SEQUENCES AND SERIES INTRODUCTION

A sequence can be defined at least two ways. The first way is that they may be defined as an ordered list of numbers. Numbers may repeat throughout the sequence. Patterns may be evident in the sequence or the numbers may be random. The sequence may always be increasing or always decreasing. The numbers in the sequence may oscillate. The numbers in the sequence may be bounded or they may grow to infinity. There is no set pattern. Any type of behavior is possible. It is just a list of numbers.

The numbers in the sequence may tend to a limit, so that all the numbers start approaching one specific value. The numbers in the sequence may group around two or more numbers – always staying close to those numbers – always getting closer to them (cluster points).

The second definition of a sequence is to define it as a function where the domain of the function is the positive integers. Sometimes zero is included in the domain. It is also possible to define the domain to include some negative integers.

We do not like to make the domain of the sequence function to be the set of all integers. This causes some problems. The starting point of the sequence becomes negative infinity and then writing the sequence in order becomes impossible. Usually the starting point of a sequence is zero or one and writing the sequence in order is easy.

Some people like to think of a sequence as an ordered set but this requires new definitions. Sets of numbers do not have any order. And for sets, repetitions of numbers are not allowed. A number is either in a set or it is not, and it is not repeated. For those of you studying computer science, you will come across POSETS, Partially Ordered Sets. There are similarities between POSETS and sequences.

IMPORTANT SEQUENCES

Arithmetic Sequence: a sequence defined by adding the same number

Ex: 1,3,5,7,9… just keep on adding 2 to get the next term

Formula: $a_{n+1} = a_n + 2$

Ex: 5,9,13,17,21… just keep on adding 4 to get the next term

Formula: $a_{n+1} = a_n + 4$

Geometric Sequence: a sequence defined by multiplying by the same number

Ex: 3,6,12,24,48,96… just keep multiplying by 2 to get the next term

Formula: $a_{n+1} = a_n * 2$

Ex: 4,12,36,108,324… just keep multiplying by 3 to get the next term

Formula: $a_{n+1} = a_n * 3$

Ex: 1, ½, ¼ , 1/8 , 1/16… just keep multiplying by ½ to get the next term

Fibonacci Sequence: a sequence generated by adding adjacent elements together

Ex: 1,2,3,5,8,13,21… 1+2=3 2+3=5 3+5=8 8+13=21 13+21=34…

Formula: $a_n = a_{n-1} + a_{n-2}$

Ex: 2,7,9,16,25,41… 2+7=9 7+9=16 16+25=41 25+41=66…

Formula: $a_n = a_{n-1} + a_{n-2}$

Harmonic Sequence: the reciprocal of the positive integers

Ex: 1, 1/2 , 1/3 , 1/4 , 1/5 ….

Formula: $a_n = 1/n$

Alternating sequence: -1,1,-1,1,-1,1…

Formula: $a_n = (-1)^n$ or $a_n = (-1)^{n+1}$

Another formula: $a_n = \cos n\pi$

Oscillating sequences:

Ex: 0,1,0,-1,0,1,0,-1,0,1,-1,0…

Formula: $a_n = \sin\left(\frac{n\,\pi}{2}\right)$

LIMITS OF SEQUENCES

When we talk about the limit of the sequence, we are talking about the behavior of a_n as n goes to infinity. We are talking about $\lim_{n\to\infty} a_n$. Sometimes this limit exists. Sometimes it does not. Sometimes a_n can go to positive or negative infinity so the limit diverges, or the limit does not exist. Sometimes the limit just bounces back and forth and between two numbers – this is what happens with oscillating series – this is divergence by oscillation.

If the sequence a_n heads towards one particular number and gets closer and closer to that number, then the limit exists. If this does not happen, then no matter what the behavior of the sequence, the limit does not exist. In order for the limit of a sequence to exist, a_n must get closer and closer to one single number. Once it is in the vicinity of that number it never leaves the neighborhood, and always gets closer to the target. The distance between the sequence and its limit goes to zero.

Examples:

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ the limit of the harmonic sequence is zero

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos n\pi$ does not exist – it just oscillates back and forth between

the values +1 and -1

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$ it just keeps getting closer and closer to 1

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin\frac{n\pi}{2}$ does not exist – the sequence just bounces around between -1,0 and 1

It does not get close to one single number.

PRECISE DEFINITION OF LIMIT OF A SEQUENCE

Let ϵ be any positive number, no matter how small.

Let N be any positive integer, no matter how big.

Let the index of the sequence n be such that n > N so that n is really large

If the above is true and if $|L - a_n| < \epsilon$ (where epsilon is incredibly small) then the limit of the above sequence exists and it equals the number L: $\lim_{n \to \infty} a_n = L$

The above definition is a mathematical way of saying that if you go way, way out into the sequence, it gets really, really close to a single number.

RELATIONSHIPS BETWEEN FUNCTIONS AND SEQUENCES

Let y = f(x) and for the sake of argument let the domain of the function be all real numbers. The domain does not have to be all real numbers but it makes the argument easier. We can transform y = f(x) into a sequence by changing the domain from all real numbers into the positive integers. So we can define $a_n = f(n)$.

Mathematicians work on the convention that x is usually a continuous variable and that n is usually a positive integer. By changing f(x) into f(n) we are changing a function of a continuous variable into a sequence with a discrete variable.

If y = f(x) and $\lim_{x \to \infty} f(x) = L$ then $\lim_{n \to \infty} a_n = L$

So the limit of the function is also the limit of the sequence.

DIVERGENCE TO INFINITY

If a_n just keeps growing and growing and does not stop, then we can write $\lim_{n \to \infty} a_n = \infty$

We need a precise definition of this statement. This is how mathematicians make a precise statement of a sequence going to infinity:

Let N be a very large value of the index n

Let M be any large positive number – no matter how large – no matter how many times you pick another one.

If choosing n>N makes $a_n > M$ then a_n goes to infinity: $\lim_{n \to \infty} a_n = \infty$

A USEFUL TOOL

If $\lim_{n \to \infty} |a_n| = 0$ then $\lim_{n \to \infty} a_n = 0$

If the absolute value of a sequence has a zero limit, then the sequence has a zero limit. This is very useful since absolute value signs are used a lot in the theorems and proofs. It also helps when we are dealing with alternating series – it allows us to solve problems without those pesky minus signs.

A THEOREM GOING BACK TO BASIC THEORY OF FUNCTIONS: CONTINUITY

We discussed continuity of functions back in calculus one. One way to define it was to state three criteria:

1. The function exists; f(a) exists and equals a real number
2. The limit $\lim_{x \to a} f(x)$ exists and equals a real number
3. The value of the function equals the value of the limit: $\lim_{x \to a} f(x) = f(a)$

The other way to define continuity was with the epsilon delta method: a function is continuous when the following is true: The function f(x) is close in value to f(a)

$|f(x) - f(a)| < \epsilon$

whenever x is close to a

$|x - a| < \delta$

Here, both epsilon and delta are very small (positive) numbers.

There is a third way to define continuity and it uses sequences. Here we define a sequence x_n and we let it equal a sequence of numbers on the x axis. We can denote the sequence as x_n and we let it approach a. It can be ANY sequence you want so long as the sequence approaches x = a. Then we can define the limit

$$\lim_{x \to a} f(x) = \lim_{n \to \infty} f(x_n) = f(a) = L$$

This definition – of approaching the point x=a along any/all such sequences – suffices for continuity.

