# ALTERNATING SERIES TEST ABSOLUTE AND CONDITIONAL CONVERGENCE

An alternating series is of the form $\sum_{k=1}^{\infty} (-1)^k a_k$ or $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ where $a_k > 0$.

It looks like this: $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots$

An alternating series will converge if two things holds true:

1. $\lim_{k \to \infty} a_k = 0$
2. $a_k \geq a_{k+1}$

This means that the sequence of $a_k$ is a monotonic decreasing sequence that approaches zero in the limit.

There are two ways to show that $a_k > a_{k+1}$. One way is work it out algebraically. The other way is to take the derivative $\frac{d\, a_k}{dk}$ and show that the derivative is always negative. Either way is acceptable.

ABSOLUTE CONVERGENCE AND CONDITIONAL CONVERGENCE

It can turn out that that the series $\sum_{k=1}^{\infty} (-1)^k a_k$ is convergent and so is the series $\sum_{k=1}^{\infty} a_k$. So even without the oscillating minus signs, the series is convergent. If $\sum_{k=1}^{\infty} a_k$ is also convergent, we say that the series in question is absolutely convergent.

Many text books spell it out like this: they set $u_k = (-1)^k a^k$. They then state that if $\sum_{k=1}^{\infty} u_k$ and $\sum_{k=1}^{\infty} |u_k|$ both converge, then the original series is absolutely convergent.

On the other hand if $\sum_{k=1}^{\infty} u_k$ is convergent but $\sum_{k=1}^{\infty} |u_k|$ is divergent, then the original series is conditionally convergent. This happens quite a lot.

**Here is a major theorem and sometimes a powerful shortcut: if a series is absolutely convergent, then it is also convergent. This means that if $\sum |u_k|$ converges, then $\sum u_k$ converges.** This theorem can save a lot of time and a lot of work.

So there are two ways to show that an alternating series converges. The first is to show that $\lim_{k \to \infty} a_k = 0$ along with $a_k \geq a_{k+1}$. This is the long way. The second way is to show that the original series is absolutely convergent. This is the short way. Either way you do it is ok.

To be absolutely clear, sometimes the short way does not work. When it does, it saves a lot of time.

## ALTERNATING SERIES TEST AND THE REMAINDER THEOREM

So what is the remainder theorem?

Sometimes we are lucky and there is a formula that predicts the exact value for a series. Most times such a formula does not exist. In these cases, we need to form a partial sum to see what the series equals, approximately.

So we cannot get an exact answer for $\sum_{k=1}^{\infty} (-1)^k a_k$ but we can always add up some numbers by computer to get an answer for $\sum_{k=1}^{n} (-1)^k a_k$ (where n is some positive number).

For the infinite series we call the exact answer S: $S = \sum_{k=1}^{\infty} (-1)^k a_k$

For the finite series we call the exact answer $s_n$: $s_n = \sum_{k=1}^{n} (-1)^k a_k$

$s_n$ is called the $n^{th}$ partial sum of the series.

The question is, how close is $s_n$ to S? What is the error? What is the remainder?

The remainder theorem tells us how close $s_n$ is to S.

---

The remainder theorem states $|S - s_n| < a_{n+1}$

---

The first "ignored" term in the partial sum tells you how close you are to the true answer.

PROOF OF THE REMAINDER THEOREM

Let $S = \sum_{k=1}^{\infty} (-1)^k a_k$ and let $s_n = \sum_{k=1}^{n} (-1)^k a_k$

Remember that for odd n, the sequence is monotonic decreasing and is bounded below. So $s_n > S$.

For even n, the sequence is monotonic increasing and is bounded above. So $s_n < S$.

If n is odd then $s_n > S > s_{n+1}$ or $0 > S - s_n > s_{n+1} - s_n$

If n is even then $s_n < S < s_{n+1}$ or $0 < S - s_n < s_{n+1} - s_n$

Both these inequalities can be combined into one statement:

$|S - s_n| < |s_{n+1} - s_n| = |a_{n+1}|$

The statement $|S - s_n| \leq |a_{n+1}|$ is the remainder theorem. We have proven it.

1. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k+1}$. Does the series converge? If it does converge, does it converge absolutely or conditionally?

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k+1}$ $\qquad a_k = \frac{1}{2k+1}$

First take the limit and see if it is zero:

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{1}{2k+1} = 0$$

Second take the derivative and see if it is always negative:

$$\frac{da_k}{dk} = \frac{d}{dk}\left(\frac{1}{2k+1}\right) = -\frac{2}{(2k+1)^2} < 0$$

Both conditions hold true. Since the derivative is negative, the sequence $a_k$ is a monotonic decreasing sequence that approaches zero. The alternating series test says that the series converges.

Test for absolute convergence:

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k+1}$ $\qquad u_k = \frac{(-1)^{k+1}}{2k+1}$ $\qquad |u_k| = \frac{1}{2k+1}$

$\sum_{k=1}^{\infty} |u_k| = \sum_{k=1}^{\infty} \frac{1}{2k+1}$ diverges by integral test.

The original series is not absolutely convergent. It is conditionally convergent.

2. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\, k}{3^k}$. Does the series converge? If it does converge, does it converge absolutely or conditionally?

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\, k}{3^k}$ $\qquad a_k = \frac{k}{3^k}$

First take the limit and see if it is zero:

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{k}{3^k} = \lim_{k \to \infty} \frac{1}{3^k \ln 3} = 0$$ (we used LHospital's rule)

Second take the derivative and see if it is always negative:

$$\frac{da_k}{dk} = \frac{d}{dk}\left(\frac{k}{3^k}\right) = \frac{3^k - k\, 3^k \ln 3}{3^{2k}} = \frac{3^k\,(1 - k \ln 3)}{3^{2k}} = \frac{(1 - k \ln 3)}{3^k} < 0$$

Both conditions hold true. Since the derivative is negative, the sequence $a_k$ is a monotonic decreasing sequence that approaches zero. The alternating series test says that the series converges.

Test for absolute convergence:

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\, k}{3^k}$ $\qquad u_k = \frac{(-1)^{k+1}\, k}{3^k}$ $\qquad |u_k| = \frac{k}{3^k}$

$\sum_{k=1}^{\infty} |u_k| = \sum_{k=1}^{\infty} \frac{k}{3^k}$ converges by ratio test.

The original series is absolutely convergent. It is not conditionally convergent.

3. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\ (k+1)}{(3k+1)}$. Does the series converge? If it does converge, does it converge absolutely or conditionally?

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\ (k+1)}{(3k+1)} \qquad a_k = \frac{(k+1)}{(3k+1)}$

First take the limit and see if it is zero:

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{(k+1)}{(3k+1)} = 3$$

The limit does not converge to zero. The series diverges.

The series not convergent. It is not absolutely convergent. It is not conditionally convergent. The series is divergent.

4. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\ (k+4)}{k^2+k}$. Does the series converge? If it does converge, does it converge absolutely or conditionally?

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\ (k+4)}{k^2+k} \qquad a_k = \frac{(k+4)}{k^2+k}$

First take the limit and see if it is zero:

$\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{(k+4)}{k^2+k} = \lim_{k \to \infty} \frac{1}{2k+1} = 0$ (we used LHospital's rule)

Second take the derivative and see if it is always negative:

$$\frac{da_k}{dk} = \frac{d}{dk}\left( \frac{(k+4)}{k^2+k} \right) = \frac{(k^2+k) - (k+4)(2k+1)}{(k^2+k)^2} = \frac{(k^2+k) - (2k^2+9k+4)}{(k^2+k)^2} = \frac{(-k^2-8k-4)}{(k^2+k)^2} < 0$$

Both conditions hold true. Since the derivative is negative, the sequence $a_k$ is a monotonic decreasing sequence that approaches zero. The alternating series test says that the series converges.

Test for absolute convergence:

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\ (k+4)}{k^2+k} \qquad u_k = \frac{(-1)^{k+1}\ (k+4)}{k^2+k} \qquad |u_k| = \frac{(k+4)}{k^2+k}$

$\sum_{k=1}^{\infty} |u_k| = \sum_{k=1}^{\infty} \frac{(k+4)}{k^2+k}$ diverges. This can be shown by integral test or comparison test or limit comparison test. If you use the comparison test, or the limit comparison test, compare it to the harmonic series with $b_k = 1/k$.

The original series is not absolutely convergent. It is conditionally convergent.

5. You are given the series $\sum_{k=1}^{\infty} (-1)^{k+1} e^{-k}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

$\sum_{k=1}^{\infty} (-1)^{k+1} e^{-k}$ $\qquad a_k = e^{-k}$

First take the limit and see if it is zero:

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} e^{-k} = \lim_{k \to \infty} \frac{1}{e^k} = 0$$

Second take the derivative and see if it is always negative:

$$\frac{da_k}{dk} = \frac{d}{dk}\left( e^{-k} \right) = -e^{-k} < 0$$

Both conditions hold true. Since the derivative is negative, the sequence $a_k$ is a monotonic decreasing sequence that approaches zero. The alternating series test says that the series converges.

Test for absolute convergence:

$\sum_{k=1}^{\infty} (-1)^{k+1} e^{-k}$ $\qquad u_k = (-1)^{k+1} e^{-k}$ $\qquad |u_k| = e^{-k}$

$\sum_{k=1}^{\infty} |u_k| = \sum_{k=1}^{\infty} e^{-k}$ is a geometric series with r = 1/e. r is in between -1 and +1. The series converges.

The original series is absolutely convergent. It is not conditionally convergent.

6. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} \ln k}{k}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} \ln k}{k}$ $\qquad a_k = \frac{\ln k}{k}$

First take the limit and see if it is zero:

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{\ln k}{k} = \lim_{k \to \infty} \frac{1}{k} = 0 \quad \text{(using LHospital's rule)}$$

Second take the derivative and see if it is always negative:

$$\frac{da_k}{dk} = \frac{d}{dk}\left( \frac{\ln k}{k} \right) = \frac{1-\ln k}{k^2} < 0 \text{ for } k > e$$

Both conditions hold true when k, an integer, is greater than or equal to 3. The series $\sum_{k=3}^{\infty} \frac{(-1)^{k+1} \ln k}{k}$ converges. The convergence of a series is not affected by the removal or the value of the first several terms. So the original series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} \ln k}{k}$ also converges.

