EULER'S FORMULA

Euler's formula states:

$$e^{ix} = \cos x + i \sin x$$

This is one of the most useful formulas in mathematics.

It is used in all of wave theory - sounds waves, optics, quantum mechanics, Fourier analysis. It helps form a mathematical basis for new types of functions called "generalized functions" (integral representation of the Dirac Delta function). It establishes the preferred method for analyzing oscillations and vibrations, especially circuit theory with alternating current.

It is a basic, fundamental theorem to the calculus of complex variables and complex functions.

It is difficult to overstate the importance of this formula.

The proof of the above theorem comes from Taylor series. We start with the three series for sine, cosine and the exponential function.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

We look at the series for e^{ix}:

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots$$

This becomes:

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!}$$

We group real and imaginary terms separately:

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \left(ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + \cdots\right)$$

Factoring out an "i" from the second parentheses we get

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

We recognize the series in parentheses and we get:

$$e^{ix} = \cos x + i \sin x$$

NOTE: if we let $x = \pi$ then we get the equations $e^{i\pi} = \cos \pi + i \sin \pi$.

This becomes $e^{i\pi} = -1 + 0i$.

Rearranging this becomes $e^{i\pi} + 1 = 0$

This last equation is considered to be the most amazing equation of mathematics. Polls were taken from mathematicians and physicists and other scientists about the most fascinating equation on math. This equation ranked number 1. It relates some pretty disparate numbers. The numbers e and pi are the most frequent irrational numbers in math and science. The numbers 0 and 1 are the identity elements for addition and multiplication. The number i, is the most basic imaginary number. All of them are related to each other by this equation.

SAME DERIVATION: DIFFERENT NOTATION:

We start with the Taylor series expansions for sine, cosine and the exponential function:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

The exponential function is absolutely convergent so the order in which we add the terms will not affect the sum. So we can break the series into two series – one odd and one even:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{m \to \infty} \sum_{n=0}^{2m+1} \frac{x^n}{n!}$$

$$e^x = \lim_{m \to \infty} \sum_{n=0}^{m} \frac{x^{2n}}{(2n)!} + \lim_{m \to \infty} \sum_{n=0}^{m} \frac{x^{2n+1}}{(2n+1)!}$$

$$e^{ix} = \lim_{m \to \infty} \sum_{n=0}^{m} \frac{(ix)^{2n}}{(2n)!} + \lim_{m \to \infty} \sum_{n=0}^{m} \frac{(ix)^{2n+1}}{(2n+1)!}$$

$$e^{ix} = \lim_{m \to \infty} \sum_{n=0}^{m} \frac{(i)^{2n} (x)^{2n}}{(2n)!} + \lim_{m \to \infty} \sum_{n=0}^{m} \frac{(i)^{2n+1} (x)^{2n+1}}{(2n+1)!}$$

$$e^{ix} = \lim_{m \to \infty} \sum_{n=0}^{m} \frac{(-1)^n (x)^{2n}}{(2n)!} + \lim_{m \to \infty} \sum_{n=0}^{m} \frac{(-1)^n \, i \, x^{2n+1}}{(2n+1)!}$$

$$e^{ix} = \lim_{m \to \infty} \sum_{n=0}^{m} \frac{(-1)^n (x)^{2n}}{(2n)!} + i \lim_{m \to \infty} \sum_{n=0}^{m} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$e^{ix} = \cos x + i \sin x$$

This is essentially the exact same derivation as before but it has more formal notation.

REPEATING THE PREVIOUS NOTE: if we let $x = \pi$ then we get the equations $e^{i\pi} = \cos \pi + i \sin \pi$. This becomes $e^{i\pi} = -1 + 0i$. Rearranging this becomes $e^{i\pi} + 1 = 0$. This last equation is considered to be the most amazing equation of mathematics. Polls were taken from mathematicians and physicists and other scientists about the most fascinating equation on math. This equation ranked first. It relates some pretty disparate numbers. The numbers e and pi are the most frequent irrational numbers in math and science. The numbers 0 and 1 are the identity elements for addition and multiplication. The number i, is the most basic imaginary number. All of them are related to each other by this equation.

