

# Connected Components & Biconnected Components

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# Outline

## 1 Connected Components

- Spanning Trees
- Articulation Points & Biconnected Graph
- Finding the articulation points



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List all connected components of an (un)directed graph.

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## Problem II

List all connected components of an (un)directed graph.

This can be done by making repeated calls to either  $\text{dfs}(v)$  or  $\text{bfs}(v)$  where  $v$  is an **unvisited vertex**.



```
void connected(void) { // dfs(0) or bfs(0)
/* determine the connected components of a graph */
    int i;
    for (i=0; i<n; i++) {
        if (!visited[i]) {
            dfs(i);
            printf("\n");
        }
    }
}
```

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- If  $G$  is represented by an **adjacency matrix**, then the time needed to determine the connected components is  $O(n^2)$ .

# Spanning Trees

## Spanning Trees

A tree  $T$  is said to be a *spanning tree* of a connected graph  $G$  if  $T$  is a subgraph of  $G$  and  $T$  contains **all vertices** of  $G$ .

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# Spanning Trees

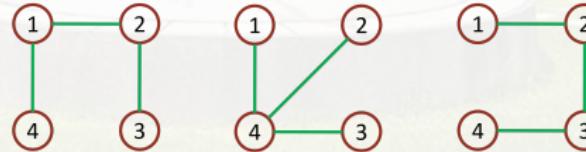
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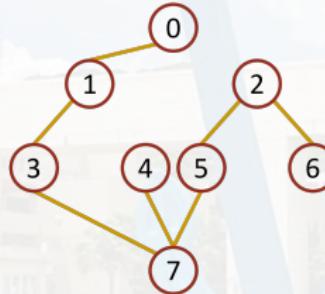
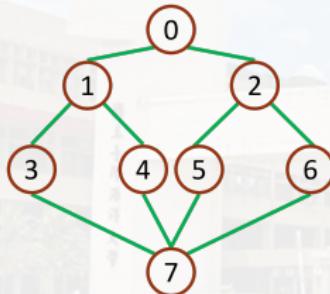
- When graph  $G$  is connected, a DFS or BFS implicitly partitions the edges in  $G$  into two sets:
  - Tree edges:** the set of edges used or traversed during the search.
  - Nontree edges:** the set of remaining edges.



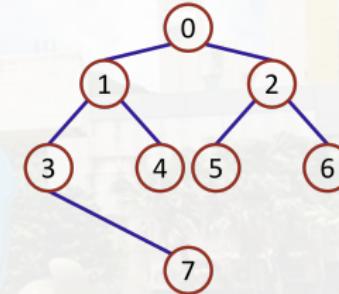
Three spanning trees of  $G_1$ .



# DFS Spanning Trees & BFS Spanning Trees



DFS (0)  
spanning tree

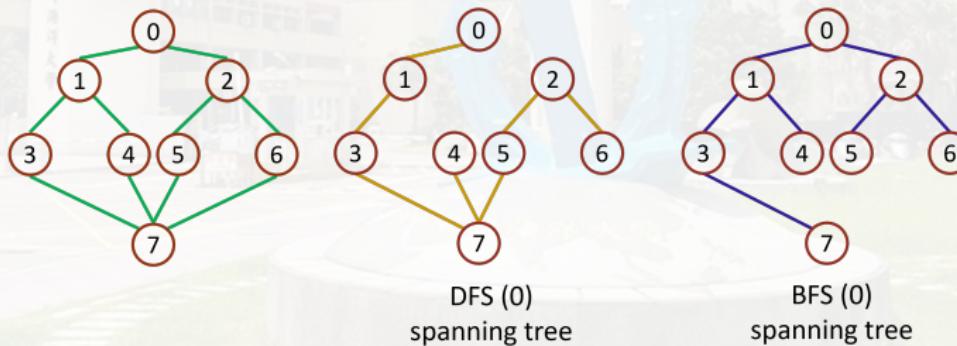


BFS (0)  
spanning tree

# Properties of (DFS or BFS) Spanning Trees

## Property I

Suppose we **add a nontree edge**,  $(v, w)$ , into any spanning tree,  $T$ . Then there will be a cycle that consists of the edge  $(v, w)$  and all the edges on the path from  $w$  to  $v$  in  $T$ .

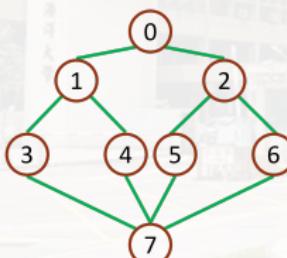


# Properties of (DFS or BFS) Spanning Trees

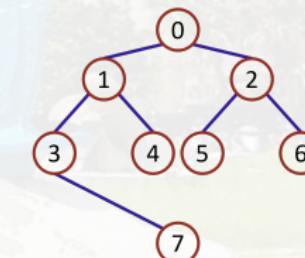
## Property II

A spanning tree is a **minimal** subgraph,  $G'$ , of  $G$  such that  $V(G') = V(G)$  and  $G'$  is connected.

- A spanning tree has  $n - 1$  edges.



DFS (0)  
spanning tree



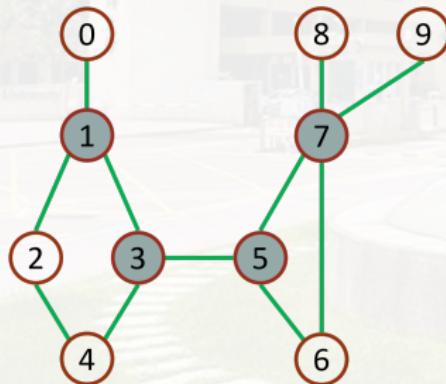
BFS (0)  
spanning tree

# Articulation Point (cut vertex)

## Articulation Points

An **articulation point** is a vertex  $v$  of  $G$  such that the deletion of  $v$ , together with all edges incident on  $v$ , produces a graph,  $G'$ , that has  $\geq 2$  connected components.

a connected graph



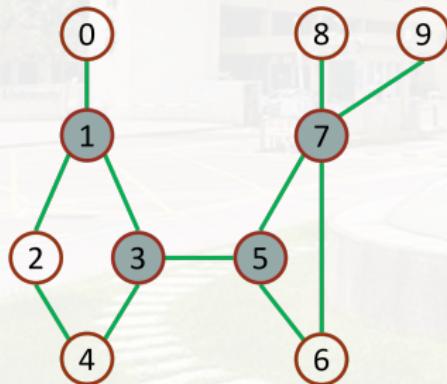
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- Four articulation points:

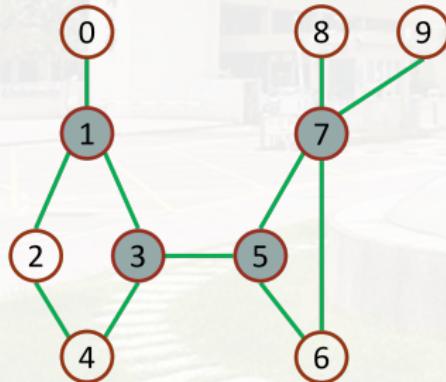


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a connected graph



- Four articulation points:  
1, 3, 5, 7.

# Biconnected Graph (雙連通圖)

## Biconnected Graph

A biconnected graph is a connected graph that has **NO** articulation points.

## Biconnected Component

A biconnected component of a connected graph  $G$  is a **maximal biconnected subgraph  $H$  of  $G$ .**

- $H$  is “maximal”: no other subgraph that is both biconnected and properly contains  $H$ .



# Biconnected Graph (雙連通圖)

## Biconnected Graph

A biconnected graph is a connected graph that has **NO** articulation points.

- A connected graph  $G$  which has an articulation point  $\Rightarrow G$  is NOT biconnected.

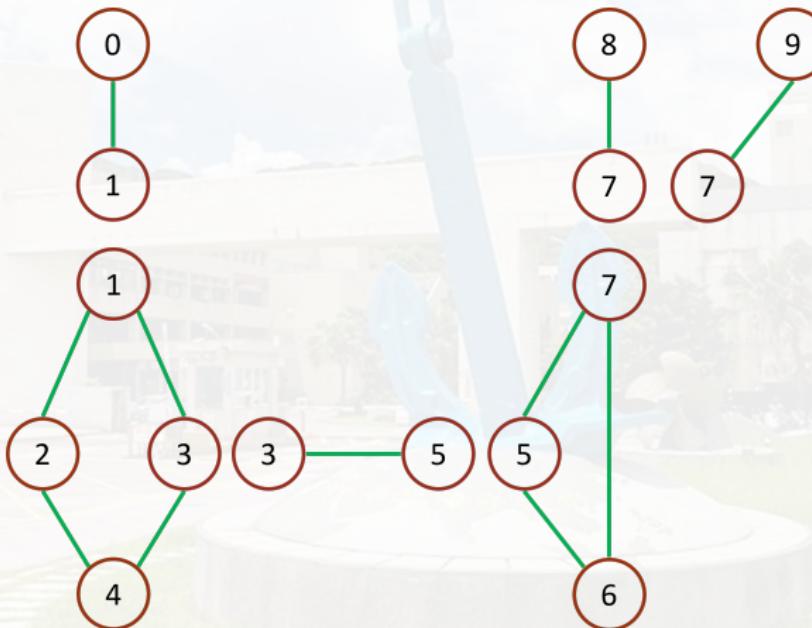
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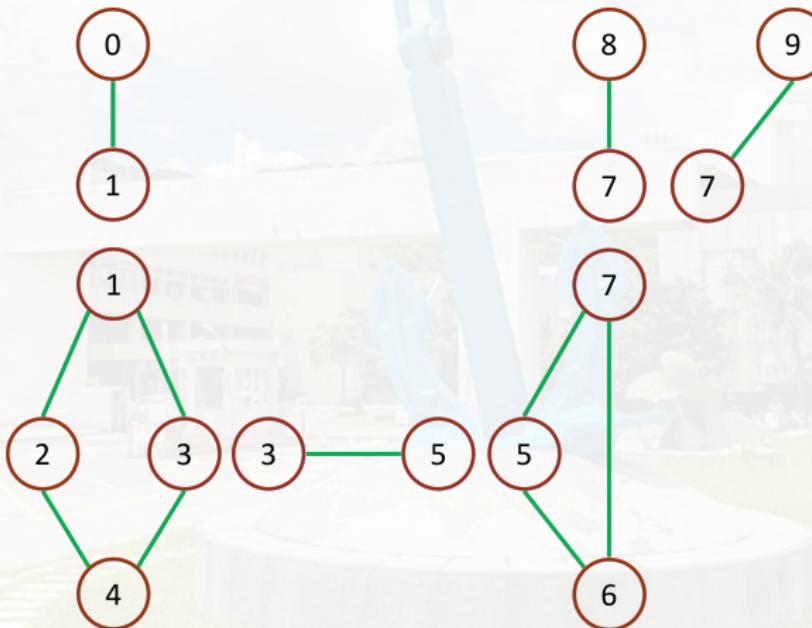
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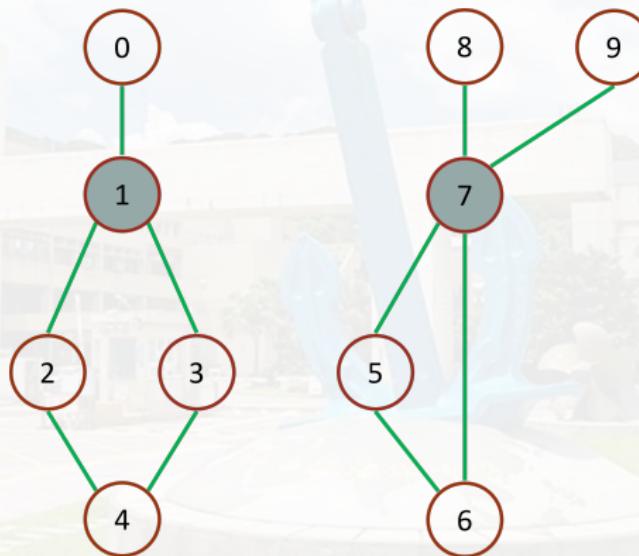
# Connected Components



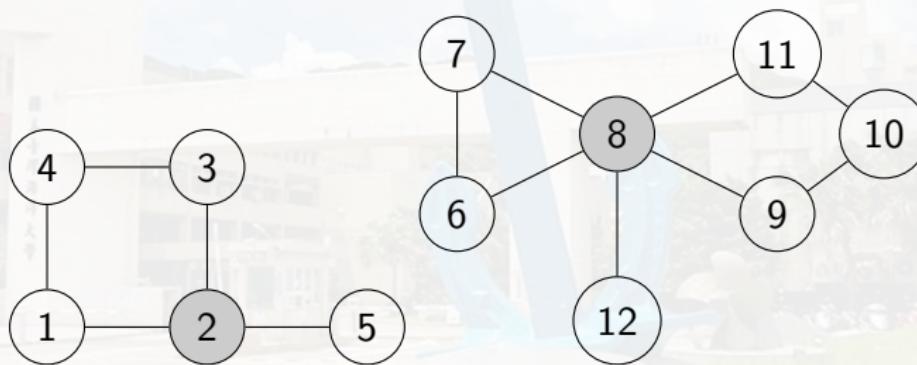
# Biconnected Components (Example)



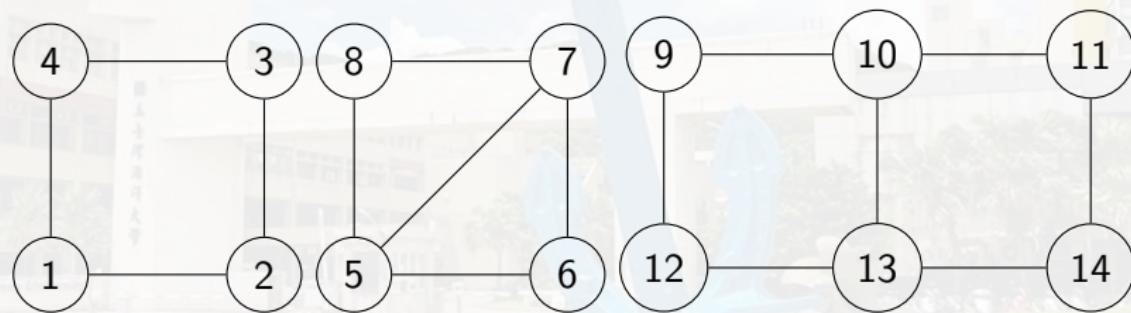
# Components but Not Biconnected



# Components but Not Biconnected



# Biconnected components

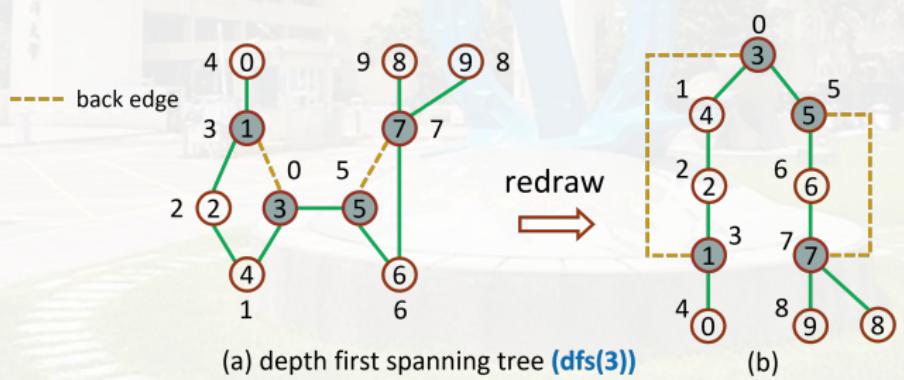


# Finding articulation points (1/3)

We can find biconnected components of a graph  $G$  using any depth-first spanning of  $G$ .

