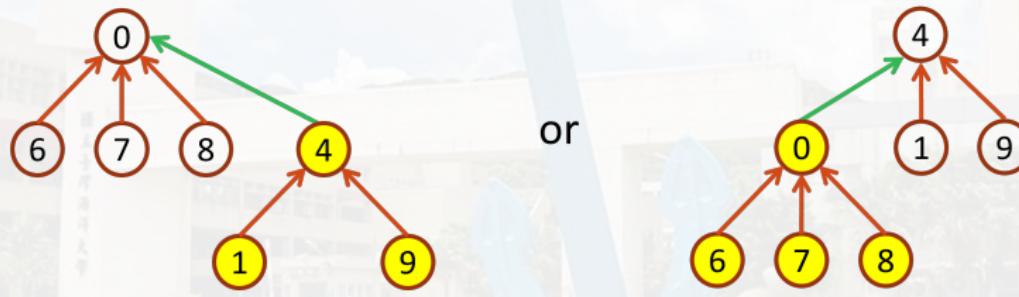
Set Representation (2/2)

- The operations that we wish to perform on these sets are:
 - **Disjoint set union:** If S_i and S_j are two disjoint sets, then **their union** $S_i \cup S_j = \{x \mid x \in S_i \text{ or } x \in S_j\}$.
 - **Find(i):** find the set containing the element i .

Possible Representations of the Union of Two Sets



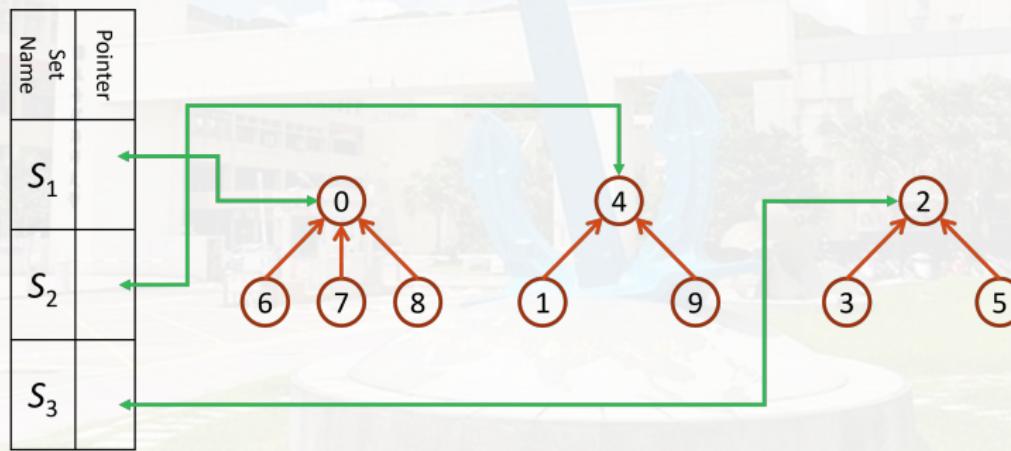
Possible Representations of the Union of Two Sets



- Since we have linked the nodes from the children to parent, we simply make one of the trees a subtree of the other.

Find() Operation

- We can find which set an element is in by following the parent links to the root and then returning the pointer to the set name.



Array Representation of Sets

- We identify the sets **by the roots** of the trees representing them.
- We can use the node's number as the index in our simplified example.
- This means that each node needs only one field: the index of its parents, to link to its parent,

i	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
parent	-1	4	-1	2	-1	2	0	0	0	4

- **Note:** The root nodes have a parent of -1.

Union and Find Operations

We can now find element i by simply following the **parent values starting at i** and continuing until we reach a negative parent value.

- For example, to **find** 5, we start at 5, and then move to 5's parent, 2. Since node 2 has a negative parent value, we have reached the root.

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- To **union** two trees with root i and j , we can simply **set $\text{parent}[i] = j$** .



