TAYLOR SERIES REMAINDER FORMULA

We write down the general formula for Taylor's series:

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \frac{1}{3!} f'''(a)(x-a)^3 + \cdots$$

Or

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(a)(x-a)^n$$

In many applications, we do not need the entire Taylor series. We only require a finite number of terms. In this case we write

$$f(x) \cong f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \frac{1}{3!} f'''(a)(x-a)^3 + \cdots + \frac{1}{n!} f^n(a)(x-a)^n$$

The question becomes, what is the error in doing this? We would like to know how close this approximation is to the true value. The error is expressed as an integral and it is called the "remainder". It is given by

$$R_n(x) = \frac{1}{n!} \int_a^x (x-t)^n f^{n+1}(t)\, dt$$

One property of the remainder is that it goes to zero as n goes to infinity.

$$\lim_{n \to \infty} R_n(x) = 0$$

We want to calculate the remainder for several Taylor series.

In evaluating the remainder, we use a specialized form of the mean value theorem.

This modified theorem states that if g(x) is positive them $\int_a^b f(x)\, g(x)\, dx = f(c) \int_a^b g(x)\, dx$ where c is between a and b.

In this case it becomes $R_n(x) = \frac{1}{n!} \int_a^x (x-t)^n f^{n+1}(t)\, dt = \frac{1}{n!} f^{n+1}(c) \int_a^x (x-t)^n\, dt$ where c is in between a and x.

Sometimes it will happen the calculating the derivative $f^{n+1}(c)$ is extremely difficult, so much so that it makes the question intractable. In this case, if the series is alternating, we will be able to use the remainder theorem for alternating series. To reiterate, this theorem says that the remainder is less than the value of the first missing term. So we use the equation R = u_{n+1}. This method is relatively easy to use. We will use this and the integral form of the remainder for the questions herein.

EXAMPLE

Find the remainder for the Taylor series of e^x

Start with the general formula for Taylor series:

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{k}(a)\ (x-a)^{k}$$

$$e^{x} = e^{a} + (x-a) + e^{a}\frac{(x-a)^{2}}{2!} + e^{a}\frac{(x-a)^{3}}{3!} + e^{a}\frac{(x-a)^{4}}{4!} + \cdots$$

$$e^{x} = \sum_{k=0}^{\infty} \frac{1}{k!} e^{a}\ (x-a)^{k}$$

If we stop at the nth term this becomes:

$$e^{x} \cong e^{a} + (x-a) + e^{a}\frac{(x-a)^{2}}{2!} + e^{a}\frac{(x-a)^{3}}{3!} + e^{a}\frac{(x-a)^{4}}{4!} + \cdots + e^{a}\frac{(x-a)^{n}}{n!}$$

$$e^{x} \cong \sum_{k=0}^{n} \frac{1}{k!} e^{a}\ (x-a)^{k}$$

The remainder integral is given by

$$R_n(x) = \frac{1}{n!}\int_a^x f^{n+1}(t)\,(x-t)^n\ dt$$

$$R_n(x) = \frac{1}{n!}\int_a^x e^{t}\,(x-t)^n\ dt$$

This is a tedious integral. We can evaluate it by using a modified form of the mean value theorem of integral calculus. As such we write

$$R_n(x) = \frac{1}{n!} f^{n+1}(c)\int_a^x (x-t)^n\ dt$$

$$R_n(x) = e^{c}\frac{1}{n!}\int_a^x (x-t)^n\ dt \qquad \text{where } a < c < x$$

Integrating we get:

$$R_n(x) = e^{c}\frac{1}{n!}(-1)\left.\frac{(x-t)^{n+1}}{n+1}\right|_a^x = -e^{c}\left.\frac{(x-t)^{n+1}}{(n+1)!}\right|_a^x = e^{c}\frac{(x-a)^{n+1}}{(n+1)!}$$

We do not know the exact value for c, although we know it is between x and a.

We can use the fact that $e^{c} < e^{a}$ to write an inequality:

$$R_n(x) < e^{a}\frac{(x-a)^{n+1}}{(n+1)!}$$

This is the expression we want. We have an upper bound for the error. All we have to do is plug in x and a. x is the value we plug into the series. a is the point of expansion.

EXAMPLE

Find the remainder for the Taylor series of sin x

Start with the general formula for Taylor series:

$f(x) \; = \; \sum_{k=\,0}^{\infty} \frac{1}{k!} \; f^{k}\,(a) \;\; (x-a)^{k}$

$\sin x \; = \; \sin a \; + \; \cos a \;\, (x-a) \;\; - \sin a \;\; \frac{(x-a)^2}{2!} \;\; - \;\; \cos a \;\; \frac{(x-a)^3}{3!} \; + \; \sin a \;\; \frac{(x-a)^4}{4!} \; + \cdots$

We can write this series in the somewhat abbreviated way

$\sin x \; = \; \sum_{n\,=\,0}^{\infty} \binom{+}{-} \; \binom{\sin a}{\cos a} \; \frac{(x-a)^n}{n!}$

We write the remainder integral

$R_n(x) \;\; = \; \frac{1}{n!} \int_a^x \;\; f^{n+1}(t) \;\; (\; x - t\,)^n \;\; dt$

Using the mean value theorem of integral calculus this becomes:

$R_n(x) \;\; = \; \frac{1}{n!} \;\; f^{n+1}(c) \int_a^x \;\;\; (\; x - t\,)^n \;\; dt$ where $a < c < x$

For this integral the nth term is present in the series and the (n+1)st term is missing.

