METHOD OF TRIGONOMETRIC SUBSTITUTIONS

In dealing with integrals, we will often come across the following forms:

$a^2 - x^2$

$a^2 + x^2$

$x^2 - a^2$

When these forms come about, we can still integrate the integrand using trigonometric substitutions. In this case, we let x equal some trigonometric function.

The substitution rules work as follows:

$a^2 - x^2$ we make the substitution $x = a\sin\theta$ or $x = a\cos\theta$ - either substitution will work

$a^2 + x^2$ we make the substitution $x = a\tan\theta$ or $x = a\cot\theta$ - either substitution will work

$x^2 - a^2$ we make the substitution $x = a\sec\theta$ or $x = a\csc\theta$ - either substitution will work

Although the first and the third forms for x look almost identical (they only differ by a minus sign), in practice they may fall under a square root sign, making the situations quite different.

The key to having these substitutions work are two fold – the first is the Pythagorean relations – the second are the forms for the derivatives of the relevant trig functions.

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\frac{d(\sin\theta)}{d\theta} = \cos\theta \qquad \frac{d(\cos\theta)}{d\theta} = -\sin\theta$$

$$\frac{d(\tan\theta)}{d\theta} = \sec^2\theta \qquad \frac{d(\cot\theta)}{d\theta} = -\csc^2\theta$$

$$\frac{d(\sec\theta)}{d\theta} = \sec\theta\ \tan\theta \qquad \frac{d(\csc\theta)}{d\theta} = -\csc\theta\ \cot\theta$$

It turns out that these two sets of relations overlap in such a way that makes the integrals solvable. The best way to see this (and maybe the only way) is to solve questions.

1. Evaluate

$$\int \frac{1}{x^2\sqrt{x^2-4}}\ dx$$

$x^2 - a^2 = \ x^2 - 4$

So we use $\quad x = a\sec t \ = \ 2\sec t \qquad$ and $dx = 2\sec t\ \tan t\ dt$

$$\int \frac{1}{x^2\sqrt{x^2-4}}\ dx \ = \ \int \frac{2\sec t\tan t}{4\ \sec^2 t\sqrt{4\sec^2 t-4}}\ dt \ = \ \int \frac{2\sec t\tan t}{4{\cdot}2\ \sec^2 t\sqrt{\sec^2 t-1}}\ dt$$

In the last step, we factored out a 4 from the radical.

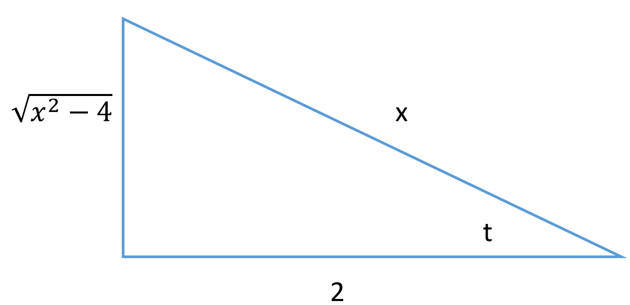
At this point the Pythagorean relation comes in to change the radical $\sec^2 t - 1 = \tan^2 t$

$$\frac{1}{4}\int \frac{\sec t\tan t}{\sec^2 t\sqrt{\tan^2 t}}\ dt \ = \frac{1}{4}\int \frac{\sec t\tan t}{\sec^2 t\tan t}\ dt \ = \frac{1}{4}\int \frac{1}{\sec t}\ dt$$

The reciprocal of secant is cosine – so the integral becomes

$$\frac{1}{4}\int\ \cos t\ \ dt \ \ = \frac{1}{4}\sin t\ +\ C$$

We now have to draw a right triangle to go back to the variable x. The triangle we draw is based on the initial definition we made: x = 2 sec t or better that sec t = x/2. The third side is found by Pythagorean theorem.