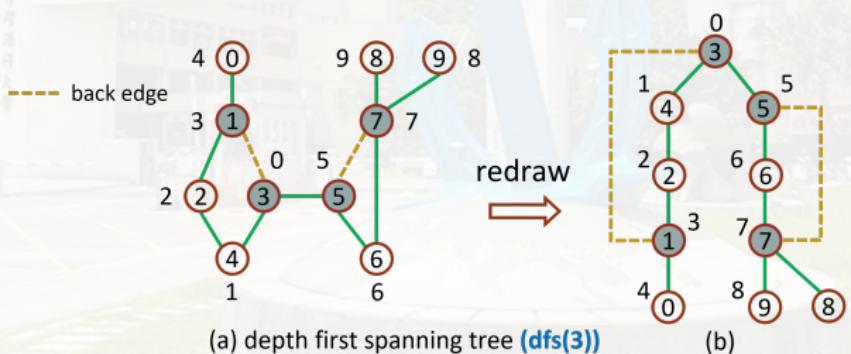
## Back edges

- Tree edges: DFS
- Nontree edges: we call them **back edges**



# Observations

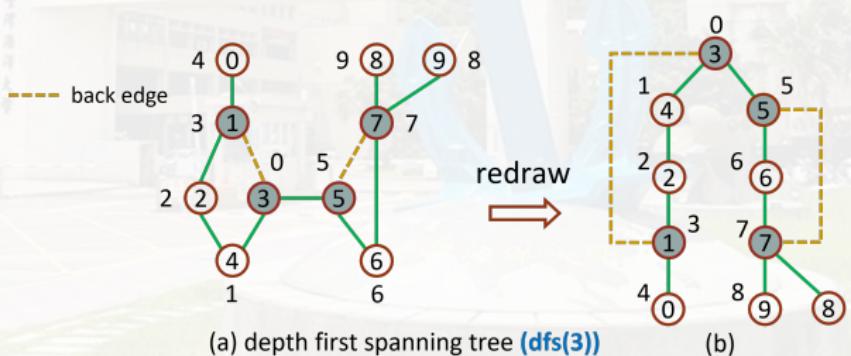
- The root of a depth first spanning tree is an articulation point if and only if it has  $\geq 2$  children.
- Any other vertex  $u$  is an articulation point if and only if it has  $\geq 1$  child  $w$  such that we cannot reach an ancestor of  $u$  using that consists of only  $w$ , descendants of  $w$ , and a single back edge.



- $v_5$  is an articulation point

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- $v_5$  is an articulation point, but  $v_6$  is NOT.

## Finding articulation points (2/3)

$\text{dfn}(v)$

The depth first numbers, or  $\text{dfn}$ , of the vertices give the sequence in which the vertices are visited during the depth first search.

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- If  $u$  is an ancestor of  $v$  in the depth first spanning tree, then  $\text{dfn}(u) < \text{dfn}(v)$ .

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### $\text{dfn}(v)$

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- If  $u$  is an ancestor of  $v$  in the depth first spanning tree, then  $\text{dfn}(u) < \text{dfn}(v)$ .

### $\text{low}(v)$

The  $\text{low}(u)$  value of vertex  $u$  is the lowest depth first number that we can reach from  $u$  using a path of descendants followed by at most 1 back edge.

$$\text{low}(u) = \min \left\{ \begin{array}{l} \text{dfn}(u), \\ \min\{\text{low}(w) \mid w \text{ is a child of } u\}, \dots (*) \\ \min\{\text{dfn}(w) \mid (u, w) \text{ is a back edge}\} \dots (**) \end{array} \right.$$

## Finding articulation points (3/3)

### articulation points

$u$  is an articulation point iff one of the following conditions are satisfied:

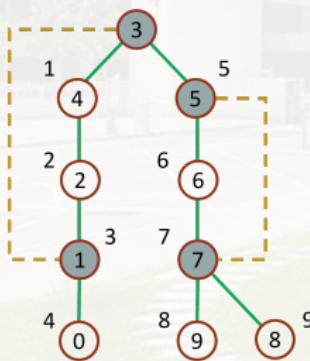
- $u$  is the **root** of the spanning tree and **has two or more children**.
- $u$  is not the root of the spanning tree and  $u$  **has a child  $w$**  such that  $\text{low}(w) \geq \text{dfn}(u)$ .

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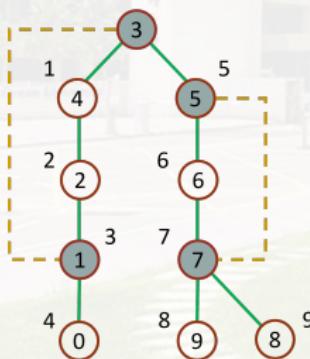
| vertex | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|--------|---|---|---|---|---|---|---|---|---|---|
| dfn    | 4 | 3 | 2 | 0 | 1 | 5 | 6 | 7 | 9 | 8 |
| low    | 4 | 0 | 0 | 0 | 0 | 5 | 5 | 5 | 9 | 8 |

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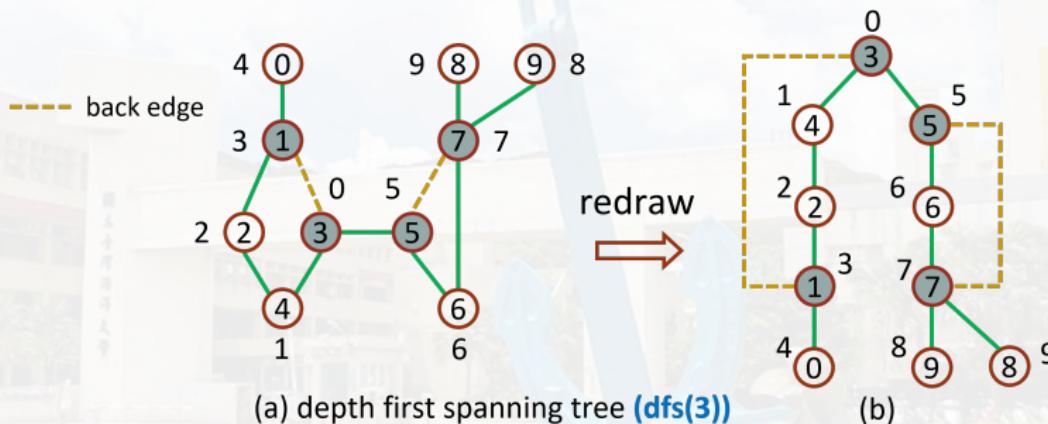
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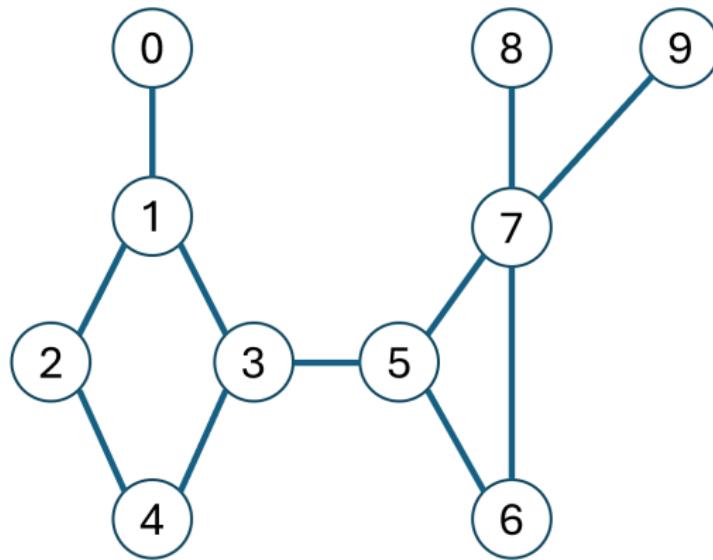


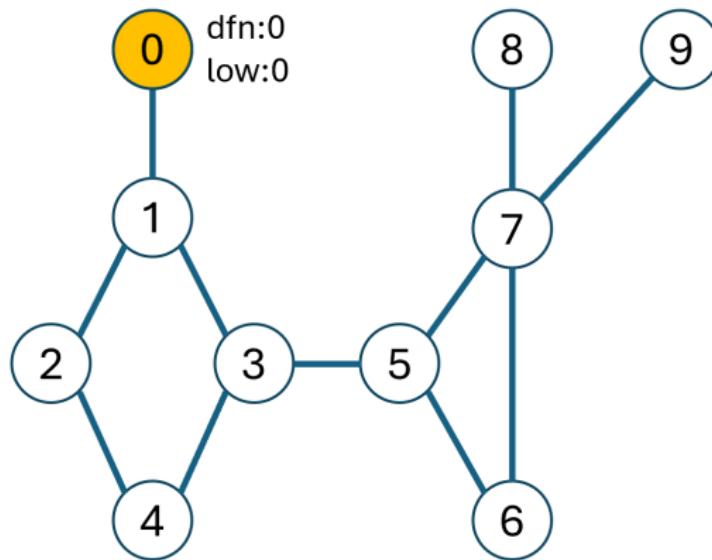
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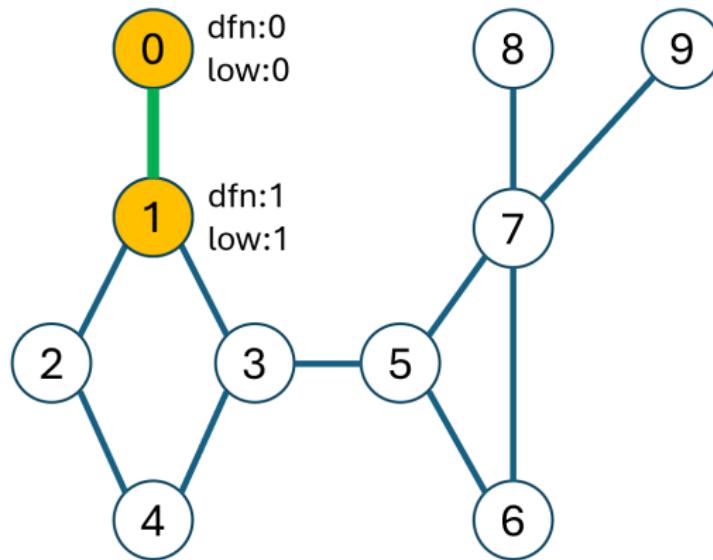
- articulation points: 1, 3, 5, 7.

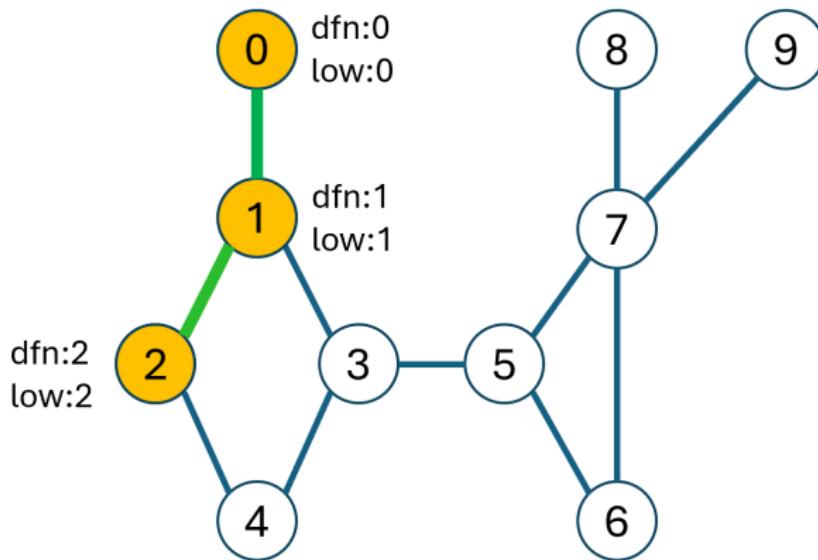
# Example of Computing dfn and low values