Test for absolute convergence:

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} \ln k}{k}$ $\qquad u_k = \frac{(-1)^{k+1} \ln k}{k}$ $\qquad |u_k| = \frac{\ln k}{k}$

$\sum_{k=1}^{\infty} |u_k| = \sum_{k=1}^{\infty} \frac{\ln k}{k}$ diverges by the integral test. The original series is not absolutely convergent. It is conditionally convergent.

7. You are given the series $\sum_{k=1}^{\infty} \left(-\frac{3}{5}\right)^k$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

   This series is an alternating series. It is also a geometric series: r= -1/3. Since r is between -1 and +1, the geometric series converges. If we test for absolute convergence, then we get a geometric series with r= + 1/3. The "absolute" series will also converge. The original series is both convergent and absolutely convergent.

8. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^k\, 2^k}{k!}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

   $\sum_{k=1}^{\infty} \frac{(-1)^k\, 2^k}{k!}$ $\qquad a_k = \frac{2^k}{k!}$

   We will solve this question differently. Rather than use the alternating series test directly, which is difficult, we will use a theorem regarding absolute convergence: **if a series converges absolutely, then it converges. So if $\sum|u_k|$ converges, so does $\sum u_k$.**

   We established in a previous handout that $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{2^k}{k!} = 0$

   To repeat that argument, we let $b_k = \frac{1}{a_k} = \frac{k!}{2^k}$

   $b_k = \frac{1}{2}\,\frac{2}{2}\,\frac{3}{2}\,\frac{4}{2}\,\cdots\,\frac{k}{2}$

   Look at the product for all the fractions preceding k/2. From these products we see that for

   $k \geq 4$ we have $b_k > \frac{k}{2}$.

   So $\lim_{k\to\infty} b_k = \lim_{k\to\infty} \frac{k!}{2^k} > \lim_{k\to\infty} \frac{k}{2} = \infty$

   If $b_k \to \infty$ then $a_k = \frac{1}{b_k} \to 0$

   So now we know that $a_k$ goes to 0. We can use the ratio test to establish absolute convergence.

   $u_k = \frac{(-1)^k\, 2^k}{k!} \qquad u_{k+1} = \frac{(-1)^{k+1}\, 2^{k+1}}{(k+1)!}$

   $\lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty} \left| \frac{2^{k+1}}{(k+1)!} \cdot \frac{k}{2^k} \right| = \lim_{k\to\infty} \left|\frac{2}{k+1}\right| = 0$

   Since $\lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| < 1$ the series converges absolutely.

   Since the series converges absolutely is also convergent in its original form.

9. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^k \, 3^k}{k^2}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

$\sum_{k=1}^{\infty} \frac{(-1)^k \, 3^k}{k^2}$ $\qquad a_k = \frac{3^k}{k^2}$

Take limit of $a_k$: $\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{3^k}{k^2} = \lim_{k \to \infty} \frac{3^k \ln 3}{2k} = \lim_{k \to \infty} \frac{3^k \ln^2 3}{2} = \infty$

$a_k$ does not approach zero. The series diverges.

10. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^k \, k}{5^k}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

$\sum_{k=1}^{\infty} \frac{(-1)^k \, k}{5^k}$ $\qquad a_k = \frac{k}{5^k}$

First take the limit and see if it is zero:

$\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{k}{5^k} = \lim_{k \to \infty} \frac{1}{5^k \ln 5} = 0$ (using LHospital's rule)

Second take the derivative and see if it is always negative:

$\frac{da_k}{dk} = \frac{d}{dk}\left(\frac{k}{5^k}\right) = \frac{5^k - k\,5^k \ln 5}{5^{2k}} = \frac{5^k(1 - k \ln 5)}{5^{2k}} < 0$ for k > 0

Both conditions hold true. Since the derivative is negative, the sequence $a_k$ is a monotonic decreasing sequence that approaches zero. The alternating series test says that the series converges.

The series $\sum_{k=1}^{\infty} \frac{(-1)^k \, k}{5^k}$ converges.

Test for absolute convergence:

$\sum_{k=1}^{\infty} \frac{(-1)^k \, k}{5^k}$ $\qquad u_k = \frac{(-1)^k \, k}{5^k}$ $\qquad |u_k| = \frac{k}{5^k}$

$\sum_{k=1}^{\infty} |u_k| = \sum_{k=1}^{\infty} \frac{k}{5^k}$

We can show that this series converges by the ratio test.

$\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k \to \infty} \left|\frac{k+1}{5^{k+1}} \frac{5^k}{k}\right| = \lim_{k \to \infty} \left|\frac{1}{5} \frac{k}{k+1}\right| = \frac{1}{5}$

Since the limit is less than 1, the "absolute" series converges.

The original series is absolutely convergent. It is not conditionally convergent.

11. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^k k^3}{e^k}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

$\sum_{k=1}^{\infty} \frac{(-1)^k k^3}{e^k} \qquad a_k = \frac{k^3}{e^k}$

First take the limit and see if it is zero:

$\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{k^3}{e^k} = \lim_{k \to \infty} \frac{6}{e^k} = 0$ (using LHospital's rule three times!)

Second take the derivative and see if it is always negative:

$\frac{da_k}{dk} = \frac{d}{dk}\left(\frac{k^3}{e^k}\right) = \frac{e^k - k^3 e^k}{e^{2k}} = \frac{e^k(1 - k^3)}{e^{2k}} < 0$ for k > 0

Both conditions hold true. Since the derivative is negative, the sequence $a_k$ is a monotonic decreasing sequence that approaches zero. The alternating series test says that the series converges.

The series $\sum_{k=1}^{\infty} \frac{(-1)^k k^3}{e^k}$ converges.

Test for absolute convergence:

$\sum_{k=1}^{\infty} \frac{(-1)^k k^3}{e^k} \qquad u_k = \frac{(-1)^k k^3}{e^k} \qquad |u_k| = \frac{k^3}{e^k}$

$\sum_{k=1}^{\infty} |u_k| = \sum_{k=1}^{\infty} \frac{k^3}{e^k}$

We can show that this series converges by the ratio test.

$\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k \to \infty} \left|\frac{(k+1)^3}{e^{k+1}} \frac{e^k}{k^3}\right| = \lim_{k \to \infty} \left|\frac{1}{e}\left(\frac{k}{k+1}\right)^3\right| = \frac{1}{e}$

Since the limit is less than 1, the "absolute" series converges.

The original series is absolutely convergent. It is not conditionally convergent.

12. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k^k}{k!}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k^k}{k!} \qquad a_k = \frac{k^k}{k!}$

First take the limit and see if it is zero:

Let's write out $a_k$:

$a_k = \frac{k}{1}\frac{k}{2}\frac{k}{3} \cdots \frac{k}{k} = \frac{k}{1}\frac{k}{2}\frac{k}{3} \cdots 1$

We see that $a_k > 1$ for k > 1

The limit of $a_k$ cannot be zero. The series does not converge.

13. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{3k}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{3k} \qquad a_k = \frac{1}{3k}$

First take the limit and see if it is zero:

$\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{1}{3k} = 0$

Second take the derivative and see if it is always negative:

$\frac{da_k}{dk} = \frac{d}{dk}\left(\frac{1}{3k}\right) = -\frac{1}{3k^2} < 0$ for k > 0

Both conditions hold true. Since the derivative is negative, the sequence $a_k$ is a monotonic decreasing sequence that approaches zero. The alternating series test says that the series converges.

The series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{3k}$ converges.

Test for absolute convergence:

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{3k} \qquad u_k = \frac{(-1)^{k+1}}{3k} \qquad |u_k| = \frac{1}{3k}$

$\sum_{k=1}^{\infty} |u_k| = \sum_{k=1}^{\infty} \frac{1}{3k}$

This series is a harmonic series. It diverges by the p test or the integral test.

The original series is not absolutely convergent. It is conditionally convergent.

14. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{4/3}}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

In this question, we are going to use the theorem that **if a series converges absolutely, then it converges.**

Test for absolute convergence:

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{4/3}} \qquad u_k = \frac{(-1)^{k+1}}{k^{4/3}} \qquad |u_k| = \frac{1}{k^{4/3}}$

$\sum_{k=1}^{\infty} |u_k| = \sum_{k=1}^{\infty} \frac{1}{k^{4/3}}$

This series converges by the p test. So the series is absolutely convergent. Therefore the original series is also convergent.

15. You are given the series $\sum_{k=1}^{\infty} \frac{(-4)^k}{k^2}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

$\sum_{k=1}^{\infty} \frac{(-4)^k}{k^2} \qquad a_k = \frac{4^k}{k^2}$

First take the limit and see if it is zero:

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{4^k}{k^2} = \lim_{k \to \infty} \frac{4^k \ln^2 4}{2} = \infty \quad \text{(using LHospital's rule twice)}$$

Game over – $a_k$ does not approach zero. The series does not converge.

16. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

We will use the theorem that **if a series converges absolutely, then it converges**.

$u_k = \frac{(-1)^{k+1}}{k!} \qquad u_{k+1} = \frac{(-1)^{k+2}}{(k+1)!}$

We can show that this series converges absolutely by the ratio test.

$$\lim_{k \to \infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k \to \infty} \left|\left(\frac{k!}{(k+1)!}\right)\right| = \lim_{k \to \infty} \left|\frac{1}{k+1}\right| = 0$$

Since the limit is less than 1, the "absolute" series converges.

The original series is therefore also convergent.

17. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} \ln k}{k}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} \ln k}{k} \qquad a_k = \frac{\ln k}{k}$

Does $a_k \to 0$ ?

