INFINITE SERIES P TEST AND DIRECT COMPARISON TEST

P TEST STATEMENT

You are given an infinite series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ where p is any real number. If $p > 1$ the series converges. If $p \leq 1$ then the series diverges.

COMPARISON TEST STATEMENT

You are a given an infinite series $\sum_{n=1}^{\infty} a_n$ with $a_n > 0$. Let $\sum_{n=1}^{\infty} b_n$ be a known series of positive terms, of your own choosing, that converges. If $b_n \geq a_n$ for all n, then the original series converges .

Alternately, let $\sum_{n=1}^{\infty} b_n$ be a known series of positive terms, of your own choosing, that diverges. If $b_n \leq a_n$ for all n, then the original series diverges .

The test can be relaxed a little bit. Instead of the inequalities holding <u>for all n</u>, they must hold for some N whenever n > N. So the inequality between a_n and b_n does not have to hold for all n – but it must kick in totally for large values of n.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

   The power p = 2 and is greater than 1 so the series converges

2. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

   The power p = 3 and is greater than 1 so the series converges

3. $\sum_{n=1}^{\infty} \frac{1}{n^4}$

   The power p = 4 and is greater than 1 so the series converges

4. $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$

   The power p = 1.1 and is greater than 1 so the series converges

5. $\sum_{n=1}^{\infty} \frac{1}{n}$

The power p = 1 so the series diverges

6. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

The power p = 1/2 and is less than 1 so the series diverges

7. $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$

The power p = 1/3 and is less than 1 so the series diverges

8. $\sum_{n=1}^{\infty} \frac{1}{n^{-2}}$

The power p = -2 and is less than 1 so the series diverges

9. $\sum_{n=1}^{\infty} \frac{1}{n^{0.991}}$

The power p = 0.991 and is less than 1 so the series diverges

10. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$a_n = \frac{1}{n^2+1}$ and let $b_n = \frac{1}{n^2}$

$n^2 + 1 > n^2$ for all n. Take reciprocals of this.

$\frac{1}{n^2+1} < \frac{1}{n^2}$ for all n

This means that $\sum_{n=1}^{\infty} \frac{1}{n^2+1} < \sum_{n=1}^{\infty} \frac{1}{n^2}$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then by the comparison test $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ also converges.

11. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$

$a_n = \frac{\sin^2 n}{2^n} > 0$ and let $b_n = \frac{1}{2^n} > 0$

We know $\sin^2 n < 1$

Divide both sides by 2^n

$\frac{\sin^2 n}{2^n} < \frac{1}{2^n}$ for all n

This means that $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n} < \sum_{n=1}^{\infty} \frac{1}{2^n}$

The series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series and converges to 1

By the comparison test the series $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$ must converge.

12. $\sum_{n=1}^{\infty} \frac{n}{2^n (n+1)}$

$a_n = \frac{n}{2^n (n+1)} > 0$ for all n and let $b_n = \frac{1}{2^n}$

We have the inequality $\frac{n}{n+1} < 1$ for all positive n

Divide both sides of the inequality by 2^n

$\frac{n}{2^n (n+1)} < \frac{1}{2^n}$

This means that $\sum_{n=1}^{\infty} \frac{n}{2^n (n+1)} < \sum_{n=1}^{\infty} \frac{1}{2^n}$

The series on the right is a geometric series and converges to 1.

By the comparison test, the original series converges. $\sum_{n=1}^{\infty} \frac{n}{2^n (n+1)}$ converges

13. $\sum_{n=1}^{\infty} \frac{n}{5\,n^2-4}$

$a_n = \frac{n}{5\,n^2-4} > 0$ for n > 1 and let $b_n = \frac{n}{5n^2} = \frac{1}{5n} > 0$ for n > 1

We have the inequality $5n^2 - 4 < 5n^2$ for n > 1

Take the reciprocal. We get $\frac{1}{5n^2-4} > \frac{1}{5n^2}$

Multiply both sides by n (where n>1)

$$\frac{n}{5n^2-4} > \frac{n}{5n^2} = \frac{1}{5n}$$

So the series have the inequality $\sum_{n=1}^{\infty} \frac{n}{5\,n^2-4} > \sum_{n=1}^{\infty} \frac{1}{5n} = \frac{1}{5}\sum_{n=1}^{\infty} \frac{1}{n}$

The series on the right is the harmonic series and is known to diverge. Its divergence is established by the p test or the integral test.

By the comparison test, the original series diverges. $\sum_{n=1}^{\infty} \frac{n}{5\,n^2-4}$ diveges

14. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{4n^2-7}}$

$a_n = \frac{1}{\sqrt{4n^2-7}}$ and let $b_n = \frac{1}{2n}$ both greater than zero for n > 2

We have the inequality: $4n^2 - 7 < 4n^2$ for n > 2

Take square roots: $\sqrt{4n^2 - 7} < 2n$ for n > 2

Take reciprocals: $\frac{1}{\sqrt{4n^2-7}} > \frac{1}{2n}$ for n > 2

This gives the inequality $\sum_{n=2}^{\infty} \frac{1}{\sqrt{4n^2-7}} > \sum_{n=2}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{n}$

The series on the right is the harmonic series (with the first term missing). The starting of a series does not affect convergence so the series on the right diverges.

