Chaos Project: The Newton-Raphson Method & Newton-Raphson Fractals

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1 Introduction

This project explores the Newton-Raphson iterative root-finding method, particularly some of its chaotic behaviour in the complex plane. Fractal patterns emerge when examining which starting points land on which roots. The project was inspired by 3Blue1Brown's video on the subject, *Newton's fractal (which Newton knew nothing about)*.

All MATLAB work and *Desmos* interactive figures were created from scratch, without the use of generative AI tools.

2 What is the Newton-Raphson Method?

For some differentiable function $f(x)$, the Newton-Raphson Method (often referred to as Newton's method, though I prefer to give Joseph Raphson his deserved credit for simplifying Isaac Newton's version years later into the iterative form used today) is a process used to find the solutions to $f(x) = 0$, particularly when closed-form solutions are difficult or even impossible.

The method takes some initial value x_0 (usually a simple guess at where the desired root of $f(x)$ may lie, based on graphs or other information), and calculates the next iterate using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

where $f'(x_n)$ denotes f's first derivative. As this process is repeated, the values should converge to one of the function's roots.

Given that this can be derived from the linear Taylor approximation of $f(x)$, the process can be visualised on the x-$f(x)$ plane using tangent lines at every iterate x_n. The tangent line to the graph of $f(x)$ at some initial root guess x_0 will cross the x-axis at a new value, which is then used as the next value from which to extend a tangent line. These steps would be repeated until a root is reached.

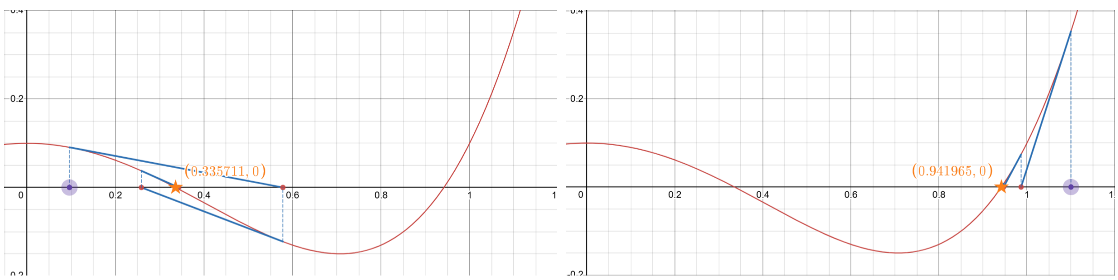

Figure 1: Examples of the Newton-Raphson method being used at different initial values to find the roots of $f(x) = x^4 - x^2 + 0.1$. Created in *Desmos*.

It's not always obvious which root the method will land on - looking at the processes in action in Figure 1, an iterate x_n that satisfies $f'(x_n) \approx 0$ may send x_{n+1} far away from the expected root, alluding to chaotic behaviour.

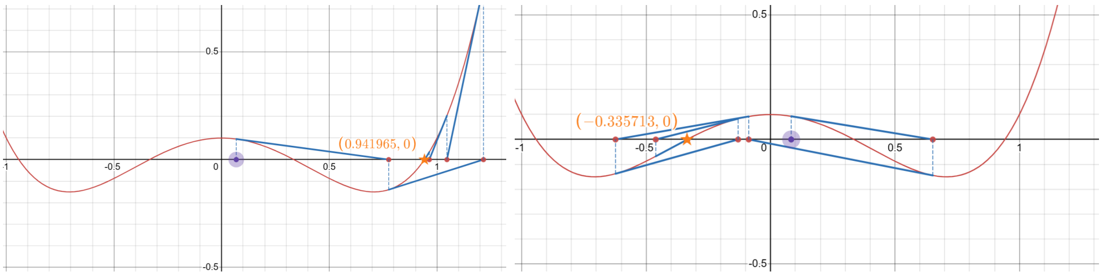

Figure 2: The Newton-Raphson method's sensitivity to initial values if the size of $f'(x_0)$ is small. Created in *Desmos*.

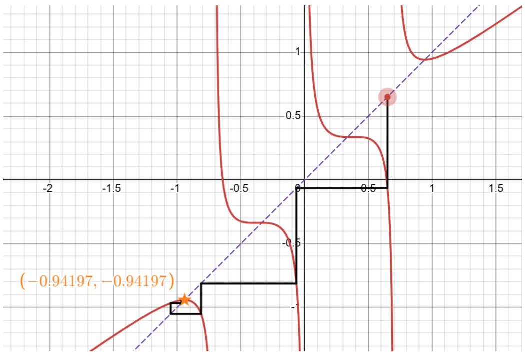

Figure 3: The Newton-Raphson method may also be represented as a cobweb diagram, with the function in red defined as $x - \frac{f(x)}{f'(x)}$, pictured here with the same $f(x)$ as defined in Figure 1. Created in *Desmos*.

