TAYLOR SERIES EXAMPLES

TAYLOR SERIES EXPANSION ABOUT x=a:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

MACLAURIN EXPANSION

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

EXAMPLE

Find the interval of convergence for e^x and find an expression for e itself

First let's get an expression for e: Start with the Maclaurin formula for e^x.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Or

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Set x = 1

$$e^1 = e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

Or

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

This is a nice formula for e.

We now have two expressions for e. The other one is $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$.

As a note, it is possible to take the limit expression for e and transform it into the factorial expression by using the binomial theorem.

Now let's get the radius of convergence. We use the ratio test. The series converges when the limit in the ratio test is less than 1.

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$u_n = \frac{x^n}{n!} \qquad u_{n+1} = \frac{x^{n+1}}{n+1}$$

$$\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \to \infty} \left|\frac{x^{n+1}}{n!} \frac{n!}{x^n}\right| = \lim_{n \to \infty} \left|\frac{x^{n+1}}{x^n} \frac{n!}{(n+1)!}\right| = \lim_{n \to \infty} \left|\frac{x}{n+1}\right| = 0$$

The limit is 0 for all values of x. So the limit is less than 1 for all values of x. This means that the series converges for all values of x. The interval of convergence is from minus infinity to positive infinity .

EXAMPLE

Find the Taylor series for e^x centered about a = 2

Start with Taylor's formula:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

We were told that a=2 so this becomes

$$f(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \cdots$$

All derivatives of e^x are equal to e^x – it is the only function that equals its own derivatives. So all derivatives in the formula equal $e^a = e^2$.

$$f(x) = e^x = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3 + \cdots$$

This can be written as

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$$

The results from the ratio test will not change – this series will also converge for all x

EXAMPLE

Find the Taylor series for $f(x) = \sin x$ centered about $a = \frac{\pi}{3}$

Start with Taylor's formula:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

We were told that $a = \frac{\pi}{3}$ so this becomes

$$f(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x-\frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{f'''\left(\frac{\pi}{3}\right)}{3!}\left(x-\frac{\pi}{3}\right)^3 + \cdots$$

We now have to get the derivatives:

$f(x) = \sin x$ $\qquad f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$f'(x) = \cos x$ $\qquad f'\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

$f''(x) = -\sin x$ $\qquad f''\left(\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

$f'''(x) = -\cos x$ $\qquad f'''\left(\frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$

$f^{iv}(x) = \sin x$ $\qquad f^{iv}\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\vdots$

And it repeats

$$\sin x = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x-\frac{\pi}{3}\right) - \frac{1}{2!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)^2 - \frac{1}{3!}\frac{1}{2}\left(x-\frac{\pi}{3}\right)^3 + \frac{1}{4!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)^4 + \frac{1}{5!}\frac{1}{2}\left(x-\frac{\pi}{3}\right)^5 - \frac{1}{6!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)^6 - \cdots$$

There are patterns here so we group the terms by even and odd exponents:

$$\sin x = \left[\frac{\sqrt{3}}{2} - \frac{1}{2!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)^2 + \frac{1}{4!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)^4 - \frac{1}{6!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)^6 + \cdots\right] + \left[\frac{1}{2}\left(x-\frac{\pi}{3}\right) - \frac{1}{3!}\frac{1}{2}\left(x-\frac{\pi}{3}\right)^3 + \frac{1}{5!}\frac{1}{2}\left(x-\frac{\pi}{3}\right)^5 - \cdots\right]$$

$$\sin x = \frac{\sqrt{3}}{2}\left[1 - \frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{3}\right)^4 - \frac{1}{6!}\left(x-\frac{\pi}{3}\right)^6 + \cdots\right] + \frac{1}{2}\left[\left(x-\frac{\pi}{3}\right) - \frac{1}{3!}\left(x-\frac{\pi}{3}\right)^3 + \frac{1}{5!}\left(x-\frac{\pi}{3}\right)^5 - \cdots\right]$$

This can be written in sigma notation:

$$\sin x = \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{3}\right)^{2n} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(x-\frac{\pi}{3}\right)^{2n+1}$$

EXAMPLE

Find the Taylor series for $f(x) = \sin x$ centered about $a = \frac{\pi}{4}$

Start with Taylor's formula:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

We were told that $a = \frac{\pi}{4}$ so this becomes

$$f(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(x-\frac{\pi}{4}\right)^3 + \cdots$$

We now have to get the derivatives:

$f(x) = \sin x \qquad f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$f'(x) = \cos x \qquad f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

$f^{iv}(x) = \sin x \qquad f^{iv}\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\vdots$

And it repeats

$$\sin x = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{1}{2!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^2 - \frac{1}{3!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^3 + \frac{1}{4!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^4 + \frac{1}{5!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^5 - \frac{1}{6!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^6 - \cdots$$

There are patterns here so we group the terms by even and odd exponents:

$$\sin x = \left[\frac{\sqrt{2}}{2} - \frac{1}{2!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^2 + \frac{1}{4!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^4 - \frac{1}{6!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^6 + \cdots\right] + \left[\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{1}{3!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^3 + \frac{1}{5!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^5 - \cdots\right]$$

$$\sin x = \frac{\sqrt{2}}{2}\left[1 - \frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{4}\right)^4 - \frac{1}{6!}\left(x-\frac{\pi}{4}\right)^6 + \cdots\right]$$

$$+ \frac{\sqrt{2}}{2}\left[\left(x-\frac{\pi}{4}\right) - \frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3 + \frac{1}{5!}\left(x-\frac{\pi}{4}\right)^5 - \cdots\right]$$

This can be written in sigma notation:

$$\sin x = \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{4}\right)^{2n} + \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(x-\frac{\pi}{4}\right)^{2n+1}$$

EXAMPLE

Find the Taylor series for $f(x) = \sin x$ centered about $a = \frac{\pi}{6}$

Start with Taylor's formula:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

We were told that $a = \frac{\pi}{6}$ so this becomes

$$f(x) = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x-\frac{\pi}{6}\right) + \frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x-\frac{\pi}{6}\right)^2 + \frac{f'''\left(\frac{\pi}{6}\right)}{3!}\left(x-\frac{\pi}{6}\right)^3 + \cdots$$

We now have to get the derivatives:

$f(x) = \sin x$	$f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$
$f'(x) = \cos x$	$f'\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$
$f''(x) = -\sin x$	$f''\left(\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}$
$f'''(x) = -\cos x$	$f'''\left(\frac{\pi}{6}\right) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$
$f^{iv}(x) = \sin x$	$f^{iv}\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$

$\vdots$

And it repeats

$$\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right) - \frac{1}{2!}\frac{1}{2}\left(x-\frac{\pi}{6}\right)^2 - \frac{1}{3!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)^3 + \frac{1}{4!}\frac{1}{2}\left(x-\frac{\pi}{6}\right)^4 + \frac{1}{5!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)^5 - \frac{1}{6!}\frac{1}{2}\left(x-\frac{\pi}{6}\right)^6 - \cdots$$

There are patterns here so we group the terms by even and odd exponents:

$$\sin x = \left[\frac{1}{2} + -\frac{1}{2!}\frac{1}{2}\left(x-\frac{\pi}{6}\right)^2 + \frac{1}{4!}\frac{1}{2}\left(x-\frac{\pi}{6}\right)^4 - \frac{1}{6!}\frac{1}{2}\left(x-\frac{\pi}{6}\right)^6 - \cdots\right] + \left[\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right) - \frac{1}{3!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)^3 + \frac{1}{5!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)^5 - \cdots\right]$$

$$\sin x = \frac{1}{2}\left[1 - \frac{1}{2!}\left(x-\frac{\pi}{6}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{6}\right)^4 - \frac{1}{6!}\left(x-\frac{\pi}{6}\right)^6 - \cdots\right] + \frac{\sqrt{3}}{2}\left[\left(x-\frac{\pi}{6}\right) - \frac{1}{3!}\left(x-\frac{\pi}{6}\right)^3 + \frac{1}{5!}\left(x-\frac{\pi}{6}\right)^5 - \cdots\right]$$

This can be written in sigma notation:

$$\sin x = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{6}\right)^{2n} + \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(x-\frac{\pi}{6}\right)^{2n+1}$$

EXAMPLE

Find the Taylor series for $f(x) = \sin x$ centered about $a = \frac{\pi}{2}$

Start with Taylor's formula:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

We were told that $a = \frac{\pi}{2}$ so this becomes

$$f(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x-\frac{\pi}{2}\right) + \frac{f''\left(\frac{\pi}{2}\right)}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{f'''\left(\frac{\pi}{2}\right)}{3!}\left(x-\frac{\pi}{2}\right)^3 + \cdots$$

We now have to get the derivatives:

$f(x) = \sin x$ $\qquad f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$

$f'(x) = \cos x$ $\qquad f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$

$f''(x) = -\sin x$ $\qquad f''\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$

$f'''(x) = -\cos x$ $\qquad f'''\left(\frac{\pi}{2}\right) = -\cos\frac{\pi}{2} = 0$

$f^{iv}(x) = \sin x$ $\qquad f^{iv}\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$

$\vdots$

And it repeats

$$\sin x = 1 - \frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x-\frac{\pi}{2}\right)^6 + \cdots$$

This can be written in sigma notation:

$$\sin x = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{2}\right)^{2n}$$

EXAMPLE

Find the Taylor series for $f(x) = \cos x$ centered about $a = \frac{\pi}{6}$

Start with Taylor's formula:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

We were told that $a = \frac{\pi}{6}$ so this becomes

$$f(x) = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x-\frac{\pi}{6}\right) + \frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x-\frac{\pi}{6}\right)^2 + \frac{f'''\left(\frac{\pi}{6}\right)}{3!}\left(x-\frac{\pi}{6}\right)^3 + \cdots$$

