

# Heaps

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# Outline

## 1 Introduction

- Building a heap



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# Heaps

## Max Tree

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# Heaps

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## Max Heap

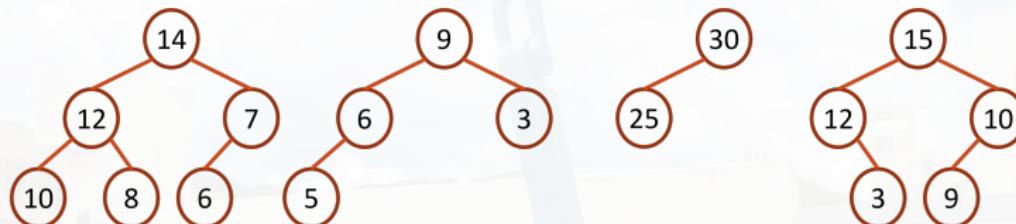
A complete binary tree that is also a max tree.

## Min Heap

A complete binary tree that is also a min tree.



# Examples: Max & Min Trees

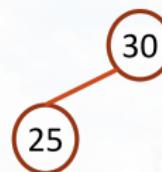
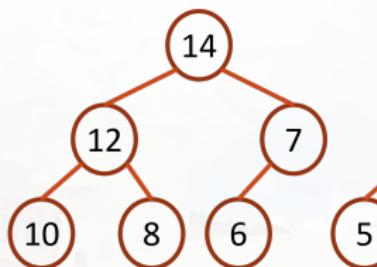


Max Trees



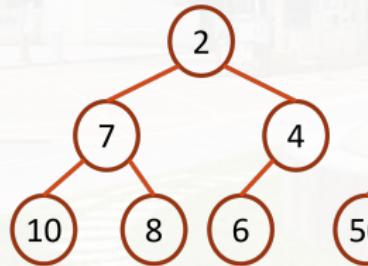
Min Trees

# Examples: Max & Min Heaps



Max Heaps

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Min Heaps

# The Key Application: Priority Queues

- Heaps are frequently used to implement **priority queues**.
- In this kind of queue,
  - the element to be **deleted** is the one with **highest** (or **lowest**) priority.
  - at **any time**, an element with **arbitrary priority** can be **inserted** into the queue.

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- A heap is a **complete** binary tree.

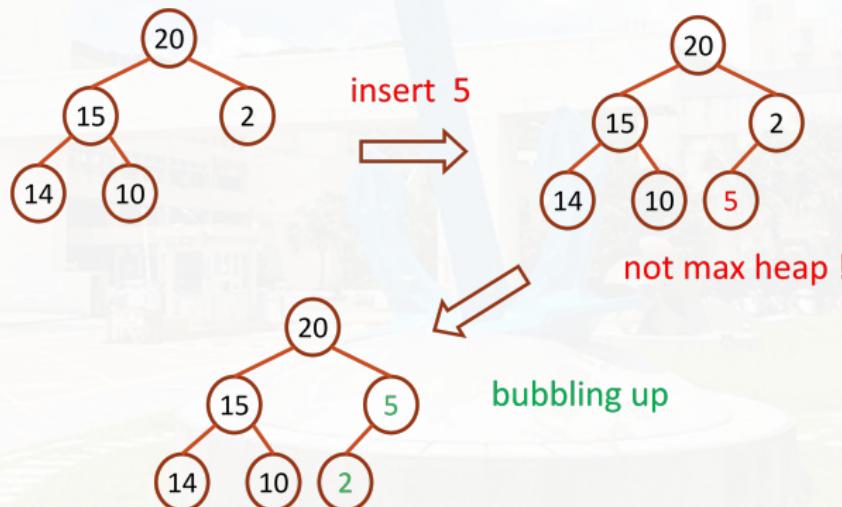
# Insertion into a Max Heap

- The **bubbling process**.
  - It begins at the new node of the tree and moves toward the root.