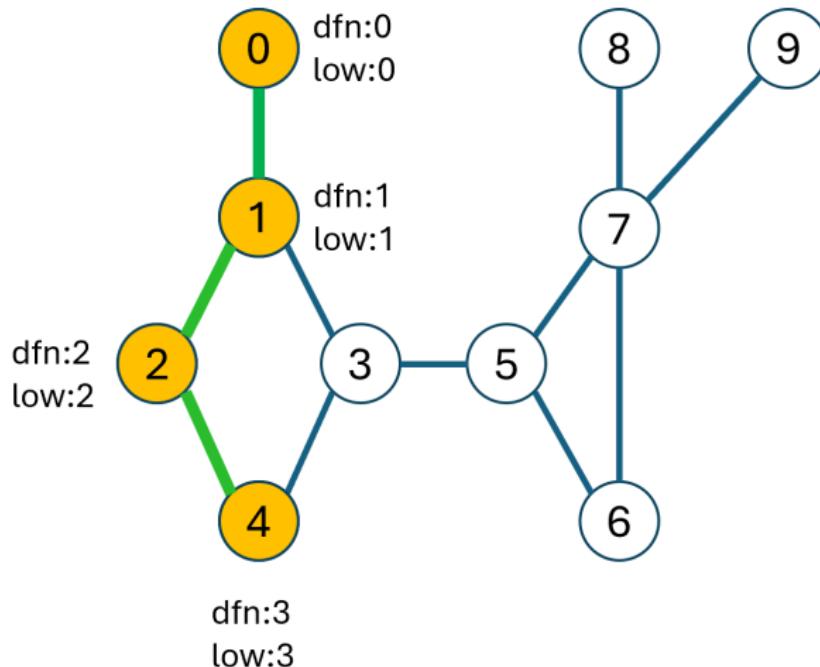


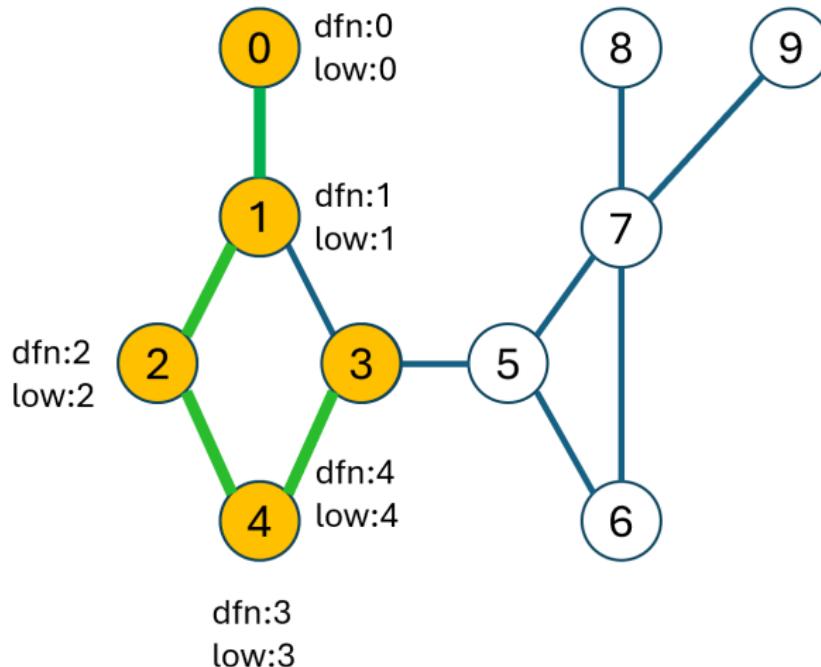


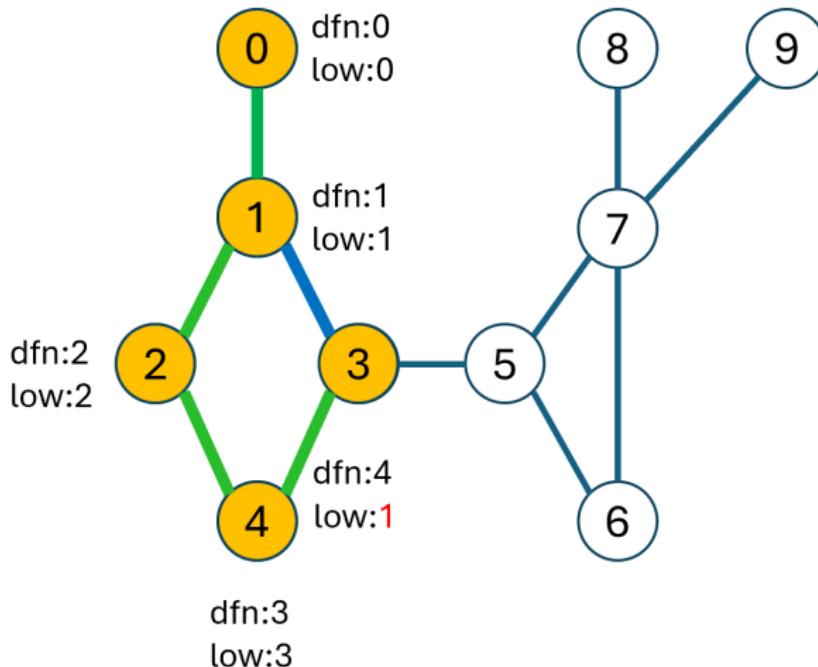


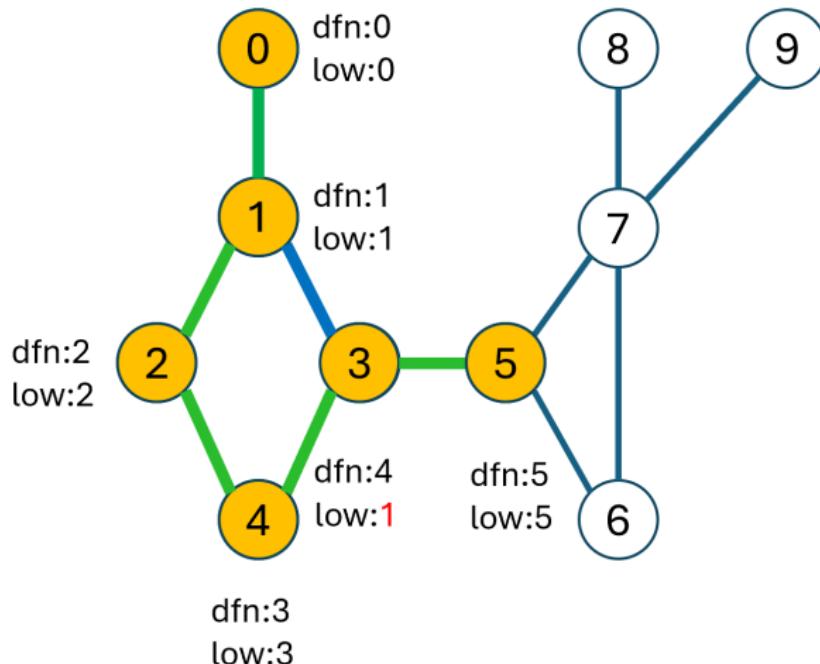


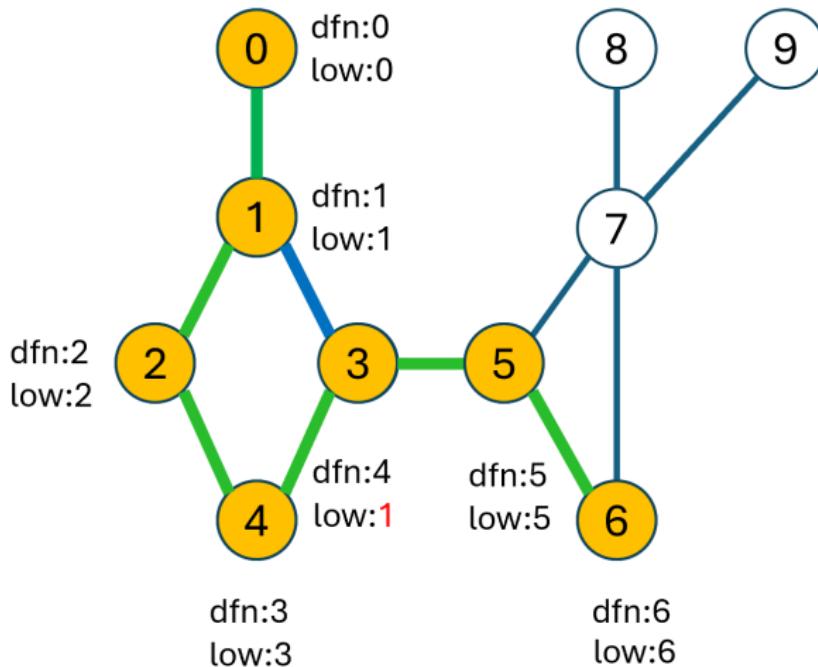


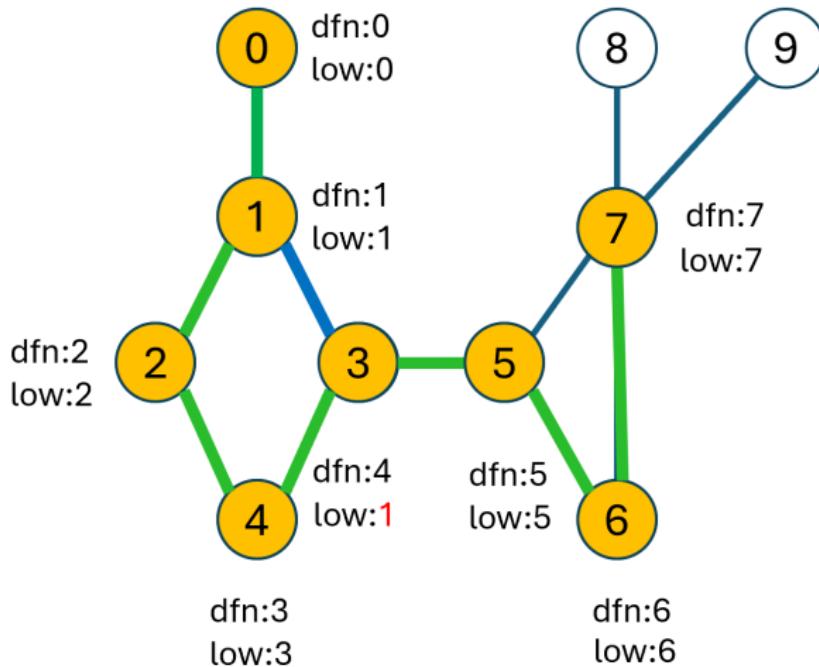


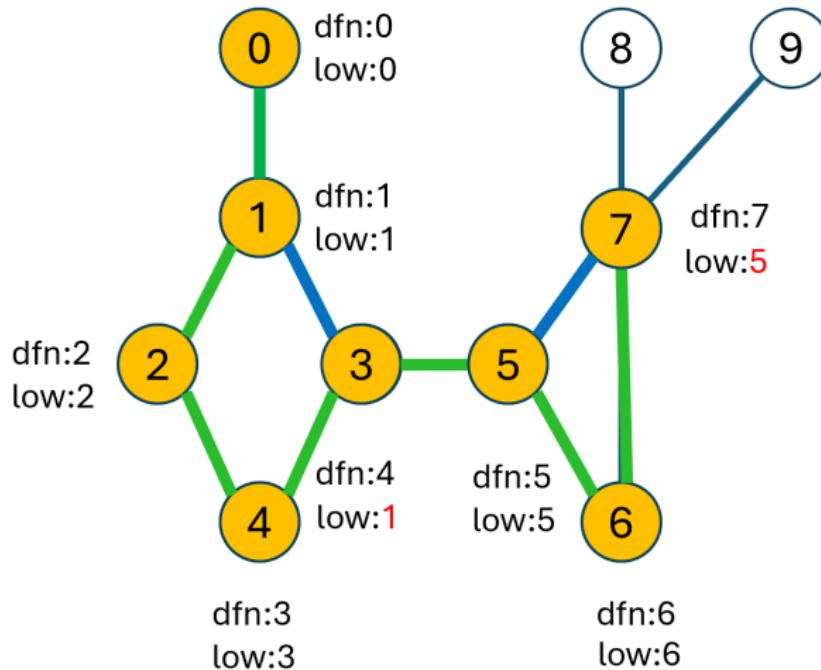


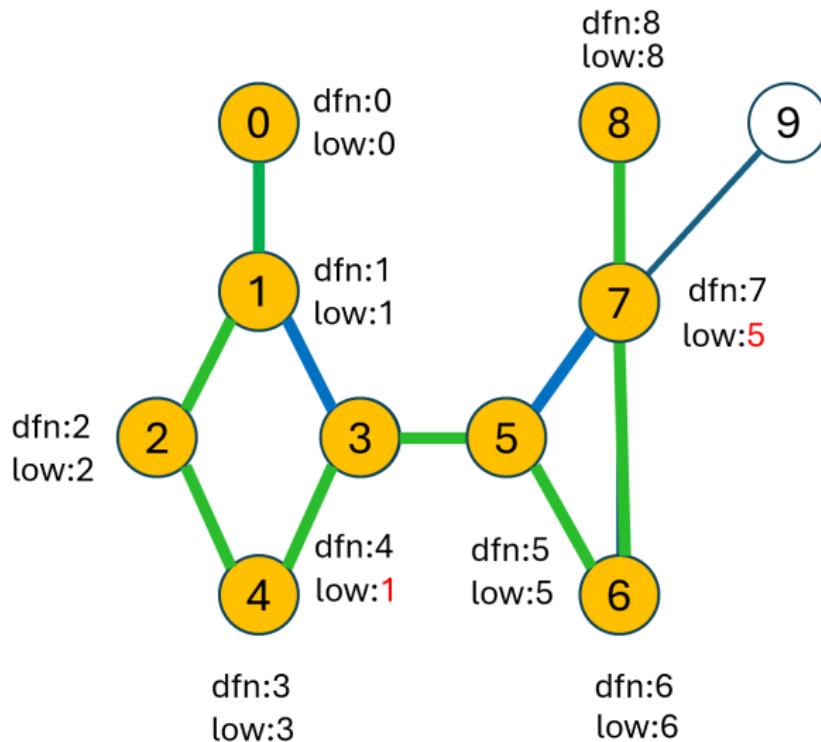


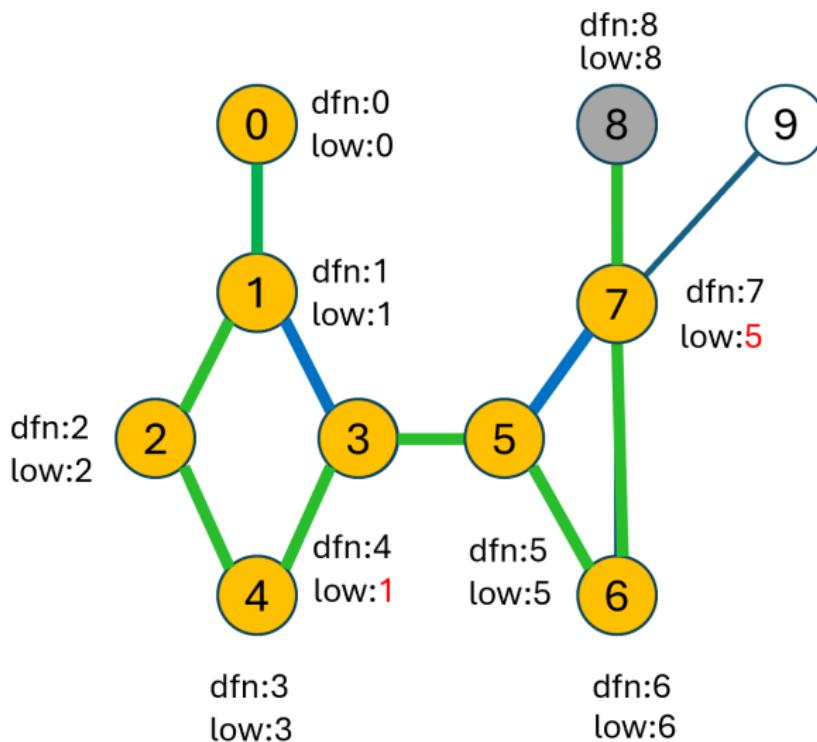


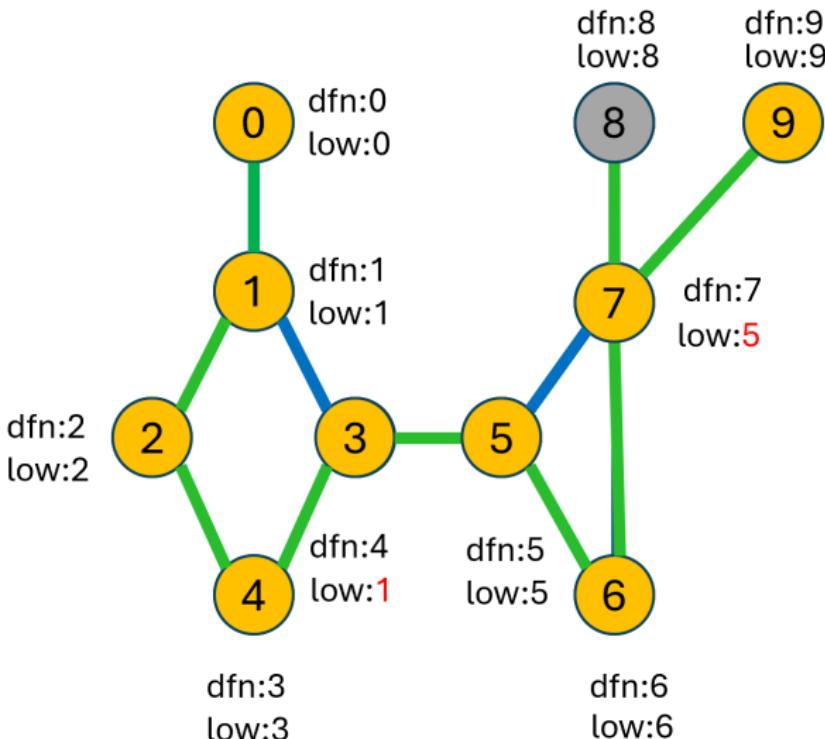


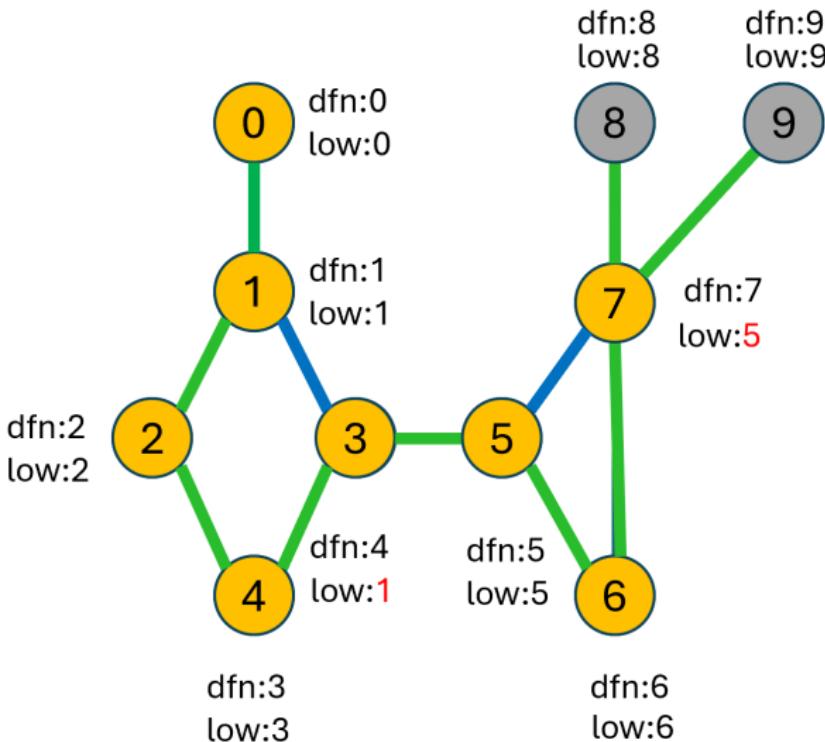


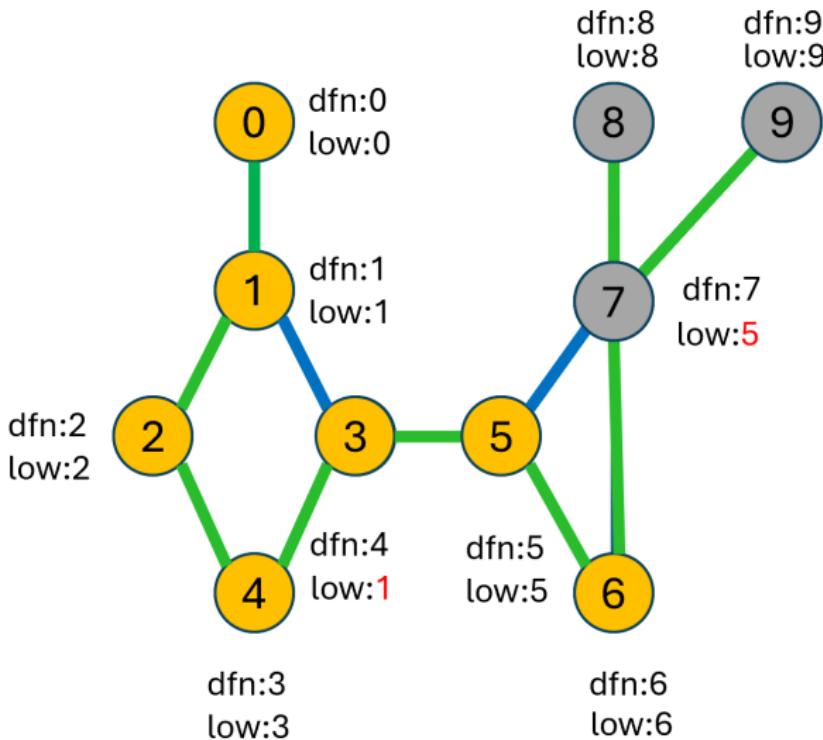


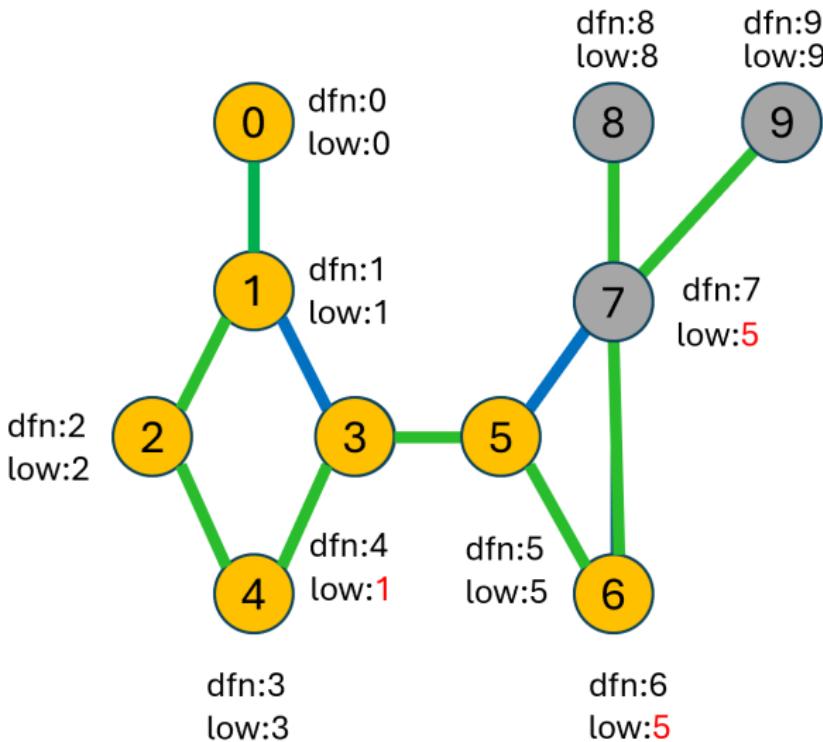


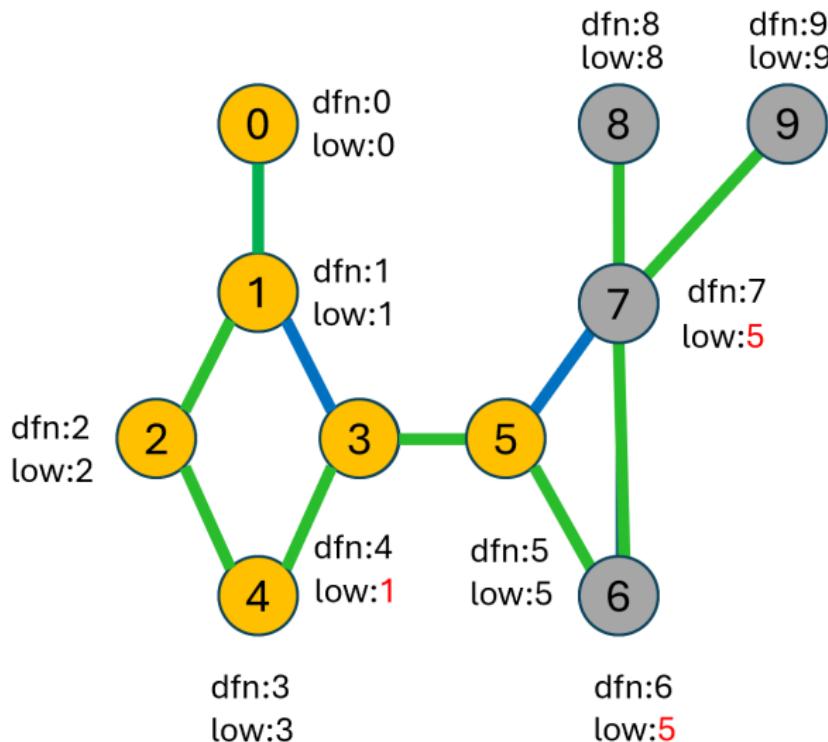


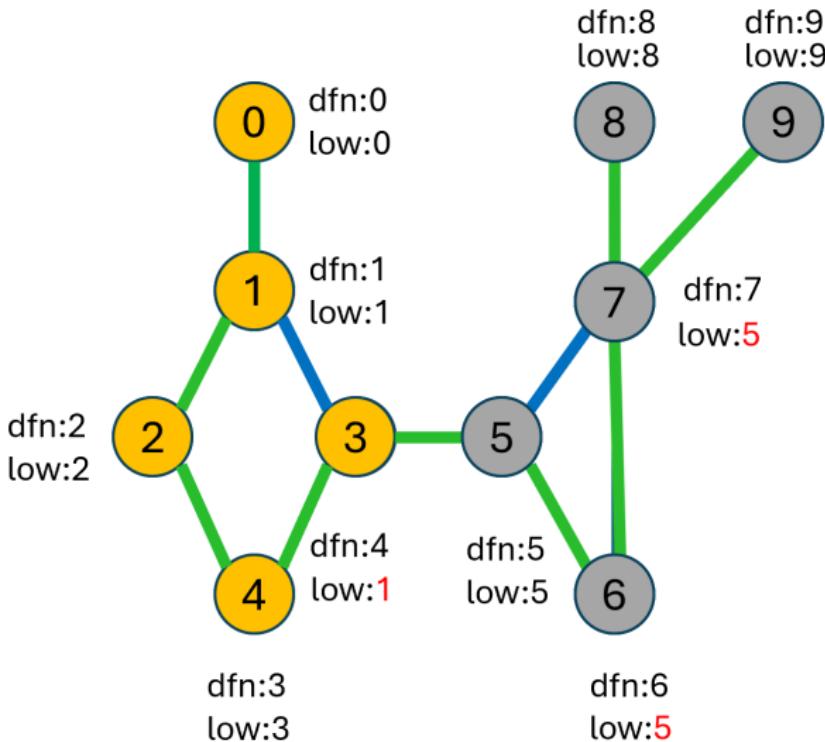


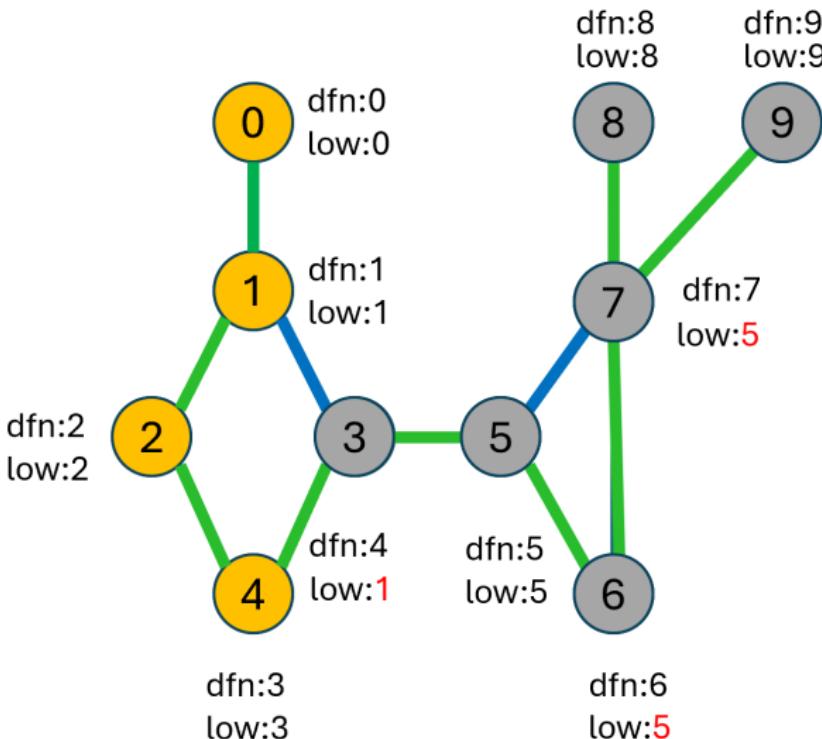


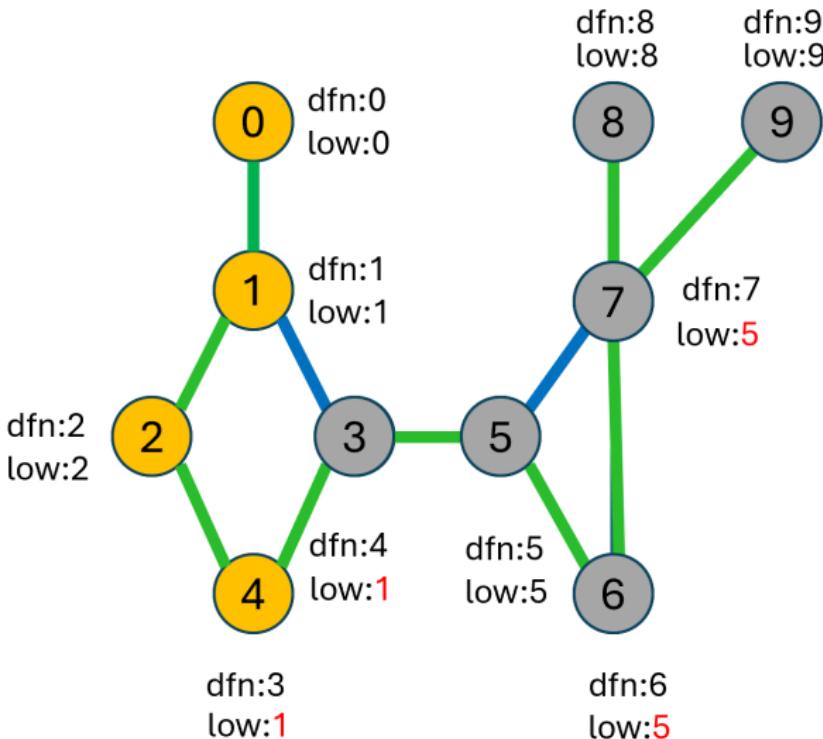


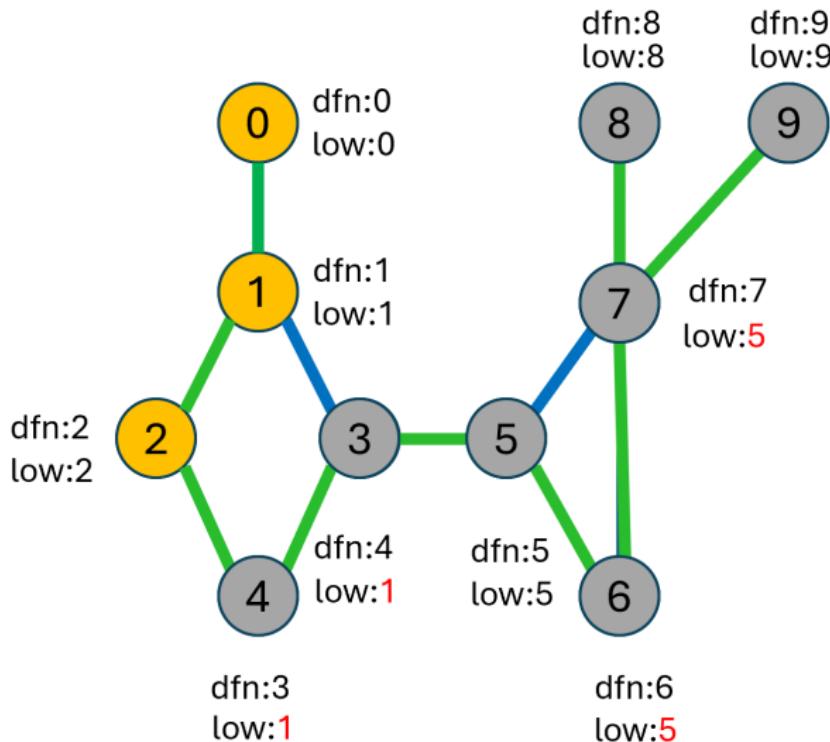


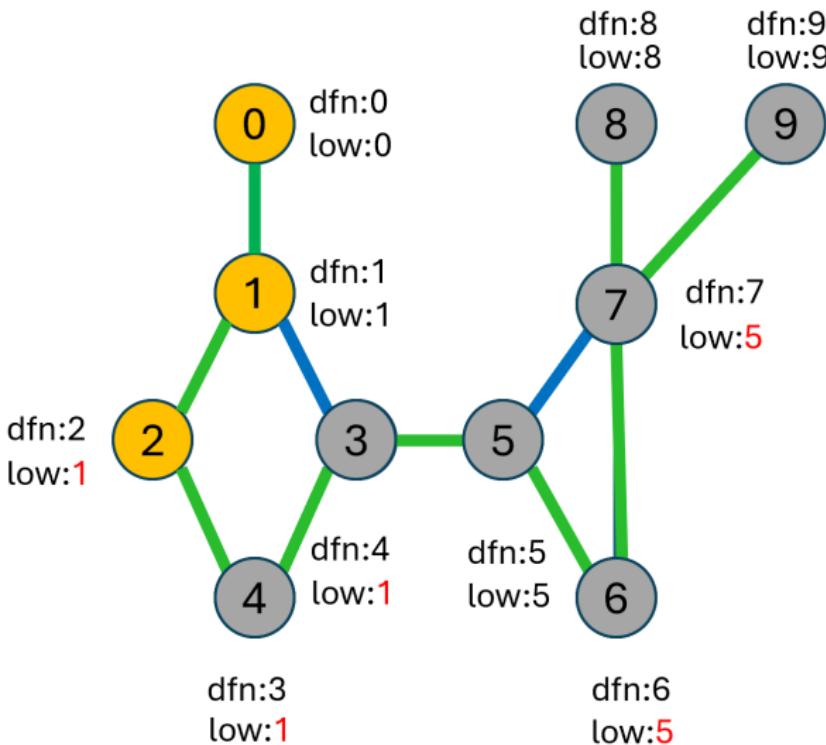


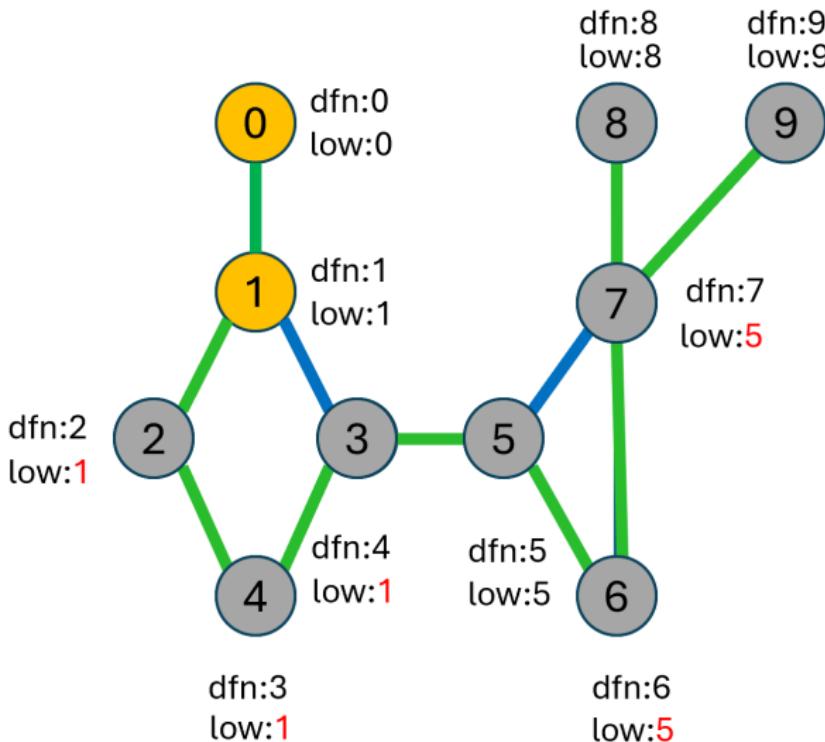


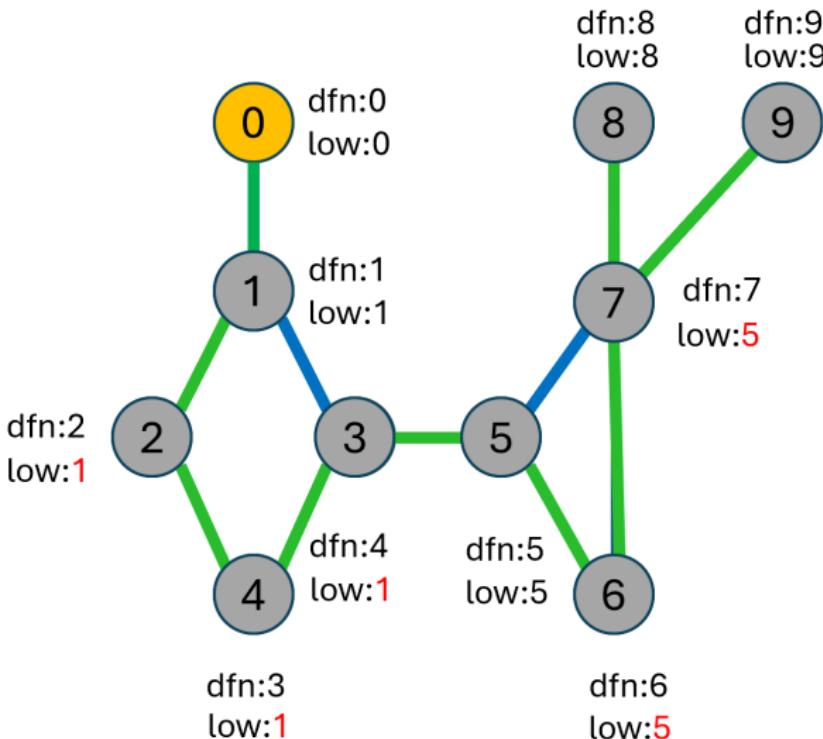


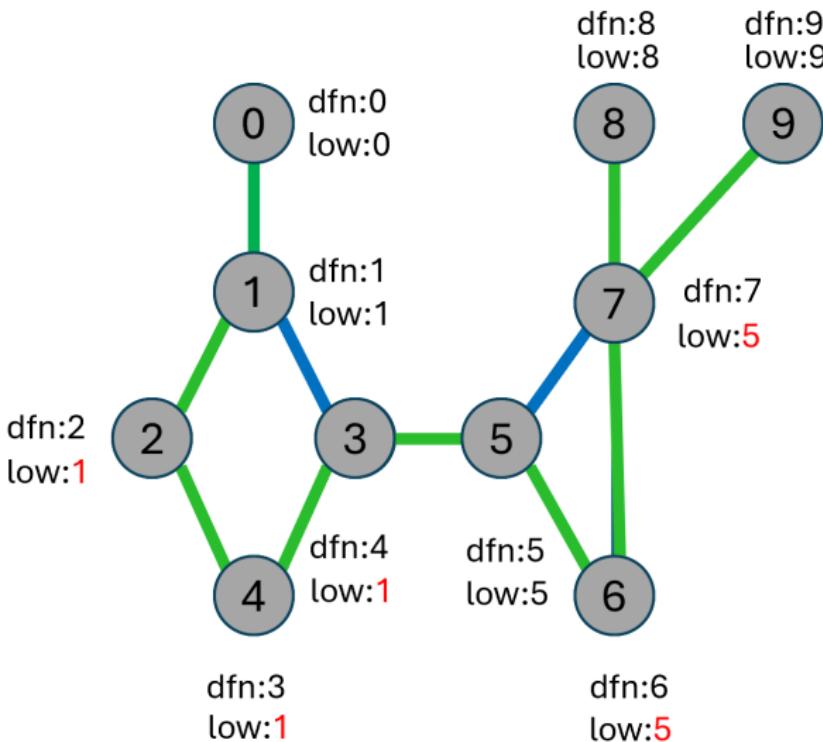












# The codes for computing dfn and low

Time complexity:  $O(e)$ .

```
void dfn_low(int u, int v) {
