INFINITE SERIES CONVERGENCE TESTS

When we study infinite series, the first thing we study is whether the series converges or diverges. Finding the exact value of the series is a study for another subject. To be sure, there are formulas and methods to find the sum of some infinite series – there are formulas that exist – but we are not interested in them. The only thing we are concerned about is whether the series converges or diverges. In the beginning, we will only concern ourselves with infinite series with all terms positive – we are only adding positive numbers. At some point we will look at alternating series (series where the terms are consecutively positive and negative).

Here are a list of the tests

1. Integral test
2. P test
3. Comparison test
4. Limit comparison test
5. Cauchy's n^{th} root test
6. Ratio test
7. Alternating series test

INTEGRAL TEST

Let $a_n = f(n)$ be positive (non-negative). And let $\lim_{n \to \infty} a_n = 0$. Then the summation $\sum_{i=1}^{\infty} a_n$ converges if and only if the integral $\int_1^{\infty} f(x)\, dx$ converges. If the integral diverges, so does the series. Important note – the lower bound on the integral does not have to equal one. You can make it any positive number you wish. *Do not make the lower bound zero as this can cause problems at the origin – there may be a vertical asymptote at the origin and we wish to avoid this complication.*

The message of this test – if you want to see if $\sum_{n=1}^{\infty} f(n)$ converges, just evaluate $\int_1^{\infty} f(x)\, dx$. Whatever the integral does, the series does the same.

P TEST

If you are given the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ the series will converge if p>1. The series will diverge if $p \geq 1$.

All you have to do is look at the exponent. It is that easy!

COMPARISON TEST TO PROVE CONVERGENCE

You are given a series $\sum_{n=1}^{\infty} a_n$ where $a_n \geq 0$ and you want to know if it converges.

Find a series of your own choosing $\sum_{n=1}^{\infty} b_n$ where $b_n \geq 0$ that is known to converge.

If $b_n \geq a_n$ for all n, then the first series must converge.

The last statement can be relaxed a little bit. The inequality $b_n \geq a_n$ does not have to start immediately at n=1. It could begin later on in the series – maybe at n =10 or at n = 7. So long as there is some point where b_n dominates a_n , the test will work.

COMPARISON TEST TO PROVE DIVERGENCE

You are given a series $\sum_{n=1}^{\infty} a_n$ where $a_n \geq 0$ and you want to know if it diverges.

Find a series of your own choosing $\sum_{n=1}^{\infty} b_n$ where $b_n \geq 0$ that is known to diverge.

If $b_n \leq a_n$ for all n, then the first series must diverge.

Again , the last statement can be relaxed a little bit. The inequality $b_n \leq a_n$ does not have to start immediately at n=1.

LIMIT COMPARISON TEST

You are given a series $\sum_{n=1}^{\infty} a_n$ where $a_n \geq 0$ and you want to know if it converges or diverges.

Find a series of your own choosing $\sum_{n=1}^{\infty} b_n$ where $b_n \geq 0$ that is known to converge or known to diverge (either way).

Form the ration $\frac{a_n}{b_n}$.

Note that the comparison series is in the denominator.

Take the limit of the ratio: find the limit $\lim_{n \to \infty} \frac{a_n}{b_n}$

If $\lim_{n \to \infty} \frac{a_n}{b_n} = c$ where c is any positive real number, then $\sum_{n=1}^{\infty} a_n$ will have the same behavior as $\sum_{n=1}^{\infty} b_n$.

If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

LIMIT COMPARISON TEST EXCEPTIONAL CASE C = 0

You are given a series $\sum_{n=1}^{\infty} a_n$ where $a_n \geq 0$ and you want to know if it converges or diverges.

Find a series of your own choosing $\sum_{n=1}^{\infty} b_n$ where $b_n \geq 0$ that is known to **converge**.

If $\lim_{n \to \infty} \frac{a_n}{b_n} = c = 0$ then $\sum_{n=1}^{\infty} a_n$ also converges.

