Myerson's Lemma

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Outline

1. Myerson's Lemma
   - Single-Parameter Environments
   - The Lemma
   - Application to the Sponsored Search Auction

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Single-Parameter Environments

Consider a more generalized and abstract setting:

Single-Parameter Environments

- n agents (e.g., bidders).
- A private valuation $v_i \geq 0$ for each agent i (per unit of stuff).
- A feasible set $X = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\} \subseteq \mathbb{R}^n$.
  - x_i: amount of stuff given to agent i.

Single-Parameter Environments (Examples)

- Single-item auction:
  - $\sum_{i=1}^{n} X_i \leq 1$, and $x_i \in \{0, 1\}$ for each i.

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  - k identical items, $\sum_{i=1}^{n} X_i \leq k$, and $x_i \in \{0, 1\}$ for each i.
- Sponsored Search Auction:
  - X: the set of n-vectors $\Leftrightarrow$ assignments of bidders to slots.
  - Each slot (resp., bidder) is assigned to ≤ 1 bidder (resp., slot).
  - The component $x_i = \alpha_j$: bidder i is assigned to slot j.
    - α_j: the click-through rate of slot j.
    - Assume that the quality score $\beta_i = 1$ for all i.

Allocation and Payment Rules

Choices to make in a sealed-bid auction

- Collect bids $\boldsymbol{b} = (b_1, \ldots, b_n)$.
- Allocation Rule: Choose a feasible $\boldsymbol{x}(\boldsymbol{b}) \in X \subseteq \mathbb{R}^n$.
- Payment Rule: Choose payments $\boldsymbol{p}(\boldsymbol{b}) \in \mathbb{R}^n$.

- A *direct-revelation mechanism*.

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- A *direct-revelation mechanism*.
- Example of *indirect mechanism*: iterative ascending auction.

Allocation and Payment Rules (contd.)

With allocation rule $\boldsymbol{x}$ and payment rule $\boldsymbol{p}$,

- agent i receives utility $u_i(\boldsymbol{b}) = v_i \cdot x_i(\boldsymbol{b}) - p_i(\boldsymbol{b})$.
- $p_i(\boldsymbol{b}) \in [0, b_i \cdot x_i(\boldsymbol{b})]$.
  - $p_i(\boldsymbol{b}) \geq 0$: prohibiting the seller from paying the agents.
  - $p_i(\boldsymbol{b}) \leq b_i \cdot x_i(\boldsymbol{b})$: a truthful agent receives nonnegative utility.

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  - $p_i(\boldsymbol{b}) \leq b_i \cdot x_i(\boldsymbol{b})$: a truthful agent receives nonnegative utility. Why?

The Myerson's Lemma

Definition (Implementable Allocation Rule)

An allocation rule $\boldsymbol{x}$ for a single-parameter environment is implementable if there is a payment rule $\boldsymbol{p}$ such that the direct-revelation mechanism $(\boldsymbol{x}, \boldsymbol{p})$ is DSIC.

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Definition (Monotone Allocation Rule)

An allocation rule $\boldsymbol{x}$ for a single-parameter environment is monotone if for every agent i and bids $\boldsymbol{b}_{-i}$ by other agents, the allocation $x_i(z, \boldsymbol{b}_{-i})$ to i is nondecreasing in her bid z.

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So, how about awarding the item to the second-highest bidder?
You raise your bid, you might lose the chance of getting it!

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The Myerson's Lemma

Theorem (Myerson's Lemma)

Fix a single-parameter environment.

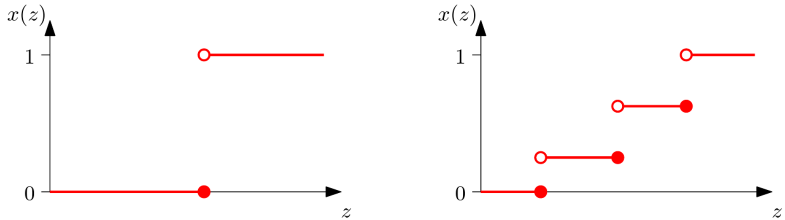
(i) An allocation rule $\boldsymbol{x}$ is implementable if and only if it is monotone.

