INTEGRATION BY PARTS
PURCELL

1. Evaluate $\int x \cos x \; dx$

2. Evaluate $\int \ln x \;\; dx$

3. Evaluate $\int x^2 \cos x \; dx$

4. Evaluate $\int \cot^{-1} x \;\; dx$

5. Evaluate $\int e^x \sin x \; dx$

6. Evaluate $\int e^x \sin x \; dx$

7. Evaluate $\int \sin^{-1} x \; dx$

8. Evaluate $\int \sec^{-1} x \;\; dx$

9. Evaluate $\int \sec^3 x \; dx$

10. $\int x \, e^x \, dx$

11. Evaluate $\int y \;\, e^{3y} \;\, dy$

12. Evaluate $\int \; x \sin 3x \;\; dx$

13. Evaluate $\int \; \ln(3x) \;\; dx$

14. Evaluate $\int \; \tan^{-1} x \;\; dx$

15. Evaluate $\int x \;\; \sqrt{x+1} \;\; dx$

16. Evaluate $\int t \sec^2 5t \;\; dt$

17. Evaluate $\int \sqrt{x} \; \ln x \;\; dx$

18. Evaluate $\int \; z^3 \; \ln z \;\; dz$

19. Evaluate $\int \; x \; \tan^{-1} x \;\; dx$

20. Evaluate $\int \; t \cos 4t \;\; dt$

21. Evaluate $\int \; x \; \cos^2 x \; \sin x \;\; dx$

22. Evaluate $\int \; x \;\; a^x \;\; dx$

23. Evaluate $\int \; \sin^2 x \;\; dx$

INTEGRATION BY PARTS
PURCELL

1. Evaluate $\int x\cos x \; dx$

$let\ u = x \qquad dv = \cos x \;\; dx$

$du = dx \qquad v = \sin x$

$\int u\,dv \; = \; uv - \int v\,du$

$$\int x\cos x \; dx \; = x\sin x \; - \; \int \sin x \; dx$$

$$\int x\cos x \; dx \; = x\sin x \; + \cos x \; + \; C$$

2. Evaluate $\int \ln x \;\; dx$

$let\ u = \ln x \qquad dv = dx$

$du = \frac{1}{x}\, dx \qquad v = x$

$\int u\,dv \; = \; uv - \int v\,du$

$$\int \ln x \;\; dx \; = x\ln x - \int x \cdot \frac{1}{x} \;\; dx$$

$$\int \ln x \;\; dx \; = x\ln x - \int \; dx$$

$$\int \ln x \;\; dx \; = x\ln x - \; x \; + \; C$$

3. Evaluate $\int x^2 \cos x \ dx$ $\qquad$ $let\ u = x^2$ $\qquad$ $dv = \cos x \ dx$

$du = 2x\, dx$ $\qquad$ $v = \sin x$

$\int u\, dv = uv - \int v\, du$

$$\int x^2 \cos x \ dx = x^2 \sin x - \int \sin x \ \ 2x\, dx$$

$$\int x^2 \cos x \ dx = x^2 \sin x - \int 2 \ x \ \sin x \ \ dx$$

We have to use method of parts again:

$let\ u = 2x$ $\qquad$ $dv = \sin x \ dx$

$du = 2\, dx$ $\qquad$ $v = -\cos x$

$$\int x^2 \cos x \ dx = x^2 \sin x - \left(-2x \cos x - 2 \int -\cos x \ \ dx \right)$$

$$\int x^2 \cos x \ dx = x^2 \sin x + 2x \cos x - 2 \int \cos x \ \ dx$$

$$\int x^2 \cos x \ dx = x^2 \sin x + 2x \cos x - 2 \sin x + C$$

4. Evaluate $\int \cot^{-1} x \ \ dx$

$$let\ u = \cot^{-1} x \qquad dv = dx$$

$$du = -\frac{1}{1+x^2}\ dx \qquad v = x$$

$\int u\, dv \ = \ uv - \int v\, du$

$$\int \cot^{-1} x \ \ dx \ = \ x\cot^{-1} x \ - \ \int -\frac{1}{1+x^2} \ \ x \ dx$$

$$\int \cot^{-1} x \ \ dx \ = \ x\cot^{-1} x \ + \ \int \frac{x}{1+x^2} \ \ dx$$

For the remaining integral on the right, it has the form for ln x – this can be seen by a u substitution. Making the necessary modifications we have:

$$\int \cot^{-1} x \ \ dx \ = \ x\cot^{-1} x \ + \ \frac{1}{2}\int \frac{2x}{1+x^2} \ \ dx$$

$$\int \cot^{-1} x \ \ dx \ = \ x\cot^{-1} x \ + \ \frac{1}{2}\ln(1+x^2) \ + \ C$$

