

# Existence of Pure-Strategy Nash Equilibria in a Two-Party Policy Competition Game Extending to the General Case

Speaker: Chuang-Chieh Lin (Joseph)  
NTOU, TW

a joint work with

Chi-Jen Lu  
Academia Sinica, TW

Po-An Chen  
NYCU, TW

Chih-Chieh Hung  
NCHU, TW

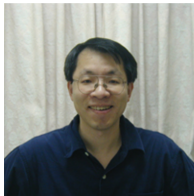
9 May 2025



# Authors



Chuang-Chieh Lin



Chi-Jen Lu



Po-An Chen



Chih-Chieh Hung



# Outline

- 1 Motivations
- 2 The Setting
- 3 Our Contribution
- 4 Future and Ongoing Work

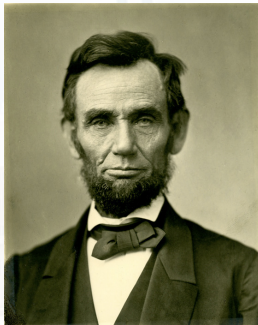


# Outline

- 1 Motivations
- 2 The Setting
- 3 Our Contribution
- 4 Future and Ongoing Work



# The Inspiration (an EC'17 paper)

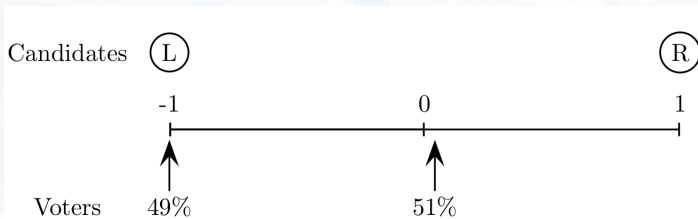


*"[...] and that government of the people, by the people, for the people, shall not perish from the earth."*

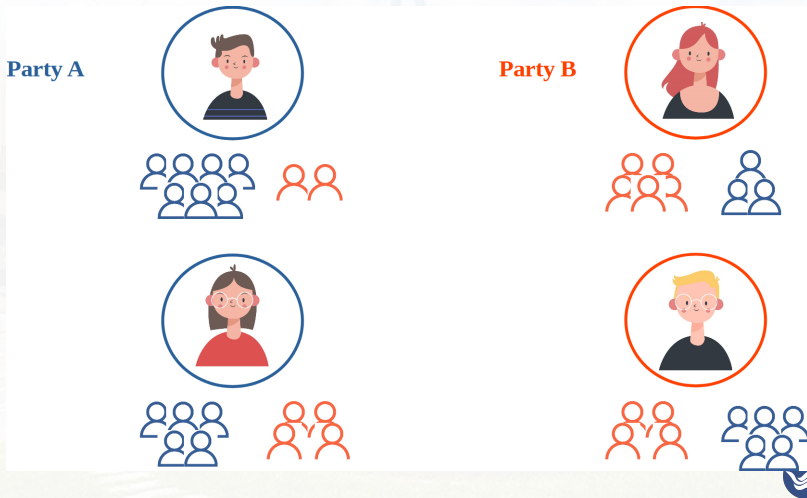
*— Abraham Lincoln, 1863.*



# Previous Work (I): Distortion of Social Choice Rules



## Previous Work (II): Two-Party Election Game



## Previous Work (II): Two-Party Election Game

- Parties are players.
- Strategies: their candidates (or policies).
- A candidate beats the other candidates from other candidates of other parties with **uncertainty**.
- The payoff of each party: **expected utility** its supporters can get.





## Previous Work (II): Two-Party Election Game (contd.)

- Party  $A$ :  $m$  candidates, party  $B$ :  $n$  candidates.
- Candidate  $A_i$  can bring social utility  $u(A_i) = u_A(A_i) + u_B(A_i) \in [0, \beta]$  for some real  $\beta \geq 0$ .
- $p_{i,j}$ :  $\Pr[A_i \text{ wins over } B_j]$ .
  - E.g., **Linear**:  $p_{i,j} := (1 + (u(A_i) - u(B_j))/\beta)/2$
- Payoff (reward)  $r_A = p_{i,j}u_A(A_i) + (1 - p_{i,j})u_A(B_j)$ .



