POWER SERIES AND THE FUNCTIONS THEY REPRESENT
GEOMETRIC SERIES AS THE STARTING POINT

When we say that $f(x) = \sum_{n=1}^{\infty} a_n \, x^n$ we really mean that they are identically equal. It means that many functions can be represented as power series. They are actually equal to each other. This gives us a brand new way to view functions. An important question in mathematics is to find the power series for a given function.

There is an entire theory built around this and many important functions (Trigonometric functions, exponential functions, logarithmic functions, Bessel Functions, Hypergeometric functions, Confluent Hypergeometric functions) can be represented by power series.

Before we get into this topic full force, we start with the simplest case of a power series and the function it represents. The power series we are talking about is the geometric series.

We know from high school that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 \ldots$$

This is a geometric series with r=x.

In other words, if we have the power series $\sum_{n=0}^{\infty} a_n \, x^n$ and we let all the $a_n = 1$, we get a geometric series.

We could modify the above geometric to series to get:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n \, x^n = 1 - x + x^2 - x^3 + x^4 \ldots$$

This is also a geometric series but with r = -x.

Even with this simple case there is a lot we can accomplish! If we recall that power series can differentiated and integrated (term by term) we will see just how much this simple series gives us. As a note, integrating or differentiating a power series will not change the interval of convergence – it might change convergence at the endpoints of the interval – but that is the only change. So when integrating or differentiating a power series, convergence at the endpoints must be checked.

EXAMPLE:

Consider $f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 + \cdots$

Let's integrate both sides:

$$\int \frac{1}{1+x}\, dx = \int (1 - x + x^2 - x^3 + x^4 + \cdots)\, dx$$

$$\ln|1+x| = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots + C$$

We can show that the constant of integration, C must equal zero by setting x = 0 in the above equation.

We could write this in a very formal way as follows:

$$\int \frac{1}{1+x}\, dx = \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx$$

We are allowed to bring the integral inside the summation:

$$\int \frac{1}{1+x}\, dx = \sum_{n=0}^{\infty} \int (-1)^n x^n \, dx$$

$$\int \frac{1}{1+x}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$\ln|1+x| = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

We now have a power series representation for the natural logarithm.

A note on the interval of convergence: for the power series of $\frac{1}{1+x}$ the interval is -1<x<1

The interval of convergence for ln(1+x) is $-1 < x \leq 1$. Integrating the series can only affect the convergence at the endpoints of the interval – otherwise the interval of convergence is unchanged. **I really do not want to obfuscate the primary message of this handout – which is the ability to get new power series using derivatives and integrals of the geometric series. But the textbooks do talk about the interval of convergence quite a bit – that the interval of convergence is essentially unchanged under integration or differentiation – except possibly at the endpoints.**

EXAMPLE

We can consider the power series for the function

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

If we integrate both sides we get

$$\int \frac{1}{1-x}\, dx = \int (1 + x + x^2 + x^3 + x^4 + \cdots)\, dx$$

$$-\ln|1-x| = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$$

$$\ln|1-x| = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} +$$

Again the constant of integration can be shown to equal 0.

If we did it formally it would look like this:

$$\int \frac{1}{1-x}\, dx = \int \sum_{n=0}^{\infty} x^n \, dx$$

We are allowed to bring the integral inside the summation:

$$\int \frac{1}{1-x}\, dx = \sum_{n=0}^{\infty} \int x^n\, dx$$

$$\int \frac{1}{1-x}\, dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$-\ln|1-x| = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$\ln|1-x| = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

We could have derived this same expression from the previous result, by changing x into -x.

The interval of convergence for $\frac{1}{1-x}$ is $\quad -1 < x < 1$

The interval of convergence for $\ln|1-x|$ is $\quad -1 \leq x < 1$

EXAMPLE

Something numerical: consider the result for ln (1+x).

$$\ln|\ 1\ +\ x\ |\ =\ \sum_{n\,=\,0}^{\infty}\ (-1)^n \frac{x^{n+1}}{n+1}\ =\ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$$

Now let x = 1:

$$\ln|\ 2\ |\ =\ \sum_{n\,=\,0}^{\infty}\ (-1)^n \frac{1}{n+1}\ =\ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$$

The value of the natural logarithm of 2 is equal to the alternating harmonic series.

