TAYLOR SERIES DERIVATIONS

BY MEAN VALUE THEOREM

Before we begin with the mean value theorem, there is a very useful and extremely relevant statement about derivatives that should be mentioned at the beginning: $\frac{d^n(x^n)}{dx^n} = n!$ It is not obvious why this is important, but it will be evident later.

First use Rolle's theorem to get mean value theorem. f(x) is continuous and differentiable. We create a test function H(x):

$$H(x) = (b-a)\big(f(x) - f(a)\big) - (x-a)\big(f(b) - f(a)\big)$$

Creating a test function H(x) to use Rolle's theorem.

$H(a) = 0 \quad H(b) = 0$

H(x) is continuous and differentiable. H'(c) = 0.

$$H'(x) = (b-a)\, f'(x) - \big(f(b) - f(a)\big)$$

$$H'(c) = 0 = (b-a)\, f'(c) - \big(f(b) - f(a)\big)$$

$f'(c) = \frac{f(b) - f(a)}{b - a}$ Mean value theorem established.

We can write this as:

$$f(b) = f(a) + f'(c)\,(b-a)$$

$f(b) \approx f(a) + f'(a)\,(b-a)$ approximation changing f'(c) into f'(a) since we know where a is

$f(b) = f(a) + f'(a)\,(b-a) + K\,(b-a)^2$ unknown correction term added

This formula is exact for f(b) but only approximate for other values of x:

$$f(x) \approx f(a) + f'(a)(x-a) + K(x-a)^2$$

Create another test function (an error function) to use Rolle's theorem.
Use the above approximation to create the error function.

So we use the function $f(x) \approx f(a) + f'(a)(x-a) + K(x-a)^2$
To create an error function: we take the difference between the left side and the right side of the equation and note the difference:

$$H_1(x) = f(x) - [f(a) + f'(a)(x-a) + K(x-a)^2]$$

We will now examine this error function. We are about to see that both $H_1(x)$ and $H'_1(x)$ are both subject to Rolle's theorem.

To see this, there are two things to note:

First that $H_1(x)$ has a double root at x = a. This means that $H_1(a) = H'_1(a) = 0$.

Second $H_1(a) = H_1(b) = 0$. So we can Invoke Rolle's theorem on $H_1(x)$. There is some point c_1 such that $H'(c_1) = 0$. The resulting equation from this derivative is not important. We could find it, but it would not yield any information for this derivation.

The important result is now this: both $H_1'(a)$ and $H_1'(c_1)$ equal zero. Since H_1' is continuous and differentiable, we can invoke Rolle's theorem on H_1':

$$H_1'(x) = f'(x) - [\, f'(a) + 2K(x-a)\,]$$

There exists c_2 such that $H_1''(c_2) = 0$

$$H_1''(x) = f''(x) - 2K$$

$$H_1''(c_2) = f''(c_2) - 2K = 0$$

We now have a value of K: $K = \frac{1}{2} f''(c_2)$

We can go back and rewrite some equations:

$f(b) = f(a) + f'(a)(b-a) + K(b-a)^2$ will become

$f(b) \; = \; f(a) \; + \; f'(a)\,(b-a) \; + \; \frac{1}{2}\, f''(c_2)\,(b-a)^2$

Also:

$f(x) \; \approx \; f(a) \; + \; f'(a)\,(x-a) \; + \; K\;(x-a)^2$ will become

$f(x) \; \approx \; f(a) \; + \; f'(a)\,(x-a) \; + \; \frac{1}{2}\, f''(c_2)\;(x-a)^2$ will become

There is an ordered sequence that Rolle's theorem is creating: we have the following inequalities:

$a < \; c_2 \; < \; c_1 \; < b$

Every time we use Rolle's theorem, a new constant moves closer to a.

We do this procedure one more time to see the pattern.

We start with exact expression

$f(b) \; = \; f(a) \; + \; f'(a)\,(b-a) \; + \; \frac{1}{2}\, f''(c_2)\,(b-a)^2$

and change it into

$f(b) \; \cong \; f(a) \; + \; f'(a)\,(b-a) \; + \; \frac{1}{2}\, f''(a)\,(b-a)^2$

We do this because we know where a is and we do not know where c_2 is (it would take some work to find it).

We generalize the above approximation to get

$f(x) \; \cong \; f(a) \; + \; f'(a)\,(x-a) \; + \; \frac{1}{2}\, f''(a)\,(x-a)^2$

We create an error term of f(b) to make the equation exact:

We start with $f(b) \; \cong \; f(a) \; + \; f'(a)\,(b-a) \; + \; \frac{1}{2}\, f''(a)\,(b-a)^2$

and we assume a correction

$f(b) \; = \; f(a) \; + \; f'(a)\,(b-a) \; + \; \frac{1}{2}\, f''(a)\,(b-a)^2 + \; K\,(b-a)^3$

This equation is exact and works only for f(b).

