INFINITE SERIES INTEGRAL TEST SOLVED PROBLEMS

Statement of integral test: we are given the infinite series $\sum_{k=1}^{\infty} u_k$ where $u_k > 0$. We are told that $u_k = f(k)$ where f(k) is a known function of the positive integer k. If the integral of f(x) converges, that is if $\int_1^{\infty} f(x)\, dx$ converges (to any number) then the series converges. If the integral diverges, the series also diverges.

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Show that the following series converge or diverge by integral test

1. $\sum_{k=1}^{\infty} \frac{1}{k+6}$

   The corresponding integral is $\int_1^{\infty} \frac{1}{x+6}\, dx$

$$\int_1^{\infty} \frac{1}{x+6}\, dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{x+6}\, dx = \lim_{b \to \infty} \ln|x+6|\Big|_1^b$$

$$\int_1^{\infty} \frac{1}{x+6}\, dx = \lim_{b \to \infty} \ln|b+6| - \ln 7 = \ln|\infty| - \ln 7 = \infty - \ln 7 = \infty$$

   The integral diverges. This means the series diverges.

2. $\sum_{k=1}^{\infty} \frac{3}{5k}$

   The corresponding integral is $\int_1^{\infty} \frac{3}{5x}\, dx$

$$\int_1^{\infty} \frac{3}{5x}\, dx = \lim_{b \to \infty} \frac{3}{5} \int_1^{b} \frac{1}{x}\, dx = \frac{3}{5} \lim_{b \to \infty} \ln|x|\Big|_1^b$$

$$\int_1^{\infty} \frac{3}{5x}\, dx = \frac{3}{5} \lim_{b \to \infty} \ln|b| - \ln 1 = \ln|\infty| - \ln 1 = \infty - 0 = \infty$$

   The integral diverges. This means the series diverges.

3. $\sum_{k=1}^{\infty} \frac{1}{5k+2}$

The corresponding integral is $\int_1^{\infty} \frac{1}{5x+2}\, dx$

$$\int_1^{\infty} \frac{1}{5x+2}\, dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{5x+2}\, dx = \frac{1}{5} \lim_{b \to \infty} \ln|5x+2|\Big|_1^b$$

$$\int_1^{\infty} \frac{1}{5x+2}\, dx = \frac{1}{5} \lim_{b \to \infty} \ln|5b+2| - \ln 7 = \ln|\infty| - \ln 7 = \infty - \ln 7 = \infty$$

The integral diverges. This means the series diverges.

4. $\sum_{k=1}^{\infty} \frac{k}{1+k^2}$

The corresponding integral is $\int_1^{\infty} \frac{x}{1+x^2}\, dx$

$$\int_1^{\infty} \frac{x}{x^2+1}\, dx = \lim_{b \to \infty} \int_1^{b} \frac{x}{x^2+1}\, dx = \frac{1}{2} \lim_{b \to \infty} \ln|x^2+1|\Big|_1^b$$

$$\int_1^{\infty} \frac{x}{x^2+1}\, dx = \frac{1}{2} \lim_{b \to \infty} \ln|b^2+1| - \ln 2 = \ln|\infty| - \ln 2 = \infty - \ln 2 = \infty$$

The integral diverges. This means the series diverges.

5. $\sum_{k=1}^{\infty} \frac{1}{1+9k^2}$

The corresponding integral is:

$$\int_1^{\infty} \frac{1}{1 + 9x^2} dx = \int_1^{\infty} \frac{1}{1 + (3x)^2} dx = \frac{1}{3} \int_1^{\infty} \frac{1}{1 + (3x)^2} 3dx$$

$$\frac{1}{3} \int_1^{\infty} \frac{1}{1 + (3x)^2} 3dx = \lim_{b \to \infty} \frac{1}{3} \int_1^{b} \frac{1}{1 + (3x)^2} d(3x) = \lim_{b \to \infty} \frac{\tan^{-1} 3x}{3} \bigg|_1^b$$

$$\int_1^{\infty} \frac{1}{1 + (3x)^2} 3dx = \lim_{b \to \infty} \frac{\tan^{-1} b}{3} - \frac{\tan^{-1} 3}{3} = \frac{\tan^{-1} \infty}{3} - \frac{\tan^{-1} 3}{3}$$

$$\int_1^{\infty} \frac{1}{1 + (3x)^2} 3dx = \frac{\pi}{6} - \frac{\tan^{-1} 3}{3}$$

The integral converges. The series converges.

