INTEGRATING POWERS OF SINE, COSINE, TANGENT AND SECANT BY REDUCTION FORMULAS

HANDOUT

Reduction formulas:

$$\int sin^n x \; dx = -\frac{(sin^{n-1} x \; cos\, x)}{n} + \frac{(n-1)}{n} \int sin^{n-2} x \; dx$$

$$\int cos^n x \; dx = \frac{(cos^{n-1} x \, sin\, x)}{n} + \frac{(n-1)}{n} \int cos^{n-2} x \; dx$$

$$\int sec^n x \; dx = \frac{(sec^{n-2} x \; tan\, x)}{n-1} + \frac{(n-2)}{n-1} \int sec^{n-2} x \; dx$$

$$\int tan^n x \; dx = \frac{tan^{n-1} x}{n-1} - \int tan^{n-2} x \; dx$$

These reduction formulas will work for any powers of the above trig functions. They are derived via integration by parts.

DERIVATION OF REDUCTION FORMULAS VIA INTEGRATION BY PARTS

$$\int sin^n x\ dx = \int sin^{n-1} x\ sin\, x\ \ dx$$

$let\ u = sin^{n-1} x \qquad\qquad dv = sin\, x\ \ dx$

$du = (n-1)\, sin^{n-2} x\ cos\, x\ \ dx \qquad\qquad v = -cos\, x$

$$\int u\ dv = uv - \int v\, du$$

$$\int sin^n x\ dx = -cos\, x\ sin^{n-1} x - \int (-cos\, x)\,(n-1)\, sin^{n-2} x\ cos\, x\ \ dx$$

$$\int sin^n x\ dx = -cos\, x\ sin^{n-1} x + (n-1) \int sin^{n-2} x\ cos^2 x\ \ dx$$

$$\int sin^n x\ dx = -cos\, x\ sin^{n-1} x + (n-1) \int sin^{n-2} x\ \ (1 - sin^2 x)\ \ dx$$

$$\int sin^n x\ dx = -cos\, x\ sin^{n-1} x + (n-1) \int sin^{n-2} x\ dx - (n-1) \int sin^n x \quad dx$$

We note a recursion in the integrals:

$$n \int sin^n x\ dx = -cos\, x\ sin^{n-1} x + (n-1) \int sin^{n-2} x\ dx$$

$$\int sin^n x\ dx = \frac{(-cos\, x\ sin^{n-1} x)}{n} + \frac{(n-1)}{n} \int sin^{n-2} x\ dx$$

$$\int cos^n x \; dx \;=\; \int \; cos^{n-1} x \; cos\,x \;\; dx$$

$let\; u = cos^{n-1} x$ $\qquad\qquad$ $dv = cos\,x \; dx$

$du = \; -(n-1)\, cos^{n-2} x \;\; sin\,x \;\; dx$ $\qquad\qquad$ $v = sin\,x$

$$\int u \; dv \;=\; uv \;-\; \int v \, du$$

$$\int cos^n x \; dx \;=\; cos^{n-1} x \; sin\,x \;-\; \int sin\,x \; \big(-(n-1)\big) \; cos^{n-2} x \, sin\,x \;\; dx$$

$$\int cos^n x \; dx \;=\; cos^{n-1} x \; sin\,x \;+\; (n-1) \int \; cos^{n-2} x \;\; sin^2 x \;\; dx$$

$$\int cos^n x \; dx \;=\; cos^{n-1} x \; sin\,x \;+\; (n-1) \int \; cos^{n-2} x \;\; (1 - cos^2 x) \;\; dx$$

$$\int cos^n x \; dx \;=\; cos^{n-1} x \; sin\,x \;+\; (n-1) \int \; cos^{n-2} x \;\; dx \;-\; (n-1) \int \; cos^n x \;\; dx$$

We note a recursion has occurred in the integrals:

$$n \int cos^n x \; dx \;=\; cos^{n-1} x \; sin\,x \;+\; (n-1) \int \; cos^{n-2} x \;\; dx$$

$$\int cos^n x \; dx \;=\; \frac{(cos^{n-1} x \; sin\,x)}{n} \;+\; \frac{(n-1)}{n} \int \; cos^{n-2} x \;\; dx$$

$$\int sec^n\, x\ dx \ = \ \int \ sec^{n-2}\, x \quad sec^2\, x \ \ dx$$

$let\ u = sec^{n-2}\, x$ $\qquad\qquad$ $dv = sec^2\, x \ \ dx$

$du = (n-2)\, sec^{n-3}\, x\ \ sec\, x \quad tan\, x \ \ dx$ $\qquad\qquad$ $v = tan\, x$