Another way to write this is: if $\lim_{n \to \infty} x_n = a,\ then\ \lim_{n \to \infty} f(x_n) = f\left(\lim_{n \to \infty} x_n\right) = f(a)$

Some authors define continuity as the ability to interchange the order of finding the limit with finding the value of the function. The way Stewart presents it, he says if f(x) is continuous, then the above is legal.

MONOTONIC SEQUENCES

A sequence is monotonic increasing if $a_{n+1} > a_n$

A sequence is monotonic decreasing if $a_{n+1} < a_n$

A NICE THEOREM FOR MONOTONIC SEQUENCES

Let the sequence a_n be represented by a function $a_n = f(n)$
Let the corresponding (continuous) function be represented by y = f(x).
If f'(x) is always positive then the sequence a_n is monotonic increasing
If f'(x) is always negative then the sequence a_n is monotonic decreasing

So we can determine if the sequence is monotonic by taking the derivative.

BOUNDED SEQUENCES

Just like functions can be bounded, sequences can also be bounded. Let a_n be a sequence where n is a positive integer. The sequence is bounded above if $a_n < M$ for all n, where M is any. number. A sequence is bounded below of $a_n > L$ for all n, where L is any number. If a sequence is bounded above and below, then the sequence is said to be 'bounded'.

MAJOR THEOREM:

If a sequence is monotonic increasing and bounded above, then that sequence has a limit – the limit $\lim\limits_{n \to \infty} a_n$ exists and equals a real number. Also, if a monotonic decreasing sequence is bounded below, the limit also exists. Such a sequence is said to converge or is convergent.

A form of this theorem was used to justify the comparison test for improper integrals.

QUESTIONS STEWART PAGE 704

1. What is a sequence?

   A sequence is an ordered list of numbers or it is a function where the domain is the positive integers (or the non-negative integers).

1b. What does it mean to say that $\lim_{n \to \infty} a_n = 8$?

   It means that as n gets larger and larger, the values of the sequence get closer and closer to 8, such that the distance between a_n and 8 goes to zero.

   Let N be any integer no matter now large and let n be the index of the sequence a_n.
   Let ϵ be a positive number, no matter how small.

   If $|a_n - 8| < \epsilon$ when n > N then $\lim_{n \to \infty} a_n = 8$

1c. What does it mean to say that $\lim_{n \to \infty} a_n = \infty$?

   It means that as the index of the series gets larger and larger, the values of the sequence keep growing without any bound .

   It means that:
   If N is any large positive integer, no matter how large
   If n is the index of the series
   If M is any positive number, no matter how large

   Then if n>N we have that $a_n > M$

2. What is a convergent series? Give two examples
   A convergent series is one where $\lim_{n \to \infty} a_n$ exists and equals a real number

$$a_n = \frac{1}{n} \qquad \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0$$

$$a_n = \frac{1}{n^2} \qquad \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n^2} = 0$$

$$a_n = e^{-1/n} \qquad \lim_{n \to \infty} a_n = \lim_{n \to \infty} e^{-1/n} = 0$$

$$a_n = \frac{n}{n+2} \qquad \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n}{n+2} = 1$$

$$a_n = \sin\left(\frac{\pi}{4} + \frac{1}{n}\right) \qquad \lim_{n \to \infty} a_n = \lim_{n \to \infty} \sin\left(\frac{\pi}{4} + \frac{1}{n}\right) = \frac{\sqrt{2}}{2}$$

2b. What is a divergent series? Give examples

$a_n = n \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} n = \infty$

$a_n = n^2 \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \infty$

$a_n = (-1)^n \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n$ diverges oscillation

$a_n = \sin\left(\frac{n\pi}{2}\right) \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin\left(\frac{n\pi}{2}\right)$ diverges oscillation

$a_n = \sin n \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin(n)$ diverges – not pure oscillation but oscillatory behavior. Cluster points but no limit.

3. List the first five terms of the sequence

a. $a_n = \frac{2^n}{2n+1} \quad a_0 = 1 \quad a_1 = \frac{2}{3} \quad a_2 = \frac{4}{5} \quad a_3 = \frac{8}{7} \quad a_4 = \frac{16}{9} \quad a_5 = \frac{32}{11}$

b. $a_n = \frac{(n^2-1)}{n^2+1} \quad a_0 = -1 \quad a_1 = 0 \quad a_2 = \frac{3}{5} \quad a_3 = \frac{8}{10} \quad a_4 = \frac{15}{17} \quad a_5 = \frac{24}{26}$

c. $a_n = \frac{(-1)^n}{5^n} \quad a_0 = 1 \quad a_1 = -\frac{1}{5} \quad a_2 = \frac{1}{25} \quad a_3 = -\frac{1}{125} \quad a_4 = \frac{1}{625} \quad a_5 = -\frac{1}{3125}$

d. $a_n = \cos\left(\frac{n\pi}{2}\right) \quad a_0 = 1 \quad a_1 = 0 \quad a_2 = -1 \quad a_3 = 0 \quad a_4 = 1 \quad a_5 = 0$

e. $a_n = \frac{1}{n!} \quad a_0 = 1 \quad a_1 = 1 \quad a_2 = \frac{1}{2} \quad a_3 = \frac{1}{6} \quad a_4 = \frac{1}{24} \quad a_5 = \frac{1}{120}$

f. $a_n = \frac{(-1)^n n}{n!+1} \quad a_0 = 0 \quad a_1 = -\frac{1}{2} \quad a_2 = \frac{2}{3} \quad a_3 = -\frac{3}{7} \quad a_4 = \frac{4}{25} \quad a_5 = -\frac{5}{121}$

g. $a_1 = 1 \quad a_{n+1} = 5a_n - 3$ this is called a recurrence relation

$a_2 = 7 \quad a_3 = 32 \quad a_4 = 157 \quad a_5 = 782$

h. $a_1 = 6 \qquad a_{n+1} = \frac{a_n}{n}$

$a_2 = 6 \quad a_3 = 3 \quad a_4 = 1 \quad a_5 = \frac{1}{4} \quad a_6 = \frac{1}{20}$

i. $a_1 = 2 \qquad a_{n+1} = \frac{a_n}{1+a_n}$

$a_2 = \frac{2}{3} \quad a_3 = \frac{2}{5} \quad a_4 = \frac{2}{7} \quad a_5 = \frac{2}{9} \quad a_6 = \frac{2}{11} \quad a_7 = \frac{2}{13}$

starting to generate the odd harmonic sequence – cute

j. $a_1 = 2 \quad a_2 = 1 \quad a_{n+1} = a_n - a_{n-1}$

$a_3 = 1 \quad a_4 = 0 \quad a_5 = -1 \quad a_6 = -1 \quad a_7 = 0 \quad a_8 = 1 \quad a_9 = 1$

$a_{10} = 0 \quad a_{11} = -1$

4. Given the sequence find a_n

a. $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \cdots\right\}$

$a_n = \frac{1}{2n}$

b. $\left\{4, -1, \frac{1}{4}, -\frac{1}{16}, \cdots\right\}$

$a_n = \frac{4\,(-1)^{n+1}}{4^{n-1}}$

c. $\left\{-3, 2, -\frac{4}{3}, \frac{8}{9}, -\frac{16}{27} \cdots\right\}$

$a_n = -3\left(-\frac{2}{3}\right)^{n-1}$

d. $\{5, 8, 11, 14, 17, \cdots\}$

$a_1 = 5 \quad a_n = a_{n-1} + 3$ arithmetic sequence

e. $\left\{\frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6} \cdots\right\}$