Initial Attempt for the Union-Find Functions

```
int simpleFind (int i) {
    for (; parent[i] >= 0; i = parent[i])
        ;
    return i;
}

void simpleUnion (int i, int j) {
    parent [i] = j;
}
```

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parent	-1	4	-1	2	-1	2	0	0	0	4

Initial Attempt for the Union-Find Functions

```
int simpleFind (int i) {  
    // try the while-loop instead  
    while (parent[i] >= 0)  
        i = parent[i];  
    return i;  
}  
  
void simpleUnion (int i, int j) {  
    parent [i] = j;  
}
```

i	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
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 - The following sequence of union-find operations produces the **degenerate tree** (退化樹):
 - $\text{union}(0,1), \text{find}(0)$.
 - $\text{union}(1,2), \text{find}(0)$.
 - \vdots
 - $\text{union}(n-2, n-1), \text{find}(0)$.
- ▷ The time complexity of **union operations** is $O(n)$.
- ▷ The time complexity of **find operations** is $\sum_{i=2}^n i = O(n^2)$.

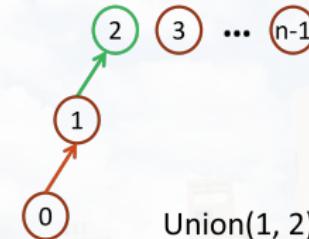




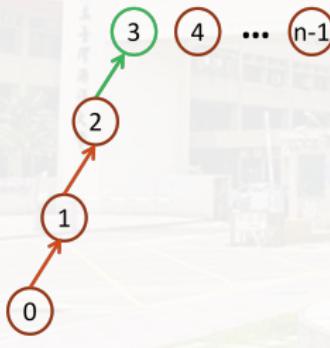
initial



Union(0, 1)



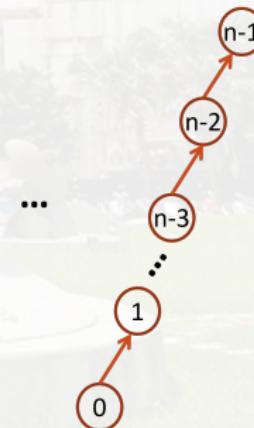
Union(1, 2)



Union(2, 3)



Union(3, 4)



Weighting Rule for Union comes to the Rescue!

Goal: Avoiding the creation of degenerate trees.

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Weighting Rule for union(i, j)

- If the number of nodes in tree i is less than the number of nodes in tree j , make j the parent of i ;
- If the number of nodes in tree i is greater than the number of nodes in tree j , make i the parent of j .
- **Note:** If i is a root node, we set $\text{parent}[i]$ to be the negative number of nodes in that tree.

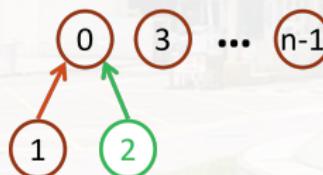
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Example of Using the Weight Rule



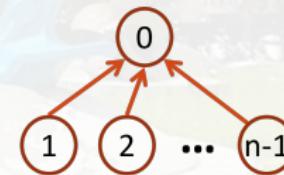
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Union(0, 1)



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Union(0, 2)

Union(0, n-1)

