INFINITE SERIES RATIO TEST

Statement of ratio test:

We are given the series $\sum_{k=1}^{\infty} u_k$ where $u_k > 0$. We form the limit $\lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right|$. If this limit is less than 1, then the series converges. If this limit is greater than 1, the series diverges. If this limit equals 1, then the test has failed – we do not know if the series converges or diverges. Another test must be used.

ANTON PAGE 604

1. $\sum_{k=1}^{\infty} \frac{3^k}{k!}$

   $u_k = \frac{3^k}{k!} \qquad u_{k+1} = \frac{3^{k+1}}{(k+1)!}$

   $\lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \to \infty} \left| \frac{3^{k+1}}{(k+1)!} \frac{k!}{3^k} \right| = \lim_{k \to \infty} \left| \frac{3}{(k+1)} \right| = 0$

   Since $\lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right| < 1$ then the series converges

2. $\sum_{k=1}^{\infty} \frac{4^k}{k^2}$

   $u_k = \frac{4^k}{k^2} \qquad u_{k+1} = \frac{4^{k+1}}{(k+1)^2}$

   $\lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \to \infty} \left| \frac{4^{k+1}}{(k+1)^2} \frac{k^2}{4^k} \right| = \lim_{k \to \infty} \left| \frac{4\,k^2}{(k+1)^2} \right| = 4$

   Since $\lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right| > 1$ the series diverges

3. $\sum_{k=1}^{\infty} \frac{1}{5\,k}$

   $u_k = \frac{1}{5k} \qquad u_{k+1} = \frac{1}{5(k+1)}$

   $\lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \to \infty} \left| \frac{1}{5(k+1)} \frac{5k}{1} \right| = \lim_{k \to \infty} \left| \frac{k}{k+1} \right| = 1$

   Since $\lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right| = 1$ the test has failed. The test does not tell us if the series converges or diverges. We must use another test. The integral test will tell us that it diverges.

4. $\sum_{k=1}^{\infty} \frac{k}{2^k}$

$u_k = \frac{k}{2^k} \qquad u_{k+1} = \frac{(k+1)}{2^{k+1}}$

$$\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k \to \infty} \left| \frac{k+1}{2^{k+1}} \; \frac{2^k}{\text{k}} \right| = \lim_{k \to \infty} \left|\frac{k+1}{2k}\right| = \frac{1}{2}$$

Since $\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| < 1$ the series converges.

5. $\sum_{k=1}^{\infty} \frac{k!}{k^3}$

$u_k = \frac{k!}{k^3} \qquad u_{k+1} = \frac{(k+1)!}{(k+1)^3}$

$$\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k \to \infty} \left| \frac{(k+1)!}{(k+1)^3} \; \frac{\text{k}^3}{\text{k}\,!} \right| = \lim_{k \to \infty} \left|\frac{k \cdot k^3}{(k+1)^3}\right| = \lim_{k \to \infty} \left|\frac{k^4}{(k+1)^3}\right| = \infty$$

Since the limit $\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| > 1$ the series diverges.

6. $\sum_{k=1}^{\infty} \frac{k}{k^2 + 1}$

$u_k = \frac{k}{k^2 + 1} \qquad u_{k+1} = \frac{k+1}{(k+1)^2+1}$

$$\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k \to \infty} \left| \frac{k+1}{(k+1)^2+1} \; \frac{\text{k}^2+1}{\text{k}} \right| = \lim_{k \to \infty} \left| \frac{k+1}{k} \; \frac{\text{k}^2+1}{(k+1)^2+1} \right|$$

$$\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k \to \infty} \left| \frac{k+1}{k} \right| \lim_{k \to \infty} \left| \frac{\text{k}^2+1}{(k+1)^2+1} \right| = 1 \cdot 1 = 1$$

Since the limit $\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| = 1$ the test has failed. The test did not tell us if the series converged. We will need to use another test. The integral test will show that the series diverges.