$a_n = \frac{(-1)^{n+1}\, n^2}{n+1}$

f. $\{1\,,\ 0\ ,\ -1\ ,\ 0\ ,\ 1\ ,\ 0\ ,\ -1,\ 0\cdots\}$

$a_n = \sin\left(\frac{n\pi}{2}\right)$

5. DECIMAL PATTERNS TO SEE LIMITS. Evaluate each of the sequences numerically to see if you can determine the limit by the numbers

a. $a_n = \frac{3n}{1+6n}$

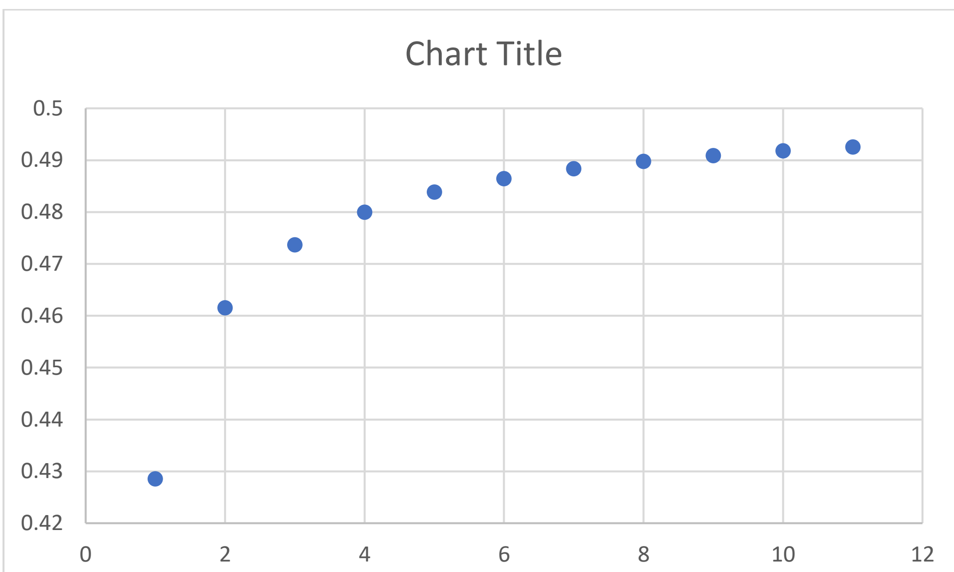
$a_1 = \frac{3}{7} = 0.4286 \qquad a_2 = \frac{6}{13} = 0.4615 \qquad a_3 = \frac{9}{19} = 0.4737 \qquad a_4 = \frac{12}{25} = 0.48$

$a_5 = \frac{15}{31} = 0.4834 \qquad a_6 = \frac{18}{37} = 0.4865 \qquad a_7 = \frac{21}{43} = 0.4884 \qquad a_8 = \frac{24}{49} = 0.4898$

$a_9 = \frac{27}{55} = 0.4909 \qquad a_{10} = \frac{30}{61} = 0.4918$

The limit is going to ½ or 0.5 but the sequence is converging very slowly

The graph looks like this:



The graph more clearly shows the limiting behavior – it looks like it is approaching a horizontal asymptote (approaching a limit).

b. $a_n = 2 + \frac{(-1)^n}{n}$

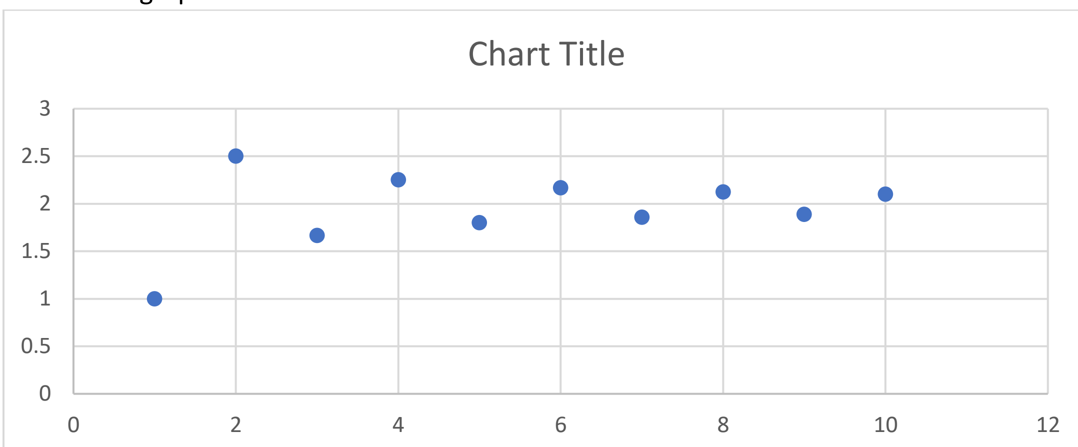
$a_1 = 1 \quad a_2 = \frac{5}{2} = 2.5 \quad a_3 = \frac{5}{3} = 1.6667 \quad a_4 = \frac{9}{4} = 2.25 \quad a_5 = \frac{9}{5} = 1.8$

$a_6 = \frac{13}{6} = 2.16667 \quad a_7 = \frac{13}{7} = 1.857 \quad a_8 = \frac{17}{8} = 2.125 \quad a_9 = \frac{17}{9} = 1.8888$

$a_{10} = 2.1$

The limit is going to 2 but the decimals do not show it that clearly. Slow convergence.

Here is the graph:



You can see the sequence is heading towards 2 but it is not very clear.

c. $a_n = 1 + \left(-\frac{1}{2}\right)^n$

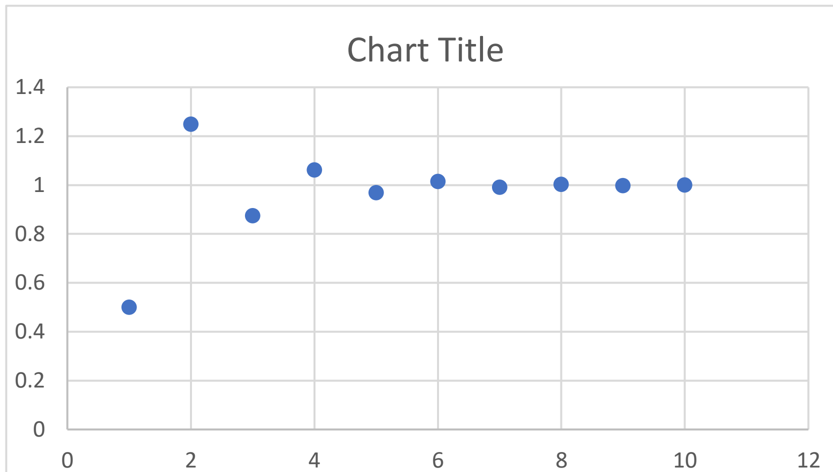
$a_1 = \frac{1}{2} = 0.5 \quad a_2 = \frac{5}{4} = 1.25 \quad a_3 = \frac{7}{8} = 1.875 \quad a_4 = \frac{17}{16} = 1.0625$

$a_5 = \frac{31}{32} = 0.96875 \quad a_6 = \frac{65}{64} = 1.015625 \quad a_7 = \frac{127}{128} = 0.9921875$

$a_8 = \frac{257}{256} = 1.004 \quad a_9 = \frac{511}{512} = 0.998 \quad a_{10} = \frac{1025}{1024} = 1.001$

The numbers here show it more clearly – they are getting closer to 1.

