CONVERGENCE OF POWER SERIES BY RATIO TEST

A power series is an infinite series of the form $\sum_{n=1}^{\infty} c_n x^n$. Up to this point, we have only considered series of the form $\sum_{n=1}^{\infty} c_n$. We discussed the convergence of these easier series using various tests; integral test, comparison test, p test and so on.

When we consider power series, our question on convergence changes. A power series must always converge, but there are three possible ways this can happen. The first way is that it only converges for a single value of x (that value is x = 0). The second way is that it can converge when x is confined to an interval: $-R < x < +R$ where R is a constant known as the radius of convergence. The series can also converge when x = R or when x = -R or both, but this needs further testing. The third possibility is that the power series converges for all values of x.

We can think of a power series as defining a function of x: $f(x) = \sum_{n=1}^{\infty} c_n x^n$. So according to the above, the domain of this function could be x=0, -R < x < R (with the endpoints possibly included) or the domain could be all real numbers.

We are concerned with finding the values of x for which the power series converges. The main tool to do this is the ratio test. We will restate it now: A series $\sum_{n=1}^{\infty} u_n$ converges if $\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$. For us, $u_n = c_n x^n$.

There are three more important properties of power series that are of great use. Let's consider the cases where $f(x) = \sum_{n=1}^{\infty} c_n x^n$ and it converges for -R < x < + R (and possibly the endpoints), or the case where the power series converges for all x. In these two cases we have the following properties for power series:

1. The function f(x) defined by the power series is continuous at any point where the power series converges. So power series only produce continuous functions. Any point of convergence is a point of continuity.
2. Power series are differentiable. We can take their derivatives. We say that they are "term by term" differentiable. It works like this:

$$f(x) = \sum_{n=1}^{\infty} c_n x^n$$

$$f'(x) = \frac{d}{dx} \sum_{n=1}^{\infty} c_n x^n = \sum_{n=1}^{\infty} \frac{d}{dx} (c_n x^n) = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

If we write this out, it looks like this:

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$$f'(x) = \frac{d}{dx} (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots)$$

$$f'(x) = 0 + c_1 + 2 c_2 x + 3 c_3 x^2 + \cdots$$

3. Power series are integrable – they can be integrated. We say that they too can be integrated "term by term".

$$f(x) = \sum_{n=1}^{\infty} c_n x^n$$

$$\int f(x)\, dx = \int \left(\sum_{n=1}^{\infty} c_n x^n \right) dx = \sum_{n=1}^{\infty} \left(\int c_n x^n \, dx \right)$$

$$\int f(x)\, dx = \sum_{n=1}^{\infty} c_n \frac{x^{n+1}}{n+1} + C$$

The fact that power series are representations of continuous functions is extremely important, but it is not the focus of this handout. The ability to integrate and differentiate power series term by term, is also remarkably important. But these properties are only mentioned here. They will not be explored. This handout only addresses the convergence of power series, nothing else.

1. Find the interval of convergence for $\sum_{n=0}^{\infty} x^n$

$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

$u_n = x^n \qquad u_{n+1} = x^{n+1}$

Use the ratio test:

The series converges when $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$

$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty} |x| = |x| < 1$

$|x| < 1 \qquad\qquad -1 < x < 1$

We must check endpoints:

When x = 1 we have $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1 = 1 + 1 + 1 + 1 \ldots = \infty$

Series does not converge when x = 1

When x = -1 we have $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 \ldots$ diverges by oscillation

Series does not converge when x = -1

Final answer: series converges when -1 < x < 1

2. Find the interval of convergence for $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Write the series out: $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

$u_n = \frac{x^n}{n!}$ $\qquad$ $u_{n+1} = \frac{x^{n+1}}{(n+1)!}$

Use the ratio test:

The series converges when $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \ \frac{\text{n!}}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n} \ \frac{\text{n!}}{(n+1)!}\right| =$$
$$\lim_{n\to\infty} \left|x \ \frac{\text{n!}}{(n+1)!}\right|$$

$$= \lim_{n\to\infty} \left|x \ \frac{1}{(n+1)}\right| = |x| \lim_{n\to\infty} \left|\frac{1}{(n+1)}\right| = 0$$

We see that the value is always zero, no matter what value of x you use. The limit test produces a value that is always less than 1, no matter what value of x you use. This means that the series converges for all values of x.