6. $\sum_{k=1}^{\infty} \frac{1}{(2k+4)^{3/2}}$

The corresponding integral is:

$$\int_1^{\infty} \frac{1}{(2x+4)^{3/2}} dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{(2x+4)^{3/2}} dx = \frac{1}{2} \lim_{b \to \infty} \int_1^{b} \frac{1}{(2x+4)^{3/2}} d(2x)$$

$$\int_1^{\infty} \frac{1}{(2x+4)^{3/2}} dx = \frac{1}{2} \lim_{b \to \infty} \int_1^{b} \frac{1}{(2x+4)^{\frac{3}{2}}} d(2x)$$

$$\int_1^{\infty} \frac{1}{(2x+4)^{3/2}} dx = - \lim_{b \to \infty} \frac{1}{(2x+1)^{\frac{1}{2}}} \bigg|_1^b = - \lim_{b \to \infty} \left[\frac{1}{(2b+1)^{\frac{1}{2}}} - \frac{1}{(3)^{\frac{1}{2}}} \right]$$

$$\int_1^{\infty} \frac{1}{(2x+4)^{3/2}} dx = - \left[\frac{1}{\infty} - \frac{1}{(3)^{\frac{1}{2}}} \right] = \frac{1}{\sqrt{3}}$$

The integral converges. The series converges.

7. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+5}}$

The corresponding integral is:

$$\int_1^{\infty} \frac{1}{\sqrt{x+5}}\, dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{\sqrt{x+5}}\, dx$$

$$\int_1^{\infty} \frac{1}{\sqrt{x+5}}\, dx = \lim_{b \to \infty} 2\sqrt{x+5}\,\Big|_1^b = \lim_{b \to \infty} \left[2\sqrt{b+5} - 2\sqrt{6}\right]$$

$$\int_1^{\infty} \frac{1}{\sqrt{x+5}}\, dx = 2\sqrt{\infty} - 2\sqrt{6} = \infty - 2\sqrt{6}$$

The integral diverges. The series diverges.

8. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{e}}$

This is a trick question. Look at $a_k = \frac{1}{\sqrt[k]{e}}$.

If we take the limit we get $\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{1}{\sqrt[k]{e}} = 1$

It is imperative that the limit of a_k go to zero in order for the series to converge.

This limit goes to 1.

There is no need for an integral test – nor any test – the most fundamental part of convergence is already violated.

The series diverges. No tests needed.

9. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{2k-1}}$

The corresponding integral is:

$$\int_1^{\infty} \frac{1}{\sqrt[3]{2x-1}}\, dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{\sqrt[3]{2x-1}}\, dx$$

$$\int_1^{\infty} \frac{1}{\sqrt[3]{2x-1}}\, dx = \frac{1}{2} \lim_{b \to \infty} \frac{3}{2} (2x-1)^{\frac{2}{3}}\Big|_1^b = \frac{3}{4} \lim_{b \to \infty} \left[(2b-1)^{2/3} - 1 \right]$$

$$\int_1^{\infty} \frac{1}{\sqrt[3]{2x-1}}\, dx = \frac{3}{4} [\infty - 1] = \infty$$

The integral diverges. The series diverges.

10. $\sum_{k=3}^{\infty} \frac{\ln k}{k}$

The corresponding integral is:

$$\int_1^{\infty} \frac{\ln x}{x}\, dx = \lim_{b \to \infty} \int_1^{b} \frac{\ln x}{x}\, dx = \lim_{b \to \infty} \frac{\ln^2 x}{2}\Big|_1^b$$

$$\int_1^{\infty} \frac{\ln x}{x}\, dx = \lim_{b \to \infty} \frac{\ln^2 b}{2} = \infty$$

The integral diverges. The series diverges.

11. $\sum_{k=1}^{\infty} \frac{k}{\ln(k+1)}$

This is another trick question. Look at the summand: $a_k = \frac{k}{\ln(k+1)}$

If we take the limit we get $\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{k}{\ln(k+1)} = \lim_{k \to \infty} k + 1 = \infty$

We used LHospitals rule in the above limit.

In order for an series to converge is it imperative that $\lim_{k \to \infty} a_k = 0$

This does not happen. So the series must diverge because it has broken the most fundamental rule.

There is no need for tests – no integral test or any other test.

The series diverges.

12. $\sum_{k=1}^{\infty} k\ e^{-k^2}$

The corresponding integral is:

$$\int_1^{\infty} x\,e^{-x^2}\ dx = \lim_{b\to\infty} \int_1^{b} x\,e^{-x^2}\ dx$$

$$\int_1^{\infty} x\,e^{-x^2}\ dx = -\lim_{b\to\infty} \frac{1}{2}\,e^{-x^2}\Big|_1^b = \frac{1}{2}\lim_{b\to\infty}\left[\ e^{-1} - e^{-b^2}\right]$$

$$\int_1^{\infty} x\,e^{-x^2}\ dx = \frac{1}{2}\,[e^{-1} - \ 0] = \frac{2}{e}$$

The integral converges. The series converges.

