

A Sketch of Nash's Theorem from Fixed Point Theorems

Joseph Chuang-Chieh Lin

Department of Computer Science & Engineering,
National Taiwan Ocean University

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Reference

- Lecture Notes in 6.853 Topics in Algorithmic Game Theory [[link](#)].
- *Fixed Point Theorems and Applications to Game Theory*. Allen Yuan. The University of Chicago Mathematics REU 2017. [[link](#)].
 - REU = Research Experience for Undergraduate students.



Outline

- 1 Brouwer's Fixed Point Theorem
 - Sketch of the Proof of Nash's Theorem (from Brouwer's Theorem)
- 2 Kakutani's Fixed Point Theorem
 - Pure Nash Equilibria of Pure Strategic Games
 - Preliminaries
 - Main Theorem I & The Proof
 - Mixed Nash Equilibria of Finite Strategies Games
 - Preliminaries & Assumptions
 - Main Theorem II & the Proof



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The Setting

- A set N of n players.
- Strategy set $S_i = \{s_{i,1}, \dots, s_{i,k_i}\}$ for each player $i \in N$, k_i is bounded.
- Utility function: u_i for each player i .
- $\Delta := \Delta_1 \times \Delta_2 \times \dots \times \Delta_n$: a Cartesian product of $(\Delta_i)_{i \in N}$.
 - For $\mathbf{x} \in \Delta$, $x_i(s)$ denotes the probability mass on strategy $s \in S_i$.
 - $\Delta_i = \{(x_i(s_{i,1}), x_i(s_{i,2}), \dots, x_i(s_{i,k_i})) \mid x_i(s_{i,j}) \geq 0 \ \forall j; \sum_j x_i(s_{i,j}) = 1\}$.
 - $x_i \in \Delta_i$: a **mixed strategy**.



Nash's Theorem

Nash (1950)

Every game $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ has a Nash equilibrium.

- **Note:** $u_i(\mathbf{x}) := \sum_{s \in S_i} x_i(s) \cdot u_i(s; \mathbf{x}_{-i})$.



Nash's Theorem

Nash (1950)

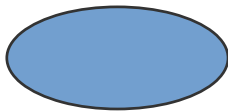
Every game $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ has a Nash equilibrium.

- **Note:** $u_i(\mathbf{x}) := \sum_{s \in S_i} x_i(s) \cdot u_i(s; \mathbf{x}_{-i})$.
- No player wants to deviate to the other strategy unilaterally.

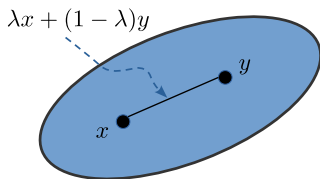




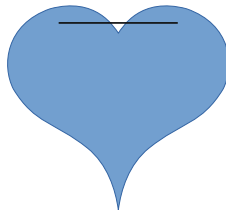
open &
bounded



closed &
bounded

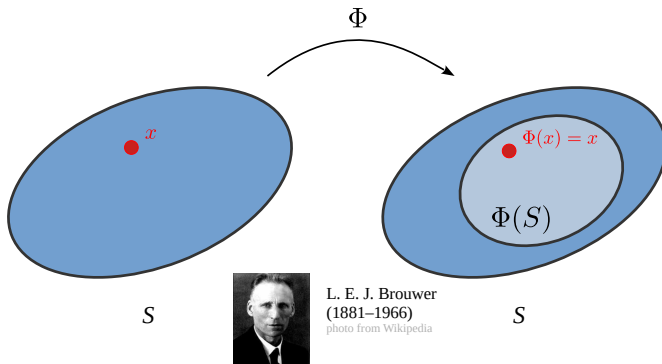


convex



not convex

Fixed Point



Brouwer's Fixed Point Theorem

Brouwer's Fixed-Point Theorem

Let D be a convex, compact (closed and bounded) subset of the Euclidean space. If $f : D \rightarrow D$ is continuous, then there exists $x \in D$ such that

$$f(x) = x.$$

- **Idea:** We want the function f to satisfy the conditions of Brouwer's fixed point theorem.



Brouwer's Fixed Point Theorem

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Let D be a convex, compact (closed and bounded) subset of the Euclidean space. If $f : D \rightarrow D$ is continuous, then there exists $x \in D$ such that

$$f(x) = x.$$

- **Idea:** We want the function f to satisfy the conditions of Brouwer's fixed point theorem.
- Try to relate utilities of players to a function f like above.



The Gain function

Gain

Suppose that $\mathbf{x}' \in \Delta$ is given. For a player i and strategy $s_i \in S_i$ (or $s_i \in \Delta_i$), we define the **gain** as

$$\text{Gain}_{i,s_i}(\mathbf{x}') = \max\{u_i(s_i; \mathbf{x}'_{-i}) - u_i(\mathbf{x}), 0\},$$

which is non-negative.

- $\mathbf{x}'_{-i} := (x_j)_{j \in N}, (x_{-i}, x_i) = \mathbf{x}$.
- It's equal to the increase in payoff for player i if he/she were to switch to pure strategy s_i .



