U SUBSTITUTIONS Smith & Minton pg 350

1. $\int x^3 \sqrt{x^4+3}\ dx$

2. $\int \frac{\sin x}{\sqrt{\cos x}}\ dx$

3. $\int \sqrt{1+10x}\, dx$

4. $\int (\sin x)^3 \cos x\ dx$

5. $\int t^2 \cos t^3\ dt$

6. $\int \sin t\ (\cos t + 3)^{3/4}\, dt$

7. $\int x\ e^{x^2+1}\ dx$

8. $\int e^x \sqrt{e^x+4}\ dx$

9. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx$

10. $\int \frac{\cos\left(\frac{1}{x}\right)}{x^2}\ dx$

11. $\int \frac{\sqrt{\ln x}}{x}\, dx$

12. $\int \sec^2 x \;\sqrt{\tan x}\, dx$

13. $\int \frac{1}{\sqrt{u}\,(\sqrt{u}+1)}\, du$

14. $\int \frac{v}{v^2+4}\, dv$

15. $\int \frac{4}{x\,(\ln x+1)^2}\, dx$

16. $\int \tan 2x\, dx$

17. $\int \frac{(arc\sin x)^3}{\sqrt{1-x^2}}\, dx$

18. $\int x^2 \sec^2 x^3\, dx$

19. $\int \frac{x}{\sqrt{1-x^4}}\, dx$

20. $\int \frac{x^3}{\sqrt{1-x^4}}\, dx$

21. $\int \frac{x^2}{1+x^6}\, dx$

22. $\int \frac{x^5}{1+x^6}\ dx$

23. $\int \frac{(1+x)}{1+x^2}\ dx$

24. $\int \frac{(1+x)}{1-x^2}\ dx$

25. $\int \frac{3\sqrt{x}}{1+x^3}\ dx$

26. $\int \frac{x\sqrt{x}}{1+x^5}\ dx$

27. $\int \frac{t^2}{\sqrt[3]{t+3}}\ dt$

28. $\int \frac{1}{\sqrt{1+\sqrt{x}}}\ dx$

29. $\int \frac{1}{x\sqrt{x^4-1}}\ dx$

SOLUTIONS

1. $\int x^3 \sqrt{x^4+3}\ dx$

$\int x^3 (x^4+3)^{\frac{1}{2}}\ dx$

$let\ u = x^4 + 3$

$\frac{du}{dx} = 4x^3$

$du = 4x^3\ dx$

$\frac{du}{4} = x^3\ dx$

$\int x^3 (x^4+3)^{\frac{1}{2}}\ dx = \int (x^4+3)^{\frac{1}{2}}\ x^3\ dx = \int u^{1/2}\ \frac{1}{4}\ du$

$= \frac{1}{4} \int u^{1/2}\ du = \frac{1}{4}\frac{2}{3}\ u^{3/2} + C$

$= \frac{1}{6}\ u^{3/2} + C$

$= \frac{1}{6}\ (x^4+3)^{3/2} + C$

2. $\int \frac{\sin x}{\sqrt{\cos x}}\ dx$

let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$du = -\sin x\ \ dx$

$-du = \sin x\ \ dx$

$= \int \frac{-1}{\sqrt{u}}\ du = -\int \frac{1}{u^{1/2}}\ du = -2\,u^{1/2} + C = -2\sqrt{\cos x} + C$

3. $\int \sqrt{1+10x}\, dx$

let $u = 1 + 10x$

$\frac{du}{dx} = 10$

$du = 10\, dx$

$\frac{du}{10} = dx$

$= \int \sqrt{u}\, \frac{du}{10} = \frac{1}{10} \int \sqrt{u}\, du$

$= \frac{1}{10} \frac{2}{3} u^{3/2} + C$

$= \frac{1}{15} (1 + 10x)^{3/2} + C$

4. $\int (\sin x)^3 \cos x\, dx$

let $u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\, dx$

$\int (\sin x)^3 \cos x\, dx = \int u^3\, du$

$= \frac{u^4}{4} + C$

$= \frac{\sin^4 x}{4} + C$

5. $\int t^2 \cos t^3\, dt$

let $u = t^3$

$\frac{du}{dt} = 3\, t^2$

$du = 3\, t^2\, dt$

$\frac{du}{3} = t^2\, dt$

$\int t^2 \cos t^3\, dt = \int \cos u\, \frac{du}{3}$

$= \frac{1}{3} \int \cos u\, du = \frac{1}{3} \sin u + C = \frac{1}{3} \sin t^3 + C$

