IMPORTANT THEOREMS FOR INFINITE SERIES

The handouts on infinite series have dealt with specific theorems to determine the convergence of a an infinite series. There are many other statements regarding the properties of infinite series that have not been mentioned. These statements are definitions and additional theorems. They are extremely important. These definitions and additional theorems do not directly affect tests for determination of convergence, but they do state what can and what cannot be done with infinite series. They also state the exact meanings of properties and operations.

DEFINITION OF CONVERGENCE.

We consider the infinite series $\sum_{k=1}^{\infty} u_k$ where u_k is a real number, positive, negative or zero. We define the partial sums of the series s_n to equal $s_n = \sum_{k=1}^{n} u_k = u_1 + u_2 + \cdots + u_n$. The partial sums form their own sequence. If the sequence of s_n converge to a limit which we denote as S, we say that the series converges to the sum S.

If $\lim_{n \to \infty} s_n = S$ then $S = \sum_{k=1}^{\infty} u_k$

If the limit $\lim_{n \to \infty} s_n$ fails to converge, then we say that the infinite series diverges.

THE HARMONIC SERIES

The series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is called the harmonic series. This series diverges to positive infinity. Although it diverges it is an extremely important series used in the study of infinite series. The divergence of the harmonic series can be proven using calculus or by elementary number arguments.

THE GEOMETRIC SERIES

The series $\sum_{k=1}^{\infty} a\, r^k = ar + ar^2 + ar^3 + \cdots$ is called an infinite geometric series. The series converges if $-1 < r < 1$. The series diverges otherwise. The geometric series is extremely important to developing the theory of infinite series. It is the basis of the ratio test and the nth root test. There is a formula for the sum of a geometric series:

$$S = \sum_{k=1}^{\infty} a\, r^k = \frac{ar}{1 - r}$$

THE NECESSARY CONDITION FOR CONVERGENCE

The series $\sum_{k=1}^{\infty} u_k$ will only converge if $\lim_{k \to \infty} u_k = 0$. If this limit is not zero the series cannot converge. This limit is NECESSARY. It cannot be violated. If this limit is zero, then the series might converge or it might not. Further testing is necessary.

THEOREM ON ADDITION AND SUBTRACTION OF INFINITE SERIES

If $\sum u_k$ and $\sum v_k$ both converge then we add and subtract series using the associative law:

$$\sum(u_k + v_k) = \sum u_k + \sum v_k$$

$$\sum(u_k - v_k) = \sum u_k - \sum v_k$$

THEOREM ON MULTIPLICATION BY A CONSTANT

If c is a real constant, then multiplication by c does not affect convergence. If $\sum u_k$ converges, then $\sum c\, u_k$ also converges. The same holds true if they diverge.

DISTRIBUTIVE LAW

If $\sum u_k$ converges then the distributive law holds true: $\sum c\, u_k = c \sum u_k$

INITIAL VALUE OF INDEX THEOREM

If $\sum_{k=1}^{\infty} u_k$ converges, then $\sum_{k=N}^{\infty} u_k$ also converges, where N is any positive integer greater than 1. The starting point of the series does not affect convergence.

BOUNDED MONOTONIC SEQUENCE THEOREM FOR SERIES

Let $u_k > 0$ for the infinite series $\sum_{k=1}^{\infty} u_k$. Consider the sequence of partial sums $s_n = \sum_{k=1}^{n} u_k$.

The sequence of partial sums s_n forms a monotonic increasing sequence. If this sequence is bounded above by some number M; $s_n < M \quad \forall n$ then the sequence of partial sums converges to some limiting value S and the infinite series converges to the same value S.

ABSOLUTE CONVERGENCE OF A SERIES

If a series converges in absolute value, then the original series converges.

If $\sum |u_k|$ converges then $\sum u_k$ converges. This is true for all kinds of series – not just series with positive terms or series with alternating terms. This is true for all series.

CONDITIONALLY CONVERGENT SERIES

This is the strangest and most remarkable theorem on infinite series in a first year calculus course.

If $\sum |u_k|$ diverges but $\sum u_k$ converges (this happens a lot – particularly with alternating series), then the series is called conditionally convergent. If a series is conditionally convergent, then it is NOT absolutely convergent (they are opposites).

If a series is conditionally convergent, then changing the order in which you add the terms will change the sum of the series!!!!! The associative law of addition <u>fails</u> when you have conditionally convergent series.

The sums $S_1 = u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 + u_8 + \ldots$

and $S_2 = (u_1 + u_3 + u_5) + u_2 + (u_7 + u_9 + u_{11}) + u_4$

will be different! This is covered in much greater detail in analysis. It is only mentioned in this course.