

The Graph Abstract Data Type

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Outline

1 Introduction

- Motivating Examples
- Graphs

2 Graph Representations

Outline

1 Introduction

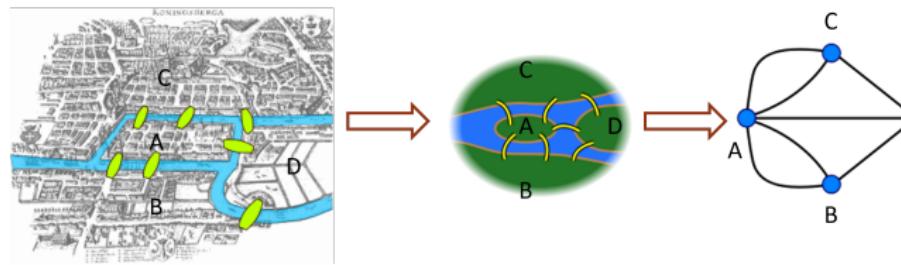
- Motivating Examples
- Graphs

2 Graph Representations

Königsberg Bridge Problem

Question

Can we walk across all the bridges **exactly once** in returning back to the starting land area?



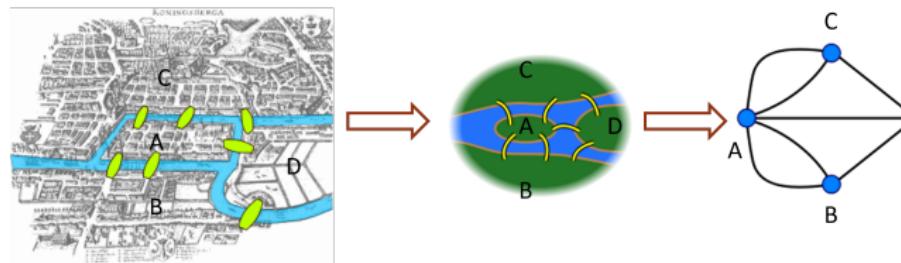
- Recall **graphs** from the Discrete Mathematics course.

- Land \mapsto vertex
- Bridge \mapsto edge

Königsberg Bridge Problem

Question (Eulerian Walk)

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- Recall **graphs** from the Discrete Mathematics course.
 - Land \mapsto vertex
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Eulerian Walk

Euler's Theorem

A connected graph has an Euler cycle if and only if every vertex has even degree.

- **degree** of vertex v : number of neighbors of v in the graph.
- **connected**: there is a path connecting every two vertices in the graph.

Eulerian Walk

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So, what about the answer to the Königberg Bridge Problem?

Definition of a Graph

Graph

A graph $G = (V, E)$ consists of two sets V and E , such that

- V : a finite, nonempty set of **vertices**;
- E : a set of vertex pairs which are called **edges**.



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 - **Directed** graph: a directed vertex pair $< u, v >$ has u as the tail and v as the head.

Definition of a Graph

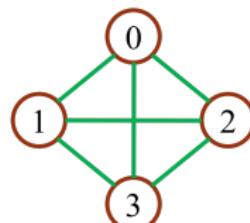
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 - **Directed** graph: a directed vertex pair $< u, v >$ has u as the tail and v as the head.
 - $< u, v >$ and $< v, u >$ indicate different edges.