It will be approximate for x:

$$f(x) \cong f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + K(x-a)^3$$

To determine the form for K, we create another error function – taking the difference between the left side and the right side of the last approximation:

$$H(x) = f(x) - \left[f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + K(x-a)^3 \right]$$

This error function has all the properties we could hope for:

$H(a) = 0 \qquad H'(a) = 0 \qquad H''(a) = 0$

$H(b) = 0$

Take $H(a) = 0$ and $H(b) = 0$. Rolle's theorem applies so $H'(c_1) = 0$.

Take $H'(a) = 0$ and $H'(c_1) = 0$. Rolle's theorem applies so $H''(c_2) = 0$.

Take $H''(a) = 0$ and $H''(c_2) = 0$. Rolle's theorem applies so $H'''(c_3) = 0$.

$$H(x) = f(x) - \left[f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + K(x-a)^3 \right]$$

$$H'''(x) = f'''(x) - 3!\, K$$

$$H'''(c_3) = f'''(c_3) - 3!\, K = 0$$

$$K = \frac{1}{3!} f'''(c_3)$$

We can now rewrite our two expressions for f(b) and f(x)

These equations:

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2} f''(a)(b-a)^2 + K(b-a)^3$$

$$f(x) \cong f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + K(x-a)^3$$

Now become

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2!} f''(a)(b-a)^2 + \frac{1}{3!} f'''(c_3)(b-a)^3$$

$$f(x) \cong f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \frac{1}{3!} f'''(c_3)(x-a)^3$$

Rolle's theorem has produced a sequence of inequalities:
$a < c_3 < c_2 < c_1 < b$

The final generalization is this:

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2!} f''(a)(b-a)^2 + \frac{1}{3!} f'''(a)(b-a)^3 + \frac{1}{4!} f''''(a)(b-a)^4 + \ldots + \frac{1}{n!} f^n(a)(b-a)^n + K(b-a)^{n+1}$$

K will be given by $K = \frac{f^{n+1}(c_{n+1})}{(n+1)!}$

The error term is $error = K(b-a)^{n+1} = \frac{f^{n+1}(c_{n+1})}{(n+1)!}(b-a)^{n+1}$

The sequence of constants is $a < c_{n+1} < c_n < \cdots < c_2 < c_1 < b$

The error term is approaching zero. The factorial in the denominator is making the fraction very small.

DERIVATION USING INTEGRATION BY PARTS

We start with the statement

$$f(b) - f(a) = \int_a^b f'(t)\ dt$$

Now we do integration by parts on the integral:

$\int_a^b f'(t)\ dt \qquad u = f'(t) \qquad dv = dt$

$\qquad du = f''(t)\,dt \qquad v = t - b = -(b - t)$

$$\int u\,dv = u\,v - \int v\ du$$

$$\int_a^b f'(t)\ dt = f'(t)\,(t-b)\Big|_a^b - \int_a^b f''(t)\,[\,-(b-t)\,]\,dt$$

$$\int_a^b f'(t)\ dt = f'(t)\,(t-b)\Big|_a^b + \int_a^b f''(t)\,[\,(b-t)\,]\,dt$$

$$\int_a^b f'(t)\ dt = f'(b)\,(b-b) - f'(a)\,(a-b) + \int_a^b f''(t)\,(b-t)\ dt$$

$$\int_a^b f'(t)\ dt = -f'(a)\,(a-b) + \int_a^b f''(t)\,(\,b-t\,)\ dt$$

$$\int_a^b f'(t)\ dt = f'(a)\,(b-a) + \int_a^b f''(t)\,(\,b-t\,)\ dt$$

$$f(b) - f(a) = f'(a)\,(b-a) + \int_a^b f''(t)\,(\,b-t\,)\ dt$$

$$f(b) = f(a) + f'(a)\,(b-a) + \int_a^b f''(t)\,(\,b-t\,)\ dt$$

$\int u\,dv = u\,v - \int v\ du \qquad u = f''(t) \qquad dv = (b-t)\,dt$

$\qquad du = f'''(t)\,dt \qquad v = \frac{(b-t)^2}{2}$

$$f(b) = f(a) + f'(a)\,(b-a) - f''(t)\,\frac{(b-t)^2}{2}\Big|_a^b + \int_a^b f'''(t)\,\frac{(\,b-t\,)^2}{2}\ dt$$

$$f(b) = f(a) + f'(a)\,(b-a) + f''(a)\,\frac{(\,b\ -\ a)^2}{2} + \int_a^b f'''(t)\,\frac{(b-t)^2}{2}\ dt$$

$$f(b) = f(a) + f'(a)(b-a) + f''(a)\frac{(b-a)^2}{2} + \int_a^b f'''(t)\frac{(b-t)^2}{2}\ dt$$

