COMPARISON TEST FOR CONVERGNECE OF IMPROPER INTEGRALS

When we do improper integrals we take the integral and then take the limit:

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx = \lim_{b \to \infty} F(b) - F(a)$$

In order to determine the value of the integral, or whether it diverges (goes to infinity or oscillates), we have to be able to do the integral.

What if we had an integral that we could not integrate? We could not find the value of the integral. So an exact answer is out of the question. The only question we could answer is whether the integral converges or diverges. That is the purpose of this handout. We want to look at integrals that cannot be evaluated. We want to see if such integrals converge or diverge.

For example, consider the integral $\int_0^\infty e^{-x^2}\,dx$. At this stage of our knowledge, we do not have a method to find the exact value of this integral. But we can prove that the integral converges to a number (it turns out to be $\frac{\sqrt{\pi}}{2}$). How do we prove this integral converges? Again that is the purpose of this handout.

A BOUNDED FUNCTION

A function is said to be bounded above if the function is less than some number, for all values of x. For example f(x) = sin x is bounded above by 2. The sine of x is always less than 2.

A function is said to be bounded below if there is a number that is always less than the function. Again, f(x) = sin x is bounded below because -2 is always less than sin x.

A function that is both bounded above and below is just said to be 'bounded'.

DOMINATING FUNCTION

Let f(x) and g(x) be defined on some interval (finite or infinite). If $|f(x)| \leq \; g(x)$ for all x, then g(x) is said to dominate f(x). Note that f(x) is in absolute values and g(x) is not.

Now that we have introduced these definitions, we can introduce some theorem that are going to allow us to show that improper integrals converge. The use of these theorems might not be immediately apparent but we will see that they are quite transparent once we understand them.

THEOREM ON THE BOUNDS OF AN IMPROPER INTEGRAL

Let $f(x)$ be defined on the interval $x \; \epsilon \; [a, \infty)$. Consider the improper integral $\int_a^\infty f(x)\,dx$. First the equation $\int_a^\infty f(x)\,dx \; = \; \int_a^c f(x)\,dx \; + \; \int_c^\infty f(x)\,dx$ still holds (where c is some number greater than a). Second, the integrals $\int_a^\infty f(x)\,dx$ & $\int_c^\infty f(x)\,dx$ will either both converge or diverge.

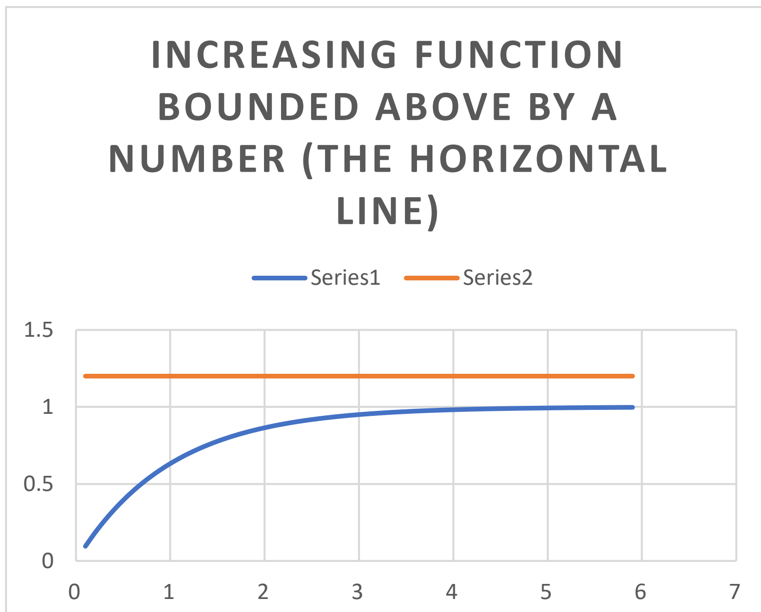
This theorem is telling us that the value of an integral $\int_a^c f(x)\,dx$ on some finite subinterval does not affect the <u>convergence</u> of the original integral. The starting point of the integral, whether a or c or any other number does not affect the <u>convergence</u> of the integral. The actual value of the integral might depend on $\int_a^c f(x)\,dx$ but the convergence of the integral does not.

So when testing the convergence of an improper integral, the value of the lower bound actually does not make a difference! This theorem is useful and will come up again when we study infinite series.

I am now going to state a theorem that is the backbone to proving whether an improper integral converges. The relevance of the theorem might not be apparent but it is the basic idea behind any proof for convergence of an improper integral.

THEOREM: Let $F(x)$ be a monotonic increasing function (a non-decreasing function) that is defined on an infinite interval $x \epsilon [a, \infty)$. Let U equal some real number. If F(x) is bounded above by U (if F(x) < U for all values of x) then the limit $\lim_{x \to \infty} F(x)$ exists and it will be less than or equal to U.

