ARC LENGTH IN CARTESIAN FORM

1. Find the arc length of the curve y = 3 x where x goes from 1 to 4.

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \; dx \qquad y = 3x \qquad \frac{dy}{dx} = 3$$

$$s = \int_{x=1}^{x=4} \sqrt{1 + (3)^2} \; dx$$

$$s = \sqrt{10} \int_1^4 dx$$

$$s = 3\sqrt{10}$$

2. Find the arc length of the curve $y = \frac{2}{3} \; x^{3/2}$ as x goes from 0 to 1.

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \; dx \qquad y = \frac{2}{3} \; x^{3/2} \qquad \frac{dy}{dx} = x^{1/2}$$

$$s = \int_{x=0}^{x=1} \sqrt{1 + (x^{1/2})^2} \; dx$$

$$s = \int_{x=0}^{x=1} \sqrt{1 + x} \; dx$$

$$s = \frac{2}{3} \, (1 + x)^{3/2} \Big|_0^1$$

$$s = \frac{2}{3} \left[2^{3/2} \; - \; 1\right]$$

VERY DIFFICULT QUESTION

3. Find the arc length of the parabola $y = \frac{1}{2}x^2$ as x goes from 0 to 1.

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad y = \frac{1}{2}\, x^2 \qquad \frac{dy}{dx} = x$$

$$s = \int_{x=0}^{x=1} \sqrt{1 + (x)^2}\, dx$$

Let $x = \tan t$ $\qquad dx = \sec^2 t\ dt$

$$s = \int_{t=0}^{t=\pi/4} \sqrt{1 + (\tan t)^2}\, \sec^2 t\ dt$$

$$s = \int_{t=0}^{t=\pi/4} \sqrt{\sec^2 t}\, \sec^2 t\ dt$$

$$s = \int_{t=0}^{t=\pi/4} \sec^3 t\ dt$$

Integrate by parts:

$u = \sec t$ $\qquad dv = \sec^2 t\ dt$

$du = \sec t \tan t\ \ dt$ $\qquad v = \tan t$

$$\int_{t=0}^{t=\pi/4} \sec^3 t\ dt = \sec t \tan t\Big|_{t=0}^{t=\frac{\pi}{4}} - \int_{t=0}^{t=\pi/4} \sec t\ \tan^2 t\ \ dt$$

$$\int_{t=0}^{t=\pi/4} \sec^3 t\ dt = \sec t \tan t\Big|_{t=0}^{t=\frac{\pi}{4}} - \int_{t=0}^{t=\pi/4} \sec t\ (\sec^2 t - 1)\ dt$$

$$\int_{t=0}^{t=\pi/4} \sec^3 t\ dt = \sec t \tan t\Big|_{t=0}^{t=\frac{\pi}{4}} - \int_{t=0}^{t=\frac{\pi}{4}} \sec^3 t\ \ dt + \int_{t=0}^{t=\frac{\pi}{4}} \sec t\ \ dt$$

$$2\int_{t=0}^{t=\pi/4} \sec^3 t \, dt = \sec t \tan t\Big|_{t=0}^{t=\frac{\pi}{4}} + \int_{t=0}^{t=\frac{\pi}{4}} \sec t \, dt$$

$$2\int_{t=0}^{t=\pi/4} \sec^3 t \, dt = \sec t \tan t + \ln|\sec t + \tan t| \Big|_{t=0}^{t=\frac{\pi}{4}}$$

$$\int_{t=0}^{t=\pi/4} \sec^3 t \, dt = \frac{1}{2}[\sec t \tan t + \ln|\sec t + \tan t|]\Big|_{t=0}^{t=\frac{\pi}{4}}$$

$$\int_{t=0}^{t=\pi/4} \sec^3 t \, dt = \frac{1}{2}\left[\sec\frac{\pi}{4}\tan\frac{\pi}{4} + \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| \right] - 0$$

$$\int_{t=0}^{t=\pi/4} \sec^3 t \, dt = \frac{1}{2}[\sqrt{2} + \ln|\sqrt{2} + 1|]$$