If n is even this becomes

$R_n(x) \;\; = \; \frac{1}{n!} \int_a^x \cos t \;\;\, (\; x \; - \;\; t\,)^n \;\; dt$ ignoring the + or - sign

$R_n(x) \;\; = \;\; \frac{1}{n!} \cos c \; \int_a^x \;\;\, (x \; - \;\; t\,)^n \;\; dt$

$R_n(x) \;\; = \;\; (-1)^n \;\; \frac{1}{n!} \cos c \; \int_a^x \;\;\, (\,t - \;\; x\;\,)^n \;\; dt$

$R_n(x) \;\; = \;\; (-1)^n \;\; \frac{1}{n!} \cos c \;\;\; \frac{(\,t - \,x\;)^{n+1}}{n+1} \; \Big|_a^x$

$R_n(x) \;\; = \;\; (-1)^{\,n\,+\,1} \;\; \frac{1}{n!} \cos c \;\;\; \frac{(\,a - \,x\;)^{n+1}}{n+1}$

$R_n(x) \;\; = \;\; (-1)^{\,n\,+\,1} \,(-1)^{n+1} \;\, \frac{1}{n!} \cos c \;\;\; \frac{(\;x\;\,-\;\,a\;\,)^{n+1}}{n+1}$

$R_n(x) \;\; = \;\; \cos c \;\;\; \frac{(\;x\;\,-\;\,a\;\,)^{n+1}}{(\,n+1\,)\,!}$ where $a < c < x$

If n is odd this becomes

$R_n(x) \;\; = \; \frac{1}{n!} \int_a^x \sin t \;\;\, (\; x \; - \;\; t\,)^n \;\; dt$

The integration is almost identical and the result will be

$R_n(x) \;\; = \;\; \sin c \;\;\; \frac{(\;x\;\,-\;\,a\;\,)^{n+1}}{(\,n+1\,)\,!}$ ignoring the +/- sign

EXAMPLE

Find the remainder for the Taylor series of cos x

Start with the general formula for Taylor series:

$$f(x) \;=\; \sum_{k=0}^{\infty} \frac{1}{k!}\; f^{k}\,(a)\;\,(x-a)^{k}$$

$$\cos x \;=\; \cos a - \;\sin a\;\,(x-a) \;-\; \cos a\; \frac{(x-a)^2}{2!} \;-\; \sin a\; \frac{(x-a)^3}{3!} \;+\; \cos a\; \frac{(x-a)^4}{4!} \;+\cdots$$

We can write this series with the shorthand notation:

$$\cos x \;=\; \sum_{n=0}^{\infty} \binom{+}{-}\; \binom{\sin a}{\cos a}\; \frac{(x-a)^n}{n!}$$

We write the remainder integral:

$$R_n(x) \;=\; \frac{1}{n!} \int_a^x \; f^{n+1}(t)\,(x-t)^n \;\, dt$$

Using the mean value theorem of integral calculus this becomes

$$R_n(x) \;=\; \frac{1}{n!}\; f^{n+1}(c) \int_a^x \; f^{n+1}(t)\,(x-t)^n \;\, dt \quad \text{where } a < c < x$$

This integral represents the situation where the nth term is present in the series and the n+1 st term is missing.

If n is even this becomes

$$R_n(x) \;=\; \frac{1}{n!} \int_a^x \pm \sin t \;\;(x-t)^n \;\, dt$$

Ignore the +/- and we write

$$R_n(x) \;=\; \frac{1}{n!} \int_a^x \;\sin t \;\;(x-t)^n \;\, dt$$

Using the mean value theorem of integral calculus this becomes

$$R_n(x) \;=\; \frac{1}{n!}\,\sin c \int_a^x \;(x-t)^n \;\, dt \quad \text{where } a < c < x$$

$$R_n(x) \;=\; \frac{1}{n!}\,\sin c\; \frac{(x-a)^{n+1}}{n+1}$$

$$R_n(x) \;=\; \sin c\; \frac{(x-a)^{n+1}}{(n+1)!}$$

If n were odd the result would be

$$R_n(x) \;=\; \cos c\; \frac{(x-a)^{n+1}}{(n+1)!}$$

EXAMPLE

Find the remainder for the Taylor series of $\ln(1+x)$ expanded about x=0.

First write out Taylors formula:

$$f(x) = f(0) + f'(0)\,x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots$$

Or

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(0)\ x^n$$

Starting with the series for $\ln(1+x)$

$$\ln\ (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\ln(\,1+x\,) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

The remainder integral is

$$R_n(x) = \frac{1}{n!} \int_a^x f^{n+1}(t)\ (x-t)^n\,dt$$

Using the mean value theorem of integral calculus this can be written as

$$R_n(x) = \frac{1}{n!} f^{n+1}(c) \int_a^x (x-t)^n\,dt \qquad \text{where } a < c < x$$

The (n+1)st derivative can be written as $f^{n+1}(c) = \frac{(-1)^n\ n!}{(1+c)^{n+1}}$. This can be found by actually taking derivatives of ln(1+x) and seeing the pattern.

$$R_n(x) = \frac{1}{n!} \frac{(-1)^n\ n!}{(1+c)^{n+1}} \int_a^x (x-t)^n\,dt$$

$$R_n(x) = \frac{(-1)^n}{(1+c)^{n+1}} \int_a^x (x-t)^n\,dt$$

$$R_n(x) = \frac{-(-1)^n}{(1+c)^{n+1}} \left.\frac{(x-t)^{n+1}}{n+1}\right|_a^x$$

$$R_n(x) = \frac{(-1)^n}{(1+c)^{n+1}} \frac{(x-a)^{n+1}}{n+1}$$

For us, a = 0 so this becomes:

$$R_n(x) = \frac{(-1)^n}{(1+c)^{n+1}} \frac{x^{n+1}}{n+1}$$

The remainder is a maximum when c = 0 so this becomes $R_n(x) = (-1)^n \frac{x^{n+1}}{n+1}$. This is an upper bound for the error. The error must be less that this expression.