That is a pretty remarkable result.

What is really interesting about it (in my opinion) is that an irrational number like ln 2 can be represented as a sum of rational numbers (a sum of fractions). It turns out that this happens all the time when dealing with infinite sums – an irrational number can be represented as an infinite series of fractions. There is an irony to it in that we emphasize that an irrational number is NOT a fraction and cannot be represented as a ratio – but it turns out that irrational numbers can be represented as an infinite sum of fractions.

EXAMPLE

$f(x) = \frac{1}{1+\,x^2}$

At first thought, this is not a geometric series – but in fact it is. For this geometric series r = - x^2. We

can write it like this:

$$\frac{1}{1+\,x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

$$\frac{1}{1+\,x^2} = \sum_{n=0}^{\infty} (-1)^n \; x^{2n}$$

A geometric series converges when -1 < r < 1. In this case the interval becomes $0 \le x^2 < 1$. This produces the interval $-1 < x < 1$. The same result could be found by the ratio test.

If we integrate both sides we get:

$$\int \frac{1}{1+\,x^2}\; dx = \int (1 - x^2 + x^4 - x^6 + x^8 -)\, dx$$

$$\int \frac{1}{1+\,x^2}\; dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} -$$

If we do this formally the derivation would look like:

$$\int \frac{1}{1+\,x^2}\; dx = \int \sum_{n=0}^{\infty} (-1)^n \; x^{2n}\; dx$$

We are allowed to bring the integral inside the summation:

$$\int \frac{1}{1+\,x^2}\; dx = \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\; dx$$

$$\int \frac{1}{1+\,x^2}\; dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

The integral $\int \frac{1}{1+\,x^2}\; dx$ is equal to arc tan x. We now have a power series for arc tan x.

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} -$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

The interval of convergence is $-1 < \; x \le \; 1$

In the above, we neglected the constant of integration. It turns out that it is zero. This can be seen using arc tan(0) = 0. Plug x=0 into the above and this will show that the constant of integration is zero.

EXAMPLE

Numerical result:

Let's take the previous result $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} -$

Plug in x = 1

$\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$

The series on the right converges by the alternating series test.

But we know that $\tan^{-1} 1 = \frac{\pi}{4}$

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$

This result is also pretty amazing. Pi can be expressed as an infinite sum of fractions. The series on the right is called the odd, alternating harmonic series.

This equation is called Leibnitz formula for pi. Many text books make note that the series on the right converges VERY slowly so this equation for pi is not very useful to calculate its value.

EXAMPLE

When we take the derivative of an integral of a power series, it does not change the open interval of convergence. It might change convergence at the endpoints of the interval.

If we take the length of this interval and divide it by 2, this is called the radius of convergence.

The radius of convergence is denoted by R.

The radius of convergence is directly related to the interval of convergence.

We can write the open interval of convergence in terms of R: -R < X < +R

In this question we are asked to show that the radius of convergence does not change when we take the derivative of the power series (the open interval of convergence does not change – the convergence at the endpoints might change).

Consider the series $f(x) = \sum_{n=0}^{\infty} c_n x^n$

We will look at the ratio test for convergence. It tells us that the series converges when

$\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$. The actual value of this limit is denoted by ρ.

$\rho = \lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right|$

It turns out that ρ is the reciprocal of the radius of convergence: $\rho = \frac{1}{R}$

For the original series we have

$$\rho = \lim_{n \to \infty} \left|\frac{c_{n+1}\ x^{n+1}}{c_n\ x^n}\right| = \lim_{n \to \infty} \left|\frac{c_{n+1}}{c_n}\ x\right| = |x| \lim_{n \to \infty} \left|\frac{c_{n+1}}{c_n}\right|$$