We now have to get the derivatives:

$f(x) = \cos x$ $\qquad f\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$f'(x) = -\sin x$ $\qquad f'\left(\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}$

$f''(x) = -\cos x$ $\qquad f''\left(\frac{\pi}{6}\right) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$

$f'''(x) = \sin x$ $\qquad f'''\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$

$f^{iv}(x) = \cos x$ $\qquad f^{iv}\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\vdots$

And it repeats

$$\cos x = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x-\frac{\pi}{6}\right) - \frac{1}{2!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)^2 + \frac{1}{3!}\frac{1}{2}\left(x-\frac{\pi}{6}\right)^3 + \frac{1}{4!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)^4 - \frac{1}{5!}\frac{1}{2}\left(x-\frac{\pi}{6}\right)^5 - \frac{1}{6!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)^6 - \cdots$$

Again there are patterns here that are similar to the ones for sine so we group the terms by even and odd exponents:

$$\cos x = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x-\frac{\pi}{6}\right) - \frac{1}{2!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)^2 + \frac{1}{3!}\frac{1}{2}\left(x-\frac{\pi}{6}\right)^3 + \frac{1}{4!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)^4 - \frac{1}{5!}\frac{1}{2}\left(x-\frac{\pi}{6}\right)^5 - \frac{1}{6!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)^6 - \cdots$$

$$\cos x = \frac{\sqrt{3}}{2}\left[1 - \frac{1}{2!}\left(x-\frac{\pi}{6}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{6}\right)^4 - \frac{1}{6!}\left(x-\frac{\pi}{6}\right)^6 - \cdots\right] + \frac{1}{2}\left[-\left(x-\frac{\pi}{6}\right) + \frac{1}{3!}\left(x-\frac{\pi}{6}\right)^3 - \frac{1}{5!}\left(x-\frac{\pi}{6}\right)^5 +\right]$$

This can be written in sigma notation:

$$\sin x = \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{6}\right)^{2n} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(x-\frac{\pi}{6}\right)^{2n+1}$$

EXAMPLE

Find the Taylor series for $f(x) = \cos x$ centered about $a = \frac{\pi}{4}$

Start with Taylor's formula:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

We were told that $a = \frac{\pi}{4}$ so this becomes

$$f(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(x-\frac{\pi}{4}\right)^3 + \cdots$$

We now have to get the derivatives:

$f(x) = \cos x$ $\qquad f\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$f'(x) = -\sin x$ $\qquad f'\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

$f''(x) = -\cos x$ $\qquad f''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

$f'''(x) = \sin x$ $\qquad f'''\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$f^{iv}(x) = \cos x$ $\qquad f^{iv}\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\vdots$

And it repeats

$$\cos x = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{1}{2!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^2 + \frac{1}{3!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^3 + \frac{1}{4!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^4 - \frac{1}{5!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^5 - \frac{1}{6!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^6 - \cdots$$

Once again there are patterns here so we group the terms by even and odd exponents:

$$\cos x = \left[\frac{\sqrt{2}}{2} - \frac{1}{2!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^2 + \frac{1}{4!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^4 - \frac{1}{6!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^6 + \cdots\right] + \left[-\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) + \frac{1}{3!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^3 - \frac{1}{5!}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^5\right]$$

$$\cos x = \frac{\sqrt{2}}{2}\left[1 - \frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{4}\right)^4 - \frac{1}{6!}\left(x-\frac{\pi}{4}\right)^6 + \cdots\right]$$

$$-\frac{\sqrt{2}}{2}\left[\left(x-\frac{\pi}{4}\right) - \frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3 + \frac{1}{5!}\left(x-\frac{\pi}{4}\right)^5 - \cdots\right]$$

This can be written in sigma notation:

$$\sin x = \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{4}\right)^{2n} - \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(x-\frac{\pi}{4}\right)^{2n+1}$$

EXAMPLE

Find the Taylor series for $f(x) = \cos x$ centered about $a = \frac{\pi}{3}$

Start with Taylor's formula:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

We were told that $a = \frac{\pi}{3}$ so this becomes

$$f(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x-\frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{f'''\left(\frac{\pi}{3}\right)}{3!}\left(x-\frac{\pi}{3}\right)^3 + \cdots$$

We now have to get the derivatives:

$f(x) = \cos x$ $\qquad f\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

$f'(x) = -\sin x$ $\qquad f'\left(\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

$f''(x) = -\cos x$ $\qquad f''\left(\frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$

$f'''(x) = \sin x$ $\qquad f'''\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$f^{iv}(x) = \cos x$ $\qquad f^{iv}\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

$\vdots$

And it repeats

$$\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right) - \frac{1}{2!}\frac{1}{2}\left(x-\frac{\pi}{3}\right)^2 + \frac{1}{3!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)^3 + \frac{1}{4!}\frac{1}{2}\left(x-\frac{\pi}{3}\right)^4 - \frac{1}{5!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)^5 - \frac{1}{6!}\frac{1}{2}\left(x-\frac{\pi}{3}\right)^6 - \cdots$$

Once again there are patterns here so we group the terms by even and odd exponents:

$$\begin{aligned}\cos x = &\left[\frac{1}{2} - \frac{1}{2!}\frac{1}{2}\left(x-\frac{\pi}{3}\right)^2 + \frac{1}{4!}\frac{1}{2}\left(x-\frac{\pi}{3}\right)^4 - \frac{1}{6!}\frac{1}{2}\left(x-\frac{\pi}{3}\right)^6 + \cdots\right] \\ &+\left[-\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right) + \frac{1}{3!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)^3 - \frac{1}{5!}\frac{1}{2}\left(x-\frac{\pi}{3}\right)^5 + \cdots\right]\end{aligned}$$

$$\begin{aligned}\cos x = &\frac{1}{2}\left[1 - \frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{3}\right)^4 - \frac{1}{6!}\left(x-\frac{\pi}{3}\right)^6 + \cdots\right] \\ &+\frac{\sqrt{3}}{2}\left[-\left(x-\frac{\pi}{3}\right) + \frac{1}{3!}\left(x-\frac{\pi}{3}\right)^3 - \frac{1}{5!}\frac{1}{2}\left(x-\frac{\pi}{3}\right)^5 + \cdots\right]\end{aligned}$$

This can be written in sigma notation:

$$\cos x = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{3}\right)^{2n} - \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(x-\frac{\pi}{3}\right)^{2n+1}$$

EXAMPLE

Find the Taylor series for $f(x) = \cos x$ centered about $a = \frac{\pi}{2}$

Start with Taylor's formula:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

We were told that $a = \frac{\pi}{3}$ so this becomes

$$f(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x-\frac{\pi}{2}\right) + \frac{f''\left(\frac{\pi}{2}\right)}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{f'''\left(\frac{\pi}{2}\right)}{3!}\left(x-\frac{\pi}{2}\right)^3 + \cdots$$

We now have to get the derivatives:

$f(x) = \cos x$ $\qquad f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$

$f'(x) = -\sin x$ $\qquad f'\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$

$f''(x) = -\cos x$ $\qquad f''\left(\frac{\pi}{2}\right) = -\cos\frac{\pi}{2} = 0$

$f'''(x) = \sin x$ $\qquad f'''\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$

$f^{iv}(x) = \cos x$ $\qquad f^{iv}\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$

$\vdots$

And it repeats

$$\cos x = -\left(x-\frac{\pi}{2}\right) + \frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3 - \frac{1}{5\,!}\left(x-\frac{\pi}{2}\right)^5 + \cdots$$

This can be written in sigma notation:

$$\cos x = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(x-\frac{\pi}{2}\right)^{2n+1}$$

EXAMPLE
Find the first three terms of the Taylor series for tangent about a = 0

Unlike the previous functions, there is no nice neat formula for Taylor series of tan(x). It has to be calculated term by term. There are some advanced ways to calculate the series for tangent and they involve Bernoulli numbers. The Taylor series for tangent does have a nice neat form in terms of Bernoulli numbers, but the Bernoulli numbers themselves have to be calculated or found on a table.

Start with Taylor's formula:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

We were told that $a = 0$ so this becomes

$$f(x) = f(0) + f'(0)\, x + \frac{f''(0)}{2!}\, x^2 + \frac{f'''(0)}{3!}\, x^3 + \cdots$$

We now have to get the derivatives:

$f(x) = \tan x$ $\qquad f(0) = 0$

$f'(x) = \sec^2 x$ $\qquad f'(0) = 1$

$f''(x) = 2\sec^2 x \tan x$ $\qquad f''(0) = 0$

$f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$ $\qquad f'''(0) = 2$

$f^{iv}(x) = (24\sec^4 x - 8\sec^2 x)\tan x$ $\qquad f^{iv}(0) = 0$

$f^{v}(x) = 120\sec^6 x - 120\sec^4 x + 16\sec^2 x$ $\qquad f^{v}(0) = 16$

$\vdots$

These derivatives are tedious and awkward. Sadly there is no faster way to get the series coefficients.

$$f(x) = f(0) + f'(0)\, x + \frac{f''(0)}{2!}\, x^2 + \frac{f'''(0)}{3!}\, x^3 + \cdots$$

$$\tan x = x + \frac{2}{3!}x^3 + \frac{16}{5!}x^5 + \cdots$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$

This is correct and can be verified by looking online or at text books.

