Preliminary Probability Theory for Randomized Algorithms

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- For convenience, we omit the contents of set theory concepts, counting techniques, and the discussion of sample spaces and probability measures.

Outline

- Probability
  - Some useful theorems and principles
  - Conditional probability
  - Theorem of total probability
  - Bayes’ theorem
  - Independent events
  - Independent trails

Kolmogorov axioms

- 1. For any event $A \subset S$, $P(A) \geq 0$.

- 2. $\Pr[S] = 1$.

- 3. If $A_1, A_2, \ldots$ are mutually exclusive events, then

$$\Pr[A_1 \cup A_2 \cup \ldots] = \Pr[A_1] + \Pr[A_2] + \cdots$$

Useful theorems

- $\Pr[\varnothing] = 0$ for any experiment.

- For any event $A \subseteq S$, $\Pr[A] = 1 - \Pr[\overline{A}]$.

- If $A \subseteq S$, $B \subseteq S$ are any two events, then

  $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$.

- If $A \subset B$, then $\Pr[A] \leq \Pr[B]$.

Conditional probability

- In an experiment with sample space S, let B be any event such that $\Pr[B] > 0$. Then the conditional probability of A occurring, given that B has occurred, is

$$\Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

for any $A \subset S$.

- Suppose you are going to buy milk in a supermarket.
  - There are a total of 40 boxes for you to choose from.
  - 10 of them are corrupted (not visible on the outside).
  - Then, you are asked to buy two boxes of milk.

    What is the probability that both boxes are good?

- A: the event that the first box you choose is good.

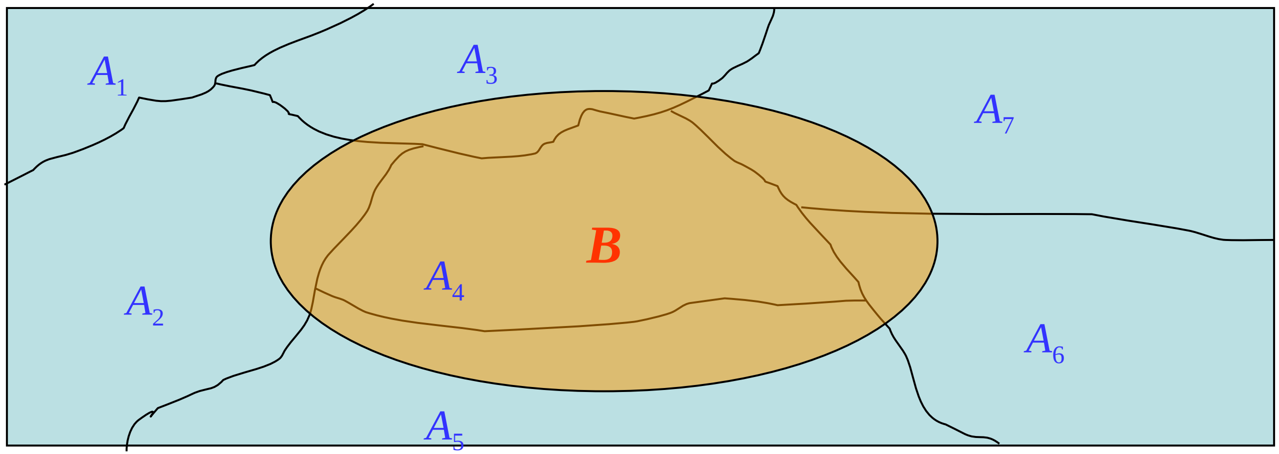
  B: the event that the second box you choose is good.

  Then
$$\Pr[A] = \frac{30}{40}$$
$$\Pr[B \mid A] = \frac{29}{39}$$
$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B \mid A] = \frac{30}{40} \cdot \frac{29}{39} = \frac{87}{156}.$$

Theorem of total probability

- If A_1, A_2,…, A_n is a partition of S, and B is any event, then

$$\Pr[B] = \sum_{i=1}^{n} \Pr[B \mid A_i] \Pr[A_i].$$



- From the theorem of total probability, and granted that $A_1, A_2, \ldots, A_n$ is a partition of S, we have

$$\begin{aligned}\Pr[A_i \mid B] &= \frac{\Pr[A_i \cap B]}{\Pr[B]} \\ &= \frac{\Pr[A_i] \cdot \Pr[B \mid A_i]}{\sum_{i=1}^{n} \Pr[A_i] \cdot \Pr[B \mid A_i]}\end{aligned}$$

- This result is known as *Bayes' theorem*.

- Assuming a jury selected to participate in a criminal trial, whether the defendant is guilty or not guilty, there is a 95% chance of making the correct verdict.
- It is also assumed that the local police law enforcement is very strict, such that 99% of the people being tried are actually guilty.
- **If a jury is known to sentence a defendant not guilty**, **what is the probability that the defendant is really not guilty?**

- Let A_1 be the event that the defendant is guilty, and let $A_2 = \bar{A}_1$ denote the event that he is not guilty.
- Let B be the event that the defendant is sentenced to unguilty.
- We want to know $\mathbf{Pr}[A_2 \mid B]$.

- $$\Pr[A_2|B] = \frac{\Pr[A_2]\Pr[B|A_2]}{\Pr[A_1]\Pr[B|A_1] + \Pr[A_2]\Pr[B|A_2]}$$
$$= \frac{(0.01)(0.95)}{(0.99)(0.05) + (0.01)(0.95)}$$
$$= 0.161$$

- Before the sentence, this defendant is supposed to be unguilty with probability 1%.
- After the sentence of unguilty, the probability is increased to be 16.1%.

Independent events

If $A \subset S$ and $B \subset S$ are any two events with nonzero probabilities, A and B are called independent if and only if $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$

That is, $\Pr[A] = \Pr[A \mid B]$ and $\Pr[B] = \Pr[B \mid A]$.

Independent trials

- An experiment is said to consist of n ***independent*** trials if and only if
  - $S = T_1 \times T_2 \times \cdots \times T_n$.
  - For every $(x_1, x_2, \ldots, x_n) \in S$,

    $\Pr[\{(x_1, x_2, \ldots, x_n)\}] = \Pr[\{x_1\}] \cdot \Pr[\{x_2\}] \cdots \Pr[\{x_n\}]$,

    where $\Pr[\{x_i\}]$ is the probability of $x_i \in T_i$ occurring on trial i.