IMPROPER INTEGRALS – CONTINUITY

When we went over the fundamental theorem of calculus, we noted that a function only has to be continuous in order for the fundamental theorem to work.

The fundamental theorem tells us that the Riemann sum $\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\ \Delta x$ equals the integral $\int_a^b f(x)\, dx = \ F(b) - F(a)$ so long as f(x) is continuous on the closed interval [a,b].

What if a function is not continuous? What if it has a vertical asymptote? Can we still use the fundamental theorem to find the integral? Certainly there is still an area in the region we are considering. And so long as f(x) is positive, can't we still find that area? And even if f(x) is both positive and negative, can't we still interpret the integral as signed area? And the answer is YES. In many circumstances we can do the integral even if it has a vertical asymptote.

This is a major extension to our ability to do integrals. Continuity of the function is sufficient to guarantee that the integral exists. But there are many functions that are not continuous and even though they have a vertical asymptote, causing the discontinuity, we can still do the integral. Our ability to do integrals has been extended to functions that are not continuous. This is a major statement.

So we have a function that is not continuous. Is it guaranteed that we can do the integral? The answer is NO. Sometimes the integral will exist and sometimes it won't. This is what we are going to look at in this handout.

Mathematics is a rigorous subject and everything has to be precisely defined. So we have to be more exact as to what is going on.

Let f(x) be continuous on the interval [a,b) and let there be a vertical asymptote at x = b. So as the function approaches x=b, the function shoots off to infinity. It could be positive infinity or negative infinity, it does not make a difference. We can still ask if $\int_a^b f(x)\,dx$ exists if we define it like this:

Let $t \epsilon [a,b)$. Then the integral $\int_a^b f(x)\,dx \equiv \lim_{t \to b^-} \int_a^t f(x)\,dx$.

This is actually very smart. The function is definitely continuous on [a,t]. This means that the integral on the right side has to exist. We can consider what happens at the discontinuity by letting t approach b with a limit.

If the asymptote exists at x = a, then we can modify the above like this:

Let $t \epsilon (a,b]$. Then the integral $\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$

Either way, if there is an asymptote, we define the integral up to some point t, and then let t approach the asymptote.

So what is the answer? The answer can be a number, like 3 or pi, or the integral can diverge – the integral can go to infinity (positive or negative). So there are only two possible answers – the integral equals a number (like 3) or the integral diverges. Those are the only two possible answers.

There is a third scenario we need to discuss. What if the vertical asymptote is somewhere in the middle of the interval [a,b]? What if the vertical asymptote is at x = c, where $c \epsilon (a,b)$? In this case we have to evaluate two integrals and we do it like this:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Where $\int_a^c f(x)\,dx = \lim_{t \to c^-} \int_a^t f(x)\,dx$

and $\int_c^b f(x)\,dx = \lim_{u \to c^+} \int_u^b f(x)\,dx$

If both integrals on the right converge and produce actual numbers, then the original integral on the left converges and is the sum of those two numbers. <u>If either integral on the right side diverges, then the original integral diverges. The moment one of the two integrals on the right goes to infinity, the question is over.</u> It does not make a difference what the other integral does. Even if the other integral seems to cancel the first, it does make a difference; the original integral diverges.

Many students complain about this – sometimes one integral goes to positive infinity and the other goes to negative infinity – shouldn't they cancel? The answer is no, they do not cancel. The expression

$\infty - \infty$ is an indeterminate form and it can actually come out to equal anything. Students still complain. Sometimes the regions of integration are completely symmetrical – they are congruent – they are the exact same shape – but one is positive and the other is negative – shouldn't the integrals cancel? The answer is no. They do not cancel – the problem is with the limits – the indeterminate form $\infty - \infty$ is tricky and simple cancellation based on geometric congruence does not work. This can be demonstrated but time and space won't allow for it here.

You will note that I used t and u as the variables for the limits – I used two different variables. Many text books do not do this. I think this is a mistake. If the text book uses the same variable for both limits, just accept it. By using the same variable for both limits, they are intimating that cancellation based on geometric congruence does work – and this is wrong. The authors know that such cancellations are wrong. But they use the same variable anyway. Woof! There is also a technicality that occurs in a higher course that I wish to avoid (The Cauchy Principle Value).

Main point of the above: if one of the two integrals on the right side diverges, do not even look at the other integral. The answer to the original question is that the original integral diverges.