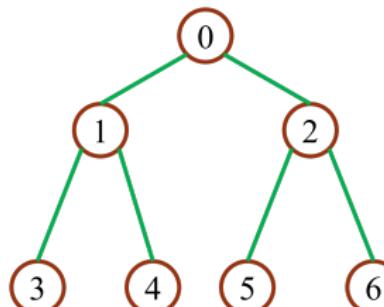


Examples

 G_1

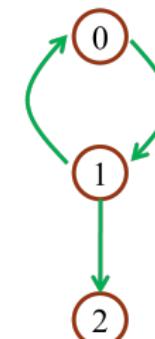
$$V(G_1)=\{0, 1, 2, 3\}$$

$$E(G_1)=\{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}$$

 G_2

$$V(G_2)=\{0, 1, 2, 3, 4, 5, 6\}$$

$$E(G_2)=\{(0, 1), (0, 2), (1, 3), (1, 4), (2, 5), (2, 6)\}$$

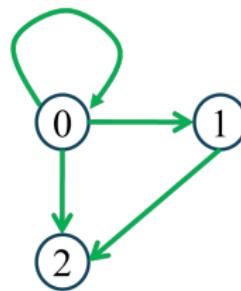
 G_3

$$V(G_3)=\{0, 1, 2\}$$

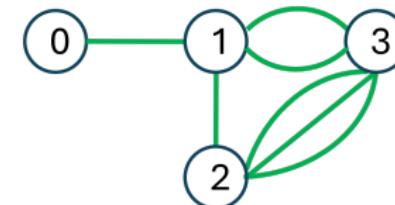
$$E(G_3)=\{<0, 1>, <1, 0>, <1, 2>\}$$



Self-Loop & Multigraph



Graph with
a self-loop



multigraph

- (v, v) : self-loop.
- multigraph: a graph with multiple occurrence of some edges.

Complete Graph

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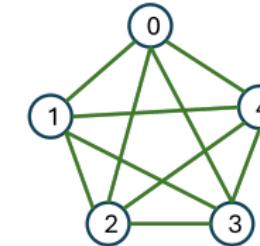
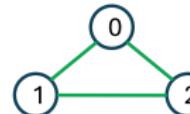
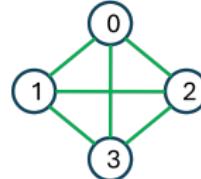
- The number of edges in a complete undirected graph of n vertices:
 $n(n - 1)/2$.
- The number of edges in a complete **directed** graph of n vertices:

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- The number of edges in a complete undirected graph of n vertices: $n(n - 1)/2$.
- The number of edges in a complete **directed** graph of n vertices: $n(n - 1)$.



Subgraph and Induced Subgraph

- If (u, v) is an edge in $E(G)$, then the vertices u and v are adjacent and that the edge (u, v) is incident on vertices u and v .

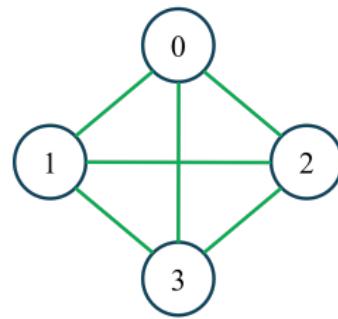
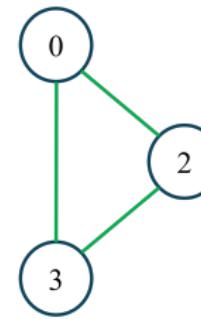
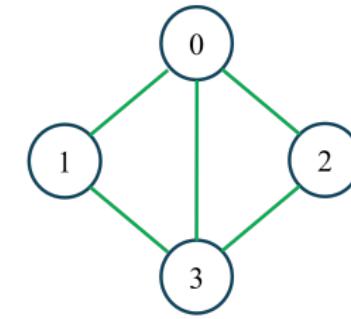
Subgraph

A subgraph of G is a graph G' such that $V(G') \subseteq V(G)$ and $E(G') \subseteq EG$.

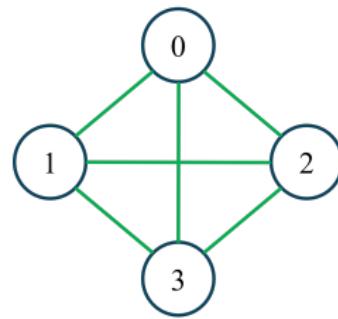
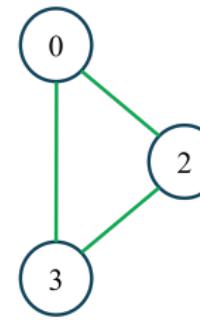
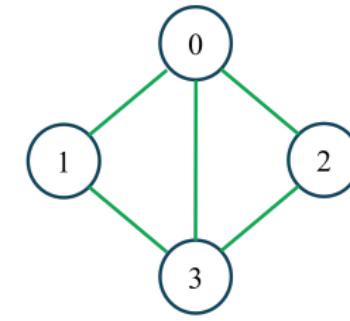
Induced Subgraph

A graph G' is an induced subgraph of G if G' is a subgraph of G and for any two vertices $u, v \in V(G')$, $(u, v) \in E(G)$ if and only if $(u, v) \in E(G')$.

