Algorithmic Mechanism Design

Knapsack Auctions

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Outline

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Whenever there is a shared resource with *limited* capacity, you have a knapsack problem.

Definition

- We study about another example of single-parameter environments.

Knapsack Auctions

- Each bidder i has a publicly known size w_i and a private valuation.
- The seller has a capacity W.
- The feasible set X is defined as the 0-1 vectors $(x_1, \ldots, x_n)$ such that $\sum_{i=1}^{n} w_i x_i \leq W$.
    - $x_i = 1$: i is a winning bidder.

Explanations

- Each bidder's size could represent
  - the duration of a company's television ad;
  - the valuation its willingness-to-pay for its ad being shown;
  - the seller capacity the length of a commercial break.
- The situation that bidders who want
  - files stored on a *shared server*,
  - data streams sent through a shared communication channel
  - processes to be executed on a *shared* supercomputer.

  $\vdots$

Assumptions

- We receive truthful bids and decide on our allocation rule.
- **Goal:** Devise a payment rule that extends the allocation rule to a DSIC mechanism.

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To maximize the welfare:

$$\mathbf{x}(\mathbf{b}) = \arg\max_{X} \sum_{i=1}^{n} b_i x_i .$$

The goal is to compute the subset of items of maximum total value that has total size bounded by W.

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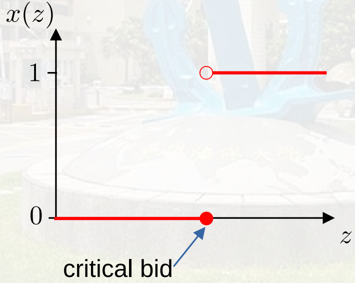
- ⋆ Check that the allocation rule $\mathbf{x}(\cdot)$ is monotone.
  - Bidding higher can only get her more stuff.

Outline

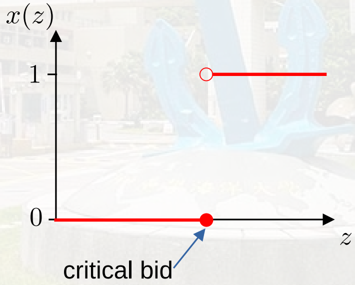
The Guarantee from Myerson's Lemma

- Myerson's lemma guarantees the existence of a payment rule **p** such that the mechanism $(\mathbf{x}, \mathbf{p})$ is DSIC.
- Since the allocation rule is monotone and assigns 0 or 1 to each bidder, the allocation curve $x_i(\cdot, \mathbf{b}_{-i})$ is 0 until some "breakpoint" z.
  - At z, the allocation jumps to 1.



The Guarantee from Myerson's Lemma (contd.)

- If i bids less than z, she loses and pays 0.
- If i bids more than z, she pays $\geq z \cdot (1 - 0) = z$.
  - z is the infimum bid that she could make and continue to win (holding $\mathbf{b}_{-i}$ fixed).



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(Recall) An ideal mechanism

Properties of an Ideal Mechanism

- DSIC
- welfare maximizing (assuming truthful bids).
- runs in polynomial time in the input size (e.g., bids, sizes, the capacity).

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- The knapsack problem is a notorious **NP**-hard problem.
  - No polynomial time implementation of the allocation rule unless $\mathbf{NP} = \mathbf{P}$.
- Hence, we would like to consider relaxing at least one of the three goals.

An ideal mechanism

Properties of an Ideal Mechanism

- DSIC
- welfare maximizing (assuming truthful bids).
- runs in polynomial time in the input size (e.g., bids, sizes, the capacity).

- Relax the second requirement as little as possible.
- Design a polynomial time and monotone allocation rule that comes as close as possible to the maximum possible social welfare.

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- The primary goal in approximation algorithms is to design polynomial-time algorithms for **NP**-hard optimization problems that are as close to the optimal solution as possible.
- Algorithmic mechanism design has exactly the same goal, except that the algorithms must additionally obey a monotonicity constraint.
- The incentive constraints of the mechanism design goal are neatly compiled into a relatively intuitive extra constraint on the allocation rule.