## Previous Work (II): Two-Party Election Game (contd.)

- Party  $A$ :  $m$  candidates, party  $B$ :  $n$  candidates.
- Candidate  $A_i$  can bring social utility  $u(A_i) = u_A(A_i) + u_B(A_i) \in [0, \beta]$  for some real  $\beta \geq 0$ .
- $p_{i,j}$ :  $\Pr[A_i \text{ wins over } B_j]$ . more utility for all the people, more likely to win
  - E.g., **Linear**:  $p_{i,j} := (1 + (u(A_i) - u(B_j))/\beta)/2$
- Payoff (reward)  $r_A = p_{i,j}u_A(A_i) + (1 - p_{i,j})u_A(B_j)$ .

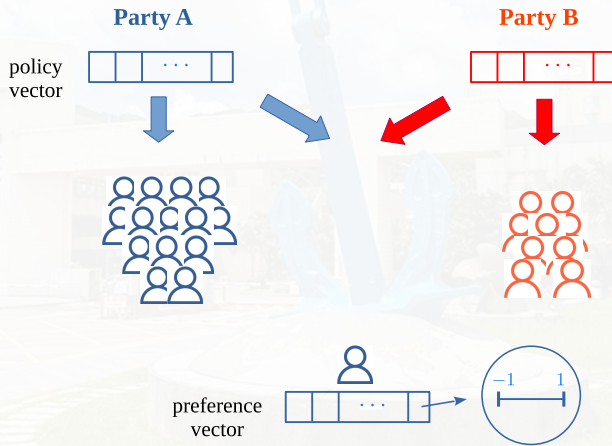


# Outline

- 1 Motivations
- 2 The Setting
- 3 Our Contribution
- 4 Future and Ongoing Work



# Policies and Preferences



# The Setting

- Policy vectors:  $\mathbf{z}_A, \mathbf{z}_B \in S \subset \mathbb{R}^k$ .
  - $\|\mathbf{z}_A\| \leq 1$  and  $\|\mathbf{z}_B\| \leq 1$ .
  - State (or profile):  $\mathbf{z} := (\mathbf{z}_A, \mathbf{z}_B)$ .

# The Setting

- Policy vectors:  $\mathbf{z}_A, \mathbf{z}_B \in S \subset \mathbb{R}^k$ .
  - $\|\mathbf{z}_A\| \leq 1$  and  $\|\mathbf{z}_B\| \leq 1$ .
  - State (or profile):  $\mathbf{z} := (\mathbf{z}_A, \mathbf{z}_B)$ .
- $V_A$  and  $V_B$ : the supporters of  $A$  and  $B$ .
  - $V := V_A \dot{\cup} V_B$ ,  $|V| = n$ .
- Preference vector of a voter  $v \in V$ :  $\mathbf{q}_v$ .
- $Q_A := \sum_{v \in V_A} \mathbf{q}_v$ ,  $Q_B := \sum_{v \in V_B} \mathbf{q}_v$ ,  $Q := Q_A + Q_B$ ,  $\|Q_A\|, \|Q_B\| \leq 1$ .

# The Setting

- Policy vectors:  $\mathbf{z}_A, \mathbf{z}_B \in S \subset \mathbb{R}^k$ .
  - $\|\mathbf{z}_A\| \leq 1$  and  $\|\mathbf{z}_B\| \leq 1$ .
  - State (or profile):  $\mathbf{z} := (\mathbf{z}_A, \mathbf{z}_B)$ .
- $V_A$  and  $V_B$ : the supporters of  $A$  and  $B$ .
  - $V := V_A \dot{\cup} V_B$ ,  $|V| = n$ .
- Preference vector of a voter  $v \in V$ :  $\mathbf{q}_v$ .
- $Q_A := \sum_{v \in V_A} \mathbf{q}_v$ ,  $Q_B := \sum_{v \in V_B} \mathbf{q}_v$ ,  $Q := Q_A + Q_B$ ,  $\|Q_A\|, \|Q_B\| \leq 1$ .
- The utility

$$u_A(\mathbf{z}_A) = \mathbf{z}_A^\top Q_A, \quad u_B(\mathbf{z}_A) = \mathbf{z}_A^\top Q_B.$$

$$u_A(\mathbf{z}_B) = \mathbf{z}_B^\top Q_A, \quad u_B(\mathbf{z}_B) = \mathbf{z}_B^\top Q_B.$$



