L'HOSPITAL'S RULE FORM ZERO x INFINITY

When L'Hospital's rule gives us the form $0 * \infty$ the way to solve it is to change it into the form 0/0 or possibly into the form $\frac{\infty}{\infty}$. If the limit is solvable, then it will be solvable one of these two ways. If you try one way and it does not work, try the other way.

If neither way works, then there are other ways to solve the limit that go past L'Hospital's theorem – some of them involve what is called a Taylor expansion (we should get to Taylor Series by the end of the semester – hopefully).

We are not concerned about these more difficult limits right now. I only mention them in case you come across a situation where L'Hospital's theorem does not lead to an answer. This does happen from time to time. Otherwise L'Hospital's rule does work most of the time and is a powerful tool to solving limits.

ANTON PAGE 554

1. $\lim_{x \to 0^+} \sin x \ \ln x$

$\lim_{x \to 0^+} \sin x \ \ln x \ = \ \sin 0 \ \ln 0 \ = \ 0\,(-\infty)$ indeterminate form

Try the form ∞/∞:

$\lim_{x \to 0^+} \sin x \ \ln x \ = \ \lim_{x \to 0^+} \frac{\ln x}{\csc x} \ = \ -\frac{\infty}{\infty}$ indeterminate form second type

Now use LHospitals theorem on $\lim_{x \to 0^+} \frac{\ln x}{\csc x}$:

$\lim_{x \to a} \frac{f(x)}{g(x)} \ = \ \lim_{x \to a} \frac{f'(x)}{g'(x)}$ L'Hospital's Rule

$\lim_{x \to 0^+} \frac{\ln x}{\csc x} \ = \ \lim_{x \to 0^+} \frac{x^{-1}}{-\csc x \cot x}$ L'Hospital's rule

Now we have to rearrange the expression:

$\lim_{x \to 0^+} \frac{x^{-1}}{-\csc x \cot x} \ = \ \lim_{x \to 0^+} \frac{\sin x}{-x} \tan x \ = \ \lim_{x \to 0^+} \frac{\sin x}{-x} \quad \lim_{x \to 0^+} \tan x$

$= (-1) \tan 0 \ = \ (-1)\,(0) \ = \ 0$

$\lim_{x \to 0^+} \sin x \ \ln x \ = \ 0$

2. $\lim_{x \to 0^+} x \ln x$

$\lim_{x \to 0^+} x \ln x \;=\; 0 \ln(0) \;=\; 0\,(-\infty)$ indeterminate form

Try the form ∞/∞

$\lim_{x \to 0^+} x \ln x \;=\; \lim_{x \to 0^+} \frac{\ln x}{x^{-1}} \;=\frac{\ln 0}{1/0} \;=\; -\infty/\infty$ indeterminate form second type

Now use LHospitals theorem on $\lim_{x \to 0^+} \frac{\ln x}{x^{-1}}$:

$\lim_{x \to a} \frac{f(x)}{g(x)} \;=\; \lim_{x \to a} \frac{f'(x)}{g'(x)}$ L'Hospital's Rule

$\lim_{x \to 0^+} \frac{\ln x}{x^{-1}} \;=\; \lim_{x \to 0^+} \frac{x^{-1}}{-x^{-2}} \;=\; \lim_{x \to 0^+} x \;=\; 0$

$\lim_{x \to 0^+} x \ln x \;=\; 0$

3. $\lim_{x \to \frac{\pi}{4}} (1 - \tan x)(\sec 2x)$

$$\lim_{x \to \frac{\pi}{4}} (1 - \tan x)(\sec 2x) = \left(1 - \tan\frac{\pi}{4}\right)\sec\frac{\pi}{2} = (1-1)\sec\frac{\pi}{2} = 0 \cdot \infty$$

Try the form 0/0

$$\lim_{x \to \frac{\pi}{4}} \frac{(1-\tan x)}{\cos 2x} = \frac{0}{0}$$

Now use LHospitals theorem on $\lim_{x \to \frac{\pi}{4}} \frac{(1-\tan x)}{\cos 2x}$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$ L'Hospital's Rule

$$\lim_{x \to \frac{\pi}{4}} \frac{(1-\tan x)}{\cos 2x} = \lim_{x \to \frac{\pi}{4}} \frac{-\sec^2 x}{-2\sin 2x} = \frac{\sec^2\frac{\pi}{4}}{2\sin \pi/2} = \frac{2}{2(1)} = 1$$

$$\lim_{x \to \frac{\pi}{4}} (1 - \tan x)(\sec 2x) = 1$$

4. $\lim_{x \to \infty} x\ e^{-x}$

$\lim_{x \to \infty} x\ e^{-x} = \infty \cdot 0$ indeterminate form

Try the form ∞/∞ and see if it works

$\lim_{x \to \infty} \frac{x}{e^x} = \frac{\infty}{\infty}$

Now use LHospital's theorem on $\lim_{x \to \infty} \frac{x}{e^x}$:

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ L'Hospital's Rule

$\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$

$\lim_{x \to \infty} x\ e^{-x} = 0$

5. $\lim_{x \to \infty} x \sin\frac{\pi}{x}$

$$\lim_{x \to \infty} x \sin\frac{\pi}{x} = \infty \cdot \sin 0 = \infty \cdot 0$$ indeterminate form

Try the form 0/0

$$\lim_{x \to \infty} x \sin\frac{\pi}{x} = \lim_{x \to \infty} \frac{\sin\frac{\pi}{x}}{\frac{1}{x}} = \frac{0}{0}$$

Now use LHospital's Theorem on $\lim_{x \to \infty} \frac{\sin\frac{\pi}{x}}{\frac{1}{x}}$:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$ L'Hospital's Rule

$$\lim_{x \to \infty} \frac{\sin\frac{\pi}{x}}{\frac{1}{x}} = \lim_{x \to \infty} \frac{\cos\frac{\pi}{x}\left(-\frac{\pi}{x^2}\right)}{\frac{-1}{x^2}}$$

$$\lim_{x \to \infty} (\pi) \cos\frac{\pi}{x} = \pi \cos 0 = \pi\,(1) = \pi$$

6. $\lim_{x \to \infty} x \sin\frac{\pi}{x}$

Another way to do this question is to let $u = 1/x$. If you make this substitution, then the limit becomes

$\lim_{u \to 0} \frac{\sin \pi u}{u}$ which is of the form 0/0. L'Hospital's rule will produce pi.

7. $\lim_{x \to \pi^-} (x - \pi) \tan \frac{x}{2}$

$$\lim_{x \to \pi^-} (x - \pi) \tan \frac{x}{2} = 0 \cdot \tan \frac{\pi^-}{2} = 0 \cdot \infty$$

Try the form 0/0

$$\lim_{x \to \pi^-} (x - \pi) \tan \frac{x}{2} = \lim_{x \to \pi^-} \frac{(x - \pi)}{\cot \frac{x}{2}} = \frac{0}{0}$$

Now use L'Hospital's theorem on $\lim_{x \to \pi^-} \frac{(x - \pi)}{\cot \frac{x}{2}}$:

$$\lim_{x \to \pi^-} \frac{(x - \pi)}{\cot \frac{x}{2}} = \lim_{x \to \pi^-} \frac{1}{-\frac{1}{2} \csc^2 \frac{x}{2}} = \lim_{x \to \pi^-} -2 \sin^2 \frac{x}{2} = -2 \sin^2 \frac{\pi}{2} = -2$$

$$\lim_{x \to \pi^-} (x - \pi) \tan \frac{x}{2} = -2$$

8. $\lim_{x \to 0^+} \tan x \ \ln x$

$$\lim_{x \to 0^+} \tan x \ \ln x = \tan 0 \cdot \ln 0 = 0(-\infty)$$ indeterminate form

Try the form ∞/∞

$$\lim_{x \to 0^+} \frac{\ln x}{\cot x} = -\frac{\infty}{\infty}$$

Now use L'Hospital's theorem on $\lim_{x \to 0^+} \frac{\ln x}{\cot x}$

$$\lim_{x \to 0^+} \frac{\ln x}{\cot x} = \lim_{x \to 0^+} \frac{x^{-1}}{-\csc^2 x} = - \lim_{x \to 0^+} \frac{\sin^2 x}{x}$$

The last limit above still produces 0/0 – so we do LHospital's rule once more:

$$- \lim_{x \to 0^+} \frac{\sin^2 x}{x} = - \lim_{x \to 0^+} \frac{2 \sin x \cos x}{1} = 2 \sin 0 \ \cos 0 = 2(0)\,(1) = 0$$

$$\lim_{x \to 0^+} \tan x \ \ln x = 0$$

9. $\lim_{x \to 0^+} x^2 \ e^{1/x}$

$$\lim_{x \to 0^+} x^2 \ e^{1/x} = 0 \cdot e^{+\infty} = 0 \cdot \infty \quad \text{indeterminate form}$$