EXAMPLE

You are given the Taylor series

$$\ln x \;=\; \ln a \;+\; \frac{1}{a}\,(x-a) \;-\; \frac{1}{2}\,\frac{1}{a^2}\,(x-a)^2 \;+\; \frac{1}{3}\,\frac{1}{a^3}\,(x-a)^3$$

Find the remainder integral for the Taylor series of ln x expanded about x = a.

Let's write the series as

$$\ln x \;=\; \ln a \;+\; \frac{1}{a}\,(x-a) \;-\; \frac{1}{2}\,\frac{1}{a^2}\,(x-a)^2 \;+\; \frac{1}{3!}\,\frac{2!}{a^3}\,(x-a)^3 \;-\; \frac{1}{4!}\,\frac{3!}{a^4}\,(x-a)^4$$

From this series we can see that $f^n(a) = (-1)^{\,n+1}\,\frac{(n-1)!}{a^n}$

Write the remainder integral

$$R_n(x) \;=\; \frac{1}{n!}\int_a^x f^{n+1}(t)\,(x-t)^n\;dt$$

$$R_n(x) \;=\; \frac{1}{n!}\int_a^x (-1)^{\,n+2}\,\frac{(n)!}{t^{\,n+1}}\,(x-t)^n\;dt$$

Modify the integral using the mean value theorem for integrals:

$$R_n(x) \;=\; \frac{1}{n!}\,f^{n+1}(c)\int_a^x (x-t)^n\;dt \qquad \text{where } a < c < x$$

$$R_n(x) \;=\; (-1)^{\,n+2}\,\frac{1}{c^n}\int_a^x (x-t)^n\;dt$$

$$R_n(x) \;=\; (-1)^{\,n}\,\frac{1}{c^n}\int_a^x (x-t)^n\;dt$$

$$R_n(x) \;=\; -(-1)^{\,n}\,\frac{1}{c^n}\,\left.\frac{(x-t)^{\,n+1}}{n+1}\right|_a^x$$

$$R_n(x) \;=\; (-1)^{\,n}\,\frac{1}{c^n}\,\frac{(x-a)^{\,n+1}}{n+1}$$

The remainder is a maximum when c = a so this becomes $R_n(x) \;=\; (-1)^{\,n}\,\frac{1}{a^n}\,\frac{(x-a)^{\,n+1}}{n+1}$
This is an upper bound on the error.

EXAMPLE

You are given the Maclaurin series $e^x = \quad 1 + x\ + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
Let x = -1/10.
Find the result accurate to 5 decimal places.

We need the remainder integral to find out how many terms we need in the Maclaurin series.

We write $R_n(x)\ = \frac{1}{n!} \int_a^x f^{n+1}(t)\ (x-t)^n\ dt$

For us, a = 0 and x = -1/10

Using the mean value theorem for integrals we get

$$R_n(x)\ = \frac{1}{n!}\ e^c \int_0^{-1/10} \left(-\frac{1}{10} - t\right)^n\ dt$$

$$R_n(x)\ = \frac{(-1)^n}{n!}\ e^c \int_0^{-1/10} \left(\frac{1}{10} + t\right)^n\ dt$$

$$R_n(x)\ = \frac{(-1)^n}{n!}\ e^c \quad \frac{(\ 0.1 + t)^{\,n+1}}{n+1}\ \Big|_0^{-\,0.1}$$

$$R_n(x)\ = \frac{(-1)^n}{n!}\ e^c\ \frac{(\ 0.1\ )^{\,n+1}}{n+1}$$

Ignore the -1 in front and set c = 0 to get an upper bound on the error.

$$R_n(x)\ = \quad \frac{(\ 0.1\ )^{\,n+1}}{(n+\ 1)!}$$

We want 5 decimal accuracy so we want set $R_n(x)\ = \quad \frac{(\ 0.1\ )^{\,n+1}}{(n+\ 1)!} < \ 10^{-5}$

$$10^{-5}\ > \ \frac{1}{10^{\,n+1}\ \ (n+\ 1)!}$$

Rearranging we get $(n+1)!\ >\ \ 10^{4-n}$

This equation has to be solved numerically:
$n = 1 \qquad 2!\ <\ \ 10^3$ no good
$n = 2 \qquad 3!\ <\ \ 100$ no good
$n = 3 \qquad 4!\ >\ 10$ GOOD!

$$e^x\ \ \cong\ \ 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$e^{-\,0.1}\ \ \cong\ \ 1\ -\ 0.1\ +\ \frac{0.01}{2!}\ -\ \frac{0.001}{3!}$$

$$e^{-\,0.1}\ \ \cong\ \ 0.904833333$$

Calculator answer: 0.904837418

The difference is 0.000004085
EXAMPLE

You are given the binomial expression $(1+x)^{-\frac{1}{4}}$ expanded about a = 0. Let x = 0.1.
Use the binomial expansion to find $(1.1)^{-1/4}$ accurate to 3 decimals

We start with the binomial theorem

$$(1+x)^n = 1 + nx + \frac{1}{2!}n(n-1)x^2 + \frac{1}{3!}n(n-1)(n-2)x^3 + \frac{1}{4!}n(n-1)(n-2)(n-3)x^4 + \cdots$$

$$(1+x)^{-1/4} = 1 - \frac{1}{4}x + \frac{1}{2!}\left(\frac{1}{4}\right)\left(\frac{5}{4}\right)x^2 - \frac{1}{3!}\left(\frac{1}{4}\right)\left(\frac{5}{4}\right)\left(\frac{9}{4}\right)x^3 + \frac{1}{4!}\left(\frac{1}{4}\right)\left(\frac{5}{4}\right)\left(\frac{9}{4}\right)\left(\frac{13}{4}\right)x^4 + \cdots$$

