ME 712 FEA Spring 2012 Exam 2 Soln

1) $N = a + b\xi + c\xi^2$

$$N_1 = \begin{cases} 1 & \xi = -1 \\ 0 & \xi = 0 \\ 0 & \xi = 1 \end{cases} \quad N_2 = \begin{cases} 0 \\ 1 \\ 0 \end{cases} \quad N_3 = \begin{cases} 0 \\ 0 \\ 1 \end{cases}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Columns of inverse of this are coefficients for N_1, N_2, N_3

$$N_1 = \frac{1}{2}(\xi^2 - \xi)$$
$$N_2 = 1 - \xi^2$$
$$N_3 = \frac{1}{2}(\xi^2 + \xi)$$

2) $x = N_1 x_1 + N_2 x_2 + N_3 x_3$ $\quad x_1 = 0, x_2 = \frac{3}{4}\ell$, $x_3 = \ell$

$= \ell\left(N_2 \frac{3}{4} + N_3\right)$

$= \ell\left(\frac{3}{4} - \frac{3}{4}\xi^2 + \frac{1}{2}\xi + \frac{1}{2}\xi^2\right)$

$= \ell\left(\frac{3}{4} + \frac{1}{2}\xi - \frac{1}{4}\xi^2\right)$

$\frac{dx}{d\xi} = \ell\left(\frac{1}{2} - \frac{1}{2}\xi\right)$

$J = \ell\left(\frac{1}{2} - \frac{1}{2}\xi\right)$

$$K = \int_0^\ell \left(\frac{dN}{dx}\right)^T E A \frac{dN}{dx}\, dx$$

$$= \sum_{i=1}^{3} \left(\frac{dN(\xi_i)}{d\xi}\frac{1}{J}\right)^T E A \left(\frac{dN(\xi_i)}{d\xi}\frac{1}{J}\right) J\, W_i$$

i	W	ξ
1	$\frac{5}{9}$	$-\sqrt{\frac{3}{5}}$
2	$\frac{8}{9}$	0
3	$\frac{5}{9}$	$\sqrt{\frac{3}{5}}$

See code

$$K = \frac{EA}{\ell}\begin{bmatrix} 1.83\bar{3} & -3.33\bar{3} & 1.50 \\ & 13.3\bar{3} & -10.00 \\ \text{sym} & & 8.50 \end{bmatrix}$$

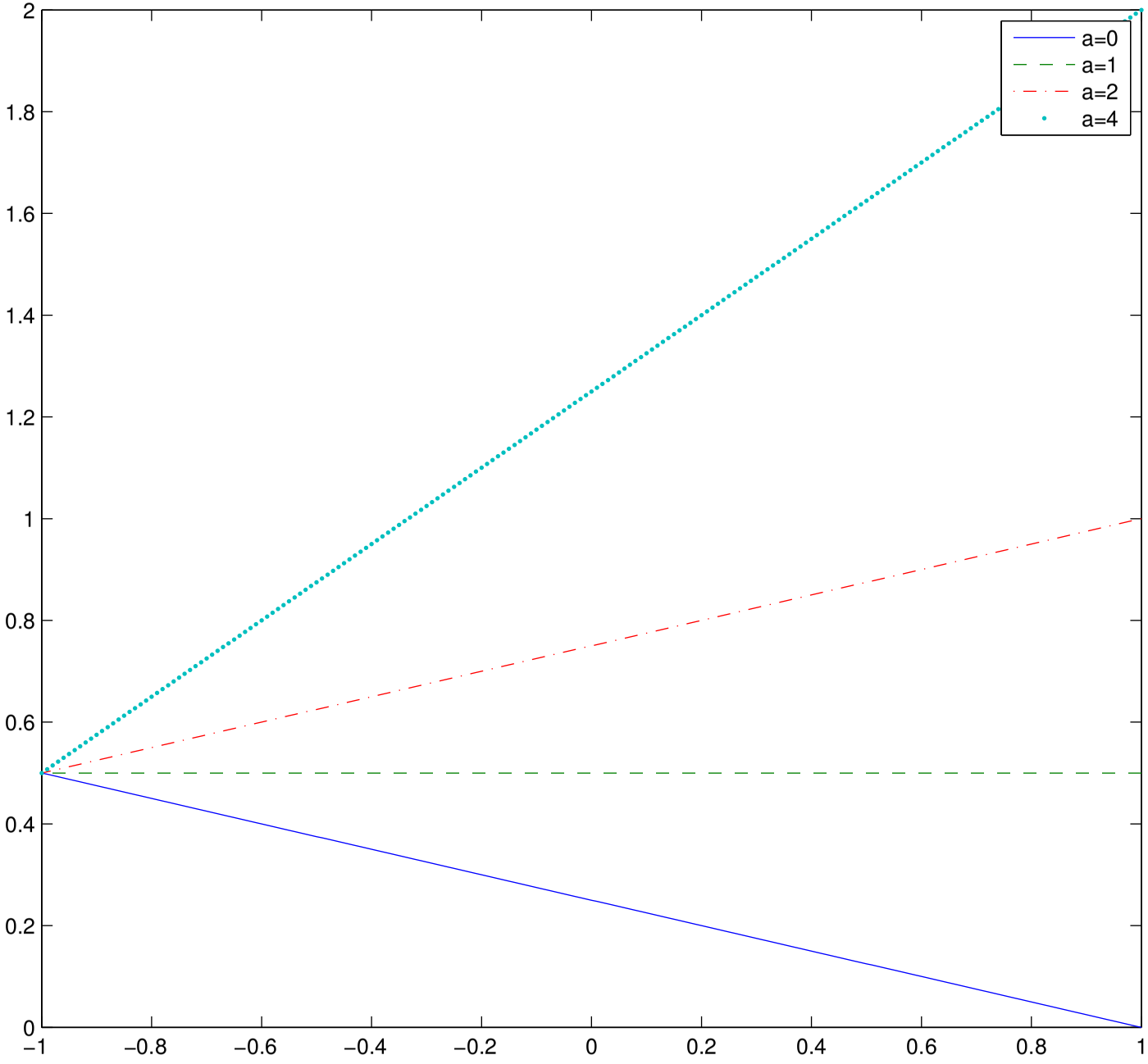
3) See soln to corresponding problem, exam 1.

$$J = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 1-\eta & 1+\eta & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & 1+\xi & 1-\xi \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & a \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Along the y axis, $\xi = -\eta$.
Multiplying, J becomes

$$J = \frac{1}{4}\begin{bmatrix} -2 & a-\eta a+1+\eta \\ -2 & -a+\eta a-1-\eta \end{bmatrix}$$

See plot. det J goes to 0 as a goes to zero. This will result in divide by zero errors near node 2.


a=0
a=1
a=2
a=4
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

4) First, fix left end, M becomes

$$M = \frac{\rho A \ell}{420}\begin{bmatrix} 4\ell^2 & & \text{sym} \\ 13\ell & 156 & \\ -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix}$$

The constraint eqns are

$$v_2 = \theta_1 \ell$$

$$\theta_2 = \theta_1$$

$$\begin{bmatrix} \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \ell \\ 1 \end{bmatrix} \theta_1$$

$$M_{red} = \frac{\rho A \ell^3}{420}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 13 & -3 \\ 13 & 156 & -22 \\ -3 & -22 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{(\rho A \ell)\ell^2}{420}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 14 \\ 147 \\ -21 \end{bmatrix}$$

$$= \frac{m\ell^2}{420}(140) = \underline{\frac{1}{3} m\ell^2}$$