5. Evaluate $\int e^x \sin x \ dx$

For this integral we have a choice as to which function is u and which is dv. You can let u be either function and it will still work. However, once you let u = e^x (let's say) then you must continue with this throughout the solution.

$$\int e^x \sin x \ dx \qquad let\ u = e^x \qquad dv = \sin x \ \ dx$$

$$du = e^x \ \ dx \qquad v = -\cos x$$

$$\int u\, dv \ = \ uv - \int v\, du$$

$$\int e^x \sin x \ dx \ = \ -e^x \cos x \ - \ \int -\cos x \ \ e^x \, dx$$

$$\int e^x \sin x \ dx \ = \ -e^x \cos x \ + \ \int \cos x \ \ e^x \, dx$$

Now we have to do it again: $let\ u = e^x$ $dv = \cos x\ dx$

$du = e^x\ dx$ $v = \sin x$

$$\int e^x \sin x\ dx = -e^x \cos x + \left(e^x \sin x - \int \sin x\ e^x\ dx \right)$$

$$\int e^x \sin x\ dx = -e^x \cos x + e^x \sin x - \int \sin x\ e^x\ dx$$

$$\int e^x \sin x\ dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\ dx$$

We now have a recursion. We have the same integral we started with. We can combine them into one integral.

$$2 \int e^x \sin x\ dx = -e^x \cos x + e^x \sin x$$

$$\int e^x \sin x\ dx = \frac{1}{2} \left(-e^x \cos x + e^x \sin x \right) + C$$

6. Evaluate $\int e^x \sin x \; dx$ but now do it the other choice $let\ u = \sin x \qquad dv = e^x dx$

$du = \cos x \;\; dx \qquad v = e^x$

$$\int u\, dv = uv - \int v\, du$$

$$\int e^x \sin x \; dx = \sin x \; e^x - \int e^x \cos x \; dx$$

Do it again: $let\ u = \cos x \qquad dv = e^x\, dx$

$du = -\sin x \;\; dx \qquad v = e^x$

$$\int e^x \sin x \; dx = \sin x \; e^x - \left(\cos x \;\; e^x - \int e^x \,(-\sin x)\, dx\right)$$

$$\int e^x \sin x \; dx = \sin x \; e^x - \cos x \;\; e^x - \int e^x \sin x \; dx$$

We have recursion. The last integral is the same as the integral we started with. Combine the into one integral.

$$2\int e^x \sin x \; dx = \sin x \; e^x - \cos x \;\; e^x$$

$$\int e^x \sin x \; dx = \frac{1}{2}\; (\sin x \; e^x - \cos x \;\; e^x\,) + C$$

7. Evaluate $\int \sin^{-1} x \ dx$ $\qquad$ $let\ u = \sin^{-1} x$ $\qquad$ $dv = dx$

$du = \frac{1}{\sqrt{1 - x^2}}\, dx$ $\qquad$ $v = x$

$$\int u\, dv = uv - \int v\, du$$

$$\int \sin^{-1} x \ dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}}\, dx$$