For the derivative of the series we have

$$f'(x) = \sum_{n=0}^{\infty} n\ c_n\ x^{n-1}$$

If we consider rho for this series we get:

$$\rho = \lim_{n \to \infty} \left|\frac{(n+1)\, c_{n+1}\ x^{n+1}}{n\ c_n\ x^n}\right| = \lim_{n \to \infty} \left|\frac{(n+1)\, c_{n+1}}{n\, c_n}\ x\right| =$$
$$|x| \lim_{n \to \infty} \left|\frac{c_{n+1}}{c_n}\ \frac{n+1}{n}\right|$$

$$\rho = |x| \lim_{n \to \infty} \left|\frac{c_{n+1}}{c_n}\right| \lim_{n \to \infty} \left|\frac{n+1}{n}\right| = |x| \lim_{n \to \infty} \left|\frac{c_{n+1}}{c_n}\right| \cdot 1$$

$$\rho = |x| \lim_{n \to \infty} \left|\frac{c_{n+1}}{c_n}\right|$$

We see that the expressions for rho are the same for both f(x) and f'(x). The radius of convergence is the same for both series. Taking a derivative does not change the radius of convergence. The same result holds true when you take the integral of a power series. The radius of convergence does not change. The proof is similar. Again, convergence at the endpoints of the interval might change. This must be checked.

EXAMPLE

Find a power series for the function $f(x) = \frac{5}{1 - 4x^2}$

Let's write this as $f(x) = 5 \cdot \frac{1}{1 - 4x^2}$

The fraction is a geometric series sum with r = $4x^2$

$$f(x) = 5 \ (1 + (4x^2) + (4x^2)^2 + (4x^2)^3 + (4x^2)^3 + \cdots)$$

$$f(x) = 5 \ \sum_{n=0}^{\infty} (4x^2)^n$$

Or

$$f(x) = 5 \ \sum_{n=0}^{\infty} 4^n x^{2n}$$

A geometric series converges when -1 < r < 1. In this case, this translates to $0 \leq 4x^2 < 1$.

This becomes $0 \leq x^2 < \frac{1}{4}$. This translates to the interval $-\frac{1}{2} < x < \frac{1}{2}$. The same result could be found by the ratio test. The series does not converge at the endpoints.

EXAMPLE

Find a power series for the function $f(x) = \frac{2}{3 - x}$

To find the power series we rewrite the function: $f(x) = \frac{2}{3} \frac{1}{1 - \frac{x}{3}}$

This is now seen as a geometric series with r = x/3

$$f(x) = \frac{2}{3} \frac{1}{1 - \frac{x}{3}} = \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

A geometric series converges when -1 < r < 1 or -1 < x/3 < 1

The interval of convergence is -3 < x < 3

The series does not converge at the endpoints.

EXAMPLE

Find a power series for the function $f(x) = \frac{4}{2x + 3}$

We rewrite the function as $f(x) = \frac{4}{3} \frac{1}{1 + \frac{2x}{3}}$

This is a geometric series with r = 2x/3

We can expand this as a geometric series:

$$f(x) = \frac{4}{3}\left(1 - \frac{2x}{3} + \left(\frac{2x}{3}\right)^2 - \left(\frac{2x}{3}\right)^3 \cdots\right)$$

$$f(x) = \frac{4}{3} \sum_{n=0}^{\infty} \left(-\frac{2x}{3}\right)^n$$

A geometric series converges when -1 < r < 1 or $-1 < \frac{2x}{3} < 1$

The interval of convergence is $-\frac{3}{2} < x < \frac{3}{2}$

The series does not converge at the endpoints.

EXAMPLE $f(x) = \frac{x^2}{x^4 + 16}$ find the power series

Write this as $f(x) = \frac{x^2}{16} \frac{1}{\left(1+\frac{x^4}{16}\right)}$

$$f(x) = \frac{x^2}{16}\left(1 - \frac{x^4}{16} + \left(\frac{x^4}{16}\right)^2 - \left(\frac{x^4}{16}\right)^3 + \left(\frac{x^4}{16}\right)^4 \cdots\right)$$

$$f(x) = \frac{x^2}{16} \sum_{n=0}^{\infty} \left(-\frac{x^4}{16}\right)^n$$

This is the answer. It can be rewritten in the following form as well:

$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x)^{4n+2}}{16^{n+1}}$ or

$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x)^{4n+2}}{2^{4n+4}}$ or

$$f(x) = \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^{4n+2}$$

When we write $f(x) = \frac{x^2}{16} \frac{1}{\left(1+\frac{x^4}{16}\right)}$ we see a geometric series with $r = \frac{x^4}{16}$

In this case we see that r must be non-negative so we have convergence when $0 \leq r < 1$

This means $0 \leq \frac{x^4}{16} < 1$ or $0 \leq x^4 < 16$. This seems to become $0 \leq x < 2$. Since the power of x is even, it does allow negative values of x. The true interval is $-2 < x < 2$.