We write the remainder integral

$$R_n(x) = \frac{1}{n!}\int_a^x f^{n+1}(t)\,(x-t)^n\,dt$$

We use the mean value theorem for integrals to get:

$$R_n(x) = \frac{1}{n!}f^{n+1}(c)\int_a^x (x-t)^n\,dt$$

$$R_n(x) = \frac{1}{n!}f^{n+1}(c)\frac{(x-a)^{n+1}}{n+1}$$

For us, x = 0.1 and a = 0

$$R_n(0.1) = \frac{1}{n!}f^{n+1}(c)\frac{(0.1)^{n+1}}{n+1} = f^{n+1}(c)\frac{(0.1)^{n+1}}{(n+1)!}$$

We need an expression for the (n+1)st derivative.

$$f^n(x) = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}-1\right)\left(-\frac{1}{4}-2\right)\left(\frac{1}{4}-3\right)\ldots\left(-\frac{1}{4}-(n-1)\right)(1+x)^{-\frac{1}{4}-n}$$

$$f^{n+1}(x) = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}-1\right)\left(-\frac{1}{4}-2\right)\left(\frac{1}{4}-3\right)\ldots\left(-\frac{1}{4}-n\right)(1+x)^{-\frac{1}{4}-n-1}$$

$$f^{n+1}(c) = \pm\frac{1}{4^{n+1}}(1)(5)(9)(1+4n)(1+c)^{-\frac{1}{4}-n-1}$$

$$f^{n+1}(c) = \pm\frac{1}{4^{n+1}}(1)(5)(9)(1+4n)\frac{1}{(1+c)^{\frac{5}{4}+n}}$$

$$R_n(0.1) = \frac{1}{4^{n+1}}(1)(5)(9)(1+4n)\frac{1}{(1+c)^{\frac{5}{4}+n}}\frac{(0.1)^{n+1}}{(n+1)!}$$ where we dropped the +/- signs.

Set c = 0. This will give us an upper bound on the error.

$$R_n(0.1) = \frac{1}{4^{n+1}}(1)(5)(9)(1+4n)\frac{(0.1)^{n+1}}{(n+1)!}$$

We want our result to be accurate to three decimals. The error must be in the 4th decimal place, so we want the error to be less than 0.001

$$R_n(0.1) = \frac{1}{4^{n+1}}(1)(5)(9)(1+4n)\frac{(0.1)^{n+1}}{(n+1)!} < 0.001$$

We must solve this numerically:

$n = 1 \qquad R_1(0.1) = \frac{1}{4^2}(1)(5)\frac{(0.01)}{2!} = 0.00125 > 0.001$ NO GOOD!

$n = 2 \qquad R_2(0.1) = \frac{1}{4^3}(1)(5)(9)\frac{(0.001)}{3!} = 0.0009375 < 0.001$ GOOD!

$$(1+x)^n \cong 1 + n\,x + \frac{1}{2!}\,n(n-1)x^2$$

$$(1+0.1)^{-1/4} \cong 1 + \left(-\frac{1}{4}\right)(0.1) + \frac{1}{2!}\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)(0.1)^2$$

$$(1+0.1)^{-1/4} \cong 0.9765625$$

Calculator answer: 0.976454

The difference is 0.0001085

EXAMPLE

Find the square root of 5 accurate to 0.00001

We need the formula for the (n+1)st derivative of the square root function. This was solved in a previous handout. We have:

$$f^n(x) = \frac{d^n(\sqrt{x})}{dx^n} = (-1)^{n-1} \frac{1}{2^{2n-1}} \frac{(2n-2)!}{(n-1)!} x^{-(2n-1)/2}$$

$$f^{n+1}(x) = (-1)^n \frac{1}{2^{2n+1}} \frac{(2n)!}{(n)!} x^{-(2n+1)/2}$$

We now compute the remainder integral:

$$R_n(x) = \frac{1}{n!} \int_a^x f^{n+1}(t)\,(x-t)^n\,dt$$

Using the mean value theorem of integral calculus this becomes:

$$R_n(x) = \frac{1}{n!} f^{n+1}(c) \int_a^x (x-t)^n\,dt \qquad \text{where } a < c < x$$

$$R_n(x) = \frac{1}{n!} f^{n+1}(c) \frac{(x-a)^{n+1}}{n+1}$$

$$R_n(x) = f^{n+1}(c) \frac{(x-a)^{n+1}}{(n+1)!}$$

$$R_n(x) = (-1)^n \frac{1}{2^{2n+1}} \frac{(2n)!}{(n)!} \left(c^{-\frac{2n+1}{2}} \right) \frac{(x-a)^{n+1}}{(n+1)!}$$

We have x = 5 and a = 4 with c in between. To maximize the expression (to get an upper bound for the remainder), we let c = 4. This will give us an upper bound on the error.

$$R_n(5) = (-1)^n \frac{1}{2^{2n+1}} \frac{(2n)!}{(n)!} \left(4^{-\frac{2n+1}{2}} \right) \frac{(5-4)^{n+1}}{(n+1)!}$$

$$R_n(5) = (-1)^n \frac{1}{2^{2n+1}} \frac{(2n)!}{(n)!} \left(\frac{1}{2^{2n+1}} \right) \frac{1}{(n+1)!}$$

$$R_n(4) = (-1)^n \frac{1}{4^{2n+1}} \frac{(2n)!}{(n)!} \frac{1}{(n+1)!}$$

$$R_n(3) = (-1)^n \frac{1}{4^{2n+1}} \frac{(2n)!}{(n)!} \frac{1}{(n+1)!}$$

This magnitude of this expression must be less than 0.00001

$$R_n(3) = \frac{1}{4^{2n+1}} \frac{(2n)!}{(n)!} \frac{1}{(n+1)!} < 0.00001$$

We must solve this numerically:

$n = 1$ $\quad R_1(3) = 0.015625 > 0.00001$ NO GOOD

$n = 2$ $\quad R_2(3) = 0.001953 > 0.00001$ NO GOOD
$n = 3$ $\quad R_3(3) = 0.0003052 > 0.00001$ NO GOOD

$n = 4$ $\quad R_4(3) = 0.00005341 > 0.00001$ NO GOOD

$n = 5$ $\quad R_5(3) \; = \; 0.00001001 > \; 0.00001$ NO GOOD (BUT REALLY CLOSE)

$n = 6$ $\quad R_6(3) \; = \; 0.000002 \; < \; 0.00001$ GOOD!

