METHOD OF WASHERS

1. You are given the curves y = x and y = x^2. Consider the region bounded by these two curves in the first quadrant. It is rotated about the y axis. Find the volume generated.

2. You are given the functions $y = x^2$ and $y = x$. Consider the region bounded by the two functions in the first quadrant. The region is rotated about the x axis. Find the volume generated.

3. You are given the region bounded by the parabola $y = x^2$, $x = 2$ and the x axis. This region is rotated about the y axis. Find the volume generated.

4. You are given the functions $y = x^2 + \frac{1}{2}$ and the function $y = x$ over the interval $x \in [0,2]$. The region is revolved about the x axis. Find the volume generated.

5. You are given the region bounded by the curve $y = x^2 \quad x = 2$ and the x axis. The region is rotated about the line y = 4. Find the volume of generated.

6. You are given the region bounded by the curves $y = x^2$ $x = 2$ and the x axis. The region is rotated about the vertical line x = 5. Find the volume of the region.

7. Find the volume of the solid that results when the region bounded by the function $y = x^2$ and $y = 9$ is revolved about the x axis.

8. Find the volume of the solid that results when the region enclosed by $y = x^2 + 1$ and $y = x + 3$ is revolved about the x axis.

9. You are given a sphere of radius R. A cylindrical hole is bored out of the sphere through its center. Find the volume of the remaining solid.

DIFFUCULT QUESTION VOLUME OF TORUS

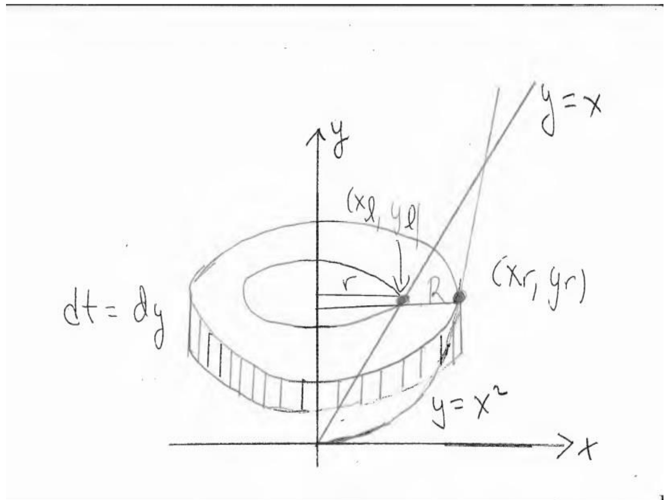
10. You are given the circle $(x - b)^2 + y^2 = b^2$ centered at (b,0). It is rotated about the line $x = a$ where $a > 2b$. Find the volume of the torus.

DIFFUCULT QUESTION

11. You are given the parabola $y = x^2$ bounded above by the line $y = 1$. This region is revolved around the vertical line x = 2. Find the volume of the solid generated. It looks like an upside down Bundt cake.

METHOD OF WASHERS

1. You are given the curves y = x and y = x^2. Consider the region bounded by these two curves in the first quadrant. It is rotated about the y axis. Find the volume generated.



Right function | Left function

$dt = dy$

$y = x^2$ $\qquad$ $y = x$

$R = x_r = \sqrt{y}$ $\qquad$ $r = x_l = y$

The curves intersect at the points (0,0) and (1,1)