4. Find the arc length of the curve $y = \frac{3}{2} x^{2/3}$ as x goes from 0 to 1.

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \qquad y = \frac{3}{2} x^{2/3} \qquad \frac{dy}{dx} = \frac{1}{x^{1/3}}$$

$$s = \int_{x=0}^{x=1} \sqrt{1 + \left(\frac{1}{x^{1/3}}\right)^2} \, dx$$

$$s = \int_{x=0}^{x=1} \sqrt{1 + \frac{1}{x^{2/3}}} \, dx$$

$$s = \int_{x=0}^{x=1} \sqrt{\frac{x^{2/3}}{x^{2/3}} + \frac{1}{x^{2/3}}} \, dx$$

$$s = \int_{x=0}^{x=1} \sqrt{\frac{x^{2/3} + 1}{x^{2/3}}}\ dx$$

$$s = \int_{x=0}^{x=1} \sqrt{x^{2/3} + 1}\ \frac{1}{x^{1/3}}\ dx$$

Let $u = x^{2/3} + 1$ $\quad du = \frac{2}{3}\ \frac{1}{x^{1/3}}\ dx$ $\qquad \frac{3}{2}\ du = \frac{1}{x^{1/3}}\ dx$

$$s = \int_{u=1}^{u=2} \sqrt{u}\ \frac{3}{2}\ du$$

$$s = \frac{3}{2} \int_{u=1}^{u=2} \sqrt{u}\ du$$

$$s = \frac{3}{2}\ \frac{2}{3}\ u^{3/2}\Big|_1^2$$

$$s = u^{3/2}\Big|_1^2$$

$$s = 2\sqrt{2} - 1$$

5. You are given the parametric equations $x = a\cos t$ $\qquad y = a\sin t$. The parameter t goes from 0 to 2pi. Find the length of the curve. (The curve is a circle of radius a centered at the origin travelling in a counter clockwise path).

$$s = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$\frac{dx}{dt} = -a\sin t$ $\qquad \frac{dy}{dt} = a\cos t$

$$s = \int_{t=0}^{t=2\pi} \sqrt{(-a\sin t)^2 + (a\cos t)^2}\ dt$$

$$s = a \int_{t=0}^{t=2\pi} \sqrt{\sin^2 t + \cos^2 t}\ dt$$

$\sin^2 t + \cos^2 t = 1$

$$s = a \int_{t=0}^{t=2\pi} dt$$

$$s = 2\pi a$$

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6. You are given the parametric equations $x = a\cos^3 t \quad y = a\sin^3 t$. Find the arc length of the curve as to goes from 0 to ½ PI.

$$s = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = 3a\cos^2 t\ (-\sin t) \qquad \frac{dy}{dt} = 3a\sin^2 t\ (\cos t)$$

$$s = \int_{t=0}^{t=\pi/2} \sqrt{\left(3a\cos^2 t\ (-\sin t)\right)^2 + \left(3a\sin^2 t\ (\cos t)\right)^2}\, dt$$

$$s = \int_{t=0}^{t=\pi/2} \sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t}\, dt$$

$$s = \int_{t=0}^{t=\pi/2} 3a\cos t\sin t\ \sqrt{\cos^2 t + \sin^2 t}\, dt$$

$$\cos^2 t + \sin^2 t = 1$$

$$s = 3a \int_{t=0}^{t=\pi/2} \cos t\ \sin t\ dt$$

Let $u = \sin t \qquad du = \cos t\ dt$

$$s = 3a \int_{u=0}^{u=1} u\ du$$

$$s = 3a \left.\frac{u^2}{2}\right|_0^1$$

$$s = \frac{3a}{2}$$

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7. Find the length of the curve $y = \frac{1}{3}\,(x^2 + 2)^{3/2}$ as x goes from 0 to 3.