    /* compute dfn and low while performing a dfs
    search beginning at vertex u, v is the parent
    of u (if any) */
    node_pointer ptr;
    int w;
    dfn[u] = low[u] = num++;
    for (ptr = graph[u]; ptr; ptr = ptr->link) {
        w = ptr->vertex;
        if (dfn[w] < 0) {
            /* w is an unvisited vertex */
            dfn_low(w, u); // visit w from u
            low[u] = MIN2(low[u], low[w]); // (*)
        } else if (w != v)
            low[u] = MIN2(low[u], dfn[w]); // (**)
    }
}
```

```
short int dfn [MAX_VERTICES];
short int low[MAX_VERTICES];
int num = 0;

void init(void) {
    int i;
    for(i = 0; i < n; i++) {
        visited[i] = false;
        dfn[i] = low[i] = -1;
    }
    num = 0;
}
```

bootstrapping by

`dfn_low(x, -1)`



## Connected Components

Finding the articulation points

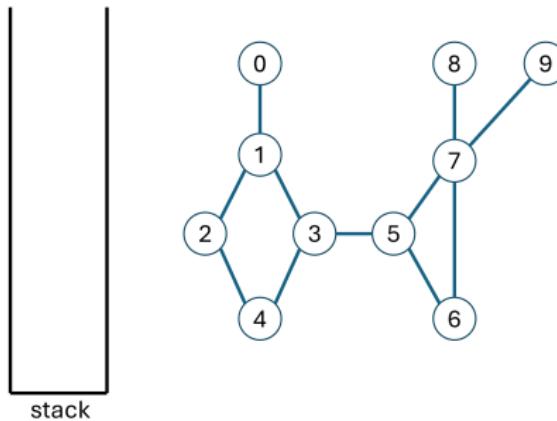
Code for Biconnected Components ( $O(n + e)$  time)

```

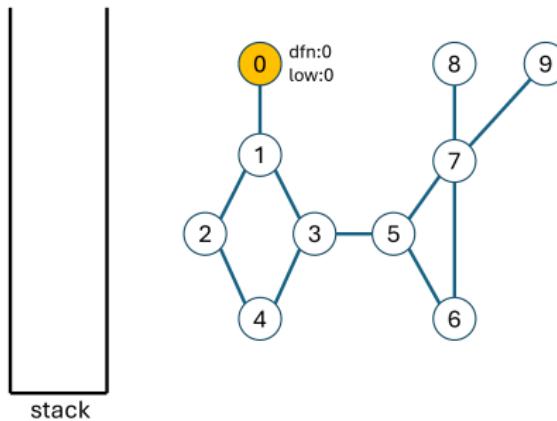
void bicon(int u, int v) { /* dfn[] = -1, num = 0, s is an empty stack initially*/
/* v is the parent node of u in the DFS spanning tree */
    nodePointer ptr;
    int w, x, y;
    dfn[u] = low[u] = num++;
    for (ptr = graph[u]; ptr; ptr = ptr->link) { // loop for u's neighbors
        w = ptr->vertex;
        if (v != w && dfn[w] < dfn[u]) {
            push(u,w); /* add edge (u,w) into stack s */
            if (dfn[w] < 0) { /* w is not visited yet */
                bicon(w, u);
                low[u] = MIN2(low[u],low[w]);
                if (low[w] >= dfn[u]) {
                    /* If we have low(w)>= dfn(v) whenever dfn_low(u,w) returns,
                     * then a new biconnected component is found! */
                    printf("New biconnected component:");
                    do { /* pop an edge from stack s */
                        pop(&x, &y);
                        printf("<%d,%d>", x, y);
                    } while (!(x == u) && (y == w));
                    printf("\n");
                }
            } else if (w != v)
                low[u] = MIN2(low[u],dfn[w]);
        }
    }
}