The triangle says that $\sin t \ = \ \frac{\sqrt{4-x^2}}{x}$

So $\int \frac{1}{x^2\sqrt{x^2-4}}\ dx \ = \frac{1}{4}\ \frac{\sqrt{4-x^2}}{x}\ +\ C$

2. Evaluate

$$\int \frac{1}{x^2\sqrt{x^2-4}}\; dx$$

$x^2 - a^2 = \; x^2 - 4$

So we use $\quad x = a\csc t = \; 2\csc t \qquad$ and $\quad dx = -2\csc t\; \cot t\; dt$

$$\int \frac{1}{x^2\sqrt{x^2\; - \; 4}}\; dx \; = \; \int \frac{-2\csc t\cot t}{4\csc^2 t\; \sqrt{4\csc^2 t\; - \; 4}}\; dx$$

Pull the four out of the radical:

$$\int \frac{-2\csc t\cot t}{4\cdot 2\csc^2 t\; \sqrt{\csc^2 t - 1}}\; dx \; = \; -\frac{1}{4}\int \frac{\csc t\cot t}{\csc^2 t\; \sqrt{\csc^2 t - 1}}\; dx$$

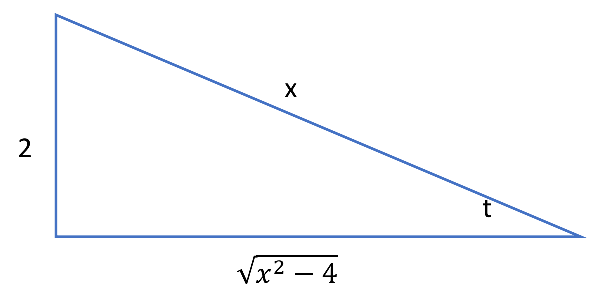
Use the Pythagorean relation $\csc^2 t - 1 \; = \; \cot^2 t$

$$-\frac{1}{4}\int \frac{\csc t\cot t}{\csc^2 t\; \sqrt{\cot^2 t}}\; dx \; = \; -\frac{1}{4}\int \frac{\csc t\cot t}{\csc^2 t\cot t}\; dx \; = \; -\frac{1}{4}\int \frac{1}{\csc t}\; dx$$

The reciprocal of cosecant is sine – so we get

$$-\frac{1}{4}\int \sin t\;\; dx \; = \; \frac{1}{4}\cos t \; + \; C$$

Draw a right triangle to go back from t into x. We use the original definition x = 2 csc t or better that csc t = x/2



$\int \frac{1}{x^2\sqrt{4-x^2}}\; dx \; = \frac{1}{4}\frac{\sqrt{x^2-4}}{x} \; + \; C$ Note that this answer is the same as in question 1. The solution is independent of the substitution.

3. Evaluate $\int \frac{1}{x^2\sqrt{4-x^2}}\ dx$

Here we have the expression $4 - x^2$ which is of the form $a^2 - x^2$

This indicates the substitution $x = a\sin t \ = \ 2\sin t$

$$\int \frac{1}{x^2\sqrt{4-x^2}}\ dx \ = \ \int \frac{2\cos t\ dt}{4\sin^2 t\ \sqrt{4\ -\ 4\sin^2 t}}$$

$$= \int \frac{2\cos t\ dt}{4{\cdot}2\ \sin^2 t\ \sqrt{1\ -\ \sin^2 t}} \ = \ \frac{1}{4}\ \int \frac{\cos t\ dt}{\sin^2 t\ \sqrt{1\ -\ \sin^2 t}}$$

Use the Pythagorean relation $1 - \sin^2 t \ = \ \cos^2 t$

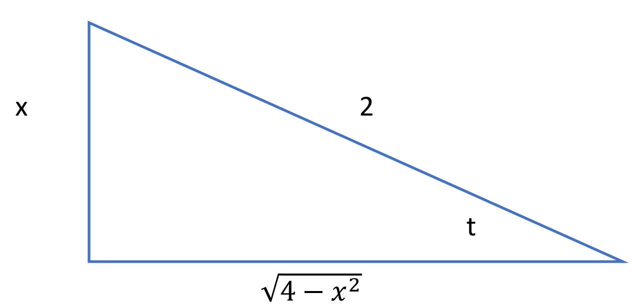
$$= \frac{1}{4}\ \int \frac{\cos t\ dt}{\sin^2 t\ \sqrt{1\ -\ \sin^2 t}} \ = \ \frac{1}{4}\ \int \frac{\cos t\ dt}{\sin^2 t\ \sqrt{\cos^2 t}} \ = \ \frac{1}{4}\ \int \frac{dt}{\sin^2 t}$$

The reciprocal of sine is cosecant so this becomes:

$$\frac{1}{4}\int \csc^2 t\ dt \ = \ -\frac{1}{4}\cot(t)\ +\ C$$

We must now use a right triangle to go back to the variable x.