# The Setting (Winning Prob. & Payoff)

- Winning probability:

$$p_{A \succ B} = \frac{1}{2} + \frac{1}{4}(\mathbf{z}_A - \mathbf{z}_B)^\top \mathbf{Q},$$

$$p_{B \succ A} = \frac{1}{2} + \frac{1}{4}(\mathbf{z}_B - \mathbf{z}_A)^\top \mathbf{Q}.$$

- 1/4: a normalization factor.
- The payoffs:

$$R_A(\mathbf{z}) = p_{A \succ B} \cdot \mathbf{z}_A^\top \mathbf{Q}_A + p_{B \succ A} \cdot \mathbf{z}_B^\top \mathbf{Q}_A,$$

$$R_B(\mathbf{z}) = p_{B \succ A} \cdot \mathbf{z}_B^\top \mathbf{Q}_B + p_{A \succ B} \cdot \mathbf{z}_A^\top \mathbf{Q}_B.$$





So, we can compute the gradients and Hessian...

$$\frac{\partial R_A(\mathbf{z})}{\partial \mathbf{z}_A} = \frac{1}{2}Q_A + \frac{(\mathbf{z}_A - \mathbf{z}_B)^\top Q}{4}Q_A + \frac{(\mathbf{z}_A - \mathbf{z}_B)^\top Q_A}{4}Q.$$

$$\frac{\partial R_B(\mathbf{z})}{\partial \mathbf{z}_B} = \frac{1}{2}Q_B + \frac{(\mathbf{z}_B - \mathbf{z}_A)^\top Q}{4kn}Q_B + \frac{(\mathbf{z}_B - \mathbf{z}_A)^\top Q_B}{4kn}Q.$$

$$\frac{\partial^2 R_A(\mathbf{z})}{\partial \mathbf{z}_A^2}[i,j] = \frac{1}{4}(Q[i]Q_A[j] + Q[j]Q_A[i]),$$

$$\frac{\partial^2 R_B(\mathbf{z})}{\partial \mathbf{z}_B^2}[i,j] = \frac{1}{4}(Q[i]Q_B[j] + Q[j]Q_B[i]).$$



# Outline

- 1 Motivations
- 2 The Setting
- 3 Our Contribution
- 4 Future and Ongoing Work



# Previous Contributions

## [Nash 1950]

Every FINITE game has a **mixed-strategy** Nash equilibrium.

## Our Contribution

In this work, we show that there **exists** a **pure-strategy Nash equilibrium (PSNE)** in the two-party policy competition game for

- the degenerate case:  $k = 1$
- the general case  $k \geq 1$  under the **consensus-reachable** condition
- The two-party policy competition game is NOT a finite game.
- The above PSNE consists of dominant-strategies.



# Our Contributions

[Nash 1950]

Every FINITE game has a **mixed-strategy** Nash equilibrium.

## Our Contribution

In this work, we show that there **exists** a **pure-strategy Nash equilibrium (PSNE)** in the two-party policy competition game for

- the degenerate case:  $k = 1$
  - the general case  $k \geq 1$  under the **consensus-reachable** condition
  - the general case  $k \geq 1$  for **non-consensus-reachable condition yet under a mild assumption**.
- 
- The two-party policy competition game is NOT a finite game.
  - The above PSNE consists of dominant-strategies.



# Claim of the Egoistic Property

## Claim

The egoistic property must hold in the two-party policy competition game.

- $\mathbf{z}_A^\top Q_A \geq \mathbf{z}_B^\top Q_A$  and  $\mathbf{z}_B^\top Q_B \geq \mathbf{z}_A^\top Q_B$ .