ANOTHER DERIVATION OF EULER'S THEOREM USING INTEGRALS

We use the integral for the natural logarithm:

$$\int \frac{1}{x}\, dx = \ln x + C$$

If we let x = cos u + i sin u then dx = (- sin u + i cos u) du

$$\int \frac{1}{\cos u + i \sin u} (-\sin u + i \cos u)\, du = \ln(\cos u + i \sin u) + C$$

$$\int \frac{(-\sin u + i \cos u)}{\cos u + i \sin u}\, du = \ln(\cos u + i \sin u) + C$$

Looking at the integral, factor out an i from the numerator:

$$i \int \frac{(i \sin u + \cos u)}{\cos u + i \sin u}\, du = \ln(\cos u + i \sin u) + C$$

The fraction inside the integral now equals one:

$$i \int 1\, du = \ln(\cos u + i \sin u) + C$$

$$i\, u = \ln(\cos u + i \sin u) + C$$

We can write as $\ln(\cos u + i \sin u) = i\, u + C$

Using inverse functions this can be written as

$$e^{iu+C} = \cos u + i \sin u$$

$$e^{C}\, e^{iu} = \cos u + i \sin u$$

We can write this as $C_1\, e^{iu} = \cos u + i \sin u$

To figure out the constant, let u = 0. This will show that $C_1 = 1$.

Finally we have $e^{iu} = \cos u + i \sin u$

A careful student might complain about the use of the formula $\int \frac{1}{x}\, dx = \ln x + C$. Technically the formula should be $\int \frac{1}{x}\, dx = \ln|x| + C$. In the theory of real variables, the second formula is the correct one. In the theory of complex variables, the first one is correct. It turns out that taking the absolute value of a complex number, or a complex variable, destroys the complex aspect of the number and changes it into a real number. The absolute value sign is dropped in complex variables.

EXAMPLE

Find e^{-ix}

We know that $e^{it} = \cos t + i \sin t$

Replace t with – x:

$$e^{-ix} = \cos(-x) + i \sin(-x)$$

$$e^{-ix} = \cos x - i \sin x$$

EXAMPLE

Find sin x in terms of the exponential function

We have the following two equations:

$e^{ix} = \cos x + i \sin x$

$e^{-ix} = \cos x - i \sin x$

Subtract the two equations:

$e^{ix} - e^{-ix} = 2i \sin x$

Divide by 2i

$$\sin x = \frac{(e^{ix} - e^{-ix})}{2i}$$

EXAMPLE

Express cos x in terms of the exponential function

We have the two equations:

$e^{ix} = \cos x + i \sin x$

$e^{-ix} = \cos x - i \sin x$

Add the two equations

$e^{ix} + e^{-ix} = 2 \cos x$

Divide by 2:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

EXAMPLE
Express tangent in terms of the exponential function

We have the two equations

$\sin x = \frac{(e^{ix} - e^{-ix})}{2i}$

$\cos x = \frac{e^{ix} + e^{-ix}}{2}$

Use $\tan x = \frac{\sin x}{\cos x}$

We get $\tan x = \frac{(e^{ix} - e^{-ix})}{i\ (e^{ix} + e^{-ix})}$

EXAMPLE
Express cotangent in terms of the exponential function

We have the equation $\tan x = \frac{(e^{ix} - e^{-ix})}{i\ (e^{ix} + e^{-ix})}$

Use $\cot x = \frac{1}{\tan x}$

$\cot x = \frac{i\,(e^{ix} + e^{-ix})}{(e^{ix} - e^{-ix})}$

EXAMPLE
Express secant in terms of the exponential function.

We have the equations $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

Use $\sec x = \frac{1}{\cos x}$

We get $\sec x = \frac{2}{e^{ix} + e^{-ix}}$

EXAMPLE

Express csc x in terms of the exponential function

We start with the equation $\sin x = \frac{(e^{ix} - e^{-ix})}{2i}$

Use $\csc x = \frac{1}{\sin x}$

$$\csc x = \frac{2i}{e^{ix} - e^{-ix}}$$

EXAMPLE

Derive the formula $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

Start with the equation

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

Square both sides

$$\cos^2 x = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^2$$

$$\cos^2 x = \frac{1}{4}\left(e^{2ix} + 2 + e^{-2ix}\right)$$

$$\cos^2 x = \frac{2}{4} + \left(\frac{e^{2ix} + e^{-2ix}}{4} \right)$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\left(\frac{e^{2ix} + e^{-2ix}}{2} \right)$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

