L'HOSPITAL'S RULE:
FORM INFINITY – INFINITY

When you have the limit $\lim_{x \to a} \big(f(x) - g(x)\big) = \infty - \infty$, you can find the limit by algebraic manipulation. It is possible to get the form into 0/0 or ∞/∞. Once this is done, we can apply LHospital's rule. But we have to do the algebra to change the form.

1. $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

$$\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \infty - \infty$$

Get a common denominator:

$\lim_{x \to 0^+} \left(\frac{\sin x - x}{x \sin x} \right)$ this form is 0/0 so we can use LHospital's rule

$\lim_{x \to 0^+} \left(\frac{\sin x - x}{x \sin x} \right) = \lim_{x \to 0^+} \left(\frac{\cos x - 1}{\sin x + x \cos x} \right)$ this form is still 0/0 so we do it again

$$\lim_{x \to 0^+} \left(\frac{-\sin x}{\cos x + \cos x - x \sin x} \right) = \frac{0}{2} = 0$$

2. $\lim_{x \to 0} \frac{1}{1 - \cos x} - \frac{2}{\sin^2 x}$

$$\lim_{x \to 0} \frac{1}{1 - \cos x} - \frac{2}{\sin^2 x} = \infty - \infty$$

$$\lim_{x \to 0} \frac{1 + \cos x}{(1 - \cos x)(1 + \cos x)} - \frac{2}{\sin^2 x}$$

$$\lim_{x \to 0} \frac{1 + \cos x}{(1 - \cos x)(1 + \cos x)} - \frac{2}{1 - \cos^2 x}$$

$$\lim_{x \to 0} \frac{1+\cos x}{(1-\cos x)(1+\cos x)} - \frac{2}{(1-\cos x)(1+\cos x)}$$

$$\lim_{x \to 0} \frac{\cos x - 1}{(1-\cos x)(1+\cos x)}$$

$$\lim_{x \to 0} \frac{-1}{(1+\cos x)} = -\frac{1}{2}$$

Note that at no point did we use LHospital's rule. We were able to find the limit by algebra.

3. $\lim_{x \to 0} \frac{1}{x} - \frac{1}{e^x - 1}$

$\lim_{x \to 0} \frac{1}{x} - \frac{1}{e^x - 1} = \infty - \infty$ this will be true no matter how you approach zero

Get a common denominator: $\lim_{x \to 0} \frac{e^x - 1 - x}{x\,(e^x - 1)}$

This form yields 0/0 so we can use LHospital's rule

$\lim_{x \to 0} \frac{e^x - 1}{(e^x - 1) + xe^x}$ this form also yields 0/0 so we do LHospital's rule one more time

$$\lim_{x \to 0} \frac{e^x}{e^x + xe^x + e^x} = \frac{1}{2}$$

4. $\lim_{x \to 0} \cot x - \csc x$

$$\lim_{x \to 0} \cot x - \csc x = \infty - \infty$$

$$\lim_{x \to 0} \cot x - \csc x = \lim_{x \to 0} \frac{\cos x}{\sin x} - \frac{1}{\sin x} = \lim_{x \to 0} \frac{\cos x - 1}{\sin x}$$

This last form will yield 0/0 so we can use LHospital's rule

$$\lim_{x \to 0} -\frac{\sin x}{\cos x} = 0$$

$$\lim_{x \to 0} \cot x - \csc x = 0$$

5. $\lim_{x \to \infty} \ln x - \ln(x+1)$

$$\lim_{x \to \infty} \ln x - \ln(x+1) = \infty - \infty$$

$$\lim_{x \to \infty} \ln \tfrac{x}{x+1} = \ln 1 = 0$$

$$\lim_{x \to \infty} \ln x - \ln(x+1) = 0$$

Notice we did not need LHospital's rule. All we used was the limit of x/(x+1) is one.

6. $\lim_{x \to \infty} \ln x - \ln(x+1)$ repeat

$$\lim_{x \to \infty} \ln x - \ln\left(x\left(1 + \tfrac{1}{x}\right)\right) = \lim_{x \to \infty} \ln x - \ln x - \ln\left(1 + \tfrac{1}{x}\right)$$

$$\lim_{x \to \infty} - \ln\left(1 + \tfrac{1}{x}\right) = -\ln 1 = 0$$

An alternate algebraic method. Note no need for LHospital's rule. This happens sometimes.