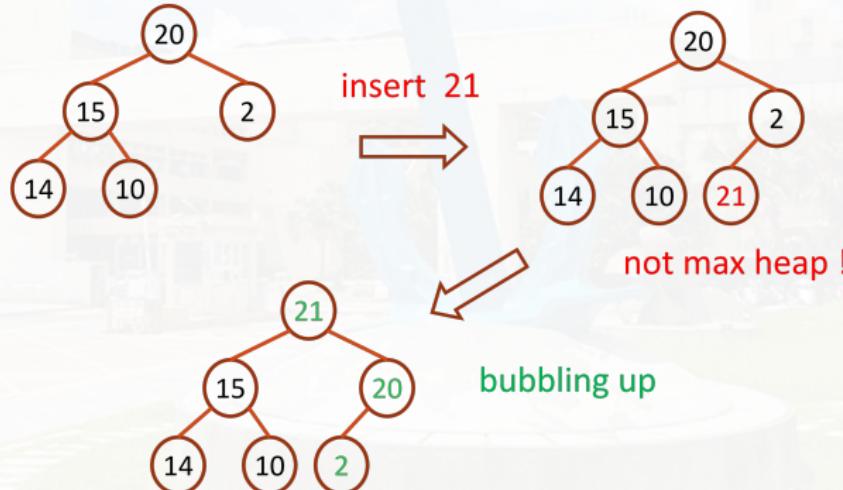
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# The Code for Insertion into a Max Heap

- Consider the following declarations:

```
#define MAX_ELEMENTS 200 /* maximum heap size+1 */  
#define HEAP_FULL (n) (n == MAX_ELEMENTS -1)  
#define HEAP_EMPTY (n) (!n)  
typedef struct {  
    int key;  
    /* other fields */  
} element;  
element heap[MAX_ELEMENTS];  
int n = 0;
```

# The Code for Insertion into a Max Heap

```
void push (element item, int *n) {
    /* insert item into a max heap of current size *n */
    int i;
    if (HEAP_FULL(*n)) {
        printf("The heap is full.\n");
        exit(EXIT_FAILURE);
    } // O(1) time
    i = ++(*n);
    while ((i != 1) && (item.key > heap[i/2].key)) {
        heap[i] = heap[i/2];
        i /= 2;
    } // O(lg n) time
    heap[i] = item; // O(1) time
}
```

- The time complexity of the insertion:  $O(\lg n)$ .

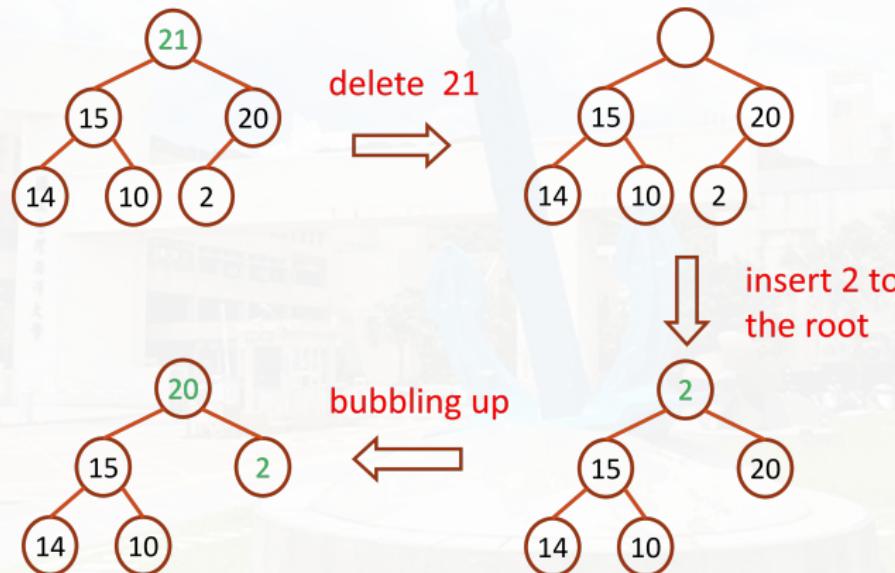
# Deletion from a Max Heap

- When an element is to be deleted from a max heap, it is **ALWAYS** taken from the root of the heap.