We use our definition x = 2 sin t or sin t = x/2



$$\frac{1}{4}\int \csc^2 t\ dt \ = \ -\frac{1}{4}\cot(t)\ +\ C$$

$$\int \frac{1}{x^2\sqrt{4-x^2}}\ dx \ = \ -\frac{1}{4}\ \frac{\sqrt{4-x^2}}{x}\ +\ C$$

4. Evaluate $\int \frac{1}{x^2\sqrt{4-x^2}}\ dx$

Here we have the expression $4 - x^2$ which is of the form $a^2 - x^2$

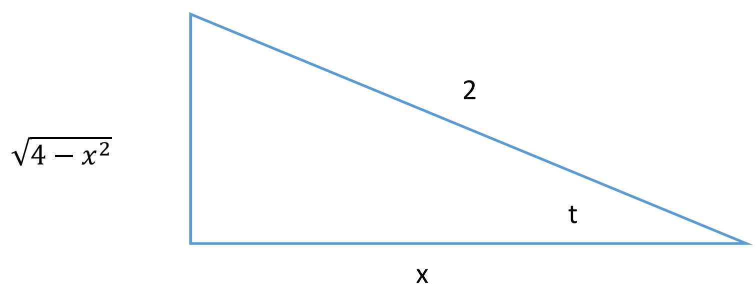
This indicates the substitution $x = a\cos t\ =\ 2\cos t$

$$\int \frac{1}{x^2\sqrt{4-\ x^2}}\ dx\ =\ \int \frac{-2\sin t\ \ dt}{4\cos^2 t\ \sqrt{4-4\cos^2 t}}\ =\ \int \frac{-2\sin t\ \ dt}{4\cdot 2\ \cos^2 t\ \sqrt{1-\cos^2 t}}$$

$$-\frac{1}{4}\int \frac{\sin t\ \ dt}{\cos^2 t\ \sqrt{1-\cos^2 t}}\ =\ -\frac{1}{4}\int \frac{\sin t\ \ dt}{\cos^2 t\ \sqrt{\sin^2 t}}\ =\ -\frac{1}{4}\int \frac{\sin t\ \ dt}{\cos^2 t\sin t}$$

$$-\frac{1}{4}\int \frac{dt}{\cos^2 t}\ =\ -\frac{1}{4}\int \sec^2 t\ \ dt\ =\ -\frac{1}{4}\tan t\ +\ C$$

Now we use a right triangle to go back to x. We need the definition x = 2 cos t



$$\int \frac{1}{x^2\sqrt{4-\ x^2}}\ dx\ =\ -\frac{1}{4}\tan t\ +\ C\ =\ -\frac{1}{4}\frac{\sqrt{4-x^2}}{x}\ +\ C$$

This is the same as for question three. The method of substitution won't affect the result.

5. Evaluate $\int \frac{1}{\sqrt{9+x^2}}\ dx$

We have the expression $9 + x^2$ which is of the form $a^2 + x^2$.

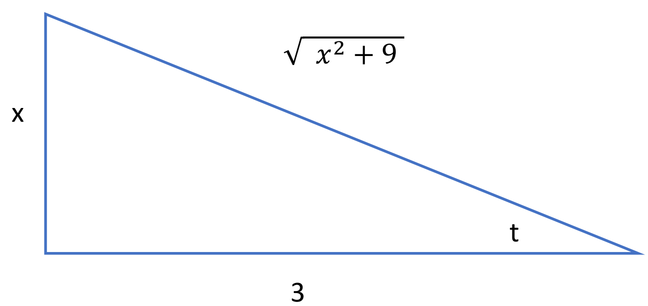
We make the substitution $x = a \tan t \ = \ 3 \tan t$.

$$\int \frac{1}{\sqrt{9+x^2}}\ dx \ = \ \int \frac{1}{\sqrt{9+9\tan^2 t}}\ 3\sec^2 t\ dt \ = \ \int \frac{1}{3\sqrt{1\ +\ \tan^2 t}}\ 3\sec^2 t\ dt$$

$$= \ \int \frac{1}{3\sqrt{1\ +\ \tan^2 t}}\ 3\sec^2 t\ dt \ = \ \int \frac{1}{3\sqrt{\sec^2 t}}\ 3\sec^2 t\ dt$$

$$= \ \int \frac{1}{\sec t}\ \sec^2 t\ dt \ = \ \int \sec t\ \ dt \ = \ \ln|\sec t\ +\ \tan t|\ +\ C$$

Now use a right triangle to go back to variable x:



$$\int \frac{1}{\sqrt{9+x^2}}\ dx \ = \ \ln|\sec t\ +\ \tan t|\ +\ C \ = \ \ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + C$$

6. Evaluate Evaluate $\int \frac{1}{\sqrt{9+x^2}}\ dx$

We have the expression $9+x^2$ which is of the form a^2+x^2.