LIMIT COMPARISON TEST EXCEPTIONAL CASE C = INFINITY

You are given a series $\sum_{n=1}^{\infty} a_n$ where $a_n \geq 0$ and you want to know if it converges or diverges.

Find a series of your own choosing $\sum_{n=1}^{\infty} b_n$ where $b_n \geq 0$ that is known to **diverge**.

If $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ then $\sum_{n=1}^{\infty} a_n$ also diverges.

CAUCHY'S NTH ROOT TEST

You are given the series $\sum_{n=1}^{\infty} a_n$ where $a_n \geq 0$

If $\lim_{n \to \infty} \sqrt[n]{a_n} < 1$ the series converges

If $\lim_{n \to \infty} \sqrt[n]{a_n} > 1$ the series diverges

If $\lim_{n \to \infty} \sqrt[n]{a_n} = 1$ then we do not know what the series does – this test has failed - pick another test

THE RATIO TEST

You are given the series $\sum_{n=1}^{\infty} a_n$ where $a_n \geq 0$

Evaluate the limit $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right|$

If $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ the series converges

If $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ the series diverges

If $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ then we do not know what the series does – this test has failed – pick another test

ALTERNATING SERIES TEST

You are given the series $\sum_{n=1}^{\infty} (-1)^n a_n$ where $a_n > 0$

If $a_n \to 0$ and if $a_{n+1} < a_n$ then the series converges.

So there are two criteria for an alternating series to converge.

The first is that a_n tends to zero as n goes to infinity.

The second is that a_n is a strictly monotonic decreasing sequence.

If these two criteria are met, the series converges.

If they are not met, then the series diverges.

PROOF OF THE INTEGRAL TEST

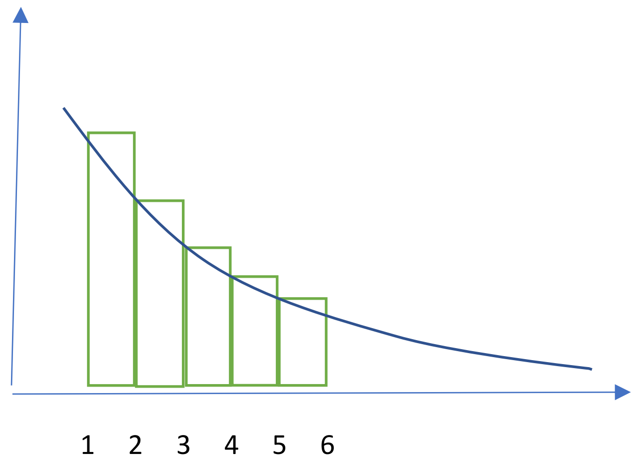
The summation $\sum_{n=1}^{\infty} a_n$ is to be

considered as a left handed sum.

This way the rectangles are

circumscribed

$\sum_{n=1}^{\infty} a_n > \int_1^{\infty} f(x)\, dx$



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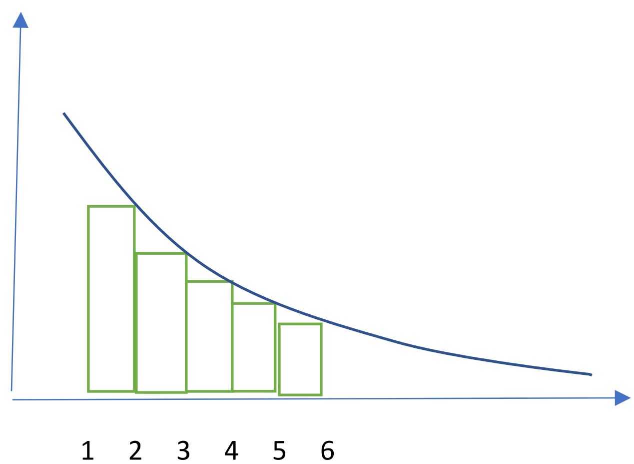

This gives us the following inequality: $\sum_{n=2}^{\infty} a_n < \int_1^{\infty} f(x)\, dx < \sum_{n=1}^{\infty} a_n$

Look at the partial sum $S_m = \sum_{n=2}^{m} a_n$. Since a_n is positive, S_m is a monotonic increasing sequence.

