INTEGRATION BY PARTIAL FRACTIONS QUADRATICS AND COMPLETING THE SQUARE

In the first handout on integration by partial fractions, we concentrated on linear factors. The expansion for such fractions was of the form:

$$\frac{1}{(x-r_1)(x-r_2)(x-r_3)} = \frac{A_1}{x-r_1} + \frac{A_2}{x-r_2} + \frac{A_3}{x-r_3}$$

The above equation was the guiding principle to all such expansions. The numerator in the fraction (on the left hand side) is not necessarily equal to 1. It can be any polynomial that is of lower degree than the denominator.

In this handout, we wish to concentrate on fractions where there are quadratics in the denominator. For these fractions the expansion will have the form:

$$\frac{1}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$$

This equation will be the guiding principle.

We could have something similar, like this:

$$\frac{1}{(x^2+b_1\,x+\,c_1)\,(x^2+\,b_2\,x+c_2)} = \frac{A_1x+B_1}{x^2+b_1\,x+\,c_1} + \frac{A_2x+B_2}{x^2+\,b_2\,x+c_2}$$

The pattern that needs to be seen is this: linear factors in the denominator get a constant in the numerator. Quadratic factors in the denominator get a linear polynomial in the numerator.

We could consider cubic polynomials in the denominator:

$$\frac{1}{(x-a)(x^3+bx^2+cx+d)} = \frac{A}{x-a} + \frac{Bx^2+Cx+D}{x^3+bx^2+cx+d}$$

We see that a cubic polynomial in the denominator gets a quadratic polynomial in the numerator.

The most general statement for partial fraction expansion is this: whatever the degree of the polynomial in the denominator, its numerator is a polynomial that is exactly one degree less.

With this we can start on the problems in this handout. We will only concentrate on polynomials with highest degree two – we will not consider cubic polynomials nor higher. They are far too difficult. This is standard practice for a second course in calculus.

One reason we do not deal with higher degree polynomials is that all such polynomials can (in principle) be factored into linear terms and quadratic terms. Another reason that we do not deal with higher degree polynomials is that it is difficult to factor a cubic (Cardano's or Tartaglia's method), extremely difficult to factor 4^{th} degree polynomials (Ferrari's method) and usually impossible to factor 5^{th} degree or higher polynomials (Galios and Abel).

So we'll stick with second degree polynomials.

In the first handout, we had a very specific technique to evaluate the unknown coefficients in the partial fraction expansion. That method will not work here. To get the A's, the B's the C's and so on we will have to use a completely different method. What we do is generate a set of simultaneous equations. This set of equations can be solved using elementary algebra (row reduction). The best way to do it is to the TI 84 and invoke the RREF command.

1. Evaluate $\int \frac{1}{x\,(x^2+1)}\ dx$

$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$ partial fraction expansion

$\frac{1}{x(x^2+1)} = \frac{A(x^2+1)}{x(x^2+1)} + \frac{(Bx+C\,)\,x}{x\,(x^2+1)}$ common denominator

$\frac{1}{x(x^2+1)} = \frac{(Ax^2+A)}{x(x^2+1)} + \frac{(Bx^2+C\,x)}{x\,(x^2+1)}$ distribute

$\frac{1}{x(x^2+1)} = \frac{(Ax^2+A)+(Bx^2+C\,x)}{x(x^2+1)}$ add fractions

$1 = (Ax^2 + A) + (Bx^2 + C\,x)$ equate the numerators

$1 = (A + B)\,x^2 + Cx + A$ rearrange terms

The polynomial on the right must equal the "polynomial" on the left. So we equate terms:

$0x^2 + 0x + 1 = (A + B)\,x^2 + Cx + A$

X^2 : 0 = A+B

X: 0 = C

Constants: 1 = A

We get the solutions A=1, B = -1, C=0 → $\frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{-x}{x^2+1}$

$$\int \frac{1}{x\,(x^2+1)}\ dx = \int \left(\frac{1}{x} + \frac{-x}{x^2+1}\right) dx = \ln|x| - \frac{1}{2}\ln(x^2+1) + const$$

2. Evaluate $\int \frac{1}{(x^2+1)\,(x^2+4)}\,dx$

$\frac{1}{(x^2+1)\,(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+\ 4}$ partial fraction expansion

$\frac{1}{(x^2+1)\,(x^2+4)} = \frac{(Ax+B)\,(x^2+\ 4\,)}{(x^2+1)} + \frac{(Cx+D)(x^2+\ 1\,)}{(x^2+\ 4\,)}$ common denominator

$\frac{1}{(x^2+1)\,(x^2+4)} = \frac{(Ax+B)\,(x^2+\ 4\,)+\ (Cx+D)(x^2+\ 1\,)}{(x^2+1)}$ add fractions