7. $\sum_{k=1}^{\infty} \frac{k}{5^k}$

$u_k = \frac{k}{5^k} \qquad u_{k+1} = \frac{k+1}{5^{k+1}}$

$$\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k \to \infty} \left| \frac{k+1}{5^{k+1}} \; \frac{5^k}{\text{k}} \right|$$

$$= \lim_{k \to \infty} \left| \frac{5^k}{5^{k+1}} \; \frac{k+1}{\text{k}} \right| = \lim_{k \to \infty} \left| \frac{1}{5} \; \frac{k+1}{\text{k}} \right| = \frac{1}{5}$$

The limit equals 1/5.

Since the limit is less than one, the series converges.

8. $\sum_{k=1}^{\infty} \frac{1}{k^2}$

$u_k = \frac{1}{k^2}$ $\qquad u_{k+1} = \frac{1}{(k+1)^2}$

$$\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k \to \infty} \left| \frac{k^2}{(k+1)^2} \right| = 1$$

Since the limit equals 1, the test has failed. We do not know if the series converges. If we use the p test, this will tell us that the series does converge.

9. $\sum_{k=1}^{\infty} \frac{7^k}{k!}$

$u_k = \frac{7^k}{k!}$ $\qquad u_{k+1} = \frac{7^{k+1}}{(k+1)!}$

$$\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k \to \infty} \left| \frac{7^{k+1}}{(k+1)!} \; \frac{k!}{7^k} \right| = \lim_{k \to \infty} \left| \frac{k!}{(k+1)!} \; \frac{7^{k+1}}{7^k} \right| = \lim_{k \to \infty} \left|\frac{1}{k+1} \cdot 7 \right| = 0$$

The limit is less than one so the series converges.

10. $\sum_{k=1}^{\infty} \frac{k^2}{5^k}$

$u_k = \frac{k^2}{5^k}$ $\qquad u_{k+1} = \frac{(k+1)^2}{5^{k+1}}$

$$\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k \to \infty} \left| \frac{(k+1)^2}{5^{k+1}} \; \frac{5^k}{k^2} \right| = \lim_{k \to \infty} \left| \frac{5^k}{5^{k+1}} \; \frac{(k+1)^2}{k^2} \right|$$

$$\lim_{k \to \infty} \left| \frac{1}{5} \; \frac{(k+1)^2}{k^2} \right| = \frac{1}{5} \lim_{k \to \infty} \left| \frac{(k+1)^2}{k^2} \right| = \frac{1}{5}$$

Since the limit is less than one, the series converges.

11. $\sum_{k=1}^{\infty} \; k^{50}\, e^{-k}$

$u_k = \; k^{50}\, e^{-k} \qquad\qquad u_{k+1} \; = \; (k+1)^{50} \;\; e^{-(k+1)}$

$$\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k \to\infty} \left|\frac{(k+1)^{50} \;\; e^{-(k+1)}}{k^{50}\, e^{-k}}\right| = \lim_{k \to\infty} \left| \frac{(k+1)^{50}}{k^{50}} \;\; \frac{e^{-(k+1)}}{e^{-k}} \right|$$

$$\lim_{k \to\infty} \left| \frac{(k+1)^{50}}{k^{50}} \;\; \frac{1}{e^k} \right| = \lim_{k \to\infty} \left| \frac{(k+1)^{50}}{k^{50}} \right| \;\; \lim_{k \to\infty} \left| \frac{1}{e^k} \right|$$

$$\lim_{k \to\infty} \left| \frac{(k+1)^{50}}{k^{50}} \;\; \frac{1}{e^k} \right| = 1 \cdot \; 0 \; = \; 0$$

Since the limit is less than 1, the series converges.

