Revenue-Maximizing Auctions

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- In previous lectures, we only focus on maximizing the social welfare, while revenue is generated only as a side effect.
  - Though, indeed, there are real-world scenarios that the primary objective is welfare maximization (i.e., government auctions)
- In this lecture, we:
  - Study mechanisms that are designed to raise as much revenue as possible.

Outline

A trivial example

- Suppose that there is one item and only one bidder, with private valuation v (hence, no other competitors).
- The direct-revelation DSIC auction: **take-it-or-leave-it**.
  - With a posted price $r \geq 0$, the auction's revenue is either

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- Maximizing **social welfare** is trivial:
  - Set $r := 0$.
  - Independent of v.

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- How should we set r in order to maximize **revenue**?
  - Note the difficulty: v is private.
  - Let's consider another point of view: Bayesian analysis.

- **Goal:** Characterize the expected revenue-maximizing mechanisms with respect to a prior distribution over agents' valuations.

Bayesian Environment

Bayesian Environment

- **A single-parameter environment.** Assume that there is a constant M such that $x_i \leq M$ for every i and feasible solution $(x_1, \ldots, x_n) \in X$.
- **Independent distributions $F_1, \ldots, F_n$ with positive and continuous density functions $f_1, \ldots, f_n$.** Assume that the private valuation v_i of participant i is drawn from F_i.
  - Also, assume that the support of every distribution F_i belongs to $[0, v_{\max}]$ for some $v_{\max} < \infty$.

⋆ The mechanism designer knows the distributions $F_1, \ldots, F_n$.

⋆ The realizations $v_1, \ldots, v_n$ of agents' valuations are still private.

The goal now

- Among all DSIC mechanisms, the optimal mechanism is the one having the highest expected revenue (assuming truthful bids).
  - The expectation is w.r.t. $F_1 \times F_2 \times \cdots \times F_n$ over valuation profiles.
- The expected revenue of a posted price r is then

$$r \cdot (1 - F(r)),$$

where r represents the revenue of a sale while $(1 - F(r))$ represents the probability of a sale.

- Solve for the best posted price $r^* \Rightarrow$ a monopoly price.

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- Solve for the best posted price $r^* \Rightarrow$ a monopoly price.
- For example, if F is the uniform distribution on $[0, 1]$, so that $F(x) = x$ on $[0, 1]$, then the monopoly price is $\frac{1}{2}$, achieving an expected revenue of $\frac{1}{4}$.

Single-Item Auction with Two Bidders

Exercise

Consider a single-item auction with two bidders with valuations drawn independently from the uniform distribution on $[0, 1]$.

a. Prove that the expected revenue obtained by a second-price auction (with no reserve) is $\frac{1}{3}$.
b. Prove that the expected revenue obtained by a second-price auction with reserve $\frac{1}{2}$ is $\frac{5}{12}$.

Outline

Goal

- An explicit description of an optimal (i.e., expected revenue-maximizing) DSIC mechanism for every single-parameter environment and distributions $F_1, \ldots, F_n$.

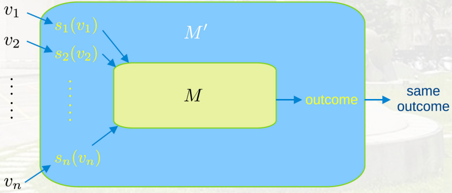
Recall

- Every DSIC mechanism is equivalent to a direct-revelation DSIC mechanism.

Theorem (Revelation Principle for DSIC Mechanisms)

For every mechanism M where every participant always has a dominant strategy, there is an equivalent direct-revelation DSIC mechanism M'.

- We use a simulation argument to construct M' as follows.



Recall

- Every DSIC mechanism is equivalent to a direct-revelation DSIC mechanism.
- Hence we can pay our attention to such mechanisms.
- Assume truthful bids for the rest of our discussions.
  - $\boldsymbol{b} = \boldsymbol{v}$.

Expected revenue of a DSIC mechanism $(\boldsymbol{x}, \boldsymbol{p})$

- The expected revenue of a DSIC mechanism $(\boldsymbol{x}, \boldsymbol{p})$ is

$$\mathbf{E}_{\boldsymbol{v} \sim \boldsymbol{F}} \left[\sum_{i=1}^{n} p_i(\boldsymbol{v}) \right],$$

where the expectation is w.r.t. $\boldsymbol{F} = F_1 \times \cdots \times F_n$ over agents' valuations.

