Eigenvalues of a Tri-diagonal Matrix. Sturm's (Meirovitch) Theorem.

$$\det\left[A - \lambda I\right] = \det\begin{bmatrix} \alpha_1 - \lambda & \beta_2 & 0 & \cdots & 0 \\ \beta_2 & \alpha_2 - \lambda & \beta_3 & & \\ \vdots & \beta_3 & \alpha_3 - \lambda & & \\ \vdots & & & \ddots & \\ 0 & & & & \alpha_n - \lambda \end{bmatrix}$$

Denoting by $p_i(\lambda)$ the principal minor determinant of order i

$$p_1(\lambda) = \alpha_1 - \lambda$$

$$p_2(\lambda) = (\alpha_1 - \lambda)(\alpha_2 - \lambda) - \beta_2^2 = p_1(\lambda)(\alpha_2 - \lambda) - p_0(\lambda)\beta_2^2$$

$$p_3(\lambda) = p_2(\lambda)(\alpha_3 - \lambda) - p_1(\lambda)\beta_3^2$$

$$p_i(\lambda) = p_{i-1}(\lambda)(\alpha_i - \lambda) - p_{i-2}(\lambda)\beta_i^2$$

where $p_0(\lambda) = 1$

The characteristic eqn is then simply

$$p_n(\lambda) = 0$$

The sequence of polynomials $p_1(\lambda)\, p_2(\lambda) \ldots p_n(\lambda)$ is called the Sturm Sequence.

Consider a given interval $a < \lambda < b$
where neither a or b is a root.
We want to find # of roots in interval

The Sturm sequence has the following properties

1. $p_0(\lambda) \neq 0$

2. If $p_{i-1}(\mu) = 0$, $p_i(\mu)$ and $p_{i-2}(\mu)$ are nonzero and of opposite signs

3. As λ passes through a zero of $p_n(\lambda)$, $p_n(\lambda)/p_{n-1}(\lambda)$ changes from positive to negative.

Theorem: The number of sign changes $s(\mu)$ in the sequence of $p_0(\mu), p_1(\mu) \ldots p_n(\mu)$ is equal to the # of roots of $p_n(\lambda)$ corresponding to $\lambda < \mu$. Then the # of zeros of the polynomial $p_n(\lambda)$ in the interval (a,b) is $s(b) - s(a)$.

In our case, the matrix will be P.D. So $a=0$ is a good choice. Then guess for b.

Exaple

Consider the matrix

$$A_3 = \begin{bmatrix} 1 & -1.732 & 0 & 0 \\ & 7.667 & 1.247 & 0 \\ & & .976 & -.124 \\ & & & .357 \end{bmatrix}$$

$p_0 = 1$

$p_1 = 1 - \lambda$

$p_2 = (7.67 - \lambda)\, p_1 - 1.732^2\, p_0$

$p_3 = (.976 - \lambda)\, p_2 - 1.247^2\, p_1$

$p_4 = (.357 - \lambda)\, p_3 - .124^2\, p_2$

$\sum_{i=1}^{4} \lambda_i = \text{trace}(A) = 10$, so max eigenvalue is 10. (If all others are zero)

λ	p_1	p_2	p_3	p_4	$s(\lambda)$
10	-9	18	-148.42	1431	4 ($p_0 = 1$)
5	-4	-13.67	61.2	-284	3
7.5	-6.5	-4.083	36.75	-262	3
8.75	-7.75	5.4	-29.9	251	4
8.125	-7.125	.265	9.18	-71	3
8.4375	-7.438	2.73	-8.2	71	4
8.28125	-7.2812	1.47	.55	-4.39	3

One eigenvalue between 5 and ten

One eigenvalue exists between 8.28125 and 8.4375

Note that p_4 tended smaller.
A guess of 8.28125 is best!

Then apply Newton-Raphson, Golden section, bisection on p_4 to find actual root.

Eigenvectors can be computed easily from here.