# L'HOSPITAL'S RULE EXPONENTIAL CASES

We have seen that L'Hospital's rule works directly when we have an indeterminate form 0/0 or $\frac{\infty}{\infty}$.

In either case we use the equation $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

L'Hospital's rule tells us that if $\lim_{x \to a} \frac{f(x)}{g(x)}$ is one of these two indeterminate forms, all we have to do is evaluate $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ and the answers will be the same.

We were able to extend this to a third indeterminate form: the form $0 \cdot \infty$.

This form looks different from the first two, but by using algebra, we can change it into one of the first two forms. So the form $0 \cdot \infty$ can be changed into 0/0 or $\infty/\infty$. Once this is done (usually by simple algebra) then we can apply L'Hospital's rule directly.

In this handout, we look at the three exponential forms $0^0, 1^\infty, \infty^0$. All of these forms are handled in the exact same way. The method of solution for all three is identical.

If $y = f(x)/g(x)$ and we take the limit of y – to get one of these three exponential forms – then we stop. Instead of taking the limit of y, we take the limit of ln y!!! The natural log saves us. By taking the natural logarithm, we can convert these three exponential forms into the third indeterminate form. The natural logarithm changes all of them into $0 \cdot \infty$

Examples:

$\ln 0^0 = 0 \ln 0 = 0\,(-\infty)$

$\ln 1^\infty = \infty\,(\ln 1) = \infty \cdot 0$

$\ln \infty^0 = 0 \ln \infty = 0 \cdot \infty$

When we evaluate the limit of ln y and get our answer, let's call it L, then the answer to the original limit is $e^L$. That's all that there is to it. Let's work some examples.

1. $\lim_{x \to 0^+} x^x$

$\lim_{x \to 0^+} x^x = 0^0$

Let $y = x^x$ then $\ln y = x \ln x$

$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} x \ln x = 0 \cdot (-\infty)$ )

$\lim_{x \to 0^+} \frac{\ln x}{x^{-1}} = -\infty/\infty$

$\lim_{x \to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x \to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x \to 0^+} -x = 0$

$\lim_{x \to 0^+} \ln y = 0$ so $\lim_{x \to 0^+} y = e^0 = 1$

$\lim_{x \to 0^+} x^x = 1$

2. $\lim_{x \to 0} (2x)^{x^2}$

$\lim_{x \to 0} (2x)^{x^2} = 0^0$

Let $y = (2x)^{x^2}$ so $\ln y = x^2 \ln 2x$

$\lim_{x \to 0} \ln y = \lim_{x \to 0} x^2 \ln 2x = 0\,(-\infty)$

$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln 2x}{x^{-2}} = -\infty/\infty$

$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{1/x}{-2\,x^{-3}} = \lim_{x \to 0} -\frac{1}{2}\, x^2 = 0$

$\lim_{x \to 0} \ln y = 0$ so $\lim_{x \to 0} y = e^0 = 1$

$\lim_{x \to 0} (2x)^{x^2} = 1$

3. $\lim_{x \to 0} (\cos x)^{\cot x}$

$$\lim_{x \to 0} (\cos x)^{\cot x} = 1^{\infty}$$

Let $y = (\cos x)^{\cot x}$

Then $\ln y = \cot x \ \ln \cos x$

$$\lim_{x \to 0} \cot x \ \ln \cos x = \cot 0 \cdot \ln \cos 0 = \infty \cdot 0$$

$$\lim_{x \to 0} \frac{\ln \cos x}{\tan x} = \frac{0}{0}$$ so use L'Hospital's rule

$$\lim_{x \to 0} -\frac{\tan x}{\sec^2 x} = -\frac{\tan 0}{\sec^2 0} = -\frac{0}{1} = 0$$

So we have $\lim_{x \to 0} \ln y = 0$ and therefore $\lim_{x \to 0} y = e^0 = 1$

$$\lim_{x \to 0} (\cos x)^{\cot x} = 1$$

4. $\lim_{x \to 0} \left(x + e^{\frac{x}{2}}\right)^{2/x}$

$$\lim_{x \to 0} \left(x + e^{\frac{x}{2}}\right)^{2/x} = (0 + e^0)^{2/0} = 1^{\infty}$$

We let $y = \left(x + e^{\frac{x}{2}}\right)^{2/x}$ so that $\ln y = \frac{2}{x} \ln\left(x + e^{x/2}\right)$

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{2}{x} \ln\left(x + e^{x/2}\right) = \frac{0}{0}$$

So use L'Hospital's rule

$$\lim_{x \to 0} 2\frac{\left(1+\frac{1}{2}e^{\frac{x}{2}}\right)}{x+e^{x/2}} = \frac{2\left(1+\frac{1}{2}\right)}{1+1} = \frac{3}{2}$$