The series converges for all real numbers.

3. Find the interval of convergence for $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n (n+1)}$

Write the series out: $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n (n+1)} = 1 - \frac{x}{3(2)} + \frac{x^2}{3^2 (3)} + \frac{x^3}{3^3 (4)} + \frac{x^4}{3^4 (5)} + \ldots$

$u_n = \frac{(-1)^n x^n}{3^n (n+1)} \qquad u_{n+1} = \frac{(-1)^{n+1} x^{n+1}}{3^{n+1} (n+2)}$

Use the ratio test:

The series converges when $\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$

$$\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \to \infty} \left|\frac{x^{n+1}}{3^{n+1} (n+2)} \; \frac{3^n (n+1)}{x^n}\right| = \lim_{n \to \infty} \left|\frac{x^{n+1}}{x^n} \; \frac{3^n (n+1)}{3^{n+1} (n+2)}\right|$$

$$= \lim_{n \to \infty} \left|x \; \frac{(n+1)}{3(n+2)}\right| = |x| \lim_{n \to \infty} \left|\frac{(n+1)}{3(n+2)}\right| = \frac{|x|}{3} < 1$$

This yields: $|x| < 3$ or $-3 < x < 3$

Test the endpoints:

When x = 3: $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n (n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{3^n (n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)}$ converges by alternating series test. The original series converges when x = 3

When x = -3: $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n (n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n (-3)^n}{3^n (n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^n}{(n+1)} =$ $\sum_{n=0}^{\infty} \frac{1}{(n+1)}$ this diverges because it is the harmonic series. The original series diverges when x = -3.

INTERVAL OF CONVERGENCE $-3 < x \leq 3$

4. Find the interval of convergence for $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

Write the series out: $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ \ldots$

$u_n = \frac{(-1)^n x^{2n}}{(2n)!} \qquad u_{n+1} = \frac{(-1)^{n+1} x^{2n+2}}{(2n+2)!}$

Use the ratio test:

The series converges whenlim $_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x^{2n+2}}{(2n+2)!} \ \frac{(2n!)}{x^{2n}}\right| = \lim_{n\to\infty} \left|\frac{x^{2n+2}}{x^{2n}} \ \frac{(2n)!}{(2n+2)!}\right|$$

$$= \lim_{n\to\infty} \left| x^2 \ \frac{1}{(2n+2)\,(2n+1)}\right| = x^2 \lim_{n\to\infty} \left| \frac{1}{(2n+2)\,(2n+1)}\right| = 0 < 1$$

The ratio test produces a value of 0. This means that the series converges for all values of x. The interval of convergence is from minus infinity to positive infinity. The series converges for all real numbers.

5. Find the interval of convergence for $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2}$

Write the series out: $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2} = (x-5) - \frac{(x-5)^2}{2^2} + \frac{(x-5)^3}{4^2} - \frac{(x-5)^6}{6^2} \ ...$

$u_n = \frac{(x-5)^n}{n^2} \qquad u_{n+1} = \frac{(x-5)^{n+1}}{(n+1)^2}$

Use the ratio test:

The series converges when $\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$

$$\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \to \infty} \left|\frac{(x-5)^{n+1}}{(n+1)^2} \quad \frac{\mathrm{n}^2}{(x-5)^n}\right| = \lim_{n \to \infty} \left|\frac{(x-5)^{n+1}}{(x-5)^n} \quad \frac{n^2}{(n+1)^2}\right|$$

$$= \lim_{n \to \infty} \left|(x-5) \quad \frac{n^2}{(n+1)^2}\right| = |x-5| \lim_{n \to \infty} \frac{n^2}{(n+1)^2} = |x-5| < 1$$

$-1 < (x-5) < 1$ or $4 < x < 6$

Test endpoints:

X = 4 $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges by alternating series test

The original series converges when x = 4

X= 6 $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p test. The original series converges when x = 6.

