INFINITE SERIES LIMIT COMPARISON TEST

LIMIT COMPARISON TEST

Let $\sum_{n=a}^{\infty} a_n$ be a series under investigation and let $\sum_{n=1}^{\infty} b_n$ be a comparison series whose behavior is know. We are told that $a_n > 0$ & $b_n > 0$.

If the limit $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a real number c > 0 then the two series either both converge or both diverge

LIMIT COMPARISON TEST EXCEPTIONAL CASE C = 0

Let $\sum_{n=a}^{\infty} a_n$ be a series under investigation and let $\sum_{n=1}^{\infty} b_n$ be a comparison series THAT CONVERGES. We are told that $a_n > 0$ & $b_n > 0$.

If the limit $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ then $\sum_{n=a}^{\infty} a_n$ converges.

LIMIT COMPARISON TEST EXCEPTIONAL CASE C = 0

Let $\sum_{n=a}^{\infty} a_n$ be a series under investigation and let $\sum_{n=1}^{\infty} b_n$ be a comparison series THAT DIVERGES. We are told that $a_n > 0$ & $b_n > 0$.

If the limit $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$ then $\sum_{n=a}^{\infty} a_n$ diverges.

LIMIT COMPARISON TEST EXPCEPTIONAL CASE C = INFINITY

Let $\sum_{n=a}^{\infty} a_n$ be a series under investigation and let $\sum_{n=1}^{\infty} b_n$ be a comparison series THAT DIVERGES. We are told that $a_n > 0$ & $b_n > 0$.

If the limit $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ then $\sum_{n=a}^{\infty} a_n$ diverges.

1. $\sum_{k=1}^{\infty} \frac{4k^2 - 2k + 6}{8k^7 + k - 9}$ ANTON PAGE 612

We have $a_k = \frac{4k^2 - 2k + 6}{8k^7 + k - 9}$. We let $b_k = \frac{1}{k^5}$.

We know that $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^5}$ must converge by the p test

Evaluate $\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{k^5 (4k^2 - 2k + 6)}{(8k^7 + k - 9)} = \lim_{k \to \infty} \frac{(4k^7 - 2k^6 + 6k^5)}{(8k^7 + k - 9)} = \frac{4}{8} = \frac{1}{2}$

Since $\lim_{k \to \infty} \frac{a_k}{b_k}$ equals a finite positive number the series $\sum_{k=1}^{\infty} a_k$ must match the behavior of the series $\sum_{k=1}^{\infty} b_k$.

Since $\sum_{k=1}^{\infty} b_k$ converges, the series $\sum_{k=1}^{\infty} a_k$ converges.

2. $\sum_{k=1}^{\infty} \frac{1}{9k+6}$ ANTON PAGE 612

We have $a_k = \frac{1}{9k+6}$. We let $b_k = \frac{1}{k}$.

We know that $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ must diverge by the p test

Evaluate $\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{k}{9k+6} = \frac{1}{9}$

Since $\lim_{k \to \infty} \frac{a_k}{b_k}$ equals a finite positive number the series $\sum_{k=1}^{\infty} a_k$ must match the behavior of the series $\sum_{k=1}^{\infty} b_k$.

Since $\sum_{k=1}^{\infty} b_k$ diverges, the series $\sum_{k=1}^{\infty} a_k$ diverges.

3. $\sum_{k=1}^{\infty} \frac{5}{3^k + 1}$ ANTON PAGE 612

We have $a_k = \frac{5}{3^k + 1}$. We let $b_k = \frac{1}{3^k}$.

We know that $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{3^k}$ must converge because it is a geometric series with r =1/3 .

Evaluate $\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{5 \cdot 3^k}{3^k + 1} = 5$

Since $\lim_{k \to \infty} \frac{a_k}{b_k}$ equals a finite positive number the series $\sum_{k=1}^{\infty} a_k$ must match the behavior of the series $\sum_{k=1}^{\infty} b_k$.

Since $\sum_{k=1}^{\infty} b_k$ converges, the series $\sum_{k=1}^{\infty} a_k$ converges.

