Randomized Algorithms

— Randomized QuickSort

Joseph Chuang-Chieh Lin

Department of Computer Science & Information Engineering,
Tamkang University

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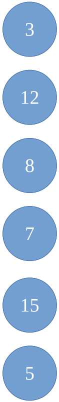
Illustration (a binary tree T demonstrating RandQS)



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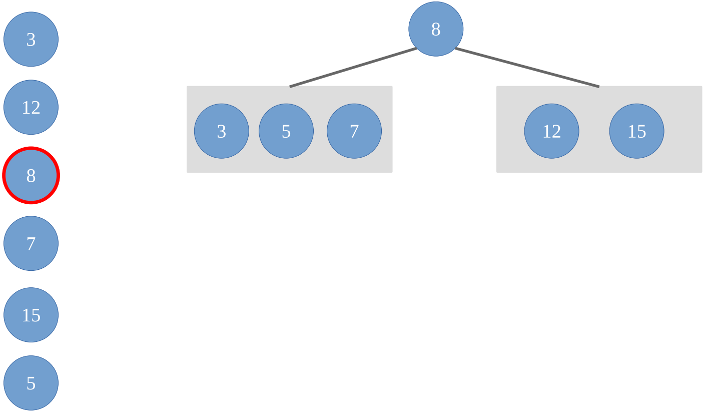

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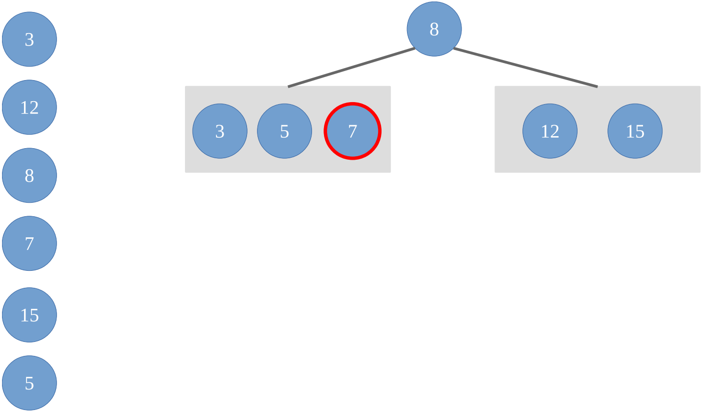

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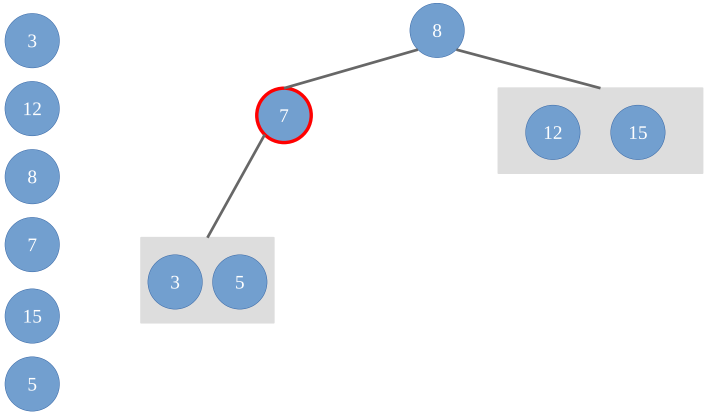

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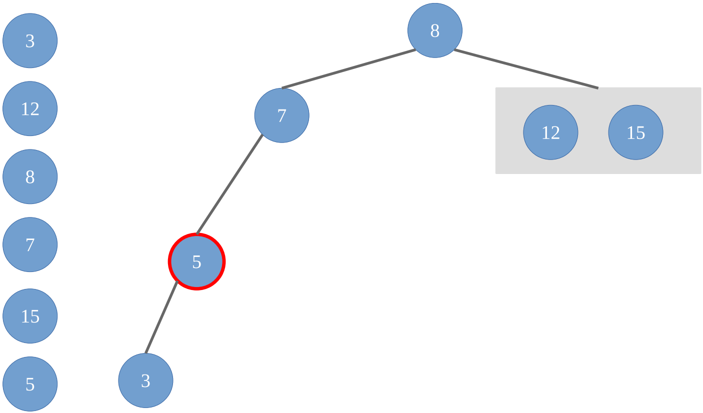