We make the substitution $x = a\cos t = 3\cot t$.

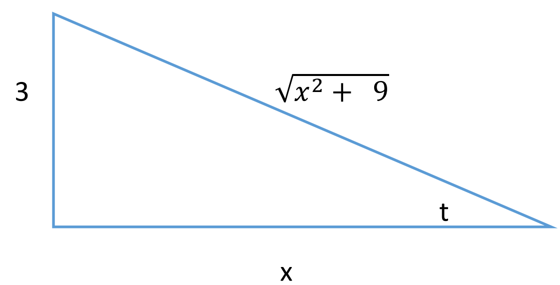
$$\int \frac{1}{\sqrt{9+x^2}}\ dx = \int \frac{1}{\sqrt{9+9\cot^2 t}}(-3\csc^2 t)\, dt = \int \frac{1}{3\sqrt{1+\cot^2 t}}(-3\, csc^2\, t)\, dt$$

$$= -\int \frac{\csc^2 t}{\sqrt{1+\cot^2 t}}\ dt = -\int \frac{\csc^2 t}{\sqrt{\csc^2 t}}\ dt = -\int \frac{\csc^2 t}{\csc t}\ dt$$

$$= -\int \csc t\ dt = \ln|\csc t + \cot t| + C$$

Now we must use a right triangle to change the variable from t back to x:

We use cot t = x/3



$$\int \frac{1}{\sqrt{9+x^2}}\ dx = \ln|\csc t + \cot t| + C = \ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + C$$

This agrees with the answer done in question 5

7. Evaluate $\int \frac{\sqrt{x^2-25}}{x}\ dx$

The expression $x^2 - 25$ is of the form $x^2 - a^2$ so we make the substitution $x = a\sec t\ = 5\sec t$

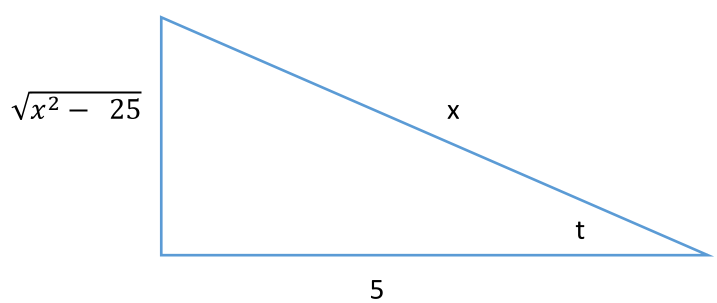
$$\int \frac{\sqrt{x^2-25}}{x}\ dx\ =\ \int \frac{\sqrt{25\sec^2 t - 25}}{5\sec t}\ 5\sec t\tan t\ \ dt$$

$$=\ \int \frac{5\ \sqrt{\sec^2 t - 1}}{5\sec t}\ 5\sec t\tan t\ \ dt\ =\ 5\int\sqrt{\sec^2 t\ -\ 1}\tan t\ dt$$

$$=\ 5\int\sqrt{\tan^2 t}\ \ \tan t\ dt = 5\int\ \tan^2 t\ \ dt$$

$$=\ 5\ \int(\sec^2 t - 1)\,dt\ =\ 5\,(\tan t - t\,) +\ C$$

Once again we draw a right triangle to go from t back to x: x = 5 sec t or sec t = x/5



$$\int \frac{\sqrt{x^2-25}}{x}\ dx\ =\ 5\,(\tan t - t\,) +\ C\ \ =\ 5\left(\frac{\sqrt{x^2-25}}{5} - \sec^{-1}\frac{x}{5}\right) + C$$

$$=\ \left(\sqrt{x^2-25}\ -\ 5\ \sec^{-1}\frac{x}{5}\right)\ +\ C$$

8. Evaluate $\int \frac{\sqrt{x^2-25}}{x}\ dx$

The expression $x^2 - 25$ is of the form $x^2 - a^2$ so we make the substitution $x = a\csc t\ =$ $5\csc t$

$$\int \frac{\sqrt{x^2-25}}{x}\ dx\ =\ \int \frac{\sqrt{25\csc^2 t-25}}{5\csc t}\ (-5\csc t\ \cot t\ \ dt)$$