Proof of Nash's Theorem (Define a response function)

- Define a function $f : \Delta \rightarrow \Delta$ that satisfies the conditions of Brouwer's fixed point theorem.
- For all $\mathbf{x} \in \Delta$, $\mathbf{y} = f(\mathbf{x})$ where for all $i \in N$ and $s_i \in S_i$,

$$y_i(s_i) := \frac{x_i(s_i) + \text{Gain}_{i;s_i}(\mathbf{x})}{1 + \sum_{s'_i \in S_i} \text{Gain}_{i;s'_i}(\mathbf{x})}.$$

- f tries to boost the probability mass where strategy switching results in gains in payoff.



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- $f : \Delta \rightarrow \Delta$ is continuous (verify this by yourself).
- Δ is a product of simplices so it is convex (verify this by yourself).
- Δ is closed and bounded, so it is compact.



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- $f : \Delta \rightarrow \Delta$ is continuous (verify this by yourself).
 - Δ is a product of simplices so it is convex (verify this by yourself).
 - Δ is closed and bounded, so it is compact.
- ★ Brouwer's fixed point theorem guarantees the existence of a fixed point of f .



Claim: Any fixed point of f is a Nash equilibrium

- It suffices to prove that a fixed point $\mathbf{x} = f(\mathbf{x})$ satisfies:
 - $\text{Gain}_{i;s_i}(\mathbf{x}) = 0$, for each $i \in N$ and each $s_i \in S_i$.



Claim: Any fixed point of f is a Nash equilibrium

Prove it by contradiction.

- Assume that there is some player p who can improve his/her payoff by switching to a pure strategy, say s_p :
 - $\text{Gain}_{p;s_p}(\mathbf{x}) > 0$.



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Prove it by contradiction.

- Assume that there is some player p who can improve his/her payoff by switching to a pure strategy, say s_p :
 - $\text{Gain}_{p;s_p}(\mathbf{x}) > 0$.
- Note that we must have $x_p(s_p) > 0$, otherwise \mathbf{x} cannot be a fixed point of f .
 - From the definition of f ; the numerator would be > 0 .

$$y_p(s_p) := \frac{x_p(s_p) + \text{Gain}_{p;s_p}(\mathbf{x})}{1 + \sum_{s'_p \in S_p} \text{Gain}_{p;s'_p}(\mathbf{x})}.$$



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Prove it by contradiction.

- Assume that there is some player p who can improve his/her payoff by switching to a pure strategy, say s_p :
 - $\text{Gain}_{p;s_p}(\mathbf{x}) > 0 \Rightarrow u_p(s_p; \mathbf{x}_{-p}) - u_p(\mathbf{x}) > 0.$



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Prove it by contradiction.

- Assume that there is some player p who can improve his/her payoff by switching to a pure strategy, say s_p :
 - $\text{Gain}_{p;s_p}(\mathbf{x}) > 0 \Rightarrow u_p(s_p; \mathbf{x}_{-p}) - u_p(\mathbf{x}) > 0$.
- We argue that there must be some other pure strategy \hat{s}_p such that:
 - $x_p(\hat{s}_p) > 0$ and
 - $u_p(\hat{s}_p; \mathbf{x}_{-p}) - u_p(\mathbf{x}) < 0$

★ Notice that

$$u_p(\mathbf{x}) := \sum_{s \in S_p} x_p(s) \cdot u_p(s; \mathbf{x}_{-p}).$$



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Prove it by contradiction.

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- We argue that there must be some other pure strategy \hat{s}_p such that:
 - $x_p(\hat{s}_p) > 0$ and
 - $u_p(\hat{s}_p; \mathbf{x}_{-p}) - u_p(\mathbf{x}) < 0 \Rightarrow \text{Gain}_{p;\hat{s}_p}(\mathbf{x}) = 0.$

★ Notice that

$$u_p(\mathbf{x}) := \sum_{s \in S_p} x_p(s) \cdot u_p(s; \mathbf{x}_{-p}).$$

- We obtain that (\mathbf{x} is not a fixed point $\Rightarrow \nRightarrow$)

$$y_p(\hat{s}_p) := \frac{x_p(\hat{s}_p) + \text{Gain}_{p;\hat{s}_p}(\mathbf{x})}{1 + \sum_{s'_p \in S_p} \text{Gain}_{p;s'_p}(\mathbf{x})} < x_p(\hat{s}_p).$$



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An Extension of Brouwer's work

- Focus: **set-valued** functions.
 - Refer here for further readings.
 - Why do we consider set-valued functions?



An Extension of Brouwer's work

- Focus: **set-valued** functions.
 - Refer here for further readings.
 - Why do we consider set-valued functions?
 - Best-responses.



Upper Semi-Continuous (having a closed graph)

Upper semi-continuous functions

Let

- $\mathbb{P}(X)$: all nonempty, closed, convex subsets of X .
- S : a nonempty, compact, and convex set.

