

Randomized Algorithms

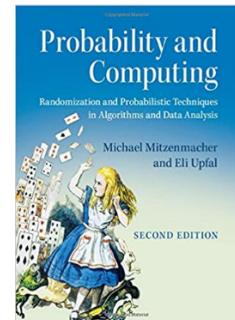
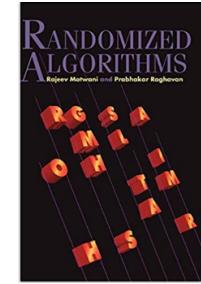
Introduction

Joseph Chuang-Chieh Lin

Dept. CSE,
National Taiwan Ocean University

Textbooks and Materials

- Textbooks:
 - *Randomized Algorithms*. Motwani, R. and Raghavan, P., 1995. Cambridge University Press.
 - Probability and Computing: Randomization and Probabilistic Techniques in Algorithms and Data Analysis. Second Edition. M. Mitzenmacher and E. Upfal, Cambridge University Press, 2017.
- Other materials:
 - Prepared slides.



Prerequisites

- Basic undergraduate courses in
 - Algorithms
 - Data structures
 - Probability theory
 - Discrete mathematics
 - Linear Algebra
- Motivation.
- Curiosity.

Topics

- Introduction
- The Min-Cut Problem
- Discrete Random Variables & Expectations
- Randomized QuickSort
- k -Smallest Number Selection
- Complexities
- Minimax Principles
- Coupon Collector's Problem
- The Secretary Problem
- Moments & Deviations
- Chernoff & Hoeffding Bounds
- Balls and Bins
- Markov Chains & Random Walks
- Continuous Distributions & The Poisson Process
- The Probabilistic Method
- Lovász Local Lemma and Applications
- The Monte Carlo Method
- Other Selected Topics

Grading Policy

- Attendance (10%)
- Assignments and Quizzes (60%)
 - $4 \times 15\%$
 - Assignments will be announced first, then we have quizzes on them.
- Final Exam/Paper Presentation (30%)
 - Depending on the teaching progress.

Grading Policy for the Presentations (If Necessary)

- Students need to group to teams
- Complete the presentation: 60 point
 - Duration for each presentation: 30~50 minutes.
- Raising questions: +4 point for each one (maximum +20 point)
- Clearly answering the teacher's 4 questions: +5 point for each one.

Deterministic Algorithms

vs.

Randomized Algorithms

Traditional deterministic algorithms



Randomized algorithms



Why?

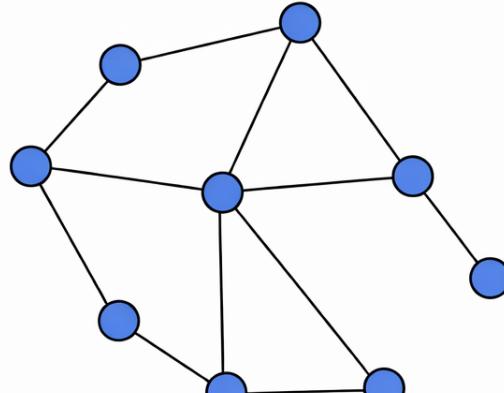
- Randomized algorithms are
 - often much ***simpler*** than the best known deterministic ones.
 - often much ***more efficient*** (faster or using less space) than the best known deterministic ones.
- Sometimes ideas from the randomized algorithms lead to good deterministic algorithms.

Comparisons

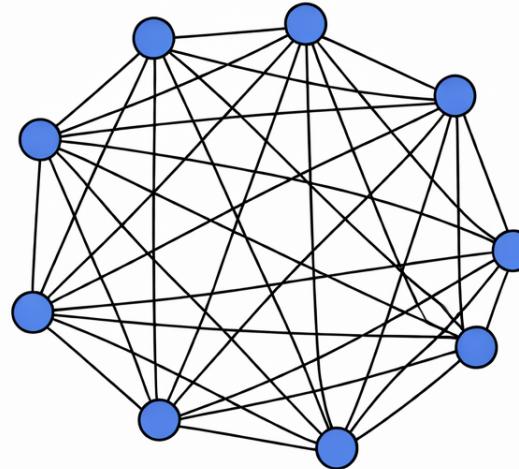
- It's different from the *average-case* analysis of deterministic algorithms.
 - e.g., expected running time of a deterministic algorithm on input sampled from a distribution.
- In most scenarios, it's NOT a heuristic algorithm.
 - The **accuracy** is guaranteed, or
 - The **running time** is guaranteed.