Note how Figure 1.1 and Figure 2 all have roughly the same starting point, yet each converge to unique roots.

3 The Newton-Raphson Method for Complex Roots

For real number initial guesses for the roots of real functions, the Newton-Raphson method's iterates are completely confined to the real numbers. However, some function $f(x)$ may also have complex values that satisfy $f(x) = 0$. Because the graph of this function won't cross the x-axis, the Newton-Raphson method may tend towards one of the function's local minima instead, which in turn 'blows up' the next iterate and sends it far from previous iterations.

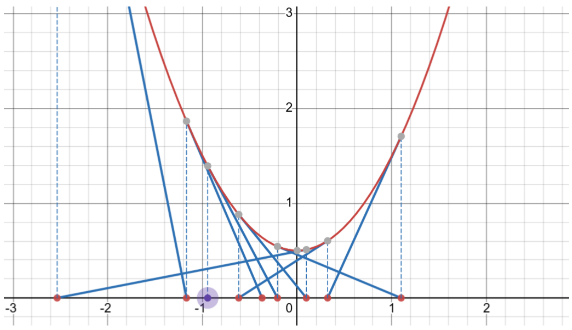

Figure 4: The Newton-Raphson method being used on $f(x) = x^2 + 0.5$ with a real starting point, failing to converge to a real value. Created in *Desmos*.

If, instead of focusing on the reals and dealing in graphs and tangents, a complex number is chosen as the initial guess and the Newton-Raphson method is considered as an iterative complex number map, these previously hidden complex roots may now be reached. The process can instead be thought of as playing a game of Marco Polo on the complex plane, with the stepping forward of each new iteration being guided by f and its derivative.

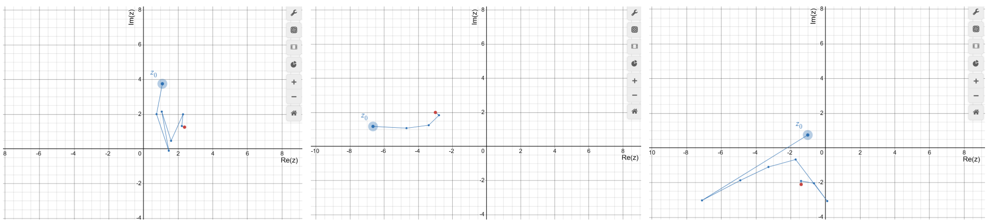

Figure 5: Examples of the Newton-Raphson method converging on a function's various complex roots. Created in *Desmos*.

In the third example shown in Figure 5, the same type of small-derivative leap that had been seen before in the purely real case is also present here.

4 Generating Newton-Raphson Fractals

When looking at the entire complex plane, which starting points will eventually end up at which roots for some given function?

I've written a MATLAB script that takes a function $f(z)$, and creates a grid of points at which to begin the Newton-Raphson method. Once all points are sufficiently close to the roots of $f(z)$, the original grid of points is coloured according to which root was landed on. Finer and finer grids that replicate the pixels of an image reveal beautiful fractal patterns full of self-similarity.