Then the set-valued function $\Phi : S \rightarrow \mathbb{P}(S)$ is **upper semi-continuous** if

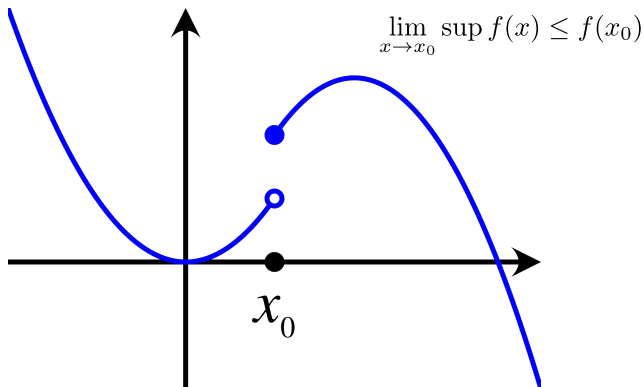
for arbitrary sequences $(\mathbf{x}_n)_{n \in \mathbb{N}}, (\mathbf{y}_n)_{n \in \mathbb{N}}$ in S , we have

- $\lim_{n \rightarrow \infty} \mathbf{x}_n = \mathbf{x}_0$,
- $\lim_{n \rightarrow \infty} \mathbf{y}_n = \mathbf{y}_0$,
- $\mathbf{y}_n \in \Phi(\mathbf{x}_n)$ for all $n \in \mathbb{N}$,

imply that $\mathbf{y}_0 \in \Phi(\mathbf{x}_0)$.

Removable discontinuity, Sequentially compact, Bolzano–Weierstrass theorem.





(Figure from Wikipedia)

Fixed Point of Set-Valued Functions

Fixed Point (Set-Valued)

A fixed point of a set-valued function $\Phi : S \rightarrow \mathbb{P}(S)$ is a point $\mathbf{x}^* \in S$ such that $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$.



Kakutani's Theorem for Simplices

Kakutani's Theorem for Simplices (1941)

If S is an r -dimensional closed simplex in a Euclidean space and $\Phi : S \rightarrow \mathbb{P}(S)$ is upper semi-continuous, then Φ has a fixed point.



Kakutani's Fixed-Point Theorem

Kakutani's Fixed-Point Theorem (1941)

If S is a **nonempty, compact, convex set** in a Euclidean space and $\Phi : S \rightarrow \mathbb{P}(S)$ is upper semi-continuous, then Φ has a fixed point.



Kakutani's Fixed-Point Theorem

Kakutani's Fixed-Point Theorem (1941)

If S is a **nonempty, compact, convex set** in a Euclidean space and $\Phi : S \rightarrow \mathbb{P}(S)$ is upper semi-continuous, then Φ has a fixed point.

- We won't go over its proof.
- Instead, we will delve into how it can be used to prove Nash's Theorem from the perspectives of set-valued functions and best-responses.



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Cartesian product of Sets

Cartesian Product

For a family of sets $\{A_i\}_{i \in N}$, $\prod_{i \in N} A_i = A_1 \times A_2 \times \cdots \times A_n$ denotes the Cartesian product of A_i for $i \in N$.

Profile

for $x_i \in A_i$, then $(x_i)_{i \in N}$ is called a (strategy) profile.



Binary Relation

Binary Relation

- A binary relation on a set A is a subset of $A \times A$ consisting of all pairs of elements.
- For $a, b \in A$, we denote by $R(a, b)$ if a is related to b .



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Properties on Binary Relations

- **Completeness:** For all $a, b \in A$, we have $R(a, b)$, $R(b, a)$, or both.
- **Reflexivity:** For all $a \in A$, we have $R(a, a)$.
- **Transitivity:** For $a, b, c \in A$, if $R(a, b)$ and $R(b, c)$, then we have $R(a, c)$.



Preference Relation

Preference Relation

A preference relation is a **complete, reflexive, and transitive** binary relation.

- Denote by $a \succsim b$ if a is related to b .
- Denote by $a \succ b$ if $a \succsim b$ but $b \not\succsim a$.
- Denote by $a \sim b$ if $a \succsim b$ and $b \succsim a$.



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 - Denote by $a \sim b$ if $a \succsim b$ and $b \succsim a$.
-
- $a \succsim b$: a is **weakly preferred to** b .
 - $a \sim b$: agent is indifferent between a and b .



Continuity on a Preference relation

Continuous Preference Relation

A preference relation is **continuous** if:

whenever there exist sequences $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ in A such that

- $\lim_{k \rightarrow \infty} a_k = a$,
- $\lim_{k \rightarrow \infty} b_k = b$,
- and $a_k \succsim b_k$ for all $k \in \mathbb{N}$

we have $a \succsim b$.



Strategic Games

Strategic Games

A strategic game is a tuple $\langle N, (A_i), (\succsim_i) \rangle$ consisting of

- a finite set of **players** N .
 - for each player $i \in N$, a nonempty set of **actions** A_i .
 - for each player $i \in N$, a **preference relation** \succsim_i on $A = \prod_{j \in N} A_j$.
-
- A strategic game is **finite** if A_i is finite for all $i \in N$.



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- A strategic game is **finite** if A_i is finite for all $i \in N$.
 - **Note:** \succsim_i is not defined on A_i only, but instead on the set of all $(A_j)_{j \in N}$.