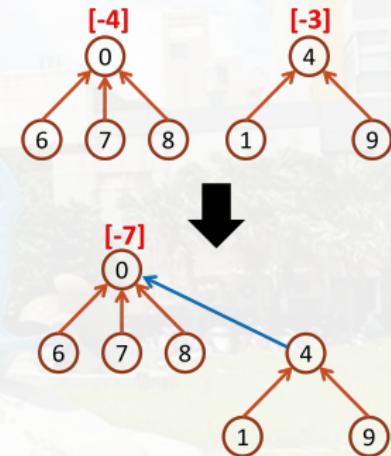
The Code of Union Function Using Weighting Rule

```

void weightedUnion(int i, int j) {
/* union the sets with roots i and j,
   i != j, using the weighting rule.
   parent [i] = -count [i]
   and parent [j] = -count[j] */

    int temp = parent[i] + parent[j];
    if (parent[i] > parent[j]) {
        parent[i] = j; /*make j the new root */
        parent[j] = temp;
    } else {
        parent[j] = i; /*make i the new root */
        parent[i] = temp;
    }
}

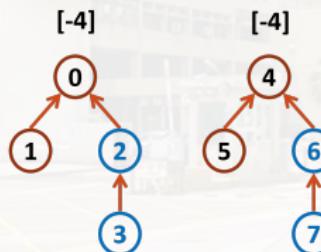
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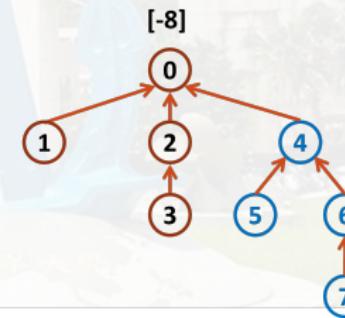
Trees Achieving the Worst Case



(b) Height-2 trees following Union(0, 1), (2, 3), (4, 5) and (6, 7)



(c) Height-3 trees following
Union(0, 2) and (4, 6)



(d) Height-4 tree following
Union(0, 4)

Worst depth of the tree by the Weighted Union

Lemma

Let T be a tree with n nodes created by `weightedUnion`, then no node in T has level greater than $\lfloor \lg n \rfloor + 1$.

- Proof by induction on n (Exercise).

Another Rule: Collapsing Rule

Collapsing Rule for union(i, j)

- If j is a node on the path from i to its root and $\text{parent}[j] \neq \text{root}(i)$, then set $\text{parent}[j]$ to $\text{root}(i)$.
- Consider the previous example, and process the following eight `find()`:

$\overbrace{\text{find}(7), \text{find}(7), \dots, \text{find}(7)}^{8 \text{ times}}$.

- The `SimpleFind()` needs $3 \times 8 = 24$ moves.



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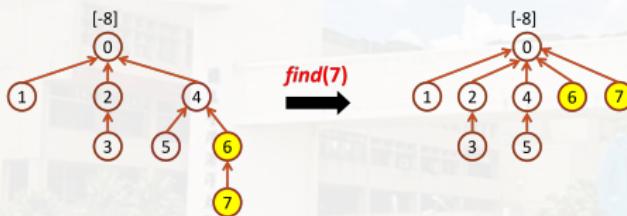
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- The `SimpleFind()` needs $3 \times 8 = 24$ moves.
- The `CollapsingFind()` needs $3 + 3 + 7 = 13$ moves.
 - first `find(7)`: 3 moves.
 - reset 3 links: 3 moves.
 - remaining 7 finds: 7 moves.



Collapsing Rule (contd.)



- When `collapsingFind` is used, the first `find(7)` requires going up three links and then resetting two links.
- Note:** Even though only two parent links need to be reset, `collapsingFind` will actually reset three (the parent of 4 is reset to 0).

The Code for the Collapsing Rule

```

int collapsingFind (int i) {
    /* find the root of the tree containing element i.
     * Use the collapsing rule to collapse all nodes
     * from i to root */
    int root, trail, lead;
    for (root=i; parent[root]>=0; root=parent[root])
        ;
    for (trail=i; trail != root; trail=lead) {
        lead = parent[trail];
        parent[trail] = root;
    }
    return root;
}

```



Consider $i = 7$:

$\text{root} = 0$ (after the 1st for-loop)

$\text{trail} = 7$

$\text{lead} = \text{parent}[7] = 6$

$\text{parent}[\text{trail}] = \text{parent}[7] = 0$

$\text{trail} = 6$

$\text{lead} = \text{parent}[6] = 4$

$\text{parent}[6] = 0$

$\text{trail} = 4$

$\text{lead} = \text{parent}[4] = 0$

$\text{parent}[4] = 0$

$\text{trail} = 0$

Discussions