1. Evaluate $\int_0^1 \frac{1}{x^2}\,dx$

The function $f(x) = \frac{1}{x^2}$ has a vertical asymptote at x = 0 so the integral is improper.

We write $\int_0^1 \frac{1}{x^2}\,dx \;=\; \lim_{a \to 0^+} \int_a^1 \frac{1}{x^2}\,dx$

$$\int_0^1 \frac{1}{x^2}\,dx \;=\; \lim_{a \to 0^+} \left. -\frac{1}{x}\right|_a^1 \;=\; \lim_{a \to 0^+} \frac{1}{a} - \frac{1}{1} \;=\; \lim_{a \to 0^+} \frac{1}{a} - 1$$

Since $\lim_{a \to 0^+} \frac{1}{a} = +\infty$ the limit diverges – so does the integral

$\int_0^1 \frac{1}{x^2}\,dx$ diverges to positive infinity

2. Evaluate $\int_0^1 \frac{1}{(x-1)^3}\,dx$

The function $f(x) = \frac{1}{(x-1)^3}$ has a vertical asymptote at x = 1 so the integral is improper.

We write $\int_0^1 \frac{1}{(x-1)^3}\,dx \;=\; \lim_{b \to 1^-} \int_0^b \frac{1}{(x-1)^3}\,dx$

$$\int_0^1 \frac{1}{(x-1)^3}\,dx \;=\; \lim_{b \to 1^-} \left. -\frac{1}{(x-1)^2}\right|_0^b \;=\; \lim_{b \to 1^-} \frac{1}{1^2} - \frac{1}{(b-1)^2}$$

The limit $\lim_{b \to 1^-} \frac{1}{(b-1)^2}$ diverges to positive infinity – and so does the integral

$\int_0^1 \frac{1}{(x-1)^3}\,dx$ diverges to positive infinity

3. Evaluate $\int_1^{10} \frac{1}{\sqrt{x-1}}\ dx$

The function $f(x) = \frac{1}{\sqrt{x-1}}$ has a vertical asymptote at x = 1. This makes the integral improper.

So we write the integral as $\int_1^{10} \frac{1}{\sqrt{x-1}}\ dx\ =\ \lim_{a \to 1^-} \int_a^{10} \frac{1}{\sqrt{x-1}}\ dx$

$$\int_1^{10} \frac{1}{\sqrt{x-1}}\ dx\ =\ \lim_{a \to 1^-} 2\sqrt{x-1}\Big|_a^{10}\ =\ \lim_{a \to 1^-} 2\sqrt{10-1}\ -\ 2\sqrt{a-1}$$

$$\int_1^{10} \frac{1}{\sqrt{x-1}}\ dx\ = 2\sqrt{9}\ -\ 2\sqrt{0}\ =\ 6$$

In this case the integral converges. Even though there is a vertical asymptote and the function shoots up to positive infinity, there is still a finite amount of area present. The area is 6.

4. Evaluate the integral $\int_{-1}^{1} \frac{1}{x}\ dx$

The function $f(x)\ =\ \frac{1}{x}$ has a vertical asymptote at x = 0 so the integral is improper. The function has a discontinuity in the middle of the integral.

One might thing that the function $f(x) = \frac{1}{x}$ has odd symmetry and that the integral of such an odd function is automatically zero. We cannot think like this – that only works if the function is continuous. So we have to throw that idea out.

Since the integral is improper we write

$$\int_{-1}^{1} \frac{1}{x}\ dx\ =\ \lim_{t \to 0^-} \int_{-1}^{t} \frac{1}{x}\ dx\ +\ \lim_{u \to 0^+} \int_{u}^{1} \frac{1}{x}\ dx$$

Let's look at the integral $\lim_{u \to 0^+} \int_u^1 \frac{1}{x}\ dx$. We could have looked at the other one but we picked this one first.

$$\lim_{u \to 0^+} \int_{u}^{1} \frac{1}{x}\ dx\ =\ \lim_{u \to 0^+}\ \ln|x|\Big|_u^1\ =\ \lim_{u \to 0^+} \ln 1 - \ln|u|$$

Since the limit of ln u goes to negative infinity, the integral diverges to positive infinity.

Since this integral diverges, we do not have to look at the other one. The question is answered. If one of the integrals diverges, the original integral diverges.

$\int_{-1}^{1} \frac{1}{x}\ dx$ diverges

Some students may ask, does it diverge to positive infinity or negative infinity? It turns out that we cannot answer that question. It might be better to say that the original integral FAILS TO CONVERGE – it does not produce a finite number. But it is ok to say that the original integral diverges, even though it is not necessarily going either infinity.