Interval of convergence $4 \leq x \leq 6$

6. Find the interval of convergence for $\sum_{n=1}^{\infty} \frac{x^n}{n+1}$

Write the series out: $\sum_{n=1}^{\infty} \frac{x^n}{n+1} = \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \cdots$

$u_n = \frac{x^n}{n+1} \qquad u_{n+1} = \frac{x^{n+1}}{n+2}$

Use the ratio test:

The series converges when $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{n+2}\,\frac{n+1}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\,\frac{n+1}{n+2}\right|$$

$$= \lim_{n\to\infty} \left|x\,\frac{n+1}{n+2}\right| = |x| \lim_{n\to\infty} \left|\frac{n+1}{n+2}\right| = |x| < 1$$

So -1 < x < 1

Test end points:

X=1 $\sum_{n=1}^{\infty} \frac{x^n}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges by integral test – harmonic series. The original series diverges when x = 1.

X=-1 $\sum_{n=1}^{\infty} \frac{x^n}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$ converges by alternating series test. The original series converges when x = -1.

The interval of convergence is $-1 \leq x < 1$

7. Determine the interval of convergence for $\sum_{k=0}^{\infty} \ 3^k \ x^k$

$u_k = \ 3^k \ x^k \qquad\qquad u_{k+1} \ = \ 3^{\,k+1} \ x^{\,k+1}$

Use the ratio test: series converges when $\lim_{\,k \to \infty} \left|\frac{u_{\,k+1}}{u_{\,k}}\right| \ < \ 1$

$$\lim_{\,k \to\infty} \left|\frac{3^{\,k+1} \ x^{\,k+1}}{3^{\,k} \ x^{\,k}}\right| \ = \ \lim_{\,k \to\infty} |\,3\,x\,| \ = \ |3x| \ < \ 1$$

$$|x| < \frac{1}{3} \qquad -\frac{1}{3} < \ x \ < \frac{1}{3}$$

Test endpoint x = 1/3

$$\sum_{k=0}^{\infty} \ 3^k \ x^k \ = \ \sum_{k=0}^{\infty} \ 3^k \left(\frac{1}{3}\right)^k \ = \ \sum_{k=0}^{\infty} \ 1 \ = \ 1+1+1+1+1 \ldots \ = \infty$$

Series diverges to infinity. The original series diverges when x = 1/3.

Test endpoint x = - 1/3

$$\sum_{k=0}^{\infty} \ 3^k \ x^k \ = \ \sum_{k=0}^{\infty} \ 3^k \left(\frac{-1}{3}\right)^k \ = \ \sum_{k=0}^{\infty} \ (-1)^k \ = \ 1-1+1-1+1-1 \ldots$$

Series diverges by oscillation. The original series diverges when x = -1/3.

The series converges on the open interval $-\frac{1}{3} < \ x \ < \frac{1}{3}$

8. Find the interval of convergence for the series $\sum_{k=0}^{\infty} \frac{(-1)^k\ x^k}{k!}$

$u_k \ = \ \frac{(-1)^k\ x^k}{k!} \qquad\qquad u_{k+1} \ = \ \frac{(-1)^{k+1}\ x^{k+1}}{(k+1)!}$

Determine convergence by ratio test:

Series converges when $\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| < 1$

$$\lim_{k \to \infty} \left| \frac{x^{k+1}}{(k+1)!}\ \frac{k!}{x^k} \right| = \lim_{k \to \infty} \left| \frac{x^{k+1}}{x^k}\ \frac{k!}{(k+1)!} \right| = \lim_{k \to \infty} \left| x\ \frac{1}{(k+1)} \right| =$$

$|x| \lim_{k \to \infty} \left| \frac{1}{(k+1)} \right| = 0$ for all x

The limit is always less than 1 so the series converges for all x.