An illustrating example

- Determining if a graph is very dense (or sparse)



Sparse



Dense

Another illustrating example

- **Problem:** find a grade-‘A’ student in a class of n students where half of them get ‘A’.
- What is the time complexity for the best deterministic algorithm?
 - I mean, in the “worst case”.

A randomized algorithm (from Wikipedia)

```
findingA_LV(array  $L$ ,  $n$ ) // Las Vegas
begin
    repeat
        Randomly select one element out of  $n$  elements.
    until 'A' is found
```

```
end
```

Assignment: Prove that the expected number of iterations is $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i}{2^i} \leq 2$.

A randomized algorithm (from Wikipedia)

```
findingA_MC(array  $L$ ,  $n$ ,  $k$ ) // Monte Carlo
begin
     $i \leftarrow 0$ 
    repeat
        Randomly select one element out of  $n$  elements.
         $i \leftarrow i + 1$ 
    until  $i = k$  or 'A' is found
end
```

After k iterations, $\Pr[\text{find } A] = 1 - (1/2)^k$.

Randomization Helps in Making Decisions

Birthday problem (paradox)

- There are n randomly chosen people in a room.
- How *large* should n be such that there is at least one pair of them having the same birthday (mm/dd)?
- By the pigeonhole principle, $n = 367$? or 366 ?

Birthday problem (paradox)

- There are n randomly chosen people in a room.
- How *large* should n be such that there is at least one pair of them having the same birthday (mm/dd)?
- By the pigeonhole principle, $n = 367?$ or $366?$
- Let us consider this problem in the other way around.
How *large* should n be such that there is at least one pair of them having the same birthday (mm/dd) with probability ≥ 0.5 ?

Birthday problem (paradox)

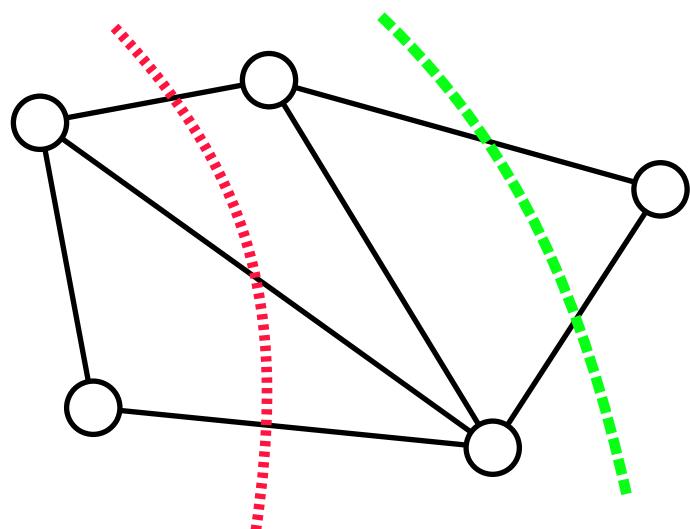
- n people: x_1, x_2, \dots, x_n
- Event i : some pair of x_1, x_2, \dots, x_i have the same birthday.
- $\Pr[\text{Event2}] = 1 - \frac{364}{365}$
- $\Pr[\text{Event3}] = 1 - \frac{364}{365} \cdot \frac{363}{365}$
- ...
- $\Pr[\text{Event23}] = 1 - \frac{364}{365} \cdot \frac{363}{365} \cdot \dots \cdot \frac{343}{365} \approx 0.507297.$
- 23 is much less than 366 or 367.

Birthday problem (paradox)

- **Assignment:** Compute n such that there is at least one pair of them having the same birthday with probability ≥ 0.9 .