## The General Case: $k \geq 1$ — Simplification by Polar Coordinates

- It is sufficient for party  $A$  and  $B$  to consider the space  $\text{span}(\{Q_A, Q_B\})$ .
- Represent  $\mathbf{z}_A$  (resp.,  $\mathbf{z}_B$ ) in terms of **polar coordinates**  $(r_A, \theta_A)$  (resp.,  $(r_B, \theta_B)$ ).
  - $r_A = \|\mathbf{z}_A\|, r_B = \|\mathbf{z}_B\|$
  - $\theta_A$  (resp.,  $\theta_B$ ) is the angle b/w  $Q_A$  and  $\mathbf{z}_A$  (resp.,  $Q_B$  and  $\mathbf{z}_B$ ).

For any two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^k, k \geq 1$ ,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \|\mathbf{u}\| \|\mathbf{v}\| \cos(\theta),$$

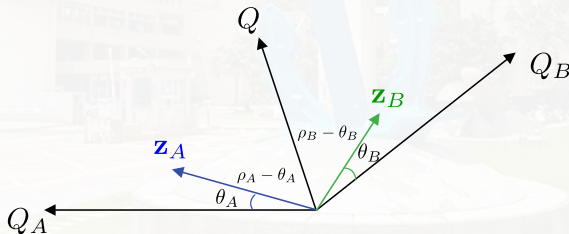
where  $\theta$  is the angle b/w  $\mathbf{u}$  and  $\mathbf{v}$ .



# A Good Condition

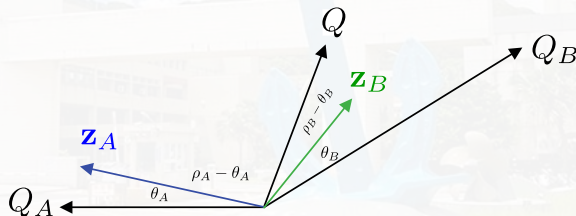
## Consensus-Reachable

A two-party policy competition game is *consensus-reachable* if  $Q_A^\top Q \geq 0$  and  $Q_B^\top Q \geq 0$ .



# An Example of Not Consensus-Reachable

$$\rho_A > \pi/2.$$





## About the Norms $r_A, r_B$

- A mild assumption:  $\theta_A, \theta_B \leq \pi/2$ .

### Lemma

For the nonconsensus-reachable case where  $\rho_A > \pi/2$  and  $\theta_A, \theta_B \in [0, \pi/2]$ , the best responses of the two players always set  $r_A = r_B = 1$ .

Sketch of the proof:

- Show that  $\frac{\partial R_A(\mathbf{r}, \theta)}{\partial r_A} \big|_{\theta_A = \rho_A - \pi/2} \geq 0$ .
- For  $\theta \in [0, \rho_A - \theta_A]$ , we can show that  $\frac{\partial^2 R_A(\mathbf{r}, \theta)}{\partial r_A^2} \leq 0$ .
- Hence, it follows that  $\frac{\partial R_A(\mathbf{r}, \theta)}{\partial r_A} \geq 0$  for  $\theta_A \in [0, \rho_A - \pi/2]$ .
- Combining  $\frac{\partial^2 R_A(\mathbf{r}, \theta)}{\partial r_A^2} \geq 0$  for  $\theta_A \in [\rho_A - \pi/2, \pi/2]$ ,  
we have  $\frac{\partial R_A(\mathbf{r}, \theta)}{\partial r_A} \geq 0$  for  $\theta_A \in [0, \pi/2]$ .



## About the angles: $\theta_A, \theta_B$

- Set  $x := \cos(\theta_A)$  and  $y := \cos(\theta_B)$ .
- Let  $f(x) := R_A(r_A = 1, \theta_A)$  and  $g(y) := R_B(r_B = 1, \theta_B)$ ,  $x, y \in [0, 1]$ .

