Strong Price of Anarchy, Utility Games and Coalitional Dynamics

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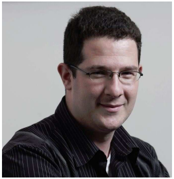
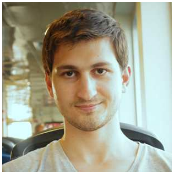


Yoram Bachrach



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Outline

Strong Nash Equilibrium [Aumann *et al.* 1959]

A strategy profile $s \in (S_i)_{i\in[n]}$ is a strong Nash equilibrium (strong NE) if

- for any coalition $C \subseteq [n]$ and
- for any coalitional strategy $s_C \in S_C = (S_j)_{j\in C}$,

there exists a player $i \in C$ such that $u_i(s) \geq u_i(s_C, s_{-C})$.

- *Strong price of anarchy* (Strong PoA) measures the quality degradation of strong NE in games [Andelman *et al.* 2009].

Battle of Sex

girl \ boy	baseball	movie
baseball	(2,2)	(0,0)
movie	(0,0)	(1,1)

Prisoner's Dilemma

Alice \ Bob	cooperate	defect
cooperate	(−1,−1)	(−10,0)
defect	(0,−10)	(−8,−8)

Contribution of this paper

- The coalitional smoothness framework.
- Bounding the strong PoA of (λ, μ)-coalitionally smooth games.
- A monotone utility maximization game has strong PoA ≤ 2 if each player's utility $\geq$ his marginal contribution to the welfare.
- The strong PoA is close to PoS (price of stability) for the potential games with potential function similar to the social welfare function.
- The strong PoA of coalitional sink equilibria.

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Preliminaries...

- Utility maximization games (can extend to *cost minimization games*).
    - S_i: strategy space of player $i \in [n]$.
    - $u_i : S_1 \times \ldots S_n \mapsto \mathbf{R}_+$: the utility of player i.
    - For $C \subseteq [n]$:
        - $S_C = (S_i)_{i \in C}$ the joint strategy space;
        - $\Delta(S_C)$: the space of distributions over S_C.
    - The social welfare: $SW(s) = \sum_{i \in [n]} u_i(s)$.

Coalitional Smoothness

Coalitional Smoothness

A utility maximization game is (λ, μ)-coalitionally smooth if there exists a strategy profile s^* such that for any strategy profile s and for any permutation π of the players:

$$\sum_{i=1}^{n} u_i(s^*_{N_{\pi(i)}}, s_{-N_{\pi(i)}}) \geq \lambda \cdot SW(s^*) - \mu \cdot SW(s),$$

where

- $N_{\pi(i)} = \{j \in [n] : \pi(j) \geq \pi(i)\}$: the set of all players succeeding i in π;
- $(s^*_{N_t}, s_{-N_t})$: all players in N_t play s^* and the others play s.

In cost minimization games, we require:

$$\sum_{i=1}^{n} c_i(s^*_{N_{\pi(i)}}, s_{-N_{\pi(i)}}) \leq \lambda \cdot SC(s^*) + \mu \cdot SC(s).$$

Theorem 3

If a game is (λ, μ)-coalitional smooth for some $\lambda, \mu \geq 0$, then its every strong NE has social welfare $\geq \frac{\lambda}{1+\mu}$ of the optimal.

Proof:

- s: strong NE strategy profile;

  s^*: the optimal strategy profile.

- ALL players coalitionally deviate to s^* $\Rightarrow$ $\exists i$ blocking the deviation.
  - Reorder the players s.t. this one is player 1.

- Similarly, we can reorder the players s.t. if players $\{i, \ldots, n\}$ deviate to s^* $\Rightarrow$ i is the one blocking the deviation.

Proof of Theorem 3 (contd.)

- Player i's utility at s is at least the one in the deviating profile.
  - $\star$ $\forall i \in [n]$, $u_i(s) \geq u_i(s^*_{N_i}, s_{-N_i})$.
-
$$SW(s) = \sum_{i=1}^{n} u_i(s) \geq \sum_{i=1}^{n} u_i(s^*_{N_i}, s_{-N_i}) \geq \lambda \cdot SW(s^*) - \mu \cdot SW(s).$$

On Monotone Utility Games

- Every player has an s_i^{out} strategy (i.e., not entering the game).
- Monotone (w.r.t. participation):
  - No player can decrease the social welfare by entering the game.
  - ⋆ $\forall i \in [n], \forall s : SW(s) \geq SW(s_i^{out}, s_{-i})$.