$$V = \int_a^b \pi(R^2 - r^2)\, dt$$

$$V = \int_{y=0}^{y=1} \pi\left(x_r^2 - x_l^2\right) dy$$

$$V = \int_{y=0}^{y=1} \pi\left(\left(\sqrt{y}\right)^2 - y^2\right) dy$$

$$V = \pi \int_{y=0}^{y=1} (y - y^2)\, dy$$

$$V = \pi\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^1$$

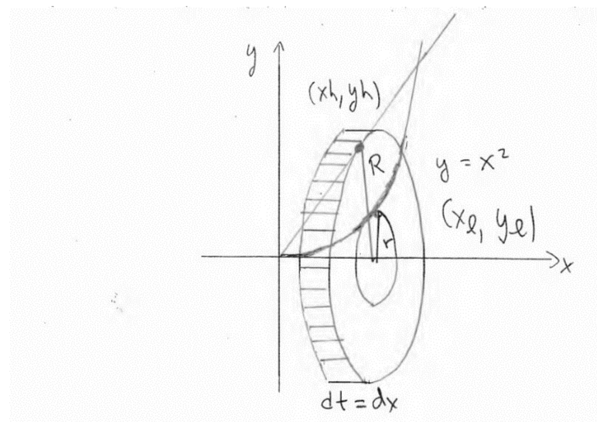
$$V = \pi\left(\frac{1}{2} - \frac{1}{3}\right) - 0 = \frac{\pi}{6}$$

2. You are given the functions $y = x^2$ and $y = x$. Consider the region bounded by the two functions in the first quadrant. The region is rotated about the x axis. Find the volume generated.

High function Low function dt=dx

$R = y_h = x \qquad r = y_l = x^2$

The points of intersection are (0,0) and (1,1)



$$V = \int_a^b \pi(R^2 - r^2)\, dt$$

$$V = \int_{x=0}^{x=1} \pi(\, y_h^2 - y_l^2 \,)\, dx$$

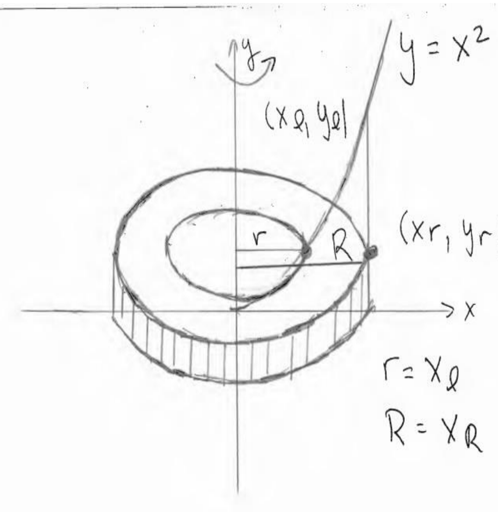
$$V = \int_{x=0}^{x=1} \pi(\, x^2 - x^4 \,)\, dx$$

$$V = \pi \left(\frac{x^3}{3} - \frac{x^5}{5} \right) \Bigg|_0^1$$

$$V = \pi \left(\frac{1}{3} - \frac{1}{5} \right)$$

$$V = \frac{2\pi}{15}$$

3. You are given the region bounded by the parabola $y = x^2$, $x = 2$ and the x axis. This region is rotated about the y axis. Find the volume generated.



Right function   Left function

dt = dy

$R = x_r = 2$ $r = x_l = \sqrt{y}$

The bounds of integration are y = 0 to y = 4

$$V = \int_a^b \pi(R^2 - r^2)\,dt$$

$$V = \int_{y=0}^{y=4} \pi(x_r^2 - x_l^2)\,dy$$

$$V = \int_{y=0}^{y=4} \pi\left(4 - \left(\sqrt{y}\right)^2\right)\,dy$$

$$V = \int_{y=0}^{y=4} \pi(4 - y)\,dy$$

$$V = \pi\left(4y - \frac{y^2}{2}\right)\Big|_0^4$$

$$V = \pi\left(16 - \frac{16}{2}\right) - (0)$$

$$V = 8\pi$$

4. You are given the functions $y = x^2 + \frac{1}{2}$ and the function $y = x$ over the interval $x \in [0,2]$. The region is revolved about the x axis. Find the volume generated.

High function | Low function | dt = dx

$y = x^2 + \frac{1}{2}$ | $y = x$

$R = y_h = x^2 + \frac{1}{2}$ | $r = y_l = x$

$$V = \int_a^b \pi(R^2 - r^2)\, dt$$

$$V = \int_{x=0}^{x=2} \pi\left(y_h^2 - y_l^2\right) dx$$

$$V = \int_{x=0}^{x=2} \pi\left(\left(x^2 + \frac{1}{2}\right)^2 - x^2\right) dx$$