$1 = \ (Ax+B)\,(x^2+\ \ 4\,)\ +\ \ (Cx+D)(x^2+\ \ 1\,)$ equate numerators

$1 = (Ax^3+Bx^2+4Ax+4B)\ +\ \ (Cx^3+Dx^2+Cx+D)$ expand

$1 = (A+B)x^3\ \ +\ (B+D)\,x^2\ +\ \ (4A+C)x\ \ +\ \ 4B+D$ group terms

The polynomial on the left equals the polynomial on the right – so equate them term by term:

$0x^3+0x^2+0x+\ 1 = (A+B)x^3\ \ +\ (B+D)\,x^2\ +\ \ (4A+C)x\ \ +\ \ 4B+D$

X^3: 0 = A+B

X^2: 0 = B+D

X: 0 = 4A + C

Const: 1 = 4B+ D

Solving these equations we get A = -1/3 B = 1/3 C = 4/3 D =-1/3

$$\frac{1}{(x^2+1)\,(x^2+4)} = \frac{1}{3}\,\frac{-x+1}{x^2+1} + \frac{1}{3}\,\frac{4x\ -1}{x^2+\ 4}$$

$$\int \frac{1}{(x^2+1)\,(x^2+4)}\,dx = \int \left(\frac{1}{3}\,\frac{-x+1}{x^2+1} + \frac{1}{3}\,\frac{4x\ -1}{x^2+\ 4}\right) dx$$

$$= \int \left(-\frac{1}{3}\,\frac{x}{x^2+1} + \frac{1}{3}\,\frac{1}{x^2+1} + \frac{4}{3}\,\frac{x}{x^2+\ 4} - \frac{1}{3}\,\frac{1}{x^2+\ 4}\right) dx$$

Each of the individual integrals is relatively easy to evaluate. We get:

$$= -\frac{1}{6}\ln(x^2+1) + \frac{1}{3}\tan^{-1}x + \frac{2}{3}\ln(x^2+4) - \frac{1}{6}\tan^{-1}\frac{x}{2} + const.$$

It helps to recall that $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

3. Evaluate $\int \frac{2x+3}{(x+1)\,(x^2+9)}\,dx$

$\frac{2x+3}{(x+1)\,(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$ partial fraction expansion

$\frac{2x+3}{(x+1)\,(x^2+9)} = \frac{A\,(x^2+9)}{(x+1)\,(x^2+9)} + \frac{(Bx+C)\,(x+1)}{(x+1)(x^2+9)}$ common denominator

$\frac{2x+3}{(x+1)\,(x^2+9)} = \frac{A\,(x^2+9) + (Bx+C)\,(x+1)}{(x+1)\,(x^2+9)}$ add fractions

$2x+3 = A\,(x^2+9) + (Bx+C)\,(x+1)$ equate numerators

$2x+3 = Ax^2 + 9A + Bx^2 + Bx + Cx + C$ distribute

$2x+3 = (A+B)\,x^2 + (B+C)\,x + C$ group like terms

Since the polynomials are equal, we equate coefficients:

$0x^2 + 2x + 3 = (A+B)\,x^2 + (B+C)\,x + C$

X^2: 0 = A+B

X: 2 = B+C

Const: C = 3

Solving these we get: A = 1 B = -1 C=3

$$\frac{2x+3}{(x+1)\,(x^2+9)} = \frac{1}{x+1} + \frac{-x+3}{x^2+9}$$

$$\int \frac{2x+3}{(x+1)\,(x^2+9)}\,dx = \int \left(\frac{1}{x+1} + \frac{-x+3}{x^2+9}\right) dx = \int \left(\frac{1}{x+1} - \frac{x}{x^2+9} + \frac{3}{x^2+9}\right) dx$$

The first two integrals on the right side will give us logarithms. The last integral will give us an inverse tangent. Going through the steps should yield.

$$\int \frac{2x+3}{(x+1)\,(x^2+9)}\,dx = \ln|x+1| - \frac{1}{2}\ln(x^2+9) + \tan^{-1}\frac{x}{3} + Const.$$

4. Evaluate $\int \frac{x^4}{(x+3)\,(x^2+2)}\,dx$

The first thing we notice is that the numerator is a polynomial of 4th degree and the denominator is a polynomial of 3rd degree. So we must do a long division. Doing so will yield:

$$\frac{x^4}{(x+3)\,(x^2+2)} = x-3 + \frac{x-3}{(x+3)\,(x^2+2)}$$

$$\int \frac{x^4}{(x+3)\,(x^2+2)}\,dx = \int \left(x-3 + \frac{x-3}{(x+3)\,(x^2+2)}\right) dx$$

$$= \frac{x^2}{2} - 3x + \int \frac{x-3}{(x+3)\,(x^2+2)}\,dx$$

Now we do a partial fraction expansion on $\frac{x-3}{(x+3)\,(x^2+2)}$

$\frac{x-3}{(x+3)\,(x^2+2)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+2}$ partial fraction expansion