Approximation Algorithms come to rescue? (contd.)

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- The design space of polynomial-time DSIC mechanisms is only smaller than that of polynomial-time approximation algorithms.
- (Imagine) The best-case scenario: DSIC constraint causes no additional welfare loss (beyond the loss from the polynomial-time requirement).
- Exact welfare maximization automatically yields a monotone allocation rule.
- Is that true for *approximate* welfare maximization?

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Greedy approach

- Say S be a set of winners with total size $\sum_{i \in S} w_i \leq W$.
- We choose such a set S via a simple greedy algorithm.

⋆ We can assume that $w_i \leq W$ for all i (why?)

A Greedy Knapsack Heuristic

A Greedy Algorithm

1. Sort and re-index the bidders so that
$$\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq \cdots \geq \frac{b_n}{w_n}.$$
2. Pick winners in this order until one doesn't fit, and then halt.
3. Return either the solution from Step 2 or the highest bidder: $\arg\max_i b_i$, whichever has larger social welfare.

Theorem (Knapsack Approximation Guarantee)

Assuming truthful bids, the social welfare achieved by the greedy allocation is $\geq \frac{1}{2} \times$ (maximum social welfare).

Sketch of proving the theorem

- To have an upper bound on the maximum social welfare, allow bidders to be chosen fractionally, with the value prorated accordingly.
  - E.g., if 70% of a bidder with value 10 is chosen, then it contributes 7 to the welfare.
- Sort the bidders according to the step above, and pick winners in this order until the the capacity W is fully exhausted.
  - You can verify that this maximizes the welfare over all feasible solutions.

Sketch of proving the theorem (contd.)

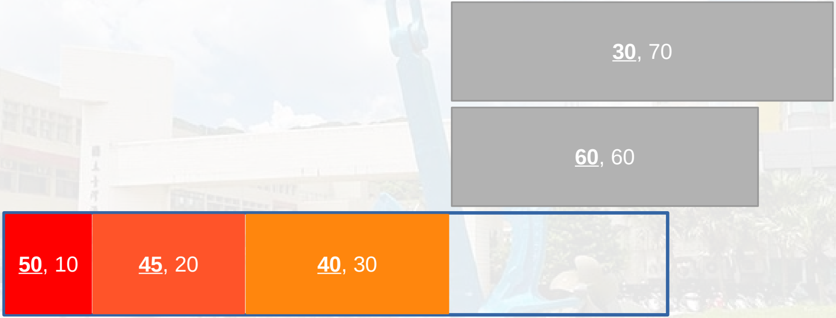
- In the optimal *fractional* solution, suppose that the first k bidders in the sorted order win and that the $(k+1)$th bidder *fractionally* wins.

⋆ The welfare achieved by steps ① and ② in the greedy allocation rule $=$ the total value of the first k bidders.

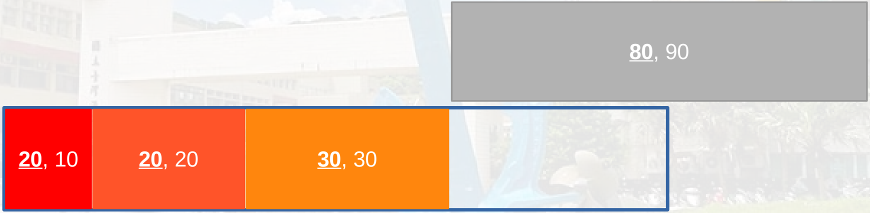
⋆ The welfare consisting only the highest bidder $\geq$ the fractional value of the $(k+1)$th bidder.

- The better of these two solutions $\geq \frac{1}{2}\times$ the welfare of the optimal fractional solution.

  $\Rightarrow$ Exercise!


30, 70
60, 60
50, 10
45, 20
40, 30


20, 50
30, 45
60, 60


80, 90
20, 10
20, 20
30, 30

Sum up

- The greedy allocation rule is monotone (check by yourself).
- Using Myerson's lemma, we can extend it to a DSIC mechanism that runs in polynomial time and, assuming truthful bids, achieves social welfare at least 50% of the maximum possible.