Do a variable substitution for the last integral: $\qquad$ $let\ t = 1 - x^2$ $\qquad$ $dt = -2x\, dx$

$$\int \sin^{-1} x \ dx = x \sin^{-1} x + \frac{1}{2} \int \frac{-2x}{\sqrt{1 - x^2}}\, dx$$

$$\int \sin^{-1} x \ dx = x \sin^{-1} x + \frac{1}{2} \cdot 2\sqrt{1 - x^2} + C$$

$$\int \sin^{-1} x \ dx = x \sin^{-1} x + \sqrt{1 - x^2} + C$$

8. Evaluate $\int \sec^{-1} x \ dx$ $\qquad$ $let\ u = \sec^{-1} x$ $\qquad$ $dv = dx$

$du = \frac{1}{x\sqrt{x^2 - 1}}\, dx$ $\qquad$ $v = x$

$$\int u\, dv = uv - \int v\, du$$

$$\int \sec^{-1} x \ dx = x \sec^{-1} x - \int \frac{x}{x\sqrt{x^2 - 1}}\, dx$$

$$\int \sec^{-1} x \ dx = x \sec^{-1} x - \int \frac{1}{\sqrt{x^2 - 1}}\, dx$$

Make a trigonometric substitution $\qquad$ $let\ x = \sec\theta$ $\qquad$ $dx = \sec\theta \ \tan\theta \ d\theta$

$$\int \sec^{-1} x \ dx = x \sec^{-1} x - \int \frac{1}{\sqrt{\sec^2\theta - 1}} \sec\theta \ \tan\theta \ d\theta$$

$$\int \sec^{-1} x \;\; dx \;\; = x \sec^{-1} x \; - \int \frac{1}{\sqrt{\tan^2 \theta}} \;\; \sec\theta \; \tan\theta \; d\theta$$

$$\int \sec^{-1} x \;\; dx \;\; = x \sec^{-1} x \; - \int \;\; \sec\theta \;\; d\theta$$

$$\int \sec^{-1} x \;\; dx \;\; = x \sec^{-1} x \; - \; \ln|\sec\theta + \tan\theta| \; + \; C$$

Since we let $x = \sec\theta$ we can also find that tangent. It is given by $\tan\theta \; = \; \sqrt{x^2 - 1}$

$$\int \sec^{-1} x \;\; dx \;\; = x \sec^{-1} x \; - \; \ln\left| \; x \; + \; \sqrt{x^2 - 1} \; \right| + \; C$$

The only thing bad about this solution is that we mixed variables – we try not to do this. We try to keep all equations to just one variable. So expressions, where x and theta are both present, are frowned upon.

9. Evaluate $\int \sec^3 x \;\; dx$

$let\; u = \sec x \qquad dv \; = \; \sec^2 x \;\; dx$

$du = \sec x \;\; \tan x \;\; dx \qquad v = \tan x$

$$\int u \, dv \; = \; uv - \int \; v \, du$$

$$\int \sec^3 x \;\; dx \; = \; \sec x \;\; \tan x \; - \int \; \tan^2 x \;\; \sec x \;\; dx$$

$$\int \sec^3 x \;\; dx \; = \; \sec x \;\; \tan x \; - \int \; (\sec^2 x \; - \; 1) \;\; \sec x \;\; dx$$

$$\int \sec^3 x \;\; dx \; = \; \sec x \;\; \tan x \; - \int \; (\sec^3 x \; - \sec x \,) \;\; dx$$

$$\int \sec^3 x \;\; dx \; = \; \sec x \;\; \tan x \; - \int \; \sec^3 x \;\;\; dx \;\; + \int \; \sec x \;\;\; dx$$

We have a recursion. We combine the two like integrals.

$$2\int \sec^3 x \; dx = \sec x \tan x + \int \sec x \; dx$$

$$2\int \sec^3 x \; dx = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\int \sec^3 x \; dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

10. $\int x\, e^x \, dx$ $\qquad$ $let\; u = x \qquad dv = e^x \; dx$

$du = dx \qquad v = e^x$

$$\int u\, dv = uv - \int v\, du$$

$$\int x\, e^x \, dx = x\, e^x - \int e^x \, dx$$

$$\int x\, e^x \, dx = x\, e^x - e^x + C$$

11. Evaluate $\int y\; e^{3y} \; dy$ $\qquad$ $let\; u = y \qquad dv = e^{3y} \; dy$

$du = dy \qquad v = \frac{1}{3}\, e^{3y}$

$$\int u\, dv = uv - \int v\, du$$

$$\int y\; e^{3y} \; dy = \frac{1}{3}\, y\; e^{3y} - \int \frac{1}{3}\; e^{3y} \; dy$$

