Auctions & Mechanism Design Basics

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- We study about a kind of science of *rule-making*.
- To make it simple, we first consider single-item auctions.
- We will go over some basics about first-price auctions and second-price auctions.
- Also, we will talk about
  - incentive guarantees,
  - strong performance guarantees, and
  - computational efficiency

  in an auction.
- We will end the discussion with Myerson's Lemma.

Outline

1. Single-Item Auctions
2. Sealed-Bid Auctions
   - First-Price Auctions
   - Second-Price Auctions
   - Case Study: Sponsored Search Auctions

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  - v_i is private.
    - Unknown to the seller and other bidders.

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  - If she wins $\Rightarrow$ $v_i - p$.

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- Step (ii): The selection rule. We consider giving the item to the **highest** bidder.

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- For a bidder: Hard to figure how to bid.
- For the seller: Hard to predict what will happen.

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  - Would your answer change if you knew there were two other bidders rather than one?

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eBay/Yahoo auction

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    whichever comes first.
  - For example, if the highest other bid is \$90.
    You only pay $\$90 + \epsilon$ for some small increment ϵ.

  $\approx$ highest other bid!

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- Is such a strategy a dominant strategy?
  - The strategy is guaranteed to **maximize** a bidder's utility **no matter what other bidders do**.

Truthfully Bidding Is Dominant Here

Proposition (Incentives in Second-Price Auctions)

In a second-price auction, every bidder i has a dominant strategy: set the bid $b_i = v_i$, equal to her private valuation.

Proof of the Proposition

- Fix a bidder i with valuation v_i.
- $\boldsymbol{b}$: the vector of all bids.
- $\boldsymbol{b}_{-i}$: the vector of $\boldsymbol{b}$ with b_i removed.

* **Goal**: Show that bidder i's utility is maximized by setting $b_i = v_i$.

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Second-Price Single-Item Auctions are "ideal"

Definition (Dominant-Strategy Incentive Compatible)

An auction is dominant-strategy incentive compatible (DSIC) if

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Social Welfare

The social welfare of an outcome of a single-item auction is

$$\sum_{i=1}^{n} v_i x_i .$$

where $\sum_{i=1}^{n} x_i \leq 1$; $x_i = 1$ if bidder i wins and 0 if she loses.

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- So such an auction is welfare maximizing if bids are truthful.

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Theorem

A second-price single-item auction satisfies:

(1) DSIC.

(2) Welfare maximizing.

(3) It can be implemented in polynomial time.

In fact, (3) is linear.

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Theorem

A second-price single-item auction satisfies:

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Background

The Social Dilemma (2020) - Trailer

- Web search results:
  - relevant to your query (by an algorithm, e.g., PageRank).
  - pops out a list of sponsored links.
    - They are paid by advertisers.
- Every time you give a search query into a search engine, an auction is run in real time to decide
  - which advertiser's links are shown,
  - how these links are arranged visually,
  - what the advertisers are charged.

Multiple Items for Sponsored Search Auctions

- Let's say the items for sale are k "slots" on a search results page.
- Bidders: the advertisers who have a bid on the keyword that was searched on.
  - On the keyword, "university", NTU, NYCU, NCKU, TKU, etc., might be the bidders.

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  - On the keyword, "SUV", Toyota, Ford, Honda, Porsche, etc., might be the bidders.
- Let's say the items are not identical.
  - Higher slots are more valuable. What do you think?

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    - The probability that the user clicks on this slot.
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- The expected value derived by advertiser i from slot j: $v_i \alpha_j$
- The social welfare is $\sum_{i=1}^{n} v_i x_i$.
  - x_i: the CTR of the slot to which bidder i is assigned.
    - $x_i = 0$: bidder i is not assigned to a slot.
  - Each slot can only be assigned to one bidder;
    each bidder gets only one slot.

Our Design Approach

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  - If the payments are not just right, then the strategic bidders will game the system.

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Design Steps

(a): Assume that the bidders bid truthfully. Then, how should we assign bidders to slots so that property (2) and (3) holds?

(b): Given the answer of above, how should we set selling prices so that property (1) holds?

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- You can prove that this assignment achieves the maximum social welfare as an exercise.

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- There is an analog of the second-price rule.
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    - A powerful and general tool for implementing this second step.