$$V = \pi \int_{x=0}^{x=2} \left(x^4 + x^2 + \frac{1}{4} - x^2\right) dx$$

$$V = \pi \int_{x=0}^{x=2} \left(x^2 + \frac{1}{4}\right) dx$$

$$V = \pi\left(\frac{x^3}{3} + \frac{x}{4}\right)\Bigg|_0^2$$

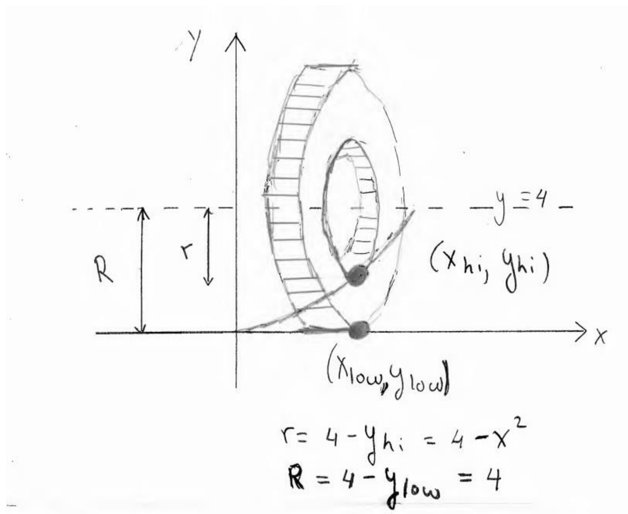
$$V = \pi\left(\frac{8}{3} + \frac{1}{2}\right) = \frac{19\pi}{3}$$

5. You are given the region bounded by the curve $y = x^2$ $x = 2$ and the x axis. The region is rotated about the line y = 4. Find the volume of generated.

Large radius $R = 4$

Small radius $r = 4 - y = 4 - x^2$

dt = dx



$$V = \int_a^b \pi(R^2 - r^2)\, dt$$

$$V = \int_{x=0}^{x=2} \pi(4^2 - (4 - x^2)^2)\, dx$$

$$V = \int_{x=0}^{x=1} \pi\big(16 - (16 - 8x^2 + x^4)\big)\, dx$$

$$V = \int_{x=0}^{x=1} \pi(8x^2 - x^4)\, dx$$

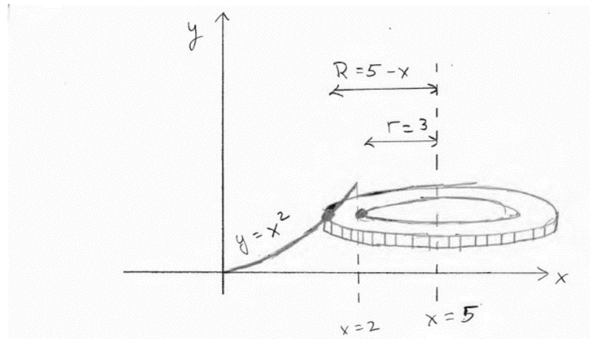
$$V = \pi\left(\frac{8x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^1$$

$$V = \pi\left(\frac{8}{3} - \frac{1}{5}\right) = \frac{37\pi}{15}$$

6. You are given the region bounded by the curves $y = x^2 \quad x = 2$ and the x axis. The region is rotated about the vertical line x = 5. Find the volume of the region.

$R = 5 \qquad\qquad r = 5 - x \;=\; 5 - \sqrt{y} \qquad\qquad$ dt = dy

The bounds of integration are y = 0 and y = 4.



