INTEGRATION BY METHOD OF PARTIAL FRACTIONS: LINEAR FACTORS

The method of partial fractions is based on the concept of finding a common denominator.

If we are told to add $\frac{1}{5} + \frac{1}{7}$ we find the common denominator (35) and we (hopefully) get $\frac{12}{35}$.

The question regarding partial fractions is the reverse question: If we are given the fraction $\frac{12}{35}$, then what were the original fractions that we were adding?

The way we look at this is like this: $\frac{12}{35} = \frac{12}{5 \cdot 7} = \frac{A}{5} + \frac{B}{7}$

Our goal is to find A and B. Of course we already see that A equals 1 and B equals 1.

Now for actual, numerical fractions there are an infinite number of answers. There is no unique answer. And that is ok. Let's look at this numerical example to see why.

For the above question, we would end up with $\frac{12}{5 \cdot 7} = \frac{7A}{5 \cdot 7} + \frac{5B}{5 \cdot 7}$

This boils down to finding a solution to the equation 12 = 7A + 5B.

We already know that one solution is A=1 and B=1. Another answer is A=-4 and B=8. Another answer is A=-9 and B=15. In fact, drop A by 5 each time and increase B by 7 each time, and you will always get another integer solution. So for fractions it is possible that there is more than one integer answer. Equations like this, where we want integer answers, are called Linear Diophantine equations. They are pretty important but we are not worried about them at this point.

What we want to do is take this idea of decomposing fractions into their summands, and apply them to rational functions (functions where one polynomial is divided by another polynomial) . There is a difference when applying this decomposition to rational functions – there is only one answer. The answer is unique. When we do this actual fractions, with numbers, there are an infinite number of solutions. When we do this rational functions, there is one unique answer.

For example what if we were given the function $\frac{1}{(x-1)\,(x+1)}$ and we wanted to find the original fractions that add up to this result? This is how we would do it:

$\frac{1}{(x-1)\,(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$ this is the fraction decomposition. We have to find A and B.

The way to find the A and B is to find the common denominator. That's how it works. It is all about the common denominator. Let's proceed:

$$\frac{1}{(x-1)\,(x+1)} = \frac{A\,(x+1)}{(x-1)\,(x+1)} + \frac{B\,(x-1)}{(x+1)\,(x-1)}$$

Each fraction on the right hand side now has the same denominator. We can add them. Doing so, we get

$$\frac{1}{(x-1)\,(x+1)} = \frac{A\,(x+1) + B(x-1)}{(x-1)\,(x+1)}$$

The fractions (the rational functions) on the left and the right are equal and already have the same denominator. This means that the numerators must be equal! So we have the equation

$$1 = A(x+1) + B(x-1)$$

To solve this for A and B, first let x equal -1 and plug this in. We will get

$1 = A(-1+1) + B(-1-1)$

$1 = A(0) + B(-2)$

$1 = 0 - 2\,B$

$1 = -2B$ or $B = -\frac{1}{2}$

Next let x equal 1 and plug this in. We will get

$1 = A(1+1) + B(1-1)$

$1 = 2A + 0$

$1 = 2A$ or $A = \frac{1}{2}$

These values of A and B are unique. They are the only ones that will work. No other values of A and B will work.

We have shown that

$$\frac{1}{(x-1)\,(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{\left(\frac{1}{2}\right)}{x-1} + \frac{\left(\frac{1}{2}\right)}{x+1}$$

Or preferably

$$\frac{1}{(x-1)\,(x+1)} = \frac{1}{2}\,\frac{1}{x-1} - \frac{1}{2}\,\frac{1}{x+1}$$

We did it! We decomposed a fraction into its constituent summands.

So now there is a very big question that faces us: What does this have to do with calculus? The reason why this is relevant is that we actually want to integrate these things – we want to evaluate the integral $\int \frac{1}{(x-1)\,(x+1)}\,dx$. Now we cannot do this as it stands but we can do $\int \frac{A}{x-1}\,dx$ and we can do $\int \frac{B}{x+1}\,dx$. In fact these last two integrals are easy (they are both natural logs).