EXAMPLE

Find the first six terms of the Taylor series for tangent about a = pi/6

Start with Taylor's formula:

$$f(x) \;=\; f(a) \;+\; f'(a)\,(x-a) \;+\; \frac{f''(a)}{2!}\;(x-a)^2 \;+\; \frac{f'''(a)}{3!}\;(x-a)^3 + \cdots$$

We were told that $a = \pi/6$ so this becomes

$$f(x) \;=\; f\left(\frac{\pi}{6}\right) \;+\; f'\left(\frac{\pi}{6}\right)\left(x-\frac{\pi}{6}\right) \;+\; \frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x-\frac{\pi}{6}\right)^2 \;+\; \frac{f'''\left(\frac{\pi}{6}\right)}{3!}\left(x-\frac{\pi}{6}\right)^3 + \cdots$$

We now have to get the derivatives:

$f(x) \;=\; \tan x$ $\qquad\qquad f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

$f'(x) \;=\; \sec^2 x$ $\qquad\qquad f'\left(\frac{\pi}{6}\right) = \frac{4}{3}$

$f''(x) = 2\sec^2 x \tan x$ $\qquad\qquad f''\left(\frac{\pi}{6}\right) = \frac{8\sqrt{3}}{9}$

$f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$ $\qquad\qquad f'''\left(\frac{\pi}{6}\right) = \frac{1936}{9}$

$f^{iv}(x) \;=\; (24\sec^4 x \;-8\sec^2 x)\tan x$ $\qquad\qquad f^{iv}\left(\frac{\pi}{6}\right) = \frac{32\sqrt{3}}{3}$

$f^{v}(x) = 120\sec^6 x - 120\sec^4 x + 16\sec^2 x$ $\qquad\qquad f^{v}\left(\frac{\pi}{6}\right) = \frac{832}{9}$

$\vdots$

These derivatives are tedious and awkward. Sadly there is no faster way to get the series coefficients.

$$f(x) \;=\; f\left(\frac{\pi}{6}\right) \;+\; f'\left(\frac{\pi}{6}\right)\left(x-\frac{\pi}{6}\right) \;+\; \frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x-\frac{\pi}{6}\right)^2 \;+\; \frac{f'''\left(\frac{\pi}{6}\right)}{3!}\left(x-\frac{\pi}{6}\right)^3 + \cdots$$

$$\tan x \;=\; f\left(\frac{\pi}{6}\right) \;+\; f'\left(\frac{\pi}{6}\right)\left(x-\frac{\pi}{6}\right) \;+\; \frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x-\frac{\pi}{6}\right)^2 \;+\; \frac{f'''\left(\frac{\pi}{6}\right)}{3!}\left(x-\frac{\pi}{6}\right)^3 + \cdots$$

$$\tan x \;=\; \frac{\sqrt{3}}{3} + \frac{4}{3}\left(x-\frac{\pi}{6}\right) + \frac{8\sqrt{3}}{9}\left(x-\frac{\pi}{6}\right)^2 + \frac{1936}{9}\left(x-\frac{\pi}{6}\right)^3 + \frac{32\sqrt{3}}{3}\left(x-\frac{\pi}{6}\right)^4 + \frac{832}{9}\left(x-\frac{\pi}{6}\right)^5 + \cdots$$

I was not able to check these numbers. If you got the first three terms correct, then that is good enough.

EXAMPLE

Find the first six terms of the Taylor series for tangent about a = pi/4

Start with Taylor's formula:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

We were told that $a = \frac{\pi}{4}$ so this becomes

$$f(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(x-\frac{\pi}{4}\right)^3 + \cdots$$

We now have to get the derivatives:

$f(x) = \tan x$ $\qquad f\left(\frac{\pi}{4}\right) = 1$

$f'(x) = \sec^2 x$ $\qquad f'\left(\frac{\pi}{4}\right) = 2$

$f''(x) = 2\sec^2 x \tan x$ $\qquad f''\left(\frac{\pi}{4}\right) = 4$

$f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$ $\qquad f'''\left(\frac{\pi}{4}\right) = 16$

$f^{iv}(x) = (24\sec^4 x - 8\sec^2 x)\tan x$ $\qquad f^{iv}\left(\frac{\pi}{4}\right) = 80$

$f^{v}(x) = 120\sec^6 x - 120\sec^4 x + 16\sec^2 x$ $\qquad f^{v}\left(\frac{\pi}{4}\right) = 512$

$\vdots$

Again, these derivatives are tedious and awkward. Sadly there is no faster way to get the series coefficients. I calculated these numbers but could not check them.

$$f(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(x-\frac{\pi}{4}\right)^3 + \cdots$$

$$\tan x = 1 + 2\left(x-\frac{\pi}{4}\right) + \frac{4}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{16}{3!}\left(x-\frac{\pi}{4}\right)^3 + \frac{80}{4!}\left(x-\frac{\pi}{4}\right)^4 + \frac{512}{5!}\left(x-\frac{\pi}{4}\right)^5 \ldots$$

I was not able to check these numbers. I think they are correct. If you got the first three terms correct, that is ok.

EXAMPLE

Find the first six terms of the Taylor series for tangent about a = pi/3

Start with Taylor's formula:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

We were told that $a = \pi/3$ so this becomes

$$f(x) = f(\pi/3) + f'(\pi/3)(x-\pi/3) + \frac{f''(\pi/3)}{2!}(x-\pi/3)^2 + \frac{f'''(\pi/3)}{3!}(x-\pi/3)^3 + \cdots$$

We now have to get the derivatives:

$f(x) = \tan x$ $\qquad f\left(\frac{\pi}{3}\right) = \sqrt{3}$

$f'(x) = \sec^2 x$ $\qquad f'\left(\frac{\pi}{3}\right) = 4$

$f''(x) = 2\sec^2 x \tan x$ $\qquad f''\left(\frac{\pi}{3}\right) = 8\sqrt{3}$

$f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$ $\qquad f'''\left(\frac{\pi}{3}\right) = 32 + 16\sqrt{3}$

$f^{iv}(x) = (24\sec^4 x - 8\sec^2 x)\tan x$ $\qquad f^{iv}\left(\frac{\pi}{3}\right) = 452\sqrt{3}$

$f^{v}(x) = 120\sec^6 x - 120\sec^4 x + 16\sec^2 x$ $\qquad f^{v}\left(\frac{\pi}{3}\right) = 6240$

$\vdots$

These derivatives are tedious and awkward. Sadly there is no faster way to get the series coefficients. I calculated these numbers but could not check them.

$$f(x) = f(\pi/3) + f'(\pi/3)(x-\pi/3) + \frac{f''(\pi/3)}{2!}(x-\pi/3)^2 + \frac{f'''(\pi/3)}{3!}(x-\pi/3)^3 + \cdots$$

$$\tan x = \sqrt{3} + 4\left(x-\frac{\pi}{3}\right) + \frac{8\sqrt{3}}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{32+16\sqrt{3}}{3!}\left(x-\frac{\pi}{3}\right)^3$$
$$+ \frac{452\sqrt{3}}{4!}\left(x-\frac{\pi}{3}\right)^4 + \frac{6240}{5!}\left(x-\frac{\pi}{3}\right)^5 \ldots$$

I was not able to check these numbers. If you got the first 4 terms correct, that is ok.

EXAMPLE

Find the Taylor series for secant about a = 0. Get the first three terms.

Like tangent, secant does not have a nice neat formula for its Taylor series. So we have to calculate each coefficient one by one.

Start with Taylor's formula:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

We were told that $a = 0$ so this becomes

$$f(x) = f(0) + f'(0)\, x + \frac{f''(0)}{2!}\, x^2 + \frac{f'''(0)}{3!}\, x^3 + \cdots$$

We now have to get the derivatives:

$f(x) = \sec x$ $\qquad f(0) = 1$

$f'(x) = \sec x \tan x$ $\qquad f'(0) = 0$

$f''(x) = \sec x \tan^2 x + \sec^3 x$

$f''(x) = 2\sec^3 x - \sec x$ $\qquad f''(0) = 1$

$f'''(x) = 6\sec^3 x \tan x - \sec x \tan x$ $\qquad f'''(0) = 0$

$f^{iv}(x) = (24\sec^5 x - 20\sec^3 x + \sec x)$ $\qquad f^{iv}(0) = 5$

$$f(x) = f(0) + f'(0)\, x + \frac{f''(0)}{2!}\, x^2 + \frac{f'''(0)}{3!}\, x^3 + \cdots$$

$$\sec x = 1 + \frac{1}{2!}\, x^2 + \frac{5}{4!}\, x^4 + \cdots$$

These terms are correct.

EXAMPLE

Find the Taylor series for $f(x) = (1+x)^n$

<u>This question is a derivation of the binomial theorem for an arbitrary exponent.</u>

Start with Taylor's formula:

$$f(x) = f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 + \frac{f'''(a)}{3!}\,(x-a)^3 + \cdots$$

We were told that $a = 0$ so this becomes

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

We now have to get the derivatives:

$f(x) = (1+x)^n$ $\qquad$ $f(0) = 1$

$f'(x) = n\,(1+x)^{n-1}$ $\qquad$ $f'(0) = n$

$f''(x) = n(n-1)(1+x)^{n-2}$ $\qquad$ $f''(0) = n(n-1)$

$f'''(x) = n\,(n-1)\,(n-2)\,(1+x)^{n-3}$ $\qquad$ $f'''(0) = n(n-1)\,(n-2)$

$f^{iv}\,(x) = n\,(n-1)\,(n-2)\,(n-3\,)\,(1+x)^{n-4}$ $\qquad$ $f^{iv}\,(0) = n\,(n-1)\,(n-2)\,(n-3\,)$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

$$(1+x)^n = 1 + n\,x + \frac{n\,(n-1)}{2!}\,x^2 + \frac{n\,(n-1)\,(n-2)}{3!}\,x^3 + \frac{n\,(n-1)\,(n-2)\,(n-3)}{4!}\,x^4 + \cdots$$

This can be written in sigma form

$$(1+x)^n = \sum_{k=0}^{\infty} \frac{n\,(n-1)(n-2)\cdots\big(n-(k-1)\big)\,x^k}{k!}$$

Notes: When n is an integer the series is finite and stops – it is not infinite.