$$f(x) = \left( \frac{1}{2} + D_0(C_1 x + \sqrt{1 - C_1^2} \sqrt{1 - x^2} - C_3) \right) D_1 x \\ + \left( \frac{1}{2} - D_0(C_1 x + \sqrt{1 - C_1^2} \sqrt{1 - x^2} - C_3) \right) D_1 C_4,$$

$$g(y) = \left( \frac{1}{2} + D_0(C_2 y + \sqrt{1 - C_2^2} \sqrt{1 - y^2} - C'_3) \right) D_2 y \\ + \left( \frac{1}{2} - D_0(C_2 y + \sqrt{1 - C_2^2} \sqrt{1 - y^2} - C'_3) \right) D_2 C'_4,$$

$$\text{subject to: } 0 \leq D_0 \leq \frac{1}{2}, 0 \leq D_1 \leq 1, -1 \leq C_1 < 0, 0 \leq C_2 \leq 1, \\ 0 \leq C_3 \leq 1, -1 \leq C_4 < 0, -1 \leq C'_3 \leq 1, -1 \leq C'_4 \leq 1, \\ C_4 \leq C_1, 0 \leq D_2 \leq 1, D_0 \leq D_2 \text{ and } C'_4 \leq C_2,$$

where  $D_0 := \|Q\|/4$ ,  $D_1 := \|Q_A\|$ ,  $D_2 := \|Q_B\|$ ,  $C_1 := \cos \rho_A$ ,  $C_2 := \cos \rho_B$ ,  
 $C_3 := \cos(\rho_B - \theta_B)$ ,  $C'_3 := \cos(\rho_A - \theta_A)$ ,  $C_4 := \cos(\rho_A + \rho_B - \theta_B)$ ,  
 $C'_4 := \cos(\rho_A + \rho_B - \theta_A)$ .



## About the angles: $\theta_A, \theta_B$ (contd.)

### Lemma

$f(x)$  is concave and  $g(y)$  is unimodal (quasi-concave).

- Therefore, Kakutani's Fixed-Point Theorem can be applied to guarantee the existence of a PSNE of the game even when it is NOT consensus-reachable.

### Theorem

Under the mild condition that  $\mathbf{z}_A^\top Q_A \geq 0$  and  $\mathbf{z}_B^\top Q_B \geq 0$ , the two-party policy competition game has at least one PSNE even when the game is not consensus-reachable.



# Outline

- 1 Motivations
- 2 The Setting
- 3 Our Contribution
- 4 Future and Ongoing Work



# Future and Ongoing Work (1/3)

## Monotone Game

A pseudo-gradient mapping of the game

$$F : \mathcal{D} \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

is said to be *monotone* if for all  $\mathbf{u} = (x_1, y_1)$  and  $\mathbf{v} = (x_2, y_2)$  in the domain one has

$$(F(\mathbf{u}) - F(\mathbf{v}))^\top (\mathbf{u} - \mathbf{v}) \geq 0.$$



## Future and Ongoing Work (2/3)

### Cocoercivity

A pseudo-gradient mapping of the game

$$F : \mathcal{D} \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

is said to be  $\lambda$ -cocoercive (for some  $\lambda > 0$ ) if for all  $\mathbf{u} = (x_1, y_1)$  and  $\mathbf{v} = (x_2, y_2)$  in the domain one has

$$(F(\mathbf{u}) - F(\mathbf{v}))^\top (\mathbf{u} - \mathbf{v}) \geq \lambda \|F(\mathbf{u}) - F(\mathbf{v})\|^2.$$



## Future and Ongoing Work (3/3)

### Counterexample of Monotonicity

The two-party policy competition game is NOT monotone in general, and hence not cocoercive for any  $\lambda \leq 1$ .

### Theorem

The two-party policy competition game is  $\lambda$ -cocoercive for  $\lambda = 1/\|Q_A\|^2$  if it is voter-symmetric.

- Voter-symmetric:  $Q_A = Q_B$ .
- E.g., Pre-Election within-in the party.



## Future and Ongoing Work (3/3)

### Counterexample of Monotonicity

The two-party policy competition game is NOT monotone in general, and hence not cocoercive for any  $\lambda \leq 1$ .

### Theorem

The two-party policy competition game is  $\lambda$ -cocoercive for  $\lambda = 1/\|Q_A\|^2$  if it is voter-symmetric.

- Voter-symmetric:  $Q_A = Q_B$ .
- E.g., Pre-Election within-in the party.
- One can apply gradient-based algorithms to find a PSNE with convergence rate  $O(1/T)$  in this case.





Thanks for your attention!

Q & A