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- It's unclear how to maximize this expression...
- Later we will consider an alternative formula which only references the allocation rule of a mechanism.

Outline

Virtual Valuations

Virtual Valuation

For an agent i with valuation distribution F_i and valuation v_i (drawn from F_i), her virtual valuation is define as

$$\varphi(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

- For example, if F_i is the uniform distribution on $[0, 1]$.

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  - $F_i(z) = z$ for $z \in [0, 1]$.
  - $f_i(z) = 1$.
  - $\varphi_i(z) = z - \frac{1-z}{1} = 2z - 1$ on $[0, 1]$.

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  - $\varphi_i(z) = z - \frac{1-z}{1} = 2z - 1$ on $[0,1]$.
- It is always at most the corresponding valuation.
- It could be *negative*.

What do virtual valuations mean?

$$\varphi(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

- One possible interpretation:
  - v_i: what you'd like to charge
  - $\frac{1-F_i(v_i)}{f_i(v_i)}$: inevitable revenue loss caused by not knowing v_i in advance.

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- Second interpretation:
  - $\varphi(v_i)$: the slope of a revenue curve at v_i.

The Crucial Lemma

Lemma (5.1 in the Textbook)

For every single-parameter environment with valuation distributions $F_1, \ldots, F_n$, every DSIC mechanism $(\boldsymbol{x}, \boldsymbol{p})$, every agent i, and every value $\boldsymbol{v}_{-i}$ of the valuations of the other agents,

$$\mathbf{E}_{v_i \sim F_i}[p_i(\boldsymbol{v})] = \mathbf{E}_{v_i \sim F_i}[\varphi_i(v_i) \cdot x_i(\boldsymbol{v})].$$

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- Note: the identity holds in expectation over v_i, and not pointwise.
  - $\varphi_i(v_i)$ could be negative for some i.

The Main Theorem

Theorem (5.2 in the Textbook)

For every single-parameter environment with valuation distributions $F_1, \ldots, F_n$ and every DSIC mechanism $(\boldsymbol{x}, \boldsymbol{p})$,

$$\mathbf{E}_{\boldsymbol{v} \sim \boldsymbol{F}}\left[\sum_{i=1}^{n} p_i(\boldsymbol{v})\right] = \mathbf{E}_{\boldsymbol{v} \sim \boldsymbol{F}}\left[\sum_{i=1}^{n} \varphi_i(v_i) \cdot x_i(\boldsymbol{v})\right].$$

- That is, the expected **revenue** equals the expected **virtual welfare**!.

Proof of Theorem 5.2

- Taking the expectation, with respect to $\boldsymbol{v}_{-i} \sim \boldsymbol{F}_{-i}$, of both sides of the equation in Lemma 5.1: (i.e., $\mathbf{E}_{v_i \sim F_i}[p_i(\boldsymbol{v})] = \mathbf{E}_{v_i \sim F_i}[\varphi_i(v_i) \cdot x_i(\boldsymbol{v})]$)[1]

$$\mathbf{E}_{\boldsymbol{v} \sim \boldsymbol{F}}[p_i(\boldsymbol{v})] = \mathbf{E}_{\boldsymbol{v} \sim \boldsymbol{F}}[\varphi_i(v_i) \cdot x_i(\boldsymbol{v})].$$

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- Applying the linearity of expectation twice:

$$\begin{aligned} \mathbf{E}_{\boldsymbol{v} \sim \boldsymbol{F}}\left[\sum_{i=1}^{n} p_i(\boldsymbol{v})\right] &= \sum_{i=1}^{n} \mathbf{E}_{\boldsymbol{v} \sim \boldsymbol{F}}[p_i(\boldsymbol{v})] \\ &= \sum_{i=1}^{n} \mathbf{E}_{\boldsymbol{v} \sim \boldsymbol{F}}[\varphi_i(v_i) \cdot x_i(\boldsymbol{v})] \\ &= \mathbf{E}_{\boldsymbol{v} \sim \boldsymbol{F}}\left[\sum_{i=1}^{n} \varphi_i(v_i) \cdot x_i(\boldsymbol{v})\right]. \end{aligned}$$

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The Crucial Lemma

Lemma 5.1

For every single-parameter environment with valuation distributions $F_1, \ldots, F_n$, every DSIC mechanism $(\boldsymbol{x}, \boldsymbol{p})$, every agent i, and every value $\boldsymbol{v}_{-i}$ of the valuations of the other agents,

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Sketch of the Proof (1/4)

- Assume that we have
  - a DSIC mechanism $(\boldsymbol{x}, \boldsymbol{p})$;
  - the allocation rule: $\boldsymbol{x}$
  - the valuation profile: $\boldsymbol{v}$.
- Recall Myerson's payment formula:

$$p_i(v_i, \boldsymbol{v}_{-i}) = \int_0^{v_i} z \cdot x_i'(z, \boldsymbol{v}_{-i})dz.$$

  for the payment made by agent i.
- Assume that $x_i(z, \boldsymbol{v}_{-i})$ is differentiable.