EXAMPLE

Find a power series for $f(x) = \frac{x}{2x^2 + 1}$

Write this as $f(x) = x \frac{1}{1 + 2x^2}$

We see a geometric series with r = - 2x^2

$$f(x) = x\ (1 - 2x^2 + (2x^2)^2 - (2x^2)^3 + (2x^2)^4 \cdots)$$

$$f(x) = x \sum_{n=0}^{\infty} (-1)^n (2x^2)^n$$

This can also be written as

$$f(x) = \sum_{n=0}^{\infty} (-1)^n (2)^n x^{2n+1}$$

For a geometric series convergence occurs when -1 < r < 1. In this case, r cannot be positive so we have $-1 < r \leq 0$. This becomes $0 \leq 2x^2 < 1$ or $0 \leq x^2 < \frac{1}{\sqrt{2}}$.

The interval of convergence for x becomes $-\frac{1}{\sqrt[4]{2}} < x < \frac{1}{\sqrt[4]{2}}$

EXAMPLE

Find a power series expansion for the function $f(x) = \frac{1}{1 + x}$ that is centered about x = 3

We answered this question previously and found a power series centered about x = 0.

The answer was $f(x) = \sum_{n=0}^{\infty} (-1)^n x^n$

The interval of convergence for this series was -1 < x < 1

This interval is centered about x = 0.

To get a new power series (for the same function) we do the following:

$$f(x) = \frac{1}{1+x} = \frac{1}{(1+3) + (x-3)} = \frac{1}{4 + (x-3)}$$

$$f(x) = \frac{1}{4} \frac{1}{1 + \frac{(x-3)}{4}}$$

This is a geometric series with r = - (x-3)/4

$$f(x) = \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x-3}{4}\right)^n$$

A geometric series converges when -1 < r <1 so we have $-1 < \frac{1}{4}(x-3) < 1$

Or $-4 < (x-3) < 4$

The interval of convergence for this series is 1 < x < 7

The series does not converge at the endpoints.

EXAMPLE

Find a power series expansion for the function $f(x) = \frac{2}{3 - x}$ centered about x = 1

We answered this question before and found a power series centered about x = 0. The answer was

$$f(x) = \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

The interval of convergence was -3 < x < 3

We want a new power series centered about x=1

We do the following:

$$f(x) = \frac{2}{3 - x} = f(x) = \frac{2}{2 - (x-1)} = \frac{1}{1 - \frac{(x-1)}{2}}$$

We have a geometric series with r = (x-1)/2

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{x-1}{2}\right)^n$$

A geometric series converges when -1 < r < 1

This means that $-1 < \frac{(x-1)}{2} < 1$ or $-2 < (x-1) < 2$

The interval of convergence is -1 < x < 3

The series does not converge at the endpoints.

EXAMPLE

Find a power series expansion for the function $f(x) = \frac{4}{2x + 3}$ centered about x = 1

We answered this question before and we found a power series centered at x = 0

$$f(x) = \frac{4}{3} \sum_{n=0}^{\infty} \left(-\frac{2x}{3}\right)^n$$

The interval of convergence for this power series is -3/2 < x < 3/2

We want a new power series so we write

$$f(x) = \frac{4}{2x + 3} = \frac{4}{2(x-1 + 1) + 3} = \frac{4}{2(x-1) + 5} = \frac{4}{5} \frac{1}{1 + \frac{2}{5}(x-1)}$$

This is a geometric series with r = -(2/5) (x-1)

$$f(x) = \frac{4}{5} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{5}\right)^n (x - 1)^n$$

A geometric series converges when -1 < r < 1 so $-1 < \frac{2}{5}(x-1) < 1$

So $-\frac{5}{2} < (x-1) < \frac{5}{2}$

The interval of convergence of this series is -3/2 < x < 7/2

EXAMPLE

Find A power series for the function $f(x) = \frac{x - 1}{x + 2}$ centered about x = 1

We write $f(x) = \frac{x - 1}{x + 2} = \frac{x - 1}{(x - 1) + 3} = \frac{(x-1)}{3} \frac{1}{\frac{(x-1)}{3} + 1}$

We have a geometric series with r = - (x-1)/3

$$f(x) = \frac{(x-1)}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x - 1}{3}\right)^n = \sum_{n=0}^{\infty} (-1)^n \left(\frac{x - 1}{3}\right)^{n+1}$$

A geometric series converges when -1 < r < 1 so we have $-1 < \frac{(x-1)}{3} < 1$

This gives $-3 < (x-1) < 3$

This becomes $-2 < x < 4$

The series diverges at the endpoints.

