L'HOSPITAL'S RULE

L'Hospital's rule is a very nice way to evaluate certain limits – limits that result in an indeterminate form.

The most basic indeterminate form is 0/0. This form results from the definition of derivative. If we recall the definition of derivative is given by the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If we plug in h = 0 without doing the appropriate algebra we get 0/0:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{f(x+0) - f(x)}{0} = \frac{f(x) - f(x)}{0} = \frac{0}{0}$$

There are 7 indeterminate forms we study. They are

1. $\frac{0}{0}$
2. $\frac{\infty}{\infty}$
3. $0 \cdot \infty$
4. 1^{∞}
5. 0^{0}
6. ∞^{0}
7. $\infty - \infty$

The first two (and the most important) deal with division. The third deals with multiplication. The next three deal with exponentiation. The last deals with subtraction.

If we take limits, and we get any one of these 7 forms, then we can solve them using L'Hospital's rule. We have to concentrate on the first two types, $\frac{0}{0}$ and $\frac{\infty}{\infty}$. Once we can solve these first two types, then we can solve the rest of them with this rule.

First form of L'Hospital's rule: if $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ and if $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

This tells us the following. Say we take the limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ and we get 0/0. We can get the answer to the limit by taking the derivative of f, taking the derivative of g and finding the limit $\lim_{x \to a} \frac{f'(x)}{g'(x)}$. The limit of the ratio of the derivatives is equal to the original limit.

Second form of L'Hospital's rule: : if $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$ and if $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

So it doesn't matter which indeterminate form you get. Whether you get 0/0 or ∞/∞, you evaluate

$\lim_{x \to a} \frac{f'(x)}{g'(x)}$ to get the answer.

INSTRUCTIONS: THERE ARE MANY SOLVED PROBLEMS HERE. YOU ARE NOT EXPECTED TO DO ALL OF THEM. LOOK AT AS MANY AS YOU NEED TO FEEL COMFORTABLE.

PROBLEMS INVOLVING 0/0

1. Find $\lim_{x \to 0} \frac{\sin x}{x}$

$$\lim_{x \to 0} \frac{\sin x}{x} = \frac{\sin 0}{0} = \frac{0}{0} \quad \text{indeterminate form}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad \text{L'Hospital's rule}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = \frac{1}{1} = 1$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

2. Find $\lim_{x \to 0} \frac{e^x - 1}{x}$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \frac{e^0 - 1}{0} = \frac{1-1}{0} = \frac{0}{0} \quad \text{indeterminate form}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad \text{L'Hospital's rule}$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = \frac{e^0}{1} = \frac{1}{1} = 1$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

3. Find $\lim_{x \to 0} \frac{1 - \cos x}{x}$

$$\lim_{x \to 0} \frac{1-\cos x}{x} = \frac{1-\cos 0}{0} = \frac{1-\cos 0}{0} = \frac{1-1}{0} = \frac{0}{0} \quad \text{indeterminate form}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad \text{L'Hospital's rule}$$

$$\lim_{x \to 0} \frac{1-\cos x}{x} = \lim_{x \to 0} \frac{\sin x}{1} = \frac{\sin 0}{1} = \frac{0}{1} = 0$$

$$\lim_{x \to 0} \frac{1-\cos x}{x} = 0$$

4. $\lim_{x \to 0} \frac{1-\cos x}{x^2}$ INTERESTING QUESTION – REPETITION OF METHOD

$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1-\cos 0}{0} = \frac{1-1}{0} = \frac{0}{0}$ indeterminate form

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ L'Hospital's rule

$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \frac{\sin 0}{0} = \frac{0}{0} = \frac{0}{0}$ indeterminate form again!

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ L'Hospital's rule again!

$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{\cos 0}{2} = \frac{1}{2}$

$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

5. $\lim_{x \to \pi} \frac{\sin x}{x - \pi}$

$\lim_{x \to \pi} \frac{\sin x}{x - \pi} = \frac{\sin \pi}{\pi - \pi} = \frac{0}{0}$ indeterminate form

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ L'Hospital's rule

$\lim_{x \to \pi} \frac{\sin x}{x - \pi} = \lim_{x \to \pi} \frac{\cos x}{1} = \cos \pi = -1$

$\lim_{x \to \pi} \frac{\sin x}{x - \pi} = -1$

6. $\lim_{x \to 1} \frac{\ln x}{x-1}$

$\lim_{x \to 1} \frac{\ln x}{x-1} = \frac{\ln 1}{1-1} = \frac{0}{0}$ indeterminate form

