

Auctions & Mechanism Design Basics

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- We study about a kind of science of *rule-making*.
- To make it simple, we first consider single-item auctions.
- We will go over some basics about first-price auctions and second-price auctions.
- Also, we will talk about
 - incentive guarantees,
 - strong performance guarantees, and
 - computational efficiencyin an auction.
- We will end the discussion with Myerson's Lemma.



Outline

- 1 Single-Item Auctions
- 2 Sealed-Bid Auctions
 - First-Price Auctions
 - Second-Price Auctions
 - Case Study: Sponsored Search Auctions



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 - Her maximum willingness-to-pay for it.
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 - Unknown to the seller and other bidders.



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- (i) Each bidder i **privately** communicates a bid b_i to the seller—in a sealed envelope.
- (ii) The seller **decides who** gets the item (if any).
- (iii) The seller **decides the selling price**.



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- Step (ii): The selection rule. We consider giving the item to the **highest** bidder.



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The winning bidder **pays her bid.**

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- Why?



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- For a bidder: Hard to figure how to bid.
- For the seller: Hard to predict what will happen.



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 - Would it help to know your opponent's birthday?
 - Would your answer change if you knew there were two other bidders rather than one?



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eBay/Yahoo auction

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- If you bid \$100 and win, do you pay \$100?
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- If you bid \$100 and win, do you pay \$100?
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 - For example, if the highest other bid is \$90.
You only pay $\$90 + \epsilon$ for some small increment ϵ .
 \approx highest other bid!



Second-Price auction

Second-Price/Vickrey Auction

The highest bidder wins and pays a price equal to the **second-highest bid**.

- Is such a strategy a **dominant strategy**?



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- Is such a strategy a **dominant strategy**?
 - The strategy is guaranteed to **maximize** a bidder's utility **no matter what other bidders do**.



Truthfully Bidding Is Dominant Here

Proposition (Incentives in Second-Price Auctions)

In a second-price auction, every bidder i has a dominant strategy: set the bid $b_i = v_i$, equal to her private valuation.



Proof of the Proposition

- Fix a bidder i with valuation v_i .
 - \mathbf{b} : the vector of all bids.
 - \mathbf{b}_{-i} : the vector of \mathbf{b} with b_i removed.
- * **Goal:** Show that bidder i 's utility is maximized by setting $b_i = v_i$.



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Second-Price Single-Item Auctions are “ideal”

Definition (Dominant-Strategy Incentive Compatible)

An auction is **dominant-strategy incentive compatible (DSIC)** if

- truthful bidding is a dominant strategy for every bidder, and
- truthful bidders always obtain nonnegative utility.



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Social Welfare

The **social welfare** of an outcome of a single-item auction is

$$\sum_{i=1}^n v_i x_i.$$

where $\sum_{i=1}^n x_i \leq 1$; $x_i = 1$ if bidder i wins and 0 if she loses.

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- So such an auction is welfare maximizing if bids are truthful.



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Theorem

A second-price single-item auction satisfies:

- (1) DSIC.
- (2) Welfare maximizing.
- (3) It can be implemented in polynomial time.

In fact, (3) is linear.



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Theorem

A second-price single-item auction satisfies:

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Background

The Social Dilemma (2020) - Trailer

- Web search results:
 - relevant to your query (by an algorithm, e.g., PageRank).
 - pops out a list of sponsored links.
 - They are paid by advertisers.
- Every time you give a search query into a search engine, an auction is run in real time to decide
 - which advertiser's links are shown,
 - how these links are arranged visually,
 - what the advertisers are charged.



Multiple Items for Sponsored Search Auctions

- Let's say the items for sale are k “slots” on a search results page.
- Bidders: the advertisers who have a bid on the keyword that was searched on.
 - On the keyword, “university”, NTU, NYCU, NCKU, NTOU, etc., might be the bidders.



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- Let's say the items are not identical.
 - Higher slots are more valuable. What do you think?



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 - Here we simply assume that $\beta_i = 1$ for each i .
- The expected value derived by advertiser i from slot j : $v_i \alpha_j$
- The social welfare is $\sum_{i=1}^n v_i x_i$.
 - x_i : the CTR of the slot to which bidder i is assigned.
 - $x_i = 0$: bidder i is not assigned to a slot.
 - Each slot can only be assigned to one bidder; each bidder gets (at most) one slot.



Our Design Approach

- Who wins what?
- Who pays what?
- The payment.



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- Who wins what?
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- The payment.
 - If the payments are not just right, then the strategic bidders will game the system.



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Design Steps

- (a): Assume that the bidders bid truthfully. Then, how should we assign bidders to slots so that property (2) and (3) holds?
- (b): Given the answer of above, how should we set selling prices so that property (1) holds?

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- You can prove that this assignment achieves the maximum social welfare as an exercise.



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 - DSIC.
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 - A powerful and general tool for implementing this second step.
 - Why not just ask the bidders to **pay the bid which is next (lower) to its own?**
 - ▷ Generalized Second Price Auction.