We need the Taylor series for the square root centered about x = 4. We will also need the a formula for the (n+1)st derivative.

This was solved previously.

We have

$$f(x) \; = \; 2 \; + \frac{1}{2^2}\,(x-4) \; + \; \sum_{n=2}^{\infty} \; (-1)^{n-1}\,\frac{1}{n!}\,\frac{1}{2^{4n-2}}\frac{(2n-2)!}{(n-1)!}\,(x-4)^{\mathrm{n}}$$

$$f(5) \; = \; 2 \; + \frac{1}{2^2} \; + \; \sum_{n=2}^{\infty} \; (-1)^{n-1}\,\frac{1}{n!}\,\frac{1}{2^{4n-2}}\frac{(2n-2)!}{(n-1)!}$$

We take the first 6 terms

$$f(5) \; \cong \; 2 \; + \frac{1}{4} - \frac{1}{2!}\frac{1}{2^6}\frac{2!}{1!} + \frac{1}{3!}\frac{1}{2^{10}}\frac{4!}{2!} - \frac{1}{4!}\frac{1}{2^{14}}\frac{6!}{3!} + \frac{1}{5!}\frac{1}{2^{18}}\frac{8!}{4!} - \frac{1}{6!}\frac{1}{2^{22}}\frac{10!}{5!}$$

$$f(5) \; \cong \; 2.236066341$$

Calculator answer: 2.236067977

The difference is 0.000001636. We have an extra decimal accuracy. This came about by approximating c with 4.

EXAMPLE

Evaluate the integral $\int_0^{1/2} \; x^3 \; \tan^{-1} x \;\; dx$ accurate to 4 decimal places.
This means that the error is in the 5^{th} decimal. So we want the error to be less than 0.00001

We start with the Taylor series for the integrand:

$$x^3 \tan^{-1} x \;=\; x^3 \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)$$

$$x^3 \tan^{-1} x \;=\; x^3 \;\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$x^3 \tan^{-1} x \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{2n+1}$$

Now that we have the Taylor series, we can evaluate the integral:

$$\int_0^{1/2} x^3 \tan^{-1} x \; dx \;=\; \int_0^{1/2} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{2n+1} \; dx$$

$$\int_0^{1/2} x^3 \tan^{-1} x \; dx \;=\; \sum_{n=0}^{\infty} (-1)^n \int_0^{1/2} \frac{x^{2n+4}}{2n+1} \; dx$$

$$\int_0^{1/2} x^3 \tan^{-1} x \; dx \;=\; \sum_{n=0}^{\infty} (-1)^n \left.\frac{x^{2n+5}}{(2n+1)\,(2n+5)}\right|_0^{1/2}$$

$$\int_0^{1/2} x^3 \tan^{-1} x \; dx \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n+5}\,(2n+1)\,(2n+5)}$$

Our choice now, to determine the number of terms, is twofold. We can compute a formula for $D^{n+1}(x^3 \tan^{-1} x)$. This derivative is ridiculously difficult. The other choice is to use the remainder theorem for alternating series. This theorem says that the remainder is less than the value of the first neglected term of the series. This is the way we will go.

$$R_n\left(\frac{1}{2}\right) \;=\; u_{n+1} \;=\; \frac{1}{2^{2n+7}\,(2n+3)\,(2n+7)}$$

We want this to be less than 0.00001

$$\frac{1}{2^{2n+7}\,(2n+3)\,(2n+7)} \;<\; 0.00001$$

Take reciprocals:

$$2^{2n+7}\;(2n+3)\,(2n+7) \;>\; 100{,}000$$

This must be solved numerically

$n = 1$ $\quad 23{,}040 \;<\; 100{,}000$ NO GOOD

$n = 2$ $\quad 157{,}696 \;>\; 100{,}000$ GOOD!

$$\int_0^{1/2} x^3 \tan^{-1} x \; dx \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n+5}\,(2n+1)\,(2n+5)} \;\cong\; \frac{1}{2^5\,(1)\,(5)} - \frac{1}{2^7\,(3)\,(7)} + \frac{1}{2^9\,(5)\,(9)}$$

$$\int_0^{1/2} x^3 \tan^{-1} x \; dx \;\cong\; 0.005921379$$

Calculator answer: 0.005915925 Difference is 0.000005454

EXAMPLE

Evaluate the integral $\int_0^1 \sin x^4 \; dx$ accurate to 4 decimals

This means that the first 4 decimals must be correct, so the error must be in the 5th decimal so the error must be < 0.00001. We start with the Taylor series for sin u

$\sin u \ = \ u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \cdots$ Or

$\sin u \ = \ \sum_{n=0}^{\infty} (-1)^n \ \frac{u^{2n+1}}{(2n+1)!}$

Substitute u = x^4

$\sin x^4 \ = \ \sum_{n=0}^{\infty} (-1)^n \ \frac{(x^4)^{2n+1}}{(2n+1)!} \ = \ \sum_{n=0}^{\infty} (-1)^n \ \frac{(x)^{8n+4}}{(2n+1)!}$

We now evaluate the given integral:

$\int_0^1 \ \sin x^4 \ \ dx \ = \ \int_0^1 \ \sum_{n=0}^{\infty} (-1)^n \ \frac{(x)^{8n+4}}{(2n+1)!} \ dx$

$\int_0^1 \ \sin x^4 \ \ dx \ = \ \sum_{n=0}^{\infty} (-1)^n \ \int_0^1 \ \frac{(x)^{8n+4}}{(2n+1)!} \ dx$

$\int_0^1 \ \sin x^4 \ \ dx \ = \ \sum_{n=0}^{\infty} (-1)^n \ \frac{1}{(2n+1)! \ (8n+5)}$

We now have to figure out how many terms we need from the series on the right.