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- Assume that $x_i(z, \boldsymbol{v}_{-i})$ is differentiable.
  - The same formula holds more generally for an arbitrary *monotone function* $x_i(z, \boldsymbol{v}_{-i})$, including piecewise constant functions.
  - A suitable interpretation of $x_i'(z, \boldsymbol{v}_{-i})$ + the corresp. integral.

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for the payment made by agent i.

- Assume that $x_i(z, \boldsymbol{v}_{-i})$ is differentiable.
  - The payments are fully dictated by the allocation rule.

Sketch of the Proof (2/4)

- Fix an agent i. We have

$$\begin{aligned}\mathbf{E}_{v_i\sim F_i}[p_i(\boldsymbol{v})] &= \int_0^{v_{\max}} p_i(\boldsymbol{v}) f_i(v_i)dv_i \\ &= \int_0^{v_{\max}} \left[\int_0^{v_i} z\cdot x_i'(z_i,\boldsymbol{v}_{-i})dz\right] f_i(v_i)dv_i\end{aligned}$$

  - 1st equality exploits the independence of agents' valuations.

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\end{aligned}$$

  - 1st equality exploits the independence of agents' valuations.
  - That is, the fixed value $\boldsymbol{v}_{-i}$ has no bearing on the distrbution F_i.

Reference

4.2: Expected Value and Variance of Continuous Random Variables

Last updated: Feb 28, 2020

◀ 4.1: Probability Density Functions (PDFs) and Cumulati... | 4.3: Uniform Distributions ▶

Kristin Kuter
Saint Mary's College

We now consider the expected value and variance for continuous random variables. Note that the interpretation of each is the same as in the discrete setting, but we now have a different method of calculating them in the continuous setting.

Definition 4.2.1

If X is a continuous random variable with pdf $f(x)$, then the ***expected value*** (or ***mean***) of X is given by

$$\mu = \mu_X = \mathrm{E}[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx.$$

Sketch of the Proof (3/4)

- Reversing the order of integration in

$$\int_0^{v_{\max}} \left[\int_0^{v_i} z \cdot x_i'(z_i, \mathbf{v}_{-i}) dz \right] f_i(v_i) dv_i$$

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$$\begin{aligned}
&\int_0^{v_{\max}} \left[\int_z^{v_{\max}} f_i(v_i)dv_i\right] z \cdot x_i'(z, \mathbf{v}_{-i})dz \\
= &\int_0^{v_{\max}} (1 - F_i(z)) \cdot z \cdot x_i'(z, \mathbf{v}_{-i})dz.
\end{aligned}$$

Sketch of the Proof (4/4)

- Using **integration by parts**:

$$\int_0^{v_{\max}} \underbrace{(1 - F_i(z)) \cdot z}_{g(z)} \cdot \underbrace{x_i'(z, \mathbf{v}_{-i})}_{h'(z)} \, dz.$$

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= \quad & (1 - F_i(z)) \cdot z \cdot x_i(z, \mathbf{v}_{-i}) \Big|_0^{v_{\max}} \\
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Outline

Maximization concerning only the allocation rule

- Theorem 5.2 says that: even though we only care about “payments”, we can still focus on an optimization problem concerning only the allocation rule of the mechanism.

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- Theorem 5.2 says that: even though we only care about "payments", we can still focus on an optimization problem concerning only the allocation rule of the mechanism.
- So, how should we choose the allocation rule $\boldsymbol{x}$ to maximize

$$\mathbf{E}_{\boldsymbol{v}\sim \boldsymbol{F}}\left[\sum_{i=1}^{n} \varphi_i(v_i)\cdot x_i(\boldsymbol{v})\right]?$$

- An obvious approach: maximize pointwise:
  - For each $\boldsymbol{v}$, choose $\boldsymbol{x}(\boldsymbol{v})$ to maximize the virtual welfare obtained on input $\boldsymbol{v}$, subject to feasibility of the allocation.

Well, not so obvious...