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ L'Hospital's rule

$\lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1} \frac{1/x}{1} = \lim_{x \to 1} \frac{1}{x} = 1$

$\lim_{x \to 1} \frac{\ln x}{x-1} = 1$

7. PURCELL PG525

$$\lim_{x \to 0} \frac{\tan 2x}{\ln(1+x)}$$

$$\lim_{x \to 0} \frac{\tan 2x}{\ln(1+x)} = \frac{\tan 0}{\ln 1} = \frac{0}{0}$$ indeterminate form

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$ L'Hospital's rule

$$\lim_{x \to 0} \frac{\tan 2x}{\ln(1+x)} = \lim_{x \to 0} \frac{2\sec^2 2x}{(1+x)^{-1}} = \frac{2\sec^2 0}{1} = 2$$

$$\lim_{x \to 0} \frac{\tan 2x}{\ln(1+x)} = 2$$

8. PURCELL PG 526

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}$$

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} = \frac{\sin 0 - 0}{0} = \frac{0}{0}$$ indeterminate form

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$ L'Hospital's rule

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2} = \frac{\cos 0 - 1}{0} = \frac{1-1}{0} = \frac{0}{0}$$ indeterminate again

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$ L'Hospital's rule again

$$\lim_{x \to 0} \frac{\cos x - 1}{3x^2} = \lim_{x \to 0} -\frac{\sin x}{6x} = \frac{\sin 0}{0} = \frac{0}{0}$$ again!!

$$\lim_{x \to 0} -\frac{\sin x}{6x} = \lim_{x \to 0} \frac{-\cos x}{6} = -\frac{\cos 0}{6} = -\frac{1}{6}$$

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}$$

9. PURCELL PAGE 527

$$\lim_{x \to 0} \frac{\sin x - 2x}{x}$$

$$\lim_{x \to 0} \frac{\sin x - 2x}{x} = \frac{\sin 0 - 0}{0} = \frac{0}{0} \quad \text{indeterminate form}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \qquad \text{L'Hospital's rule}$$

$$\lim_{x \to 0} \frac{\sin x - 2x}{x} = \lim_{x \to 0} \frac{\cos x - 2}{1} = \frac{\cos 0 - 2}{1} = \frac{1-2}{1} = -1$$

$$\lim_{x \to 0} \frac{\sin x - 2x}{x} = -1$$

10. PURCELL PAGE 527

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\left(x - \frac{\pi}{2}\right)} = \frac{\cos\frac{\pi}{2}}{0} = \frac{0}{0} \quad \text{indeterminate form}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \qquad \text{L'Hospital's rule}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}} = \lim_{x \to \frac{\pi}{2}} \frac{-\sin x}{1} = -\sin\frac{\pi}{2} = -1$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}} = -1$$

11. PURCELL PAGE 527

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{4\sin x}$$

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{4\sin x} = \frac{e^0 - e^0}{4\sin 0} = \frac{0}{0}$$ indeterminate form

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$ L'Hospital's rule

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{4\sin x} = \lim_{x \to 0} \frac{e^x + e^{-x}}{4\cos x} = \frac{e^0 + e^0}{4\cos 0} = \frac{1+1}{4} = \frac{1}{2}$$

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{4\sin x} = \frac{1}{2}$$

12. PURCELL PAGE 527

$$\lim_{t \to 1} \frac{\sqrt{t} - t}{\ln t}$$

$$\lim_{t \to 1} \frac{\sqrt{t} - t}{\ln t} = \frac{1-1}{\ln 1} = \frac{0}{0}$$ indeterminate form

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$ L'Hospital's rule

$$\lim_{t \to 1} \frac{\sqrt{t} - t}{\ln t} = \lim_{t \to 1} \frac{\frac{1}{2}\, t^{-1/2} - 1}{t^{-1}} = \frac{\frac{1}{2} - 1}{1} = -\frac{1}{2}$$

$$\lim_{t \to 1} \frac{\sqrt{t} - t}{\ln t} = -\frac{1}{2}$$

13. PURCELL PAGE 527

$$\lim_{x \to 0^+} 2\frac{\sin x}{\sqrt{x}}$$

$$\lim_{x \to 0^+} 2\frac{\sin x}{\sqrt{x}} = \frac{2\sin 0}{0} = \frac{0}{0}$$ indeterminate form

