IMPROPER INTEGRALS INFINITE BOUNDS

In the previous handout, we discussed improper integrals where the integrand had a discontinuity. The discontinuity was a vertical asymptote, somewhere in the interval of integration. We saw that sometimes these integrals converged to a number. Other times they diverged to positive or negative infinity. Those were the only two possibilities.

Another form of improper integral is one where the function is continuous, but the limits of integration are infinite. So integrals of the form $\int_a^\infty f(x)\,dx$, $\int_{-\infty}^b f(x)\,dx$ and $\int_{-\infty}^\infty f(x)\,dx$ are all improper integrals. We are going to discuss these integrals in this handout.

You might ask yourself what this second type of integral has to do with the first? Why do call both types integrals improper? They seem totally unrelated. In fact there is a relationship. If these new, improper integrals converge, then f(x) must approach the x axis as a horizontal asymptote. If f(x) does not approach the x axis very quickly, these new improper integrals will diverge. So both types of improper integrals have asymptotes. The first kind have vertical asymptotes. The new kind have horizontal asymptotes (if they converge). For these new integrals, if f(x) does not approach the x axis quickly, if the x axis is NOT a horizontal asymptote, then these integrals will diverge like mad!

As with the original improper integrals, we say that these new types either converge or diverge.

This new type of integral can converge to a number. They can diverge to positive infinity. They can diverge to negative infinity. But they have a new behavior that did not exist previously: these integrals can “oscillate” between two values and never settle down to a single value. We call this divergence by oscillation. An example will make this clear. For now just accept that this new situation exists. When you see how this oscillation works, it will make sense.

We need some rigorous definitions so that we have rules to fall back on:

$$\int_a^{\infty} f(x)\,dx \equiv \lim_{b \to \infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^{b} f(x)\,dx \equiv \lim_{a \to -\infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a \to -\infty} \int_a^c f(x)\,dx + \lim_{b \to \infty} \int_c^b f(x)\,dx$$

$$where\ c\ is\ any\ real\ number$$

These are the definitions of the three types of infinite integrals. When there are any doubts, go back to these definitions. In all these integrals, f(x) is continuous.

As with the first type of improper integral, limits are used to define approaching the bound. In the first type of integral, we approached a vertical asymptote. In this type of integral, we approach infinity.

1. Evaluate $\int_0^{\infty} e^{-x}\,dx$

$$\int_0^{\infty} e^{-x}\,dx = \lim_{b \to \infty} \int_0^b e^{-x}\,dx$$

$$\int_0^{\infty} e^{-x}\,dx = \lim_{b \to \infty} -e^{-x}\big|_0^b$$

$$\int_0^{\infty} e^{-x}\,dx = \lim_{b \to \infty} 1 - e^{-b} = 1 - 0 = 1$$

$$\int_0^{\infty} e^{-x}\,dx = 1$$

2. Evaluate $\int_1^{\infty} \frac{1}{x^2}\, dx$

$$\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{x^2}\, dx$$

$$\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{b \to \infty} -\frac{1}{x}\Big|_1^b$$

$$\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{b \to \infty} \frac{1}{1} - \frac{1}{b} = 1 - 0 = 1$$

$$\int_1^{\infty} \frac{1}{x^2}\, dx = 1$$

3. Evaluate $\int_0^{\infty} \frac{1}{1+\, x^2}\, dx$

$$\int_0^{\infty} \frac{1}{1 + x^2}\, dx = \lim_{b \to \infty} \int_0^{b} \frac{1}{1 + x^2}\, dx$$

$$\int_0^{\infty} \frac{1}{1 + x^2}\, dx = \lim_{b \to \infty} \tan^{-1} x|_0^b$$

$$\int_0^{\infty} \frac{1}{1 + x^2}\, dx = \lim_{b \to \infty} \tan^{-1} b - \tan^{-1} 0$$

$$\int_0^{\infty} \frac{1}{1 + x^2}\, dx = \lim_{b \to \infty} \tan^{-1} b - 0 = \lim_{b \to \infty} \tan^{-1} b$$

$$\int_0^{\infty} \frac{1}{1 + x^2}\, dx = \frac{\pi}{2}$$

4. Evaluate $\int_3^{\infty} \frac{1}{(x-2)^{3/2}}\ dx$

$$\int_3^{\infty} \frac{1}{(x-2)^{3/2}}\ dx = \lim_{b \to \infty} \int_3^{b} \frac{1}{(x-2)^{3/2}}\ dx$$