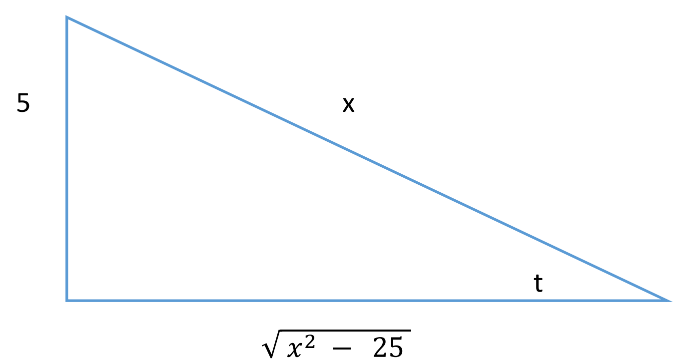
$$= \int \frac{5\sqrt{\csc^2 t-1}}{5\csc t}\ (-5\csc t\ \cot t\ \ dt)\ =\ -5\int \frac{\sqrt{\csc^2 t-1}}{\csc t}\ (\csc t\ \cot t\ \ dt)$$

$$=\ -5\ \int\ \sqrt{\csc^2 t\ -\ 1}\ \cot t\ \ dt\ =\ -5\ \int \sqrt{\cot^2 t}\ \cot t\ \ dt = \ -5\ \int\ \cot t \cot t\ \ dt$$

$$=\ -5\ \int \cot^2 t\ dt\ =\ -5\ \int(\csc^2 t\ -\ 1)\ dt\ =\ -5\,(-\cot t\ - t)\ +\ C$$

$$= 5\,(\cot t + t)\ +\ C$$

Go to the right triangle to express the answer in x: x = 5 csc t or csc t = x/5



$$\int \frac{\sqrt{x^2-25}}{x}\ dx\ =\ 5\,(\cot t + t)\ +\ C\ =\ 5\left(\frac{\sqrt{x^2-25}}{5} + \csc^{-1}\frac{x}{5}\right)\ +\ C$$

$$=\ \left(\sqrt{x^2-25}\ +\ 5\csc^{-1}\frac{x}{5}\right)\ +\ C$$

This answer does not look the same as the previous one, but they are equal. This is true since $\sec^{-1} b\ +\ \csc^{-1} b\ =\ \frac{\pi}{2}$. This identity will bring them into the exact same form. The constant of integration, C will absorb any other constants.

9. Show that $\int \frac{1}{\sqrt{a^2-x^2}}\ dx\ =\ \sin^{-1}\frac{x}{a}\ +\ C$

Let x = a sin t so that dx = a cos t dt

$$\int \frac{1}{\sqrt{a^2-x^2}}\ dx\ =\ \int \frac{1}{\sqrt{a^2-a^2\sin^2 t}}\ a\cos t\ \ dt\ =\ \int \frac{1}{a\sqrt{1-\sin^2 t}}\ a\cos t\ \ dt$$

$$\int \frac{1}{\sqrt{1-\sin^2 t}}\ \cos t\ \ dt\ =\ \int \frac{\cos t}{\sqrt{\cos^2 t}}\ dt\ =\ \int \frac{\cos t}{\cos t}\ dt$$

$$\int dt\ =\ t\ +\ C$$

Since $x = a\sin t$ then $t = \sin^{-1}\frac{x}{a}$ so we end up with $\int dt\ =\ t\ +\ C\ =\ \sin^{-1}\frac{x}{a} + C$

10. Show that $\int \frac{1}{\sqrt{a^2-x^2}}\ dx\ =\ -cos^{-1}\ \frac{x}{a}\ +\ C$

We let x = a cos t along with dx = - a sin t dt

$$\int \frac{1}{\sqrt{a^2-x^2}}\ dx\ =\ \int \frac{1}{\sqrt{a^2-a^2\cos^2 t}}\ -a\sin t\ dt\ =\ -\frac{a}{a}\ \int \frac{\sin t\ \ dt}{\sqrt{1-\cos^2 t}}$$

$$-\int \frac{\sin t\ \ dt}{\sqrt{1-\cos^2 t}}\ =\ -\int \frac{\sin t\ \ dt}{\sqrt{1-\cos^2 t}}$$

$$=\ -\int \frac{\sin t\ \ dt}{\sqrt{\sin^2 t}}\ =\ -\int \frac{\sin t}{\sin t}\ dt\ =\ -\int dt\ \ =\ -t + C$$