The Min-Cut Problem

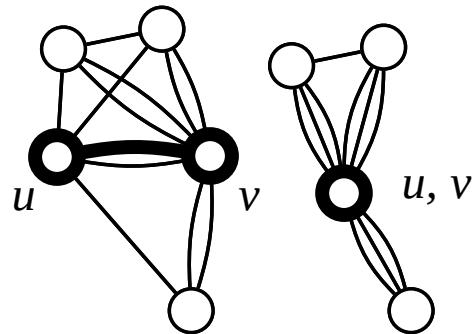
Min-Cut



$$|V| = n, |E| = m$$

- A graph $G = (V, E)$ and its two “cuts”.
 - **Cut:** a partition of the vertices in V into two non-empty, disjoint sets S and T such that
 - $S \cup T = V$
- The **cutset** of a cut:
 - $\{uv \in E \mid u \in S, v \in T\}$.
- The size of the cut:
 - the cardinality of its cutset.

Edge contraction



$$e = (u, v)$$

$$G \rightarrow G/e$$

By Thore Husfeldt - Python, networkx, neato and TikZ, CC BY-SA 3.0, <https://commons.wikimedia.org/w/index.php?curid=21117500>

Karger's edge-contraction algorithm (1993)

Procedure contract ($G = (V, E)$):

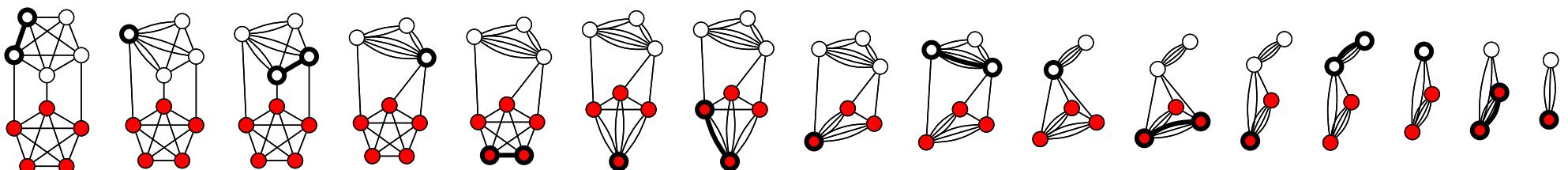
while $|V| > 2$:

 choose $e \in E$ uniformly at random

$G \leftarrow G/e$

return the only cut in G

Time complexity: $O(m)$ or $O(n^2)$.



By Thore Husfeldt - Created in python using the networkx library for graph manipulation, neato for layout, and TikZ for drawing., CC BY-SA 3.0, <https://commons.wikimedia.org/w/index.php?curid=21103489>

Analysis

- C : a specific cut of G .
- k : the size of the cut C .
- The minimum degree of G must be $\geq k$. (WHY?)
 - So, $|E| \geq nk/2$.
- The probability that the algorithm picks an edge from C to contract is

$$\frac{k}{|E|} \leq \frac{k}{nk/2} = \frac{2}{n}.$$

Analysis (contd.)

- Let p_n be the probability that the algorithm on an n -vertex graph avoids C .
- Then,

$$p_n \geq \left(1 - \frac{2}{n}\right) \cdot p_{n-1}$$

- The recurrence can be expanded as

$$p_n \geq \prod_{i=0}^{n-3} \left(1 - \frac{2}{n-i}\right) = \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdots \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{\binom{n}{2}}.$$

Analysis (contd.)

- The probability of “success” is $\binom{n}{2}^{-1}$
 - A bit too low, isn’t it?
- How about repeating it for $T = \binom{n}{2} \ln n$ times, and then choose the *minimum* of them?
- The probability of NOT finding a min-cut is $\left[1 - \binom{n}{2}^{-1}\right]^T \leq \frac{1}{e^{\ln n}} = \frac{1}{n}$.
- Total running time: $O(Tm)$ or $O(Tn^2)$.

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A classic *deterministic* bound: $O(mn + n^2 \log n)$
[Stoer–Wagner (1995)]

The best randomized bound (whp): $O(m \log^2 n)$
[Gawrychowski et al. (TOCS 2024)]

An Coding Example

- The demo code: <https://onlinegdb.com/6byIpav5uJ>

Key Take-Away Question

- Can we apply the randomized edge-contraction algorithm to the **max-cut** problem?