7. $\lim_{x \to 0} \csc^2 x - \frac{1}{x^2}$

$$\lim_{x \to 0} \csc^2 x - \tfrac{1}{x^2} = \infty - \infty$$

$$\lim_{x \to 0} \csc^2 x - \tfrac{1}{x^2} = \lim_{x \to 0} \frac{1}{\sin^2 x} - \frac{1}{x^2}$$

$$\lim_{x \to 0} \frac{x^2}{x^2 \sin^2 x} - \frac{\sin^2 x}{x^2 \sin^2 x} = \lim_{x \to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x}$$ this last form yields 0/0 so we use

LHospital's rule

$$\lim_{x \to 0} \frac{2x - 2\sin x \cos x}{2x \sin^2 x + 2x^2 \sin x \cos x} = \lim_{x \to 0} \frac{2x - \sin 2x}{2x \sin^2 x + x^2 \sin 2x}$$

This last form still yields 0/0 so we use LHospital's rule again:

$$\lim_{x \to 0} \frac{2 - 2\cos 2x}{2\sin^2 x + 2x\sin 2x + 2x\sin 2x + 2x^2\cos 2x}$$

Simplify the expression a little bit to get:

$$\lim_{x \to 0} \frac{2 - 2\cos 2x}{2\sin^2 x + 4x\sin 2x + 2x^2\cos 2x}$$

The limit still yields 0/0 so we do LHospital's rule again

$$\lim_{x \to 0} \frac{4\sin 2x}{2\sin 2x + 4\sin 2x + 8x\cos 2x + 4x\cos 2x - 4x^2\sin 2x}$$

Simplify the expression so it now becomes

$$\lim_{x \to 0} \frac{4\sin 2x}{6\sin 2x + 12x\cos 2x - 4x^2\sin 2x}$$

This is still going to give 0/0 so we do LHospital's rule one more time:

$$\lim_{x \to 0} \frac{8\cos 2x}{12\cos 2x + 12\cos 2x - 24\sin 2x - 8x\sin 2x - 8x^2\cos 2x}$$

Simplify the above and it becomes:

$$\lim_{x \to 0} \frac{8\cos 2x}{24\cos 2x - 24\sin 2x - 8x\sin 2x - 8x^2\cos 2x}$$

Since cos 0 = 1 and sin 0 = 0, we finally get limit = 8/24 = 1/3

$$\lim_{x \to 0} \csc^2 x - \frac{1}{x^2} = \frac{1}{3}$$

8. $\lim_{x \to 0} \csc x - \frac{1}{x}$

$$\lim_{x \to 0} \csc x - \frac{1}{x} = \lim_{x \to 0} \frac{1}{\sin x} - \frac{1}{x} = \lim_{x \to 0} \frac{x}{x \sin x} - \frac{\sin x}{x \sin x}$$

$\lim_{x \to 0} \frac{x - \sin x}{x \sin x}$ this gives 0/0 so we use LHospital's rule

$\lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x}$ we still have the form 0/0 so we do LHospital's rule again

$$\lim_{x \to 0} \frac{\sin x}{\cos x + \cos x - x \sin x} = \lim_{x \to 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0$$

$$\lim_{x \to 0} \csc x - \frac{1}{x} = 0$$

9. $\lim_{x \to 1} \frac{1}{x-1} - \frac{x}{\ln x}$

$$\lim_{x \to 1} \frac{\ln x}{(x-1) \ln x} - \frac{x(x-1)}{(x-1) \ln x} = \lim_{x \to 1} \frac{\ln x - x(x-1)}{(x-1) \ln x}$$

This will yield 0/0 so we use LHospital's rule

$\lim_{x \to 1} \frac{x^{-1} - 2x + 1}{\ln x + (x-1)/x} = \lim_{x \to 1} \frac{1 - 2x^2 + x}{x \ln x + (x-1)}$ this last form still produces 0/0 so we do LHospital's rule again

$$\lim_{x \to 1} \frac{1 - 2x^2 + x}{x \ln x + (x-1)} = \lim_{x \to 1} \frac{-4x + 1}{\ln x + 1 + 1} = -\frac{3}{2}$$

$$\lim_{x \to 1} \frac{1}{x-1} - \frac{x}{\ln x} = -\frac{3}{2}$$