This theorem tells us the following: We have a function that is always increasing, always going up in value. It never decreases. Such a function has two choices – it can go to positive infinity, or it will have a horizontal asymptote; it must approach that horizontal asymptote but never reach it. These are the two choices for increasing functions. If we know that the function is bounded above, by <u>any</u> number, then it has that horizontal asymptote; it will always be less than that horizontal asymptote and the limit $\lim_{x \to \infty} F(x)$ exists. This limit will equal the value of that horizontal asymptote.



The diagram shows an increasing function bounded above by a number; the number is represented by the horizontal line. The value of this number is y = 1.2 Since the increasing function is bounded above by 1.2, the function must have some limit (less than or equal to 1.2). It turns out that the limit equals = 1; $\lim_{x \to \infty} f(x) = 1$

Finally we get to the test that tells us whether an improper integral converges.

COMPARISON TEST FOR CONVERGENCE OF IMPROPER INTEGRALS

Let f(x) and g(x) be non-negative functions defined on the interval $x \in [a, \infty)$.

The integrals $\int_a^b f(x)\, dx$ and $\int_a^b g(x)\, dx$ both exist on any interval $x \in [a, b]$ where a<b.

Let g(x) dominate f(x).

Then if $\int_a^\infty g(x)\, dx$ converges to a real number, then the integral $\int_a^\infty f(x)\, dx$ also converges.

On the other hand, if $\int_a^\infty f(x)\, dx$ diverges to positive infinity, then the integral $\int_a^\infty g(x)\, dx$ also diverges.

To put this in simple words, both functions are positive (or sometimes zero). If the integral of the bigger function converges, the integral of the smaller function converges. If the integral of the smaller function diverges, the integral of the bigger function also diverges.

We will use the comparison test directly as our tool to see if improper integrals converge or diverge.

1. $\int_1^\infty e^{-x^2}\, dx$ converge or diverge?

   Let $f(x) = e^{-x^2}$ and let $g(x) = e^{-x}$

   Let's write the functions as $f(x) = \frac{1}{e^{x^2}}$ and $g(x) = \frac{1}{e^x}$

   For the interval $x \in [1, \infty)$ we know that $x^2 \geq x$

   Since $x^2 \geq x$ we can state $e^{x^2} \geq e^x$

   Taking reciprocals we get $\frac{1}{e^{x^2}} \leq \frac{1}{e^x}$ or $e^{-x^2} \leq e^{-x}$

   We see that $g(x)$ dominates $f(x)$

   $\int_1^\infty e^{-x^2}\, dx \leq \int_1^\infty e^{-x}\, dx$

   The integral $\int_1^\infty g(x)\, dx = \int_1^\infty e^{-x}\, dx$ converges and we can show it equals 1/e.

   By the comparison test, if the integral of g converges, then the integral of f converges.

   $\int_1^\infty e^{-x^2}\, dx$ converges

2. $\int_1^\infty \frac{1}{\sqrt{x^3+x}}\ dx$ converge or diverge?

Let $f(x) = \frac{1}{\sqrt{x^3+x}}$ and let $g(x) = \frac{1}{x^{3/2}}$ on the interval $x \epsilon [1, \infty)$

First we have $x^3 + x > x^3$ for $x \epsilon [1, \infty)$

Second we have $\sqrt{x^3 + x} > \sqrt{x^3}$

Third, taking reciprocals we have $\frac{1}{\sqrt{x^3 + x}} < \frac{1}{\sqrt{x^3}}$

This means that g(x) dominates f(x) on the interval. If the integral for g converges on the interval, then the integral of f also converges.

$\int_1^\infty g(x)\, dx = \int_1^\infty \frac{1}{x^{3/2}}\ dx = 2$ the integral for g converges.

We conclude that $\int_1^\infty f(x)\, dx = \int_1^\infty \frac{1}{\sqrt{x^3+x}}\ dx$ converges

3. $\int_2^\infty \frac{1}{\sqrt{x^3 + 1}}\ dx$ converge or diverge?

Let $f(x) = \frac{1}{\sqrt{x^3 + 1}}$ and let $g(x) = \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}$ on the interval $x \epsilon [2, \infty)$

For $x \geq 2$ we have $x^3 + 1 > x^3$

It follows that $\sqrt{x^3 + 1} > x^3$

Taking reciprocals it follows that $\frac{1}{\sqrt{x^3 + 1}} < \frac{1}{\sqrt{x^3}}$

So $|f(x)| \leq g(x)$ for $x \geq 2$ and g dominates f on this interval.