PNE w.r.t. a Preference Relation

Pure Nash Equilibrium (PNE) with (\succsim_i)

A (pure) Nash equilibrium (PNE) of a strategic game $\langle N, (A_i), (\succsim_i) \rangle$ is a profile $\mathbf{a}^* := (a_i)_{i \in N}$ such that for all $i \in N$, we have

$$(\mathbf{a}_{-i}^*, a_i^*) \succsim_i (\mathbf{a}_{-i}^*, a'_i) \text{ for all } a'_i \in A_i.$$



Best-Response Function

Best-Response Functions

The **best-response** function of player i ,

$$BR_i : \prod_{j \in N \setminus \{i\}} A_j \rightarrow \mathbb{P}(A_i),$$

is given by

$$BR_i(\mathbf{a}_{-i}) = \{a_i \in A_i \mid (\mathbf{a}_{-i}, a_i) \succeq_i (\mathbf{a}_{-i}, a'_i) \text{ for all } a'_i \in A_i\}.$$



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- BR_i is set-valued.
- Recall: $\mathbb{P}(X)$ includes all nonempty, closed, and convex subsets of X .



PNE w.r.t. a Preference Relation

- Alternative definition of NE.

Pure Nash Equilibrium (PNE) with (\succsim_i)

A Nash equilibrium of a strategic game $\langle N, (A_i), (\succsim_i) \rangle$ is a profile $\mathbf{a}^* := (a_i)_{i \in N}$ such that $\mathbf{a}_i^* \in BR_i(\mathbf{a}_{-i}^*)$ for all $i \in N$.

- Thus, to prove the existence of a PNE for a strategic game $\langle N, (A_i), (\succsim_i) \rangle$, it suffices to show that:



PNE w.r.t. a Preference Relation

- Alternative definition of NE.

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- Thus, to prove the existence of a PNE for a strategic game $\langle N, (A_i), (\succsim_i) \rangle$, it suffices to show that:
 - There exists a profile $\mathbf{a}^* \in A$ such that for all $i \in N$ we have $\mathbf{a}_i^* \in BR_i(\mathbf{a}_{-i}^*)$.



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General Idea

- Let $BR : A \rightarrow \mathbb{P}(A)$ be

$$BR(\mathbf{a}) = \prod_{i \in N} BR_i(\mathbf{a}_{-i}).$$

- We seek for some $\mathbf{a}^* \in A$ such that $\mathbf{a}^* \in BR(\mathbf{a}^*)$.



General Idea

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$$BR(\mathbf{a}) = \prod_{i \in N} BR_i(\mathbf{a}_{-i}).$$

- We seek for some $\mathbf{a}^* \in A$ such that $\mathbf{a}^* \in BR(\mathbf{a}^*)$.
- We can then use Kakutani's Fixed-Point Theorem to show that \mathbf{a}^* exists.



General Idea

- Let $BR : A \rightarrow \mathbb{P}(A)$ be

$$BR(\mathbf{a}) = \prod_{i \in N} BR_i(\mathbf{a}_{-i}).$$

- We seek for some $\mathbf{a}^* \in A$ such that $\mathbf{a}^* \in BR(\mathbf{a}^*)$.
- We can then use Kakutani's Fixed-Point Theorem to show that \mathbf{a}^* exists.
- Yet, we need to verify the conditions under which Kakutani's Fixed-Point Theorem holds.



Quasi-Concave

Quasi-Concave of \succsim_i

A preference relation \succsim_i over A is **quasi-concave** on A_i if for all $\mathbf{a} \in A$, the set

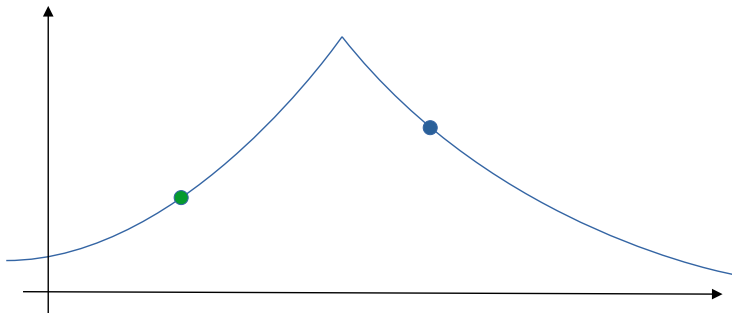
$$\{a'_i \in A_i \mid (\mathbf{a}_{-i}, a'_i) \succsim_i (\mathbf{a}_{-i}, a_i)\}$$

is **convex**.

- Then, we can consider the following theorem which guarantees the condition of a PNE.



An example of quasi-concave function.



$$f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}, \text{ for } \lambda \in [0, 1]$$



The Main Theorem I

Main Theorem I

The strategic game $\langle N, (A_i), (\succsim_i) \rangle$ has a (pure) Nash equilibrium if

- A_i is a nonempty, compact, and convex subset of a Euclidean space
 - \succsim_i is continuous and quasi-concave on A_i for all $i \in N$.
-
- We will show that A (cf. S) and BR (cf. Φ) satisfy the conditions to apply Kakutani's Fixed-Point Theorem.



Requirements for A & BR

- A_i is nonempty, compact and convex for all $i \in N$, so their Cartesian product (i.e., A) must also be nonempty, compact and convex.



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- We need to show that $BR_i(\mathbf{a}_{-i})$ is nonempty, closed, and convex for all $\mathbf{a}_{-i} \in \prod_{j \in N \setminus \{i\}} A_j$.



