PARTIAL FRACTIONS REPEATED TERMS

So far we have seen that if we have linear terms and quadratic terms, they have an expansion that is governed by the following:

$$\frac{1}{(x-a_1)\,(x-a_2)\,(x^2+b_1x+c_1)\,(x^2+b_2x+c_2)}$$

$$=\frac{A_1}{(x-a_1)}+\frac{A_2}{(x-a_2)}+\frac{B_1x+C_1}{(x^2+b_1x+c_1)}+\frac{B_2\,x+C_2}{(x^2+b_2x+c_2)}$$

Linear terms in the expansion have a numerator that is a constant. Quadratic terms in the expansion get a linear polynomial in the numerator.

If there were a cubic polynomial in the denominator it would have a numerator that was quadratic:

$$\frac{1}{(x-a_1)\,(x^3+b_1x^2+c_1x+d_1)}=\frac{A_1}{x-a_1}+\frac{B_1x^2+C_1x+D_1}{x^3+b_1x^2+c_1x+d_1}$$

Whatever the polynomial in the denominator, its numerator will be a polynomial that is one degree less. This is a universal rule for the expansions.

The above covers everything that we have done up to this point. There is only one more case to cover – the case of repeated factors. This is what we are going to review in this handout. A repeated factor is one that has an positive integer exponent. The expansion rule is straight forward. Here it is:

$$\frac{1}{(x+1)\,(x+2)^3}=\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}+\frac{D}{(x+2)^3}$$

The repeated term is $(x+2)^3$. The expansion contains this term starting with exponent one and going all the way up to exponent 3.

Another example to demonstrate repeated terms involves a quadratic:

$$\frac{1}{(x+1)(x^2+5)^2} = \frac{A}{x+1} + \frac{Bx+C}{(x^2+5)} + \frac{Dx+E}{(x^2+5)^2}$$

Here we see that the numerator always is a linear polynomial (one degree less than the repeated polynomial). This is true for all repeated cases. The polynomial in the numerator is always one degree less than the repeated polynomial (the polynomial inside the parentheses) no matter what the outside exponent is.

1. Evaluate $\int \frac{1}{(x+1)\,(x+2)^2}\,dx$

$\frac{1}{(x+1)\,(x+2)^2} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$ this is the correct expansion

$\frac{1}{(x+1)\,(x+2)^2} = \frac{A\,(x+2)^2}{(x+1)\,(x+2)^2} + \frac{B(x+1)(x+2)}{(x+2)\,(x+1)\,(x+2)} + \frac{C(x+1)}{(x+1)(x+2)^2}$ common denominator

$1 = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$ equate numerators

$1 = A(x^2+4x+4) + B(x^2+3x+2) + C(x+1)$ expand

$1 = Ax^2 + 4Ax + 4A + Bx^2 + 3Bx + 2B + Cx + C$ expand

$1 = (A+B)x^2 + (4A+3B+C)x + (4A+2B+C)$ group like terms

Equate polynomials to generate simultaneous equations:

$$0x^2 + 0x + 1 = (A + B)x^2 + (4A + 3B + C)x + (4A + 2B + C)$$

X^2: 0 = A+B

X: 0 = 4A+3B+C

X^0: 1 = 4A+2B+C

Solve the system of equations. I use the TI84 with rref command.

A=1 B=-1 C=-1

$$\frac{1}{(x+1)\,(x+2)^2} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$$

$$\frac{1}{(x+1)\,(x+2)^2} = \frac{1}{(x+1)} - \frac{1}{(x+2)} - \frac{1}{(x+2)^2}$$

$$\int \frac{1}{(x+1)\,(x+2)^2}\,dx = \int \left(\frac{1}{(x+1)} - \frac{1}{(x+2)} - \frac{1}{(x+2)^2}\right) dx$$

$$\int \frac{1}{(x+1)\,(x+2)^2}\,dx = \ln|x+1| - \ln|x+2| + \frac{1}{x+2} + C$$

2. Evaluate $\int \frac{1}{(x-2)\,(x+4)^2}\,dx$

$\frac{1}{(x-2)\,(x+4)^3} = \frac{A}{(x-2)} + \frac{B}{(x+4)} + \frac{C}{(x+4)^2}$ partial fraction expansion