$du = (n-2)\, sec^{n-2}\, x\ \ tan\, x \ \ dx$ $\qquad\qquad$ $v = tan\, x$

$$\int u\, dv \ = \ uv - \int \ v\, du$$

$$\int sec^n\, x\ dx \ = \ sec^{n-2}\, x\ tan\, x \ - \ \int tan\, x\ \big((n-2)\, sec^{n-2}\, x\ tan\, x\big)\, dx$$

$$\int sec^n\, x\ dx \ = \ sec^{n-2}\, x\ tan\, x \ - \ (n-2) \int \ (\, sec^{n-2}\, x\ tan^2\, x)\, dx$$

Use Pythagorean relation on tangent squared:

$$\int sec^n\, x\ dx \ = \ sec^{n-2}\, x\ tan\, x \ - \ (n-2) \int \ \big(\, sec^{n-2}\, x \ \ (sec^2\, x \ - \ 1)\big)\, dx$$

$$\int sec^n\, x\ dx \ = \ sec^{n-2}\, x\ tan\, x \ - \ (n-2) \int \ (\ sec^n\, x \ - \ sec^{n-2}\, x\,)\, dx$$

$$\int sec^n\, x\ dx \ = \ sec^{n-2}\, x\ tan\, x \ - \ (n-2) \int \ sec^n\, x\ dx \ + \ (n-2) \int \ sec^{n-2}\, x\ dx$$

We note a recursion has occurred in the integrations:

$$(n-1) \int sec^n\, x\ dx \ = \ sec^{n-2}\, x\ tan\, x \ + \ (n-2) \int \ sec^{n-2}\, x\ dx$$

$$\int sec^n\, x\ dx \ = \frac{(sec^{n-2}\, x\ tan\, x)}{n-1} \ + \frac{(n-2)}{n-1} \int \ sec^{n-2}\, x\ dx$$

$$\int tan^n x \; dx = \int tan^{n-2} x \;\; tan^2 x \;\; dx$$

$$\int tan^n x \; dx = \int tan^{n-2} x \;\; (sec^2 x - 1) \; dx$$

$$\int tan^n x \; dx = \int tan^{n-2} x \;\; sec^2 x \;\; dx - \int tan^{n-2} x \quad dx$$

$let\; u = tan^{n-2} x \qquad\qquad dv = sec^2 x \; dx$

$du = (n-2)\, tan^{n-3} x \;\; sec^2 x \; dx \qquad\qquad v = tan\, x$

$$\int u\, dv = uv - \int v\, du$$

$$\int tan^n x \; dx = tan^{n-2} x \; tan\, x - \int tan\, x \; (n-2)\, tan^{n-3} x \;\; sec^2 x \; dx \quad - \int tan^{n-2} x \quad dx$$

$$\int tan^n x \; dx = tan^{n-1} x - (n-2) \int tan^{n-2} x \;\; sec^2 x \; dx \quad - \int tan^{n-2} x \quad dx$$

Use Pythagorean relation on secant squared – middle term –

$$\int tan^n x \; dx = tan^{n-1} x - (n-2) \int tan^{n-2} x \;\; (tan^2 x + 1) \; dx \quad - \int tan^{n-2} x \quad dx$$

$$\int tan^n x \; dx = tan^{n-1} x - (n-2) \int tan^n x \; dx - (n-2) \int tan^{n-2} x \; dx - \int tan^{n-2} x \quad dx$$

$$\int tan^n x \; dx = tan^{n-1} x - (n-2) \int tan^n x \; dx - (n-1) \int tan^{n-2} x \quad dx$$

We note a recursion -

$$(n-1) \int tan^n x \; dx = tan^{n-1} x - (n-1) \int tan^{n-2} x \quad dx$$

$$\int tan^n x \; dx = \frac{(\, tan^{n-1} x \,)}{n-1} - \int tan^{n-2} x \quad dx$$