Theorem 4

Any monotone utility maximization game is (γ, γ)-coalitional smooth, if each player is guaranteed at least a γ fraction of his marginal contribution to the social cost, i.e.,

$$\forall s : \ u_i(s) \geq \gamma(SW(s) - SW(s_i^{out}, s_{-i})).$$

Some remarks

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[Vetta @FOCS 2002] & [Roughgarden @STOC 2009]

For any monotone utility-maximization game $\mathcal{G}$ with a submodular welfare function, if each player receives a γ fraction of the marginal contribution to the welfare, then $\mathcal{G}$ is (γ, γ)-smooth.

$\Rightarrow$ Every NE achieves a $\frac{\gamma}{\gamma+1}$ fraction of the optimal welfare.

- Theorem 4 complement Vetta's result.

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Proof of Theorem 4

s^*: the optimal strategy profile.

- By the marginal contribution property,

$$\sum_{i=1}^{n} u_i(s^*_{N_i}, s_{-N_i}) \geq \gamma \cdot \sum_{i=1}^{n} \left(SW\left(s^*_{N_i}, s_{-N_i}\right) - SW(s_i^{out}, s^*_{N_{i+1}}, s_{-N_i})\right).$$

- By the monotonicity assumption,

$$SW(s_i^{out}, s^*_{N_{i+1}}, s_{-N_i}) \leq SW(s_i, s^*_{N_{i+1}}, s_{-N_i}) = SW(s^*_{N_{i+1}}, s_{-N_{i+1}}).$$

- Thus we have

$$\begin{aligned} \sum_{i=1}^{n} u_i(s^*_{N_i}, s_{-N_i}) &\geq \gamma \cdot \sum_{i=1}^{n} \left(SW(s^*_{N_i}, s_{-N_i}) - SW(s^*_{N_{i+1}}, s_{-N_{i+1}})\right) \\ &\geq \gamma \cdot SW(s^*) - \gamma \cdot SW(s) \quad (\because \text{telescoping}). \end{aligned}$$

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Best Nash vs. Worst Strong Nash Equilibrium

Best Nash vs. Worst Strong Nash Equilibrium

- Strong Nash equilibria $\subseteq$ Nash equilibria.
  - Strong PoA cannot be better than PoS (when Strong NE exists).
- A strong connection between the Strong PoA and PoS exists in certain potential games!
  - Through the lens of coalitional smoothness.

Best Nash vs. Worst Strong Nash Equilibrium (contd.)

(λ, μ)-close

A potential function of a potential game is (λ, μ)-close to the social welfare if:

$$\lambda \cdot SW(s) \leq \Phi(s) \leq \mu \cdot SW(s),$$

for $\lambda, \mu > 0$ and strategy profile s.

- The best NE achieves $\geq \frac{\lambda}{\mu}$ of the optimal social welfare.
  - $SW(\tilde{s}) \geq \frac{1}{\mu} \cdot \Phi(\tilde{s}) \geq \frac{1}{\mu} \cdot \Phi(s^*) \geq \frac{\lambda}{\mu} \cdot SW(s^*)$.
    - ▷ $\tilde{s}$: maximizer of Φ.
    - ▷ s^*: optimal of $SW(\cdot)$.

Best Nash vs. Worst Strong Nash Equilibrium (contd.)

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    - ▷ $\tilde{s}$: maximizer of Φ.
    - ▷ s^*: optimal of $SW(\cdot)$.

The potential games whose are PoS very close to the strong PoA

Theorem 6

In a utility-maximization potential game $\mathcal{G}$ with non-negative utilities, the potential is (λ, μ)-close to SW $\Rightarrow$ $\mathcal{G}$ is (λ, μ)-coalitionally smooth.

$\Rightarrow$ Every strong Nash equilibrium achieves $\geq \frac{\lambda}{1+\mu} \times$ optimal SW (by Theorem 3).

Proof of Theorem 6

Theorem 6

In a utility-maximization potential game $\mathcal{G}$ with non-negative utilities, the potential is (λ, μ)-close to SW $\Rightarrow$ $\mathcal{G}$ is (λ, μ)-coalitionally smooth.

Proof:

- Consider an arbitrary order of the players and some strategy profile s.

$$\begin{aligned} u_i(s^*_{N_i}, s_{-N_i}) &= \Phi(s^*_{N_i}, s_{-N_i}) - \Phi(s^*_{N_{i+1}}, s_{-N_{i+1}}) + u_i(s^*_{N_{i+1}}, s_{-N_{i+1}}) \\ &\geq \Phi(s^*_{N_i}, -s_{N_i}) - \Phi(s^*_{N_{i+1}}, s_{-N_{i+1}}) \end{aligned}$$

$$\begin{aligned} \therefore \sum_{i=1}^{n} u_i(s^*_{N_i}, s_{-N_i}) &\geq \sum_{i=1}^{n} (\Phi(s^*_{N_i}) - \Phi(s^*_{N_{i+1}}, s_{-N_{i+1}})) \\ &= \Phi(s^*) - \Phi(s) \quad (\because \text{telescoping}) \\ &\geq \lambda \cdot SW(s^*) - \mu \cdot SW(s) \quad (\because (\lambda, \mu)\text{-close}). \end{aligned}$$

Non-negative externalities

A utility maximization game has non-negative externalities if for any strategy profile s and for any pair of players i, j we have

$$u_i(s) \geq u_i(s_j^{out}, s_{-j}).$$

Theorem 7

A utility-maximization potential game with non-negative externalities and such that $\Phi(s) \geq \lambda \cdot SW(s)$ is $(\lambda, 0)$-coalitionally smooth.