Instead of evaluating the remainder integral, we will use the remainder theorem for alternating series. The error in truncating the series is less than the first missing terms – so we set the remainder equal to the first missing term:

$R = \ u_{n+1}$ ignoring the sign of the term

From the series we see $u_n = \ \frac{1}{(2n+1)! \ (8n+5)}$ so $u_{n+1} = \ \frac{1}{(2n+3)! \ (8n+13)}$

We now have $R = \ \frac{1}{(2n+3)! \ (8n+13)} \ < \ 0.00001$

This inequality must be solved numerically:
$n = \ 1 \qquad R = 0.000397 \ > \ 0.00001$ NO GOOD
$n = 2 \qquad R = \ 0.00000684 < 0.00001$ GOOD!

$\int_0^1 \ \sin x^4 \ \ dx \ = \ \sum_{n=0}^{\infty} (-1)^n \ \frac{1}{(2n+1)! \ (8n+5)} \ \cong \ \frac{1}{5} - \frac{1}{(3!)\,(13)} + \frac{1}{(5!)\,(21)}$

$\int_0^1 \ \sin x^4 \ \ dx \ \cong \ 0.1875763126$

Calculator answer: 0.1875695447 Difference = 0.000006768

EXAMPLE

Evaluate the integral $\int_0^{0.4} \ \sqrt{\ 1 \ + \ x^4} \ \ dx$ so that it is accurate to 5 X 10^{-6}

We need the binomial theorem

$$(1+u)^n = 1 + n\,u + \frac{n\,(n-1)}{2!}\,u^2 + \frac{n\,(n-1)\,(n-2)}{3!}\,u^3 + \frac{n\,(n-1)\,(n-2)\,(n-3)}{4!}\,u^4 + \cdots$$

or

$$(1+u)^n = \sum_{k=0}^{\infty} \frac{n\,(n-1)(n-2)\cdots(n-(k-1))\,u^k}{k!}$$

For us, n = ½ and u = x^4

This yields

$$(1+x^4)^{1/2} = 1 + \frac{1}{2}x^4 - \frac{1}{2!}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x^8 + \frac{1}{3!}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)x^{12} - \frac{1}{4!}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)x^{16} + \frac{1}{5!}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)x^{20} + \cdots + (-1)^{n+1}\frac{1}{n!}\frac{1}{2^n}(2n-3)!!\,x^{4n} + \cdots$$

$$(1+x^4)^{1/2} = 1 + \frac{1}{2}x^4 - \frac{1}{2!}\left(\frac{1}{2^2}\right)x^8 + \frac{1}{3!}\left(\frac{3!!}{2^3}\right)x^{12} - \frac{1}{4!}\left(\frac{5!!}{2^4}\right)x^{16} + \frac{1}{5!}\left(\frac{7!!}{2^5}\right)x^{20} + \cdots + (-1)^{n+1}\frac{1}{n!}\frac{1}{2^n}(2n-3)!!\,x^{4n} + \cdots$$

Using the double factorial formula (2n)! = (2n-1)!! (2n)!! we transform the above:

$$(1+x^4)^{1/2} = 1 + \frac{1}{2}x^4 - \frac{1}{2!}\left(\frac{1}{2^2}\right)x^8 + \frac{1}{3!}\left(\frac{4!}{2^3}\right)\left(\frac{1}{4!!}\right)x^{12} - \frac{1}{4!}\left(\frac{6!}{2^4}\right)\left(\frac{1}{6!!}\right)x^{16} + \frac{1}{5!}\left(\frac{8!}{2^5}\right)\left(\frac{1}{8!!}\right)x^{20} + \cdots + (-1)^{n+1}\frac{1}{n!}\frac{1}{2^n}\frac{(2n-2)!}{(2n-2)!!}\,x^{4n} + \cdots$$

Now we use the formula (2n)!! = 2^n n!

$$(1+x^4)^{1/2} = 1 + \frac{1}{2}x^4 - \frac{1}{2!}\left(\frac{1}{2^2}\right)x^8 + \frac{1}{3!}\left(\frac{4!}{2^3}\right)\left(\frac{1}{2^2}\right)\left(\frac{1}{2!}\right)x^{12} - \frac{1}{4!}\left(\frac{6!}{2^4}\right)\left(\frac{1}{2^3}\right)\left(\frac{1}{3!}\right)x^{16} + \frac{1}{5!}\left(\frac{8!}{2^5}\right)\left(\frac{1}{2^4}\right)\left(\frac{1}{4!}\right)x^{20} + \cdots + (-1)^{n+1}\frac{1}{n!}\frac{1}{2^n}\frac{(2n-2)!}{(n-1)!\;2^{n-1}}\,x^{4n} + \cdots$$

Rewriting and combining terms this becomes:

$$(1+x^4)^{1/2} = 1 + \frac{1}{2}x^4 - \frac{1}{2!}\left(\frac{1}{2^2}\right)x^8 + \frac{1}{3!}\left(\frac{1}{2!}\right)\left(\frac{4!}{2^5}\right)x^{12} - \frac{1}{4!}\left(\frac{1}{3!}\right)\left(\frac{6!}{2^7}\right)x^{16} + \frac{1}{5!}\left(\frac{1}{4!}\right)\left(\frac{8!}{2^9}\right)x^{20} + \cdots + (-1)^{n+1}\frac{1}{n!}\frac{1}{(n-1)!}\frac{(2n-2)!}{2^{2n-1}}\,x^{4n} + \cdots$$

This can be written in sigma form:

$$(1+x^4)^{1/2} = 1 + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n!}\frac{1}{(n-1)!}\frac{(2n-2)!}{2^{2n-1}}\,x^{4n}$$