Since $x = a\cos t$ then we have $t = \cos^{-1}\frac{x}{a}$

We get that the integral is $\int \frac{1}{\sqrt{a^2-x^2}}\ dx\ =\ -cos^{-1}\ \frac{x}{a}\ +\ C$

11. Show that $\int \frac{1}{x^2 + a^2}\ dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

We have the form $x^2 + a^2$ so we let $x = a\tan t$

We get $dx = a\sec^2 t\ dt$

$$\int \frac{1}{x^2 + a^2}\ dx = \int \frac{1}{a^2\tan^2 t + a^2}\ a\sec^2 t\ dt$$

$$\frac{a}{a^2}\int \frac{1}{\tan^2 t + 1}\ \sec^2 t\ dt = \frac{1}{a}\int \frac{1}{\sec^2 t}\ \sec^2 t\ dt$$

$$\frac{1}{a}\int 1\ dt = \frac{1}{a}\ t + c$$

Since $x = a\tan t$ then $\tan t = \frac{x}{a}$

Taking the inverse function this becomes $t = \tan^{-1}\frac{x}{a}$

This is our answer.

12. Show that $\int \frac{1}{x^2 + a^2}\ dx = -\frac{1}{a}\cot^{-1}\frac{x}{a} + C$

We have the form $x^2 + a^2$ so we let $x = a\cot t$

We will get $dx = -a\csc^2 t\ dt$

$$\int \frac{1}{x^2 + a^2}\ dx = \int \frac{1}{a^2\cot^2 t + a^2}\ (-a\csc^2 t\ dt) = -\frac{a}{a^2}\int \frac{\csc^2 t}{1 + \cot^2 t}\ dt$$

$$= -\frac{1}{a}\int \frac{\csc^2 t}{\csc^2 t}\ dt = -\frac{1}{a}\int dt = -\frac{1}{a}\ t + C$$

Since x = a cot t then cot t = x/a and $t = \cot^{-1}\frac{x}{a}$

$$\int \frac{1}{x^2 + a^2}\ dx = -\frac{1}{a}\cot^{-1}\frac{x}{a} + C$$

13. Evaluate $\int \sqrt{1-\ x^2}\ \ dx$

Let $x = \sin t$ so that $dx = \cos t\ \ dt$

$$\int \sqrt{1-\ x^2}\ \ dx = \int \sqrt{1-\ \sin^2 t}\ \cos t\ dt = \int \sqrt{\cos^2 t}\ \cos t\ \ dt$$

$$= \int \cos t\ \cos t\ \ dt = \int \cos^2 t\ dt$$

Use the double angle formula $\cos 2t = 2\cos^2 t - 1$ so that $\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$

$$= \int \left(\frac{1}{2} + \frac{1}{2}\cos 2t\right)\ \ dt = \int \frac{1}{2}\, dt + \frac{1}{2}\int \cos 2t\ \ dt$$

$$= \frac{1}{2}\, t + \frac{1}{2}\,\frac{1}{2}\int \cos 2t\ \ 2\, dt$$

For the last integral we are shooting for the form cos u du. This is why we multiplied and divided by 2.

$$= \frac{1}{2}\, t + \frac{1}{4}\sin 2t + C$$

We now use the double angle formula for sine: sin(2t) = 2 sin t cos t

$$= \frac{1}{2}\, t + \frac{1}{4}\cdot\ 2\sin t\cos t\ \ + C = \frac{t}{2} + \frac{1}{2}\sin t\ \cos t + C$$