Examples

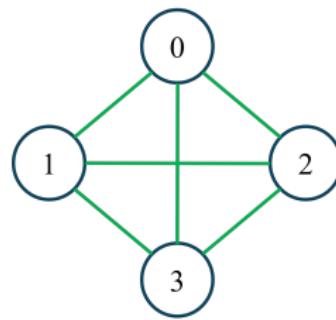
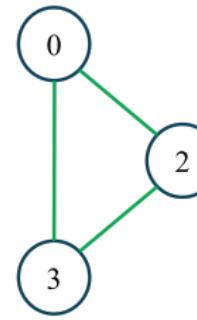
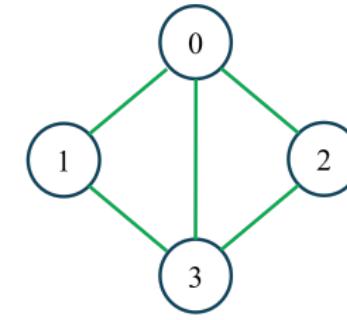
 G  H_1  H_2

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- H_1, H_2 : subgraphs of G .
- H_1 is an induced subgraph of G , but H_2 is NOT.

Path (1/2)

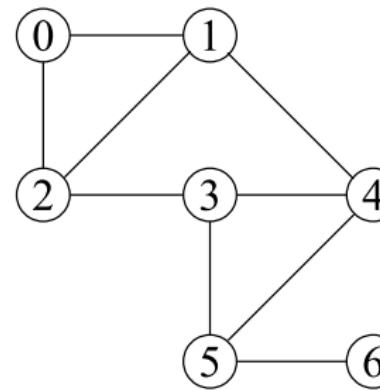
Path

A (directed or undirected) path from vertex u to vertex v in graph G is a sequence of vertices $u, i_1, i_2, \dots, i_k, v$, such that $(u, i_1), (i_1, i_2), \dots, (i_k, v)$ are edges in $E(G)$.

Path (1/2)

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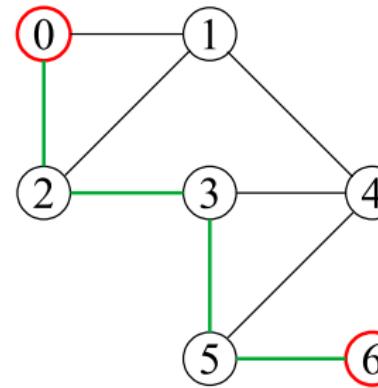
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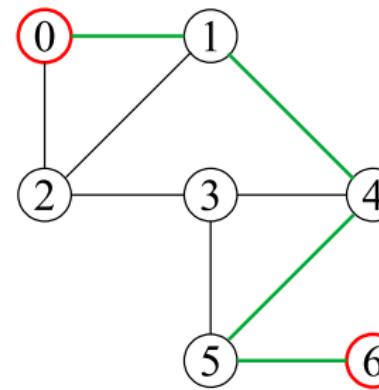
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Path (2/2)

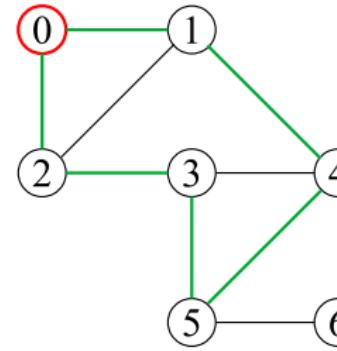
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- A **simple path** a path in which all **vertices**, except possibly the first and the last, are distinct.