$\frac{1}{(x-2)\,(x+4)^2} = \frac{A(x+4)^2}{(x-2)\,(x+4)^2} + \frac{B\,(x-2)\,(x+4)}{(x-2)\,(x+4)^2} + \frac{C(x-2)}{(x-2)(x+4)^2}$ Common denominator

$1 = A(x+4)^2 + B(x-2)(x+4) + C(x-2)$ Equate numerators

$1 = A\,(x^2 + 8x + 16) + B(x^2 + 2x - 8) + C(x-2)$ Expand

$1 = (A+B)x^2 + (8A + 2B + C)x + (16A - 8B - 2C)$ group like terms

Equate polynomials to generate simultaneous equations:

A+B=0

8A+2B+C = 0

16A – 8B – 2C = 1

Solve simultaneous equations – again I use TI84 with rref command:

A = 1/36 B = -1/36 C = -1/6

$\frac{1}{(x-2)\,(x+4)^3} = \frac{A}{(x-2)} + \frac{B}{(x+4)} + \frac{C}{(x+4)^2}$

$\frac{1}{(x-2)\,(x+4)^3} = \frac{1}{36}\,\frac{1}{(x-2)} - \frac{1}{36}\,\frac{1}{(x+4)} - \frac{1}{6}\,\frac{1}{(x+4)^2}$

$$\int \frac{1}{(x-2)\,(x+4)^2}\,dx \;=\; \int \left(\frac{1}{36}\,\frac{1}{(x-2)} \;-\frac{1}{36}\,\frac{1}{(x+4)} \;-\frac{1}{6}\,\frac{1}{(x+4)^2}\right) dx$$

$$\int \frac{1}{(x-2)\,(x+4)^2}\,dx \;=\; \frac{1}{36}\ln|x-2| \;-\; \frac{1}{36}\ln|x+4| \;+\frac{1}{6}\cdot\frac{1}{x+4} \;+\; Const.$$

3. Evaluate $\int \frac{2x+6}{(x^2+4)\,(x-1)^2}\ dx$

$$\frac{2x+6}{(x^2+4)\,(x-1)^2} = \frac{Ax+B}{(x^2+4)} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2}$$ partial fraction expansion

$$\frac{2x+6}{(x^2+4)\,(x-1)^2} = \frac{(Ax+B)\ (x-1)^2}{(x^2+4)\,(x-1)^2} + \frac{C\,(x-1)\,(x^2+4)}{(x-1)^2\,(x^2+4)} + \frac{D\,(x^2+4)}{(x-1)^2\ (x^2+4)}$$ common denominator

$$2x+6 = (Ax+B)\ (x-1)^2 + C\,(x-1)\,(x^2+4) + D\,(x^2+4)$$ equate numerators

$$2x+6 = (Ax+B)\ (x^2-2x+1) + C\,(x^3-x^2+4x-4) + D\,(x^2+4)$$
$$2x+6 = (Ax^3-2Ax^2+Bx^2+Ax-2Bx+B) + (Cx^3-Cx^2+4Cx-4C) + (D\,x^2+4D)$$

$$2x+6 = (A+C)\ x^3 + (-2A+B-C+D)x^2 + (A-2B+4C)x + (B-4C+4D)$$

A+C=0

-2A+B-C+D = 0

A-2B + 4C = 2

B-4C+4D = 6

A=6/25 B = -34/25 C = -6/25 D = 8/5

$$\frac{2x+6}{(x^2+4)\,(x-1)^2} = \frac{Ax+B}{(x^2+4)} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2}$$

$$\frac{2x+6}{(x^2+4)\,(x-1)^2} = \frac{1}{25}\,\frac{6x-34}{(x^2+4)} - \frac{6}{25}\,\frac{1}{(x-1)} + \frac{8}{5}\,\frac{1}{(x-1)^2}$$

$$\int \frac{2x+6}{(x^2+4)\,(x-1)^2}\ dx \ = \int \left(\frac{1}{25}\,\frac{6x-34}{(x^2+4)} - \frac{6}{25}\,\frac{1}{(x-1)} + \frac{8}{5}\,\frac{1}{(x-1)^2}\right) dx$$

$$\int \frac{2x+6}{(x^2+4)\,(x-1)^2}\ dx \ = \frac{6}{25}\cdot\frac{1}{2}\ \ln(x^2+4)\ - \frac{34}{25}\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} - \frac{6}{25}\ln|x-1| - \frac{8}{5}\frac{1}{x-1}\ + C$$

$$\int \frac{2x+6}{(x^2+4)\,(x-1)^2}\ dx \ = \frac{3}{25}\ \ln(x^2+4)\ - \frac{17}{25}\ \tan^{-1}\frac{x}{2} - \frac{6}{25}\ln|x-1| - \frac{8}{5}\frac{1}{x-1}\ + C$$