6. $\int \sin t \ \ (\cos t + 3)^{3/4}\, dt$

let $u = \cos t \ + \ 3$

$\frac{du}{dt} = -\sin t$

$du = -\sin t \ dt$

$-du = \sin t \ \ dt$

$$= \int u^{3/4} \ (-du) \ = -\int u^{3/4}\, du \ = -\frac{4}{7}\ u^{7/4} \ + \ C \ = -\frac{4}{7}\ (\cos t \ + \ 3\,)^{7/4} \ + \ C$$

7. $\int x \ e^{x^2+1} \ dx$

let $u = x^2 + 1$

$\frac{du}{dx} = 2x$

$du = 2x\, dx$

$\frac{1}{2}\, du = x\, dx$

$$= \int e^u \frac{1}{2}\, du \ = \ \frac{1}{2} \int e^u\, du \ = \ \frac{1}{2}\ e^u + \ C \ = \ \frac{1}{2}\ e^{x^2+1} \ + \ C$$

8. $\int e^x \sqrt{e^x + 4} \ \ dx$

let $u = e^x + \ 4$

$\frac{du}{dx} = e^x$

$du = e^x\, dx$

$$\int \sqrt{u}\, du \ = \ \int u^{1/2}\, du \ = \ \frac{2}{3}\ u^{3/2} \ + \ C \ = \ \frac{2}{3}\ (e^x + 4)^{3/2} \ + \ C$$

9. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx$

let $u = \sqrt{x}$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$du = \frac{1}{2\sqrt{x}}\ dx$$

$$2\ du = \frac{1}{\sqrt{x}}\ dx$$

$$\int e^u\,(2\,du) = 2\int e^u\,du = 2\,e^u + C = 2\,e^{\sqrt{x}} + C$$

10. $\int \frac{\cos\left(\frac{1}{x}\right)}{x^2}\ dx$

let $u = \frac{1}{x}$

$$\frac{du}{dx} = -\frac{1}{x^2}$$

$$du = -\frac{1}{x^2}\ dx$$

$$-\,du = \frac{1}{x^2}\ dx$$

$$= -\int \cos u\ du = -\sin u + C = -\sin\frac{1}{x} + C$$

11. $\int \frac{\sqrt{\ln x}}{x}\ dx$

let $u = \ln x$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{dx}{x}$$

$$= \int \sqrt{u}\,du = \frac{2}{3}\ u^{3/2} + C = \frac{2}{3}\ (\ln x)^{3/2} + C$$

12. $\int \sec^2 x\ \sqrt{\tan x}\ dx$

let $u = \tan x$

$$\frac{du}{dx} = \sec^2 x$$

$$du = \sec^2 x\ \ dx$$

$$\int \sqrt{u}\,du = \frac{2}{3}\ u^{3/2} + C = \frac{2}{3}\ (\tan x)^{3/2} + C$$

13. $\int \frac{1}{\sqrt{u}\,(\sqrt{u}+1)}\ du$

let $x = \sqrt{u} + 1$

$\frac{dx}{du} = \frac{1}{2\sqrt{u}}$

$dx = \frac{1}{2\sqrt{u}}\ du$

$2\ dx = \frac{1}{\sqrt{u}}\ du$

$= 2 \int \frac{1}{x}\ dx = 2\ln x + C = 2\ln\left(\sqrt{u} + 1\right) + C$

14. $\int \frac{v}{v^2+4}\ dv$

let $u = v^2 + 4$

$\frac{du}{dv} = 2v$

$du = 2\ v\ dv$

$\frac{1}{2}\ du = v\ dv$

$= \frac{1}{2}\int \frac{1}{u}\ du = \frac{1}{2}\ln u + C = \frac{1}{2}\ln(v^2+4) + C$