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{\ln k}{k} = \lim_{k \to \infty} \frac{1}{k} = 0$$

Does $a_k$ form a decreasing sequence?

$$\frac{d}{dk}\left(\frac{\ln k}{k}\right) = \frac{1-\ln^2 k}{k^2} < 0 \quad \text{for integer k > 2}$$

So the alternating series $\sum_{k=3}^{\infty} \frac{(-1)^{k+1} \ln k}{k}$ converges. Since the inclusion, exclusion or the values of the first several terms does not affect convergence, we have $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} \ln k}{k}$ also converges.

Test for absolute convergence:

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} \ln k}{k} \qquad u_k = \frac{(-1)^{k+1} \ln k}{k}$

The absolute series is $\sum_{k=1}^{\infty} |u_k| = \sum_{k=1}^{\infty} \frac{\ln k}{k}$

This series diverges by the integral test.

You can also show divergence by the comparison test or the limit comparison test where $b_k$ = 1/k for both tests.

The series converges but is not absolutely convergent. It is conditionally convergent.

18. You are given the series $\sum_{k=1}^{\infty} \frac{\cos \pi k}{k}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

It helps to remember that $\cos \pi k = (-1)^k$

$\sum_{k=1}^{\infty} \frac{\cos \pi k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \qquad a_k = \frac{1}{k}$

First take the limit and see if it is zero:
$\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{1}{k} = 0$

Second take the derivative and see if it is always negative:

$\frac{da_k}{dk} = \frac{d}{dk}\left(\frac{1}{k}\right) = -\frac{1}{k^2} < 0$ for k > 0

Both conditions hold true. The series $\sum_{k=1}^{\infty} \frac{\cos \pi k}{k}$ converges.

Test for absolute convergence:

$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \qquad u_k = \frac{(-1)^k}{k} \qquad |u_k| = \frac{1}{k}$

$\sum_{k=1}^{\infty} |u_k| = \sum_{k=1}^{\infty} \frac{1}{k}$

This series is a harmonic series and diverges by the p test or the integral test.

The original series is not absolutely convergent. It is conditionally convergent.

19. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2+1}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

We will make use of the theorem: if a series is absolutely convergent, then it is convergent.

$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2+1} \qquad u_k = \frac{(-1)^k}{k^2+1}$

$\sum_{k=1}^{\infty} |u_k| = \sum_{k=1}^{\infty} \frac{1}{k^2+1}$

This series converges by the integral test. It also converges by the comparison test where $b_k = \frac{1}{k^2}$.

The series is absolutely convergent. Therefore the original series is also convergent.

20. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^k\ (k+2)}{k\ (k+3)}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

We need to test to see if $a_k$ goes to zero and if $a_k$ forms a monotonic decreasing sequence.

$\sum_{k=1}^{\infty} \frac{(-1)^k\ (k+2)}{k\ (k+3)} \qquad a_k = \frac{(k+2)}{k\ (k+3)}$

Take limit of $a_k$

$\lim_{k \to \infty} \frac{(k+2)}{k\ (k+3)} = \lim_{k \to \infty} \frac{(k+2)}{(k^2+3k)} = \lim_{k \to \infty} \frac{1}{2k+3} = 0$ (we used L'hospital's rule)

Take the derivative of $a_k$:

$\frac{d}{dk}\left(\frac{(k+2)}{(k^2+3k)}\right) = \frac{k^2+3k-\ (k+2)\,(2k+3)}{(k^2+3k)^2}$

$\frac{d}{dk}\left(\frac{(k+2)}{(k^2+3k)}\right) = \frac{k^2+3k-\ \left(2k^2+6k+6\right)}{(k^2+3k)^2} = \frac{-k^2-3k}{(k^2+3k)^2} < 0$ for $k \geq 1$

So $a_k$ is a monotonic decreasing sequence that approaches zero. The alternating series test says the series converges.

Test for absolute convergence:

$\sum_{k=1}^{\infty} \frac{(-1)^k\ (k+2)}{k\ (k+3)} \qquad u_k = \frac{(-1)^k\ (k+2)}{k\ (k+3)}$

$\sum_{k=1}^{\infty} |u_k| = \sum_{k=1}^{\infty} \frac{(k+2)}{k\ (k+3)}$

This series diverges by the limit comparison test. Use the series where $b_k = 1/k$.

So the series is not absolutely convergent. The series is conditionally convergent.

21. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^k\ k^2}{k^3+1}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

We need to test to see if $a_k$ goes to zero and if $a_k$ forms a monotonic decreasing sequence.

$\sum_{k=1}^{\infty} \frac{(-1)^k\ k^2}{k^3+1} \qquad a_k = \frac{k^2}{k^3+1}$

Take limit of $a_k$

$\lim_{k \to \infty} \frac{k^2}{k^3+1} = \lim_{k \to \infty} \frac{2k}{3k^2} = \lim_{k \to \infty} \frac{2}{3k} = 0$ (we used L'hospital's rule)

Take the derivative of $a_k$:

$\frac{d}{dk}\left( \frac{k^2}{k^3+1} \right) = \frac{2k\,(k^3+1) - k^2\,(3k^2)}{(k^3+1)^2} = \frac{-k^4 + 2k}{(k^3+1)^2} < 0$ for $k \geq 2$

So $a_k$ is a monotonic decreasing sequence that approaches zero. The alternating series test says the series converges.