4. $\sum_{k=1}^{\infty} \frac{k\,(k+3)}{(k+1)\,(k+2)\,(k+5)}$ ANTON PAGE 612

We have $a_k = \frac{k\,(k+3)}{(k+1)\,(k+2)\,(k+5)}$. We let $b_k = \frac{1}{k}$.

We know that $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ must diverge because it is a harmonic series.

Evaluate $\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{k \cdot k\,(k+3)}{(k+1)\,(k+2)\,(k+5)} = \lim_{k \to \infty} \frac{k^3 + 3\,k^2}{k^3+8\,k^2+17k+10} =$ 1

Since $\lim_{k \to \infty} \frac{a_k}{b_k}$ equals a finite positive number the series $\sum_{k=1}^{\infty} a_k$ must match the behavior of the series $\sum_{k=1}^{\infty} b_k$.

Since $\sum_{k=1}^{\infty} b_k$ diverges, the series $\sum_{k=1}^{\infty} a_k$ diverges.

5. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{8k^2-3k}}$ ANTON PAGE 612

We have $a_k = \frac{1}{\sqrt[3]{8k^2-3k}}$. We let $b_k = \frac{1}{\mathrm{k}^{2/3}}$.

We know that $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^{2/3}}$ must diverge by the p test.

Evaluate $\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{k^{2/3}}{\sqrt[3]{8k^2-3k}} = \frac{1}{2}$

Since $\lim_{k \to \infty} \frac{a_k}{b_k}$ equals a finite positive number the series $\sum_{k=1}^{\infty} a_k$ must match the behavior of the series $\sum_{k=1}^{\infty} b_k$.

Since $\sum_{k=1}^{\infty} b_k$ diverges, the series $\sum_{k=1}^{\infty} a_k$ diverges.

6. $\sum_{k=1}^{\infty} \frac{1}{(2k+3)^{17}}$ ANTON PAGE 612

We have $a_k = \frac{1}{(2k+3)^{17}}$. We let $b_k = \frac{1}{k^{17}}$.

We know that $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^{17}}$ must converge by the p test.

Evaluate $\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{k^{17}}{(2k+3)^{17}} = \frac{1}{2^{17}}$

Since $\lim_{k \to \infty} \frac{a_k}{b_k}$ equals a finite positive number the series $\sum_{k=1}^{\infty} a_k$ must match the behavior of the series $\sum_{k=1}^{\infty} b_k$.

Since $\sum_{k=1}^{\infty} b_k$ converges, the series $\sum_{k=1}^{\infty} a_k$ converges.

7. $\sum_{k=1}^{\infty} \frac{1}{k^3 + 2k + 1}$ ANTON PAGE 612

We have $a_k = \frac{1}{k^3 + 2k + 1}$. We let $b_k = \frac{1}{k^3}$.

We know that $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^3}$ must converge by the p test.

Evaluate $\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{k^3}{k^3 + 2k + 1} = 1$

Since $\lim_{k \to \infty} \frac{a_k}{b_k}$ equals a finite positive number the series $\sum_{k=1}^{\infty} a_k$ must match the behavior of the series $\sum_{k=1}^{\infty} b_k$.

Since $\sum_{k=1}^{\infty} b_k$ converges, the series $\sum_{k=1}^{\infty} a_k$ converges.

8. $\sum_{k=1}^{\infty} \frac{1}{(3+k)^{2/5}}$ ANTON PAGE 612

We have $a_k = \frac{1}{(3+k)^{2/5}}$. We let $b_k = \frac{1}{k^{2/5}}$.

We know that $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^{2/5}}$ must diverge by the p test.

Evaluate $\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{k^{2/5}}{(3+k)^{2/5}} = 1$

Since $\lim_{k \to \infty} \frac{a_k}{b_k}$ equals a finite positive number the series $\sum_{k=1}^{\infty} a_k$ must match the behavior of the series $\sum_{k=1}^{\infty} b_k$.

Since $\sum_{k=1}^{\infty} b_k$ diverges, the series $\sum_{k=1}^{\infty} a_k$ diverges.