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- We need to show that $BR_i(\mathbf{a}_{-i})$ is nonempty, closed, and convex for all $\mathbf{a}_{-i} \in \prod_{j \in N \setminus \{i\}} A_j$.
 - Their Cartesian product $BR(\mathbf{a})$ is then nonempty, closed and convex, too.
 - We then have $BR : A \rightarrow \mathbb{P}(A)$.



$BR_i(\mathbf{a}_{-i})$ is nonempty

- Assume that we can construct a continuous function (utility function) $u_i : A_i \rightarrow \mathbb{R}$ such that for $a_i, a'_i \in A_i$, $(\mathbf{a}_{-i}, a_i) \succsim (\mathbf{a}_{-i}, a'_i)$ if and only if $u_i(a_i) \geq u_i(a'_i)$.



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- Since A_i is compact and u_i is continuous, $u_i(A_i)$ is compact as well.
- By the [Extreme Value Theorem](#), there must exist some $a_i^* \in A_i$ such that $u_i(a_i^*) \geq u_i(a_i)$ for all $a_i \in A_i$.



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- Since A_i is compact and u_i is continuous, $u_i(A_i)$ is compact as well.
- By the [Extreme Value Theorem](#), there must exist some $a_i^* \in A_i$ such that $u_i(a_i^*) \geq u_i(a_i)$ for all $a_i \in A_i$.
- By definition of u_i , it follows that $(\mathbf{a}_{-i}, a_i^*) \succsim (\mathbf{a}_{-i}, a_i)$ for all $a_i \in A_i$, thus $a_i^* \in BR_i(\mathbf{a}_{-i})$.



$BR_i(\mathbf{a}_{-i})$ is nonempty

- Assume that we can construct a continuous function (utility function) $u_i : A_i \rightarrow \mathbb{R}$ such that for $a_i, a'_i \in A_i$, $(\mathbf{a}_{-i}, a_i) \succsim (\mathbf{a}_{-i}, a'_i)$ if and only if $u_i(a_i) \geq u_i(a'_i)$.
- Since A_i is compact and u_i is continuous, $u_i(A_i)$ is compact as well.
- By the [Extreme Value Theorem](#), there must exist some $a_i^* \in A_i$ such that $u_i(a_i^*) \geq u_i(a_i)$ for all $a_i \in A_i$.
- By definition of u_i , it follows that $(\mathbf{a}_{-i}, a_i^*) \succsim (\mathbf{a}_{-i}, a_i)$ for all $a_i \in A_i$, thus $a_i^* \in BR_i(\mathbf{a}_{-i})$.
- So $BR_i(\mathbf{a}_{-i})$ is nonempty.



$BR_i(\mathbf{a}_{-i})$ is closed

- Take an arbitrary $p \in \overline{BR_i(\mathbf{a}_{-i})}$.
- There must exist some sequence $(p_k)_{k \in \mathbb{N}}$ such that $p_k \in BR_i(\mathbf{a}_{-i})$ for all $k \in \mathbb{N}$ and $\lim_{k \rightarrow \infty} p_k = p$.



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- By the definition of BR_i , we know that $(\mathbf{a}_{-i}, p_k) \succsim_i (\mathbf{a}_{-i}, a_i)$ for all $a_i \in A_i$.



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- By the definition of BR_i , we know that $(\mathbf{a}_{-i}, p_k) \succsim_i (\mathbf{a}_{-i}, a_i)$ for all $a_i \in A_i$.
- For each $a_i \in A_i$, we can construct
 - a sequence $((\mathbf{a}_{-i}, p_k))_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} (\mathbf{a}_{-i}, p_k) = (\mathbf{a}_{-i}, p)$.
 - a sequence $((\mathbf{a}_{-i}, a_i))_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} (\mathbf{a}_{-i}, a_i) = (\mathbf{a}_{-i}, a_i)$.



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- Note that $(\mathbf{a}_{-i}, p_k) \succsim_i (\mathbf{a}_{-i}, a_i)$ for all $k \in \mathbb{N}$.
 - By the continuity of \succsim_i , we have $(\mathbf{a}_{-i}, p) \succsim_i (\mathbf{a}_{-i}, a_i)$ for all $a_i \in A_i$.



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 - By the definition of BR_i , we know that $(\mathbf{a}_{-i}, p_k) \succsim_i (\mathbf{a}_{-i}, a_i)$ for all $a_i \in A_i$.
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 - By the continuity of \succsim_i , we have $(\mathbf{a}_{-i}, p) \succsim_i (\mathbf{a}_{-i}, a_i)$ for all $a_i \in A_i$.
- $\Rightarrow p \in BR_i(\mathbf{a}_{-i})$



$BR_i(\mathbf{a}_{-i})$ is closed

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 - There must exist some sequence $(p_k)_{k \in \mathbb{N}}$ such that $p_k \in BR_i(\mathbf{a}_{-i})$ for all $k \in \mathbb{N}$ and $\lim_{k \rightarrow \infty} p_k = p$.
 - By the definition of BR_i , we know that $(\mathbf{a}_{-i}, p_k) \succsim_i (\mathbf{a}_{-i}, a_i)$ for all $a_i \in A_i$.
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 - Note that $(\mathbf{a}_{-i}, p_k) \succsim_i (\mathbf{a}_{-i}, a_i)$ for all $k \in \mathbb{N}$.
 - By the continuity of \succsim_i , we have $(\mathbf{a}_{-i}, p) \succsim_i (\mathbf{a}_{-i}, a_i)$ for all $a_i \in A_i$.
- $\Rightarrow p \in BR_i(\mathbf{a}_{-i})$ ($\therefore BR_i(\mathbf{a}_{-i})$ is closed).