Try to get the form ∞/∞

$$\lim_{x \to 0^+} x^2 \ e^{1/x} = \lim_{x \to 0^+} \frac{e^{x^{-1}}}{x^{-2}} = \lim_{x \to 0^+} \frac{e^{1/x}}{1/x^2} = \infty/\infty$$

$$\lim_{x \to 0^+} \frac{e^{\frac{1}{x}}}{\frac{1}{x^2}} = \lim_{x \to 0^+} \frac{e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-2\left(\frac{1}{x^3}\right)} \quad \text{clean this up a bit}$$

$$\lim_{x \to 0^+} \frac{x \ e^{1/x}}{-2} = -\frac{1}{2} \lim_{x \to 0^+} x \ e^{1/x}$$

The last limit is still zero times infinity but we see the power of x has decreased. We do LHospital's rule once more:

$$-\frac{1}{2} \lim_{x \to 0^+} \frac{e^{x^{-1}}}{x^{-1}} = -\frac{1}{2} \lim_{x \to 0^+} \frac{e^{x^{-1}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

$$-\frac{1}{2} \lim_{x \to 0^+} e^{x^{-1}} = -\frac{1}{2} e^{\frac{1}{0^+}} = -\infty$$

So $\lim_{x \to 0^+} x^2 \ e^{1/x} = -\infty$ or we say the limit does not exist.

10. $\lim_{x \to \infty} \sqrt{x}\ e^{-x/2}$

$$\lim_{x \to \infty} \sqrt{x}\ e^{-x/2} = \infty \cdot 0$$

Try the form ∞/∞

$$\lim_{x \to \infty} \sqrt{x}\ e^{-x/2} = \lim_{x \to \infty} \frac{\sqrt{x}}{e^{x/2}} = \infty/\infty$$

Now use L'Hospital's rule:

$$\lim_{x \to \infty} \frac{\sqrt{x}}{e^{x/2}} = \lim_{x \to \infty} \frac{1}{2\sqrt{x}\left(\frac{1}{2}\right) e^{x/2}} = \lim_{x \to \infty} \frac{1}{\sqrt{x}\ e^{x/2}} = 0$$

11. $\lim_{x \to 0} \sin 3x\ \csc 5x$

$$\lim_{x \to 0} \sin 3x\ \csc 5x = \sin 0\ \ \csc 0 = (0)\,(\pm\ \infty)$$

$$\lim_{x \to 0} \sin 3x\ \csc 5x = \lim_{x \to 0} \frac{\sin 3x}{\sin 5x} = \frac{\sin 0}{\sin 0} = \frac{0}{0}$$

$$\lim_{x \to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x \to 0} \frac{3\cos 3x}{5\cos 5x} = \frac{3\cos 0}{5\cos 0} = \frac{3}{5}$$

12. $\lim_{x \to \frac{\pi}{2}^-} \cos x\ \sec 5x$

$$\lim_{x \to \frac{\pi}{2}^-} \cos x\ \sec 5x = \cos\frac{\pi}{2}^-\ \sec\frac{5\pi}{2}^- = 0 \cdot \infty$$

$$\lim_{x \to \frac{\pi}{2}^-} \frac{\cos x}{\cos 5x} = \frac{\cos\frac{\pi}{2}}{\cos\frac{5\pi}{2}} = \frac{0}{0}$$

$$\lim_{x \to \frac{\pi}{2}^-} \frac{\cos x}{\cos 5x} = \lim_{x \to \frac{\pi}{2}^-} \frac{-\sin x}{-5\ \sin 5x} = \lim_{x \to \frac{\pi}{2}^-} \frac{\sin x}{5\ \sin 5x} = \frac{\sin\frac{\pi}{2}}{5\sin\frac{5\pi}{2}} = \frac{1}{5}$$

$$\lim_{x \to \frac{\pi}{2}^-} \cos x\ \sec 5x = \frac{1}{5}$$

13. $\lim_{x \to \infty} x^3 e^{-x^2}$

$$\lim_{x \to \infty} x^3 e^{-x^2} = \infty \cdot 0$$

Try the form ∞/∞

$$\lim_{x \to \infty} x^3 e^{-x^2} = \lim_{x \to \infty} \frac{x^3}{e^{x^2}} = \frac{\infty}{\infty}$$

$$\lim_{x \to \infty} \frac{x^3}{e^{x^2}} = \lim_{x \to \infty} \frac{3x^2}{2x e^{x^2}} = \lim_{x \to \infty} \frac{3x}{2e^{x^2}} = \frac{\infty}{\infty}$$

$$\lim_{x \to \infty} \frac{3x}{2e^{x^2}} = \lim_{x \to \infty} \frac{3}{4x e^{x^2}} = \frac{3}{\infty} = 0$$

$$\lim_{x \to \infty} x^3 e^{-x^2} = 0$$