```

# Illustration

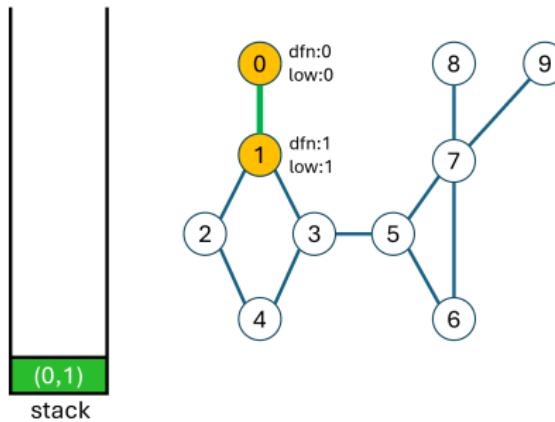


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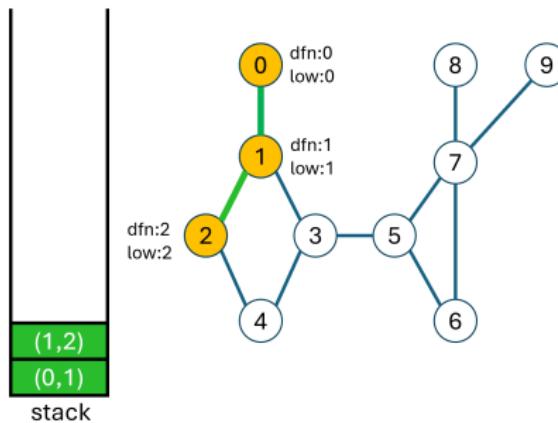


stack

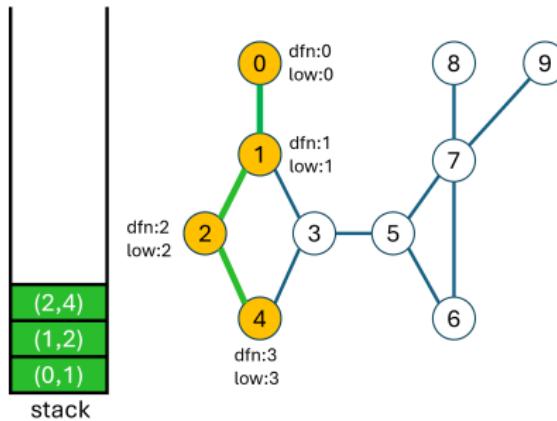
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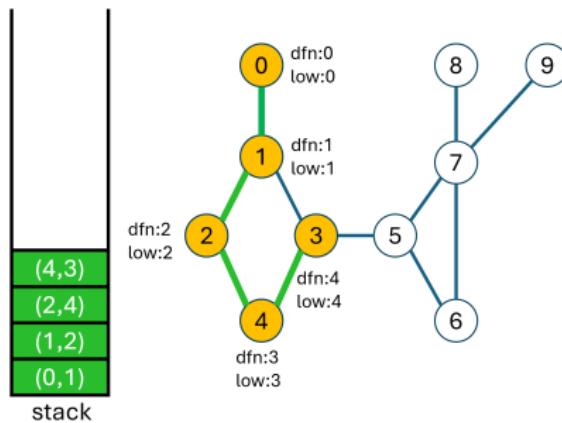
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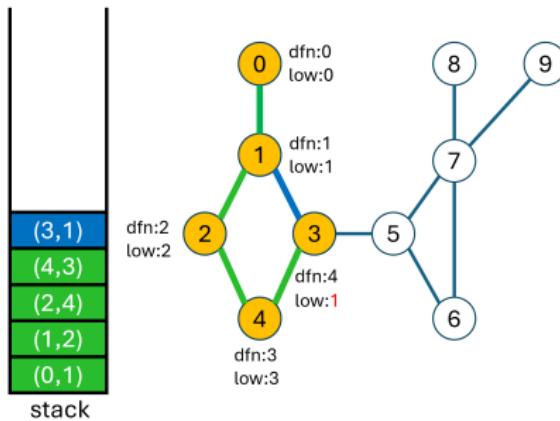
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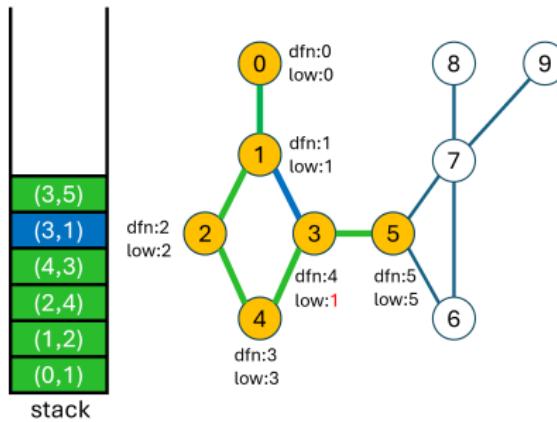
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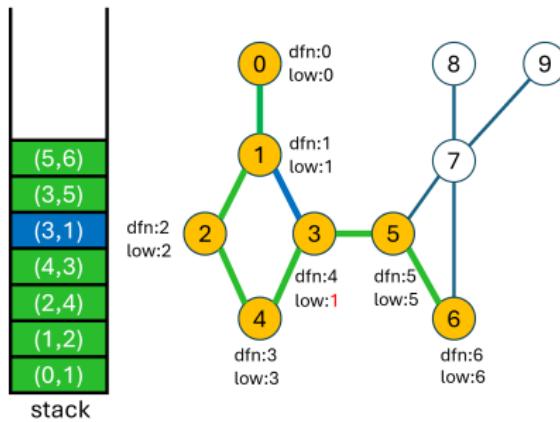
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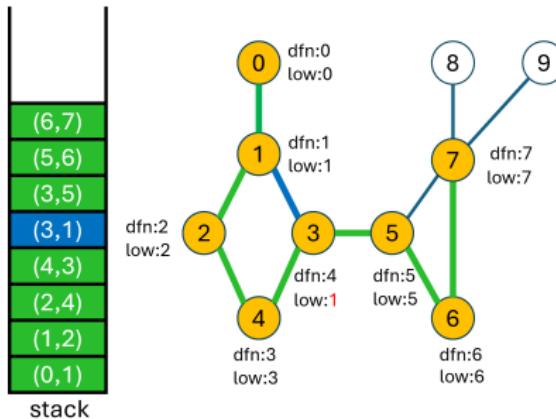
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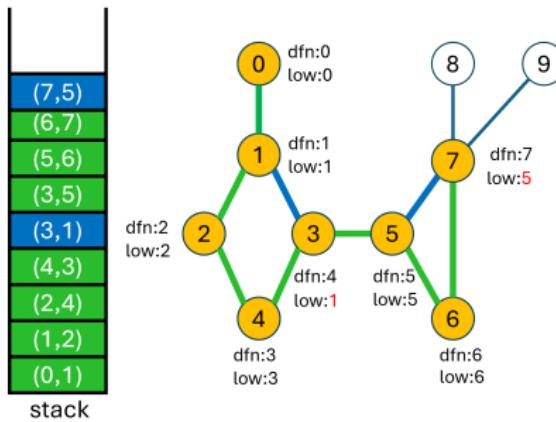
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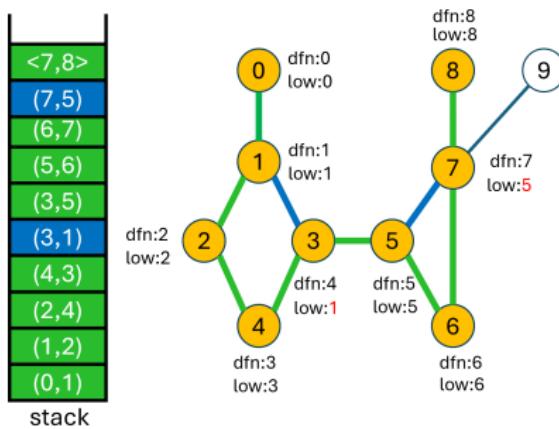
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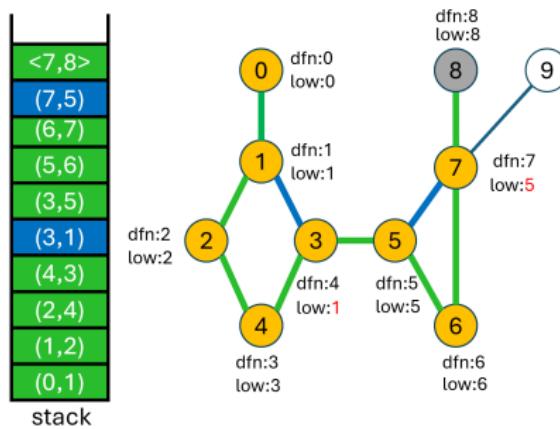
# Illustration



# Illustration



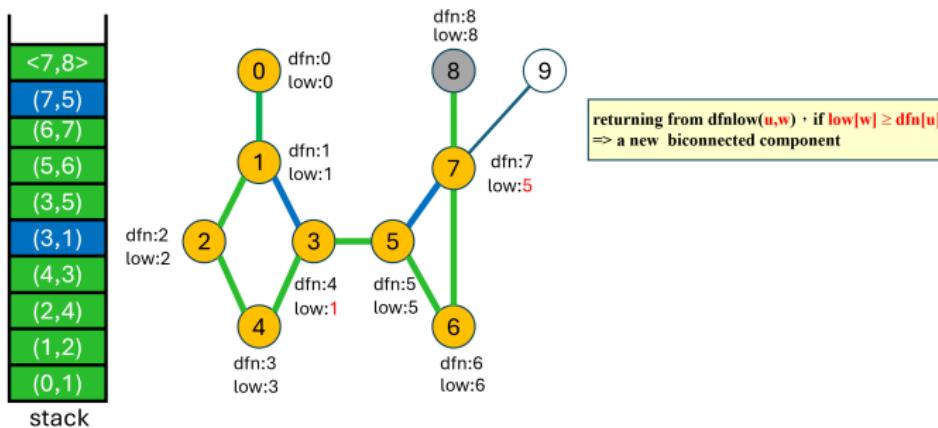
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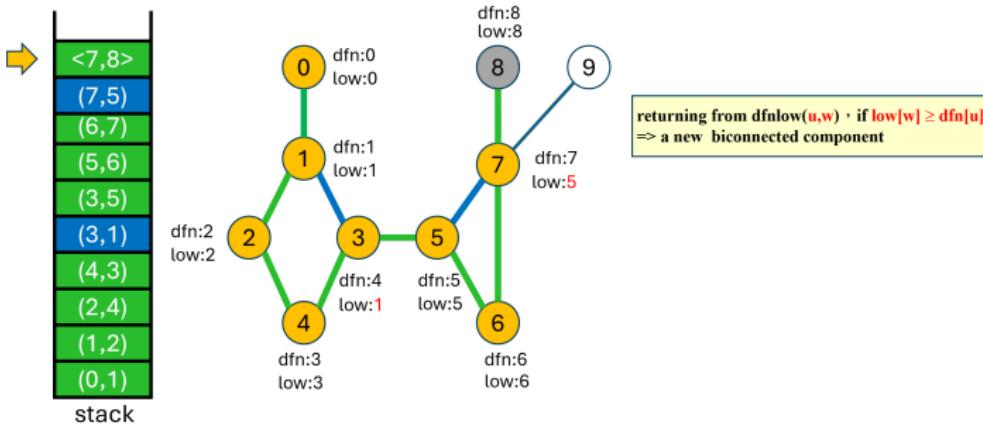
## Connected Components