$\int u\,dv = u\,v - \int v\ du$ $\qquad u = f'''(t) \qquad dv = \frac{(b-t)^2}{2}\,dt$

$\qquad du = f''''(t)\ dt \qquad v = -\frac{(b-t)^3}{3!}\ dt$

$$f(b) = f(a) + f'(a)(b-a) + f''(a)\frac{(b-a)^2}{2} - f'''(t)\frac{(b-t)^3}{3!}\Big|_a^b + \int_a^b f''''(t)\frac{(b-t)^3}{3!}\ dt$$

$$f(b) = f(a) + f'(a)(b-a) + f''(a)\frac{(b-a)^2}{2} + f'''(a)\frac{(b-a)^3}{3!} + \int_a^b f''''(t)\frac{(b-t)^3}{3!}\ dt$$

$$f(b) = f(a) + f'(a)(b-a) + f''(a)\frac{(b-a)^2}{2} + f'''(a)\frac{(b-a)^3}{3!} + \int_a^b f''''(t)\frac{(b-t)^3}{3!}\ dt$$

If we go to the nth power we get:

$$f(b) = f(a) + f'(a)(b-a) + f''(a)\frac{(b-a)^2}{2} + f'''(a)\frac{(b-a)^3}{3!} + \dots + f^n(a)\frac{(b-a)^n}{n!} + \int_a^b f^{n+1}(t)\frac{(b-t)^n}{n!}\ dt$$

This series is the Taylor series but with b not x.

The remainder term R is the integral $\int_a^b f^{n+1}(t)\frac{(b-t)^n}{n!}\ dt$

We can use the mean value theorem of integral calculus to write it as

$$R = \int_a^b f^{n+1}(t)\frac{(b-t)^n}{n!}\ dt = f^{n+1}(c)\frac{(b-c)^n}{n!}(b-a)$$

where c is somewhere in between a and b. To find the largest possible error, we replace $(b-c)^n$ with $(b-a)^n$ so that $R = \int_a^b f^{n+1}(t)\frac{(b-t)^n}{n!}\ dt = f^{n+1}(c)\frac{(b-a)^{n+1}}{n!}$

Another way to evaluate the remainder is to pull out $f^{n+1}(t)$ from the integral at some constant value.

Evaluate the remaining integral: $R \;=\; f^{\,n+1}(c)\int_a^b \frac{(b-t)^{\,n}}{n!}\; dt$.

This will yield

$R = \; f^{n+1}(c)\;\; \frac{-(b-t)^{n+1}}{(n+1)!}\;\Big|_a^b \;=\; f^{n+1}(c)\;\frac{(\,b-a\,)^{n+1}}{(n+1)!}$

DERIVATION BY N FOLD INTEGRATION

We start with the integral $\int_a^x \int_a^x \cdots \int_a^x f^n(x)\,dx \cdots\; dx\;\; dx$ where there are n integrals and the integrand is $f^n(x) \;=\; \frac{d^n f(x)}{dx^n}$

Technically the notation on this is slightly unorthodox and a careful mathematician would write it as $\int_a^{x_{n+1}} \;\cdots\; \int_a^{x_3} \int_a^{x_2} \int_a^{x_1} \; f(t)\;dt\;\;dx_1\;\;dx_2\;dx_3\;\cdots\;\;dx_n$.

In this case however we will forego the technicality – it won't get us into trouble – and we will stick with the simpler notation of the first integral.

We must now do n integrations.

$$\int_a^x f^n(x)\,dx \;=\; f^{n-1}(x)\,|_a^x \;=\; f^{n-1}(x) - \; f^{n-1}(a)$$

Now 2 integrals:

$$\int_a^x \int_a^x f^n(x)\,dx\;\;dx \;\;=\;\; \int_a^x \left(f^{n-1}(x) - \; f^{n-1}(a)\right)\,dx$$

$$\int_a^x \int_a^x f^n(x)\,dx\;\;dx \;\;=\;\; f^{n-2}(x)|_a^x \;-\; f^{n-1}(a)\,(x-a)$$

$$\int_a^x \int_a^x f^n(x)\,dx\;\;dx \;\;=\;\; f^{n-2}(x) - \; f^{n-2}(a) \;-\; f^{n-1}(a)\,(x-a)$$