$BR_i(\mathbf{a}_{-i})$ is convex

- Consider $a_i \in BR_i(\mathbf{a}_{-i})$.
- \succsim_i is quasi-concave on $A_i \Rightarrow$



$BR_i(\mathbf{a}_{-i})$ is convex

- Consider $a_i \in BR_i(\mathbf{a}_{-i})$.
- \succsim_i is quasi-concave on $A_i \Rightarrow$

$$S = \{a'_i \in A_i \mid (\mathbf{a}_{-i}, a'_i) \succsim_i (\mathbf{a}_{-i}, a_i)\} \text{ is convex}$$

- Since a_i is a best response, the responses a'_i weakly preferable to a_i must be also best responses.



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- Since a_i is a best response, the responses a'_i weakly preferable to a_i must be also best responses. $\Rightarrow S \subseteq BR_i(\mathbf{a}_{-i})$.



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- \succsim_i is quasi-concave on $A_i \Rightarrow$

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- Since a_i is a best response, the responses a'_i weakly preferable to a_i must be also best responses. $\Rightarrow S \subseteq BR_i(\mathbf{a}_{-i})$.
- Any other best response $a_i^* \in BR_i(\mathbf{a}_{-i})$ must be at least good as a_i



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- \succsim_i is quasi-concave on $A_i \Rightarrow$

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- Since a_i is a best response, the responses a'_i weakly preferable to a_i must be also best responses. $\Rightarrow S \subseteq BR_i(\mathbf{a}_{-i})$.
- Any other best response $a_i^* \in BR_i(\mathbf{a}_{-i})$ must be at least good as a_i
 $\Rightarrow BR_i(\mathbf{a}_{-i}) \subseteq S$.
- Hence, we have $BR_i(\mathbf{a}_{-i}) = S$, so $BR_i(\mathbf{a}_{-i})$ is convex.



- Next, we will show that BR is upper semi-continuous.



Recall: Upper Semi-Continuous

Upper semi-continuous functions

Let

- $\mathbb{P}(X)$: all nonempty, closed, convex subsets of X .
- S : a nonempty, compact, and convex set.

Then the set-valued function $\Phi : S \rightarrow \mathbb{P}(S)$ is **upper semi-continuous** if

for arbitrary sequences $(\mathbf{x}_n)_{n \in \mathbb{N}}, (\mathbf{y}_n)_{n \in \mathbb{N}}$ in S , we have

- $\lim_{n \rightarrow \infty} \mathbf{x}_n = \mathbf{x}_0$,
 - $\lim_{n \rightarrow \infty} \mathbf{y}_n = \mathbf{y}_0$,
 - $\mathbf{y}_n \in \Phi(\mathbf{x}_n)$ for all $n \in \mathbb{N}$,
- imply that $\mathbf{y}_0 \in \Phi(\mathbf{x}_0)$.



BR is upper semi-continuous

- Consider two sequences $(\mathbf{x}^k), (\mathbf{y}^k)$ in A such that

$$\lim_{k \rightarrow \infty} \mathbf{x}^k = \mathbf{x}^0,$$

$$\lim_{k \rightarrow \infty} \mathbf{y}^k = \mathbf{y}^0.$$

$$\mathbf{y}^k \in BR_i(\mathbf{x}^k) \text{ for all } k \in \mathbb{N}.$$

- Then we have $\mathbf{y}_i^k \in BR_i(\mathbf{x}_{-i}^k)$ for all $i \in N, k \in \mathbb{N}$.



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$$\mathbf{y}^k \in BR_i(\mathbf{x}^k) \text{ for all } k \in \mathbb{N}.$$

- Then we have $y_i^k \in BR_i(\mathbf{x}_{-i}^k)$ for all $i \in N, k \in \mathbb{N}$.
- For an arbitrary $i \in N$, we have $(\mathbf{x}_{-i}^k, y_i^k) \succsim_i (\mathbf{x}_{-i}^k, a_i)$ for all $a_i \in A_i$ and $k \in \mathbb{N}$ (\because best response).



BR is upper semi-continuous (contd.)

- For each $a_i \in A_i$, we can construct:
 - a sequence $((\mathbf{x}_{-i}^k, y_i^k))_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} (\mathbf{x}_{-i}^k, y_i^k) = (\mathbf{x}_{-i}^0, y_i^0)$.
 - a sequence $((\mathbf{x}_{-i}^k, a_i))_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} (\mathbf{x}_{-i}^k, a_i) = (\mathbf{x}_{-i}^0, a_i)$.
- Note that we have $(\mathbf{x}_{-i}^k, y_i^k) \succsim_i (\mathbf{x}_{-i}^k, a_i)$ for all $k \in \mathbb{N}$.
 - By continuity of \succsim_i , we have $(\mathbf{x}_{-i}^0, y_i^0) \succsim_i (\mathbf{x}_{-i}^0, a_i)$ for all $a_i \in A_i$.



BR is upper semi-continuous (contd.)