$$V = \int_a^b \pi\,(R^2 - r^2)\,dt$$

$$V = \int_{y=0}^{y=5} \pi\,(5^2 - (5-x)^2)\,dy$$

$$V = \int_{y=0}^{y=5} \pi\left(25 - \left(5-\sqrt{y}\right)^2\right)dy$$

$$V = \int_{y=0}^{y=5} \pi\left(25 - \left(25 - 10\sqrt{y} + y\right)\right)dy$$

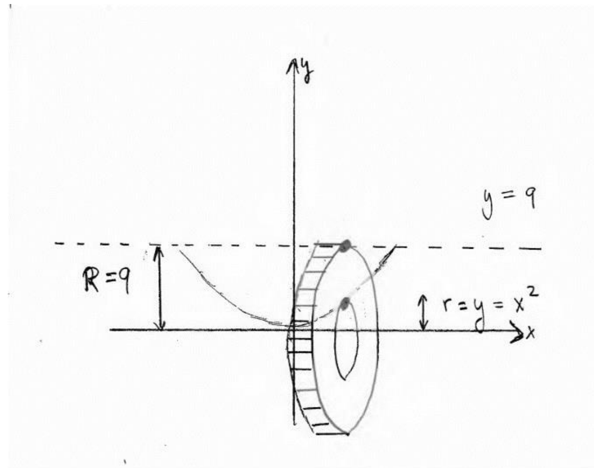
$$V = \int_{y=0}^{y=5} \pi\left(10\sqrt{y} - y\right)dy$$

$$V = \pi\left(\frac{20}{3}\,y^{3/2} - \frac{y^2}{2}\right)\Bigg|_0^5$$

$$V = \pi\left(\frac{20\sqrt{125}}{3} - \frac{25}{2}\right)$$

7. Find the volume of the solid that results when the region bounded by the function $y = x^2$ and $y = 9$ is revolved about the x axis.

High function	Low function
$y = 9$	$y = x^2$
$R = y_h = 9$	$r = y_l = x^2$



$$V = \int_a^b \pi\,(R^2 - r^2)\,dt$$

$$V = \int_a^b \pi\,(y_h^2 - y_l^2)\,dx$$

$$V = \int_{x=-3}^{x=3} \pi\,(9^2 - (x^2)^2)\,dx$$

$$V = \pi \int_{x=-3}^{x=3} (81 - x^4)\,dx$$

Even symmetry:

$$V = 2\pi \int_{x=0}^{x=3} (81 - x^4)\,dx$$

$$V = 2\pi\left(81\,x - \frac{x^5}{5}\right)\Bigg|_0^3$$

$$V = 2\pi\left(243 - \frac{243}{5}\right)$$

$$V = \frac{1944\pi}{5}$$

8. Find the volume of the solid that results when the region enclosed by $y = x^2 + 1$ and $y = x + 3$ is revolved about the x axis.

High function	Low function
$y_h = x + 3$	$y_l = x^2 + 1$
$R = y_h$	$r = y_l$
$R = x + 3$	$r = x^2 + 1$

Bounds: $x^2 + 1 = x + 3$ $\quad x^2 - x - 2 = 0 \quad x = -1$ & $x = 2$

$$V = \int_a^b \pi\,(R^2 - r^2)\,dt$$

$$V = \int_{x=-1}^{x=2} \pi\left(y_h^2 - y_l^2\right)dx$$

$$V = \int_{x=-1}^{x=2} \pi\,(\,(x+3)^2 - (x^2 + 1\,)^2)\,dx$$

$$V = \int_{x=-1}^{x=2} \pi\,(\;(x^2 + 6x + 9) - (x^4 + 2x^2 + 1)\,)\,dx$$

$$V = \int_{x=-1}^{x=2} \pi\,(-x^4 - x^2 + 6x + 8\,)\,dx$$

$$V = \pi\left(-\frac{x^5}{5} - \frac{x^3}{3} + 3x^2 + 8x\right)\Bigg|_{-1}^{2}$$

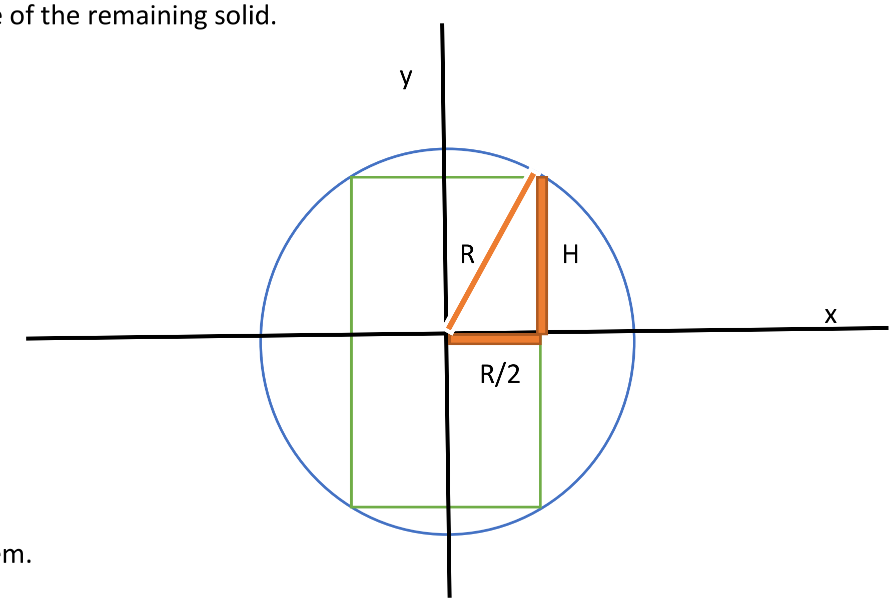
$$V = \pi\left[\left(-\frac{32}{5} - \frac{8}{3} + 12 + 16\right) - \left(-\frac{-1}{5} - \frac{-1}{3} + 3 - 8\right)\right]$$