Here is the graph:



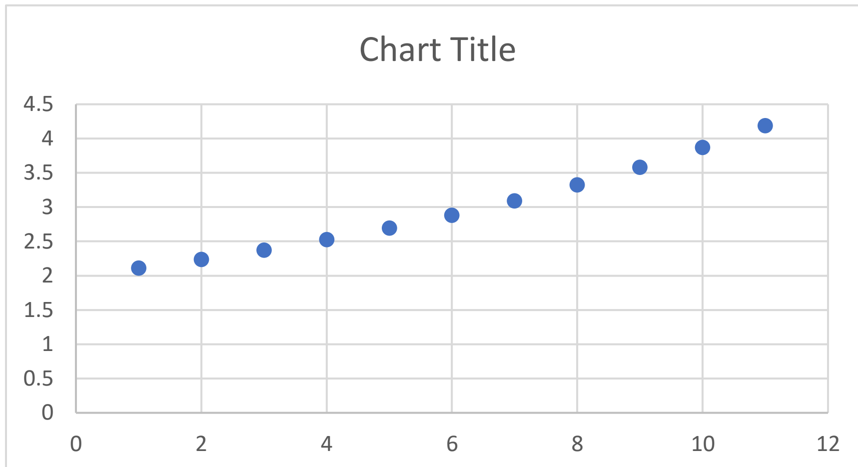
d. $a_n = \ 1 + \frac{10^n}{9^n}$

$a_1 = \ 2.111 \quad a_2 = 2.23456 \quad a_3 = 2.3717 \quad a_4 \ = \ 2.5242 \quad a_5 = 2.694$

$a_6 = 2.88167 \quad a_7 = \ 3.0908 \quad a_8 = 3.323 \quad a_9 = 3.58117 \quad a_{10} = 3.868$

The sequence of numbers is growing but very slowly. The series will diverge to positive infinity but this is difficult to see from the numerical patterns.

Here is the graph:



We can see that the sequence is increasing and the rate of increase is growing. It has the shape of exponential growth.

6. EVALUATING LIMITS USING ALGEBRA AND CALCULUS: FIND $\lim\limits_{n \to \infty} a_n$

a. $a_n = \frac{(3+5n^2)}{(n+n^2)}$

By LHospital's rule: $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \frac{(3+5n^2)}{(n+n^2)} = \lim\limits_{n \to \infty} \frac{10n}{2n} = \lim\limits_{n \to \infty} \frac{10}{2} = 5$

By algebra: $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \frac{(3+5n^2)}{(n+n^2)} = \lim\limits_{n \to \infty} \frac{\left(\frac{3}{n^2}+5\right)}{\left(\frac{1}{n}+1\right)} = \frac{(0+5)}{(0+1)} = 5$

b. $a_n = \frac{(3+5n^2)}{(1+n)}$

By LHospital's rule: $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \frac{(3+5n^2)}{(1+n)} = \lim\limits_{n \to \infty} \frac{10n}{1} = \infty$ diverges

By algebra: $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \frac{(3+5n^2)}{(1+n)} = \lim\limits_{n \to \infty} \frac{n^2\left(\frac{3}{n^2}+5\right)}{n\left(\frac{1}{n}+1\right)} = \lim\limits_{n \to \infty} \frac{n\,(0+5)}{(0+1)}$

$\lim\limits_{n \to \infty} 5n = \infty$ diverges

c. $a_n = \frac{n^4}{n^3-2n}$

By LHospital's rule: $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \frac{n^4}{n^3-2n} = \lim\limits_{n \to \infty} \frac{4\,n^3}{3n^2} = \lim\limits_{n \to \infty} \frac{4\,n}{3} = \infty$ diverges

By algebra: $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \frac{n^4}{n^3-2n} = \lim\limits_{n \to \infty} \frac{n^4}{n^3\left(1-\frac{2}{n^2}\right)} = \lim\limits_{n \to \infty} \frac{n}{\left(1-\frac{2}{n^2}\right)} =$

$\lim\limits_{n \to \infty} n = \infty$ diverges

d. $a_n = 2 + (0.86)^n$

$\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} 2 + (0.86)^n = 2 + 0 = 2$

e. $a_n = 3^n\ 7^{-n}$

$\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} 3^n\ 7^{-n} = \lim\limits_{n \to \infty} \left(\frac{3}{7}\right)^n = 0$

f. $a_n = \frac{\sqrt{n}}{\sqrt{n}+2}$

By LHospital's rule: $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n}+2} = \lim\limits_{n \to \infty} \frac{\frac{1}{2}\,n^{-1/2}}{\frac{1}{2}n^{-1/2}} = \lim\limits_{n \to \infty} 1 = 1$

By algebra: $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n}+2} = \lim\limits_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n}\,(1+2/\sqrt{n})} = \lim\limits_{n \to \infty} \frac{1}{(1+2/\sqrt{n})} = \frac{1}{1+0} = 1$

g. $a_n = e^{-1/\sqrt{n}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{-1/\sqrt{n}} = \lim_{n\to\infty} \frac{1}{e^{\frac{1}{\sqrt{n}}}} = \frac{1}{e^0} = 1$$

h. $a_n = \frac{4^n}{1+9^n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4^n}{1+9^n} = \lim_{n\to\infty} \frac{4^n}{9^n\left(\frac{1}{9^n}+1\right)} = \lim_{n\to\infty} \frac{4^n}{9^n\ (0+1)} = \lim_{n\to\infty} \left(\frac{4}{9}\right)^n = 0$$

i. $a_n = \sqrt{\frac{1+4n^2}{1+n^2}}$

By LHospital's rule:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{1+4n^2}{1+n^2}} = \sqrt{\lim_{n\to\infty} \frac{1+4n^2}{1+n^2}} = \sqrt{\lim_{n\to\infty} \frac{8n}{2n}} = \sqrt{\lim_{n\to\infty} 4} = 2$$

By algebra:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{1+4n^2}{1+n^2}} = \sqrt{\lim_{n\to\infty} \frac{1+4n^2}{1+n^2}} = \sqrt{\lim_{n\to\infty} \frac{n^2\left(\frac{1}{n^2}+4\right)}{n^2\left(\frac{1}{n^2}+1\right)}} =$$

$$\sqrt{\lim_{n\to\infty} \frac{\left(\frac{1}{n^2}+4\right)}{\left(\frac{1}{n^2}+1\right)}} = \sqrt{\frac{(0+4)}{(0+1)}} = 2$$

For both solutions we used the idea that $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$

This works when f(x) is a continuous function and makes life a bit easier.

j. $a_n = \cos\left(\frac{n\,\pi}{n+1}\right)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\left(\frac{n\,\pi}{n+1}\right) = \cos\left(\lim_{n\to\infty} \frac{n\,\pi}{n+1}\right) = \cos\pi = -1$$

We used the idea that $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$
Again this works when f(x) is a continuous function.