(ii) If $\boldsymbol{x}$ is monotone, then there is a unique payment rule for which the direct-revelation mechanism $(\boldsymbol{x}, \boldsymbol{p})$ is DSIC and $p_i(\boldsymbol{b}) = 0$ whenever $b_i = 0$.

(iii) The payment rule in (ii) is given by an explicit formula.

"Monotone" is more operational.

Allocation curves: allocation as a function of bids



(a) 0-1 Monotone Curve (b) Piecewise Constant Monotone Curve

Figures from Tim Roughgarden's lecture notes.

Constraints from DSIC

Consider $0 \leq z < y$.

Say agent i has a private valuation z and free to submit a false bid y or agent i has a private valuation y and free to submit a false bid z

DSIC: Bidding truthfully brings maximum utility.

$$\begin{aligned} z \cdot x(z) - p(z) &\geq z \cdot x(y) - p(y) \\ y \cdot x(y) - p(y) &\geq y \cdot x(z) - p(z) \end{aligned}$$

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So

$$z \cdot (x(y) - x(z)) \le p(y) - p(z) \le y \cdot (x(y) - x(z)).$$

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$p(y) - p(z)$ can be bounded below and above.

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$\Rightarrow$ every implementable allocation rule is monotone (why?)

Case: x is a piecewise constant function

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- Try: fix z and let y tend to z.

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  $\Rightarrow$ left-hand and right-hand sides $\to 0$ if there is no jump in x at z.

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$$p_i(b_i, \boldsymbol{b}_{-i}) = \sum_{j=1}^{\ell} z_j \cdot [\text{ jump in } x_i(\cdot, \boldsymbol{b}_{-i}) \text{ at } z_j],$$

where $z_1, \ldots, z_\ell$ are breakpoints of $x_i(\cdot, \boldsymbol{b}_{-i})$ in the range $[0, b_i]$.

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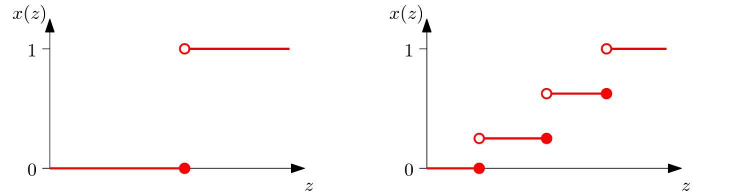

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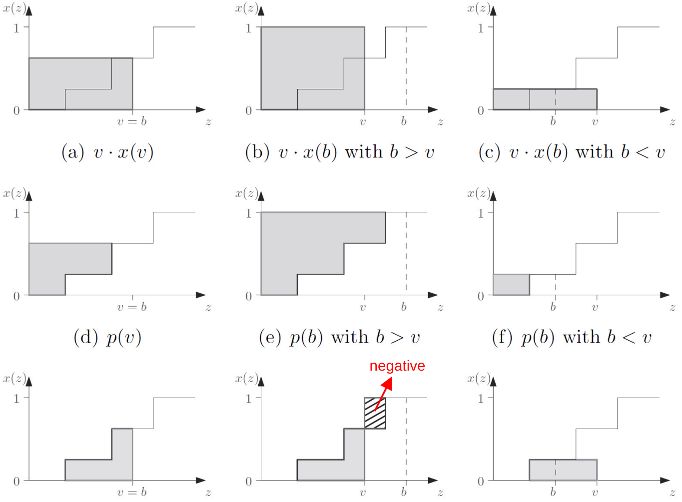

(a) $v \cdot x(v)$ (b) $v \cdot x(b)$ with $b > v$ (c) $v \cdot x(b)$ with $b < v$

(d) $p(v)$ (e) $p(b)$ with $b > v$ (f) $p(b)$ with $b < v$

(g) utility with $b = v$ (h) utility with $b > v$ (i) utility with $b < v$

Case: x is a monotone function

$$z \cdot (x(y) - x(z)) \leq p(y) - p(z) \leq y \cdot (x(y) - x(z)).$$

- Suppose x is differentiable.
- Dividing the inequalities by $y - z$:

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$$p'(z) = z \cdot x'(z).$$

$$p_i(b_i, \boldsymbol{b}_{-i}) = \int_0^{b_i} z \cdot \frac{d}{dz} x_i(z, \boldsymbol{b}_{-i}) dz.$$