First, consider the case where integral $\int_1^{\infty} f(x)\, dx$ converges to a number L.

Then S_m is a monotonic sequence that is bounded above. We know from our study of sequences that such a sequence converges. So the infinite series $\sum_{n=2}^{\infty} a_n$ converges. If $\sum_{n=2}^{\infty} a_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

Second, consider the integral $\int_1^\infty f(x)\,dx$. This integral is a lower bound for $\sum_{n=1}^\infty a_n$. As a lower bound for the series, if the integral diverges, so must the series.

We have just shown that the integral converges when the series does. We have also shown that the integral diverges when the series diverges. So we have proven the test. The convergence of the series matches the convergence of the integral.

PROOF OF THE P TEST

Consider the infinite series $\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty \frac{1}{n^p}$ where p is a positive number.

Use the integral test: the series $\sum_{n=1}^\infty \frac{1}{n^p}$ converges or diverges as $\int_1^\infty \frac{1}{x^p}\,dx$

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx = \lim_{b\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^b = \lim_{b\to\infty}\frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1}$$

If p > 1 then the exponent on b is negative and we have

$\lim_{b\to\infty} \frac{1}{(-p+1)\ b^{p\,-\,1}} - \frac{1}{-p+1} = 0 - \frac{1}{-p+1} = \frac{1}{-p+1}$ so the integral converges

If p < 1 then the exponent on b is positive and we have:

$\lim_{b\to\infty}\frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1} = \infty - \frac{1}{-p+1} = \infty$ and the integral diverges

If p = 1 then the original integral gives un ln(x) and this diverges as x goes to infinity.

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^1}\,dx = \lim_{b\to\infty} \ln x\ \Big|_1^b = \lim_{b\to\infty} \ln b = \infty$$

COMPARISON TEST PROOF

Let $\sum_{n=1}^\infty a_n$ be a series of positive terms whose convergence is to be determined. Let $\sum_{n=1}^\infty b_n$ be a known series whose behavior is known – to either converge of diverge. And $a_n < b_n$ for all n.

We use the theorem on bounded monotonic sequences. A sequence that is monotonic increasing and that is bounded above, must converge to a limiting value. The sequence that is of interest to us is the partial sums of the series $\sum_{n=1}^\infty a_n$.

We consider the case where the test series $\sum_{n=1}^{\infty} b_n$ is known to converge. We will use this to prove that $\sum_{n=1}^{\infty} a_n$ converges.

Let $S_k = \sum_{n=1}^{k} a_n$

The sequence of partial sums S_k is a monotonic increasing sequence: $S_{k+1} > S_k$

This is seen by realizing that $S_{k+1} = S_k + a_{k+1}$. But we were told that the term a_{k+1} is positive so

$S_{k+1} - S_k = a_{k+1} > 0$ or that $S_{k+1} > S_k$. So the sequence S_k is seen to be increasing.

The next thing we have is that $S_k = \sum_{n=1}^{k} a_n < \sum_{n=1}^{\infty} a_n$

But we already know that $\sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} b_n$

$S_k < \sum_{n=1}^{\infty} b_n$

Since $\sum_{n=1}^{\infty} b_n$ is known to converge, then S_k is bounded above by a real positive number. A bounded monotonic sequence must converge so $\lim_{k \to \infty} S_k = L_a$ or $\sum_{n=1}^{\infty} a_n = L_a$

So the convergence part of the theorem has been proved.

To prove the divergence part of the theorem let the comparison series $\sum_{n=1}^{\infty} b_n$ be known to diverge. Let this comparison series be chosen so that $b_n < a_n$ for all n.

