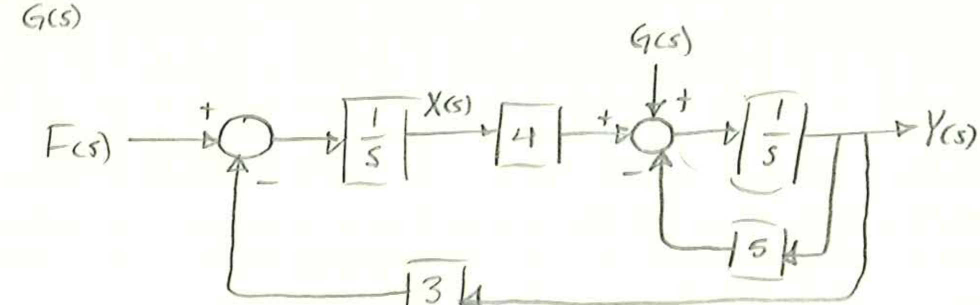
ME 3210 - System Dynamics | Exam II Solution | 11/7/13 Pc

(1) Find: $\dfrac{Y(s)}{F(s)}$, $\dfrac{X(s)}{G(s)}$

given:



Solution:

Considering each sum block:

$$sX(s) = F(s) - 3Y(s)$$

$$sY(s) = G(s) + 4X(s) - 5Y(s)$$

moving X(s) and Y(s) to the left, and writing in matrix form

$$\begin{bmatrix} s & 3 \\ -4 & s+5 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix}$$

then:

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} s & 3 \\ -4 & s+5 \end{bmatrix}^{-1} \begin{bmatrix} F \\ G \end{bmatrix} = \frac{1}{s^2+5s+12} \begin{bmatrix} s+5 & -3 \\ 4 & s \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix}$$

$$X(s) = \frac{s+5}{s^2+5s+12} F(s) + \frac{-3}{s^2+5s+12} G(s)$$

$$Y(s) = \frac{4}{s^2+5s+12} F(s) + \frac{s}{s^2+5s+12} G(s)$$

$$\therefore \boxed{\frac{Y(s)}{F(s)} = \frac{4}{s^2+5s+12} \quad , \quad \frac{X(s)}{G(s)} = \frac{-3}{s^2+5s+12}}$$

(2) write the state space matrices:

(a) $x_1 - 5u$ and x_2 are outputs

$$\dot{x}_1 = -5x_1 + 3x_2$$
$$\dot{x}_2 = x_1 - 4x_2 + 5u$$

$$A = \begin{bmatrix} -5 & 3 \\ 1 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \quad \text{for } \dot{\underline{x}} = A\underline{x} + Bu$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -5 \\ 0 \end{bmatrix} \quad \text{for } \underline{y} = C\underline{x} + Du$$

(b) $\dot{y}$ is the output

$$2\dddot{y} + 5\ddot{y} + 4\dot{y} + 7y = f$$

let $\underline{z} = \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix}$, $\dot{\underline{z}} = \begin{bmatrix} \dot{y} \\ \ddot{y} \\ \dddot{y} \end{bmatrix}$, $\dot{\underline{z}} = A\underline{z} + Bf$, $\underline{x} = C\underline{z} + Df$

then:

$$\dot{\underline{z}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{7}{2} & -2 & -\frac{5}{2} \end{bmatrix} \underline{z} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} f$$

$$\underline{x} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \underline{z} + \begin{bmatrix} 0 \end{bmatrix} f$$

$$\therefore \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{7}{2} & -2 & -\frac{5}{2} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad D = 0$$

(2) - Continued

(c) $y(t)$ and $\dot{y}(t)$ are outputs

$$\frac{Y(s)}{F(s)} = \frac{6}{3s^3 + 63 + 10} = \frac{6}{3s^3 + 73}$$

$$\left(3s^3 + 73\right)Y(s) = 6F(s)$$

taking inverse Laplace :

$$3\dddot{y} + 73y = 6f$$

let $\underline{z} = \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix}$,

$$\dot{\underline{z}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{73}{3} & 0 & 0 \end{bmatrix} \underline{z} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} f$$

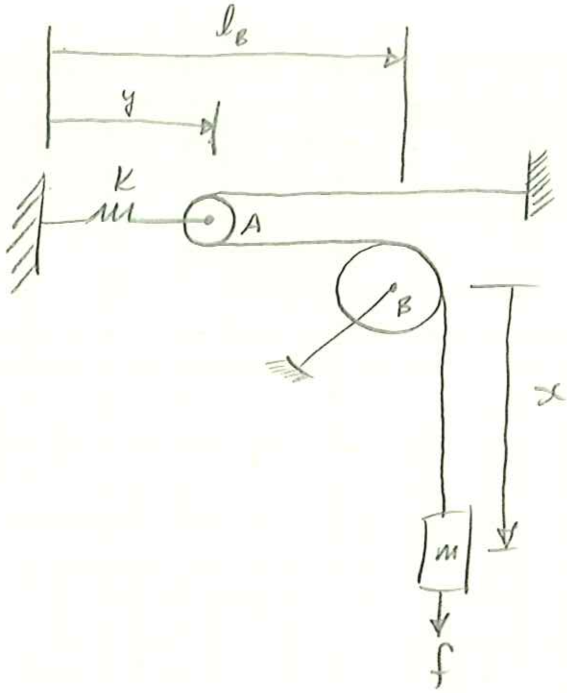
$$\underline{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \underline{z} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} f$$