$\frac{x-3}{(x+3)\,(x^2+2)} = \frac{A\,(x^2+2)}{(x+3)\,(x^2+2)} + \frac{(x+3)\,(Bx+C)}{(x+3)(x^2+2)}$ common denominator

$\frac{x-3}{(x+3)\,(x^2+2)} = \frac{A\,(x^2+2) + (x+3)\,(Bx+C)}{(x+3)\,(x^2+2)}$ add fractions

$x-3 = A\,(x^2+2) + (x+3)\,(Bx+C)$ equate numerators

$x-3 = Ax^2 + 2A + Bx^2 + 3Bx + Cx + 3C$ expand

$x-3 = (A+B)\,x^2 + (2A+3B)\,x + (2A+3C)$ group like terms

Now we equate coefficients

$0x^2 + 1\,x - 3 = (A+B)\,x^2 + (2A+3B)\,x + (2A+3C)$

X^2: 0 = A+B

X: 1 = 2A + 3B

X^0: -3 = 2A + 3C

Solving these we get A = -1 B = 1 C = -1/3

$$\frac{x-3}{(x+3)\,(x^2+2)} = \frac{-1}{x+3} + \frac{x-\frac{1}{3}}{x^2+2} = -\frac{1}{x+3} + \frac{x}{x^2+2} - \frac{1}{3}\frac{1}{x^2+2}$$

$$\int \frac{x^4}{(x+3)\,(x^2+2)}\,dx = \frac{x^2}{2} - 3x + \int \frac{x-3}{(x+3)\,(x^2+2)}\,dx$$

$$\int \frac{x^4}{(x+3)\,(x^2+2)}\,dx = \frac{x^2}{2} - 3x + \int \left(-\frac{1}{x+3} + \frac{x}{x^2+2} - \frac{1}{3}\frac{1}{x^2+2}\right) dx$$

The first two integrals on the right side will be logarithms. The last one will be inverse tangent. If we go through the evaluation, we should get:

$$\int \frac{x^4}{(x+3)\,(x^2+2)}\,dx = \frac{x^2}{2} - 3x - \ln|x+3| + \frac{1}{2}\ln(x^2+2) - \frac{1}{3\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

5. COMPLETING THE SQUARE

Evaluate the integral $\int \frac{1}{x^2+2x+2}\ dx$

This integral is not a partial fraction integral. However the integrand is of a type that can occur in partial fraction expansions.

The way to handle such integrals is to complete the square.

Take the middle coefficient: 2

Take half of it: 1

Square the current number: $1^2 = 1$

Add and subtract 1 from the quadratic within the indicated parentheses:

$$x^2 + 2x + 2 \ = \ (x^2 + 2x) + (2) = (x^2 + \ 2x \ + \ 1) \ + \ (2 - 1)$$

The first parentheses on the right is a perfect square:

$x^2 + 2x + 2 \ = \ (x + 1)^2 + 1$ this is what we want

$\int \frac{1}{x^2+2x+2}\ dx \ = \ \int \frac{1}{(x+1)^2+1}\ dx$ now let u = x+1 so that du=dx

$$\int \frac{1}{x^2+2x+2}\ dx \ = \ \int \frac{1}{u^2+1}\ du \ = \tan^{-1} u \ + \ C$$

6. Evaluate $\int \frac{1}{x^2+4x+5}\, dx$

This integral is not a partial fraction integral. However the integrand is of a type that can occur in partial fraction expansions.

The way to handle such integrals is to complete the square.

Take the middle coefficient: 4

Take half of it: 2

Square the current number: $2^2 = 4$

Add and subtract 4 from the quadratic within the indicated parentheses:

$$x^2 + 4x + 5 \;=\; (x^2 + 4x) + (5) = (x^2 + \;4x \;+\; 4) \;+\; (5 - 4)$$

The first parentheses on the right is a perfect square:

$x^2 + 4x + 5 \;=\; (x+2)^2 + 1$ this is what we want

$\int \frac{1}{x^2+4x+5}\, dx = \int \frac{1}{(x+2)^2+1}\, dx$ let u = x+2 do that du = dx

$$\int \frac{1}{x^2+4x+5}\, dx = \int \frac{1}{u^2+1}\, du \;=\; \tan^{-1} u \;+\; C \;= \tan^{-1}(x+2) \;+\; c$$

7. Evaluate $\int \frac{1}{x^2+6x+13}\, dx$

This integral is not a partial fraction integral. However the integrand is of a type that can occur in partial fraction expansions.

The way to handle such integrals is to complete the square.