EXAMPLE

Express $\cos^3 x$ in terms of cos x, cos 2x and cos 3x

Start with the equation:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

Cube both sides

$$\cos^3 x = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^3$$

$$\cos^3 x = \frac{1}{8}\left(e^{3ix} + 3\,e^{ix} + 3\,e^{-ix} + e^{-3ix}\right)$$

$$\cos^3 x = \frac{1}{8}\left(\left(e^{3ix} + e^{-3ix}\right) + 3\left(e^{-ix} + e^{ix}\right)\right)$$

$$\cos^3 x = \frac{1}{4}\left(\frac{\left(e^{3ix} + e^{-3ix}\right)}{2} + 3\,\frac{\left(e^{ix} + e^{-ix}\right)}{2}\right)$$

$$\cos^3 x = \frac{1}{4}\left(\cos 3x + 3\cos x\right)$$

EXAMPLE

Express $\cos^4 x$ in terms of cos x, cos 2x, cos 3x cos 4x

We start with the equation

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

Raise both sides to the 4th power:

$$\cos^4 x = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^4$$

$$\cos^4 x = \frac{1}{16}\left(e^{4ix} + 4e^{2ix} + 6 + 4e^{-2ix} + e^{-4ix}\right)$$

Group like terms:

$$\cos^4 x = \frac{1}{16}\left(e^{4ix} + e^{-4ix} + 4e^{2ix} + 4e^{-2ix} + 6\right)$$

$$\cos^4 x = \frac{1}{16}\left(\left(e^{4ix} + e^{-4ix}\right) + 4\left(e^{2ix} + e^{-2ix}\right) + 6\right)$$

$$\cos^4 x = \frac{1}{8}\left(\frac{\left(e^{4ix} + e^{-4ix}\right)}{2} + 4\,\frac{\left(e^{2ix} + e^{-2ix}\right)}{2} + 3\right)$$

$$\cos^4 x = \frac{1}{8}\left(\cos 4x + 4\cos 2x + 3\right)$$

EXAMPLE

Express $\sin^2 x$ in terms of cos nx

We start with the expression

$$\sin x = \frac{(e^{ix} - e^{-ix})}{2i}$$

Square both sides:

$$\sin^2 x = \left(\frac{(e^{ix} - e^{-ix})}{2i} \right)^2$$

$$\sin^2 x = \left(\frac{(e^{2ix} - 2 + e^{-2ix})}{-4} \right)$$

$$\sin^2 x = -\frac{1}{4}(e^{2ix} + e^{-2ix} - 2)$$

$$\sin^2 x = -\frac{1}{4}(2 \cos 2x - 2)$$

$$\sin^2 x = \left(-\frac{1}{2} \cos 2x + \frac{1}{2} \right)$$

EXAMPLE

Express $\sin^3 x$ in terms of sin nx

$$\sin x = \frac{(e^{ix} - e^{-ix})}{2i}$$

Raise both sides to the 3rd power:

$$\sin^3 x = \left(\frac{(e^{ix} - e^{-ix})}{2i} \right)^3$$

$$\sin^3 x = -\frac{1}{8i} (e^{3ix} - 3e^{ix} + 3 e^{-ix} - e^{3ix})$$

Group like terms:

$$\sin^3 x = -\frac{1}{8i} (e^{3ix} - e^{3ix} + 3 e^{-ix} - 3e^{ix})$$

$$\sin^3 x = -\frac{1}{4} \left(\frac{(e^{3ix} - e^{3ix})}{2i} + 3 \frac{(e^{-ix} - e^{ix})}{2i} \right)$$

$$\sin^3 x = -\frac{1}{4} (\sin 3x + 3 \sin x)$$

EXAMPLE

Express $\sin^4 x$ in terms of $\cos nx$

We start with the expression:

$$\sin x = \frac{(e^{ix} - e^{-ix})}{2i}$$

Raise both sides to the 4^{th} power:

$$\sin^4 x = \left(\frac{(e^{ix} - e^{-ix})}{2i} \right)^4$$

$$\sin^4 x = \frac{1}{16} \left(e^{ix} - e^{-ix} \right)^4$$

$$\sin^4 x = \frac{1}{16} \left(e^{4ix} - 4e^{2ix} + 6 - 4\,e^{-2ix} + e^{-4ix} \right)$$