12. $\sum_{k=1}^{\infty} \; k \left(\frac{2}{3}\right)^k$

$u_k \; = \; k \left(\frac{2}{3}\right)^k \qquad\qquad u_{k+1} \; = \; (k+1) \left(\frac{2}{3}\right)^{k+1}$

$$\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k \to\infty} \left| (k+1) \left(\frac{2}{3}\right)^{k+1} \frac{1}{k} \left(\frac{3}{2}\right)^k \right| = \lim_{k \to\infty} \left| \frac{(k+1)}{k} \;\; \frac{2}{3} \right|$$

$$\frac{2}{3} \; \lim_{k \to\infty} \left| \frac{(k+1)}{k} \right| \; = \frac{2}{3} \cdot \; 1 \; = \; \frac{2}{3}$$

Since the limit is less than one, the series converges.

13. $\sum_{k=1}^{\infty} \; \frac{1}{k \; \ln k}$

$u_k \; = \; \frac{1}{k \; \ln k} \qquad\qquad u_{k+1} \; = \; \frac{1}{(k+1) \; \ln(k+1)}$

$$\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k \to\infty} \left| \frac{1}{(k+1) \; \ln(k+1)} \cdot \frac{k \; \ln k}{1} \right| = \lim_{k \to\infty} \left| \frac{k}{(k+1)} \cdot \frac{\ln k}{\ln(k+1)} \right|$$

$$\lim_{k \to\infty} \left| \frac{k}{(k+1)} \right| \; \lim_{k \to\infty} \left| \frac{\ln k}{\ln(k+1)} \right| = 1 \cdot \; \lim_{k \to\infty} \left| \frac{\frac{1}{k}}{\frac{1}{(k+1)}} \right|$$

$$1 \cdot \; \lim_{k \to\infty} \left| \frac{k+1}{k} \right| \; = \; 1 \cdot 1 \; = 1$$

Since the limit equals one, the test fails. The test does not tell us if the series converges. The integral test will work and it will show that the series diverges.

14. $\sum_{k=1}^{\infty} \left(\frac{4}{7k-1}\right)^k$

$$u_k = \left(\frac{4}{7k-1}\right)^k \qquad u_{k+1} = \left(\frac{4}{7\,(k+1)-1}\right)^{k+1}$$

$$\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k \to \infty} \left| \left(\frac{4}{7\,(k+1)-1}\right)^{k+1} \cdot \left(\frac{7k-1}{4}\right)^k \right|$$

$$\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k \to \infty} \left| 4 \left(\frac{7k-1}{7\,(k+1) - 1}\right)^k \right| = 4 \lim_{k \to \infty} \left| \left(\frac{7k-1}{7\,(k+1) - 1}\right)^k \right|$$

The expression inside the absolute values is an indeterminate form 1^{∞}. We will need LHospital's rule.

$$y = \left(\frac{7k-1}{7\,(k+1) - 1}\right)^k \qquad \ln y = k \ln\left(\frac{7k-1}{7\,(k+1) - 1}\right)$$

$$\lim_{k \to \infty} \ln y = \lim_{k \to \infty} k \ln\left(\frac{7k-1}{7\,(k+1) - 1}\right) = \infty \cdot \ln 1 = \infty \cdot 0$$

$$\lim_{k \to \infty} \ln y = \lim_{k \to \infty} \frac{\ln\,(7k-1) - \ln(7k+6)}{k^{-1}}$$

$$\lim_{k \to \infty} \ln y = \lim_{k \to \infty} \frac{\frac{7}{7k-1} - \frac{7}{7k+6}}{-\frac{1}{k^2}} = \lim_{k \to \infty} -k^2 \left(\frac{7}{7k-1} - \frac{7}{7k+6}\right)$$

$$\lim_{k \to \infty} \ln y = \lim_{k \to \infty} -k^2 \left(\frac{49\,k+42 - 49k + 7}{49k^2+35k-6}\right) = \lim_{k \to \infty} -k^2 \left(\frac{49}{49k^2+35k-6}\right)$$

$$\lim_{k \to \infty} \ln y = \lim_{k \to \infty} \left(\frac{-\,49\;k^2}{49k^2+35k-6}\right) = -\frac{49}{49} = -1$$

So $y \to e^{-1}$ which is less than 1. Since the limit $\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| = e^{-1}$ and is less than one, the series converges.