So a rational function cannot be integrated as it stands, but its constituent summands can be integrated. And this is why we study the method.

1. Evaluate $\int \frac{1}{x^2-x}\ dx$

$$\int \frac{1}{x^2-x}\ dx\ =\ \int \frac{1}{x\,(x-1)}\ dx$$

$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$ decompose the fraction

$\frac{1}{x(x-1)} = \frac{A\,(x-1)}{x\ (x-1)} + \frac{B\,x}{x\,(x-1)}$ get common denominator

$\frac{1}{x(x-1)} = \frac{A\,(x-1)\ +\ Bx}{x\ (x-1)}$ add into one fraction

$1 = A(x-1)\ +\ Bx$ equate numerators

NOW FIND A AND B: let x = 1

$1 = A(1-1)\ +\ B(1)$

$1 = A(0) + B$ so B=1

Let x = 0

$1 = A(0-1)\ +\ B(0)$ so A = -1

$\frac{1}{x(x-1)} = \frac{-1}{x} + \frac{1}{x-1}$

$$\int \frac{1}{x^2-x}\ dx\ =\ \int\ \left(\frac{-1}{x} + \frac{1}{x-1}\right) dx\ \ =\ -\ln|x|\ +\ \ln|x-1|\ +\ C$$

2. Evaluate $\int \frac{1}{(x+1)\,(x+2)}\ dx$

$\frac{1}{(x+1)\,(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$ decompose the fractions

$\frac{1}{(x+1)\,(x+2)} = \frac{A\,(x+2)}{(x+1)\,(x+2)} + \frac{B\,(x+1)}{(x+1)\,(x+2)}$ get common denominator

$\frac{1}{(x+1)\,(x+2)} = \frac{A\,(x+2) +\ B\,(x+1)}{(x+1)\,(x+2)}$ add fractions

$1 = A(x+2) + \ B(x+1)$ equate numerators

Let x = -2. This will yield B=-1.

Let x = -1. This will yield A = 1.

$\frac{1}{(x+1)\,(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$ fraction is decomposed

$$\int \frac{1}{(x+1)\,(x+2)}\ dx = \int \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx = \ln|x+1| - \ln|x+2| + C$$

3. Evaluate $\int \frac{1}{(x+1)\,(x+2)\,(x+3)}\,dx$

$\frac{1}{(x+1)\,(x+2)\,(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+3)}$ decompose fraction to constituents

$\frac{1}{(x+1)\,(x+2)\,(x+3)} = \frac{A\ (x+2)(x+3)}{(x+1)\,(x+2)(x+3)} + \frac{B\,(x+1)(x+3)}{(x+1)\,(x+2)(x+3)} + \frac{C\,(x+1)(x+2)}{(x+1)\,(x+2)(x+3)}$ get common denominator

$\frac{1}{(x+1)\,(x+2)\,(x+3)} = \frac{A\,(x+2)(x+3)\ +\ B\,(x+1)(x+3)\ +\ C\,(x+1)(x+2)}{(x+1)\,(x+2)(x+3)}$ add fractions

$1 = A(x+2)(x+3)\ +\ B\,(x+1)(x+3)\ +\ C\,(x+1)(x+2)$ equate numerators

Let x = -1. The last two terms will vanish and we will get A = ½

Let x = -2. The first and third terms will vanish and we will get B = -1

Let x = -3. The first and second terms will vanish and we will get C = ½

$\frac{1}{(x+1)\,(x+2)\,(x+3)} = \frac{1}{2}\,\frac{1}{(x+1)} - \frac{1}{(x+2)} + \frac{1}{2}\,\frac{1}{(x+3)}$

$$\int \frac{1}{(x+1)\,(x+2)\,(x+3)}\,dx \;=\; \int \left(\frac{1}{2}\,\frac{1}{(x+1)} - \frac{1}{(x+2)} + \frac{1}{2}\,\frac{1}{(x+3)}\right) dx$$

$$= \frac{1}{2}\ln|x+1| - \ln|x+2| + \frac{1}{2}\ln|x+3| + C$$

Note that the integrals are actually easy. The work exists in doing the algebra.