Path (2/2)

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Path (2/2)

- The **length** of a path is the **number of edges** on it.
- A **simple path** is a path in which all **vertices**, except possibly the first and the last, are distinct.
- A **cycle** is a **simple path** in which the **first** and **last** vertices are the same.
 - A simple path from v to v .



Connected and Connected Component

Connected

- In an undirected graph G , two vertices u and v are connected iff there is a path in G from u to v .
- An undirected graph is **connected** iff for **every pair of distinct vertices** u and v in $V(G)$ there is a path from u to v in G .

Connected Component

A **connected component** (or simply a component) H of an undirected graph is a **maximal** connected subgraph.



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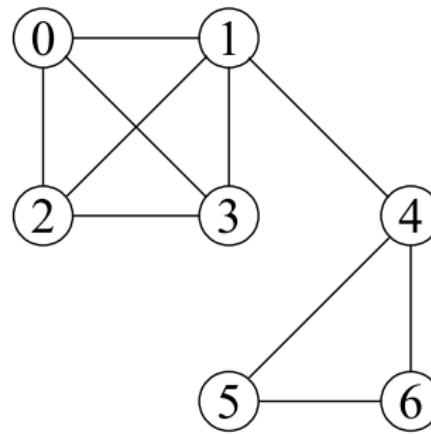
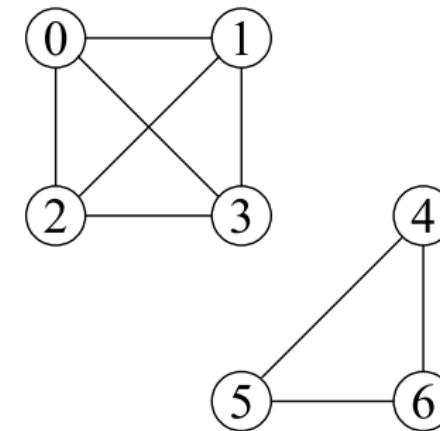
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tree

A tree is a connected **acyclic** (i.e., has no cycles) graph.



Example of Connected Components

 G_1  G_2

Strongly Connected Graph

Strongly Connected Graph

A directed graph G is said to be **strongly connected** iff for **every pair** of distinct vertices $u, v \in V(G)$, there is **directed path** from u to v and also from v to u .

Strongly Connected Graph

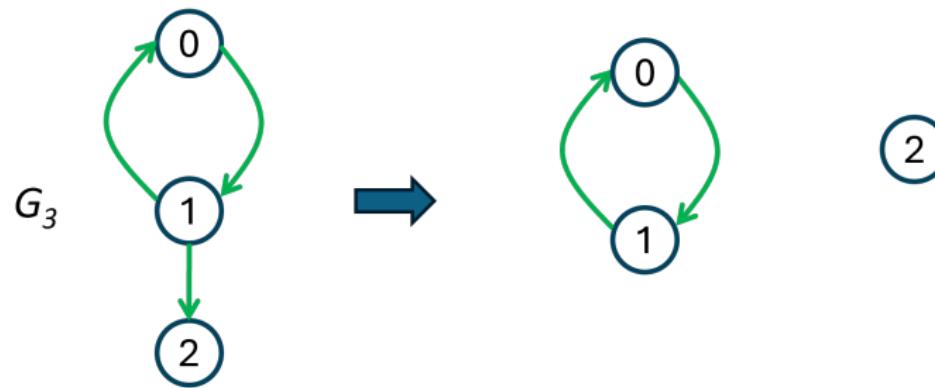
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Strongly Connected Component

A strongly connected component is a maximal subgraph that is strongly connected.

Strongly Connected Components



Vertex Degree

- The degree of a vertex is the **number of edges incident to** that vertex.
- For a **directed** graph G ,
 - The **in-degree** of a vertex is the number of edges for which vertex is **head**.
 - the **out-degree** of a vertex is the number of edges for which the vertex is the **tail**.