In this case, the expression $\frac{n\,(n-1)(n-2)\cdots\big(n-(k-1)\big)}{k!}$ can be written in terms of factorials:

$\frac{n!}{(n-k)!\;k!} = \frac{n\,(n-1)(n-2)\cdots\big(n-(k-1)\big)}{k!}$. This is abbreviated as $C_k^n = \binom{n}{k} = \frac{n!}{(n-k)!\;k!} =$

$\frac{n\,(n-1)(n-2)\cdots\big(n-(k-1)\big)}{k!}$. These are the binomial coefficients. They are the numbers in Pascal's triangle. The ratio test shows that this converges for $-1 < x < 1$. The series may or may not converge at the endpoints. When n is positive, it will converge at the endpoints. When n is between -1 and 0, the series will converge at -1 but not at +1.

EXAMPLE
Find the Taylor series for $f(x) = (1 - x)^n$

Take the result from the previous question and replace x with -x

$$(1+x)^n = \sum_{k=0}^{\infty} \frac{n(n-1)(n-2)\cdots(n-(k-1))\,x^k}{k!}$$

$$(1-x)^n = \sum_{k=0}^{\infty} \frac{n(n-1)(n-2)\cdots(n-(k-1))\,(-x)^k}{k!}$$

$$(1-x)^n = \sum_{k=0}^{\infty} \frac{(-1)^k\, n(n-1)(n-2)\cdots(n-(k-1))\,x^k}{k!}$$

This will also converge for -1 < x < +1. If n is positive the interval will include the endpoints. If n is between -1 and zero inclusive, then the series will converge for x = -1 but not for x = +1.

If you write the terms out it will be almost identical to the original binomial series but this one will be alternating.

$$(1-x)^n = 1 - n\,x + \frac{n(n-1)}{2!}\,x^2 - \frac{n(n-1)(n-2)}{3!}\,x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}\,x^4 + \cdots$$

PERSONAL NOTE: When I solve questions using binomial expansion, this last form is the form that I use. I do not use the sigma form.

EXAMPLE

Find the Taylor series for $f(x) = \sqrt{9-x}$ by using the binomial theorem.

Write this as $f(x) = 3\sqrt{1 - \frac{x}{9}}$

Note that the domain of the function is $x \leq 9$.

We can now invoke the binomial theorem (which is the Taylor series for this function).

$$(1 - u)^n = \sum_{k=0}^{\infty} \frac{(-1)^k n(n-1)(n-2)\cdots(n-(k-1))\, u^k}{k!}$$

This form is a little intimidating – the easier form is

$$(1 - u)^n = 1 - n\,u + \frac{n(n-1)}{2!}u^2 + \frac{n(n-1)(n-2)}{3!}u^3 - \frac{n(n-1)(n-2)(n-3)}{4!}u^4 + \cdots$$

Replace u with x/9:

$$f(x) = 3\sqrt{1 - \frac{x}{9}} = 3\sum_{k=0}^{\infty} \frac{(-1)^k n(n-1)(n-2)\cdots(n-(k-1))\,(x/9)^k}{k!}$$

Using the easier form, we can write out the first few terms:

$$f(x) = 3\left(1 + \left(\frac{1}{2}\right)\left(-\frac{x}{9}\right) + \frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{x}{9}\right)^2 + \frac{1}{3!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{x}{9}\right)^3 + \right.$$
$$\left. + \frac{1}{4!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{x}{9}\right)^4 + \frac{1}{5!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\left(-\frac{x}{9}\right)^5 \ldots\right)$$

$$f(x) = 3\left(1 - \left(\frac{1}{2}\right)\left(\frac{x}{9}\right) - \frac{1}{2!}\left(\frac{1}{2^2}\right)\left(\frac{x}{9}\right)^2 - \frac{1}{3!}\left(\frac{3}{2^3}\right)\left(\frac{x}{9}\right)^3 - \frac{1}{4!}\left(\frac{3\cdot 5}{2^4}\right)\left(\frac{x}{9}\right)^4 - \right.$$
$$\left. - \frac{1}{5!}\left(\frac{3\cdot 5\cdot 7}{2^5}\right)\left(\frac{x}{9}\right)^5 \ldots\right)$$

$$f(x) = 3\left(1 - \frac{x}{18} - \frac{x^2}{648} - \frac{x^3}{34992} - \cdots\right)$$

The interval of convergence for this series is -9 < x < 9. The series will converge at both endpoints as well.

EXAMPLE STEWART

Find the Taylor series for the function $f(x) = \sqrt[3]{8 + x}$ by using the binomial theorem.

Write this as $f(x) = 2\sqrt[3]{1 + \frac{x}{8}} = 2\left(1 + \frac{x}{8}\right)^{1/3}$

Note that the domain of the function is $x \geq -8$.

We can now invoke the binomial theorem (which is the Taylor series for this function).

$$(1 + u)^n = \sum_{k=0}^{\infty} \frac{n(n-1)(n-2)\cdots(n-(k-1))\, u^k}{k!}$$

Again this form is a little intimidating – the easier form is

$$(1 + u)^n = 1 + n\,u + \frac{n(n-1)}{2!}u^2 + \frac{n(n-1)(n-2)}{3!}u^3 + \frac{n(n-1)(n-2)(n-3)}{4!}u^4 + \cdots$$

Replace u with x/8:

$$f(x) = 2\left(1 + \frac{x}{8}\right)^{1/3} = 2\sum_{k=0}^{\infty} \frac{n(n-1)(n-2)\cdots(n-(k-1))\,(x/8)^k}{k!}$$

We can write out the first few terms

$$f(x) = 2\left(1 + \left(\tfrac{1}{3}\right)\left(\tfrac{x}{8}\right) + \tfrac{1}{2!}\left(\tfrac{1}{3}\right)\left(-\tfrac{2}{3}\right)\left(\tfrac{x}{8}\right)^2 + \tfrac{1}{3!}\left(\tfrac{1}{3}\right)\left(-\tfrac{2}{3}\right)\left(-\tfrac{5}{3}\right)\left(\tfrac{x}{8}\right)^3 + \tfrac{1}{4!}\left(\tfrac{1}{3}\right)\left(-\tfrac{2}{3}\right)\left(-\tfrac{5}{3}\right)\left(-\tfrac{8}{3}\right)\left(\tfrac{x}{8}\right)^4 \ldots\right)$$

$$f(x) = 2\left(1 + \left(\tfrac{1}{3}\right)\left(\tfrac{x}{8}\right) - \tfrac{1}{2!}\left(\tfrac{2}{3^2}\right)\left(\tfrac{x}{8}\right)^2 + \tfrac{1}{3!}\left(\tfrac{2\cdot 5}{3^3}\right)\left(\tfrac{x}{8}\right)^3 - \tfrac{1}{4!}\left(\tfrac{2\cdot 5\cdot 8}{3^4}\right)\left(\tfrac{x}{8}\right)^4 - \cdots\right)$$

$$f(x) = 2\left(1 - \frac{x}{24} + \frac{x^2}{1152} - \frac{x^3}{73728} - \cdots\right)$$

The interval of convergence for this series is -8 < x < 8. The series will converge at both endpoints as well.

EXAMPLE

Find the Taylor series for $f(x) = \frac{1}{(1+x)^4}$ by using the binomial theorem.

Write the function as $f(x) = (1+x)^{-4}$

Invoke the binomial theorem (which is the Taylor series for this function)

$$(1+x)^n = \sum_{k=0}^{\infty} \frac{n(n-1)(n-2)\cdots(n-(k-1))\,x^k}{k!}$$

$$(1+x)^{-4} = \sum_{k=0}^{\infty} \frac{-4(-5)(-6)\cdots(4-(k-1))\,x^k}{k!}$$

We write out the more transparent form:

$$(1+u)^n = 1 + n\,u + \frac{n(n-1)}{2!}u^2 + \frac{n(n-1)(n-2)}{3!}u^3 + \frac{n(n-1)(n-2)(n-3)}{4!}u^4 + \cdots$$

We can write out the first few terms of the series:

$$(1+x)^{-4} = 1 + (-4)\,x + \frac{1}{2!}(-4)(-5)\,x^2 + \frac{1}{3!}(-4)(-5)(-6)\,x^3 + \frac{1}{4!}(-4)(-5)(-6)(-7)\,x^4 + \cdots$$

This series will converge for -1 < x < +1. The exponent, n is -4. Since this is less than -1 the series will not converge at the endpoints.

EXAMPLE

Find a Taylor series for $f(x) = (a+x)^n$ by using the binomial theorem.

We can do this by algebraic transformation and then using the binomial theorem. The binomial series is a Taylor series for expressions of the form $(1+u)^n$.