$$\int_3^{\infty} \frac{1}{(x-2)^{3/2}}\ dx = \lim_{b \to \infty} \left. -2\frac{1}{(x-2)^{\frac{1}{2}}} \right|_3^b$$

$$\int_3^{\infty} \frac{1}{(x-2)^{3/2}}\ dx = \lim_{b \to \infty} 2\frac{1}{(3-2)^{\frac{1}{2}}} - 2\frac{1}{(b-2)^{\frac{1}{2}}}$$

$$\int_3^{\infty} \frac{1}{(x-2)^{3/2}}\ dx = \lim_{b \to \infty} 2 - 2\frac{1}{(b-2)^{\frac{1}{2}}} = 2 - 0 = 2$$

$$\int_3^{\infty} \frac{1}{(x-2)^{3/2}}\ dx = 2$$

5. $\int_1^{\infty} \frac{1}{x^{1.001}}\ dx$

$$\int_1^{\infty} \frac{1}{x^{1.001}}\ dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{x^{1.001}}\ dx$$

$$\int_1^{\infty} \frac{1}{x^{1.001}}\ dx = \lim_{b \to \infty} -\frac{1}{0.001}\ \frac{1}{x^{0.001}}\bigg|_1^b$$

$$\int_1^{\infty} \frac{1}{x^{1.001}}\ dx = \lim_{b \to \infty} \frac{-1000}{x^{0.001}}\bigg|_1^b$$

$$\int_1^{\infty} \frac{1}{x^{1.001}}\ dx = \lim_{b \to \infty} \frac{1000}{x^{0.001}}\bigg|_b^1$$

$$\int_1^{\infty} \frac{1}{x^{1.001}}\ dx = \lim_{b \to \infty} 1000 - \frac{1000}{b^{0.001}}$$

$$\int_1^{\infty} \frac{1}{x^{1.001}}\ dx = 1000$$

6. $\int_1^{\infty} \frac{1}{x^{0.999}}\ dx$

$$\int_1^{\infty} \frac{1}{x^{0.999}}\ dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{x^{0.999}}\ dx$$

$$\int_1^{\infty} \frac{1}{x^{0.999}}\ dx = \lim_{b \to \infty} \frac{x^{0.001}}{0.001}\bigg|_1^b$$

$$\int_1^{\infty} \frac{1}{x^{0.999}}\ dx = \lim_{b \to \infty} 1000\ b^{0.001} - 1000$$

$$\int_1^{\infty} \frac{1}{x^{0.999}}\ dx = \infty$$

The integral $\int_1^{\infty} \frac{1}{x^{0.999}}\ dx$ diverges to positive infinity.

7. $\int_0^{\infty} e^{-x}\cos x \;\; dx$

$$\int_0^{\infty} e^{-x}\cos x \;\; dx = \lim_{b \to \infty} \int_0^{b} e^{-x}\cos x \;\; dx$$

Use integration by parts:

$$\int_0^{\infty} e^{-x}\cos x \;\; dx = \lim_{b \to \infty} e^{-x}\sin x|_0^b + \lim_{b \to \infty} \int_0^{b} e^{-x}\sin x \;\; dx$$

Do it again:

$$\int_0^{\infty} e^{-x}\cos x \;\; dx = \lim_{b \to \infty} e^{-x}\sin x|_0^b - \lim_{b \to \infty} e^{-x}\cos x|_0^b - \lim_{b \to \infty} \int_0^{b} e^{-x}\cos x \;\; dx$$

$$2\int_0^{\infty} e^{-x}\cos x \;\; dx = \lim_{b \to \infty} e^{-x}\sin x|_0^b - \lim_{b \to \infty} e^{-x}\cos x|_0^b$$

$$2\int_0^{\infty} e^{-x}\cos x \;\; dx = \lim_{b \to \infty} e^{-b}\sin b - e^{0}\sin 0 - \lim_{b \to \infty} e^{-b}\cos b + e^{0}\cos 0$$

$$2\int_0^{\infty} e^{-x}\cos x \;\; dx = \lim_{b \to \infty} e^{-b}\sin b - 0 - \lim_{b \to \infty} e^{-b}\cos b + 1$$

sin 0 = 0 cos 0 = 1 $e^0 = 1$

$\lim_{b \to \infty} e^{-b}\sin b = 0$ this happens because the exponential is going to zero. Sine is fluctuating between -1 and 1. Zero times any number (between -1 and 1) will still be zero.