$$V = \pi\left[28 - \frac{32}{5} - \frac{8}{3} - \frac{1}{5} - \frac{1}{3} + 5\right]$$

$$V = \pi\left[33 - \frac{33}{5} - \frac{9}{3}\right] = \frac{117\pi}{5}$$

9. You are given a sphere of radius R. A cylindrical hole is bored out of the sphere through its center. Find the volume of the remaining solid.

$$x^2 + y^2 = R^2$$



We need to get H.

Use Pythagorean theorem.

$$R^2 = H^2 + \left(\frac{R}{2}\right)^2 \qquad H^2 = \frac{3R^2}{4} \qquad H = \frac{\sqrt{3}\,R}{2}$$

$$V = \int_a^b \pi\,(r_2^2 - r_1^2)\,dt$$

$$V = \int_{y=-H}^{y=H} \pi\left(x^2 - \left(\frac{R}{2}\right)^2\right) dy$$

$$V = \int_{y=-H}^{y=H} \pi\left(R^2 - y^2 - \left(\frac{R}{2}\right)^2\right) dy$$

$$V = 2\pi \int_{y=0}^{y=H} \left(\frac{3R^2}{4} - y^2\right) dy$$

$$V = 2\pi\left(\frac{3R^2\,y}{4} - \frac{1}{3}\,y^3\right)\Bigg|_0^H$$

$$V = 2\pi\left(\frac{3R^2 H}{4} - \frac{H^3}{3}\right)$$

DIFFUCULT QUESTION VOLUME OF TORUS

10. You are given the circle $(x-b)^2 + y^2 \;=\; b^2$ centered at (b,0). It is rotated about the line $x = a$ where $a > 2b$. Find the volume of the torus.

Rewrite the equation:

$$x - b \;=\; \pm\sqrt{b^2 - y^2}$$

$$x = b \pm \sqrt{b^2 - y^2}$$

Left

$$R \;=\; a - x_l = a - \left(b - \sqrt{b^2 - y^2} \right)$$

$$R \;= a - b + \sqrt{b^2 - y^2}$$

$$R \;= (a-b) + \sqrt{b^2 - y^2}$$

Right

$$r = a - x_r = a - \left(b + \sqrt{b^2 - y^2}\right)$$

$$r = \; a - \; b - \; \sqrt{b^2 - y^2}$$

$$r = (\,a - \; b) - \; \sqrt{b^2 - y^2}$$

$$V = \int_{y\,=\,-b}^{y\,=\,b} (\pi R^2 - \pi r^2)\; dy$$

$$V = \int_{y\,=\,-b}^{y\,=\,b} \left(\pi \left((a-b) + \sqrt{b^2 - y^2}\;\right)^2 - \pi\left((\,a - \; b) - \; \sqrt{b^2 - y^2}\right)^2\right)\; dy$$

$$V = \;\pi \int_{y\,=\,-b}^{y\,=\,b} \left(\left((a-b) + \sqrt{b^2 - y^2}\;\right)^2 - \left((\,a - \; b) - \; \sqrt{b^2 - y^2}\right)^2\right)\; dy$$

Expanding and cancelling – there is a lot of cancellation:

$$V = \;\pi \int_{y\,=\,-b}^{y\,=\,b} \left(\; 4(a-b)\sqrt{b^2 - y^2}\right)\; dy$$