Test for absolute convergence:

$\sum_{k=1}^{\infty} \frac{(-1)^k\ k^2}{k^3+1} \qquad u_k = \frac{(-1)^k\ k^2}{k^3+1}$

$\sum_{k=1}^{\infty} |u_k| = \sum_{k=1}^{\infty} \frac{k^2}{k^3+1}$

This series diverges by the integral test. You could also show divergence using either comparison test using $b_k = 1/k$.

So the series is not absolutely convergent. The series is conditionally convergent.

22. You are given the series $\sum_{k=1}^{\infty} \sin\frac{k\pi}{2}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

$\sum_{k=1}^{\infty} \sin\frac{k\pi}{2} \qquad u_k = \sin\frac{k\pi}{2}$

The sequence for $u_k$ oscillates: 1,0,-1,0,1,0,-1,0…

The limit does not approach zero: $\lim_{k \to \infty} u_k = \lim_{k \to \infty} \sin\frac{k\pi}{2}$ does not converge to anything.

The limit does not exist. So the series does not converge.

23. You are given the series $\sum_{k=1}^{\infty} \frac{\sin k}{k^3}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

We need to test to see if $a_k$ goes to zero and if $a_k$ forms a monotonic decreasing sequence.

$\sum_{k=1}^{\infty} \frac{\sin k}{k^3}$ $\qquad a_k = \frac{\sin k}{k^3}$

Technically this series is out of our domain. We are supposed to be dealing with series with only positive values or strictly alternating series. This series is neither. Sin(k) oscillates but is not strictly alternating. But we can still analyze the series. We will make use of the theorem that if a series converges absolutely, then it converges.

Look at the series $\sum_{k=1}^{\infty} \frac{|\sin k|}{k^3}$

Since sin k is bounded between -1 and 1, |sin k| is bounded between 0 and 1.

We have the inequality: $0 < \sum_{k=1}^{\infty} \frac{|\sin k|}{k^3} < \sum_{k=1}^{\infty} \frac{1}{k^3}$

The series on the right converges by the p test. The series $\sum_{k=1}^{\infty} \frac{|\sin k|}{k^3}$ converges by the comparison test.

The given series $\sum_{k=1}^{\infty} \frac{\sin k}{k^3}$ converges since it converges absolutely.

24. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k \ln k}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

We need to test to see if $a_k$ goes to zero and if $a_k$ forms a monotonic decreasing sequence.

$\sum_{k=1}^{\infty} \frac{(-1)^k}{k \ln k}$ $\qquad a_k = \frac{1}{k \ln k}$

Take limit of $a_k$

$\lim_{k \to \infty} \frac{1}{k \ln k} = 0$

Take the derivative of $a_k$:

$\frac{d}{dk}\left(\frac{1}{k \ln k}\right) = -\frac{1+\ln k}{(k \ln k)^2} < 0 \quad \text{for } k \geq 1$

So $a_k$ is a monotonic decreasing sequence that approaches zero. The alternating series test says the series converges.

$\sum_{k=1}^{\infty} \frac{(-1)^k}{k \ln k}$ converges

Test for absolute convergence:

$\sum_{k=1}^{\infty} \frac{(-1)^k}{k \ln k}$ $\quad u_k = \frac{(-1)^k}{k \ln k}$

$\sum_{k=1}^{\infty} |u_k| = \sum_{k=1}^{\infty} \frac{1}{k \ln k}$

This series diverges by the integral test.

So the series is not absolutely convergent.

The series is conditionally convergent.

25. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k\,(k+1)}}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

We need to test to see if $a_k$ goes to zero and if $a_k$ forms a monotonic decreasing sequence.

$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k\,(k+1)}}$ $\quad a_k = \frac{1}{\sqrt{k\,(k+1)}}$

Take limit of $a_k$

$\lim_{k \to \infty} \frac{1}{\sqrt{k\,(k+1)}} = 0$

Take the derivative of $a_k$:

$\frac{d}{dk}\left(\frac{1}{\sqrt{k\,(k+1)}}\right) = -\frac{2k+1}{(k^2+k)^{3/2}} < 0$ for $k \geq 1$

So $a_k$ is a monotonic decreasing sequence that approaches zero. The alternating series test says the series converges.

$\sum_{k=1}^{\infty} \frac{(-1)^k}{k \ln k}$ converges

Test for absolute convergence:

$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k\,(k+1)}}$ $\quad u_k = \frac{(-1)^k}{\sqrt{k\,(k+1)}}$

$\sum_{k=1}^{\infty} |u_k| = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k\,(k+1)}}$

This series diverges by the limit comparison test. Use $b_k = 1/k$.

So the series is not absolutely convergent.

The series is conditionally convergent.