The integral $\int_2^\infty g(x)\, dx = \int_2^\infty \frac{1}{x^{3/2}}\ dx = 2$ so it converges.

By the comparison theorem, the integral $\int_2^\infty f(x)\, dx = \int_2^\infty \frac{1}{\sqrt{x^3 + 1}}\ dx$ also converges.

4. $\int_1^\infty \frac{|\sin x|}{x^2}\ dx$ converge or diverge?

We start with three definitions. Let $f(x) = \frac{\sin x}{x^2}$ and $g(x) = \frac{1}{x^2}$.

Finally we let $h(x) = \sin x$

We know that sin x is a bounded function: $-1 \leq \sin x \leq 1$

Taking absolute values we have $|h(x)| = |\sin x| \leq 1$

Dividing by x^2 we have $\frac{|\sin x|}{x^2} \leq \frac{1}{x^2}$

This tells us that $|f(x)| \leq g(x)$ so f is dominated by g.

The integral of g is $\int_1^\infty g(x)\, dx = \int_1^\infty \frac{1}{x^2}\ dx = 1$ so it converges.

By the comparison test, the integral of f is $\int_1^\infty f(x)\, dx = \int_1^\infty \frac{|\sin x|}{x^2}\ dx$ and it converges.

5. $\int_0^\infty \frac{1}{2+e^x}\ dx$ converge or diverge?

For the interval $x \geq 2$ we have $e^x + 2 \geq e^x$

Taking reciprocals we have $\frac{1}{e^x+2} \leq \frac{1}{e^x}$

Let $f(x) = \frac{1}{e^x+2}$ and let $g(x) = \frac{1}{e^x} = e^{-x}$

So $|f(x)| \leq g(x)$ so g dominates f on the given interval.

The integral of g is given by $\int_0^\infty g(x)\ dx = \int_0^\infty e^{-x}\, dx = 1$

The integral of g converges. By the comparison test, the integral of f converges.

$\int_0^\infty \frac{1}{2+e^x}\ dx$ converges.

6. $\int_0^{\infty} e^{-x^4}\,dx$ converge or diverge?

We start with the statement that $\int_0^{\infty} e^{-x^4}\,dx \;=\; \int_0^{1} e^{-x^4}\,dx \;+\; \int_1^{\infty} e^{-x^4}\,dx$

The two integrals $\int_0^{\infty} e^{-x^4}\,dx$ and $\int_1^{\infty} e^{-x^4}\,dx$ will either both converge or both diverge. The integral $\int_0^{1} e^{-x^4}\,dx$ is finite in value and will not affect overall convergence.

For the interval $x \geq 1$ we have $x^4 \geq \; x$

We also have $e^{x^4} \geq \; e^x$

Taking reciprocals we have $\frac{1}{e^{x^4}} \leq \frac{1}{e^x}$ or $e^{-x^4} \leq \; e^{-x}$

So for the interval we have that e^{-x} dominates e^{-x^4}.

The integral $\int_1^{\infty} e^{-x}\,dx \;=\; \frac{1}{e}$ so it converges.

By the comparison test we have $\int_1^{\infty} e^{-x^4}\,dx$ must converge.

The original integral $\int_0^{\infty} e^{-x^4}\,dx$ must also converge.

7. $\int_1^{\infty} \frac{1}{\sqrt{x^6+x^4+1}}\,dx$ converge or diverge?

For the interval $x \geq 1$ we have $x^6 + x^4 + 1 \geq \; x^6$

Taking square roots we have $\sqrt{x^6 + x^4 + 1} \geq \; x^3$

Taking reciprocals we have $\frac{1}{\sqrt{x^6+x^4+1}} \leq \frac{1}{x^3}$

The integrand is dominated.

The integral $\int_1^{\infty} \frac{1}{x^3}\,dx \;= \frac{1}{2}$ so it converges.

By the comparison test the integral $\int_1^{\infty} \frac{1}{\sqrt{x^6+x^4+1}}\,dx$ also converges.

8. $\int_1^\infty \frac{1}{\sqrt{x^4+2x}}\ dx$ converge or diverge?

For the interval $x \geq 1$ we have $x^4 + 2x \geq\ x^4$

Taking square roots we have $\sqrt{x^4 + 2x} \geq\ x^2$

Taking reciprocals we have $\frac{1}{\sqrt{x^4+2x}} \ \leq\ \frac{1}{x^2}$

The integral $\int_1^\infty \frac{1}{x^2}\, dx \ =\ 1$ so it converges

By the comparison test $\int_1^\infty \frac{1}{\sqrt{x^4+2x}}\ dx$ also converges.

9. $\int_1^\infty x^2\, e^{-x^2}\, dx$ converge or diverge?