EXAMPLE

Find a power series for $f(x) = \frac{1}{(1+x)^2}$

We know that $g(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

If we take a derivative of both sides we get

$$-\frac{1}{(1+x)^2} = \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n x^n$$

$$-\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} (-1)^n \frac{d}{dx} x^n$$

$$-\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} (-1)^n \; n \; x^{n-1}$$

$$\frac{1}{(1+x)^2} = -\sum_{n=0}^{\infty} (-1)^n \; n \; x^{n-1}$$

$$\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} (-1)^{n+1} \; n \; x^{n-1}$$

This can also be written as $\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n+1} \; n \; x^{n-1}$

The interval of convergence for both series is $-1 < x < 1$

This can be determined by the ratio test.

EXAMPLE

Find a power series for $f(x) = \ln\ (5 - x)$

We start with the function $\frac{1}{5-x} = \frac{1}{5}\frac{1}{1-\frac{x}{5}} = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n$

This becomes $\frac{1}{5-x} = \sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^{n+1} x^n$

We integrate both sides: $\int \frac{1}{5-x}\,dx = \int\left(\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^{n+1} x^n\right)dx$

We are allowed to bring the integral inside the summation:

$$\int \frac{1}{5-x}\,dx = \sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^{n+1}\int x^n\,dx$$

$$-\ln|5-x| = \sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^{n+1}\frac{x^{n+1}}{n+1}$$

$$\ln|5-x| = -\sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{x}{5}\right)^{n+1}$$

Interval of convergence for $\frac{1}{5-x}$ $\quad -5 < x < 5 \quad$ by geometric series

Interval of convergence for $\ln|5-x|$ $\quad -5 \le x < 5 \quad$ by ratio test

Note that the interval of convergence only changes at the end point.

EXAMPLE

Find a power series for $f(x) = x^2 \tan^{-1} x^3$

We know a power series for inverse tangent: $\tan^{-1} u = u - \frac{u^3}{3} + \frac{u^5}{5} - \frac{u^7}{7} + \cdots$

$$\tan^{-1} u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{2n+1}$$

$$\tan^{-1} x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{2n+1}$$

$$\tan^{-1} x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1}$$

$$x^2 \tan^{-1} x^3 = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1}$$

$$x^2 \tan^{-1} x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+5}}{2n+1}$$

Interval of convergence $-1 < x \leq 1$ by ratio test

EXAMPLE

Find a power series for $f(x) = \frac{x}{(1+4x)^2}$

Start with $\frac{1}{1+4x} = \sum_{n=0}^{\infty} (-1)^n (4)^n x^n$

Take the derivative: $\frac{-4}{(1+4x)^2} = \sum_{n=0}^{\infty} (-1)^n (4)^n \, n \, x^{n-1}$

Divide both sides by -4: $\frac{1}{(1+4x)^2} = \sum_{n=0}^{\infty} (-1)^{n-1} (4)^{n-1} \, n \, x^{n-1}$

Multiply both sides by x: $\frac{x}{(1+4x)^2} = \sum_{n=0}^{\infty} (-1)^{n-1} (4)^{n-1} \, n \, x^{n}$

Alternate: $\frac{x}{(1+4x)^2} = \sum_{n=1}^{\infty} (-1)^{n-1} (4)^{n-1} \, n \, x^{n}$

EXAMPLE

Find a power series for $f(x) = \left(\frac{x}{2 - x}\right)^3$

Start with:

$$\frac{1}{2 - x} = \frac{1}{2}\frac{1}{1 - \frac{x}{2}} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} x^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} x^n$$

Take a derivative:

$$\frac{1}{(2-x)^2} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} n\, x^{n-1}$$

Take another derivative:

$$\frac{2}{(2-x)^3} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} n\,(n-1)\, x^{n-1}$$

Multiply both sides by ½ :

$$\frac{1}{(2-x)^3} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+2} n\,(n-1)\, x^{n-1}$$