Take the middle coefficient: 6

Take half of it: 3

Square the current number: $3^2 = 9$

Add and subtract 9 from the quadratic within the indicated parentheses:

$$x^2 + 6x + 13 \;=\; (x^2 + 6x) + (13) = (x^2 + \;6x \;+\; 9) \;+\; (13 - 9)$$

One of the terms now is a perfect square so we can write:

$$x^2 + 6x + 13 \;=\; (x+3)^2 \;+\; 4$$

$\int \frac{1}{x^2+6x+13}\, dx \;=\; \int \frac{1}{(x+3)^2 \,+\, 4}\, dx$ let u = x+3 so that du = dx

$$\int \frac{1}{x^2 + 6x + 13}\, dx \;=\; \int \frac{1}{u^2 \;+\; 2^2}\, du \;=\; \frac{1}{2}\tan^{-1}\frac{u}{2} \;+\; C \;=\; \frac{1}{2}\tan\frac{x+3}{2} \;+\; C$$

8. Evaluate $\int \frac{1}{(x+1)\,(x^2+2x+10)}\;dx$

$$\frac{1}{(x+1)\,(x^2+2x+10)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+2x+10)}$$ Partial fraction expansion

$$\frac{1}{(x+1)\,(x^2+2x+10)} = \frac{A\left(x^2+2x+10\right)}{(x+1)(x^2+2x+10)} + \frac{(Bx+C)(x+1)}{(x+1)\,(x^2+2x+10)}$$ Common denominator

$$\frac{1}{(x+1)\,(x^2+2x+10)} = \frac{A\left(x^2+2x+10\right) + (Bx+C)(x+1)}{(x+1)(x^2+2x+10)}$$ Add fractions

$$1 = A(x^2+2x+10) + (Bx+C)(x+1)$$ Equate numerators

$$1 = Ax^2 + 2Ax + 10A + Bx^2 + Bx + Cx + C$$ Expand and distribute

$$1 = (A+B)x^2 + (2A+B+C)x + (10A+C)$$ Group like terms

Equate coefficients of polynomials on left side and right side

$0x^2 + 0x + 1 = (A+B)x^2 + (2A+B+C)x + (10A+C)$

X^2: 0 = A+B

X: 0 = 2A+B+C

X^0: 1 = 10A+C

Solving these we should get A = 1/9 B=-1/9 C=-1/9

$$\frac{1}{(x+1)\,(x^2+2x+10)} = \frac{1}{9}\frac{1}{(x+1)} - \frac{1}{9}\frac{x+1}{(x^2+2x+10)}$$

$$\int \frac{1}{(x+1)\,(x^2+2x+10)}\;dx = \int \left(\frac{1}{9}\frac{1}{(x+1)} - \frac{1}{9}\frac{x+1}{(x^2+2x+10)}\right)\;dx$$

$$\int \frac{1}{(x+1)\,(x^2+2x+10)}\;dx = \frac{1}{9}\ln|x+1| - \frac{1}{9}\int \frac{x+1}{(x^2+2x+10)}\;dx$$

Complete the square on the quadratic in the right side integral. Doing so will yield:

$x^2 + 2x + 10 \;=\; (x+1)^2 + 9$

$$\int \frac{1}{(x+1)\,(x^2+2x+10)}\;dx \;=\; \frac{1}{9}\ln|x+1| \;-\; \frac{1}{9}\int \frac{x+1}{(x+1)^2+9}\;dx$$

Let u = x+1 so that du = dx

$$\int \frac{1}{(x+1)\,(x^2+2x+10)}\;dx \;=\; \frac{1}{9}\ln|x+1| \;-\; \frac{1}{9}\int \frac{u}{u^2+9}\;du$$

The last integral is a natural logarithm:

$$\int \frac{1}{(x+1)\,(x^2+2x+10)}\;dx \;=\; \frac{1}{9}\ln|x+1| \;-\; \frac{1}{9}\frac{1}{2}\,\ln(u^2+9) \;+\; C$$

$$\int \frac{1}{(x+1)\,(x^2+2x+10)}\;dx \;=\; \frac{1}{9}\ln|x+1| \;-\; \frac{1}{18}\,\ln((x+1)^2+9) \;+\; C$$

9. Evaluate $\int \frac{2x+5}{(x^2+1)(x^2+4x+8)}\ dx$

This is a long and difficult question. But it is standard.

$\frac{2x+5}{(x^2+1)(x^2+4x+8)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4x+8}$ partial fraction expansion

$\frac{2x+5}{(x^2+1)(x^2+4x+8)} = \frac{(Ax+B)\,(x^2+4x+8)}{(x^2+1\,)\,(x^2+4x+8)} + \frac{(Cx+D)\,(x^2+1\,)}{(x^2+4x+8)\,(x^2+1\,)}$ common denominator

$2x+5 = (Ax+B)\,(x^2+4x+8) + (Cx+D)\,(x^2+1\,)$ equate numerators

$2x+5 = Ax^3+4Ax^2+8Ax+Bx^2+4Bx+\ 8B+\ Cx^3+Dx^2+Cx+D$ expand

$2x+5 = (A+C)x^3 + (4A+B+D)\,x^2 + (8A+4B+C)x + (8B+D)$ group

Equate polynomials:

$$0x^3+0x^2+\ 2x+5 = (A+C)x^3 + (4A+B+D)\,x^2 + (8A+4B+C)x$$
$$+(8B+D)$$

X^3: 0 = A+C

X^2: 0 = 4A + B + D

X: 2 = 8A + 4B + C

X^0: 5 = 8B+D

We solve these equations. I use TI84 using the RREF command.