Finding the articulation points

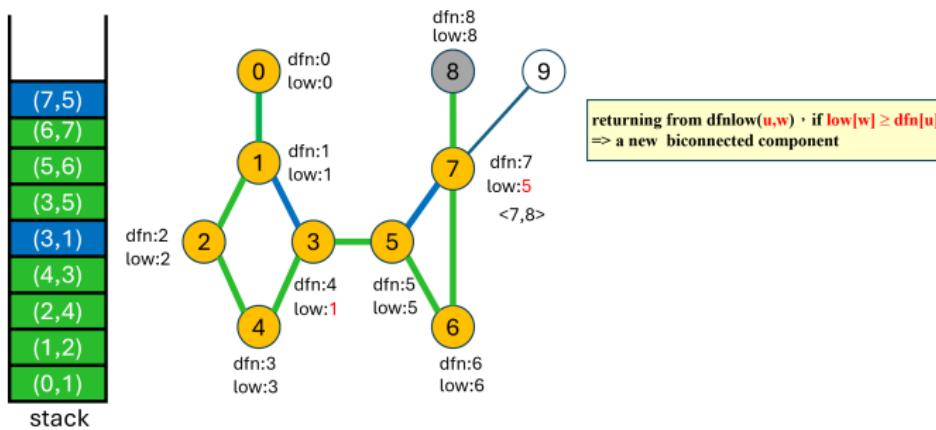
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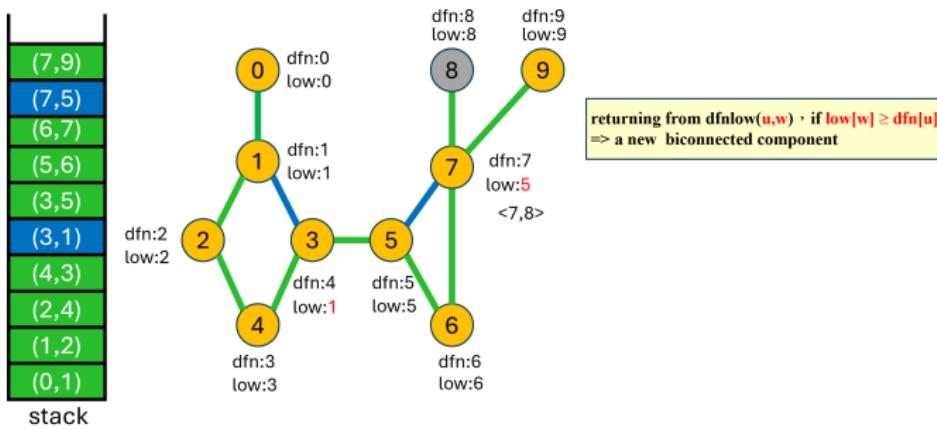
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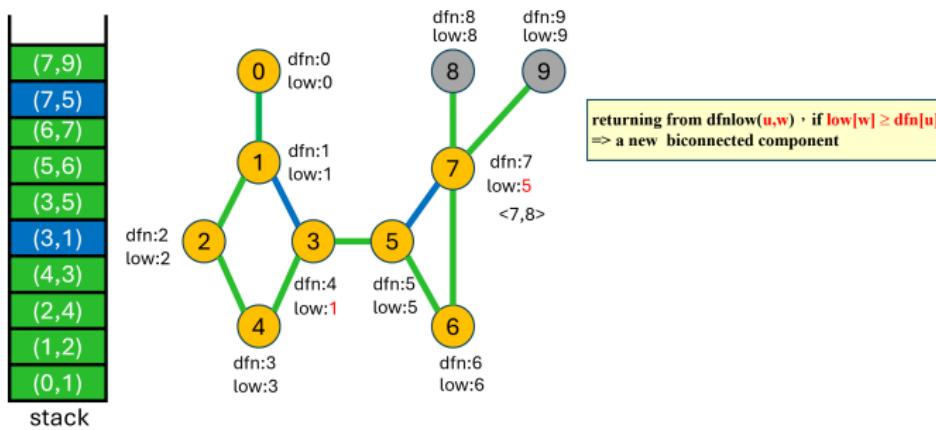
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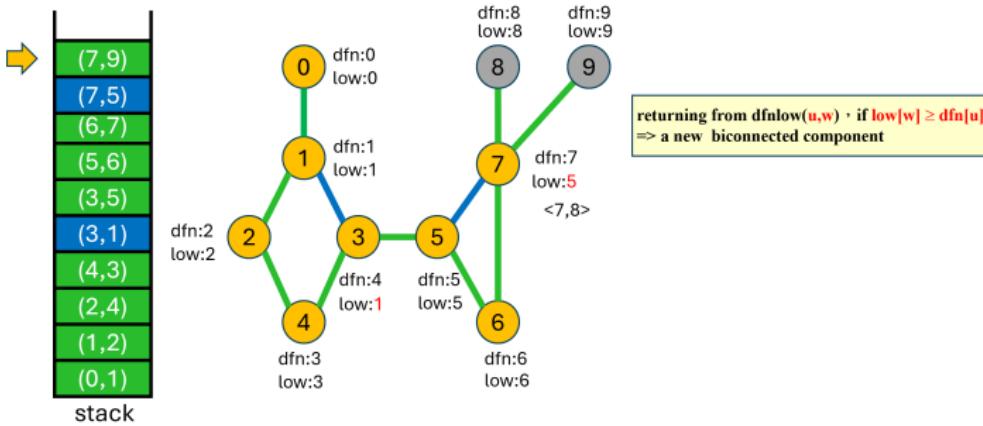
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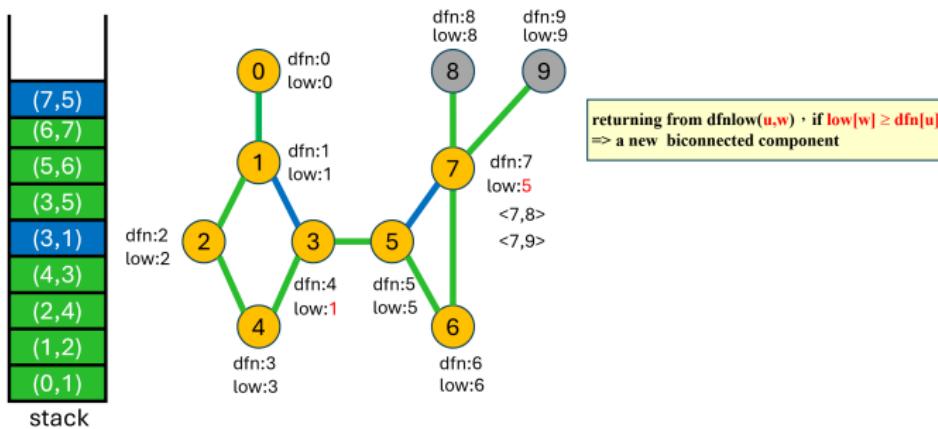
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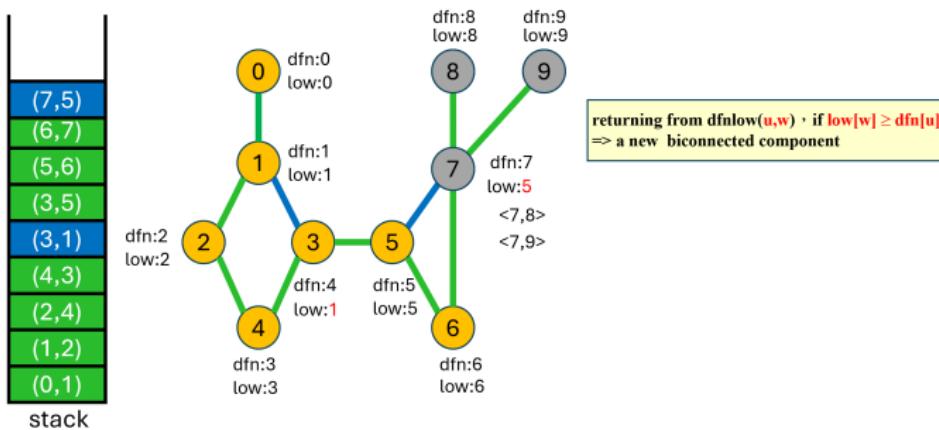
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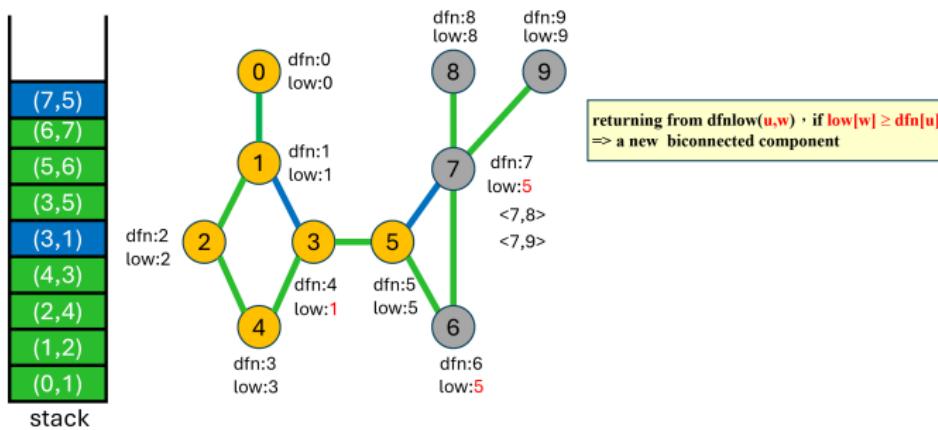
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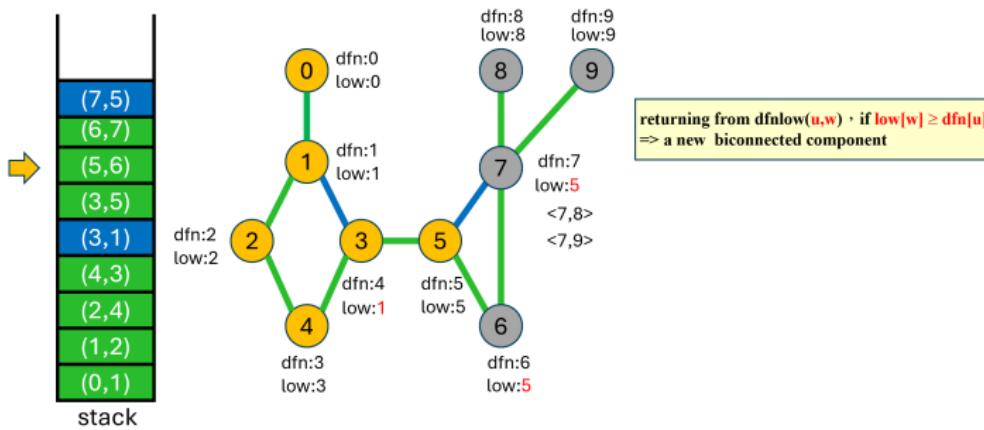
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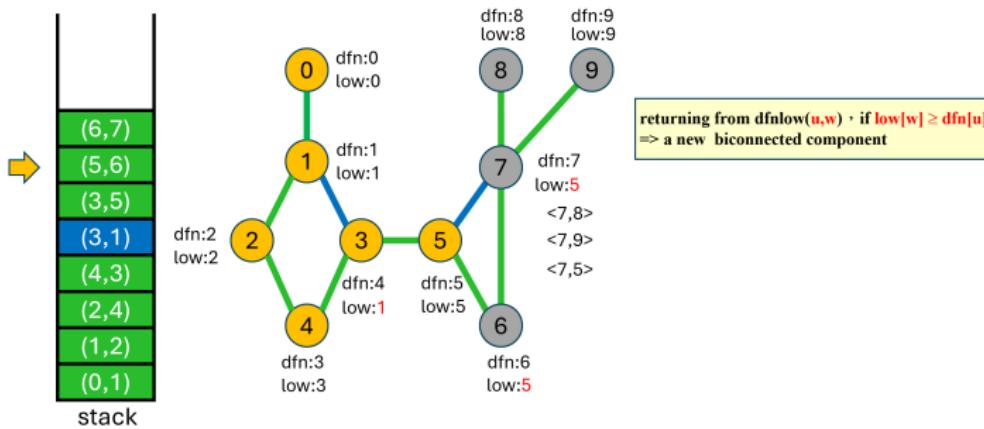
# Illustration



# Illustration



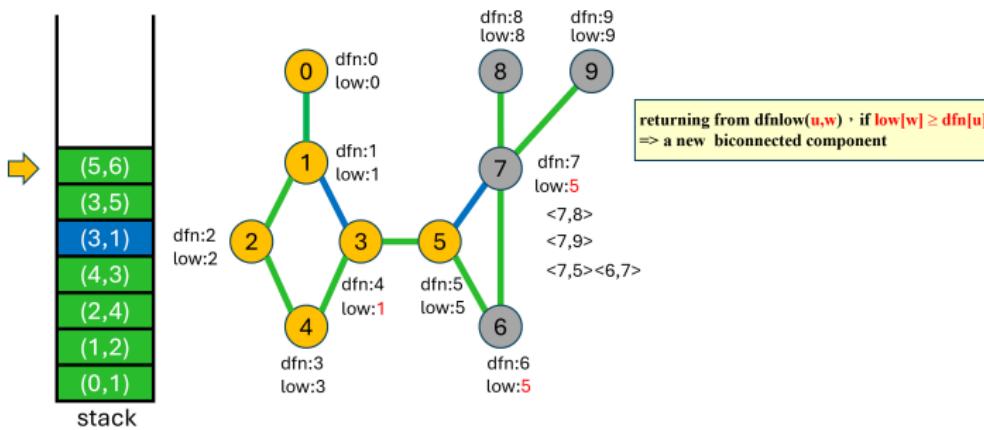
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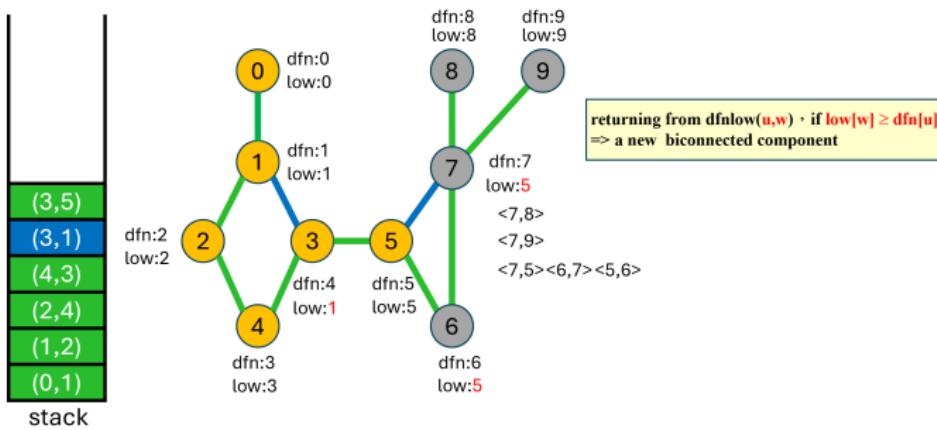
## Connected Components

Finding the articulation points

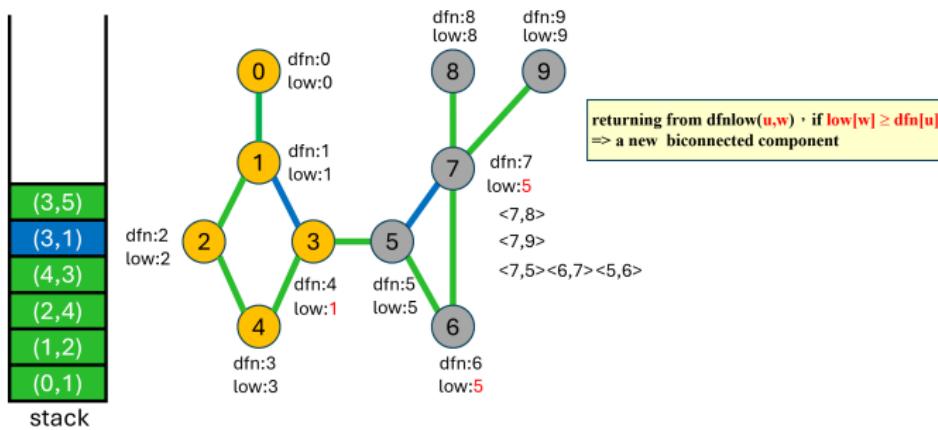
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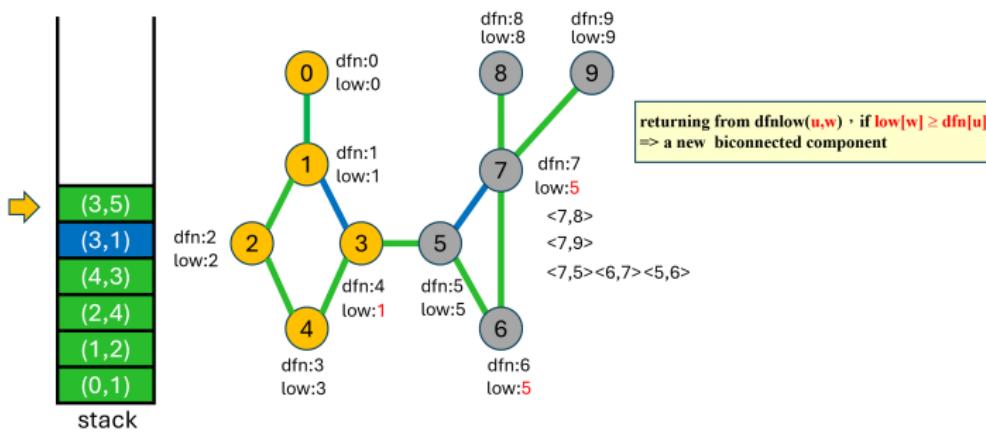
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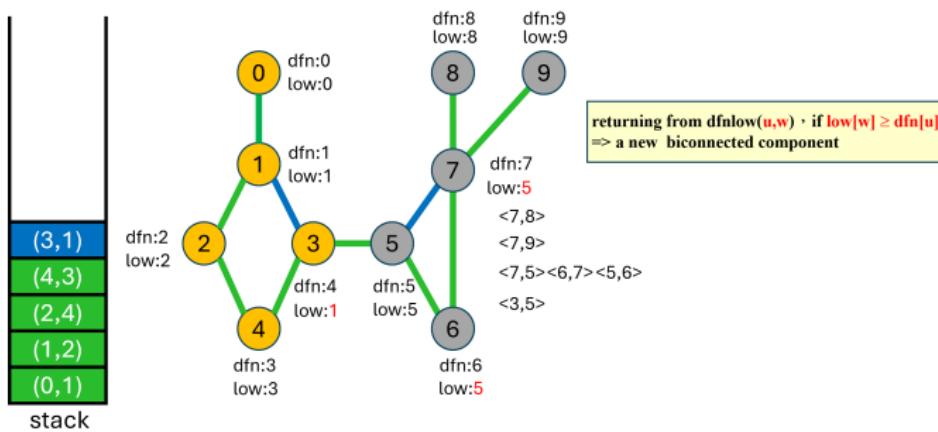
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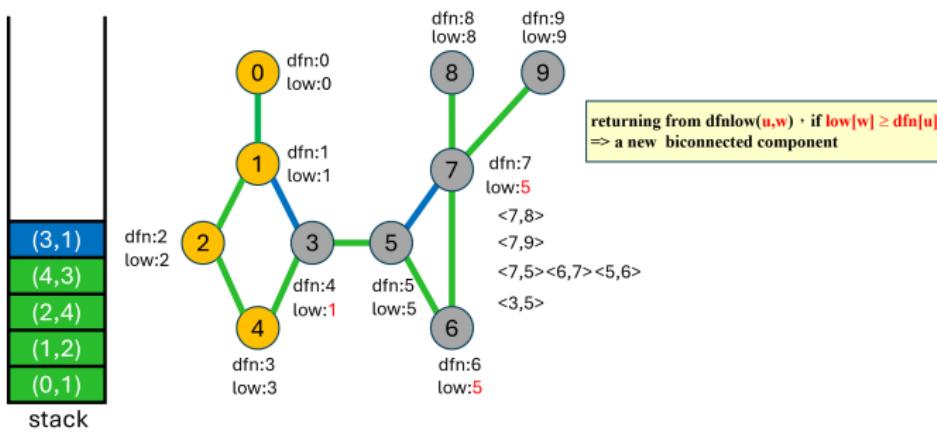
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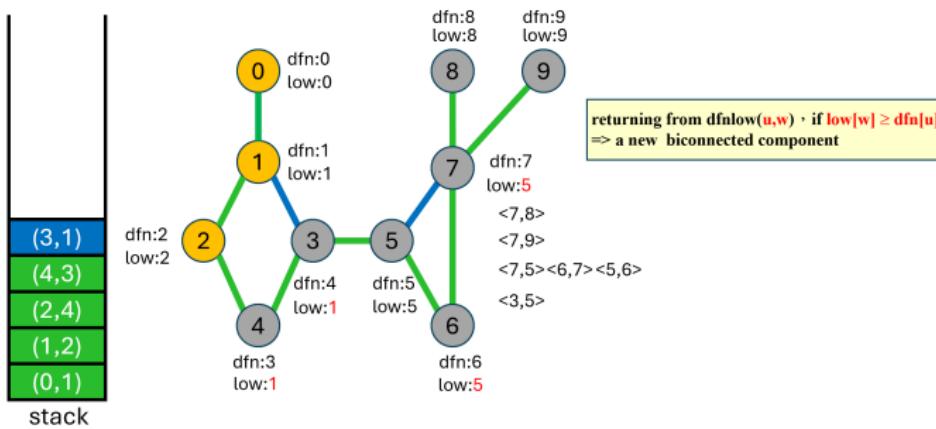
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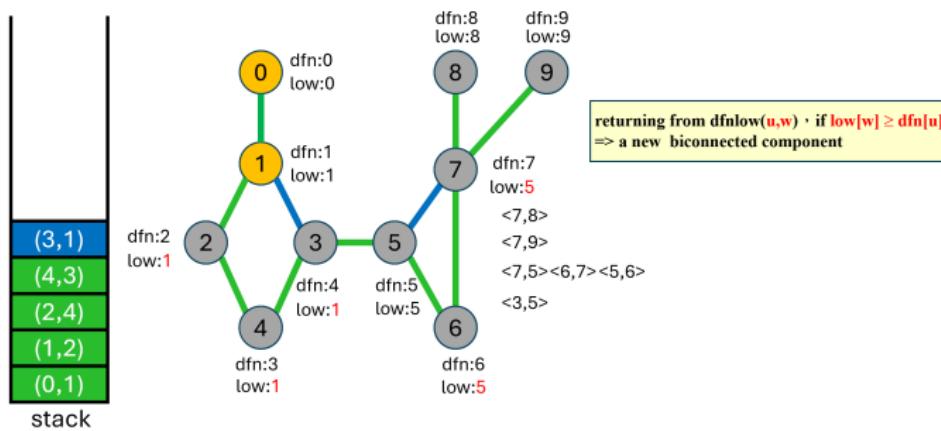
# Illustration