4. Evaluate $\int \frac{3x+2}{x(x+1)(x+2)}\,dx$

$\frac{3x+2}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+2)}$ decompose fraction

Get common denominator:

$$\frac{3x+2}{x(x+1)(x+2)} = \frac{A\,(x+1)(x+2)}{x(x+1)(x+2)} + \frac{B\,(x)(x+2)}{x(x+1)(x+2)} + \frac{C\,x(x+1)}{x(x+1)(x+2)}$$

Equate numerators:

$3x+2 = A\,(x+1)(x+2) + B\,(x)(x+2) + C\,x(x+1)$

Now get A,B, C:

Let x = 0 2 = 2A+0+0 so that A=1

Let x = -1 -1 = 0 –B + 0 so that B=1

Let x = -2 -4 = 0 + 0 +2C so that C = -2

$\frac{3x+2}{x(x+1)(x+2)} = \frac{1}{x} + \frac{1}{(x+1)} - \frac{2}{(x+2)}$

$\int \frac{3x+2}{x(x+1)(x+2)}\,dx = \int \left(\frac{1}{x} + \frac{1}{(x+1)} - \frac{2}{(x+2)}\right)dx$

$$= \ln|x| + \ln|x+1| - 2\ln|x+2| + C$$

5. Evaluate $\int \frac{x^3+2}{x(x+1)}\,dx$

This is an exceptional case. Note that the polynomial in the numerator has a higher degree than the polynomial in the denominator. When this happens, decomposition cannot be the first step. In order to decompose the fraction, the polynomial in the numerator must have LOWER degree than the denominator. To do this question, we must first do long division.

If we do, then we will get $\frac{x^3+2}{x(x+1)} = \frac{x^3+2}{(x^2+\ x)} = x-1+\frac{x+2}{x(x+1)}$

$$\int \frac{x^3+2}{x(x+1)}\,dx = \int \left(x-1+\frac{x+2}{x(x+1)}\right)dx = \frac{x^2}{2} - x + \int \left(\frac{x+2}{x(x+1)}\right)dx$$

We can now do the decomposition on $\frac{x+2}{x(x+1)}$:

$\frac{x+2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

Going through the procedure we will get the equation

$x+2 = A(x+1) + Bx$

Let x = 0 and we get A=2

Let x = -1 and we get B=-1

$\frac{x+2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{2}{x} - \frac{1}{x+1}$

$\int \frac{x^3+2}{x(x+1)}\,dx = \frac{x^2}{2} - x + \int \left(\frac{2}{x} - \frac{1}{x+1}\right)dx$

$\int \frac{x^3+2}{x(x+1)}\,dx = \frac{x^2}{2} - x + 2\ln|x| - \ln|x+1| + C$

6. Evaluate $\int \frac{x^2}{x^2-1}\ dx$

Again this integral is an exceptional case. The polynomial in the numerator must be of lower degree than the denominator. So we must do long division. Doing so we will get:

$$\frac{x^2}{x^2-1} = 1 + \frac{1}{x^2-1}$$

$$\int \frac{x^2}{x^2-1}\ dx = \int \left(1 + \frac{1}{x^2-1}\right) dx = x + \int \left(\frac{1}{x^2-1}\right) dx$$

We now have to do a decomposition on the remaining integral:

$$\frac{1}{x^2-1} = \frac{1}{(x+1)\,(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

Going through the procedure to get a common denominator on the right side and equating the numerators will yield:

$$1 = A\,(x-1) + B(x+1)$$

To get A and B:

Let x = 1 so that B= ½

Let x = -1 so that A = - ½

$$\int \frac{x^2}{x^2-1}\ dx = x + \int \left(-\frac{1}{2}\,\frac{1}{x+1} + \frac{1}{2}\,\frac{1}{x-1}\right) dx$$

$$\int \frac{x^2}{x^2-1}\ dx = x - \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$

7. Evaluate $\int \frac{(2x+7)}{x^2+5x+6}\ dx$

$$\frac{(2x+7)}{x^2 + 5x + 6} = \frac{(2x+7)}{(x+2)\ (x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$2x + 7 = A\,(x+3) + B(x+2)$$