- For example, consider a single-item auction, where the feasible constraint is $\sum_{i=1}^{n} x_i(\boldsymbol{v}) \leq 1$ for every $\boldsymbol{v}$.

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- What's the virtual welfare-maximizing rule?
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  - ⋆ **Note:** virtual valuations can be negative (e.g., consider $\varphi_i(v_i) = 2v_i - 1$ for v_i uniformly drawn from $[0, 1]$).

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  - ⋆ **Note:** virtual valuations can be negative (e.g., consider $\varphi_i(v_i) = 2v_i - 1$ for v_i uniformly drawn from $[0, 1]$).
  - The virtual welfare is maximized by not awarding the item to anyone.

An Issue/Key Question

- Such a virtual welfare-maximizing allocation rule maximizes the expected virtual welfare over all allocation rules.

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Is the virtual welfare-maximizing allocation rule monotone?

- If so, Myerson's lemma can be applied and the rule can be extended to a DSIC mechanism, hence the mechanism results in the **maximum possible expected revenue** by Theorem 5.2.

Regularity Comes to the Rescue

Regular Distribution

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- For example, consider F to be the uniform distribution on $[0, 1]$.
- It's regular since the corresponding $\varphi(v) = 2v - 1$ which is nondecreasing in v.

Virtual Welfare Maximizer

Assume that F_i is regular for each i.

1. Transform the (truthfully reported) valuation v_i of agent i into $\varphi_i(v_i)$.
2. Choose the feasible allocation $(x_1, \ldots, x_n)$ that maximizes the virtual welfare $\sum_{i=1}^{n} \varphi_i(v_i)x_i$.
3. Charge payments according to Myerson's payment formula (refer to previous lectures).

Virtual Welfare Maximizers Are Optimal

Theorem 5.4

For every single-parameter environment and **regular distributions** $F_1, \ldots, F_n$, the corresponding virtual welfare maximizer is a DSIC mechanism with the maximum-possible expected revenue.

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- Here revenue-maximizing mechanisms are almost the same as welfare-maximizing ones.
- They differ only in using *virtual* valuations in place of valuations.

Outline

Any familiar mechanisms?

- Let's consider single-item auctions.

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- Assume bidders are i.i.d. with a common valuation distribution F (hence a common virtual valuation φ).
- Assume that F is strictly regular (hence φ is strictly increasing).
- The virtual-welfare-**maximizing** mechanism awards the item to the bidder with the highest **nonnegative** virtual valuation (if any).
  - That is, the bidder with the highest valuation.
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- The allocation rule: the same as that of a second-price auction with a reserve price of $\varphi^{-1}(0)$.

- eBay is (roughly) the optimal auction format!

Theorem (Myerson's Lemma)

Fix a single-parameter environment.

(i) An allocation rule $\boldsymbol{x}$ is implementable if and only if it is monotone.

(ii) If $\boldsymbol{x}$ is monotone, then there is a unique payment rule for which the direct-revelation mechanism $(\boldsymbol{x}, \boldsymbol{p})$ is DSIC and $p_i(\boldsymbol{b}) = 0$ whenever $b_i = 0$.

(iii) The payment rule in (ii) is given by an explicit formula.

Exercise

- Consider a virtual valuation $\varphi(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ where F is a strictly increasing distribution function with a strictly positive density function f on the interval $[0, v_{\max}]$, with $v_{\max} < \infty$.
- For a single bidder with valuation drawn from F, for $q \in [0, 1]$, define $V(q) = F^{-1}(1-q)$ as the posted price that yields a probability q of a sale.
- Define $R(q) = q \cdot V(q)$ as the expected revenue obtained from a single bidder when the probability of a sale is q.
- The function $R(q)$, for $q \in [0, 1]$, is the revenue curve of F. Note that $R(0) = R(1) = 0$.

⋆ Please prove that the slope of the revenue curve at q (i.e., $R'(q)$) is precisely $\varphi(v_i)$.

Hint

Theorem [Derivative of an Inverse Function]

Given an invertible function $f(x)$, the derivative of its inverse function $f^{-1}(x)$ evaluated at $x = a$ is

$$[f^{-1}]'(a) = \frac{1}{f'[f^{-1}(a)]}.$$

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$$1 = f'(y) \cdot \frac{dy}{dx}.$$

- Thus, $\dfrac{dy}{dx} = \dfrac{1}{f'(y)} \quad \Rightarrow \quad [f^{-1}]'(x) = \dfrac{1}{f'[f^{-1}(x)]}.$