A=-6/65 B=43/65 C= 6/65 D= -19/65

$\frac{2x+5}{(x^2+1)(x^2+4x+8)} = \frac{1}{65}\frac{-6x\ +\ 43}{x^2+1} + \frac{1}{65}\frac{6x\ -\ 19}{x^2+4x+8}$

$$\int \frac{2x+5}{(x^2+1)(x^2+4x+8)}\,dx = \int \left(\frac{1}{65}\,\frac{-6x + 43}{x^2+1} + \frac{1}{65}\,\frac{6x - 19}{x^2+4x+8}\right) dx$$

$$\int \frac{2x+5}{(x^2+1)(x^2+4x+8)}\,dx = \frac{1}{65}\int \frac{-6x + 43}{x^2+1}\,dx + \frac{1}{65}\int \frac{6x - 19}{x^2+4x+8}\,dx$$

$$\int \frac{2x+5}{(x^2+1)(x^2+4x+8)}\,dx = \frac{1}{65}\int \left(\frac{-6x}{x^2+1} + \frac{43}{x^2+1}\right) dx + \frac{1}{65}\int \frac{6x - 19}{x^2+4x+8}\,dx$$

We evaluate the first integral. The first term is a log and the second term is inverse tangent.

$$\int \frac{2x+5}{(x^2+1)(x^2+4x+8)}\,dx = \frac{-6}{65}\cdot\frac{1}{2}\ln(x^2+1) + \frac{43}{65}\tan^{-1}x + \frac{1}{65}\int \frac{6x - 19}{x^2+4x+8}\,dx$$

$$\int \frac{2x+5}{(x^2+1)(x^2+4x+8)}\,dx = \frac{-3}{65}\ln(x^2+1) + \frac{43}{65}\tan^{-1}x + \frac{1}{65}\int \frac{6x - 19}{x^2+4x+8}\,dx$$

For the second integral we have to complete the square:

$x^2 + 4x + 8 = (x+2)^2 + 2^2$

$\frac{1}{65}\int \frac{6x - 19}{(x+2)^2+2^2}\,dx$ let u = x+2 so that x = u-2 and du = dx

$$\frac{1}{65}\int \frac{6(u-2) - 19}{u^2+2^2}\,dx = \frac{1}{65}\int \frac{6u-12-19}{u^2+2^2}\,du = \frac{1}{65}\int \frac{6u-31}{u^2+2^2}\,du$$

$$\frac{1}{65}\int \frac{6x - 19}{(x+2)^2+2^2}\,dx = \frac{6}{65}\int \frac{u}{u^2+2^2}\,du - \frac{31}{65}\int \frac{1}{u^2+2^2}\,du$$

On the right, the first integral is a logarithm. The second integral will give inverse tangent. Use the formula $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$ or work out the details with an appropriate substitution

$$\frac{1}{65}\int \frac{6x \;-\; 19}{(x+2)^2+2^2}\,dx \;=\; \frac{6}{65}\cdot\frac{1}{2}\,\ln(u^2+2^2) \;-\; \frac{31}{65}\cdot\,\frac{1}{2}\tan\frac{u}{2}\;+C$$

$$\frac{1}{65}\int \frac{6x \;-\; 19}{(x+2)^2+2^2}\,dx \;=\; \frac{3}{65}\,\ln(x^2+4x+8) \;-\; \frac{31}{130}\,\tan\frac{x^2+4x+8}{2}\;+C$$

Finally we get:

$$\int \frac{2x+5}{(x^2+1)(x^2+4x+8)}\,dx \;=\frac{-3}{65}\ln(x^2+1)+\frac{43}{65}\,\tan^{-1}x+\frac{1}{65}\int \frac{6x\;-\;19}{x^2+4x+8}\,dx$$

$$\int \frac{2x+5}{(x^2+1)(x^2+4x+8)}\,dx \;=\frac{-3}{65}\,\ln(x^2+1)+\frac{43}{65}\tan^{-1}x+\frac{3}{65}\,\ln(x^2+4x+8)$$
$$-\;\frac{31}{130}\,\tan\frac{x^2+4x+8}{2}\;+C$$

