TAYLOR SERIES
INTEGRALS AND LIMITS

Many integrals cannot be done using the known forms of integration. In many of these cases the integration can be done using term by term integration of Taylor series.

There are also limits that can be done with LHospital's rule but these can be quite difficult and require many iterations of the method. These limits can be performed more easily using Taylor series.

We will look at both kinds of problems.

EXAMPLE

Find $\int \sin(x^2)\ dx$

We first write the Taylor series for sin(u)

$$\sin u \ = \ \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!}$$

Substitute u = x^2

$$\sin x^2 \ = \ \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} \ = \ \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$$

$$\int \sin(x^2)\ dx \ = \ \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}\ dx$$

$$\int \sin(x^2)\ dx \ = \ \sum_{n=0}^{\infty} (-1)^n \int \frac{x^{4n+2}}{(2n+1)!}\ dx$$

$$\int \sin(x^2)\ dx \ = \ \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)\,(2n+1)!} + C$$

EXAMPLE

Find $\int e^{-x^2}\, dx$

Start with the Taylor series for e^u

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$$

Substitute $u = -x^2$

$$e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^n}{n!}$$

$$e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

$$\int e^{-x^2}\, dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}\, dx$$

$$\int e^{-x^2}\, dx = \sum_{n=0}^{\infty} (-1)^n \int \frac{x^{2n}}{n!}\, dx$$

$$\int e^{-x^2}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)\, n!} + C$$

EXAMPLE

Find the integral $\int \ln(1 + x^3)\, dx$

Start with the Taylor series for ln(1+u)

$$\ln(1+u) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{u^n}{n}$$

Substitute $u = x^3$

$$\ln(1+x^3) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{3n}}{n}$$

$$\int \ln(1+x^3)\, dx = \int \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{3n}}{n}\, dx$$

$$\int \ln(1+x^3)\, dx = \sum_{n=1}^{\infty} (-1)^{n+1} \int \frac{x^{3n}}{n}\, dx$$

$$\int \ln(1+x^3)\, dx = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{3n+1}}{n\,(3n+1)} + C$$

EXAMPLE

Evaluate the integral $\int \frac{\sin x}{x}\ dx$

Start with the Taylor series for sin x

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$$

$$\int \frac{\sin x}{x}\ dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}\ dx$$

$$\int \frac{\sin x}{x}\ dx = \sum_{n=0}^{\infty} (-1)^n \int \frac{x^{2n}}{(2n+1)!}\ dx$$

$$\int \frac{\sin x}{x}\ dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!} + C$$

EXAMPLE

Evaluate the integral $\int \cos(e^x)\ dx$

Start with the Taylor series for cos (u)

$$\cos u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n}}{(2n)!}$$

Substitute u = e^x

$$\cos(e^x) = \sum_{n=0}^{\infty} (-1)^n \frac{e^{2n\,x}}{(2n)!}$$

$$\int \cos(e^x)\ dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{e^{2n\,x}}{(2n)!}\ dx$$

$$\int \cos(e^x)\ dx = \sum_{n=0}^{\infty} (-1)^n \int \frac{e^{2n\,x}}{(2n)!}\ dx$$

$$\int \cos(e^x)\ dx = \sum_{n=0}^{\infty} (-1)^n \frac{e^{2n\,x}}{(2n)\,(2n)!} + C$$

The answer is an infinite series but it is not a Taylor series. We used Taylor series to get this form but the answer itself is not a Taylor series.

EXAMPLE STEWART

Evaluate $\int \sqrt{1+\ x^3}\ dx$

This is an ugly question. It is difficult. If you want to skip it, you can. There are a lot of factorials and double factorials in it and the algebra gets tedious. It takes a lot of work to find mathematically representable patterns.

There are two factorial formulas the dominate the notation:

(2n)! = (2n)!! (2n-1)!!

(2n)!! = 2^n n!