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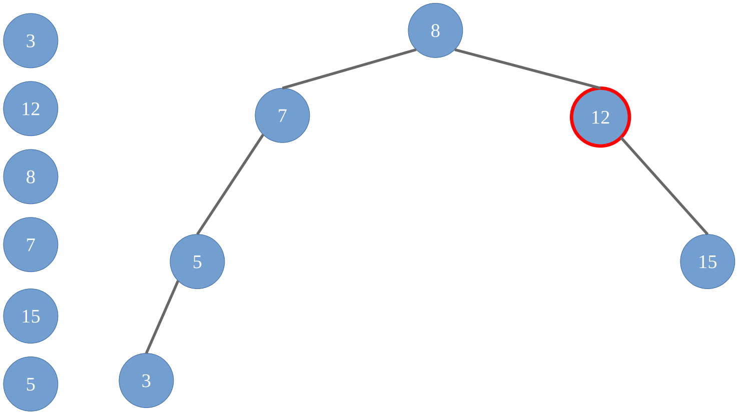

Algorithm RandQS

Input: A set of (distinct) numbers S
Output: The elements of S sorted in increasing order.

1. Choose an element $y \in S$ uniformly at random;
2. By comparing each element of S with y, compute
   - $S_1 := \{x \in S : x < y\}$;
   - $S_2 := \{x \in S : x > y\}$;
3. Recursively sort S_1 (i.e., run RandQS(S_1)) and S_2 (i.e., run RandQS(S_2)), and output the sorted version of S_1, followed by y, and then the sorted version of S_2.

Analysis (Expected Number of Comparisons)

- Comparisons are performed in Step 2.
- Let $S_{(i)}$ denote the element of rank i (i.e., the ith smallest in S).
- Define X_{ij}:
  - $X_{ij} = 1$ if $S_{(i)}$ and $S_{(j)}$ are compared in an execution.
  - $X_{ij} = 0$ otherwise.

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$$\mathbb{E}\left[\sum_{i=1}^{n} \sum_{j>i} X_{ij}\right] = \sum_{i=1}^{n} \sum_{j>i} \mathbb{E}[X_{ij}].$$

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- Note: $S_{(i)}$ and $S_{(j)}$ are compared in an execution only when one of them is an ancestor of the other in the binary tree T.

Analysis (contd.)

$$\sum_{i=1}^{n} \sum_{j>i} p_{ij} = \sum_{i=1}^{n} \sum_{j>i} \frac{2}{j-i+1}$$

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$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{j>i} p_{ij} &= \sum_{i=1}^{n}\sum_{j>i}\frac{2}{j-i+1} \\
&= \sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{2}{j-i+1}
\end{aligned}
$$

Analysis (contd.)

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&= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n} \sum_{k=2}^{n-i+1} \frac{2}{k}
\end{aligned}$$

Analysis (contd.)

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&= \sum_{i=1}^{n}\sum_{k=2}^{n-i+1}\frac{2}{k} \\
&\le 2\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{1}{k} \qquad .
\end{aligned}$$

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\end{aligned}
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- Note that $H_n = \sum_{k=1}^{n} 1/k$

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&= \sum_{i=1}^{n}\sum_{k=2}^{n-i+1}\frac{2}{k} \\
&\leq 2\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{1}{k} = O(n\log n).
\end{aligned}
$$

- Note that $H_n = \sum_{k=1}^{n} 1/k \approx \Theta(\ln n)$.

Exercise (3%)

Using $O(n)$ Median-of-Medians Algorithm

- **Remark:** The Median-of-Medians algorithm (reference here) by Blum et al. can compute a median of an array of n numbers in a list in $O(n)$ time deterministically.
- Please prove that Algorithm MedianQS (next page) can sort an array of n numbers in $O(n \log n)$ time deterministically.

Algorithm MedianQS

Input: A set of (distinct) numbers S
Output: The elements of S sorted in increasing order.

1. Compute the median y of S using the Median-of-Medians algorithm;
2. By comparing each element of S with y, compute
   - $S_1 := \{x \in S : x < y\}$;
   - $S_2 := \{x \in S : x > y\}$;
3. Recursively sort S_1 (i.e., run MedianQS(S_1)) and S_2 (i.e., run MedianQS(S_2)), and output the sorted version of S_1, followed by y, and then the sorted version of S_2.

Discussions