26. You are given the series $\sum_{k=2}^{\infty} \; (-1)^k \, \frac{1}{(\ln k)^k}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

We need to test to see if $a_k$ goes to zero and if $a_k$ forms a monotonic decreasing sequence.

$$\sum_{k=2}^{\infty} \; (-1)^k \, \frac{1}{(\ln k)^k} \qquad a_k = \frac{1}{(\ln k)^k}$$

Take limit of $a_k$

$$\lim_{k \to \infty} \; \frac{1}{(\ln k)^k} \; = \; 0$$

Take the derivative of $a_k$:

$$\frac{d}{dk}\left(\frac{1}{(\ln k)^k}\right) = -\frac{k}{k \; (\ln k)^{k+1}} = -\frac{1}{(\ln k)^{k+1}} \; < \; 0 \quad \text{for } k \geq \; 2$$

So $a_k$ is a monotonic decreasing sequence that approaches zero. The alternating series test says the series converges.

$$\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln k} \quad \text{converges}$$

Test for absolute convergence:

$$\sum_{k=2}^{\infty} \; (-1)^k \, \frac{1}{(\ln k)^k} \qquad u_k = \frac{1}{(\ln k)^k}$$

$$\sum_{k\,=\,2}^{\infty} |u_k| \; = \; \sum_{k\,=\,2}^{\infty} \; \frac{1}{(\ln k)^k}$$

This series converges by the Cauchy nth root test. You could also prove convergence by the comparison test to a geometric series. Choose r = ½ .

So the series is absolutely convergent.

27. You are given the series $\sum_{k=2}^{\infty} \ (-1)^k \ \frac{1}{\sqrt{k+1}\ +\ \sqrt{k}}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

We need to test to see if $a_k$ goes to zero and if $a_k$ forms a monotonic decreasing sequence.

$\sum_{k=2}^{\infty} \ (-1)^k \ \frac{1}{\sqrt{k+1}\ +\ \sqrt{k}} \qquad a_k = \frac{1}{\sqrt{k+1}\ +\ \sqrt{k}}$

Take limit of $a_k$

$\lim_{k \to \infty} \ \frac{1}{\sqrt{k+1}\ +\ \sqrt{k}} \ = \ 0$

Take the derivative of $a_k$:

$\frac{d}{dk}\left(\frac{1}{\sqrt{k+1}\ +\ \sqrt{k}}\right) = -\left(\frac{1}{\sqrt{k+1}\ +\ \sqrt{k}}\right)^2\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{k+1}} + \frac{1}{\sqrt{k}}\right) \ < \ 0 \quad \text{for } k \geq \ 1$

So $a_k$ is a monotonic decreasing sequence that approaches zero. The alternating series test says the series converges.

$\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln k}$ converges

Test for absolute convergence:

$\sum_{k=2}^{\infty} \ (-1)^k \ \frac{1}{(\ln k)^k} \qquad u_k = \frac{1}{(\ln k)^k}$

$\sum_{k\ =\ 2}^{\infty} |u_k| \ = \ \sum_{k\ =\ 2}^{\infty} \ \frac{1}{(\ln k)^k}$

This series converges by the Cauchy nth root test. You could also prove convergence by the comparison test to a geometric series. Choose r = ½ .

So the series is absolutely convergent.

28. You are given the series $\sum_{k=2}^{\infty} \ (-1)^k \ \frac{(k^2\ +\ 1)}{k^3+2}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

We need to test to see if $a_k$ goes to zero and if $a_k$ forms a monotonic decreasing sequence.

$$\sum_{k=2}^{\infty} \ (-1)^k \ \frac{(k^2\ +\ 1)}{k^3+2} \qquad a_k \ = \ \frac{(k^2\ +\ 1)}{k^3+2}$$

Take limit of $a_k$

$$\lim_{k \to \infty} \frac{(k^2\ +\ 1)}{k^3+2} \ = \ \lim_{k \to \infty} \frac{2k}{3\,k^2} \ = \ \lim_{k \to \infty} \frac{2}{3\,k} \ = \ 0 \quad \text{(we used LHospital's rule)}$$

Take the derivative of $a_k$:

$$\frac{d}{dk}\left(\frac{(k^2\ +\ 1)}{k^3+2}\right) \ = \ \frac{2k\,(k^3+\ 2) -\ 3k^2\,(k^2+1)}{(k^3+2)^2} \ = \ \frac{-\,k^4\ -\ 3k^2+\ 4k}{(k^3+2)^2} \ = \ -\ \frac{k^4\ +\ 3k^2-\ 4k}{(k^3+2)^2} \ < \ 0$$

for $k \geq \ 1$

So $a_k$ is a monotonic decreasing sequence that approaches zero. The alternating series test says the series converges.

$\sum_{k=2}^{\infty} (-1)^k \ \frac{(k^2\ +\ 1)}{k^3+2}$ converges

Test for absolute convergence:

$$\sum_{k=2}^{\infty} (-1)^k \ \frac{(k^2\ +\ 1)}{k^3+2} \qquad u_k \ = \ \frac{(k^2\ +\ 1)}{k^3+2}$$

$$\sum_{k\,=\,2}^{\infty} |u_k| \ = \ \sum_{k\,=\,2}^{\infty} \ \frac{(k^2\ +\ 1)}{k^3+2}$$

This series diverges by the by the limit comparison test. Use $b_k$ = 1/k.

So the series is conditionally convergent.