We can now evaluate the integral

$$\int_0^{0.4} \sqrt{1 + x^4}\, dx = \int_0^{0.4} \left(1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!} \frac{1}{(n-1)!} \frac{(2n-2)!}{2^{2n-1}} x^{4n}\right) dx$$

$$\int_0^{0.4} \sqrt{1 + x^4}\, dx = 0.4 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!} \frac{1}{(n-1)!} \frac{(2n-2)!}{2^{2n-1}} \int_0^{0.4} x^{4n}\, dx$$

$$\int_0^{0.4} \sqrt{1 + x^4}\, dx = 0.4 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!} \frac{1}{(n-1)!} \frac{(2n-2)!}{2^{2n-1}} \frac{0.4^{4n+1}}{4n+1}$$

We will use the remainder theorem for alternating series to determine how many terms to take in the series. The theorem states that the error is less than the first missing term. We set the remainder R = u_{n+1}.

$$u_n = \frac{1}{n!} \frac{1}{(n-1)!} \frac{(2n-2)!}{2^{2n-1}} \frac{0.4^{4n+1}}{4n+1}$$

$$R = u_{n+1} = \frac{1}{(n+1)!} \frac{1}{(n)!} \frac{(2n)!}{2^{2n+1}} \frac{0.4^{4n+5}}{4n+5} < 0.000005$$

This inequality must be solved numerically

$n = 1$ $\qquad R = 0.00000364 < 0.000005$ GOOD! ON THE FIRST TRY!

$$\int_0^{0.4} \sqrt{1 + x^4}\, dx = 0.4 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!} \frac{1}{(n-1)!} \frac{(2n-2)!}{2^{2n-1}} \frac{0.4^{4n+1}}{4n+1}$$

$$\int_0^{0.4} \sqrt{1 + x^4}\, dx \cong 0.4 + \frac{1}{1!} \frac{1}{0!} \frac{0!}{2} \frac{0.4^5}{5}$$

$$\int_0^{0.4} \sqrt{1 + x^4}\, dx \cong 0.4 + \frac{1}{1!} \frac{1}{0!} \frac{0!}{2} \frac{0.4^5}{5}$$

$$\int_0^{0.4} \sqrt{1 + x^4}\, dx \cong 0.401024$$

Calculator answer: 0.401020391

Difference: 3.609×10^{-6}

EXAMPLE

Evaluate the integral $\int_0^{0.5} x^2\, e^{-x^2}\ dx$ accurate to within 0.00001

We start with the Taylor series for e^u

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \cdots$$

Substitute u = - x^2.

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} -$$

In sigma form this becomes

$$e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

The integrand can be written as

$$x^2\, e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n!}$$

We can now evaluate the integral:

$$\int_0^{0.5} x^2\, e^{-x^2}\ dx = \int_0^{0.5} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n!}\ dx$$

$$\int_0^{0.5} x^2\, e^{-x^2}\ dx = \sum_{n=0}^{\infty} (-1)^n \int_0^{0.5} \frac{x^{2n+2}}{n!}\ dx$$

$$\int_0^{0.5} x^2\, e^{-x^2}\ dx = \sum_{n=0}^{\infty} (-1)^n \frac{0.5^{2n+3}}{n!\ (2n+3)}$$

The derivative $f^{n+1}(x)$ is extremely difficult even with Leibnitz rule. Instead we use the fact that the series is alternating and state the remainder is R = u_{n+1}

$$u_n = \frac{0.5^{2n+3}}{n!\ (2n+3)} \qquad u_{n+1} = \frac{0.5^{2n+5}}{(n+1)!\ (2n+5)}$$

$$R = \frac{0.5^{2n+5}}{(n+1)!\ (2n+5)} < 0.00001$$

We solve this numerically:

$n = 1 \quad R = 0.000558 > 0.00001$ NO GOOD

$n = 2 \quad R = 0.0000362 > 0.00001$ NO GOOD

$n = 3 \quad R = 0.00000184 < 0.00001$ GOOD!

$$\int_0^{0.5} x^2\, e^{-x^2}\, dx \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{0.5^{2n+3}}{n!\,(2n+3)}$$

$$\int_0^{0.5} x^2\, e^{-x^2}\, dx \;\cong\; \frac{0.5^3}{3} - \frac{0.5^5}{1!\,(5)} + \frac{0.5^7}{2!\,(7)} - \frac{0.5^9}{3!\,(9)}$$

$$\int_0^{0.5} x^2\, e^{-x^2}\, dx \;\cong\; 0.0359385334$$

Calculator answer: 0.0359403074 Difference: 0.000001774

EXAMPLE

You are given f(x) = e^{2x}. You expand about a = 0 and you let n=5. Find the sum. Find the remainder. Let x=0.2 and find how close you are to the true answer.

Start with the Taylor series for e^u.

$$e^u \cong 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!}$$

$$e^u \cong \sum_{n=0}^{5} \frac{u^n}{n!}$$

Substitute u = 2x

$$e^{2x} \cong 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \frac{(2x)^5}{5!}$$

$$e^{2x} \cong \sum_{n=0}^{5} \frac{(2x)^n}{n!}$$

The remainder is given by the integral

$$R_n(x) = \frac{1}{n!} \int_a^x f^{n+1}(t)\ (x-t)^n\, dt$$

$$R_n(x) = \frac{1}{n!} f^{n+1}(c) \int_a^x (x-t)^n\, dt$$

$$R_n(x) = \frac{1}{n!} f^{n+1}(c) \frac{(x-a)^{n+1}}{n+1} = f^{n+1}(c) \frac{(x-a)^{n+1}}{(n+1)!} \qquad a < c < x$$

$$R_5(x) = f^6(c) \frac{(x)^6}{(6)!}$$

There is a formula for the derivative of f(x) = e^{2x}: $f^{n+1}(x) = 2^{n+1} e^{2x}$

$$R_5(x) = 2^6 e^{2c} \frac{(x)^6}{(6)!}$$

We can get an upper bound on the error by setting c = x

$$R_5(x) = 2^6 e^{2x} \frac{(x)^6}{(6)!}$$

At x = 0.2 the remainder is $R_5(0.4) = 2^6 e^{0.2} \frac{(0.2)^6}{6!} = 6.948\ x\ 10^{-6}$

