

All Pairs Shortest Path

Floyd-Warshall Algorithm

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Outline

1 Introduction

2 Floyd-Warshall algorithm

3 Transitive Closure

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All Pairs Shortest Paths

Goal

Find the shortest path between all pairs of vertices in the graph.

- Naïve approach:

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- Naïve approach:

Running Bellman-Ford algorithm for n vertices $\Rightarrow n \cdot O(ne) = O(n^2e)$ time.

- $|V| = n, |E| = e.$



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1 Introduction

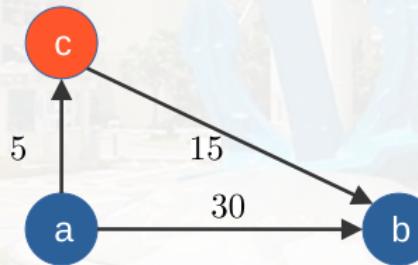
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Floyd-Warshall algorithm ($O(n^3)$ time)

Idea

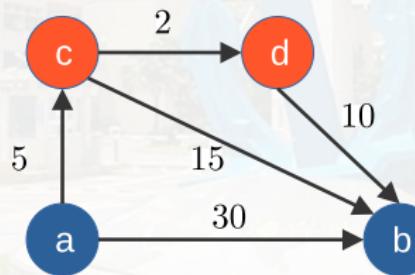
We can gradually consider “all” intermediate routes between two vertices i and j .



Floyd-Warshall algorithm ($O(n^3)$ time)

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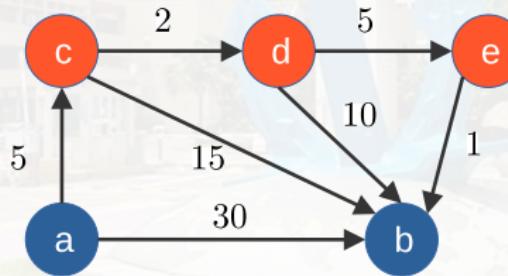
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Floyd-Warshall algorithm (contd.)

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Compute $\text{dist}^k[i][j]$: shortest path (length) from i to j routing through vertices $\{0, 1, \dots, k - 1, k\}$.

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Compute $\text{dist}^k[i][j]$: shortest path (length) from i to j routing through vertices $\{0, 1, \dots, k-1, k\}$.

- Initialization: $\text{dist}^0[i][j] = \text{length}[i][j]$ for all i, j .
- For $k \geq 1$:

$$\text{dist}^k[i][j] = \min\{\text{dist}^{k-1}[i][j], \text{dist}^{k-1}[i][k] + \text{dist}^{k-1}[k][j]\}.$$

- Effectively reuse the calculated information (DP).



Initial Setup

```
for (i=0; i<n; i++) {  
    for (j=0; j<n; j++) {  
        dist[i][j] = length[i][j]  
        if (length[i][j] != INT_MAX) {  
            next[i][j] = j; // for constructing the path  
            // next step from i to j: go to j  
        }  
    }  
}
```



The Main Function of FW

```
for (k=0; k<n; k++) {  
    for (i=0; i<n; i++) {  
        for (j=0; j<n; j++) {  
            if (dist[i][k] + dist[k][j] < dist[i][j]) {  
                dist[i][j] = dist[i][k] + dist[k][j];  
                next[i][j] = next[i][k];  
                // next step from i to j: go to k  
            }  
        }  
    }  
}  
  
// Then we deal with negative cycles!  
dist = propagateNegativeCycles(dist, n);
```

Dealing with negative cycles

```
propagateNegativeCycles(dist, n) {  
    for (k=0; k<n; k++) {  
        for (i=0; i<n; i++) {  
            for (j=0; j<n; j++) {  
                if (dist[i][k] + dist[k][j] < dist[i][j]) {  
                    // the distance is still updatable by a negative cycle  
                    dist[i][j] = -INT_MAX;  
                    next[i][j] = -1  
                }  
            }  
        }  
    }  
    return dist; // return the final distance matrix  
}
```



Path Reconstruction

```
reconstructPath(dist, next, start, end) {
    node queue path[n];
    if (dist[start][end] == INT_MAX) return path;

    int current = start;
    while (current != end) {
        if (current == -1) return NULL;
        else
            enqueue(path, current)
        current = next[current][end]
    }
    if (next[current][end] == -1) return NULL;
    else enqueue(path, end);
    return path;
}
```



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Transitive Closure (1/3)

Question

Given a directed graph G with *unweighted* edges, determine if there is a path from i to j for all vertex pairs i, j .



Transitive Closure (1/3)

Question

Given a directed graph G with *unweighted* edges, determine if there is a path from i to j for all vertex pairs i, j .

- case (i): positive path lengths;
- case (ii): nonnegative path lengths.



Transitive Closure (2/3)

Transitive closure matrix

The **transitive closure matrix** A^+ of a directed graph G is a matrix such that $A^+[i][j] = 1$ if there is a path of length > 0 from i to j (otherwise $A^*[i][j] = 0$).

Reflexive transitive closure matrix

The **reflexive transitive closure matrix** A^* of a directed graph G is a matrix such that $A^*[i][j] = 1$ if there is a path of length ≥ 0 from i to j (otherwise $A^*[i][j] = 0$).

Transitive Closure (3/3)

- We can utilize the all-pairs-shortest-path algorithm (FW).
- Note:
 - $\text{length}[i][j] = 1$ if (i, j) is an edge in G .
 - $\text{length}[i][j] = +\infty$ if (i, j) is not in G .
- Change the if-statement in FW algorithm (distance updating):

```
dist[i][j] = (dist[i][j] || dist[i][k] && dist[k][j])
```

- Initialize the distance to be the **adjacency matrix** of the graph.

Discussions