First we write $f(x) = a^n \left(1 + \frac{x}{a}\right)^n$

We can use the binomial theorem in the form

$$(1+u)^n = 1 + n\,u + \frac{n\,(n-1)}{2!}\,u^2 + \frac{n\,(n-1)\,(n-2)}{3!}\,u^3 + \frac{n\,(n-1)\,(n-2)\,(n-3)}{4!}\,u^4 + \cdots$$

Replace u with x/a

$$\left(1+\frac{x}{a}\right)^n = 1 + n\,\frac{x}{a} + \frac{n\,(n-1)}{2!}\left(\frac{x}{a}\right)^2 + \frac{n\,(n-1)\,(n-2)}{3!}\left(\frac{x}{a}\right)^3 + \frac{n\,(n-1)\,(n-2)\,(n-3)}{4!}\left(\frac{x}{a}\right)^4 + \cdots$$

$$f(x) = a^n\left(1+\frac{x}{a}\right)^n = a^n\left(1 + n\,\frac{x}{a} + \frac{n\,(n-1)}{2!}\left(\frac{x}{a}\right)^2 + \frac{n\,(n-1)\,(n-2)}{3!}\left(\frac{x}{a}\right)^3 + \frac{n\,(n-1)\,(n-2)\,(n-3)}{4!}\left(\frac{x}{a}\right)^4 + \cdots\right)$$

$$f(x) = \left(a^n + n\,x\,a^{n-1} + \frac{n\,(n-1)}{2!}\,x^2\,a^{n-2} + \frac{n\,(n-1)\,(n-2)}{3!}\,x^3\,a^{n-3} + \frac{n\,(n-1)\,(n-2)\,(n-3)}{4!}\,x^4\,a^{n-4} + \cdots\right)$$

$$(x+a)^n = \left(a^n + n\,x\,a^{n-1} + \frac{n\,(n-1)}{2!}\,x^2\,a^{n-2} + \frac{n\,(n-1)\,(n-2)}{3!}\,x^3\,a^{n-3} + \frac{n\,(n-1)\,(n-2)\,(n-3)}{4!}\,x^4\,a^{n-4} + \cdots\right)$$

EXAMPLE

Find a Taylor series for $f(x) = (a+x)^n$ by making a table of derivatives.

Start with Taylor's formula centered about 0.

$$f(x) = f(0) + f'(0)\ x + \frac{f''(0)}{2!}\ x^2 + \frac{f'''(0)}{3!}\ x^3 + \cdots$$

We now have to get the derivatives:

$f(x) = (a+x)^n$ $\qquad f(0) = a^n$

$f'(x) = n\,(a+x)^{n-1}$ $\qquad f'(0) = n\ a^{n-1}$

$f''(x) = n(n-1)(a+x)^{n-2}$ $\qquad f''(0) = n(n-1)\,a^{n-2}$

$f'''(x) = n\,(n-1)\,(n-2)\,(a+x)^{n-3}$ $\qquad f'''(0) = n(n-1)\,(n-2)\,a^{n-3}$

$f^{iv}\,(x) = n\,(n-1)\,(n-2)\,(n-3\,)\,(a+x)^{n-4}$ $\qquad f^{iv}\,(0) = n\,(n-1)\,(n-2)\,(n-3\,)\,a^{n-4}$

$$f(x) = f(0) + f'(0)\ x + \frac{f''(0)}{2!}\ x^2 + \frac{f'''(0)}{3!}\ x^3 + \cdots$$

$$(a+x)^n = a^n + n\,a^{n-1}\,x + \frac{n\,(n-1)}{2!}\ a^{n-2}\,x^2 + \frac{n\,(n-1)\,(n-2)}{3!}\ a^{n-3}\,x^3 + \frac{n\,(n-1)\,(n-2)\ (n-3)}{4!}\ a^{n-4}\,x^4 + + \cdots$$

This can be written in sigma form

$$(a+x)^n = \sum_{k\,=\,0}^{\infty} \frac{1}{k!}\,n\,(n-1)\,(n-2)\cdots\big(n-(k-1)\big)\ a^{n-k}\ x^k$$

EXAMPLE

Find the Taylor series for $f(x) = \sin^{-1} x$ by making use of the binomial theorem.

Instead of using the formal mechanisms of Taylor series (forming the table of derivatives) we will use the binomial theorem (which is a Taylor series for binomial expressions).

We know that the derivative of arc sin(x) is given by $f'(x) = \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1- x^2}}$

We can write this as $\sin^{-1} x = \int \frac{1}{\sqrt{1-x^2}} \, dx + C$

$$(1+u)^n = 1 + n\,u + \frac{n\,(n-1)}{2!}\,u^2 + \frac{n\,(n-1)\,(n-2)}{3!}\,u^3 + \frac{n\,(n-1)\,(n-2)(n-3)}{4!}\,u^4 + \cdots$$

Replacing u with $-x^2$ will yield

$$(1 - x^2)^n = 1 - n\,x^2 + \frac{n\,(n-1)}{2!}\,x^4 - \frac{n\,(n-1)\,(n-2)}{3!}\,x^6 + \frac{n\,(n-1)\,(n-2)(n-3)}{4!}\,x^8 + \cdots$$

We now let n = -1/2

$$(1 - x^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}\,x^2 + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\,x^4 - \frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^6 + \frac{1}{4!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\,x^8 + \cdots$$

$$(1 - x^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}\,x^2 + \frac{1}{2!}\left(\frac{1\cdot 3}{2^2}\right)\,x^4 + \frac{1}{3!}\left(\frac{1\cdot3\cdot5}{2^3}\right)x^6 + \frac{1}{4!}\left(\frac{1\cdot3\cdot5\cdot7}{2^4}\right)\,x^8 + \cdots$$

Integrating both sides yields arc sin(x)

$$\sin^{-1} x = x + \frac{1}{2}\,\frac{x^3}{3} + \frac{1}{2!}\left(\frac{1\cdot 3}{2^2}\right)\frac{x^5}{5} + \frac{1}{3!}\left(\frac{1\cdot3\cdot5}{2^3}\right)\frac{x^7}{7} + \frac{1}{4!}\left(\frac{1\cdot3\cdot5\cdot7}{2^4}\right)\frac{x^9}{9} + \cdots + C$$

Setting x = 0 along with $\sin^{-1}(0) = 0$ shows that C=0.

$$\sin^{-1} x = x + \frac{1}{2}\,\frac{x^3}{3} + \frac{1}{2!}\left(\frac{1\cdot 3}{2^2}\right)\frac{x^5}{5} + \frac{1}{3!}\left(\frac{1\cdot3\cdot5}{2^3}\right)\frac{x^7}{7} + \frac{1}{4!}\left(\frac{1\cdot3\cdot5\cdot7}{2^4}\right)\frac{x^9}{9} + \cdots$$

This can be written in the slightly better form:

$$\sin^{-1} x = x + \frac{1\cdot 3}{2}\,\frac{x^3}{3^2} + \frac{1}{2!}\left(\frac{1\cdot3\cdot5\cdot7}{2^2}\right)\frac{x^5}{5^2} + \frac{1}{3!}\left(\frac{1\cdot3\cdot5\cdot7}{2^3}\right)\frac{x^7}{7^2} + \frac{1}{4!}\left(\frac{1\cdot3\cdot5\cdot7\cdot9}{2^4}\right)\frac{x^9}{9^2} + \cdots$$

<u>**This is the answer. You can stop here if you wish.**</u>

This last form can be written in sigma form a bit more easily. We get:

$$\sin^{-1} x \;=\; \sum_{n=0}^{\infty} \frac{(2n+1)!!\; x^{2n+1}}{n!\; 2^n\, (2n+1)^2}$$

The double factorial in the numerator is a recognized form of a special factorial.

For example, 7!! = 1 * 3 * 5 * 7 and 9!! = 1*3*5*7*9 or 8!! = 2*4*6*8

It is possible to not use the double factorial. We can transform the equation into regular factorial notation by multiplying top and bottom by (2n)!!

This will bring the Taylor series into the form

$$\sin^{-1} x \;=\; \sum_{n=0}^{\infty} \frac{(2n+1)!\; x^{2n+1}}{n!\, n!\; 2^n\, 2^n\, (2n+1)^2}$$

We used the idea that (2n+1)! = (2n+1)!! (2n)!!

We also used the idea that (2n)!! = 2^n n!

You are not expected to know these factorial identities. If you want, you can practice them and see that they really work. These factorial identities are not that difficult but if you find them daunting, you are not expected to know them. They are introduced now only for easier and more expedient notation.

As a final note, you can get a really nice formula for π by letting x = ½.

EXAMPLE

Find the Taylor series for $f(x) = e^{x^2}$ by using the existing series for e^u.

We could proceed to make a table of derivatives, but this is not necessary.

An easier way exists. We can use the Taylor series for e^u and then replace u with x^2.

We have

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \cdots$$

$$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!} + \cdots$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots$$

In sigma form this is $e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

This series converges for all real numbers

EXAMPLE

Find the Taylor series for $f(x) \ = \ e^{x^2}$ by making the table of derivatives. Find the first three nonzero terms and compare them to the previous answer.

Write the table of derivatives:

$f(x) \ = \ e^{x^2}$ $\qquad\qquad$ $f(0) \ = 1$

$f'(x) \ = (2x)e^{x^2}$ $\qquad\qquad$ $f'(0) \ = 0$

$f''(x) \ = \ 4x^2\, e^{x^2} \ + \ 2\, e^{x^2}$ $\qquad\qquad$ $f''(0) \ = 2$

$f'''(x) \ = \ 8x^3\, e^{x^2} + 12x\, e^{x^2}$

$f'''(x) \ = (\, 8x^3 \ + 12x \ )\, e^{x^2}$ $\qquad\qquad$ $f'''(0) = 0$

$f^{iv}\,(x) \ = \ \left(\, 16x^4 \ + \ 48x^2 \ + 12 \,\right) e^{x^2}$ $\qquad\qquad$ $f^{iv}(0) \ = \ 12$

$f^{v}\,(x) \ = \ \left(\, 32x^5 \ + \ 96x^3 \ + 24x \,\right) e^{x^2}$
$\qquad + \ (\, 64x^3 \ + \ 96\, x \,)\, e^{x^2}$

$f^{v}\,(x) \ = \ \left(\, 32x^5 \ + \ 160x^3 \ + 120x \,\right) e^{x^2}$ $\qquad\qquad$ $f^{v}(0) \ = \ 0$

$f^{vi}\,(x) \ = \ \left(\, 64x^6 \ + \ 320x^4 \ + 240x^2 \,\right) e^{x^2}$
$\qquad + \ (160x^4 + 480x^2 + 120)\, e^{x^2}$ $\qquad\qquad$ $f^{vi}(0) \ = 120$

$$f(x) \ = \ f(0) \ + \ f'(0)\ x \ + \ \frac{f''(0\,)}{2!}\ x^2 \ + \ \frac{f'''(0)}{3!}\ x^3 + \cdots$$

$$e^{x^2} \ = \ 1 \ + 0x \ + \frac{1}{2!}\,(2)\,x^2 \ + \ 0\,x^3 \ + \frac{1}{4!}\,(12)\,x^4 \ + \ 0\,x^5 \ + \frac{1}{6!}\,(120)\,x^6$$

$$e^{x^2} \ = \ 1 \ + \ x^2 \ + \frac{1}{2}\,x^4 \ + \frac{1}{6}\ x^6 + \cdots \ = \ 1 + x^2 \ + \frac{1}{2!}\,x^4 \ + \frac{1}{3!}\,x^3 + \cdots$$

This does agree with the previous result. It is more labor intensive but does produce the pattern somewhat efficiently. It is not as good as the previous method.