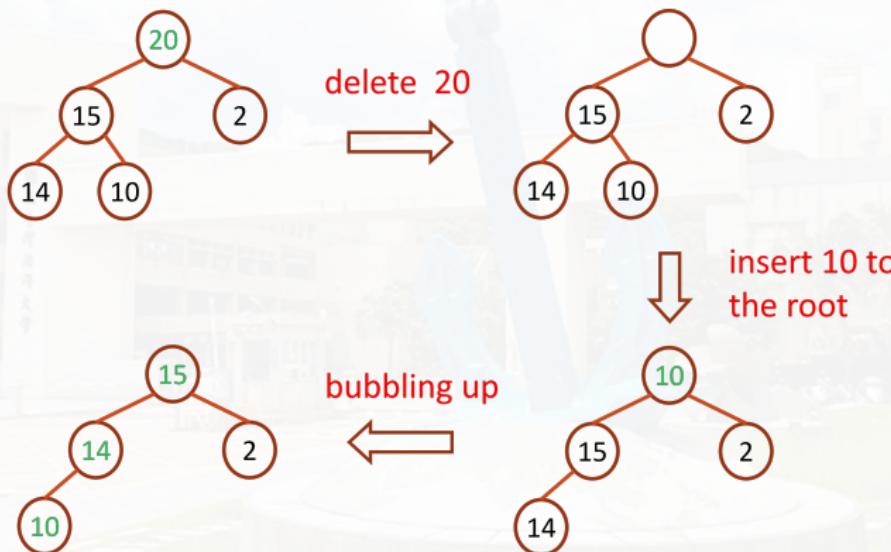
# Deletion from a Max Heap

- When an element is to be deleted from a max heap, it is **ALWAYS** taken from the root of the heap.
- The steps of deletion from a Max heap:
  - delete the root node.
  - insert the last node into the root (say  $r$ ).
  - use the **bubbling up process** to ensure that the resulting heap remains a max heap (a.k.a. **heapify** at  $r$ ).

# Illustration of Deletion from a Max Heap



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# The Code for Deletion from a Max Heap

```
element pop(int *n) {
    /* delete element with the highest key from the heap */
    int parent, child;
    element item, temp;
    if (HEAP_EMPTY(*n)) {
        fprintf(stderr, "The heap is empty\n");
        exit(EXIT_FAILURE);
    }
    /* save value of the element with the highest key */
    item = heap[1];
    /* use last element in heap to adjust heap */
    temp = heap[(*n)--];
    parent = 1;
    child = 2; // default: the left child
    while (child <= *n) { // O(lg n) time
        /* find the larger child of the current parent */
        if ((child < *n) && (heap[child].key < heap[child+1].key))
            child++; // okay, it's the right child!
        if (temp.key >= heap[child].key) break; // the new root is the maximum!
        /* if the max-child gets larger key, move to the next lower level */
        heap[parent] = heap[child];
        parent = child;
        child *= 2;
    }
    heap[parent] = temp;
    return item;
}
```

# Time Complexity of the Deletion from a Max Heap

- Delete the root node:  $O(1)$ .
- Insert the last node to the root:  $O(1)$ .
- Since the height of the heap is  $\lceil \lg(n + 1) \rceil$ , the while loop is iterated for  $O(\lg n)$  times.
- Thus, the overall time complexity: the time complexity of the deletion:  $O(\log n)$ .

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# An $O(n)$ time algorithm for building a (max) heap

Input:  $n$  numbers:  $x_1, x_2, \dots, x_n$ .

## Efficient Heap Construction

- ① For each input number  $x_i$ , insert  $x_i$  into array  $A$  at  $A[i - 1]$  one by one.
- ② For  $i = \lfloor n/2 \rfloor - 1$  down to 0:
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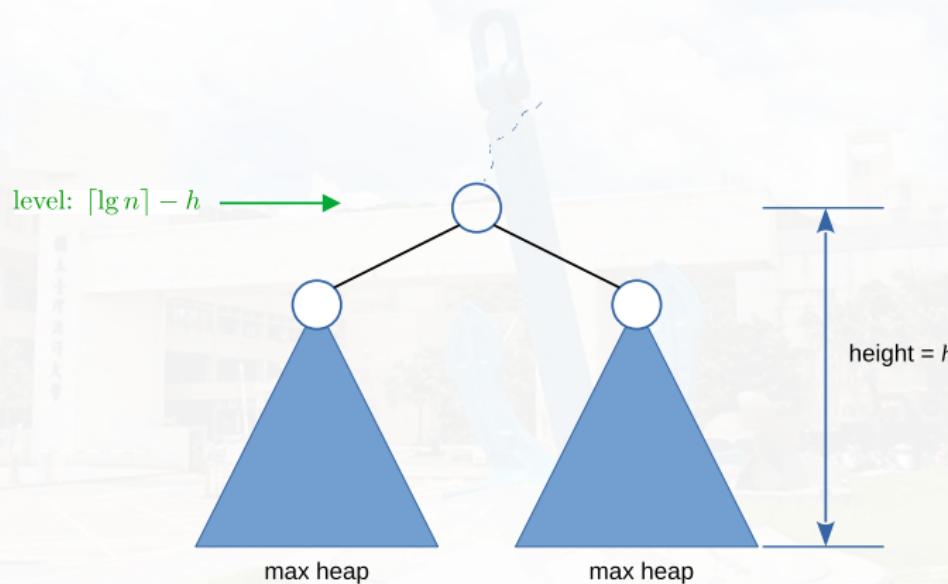
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    - Run  $\text{heapify}(A, i)$
- That is, we build a heap in a bottom-up fashion!