15. $\int \frac{4}{x\,(\ln x + 1)^2}\ dx$

let $u = \ln x + 1$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}\ dx$

$= \int \frac{4}{u^2}\ du = -4\ \frac{1}{u} + C = -\frac{4}{\ln x + 1} + C$

16. $\int \tan 2x\ dx$

let $u = 2x$

$\frac{du}{dx} = 2$

$du = 2\ dx$

$\frac{1}{2}\ du = dx$

$= \frac{1}{2}\int \tan u\ du = -\frac{1}{2}\ln|\cos u| + C = -\frac{1}{2}\ln|\cos 2x| + C$

17. $\int \frac{(arc\sin x)^3}{\sqrt{1-x^2}} \; dx$

let $u = arc\sin x$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$

$du = \frac{1}{\sqrt{1-x^2}} \, dx$

$= \int u^3 \, du \quad = \quad \frac{u^4}{4} + C \quad = \quad \frac{(arc\sin x)^4}{4} + C$

18. $\int x^2 \sec^2 x^3 \; dx$

let $u = x^3$

$\frac{du}{dx} = 3x^2$

$du = 3\,x^2 \, dx$

$\frac{1}{3} \, du = x^2 \, dx$

$\frac{1}{3} \int \sec^2 u \; du = \frac{1}{3} \tan u + C = \frac{1}{3} \tan x^3 + C$

19. $\int \frac{x}{\sqrt{1-x^4}} \; dx$

let $u = x^2$

$\frac{du}{dx} = 2x$

$du = 2\,x\,dx$

$\frac{1}{2} \, du = x \; dx$

$= \frac{1}{2} \int \frac{1}{\sqrt{1-u^2}} \, du = \frac{1}{2} \, arc\sin u + C = \frac{1}{2} \, arc\sin x^2 + C$

20. $\int \frac{x^3}{\sqrt{1-x^4}} \; dx$

let $u = 1 - x^4$

$\frac{du}{dx} = -4\,x^3$

$du = -4\,x^3 \, dx$

$-\frac{1}{4} \, du = x^3 \, dx$

$-\frac{1}{4} \int \frac{du}{\sqrt{u}} \quad = \quad -\frac{1}{4} \; 2\sqrt{u} + C \quad = \quad -\frac{1}{2} \sqrt{1-x^4} + C$

21. $\int \frac{x^2}{1+x^6}\ dx$

let $u = x^3$

$$\frac{du}{dx} = 3x^2$$

$$du = 3\,x^2\,dx$$

$$\frac{1}{3}\,du = x^2\,dx$$

$$= \frac{1}{3}\int \frac{1}{1+u^2}\ du \ = \frac{1}{3}\,arc\tan u \ + \ C \ = \frac{1}{3}\,arc\tan x^3 \ + \ C$$

22. $\int \frac{x^5}{1+x^6}\ dx$

let $u = 1 + x^6$

$$\frac{du}{dx} = 6\,x^5$$

$$du = 6\,x^5\,dx$$

$$\frac{1}{6}\ du \ = x^5\,dx$$

$$= \frac{1}{6}\int \frac{1}{u}\ du \ = \frac{1}{6}\ln u \ + \ C \ = \frac{1}{6}\ln(1+x^6) \ + C$$

23. $\int \frac{(1+x)}{1+x^2}\ dx$ the way to do this is make two integrals:

$$\int \frac{(1+x)}{1+x^2}\ dx \ = \int \frac{1}{1+x^2}\,dx \ + \ \int \frac{x}{1+x^2}\,dx$$

$$\int \frac{(1+x)}{1+x^2}\ dx \ = arc\tan x \ + \frac{1}{2}\ln(1+x^2) \ + \ C$$

24. $\int \frac{(1+x)}{1-x^2}\,dx$ this one is tricky – not so much a u substitution but simplify your expressions before you integrate:

