Sp '06 ME 712 Solutions (Final Exam)

1) By changing the rotation variables to θL the EOM is

$$M\ddot{\underline{d}} + K\underline{d} = \underline{0} \qquad d = \begin{bmatrix} v_1 \\ \theta_1 L \\ v_2 \\ \theta_2 L \end{bmatrix}$$

where

$$K = \frac{EI}{L^3}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}, \quad M = \frac{m}{420}\begin{bmatrix} 156 & & & \\ 22 & 4 & \text{diag} & \\ 54 & 13 & 156 & \\ -13 & -3 & -22 & 4 \end{bmatrix}$$

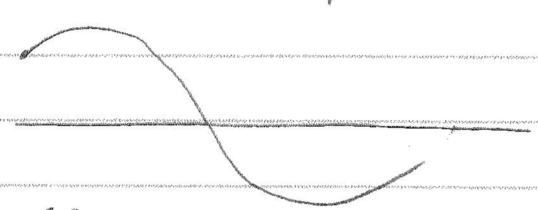
Our transformation is

$$\begin{bmatrix} v_1 \\ L\theta_1 \\ v_2 \\ L\theta_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}}_{T} L\theta_1$$

So $K_{red} = T^T K T = \dfrac{48EI}{L^3}$

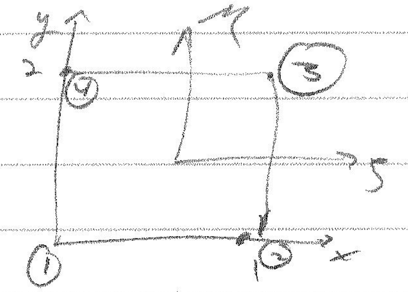
$M_{red} = T^T M T = \dfrac{71m}{420}$

b) For the reduced mass matrix: we've derived nothing special. The shape we defined looks like



If we had defined

$v_1 = -v_2 = -\theta_1 L/2$, $\theta_2 = \theta_1$, then we would have had the mass moment of inertia.



2) Shape functions are

$$N_i = \frac{1}{4}(1 - \xi\xi_i)(1 - \eta\eta_i)$$

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$
$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$
$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$\epsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = J \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} \qquad J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \quad (1)$$

By observation

$$\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \; \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \quad (2)$$

Need to change ϵ to

$$\epsilon = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}}_{\alpha} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} \qquad (3)$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} \qquad \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} \qquad (4)$$

Use (2), then (4) with (1) into (3) into

$$K = \sum_{i=1}^{2} \sum_{i=1}^{2} B^T E B \det J$$ (weights are 1 for 2 points)

$$B = \alpha \beta \gamma, \quad \beta = \begin{bmatrix} J^{-1} & 0 \\ 0 & J^{-1} \end{bmatrix}, \quad \gamma = (2) \text{ as}$$

$$\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & & 0 & & 0 & \\ 0 & -(1-\xi) & 0 & & 0 & & 0 & \end{bmatrix}$$

$$\times \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 2 \\ 2 \end{bmatrix}$$

See code

```
E=7.3084e10/(1-.09)*[1 .3 0;.3 1 0;0 0 (1-.3)/2];
alpha=[1 0 0 0;0 0 0 1;0 1 1 0];
J=[.5 0;0 1];
% Alternatively
J=1/4*[-(1-eta)   1-eta   1+eta   -1-eta  ;
       -(1-xi)   -1-xi   1+xi   1-xi ]*...
  [0 0;1 0;1 2;0 2]
beta=[inv(J) zeros(2);zeros(2) inv(J)];

K=zeros(8);
M=zeros(8);
for xi=(-1:2:1)/sqrt(3)
    for eta=(-1:2:1)/sqrt(3)
        [xi eta]
        gamma=1/4*[-(1-eta) 0        1-eta 0     1+eta 0    -1-eta 0;
                   -(1-xi)  0        -1-xi 0     1+xi  0     1-xi  0;
                   0        -(1-eta) 0     1-eta 0     1+eta 0    -1-eta ;
                   0        -(1-xi)  0    -1-xi  0     1+xi  0     1-xi ];
        B=alpha*beta*gamma;
        K=K+B'*E*B*det(J);
        N=1/4*[(1-xi)*(1-eta) (1-xi)*(1-eta) (1+xi)*(1-eta) (1+xi)*(1-eta) (1+xi)*(1+eta) (1+xi)*
(1+eta) (1-xi)*(1+eta) (1-xi)*(1+eta)
                ]
        M=M+N'*2770*N *det(J)
    end
end

%Problem 3
xi=-1;eta=0;
gamma=1/4*[-(1-eta) 0 1-eta 0 1+eta 0 -1-eta 0;
           -(1-xi) 0 -1-xi 0 1+xi 0 1-xi 0;
           0 -(1-eta) 0 1-eta 0 1+eta 0 -1-eta ;
           0 -(1-xi) 0 -1-xi 0 1+xi 0 1-xi ];

%Strain due to unit u2
epsu2=alpha*beta*gamma*[0 0 1 0 0 0 0 0]'

%Strain due to unit u2
epsv2=alpha*beta*gamma*[0 0 0 1 0 0 0 0]'
```

3) We can simply modify our code. Shape is the same. Matrices are the same.

4) See 2005