Comparing partial sums we have $\sum_{n=1}^{k} b_n < \sum_{n=1}^{k} a_n$

The limit $\lim_{k \to \infty} \sum_{n=1}^{k} b_n < \lim_{k \to \infty} \sum_{n=1}^{k} a_n$

The limit on the left diverges so the limit of the right diverges.

LIMIT COMPARISON TEST

We are told to examine the series $\sum_{n=1}^{\infty} a_n$ where $a_n > 0$. We are to compare it to $\sum_{n=1}^{\infty} b_n$ where the convergence of this series is known (either to converge or diverge).

We observe that $\lim_{n \to \infty} \frac{a_n}{b_n} = c$ where $c > 0$

For large n this means that $a_n \cong c\, b_n$

Using the form $\frac{a_n}{b_n} \cong c$ we can write this in the following useful way:

$m < \frac{a_n}{b_n} < M$ where $m < c < M$

This becomes $m\, b_n < a_n < M\, b_n$

Take a summation of this:

$m \sum_{n=1}^{k} b_n < \sum_{n=1}^{k} a_n < M \sum_{n=1}^{k} b_n$

We now invoke the original comparison test (proven above). Using the statement

$\sum_{n=1}^{k} a_n < M \sum_{n=1}^{k} b_n$

we see that if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges. .

Using the statement $m \sum_{n=1}^{k} b_n < \sum_{n=1}^{k} a_n$ we see that if

we see that if $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

So we prove the limit comparison test by invoking the original comparison test.

CAUCHY'S NTH ROOT TEST

The nth root test comes about by comparing a series to a geometric series.

We start with the statement that $\lim_{n \to \infty} \sqrt[n]{a_n} = k < 1$

This means that $\sqrt[n]{a_n} \cong k$ where k < 1 and true for all n > N where N is large.

This means that $\sqrt[n]{a_n}$ will be very close to k, hovering around k, within a vanishingly small interval.

We pick a number r outside this small interval, such that $k < r < 1$ with the property that $\sqrt[n]{a_n} < r$

Then we have $a_n < r^n$

We immediately have $\sum_{n=N+1}^{\infty} a_n < \sum_{n=N+1}^{\infty} r^n$

By the comparison test $\sum_{n=N+1}^{\infty} a_n$ converges. Since the starting point of the series does not affect convergence, we have convergence of the series $\sum_{n=1}^{\infty} a_n$.

Again, we have the statement $\lim_{n \to \infty} \sqrt[n]{a_n} = k > 1$ and we pick a number r such that k > r > 1

so that r is some number between 1 and k. We pick r so that it is below the interval containing k and a_n.

Then we can write $\lim_{n \to \infty} \sqrt[n]{a_n} > r$

We next have $a_n > r^n$

Taking summations of both sides we have $\sum_{n=N+1}^{\infty} a_n > \sum_{n=N+1}^{\infty} r^n$

Since r is greater than 1, the geometric series on the right diverges. So $\sum_{n=N+1}^{\infty} a_n$ must also diverge.

Since the starting point of the series does not affect convergence, we can state that $\sum_{n=1}^{\infty} a_n$

diverges.

Finally, if we have $\lim_{n \to \infty} \sqrt[n]{a_n} = 1$ then we cannot pick an r that is either greater nor less than 1. So we cannot make a comparison to a geometric series. In this case, the test fails. Some proofs go into much greater detail on this last case. They show that when the limit $\lim_{n \to \infty} \sqrt[n]{a_n} = 1$, some series will converge, while others diverge. We will not go into detail here. If the root test produces one in the limit, then you must pick another test to see what happens. The series may converge or diverge but the root test fails to give us this information.

D'ALEMBERT'S RATIO TEST

The ratio test is similar to the root test in one regard, both are derived by comparison to a geometric series.

We consider the series $\sum_{n=1}^{\infty} a_n$ where a_n is positive.