The interval of convergence is all real numbers.

9. Find the interval of convergence for the series $\sum_{k=1}^{\infty} \frac{k!}{2^k} x^k$

$u_k = \frac{k!}{2^k} x^k \qquad u_{k+1} = \frac{(k+1)!}{2^{k+1}} x^{k+1}$

Use the ratio test to determine the interval of convergence:

Series converges when $\lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right| < 1$

$\lim_{k \to \infty} \left| \frac{(k+1)!}{2^{k+1}} x^{k+1} \frac{2^k}{k!} \frac{1}{x^k} \right| = \lim_{k \to \infty} \left| \frac{(k+1)!}{k!} \frac{2^k}{2^{k+1}} \frac{x^{k+1}}{x^k} \right| =$
$\lim_{k \to \infty} \left| \frac{k}{2} x \right|$

This limit has only two values. If x = 0 the limit is zero. If x is any other number, the limit is infinity. The only value for which the series will converge is x = 0.

The series only converges for x = 0 – no other value(s).

10. Determine the interval of convergence for the series $\sum_{k=1}^{\infty} \frac{5^k}{k^2} x^k$

$u_k = \frac{5^k}{k^2} x^k \qquad u_{k+1} = \frac{5^{k+1}}{(k+1)^2} x^{k+1}$

Use the ratio test to determine convergence:

The series converges when $\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| < 1$

$\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k \to \infty} \left| \frac{5^{k+1}}{(k+1)^2} x^{k+1} \frac{k^2}{5^k} \frac{1}{x^k} \right| =$
$\lim_{k \to \infty} \left| \frac{k^2}{(k+1)^2} \frac{5^{k+1}}{5^k} \frac{x^{k+1}}{x^k} \right|$

$= \lim_{k \to \infty} \left| \frac{k^2}{(k+1)^2} 5x \right| = |5x| \lim_{k \to \infty} \left| \frac{k^2}{(k+1)^2} \right| = |5x| < 1$

$-\frac{1}{5} < x < \frac{1}{5}$

Test endpoint x = 1/5

$\sum_{k=1}^{\infty} \frac{5^k}{k^2} x^k = \sum_{k=1}^{\infty} \frac{5^k}{k^2} \left(\frac{1}{5}\right)^k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by p test

The original series converges when x = 1/5.

Test endpoint x = - 1/5

$\sum_{k=1}^{\infty} \frac{5^k}{k^2} x^k = \sum_{k=1}^{\infty} \frac{5^k}{k^2} \left(-\frac{1}{5}\right)^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges by alternating series

Test. The original series converges when x = -1/5.

The series converges for $-\frac{1}{5} \le x \le \frac{1}{5}$

11. Determine the interval of convergence for the series $\sum_{k=3}^{\infty} \frac{x^k}{\ln k}$

$u_k = \frac{x^k}{\ln k}$ $\qquad\qquad u_{k+1} = \frac{x^{k+1}}{\ln(k+1)}$

Use the ratio test to determine convergence:

If $\lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| < 1$ then series converges

$\lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty} \left| \frac{x^{k+1}}{\ln(k+1)} \frac{\ln k}{x^k} \right| = \lim_{k\to\infty} \left| \frac{x^{k+1}}{x^k} \frac{\ln k}{\ln(k+1)} \right| =$

$\lim_{k\to\infty} \left| x \frac{\ln k}{\ln(k+1)} \right| = |x| \lim_{k\to\infty} \left| \frac{\ln k}{\ln(k+1)} \right| = |x| \lim_{k\to\infty} \left| \frac{k+1}{k} \right| =$
$|x| < 1$

We used LHospital’s rule to evaluate the limit.

-1 < x < 1

Test the endpoint x = -1

$\sum_{k=3}^{\infty} \frac{x^k}{\ln k} = \sum_{k=3}^{\infty} \frac{(-1)^k}{\ln k}$ series converges by alternating series test. The original series converges when x = -1.