We have that sin t = x and the triangle will given $\cos t = \sqrt{1-x^2}$

$$= \frac{\sin^{-1} x}{2} + \frac{1}{2}\, x\sqrt{1-x^2} + C$$

14. Evaluate $\int \sqrt{1-\ x^2}\ \ dx$

Let $x = \cos t$ so that $dx = \ -\sin t\ \ dt$

$$\int \sqrt{1-\ x^2}\ \ dx \ = \ \int \sqrt{1-\ \cos^2 t}\ \ (-\sin t\ \ dt) \ = -\int \sqrt{\sin^2 t}\ \sin t\ \ dt$$

$$-\int\ \sin t \sin t\ \ dt \ = \ -\int \sin^2 t\ \ dt$$

Use the double angle formula for cosine $\cos 2t \ = \ 1 - 2\sin^2 t$

So that $\sin^2 t \ = \frac{1}{2} - \frac{1}{2}\cos 2t$

$$-\int \left(\frac{1}{2} - \frac{1}{2}\cos 2t\right)\ dt \ = -\frac{t}{2} + \frac{1}{4}\sin 2t + \ C$$

$$= \ -\frac{t}{2} + \frac{1}{2}\sin t \cos t \ + \ C$$

We have our definition that x = cos t and the triangle will give us $\sin t \ = \ \sqrt{\ 1\ -\ x^2}$

$$= \ -\frac{1}{2}\cos^{-1} x + \frac{1}{2}\ x\sqrt{1 - x^2}\ + \ C$$

The answer does match the previous question since $\sin^{-1} z\ + \cos^{-1} z \ = \ \frac{\pi}{2}$

With this formula we can show that both answers are the same.

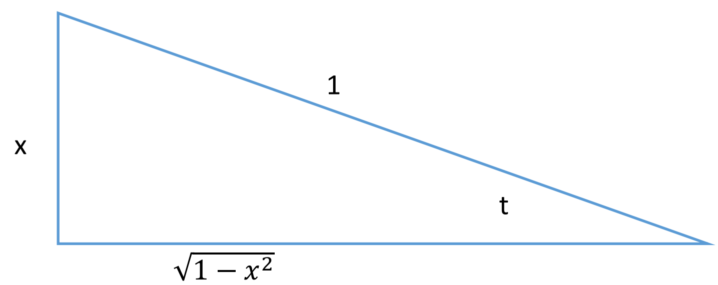
15. Evaluate $\int \frac{1}{1-x^2}\,dx$

Let x = sin t   so that   dx = cos t dt

$$= \int \frac{1}{1-\sin^2 t}\cos t\ dt = \int \frac{1}{\cos^2 t}\cos t\ \ dt$$

$$= \int \frac{\cos t}{\cos^2 t}\ dt = \int \frac{1}{\cos t}\ dt = \int \sec t\ \ dt$$

$$= \ln|\sec t + \tan t| + C$$



$$= \ln\left|\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right| + C$$

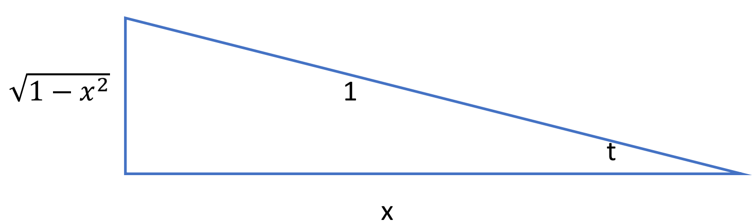
16. Evaluate $\int \frac{1}{1-x^2}\,dx$

Let x = cos t   so that   dx = - sin t dt

$$\int \frac{1}{1 - \cos^2 t}(-\sin t\ \ dt) = \int -\frac{\sin t}{\sin^2 t}\ dt = \int \frac{1}{\sin t}\ dt$$

$$= -\int \csc t\ dt = \ln|\csc t + \cot t| + C$$

$= \ln\left| \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right| + C$



The answers match.

17. Evaluate $\int \frac{1}{1-x^2}\, dx$

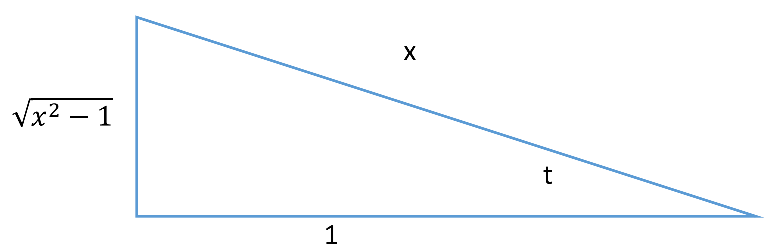
$$\int \frac{1}{1-x^2}\, dx = -\int \frac{1}{x^2 - 1}\, dx$$

Let x = sec t   so that dx = sec t tan t dt

$$= -\int \frac{1}{\sec^2 t - 1} \sec t \tan t \;\; dt = -\int \frac{1}{\tan^2 t} \sec t \tan t \;\; dt$$

$$= -\int \frac{1}{\tan t} \sec t \;\; dt = -\int \frac{\cos t}{\sin t} \frac{1}{\cos t} \;\; dt = -\int \frac{1}{\sin t}\, dt$$

$$= -\int \csc t \;\; dt = \ln|\csc t + \cot t| + C$$



$$= \ln\left|\frac{x}{\sqrt{x^2-1}} + \frac{1}{\sqrt{x^2-1}}\right| + C$$

The answers still match.