k. $a_n = \frac{n^2}{\sqrt{n^3 + 4n}}$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^2}{\sqrt{n^3 + 4n}} = \lim_{n \to \infty} \frac{n^2}{n^{\frac{3}{2}} \sqrt{1 + \frac{4}{n^2}}} = \lim_{n \to \infty} \frac{n^{\frac{1}{2}}}{\sqrt{1 + \frac{4}{n^2}}} =$$

$$\lim_{n \to \infty} \frac{n^{1/2}}{\sqrt{1+0}} = \lim_{n \to \infty} n^{1/2} = \infty \quad \text{diverges}$$

l. $a_n = e^{(2n/(n+2))} = \exp\left(\frac{2n}{n+2}\right)$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \exp\left(\frac{2n}{n+2}\right) = \exp\left(\lim_{n \to \infty} \frac{2n}{n+2}\right) = \exp 2 = e^2$$

We used the idea that $\lim_{n \to \infty} f(a_n) = f\left(\lim_{n \to \infty} a_n\right)$

m. $a_n = \frac{(-1)^n}{2\sqrt{n}}$

$$\lim_{n \to \infty} |a_n| = \lim_{n \to \infty} \frac{1}{2\sqrt{n}} = 0$$

$$\lim_{n \to \infty} a_n = 0$$

We are using the idea that if $\lim_{n \to \infty} |a_n| = 0$ then $\lim_{n \to \infty} a_n = 0$

n. $a_n = \frac{(-1)^n n}{n + \sqrt{n}}$

$$\mathrm{Lim}_{n \to \infty} a_n = \lim_{n \to \infty} \frac{(-1)^n n}{n + \sqrt{n}} = \lim_{n \to \infty} (-1)^n \cdot \lim_{n \to \infty} \frac{n}{n + \sqrt{n}} =$$

$$\lim_{n \to \infty} (-1)^n \cdot \lim_{n \to \infty} \frac{1}{1 + \frac{1}{\sqrt{n}}} = \lim_{n \to \infty} (-1)^n \cdot \frac{1}{1+0} = \lim_{n \to \infty} (-1)^n$$

diverges by oscillations

o. $\left\{\frac{(2n-1)!}{(2n+1)!}\right\}$

This is another way of saying that $a_n = \frac{(2n-1)!}{(2n+1)!}$

We have to simplify the expression before we can solve the limit.

$$a_n = \frac{1{\cdot}2{\cdot}3\cdots(2n-2)(2n-1)}{1{\cdot}2{\cdot}3\cdots(2n-2)(2n-1)\ (2n)(2n+1)}$$

Almost everything cancels

$$a_n = \frac{1}{2n\,(2n+1)}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{(2n-1)!}{(2n+1)!} = \lim_{n\to\infty}\frac{1}{2n\,(2n+1)} = \frac{1}{\infty} = 0$$

p. $\left\{\frac{\ln n}{\ln 2n}\right\}$

This is another way of writing $a_n = \frac{\ln n}{\ln 2n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\ln n}{\ln 2n} = \lim_{n\to\infty}\frac{\ln n}{\ln 2 + \ln n} = \lim_{n\to\infty}\frac{1/n}{0 + 1/n} = 1$$

We used L'Hospital's rule

q. $a_n = \sin n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\sin n$$

The truth is that we have no tools right now to prove that this limit diverges. It does diverge. It simply fails to approach any number. Instead of having a true, single limit, this sequence has cluster points – that is, the sequence aggregates and coalesces around more than one number. Each cluster point gets visited by the sequence an infinite number of times – and the sequence gets closer and closer to its cluster points. Once the sequence gets near one of its cluster points, it eventually leaves and visits other cluster points. It then hovers around them for a while and then leaves. It is like the sequence has multiple limits. But there is a difference between a limit and a cluster point. Once a sequence visits its true limit, it stays there and gets closer and closer. Once it gets near its true limit it never leaves. Sequences like sin n do not do this. They get near many cluster points. They stay near the cluster points for a while. They leave their cluster points and go visit other cluster points. And they continue this visiting and leaving routine forever.

Limits and cluster points are similar to each other because a sequence gets near to both of them an infinite number of times. But limits and cluster points are not the same. To make the case for similarity between the two, if a sequence had only a single cluster point, then that cluster point turns out to be the limit.

All this is beyond the first year of calculus but is covered in higher level courses.

r. $a_n = \frac{\tan^{-1} n}{n}$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{\tan^{-1} n}{n} = \lim_{n \to \infty} \tan^{-1} n \cdot \lim_{n \to \infty} \frac{1}{n} = \frac{\pi}{2} \lim_{n \to \infty} \frac{1}{n} = 0$$

s. $a_n = n^2 \; e^{-n}$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^2}{e^n} = \lim_{n \to \infty} \frac{2n}{e^n} = \lim_{n \to \infty} \frac{n}{e^n} = 0$$

We used LHospital's rule.

t. $a_n = \ln(n+1) - \ln n$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} [\ln(n+1) - \ln n\,] = \lim_{n \to \infty} \left[\ln \frac{(n+1)}{n} \;\right] =$$

$$\ln\left(\lim_{n \to \infty} \frac{n+1}{n} \right) = \ln 1 = 0$$

u. $a_n = \frac{\cos^2 n}{n}$

To solve this we will use the squeeze theorem (pinch theorem). We know that the square of cosine is bounded between 0 and 1: $0 \le \cos^2 n \le 1$

With this we can write the given sequence as a bounded sequence.

$$0 \le \frac{\cos^2 n}{n} \le \frac{1}{n^2}$$

Since $\lim_{n \to \infty} \frac{1}{n^2} = 0$ we must have $\lim_{n \to \infty} \frac{\cos^2 n}{n^2} = 0$

To restate the pinch theorem for sequences; if $\lim_{n \to \infty} b_n \le \lim_{n \to \infty} a_n \le \lim_{n \to \infty} c_n$ and if $\lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n = L$, then $\lim_{n \to \infty} a_n = L.$

In this case $b_n = 0$ and $c_n = 1/n^2$.

v. $a_n = \sqrt[n]{2^{3n+1}}$

$$a_n = \sqrt[n]{2^{3n+1}} = (2^{3n+1})^{1/n} = 2^{(3 + \frac{1}{n})} = 2^3 \, 2^{1/n} = 8 \cdot 2^{1/n}$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} 8 \cdot 2^{1/n} = 8 \lim_{n \to \infty} 2^{1/n} = 8 \cdot 2^0 = 8$$

w. $a_n = n \sin\frac{1}{n}$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} n \sin\frac{1}{n} = \infty \cdot 0$$

We have an indeterminate form and we need L'Hospital's rule

$$\lim_{n \to \infty} n \sin\frac{1}{n} = \lim_{n \to \infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{-\frac{1}{n^2} \cos\frac{1}{n}}{-\frac{1}{n^2}} = \lim_{n \to \infty} \cos\frac{1}{n} = \cos 0 = 1$$

x. $a_n = 2^{-n} \cos n\pi$

It turns out that $\cos n\pi = (-1)^n$. Just graph a cosine curve and this become evident. Expressions like this happen quite a bit when studying things like Fourier series so you get used to them.

$$a_n = 2^{-n} (-1)^n = \frac{(-1)^n}{2^n}$$

Instead of evaluating the sequence directly, we will evaluate its absolute value.