Test endpoint x = 1

$\sum_{k=3}^{\infty} \frac{x^k}{\ln k} = \sum_{k=3}^{\infty} \frac{1}{\ln k}$ series diverges by comparison test or limit comparison test.
Compare it with $b_k = 1/k$ for either test. The original series diverges when x = 1.

The series converges for $-1 \le x < 1$

12. Determine the interval of convergence for the series $\sum_{k=1}^{\infty} \frac{x^k}{k\,(k+1)}$

$u_k = \frac{x^k}{k\,(k+1)}$ $\qquad u_{k+1} = \frac{x^{k+1}}{(k+1)\,(k+2)}$

Use the ratio test to determine convergence of series:

Series converges when $\lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right| < 1$

$$\lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \to \infty} \left| \frac{x^{k+1}}{(k+1)\,(k+2)} \frac{k(k+1)}{x^k} \right| =$$

$$\lim_{k \to \infty} \left| \frac{x^{k+1}}{x^k} \frac{k(k+1)}{(k+1)\,(k+2)} \right| =$$

$$\lim_{k \to \infty} \left| x \frac{k}{(k+2)} \right| = |x| \lim_{k \to \infty} \left| \frac{k}{(k+2)} \right| = |x| < 1$$

$$-1 < x < 1$$

Test the endpoint x = 1

$\sum_{k=1}^{\infty} \frac{x^k}{k\,(k+1)} = \sum_{k=1}^{\infty} \frac{1}{k\,(k+1)}$ converges by comparison test. Use $b_k = 1/k^2$. Can also use limit comparison test. The original series converges when x = 1.

Test the endpoint x = -1

$\sum_{k=1}^{\infty} \frac{x^k}{k\,(k+1)} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k\,(k+1)}$ converges by alternating series test. The original series converges when x = -1.

The series converges for $-1 \le x \le 1$

13. Determine the interval of convergence for the series $\sum_{k=0}^{\infty} \frac{(-2)^k\ x^{k+1}}{k+1}$

$u_k = \frac{(-2)^k\ x^{k+1}}{k+1} \qquad u_{k+1} = \frac{(-2)^{k+1}\ x^{k+2}}{k+2}$

Use the ratio test to determine convergence

Series converges when $\lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right| < 1$

$\lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \to \infty} \left| \frac{(2)^{k+1}\ x^{k+2}}{k+2}\ \frac{k+1}{2^k\ x^{k+1}} \right| =$

$\lim_{k \to \infty} \left| \frac{k+1}{k+2}\ \frac{(2)^{k+1}\ x^{k+2}}{2^k\ x^{k+1}} \right| =$

$\lim_{k \to \infty} \left| \frac{k+1}{k+2}\ 2x \right| = |2x| \lim_{k \to \infty} \left| \frac{k+1}{k+2} \right| = |2x| < 1$

$-\frac{1}{2} < x < \frac{1}{2}$

Test endpoints x = -1/2

$\sum_{k=0}^{\infty} \frac{(-2)^k\ x^{k+1}}{k+1} = \sum_{k=0}^{\infty} \frac{(-2)^k \left(-\frac{1}{2}\right)^{k+1}}{k+1} = \sum_{k=0}^{\infty} \frac{-\frac{1}{2}}{k+1}$ diverges by integral test or by comparison test or by limit comparison test. Use $b_k = 1/k$.

Original series diverges at x = -1/2.

Test endpoint x = ½

$\sum_{k=0}^{\infty} \frac{(-2)^k\ x^{k+1}}{k+1} = \sum_{k=0}^{\infty} \frac{(-2)^k \left(\frac{1}{2}\right)^{k+1}}{k+1} = \sum_{k=0}^{\infty} \frac{\frac{1}{2}\ (-1)^k}{k+1}$

Converges by alternating series test.

Original series converges at x = 1/2.