To explain this more fully, we note that for the integrals on the right, one diverges to positive infinity. The other diverges to negative infinity. We are actually left with the form $\infty - \infty$. This is an indeterminate form. It turns out that integrals like this are extremely sensitive! Depending on how "quickly" we take the limits for t and u, we can make the integral come out to be any number we want! That is a strange phenomenon! So when the asymptote exists in the middle of the interval, it is possible to get the form $\infty - \infty$ (but this doesn't always happen – just sometimes). When this does happen, the rate at which we take the limits will determine what the indeterminate form produces. We can make it equal anything! Since the limits do not produce a unique number, we say that the integral diverges – or that the integral fails to converge.

5. Evaluate $\int_0^3 \frac{1}{(x-1)^{1/5}}\ dx$

The function $f(x) = \frac{1}{(x-1)^{1/5}}$ has a vertical asymptote in the middle of the interval; at x = 1. This makes the integral improper. So we write

$$\int_0^3 \frac{1}{(x-1)^{1/5}}\ dx = \lim_{t \to 1^-} \int_0^t \frac{1}{(x-1)^{1/5}}\ dx + \lim_{u \to 1^+} \int_u^3 \frac{1}{(x-1)^{1/5}}\ dx$$

We evaluate each integral

$$\lim_{t \to 1^-} \int_0^t \frac{1}{(x-1)^{1/5}}\ dx = \lim_{t \to 1^-} \frac{5}{4}(x-1)^{\frac{4}{5}}\Big|_0^t = \lim_{t \to 1^-} \frac{5}{4}(t-1)^{\frac{4}{5}} - \frac{5}{4}(0-1)^{\frac{4}{5}}$$

$$\lim_{t \to 1^-} \int_0^t \frac{1}{(x-1)^{1/5}}\ dx = 0 - \frac{5}{4} = -\frac{5}{4}$$

The next integral:

$$\lim_{u\to 1^+} \int_u^3 \frac{1}{(x-1)^{1/5}}\, dx = \lim_{u\to 1^-} \frac{5}{4}(x-1)^{\frac{4}{5}}\Big|_u^3 = \lim_{u\to 1^-} \frac{5}{4}(3-1)^{\frac{4}{5}} - \frac{5}{4}(u-1)^{\frac{4}{5}}$$

$$\lim_{u\to 1^+} \int_u^3 \frac{1}{(x-1)^{1/5}}\, dx = \frac{5}{4}(2)^{4/5}$$

$$\int_0^3 \frac{1}{(x-1)^{1/5}}\, dx = \lim_{t\to 1^-} \int_0^t \frac{1}{(x-1)^{1/5}}\, dx + \lim_{u\to 1^+} \int_u^3 \frac{1}{(x-1)^{1/5}}\, dx$$

$$\int_0^3 \frac{1}{(x-1)^{1/5}}\, dx = -\frac{5}{4} + \frac{5}{4}(2)^{4/5}$$

So this improper integral converges.

6. Evaluate $\int_{-1}^4 \frac{1}{x^3}\, dx$

The function has a vertical asymptote at x =0 so the integral is improper. It has a discontinuity in the middle of the interval. So we write it as

$$\int_{-1}^{4} \frac{1}{x^3}\, dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x^3}\, dx + \lim_{u\to 0^+} \int_{u}^{4} \frac{1}{x^3}\, dx$$

Evaluate $\lim_{t\to 0^-} \int_{-1}^t \frac{1}{x^3}\, dx$:

$$\lim_{t\to 0^-} \int_{-1}^t \frac{1}{x^3}\, dx = \lim_{t\to 0^-} -\frac{1}{x^2}\Big|_{-1}^t = \lim_{t\to 0^-} 1 - \frac{1}{t^2} = -\infty$$

This integral diverges to negative infinity.

So the question is solved. Don’t even look at the other integral.

The original integral diverges.