Consider $\lim_{n \to\infty} |a_n| = \lim_{n \to\infty} \frac{1}{2^n} = 0$

If $\lim_{n \to\infty} |a_n| = 0$ then $\lim_{n \to \infty} a_n = 0$

So we have $\lim_{n \to \infty} 2^{-n} \cos n\pi = 0$

As a final note, we could have solved this using the squeeze theorem (pinch theorem) since $(-1)^n$ is bounded by -1 and +1.

y. $a_n = \left(1 + \frac{2}{n}\right)^n$

This sequence is a restatement of the limiting definition of e: $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

It also turns out that $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$

Looking at the given sequence, it will give us an indeterminate form: 1^∞

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left(1 + \frac{2}{n}\right)^n = \left(1 + \frac{2}{\infty}\right)^\infty = (1+0)^\infty = 1^\infty$$

For this type of indeterminate form, we use natural logarithms and we evaluate the limit of $\ln(a_n)$.

$$\ln a_n = n \ln\left(1 + \frac{2}{n}\right) = \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}}$$

$$\lim_{n \to \infty} \ln a_n = \lim_{n \to \infty} \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}} = \frac{0}{0}$$

$$\lim_{n \to \infty} \ln a_n = \lim_{n \to \infty} \frac{\left(1+\frac{2}{n}\right)^{-1}\left(-\frac{2}{n^2}\right)}{-\frac{1}{n^2}}$$

$$\lim_{n \to \infty} \ln a_n = \lim_{n \to \infty} 2\left(1 + \frac{2}{n}\right)^{-1} = 2(1+0)^{-1} = 2$$

If the natural log of a_n approaches 2, then a_n approaches e^2.

$$\lim_{n \to \infty} \left(1 + \frac{2}{n}\right)^n = e^2$$

z. $a_n = \sqrt[n]{n}$

The limit of a_n will give an indeterminate form, in this case ∞^0

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \sqrt[n]{n} = \lim_{n \to \infty} n^{1/n} = \infty^0$$

We use the natural logarithm to evaluate this kind of indeterminate form.

$$\ln a_n = \frac{\ln n}{n}$$

$$\lim_{n \to \infty} \ln a_n = \lim_{n \to \infty} \frac{\ln n}{n} = \lim_{n \to \infty} \frac{\frac{1}{n}}{1} = \lim_{n \to \infty} \frac{1}{n} = 0$$

If $\ln a_n$ approaches zero, then a_n approaches e^0 or a_n approaches 1

$$\lim_{n \to \infty} a_n = 1$$

7. EVALUATING LIMITS USING ALGEBRA AND CALCULUS: FIND $\lim_{n \to \infty} a_n$

a. $a_n = \ln(2n^2) - \ln(n^2 + 1)$

$$a_n = \ln\left(\frac{2n^2}{n^2 + 1}\right)$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \ln\left(\frac{2n^2}{n^2 + 1}\right) = \ln\left(\lim_{n \to \infty} \frac{2n^2}{n^2 + 1}\right) = \ln 2$$

We are using the ideal that $\lim_{n \to \infty} f(a_n) = f\left(\lim_{n \to \infty} a_n\right)$
This equation is true if f(x) is continuous. This theorem is extremely useful for evaluating limits of sequences.

b. $a_n = \frac{(\ln n)^2}{n}$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{(\ln n)^2}{n} = \lim_{n \to \infty} \frac{2\frac{(\ln n)}{n}}{1} = \lim_{n \to \infty} \frac{2\,(\ln n)}{n} = \lim_{n \to \infty} \frac{2}{n} = 0$$

Here we repeatedly used L'Hospital's rule. The indeterminate form is ∞/∞ .

c. $a_n = \tan^{-1}(\ln n)$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \tan^{-1}(\ln n) = \tan^{-1}\left(\lim_{n \to \infty} \ln n\right) = \tan^{-1}\infty = \frac{\pi}{2}$$

d. $a_n = n - \sqrt{n+1}\ \sqrt{n+3}$

This is an indeterminate form $\infty - \infty$

$$a_n = n - \sqrt{(n+1)(n+3)} = n - \sqrt{n^2 + 4n + 3}$$

$$a_n = n - n\sqrt{1 + \frac{4}{n} + \frac{3}{n^2}} = n - n\sqrt{1 + \left(\frac{4}{n} + \frac{3}{n^2}\right)}$$

The only way I can think to do this is the binomial theorem

$$a_n = n - n\sqrt{1 + \left(\frac{4}{n} + \frac{3}{n^2}\right)} = n - n\left[1 + \frac{1}{2}\left(\frac{4}{n} + \frac{3}{n^2}\right) - \frac{1}{8}\left(\frac{4}{n} + \frac{3}{n^2}\right)^2 + \cdots\right]$$

Distribute the n inside the parentheses:

$$a_n = \quad n - \left[n + \frac{n}{2}\left(\frac{4}{n} + \frac{3}{n^2}\right) - \frac{n}{8}\left(\frac{4}{n} + \frac{3}{n^2}\right)^2 + \cdots \right]$$

$$a_n = \quad - \left[\ \frac{n}{2}\left(\frac{4}{n} + \frac{3}{n^2}\right) - \frac{n}{8}\left(\frac{4}{n} + \frac{3}{n^2}\right)^2 + \cdots \right]$$

We take the limit of this form of a_n

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} - \left[\ \frac{n}{2}\left(\frac{4}{n} + \frac{3}{n^2}\right) - \frac{n}{8}\left(\frac{4}{n} + \frac{3}{n^2}\right)^2 + \cdots \right]$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} - \left[\ \left(2 + \frac{3}{2n}\right) - \frac{n}{8n^2}\left(\frac{4}{1} + \frac{3}{n}\right)^2 + \cdots \right]$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} - \left[\ \left(2 + \frac{3}{2n}\right) - \frac{1}{8n}\left(\frac{4}{1} + \frac{3}{n}\right)^2 + \cdots \right]$$

We now look carefully to notice something important: all the terms with n in them will go to zero. The only term that survives is the 2 in the first parentheses.

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} n - \sqrt{n+1}\ \sqrt{n+3} \ = \ -2$$

The binomial theorem states:

$$(1+x)^n \ = \ 1 + n\,x \ + \frac{n\,(n-1)}{2!}\,x^2 + \frac{n\,(n-1)\,(n-2)}{3!}\,x^3 + \frac{n\,(n-1)\,(n-2)\,(n-3)}{4!}\,x^4 + \ \cdots$$

This series will converge for all n so long as x is in the open interval from -1 to +1. For positive integer values of n, the series will converge on the closed interval from -1 to +1.

If n is negative, then it will converge at +1 but not at -1.

The value of x that we used, $\left(\frac{4}{n} + \frac{3}{n^2}\right)$ was safe to use.

e. $\{0,\ 1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ \cdots\}$

This sequence can be written as

$a_n = 1$ if n is given by the expression $n = \frac{m^2+3m-2}{2} + 1$ where m = 1,2,3…
$a_n = 0$ otherwise

This statement was not necessary to solve the problem, but it is interesting that it can be done.

One idea that is important is that the sequence has an infinite number of 0s and an infinite number of 1s. Even though the string of 0s is getting longer and longer, there will always be a 1 popping up somewhere down the line.