The original series converges for $-\frac{1}{2} < x \leq \frac{1}{2}$

14. Find the interval of convergence for $\sum_{k=1}^{\infty} (-1)^k \frac{x^k}{\sqrt{k}}$

$u_k = (-1)^k \frac{x^k}{\sqrt{k}}$ $\qquad u_{k+1} = (-1)^{k+1} \frac{x^{k+1}}{\sqrt{k+1}}$

Use the ratio test to determine convergence

Series converges when $\lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right| < 1$

$$\lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \to \infty} \left| \frac{x^{k+1}}{\sqrt{k+1}} \quad \frac{\sqrt{k}}{x^k} \right| = \lim_{k \to \infty} \left| \frac{x^{k+1}}{x^k} \frac{\sqrt{k}}{\sqrt{k+1}} \right| =$$

$$\lim_{k \to \infty} \left| x \sqrt{\frac{k}{k+1}} \right| = |x| \lim_{k \to \infty} \sqrt{\frac{k}{k+1}} = |x| \sqrt{\lim_{k \to \infty} \frac{k}{k+1}} = |x| < 1$$

$-1 < x < 1$

Test endpoint x = 1

$\sum_{k=1}^{\infty} (-1)^k \frac{x^k}{\sqrt{k}} = \sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k}}$ converges by alternating series test. Original series converges for x = 1.

Test endpoint x = -1

$\sum_{k=1}^{\infty} (-1)^k \frac{x^k}{\sqrt{k}} = \sum_{k=1}^{\infty} (-1)^k \frac{(-1)^k}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges by p test.

Original series diverges at x = -1

Original series converges for $-1 < x \leq 1$

15. Find the interval of convergence for $\sum_{k=0}^{\infty} \frac{(x-3)^k}{2^k}$

$u_k = \frac{(x-3)^k}{2^k}$ $\qquad u_{k+1} = \frac{(x-3)^{k+1}}{2^{k+1}}$

Use the ratio test to determine convergence

Series converges when $\lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right| < 1$

$$\lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \to \infty} \left| \frac{(x-3)^{k+1}}{2^{k+1}} \frac{2^k}{(x-3)^k} \right| = \lim_{k \to \infty} \left| \frac{(x-3)}{2} \right| < 1$$

$|(x-3)| < 2$

$-2 < (x-3) < 2$

$1 < x < 5$

Test endpoints x = 1

$\sum_{k=0}^{\infty} \frac{(x-3)^k}{2^k} = \sum_{k=0}^{\infty} \frac{(1-3)^k}{2^k} = \sum_{k=0}^{\infty} \frac{(-2)^k}{2^k} = \sum_{k=0}^{\infty} (-1)^k$ diverges by oscillation

Original series diverges at x = 1

Test endpoint x = 5

$\sum_{k=0}^{\infty} \frac{(x-3)^k}{2^k} = \sum_{k=0}^{\infty} \frac{(5-3)^k}{2^k} = \sum_{k=0}^{\infty} \frac{(2)^k}{2^k} = \sum_{k=0}^{\infty} 1 = 1+1+1+1+\cdots$

Diverges to infinity

Original series diverges at x = 5

Series converges for $1 < x < 5$

16. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k} (x+1)^k$

$u_k = \frac{(-1)^k}{k} (x+1)^k \qquad u_{k+1} = \frac{(-1)^{k+1}}{k+1} (x+1)^k$

$\lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \to \infty} \left| \frac{(x+1)^{k+1}}{k+1} \frac{k}{(x+1)^k} \right| = \lim_{k \to \infty} \left| (x+1) \frac{k}{k+1} \right|$

$= |x+1| \lim_{k \to \infty} \left| \frac{k}{k+1} \right| = |x+1| < 1$

$|x+1| < 1$

$-1 < x+1 < 1$

$-2 < x < 0$

Test endpoint x = 0

$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} (x+1)^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} (0+1)^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges by alternating series test.
Original series converges at x = 0

Test endpoint at x = -2

$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} (x+1)^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} (-1)^k = \sum_{k=1}^{\infty} \frac{1}{k}$ harmonic series, diverges by integral test.
Original series diverges at x = -2

The interval of convergence is
$-2 < x \leq 0$