10. Evaluate $\int \frac{1}{x^3+8}\, dx$

At the beginning of this document we said that we would not integrate cubics but this is factorable. Using $A^3 + B^3 = (A+B)(A^2 - AB + B^2)$ we get

$$\int \frac{1}{x^3+8}\, dx \;=\; \int \frac{1}{(x+2)\,(x^2-2x+4)}\, dx$$

The question reduces to integrating a linear and a quadratic factor.

$$\frac{1}{(x+2)\,(x^2-2x+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+4}$$ partial fraction decomposition

$$\frac{1}{(x+2)\,(x^2-2x+4)} = \frac{A(x^2-2x+4)}{(x+2)\,(x^2-2x+4)} + \frac{(Bx+C)(x+2)}{(x^2-2x+4)(x+2)}$$ common denominator

$$1 = A(x^2-2x+4) + (Bx+C)(x+2)$$ equate numerators

$$1 = Ax^2 - 2Ax + 4A + Bx^2 + Bx + Cx + 2C$$ expand

$$1 = (A+B)\,x^2 + (-2A+B+C)x + (4A+2C)$$ group terms

Equate polynomials:

$$0x^2 + 0x + 1 = (A+B)\,x^2 + (-2A+B)x + (4A+2C)$$

X^2: A+B = 0

X: -2A+B + C = 0

X^0: 4A + 2C = 1

Solving these yields A = 1 B=-1 C=-3/2

$$\frac{1}{(x+2)\,(x^2-2x+4)} = \frac{1}{x+2} - \frac{x+\frac{3}{2}}{x^2-2x+4}$$

$$\int \frac{1}{x^3+8}\,dx = \int \frac{1}{(x+2)\,(x^2-2x+4)}\,dx = \int \left(\frac{1}{x+2} - \frac{x+\frac{3}{2}}{x^2-2x+4} \right) dx$$

$$\int \frac{1}{x^3+8}\,dx = \ln|x+2| - \int \left(\frac{x+\frac{3}{2}}{x^2-2x+4} \right) dx$$

For the denominator in the remaining integral we have to complete the square:

$x^2 - 2x + 4 = (x-1)^2 + 3$

$$\int \frac{1}{x^3+8}\,dx = \ln|x+2| - \int \left(\frac{x+\frac{3}{2}}{(x-1)^2+3} \right) dx$$

Let u = x-1 and x = u+1 with du = dx

$$\int \frac{1}{x^3+8}\,dx = \ln|x+2| - \int \left(\frac{u+\frac{5}{2}}{u^2+3} \right) dx$$

$$\int \frac{1}{x^3+8}\,dx = \ln|x+2| - \int \left(\frac{u}{u^2+3} \right) dx - \frac{5}{2}\int \left(\frac{1}{u^2+3} \right) dx$$

$$\int \frac{1}{x^3+8}\,dx = \ln|x+2| - \frac{1}{2}\ln|u^2+3| - \frac{5}{2\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}} + C$$

$$\int \frac{1}{x^3+8}\,dx = \ln|x+2| - \frac{1}{2}\ln|(x-1)^2+3| - \frac{5}{2\sqrt{3}}\tan^{-1}\frac{(x-1)}{\sqrt{3}} + C$$

$$\int \frac{1}{x^3+8}\,dx = \ln|x+2| - \frac{1}{2}\ln|\,x^2-2x+4| - \frac{5}{2\sqrt{3}}\tan^{-1}\frac{(x-1)}{\sqrt{3}} + C$$

11. Integrate $\int \frac{1}{x^4-1}\,dx$

The denominator can be factored.

$$x^4 - 1 = \ (x^2 - 1)\,(x^2 + 1)$$

$$x^4 - 1 = (x - 1)\,(x + 1)(x^2 + 1)$$

$$\frac{1}{x^4-1} = \frac{1}{(x-1)\,(x+1)(x^2+1)}$$

$$\int \frac{1}{x^4-1}\,dx = \int \frac{1}{(x-1)\,(x+1)(x^2+1)}\,dx$$

Now do a partial fraction expansion on the new integrand:

$$\frac{1}{(x-1)\,(x+1)(x^2+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{Cx+D}{(x^2+1\,)}$$

Partial fraction expansion:

$$\frac{1}{(x-1)\,(x+1)(x^2+1)} = \frac{A(x+1)(x^2+1\,)}{(x-1)(x+1)(x^2+1\,)} + \frac{B\,(x-1)\,(x^2+1\,)}{(x-1)(x+1)\,(x^2+1\,)} + \frac{(Cx+D)\,(x-1)(x+1)}{(x-1)(x+1)(x^2+1\,)}$$

Equate numerators:

$$1 = \ A(x+1)(x^2+1\,) + B\,(x-1)\,(x^2+1\,) + \ (Cx+D)(x-1)(x+1)$$

Expand:

$$1 = \ A\,(x^3 + x^2 + x + 1) + B\ (x^3 - x^2 + x - 1) + \ (Cx + D)\,(x^2 - 1)$$

$$1 = \ A\,(x^3 + x^2 + x + 1) + B\ (x^3 - x^2 + x - 1) + \ (\,Cx^3 + Dx^2 - Cx - D)$$