$\lim_{b \to \infty} e^{-b}\cos b$ this happens because the exponential is going to zero. Cosine is fluctuating between -1 and 1. Zero times any number (between -1 and 1) will still be zero.

$$2\int_0^{\infty} e^{-x}\cos x \;\; dx = 0 - 0 - 0 + 1 = 1$$

$$\int_0^{\infty} e^{-x}\cos x \;\; dx = \frac{1}{2}$$

8. $\int_1^\infty \frac{1}{\sqrt{x}}\ dx$

$$\int_1^\infty \frac{1}{\sqrt{x}}\ dx = \lim_{b \to \infty} \int_1^b \frac{1}{\sqrt{x}}\ dx$$

$$\int_1^\infty \frac{1}{\sqrt{x}}\ dx = \lim_{b \to \infty} \left. 2\sqrt{x}\right|_1^b$$

$$\int_1^\infty \frac{1}{\sqrt{x}}\ dx = \lim_{b \to \infty} 2\sqrt{b} - 2$$

$$\int_1^\infty \frac{1}{\sqrt{x}}\ dx = \infty$$

$\int_1^\infty \frac{1}{\sqrt{x}}\ dx$ diverges to positive infinity

9. $\int_0^\infty \frac{1}{1 + e^x}\ dx$

$$\int_0^\infty \frac{1}{1 + e^x}\ dx = \lim_{b \to \infty} \int_0^b \frac{1}{1 + e^x}\ dx$$

$$\int_0^\infty \frac{1}{1 + e^x}\ dx = \lim_{b \to \infty} \int_0^b \frac{1}{1 + e^x} \cdot \frac{e^{-x}}{e^{-x}}\ dx$$

$$\int_0^\infty \frac{1}{1 + e^x}\ dx = \lim_{b \to \infty} \int_0^b \frac{e^{-x}}{e^{-x} + 1}\ dx$$

$$\int_0^\infty \frac{1}{1 + e^x}\ dx = \lim_{b \to \infty} \left. -\ln(1 + e^{-x})\right|_0^b$$

$$\int_0^\infty \frac{1}{1 + e^x}\ dx = \lim_{b \to \infty} \left. \ln(1 + e^{-x})\right|_b^0$$

$$\int_0^\infty \frac{1}{1 + e^x}\ dx = \lim_{b \to \infty} \ln(1 + e^0) - \ln\left(1 + e^{-b}\right)$$

$$\int_0^\infty \frac{1}{1 + e^x}\ dx = \ln 2 - \ln 1 = \ln 2$$

$$\int_0^\infty \frac{1}{1 + e^x}\ dx = \ln 2$$

10. $\int_0^{\infty} e^{-x}\ x\ dx$

$$\int_0^{\infty} e^{-x}\ x\ dx = \lim_{b \to \infty} \int_0^{b} x\, e^{-x}\, dx$$

Use integration by parts:

$$\int_0^{\infty} e^{-x}\ x\ dx = \lim_{b \to \infty} -x\, e^{-x}\Big|_0^b + \lim_{b \to \infty} \int_0^{b} e^{-x}\ dx$$

$$\int_0^{\infty} e^{-x}\ x\ dx = \lim_{b \to \infty} -be^{-b} + 0 \cdot e^0 + \lim_{b \to \infty} \int_0^{b} e^{-x}\ dx$$

$$\int_0^{\infty} e^{-x}\ x\ dx = \lim_{b \to \infty} -be^{-b} + \lim_{b \to \infty} \int_0^{b} e^{-x}\ dx$$

$$\int_0^{\infty} e^{-x}\ x\ dx = \lim_{b \to \infty} -\frac{b}{e^b} + \lim_{b \to \infty} \int_0^{b} e^{-x}\ dx$$