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 - By continuity of \succsim_i , we have $(\mathbf{x}_{-i}^0, y_i^0) \succsim_i (\mathbf{x}_{-i}^0, a_i)$ for all $a_i \in A_i$.
- Thus, we have $y_i^0 \in BR_i(\mathbf{x}_{-i}^0)$ for all $i \in N$.
 - $\mathbf{y}^0 \in BR(\mathbf{x}^0)$.



BR is upper semi-continuous (contd.)

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 - By continuity of \succsim_i , we have $(\mathbf{x}_{-i}^0, y_i^0) \succsim_i (\mathbf{x}_{-i}^0, a_i)$ for all $a_i \in A_i$.
- Thus, we have $y_i^0 \in BR_i(\mathbf{x}_{-i}^0)$ for all $i \in N$.
 - $\mathbf{y}^0 \in BR_i(\mathbf{x}^0)$.
- Therefore, BR is upper semi-continuous.



BR is upper semi-continuous (contd.)

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 - By continuity of \succsim_i , we have $(\mathbf{x}_{-i}^0, y_i^0) \succsim_i (\mathbf{x}_{-i}^0, a_i)$ for all $a_i \in A_i$.
- Thus, we have $y_i^0 \in BR_i(\mathbf{x}_{-i}^0)$ for all $i \in N$.
 - $\mathbf{y}^0 \in BR(\mathbf{x}^0)$.
- Therefore, BR is upper semi-continuous.

By Kakutani's Fixed-Point Theorem, there exists some $\mathbf{a}^* \in A$ such that $\mathbf{a}^* \in BR(\mathbf{a}^*)$



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- Thus, we have $y_i^0 \in BR_i(\mathbf{x}_{-i}^0)$ for all $i \in N$.
 - $\mathbf{y}^0 \in BR(\mathbf{x}^0)$.
- Therefore, BR is upper semi-continuous.

By Kakutani's Fixed-Point Theorem, there exists some $\mathbf{a}^* \in A$ such that $\mathbf{a}^* \in BR(\mathbf{a}^*) \Rightarrow \mathbf{a}^*$ is a PNE of the strategic game.



Outline

- 1 Brouwer's Fixed Point Theorem
 - Sketch of the Proof of Nash's Theorem (from Brouwer's Theorem)
- 2 Kakutani's Fixed Point Theorem
 - Pure Nash Equilibria of Pure Strategic Games
 - Preliminaries
 - Main Theorem I & The Proof
 - Mixed Nash Equilibria of Finite Strategies Games
 - Preliminaries & Assumptions
 - Main Theorem II & the Proof



Limitations of the Previous PNE Result

- Any **finite** game cannot satisfy the conditions.



Limitations of the Previous PNE Result

- Any **finite** game cannot satisfy the conditions.
 - Each A_i cannot be convex if it is finite and nonempty.
- ★ Next, we consider extending finite games into **non-deterministic (randomized)** strategies.



Assumptions

- For a strategic game $\langle N, (A_i), (\succsim_i) \rangle$, we assume that we can construct a utility function $u_i : A \rightarrow \mathbb{R}$, where $A = \prod_{i \in N} A_i$.
- Each player's *expected utility* is coupled with the set of probability distributions over A .
- $\Delta(X)$: the set of probability distributions over X .
- If X is finite and $\delta \in \Delta(X)$, then
 - $\delta(x)$: the probability that δ assigns to $x \in X$.
 - The support of δ : $\chi(\delta) = \{x \in X \mid \delta(x) > 0\}$.



Mixed Strategy

Mixed Strategy

Given a strategic game $\langle N, (A_i), (u_i) \rangle$, we call

- $\alpha_i \in \Delta(A_i)$ a **mixed strategy**.
- $a_i \in A_i$ a **pure strategy**.

A profile of mixed strategies $\alpha = (\alpha_j)_{j \in N}$ induces a probability distribution over A .

- The probability of $\mathbf{a} = (a_j)_{j \in N}$ under α :

$$\alpha(\mathbf{a}) = \prod_{j \in N} \alpha_j(a_j). \quad (\text{a normal product})$$

(A_i is finite $\forall i \in N$ and each player's strategy is resolved independently.)



prob. = $\alpha_1(t_1) \cdot \alpha_2(s_1)$

| | | $\alpha_2(s_1)$ | $\alpha_2(s_2)$ |
|-----------------------|--|--------------------------------|--------------------------------|
| $\alpha_1(t_1)$ t_1 | | $u_1(t_1, s_1), u_2(t_1, s_1)$ | $u_1(t_1, s_2), u_2(t_1, s_2)$ |
| $\alpha_1(t_2)$ t_2 | | $u_1(t_2, s_1), u_2(t_2, s_1)$ | $u_1(t_2, s_2), u_2(t_2, s_2)$ |



Mixed Extension of $\langle N, (A_i), (u_i) \rangle$

Mixed Extension of the Strategic Games

$\langle N, (\Delta(A_i)), (U_i) \rangle$:

- $U_i : \prod_{i \in N} \Delta(A_i) \rightarrow \mathbb{R}$; expected utility over A induced by $\alpha \in \prod_{i \in N} \Delta(A_i)$.
- If A_j is finite for all $j \in N$, then

$$\begin{aligned} U_i(\alpha) &= \sum_{\mathbf{a} \in A} (\alpha(\mathbf{a}) \cdot u_i(\mathbf{a})) \\ &= \sum_{\mathbf{a} \in A} \left(\left(\prod_{j \in N} \alpha_j(a_j) \right) \cdot u_i(\mathbf{a}) \right). \end{aligned}$$



Main Theorem II

Main Theorem II

Every finite strategies game has a mixed strategy Nash equilibrium.