13. $\sum_{k=1}^{\infty} \frac{1}{(k+1)\ \ln^2(k+1)}$

The corresponding integral is $\int_1^{\infty} \frac{1}{(x+1)\ \ln^2(x+1)}\ dx = \int_2^{\infty} \frac{\ln^{-2} u}{u}\ du$

Where we made a u substitution u = x+1

$$\int_1^{\infty} \frac{1}{(x+1)\ \ln^2(x+1)}\ dx = \int_2^{\infty} \frac{\ln^{-2} u}{u}\ du = \lim_{b\to\infty} \int_2^{b} \frac{\ln^{-2} u}{u}\ du$$

$$\lim_{b\to\infty} \int_2^{b} \frac{\ln^{-2} u}{u}\ du = \lim_{b\to\infty} -\frac{1}{\ln u}\Big|_2^b = \lim_{b\to\infty} -\frac{1}{\ln b} + \frac{1}{\ln 2}$$

$$\lim_{b\to\infty} \int_2^{b} \frac{\ln^{2} u}{u}\ du = -\frac{1}{\ln\ \infty} + \frac{1}{\ln 2} = -\frac{1}{\infty} + \frac{1}{\ln 2}$$

$$\lim_{b\to\infty} \int_2^{b} \frac{\ln^{2} u}{u}\ du = 0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}$$

The integral converges and so does the series.

14. $\sum_{k=1}^{\infty} \frac{k^2+1}{k^2+3}$

This is a trick question. Look at $a_k = \frac{k^2+1}{k^2+3}$.

Take the limit: $\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{k^2+1}{k^2+3} = 1$

In order for any series to converge, this limit MUST equal zero.

This is a mandatory statement. This fundamental aspect is violated.

The series automatically does not converge. There is no need for integral test. There is no need for any test. The most fundamental behavior for a_k has been violated at the very beginning.

15. $\sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^k$

This is another trick question. Look at $a_k = \left(1 + \frac{1}{k}\right)^k$.

Take the limit: $\lim_{k \to \infty} a_k = \lim_{k \to \infty} \left(1 + \frac{1}{k}\right)^k = e$

We studied this very limit when we did LHospitals rule.

In order for any series to converge, this limit MUST equal zero.

The limit equals e!!

The series automatically does not converge. There is no need for integral test. There is no need for any test. The most fundamental behavior for a_k has been violated at the very beginning.

16. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}}$

The corresponding integral is $\int_1^{\infty} \frac{1}{\sqrt{x^2+1}}\, dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{\sqrt{x^2+1}}\, dx$

Let $x = \tan t \quad dx = \sec^2 t \; dt$

$$\lim_{b \to \infty} \int_1^b \frac{1}{\sqrt{x^2+1}}\, dx = \lim_{c \to \frac{\pi}{2}} \int_{\frac{\pi}{4}}^{c} \frac{\sec^2 t}{\sqrt{\tan^2 t + 1}}\, dt$$

Please note that we changed the bounds on the integral. The bounds have to be changed from x to t. The changes are made based on the equations $1 = \tan\frac{\pi}{4}$ and $\infty = \tan\frac{\pi}{2}$.

$$\lim_{b \to \infty} \int_1^b \frac{1}{\sqrt{x^2+1}}\, dx = \lim_{c \to \frac{\pi}{2}} \int_{\frac{\pi}{4}}^{c} \sec t \; dt$$

$$\lim_{b \to \infty} \int_1^b \frac{1}{\sqrt{x^2+1}}\, dx = \lim_{c \to \frac{\pi}{2}} \frac{\ln|\sec t + \tan t|}{1}\Bigg|_{\frac{\pi}{4}}^{c}$$

$$\lim_{b \to \infty} \int_1^b \frac{1}{\sqrt{x^2+1}}\, dx = \lim_{c \to \frac{\pi}{2}} \ln|\sec c + \tan c| - \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right|$$

$$\lim_{b \to \infty} \int_1^b \frac{1}{\sqrt{x^2+1}}\, dx = \ln|\infty + \infty| - \ln\left|\sqrt{2} + 1\right| = \infty$$

The integral diverges. The series diverges.

17. $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{k^2+1}$

The corresponding integral is $\int_1^{\infty} \frac{\tan^{-1} x}{x^2+1}\, dx = \lim_{b \to \infty} \int_1^b \frac{\tan^{-1} x}{x^2+1}\, dx$

We make a u substitution: $u = \tan^{-1} x \qquad du = \frac{1}{x^2+1}\, dx$

$$\lim_{b \to \infty} \int_1^b \frac{\tan^{-1} x}{x^2+1}\, dx = \lim_{c \to \pi/2} \int_{\pi/4}^{c} u\; du$$

$$\lim_{b \to \infty} \int_1^b \frac{\tan^{-1} x}{x^2+1}\, dx = \lim_{c \to \frac{\pi}{2}} \frac{u^2}{2}\Bigg|_{\frac{\pi}{4}}^{c}$$

$$\int_1^{\infty} \frac{\tan^{-1} x}{x^2+1}\, dx = \lim_{c \to \pi/2} \frac{c^2}{2} - \frac{\pi^2}{32} = \frac{\pi^2}{8} - \frac{\pi^2}{32} = \frac{3}{32}\pi^2$$

The integral converges so the series converges.