Use L'Hospital's rule for the limit $\lim_{b \to \infty} -\frac{b}{e^b}$

$$\lim_{b \to \infty} -\frac{b}{e^b} = \lim_{b \to \infty} \frac{1}{e^b} = 0$$

$$\int_0^{\infty} e^{-x}\ x\ dx = \lim_{b \to \infty} \int_0^{b} e^{-x}\ dx$$

$$\int_0^{\infty} e^{-x}\ x\ dx = \lim_{b \to \infty} -e^{-x}\Big|_0^b$$

$$\int_0^{\infty} e^{-x}\ x\ dx = \lim_{b \to \infty} -e^{-b} + 1 = 0 + 1 = 1$$

$$\int_0^{\infty} e^{-x}\ x\ dx = 1$$

11. Evaluate the integral $\int_0^{\infty} x^2 e^{-x}\, dx$ and use the result from question 10 that $\int_0^{\infty} x\ e^{-x}\, dx = 1$

$$\int_0^{\infty} x^2 e^{-x}\, dx = \lim_{b \to \infty} \int_0^{b} x^2 e^{-x}\, dx$$

$$\int_0^{\infty} x^2 e^{-x}\, dx = \lim_{b \to \infty} -x^2 e^{-x}\Big|_0^b + \lim_{b \to \infty} \int_0^{b} 2\,x\ e^{-x}\, dx$$

$$\int_0^{\infty} x^2 e^{-x}\, dx = \lim_{b \to \infty} -x^2 e^{-x}\Big|_0^b + 2 \lim_{b \to \infty} \int_0^{b} x\ e^{-x}\, dx$$

$$\int_0^{\infty} x^2 e^{-x}\, dx = \lim_{b \to \infty} -x^2 e^{-x}\Big|_0^b + 2 \cdot 1$$

$$\int_0^{\infty} x^2 e^{-x}\, dx = \lim_{b \to \infty} -b^2 e^{-b} + 0 + 2 \cdot 1$$

Evaluate the limit $\lim_{b \to \infty} -b^2 e^{-b}$ by L'Hospital's rule. It has the form $\infty \cdot 0$. We can get it to the form $\frac{\infty}{\infty}$ and apply L'Hospital's rule twice:

$$\lim_{b \to \infty} -b^2 e^{-b} = \lim_{b \to \infty} -\frac{b^2}{e^b} = \lim_{b \to \infty} -\frac{2b}{e^b} = \lim_{b \to \infty} -\frac{2}{e^b} = 0$$

$$\int_0^{\infty} x^2 e^{-x}\, dx = 0 + 0 + 2 \cdot 1$$

$$\int_0^{\infty} x^2 e^{-x}\, dx = 2$$

12. Show that the integral $\int_0^{\infty} x^n \ e^{-x}\, dx \ = \ n!$ where n is a positive integer

We will use the extended method of parts:

$$\int f\, g' dx \ = \ fg \ - \ f' \int g \ + \ f'' \int\int g \ - \ f''' \int\int\int g \ \cdots + \ (-1)^n f^n \int \cdots \int g$$

$f = x^n$ $\qquad\qquad$ $g = -e^{-x}$

$f' = \ n\, x^{n-1}$ $\qquad\qquad$ $\int g \ = \ e^{-x}$

$f'' \ = \ n(n-1)\, x^{n-2}$ $\qquad\qquad$ $\int\int g \ = \ -e^{-x}$

$f''' = n(n-1)(n-2)\, x^{n-3}$ $\qquad\qquad$ $\int\int\int g \ = \ e^{x}$

$\vdots$ $\qquad\qquad$ $\vdots$

$f^n = \ n!$ $\qquad\qquad$ $\int \cdots \int g \ = \ (-1)^{n-1}\, e^{-x}$

$$\int_0^b x^n \ e^{-x}\, dx \ = \ -x^n \left(e^{-x} \right) \ - n x^{n-1} e^{-x} - n(n-1) x^{n-2} e^{-x} \ - \ \cdots - n!\, e^{-x} \ \Big|_0^b$$

When the lower bound x=0 is plugged in,all the terms with x^m will equal zero. The only term that will survive is the n! term at the very end.

$$\int_0^b x^n \ e^{-x}\, dx \ = \ -b^n \left(e^{-b} \right) \ - n b^{n-1} e^{-b} - n(n-1) b^{n-2} e^{-b} \ - \ \cdots - n!\, e^{-b}$$

$$-(0 - 0 - 0 - \cdots - n!)$$

If we take the limit as b goes to infinity, L'Hospital's rule will show that all the limits are zero.

$$\lim_{b \to \infty} \frac{b^m}{e^b} \ = \ 0 \ \ for\ all\ m$$

$$\int_0^b x^n \ e^{-x}\, dx \ = \ n!$$

This integral has a name. It is called the Gamma function. It has 3 different forms. This is one of the three.