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$ L'Hospital's rule

$$\lim_{x \to 0^+} 2\frac{\sin x}{\sqrt{x}} = \lim_{x \to 0^+} 2\frac{\cos x}{\left(\frac{1}{2}\right) x^{-1/2}} = \lim_{x \to 0^+} 2\, x^{1/2} \cos x = 2(0)\cos 0 = 0$$

$$\lim_{x \to 0^+} 2\frac{\sin x}{\sqrt{x}} = 0$$

14. PURCELL PAGE 527

$$\lim_{x \to 0} \frac{\sin^{-1} x}{3\tan^{-1} x}$$

$$\lim_{x \to 0} \frac{\sin^{-1} x}{3\tan^{-1} x} = \frac{\sin^{-1} 0}{3\tan^{-1} 0} = \frac{0}{0}$$ indeterminate form

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$ L'Hospital's rule

$$\lim_{x \to 0} \frac{\sin^{-1} x}{3\tan^{-1} x} = \lim_{x \to 0} \frac{\left(1-x^2\right)^{-1/2}}{3\,(1+x^2)^{-1}} = \frac{1}{3}$$

15. PURCELL PAGE 527

$$\lim_{x \to 0^+} \frac{8^{\sqrt{x}} - 1}{3^{\sqrt{x}} - 1}$$

$$\lim_{x \to 0^+} \frac{8^{\sqrt{x}} - 1}{3^{\sqrt{x}} - 1} = \frac{8^0 - 1}{3^0 - 1} = \frac{1-1}{1-1} = \frac{0}{0}$$ indeterminate form

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$ L'Hospital's rule

$$\lim_{x \to 0^+} \frac{8^{\sqrt{x}} - 1}{3^{\sqrt{x}} - 1} = \lim_{x \to 0^+} \frac{8^{\sqrt{x}} \ln 8 \left(\frac{1}{2}\right) x^{-1/2}}{3^{\sqrt{x}} \ln 3 \left(\frac{1}{2}\right) x^{-1/2}} = \lim_{x \to 0^+} \frac{8^{\sqrt{x}} \ln 8}{3^{\sqrt{x}} \ln 3} = \frac{\ln 8}{\ln 3}$$

PROBLEMS INVOLVING INFINITY/INFINITY

16. $\lim_{x \to \infty} \frac{x}{e^x}$

$\lim_{x \to \infty} \frac{x}{e^x} = \frac{\infty}{e^\infty} = \frac{\infty}{\infty}$ indeterminate form second kind

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ L'Hospital's rule

$\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$

$\lim_{x \to \infty} \frac{x}{e^x} = 0$

17. $\lim_{x \to \infty} \frac{e^x}{x}$

$\lim_{x \to \infty} \frac{e^x}{x} = \frac{e^\infty}{\infty} = \frac{\infty}{\infty}$ indeterminate form second kind

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ L'Hospital's rule

$\lim_{x \to \infty} \frac{e^x}{x} = \lim_{x \to \infty} \frac{e^x}{1} = e^\infty = \infty$

$\lim_{x \to \infty} \frac{e^x}{x} = \infty$ or $\lim_{x \to \infty} \frac{e^x}{x}$ diverges or does not exist

18. $\lim_{x \to \infty} \frac{x^2}{e^x}$

$\lim_{x \to \infty} \frac{x^2}{e^x} = \frac{\infty^2}{e^\infty} = \frac{\infty}{\infty}$ indeterminate form of second kind

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ L'Hospital's rule

$\lim_{x \to \infty} \frac{x^2}{e^x} = 2 \lim_{x \to \infty} \frac{x}{e^x} = 2 \lim_{x \to \infty} \frac{1}{e^x} = 2\left(\frac{1}{\infty}\right) = 0$

19. IMPORTANT QUESTION

$\lim\limits_{x \to \infty} \frac{x^n}{e^x}$ where n is a positive integer

Repeated application of L'Hospital's rule will produce the equation

$$\lim_{x \to \infty} \frac{x^n}{e^x} = \lim_{x \to \infty} \frac{n!}{e^x} = \frac{n!}{e^\infty} = \frac{n!}{\infty} = 0$$

For any power of n (positive) this limit is zero. It turns out this is true for all powers of n. Exponentials approach infinity so fast they dwarf the growth of any polynomials. Compared to exponentials, polynomials move like turtles. Exponentials approach infinity like rocket ships. Polynomials grow slowly.

20. $\lim\limits_{x \to \infty} \frac{\ln x}{x}$

$$\lim_{x \to \infty} \frac{\ln x}{x} = \frac{\ln \infty}{\infty} = \frac{\infty}{\infty} \text{ indeterminate for second kind}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \qquad \text{L'Hospital's rule}$$

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{x^{-1}}{1} = \lim_{x \to \infty} \frac{1}{x} = \frac{1}{\infty} = 0$$

$$\lim_{x \to \infty} \frac{\ln x}{x} = 0$$

This shows that logarithmic approach to infinity is VERY slow. It is one of the slowest approaches to infinity that we know – perhaps the slowest approach to infinity.