$$\int \frac{(1+x)}{1-x^2}\,dx \ = \int \frac{(1+x)}{(1+x)(1-x)}\ dx$$

$$= \ \int \frac{1}{1-x}\,dx \ = - \ \int \frac{1}{x-1}\,dx$$

$$= -\ln|x-1| \ + \ C$$

25. $\int \frac{3\sqrt{x}}{1+x^3}\ dx$ let $u = \sqrt{x}\ \ or\ \ x = u^2$

$dx = 2u\ du$

$= \int \frac{3u}{1+u^6}\ 2u\ du\ =\ 6\int \frac{u^2}{1+u^6}\ du$

Now let $t = u^3$ so $dt = 3u^2\ du$ or finally $\frac{dt}{3} = u^2\ du$

$= 2\int \frac{dt}{1+t^2}\ =\ 2\ arc\tan t\ +\ C$

$=\ 2\ arc\tan u^3 + C$

$=\ 2\ arc\tan x^{3/2}\ \ +\ \ C$

26. $\int \frac{x\sqrt{x}}{1+x^5}\ dx$ let $u = \sqrt{x}\ \ or\ \ x = u^2$

$\frac{dx}{du} =\ 2u$

$dx = 2u\ du$

$\int \frac{x\sqrt{x}}{1+x^5}\ dx\ =\ \int \frac{u^3}{1+u^{10}}\ 2u\ du$

$=\ 2\int \frac{u^4}{1+u^{10}}\ du$ now let $t = u^5\ \ so\ \ dt = 5u^4\ du$

$= \frac{2}{5}\int \frac{dt}{1+t^2}\ =\ \frac{2}{5}\ arc\tan t\ +\ C$

$= \frac{2}{5}\ arc\tan u^5\ +\ C$

$=\ \frac{2}{5}\ arc\tan x^{5/2}\ +\ C$

27. $\int \frac{t^2}{\sqrt[3]{t+3}}\, dt$ let $u = t+3 \quad so \quad du = dt$

$\int \frac{(u-3)^2}{u^{1/3}}\, du$

$\int \frac{(u^2-6u+9)}{u^{1/3}}\, du = \int \left(u^{5/3} - 6\,u^{2/3} + 9\,u^{-1/3}\right) du$

$= \frac{3}{8}\, u^{8/3} - 6\left(\frac{3}{5}\right) u^{5/3} + 9\left(\frac{3}{2}\right) u^{2/3} + C$

$= \frac{3}{8}\, u^{8/3} - \frac{18}{5}\, u^{\frac{5}{3}} + \frac{27}{2}\, u^{2/3} + C$

$= \frac{3}{8}\,(t+3)^{8/3} - \frac{18}{5}\,(t+3)^{5/3} + \frac{27}{2}\,(t+3)^{2/3} + C$

28. $\int \frac{1}{\sqrt{1+\sqrt{x}}}\, dx$ let $u = \sqrt{x} \quad or \quad x = u^2$

$dx = 2u\, du$

$= \int \frac{1}{\sqrt{1+u}}\, 2u\, du = 2\int \frac{u}{\sqrt{1+u}}\, du$

Now let $t = 1+u \quad so \;\; dt = du$

$2\int \frac{t-1}{\sqrt{t}}\, dt = 2\int \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) dt$

$= 2\left(\frac{2}{3}\, t^{3/2} - 2\, t^{1/2}\right) + C$

$= \left(\frac{4}{3}\, t^{3/2} - 4\, t^{1/2}\right) + C$

$= \left(\frac{4}{3}\, (1+u)^{3/2} - 4\, (1+u)^{1/2}\right) + C$

$= \left(\frac{4}{3}\, \left(1+\sqrt{x}\right)^{3/2} - 4\, \left(1+\sqrt{x}\right)^{1/2}\right) + C$

29. $\int \frac{1}{x\sqrt{x^4-1}}\ dx$ let $u = x^2$ $so\ \ \frac{du}{2} = x\,dx$

$$\int \frac{1}{x\sqrt{x^4-1}}\ dx \ = \int \frac{x}{x^2\sqrt{x^4-1}}\ dx$$

$$= \frac{1}{2}\int \frac{du}{u\sqrt{u^2-1}} \ = \ \frac{1}{2}\ arc\sec u + C$$

$$= \frac{1}{2}\ arc\sec x^2 \ + \ C$$