Outline

Reiteration

- There are good reasons to strive for a DSIC guarantee.
  - Easy for a participant to figure out what to do in a DSIC mechanism.
  - The designer can predict the mechanism's outcome.

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The DSIC Condition

The DSIC Condition

(a) For every valuation profile, the mechanism has a dominant-strategy equilibrium.
  - ⋆ An outcome that results from every participant playing a *dominant strategy*.

(b) In this dominant-strategy equilibrium, every participant truthfully reports her private information to the mechanism.

- The revelation principle asserts that:

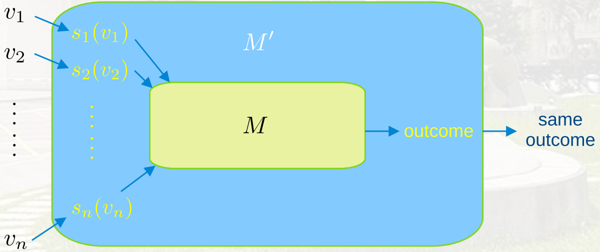
given (a), then (b) comes for free!

The Revelation Principle

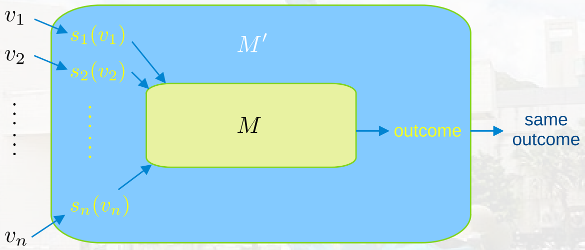
Theorem (Revelation Principle for DSIC Mechanisms)

For every mechanism M where every participant always has a dominant strategy, there is an equivalent direct-revelation DSIC mechanism M'.

- We use a simulation argument to construct M' as follows.

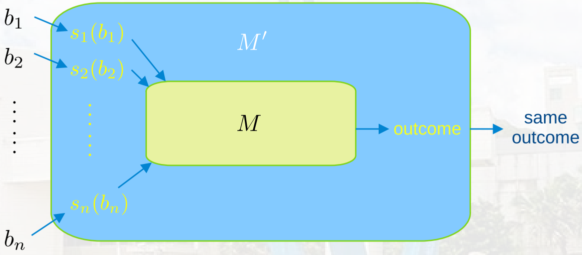


Proof



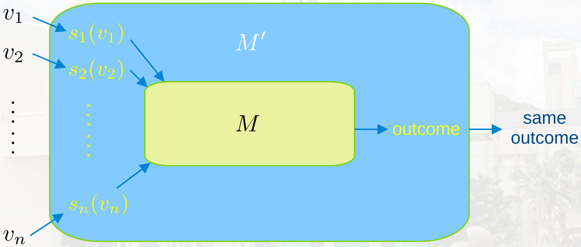
- For every participant i and its private information v_i, she has a dominant strategy $s_i(v_i)$ in mechanism M (by assumption).

Proof



- Construct M', such that participants delegate the responsibility of playing the appropriate dominant strategy to M'.
  - M' accepts bids $b_1, \ldots, b_n$.
  - Then M', which is of direct-revelation, submits the bids $s_1(b_1), \ldots, s_n(b_n)$ to the mechanism M and choose the same outcome that M does.

Proof



- Mechanism M' is DSIC:
  - If a participant i has private information v_i, then submitting a bid other than v_i can only result in M' playing a strategy other than $s_i(v_i)$ in M, which can only decrease i's utility.

What we have learned from the theorem?

- Truthfulness per se is not important.
- The difficult part is the requirement to have a dominant-strategy equilibrium.

Outline

Heads up

- DSIC and non-DSIC mechanisms are *incomparable*.
  - The former enjoys stronger incentive guarantees
  - The latter may enjoy better performance guarantees.