So we have $\lim_{x \to 0} \ln y = \frac{3}{2}$ which means that $\lim_{x \to 0} y = e^{3/2}$

$$\lim_{x \to 0} \left(x + e^{\frac{x}{2}}\right)^{2/x} = e^{3/2}$$

5. $\lim_{x \to \frac{\pi}{2}} (\tan x)^{\cos x}$

$$\lim_{x \to \frac{\pi}{2}} (\tan x)^{\cos x} = \left(\tan \frac{\pi}{2}\right)^{\cos \pi/2} = (\pm\infty)^0$$

So we let y = $(\tan x)^{\cos x}$ and ln y = cos x ln(tan x)

$$\lim_{x \to \frac{\pi}{2}} \ln y = \lim_{x \to \frac{\pi}{2}} \cos x \ln(\tan x) = 0 \cdot \infty$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\ln \tan x}{\sec x} = \frac{\infty}{\infty}$$

So now we can use L'Hospital's rule

$$\lim_{x \to \frac{\pi}{2}} \frac{\left[\frac{1}{\tan x}\right] \sec^2 x}{\sec x \tan x} = \lim_{x \to \frac{\pi}{2}} \frac{\sec x}{\tan^2 x} = \frac{\infty}{\infty}$$

So we do LHospital's rule again:

$$\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{\tan^2 x} = \lim_{x \to \frac{\pi}{2}} \frac{\sec x \tan x}{2 \tan x \sec^2 x} = \lim_{x \to \frac{\pi}{2}} \frac{1}{2 \sec x} = \frac{1}{\infty} = 0$$

So we have $\lim_{x \to \frac{\pi}{2}} \ln y = 0$ and therefore $\lim_{x \to \frac{\pi}{2}} y = e^0 = 1$

$$\lim_{x \to \frac{\pi}{2}} (\tan x)^{\cos x} = 1$$

6. $\lim_{x \to \infty} x^{1/x}$

$\lim_{x \to \infty} x^{1/x} = \infty^0$

So we let y = $x^{1/x}$ and we have ln y = 1/x ln x

So we take the limit of ln y: $\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$

Now use LHospital's rule:

$\lim_{x \to \infty} \frac{1}{x} = 0$

So $\lim_{x \to \infty} \ln y = 0$ and $\lim_{x \to \infty} y = \lim_{x \to \infty} x^{1/x} = e^0 = 1$

$\lim_{x \to \infty} x^{1/x} = 1$

7. $\lim_{x \to 0} (\cos x)^{1/x^2}$

$\lim_{x \to 0} (\cos x)^{1/x^2} = 1^\infty$

Let $y = (\cos x)^{1/x^2}$ so that $\ln y = \frac{\ln(\cos x)}{x^2}$

We now have $\lim_{x \to 0} \frac{\ln \cos x}{x^2} = \frac{0}{0}$ so we can use L'Hospital's rule

$\lim_{x \to 0} \frac{\ln \cos x}{x^2} = \lim_{x \to 0} -\frac{\tan x}{2x} = \frac{0}{0}$

So we do L'Hospital's rule again:

$\lim_{x \to 0} -\frac{\sec^2 x}{2} = -\frac{1}{2}$

$\lim_{x \to 0} \ln y = -\frac{1}{2}$ so $\lim_{x \to 0} y = e^{-1/2}$

$\lim_{x \to 0} (\cos x)^{1/x^2} = e^{-1/2}$

8. $\lim_{x \to 0} (\cot x)^x$

$\lim_{x \to 0} (\cot x)^x = \infty^0$

$y = (\cot x)^x$ & $\ln y = x \ln \cot x$

$$\lim_{x \to 0} \frac{(\ln \cot x)}{1/x} = \lim_{x \to 0} \frac{\left(\frac{-\csc^2 x}{\cot x}\right)}{-1/x^2}$$

We have to rearrange this quite a bit. It will become

$$\lim_{x \to 0} \frac{x^2}{\sin^2 x} \cdot \tan x = \lim_{x \to 0} \left(\frac{x}{\sin x}\right)^2 \lim_{x \to 0} \tan x = 1 \cdot 0 = 0$$

So $\lim_{x \to 0} (\cot x)^x = e^0 = 1$

9. Show that $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

This is a remarkably important limit. Please remember it.