Let x = -3

1 = 0 – B so B=-1

Let x = -2

3 = A + 0 so A = 3

$$\frac{(2x+7)}{x^2 + 5x + 6} = \frac{(2x+7)}{(x+2)\ (x+3)} = \frac{3}{x+2} - \frac{1}{x+3}$$

$$\int \frac{(2x+7)}{x^2 + 5x + 6}\ dx = \int \left(\frac{3}{x+2} - \frac{1}{x+3}\right) dx = 3\ln|x+2| - \ln|x+1| + C$$

8. Evaluate $\int \frac{x^4}{x^2 + 5x + 4}\, dx$

Do long division $\frac{x^4}{x^2 + 5x + 4} = x^2 - 5x + 21 - \frac{85x + 20}{x^2 + 5x + 4}$

$$\int \frac{x^4}{x^2 + 5x + 4}\, dx = \int \left(x^2 - 5x + 21 - \frac{85x + 20}{x^2 + 5x + 4}\right) dx$$

$$= \frac{x^3}{3} - \frac{5}{2}x^2 + 21x - \int \left(\frac{85x + 20}{x^2 + 5x + 4}\right) dx$$

$$= \frac{x^3}{3} - \frac{5}{2}x^2 + 21x - \int \frac{85x + 20}{(x+1)(x+4)}\, dx$$

$$\frac{85x + 20}{(x+1)(x+4)} = \frac{A}{x+1} + \frac{B}{x+4}$$

$$85x + 20 = A(x+4) + B(x+1)$$

Let x = -1

-65 = 3A + 0 so $A = -\frac{65}{3}$

Let x = -4

-320 = 0 - 3B so $B = \frac{320}{3}$

$$= \frac{x^3}{3} - \frac{5}{2}x^2 + 21x - \int \left(-\frac{65}{3}\frac{1}{x+1} + \frac{320}{3}\frac{1}{x+4}\right) dx$$

$$= \frac{x^3}{3} - \frac{5}{2}x^2 + 21x + \int \left(\frac{65}{3}\frac{1}{x+1} - \frac{320}{3}\frac{1}{x+4}\right) dx$$

$$= \frac{x^3}{3} - \frac{5}{2}x^2 + 21x + \frac{65}{3}\ln|x+1| - \frac{320}{3}\ln|x+4| + C$$

9. Evaluate $\int \sec x \; dx$

We know that there was a trick that allowed us to integrate this, but there is a more formal method. Here we show how to find the integral without depending on deep insight.

$$\int \sec x \; dx \; = \; \int \frac{1}{\cos x} \; dx \; = \; \int \frac{1}{\cos x} \, \frac{\cos x}{\cos x} \; dx \; = \; \int \frac{\cos x}{\cos^2 x} \; dx$$

$$= \int \frac{\cos x}{1-\sin^2 x} \; dx$$

Let u = sin x so that du = cos x dx

The integral becomes

$$\int \frac{\cos x}{1-\sin^2 x} \; dx \; = \; \int \frac{1}{1-u^2} \; du$$

We can do a partial fraction decomposition on this to get:

$$\frac{1}{1 - u^2} \; = \; \frac{1}{2}\frac{1}{1 - u} \; + \; \frac{1}{2}\frac{1}{1 + u}$$

$$\int \frac{1}{u^2-1} \; du \; = \; \int \left(-\frac{1}{2}\frac{1}{u-1} \; + \; \frac{1}{2}\frac{1}{u+1}\right) \; dx$$

$$= \; -\frac{1}{2}\ln|u-1| \; + \; \frac{1}{2}\ln|u+1| \; + \; c = \frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| \; + \; C$$

$$= \frac{1}{2}\ln\left|\frac{\sin x + 1}{\sin x - 1}\right| \; + \; C \; = \; \frac{1}{2}\ln\left|\frac{\sin x + 1}{\sin x - 1} \cdot \frac{\sin x + 1}{\sin x + 1}\right| + \; C$$

$$= \frac{1}{2}\ln\left|\frac{(\sin x + 1)^2}{\sin^2 x - 1}\right| \; + \; C \; = \; \frac{1}{2}\ln\left|\frac{(\sin x + 1)^2}{-\cos^2 x}\right| \; + \; C$$