With the definitions:

n! = n(n-1)(n-2)… (3)(2)(1)

(2n)!! = (2n) (2n-2) (2n-4)…(2)

(2n+1)!! = (2n+1)(2n-1)(2n-3)…(3)(1)

Start with with the Taylor series for $\sqrt{1\ +\ u}$

$$(1+u)^n = 1 + nu + \frac{1}{2!}n(n-1)u^2 + \frac{1}{3!}n(n-1)(n-2)u^3 + \frac{1}{4!}n(n-1)(n-2)(n-3)u^4 + \cdots$$

$$(1+u)^{1/2} = 1 + \left(\frac{1}{2}\right)u + \frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)u^2 + \frac{1}{3!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)u^3 + \frac{1}{4!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)u^4 + \frac{1}{5!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)u^5 \ldots$$

$$(1+u)^{1/2} = 1 + \frac{1}{2}u - \frac{1}{2!}\frac{1!!}{2^2}u^2 + \frac{1}{3!}\frac{3!!}{2^3}u^3 - \frac{1}{4!}\frac{5!!}{2^4}u^4 + \frac{1}{5!}\frac{7!!}{2^5}u^5 \ldots$$

$$(1+u)^{1/2} = 1 + \frac{1}{2}u - \frac{1}{2!}\frac{1}{2^2}\frac{(1!!)(2)!!}{(2)!!}u^2 + \frac{1}{3!}\frac{1}{2^3}\frac{(3!!)(4!!)}{4!!}u^3 - \frac{1}{4!}\frac{1}{2^4}\frac{(5!!)(6!!)}{(6!!)}u^4 - \frac{1}{5!}\frac{1}{2^5}\frac{(7!!)(8!!)}{(8!!)}u^5 \ldots$$

$$(1+u)^{1/2} = 1 + \frac{1}{2}u - \frac{1}{2!}\frac{1}{2^2}\frac{2!}{(2)!!}u^2 + \frac{1}{3!}\frac{1}{2^3}\frac{4!}{4!!}u^3 - \frac{1}{4!}\frac{1}{2^4}\frac{6!}{(6!!)}u^4 - \frac{1}{5!}\frac{1}{2^5}\frac{8!}{(8!!)}u^5 \ldots$$

$$(1+u)^{1/2} = 1 + \frac{1}{2}u + \sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n!}\frac{1}{2^n}\frac{(2n-2)!}{(2n-2)!!}u^n$$

$$(1+u)^{1/2} = 1 + \frac{1}{2}u + \sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n!}\frac{1}{2^n}\frac{(2n-2)!}{2^{n-1}(n-1)!}u^n$$

$$(1+u)^{1/2} = 1 + \frac{1}{2}u + \sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n!}\frac{1}{2^{2n-1}}\frac{(2n-2)!}{(n-1)!}u^n$$

Substitute u = x^3

$$(1+x^3)^{1/2} = 1 + \frac{1}{2}x^3 + \sum_{n=2}^{\infty} (-1)^{n+1}\frac{1}{n!}\frac{1}{2^{2n-1}}\frac{(2n-2)!}{(n-1)!} \; x^{3n}$$

$$\int \sqrt{1+x^3}\; dx = x + \frac{x^4}{8} + \int \sum_{n=2}^{\infty} (-1)^{n+1}\frac{1}{n!}\frac{1}{2^{2n-1}}\frac{(2n-2)!}{(n-1)!} \; x^{3n} \; dx$$

$$\int \sqrt{1+x^3}\; dx = x + \frac{x^4}{8} + \sum_{n=2}^{\infty} (-1)^{n+1}\frac{1}{n!}\frac{1}{2^{2n-1}}\frac{(2n-2)!}{(n-1)!}\frac{x^{3n+1}}{3n+1} + C$$

EXAMPLE STEWART

Evaluate $\int \sqrt{1+x^3}\; dx$

We can redo the previous question using a different notation. This notation is accepted in some parts of mathematics and is knowns as a generalized binomial coefficient or a negative binomial coefficient.

$$(1+u)^n = \sum_{k=0}^{\infty} \binom{n}{k} u^k$$

$$(1+u)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} u^k$$

$$(1+x^3)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^{3k}$$

$$\int \sqrt{1+x^3}\; dx = \int \sum_{k=0}^{\infty} \binom{1/2}{k} x^{3k} \; dx$$

$$\int \sqrt{1+x^3}\; dx = \sum_{k=0}^{\infty} \binom{1/2}{k} \int x^{3k} \; dx$$

$$\int \sqrt{1+x^3}\; dx = \sum_{k=0}^{\infty} \binom{1/2}{k} \frac{x^{3k+1}}{(3k+1)} + C$$

The generalized binomial coefficient can be written explicitly as

$$\binom{n}{k} = \frac{n(n-1)\,(n-2)\ldots(n-(k-1))}{k!}$$

There is obviously a lot of arithmetic in simplifying the binomial coefficient and its simplicity is hiding some tedious arithmetic. It is good as a very condensed shorthand notation.