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx$$

$y = \frac{1}{3}\,(x^2 + 2)^{3/2}$ $\quad\quad$ $\frac{dy}{dx} = \frac{1}{2}\,(x^2 + \ 2)^{1/2}\,2x$

$$\frac{dy}{dx} = \ (x^2 + \ 2)^{1/2}\ x$$

$$s = \int_{x\,=\,0}^{x\,=\,3} \sqrt{1 + \ ((x^2 + \ 2)^{1/2}\ x\ \ )^2}\ \ dx$$

$$s = \int_{x\,=\,0}^{x\,=\,3} \sqrt{1 + \ (\,(x^2 + 2)\ x^2\ \ )}\ \ dx$$

$$s = \int_{x\,=\,0}^{x\,=\,3} \sqrt{\ x^4 + \ 2\,x^2 + 1}\ \ dx$$

$$s = \int_{x\,=\,0}^{x\,=\,3} \sqrt{\ (x^2 + \ 1)^2}\ \ dx$$

$$s = \int_{x\,=\,0}^{x\,=\,3} (x^2 + \ 1\,)\ dx$$

$$s = \left.\frac{x^3}{3} + \ x\ \right|_0^3$$

$$s = \frac{27}{3} + 3 \ = \ 12$$

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8. Find the arc length of the curve $y = \frac{x^3}{3} + \frac{1}{4x}$ as x goes from 1 to 3

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$\frac{dy}{dx} = x^2 - \frac{1}{4x^2}$$

$$s = \int_{x=1}^{x=3} \sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2}\ dx$$

$$s = \int_{x=1}^{x=3} \sqrt{1 + \left(x^4 - \frac{1}{2} + \frac{1}{16x^4}\right)}\ dx$$

$$s = \int_{x=1}^{x=3} \sqrt{\left(x^4 + \frac{1}{2} + \frac{1}{16x^4}\right)}\ dx$$

$$s = \int_{x=1}^{x=3} \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}\ dx$$

$$s = \int_{x=1}^{x=3} \left(x^2 + \frac{1}{4x^2}\right)\ dx$$

$$s = \frac{x^3}{3} - \frac{1}{4x}\Bigg|_1^3$$

$$s = \left[\left(9 - \frac{1}{12}\right) - \left(\frac{1}{3} - \frac{1}{4}\right)\ \right]$$

$$s\ = 9 - \frac{1}{6} = \frac{53}{6}$$

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9. Find the arc length of the curve $x = \frac{y^4}{4} + \frac{1}{8\,y^2}$ as y goes from 1 to 2.

$$s = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

$\frac{dx}{dy} = \ y^3 - \frac{1}{4y^3}$

$$s = \int_1^2 \sqrt{1 + \left(y^3 - \frac{1}{4y^3}\right)^2}\ dy$$

$$s = \int_1^2 \sqrt{1 + \left(y^6 \ - \ \frac{1}{2} \ + \ \frac{1}{16y^6}\right)}\ dy$$

$$s = \int_1^2 \sqrt{\left(y^6 + \ \frac{1}{2} \ + \ \frac{1}{16y^6}\right)}\ dy$$

$$s = \int_1^2 \sqrt{\left(y^3 \ + \ \frac{1}{4y^3}\right)^2}\ dy$$

$$s = \int_1^2 \left(y^3 \ + \ \frac{1}{4y^3}\right)\ dy$$

$$s = \frac{y^4}{4} - \frac{1}{8y^2}\Bigg|_1^2$$

$$s = \left[\left(4 - \frac{1}{32}\right) - \left(\frac{1}{4} - \frac{1}{8}\right)\right]$$

$$s = \frac{123}{32}$$

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10. Find the length of the curve $(y+1)^2 = 4\,x^3$ as x goes from 0 to 1.