$$\int y\; e^{3y} \; dy = \frac{1}{3}\, y\; e^{3y} - \frac{1}{9}\, e^{3y} + C$$

12. $\int x \sin 3x \; dx$ $\quad$ $let\ u = x$ $\quad$ $dv = \sin 3x \; dx$

$du = dx$ $\quad$ $v = -\frac{1}{3}\cos 3x$

$$\int u\, dv = uv - \int v\, du$$

$$\int x \sin 3x \; dx = -\frac{1}{3}\, x \cos 3x - \int -\frac{1}{3} \cos 3x \; dx$$

$$\int x \sin 3x \; dx = -\frac{1}{3}\, x \cos 3x + \frac{1}{3} \int \cos 3x \; dx$$

$$\int x \sin 3x \; dx = -\frac{1}{3}\, x \cos 3x + \frac{1}{9} \sin 3x + C$$

13. Evaluate $\int \ln(3x) \; dx$ $\quad$ $let\ u = \ln 3x$ $\quad$ $dv = dx$

$du = \frac{1}{x}\, dx$ $\quad$ $v = x$

It looks like an error but the du is actually correct.

$$\int u\, dv = uv - \int v\, du$$

$$\int \ln(3x) \; dx = x \ln(3x) - \int x \cdot \frac{1}{x} \; dx$$

$$\int \ln(3x) \; dx = x \ln(3x) - \int dx$$

$$\int \ln(3x) \; dx = x \ln(3x) - x + C$$

14. $\int \tan^{-1} x \; dx$ $\qquad$ $let\; u = \tan^{-1} x$ $\qquad$ $dv = dx$

$du = \frac{1}{x^2+1}\, dx$ $\qquad$ $v = x$

$$\int u\, dv = uv - \int v\, du$$

$$\int \tan^{-1} x \; dx = x \tan^{-1} x - \int \frac{x}{x^2 + 1} \; dx$$

The remaining integral is in the form for ln x. Make a variable subsititution t = $x^2 + 1$ and observe that dt = 2x dx. Modifying the integral:

$$\int \tan^{-1} x \; dx = x \tan^{-1} x - \frac{1}{2} \int \frac{2\, x}{x^2 + 1} \; dx$$

$$\int \tan^{-1} x \; dx = x \tan^{-1} x - \frac{1}{2} \ln(x^2 + 1) + C$$

15. Evaluate $\int x \; \sqrt{x+1} \; dx$ $\qquad$ $let\; u = x$ $\qquad$ $dv = \sqrt{x + 1} \; dx$

$du = dx$ $\qquad$ $v = \frac{2}{3} \, (x + 1)^{3/2}$

$$\int u\, dv = uv - \int v\, du$$

$$\int x \; \sqrt{x+1} \; dx = \frac{2}{3} x \; (x+1)^{3/2} - \frac{2}{3} \int (x + 1)^{3/2} \; dx$$

$$\int x \; \sqrt{x+1} \; dx = \frac{2}{3} x \; (x+1)^{3/2} - \frac{2}{3} \cdot \frac{2}{5} \; (x+1)^{5/s} + C$$

$$\int x \; \sqrt{x+1} \; dx = \frac{2}{3} x \; (x+1)^{3/2} - \frac{4}{15} \; (x+1)^{5/2} + C$$

This could have been done differently. I normally solve this by u substitution. Let u = x+1

This would suffice. Method of parts is not necessary but does work.

16. Evaluate $\int t\sec^2 5t\ dt$ $\quad$ $let\ u = t$ $\quad$ $dv = \sec^2 5t\ dt$

$du = dt$ $\quad$ $v = \frac{1}{5}\tan 5t$

$$\int u\,dv = uv - \int v\,du$$

$$\int t\sec^2 5t\ dt = \frac{1}{5}\ t\ \tan 5t - \int \frac{1}{5}\tan 5t\ \ dt$$

$$\int t\sec^2 5t\ dt = \frac{1}{5}\ t\ \tan 5t - \frac{1}{5}\int \tan 5t\ \ dt$$

$$\int t\sec^2 5t\ dt = \frac{1}{5}\ t\ \tan 5t - \frac{1}{25}\left(-\ln|\cos 5t|\right) + C$$