EXAMPLE STEWART

Evaluate the integral $\int \tan^{-1} x^2 \; dx$

We start with the Taylor series for $\tan^{-1}$ u

$$\tan^{-1} u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{2n+1}$$

Replace u with x^2

$$\tan^{-1} x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

$$\int \tan^{-1} x^2 \; dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1} \; dx$$

$$\int \tan^{-1} x^2 \; dx = \sum_{n=0}^{\infty} (-1)^n \int \frac{x^{4n+2}}{2n+1} \; dx$$

$$\int \tan^{-1} x^2 \; dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)} + C$$

EXAMPLE

Evaluate the integral $\int \frac{\cos x - 1}{x} \quad dx$

Start with the Taylor series

$\cos x \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

If we write this out it becomes

$$\cos x \;=\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

We get

$$\cos x - 1 \;=\; -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

And then:

$$\frac{\cos x - 1}{x} \;=\; -\frac{x^1}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \cdots$$

This can be written in sigma form by starting the series at n = 1 and reducing the power of x by 1.

$$\frac{\cos x - 1}{x} \;=\; \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n)!}$$

$$\int \frac{\cos x - 1}{x} \quad dx \;=\; \int \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n)!} \quad dx$$

$$\int \frac{\cos x - 1}{x} \quad dx \;=\; \sum_{n=1}^{\infty} (-1)^n \int \frac{x^{2n-1}}{(2n)!} \; dx$$

$$\int \frac{\cos x - 1}{x} \quad dx \;=\; \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)\,(2n)!} \; + \; C$$

EXAMPLE

Evaluate the integral $\int_0^{\frac{1}{2}} x^3 \tan^{-1} x \; dx$

We start with the Taylor series for $\tan^{-1} x$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$x^3 \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{2n+1}$$

$$\int_0^{\frac{1}{2}} x^3 \tan^{-1} x \; dx = \int_0^{\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{2n+1} \; dx$$

$$\int_0^{\frac{1}{2}} x^3 \tan^{-1} x \; dx = \sum_{n=0}^{\infty} (-1)^n \int_0^{1/2} \frac{x^{2n+4}}{2n+1} \; dx$$

$$\int_0^{\frac{1}{2}} x^3 \tan^{-1} x \; dx = \sum_{n=0}^{\infty} (-1)^n \left. \frac{x^{2n+5}}{(2n+1)(2n+5)} \right|_0^{1/2}$$

$$\int_0^{\frac{1}{2}} x^3 \tan^{-1} x \; dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n+5}\,(2n+1)(2n+5)}$$

EXAMPLE

Evaluate the integral $\int_0^1 \sin x^4 \quad dx$

Start with the Taylor series for sin u

$$\sin u \; = \; \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!}$$

Substitute for u with x^4

$$\sin x^4 \; = \; \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+4}}{(2n+1)!}$$

$$\int_0^1 \sin x^4 \quad dx \;\; = \;\; \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+4}}{(2n+1)!} \quad dx$$

$$\int_0^1 \sin x^4 \quad dx \;\; = \;\; \sum_{n=0}^{\infty} (-1)^n \int_0^1 \frac{x^{8n+4}}{(2n+1)!} \quad dx$$

$$\int_0^1 \sin x^4 \quad dx \;\; = \;\; \sum_{n=0}^{\infty} (-1)^n \left. \frac{x^{8n+5}}{(8n+5)\,(2n+1)!} \right|_0^1$$

$$\int_0^1 \sin x^4 \quad dx \;\; = \;\; \sum_{n=0}^{\infty} (-1)^n \frac{1}{(8n+5)\,(2n+1)!}$$

EXAMPLE

Evaluate $\int_0^{0.4} \sqrt{1 + x^4}\ dx$

$\sqrt{1 + u}$

$$(1+u)^n = 1 + n\,u + \frac{1}{2!} n(n-1)\,u^2 + \frac{1}{3!}\, n\,(n-1)\,(n-2)\,u^3 + \frac{1}{4!}\, n\,(n-1)\,(n-2)\,(n-3)\,u^4 + \cdots$$

$$(1+u)^{1/2} = 1 + \left(\frac{1}{2}\right) u + \frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) u^2 + \frac{1}{3!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) u^3 + \frac{1}{4!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right) u^4 + \cdots$$