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$y = 2\,x^{3/2} - 1 \qquad y' = \ 3\,x^{1/2}$

$$s = \int_{x=0}^{x=1} \sqrt{1 + \ (3\,x^{1/2})^2}\ dx$$

$$s = \int_{x=0}^{x=1} \sqrt{1 + \ 9x}\ dx$$

Let $u = 1 + 9x \qquad du = \ 9\,dx \qquad dx = \frac{1}{9}\,du$

$$s = \int_{u=1}^{u=10} \sqrt{u}\ \frac{du}{9}$$

$$s = \frac{1}{9}\frac{2}{3}\ u^{\frac{3}{2}}\Big|_1^{10} = \frac{2}{27}\ u^{\frac{3}{2}}\Big|_1^{10}$$

$$s = \frac{2}{27}\left(10\sqrt{10} \ - \ 1\right)$$

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11. You are given the parametric equations $x = a\cos t + at\sin t \qquad y = a\sin t - a\,t\cos t$ where a is a positive constant. The parameter t goes from 0 to ½ PI. Find the distance travelled by the particle.

$$s = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$\frac{dx}{dt} = -a\sin t + a\,sint + a\,t\cos t \ = \ at\cos t$$

$$\frac{dy}{dt} = a\cos t - a\cos t + a\,t\sin t \ = \ at\sin t$$

$$s = \int_{t=0}^{t=\pi/2} \sqrt{(at\cos t)^2 + (at\sin t)^2}\ dt$$

$$s = \int_{t=0}^{t=\pi/2} at\sqrt{\cos^2 t + \sin^2 t}\ dt$$

$$s = \int_{t=0}^{t=\pi/2} at\ \ dt = \left.\frac{at^2}{2}\right|_0^{\pi/2} = \frac{a\pi^2}{8}$$

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12. You are given the parametric equations $x = \frac{t^2}{2}$ $y = \frac{1}{3}(2t+1)^{3/2}$. T goes between 0 and 4. Find the distance travelled by the particle.

$$s = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$\frac{dx}{dt} = t$ $\frac{dy}{dy} = \frac{1}{2}(2t+1)^{1/2}\,2 = (2t+1)^{1/2}$

$$s = \int_{t=0}^{t=4} \sqrt{(t)^2 + ((2t+1)^{1/2})^2}\ dt$$

$$s = \int_{t=0}^{t=4} \sqrt{t^2 + 2t + 1}\ dt$$

$$s = \int_{t=0}^{t=4} \sqrt{(t+1)^2}\ dt$$

$$s = \int_{t=0}^{t=4} (t+1)\,dt$$

$$s = \left.\frac{t^2}{2} + t\right|_0^4 = 8 + 4 = 12$$

13. The position of a particle is given by the parametric equations $x = \frac{1}{3}(2t+3)^{3/2}$ and $y = \frac{t^2}{2} + t$. Find the distance travelled by the particle as t goes from 0 to 3.

$$s = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$\frac{dx}{dt} = (2t+3)^{1/2}$ $\qquad$ $\frac{dy}{dt} = t + 1$

$$s = \int_{t=0}^{t=3} \sqrt{((2t+3)^{1/2})^2 + (t+1)^2}\, dt$$

$$s = \int_{t=0}^{t=3} \sqrt{2t + 3 + t^2 + 2t + 1}\, dt$$

$$s = \int_{t=0}^{t=3} \sqrt{t^2 + 4t + 4}\, dt$$

$$s = \int_{t=0}^{t=3} \sqrt{(t+2)^2}\, dt$$

$$s = \int_{t=0}^{t=3} (t + 2)\, dt$$

$$s = \frac{t^2}{2} + 2t \Big|_0^3$$

$$s = \frac{9}{2} + 6 = \frac{21}{2}$$

14. Find the arc length of the curve $y = \left(4 - x^{\frac{2}{3}}\right)^{3/2}$ as x goes from 1 to 8.

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$\frac{dy}{dx} = \frac{3}{2}\left(4 - x^{\frac{2}{3}}\right)^{\frac{1}{2}} \frac{2}{3}\ x^{-\frac{1}{3}}$$