$y = \left(1 + \frac{1}{x}\right)^x$ $\qquad$ $\ln y = x \ln\left(1 + \frac{1}{x}\right) = \frac{\ln(1+x^{-1})}{x^{-1}}$

We take the limit of ln y and use L'Hospital's rule:

$$\lim_{x \to \infty} \frac{\ln(1 + x^{-1})}{x^{-1}} = \lim_{x \to \infty} \frac{(1 + x^{-1})\,(-1\,x^{-2})}{-1\,x^{-2}} = \lim_{x \to \infty} (1 + x^{-1}) = 1 + 0 = 1$$

So $\lim_{x \to \infty} \ln y = 1$ so $\lim_{x \to \infty} y = e^1 = e$

Therefore $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

10. $\lim_{x \to 0^+} (\sin x)^x$

$$\lim_{x \to 0^+} (\sin x)^x = 0^0$$

$$y = (\sin x)^x \qquad \ln y = x \ln \sin x$$

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} x \ln \sin x = 0 \cdot \infty$$

Use L'Hospital's rule:

$$\lim_{x \to 0^+} \frac{\ln \sin x}{x^{-1}} = \lim_{x \to 0^+} \frac{\cot x}{-x^{-2}} = \lim_{x \to 0^+} -\frac{x^2}{\tan x} = \frac{0}{0}$$

We still have an indeterminate form so we do L'Hospital' rule one more time

$$\lim_{x \to 0^+} -\frac{x^2}{\tan x} = \lim_{x \to 0^+} -\frac{2x}{\sec^2 x} = 0$$

So $\lim_{x \to 0^+} \ln y = 0$ and $\lim_{x \to 0^+} y = e^0 = 1$

$$\lim_{x \to 0^+} (\sin x)^x = 1$$

11. $\lim_{x \to 0} (\cos x - \sin x)^{1/x}$

$\lim_{x \to 0} (\cos x - \sin x)^{1/x} = 1^{\infty}$

So let $y = (\cos x - \sin x)^{1/x}$ and we get $\ln y = \frac{\ln (\cos x - \sin x)}{x}$

$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln (\cos x - \sin x)}{x} = \frac{0}{0}$

$\lim_{x \to 0} \frac{-\sin x - \cos x}{\cos x - \sin x} = -1$

So we have $\lim_{x \to 0} \ln y = -1$ and $\lim_{x \to 0} y = e^{-1}$

$\lim_{x \to 0} (\cos x - \sin x)^{1/x} = \frac{1}{e}$

12. $\lim_{x \to \pi/2} (\cos x)^{x-\pi/2}$

$$\lim_{x \to \pi/2} (\cos x)^{x-\pi/2} = 0^0$$

We let $y = (\cos x)^{x-\pi/2}$ so that $\ln y = \left(x - \frac{\pi}{2}\right) \ln \cos x$

$$\lim_{x \to \frac{\pi}{2}} \ln y = \lim_{x \to \frac{\pi}{2}} \left(x - \frac{\pi}{2}\right) \ln \cos x = 0 \cdot \infty$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\ln \cos x}{\left(x - \frac{\pi}{2}\right)^{-1}} = \infty/\infty$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\ln \cos x}{\left(x - \frac{\pi}{2}\right)^{-1}} = \lim_{x \to \frac{\pi}{2}} \frac{-\tan x}{-\left(x - \frac{\pi}{2}\right)^{-2}} = \infty/\infty$$

Rearrange the limit: $\lim_{x \to \frac{\pi}{2}} \frac{-\tan x}{-\left(x - \frac{\pi}{2}\right)^{-2}} = \lim_{x \to \frac{\pi}{2}} \frac{\left(x - \frac{\pi}{2}\right)^2}{\cot x} = \frac{0}{0}$

L'Hospital's rule one more time:

$$\lim_{x \to \frac{\pi}{2}} \frac{\left(x - \frac{\pi}{2}\right)^2}{\cot x} = \lim_{x \to \frac{\pi}{2}} \frac{2\left(x - \frac{\pi}{2}\right)}{-\csc^2 x} = -\lim_{x \to \frac{\pi}{2}} 2\left(x - \frac{\pi}{2}\right) \sin^2 x = 0$$

So ln y approaches 0. The limit of y is $e^0 = 1$

$$\lim_{x \to \pi/2} (\cos x)^{x-\pi/2} = 1$$

13. $\lim_{x \to 0^+} (\sin x)^{\sin x}$

$$\lim_{x \to 0^+} (\sin x)^{\sin x} = 0^0$$

Let $y = (\sin x)^{\sin x}$ so $\ln y = \sin x \ \ln(\sin x)$

$$\lim_{x \to 0^+} \sin x \ \ln(\sin x) = 0\,(-\infty)$$

$$\lim_{x \to 0^+} \frac{\ln(\sin x)}{\csc x} = -\infty/\infty$$

Use L'Hospital's rule:

$$\lim_{x \to 0^+} \frac{\cot x}{-\csc x \ \cot x} = \lim_{x \to 0^+} -\sin x = 0$$

So ln y approaches 0. Then y approaches $e^0 = 1$.

$$\lim_{x \to 0^+} (\sin x)^{\sin x} = 1$$