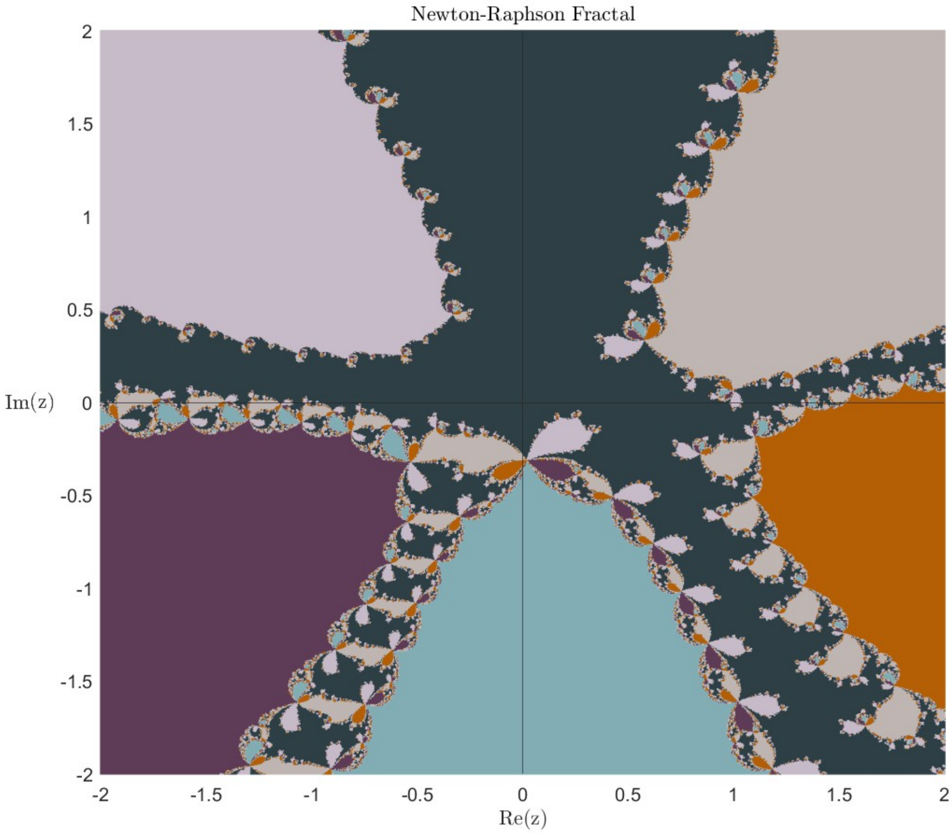

Figure 6: Newton-Raphson fractal for the function $f(z) = 1 + ix + x^2 + 0.7ix^3 - x^4 - 2x^5 + 2x^6$.

As the function $f(z)$ in Figure 6 is a sixth-degree polynomial, there exist six solutions to $f(z) = 0$, which all lie near the focal points of the parabolic shapes that appear in the fractal. With repeated Newton-Raphson iteration, any point on the complex plane that is coloured (for example) pink will eventually converge on the root associated with the pink region. The largest swathes of colour correspond to points on the plane that converge to their respective roots unbothered, away from the crests and troughs associated with polynomial graphs which we've seen have the ability to induce chaos into the process.

Amazingly, the points where each colour region meets (portrayed more clearly in the middle of Figure 7) imply that there are places the Newton-Raphson method could be started from, and an arbitrarily small perturbation applied to this starting point would result in the method zoning in on any of the other roots.

5 Links to *Desmos* Figures

- Newton Raphson Method for Real Numbers -https://www.desmos.com/calculator/ztasikgc7f
- Newton-Raphson Cobweb Diagram - https://www.desmos.com/calculator/1jthlsk0iy
- Complex Plane Newton-Raphson Iteration - https://www.desmos.com/calculator/hos8iltvsz
- Complex Plane Newton-Raphson Iteration with Fractal - https://www.desmos.com/calculator/oyh19qsugs

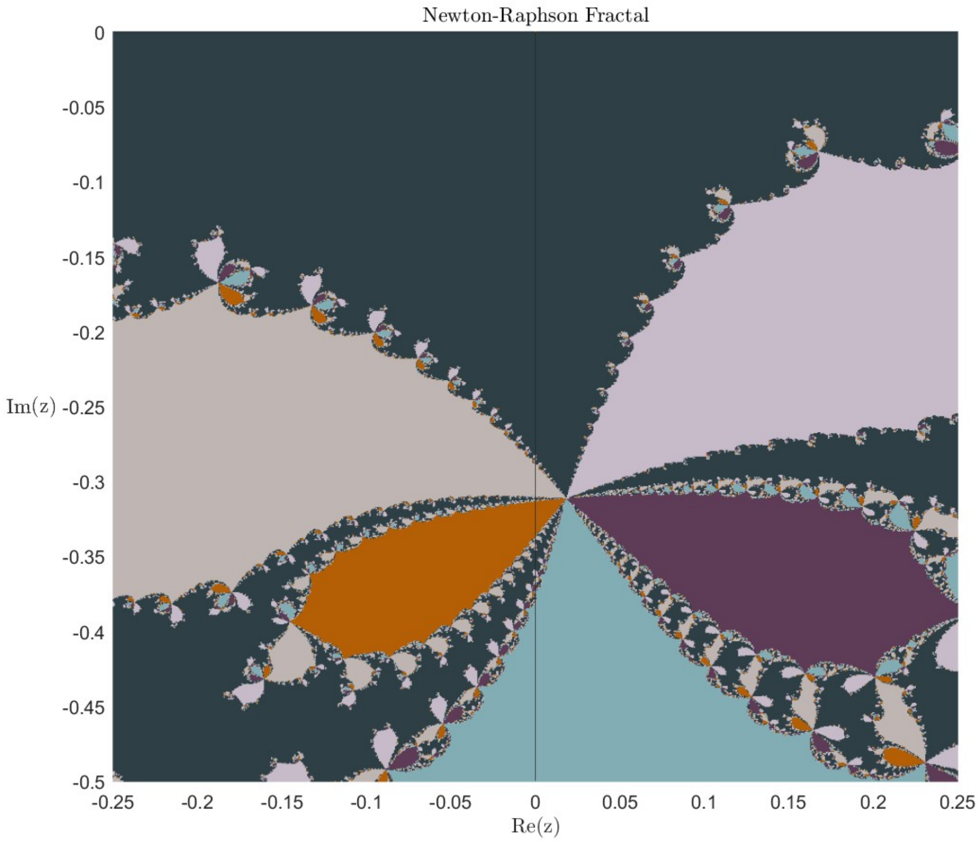

Figure 7: A magnified section of the Newton-Raphson Fractal in Figure 6.