The main idea to answering this question is meaning of the limit $\lim_{n \to \infty} a_n = L$
When this limit exists, at some point, the values of a_n get close to L and always stay close to L. Once the values of a_n are close to L, they can never get further away. They only have one choice – to get even closer to L. This is an important behavior of limits. For our sequence, the a_n are almost always close to zero – in fact they equal zero – and they stay at zero for longer and longer periods of time. But at some point, the a_n spontaneously jumps away from zero and goes to 1. This behavior violates the existence of a limit. Once a sequence is close to its limit L, it is not allowed to leave. In some ways, this sequence can be viewed as an oscillation between zero and one – so the sequence can be thought of as diverging by oscillation. It does not make a difference how long the sequence stays at the zeros or at the ones. The sequence goes back and forth between the two values and this is not "limiting" behavior.

THE SEQUENC DIVERGES OR THE SEQUENCE FAILS TO CONVERGE.

Another way to think of this is that the sequence has two cluster points – it spends an infinite amount of time near the zero and also near the one. If there are two cluster points, there cannot be a limit. Cluster points are not part of 1st year calculus. They are normally taught in the second year in a course on Analysis.

f. $\left\{\frac{1}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}, \frac{1}{5}, \right\}$

This sequence is a little strange. It is actually two distinct sequence functions combined together.

For odd indices: $a_{2n+1} = \frac{1}{n}$

For even indices: $a_{2n} = \frac{1}{n+2}$

where n is a positive integer.

The limit of both forms is zero. So the sequence converges to zero.

g. $a_n = \frac{n!}{2^n}$

This can be expanded and written as follows:

$a_n = \frac{1 \cdot 2 \cdot 3 \cdots n}{2 \cdot 2 \cdot 2 \cdots 2 \cdot 2}$

This can be written as $a_n = \frac{1}{2}\frac{2}{2}\frac{3}{2} \cdots \frac{n}{2}$

The limit becomes $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{2}\frac{2}{2}\frac{3}{2} \cdots \frac{n}{2}$

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{2} \lim_{n \to \infty} \frac{2}{2} \lim_{n \to \infty} \frac{3}{2} \cdots \lim_{n \to \infty} \frac{n}{2}$

The limits of constants are constants.

$\lim_{n \to \infty} a_n = \frac{1}{2}\frac{2}{2}\frac{3}{2} \cdots \lim_{n \to \infty} \frac{n}{2}$

The last limit diverges so $\lim_{n \to \infty} a_n = \infty$

Another way to look at the question is to consider a_n for n>3. For this interval the product of the first (n-1) fractions is greater than 1.

$$\frac{1}{2}\frac{2}{2}\frac{3}{2} \cdots \frac{n-1}{2} > 1$$

Since $a_n = \frac{1}{2}\frac{2}{2}\frac{3}{2} \cdots \frac{n-1}{2}\frac{n}{2}$ we can state that $a_n > \frac{n}{2}$

So we have $\lim_{n \to \infty} a_n > \lim_{n \to \infty} \frac{n}{2}$

The limit on the right diverges to positive infinity so $\lim_{n \to \infty} a_n$ also diverges.

h. $a_n = \frac{n!}{n^n}$ (this is an example in the text book)

The sequence is positive: a_n is always a positive number so $a_n > 0$

Write the sequence out as

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n}$$

$$a_n = \frac{1}{n}\left(\frac{2 \cdot 3 \cdots n}{n \cdot n \cdots n}\right) = \frac{1}{n}\left(\frac{2}{n}\,\frac{3}{n}\,\frac{4}{n}\,\cdots\,\frac{(n-1)}{n}\,\frac{n}{n}\right)$$

The product of fractions inside the parentheses is less than one. So we can write

$$0 < a_n < \frac{1}{n}$$

If we take the limit as n goes to infinity, the pinch theorem tells us that $\lim\limits_{n \to \infty} a_n = 0$

8. Determine if the following series converge or diverge

a. $a_n = (-1)^n \frac{n}{n+1}$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} (-1)^n \frac{n}{n+1} = \lim_{n \to \infty} (-1)^n \lim_{n \to \infty} \frac{n}{n+1}$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} (-1)^n \cdot (1) = \lim_{n \to \infty} (-1)^n$$

$a_n \rightarrow (-1)^n$ is approaching an alternating sequence so it does not coverge

b. $a_n = \frac{\sin n}{n}$

Since sin n is bounded between -1 and +1 we can write:

$-\frac{1}{n} \leq a_n \leq \frac{1}{n}$ or $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$

Since the limits of 1/n and -1/n both go to zero, the pinch theorem can be applied and we conclude that $\lim\limits_{n \to \infty} \frac{\sin n}{n} = 0$

c. $a_n = \tan^{-1} \frac{n^2}{n^2+4}$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \tan^{-1} \frac{n^2}{n^2+4} = \tan^{-1}\left(\lim_{n \to \infty} \frac{n^2}{n^2+4}\right)$$

$$\lim_{n \to \infty} a_n = \tan^{-1}(1) = \frac{\pi}{4}$$

d. $a_n = \sqrt[n]{5^n + 3^n}$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \sqrt[n]{5^n + 3^n} =$$

$$\lim_{n \to \infty} 5 \sqrt[n]{1 + \frac{3^n}{5^n}} = \lim_{n \to \infty} 5 \sqrt[n]{1 + 0} = 5$$

e. $a_n = \frac{n^2 \cos n}{n^2 + 1}$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^2 \cos n}{n^2 + 1} = \lim_{n \to \infty} \frac{n^2}{n^2 + 1} \quad \lim_{n \to \infty} \cos n$$

$$\lim_{n \to \infty} a_n = (1) \quad \lim_{n \to \infty} \cos n$$

$$a_n \to \cos n$$

The sequence is approaching the sequence for cos n. Cosine does not converge to any value and exhibits oscillatory behavior. The sequence for cosine has cluster points.

So a_n diverges by oscillatory behavior.

f. $a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots (2n-1)}{n!}$

Sometimes the following notation is used for this expression: $a_n = \frac{(2n-1)!!}{n!}$

The numerator is called a double factorial – when it skips all evens (or when it skips all odds).

$a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots (2n-1)}{n!} = \frac{1 \cdot 3 \cdot 5 \cdot \cdots (2n-1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots n}$

We don't want the double factorial. We want a full factorial. So multiply above and below:

$a_n = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots (2n-1)(2n)}{n! \left(2 \cdot 4 \cdot 6 \cdot 8 \cdots (2n)\right)}$

This can be simplified further by factoring out a 2 from all the terms in the denominator:

$a_n = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots (2n-1)(2n)}{2^n \; n! \; (1 \cdot 2 \cdot 3 \cdot 4 \cdots n)}$

We cancel n! from the numerator and the denominator.