29. You are given the series $\sum_{k=1}^{\infty} \frac{k \cos k\pi}{k^2+1}$ . Does the series converge? If it does converge, does it converge absolutely or conditionally?

We need to test to see if $a_k$ goes to zero and if $a_k$ forms a monotonic decreasing sequence.

$$\sum_{k=1}^{\infty} \frac{k \cos k\pi}{k^2+1} = \sum_{k=1}^{\infty} \frac{(-1)^k k}{k^2+1} \qquad a_k = \frac{k}{k^2+1}$$

Take limit of $a_k$

$$\lim_{k \to \infty} \frac{k}{k^2+1} = \lim_{k \to \infty} \frac{1}{2k} = 0$$ (we used LHospital's rule)

Take the derivative of $a_k$:

$$\frac{d}{dk}\left(\frac{k}{k^2+1}\right) = \frac{(k^2+1) - k\,(2k)}{(k^2+1)^2} = \frac{-k^2+1}{(k^2+1)^2} = < 0 \quad \text{for } k \geq 2$$

So $a_k$ is a monotonic decreasing sequence that approaches zero. The alternating series test says the series converges.

$$\sum_{k=1}^{\infty} \frac{k \cos k\pi}{k^2+1} \text{ converges}$$

Test for absolute convergence:

$$\sum_{k=1}^{\infty} \frac{k \cos k\pi}{k^2+1} \qquad u_k = \frac{k \cos k\pi}{k^2+1}$$

$$\sum_{k=2}^{\infty} |u_k| = \sum_{k=2}^{\infty} \frac{k}{k^2+1}$$

This series diverges by the by the integral test. You could also use the limit comparison test with $b_k = 1/k$.

So the series is conditionally convergent.

30. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ . The series converges by the alternating series test.

You add up the first seven terms.

How close are you to the true answer? What is the error? What is the remainder?

The error in the answer is $|a_8|$: $|a_8| = \frac{1}{8} = 0.125$

Technically the error is less than this, but this is considered the upper bound for the error.

31. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}$. This alternating series is convergent – it is absolutely convergent. You add up the first 5 terms. How close are you to the true answer? What is the error? What is the remainder?

The remainder is $|a_6|$: $|a_6| = \frac{1}{6!} = \frac{1}{720} = 0.0013888\ldots$

Technically the error is less than this, but this is considered the upper bound for the error.

32. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$. This alternating series is convergent – it is absolutely convergent. You add up the first 99 terms. How close are you to the true answer? What is the error? What is the remainder?

The remainder is $|a_{100}|$: $|a_6| = \frac{1}{\sqrt{100}} = \frac{1}{10} = 0.1$

Technically the error is less than this, but this is considered the upper bound for the error.

33. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k+1)\ \ln(k+1)}$. This alternating series is convergent – it is absolutely convergent. You add up the first 3 terms. How close are you to the true answer? What is the error? What is the remainder?

The remainder is $|a_4|$: $|a_4| = \frac{1}{5\ \ln 5} = 0.1243$

Technically the error is less than this, but this is considered the upper bound for the error.

34. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k+1)\ \ln(k+1)}$. This alternating series is convergent – it is absolutely convergent. You add up the first 3 terms. How close are you to the true answer? What is the error? What is the remainder?

The remainder is $|a_4|$: $|a_4| = \frac{1}{5\ \ln 5} = 0.1243$

Technically the error is less than this, but this is considered the upper bound for the error.

35. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$. This alternating series is convergent – it is absolutely convergent. You add up the first n terms. You want the error to be less than 0.0001. How many terms do you need to add?

Set the error equal to $|a_{n+1}|$

$error = |a_{n+1}| = \frac{1}{n+1}$

$0.0001 = \frac{1}{n+1}$ $\qquad n+1 = 10{,}000$

So you need to add up the first 9,999 terms. $\sum_{k=1}^{9999} \frac{(-1)^{k+1}}{k}$

36. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}$. This alternating series is convergent – it is absolutely convergent. You add up the first n terms. You want the error to be less than 0.0001. How many terms do you need to add?

Set the error equal to $|a_{n+1}|$

$error = |a_{n+1}| = \frac{1}{(n+1)!}$

$0.0001 = \frac{1}{(n+1)!}$ $\qquad (n+1)! = 10{,}000$

7! = 5040 is too small.

8! = 40,320 is ok

So n+1 = 8. You ignore the 8th term onward. You add up the first 7 terms. $\sum_{k=1}^{7} \frac{(-1)^{k+1}}{k!}$

37. You are given the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$. This alternating series is convergent – it is absolutely convergent. You add up the first n terms. You want the error to be less than 0.005. How many terms do you need to add?

Set the error equal to $|a_{n+1}|$

$error = |a_{n+1}| = \frac{1}{\sqrt{n+1}}$

$0.005 = \frac{1}{\sqrt{n+1}}$ $\qquad \sqrt{n+1} = 200 \qquad n+1 = 40{,}000$

So n+1 = 40,000. You ignore the term 40,000 onward. You add up the first 39,999 terms.

$\sum_{k=1}^{39999} \frac{(-1)^{k+1}}{\sqrt{k}}$