Multiply both side by x^3:

$$\frac{x^3}{(2-x)^3} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+2} n\,(n-1)\, x^{n+2}$$

$$\left(\frac{x}{2 - x}\right)^3 = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+2} n\,(n-1)\, x^{n+2}$$

The first two terms of the series are zero. We can write it like this:

$$\left(\frac{x}{2 - x}\right)^3 = \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^{n+2} n\,(n-1)\, x^{n+2}$$

EXAMPLE

Find a power series for $f(x) = \frac{x-1}{x+2}$

Start with:

$$\frac{1}{x+2} = \frac{1}{2}\frac{1}{1+\frac{x}{2}} = \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2}\right)^n x^n = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2}\right)^{n+1} x^n$$

Multiply both sides by x-1:

$$\frac{\mathrm{x}-1}{x+2} = (x-1)\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2}\right)^{n+1} x^n$$

$$\frac{\mathrm{x}-1}{x+2} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2}\right)^{n+1} x^{n+1} - \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2}\right)^{n+1} x^n$$

These two series can be combined into one using a technique called index shifting. For the first series we let m = n+1. We write out the first term of the 2nd series – which is just ½ .

$$\frac{\mathrm{x}-1}{x+2} = \sum_{m=1}^{\infty} (-1)^{m-1} \left(\frac{1}{2}\right)^m x^m - \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2}\right)^{n+1} x^n$$

$$\frac{\mathrm{x}-1}{x+2} = \sum_{m=1}^{\infty} (-1)^{m-1} \left(\frac{1}{2}\right)^m x^m - \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{2}\right)^{n+1} x^n - \frac{1}{2}$$

Replace all indices by k:

$$\frac{\mathrm{x}-1}{x+2} = \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{1}{2}\right)^k x^k - \sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{2}\right)^{k+1} x^k - \frac{1}{2}$$

We can now combine the two series:

$$\frac{\mathrm{x}-1}{x+2} = \sum_{k=1}^{\infty} \left[(-1)^{k-1} \left(\frac{1}{2}\right)^k - (-1)^k \left(\frac{1}{2}\right)^{k+1} \right] x^k - \frac{1}{2}$$

$$\frac{\mathrm{x}-1}{x+2} = \sum_{k=1}^{\infty} \left[(-1)^{k-1} \left(\frac{1}{2}\right)^k + (-1)^{k-1} \left(\frac{1}{2}\right)^{k+1} \right] x^k - \frac{1}{2}$$

$$\frac{\mathrm{x}-1}{x+2} = \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{1}{2}\right)^k \left(1+\frac{1}{2}\right) x^k - \frac{1}{2}$$

$$\frac{\mathrm{x}-1}{x+2} = \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{1}{2}\right)^k \left(\frac{3}{2}\right) x^k - \frac{1}{2}$$

$$\frac{\mathrm{x}-1}{x+2} = \sum_{k=1}^{\infty} (-1)^{k-1}\, 3 \left(\frac{1}{2}\right)^{k+1} x^k - \frac{1}{2}$$

EXAMPLE

Find a power series for $f(x) = \frac{x - 1}{x + 2}$

Start with:

$$\frac{1}{x + 2} = \frac{1}{2}\frac{1}{1 + \frac{x}{2}} = \frac{1}{2}\left(1 - \left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^4 + \cdots\right)$$

$$\frac{1}{x + 2} = \frac{1}{2}\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{16} + \cdots\right)$$

$$\frac{1}{x + 2} = \left(\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \frac{x^4}{32} - \cdots\right)$$

Multiply both sides by x:

$$\frac{\mathrm{x}}{x + 2} = \left(\frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{8} - \frac{x^4}{16} + \frac{x^5}{32} + \cdots\right)$$

Subtract both series:

$$\frac{x-1}{x + 2} = -\frac{1}{2} + \frac{3x}{4} - \frac{3x^2}{8} + \frac{3\,x^3}{16} \cdots$$

$$\frac{x-1}{x + 2} = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3\,(-1)^{n+1}\,x^n}{2^{n+1}}$$

Believe it or not, this is the same answer as we got above. The algebra of adding series is very tedious and can confuse the issue. Sometimes the method used above is necessary – but for some cases, writing out the terms can be much better and much easier.