This leads to a previous result:

$$(1+u)^{1/2} = 1 + \frac{1}{2} u + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n!} \frac{1}{2^n} \frac{(2n-2)!}{2^{n-1}\,(n-1)!}\ u^n$$

Substitute u = x^4

$$(1+x^4)^{1/2} = 1 + \frac{1}{2} x^4 + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n!} \frac{1}{2^n} \frac{(2n-2)!}{2^{n-1}\,(n-1)!}\ x^{4n}$$

$$\int_0^{0.4} (1+x^4)^{1/2}\ dx = \left(x + \frac{x^5}{10}\right)\Big|_0^{0.4} + \int_0^{0.4} \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n!} \frac{1}{2^n} \frac{(2n-2)!}{2^{n-1}\,(n-1)!}\ x^{4n}\ dx$$

$$\int_0^{0.4} (1+x^4)^{1/2}\ dx = \left(x + \frac{x^5}{10}\right)\Big|_0^{0.4} + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n!} \frac{1}{2^n} \frac{(2n-2)!}{2^{n-1}\,(n-1)!} \frac{x^{4n+1}}{4n+1}\Big|_0^{0.4}$$

$$\int_0^{0.4} (1+x^4)^{1/2}\ dx = \left(0.4 + \frac{(0.4)^5}{10}\right) + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n!} \frac{1}{2^n} \frac{(2n-2)!}{2^{n-1}\,(n-1)!} \frac{(0.4)^{4n+1}}{4n+1}$$

EXAMPLE

$\int_0^{0.5} x^2 \, e^{-x^2} \, dx$

Start with the Taylor series for e^u

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$$

Substitute u = - x^2

$$e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

$$x^2 \, e^{x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n!}$$

$$\int_0^{0.5} x^2 \, e^{-x^2} \, dx = \int_0^{0.5} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n!} \, dx$$

$$\int_0^{0.5} x^2 \, e^{-x^2} \, dx = \sum_{n=0}^{\infty} (-1)^n \int_0^{0.5} \frac{x^{2n+2}}{n!} \, dx$$

$$\int_0^{0.5} x^2 \, e^{-x^2} \, dx = \sum_{n=0}^{\infty} (-1)^n \left. \frac{x^{2n+3}}{n! \, (2n+3)} \right|_0^{0.5}$$

$$\int_0^{0.5} x^2 \, e^{-x^2} \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{(0.5)^{2n+3}}{n! \, (2n+3)}$$

$$\int_0^{0.5} x^2 \, e^{-x^2} \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n! \, 2^{2n+3} \, (2n+3)}$$

EXAMPLE

Evaluate the limit $\lim\limits_{x \to 0} \frac{e^x - 1 - x - \frac{1}{2}x^2}{x^3}$

Use the Taylor series for $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots$

$$\lim_{x \to 0} \frac{e^x - 1 - x - \frac{1}{2}x^2}{x^3} = \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots\right) - 1 - x - \frac{1}{2}x^2}{x^3}$$

$$\lim_{x \to 0} \frac{e^x - 1 - x - \frac{1}{2}x^2}{x^3} = \lim_{x \to 0} \frac{\left(\frac{x^3}{3!} + \frac{x^4}{4!} \ldots\right)}{x^3}$$

$$\lim_{x \to 0} \frac{e^x - 1 - x - \frac{1}{2}x^2}{x^3} = \lim_{x \to 0} \frac{1}{3!} + \frac{x}{4!} = \frac{1}{6}$$

We could do this by LHospital's rule but this provides a better method sometimes.

EXAMPLE

Evaluate the limit $\lim\limits_{x \to 0} \frac{x - \ln(1+x)}{x^2}$

Use the Taylor series for $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$

$$\lim_{x \to 0} \frac{x - \ln(1+x)}{x^2} = \lim_{x \to 0} \frac{x - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right)}{x^2}$$

$$\lim_{x \to 0} \frac{e^x - 1 - x - \frac{1}{2}x^2}{x^3} = \lim_{x \to 0} \frac{\left(\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4}\right)}{x^2}$$

$$\lim_{x \to 0} \frac{e^x - 1 - x - \frac{1}{2}x^2}{x^3} = \lim_{x \to 0} \frac{1}{2} - \frac{x}{3} + \frac{x^2}{4} = \frac{1}{2}$$

We could do this by LHospital's rule but this provides a better method sometimes.