EXAMPLE

Find the Taylor series for sin(x^2) by using the existing series for sin(u).

We could form a table of derivatives directly and develop the Taylor series step by step by following the formula. This would be disadvantageous. It is easier to use the Taylor series for sin(u) and substitute u = x^2. This is a fast and expedient way to answer the question.

We start with

$$\sin u \;=\; u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \cdots$$

Substituting $u = x^2$ yields

$$\sin(x^2) \;=\; x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

In sigma notation this becomes

$$\sin x^2 \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^2\right)^{2n+1}}{(2n+1)!}$$

Or

$$\sin x^2 \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n \,(x)^{4n+2}}{(2n+1)!}$$

That’s it. The question is solved.

EXAMPLE

Find the Taylor series for $f(x) = \sin(x^2)$ by forming a table of derivatives.

If we go about this formally and develop the table of derivatives, we get the following

$f(x) = \sin(x^2)$ $\qquad f(0) = 0$

$f'(x) = 2x\cos(x^2)$ $\qquad f'(0) = 0$

$f''(x) = 2\cos(x^2) - 4x^2\sin(x^2)$ $\qquad f''(0) = 2$

$f'''(x) = -4x\sin(x^2) - 8x\sin(x^2)$
$\quad - 8\,x^3\cos(x^2)$ $\qquad f'''(0) = 0$

$f'''(x) = -12\,x\sin x^2 - 8\,x^3\cos x^2$ $\qquad f'''(0) = 0$

$f^{iv}(x) = -12\sin(x^2) - 24x^2\cos(x^2)$
$\quad - 24\,x^2\cos(x^2) + 16\,x^4\sin(x^2)$ $\qquad f^{iv}(x) = 0$

$f^{iv}(x) = -12\sin(x^2) - 48\,x^2\cos(x^2)$
$\quad + 16\,x^4\sin(x^2)$ $\qquad f^{iv}(x) = 0$

$f^{v}(x) = -24\,x\cos(x^2) - 96\,x\cos(x^2)$
$\quad + 96\,x^3\sin(x^2) + 64\,x^3\sin(x^2)$
$\quad + 32\,x^5\cos(x^2)$ $\qquad f^{v}(x) = 0$

$f^{v}(x) = -120\,x\cos(x^2) + 160\sin(x^2)$
$\quad + 32\,x^5\cos(x^2)$ $\qquad f^{v}(x) = 0$

$f^{vi}(x) = -120\cos(x^2) + 240\,x^2\sin(x^2)$
$\quad + 320\,x\cos(x^2) + 160\,x^4\cos(x^2)$
$\quad - 64x^6\sin(x^2)$ $\qquad f^{vi}(x) = -120$

Some comments to make thus far: we had to take 6 derivatives to get 2 non-zero terms. And we can see no pattern forming. This method is tedious, prone to error, calculation intense, and very inefficient.

Proceeding to Taylor's formula we get:

$$f(x) = f(0) + f'(0)\ x + \frac{f''(0)}{2!}\ x^2 + \frac{f'''(0)}{3!}\ x^3 + \cdots$$

This becomes:

$$\sin(x^2) = 0 + 0x + \frac{2}{2!}\,x^2 + 0x^3 + 0x^4 + 0x^5 - \frac{120}{6!}\ x^6 + \cdots$$

$$\sin(x^2) = x^2 - \frac{1}{6}\,x^6 + \cdots = x^2 - \frac{1}{3!}\,x^6 + \cdots$$

This agrees with the previous result but there is no way we can see the pattern. So this way works, but is not very productive.

EXAMPLE STEWART

Find the Taylor series of f(x) = x e^x by the definition of Taylor series (by taking derivatives and evaluating them at x = a). Let a = 0.

We write Taylor's expansion

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$f(x) = x\,e^x$ $\qquad\qquad$ $f(0) = 0$

$f'(x) = xe^x + e^x$

$f'(x) = (x+1)e^x$ $\qquad\qquad$ $f'(0) = 1$

$f''(x) = (x+1)e^x + e^x$

$f''(x) = (x+2)\,e^x$ $\qquad\qquad$ $f''(0) = 2$

$f'''(x) = (x+2)e^x + e^x$

$f'''(x) = (x+3)\,e^x$ $\qquad\qquad$ $f'''(0) = 3$

$f^{iv}(x) = (x+3)e^x + e^x$

$f^{iv}(x) = (x+4)\,e^x$ $\qquad\qquad$ $f^{iv}(0) = 4$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$x\,e^x = 0 + x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{4}{4!}x^4 + \frac{5}{5!}x^5 + \cdots$$

$$x\,e^x = x + x^2 + \frac{1}{2!}x^3 + \frac{1}{3!}x^4 + \frac{1}{4!}x^5 + \cdots$$

$$x\,e^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

EXAMPLE STEWART

Find the Taylor series of f(x) = 1/(x+1) by the definition of Taylor series (by taking derivatives and evaluating them at x = a). Let a = 0.

We write Taylor's expansion:

$$f(x) = f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 + \frac{f'''(a)}{3!}\,(x-a)^3 + \cdots$$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

$$f(x) = \frac{1}{x+1} \qquad\qquad f(0) = 1$$

$$f'(x) = -\frac{1}{(x+1)^2} \qquad\qquad f'(0) = -1$$

$$f''(x) = \frac{2}{(x+1)^3} \qquad\qquad f''(0) = 2$$

$$f'''(x) = \frac{-3\cdot 2}{(x+1)^4} \qquad\qquad f'''(0) = -3!$$

$$f^{iv}(x) = \frac{4\cdot 3\cdot 2}{(x+1)^4} \qquad\qquad f^{iv}(0) = 4!$$

$$f^{v}(x) = \frac{-5\cdot 4\cdot 3\cdot 2}{(x+1)^5} \qquad\qquad f^{iv}(0) = -5!$$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

$$\frac{1}{x+1} = 1 - x + \frac{2}{2!}\,x^2 - \frac{3!}{3!}\,x^3 + \frac{4!}{4!}\,x^4 \;\ldots$$

$$\frac{1}{x+1} = 1 - x + x^2 - x^3 + x^4 \;\ldots$$

$$\frac{1}{x+1} = \sum_{n=0}^{\infty} (-1)^n\,x^n$$

EXAMPLE STEWART

Find the Taylor series of f(x) = 1/(x+1) by the definition of Taylor series (by taking derivatives and evaluating them at x = a). Let a = 2.

We write Taylor's expansion:

$$f(x) \;=\; f(a) \;+\; f'(a)\,(x-a) \;+\; \frac{f''(a)}{2!}\;(x-a)^2 \;+\; \frac{f'''(a)}{3!}\;(x-a)^3 + \cdots$$

$$f(x) \;=\; f(0) \;+\; f'(0)\,x \;+\; \frac{f''(0)}{2!}\;x^2 \;+\; \frac{f'''(0)}{3!}\;x^3 + \cdots$$

$$f(x) \;=\; \frac{1}{x+1} \qquad\qquad f(2) \;=\; \frac{1}{3}$$

$$f'(x) \;=\; -\frac{1}{(x+1)^2} \qquad\qquad f'(2) \;=\; -\frac{1}{3^2}$$

$$f''(x) = \frac{2}{(x+1)^3} \qquad\qquad f''(2) \;=\; \frac{2}{3^3}$$

$$f'''(x) \;=\; \frac{-3\cdot 2}{(x+1)^4} \qquad\qquad f'''(2) \;=\; -\frac{3!}{3^4}$$

$$f^{iv}(x) \;=\; \frac{4\cdot 3\cdot 2}{(x+1)^5} \qquad\qquad f^{iv}(2) \;=\; \frac{4!}{3^5}$$

$$f^{v}(x) \;=\; \frac{-5\cdot 4\cdot 3\cdot 2}{(x+1)^6} \qquad\qquad f^{iv}(2) \;=\; -\frac{5!}{3^6}$$

$$f(x) \;=\; f(a) \;+\; f'(a)\,(x-a) \;+\; \frac{f''(a)}{2!}\;(x-a)^2 \;+\; \frac{f'''(a)}{3!}\;(x-a)^3 + \cdots$$

$$\frac{1}{x+1} \;=\; \frac{1}{3} - \frac{1}{3^2}\,(x-2) \;+\; \frac{2}{3^3\,2!}\;(x-2)^2 \;-\; \frac{3!}{3^4\,3!}\;(x-2)^3 + \frac{4!}{3^5\,4!}\;(x-2)^4 \;+ \cdots$$

$$\frac{1}{x+1} \;=\; \frac{1}{3} - \frac{1}{3^2}\,(x-2) \;+\; \frac{1}{3^3}\;(x-2)^2 \;-\; \frac{1}{3^4}\;(x-2)^3 + \frac{1}{3^5}\;(x-2)^4 + \cdots$$

$$\frac{1}{x+1} \;=\; \sum_{n\,=\,0}^{\infty}\;(-1)^n\,\frac{x^n}{3^{n+1}}$$

EXAMPLE STEWART

Find the Taylor series of $f(x) = \sqrt[3]{x}$ by the definition of Taylor series (by taking derivatives and evaluating them at x = a). Let a = 8. Get the first several terms only.