4. Evaluating the integral $\int \frac{1}{(u^2+1)^n}\, du \qquad n > 1$

If we have a partial fraction expansion with denominator terms like $(ax^2 + bx + c)^n$

these terms can be reduced to $(u^2 + 1)^n$ by completing the square. The question is how to do the resulting integrals.

We make a trig substitution letting u = tan t along with du = sec^2 t dt

$$\int \frac{1}{(u^2+1)^n}\, du \;=\; \int \frac{1}{(\tan^2 t + 1)^n}\; \sec^2 t \;\; dt$$

$$\int \frac{1}{(u^2+1)^n}\, du \;=\; \int \frac{1}{\sec^{2n} t}\; \sec^2 t \;\; dt$$

$$\int \frac{1}{(u^2+1)^n}\, du \;=\; \int \cos^{2n-2} t \;\; dt$$

So these integral evolve into integrals of cosine to even powers. These integrals tend to be tedious. We need double angle formulas to solve.

These questions are tedious and lengthy. You should know how to do them but I will not ask them.

5. Evaluate $\int \frac{1}{(x+3)(x^2+1)^2}\, dx$

$$\frac{1}{(x+3)(x^2+1)^2} = \frac{A}{(x+3)} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$$ partial fraction expansion

$$\frac{1}{(x+3)(x^2+1)^2} = \frac{A\left(x^2+1\right)^2}{(x+3)(x^2+1)^2} + \frac{(Bx+C)\left(x^2+1\right)(x+3)}{(x+3)\,(x^2+1)\,(x^2+1)} + \frac{(Dx+E)\,(x+3)}{(x^2+1)^2\ (x+3)}$$ common denominator

$$1 = A\,(x^2+1)^2 + (Bx+C)\,(x^2+1)\,(x+3) + (Dx+E)\,(x+3)$$ equate numerators

$$1 = A(x^4+2x^2+1) + (Bx+C)\,(x^3+3x^2+x+3) + (Dx^2 + 3Dx+Ex+3E)$$ Expand

Expand

$$1 = A(x^4+2x^2+1) + (Bx^4+3Bx^3+Bx^2+3Bx+Cx^3+3Cx^2+Cx+3C\,) + (Dx^2 + 3Dx+Ex+3E)$$

Group like terms

$$1 = (A+B)x^4 + (3B+C)x^3 + (2A+B+3C+D)x^2 + (3B+C+3D+E)x + (A+3C+3E)$$

Equate polynomials to generate simultaneous equations:

A+B=0

3B+C=0

2A+B+3C+D=0

3B+C+3D +E= 0

A+3C+3E = 1

Solve these simultaneous equations – again the TI84 calculator with the RREF command.

A=1/100 B=-1/100 C = 3/100 D = -1/10 E = 3/10

$$\frac{1}{(x+3)(x^2+1)^2} = \frac{A}{(x+3)} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$$

$$\frac{1}{(x+3)(x^2+1)^2} = \frac{1}{100}\frac{1}{(x+3)} + \frac{1}{100}\frac{-x+3}{(x^2+1)} + \frac{1}{10}\frac{-x+3}{(x^2+1)^2}$$

$$\int \frac{1}{(x+3)(x^2+1)^2}\, dx = \int \left(\frac{1}{100}\frac{1}{(x+3)} + \frac{1}{100}\frac{-x+3}{(x^2+1)} + \frac{1}{10}\frac{-x+3}{(x^2+1)^2}\right) dx$$

All the integrals are straight forward except for the last one.

$$\int \frac{1}{(x+3)(x^2+1)^2}\, dx = \ln|x+3| - \frac{1}{200}\ln(x^2+1) + \frac{3}{100}\tan^{-1} x + \frac{1}{20}\left(\frac{1}{x^2+1}\right) + \frac{3}{10}\int \frac{1}{(x^2+1)^2}\, dx$$

For the last integral, let x = tan t and evaluate. I won't do it here.