By the comparison test, the original series diverges. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{4n^2-7}}$ diverges

15. $\sum_{k=1}^{\infty} \frac{1}{k!}$

$k! = k\,(k-1)\,(k-2)\ \cdots\ 3 \cdot 2 \cdot 1$ for k > 1

From the above product we can see the following inequality:

$k! > k\,(k-1) > (k-1)\ (k-1)$

$k! > (k-1)^2$ for k > 1

Taking reciprocals we have $\frac{1}{k!} < \frac{1}{(k-1)^2}$ for k > 1

We have the following inequality for the series:

$\sum_{k=2}^{\infty} \frac{1}{k!} < \sum_{k=2}^{\infty} \frac{1}{(k-1)^2}$

Change the index on the second series and let n = k-1

$\sum_{k=2}^{\infty} \frac{1}{k!} < \sum_{n=1}^{\infty} \frac{1}{n^2}$

The series is known to converge by either the p test or the integral test.

By the comparison test, the integral on the left must converge. $\sum_{k=1}^{\infty} \frac{1}{k!}$ Converges.

16. $\sum_{n=1}^{\infty} \frac{1}{n\,(n+1)}$

$a_n = \frac{1}{n\,(n+1)}$ and let $b_n = \frac{1}{n^2}$ both positive for n > 0

We start with the inequality $n(n+1) > n^2$ for n > 0

Taking reciprocals we get: $\frac{1}{n\,(n+1)} < \frac{1}{n^2}$ for n > 0

This gives the following inequality for series $\sum_{n=1}^{\infty} \frac{1}{n\,(n+1)} < \sum_{n=1}^{\infty} \frac{1}{n^2}$

The second series is known to converge by the p test or the integral test.

The series $\sum_{n=1}^{\infty} \frac{1}{n\,(n+1)}$ must converge by the comparison test.

17. $\sum_{n=2}^{\infty} \frac{n-1}{n^3}$

$a_n = \frac{n-1}{n^3}$ and $b_n = \frac{1}{n^2}$ for n > 2 (both terms are positive)

We start with the inequality that $n-1 < n$ for n > 2

Divide by sides by n^3

$\frac{(n-1)}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$ for n > 2

This gives the inequality for the series $\sum_{n=2}^{\infty} \frac{n-1}{n^3} < \sum_{n=2}^{\infty} \frac{1}{n^2}$

Since the series on the right converges by p test, the series on the left must converge by comparison test.

$\sum_{n=2}^{\infty} \frac{n-1}{n^3}$ converges

18. $\sum_{n=1}^{\infty} \frac{n+1}{n(n+2)}$

We start with the inequality $\frac{n+1}{n} > 1$ for n > 0

Divide both sides by n+2 where n > 0

$\frac{n+1}{n(n+2)} > \frac{1}{n+2}$ where n > 0

This gives us the following inequality for the series: $\sum_{n=1}^{\infty} \frac{n+1}{n(n+2)} > \sum_{n=1}^{\infty} \frac{1}{(n+2)}$

The integral on the right diverges by the integral test.

The series on the left diverges by the comparison test. $\sum_{n=1}^{\infty} \frac{n+1}{n(n+2)}$ diverges

19. $\sum_{n=1}^{\infty} \frac{n+1}{n(n+2)(n+3)}$

This is a bit of a nasty one so I am going to go directly to an inequality that will get us home:

$\frac{n+1}{n(n+2)(n+3)} < \frac{n+2}{n(n+2)(n+3)} = \frac{1}{n\ (n+3)} < \frac{1}{n^2}$ for n > 0 where n is an integer

This yields $\sum_{n=1}^{\infty} \frac{n+1}{n(n+2)(n+3)} < \sum_{n=1}^{\infty} \frac{1}{n^2}$

The series on the right converges by p test. The series on the left converges by comparison test.

$\sum_{n=1}^{\infty} \frac{n+1}{n(n+2)(n+3)}$ converges

20. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

We start with the following inequality:

$n^2 + 1 \leq 2n^2$ for $n \geq 1$ where n is an integer

Now take reciprocals:

$\frac{1}{n^2+1} \leq \frac{1}{2n^2}$ for $n \geq 1$ where n is an integer.

Take positive square roots of both sides:

$\frac{1}{\sqrt{n^2+1}} \geq \sqrt{\frac{1}{2n^2}} = \frac{\sqrt{2}}{2}\frac{1}{n}$ for $n \geq 1$ where n is an integer

$\frac{1}{\sqrt{n^2+1}} \geq \frac{\sqrt{2}}{2}\frac{1}{n}$

We can now form series: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}} \geq \frac{\sqrt{2}}{2} \sum_{n=1}^{\infty} \frac{1}{n}$

The series on the right is the harmonic series and diverges by p test or integral test.

The series on the left must diverge by comparison test.

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges

21. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

We start with the inequality $n^3 + 1 > n^3$ for $n \geq 1$ where n is an integer

Take square roots of both sides: $\sqrt{n^3+1} > n^{3/2}$ for $n \geq 1$

Take reciprocals to get: $\frac{1}{\sqrt{n^3+1}} < \frac{1}{n^{3/2}}$

Now form series: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}} < \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

The series on the right converges by the p test. So the series on the left must converge by comparison test.

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ converges

22. $\sum_{k=1}^{\infty} \frac{1}{2k^2 - k}$

This one is a bit tricky.

We start with the inequality: $k^2 \geq k$ for $k \geq 1$

Multiply both sides by -1: $-k^2 \leq -k$ for $k \geq 1$

Add $2k^2$ to both sides:

$2k^2 - k^2 \leq 2k^2 - k$

$k^2 \leq 2k^2 - k$

Take reciprocals: $\frac{1}{k^2} \geq \frac{1}{2k^2-k}$

Form the series: $\sum_{k=1}^{\infty} \frac{1}{k^2} \geq \sum_{k=1}^{\infty} \frac{1}{2k^2 - k}$

The series on the left converges by the p test. So the series on the right converges by comparison test.
$\sum_{k=1}^{\infty} \frac{1}{2k^2 - k}$ converges.