13. Evaluate $\int_0^{\infty} \sin x \ \ dx$

$$\int_0^{\infty} \sin x \ \ dx \ = \ \lim_{b \to \infty} \int_0^{b} \sin x \ \ dx$$

$$\int_0^{\infty} \sin x \ \ dx \ = \ - \lim_{b \to \infty} \cos x \ \Big|_0^b$$

$$\int_0^{\infty} \sin x \ \ dx \ = \ \lim_{b \to \infty} \cos 0 \ - \ \cos b$$

$$\int_0^{\infty} \sin x \ \ dx \ = \ \lim_{b \to \infty} 1 \ - \ \cos b$$

But $\lim\limits_{b \to \infty} \cos b$ does not exist. Cosine continues to oscillate between 1 and -1 and does not approach any number. This is divergence by oscillation.

$$\int_0^{\infty} \sin x \ \ dx \quad \textit{does not converge}$$

$$\int_0^{\infty} \sin x \ \ dx \quad \textit{diverges by oscillation}$$

14. Evaluate $\int_{-\infty}^{\infty} \cos x \ \ dx$

$$\int_{-\infty}^{\infty} \cos x \ \ dx \ = \ \lim_{a \to -\infty} \int_{a}^{0} \cos x \ \ dx \ + \ \lim_{b \to \infty} \int_{0}^{b} \cos x \ dx$$

Look at $\lim_{b \to \infty} \int_0^b \cos x \ dx$

$$\lim_{b \to \infty} \int_{0}^{b} \cos x \ dx = \lim_{b \to \infty} \ \sin x \ |_0^b$$

$$\lim_{b \to \infty} \int_{0}^{b} \cos x \ dx = \lim_{b \to \infty} \ \sin b - \sin 0$$

$$\lim_{b \to \infty} \int_{0}^{b} \cos x \ dx = \lim_{b \to \infty} \ \sin b - 0 \ = \lim_{b \to \infty} \ \sin b$$

The sine of b does not approach anything. It keeps oscillating between -1 and 1. It does not converge to any number. It just keeps fluctuating. This is divergence by oscillation.

$$\int_{0}^{b} \cos x \ dx \ \ \textit{diverges by oscillation}$$

PURCELL P534

15. $\int_0^{\infty} \frac{1}{e^x + e^{-x}} \; dx$

$$\int_0^{\infty} \frac{1}{e^x + e^{-x}} \; dx = \lim_{b \to \infty} \int_0^{b} \frac{1}{e^x + e^{-x}} \; dx$$

$$\int_0^{\infty} \frac{1}{e^x + e^{-x}} \; dx = \lim_{b \to \infty} \int_0^{b} \frac{1}{e^x + e^{-x}} \cdot \frac{e^x}{e^x} \; dx$$

$$\int_0^{\infty} \frac{1}{e^x + e^{-x}} \; dx = \lim_{b \to \infty} \int_0^{b} \frac{e^x}{e^{2x} + 1} \; dx$$

Let u = e^x and du = e^x dx and let c=e^b

$$\int_0^{\infty} \frac{1}{e^x + e^{-x}} \; dx = \lim_{c \to \infty} \int_0^{c} \frac{1}{u^2 + 1} \; du$$

$$\int_0^{\infty} \frac{1}{e^x + e^{-x}} \; dx = \lim_{c \to \infty} \tan^{-1} u \; \Big|_0^c$$

$$\int_0^{\infty} \frac{1}{e^x + e^{-x}} \; dx = \lim_{c \to \infty} \tan^{-1} c - \tan^{-1} 0$$

$$\int_0^{\infty} \frac{1}{e^x + e^{-x}} \; dx = \lim_{c \to \infty} \tan^{-1} c - 0 = \lim_{c \to \infty} \tan^{-1} c$$

$$\int_0^{\infty} \frac{1}{e^x + e^{-x}} \; dx = \frac{\pi}{2}$$

16. $\int_{-\infty}^{-2} \frac{1}{x^5}\, dx$

$$\int_{-\infty}^{-2} \frac{1}{x^5}\, dx = \lim_{b \to -\infty} \int_{b}^{-2} \frac{1}{x^5}\, dx$$

$$\int_{-\infty}^{-2} \frac{1}{x^5}\, dx = \lim_{b \to -\infty} \left. \frac{1}{-4\,x^4} \right|_{b}^{-2}$$

$$\int_{-\infty}^{-2} \frac{1}{x^5}\, dx = \lim_{b \to -\infty} \frac{1}{4b^4} - \frac{1}{64} = 0 - \frac{1}{64} = -\frac{1}{64}$$