We showed previously that $e^{-x^2} \leq\ e^{-x}$ for the interval $x \geq\ 1$

Multiplying by x[2] we have $x^2\, e^{-x^2} \leq\ x^2\, e^{-x}$

We already know that $\int_0^\infty x^2\, e^{-x}\ dx\ = 2.$

But this is note the integral we want. We want to know about $\int_1^\infty x^2\, e^{-x}\ dx$. To get this integral we use the fact that $\int_0^\infty x^2\, e^{-x}\ dx\ =\ \int_0^1 x^2\, e^{-x}\ dx\ +\ \int_1^\infty x^2\, e^{-x}\ dx$. The integral on the left converges and this implies that $\int_1^\infty x^2\, e^{-x}\ dx$ converges.

We have the comparison integral we want. We showed that $\int_1^\infty x^2\, e^{-x}\ dx$ converges. Since $x^2\, e^{-x}$ dominates $x^2\, e^{-x^2}$ we can state $\int_1^\infty x^2\, e^{-x^2}\, dx\ \leq\ \int_1^\infty x^2\, e^{-x}\ dx$.

By the comparison test, this means that $\int_1^\infty x^2\, e^{-x^2}\, dx$ converges.

10. $\int_1^\infty \frac{x}{x^3+1}\ dx$ converge or diverge?

For the interval $x \geq 1$ we have $x^3 \leq\ x^3+1$

Taking reciprocals we have $\frac{1}{x^3}\ \geq \frac{1}{x^3\ +\ 1}$

Multiplying both sides be x we get: $\frac{1}{x^2} \geq \frac{x}{x^3+1}$

We have the following integral inequality: $\int_1^\infty \frac{1}{x^2}\,dx\ \geq\ \int_1^\infty \frac{x}{x^3+1}\ dx$

Look at the integral $\int_1^\infty \frac{1}{x^2}\ dx\ =\ 1$

By the comparison test we have $\int_1^\infty \frac{x}{x^3+1}\ dx$ converges

11. $\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\ dx$ converge or diverge

We have the inequality $\sin^2 x \geq\ 0$

Add 1 to both sides: $1+\sin^2 x \geq\ 1$

Divide both sides by $\sqrt{x}$: $\frac{1+\sin^2 x}{\sqrt{x}} \geq \frac{1}{\sqrt{x}}$

We have the integral inequality $\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\ dx \geq\ \int_1^\infty \frac{1}{\sqrt{x}}\ dx$

But the integral $\int_1^\infty \frac{1}{\sqrt{x}}\ dx$ diverges to positive infinity

By the comparison test the integral $\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\ dx$ diverges

12. $\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\ dx$ converge or diverge

For the interval $x \geq 1$ we have $x^4 \geq x^4 - x$

Taking square roots we have $x^2 \geq\ \sqrt{x^4-x}$

$\frac{1}{x^2}\ \leq\ \frac{1}{\sqrt{x^4-\ x}}$

For interval $x \geq\ $ 1 we know x+1 is a positive number. Multiply both sides by x+1

$\frac{x\ +\ 1}{x^2}\ \leq\ \frac{x+\ 1}{\sqrt{x^4-\ x}}$

We have the integral inequality $\int_1^\infty \frac{x\ +\ 1}{x^2}\ dx\ \leq\ \int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\ dx$

The integral on the left $\int_1^\infty \frac{x\ +\ 1}{x^2}\ dx\ =\ \int_1^\infty \frac{1}{x} + \frac{1}{x^2}\ dx$ diverges

By the comparison test the integral on the right also diverges.

$\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\ dx$ diverges

13. $\int_0^\infty \frac{\tan^{-1} x}{2+e^x}\ dx$

For the interval $x \geq\ \ 0$ we have the inequality $e^x \leq\ \ 2\ +\ \ e^x$

Taking reciprocals we have $\frac{1}{e^x}\ \geq\ \frac{1}{2\ +\ e^x}$

On this interval $\tan^{-1} x \geq\ \ 0$ so inverse tangent is positive

$\frac{\tan^{-1} x}{e^x}\ \geq\ \frac{\tan^{-1} x}{2\ +\ e^x}$

We also know that $\frac{\pi}{2}\ >\ \tan^{-1} x$ for all x

$\frac{\pi/2}{e^x} \geq\ \frac{\tan^{-1} x}{e^x}\ \geq\ \frac{\tan^{-1} x}{2\ +\ e^x}$

$\frac{\pi}{2}\int_0^\infty \frac{1}{e^x}\ dx\ =\ \frac{\pi}{2}$ so the integral converges.

By the comparison test we have $\int_0^\infty \frac{\tan^{-1} x}{2+e^x}\ dx$ converges.