Group terms:

$$1 = (A + B + C)x^3 + \ (A - B + D)x^2 + \ (A + B - C)x + \ (A - B - D)$$

Coefficients of polynomials are equal – generate simultaneous equations:

$A + B + C = 0$

$A - B + D = 0$

$A + B - C = 0$

$A - B - D = 1$

Solve these simultaneous equations – I use the TI 84 RREF command

A = ¼ B = -1/4 C = 0 D= - ½

$$\frac{1}{(x-1)\,(x+1)(x^2+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{Cx+D}{(x^2+1\,)}$$

$$\frac{1}{(x-1)\,(x+1)(x^2+1)} = \frac{1}{4}\,\frac{1}{(x-1)} - \frac{1}{4}\,\frac{1}{(x+1)} - \frac{1}{2}\,\frac{1}{(x^2+1\,)}$$

$$\int \frac{1}{x^4-1}\,dx = \int \frac{1}{(x-1)\,(x+1)(x^2+1)}\,dx = \int \left(\frac{1}{4}\,\frac{1}{(x-1)} - \frac{1}{4}\,\frac{1}{(x+1)} - \frac{1}{2}\,\frac{1}{(x^2+1\,)}\right)dx$$

$$\int \frac{1}{x^4-1}\,dx = \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\tan^{-1}x + Const.$$

12. Evaluate $\int \frac{1}{x^4+1}\ dx$

This integral is more difficult than the previous one but the denominator can be factored. We cannot factor it into real linear factors but we can factor it into a product of quadratics.

If we allow ourselves to temporarily use complex roots, we can see that x^4+1 factors.

First we let $\quad x^4 + 1 = 0$

$x^4 = -1 \quad$ so $\quad x = (-1)^{1/4}$

Using DeMoivres theorem this becomes $x = \cos\frac{\pi}{4} + \ i\sin\frac{\pi}{4} = cis\,\frac{\pi}{4}$

Since roots occur in complex conjugate pairs, we also get

$x = \cos\frac{\pi}{4} - \ i\sin\frac{\pi}{4} = \ cis\left(-\frac{\pi}{4}\right)$

The next root must occur at $2\pi/4$ or $\frac{\pi}{2}$ away:

This is at $x = \cos\frac{3\pi}{4} + \ i\sin\frac{3\pi}{4} = \ cis\,\frac{3\pi}{4}$

The conjugate of this root is $x = \cos\frac{3\pi}{4} - \ i\sin\frac{3\pi}{4} = cis\,\frac{3\pi}{4}$

We can now factor $x^4 + 1$:

$x^4 + 1 = (x - r_1)\,(x - r_2)\,(x - r_3)\,(x - r_4)$

$x^4 + 1 = \left(x - cis\left(\frac{\pi}{4}\right)\right)\left(x - cis\left(-\frac{\pi}{4}\right)\right)\left(x - cis\left(\frac{3\pi}{4}\right)\right)\left(x - cis\left(-\frac{3\pi}{4}\right)\right)$

$x^4 + 1 = \left(x^2 - 2\cos\frac{\pi}{4}\,x + 1\right)\left(x^2 - 2\cos\frac{3\pi}{4}\,x + 1\right)$

$x^4 + 1 = \left(x^2 - \sqrt{2}\,x + 1\right)\left(x^2 + \sqrt{2}\,x + 1\right)$

The equality can be verified by distributing the right hand side.

$$\frac{1}{x^4+1} = \frac{1}{(x^2-\sqrt{2}\,x+1)\,(x^2+\sqrt{2}\,x+1)}$$

$$\frac{1}{x^4+1} = \frac{1}{(x^2-\sqrt{2}\,x+1)\,(x^2+\sqrt{2}\,x+1)} = \frac{(Ax+B)}{(x^2-\sqrt{2}\,x+1)} + \frac{(Cx+D)}{(x^2+\sqrt{2}\,x+1)}$$

$$\frac{1}{(x^2-\sqrt{2}\,x+1)\,(x^2+\sqrt{2}\,x+1)} = \frac{(Ax+B)(x^2+\sqrt{2}\,x+1)}{(x^2-\sqrt{2}\,x+1)} + \frac{(Cx+D)(x^2-\sqrt{2}\,x+1)}{(x^2+\sqrt{2}\,x+1)}$$ common denominator

$$1 = (Ax+B)(x^2+\sqrt{2}\,x+1) + (Cx+D)(x^2-\sqrt{2}\,x+1)$$ equate numerators

$$1 = (Ax^3+\sqrt{2}\,A\,x^2+Bx^2+Ax+\sqrt{2}\,B\,x+B) + (Cx^3-\sqrt{2}\,Cx^2+Dx^2+Cx-\sqrt{2}\,Dx+D)$$