We write the formula for Taylor series:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$f(x) = \sqrt[3]{x} = x^{1/3} \qquad f(2) = 2$$

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3}\frac{1}{x^{2/3}} \qquad f'(2) = \frac{1}{3}\frac{1}{2^2}$$

$$f''(x) = -\frac{2}{3^2}\frac{1}{x^{5/3}} \qquad f''(2) = -\frac{2}{3^2}\frac{1}{2^5}$$

$$f'''(x) = \frac{2\cdot 5}{3^3}\frac{1}{x^{8/3}} \qquad f'''(2) = \frac{2\cdot 5}{3^3}\frac{1}{2^8}$$

$$f^{iv}(x) = -\frac{2\cdot 5\cdot 8}{3^4}\frac{1}{x^{11/3}} \qquad f^{iv}(2) = -\frac{2\cdot 5\cdot 8}{3^4}\frac{1}{2^{11}}$$

$$f^{v}(x) = \frac{2\cdot 5\cdot 8\cdot 11}{3^5}\frac{1}{x^{14/3}} \qquad f^{v}(2) = \frac{2\cdot 5\cdot 8\cdot 11}{3^5}\frac{1}{2^{14}}$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$x^{1/3} = 2 - \frac{1}{3}\frac{1}{2^2}(x-8) - \frac{2}{3^2}\frac{1}{2^5}(x-8)^2 + \frac{2\cdot 5}{3^3}\frac{1}{2^8}(x-8)^3 - \frac{2\cdot 5\cdot 8}{3^4}\frac{1}{2^{11}}(x-8)^4 + \frac{2\cdot 5\cdot 8\cdot 11}{3^5}\frac{1}{2^{14}}(x-8)^5 + \cdots$$

$$x^{1/3} = \frac{1}{2^{-1}} - \frac{1}{3}\frac{1}{2^2}(x-8) - \frac{2}{3^2}\frac{1}{2^5}(x-8)^2 + \frac{2\cdot 5}{3^3}\frac{1}{2^8}(x-8)^3 - \frac{2\cdot 5\cdot 8}{3^4}\frac{1}{2^{11}}(x-8)^4 + \frac{2\cdot 5\cdot 8\cdot 11}{3^5}\frac{1}{2^{14}}(x-8)^5 + \cdots$$

EXAMPLE STEWART

Find the Taylor series of $f(x) = \ln x$ by the definition of Taylor series (by taking derivatives and evaluating them at x = a). Let a = 1. Get the first several terms only.

We write out Taylor's expansion:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$f(x) = \ln x$	$f(1) = 0$
$f'(x) = \frac{1}{x}$	$f'(1) = 1$
$f''(x) = -\frac{1}{x^2}$	$f''(1) = -1$
$f'''(x) = \frac{2}{x^3}$	$f'''(1) = 2$
$f^{iv}(x) = -\frac{2 \cdot 3}{x^4}$	$f^{iv}(1) = -3!$
$f^{v}(x) = \frac{2 \cdot 3 \cdot 4}{x^5}$	$f^{v}(1) = 4!$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$\ln x = (x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4 + \frac{4!}{5!}(x-1)^5 - \cdots$$

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \cdots$$

$$\ln x = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{(x-1)^n}{n}$$

EXAMPLE STEWART

Find the Taylor series of $f(x) = \cos^2 x$ by the definition of Taylor series (by taking derivatives and evaluating them at x = a). Let a = 0. Get the first several terms only.

We write out Taylor series:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

$f(x) = \cos^2 x$ $\qquad$ $f(0) = 1$

$f'(x) = -2\cos x \sin x = -\sin(2x)$ $\qquad$ $f'(0) = 0$

$f''(x) = -4\cos 2x$ $\qquad$ $f''(0) = -2^2$

$f'''(x) = 8\sin 2x$ $\qquad$ $f'''(0) = 0$

$f^{iv}(x) = 16\cos 2x$ $\qquad$ $f^{iv}(2) = 2^4$

$f^{v}(x) = -32\sin 2x$ $\qquad$ $f^{v}(2) = 0$

$f^{vi}(x) = -64\ \cos 2x$ $\qquad$ $f^{v}(2) = -2^6$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

$$\cos^2 x = 1 - \frac{1}{2!}\,2^2\,x^2 + \frac{1}{4!}\,2^4\,x^4 - \frac{1}{6!}\,2^6\,x^6 - \cdots$$

$$\cos^2 x = \sum_{n=0}^{\infty} (-1)^{n+1}\,\frac{1}{(2n)!}\,(2x)^{2n}$$

EXAMPLE

Find the Maclaurin series for $f(x) = (1+x)^{-2}$. Use the definition of the Maclaurin series (form the derivatives and use the formal series).

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

$f(x) = (1+x)^{-2}$ $\qquad$ $f(0) = 1$

$f'(x) = -2\,(1+x)^{-3}$ $\qquad$ $f'(0) = -2$

$f''(x) = (-2)\,(-3)\,(1+x)^{-4}$ $\qquad$ $f''(0) = 3!$

$f'''(x) = (-2)\,(-3)\,(-4)\,(1+x)^{-5}$ $\qquad$ $f'''(0) = -4!$

$f^{iv}(x) = (-2)\,(-3)\,(-4)\,(-5)\,(1+x)^{-6}$ $\qquad$ $f^{iv}(0) = 5!$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

$$f(x) = 1 - 2\,x + \frac{3!}{2!}\,x^2 - \frac{4!}{3!}\,x^3 + \frac{5!}{4!}\,x^4 \ldots$$

$$f(x) = 1 - 2\,x + 3\,x^2 - 4\,x^3 + 5\,x^4 \ldots$$

$$f(x) = \frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} (-1)^n\,n\,x^{n-1}$$

Get interval of convergence using ratio test.

$u_n = (-1)^n\,n\,x^{n-1}$ $\qquad$ $u_{n+1} = (-1)^{n+1}\,(n+1)\,x^n$

$\lim\limits_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$ $\qquad$ $\lim\limits_{n\to\infty}\left|\frac{(n+1)\,x^n}{n\,x^{n-1}}\right| < 1$ $\qquad$ $|x|\,\lim\limits_{n\to\infty}\left|\frac{(n+1)}{n}\right| < 1$

$|x| < 1$ $\quad$ so $\quad$ -1 < x < +1

The series will diverge at the endpoints since the summand does not approach zero.

EXAMPLE

Find the Maclaurin series for $f(x) = \ln(1+x)$. Use the definition of the Maclaurin series (form the derivatives and use the formal series).

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

$f(x) = \ln(1+x)$ $\qquad\qquad$ $f(0) = 0$

$f'(x) = (1+x)^{-1}$ $\qquad\qquad$ $f'(0) = 1$

$f''(x) = -(1+x)^{-2}$ $\qquad\qquad$ $f''(0) = -1$

$f'''(x) = 2\,(1+x)^{-3}$ $\qquad\qquad$ $f'''(0) = 2$

$f^{iv}(x) = (2)\,(-3)\,(1+x)^{-4}$ $\qquad\qquad$ $f^{iv}(0) = -3!$

$f^{v}(x) = (2)\,(3)\,(4)\,(1+x)^{-5}$ $\qquad\qquad$ $f^{v}(0) = 4!$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

$$\ln(1+x) = 0 + 1\,x - 1\,x^2 + \frac{2!}{3!}\,x^3 - \frac{3!}{4!}\,x^4 + \frac{4!}{5!}\,x^5 \ \ldots$$

$$\ln(1+x) = x - x^2 + \frac{1}{3}\,x^3 - \frac{1}{4}\,x^4 + \frac{1}{5}\,x^5 - \cdots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\,\frac{1}{n}\,x^n$$

Use the ratio test to find the interval of convergence.

$u_n = (-1)^{n+1}\,\frac{1}{n}\,x^n$ $\qquad\qquad$ $u_{n+1} = (-1)^{n+2}\,\frac{1}{(n+1)}\,x^{n+1}$

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$ $\qquad\qquad$ $\lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\,\frac{n}{x^n}\right| < 1$ $\qquad\qquad$ $|x|\,\lim_{n\to\infty}\left|\frac{n}{n+1}\right| < 1$

$|x| < 1$ $\quad$ so $\quad$ -1 < x < +1

The series will converge at x =1 since the series will become the alternating harmonic series.

It will diverge at x=-1 since it will become the harmonic series.

EXAMPLE

Find the Maclaurin series for the function $f(x) = e^{-2x}$ by using the definition of the Maclaurin series (form the table of derivatives).

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

$f(x) = e^{-2x}$	$f(0) = 1$
$f'(x) = (-2)e^{-2x}$	$f'(0) = -2$
$f''(x) = (4)e^{-2x}$	$f''(0) = 2^2$
$f'''(x) = (-8)\,e^{-2x}$	$f'''(0) = -2^3$
$f^{iv}(x) = (16)e^{-2x}$	$f^{iv}(0) = 2^4$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

$$e^{-2x} = 1 - 2\,x + \frac{2^2}{2!}\,x^2 - \frac{2^3}{3!}\,x^3 + \frac{2^4}{4!}\,x^4 - \cdots$$

In sigma notation:

$$e^{-2x} = \sum_{n=0}^{\infty}(-1)^n\frac{(2x)^n}{n!}$$

EXAMPLE

Find the Maclaurin series for the function $f(x) = 2^x$ by using the definition of the Maclaurin series (form the table of derivatives).