- Consider an arbitrary finite strategic game $\langle N, (A_i), (u_i) \rangle$, let $m_i := |A_i|$ for all $i \in N$.
- Represent each $\Delta(A_i)$ as a collection of vectors $\mathbf{p}^i = (p_1, p_2, \dots, p_{m_i})$.
 - $p_k \geq 0$ for all $k \in [m_i]$ and $\sum_{k=1}^{m_i} p_k = 1$.
 - $\Delta(A_i)$ is a standard $m_i - 1$ simplex for all $i \in N$.



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- Represent each $\Delta(A_i)$ as a collection of vectors $\mathbf{p}^i = (p_1, p_2, \dots, p_{m_i})$.
 - $p_k \geq 0$ for all $k \in [m_i]$ and $\sum_{k=1}^{m_i} p_k = 1$.
 - $\Delta(A_i)$ is a standard $m_i - 1$ simplex for all $i \in N$.
 - ★ $\Delta(A_i)$: nonempty, compact, and convex for each $i \in N$.
- U_i : continuous (\because multilinear).
- Next, we show that U_i is **quasi-concave** in $\Delta(A_i)$.



Proof of Main Theorem II (contd.)

- Consider $\alpha \in \prod_{i \in N} \Delta(A_i)$.
- **Goal:** Show that $S = \{\alpha'_i \in \Delta(A_i) \mid U_i(\alpha_{-i}, \alpha'_i) \geq U_i(\alpha_{-i}, \alpha_i)\}$ is convex.



Proof of Main Theorem II (contd.)

- Consider $\alpha \in \prod_{i \in N} \Delta(A_i)$.
- **Goal:** Show that $S = \{\alpha'_i \in \Delta(A_i) \mid U_i(\alpha_{-i}, \alpha'_i) \geq U_i(\alpha_{-i}, \alpha_i)\}$ is convex.
- Take $\beta_i, \gamma_i \in S$, $\lambda \in [0, 1]$.
- By definition of S , we have
 - $U_i(\alpha_{-i}, \beta_i) \geq U_i(\alpha_{-i}, \alpha_i)$, and
 - $U_i(\alpha_{-i}, \gamma_i) \geq U_i(\alpha_{-i}, \alpha_i)$.



Proof of Main Theorem II (contd.)

- Consider $\alpha \in \prod_{i \in N} \Delta(A_i)$.
- **Goal:** Show that $S = \{\alpha'_i \in \Delta(A_i) \mid U_i(\alpha_{-i}, \alpha'_i) \geq U_i(\alpha_{-i}, \alpha_i)\}$ is convex.
- Take $\beta_i, \gamma_i \in S, \lambda \in [0, 1]$.
- By definition of S , we have
 - $U_i(\alpha_{-i}, \beta_i) \geq U_i(\alpha_{-i}, \alpha_i)$, and
 - $U_i(\alpha_{-i}, \gamma_i) \geq U_i(\alpha_{-i}, \alpha_i)$.
- $\lambda U_i(\alpha_{-i}, \beta_i) + (1 - \lambda) U_i(\alpha_{-i}, \gamma_i) \geq \lambda U_i(\alpha_{-i}, \alpha_i) + (1 - \lambda) U_i(\alpha_{-i}, \alpha_i) = U_i(\alpha_{-i}, \alpha_i)$.



Proof of Main Theorem II (contd.)

- By the multilinearity of U_i , we have

$$\lambda U_i(\alpha_{-i}, \beta_i) + (1 - \lambda) U_i(\alpha_{-i}, \gamma_i) = U_i(\alpha_{-i}, \lambda \beta_i + (1 - \lambda) \gamma_i).$$



Proof of Main Theorem II (contd.)

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- So,

$$U_i(\alpha_{-i}, \lambda \beta_i + (1 - \lambda) \gamma_i) \geq U_i(\alpha_{-i}, \alpha_i).$$



Proof of Main Theorem II (contd.)

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$$\lambda \beta_i + (1 - \lambda) \gamma_i \in S$$



Proof of Main Theorem II (contd.)

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- So,

$$U_i(\alpha_{-i}, \lambda \beta_i + (1 - \lambda) \gamma_i) \geq U_i(\alpha_{-i}, \alpha_i).$$

$$\lambda \beta_i + (1 - \lambda) \gamma_i \in S \Rightarrow U_i \text{ is convex.}$$

- Thus, U_i is quasi-concave in $\Delta(A_i)$.

We are done.



A Question

Matching Pennies of Infinite Actions

We have two players A and B having utility functions $f(x, y) = (x - y)^2$ and $g(x, y) = -(x - y)^2$ respectively. $x, y \in [-1, 1]$.

- Does this game has a pure Nash equilibrium?
- Why can't we use Kakutani's fixed point theorem?



Thank You.