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Proof of Theorem 7

$$u_i(s^*_{N_i}, s_{-N_i}) \geq u_i(s^*_{N_i}, s^{out}_{-N_i}) \geq \Phi(s^*_{N_i}, s^{out}_{-N_i}) - \Phi(s^*_{N_{i+1}}, s^{out}_{-N_{i+1}})$$

$$\begin{aligned} \therefore \sum_{i=1}^{n} u_i(s^*_{N_i}, s_{-N_i}) &\geq \sum_{i=1}^{n} \left(\Phi(s^*_{N_i}, s^{out}_{-N_i}) - \Phi(s^*_{N_{i+1}}, s^{out}_{-N_{i+1}}) \right) \\ &= \Phi(s^*) - \Phi(s^{out}) \\ &\geq \lambda \cdot SW(s^*). \end{aligned}$$

Coalitional Best-Response Dynamics

Coalitional Best-Response Dynamics

- Particularly interesting for games NOT admitting a strong NE.
- Applying approach similar to the notion of sink equilibria [Goemans, Mirrokni & Vetta @FOCS 2005].

Sink equilibria

The sketch:

- Model the behavior of players using a state graph.
  - The vertex set: strategy profiles.
  - The arcs: corresponding to moves (i.e., best responses) of players.
- The random walks on the state graph eventually lead to a set of states having the following properties:
  - These states form a strongly connected component (a **sink equilibrium**).
  - The strongly connected component has no out-going arcs.
- The social welfare of a sink equilibrium:
  - the expected value of the stationary distribution of a random walk on the states in the sink.

Coalitional sink equilibria

Sink equilibria + coalitional deviations.

ALGORITHM 1: Coalitional Best-Response Dynamics

Let s^t be the strategy profile at iteration t. Initialize s^0 to some arbitrary strategy.
for *each iteration* t **do**
  Pick a coalitional size $k \in \{1, \ldots, n\}$ inversely proportional to k.
  Pick a coalition $C_t \subseteq [n]$ of size k uniformly at random from all possible coalitions.
  Let $s^t_{C_t} = \arg\max_{s_{C_t}} \sum_{i \in C_t} u_i(s_{C_t}, s^{t-1}_{-C_t})$ be the joint strategy profile of players in C_t that maximizes their total utility, conditional on what the rest of the players are playing.
  All players in C_t deviate to their strategy in the above optimal. Update $s^t = (s^t_{C_t}, s^{t-1}_{-C_t})$.
end

⋆ Assumption: the cooperating group can transfer utility.

Theorem 8

If a utility maximization game with non-negative utilities is (λ, μ)-coalitionally smooth, then for every coalitional sink equilibrium s:

$$\mathbf{E}[SW(s)] \geq \frac{1}{H_n} \cdot \frac{\lambda}{1+\mu} \cdot \mathrm{OPT}.$$

Proof:

- s^t: the strategy profile at time step t of the best response dynamics.
- s^*: the optimal strategy profile designated by the coalitional smoothness property.
- $\mathcal{C}_k$: the set of all possible coalitions of size k.

$\star$
$$\begin{aligned}\mathbf{E}[SW(s^t) \mid s^{t-1} = s] &= \frac{1}{H_n}\sum_{k=1}^{n}\frac{1}{k}\sum_{C\in\mathcal{C}_k}\frac{1}{\binom{n}{k}}SW(s^t_C, s_{-C}) \\ &\geq \frac{1}{H_n}\sum_{k=1}^{n}\frac{1}{k}\sum_{C\in\mathcal{C}_k}\frac{1}{\binom{n}{k}}\sum_{i\in C}u_i(s^t_C, s_{-C}).\end{aligned}$$

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Proof of Theorem 8 (contd.)