$\int_{-1}^4 \frac{1}{x^3}\, dx$ diverges

7. Evaluate $\int_0^{\frac{\pi}{2}} \tan x \; dx$

Tangent has a vertical asymptote at x = pi/2 so the integral is improper. As such we write

$$\int_0^{\frac{\pi}{2}} \tan x \; dx = \lim_{t \to \frac{\pi}{2}^-} \int_0^t \tan x \; dx$$

$$\int_0^{\frac{\pi}{2}} \tan x \; dx = \lim_{t \to \frac{\pi}{2}^-} \ln \sec x \; \Big|_0^t$$

$$\int_0^{\frac{\pi}{2}} \tan x \; dx = \lim_{t \to \frac{\pi}{2}^-} \ln \sec t - \ln \sec 0 = \lim_{t \to \frac{\pi}{2}^-} \ln \sec t - \ln 1$$

At pi/2, secant goes to infinity and so does the logarithm. So the integral diverges to positive infinity.

$\int_0^{\frac{\pi}{2}} \tan x \; dx$ diverges

8. Evaluate $\int_0^1 \frac{1}{\sqrt{1 - x^2}} \; dx$

At x = 1 the integrand has a vertical asymptote. This makes the integral improper. So we write

$$\int_0^1 \frac{1}{\sqrt{1 - x^2}} \; dx = \lim_{t \to 1^-} \int_0^t \frac{1}{\sqrt{1 - x^2}} \; dx$$

$$\int_0^1 \frac{1}{\sqrt{1 - x^2}} \; dx = \lim_{t \to 1^-} \sin^{-1} x \Big|_0^t$$

$$\int_0^1 \frac{1}{\sqrt{1 - x^2}} \; dx = \lim_{t \to 1^-} \sin^{-1} t - \sin^{-1} 0$$

$$\int_0^1 \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1} 1 - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

So the integral $\int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx = \frac{\pi}{2}$. It converges.

9. Evaluate the integral $\int_0^{\pi/2} \sec x \, dx$

Secant has a vertical asymptote at x=pi/2. This makes the integral improper. So we write

$$\int_0^{\pi/2} \sec x \, dx = \lim_{t \to \pi/2} \int_0^t \sec x \, dx$$

$$\int_0^{\pi/2} \sec x \, dx = \lim_{t \to \pi/2} \ln |\sec x + \tan x| \, |_0^t$$

$$\int_0^{\pi/2} \sec x \, dx = \lim_{t \to \pi/2} \ln |\sec t + \tan t| - \ln |\sec 0 + \tan 0|$$

$$\int_0^{\pi/2} \sec x \, dx = \lim_{t \to \pi/2} \ln |\sec t + \tan t| - \ln |1 + 0|$$

Both secant and tangent go to infinity at pi/2 so the limit on the right diverges. As such the integral diverges.

$\int_0^{\pi/2} \sec x \, dx$ diverges to positive infinity.

10. Evaluate $\int_0^2 \frac{1}{x^2-4x+3} \, dx$

The function $f(x) = \frac{1}{x^2-4x+3}$ can be written as $f(x) = \frac{1}{(x-1)\,(x-3)}$

The function has a vertical asymptote at x = 1.

$$\frac{1}{(x-1)\,(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}$$

$$1 = A(x-3) + B(x-1)$$

From this we get $A = -\frac{1}{2}$ $\quad B = \frac{1}{2}$

$$\int_0^2 \frac{1}{x^2-4x+3}\, dx = \int_0^2 \left[\frac{1}{2}\frac{1}{x-3} - \frac{1}{2}\frac{1}{x-1} \right] dx$$

$$\int_0^2 \frac{1}{x^2-4x+3}\, dx = \frac{1}{2}\int_0^2 \frac{1}{x-3}\, dx - \frac{1}{2}\int_0^2 \frac{1}{x-1}\, dx$$

$$\int_0^2 \frac{1}{x^2-4x+3}\, dx = -\frac{1}{2}\ln 3 - \frac{1}{2}\int_0^2 \frac{1}{x-1}\, dx$$

The remaining integral is improper.

$$\int_0^2 \frac{1}{x-1}\, dx = \lim_{t \to 1^-} \int_0^t \frac{1}{x-1}\, dx + \lim_{u \to 1^+} \int_u^2 \frac{1}{x-1}\, dx$$

$$\lim_{t \to 1^-} \int_0^t \frac{1}{x-1}\, dx = \lim_{t \to 1^-} \ln|x-1| \,\Big|_0^t$$

$$\lim_{t \to 1^-} \int_0^t \frac{1}{x-1}\, dx = \lim_{t \to 1^-} \ln|t-1| - \ln|-1|$$

The limit of the logarithm diverges to negative infinity. Since this integral diverges, the original integral also diverges.

$\int_0^2 \frac{1}{x^2-4x+3}\, dx$ diverges