# INTEGRATING POWERS OF SECANT BY PYTHAGOREAN IDENTITY

For even powers of secant we will need the Pythagorean relation $1 + tan^2 x = sec^2 x$. We will also need the relationship $d(tan\, x) = sec^2 x\ dx$. And with this we will also need Pascal's triangle:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

The basic idea is to peel off a secant squared and replace the remaining power of secant with tangent (via the Pythagorean relation). This method works only for even powers of secant.

We start with the two most basic forms:

$$\int sec\, x\ dx = ln|sec\, x + tan\, x| + c$$

$$\int sec^2 x\ dx = tan\, x + c$$

Now we consider the more general cases:

$$\int sec^4 x\ dx = \int sec^2 x\ sec^2 x\ dx$$

$$\int sec^4 x\ dx = \int (1 + tan^2 x)\ sec^2 x\ dx$$

$$\int sec^4 x\ dx = \int sec^2 x\ dx + \int tan^2 x\ sec^2 x\ dx$$

$$\int sec^4 x\ dx = tan\, x + \frac{1}{3} tan^3 x + c$$

$$\int sec^6 x \ \ dx = \int sec^4 x \ sec^2 x \ \ dx$$

$$\int sec^6 x \ \ dx = \int (1 + tan^2 x)^2 \ sec^2 x \ \ dx$$

$$\int sec^6 x \ \ dx = \int (1 + 2 \ tan^2 x + tan^4 x) \ \ sec^2 x \ \ dx$$ let u = tan x

$$\int sec^6 x \ \ dx = tan\, x + \frac{2}{3} tan^3 x + \frac{1}{5} tan^5 x + c$$

$$\int sec^8 x \ dx = \int sec^6 x \ sec^2 x \ dx$$

$$\int sec^8 x \ dx = \int (1 + tan^2 x)^3 \ sec^2 x \ dx$$

$$\int sec^8 x \ dx = \int (1 + 3\, tan^2 x + 3\, tan^4 x + tan^6 x) \ \ sec^2 x \ dx$$ let u = tan x

$$\int sec^8 x \ dx = tan\, x + tan^3 x + \frac{3}{5} tan^5 x + \frac{1}{7} \ tan^7 x + c$$

$$\int sec^{10} x \ dx = \int sec^8 x \ sec^2 x \ \ dx$$

$$\int sec^{10} x \ dx = \int (1 + tan^2 x)^4 \ sec^2 x \ \ dx$$

$$\int sec^{10} x \ dx = \int (1 + 4\, tan^2 x + 6\, tan^4 x + 4\, tan^6 x + tan^8 x) \ \ sec^2 x \ \ dx$$

Let u = tan x

$$\int sec^{10} x \ dx = tan\, x + \frac{4}{3} tan^3 x + \frac{6}{5} tan^5 x + \frac{4}{7} tan^7 x + \frac{1}{9} tan^9 x + c$$

For odd powers of secant, an integration by parts is needed. We will have to generate a recursion. In essence the integration by parts is recreating the reduction formula derived in a previous handout (also in the text book).

We start with the basics:

$\int sec\, x\ dx = ln|sec\, x + tan\, x| + c$

$$\int sec^3 x\ dx = \int sec\, x\ sec^2 x\ dx \qquad let\ u = sec\, x \qquad dv = sec^2 x\ dx$$

$$du = sec\, x\, tan\, x\ dx \qquad v = tan\, x$$

$\int u\, dv = uv - \int v\, du$

$\int sec^3 x\ dx = \int sec\, x\ sec^2 x\ dx = uv - \int v\, du$

$\int sec^3 x\ dx = sec\, x\, tan\, x - \int tan\, x\ sec\, x\ tan\, x\ dx$

$\int sec^3 x\ dx = sec\, x\, tan\, x - \int tan^2 x\ sec\, x\ dx$

$\int sec^3 x\ dx = sec\, x\, tan\, x - \int (sec^2 x - 1)\ sec\, x\ dx$

$\int sec^3 x\ dx = sec\, x\, tan\, x - \int sec^3 x\ dx + \int sec\, x\ dx$

$2\int sec^3 x\ dx = sec\, x\, tan\, x + \int sec\, x\ dx$

$2\int sec^3 x\ dx = sec\, x\, tan\, x + ln|sec\, x + tan\, x| + c$

$\int sec^3 x\ dx = \frac{1}{2} sec\, x\, tan\, x + \frac{1}{2} ln|sec\, x + tan\, x| + c$