9. $\sum_{k=1}^{\infty} \frac{1}{9k-2}$ ANTON PAGE 612

We have $a_k = \frac{1}{9k-2}$. We let $b_k = \frac{1}{k}$.

We know that $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ must diverge by the p test.

Evaluate $\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{k}{9k-2} = \frac{1}{9}$

Since $\lim_{k \to \infty} \frac{a_k}{b_k}$ equals a finite positive number the series $\sum_{k=1}^{\infty} a_k$ must match the behavior of the series $\sum_{k=1}^{\infty} b_k$.

Since $\sum_{k=1}^{\infty} b_k$ diverges, the series $\sum_{k=1}^{\infty} a_k$ diverges.

10. $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ ANTON PAGE 612

We have $a_k = \frac{\ln k}{\mathrm{k}}$. We let $b_k = \frac{1}{k}$.

We know that $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ must diverge by the p test.

Evaluate $\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{k \ln k}{k} = \lim_{k \to \infty} \ln k = \infty$

This is the exceptional case. Since $\lim_{k \to \infty} \frac{a_k}{b_k}$ goes to infinity, and $\sum_{k=1}^{\infty} b_k$ diverges, we must have $\sum_{k=1}^{\infty} a_k$ also diverges.

11. $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^3+1}$ ANTON PAGE 612

We have $a_k = \frac{\sqrt{k}}{k^3+1}$. We let $b_k = \frac{1}{k^{5/2}}$.

We know that $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$ must converge by the p test.

Evaluate $\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k^{5/2}\sqrt{k}}{k^3+1} = 1$

Since $\lim_{k\to\infty} \frac{a_k}{b_k}$ equals a finite positive number, then by the limit comparison test, the series $\sum_{k=1}^{\infty} a_k$ must match the behavior of the series $\sum_{k=1}^{\infty} b_k$.

Since $\sum_{k=1}^{\infty} b_k$ converges, the series $\sum_{k=1}^{\infty} a_k$ converges.

12. $\sum_{k=1}^{\infty} \frac{\ln k}{k\sqrt{k}}$ ANTON PAGE 612

We have $a_k = \frac{\ln k}{k\sqrt{k}}$. We let $b_k = \frac{1}{k^{5/4}}$.

We know that $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^{5/4}}$ must converge by the p test.

Evaluate $\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k^{5/4}\ln k}{k\sqrt{k}} = \lim_{k\to\infty} \frac{k^{5/4}\ln k}{k^{3/2}} = \lim_{k\to\infty} \frac{\ln k}{k^{1/4}}$

Use LHospital's rule:

$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\ln k}{k^{1/4}} = \lim_{k\to\infty} \frac{1/k}{1/4\ k^{-3/4}} = \lim_{k\to\infty} \frac{4\ k^{3/4}}{k}$

$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{4}{k^{1/4}} = 0$

This is the exceptional case c= 0. If $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and if $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ also converges.

The series $\sum_{k=1}^{\infty} \frac{\ln k}{k\sqrt{k}}$ converges.

13. $\sum_{k=1}^{\infty} \sin\frac{\pi}{k}$ ANTON PAGE 612

We have $a_k = \sin\frac{\pi}{k}$. We let $b_k = \frac{\pi}{k}$.

We know that $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{\pi}{k}$ must diverge by the p test.

Evaluate $\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{\sin\left(\frac{\pi}{k}\right)}{\frac{\pi}{k}} = \frac{0}{0}$

Use LHospital’s rule:

$\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{\sin\left(\frac{\pi}{k}\right)}{\frac{\pi}{k}} = \lim_{k \to \infty} \frac{-\pi/k^2 \ (\cos\frac{\pi}{k})}{-\pi/k^2} = \lim_{k \to \infty} \cos\frac{\pi}{k} = \cos 0 =$

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Since $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{\pi}{k}$ diverges, the original series must also diverge by the limit comparison test.

$\sum_{k=1}^{\infty} \sin\frac{\pi}{k}$ diverges.