$\therefore$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{73}{3} & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(3) derive EOM



$m_A = \frac{m}{2}$, $R_A = \frac{R}{2}$

$m_B = m$, $R_B = R$

$\therefore \; I_A = \frac{(m/2)(R/2)^2}{2} = \frac{mR^2}{16}$

$I_B = \frac{mR^2}{2}$

relating dof :

considering the pulley cable is a fixed length :

$$l = 2(l_B - y) + x$$

since pulley B is fixed , l_B is constant and is absorbed into l :

$$l^* = x - 2y$$

differentiating with respect to time , $\frac{d}{dt}(l^*) = 0$ and :

$$\dot{x} = 2\dot{y}$$

$$\ddot{x} = 2\ddot{y}$$

furthermore , assuming no slipping between cable/pulley :

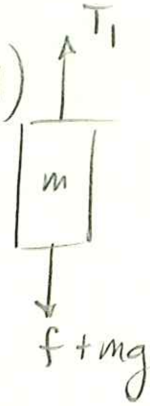
$$\dot{\theta}_A = \frac{\dot{y}}{R/2} = \frac{\dot{x}}{R} \quad , \quad \ddot{\theta} = \frac{\ddot{x}}{R}$$

$$\dot{\theta}_B = \frac{\dot{x}}{R} \quad , \quad \theta_B = \frac{\ddot{x}}{R}$$

(note that dof directions are critical , see diagrams)

(i) Newton's Method

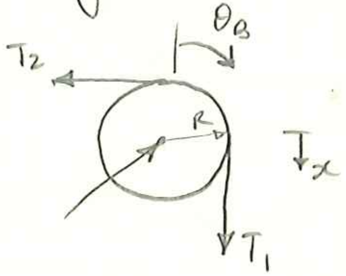
mass m: (x is not measured from equilibrium, $\therefore$ mg incl)



$$\sum F_x = m\ddot{x} = f - T_1 + mg$$

$$\therefore \; T_1 = f - m\ddot{x} + mg$$

Pulley B:



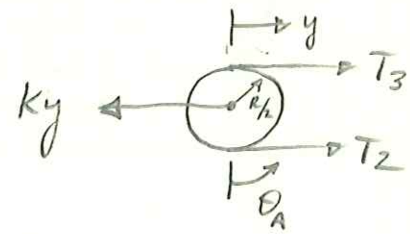
$$\circlearrowright \sum M_B = I_B \ddot{\theta}_B = R(T_1 - T_2)$$

$$\left(\frac{mR^2}{2}\right)\left(\frac{\ddot{x}}{R}\right) = R(T_1 - T_2)$$

$$\frac{m}{2}\ddot{x} = T_1 - T_2$$

$$\therefore \; T_2 = T_1 - \frac{m}{2}\ddot{x} = f - \frac{3m}{2}\ddot{x} + mg$$

Pulley A:



$$\circlearrowleft \sum M_A = I_A \ddot{\theta}_A = \left(\frac{R}{2}\right)(T_2 - T_3)$$

$$\left(\frac{mR^2}{16}\right)\left(\frac{\ddot{x}}{R}\right) = \left(\frac{R}{2}\right)(T_2 - T_3)$$

$$\frac{m\ddot{x}}{8} = T_2 - T_3$$

$$\sum F_y = m_A \ddot{y} = T_3 + T_2 - ky$$

$$T_3 = m_A \ddot{y} + ky - T_2 = \left(\frac{m}{2}\right)\left(\frac{\ddot{x}}{2}\right) + k\left(\frac{x}{2}\right) - T_2$$

Substituting:

$$\frac{m\ddot{x}}{8} = T_2 - \left(\frac{m\ddot{x}}{4} + \frac{kx}{2} - T_2\right)$$

$$\frac{3m\ddot{x}}{8} + \frac{kx}{2} = 2T_2 = 2f - 3m\ddot{x} + 2mg$$

$$\therefore \; \boxed{m\ddot{x} + \frac{4}{27}kx = \frac{16}{27}(f + mg)}$$

(ii) Energy Method

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m_A\dot{y}^2 + \frac{1}{2}I_A\dot{\theta}_A^2 + \frac{1}{2}I_B\dot{\theta}_B^2$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{m}{2}\right)\left(\frac{\dot{x}}{2}\right)^2 + \frac{1}{2}\left(\frac{mR^2}{16}\right)\left(\frac{\dot{x}}{R}\right)^2 + \frac{1}{2}\left(\frac{mR^2}{2}\right)\left(\frac{\dot{x}}{R}\right)^2$$

$$= \frac{1}{2}m\left(1 + \frac{1}{8} + \frac{1}{16} + \frac{1}{2}\right)\dot{x}^2$$

$$= \frac{1}{2}\left(\frac{27m}{16}\right)\dot{x}^2$$

$$V = \frac{1}{2}ky^2 - mgx = \frac{1}{2}k\left(\frac{x}{2}\right)^2 - mgx = \frac{kx^2}{8} - mgx$$

Lagrange:

$$\mathcal{L} = T - V = \frac{27}{32}m\dot{x}^2 - \frac{kx^2}{8} + mgx$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{d}{dt}\left(\frac{27m}{16}\dot{x}\right) = \frac{27m}{16}\ddot{x}$$

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{kx}{4} + mg$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = f$$

$$\frac{27m\ddot{x}}{16} + \frac{kx}{4} = f + mg$$

$$\therefore \boxed{m\ddot{x} + \frac{4}{27}kx = \frac{16}{27}(f + mg)}$$