The series yields:

$$e^{2(0.2)} \cong 1 + 2(0.2) + \frac{(0.4)^2}{2!} + \frac{(0.4)^3}{3!} + \frac{(0.4)^4}{4!} + \frac{(0.4)^5}{5!} = 1.491818667$$

Calculator answer: $e^{0.4} = 1.491824698$

Difference: $6.031\ x\ 10^{-6}$

EXAMPLE

You are given f(x) = cos x. You expand about a = 0 and you let n=8. Find the sum. Find the remainder. Find the value of the approximation when x = 1.

Start with the Taylor series for cos x.

$$\cos u \;\cong\; 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \frac{u^8}{8!}$$

$$\cos x \;\cong\; \sum_{n=0}^{4} (-1)^n \frac{x^{2n}}{(2n)!}$$

The remainder is given by the integral

$$R_n(x) \;=\; \frac{1}{n!}\int_a^x f^{n+1}(t)\;(x-t)^n\,dt$$

$$R_n(x) \;=\; \frac{1}{n!}\; f^{n+1}(c)\int_a^x \;(x-t)^n\,dt$$

$$R_n(x) \;=\; \frac{1}{n!}\; f^{n+1}(c)\frac{(x-a)^{n+1}}{n+1} \;=\; f^{n+1}(c)\frac{(x-a)^{n+1}}{(n+1)\,!} \qquad a < c < x$$

$$R_8(x) \;=\; f^9(c)\frac{(x)^9}{(9)\,!}$$

The 9th derivative of cosine is negative sine: $f^9(c) = -\sin c$

$$R_8(x) \;=\; -\sin c\,\frac{(x)^9}{(9)\,!}$$

The magnitude of the error is given by

$$R_8(x) \;=\; \left|\sin c\;\frac{x^9}{9!}\right| \;<\; \left|\frac{x^9}{9!}\right|$$

We would have gotten a different result had we used the remainder theorem for alternating series. That result would give $R_8(x) \;=\; \left|\frac{x^{10}}{10!}\right|$. Either result should work.

When x = 1 the original form for the remainder is $R_8(1) = \frac{1}{9!} \;=\; 2.7557\;x\;10^{-6}$

The series is approximately $\cos 1 \;\cong\; 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} \;=\; 0.5403025794$

Calculator answer: cos 1 = 0.5403023059

Difference: 2.735 X 10^{-7}

EXAMPLE

Find the Taylor polynomial for $f(x) = \frac{1}{x+1}$ for n = 4 and a = 0. Find the remainder. Find the value of the polynomial at x = 0.5. How close is this to the true answer?

$$f(x) \;\cong\; 1 - x + x^2 - x^3 + x^4$$

The nth derivative is given by $f^n(x) \;=\; \frac{(-1)^n\; n!}{(\,x+1\,)^{n+1}}$

The remainder integral is $R_n(x) \;=\; \frac{1}{n!}\int_a^x \; f^{n+1}(x)\,(x-t)^n\,dt$

$$R_n(x) \;=\; \frac{1}{n!}\; f^{n+1}(c)\int_a^x \;(x-t)^n\,dt$$

$$R_n(x) \;=\; \frac{1}{(n+1)!}\; f^{n+1}(c)\,(x-a)^{n+1}$$

$$R_4(x) \;=\; \frac{1}{5!}\; f^5(c)\,(x)^5 \;=\; \frac{1}{5!}\;\frac{(-1)^5\;5!}{(\,c+1\,)^5}\,(x)^5$$

Set c=0 to get an upper bound on the error. Ignore the minus sign.

$$R_4(x) \;=\; (x)^5$$

At x = 0.5 the series is $f(0.5) \;\cong\; 1 - 0.5 + (0.5)^2 - (0.5)^3 + (0.5)^4 \;=\; 0.6875$

The remainder is $R_4(0.5) \;=\; (0.5)^5 \;=\; 0.03125$

The true answer is 2/3 = 0.66666…

The difference is 0.020833

EXAMPLE

Find the Taylor polynomial up to n = 6 for a = 0; $f(x) = xe^x$. Find the remainder. Find the value of the Taylor polynomial at x = 1. How close are you to the true answer?

The Taylor polynomial is $x\, e^x = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \frac{x^6}{5!} + \frac{x^7}{6!}$

There is a formula for $f^n(x) = (x+n)e^x$

$$R_n(x) = \frac{1}{n!} \int_a^x f^{n+1}(t)\,(x-t)^n\, dt$$

$$R_n(x) = \frac{1}{n!} f^{n+1}(c) \int_a^x (x-t)^n\, dt$$

$$R_n(x) = f^{n+1}(c)\, \frac{(x-a)^{n+1}}{(n+1)!}$$

$$R_n(x) = (c+n)e^c\, \frac{(x)^{n+1}}{(n+1)!}$$

Let c = 1 to get an upper bound on the error.

$$R_n(x) = (1+n)e\, \frac{(x)^{n+1}}{(n+1)!} = e\, \frac{(x)^{n+1}}{n\,!}$$

At n=6 we get $R_6(x) = e\, \frac{x^7}{7!}$

At x = 1 this is $R_6(1) = \frac{e}{7!} = 5.393\ x\ 10^{-4}$

The polynomial equals $f(1) \cong 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} = 2.718$

The true answer is 2.718281828459045

The difference is 2.263 x 10^{-4}