21. $\lim\limits_{x \to \infty} \frac{\ln x}{a^x} \qquad a > 1$

$$\lim_{x \to \infty} \frac{\ln x}{a^x} = \frac{\ln \infty}{a^\infty} = \frac{\infty}{\infty} \quad \text{indeterminate form second kind}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \qquad \text{L'Hospital's rule}$$

$$\lim_{x \to \infty} \frac{\ln x}{a^x} = \lim_{x \to \infty} \frac{x^{-1}}{a^x \ln a} = \lim_{x \to \infty} \frac{1}{a^x \; x \; \ln a} = \frac{1}{\infty} = 0$$

22. $\lim\limits_{x \to \infty} \frac{\ln x^{100}}{x}$

$$\lim_{x \to \infty} \frac{\ln x^{100}}{x} = \lim_{x \to \infty} \frac{100 \ln x}{x} = \frac{\infty}{\infty}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \qquad \text{L'Hospital's rule}$$

$$\lim_{x \to \infty} \frac{\ln x^{100}}{x} = \lim_{x \to \infty} \frac{100 \ln x}{x} = 100 \lim_{x \to \infty} \frac{x^{-1}}{1} = 100 \lim_{x \to \infty} \frac{1}{x} = 0$$

$$\lim_{x \to \infty} \frac{\ln x^{100}}{x} = 0$$

23. $\lim\limits_{x \to 0^+} \frac{\ln x}{1/x}$

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} = \frac{\ln 0^+}{1/0^+} = -\frac{\infty}{\infty} \quad \text{indeterminate form second kind}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \qquad \text{L'Hospital's rule}$$

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{x^{-1}}{-\,x^{-2}} = \lim_{x \to 0^+} -x = 0$$

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} = 0$$

24. $\lim\limits_{x \to \infty} \frac{10^x}{x^{10}}$

$$\lim_{x \to \infty} \frac{10^x}{x^{10}} = \frac{10^\infty}{(\infty)^{10}} = \frac{\infty}{\infty} \quad \text{indeterminate form of second type}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \qquad \text{L'Hospital's rule}$$

$$\lim_{x \to \infty} \frac{10^x}{x^{10}} = \lim_{x \to \infty} \frac{10^x \ln 10}{10\, x^9} = \frac{\infty}{\infty} \quad \text{again!}$$

If we keep on doing this, we will get to the form

$$\lim_{x \to \infty} \frac{10^x}{x^{10}} = \lim_{x \to \infty} \frac{10^x \ (\ln 10)^{10}}{10!} = \frac{10^\infty \ (\ln 10)^{10}}{10!} = \infty$$

This is another case of exponential growth versus polynomial growth. Exponential growth completely dominates, no matter how big the polynomial.

25. $\lim_{x \to \infty} \frac{x^3+3x^2}{2x^3-x}$

$\lim_{x \to \infty} \frac{x^3+3x^2}{2x^3-x} = \frac{\infty}{\infty}$ indeterminate form L'Hospital's rule

$\lim_{x \to \infty} \frac{x^3+3x^2}{2x^3-x} = \lim_{x \to \infty} \frac{3x^2+6x}{6x^2-1} = \infty/\infty$

$\lim_{x \to \infty} \frac{x^3+3x^2}{2x^3-x} = \lim_{x \to \infty} \frac{3x^2+6x}{6x^2-1} = \lim_{x \to \infty} \frac{6x+6}{12x} = \lim_{x \to \infty} \frac{6}{12} = \frac{1}{2}$

We could have done this with methods from calculus one but this shows that L'Hospital's rule also works.

26. $\lim_{x \to 0^+} \frac{\ln x}{\csc x}$

$\lim_{x \to 0^+} \frac{\ln x}{\csc x} = \frac{\ln 0^+}{\csc 0^+} = -\frac{\infty}{\infty}$

$\lim_{x \to 0^+} \frac{\ln x}{\csc x} = \lim_{x \to 0^+} \frac{x^{-1}}{\csc x \cot x} = \lim_{x \to 0^+} \frac{\sin x}{x} \tan x$

$\lim_{x \to 0^+} \frac{\sin x}{x} \lim_{x \to 0^+} \tan x = 1 \cdot \tan 0 = 1 \cdot 0 = 0$