INTEGRATION EXAMPLES SINE

$$\int sin^n x \ dx = \frac{(-\cos x \ sin^{n-1} x)}{n} + \frac{(n-1)}{n} \int \ sin^{n-2} x \ dx$$

n=1

$$\int sin^1 x \ dx = \frac{(-\cos x \ sin^{1-1} x)}{1} + \frac{(1-1)}{1} \int \ sin^{1-2} x \ dx$$

$$\int sin^1 x \ dx = \frac{(-\cos x \ sin^0 x)}{1} + 0 \int \ sin^{-1} x \ dx$$

$$\int sin \ x \ dx = \frac{(-\cos x \ )}{1} + 0 = -\cos x + c$$

n=2

$$\int sin^2 x \ dx = \frac{(-\cos x \, sin^{2-1} x)}{2} + \frac{1}{2} \int sin^{2-2} x \ dx = \frac{x}{2} - \frac{(\cos x \sin x)}{2} + c$$

$$\int sin^2 x \ dx = \frac{(-\cos x \sin x)}{2} + \frac{1}{2} \int sin^0 x \ dx$$

$$\int sin^2 x \ dx = \frac{(-\cos x \sin x)}{2} + \frac{1}{2} \int \ dx$$

$$\int sin^2 x \ dx = \frac{(-\cos x \sin x)}{2} + \frac{x}{2} + c$$

n=3

$$\int sin^3 x \ dx = \frac{(-\cos x \ sin^{3-1} x)}{3} + \frac{(3-1)}{3} \int \ sin^{3-2} x \ dx$$

$$\int sin^3 x \ dx = \frac{(-\cos x \ sin^2 x)}{3} + \frac{2}{3} \int \ sin^1 x \ dx$$

$$\int sin^3 x \; dx \; = \frac{(-\cos x \; sin^2 x)}{3} \; - \; \frac{2}{3} \cos x \; + \; c$$

n=4

$$\int sin^n x \; dx \; = \frac{(-\cos x \; sin^{n-1} x)}{n} \; + \; \frac{(n-1)}{n} \int \; sin^{n-2} x \; dx$$

$$\int sin^4 x \; dx \; = \frac{(-\cos x \; sin^{4-1} x)}{4} \; + \; \frac{(4-1)}{4} \int \; sin^{4-2} x \; dx$$

$$\int sin^4 x \; dx \; = \frac{(-\cos x \; sin^3 x)}{4} \; + \; \frac{3}{4} \int \; sin^2 x \; dx$$

$$\int sin^4 x \; dx \; = \frac{(-\cos x \; sin^3 x)}{4} \; + \; \frac{3}{4} \; \left(\frac{x}{2} - \frac{\sin 2x}{4} \right) \; + \; c$$

INTEGRATION EXAMPLES COSINE

$$\int cos^n x \ dx = \frac{(cos^{n-1} x \sin x)}{n} + \frac{(n-1)}{n} \int cos^{n-2} x \ \ dx$$

n=1

$$\int cos^1 x \ dx = \frac{(cos^{1-1} x \sin x)}{1} + \frac{(1-1)}{1} \int cos^{1-2} x \ \ dx$$

$$\int cos \ x \ dx = \frac{(cos^0 x \ \ \sin x)}{1} + \frac{0}{1} \int cos^{-1} x \ \ dx$$

$$\int cos \ x \ dx = \sin x + \ 0 = \sin x + \ c$$

n=2

$$\int cos^2 x \ dx = \frac{(cos^{2-1} x \sin x)}{2} + \frac{(2-1)}{2} \int cos^{2-2} x \ \ dx$$

$$\int cos^2 x \ dx = \frac{(cos^1 x \sin x)}{2} + \frac{1}{2} \int cos^0 x \ \ dx$$

$$\int cos^2 x \ dx = \frac{(cos \ x \ \sin x)}{2} + \frac{1}{2} \int \ dx$$

$$\int cos^2 x \ dx = \frac{(cos \ x \ \sin x)}{2} + \frac{x}{2} + \ c$$

n=3

$$\int cos^3 x \ dx = \frac{(cos^{3-1} x \sin x)}{3} + \frac{(3-1)}{3} \int cos^{3-2} x \ \ dx$$

$$\int cos^3 x \ dx = \frac{(cos^2 x \sin x)}{3} + \frac{2}{3} \int cos^1 x \ \ dx$$

$$\int cos^3 x \ dx = \frac{(cos^2 x \sin x)}{3} + \frac{2}{3} \int \cos x \ \ dx$$