$$\int t\sec^2 5t\ dt = \frac{1}{5}\ t\ \tan 5t + \frac{1}{25}\ln|\cos 5t| + C$$

17. Evaluate $\int \sqrt{x}\ \ln x\ dx$ $\quad$ $let\ u = \ln x$ $\quad$ $dv = \sqrt{x}\ dx$

$du = \frac{1}{x}\ dx$ $\quad$ $v = \frac{2}{3}\ x^{3/2}$

$$\int u\,dv = uv - \int v\,du$$

$$\int \sqrt{x}\ \ln x\ dx = \frac{2}{3}\ x^{3/2}\ \ln x - \int \frac{2}{3}\ x^{3/2}\cdot\frac{1}{x}\ dx$$

$$\int \sqrt{x}\ \ln x\ dx = \frac{2}{3}\ x^{3/2}\ \ln x - \frac{2}{3}\int x^{1/2}\ dx$$

$$\int \sqrt{x}\ \ln x\ dx = \frac{2}{3}\ x^{3/2}\ \ln x - \frac{2}{3}\cdot\frac{2}{3}\ x^{3/2} + C$$

$$\int \sqrt{x}\ \ln x\ dx = \frac{2}{3}\ x^{3/2}\ \ln x - \frac{4}{9}\ x^{3/2} + C$$

18. Evaluate $\int z^3 \ln z \ dz$ $\quad$ $let\ u = \ln z$ $\quad$ $dv = z^3\, dz$

$du = \frac{1}{z}\ dz$ $\quad$ $v = \frac{z^4}{4}$

$$\int u\, dv = uv - \int v\, du$$

$$\int z^3 \ln z \ dz = \frac{z^4}{4} \ln z - \int \frac{z^4}{4} \cdot \frac{1}{z}\ dz$$

$$\int z^3 \ln z \ dz = \frac{z^4}{4} \ln z - \frac{1}{4} \int z^3\ dz$$

$$\int z^3 \ln z \ dz = \frac{z^4}{4} \ln z - \frac{1}{16}\ z^4 + C$$

19. Evaluate $\int x \tan^{-1} x \ dx$ $\quad$ $let\ u = \tan^{-1} x$ $\quad$ $dv = dx$

$du = \frac{1}{x^2 + 1}\ dx$ $\quad$ $v = x$

$$\int u\, dv = uv - \int v\, du$$

$$\int x \tan^{-1} x \ dx = x \tan^{-1} x - \int \frac{x}{x^2+1}\ dx$$

The last integral is a logarithm – this can be seen by a variable substitution – let t = x^2 + 1

The dt = 2x dx. So we modify the integral:

$$\int x \tan^{-1} x \ dx = x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{x^2+1}\ dx$$

$$\int x \tan^{-1} x \ dx = x \tan^{-1} x - \frac{1}{2} \ln(x^2+1) + C$$

20. Evaluate $\int t\cos 4t \ \ dt$ $\qquad$ $let\ u = t$ $\qquad$ $dv = \cos 4t \ \ dt$

$du = dt$ $\qquad$ $v = \frac{1}{4}\sin 4t \ \ dt$

$$\int u\,dv = uv - \int v\,du$$

$$\int t\cos 4t \ \ dt = \frac{1}{4}t\sin 4t - \int \frac{1}{4}\sin 4t \ \ dt$$

$$\int t\cos 4t \ \ dt = \frac{1}{4}t\sin 4t - \frac{1}{4}\int \sin 4t \ \ dt$$

$$\int t\cos 4t \ \ dt = \frac{1}{4}t\sin 4t - \frac{1}{16}\cos 4t + C$$

21. Evaluate $\int x\cos^2 x\sin x \ \ dx$ $\qquad$ $let\ u = x$ $\qquad$ $dv = \cos^2 x\sin x \ \ dx$

$du = dx$ $\qquad$ $v = -\frac{1}{3}\cos^3 x$

$$\int u\,dv = uv - \int v\,du$$

$$\int x\cos^2 x\sin x \ \ dx = -\frac{1}{3}x\cos^3 x - \int -\frac{1}{3}\cos^3 x \ \ dx$$

$$\int x\cos^2 x\sin x \ \ dx = -\frac{1}{3}x\cos^3 x + \frac{1}{3}\int \cos^3 x \ \ dx$$