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Let d_i be the degree of vertex i in an n -vertex graph $G = (V, E)$, then

$$|E| = \frac{1}{2} \sum_{i=1}^n d_i.$$

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2 Graph Representations

Graph Representations

- Two most commonly used representation for a graph:

Graph Representations

- Two most commonly used representation for a graph:
 - Adjacency Matrices
 - Adjacency Lists

Graph Representations

- Two most commonly used representation for a graph:
 - Adjacency Matrices
 - Adjacency Lists
- The choice of the representation:
 - the application
 - the functions one expects to perform on the graph
 - characteristics of the input graph

Adjacency Matrix

The adjacency matrix of an n -vertex graph G is a two-dimensional $n \times n$ array a , with the property that

- $a[i][j] = 1$ iff $(i, j) \in E(G)$;
- $a[i][j] = 0$ iff there is no such edge (i, j) in G .

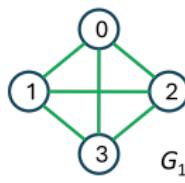
Remark

The adjacency matrix for an undirected graph is **symmetric**.

Adjacency Matrix (2/2)

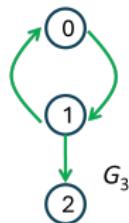
- For an undirected graph the degree of any vertex i is its row sum.
- For a directed graph the row sum is its **out-degree** and the column sum is its **in-degree**.

Adjacency Matrices (Examples)



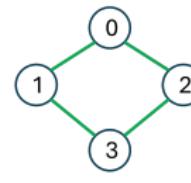
The adjacency matrix of G_1

	0	1	2	3
0	0	1	1	1
1	1	0	1	1
2	1	1	0	1
3	1	1	1	0



The adjacency matrix of G_3

	0	1	2
0	0	1	0
1	1	0	1
2	0	0	0



The adjacency matrix of G_4

	0	1	2	3	4	5	6	7
0	0	1	1	0	0	0	0	0
1	1	0	0	1	0	0	0	0
2	1	0	0	1	0	0	0	0
3	0	1	1	0	0	0	0	0
4	0	0	0	0	0	1	0	0
5	0	0	0	0	1	0	1	0
6	0	0	0	0	0	1	0	1
7	0	0	0	0	0	0	1	0

Adjacency Lists

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- One chain for each vertex in G .

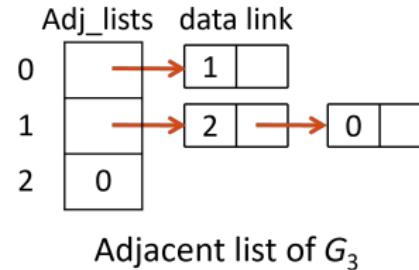
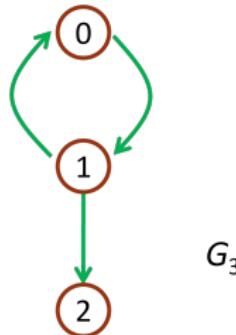
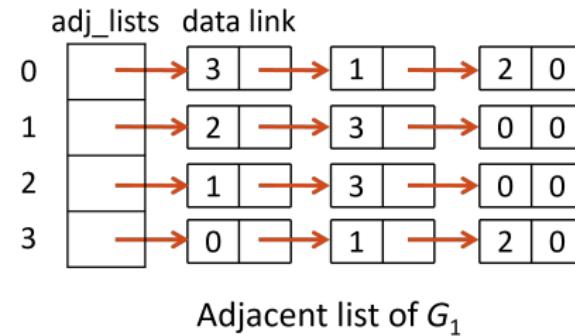
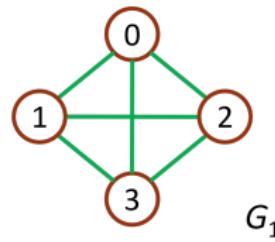
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- The data field of a chain node stores the index of an adjacent vertex.

Adjacency Lists Examples



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- In many applications, the edges of a graph have **weights** associated with them.
 - importance, costs, distance, etc.
- The adjacency matrix entries $a[i][j]$ would keep this information.
- When adjacency lists are used, we can introduce an additional field **weight** in the list nodes.

Discussions