# Illustration



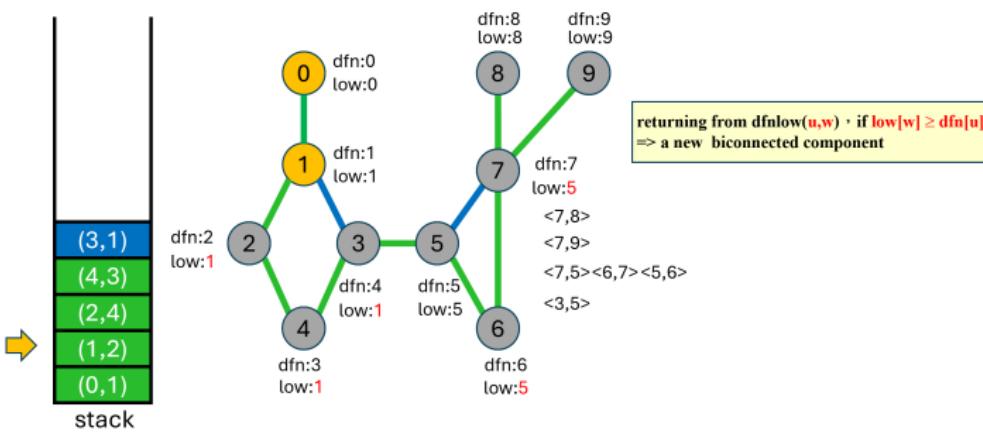
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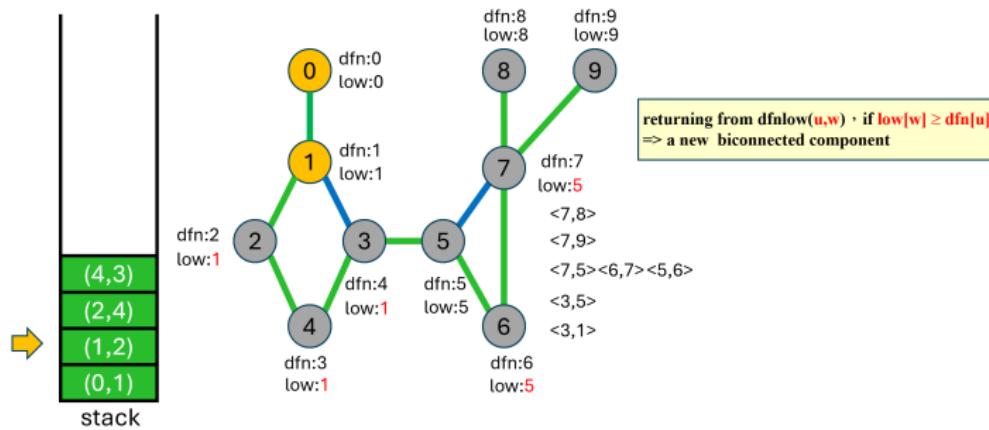
## Connected Components

Finding the articulation points

## Illustration



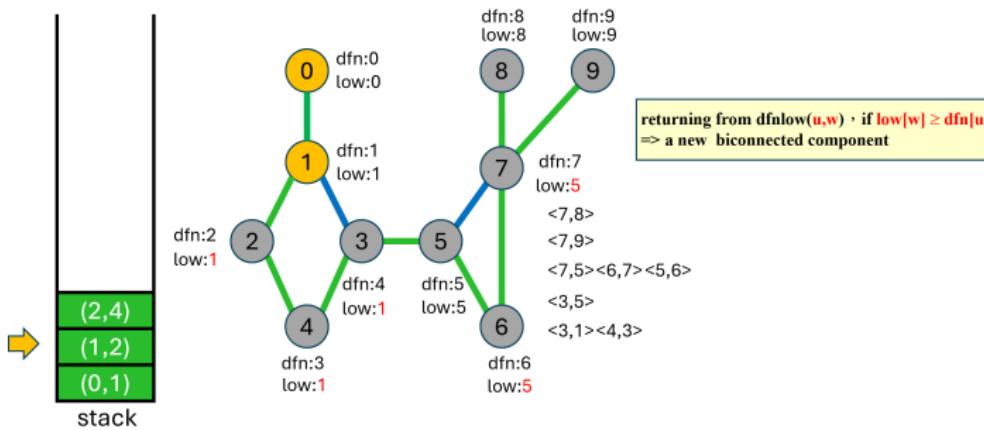
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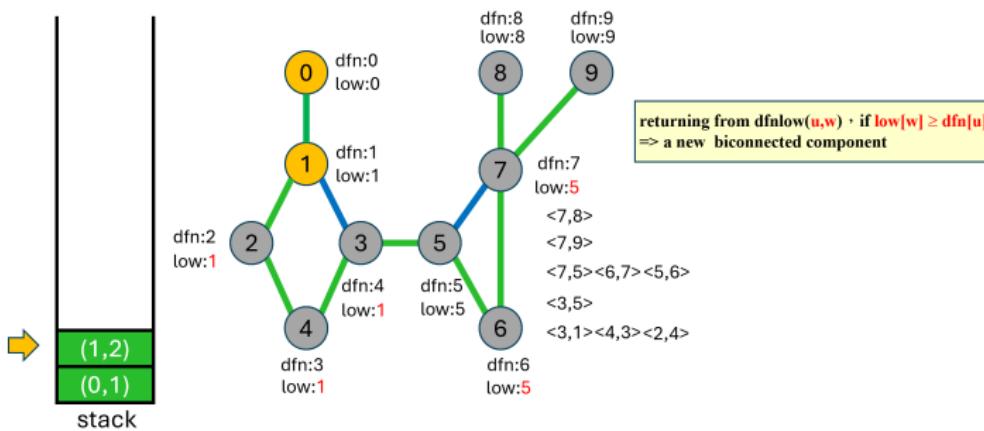
## Connected Components

Finding the articulation points

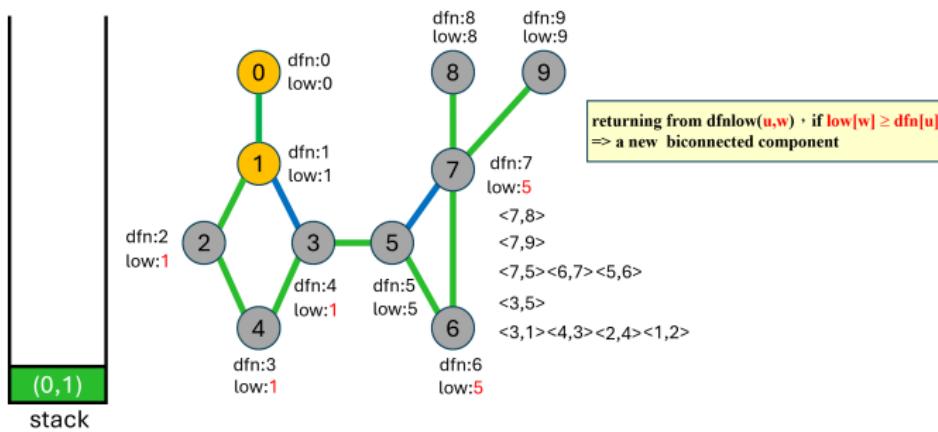
## Illustration



# Illustration



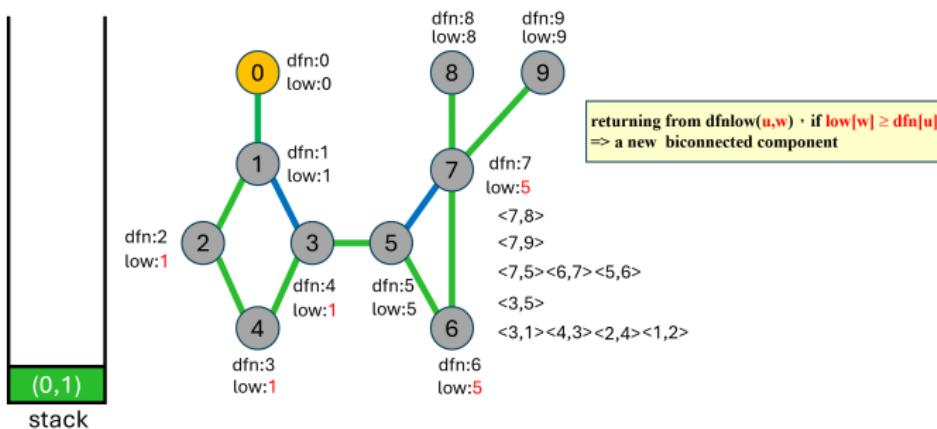
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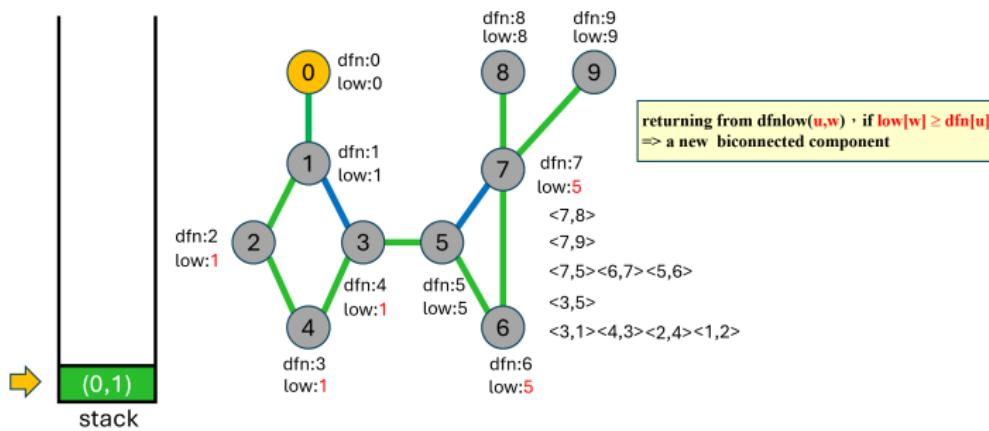
## Connected Components

Finding the articulation points

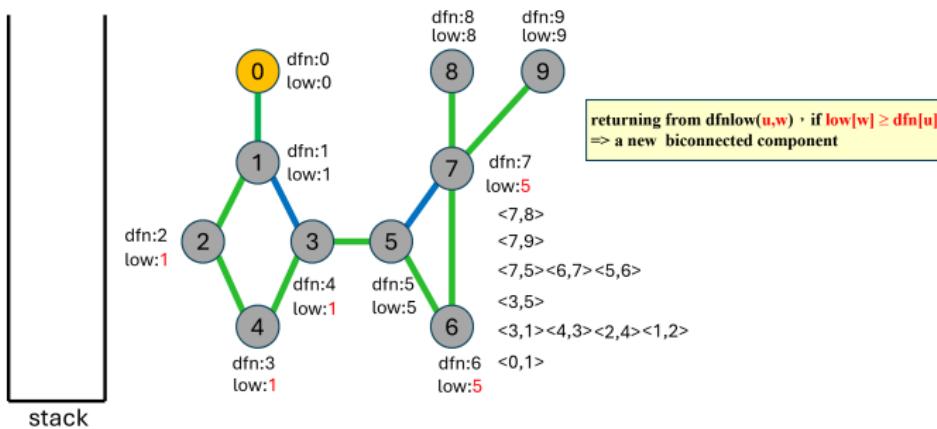
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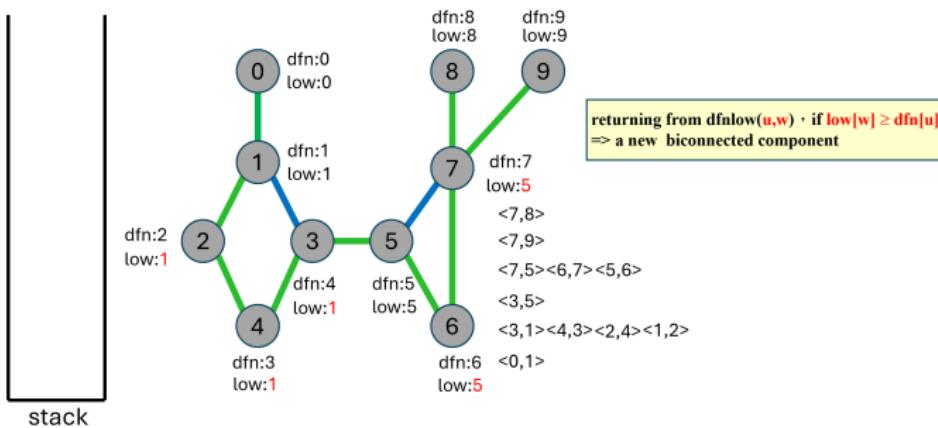
# Illustration



# Illustration



# Illustration



# Discussions