The ratio test states that if $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$ where ρ is a non-negative constant:

The series converges if $\rho < 1$

The series diverges if $\rho > 1$

The test provides no information if $\rho = 1$

We need to prove this theorem. So we let N be a positive integer, no matter how large. Let n be the index of the series. If n > N, then $\left|\frac{a_{n+1}}{a_n}\right|$ is extremely close to ρ. This means that when we go really far out into the series, like past 1,000,000, then the ratio of consecutive terms is close to this constant. The value of $\left|\frac{a_{n+1}}{a_n}\right|$ may be a hair above rho or a hair below, but it is extremely close to rho. We can write this mathematically as $\rho - \epsilon < \left|\frac{a_{n+1}}{a_n}\right| < \rho + \epsilon$. Here, epsilon is an extremely small number, vanishingly small.

If $\rho < 1$ then pick a number r where $\rho < \rho + \epsilon < r < 1$.

Then $\left|\frac{a_{n+1}}{a_n}\right| < r$ or if you wish $|a_{n+1}| < |a_n|\, r$

This statement is going to allow us to make the comparison to a geometric series. Since n > N we can write the following statements:

$|a_{N+2}| < |a_{N+1}|\, r$

$|a_{N+3}| < |a_{N+2}|\, r < |a_{N+1}|\, r^2$

$|a_{N+4}| < |a_{N+3}|\, r < |a_{N+2}|\, r^2 < |a_{N+1}|\, r^3$

$\vdots$

Condensing this list, it becomes

$|a_{N+2}| < |a_{N+1}|\, r$

$|a_{N+3}| < |a_{N+1}|\, r^2$

$|a_{N+4}| < |a_{N+1}|\, r^3$

$\vdots$

From this list we can build a series and make a comparison:

$\sum_{n=N+2}^{\infty} a_n = (|a_{N+2}| + |a_{N+3}| + |a_{N+4}| + |a_{N+5}| + \cdots)$

$\sum_{n=N+2}^{\infty} a_n < \{ |a_{N+1}| r + |a_{N+1}| r^2 + |a_{N+1}| r^3 + \cdots \}$

$\sum_{n=N+2}^{\infty} a_n < \sum_{n=1}^{\infty} |a_{N+1}| r^n$

The series on the right is a geometric series and it converges since r < 1.

By the direct comparison test, the series on the left converges.

Since the starting point of a series does not affect its convergence, we can state

$\sum_{n=1}^{\infty} a_n$ converges.

To prove divergence, go back to the statement $\rho - \epsilon < \left|\frac{a_{n+1}}{a_n}\right| < \rho + \epsilon$.

If $\rho > 1$ then then pick a number r such that $1 < r < (\rho - \epsilon) < \left|\frac{a_{n+1}}{a_n}\right|$

Then $\left|\frac{a_{n+1}}{a_n}\right| > r$ or if you wish $|a_{n+1}| > |a_n| r$

We can now repeat all the previous expressions but with the inequalities reversed:

$|a_{N+2}| > |a_{N+1}| r$

$|a_{N+3}| > |a_{N+2}| r > |a_{N+1}| r^2$

$|a_{N+4}| > |a_{N+3}| r > |a_{N+2}| r^2 > |a_{N+1}| r^3$

$\vdots$

Condensing this list, it becomes

$|a_{N+2}| > |a_{N+1}| r$

$|a_{N+3}| > |a_{N+1}| r^2$

$|a_{N+4}| > |a_{N+1}| r^3$

$\vdots$

WE can now form the series

$\sum_{n=N+2}^{\infty} a_n = (|a_{N+2}| + |a_{N+3}| + |a_{N+4}| + |a_{N+5}| + \cdots)$

$\sum_{n=N+2}^{\infty} a_n > \{ |a_{N+1}| r + |a_{N+1}| r^2 + |a_{N+1}| r^3 + \cdots \}$

$\sum_{n=N+2}^{\infty} a_n > \sum_{n=1}^{\infty} |a_{N+1}| r^n$

The series on the right is a geometric series and it diverges because r > 1.

The series on the left is greater than the series on the right.

By the comparison test, the series on the left diverges.