$$\frac{dy}{dx} = \left(4 - x^{\frac{2}{3}}\right)^{1/2} \frac{1}{x^{\frac{1}{3}}}$$

$$s = \int_{x\,=\,1}^{x\,=\,8} \sqrt{1 + \left(\left(4 - x^{\frac{2}{3}}\right)^{1/2} \frac{1}{x^{\frac{1}{3}}}\right)^2}\ dx$$

$$s = \int_{x\,=\,1}^{x\,=\,8} \sqrt{1 + \left(4 - x^{\frac{2}{3}}\right)\frac{1}{x^{\frac{2}{3}}}}\ dx$$

$$s = \int_{x\,=\,1}^{x\,=\,8} \sqrt{1 + \frac{4}{x^{\frac{2}{3}}} - 1}\ dx$$

$$s = \int_{x\,=\,1}^{x\,=\,8} \sqrt{4\ x^{-2/3}}\ dx$$

$$s = \int_{x=1}^{x=8} 2\,x^{-1/3}\ dx$$

$$s = 3\,x^{2/3}\Big|_1^8$$

$$s = 3(4-1) = 9$$

15. You are given the semicircle $y = \sqrt{9 - x^2}$. Find the length of semicircle – x goes from -3 to +3.

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \; dx$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{9-x^2}}$$

$$s = \int_{x=-3}^{x=+3} \sqrt{1 + \left(\frac{-x}{\sqrt{9-x^2}}\right)^2} \; dx$$

$$s = \int_{x=-3}^{x=+3} \sqrt{1 + \frac{x^2}{9-x^2}} \; dx$$

$$s = \int_{x=-3}^{x=+3} \sqrt{\frac{9-x^2}{9-x^2} + \frac{x^2}{9-x^2}} \; dx$$

$$s = \int_{x=-3}^{x=+3} \sqrt{\frac{9}{9-x^2}} \; dx$$

$$s = 3 \int_{x=-3}^{x=+3} \frac{1}{\sqrt{9-x^2}} \; dx$$

$$s = 6 \int_{x=0}^{x=+3} \frac{1}{\sqrt{9-x^2}} \; dx$$

$$s = 6 \, arc \sin \frac{x}{3} \Big|_0^3$$

$$s = 6 \, (arc \sin 1 - arc \sin 0)$$

$$s = 6 \left(\frac{\pi}{2} - 0 \right) = 3\pi$$

VERY DIFFICULT QUESTION

16. Find the arc length of the curve $y = 2\sqrt{x}$ as x goes from 1 to 4.

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$\frac{dy}{dx} = \frac{1}{\sqrt{x}}$

$$s = \int_{x=1}^{x=4} \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}\ dx$$

$$s = \int_{x=1}^{x=4} \sqrt{1 + \frac{1}{x}}\ dx$$

Let $u = \frac{1}{x}$ $\quad dx = -\frac{1}{u^2}\ du$

$$s = \int_{u=1}^{u=1/4} \sqrt{1+u}\left(-\frac{1}{u^2}\right) du$$

$$s = \int_{u=1/4}^{u=1} \frac{\sqrt{1+u}}{u^2}\ du$$

Let $u = \tan^2 t$ $\quad du = 2\tan t \sec^2 t\ dt$

$$s = \int_{t=\arctan 1/2}^{t=\pi/4} \frac{\sqrt{1+\tan^2 t}}{\tan^4 t}\ 2\tan t \sec^2 t\ dt$$

$$s = 2\int_{t=\arctan 1/2}^{t=\pi/4} \frac{\sec^3 t}{\tan^3 t}\ dt$$

$$s = 2\int_{t=\arctan 1/2}^{t=\pi/4} \frac{1}{\sin^3 t}\ dt$$

$$s \;=\; 2 \int_{t\,=\,arc\tan 1/2}^{t\,=\,\pi/4} \csc^3 t \;\; dt$$

Integrate by method of parts:

$$s \;=\; 2 \int_{t\,=\,\tan^{-1} 1/2}^{t\,=\,\pi/4} \csc t \;\csc^2 t \;\; dt$$

Let $u = \csc t$ $\qquad dv = \csc^2 t \;\, dt$

$du = -\csc t \cot t \;\, dt \qquad v = -\cot t$

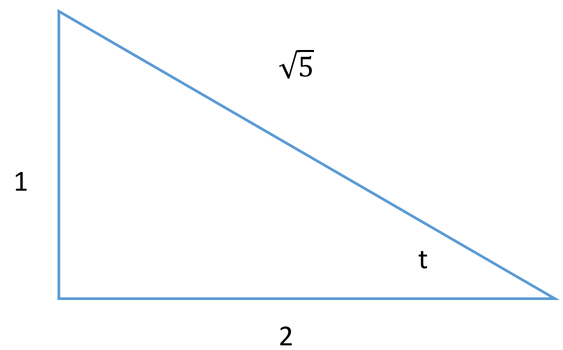
$$\int_{t\,=\,\tan^{-1} 1/2}^{t\,=\,\pi/4} \csc t \;\csc^2 t \;\; dt \;=\; -\cot t \csc t\big|_{t\,=\,\tan^{-1} 1/2}^{t\,=\,\pi/4} - \int_{t\,=\,\tan^{-1} 1/2}^{t\,=\,\pi/4} \cot^2 t \csc t \;\; dt$$

$$\int_{t\,=\tan^{-1} 1/2}^{t\,=\,\pi/4} \csc^3 t \;\; dt \;=\; -\cot t \csc t\big|_{\tan^{-1} 1/2}^{\pi/4} - \int_{\tan^{-1} 1/2}^{\pi/4} (\csc^2 t - 1) \csc t \;\; dt$$

$$\int_{t\,=\,arc\tan 1/2}^{t\,=\,\pi/4} \csc^3 t \;\; dt \;=\; -\cot t \csc t\big|_{arc\tan 1/2}^{\pi/4} - \int_{\tan^{-1}\frac{1}{2}}^{\frac{\pi}{4}} \csc^3 t \;\; dt + \int_{\tan^{-1}\frac{1}{2}}^{\frac{\pi}{4}} \csc t \;\, dt$$

$$2 \int_{t\,=\,arc\tan 1/2}^{t\,=\,\pi/4} \csc^3 t \;\; dt \;=\; -\cot t \csc t\big|_{arc\tan 1/2}^{\pi/4} + \int_{\tan^{-1}\frac{1}{2}}^{\frac{\pi}{4}} \csc t \;\, dt$$

$$2\int_{t\,=\,arc\tan 1/2}^{t\,=\,\pi/4} \csc^3 t\ \ dt \ = \ -\cot t \csc t\Big|_{arc\tan 1/2}^{\pi/4} \ - \ \ln|\csc t + \cot t|\Big|_{\tan^{-1}\frac{1}{2}}^{\frac{\pi}{4}}$$



$$2\int_{t\,=\,arc\tan 1/2}^{t\,=\,\pi/4} \csc^3 t\ \ dt \ = \left(-\sqrt{2} - \ln\left|\sqrt{2}+1\right|\right) - \ \left(-2\sqrt{5} - \ln\left|\sqrt{5}+2\right|\right)$$

$$2\int_{t\,=\,arc\tan 1/2}^{t\,=\,\pi/4} \csc^3 t\ \ dt \ = \ \left(2\sqrt{5} + \ln\left|\sqrt{5}+2\right|\right) \ - \ \left(\sqrt{2} \ + \ \ln\left|\sqrt{2}+1\right|\right)$$

$$s \ = \ \left(2\sqrt{5} + \ln\left|\sqrt{5}+2\right|\right) \ - \ \left(\sqrt{2} \ + \ \ln\left|\sqrt{2}+1\right|\right)$$