$$\int cos^3 x \ dx = \frac{(cos^2 x \sin x)}{3} + \frac{2}{3} \sin x + \ c$$

n=4

$$\int cos^n x \ dx \ = \frac{(cos^{n-1} x \, sin \, x)}{n} \ + \ \frac{(n-1)}{n} \int \ cos^{n-2} x \ \ dx$$

$$\int cos^4 x \ dx \ = \frac{(cos^{4-1} x \, sin \, x)}{4} \ + \ \frac{(4-1)}{4} \int \ cos^{4-2} x \ \ dx$$

$$\int cos^4 x \ dx \ = \frac{(cos^3 x \, sin \, x)}{4} \ + \ \frac{3}{4} \int \ cos^2 x \ \ dx$$

$$\int cos^4 x \ dx \ = \frac{(cos^3 x \, sin \, x)}{4} \ + \ \frac{3}{4} \left(\frac{x}{2} \ + \frac{sin \, 2x}{4} \right) + \ c$$

INTEGRATION EXAMPLES SECANT

$$\int sec^n x \ \ dx \ = \frac{(sec^{n-2} x \ tan\, x)}{n-1} + \frac{(n-2)}{n-1} \int sec^{n-2} x \ dx$$

We start with n = 2

$$\int sec^2 x \ \ dx \ = \frac{(sec^{2-2} x \ tan\, x)}{2-1} + \frac{(2-2)}{2-1} \int sec^{2-2} x \ dx$$

$$\int sec^2 x \ \ dx \ = \frac{(sec^0 x \ tan\, x)}{1} + \frac{0}{1} \int sec^0 x \ dx$$

$$\int sec^2 x \ \ dx \ = tan\, x + \ \ 0 \ = \ tan\, x + c$$

n=3

$$\int sec^3 x \ \ dx \ = \frac{(sec^{3-2} x \ tan\, x)}{3-1} + \frac{(3-2)}{3-1} \int sec^{3-2} x \ dx$$

$$\int sec^3 x \ \ dx \ = \frac{(sec^1 x \ tan\, x)}{2} + \frac{1}{2} \int sec^1 x \ dx$$

$$\int sec^3 x \ \ dx \ = \frac{(sec \ x \ tan\, x)}{2} + \frac{1}{2} \int sec \ x \ dx$$

$$\int sec^3 x \ \ dx \ = \frac{(sec \ x \ tan\, x)}{2} + \frac{1}{2} ln|sec\, x \ + \ tan\, x| \ + \ \ c$$

n=4

$$\int sec^4 x \ \ dx \ = \frac{(sec^{4-2} x \ tan\, x)}{4-1} + \frac{(4-2)}{4-1} \int sec^{4-2} x \ dx$$

$$\int sec^4 x \ \ dx \ = \frac{(sec^2 x \ tan\, x)}{3} + \frac{2}{3} \int sec^2 x \ dx$$

$$\int sec^4 x \ \ dx \ = \frac{(sec^2 x \ tan\, x)}{3} + \frac{2}{3} \, tan\, x \ + c$$

INTEGRATION EXAMPLES TANGENT

$$\int tan^n x \; dx = \frac{tan^{n-1} x}{n-1} - \int tan^{n-2} \; x \; dx$$

We start with n = 2

$$\int tan^2 x \; dx = \frac{tan^{2-1} x}{2-1} - \int tan^{2-2} \; x \; dx$$

$$\int tan^2 x \; dx = \frac{tan^1 x}{1} - \int tan^0 \; x \; dx$$

$$\int tan^2 x \; dx = tan\, x - \int dx$$

$$\int tan^2 x \; dx = tan\, x - x + c$$

n=3

$$\int tan^3 x \; dx = \frac{tan^{3-1} x}{3-1} - \int tan^{3-2} \; x \; dx$$

$$\int tan^3 x \; dx = \frac{tan^2 x}{2} - \int tan^1 \; x \; dx$$

$$\int tan^3 x \; dx = \frac{tan^2 x}{2} - \int tan\, x \; dx$$

$$\int tan^3 x \; dx = \frac{tan^2 x}{2} - ln|sec\, x| + c$$

N=4

$$\int tan^4 x \; dx = \frac{tan^{4-1} x}{4-1} - \int tan^{4-2} \; x \; dx$$

$$\int tan^4 x \; dx = \frac{tan^3 x}{3} - \int tan^2 \; x \; dx$$

$$\int tan^4 x \; dx = \frac{tan^3 x}{3} - (tan\, x - x) + c$$