Since the starting point of a series does not affect its convergence, we can write $\sum_{n=1}^{\infty} a_n$ diverges.

For the last case, where $\rho = 1$ we cannot choose an r that is greater nor less than one. So we cannot create a geometric series for comparison. In this case the test fails and we must use another test to determine convergence of the series. The text books go into greater detail on this last case, using a p series to demonstrate the ambiguity of this situation. We will not go into that detail here.

THE ALTERNATING SERIES TEST

We are looking at the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ where $a_n > 0$

The series converges if:

$a_n \to 0$

$a_n > a_{n+1}$

This inequality means that the a_n form a monotonic decreasing sequence

$a_1 > a_2 > a_3 > a_4 \cdots$

We also get a useful fact that $a_n - a_{n+1} > 0$

To prove the convergence test for this series we look at the partial sums of the series

$S_k = \sum_{n=1}^{k} (-1)^n a_n$

$S_k = a_1 - a_2 + a_3 - a_4 + a_5 \cdots + (-1)^{k+1} a_k$

Due to the alternating nature of the series, we have the following behavior of the partial sums

$S_1 > S_2 \qquad S_2 < S_3 \qquad S_3 > S_4 \qquad S_4 < S_5 \qquad S_5 > S_6 \quad \ldots$

The partial sums jump back and forth because we are adding and then subtracting. The partial sums do not form a monotonic sequence. This is a problem. We like having monotonic sequences. They are useful in proving convergence. Instead of looking at S_k we will look at S_{2k} and S_{2k+1}. We have to do these separately.

We have

$S_1 = a_1$

$S_3 = a_1 - a_2 + a_3 = a_1 - (a_2 - a_3) < S_1$

$S_5 = S_3 - a_4 + a_5 = S_3 - (a_4 - a_5) < S_3$

$S_7 = S_5 - a_6 + a_7 = S_5 - (a_6 - a_7) < S_5$

We see that the odd partial sums form a monotonic decreasing sequence.

$S_1 > S_3 > S_5 > S_7 \cdots$

If we could find a lower bound for this subsequence, this would suffice to prove that the subsequence converges. We can show that the lower bound is zero by rewriting the partial sums. Note that all terms in parentheses are positive.

$S_1 = a_1$ $\qquad$ $S_1 > 0$

$S_3 = (a_1 - a_2) + a_3$ $\qquad$ $S_3 > 0$

$S_5 = (a_1 - a_2) + (a_3 - a_4) + a_5$ $\qquad$ $S_5 > 0$

$S_7 = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + a_7$ $\qquad$ $S_7 > 0$

So the odd indexed partial sums are a monotonic decreasing sequence that are bounded below. This sequence converges.

Look at the even partial sums:

$S_2 = a_1 - a_2$

$S_4 = S_2 + (a_3 - a_4) > S_2$

$S_6 = S_4 + (a_5 - a_6) > S_4$

$S_8 = S_6 + (a_7 - a_8) > S_6$

We see that the even partial sums form a monotonic increasing sequence

$S_2 < S_4 < S_6 < S_8 \cdots$

We can show that this sequence is bounded above by S_1 by rewriting it. All terms in parentheses are positive:

$S_2 = a_1 - a_2 < S_1$

$S_4 = a_1 - (a_2 - a_3) - a_4 < S_1$

$S_6 = a_1 - (a_2 - a_3) - (a_4 - a_5) - a_6 < S_1$

$\vdots$

So the even partial sums are a monotonic increasing and they are bounded above. This means the sequence converges.

The last thing we have to do is show that both subsequences converge to the same number.

We use the following equation: $S_{n+1} = S_n + a_{n+1}$

Take the limit of both sides: $\lim_{n \to \infty} S_{n+1} = \lim_{n \to \infty} S_n + \lim_{n \to \infty} a_{n+1}$

But $\lim_{n \to \infty} a_{n+1} = 0$

$\lim_{n \to \infty} S_{n+1} = \lim_{n \to \infty} S_n = L$