Now 3 integrals:

$$\int_a^x \int_a^x \int_a^x f^n(x)\,dx\;\;dx\;\;dx \;\;=\; \int_a^x \left(f^{n-2}(x) - \; f^{n-2}(a) \;-\; f^{n-1}(a)\,(x-a)\right)dx$$

$$\int_a^x \int_a^x \int_a^x f^n(x)\,dx\;\;dx\;\;dx \;\;=\; f^{n-3}(x)|_a^x - \; f^{n-2}(a)\,(x-a) - \; f^{n-1}(a)\,\frac{(x-a)^2}{2}$$

$$\int_a^x \int_a^x \int_a^x f^n(x)\,dx\;\;dx\;\;dx \;=\; f^{n-3}(x) \;-\; f^{n-3}(a) - \; f^{n-2}(a)\,(x-a) - \; f^{n-1}(a)\,\frac{(x-a)^2}{2}$$

Now 4 integrals:

$$\int_a^x \int_a^x \int_a^x \int_a^x f^n(x)\, dx \cdots dx =$$

$$\int_a^x \left(f^{n-3}(x) - f^{n-3}(a) - f^{n-2}(a)\,(x-a) - f^{n-1}(a)\frac{(x-a)^2}{2} \right) dx$$

$$\int_a^x \int_a^x \int_a^x \int_a^x f^n(x)\, dx \cdots dx = f^{n-4}(x) - f^{n-4}(a) - f^{n-3}(a)\,(x-a) - f^{n-2}(a)\frac{(x-a)^2}{2} - f^{n-1}(a)\frac{(x-a)^3}{3!}$$

If we do all n integrals we get:

$$\int_a^x \int_a^x \cdots \int_a^x \int_a^x f^n(x)\, dx \cdots dx = f(x) - f(a) - f'(a)\,(x-a) - f''(a)\frac{(x-a)^2}{2} - f^{n-1}(a)\frac{(x-a)^{n-1}}{(n-1)!}$$

We see the Taylor series forming on the right.

We now have to consider the original n-fold integral one more time – but we must do it in a different way.

So once again we start with

$$\int_a^x \int_a^x \cdots \int_a^x \int_a^x f^n(x)\, dx \cdots dx$$

and once again we look at only the first integral: $\int_a^x f^n(x)\, dx$

This time, instead of doing the integral, we invoke the mean value theorem of integral calculus:

$$\int_a^x f^n(x)\, dx = f^n(c)\,(x-a)$$

Put this result into the remaining (n-1) integrals:

$$\int_a^x \int_a^x \cdots \int_a^x \int_a^x f^n(c)\,(x-a)\, dx \cdots dx$$

The derivative is a constant since it is already evaluated at some point c (in between a and x). We can bring it outside the integral

$$\int_a^x \int_a^x \cdots \int_a^x \int_a^x f^n(c)\,(x-a)\, dx \cdots dx = f^n(c) \int_a^x \int_a^x \cdots \int_a^x \int_a^x (x-a)\, dx \cdots dx$$

We have (n-1) integrals to evaluate on (x-a).

One integral on (x-a) yields

$$\int_a^x (x-a)\,dx = \frac{(x-a)^2}{2}$$

Two integrals on (x-a) yields

$$\int_a^x \int_a^x (x-a)\,dx\ dx = \frac{(x-a)^3}{3!}$$

Taking (n-1) integrals on (x-a) yields

$$\int_a^x \int_a^x \cdots \int_a^x \int_a^x (x-a)\,dx \cdots dx = \frac{(x-a)^n}{n!}$$

So the n fold integral now yields:

$$\int_a^x \int_a^x \cdots \int_a^x \int_a^x f^n(x)\,dx \cdots dx = f^n(c)\,\frac{(x-a)^n}{n!}$$

We equate the two expression for the n fold integral:

$$f^n(c)\,\frac{(x-a)^n}{n!} = f(x) - f(a) - f'(a)(x-a) - f''(a)\frac{(x-a)^2}{2}$$
$$- f^{n-1}(a)\frac{(x-a)^{n-1}}{(n-1)!}$$

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + f'''(a)\frac{(x-a)^3}{3!} + \ldots +$$
$$f^{n-1}(a)\frac{(x-a)^{n-1}}{(n-1)!} + f^n(c)\frac{(x-a)^n}{n!}$$

If we augment n by 1 we have

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + f'''(a)\frac{(x-a)^3}{3!} + \ldots +$$
$$f^n(a)\frac{(x-a)^n}{n!} + f^{n+1}(c)\frac{(x-a)^{n+1}}{(n+1)!}$$

The remainder term is $R = f^{n+1}(c)\,\frac{(x-a)^{n+1}}{(n+1)!}$. It will approach zero as n goes to infinity