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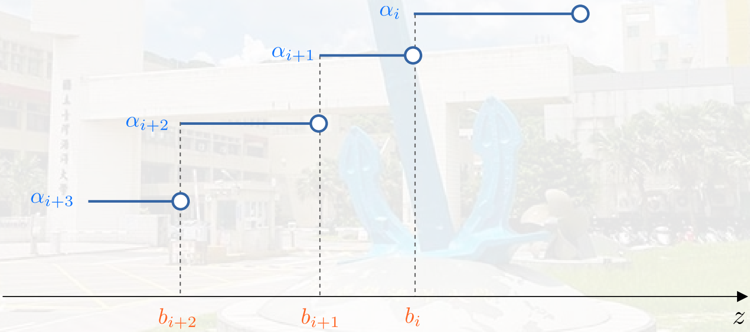
Apply to Sponsored Search Auction

The allocation rule is piecewise.

re-index the bidders: $b_1 \geq b_2 \geq \ldots \geq b_n$.



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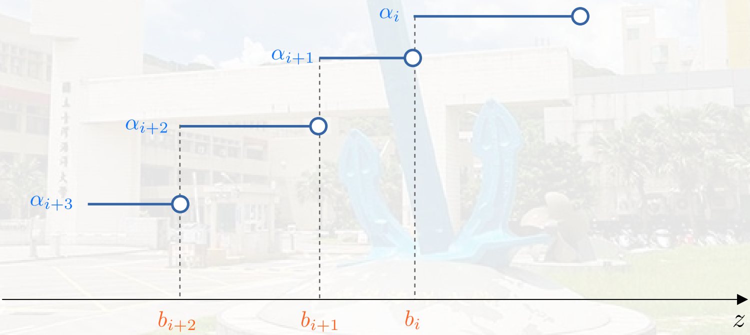

$$p_i(\boldsymbol{b}) = \sum_{j=i}^{k} b_{j+1}(\alpha_j - \alpha_{j+1}).$$

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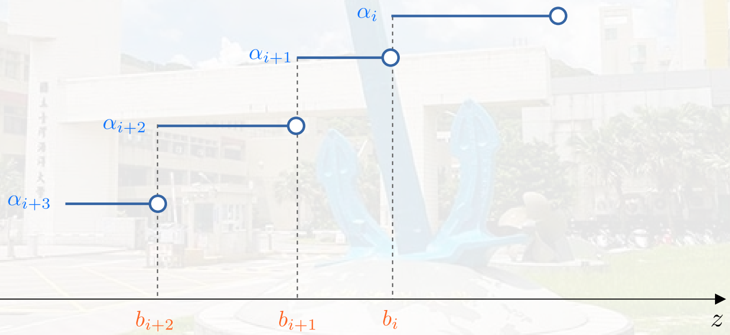

$p_i(\boldsymbol{b}) = \sum_{j=i}^{k} b_{j+1}\frac{\alpha_j - \alpha_{j+1}}{\alpha_i}$ (scaled per click).

Exercise 1 (5%)

- Recall that in the model of sponsored search auctions:
  - There are k slots, the jth slot has a click-through rate (CTR) of α_j (nonincreasing in j).
  - The utility of bidder i in slot j is $\alpha_j(v_i - p_j)$, where v_i is the private value-per-click of the bidder and p_j is the price charged per-click in slot j.
- The Generalized Second Price (GSP) Auction is defined as follows:

Exercise 1 (5%) (contd.)

The Generalized Second Price (GSP) Auction

1. Rank advertisers from highest to lowest bid; assume without loss of generality that $b_1 \geq b_2 \geq \cdots \geq b_n$.
2. For $i = 1, 2, \ldots, k$, assign the ith bidder to the i slot.
3. For $i = 1, 2, \ldots, k$, charge the ith bidder a price of b_{i+1} per click.

(a) Prove that for every $k \geq 2$ and sequence $\alpha_1 \geq \cdots \geq \alpha_k > 0$ of CTRs, the GSP auction is NOT DSIC. (*Hint: Find out an example.*)

(b) A bid profile $\boldsymbol{b}$ with $b_1 \geq \cdots \geq b_n$ is envy-free if for every bidder i and slot $j \neq i$,

$$\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_{j+1}).$$

Please verify that every envy-free bid profile is an equilibrium.