$a_n = \frac{(n+1)(n+2)(n+3) \cdots (2n-1)(2n)}{2^n \; n!}$

We now divide the expression up into separate fractions:

$$a_n = \frac{1}{2^n}\frac{(n+1)}{1}\frac{(n+2)}{2}\frac{(n+3)}{3}\frac{(n+4)}{4}\cdots\frac{(n+n)}{n}$$

$$a_n = \frac{(n+1)}{2\cdot 1}\frac{(n+2)}{2\cdot 2}\frac{(n+3)}{2\cdot 3}\frac{(n+4)}{2\cdot 4}\cdots\frac{(n+n)}{2n}$$

The last term equals 1

$$a_n = \frac{(n+1)}{2\cdot 1}\frac{(n+2)}{2\cdot 2}\frac{(n+3)}{2\cdot 3}\frac{(n+4)}{2\cdot 4}\cdots 1$$

If we take the limit now:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{(n+1)}{2\cdot 1}\frac{(n+2)}{2\cdot 2}\frac{(n+3)}{2\cdot 3}\frac{(n+4)}{2\cdot 4}\cdots 1$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{(n+1)}{2\cdot 1}\ \lim_{n\to\infty}\frac{(n+2)}{2\cdot 2}\ \lim_{n\to\infty}\frac{(n+3)}{2\cdot 3}\ \lim_{n\to\infty}\frac{(n+4)}{2\cdot 4}\cdots\ \lim_{n\to\infty} 1$$

The smallest limit on the right is $\lim_{n\to\infty} 1 = 1$. All other limits are more than this. So we can write

$$\lim_{n\to\infty} a_n > \lim_{n\to\infty}\frac{(n+1)}{2\cdot 1}$$

The limit on the right goes to infinity so $\lim_{n\to\infty} a_n = \infty$

ALTERNATE

Starting with $a_n = \frac{(n+1)}{2\cdot 1}\frac{(n+2)}{2\cdot 2}\frac{(n+3)}{2\cdot 3}\frac{(n+4)}{2\cdot 4}\cdots\frac{(n+n)}{2n}$

We can consider $b_n = 1/a_n$

We write it as $b_n = \frac{2\cdot 1}{n+1}\frac{2\cdot 2}{n+2}\frac{2\cdot 3}{n+3}\frac{2\cdot 4}{n+4}\cdots\frac{2\cdot n}{n+n}$

$$b_n = \frac{2\cdot 1}{n+1}\frac{2\cdot 2}{n+2}\frac{2\cdot 3}{n+3}\frac{2\cdot 4}{n+4}\cdots 1$$

$$b_n = \frac{2\cdot 1}{n+1}\left(\frac{2\cdot 2}{n+2}\frac{2\cdot 3}{n+3}\frac{2\cdot 4}{n+4}\cdots 1\right)$$

Of all the terms in parentheses, the largest is 1. All other fractions are smaller. So the product inside the parentheses is a fraction that is less than 1. So we can write:

$$b_n < \frac{2\cdot 1}{n+1}$$

Since b_n is positive we have $0 < b_n < \frac{2\cdot 1}{n+1}$

Take the limit as n goes to infinity. By the pinch theorem, $b_n \to 0$.

If b_n approaches zero, then a_n must approach infinity.

g. $a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)^n}$

We can write a_n as $a_n = \frac{1}{2n} \frac{3}{2n} \frac{5}{2n} \cdots \frac{(2n-1)}{2n}$

If n = 1 then a_n =1/2

If n = 2 then a_n = (1/2)(3/4)

If n = 3 then a_n = (1/2) (3/4) (5/6)

All fractions present are less than 1

So $a_n = \frac{1}{2n} \frac{3}{2n} \frac{5}{2n} \cdots \frac{(2n-1)}{2n} < \frac{1}{2n}$

Since a_n is positive we have $0 < a_n < \frac{1}{2n}$

Take the limit as n goes to infinity. By the pinch theorem, a_n goes to zero.

9. Find the limit of the sequence $\left\{\sqrt{2}\,,\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \cdots\right\}$

It is easier to see the solution if we write the sequence as

$\{2^{1/2},\ 2^{1/2}\, 2^{1/4}\ ,\ 2^{1/2}\, 2^{1/4}\, 2^{1/8},\ \cdots\}$

In this form we see that a_n is given by $a_n = 2$ ^ $\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}\right)$ where the symbol ^ means we are taking an exponent.

The exponent is a geometric series: $Exp = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$
The sum of this geometric series is

$$Exp = \frac{a(1-r^{n+1})}{1-r} = \frac{\frac{1}{2}\left(1 - \frac{1}{2^{n+1}}\right)}{1-\frac{1}{2}} = \frac{\frac{1}{2}\left(1 - \frac{1}{2^{n+1}}\right)}{\frac{1}{2}} = \left(1 - \frac{1}{2^{n+1}}\right)$$

$a_n = 2$ ^ $\left(1 - \frac{1}{2^{n+1}}\right)$

In the limit as n goes to infinity, the exponent goes to 1.

So $a_n \rightarrow 2$

10. Are the following sequences monotonic?

a. $a_n = \cos n$

One way to solve this question is to list some values of the sequence.

$a_1 = \cos 1 = 0.54 \quad a_2 = \cos 2 = -0.416 \quad a_3 = \cos 3 = -0.98999$

$a_4 = \cos 4 = -0.654 \quad a_5 = \cos 5 = 0.284$

From this list we see the sequence start to decrease and then start to increase. So the sequence is not monotonic.

b. $a_n = \frac{1}{2n+3}$

One way to solve this question is take the derivative:

$\frac{d\,(a_n)}{dn} = -\frac{1}{(2n+3)^2}$ the denominator is squared so it is always positive. The numerator is -1. So the derivative is always negative. This means that the sequence is monotonic decreasing.

Another way to solve this is to compare a_n and a_{n+1}.

$a_n = \frac{1}{2n+3} \qquad a_{n+1} = \frac{1}{2n+5}$

Since n is positive, 2n+5 is greater than 2n+3. The inequality flips when we take reciprocals. This means that $a_{n+1} < a_n$. So each term is less than the previous one. The sequence is monotonic decreasing.

c. $a_n = \frac{1-n}{2+n}$

One way to solve this is to take the derivative.

$\frac{da_n}{dn} = \frac{-(2+n)-(1-n)}{(2+n)^2} = \frac{-3-n}{(2+n)^2}$

Since n is positive, the numerator is always negative but the denominator is always positive. The derivative is always negative. The sequence is monotonic decreasing.

We could try a direct comparison between a_n and a_{n+1} .

$a_n = \frac{1-n}{2+n} \qquad a_{n+1} = \frac{-n}{3+n}$

The numerator of a_{n+1} has decreased, making it appear to be less.

The denominator of a_{n+1} has increased, also making it less.

Both effects combine to make a_{n+1} less than a_n. So the sequence is decreasing.

d. $a_n = (-1)^n\, n$

The factor $(-1)^n$ makes the sequence oscillate from positive to negative and back again. When it goes from positive to negative, that is a decrease. When it goes from negative to positive, that is an increase. So the sequence is neither strictly increasing or decreasing. The sequence is not monotonic.

e. $a_n = 2 + \frac{(-1)^n}{n}$

The expression $(-1)^n$ means that the sequence is oscillating. Oscillating sequences are neither increasing nor decreasing. The moment we see a sequence is oscillating, it is not monotonic.

f. $a_n = 3 - 2n\, e^{-n}$

If we take the derivative we get $\frac{da_n}{dn} = -2e^{-n} + 2\, n\, e^{-n} = 2\, e^{-n}(n-1)$

If n > 1 then the derivative is always positive, then the sequence is increasing.

The question remains what happens when n=1.

$a_1 = 3 - \frac{2}{e} = 2.264 \qquad a_2 = 3 - \frac{4}{e^2} = 2.459$

There is an increase from a_1 to a_2. So even for n=1, the sequence increases.

This is an increasing sequence. It is monotonic.

g. $a_n = n^3 - 3n + 3$

Take a derivative: $\frac{d\,(a_n)}{dn} = 3n^2 - 3$

For n>1 the derivative is positive indicating that the sequence is monotonic but we do not know what happens at n =1 (where the derivative is zero).

$a_1 = 1 \qquad a_2 = 5$ which is increasing

So the sequence is always increasing. The sequence is monotonic.