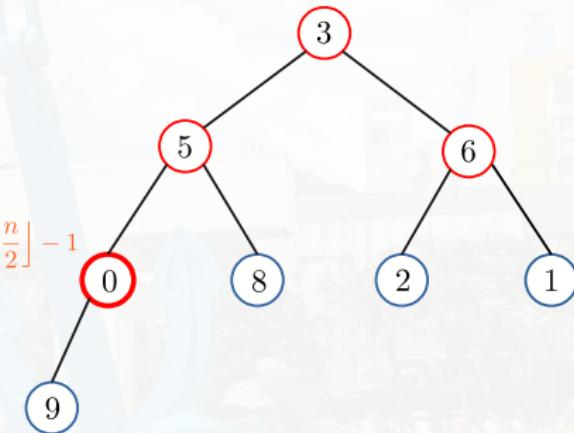
# Heap recursive view (bottom-up)



# Nodes to be Heapified

index

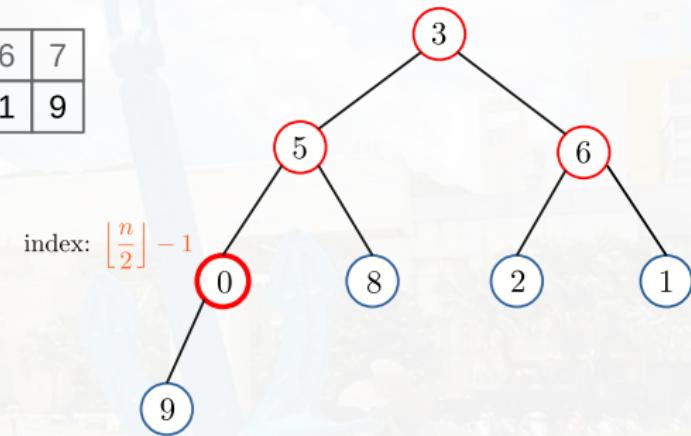
0	1	2	3	4	5	6	7
3	5	6	0	8	2	1	9

index:  $\left\lfloor \frac{n}{2} \right\rfloor - 1$ 

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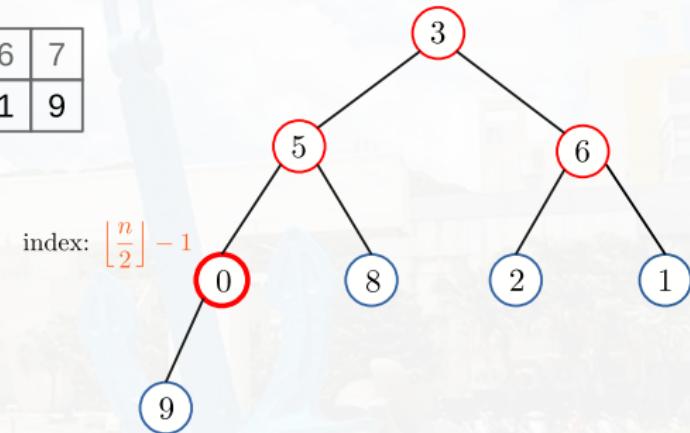


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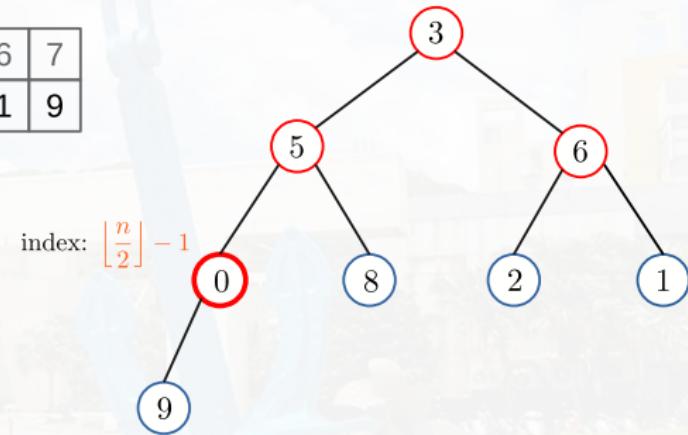
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- # Heapify steps:  $\leq \sum_{h=1}^{\lg n-1} h \cdot n_h = \sum_{h=1}^{\lg n-1} h \cdot 2^{\lceil \lg n \rceil - h} \leq 2n \sum_{h=1}^{\lg n-1} \frac{h}{2^h}$ .
- $n_h$ : the number of nodes at level  $h$ .

# Discussions