18. Evaluate $\int \frac{dx}{(a^2-x^2)^{3/2}}$

We have the form $a^2 - x^2$ so we will let x = a sin t and dx = a cos t dt

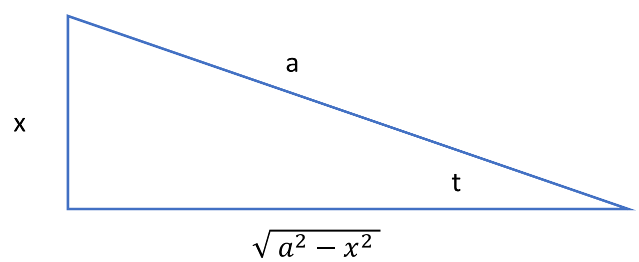
$$\int \frac{dx}{(a^2 - x^2)^{\frac{3}{2}}} = \int \frac{a\cos t \ \ dt}{(a^2 - a^2\sin^2 t)^{\frac{3}{2}}} = \frac{a}{a^3}\int \frac{\cos t}{(1 - \sin^2 t)^{\frac{3}{2}}} \ \ dt$$

$$= \frac{1}{a^2}\int \frac{\cos t}{(\cos^2 t)^{3/2}} \ \ dt = \frac{1}{a^2} \int \frac{\cos t}{\cos^3 t} = \frac{1}{a^2}\int \frac{1}{\cos^2 t} \ \ dt$$

$$= \frac{1}{a^2} \int \sec^2 t \ \ dt = \frac{1}{a^2}\tan t + C$$

Now draw a right triangle to evaluate the trig functions in terms of x:

We have x = a sin t so that sin t = x/a



$$= \frac{1}{a^2}\frac{x}{\sqrt{a^2 - x^2}} + C$$

19. Evaluate $\int \frac{dx}{(a^2-x^2)^{3/2}}$

We have the form $a^2 - x^2$ so we will let x = a cos t and dx = -a sin t dt

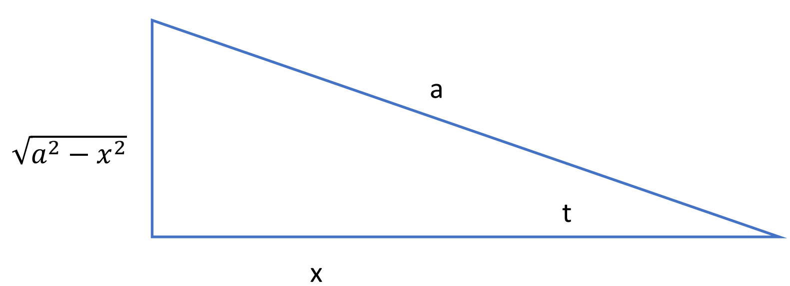
$$\int \frac{dx}{(a^2 - x^2)^{\frac{3}{2}}} = \int \frac{-a \sin t \ \ dt}{(a^2 - a^2 \cos^2 t)^{\frac{3}{2}}} = \frac{-a}{a^3} \int \frac{\sin t}{(1 - \cos^2 t)^{\frac{3}{2}}} \ dt$$

$$\frac{-a}{a^3} \int \frac{\sin t}{(sin^2 t)^{\frac{3}{2}}} \ dt = -\frac{1}{a^2} \int \frac{\sin t}{\sin^3 t} dt = -\frac{1}{a^2} \int \frac{1}{\sin^2 t} \ dt$$

$$= -\frac{1}{a^2} \int \csc^2 t \ dt = \frac{1}{a^2} \cot t + C$$

Draw a right triangle to evaluate the trig functions in terms of x:

We have x = a cos t so that cos t = x/a



$$= \frac{1}{a^2} \frac{x}{\sqrt{a^2 - x^2}} + C$$

The answer agrees with the previous question.