To solve the last integral we are going to invoke a Pythagorean relation:

$$\int x\cos^2 x\sin x \ \ dx = -\frac{1}{3}x\cos^3 x + \frac{1}{3}\int \cos x\cos^2 x \ \ dx$$

$$\int x\cos^2 x\sin x \ \ dx = -\frac{1}{3}x\cos^3 x + \frac{1}{3}\int \cos x\ (1-\sin^2 x)\ dx$$

Let $t = \sin x$ $\qquad$ $so\ \ dt = \cos x \ \ dx$

$$\int x \cos^2 x \sin x \ dx = -\frac{1}{3} \ x \cos^3 x \ + \frac{1}{3} \int \ (1-t^2) \ dt$$

$$\int x \cos^2 x \sin x \ dx = -\frac{1}{3} \ x \cos^3 x \ + \frac{1}{3} \ \left(t - \frac{t^3}{3}\right) + \ C$$

$$t = \sin x$$

$$\int x \cos^2 x \sin x \ dx = -\frac{1}{3} \ x \cos^3 x \ + \frac{1}{3} \ \left(\sin \ x \ - \frac{(\sin x)^3}{3}\right) + \ C$$

$$\int x \cos^2 x \sin x \ dx = -\frac{1}{3} \ x \cos^3 x \ + \ \frac{1}{3} \sin \ x \ - \frac{(\sin x)^3}{9} \ + \ C$$

22. Evaluate $\int x \ a^x \ dx$ $\qquad$ $let \ u = x$ $\qquad$ $dv = a^x \ dx$

$du = dx$ $\qquad$ $v = \frac{a^x}{\ln a}$

$$\int u \, dv \ = \ uv - \int \ v \, du$$

$$\int \ x \ a^x \ dx \ = \frac{x \ a^x}{\ln a} \ - \int \frac{a^x}{\ln a} \ dx$$

$$\int \ x \ a^x \ dx \ = \frac{x \ a^x}{\ln a} \ - \frac{1}{\ln a} \int \ a^x \ dx$$

$$\int \ x \ a^x \ dx \ = \frac{x \ a^x}{\ln a} \ - \ \frac{1}{\ln^2 a} \ a^x \ + \ C$$

23. Evaluate $\int \sin^2 x \; dx$

$$let\ u = \sin^2 x \qquad\qquad dv = \ dx$$

$$du = 2\sin x \cos x \;\; dx \qquad\qquad v = x$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \sin^2 x \quad dx = x\sin^2 x - \int 2x \sin x \; \cos x \; dx$$

$$\int \sin^2 x \;\; dx = x\sin^2 x - \int x \sin 2x \; dx$$

$$let\ u = x \qquad\qquad dv = \sin 2x \;\; dx$$

$$du = dx \qquad\qquad v = -\tfrac{1}{2}\cos 2x$$

$$\int \sin^2 x \;\; dx = x\sin^2 x - \left(-\frac{1}{2}\,x\cos 2x + \frac{1}{2}\int \cos 2x \; dx\right)$$

$$\int \sin^2 x \;\; dx = x\sin^2 x + \frac{1}{2}\,x\cos 2x - \frac{1}{2}\int \cos 2x \; dx$$

$$\int \sin^2 x \;\; dx = x\sin^2 x + \frac{1}{2}\,x\cos 2x - \frac{1}{4}\sin 2x + C$$

Simplify:

$$\int \sin^2 x \;\; dx = x\sin^2 x + \frac{1}{2}\,x(1 - 2\sin^2 x) - \frac{1}{4}\sin 2x + C$$

$$\int \sin^2 x \;\; dx = x\sin^2 x + \frac{1}{2}\,x - x\sin^2 x - \frac{1}{4}\sin 2x + C$$

$$\int \sin^2 x \;\; dx = \frac{1}{2}\,x - \frac{1}{4}\sin 2x + C$$

This result agrees with the answer gotten from double angle formula.