Group like terms:

$$\sin^4 x = \frac{1}{16} \left(e^{4ix} + e^{-4ix} - 4e^{2ix} - 4e^{-2ix} + 6 \right)$$

$$\sin^4 x = \frac{1}{16} \left(\left(e^{4ix} + e^{-4ix} \right) - 4 \left(e^{2ix} + e^{-2ix} \right) + 6 \right)$$

$$\sin^4 x = \frac{1}{8} \left(\frac{(e^{4ix} + e^{-4ix})}{2} - 4 \frac{(e^{2ix} + e^{-2ix})}{2} + 3 \right)$$

$$\sin^4 x = \frac{1}{8} \left(\cos 4x - 4\cos 2x + 3 \right)$$

$$\sin^4 x = \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x + \frac{3}{8}$$

EXAMPLE
Express cos 2x in terms of $\cos^n x$ and $\sin^n x$

Start with $e^{ix} = \cos x + i \sin x$

Square both sides

$$e^{2ix} = (\cos x + i \sin x)^2$$

$$e^{2ix} = (\cos^2 x - \sin^2 x) + 2i \cos x \sin x$$

Replace the left side using Euler's theorem:
$\cos 2x + i \sin 2x = (\cos^2 x - \sin^2 x) + 2i \cos x \sin x$

Equating reals and imaginaries we get

$\cos 2x = \cos^2 x - \sin^2 x \qquad \sin 2x = 2 \sin x \cos x$

EXAMPLE
Express cos 3x in terms of $\cos^n x$ and $\sin^n x$

Start with $e^{ix} = \cos x + i \sin x$

Cube both sides

$$e^{3ix} = (\cos x + i \sin x)^3$$

$$e^{3ix} = (\cos^3 x + 3i \cos^2 x \sin x - 3 \cos x \sin^2 x - i \sin^3 x)$$

Group reals and imaginaries

$$e^{3ix} = \left(\cos^3 x - 3 \cos x \sin^2 x + i\,(3 \cos^2 x \sin x - \sin^3 x)\right)$$

Replace the left side using Euler's theorem:
$\cos 3x + i \sin 3x = \left(\cos^3 x - 3 \cos x \sin^2 x + i\,(3 \cos^2 x \sin x - \sin^3 x)\right)$

Equate reals and imaginaries
$\cos 3x = \cos^3 x - 3 \cos x \sin^2 x$

$\sin 3x = 3 \cos^2 x \sin x - \sin^3 x$

These formulas could be further reduced by using the Pythogorean relation $\cos^2 x + \sin^2 x = 1$
If we do this we can express cos 3x in terms of powers of cosine only. Sin 3x can also be expressed in terms of power of sine.
$\cos 3x = 4 \cos^3 x - 3 \cos x$

$\sin 3x = 3 \sin x - 4 \sin^3 x$

EXAMPLE

Evaluate the integral $\int \sin^4 x \;\; dx$

From a previous problem we used $(\sin x)^4 = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^4$ to derive the expression

$\sin^4 x = \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x + \frac{3}{8}$

$$\int \sin^4 x \;\; dx = \int \left(\frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x + \frac{3}{8}\right) dx$$

$$\int \sin^4 x \;\; dx = \frac{1}{32}\sin 4x - \frac{1}{4}\sin 2x + \frac{3}{8}x + C$$

EXAMPLE

Evaluate the integral $\int \cos^4 x \;\; dx$

Using the equation $(\cos x)^4 = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^4$ we were able to derive the equation

$\cos^4 x = \frac{1}{8}(\cos 4x + 4\cos 2x + 3)$

$$\int \cos^4 x \;\; dx = \frac{1}{8}\int (\cos 4x + 4\cos 2x + 3)\, dx$$

$$\int \cos^4 x \;\; dx = \frac{1}{8}\left(\frac{\sin 4x}{4} + 2\sin 2x + 3x\right) + C$$