Equate coefficients of x:

A+C = 0

$\sqrt{2}\,A+B-\sqrt{2}\,C+D=0$

$A+\sqrt{2}B+C-\sqrt{2}D=0$

B+D = 1

I get the following: $A=\frac{\sqrt{2}}{4}$ $\quad B=\frac{1}{2}=\frac{2}{4}$ $\quad C=-\frac{\sqrt{2}}{4}$ $\quad D=\frac{1}{2}=\frac{2}{4}$

$$\frac{1}{x^4+1} = \frac{1}{(x^2-\sqrt{2}\,x+1)\,(x^2+\sqrt{2}\,x+1)} = \frac{(Ax+B)}{(x^2-\sqrt{2}\,x+1)} + \frac{(Cx+D)}{(x^2+\sqrt{2}\,x+1)}$$

$$\frac{1}{x^4+1} = \frac{1}{4}\,\frac{(\sqrt{2}\,x+2)}{(x^2-\sqrt{2}\,x+1)} + \frac{1}{4}\,\frac{(-\sqrt{2}\,x+2)}{(x^2+\sqrt{2}\,x+1)}$$ this is the partial fraction expansion

Now we have to complete the square:

$$\frac{1}{x^4+1} = \frac{1}{4}\,\frac{(\sqrt{2}\,x+2)}{\left(x^2-\sqrt{2}\,x+\frac{1}{2}+\frac{1}{2}\right)} + \frac{1}{4}\,\frac{(-\sqrt{2}\,x+2)}{\left(x^2+\sqrt{2}\,x+\frac{1}{2}+\frac{1}{2}\right)}$$

$$\frac{1}{x^4+1} = \frac{1}{4}\,\frac{(\sqrt{2}\,x+2)}{\left(\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}\right)} + \frac{1}{4}\,\frac{(-\sqrt{2}\,x+2)}{\left(\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}\right)}$$

$$\int \frac{1}{x^4+1}\, dx = \int \left(\frac{1}{4} \frac{(\sqrt{2}\ x+2)}{\left(\left(x-\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\right)} + \frac{1}{4} \frac{(-\sqrt{2}\, x+2)}{\left(\left(x+\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\right)} \right) du$$

$$\int \frac{1}{x^4+1}\, dx = \frac{1}{4} \int \frac{(\sqrt{2}\ x+2)}{\left(\left(x-\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\right)}\, dx + \frac{1}{4} \int \frac{(-\sqrt{2}\, x+2)}{\left(\left(x+\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\right)}\, dx$$

Let $u = x - \frac{1}{\sqrt{2}}$ $\quad x = u + \frac{1}{\sqrt{2}}$ $\qquad$ Let $v = x + \frac{1}{\sqrt{2}}$ $\quad x = v - \frac{1}{\sqrt{2}}$

$$\int \frac{1}{x^4+1}\, dx = \frac{1}{4} \int \frac{\left(\sqrt{2}\ \left(u+\frac{1}{\sqrt{2}}\right)+2\right)}{\left(u^2 + \frac{1}{2}\right)}\, du + \frac{1}{4} \int \frac{\left(-\sqrt{2}\ \left(v-\frac{1}{\sqrt{2}}\right)+2\right)}{\left(v^2 + \frac{1}{2}\right)}\, dv$$

$$\int \frac{1}{x^4+1}\, dx = \frac{1}{4} \int \frac{(\sqrt{2}\, u+3)}{\left(u^2 + \frac{1}{2}\right)}\, du + \frac{1}{4} \int \frac{(-\sqrt{2}\, v+3)}{\left(v^2 + \frac{1}{2}\right)}\, dv$$

$$\int \frac{1}{x^4+1}\, dx = \frac{\sqrt{2}}{8} \ln\left(u^2 + \frac{1}{2}\right) + \frac{3}{4}\sqrt{2}\ \tan\sqrt{2}\, u - \frac{\sqrt{2}}{8} \ln\left(v^2 + \frac{1}{2}\right) + \frac{3}{4}\sqrt{2}\ \tan\sqrt{2}\, v + C$$

$$\int \frac{1}{x^4+1}\, dx = \frac{\sqrt{2}}{8} \ln\left(\left(x-\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\right) + \frac{3}{4}\sqrt{2}\ \tan\sqrt{2}\ \left(x-\frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{8} \ln\left(\left(x+\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\right) + \frac{3}{4}\sqrt{2}\ \tan\sqrt{2}\ \left(x+\frac{1}{\sqrt{2}}\right) + C$$

NOW THAT'S AN INTEGRAL!! In fact, using this method, you can integrate anything of the form $\int \frac{1}{1+x^n}\, dx$