Symmetry:

$$V = 8\,(a-b)\;\pi \int_{y\,=\,0}^{y\,=\,b} \sqrt{b^2 - y^2}\; dy$$

$$let\; y = b\sin\theta \qquad\qquad dy = b\cos\theta\;\, d\theta$$

\

$$V = 8\,(a-b)\;\pi \int_{\theta=0}^{\theta=\pi/2} \sqrt{b^2 - b^2\sin^2\theta}\;\; b\cos\theta\;\, d\theta$$

$$V = 8\,(a-b)\,b^2\;\pi \int_{\theta=0}^{\theta=\pi/2} \sqrt{1-\;\sin^2\theta}\;\;\cos\theta\;\, d\theta$$

$$V = 8\,(a-b)\,b^2\;\pi \int_{\theta=0}^{\theta=\pi/2} \cos^2\theta\;\, d\theta$$

$$V = 8\,(a-b)\,b^2\;\pi\;\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]\Bigg|_0^{\pi/2}$$

$$V = 8\,(a-b)\,b^2\;\pi\;\left[\left(\frac{\pi}{4} - \frac{\sin\,\pi}{4}\right) - \;(0)\right]$$

$$V = 2\,\pi\,(a-b)\;\pi\,b^2$$

Note this is the circumference formed by the center of the circle – multiplied by the cross sectional area of the circle.

DIFFUCULT QUESTION

11. You are given the parabola $y = x^2$ bounded above by the line $y = 1$. This region is revolved around the vertical line x = 2. Find the volume of the solid generated. It looks like an upside down Bundt cake.

Left function

$x = -\sqrt{y}$

$x_l = -\sqrt{y}$

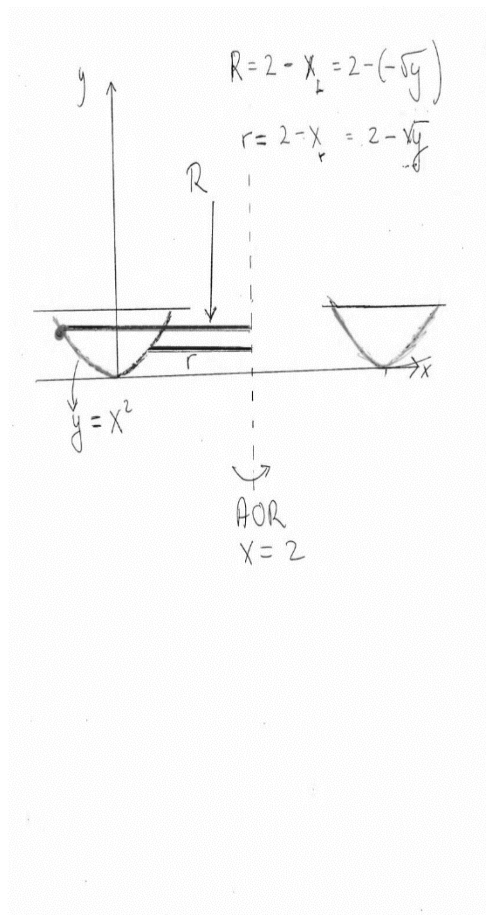
$R = 2 - x_l = 2 + \sqrt{y}$

Right function

$x = \sqrt{y}$

$x_r = \sqrt{y}$

$r = 2 - x_r = 2 - \sqrt{y}$



$$V = \int_{y=0}^{y=1} (\pi R^2 - \pi r^2)\ dy$$

$$V = \pi \int_{y=0}^{y=1} \left(\left(2 + \sqrt{y}\right)^2 - \left(2 - \sqrt{y}\right)^2 \right)\ dy$$

Expand and simplify. There is a lot of cancellation.

$$V = \pi \int_{y=0}^{y=1} \left(4\sqrt{y} \right)\ dy$$

$$V = 4\pi \int_{y=0}^{y=1} \sqrt{y}\ dy$$

$$V = \frac{8\pi}{3}\ y^{3/2} \Big|_0^1$$

$$V = \frac{8\pi}{3}$$