$\int sec^5 x\ dx = \int sec^3 x\, sec^2 x\ \ dx$ $\quad$ $let\ u = sec^3 x$ $\quad$ $dv = sec^2 x\ \ dx$

$du = 3\, sec^2 x\ \ sec\, x\ tan\, x\ \ dx$ $\quad$ $v = tan\, x$

$du = 3\, sec^3 x\ \ \ tan\, x\ \ dx$

$\int sec^5 x\ dx = sec^3 x\ tan\, x - \int\ tan\, x\ \ \ 3\, sec^3 x\ \ tan\, x\ \ dx$

$\int sec^5 x\ dx = sec^3 x\ tan\, x - 3\int\ tan^2 x\ \ sec^3 x\ dx$

$\int sec^5 x\ dx = sec^3 x\ tan\, x - 3\int\ (sec^2 x - 1)\ \ sec^3 x\ dx$

$\int sec^5 x\ dx = sec^3 x\ tan\, x - 3\int\ sec^5 x\ dx\ + 3\int sec^3 x\ \ dx$

$4\int sec^5 x\ dx = sec^3 x\ tan\, x\ + 3\int sec^3 x\ \ dx$

$\int sec^5 x\ dx\ = \frac{1}{4}\, sec^3 x\ tan\, x\ + \frac{3}{4}\int sec^3 x\ \ dx$

Now we would have to do it again for secant cubed. But we already have the result from the previous question. So we can substitute.

$\int sec^5 x\ dx\ = \frac{1}{4}\, sec^3 x\ tan\, x\ + \frac{3}{8}\ (\ sec\, x\, tan\, x\ + \ \ ln|sec\, x\ + tan\, x|\ )\ + \ c$

Obviously this is a tedious and difficult method.

$\int sec^7 x\ dx = \int sec^5 x\ sec^2 x\ dx$ $\quad$ $let\ u = sec^5 x$ $\quad$ $dv = sec^2 x\ dx$

$du = 5\, sec^4 x\, sec\, x\, tan\, x\ dx$ $\quad$ $v = tan\, x$

$du = 5\, sec^5 x\ tan\, x\ dx$

$\int sec^7 x\ dx = sec^5 x\, tan\, x - \int tan\, x\ \ 5\, sec^5 x\ tan\, x\ dx$

$\int sec^7 x\ dx = sec^5 x\, tan\, x - 5\int tan^2 x\ \ sec^5 x\ \ dx$

$\int sec^7 x\ dx = sec^5 x\, tan\, x - 5\int\ (sec^2 x - 1)\ \ sec^5 x\ \ dx$

$\int sec^7 x\ dx = sec^5 x\, tan\, x - 5\int\ \ sec^7 x\ \ dx + 5\int\ \ sec^5 x\ \ dx$

$6\int sec^7 x\ dx = sec^5 x\, tan\, x\ + 5\int\ \ sec^5 x\ \ dx$

$$\int sec^7 x \; dx \;=\frac{1}{6}\; sec^5 x \, tan\, x \;+\; \frac{5}{6}\int \; sec^5 x \;\; dx$$

Now we would have to do this for secant to the 5th power. And redo it for secant to the 3rd power. For odd powers of secant, it seems best to get the recursion formula and stick with that. These questions will be tedious, regardless how you do them, but with the formula it requires less thought on a somewhat lengthy and tedious process.