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

$f(x) = 2^x$	$f(0) = 1$
$f'^{(x)} = 2^x \ln 2$	$f'(0) = \ln 2$
$f''(x) = 2^x (\ln 2\,)^2$	$f''(0) = \ln^2 2$
$f'''(x) = 2^x (\ln 2)^3$	$f'''(0) = \ln^3 2$
$f^{iv}(x) = 2^x (\ln 2)^4$	$f^{iv}\,(0) = \ln^3 2$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

$$e^{-2x} = 1 + \ln 2\; x + \frac{\ln^2 2}{2!}\,x^2 + \frac{\ln^3 2}{3!}\,x^3 + \frac{\ln^4 2}{4!}\,x^4 - \cdots$$

In sigma notation:

$$2^x = \sum_{n=0}^{\infty} \frac{(\ln 2\;x)^n}{n!}$$

EXAMPLE STEWART

Find the Maclaurin series for the function $f(x) = x\cos x$ by using the definition of the Maclaurin series (form the table of derivatives).

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

$f(x) = x\cos x$ $\qquad f(0) = 0$

$f'^{(x)} = \cos x - x\sin x$ $\qquad f'(0) = 1$

$f''(x) = -2\sin x - x\cos x$ $\qquad f''(0) = 0$

$f'''(x) = -3\cos x + x\sin x$ $\qquad f'''(0) = -3$

$f^{iv}(x) = 4\sin x + x\cos x$ $\qquad f^{iv}(0) = 0$

$f^{v}(x) = 5\cos x - x\sin x$ $\qquad f^{iv}(0) = 5$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

$$x\cos x = 0 + 1\,x + 0\,x^2 - \frac{3}{3!}\,x^3 + 0\,x^4 + \frac{5}{5!}\,x^5 - \cdots$$

$$x\cos x = x - \frac{1}{2!}\,x^3 + \frac{1}{4!}\,x^5 - \cdots$$

In sigma notation:

$$x\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$$

EXAMPLE

Find the Maclaurin series for the function $f(x) = x^3 + 2x + 1$ by using the definition of the Maclaurin series (form the table of derivatives).

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

$f(x) = x^3 + 2x + 1$ $\qquad$ $f(0) = 1$

$f'(x) = 3x^2 + 2$ $\qquad$ $f'(0) = 2$

$f''(x) = 6\,x$ $\qquad$ $f''(0) = 0$

$f'''(x) = 6$ $\qquad$ $f'''(0) = 6$

$f^{iv}(x) = 0$ $\qquad$ $f^{iv}(0) = 0$

$f^{v}(x) = 0$ $\qquad$ $f^{v}(0) = 0$

$\vdots$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

$$f(x) = 1 + 2\,x + 0\,x^2 + \frac{6}{3!}\,x^3 + 0\,x^4 + \frac{5}{5!}\,x^5 - \cdots$$

$$f(x) = 1 + 2x + x^3$$

This question shows that the Maclaurin series of a polynomial is the same polynomial.

EXAMPLE

Find the Taylor series for the function $f(x) = x^2 + 2x + 5$ by using the definition of the Taylor series (form the table of derivatives). Expand about a = 1.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$f(x) = x^2 + 2x + 5$ $\qquad f(1) = 8$

$f'(x) = 2x + 2$ $\qquad f'(1) = 4$

$f''(x) = 2$ $\qquad f''(1) = 2$

$f'''(x) = 0$ $\qquad f'''(1) = 0$

$f^{iv}(x) = 0$ $\qquad f^{iv}(1) = 0$

$f^{v}(x) = 0$ $\qquad f^{v}(0) = 0$

$\vdots$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$
$$f(x) = 8 + 4(x-1) + 2(x-1)^2$$

If you expand everything out you will get the original polynomial.

The polynomial does not change – it only changes the way it looks – it is the identical, same polynomial.

EXAMPLE STEWART

Find the Taylor series for the function $f(x) = \ln x$ by using the definition of the Taylor series (form the table of derivatives). Expand about a = 2.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$f(x) = \ln x$ $\qquad$ $f(2) = \ln 2$

$f'(x) = \frac{1}{x}$ $\qquad$ $f'(2) = \frac{1}{2}$

$f''(x) = -\frac{1}{x^2}$ $\qquad$ $f''(2) = -\frac{1}{2^2}$

$f'''(x) = \frac{2}{x^3}$ $\qquad$ $f'''(2) = \frac{2}{2^3}$

$f^{iv}(x) = -\frac{6}{x^4}$ $\qquad$ $f^{iv}(2) = -\frac{3!}{2^4}$

$f^{v}(x) = \frac{24}{x^5}$ $\qquad$ $f^{v}(2) = \frac{4!}{2^5}$

$f^{vi}(x) = \frac{-120}{x^6}$ $\qquad$ $f^{vi}(2) = -\frac{5!}{2^6}$

$\vdots$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$\ln x = \ln 2 + \frac{1}{2}(x-2) + \frac{1}{2!}\left(-\frac{1}{2^2}\right)(x-2)^2 + \frac{1}{3!}\left(\frac{2}{2^3}\right)(x-2)^3 + \frac{1}{4!}\left(-\frac{3!}{2^4}\right)(x-2)^4$$

$$\ln x = \ln 2 + \frac{1}{2}(x-2) - \frac{1}{2}\left(\frac{1}{2^2}\right)(x-2)^2 + \frac{1}{3}\left(\frac{1}{2^3}\right)(x-2)^3 - \frac{1}{4}\left(\frac{1}{2^4}\right)(x-2)^4 + \cdots$$

In sigma notation this is:

$$\ln x = \ln 2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \frac{1}{2^n} (x-2)^n$$

EXAMPLE STEWART MODIFIED

Find the Taylor series for the function $f(x) \ = \frac{1}{x}$ by using the definition of the Taylor series (form the table of derivatives). Expand about a = 3.

$$f(x) \ = \ f(a) \ + \ f'(a)\,(x-a) \ + \ \frac{f''(a)}{2!}\ (x-a)^2 \ + \ \frac{f'''(a)}{3!}\ (x-a)^3 + \cdots$$

$f\ (x) = \frac{1}{x}$	$f\ (3) \ = \frac{1}{3}$
$f'(x) = \ -\frac{1}{x^2}$	$f'\ (3) \ = \ -\frac{1}{3^2}$
$f''(x) \ = \ \frac{2}{x^3}$	$f''(3) \ = \frac{2}{3^3}$
$f\,'''\ (x) \ = \ -\frac{6}{x^4}$	$f\,'''\ (3) \ = \ -\frac{3!}{3^4}$
$f^{\,i\,v}\,(x) \ = \ \frac{24}{x^5}$	$f^{\,i\,v}\ (3) \ = \ \frac{4!}{3^5}$
$f^{\,v}\ (x) \ = \ \frac{-120}{x^5}$	$f^{\,v}\ (3) \ = \ -\frac{5!}{3^5}$

$\vdots$

$$f(x) \ = \ f(a) \ + \ f'(a)\,(x-a) \ + \ \frac{f''(a)}{2!}\ (x-a)^2 \ + \ \frac{f'''(a)}{3!}\ (x-a)^3 + \cdots$$

$$\frac{1}{x} \ = \ \frac{1}{3} + \left(-\frac{1}{3^2}\right)(x-3) + \frac{1}{2!}\left(\frac{2}{3^3}\right)(x-3)^2 + \frac{1}{3!}\left(\frac{-3!}{3^4}\right)(x-3)^3 \ + \frac{1}{4!}\left(\frac{4!}{3^5}\right)(x-3)^4$$

$$\frac{1}{x} \ = \ \frac{1}{3} - \frac{1}{3^2}\,(x-3) \ + \ \frac{1}{3^2}\,(x-3)^2 - \frac{1}{3^4}\,(x-3)^3 \ + \frac{1}{3^5}\,(x-3)^4 \ + \cdots$$

In sigma notation this is:

$$\frac{1}{x} \ = \ \sum_{n\,=\,0}^{\infty} \ (-1)^n \ \frac{1}{3^n}\ (x-3)^n$$

EXAMPLE STEWART MODIFIED

Find the Taylor series for the function $f(x) = e^{2x}$ by using the definition of the Taylor series (form the table of derivatives). Expand about a = 3.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$f(x) = e^{2x}$	$f(3) = e^6$
$f'(x) = 2e^{2x}$	$f'(3) = 2e^6$
$f''(x) = 4e^{2x}$	$f''(3) = 2^2 e^6$
$f'''(x) = 8e^{2x}$	$f'''(3) = 2^3 e^6$
$f^{iv}(x) = 16e^{2x}$	$f^{iv}(3) = 2^4 e^6$
$f^{v}(x) = 32e^{2x}$	$f^{v}(3) = 2^5 e^6$
$\vdots$	

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$e^{2x} = e^6 + 2e^6(x-3) + \frac{1}{2!}(2^2 e^6)(x-3)^2 + \frac{1}{3!}(2^3 e^6)(x-3)^3 + \frac{1}{4!}(2^4 e^6)(x-3)^4 + \cdots$$

$$e^{2x} = e^6\left(1 + 2(x-3) + \frac{1}{2!}(2^2)(x-3)^2 + \frac{1}{3!}(2^3)(x-3)^3 + \frac{1}{4!}(2^4)(x-3)^4 + \cdots\right)$$

In sigma notation this is:

$$e^{2x} = e^6 \sum_{n=0}^{\infty} \frac{1}{n!} 2^n (x-3)^n$$