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&\geq \frac{1}{H_n} \sum_{k=1}^{n} \frac{1}{k} \sum_{C \in \mathcal{C}_k} \frac{1}{\binom{n}{k}} \sum_{i \in C} u_i(s_C^*, s_{-C}) \\
&= \frac{1}{H_n} \cdot \frac{1}{n!} \sum_{k=1}^{n} \sum_{C \in \mathcal{C}_k} \sum_{i \in C} (n-k)! \cdot (k-1)! \cdot u_i(s_C^*, s_{-C}) \\
&= \frac{1}{H_n} \cdot \frac{1}{n!} \sum_{i \in [n]} \sum_{k=1}^{n} \sum_{C \in \mathcal{C}_k, i \in C} (n-k)! \cdot (k-1)! \cdot u_i(s_C^*, s_{-C}).
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- We argue that

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&= \sum_{\pi \in \Pi} \sum_{i=1}^{n} u_i(s_{N_{\pi(i)}}^*, s_{-N_{\pi(i)}}).
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\mathbf{E}[SW(s^t) \mid s^{t-1} = s] &\geq \frac{1}{H_n} \sum_{k=1}^{n} \frac{1}{k} \sum_{C \in \mathcal{C}_k} \frac{1}{\binom{n}{k}} \sum_{i \in C} u_i(s_C^t, s_{-C}) \\
&\geq \frac{1}{H_n} \sum_{k=1}^{n} \frac{1}{k} \sum_{C \in \mathcal{C}_k} \frac{1}{\binom{n}{k}} \sum_{i \in C} u_i(s_C^*, s_{-C}) \\
&= \frac{1}{H_n} \cdot \frac{1}{n!} \sum_{k=1}^{n} \sum_{C \in \mathcal{C}_k} \sum_{i \in C} (n-k)! \cdot (k-1)! \cdot u_i(s_C^*, s_{-C}) \\
&= \frac{1}{H_n} \cdot \frac{1}{n!} \sum_{i \in [n]} \sum_{k=1}^{n} \sum_{C \in \mathcal{C}_k, i \in C} (n-k)! \cdot (k-1)! \cdot u_i(s_C^*, s_{-C}).
\end{aligned}$$

- We argue that

$$\sum_{i \in [n]} \sum_{k=1}^{n} \sum_{C \in \mathcal{C}_k, i \in C} (n-k)! \cdot (k-1)! \cdot u_i(s_C^*, s_{-C})$$

$$= \sum_{\pi \in \prod} \sum_{i=1}^{n} u_i(s^*_{N_{\pi(i)}}, s_{-N_{\pi(i)}}).$$

Proof of Theorem 8 (contd.)

- Using the coalitional smoothness property:

$$\begin{aligned}
\mathbf{E}[SW(s^t) \mid s^{t-1} = s] &\geq \frac{1}{H_n} \cdot \frac{1}{n!} \sum_{\pi \in \prod} \sum_{i=1}^{n} u_i(s^*_{N_{\pi(i)}}, s_{-N_{\pi(i)}}) \\
&\geq \frac{1}{H_n} \cdot \frac{1}{n!} \sum_{\pi \in \prod} (\lambda \cdot \mathrm{OPT} - \mu \cdot SW(s)) \\
&= \frac{1}{H_n} (\lambda \cdot \mathrm{OPT} - \mu \cdot SW(s)) .
\end{aligned}$$

- D: a steady state distribution over strategy profiles of the coalitional best response dynamics.

$$\begin{aligned}
\mathbf{E}_{s \sim D}[SW(s)] &= \mathbf{E}_{s \sim D} \mathbf{E}_{s^t}[SW(s^t) \mid s^{t-1} = s] \\
&\geq \frac{1}{H_n} (\lambda \cdot \mathrm{OPT} - \mu \cdot \mathbf{E}_{s \sim D}[SW(s)]) .
\end{aligned}$$

Social welfare at T time steps

- The Markov chain defined by the coalitional best response dynamics might take long time to converge to a steady state...

Corollary 12

The empirical distribution of play defined by doing random coalitional best responses for T time steps, achieves expected social welfare $\geq \frac{T-1}{2T} \cdot \frac{\lambda}{H_n+\mu}$ of the optimal social welfare.

Proof of Corollary 12

- In the proof of Theorem 11:

$$SW(s^t) \geq \frac{1}{H_n}\left(\lambda \cdot \mathrm{OPT} - \mu \cdot SW(s^{t-1})\right).$$

$$\Updownarrow$$

$$SW(s^t) - \frac{\lambda}{H_n + \mu}\mathrm{OPT} \geq \frac{\mu}{H_n}\left(\frac{\lambda}{H_n + \mu}\mathrm{OPT} - SW(s^{t-1})\right).$$

- Thus either $SW(s^{t-1}) \geq \frac{\lambda}{H_n+\mu}\mathrm{OPT}$ or $SW(s^t) \geq \frac{\lambda}{H_n+\mu}\mathrm{OPT}$.
- Half of the time steps have such high social welfare (in expectation).


Thank you.