$$= \frac{1}{2}\ln\left|\frac{(\sin x + 1)^2}{\cos^2 x}\right| \; + \; C \; = \; \frac{1}{2}\ln\left|\left(\frac{\sin x + 1}{\cos x}\right)^2\right| \; + \; C$$

$$= \; \frac{1}{2} \cdot 2 \; \ln\left|\frac{\sin x + 1}{\cos x}\right| \; + \; C \; = \; \ln\left|\frac{1 + \sin x}{\cos x}\right| \; + \; C$$

$$= \ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| \; + \; C \; = \; \ln|\sec x + \tan x| \; + \; C$$

10. Evaluate $\int \csc x \; dx$

$$\int \csc x \; dx = \int \frac{1}{\sin x} \; dx = \int \frac{\sin x}{\sin^2 x} \; dx = \int \frac{\sin x}{1-\cos^2 x} \; dx$$

Let u = cos x so that du = - sin x dx

$$\int \csc x \; dx = \int -\frac{du}{1-u^2} = \int \frac{du}{u^2-1}$$

$$\frac{1}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1}$$

$$1 = A(u+1) + B(u-1)$$

u = 1 gives A = ½ u = -1 gives B = - ½

$$\frac{1}{u^2-1} = \frac{1}{2}\,\frac{1}{u-1} - \frac{1}{2}\,\frac{1}{u+1}$$

$$\int \csc x \; dx = \int \left(\frac{1}{2}\,\frac{1}{u-1} - \frac{1}{2}\,\frac{1}{u+1}\right) du = \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + C$$

$$\int \csc x \; dx = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C$$

$$\int \csc x \; dx = \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C$$

$$\int \csc x \; dx = \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1} \cdot \frac{\cos x - 1}{\cos x - 1}\right| + C$$

$$\int \csc x \; dx = \frac{1}{2}\ln\left|\frac{(\cos x - 1)^2}{\cos^2 x \; - \; 1}\right| + \; C$$

$$\int \csc x \; dx = \frac{1}{2}\ln\left|\frac{(1 \; - \; \cos x)^2}{-\sin^2 x}\right| + \; C \; = \; \frac{1}{2}\ln\left|\frac{(1 \; - \; \cos x)^2}{\sin^2 x}\right| + \; C$$

$$\int \csc x \; dx \; = \; \frac{1}{2}\ln\left|\left(\frac{(1 \; - \; \cos x)}{\sin \; x}\right)^2\right| + \; C$$

$$\int \csc x \; dx \; = \; \ln\left|\frac{(1 \; - \; \cos x)}{\sin \; x}\right| + \; C$$

$$\int \csc x \; dx \; = \; \ln\left|\frac{1}{\sin \; x} \; - \frac{\cos x}{\sin x}\right| + \; C$$

$$\int \csc x \; dx \; = \; \ln|\csc x - \; \cot x \; | + \; C$$

11. Show that $\int \csc x \; dx \; = \; - \; \ln| \csc x \; + \; \cot x \; | + \; C$

$$\int \csc x \; dx \; = \; \ln|\csc x - \; \cot x \; | + \; C \; = \; \ln\left|(\csc x - \; \cot x \;) \cdot \frac{\csc x + \cot x}{\csc x + \cot x}\right| + \; C$$

$$\int \csc x \; dx \; = \; \ln\left|\frac{\csc^2 x \; - \; \cot^2 x}{\csc x \; + \; \cot x}\right| + \; C$$

$$\int \csc x \; dx \; = \; \ln\left|\frac{1}{\csc x \; + \; \cot x}\right| + \; C$$

$$\int \csc x \; dx \; = \; \ln| \; 1 \; | \; - \; \ln| \; \csc x \; + \; \cot x \; | \; + \; C$$

$$\int \csc x \; dx \; = \; 0 \; - \; \ln| \; \csc x \; + \; \cot x \; | \; + \; C$$

$$\int \csc x \; dx \; = \; - \; \ln| \; \csc x \; + \; \cot x \; | \; + \; C$$