17. $\int_{4}^{\infty} x\ e^{-x^2}\, dx$

$$\int_{4}^{\infty} x\ e^{-x^2}\, dx = \lim_{b \to \infty} \int_{4}^{b} x\ e^{-x^2}\, dx$$

If we let $u = -x^2$ so that $du = -2x\, dx$ then we have the form $e^u\, du$

$$\int_{4}^{\infty} x\ e^{-x^2}\, dx = -\frac{1}{2} \lim_{b \to \infty} \int_{4}^{b} e^{-x^2}\,(-2x)\, dx$$

$$\int_{4}^{\infty} x\ e^{-x^2}\, dx = -\frac{1}{2} \lim_{b \to \infty} \left. e^{-x^2} \right|_{4}^{b}$$

$$\int_{4}^{\infty} x\ e^{-x^2}\, dx = -\frac{1}{2} \lim_{b \to \infty} \left(e^{-b^2} - e^{-16}\right)$$

$$\int_{4}^{\infty} x\ e^{-x^2}\, dx = -\frac{1}{2} \lim_{b \to \infty} \left(0 - e^{-16}\right)$$

$$\int_{4}^{\infty} x\ e^{-x^2}\, dx = \frac{1}{2}\ e^{-16}$$

18. $\int_{-\infty}^{0} e^{3x}\ dx$

$$\int_{-\infty}^{0} e^{3x}\ dx = \lim_{a \to -\infty} \int_{a}^{0} e^{3x}\ dx$$

$$\int_{-\infty}^{0} e^{3x}\ dx = \lim_{a \to -\infty} \frac{e^{3x}}{3}\bigg|_{a}^{0}$$

$$\int_{-\infty}^{0} e^{3x}\ dx = \lim_{a \to -\infty} \frac{1}{3} - \frac{e^{ax}}{3}$$

$$\int_{-\infty}^{0} e^{3x}\ dx = \lim_{a \to -\infty} \frac{1}{3} - 0 = \frac{1}{3}$$

19. $\int_{3}^{\infty} \frac{x}{\sqrt{x^2+9}} dx$

$$\int_{3}^{\infty} \frac{x}{\sqrt{x^2+9}} dx = \lim_{b \to \infty} \int_{3}^{b} \frac{x}{\sqrt{x^2+9}} dx$$

If we let u = $x^2 + 9$ then du = 2x dx and we have the form $u^{-1/2}$ du

$$\int_{3}^{\infty} \frac{x}{\sqrt{x^2+9}} dx = \frac{1}{2} \lim_{b \to \infty} \int_{3}^{b} \frac{2x}{\sqrt{x^2+9}} dx$$

$$\int_{3}^{\infty} \frac{x}{\sqrt{x^2+9}} dx = \frac{1}{2} \lim_{b \to \infty} 2\,(x^2+9)^{1/2}\ \big|_{3}^{b}$$

$$\int_{3}^{\infty} \frac{x}{\sqrt{x^2+9}} dx = \lim_{b \to \infty} (b^2+9)^{1/2} - 18^{\frac{1}{2}}$$

$$\int_{3}^{\infty} \frac{x}{\sqrt{x^2+9}} dx = \infty$$

The integral diverges to infinity.

20. $\int_2^{\infty} \frac{1}{(1-x)^{2/3}}\ dx$

$$\int_2^{\infty} \frac{1}{(1-x)^{2/3}}\ dx \ = \ \lim_{b \to \infty} \int_2^{b} \frac{1}{(1-x)^{2/3}}\ dx$$

If we let u = 1-x then du = -dx and we have the form $u^{-2/3}$ du

$$\int_2^{\infty} \frac{1}{(1-x)^{2/3}}\ dx \ = \ \lim_{b \to \infty} -3\,(1-x)^{\frac{1}{3}}\Big|_2^b$$

$$\int_2^{\infty} \frac{1}{(1-x)^{2/3}}\ dx \ = \ \lim_{b \to \infty} 3\,(1-x)^{\frac{1}{3}}\Big|_b^2$$

$$\int_2^{\infty} \frac{1}{(1-x)^{2/3}}\ dx \ = \ \lim_{b \to \infty} 3(-1) \ - \ 3(1-b)^{1/3}$$

$$\int_2^{\infty} \frac{1}{(1-x)^{2/3}}\ dx \ = \ \infty$$

The integral diverges to positive infinity.