Sp '10 midterm soln

1) $N_1 = \frac{1}{4}(1-\xi)(1-\eta)$ $\quad N_2 = \frac{1}{4}(1-\xi)(1-\eta)$

$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$ $\quad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -(1-\eta)-(1+\eta) & (1-\eta)+(1+\eta) \\ -(1-\xi)-(1+\xi) & -(1+\xi)-(1-\xi) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$\det J = \frac{1}{2}$

For only u_3 & v_3 non-zero, we don't need all terms, so

$u = N_3 v_3$

$v = N_3 v_3$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{4}(1+\eta) \\ \frac{1}{4}(1+\xi) \end{bmatrix} d_1$$

$$= \frac{1}{4} \begin{bmatrix} -2-\eta-\xi \\ \eta-\xi \end{bmatrix} d_1 = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} d_1$$

$$\begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{4}(1+\eta) \\ \frac{1}{4}(1+\xi) \end{bmatrix} d_2$$

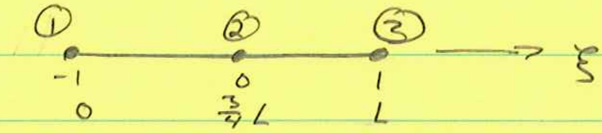
$$= \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} d_2$$

$\epsilon_x = -\frac{1}{2} d_1$

$\epsilon_y = 0$

$\gamma_{xy} = -\frac{1}{2} d_2$

2)



$$N_1 = \frac{-1}{2}\xi(1-\xi) \qquad N_2 = 1-\xi^2 \qquad N_2 = \frac{1}{2}\xi(1+\xi)$$

$$J = \frac{\partial x}{\partial \xi} = \frac{\partial}{\partial \xi} \sum N_i x_i$$

$$= \left(-\frac{1}{2} + \xi\right) x_1 - 2\xi x_2 + \left(\frac{1}{2} + \xi\right) x_3$$

$$= -2\xi\left(\frac{3}{4}L\right) + \left(\frac{1}{2} + \xi\right) L$$

$$= \frac{L}{2} - \frac{L}{2}\xi \qquad \left(\text{Note, If } \frac{3}{4} \rightarrow \frac{1}{2},\ J = \frac{L}{2}\right)$$

$$B = \left[\frac{\partial N_1}{\partial x} \quad \frac{\partial N_2}{\partial x} \quad \frac{\partial N_3}{\partial x}\right]$$

$$= \left[\frac{\partial N_1}{\partial \xi}\frac{\partial \xi}{\partial x} \quad \frac{\partial N_2}{\partial \xi}\frac{\partial \xi}{\partial x} \quad \frac{\partial N_3}{\partial \xi}\underbrace{\frac{\partial \xi}{\partial x}}_{\frac{1}{J}}\right]$$

$$= \left(\frac{L}{2}(1-\xi)\right)^{-1}\left[\left(\xi - \frac{1}{2}\right) \quad -2\xi \quad \left(\xi + \frac{1}{2}\right)\right]$$

$$K = \int_{-1}^{1} B^T E A\, B\, \det(J)\, d\xi$$

$$= EA\int_{-1}^{1} \frac{\left[\left(\xi - \frac{1}{2}\right) \quad -2\xi \quad \left(\xi+\frac{1}{2}\right)\right]^T \left[\left(\xi - \frac{1}{2}\right) \quad -2\xi \quad \left(\xi + \frac{1}{2}\right)\right]}{\left(\frac{L}{2}(1-\xi)\right)}\, d\xi$$

$$K_{11} = EA \int_{-1}^{1} \frac{\left(\xi - \frac{1}{2}\right)^2}{\left[\frac{L}{2}(1-\xi)\right]} d\xi$$

$$K_{12} = EA \int_{-1}^{1} \frac{4\xi^2}{\left(\frac{L}{2}(1-\xi)\right)} d\xi$$

$$K_{13} = EA \int_{-1}^{1} \frac{\left(\xi + \frac{1}{2}\right)^2}{\left[\frac{L}{2}(1-\xi)\right]} d\xi$$

None of which can be simplified further.
Any numerical solution is an approximation

3) $\underline{F} = \int_0^L F(x) N^T \, dx \quad , \quad J = \frac{dx}{d\xi} = \frac{L}{2}$

Polynomial is 4^{th} order, need 3 points

I can either 1) obtain shape functions in natural coordinates

2) just evaluate them at the correct locations

$\xi = 0$ is at $x = \frac{L}{2}$

$\xi = -\sqrt{0.6}$ is at $x = 0.11\,L$

$\xi = \sqrt{0.6}$ is at $x = \frac{1}{2} + \frac{1}{2}\sqrt{0.6} = 0.89\,L$

For $N_1 f$

loc	N_1	f	W	$N_1 f\, W$
$0.11L$	0.97	$0.11a$	$\frac{5}{9}$	$0.06a$
$\frac{1}{2}L$	0.5	$\frac{1}{2}a$	$\frac{8}{9}$	$0.22a$
$0.89L$	0.03	$0.89a$	$\frac{5}{9}$	$0.02a$
			sum	$0.30a$

$F_1 = 0.30a\, J = 0.15\,a\,L$

For N_2

loc	$N_2 f$	$N_2 f w$
0.11 L	0.01 aL	0.01 aL
0.5 L	0.06 aL	0.0 aL
0.89 L	0.01 aL	0.01 aL (0.0053)
	sum	0.06 aL

$\boxed{F_2 = 0.03 aL^2}$ (actual $0.033 aL^2$)

For N_3

loc	$N_3 f$	$N_3 f w$
0.11 L	0.04 a	0.002 a
0.50 L	0.25 a	0.22 a
0.89 L	0.86 a	0.48 a

$F_3 = \frac{0.70 aL}{2}$

$\boxed{F_3 = 0.35 aL}$

For N_4

	$N_4 f$	$N_4 f w$
0.11 L	-0.01 aL	-0.000 aL
0.50 L	-0.00 aL	-0.055 aL
0.89 L	-0.06 aL	-0.04 aL

$\boxed{F_4 \; -0.05 aL}$