EXAMPLE

Evaluate $\lim_{x \to 0} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{x^2}$

Use the binomial theorem:

$(1+u)^n = 1 + n\,u + \frac{1}{2!}\,n(n-1)\,u^2 + \ldots$

$$\lim_{x \to 0} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{x^2} = \lim_{x \to 0} \frac{\left(1+\frac{1}{2}x^2+\frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^4\right) - \left(1-\frac{1}{2}x^2+\frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^4\right)}{x^2}$$

$$\lim_{x \to 0} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{x^2} = \lim_{x \to 0} \frac{x^2-\frac{1}{8}\,x^4+\frac{1}{8}x^4}{x^2} = \lim_{x \to 0} \frac{x^2}{x^2} = 1$$

EXAMPLE

$$\lim_{x \to 0} \frac{1-\cos x}{1+x-e^x}$$

$$\lim_{x \to 0} \frac{1-\cos x}{1+x-e^x} = \lim_{x \to 0} \frac{1-\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)}{1+x-\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right)}$$

$$\lim_{x \to 0} \frac{1-\cos x}{1+x-e^x} = \lim_{x \to 0} \frac{\left(\frac{x^2}{2}-\frac{x^4}{24}\right)}{-\left(\frac{x^2}{2}+\frac{x^3}{6}\right)} = \lim_{x \to 0} \frac{\left(\frac{1}{2}-\frac{x^2}{24}\right)}{-\left(\frac{1}{2}+\frac{x}{6}\right)}$$

$$\lim_{x \to 0} \frac{1-\cos x}{1+x-e^x} = \lim_{x \to 0} \frac{\left(\frac{1}{2}\right)}{-\left(\frac{1}{2}\right)} = -1$$

EXAMPLE

$$\lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6}}{\sin(x^5)}$$

$$\lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6}}{\sin(x^5)} = \lim_{x \to 0} \frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\right) - x + \frac{x^3}{6}}{x^5 - \frac{x^{15}}{3!}}$$

$$\lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6}}{\sin(x^5)} = \lim_{x \to 0} \frac{\left(\frac{x^5}{5!}-\frac{x^7}{7!}\right)}{x^5 - \frac{x^{15}}{3!}} = \lim_{x \to 0} \frac{\left(\frac{1}{5!}-\frac{x^2}{7!}\right)}{1 - \frac{x^{10}}{3!}}$$

$$\lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6}}{\sin(x^5)} = \frac{1}{5!} = \frac{1}{120}$$

EXAMPLE

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{1}{2}x}{\tan^{-1}x^2}$$

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{1}{2}x}{\tan^{-1}x^2} = \lim_{x\to 0}\frac{\left(1+\frac{1}{2}x-\frac{1}{8}x^2\right)-1-\frac{1}{2}x}{x^2-\frac{x^6}{3}}$$

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{1}{2}x}{\tan^{-1}x^2} = \lim_{x\to 0}\frac{\left(-\frac{1}{8}x^2\right)}{x^2-\frac{x^6}{3}}$$

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{1}{2}x}{\tan^{-1}x^2} = \lim_{x\to 0}\frac{\left(-\frac{1}{8}\right)}{1-\frac{x^4}{3}} = -\frac{1}{8}$$

EXAMPLE

$$\lim_{x\to 0}\frac{\tan^{-1}x-\sin^{-1}x}{x^3}$$

We start by using L'Hospital's rule:

$$\lim_{x\to 0}\frac{\tan^{-1}x-\sin^{-1}x}{x^3} = \lim_{x\to 0}\frac{(1+x^2)^{-1}-(1-x^2)^{-1/2}}{3x^2}$$

Now we do Taylor series expansion (binomial expansion):

$$\lim_{x\to 0}\frac{\tan^{-1}x-\sin^{-1}x}{x^3} = \lim_{x\to 0}\frac{(1-x^2+x^4-\cdots)-\left(1+\frac{1}{2}x^2-\frac{1}{8}x^4\right)}{3x^2}$$

$$\lim_{x\to 0}\frac{\tan^{-1}x-\sin^{-1}x}{x^3} = \lim_{x\to 0}\frac{-\frac{3}{2}x^2+\frac{9}{8}x^4}{3x^2}$$

$$\lim_{x\to 0}\frac{\tan^{-1}x-\sin^{-1}x}{x^3} = \lim_{x\to 0}-\frac{3}{2}+\frac{9}{8}x^2$$

$$\lim_{x\to 0}\frac{\tan^{-1}x-\sin^{-1}x}{x^3} = -\frac{3}{2}$$