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ME460/660 Exam 2, 2010 Soln

1) $x(t) = \frac{1}{2} t$

$\bar{x} = \frac{1}{2}$ (average value)

$$a_n = \frac{2}{2} \int_0^2 \frac{t}{2} \cos n\pi t \, dt$$

$$= \frac{1}{2} \left[ \int_0^2 t \cos n\pi t \, dt \right]$$

$u = t \qquad dv = \cos n\pi t \, dt$

$du = dt \qquad v = \frac{1}{n\pi} \sin n\pi t$

$$= \frac{1}{2} \left[ \frac{t}{n\pi} \sin n\pi t \Big|_0^2 - \int_0^2 \frac{1}{n\pi} \sin n\pi t \, dt \right]$$

$$= \frac{1}{2} \left[ \frac{2}{n\pi} \underbrace{\sin 2\pi n}_{0} + \frac{1}{n^2\pi^2} \cos n\pi t \Big|_0^2 \right]$$

$$= \frac{1}{2} \left[ \cos 2\pi n - 1 \right] = 0$$

$$b_n = \frac{1}{2} \int_0^2 t \sin n\pi t \, dt = \frac{-1}{n\pi}$$

$$x(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-1}{n\pi} \sin n\pi t$$

2) Log decrement is constant for viscous damping
" increasing for Coulomb damping
decreasing for air damping

Over the 1st 3 cycles

$$\delta = \frac{1}{3} \ln \frac{1}{.63} = 0.154$$

Over the last 3

$$\delta = \frac{1}{3} \ln \frac{0.125}{0.0675} = 0.205$$

Damping is increasing, some Coulomb damping must exist with some viscous damping.

(For 1st 5 cycles, $\delta = \frac{1}{5} \ln \frac{1}{0.46} = 0.155$,

for last 5 cycles, $\delta = \frac{1}{5} \ln \frac{0.20}{0.0675} = 0.217$,

so same answer, some Coulomb, but not a straight decay envelope, so some viscous damping too)

3) $\det(K - M\omega^2) = 0$

a) and b)

$$(10 - 10\omega_n^2)(100 - 100\omega_n^2) - 100 = 0$$

$$10(1 - \omega_n^2)^2 - 1 = 0$$

$$10\omega_n^4 - 20\omega_n^2 + 9 = 0$$

$$\omega_n^2 = \frac{20 \pm \sqrt{400 - 360}}{20}$$

$$= 1 \pm \sqrt{1 - 0.9}$$

$$= 0.684,\ 1.316 \qquad \omega_1 = 0.872 \text{ rad/s}$$

$$\omega_2 = 1.147 \text{ rad/s}$$

For $\omega_n^2 = 0.684$

$$u_1 = \begin{bmatrix} 1 \\ 0.316 \end{bmatrix}, \text{ normalized, } \underline{u}_1 = \begin{bmatrix} 0.224 \\ 0.071 \end{bmatrix}$$

$$\underline{u}_1^T M \underline{u}_1 = 1$$

For $\omega_2^2 = 1.316$

$$\left[K - M\omega_2^2\right]\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underline{u}_2 = \begin{bmatrix} 1 \\ -0.316 \end{bmatrix}, \text{ normalized } \underline{u}_2 = \begin{bmatrix} 0.224 \\ -0.071 \end{bmatrix}$$

should be 1 for normalization.

c) $$S^T M S = \begin{bmatrix} 18.2 & -0.04 \\ -0.04 & 18.2 \end{bmatrix}$$

The low off-diagonal terms show orthogonality so these are mode shapes. They are not mass normalized.

4) $$\frac{EI}{\rho A} W'''' + \ddot{W} = 0$$

$$\ddot{W} = -\omega_n^2 W, \text{ so}$$

$$\frac{EI}{\rho A} W'''' - \omega_n^2 W = 0$$

$$W = T(t)\, X(x)$$

$$X(x) = A_n \cos \sigma_n x + B_n \sin \sigma_n x + C_n \cosh \sigma_n x + D_n \sinh \sigma_n x$$

$$X(0) = 0 = A_n + C_n \qquad A_n = -C_n$$

$$X'(0) = 0 = \sigma_n B_n + \sigma_n D_n \qquad B_n = -D_n$$

$$X(l) = 0 = A_n \cos \sigma_n l + B_n \sin \sigma_n l - A_n \cosh \sigma_n l - B_n \sinh \sigma_n l$$

$$X''(l) = 0 = -A_n \sigma_n^2 \cos \sigma_n l - B_n \sigma_n^2 \sin \sigma_n l - A_n \sigma_n^2 \cosh \sigma_n l - B_n \sigma_n^2 \sinh \sigma_n l$$

$$\underbrace{\begin{bmatrix} \cos \sigma_n l - \cosh \sigma_n l & \sin \sigma_n l - \sinh \sigma_n l \\ \cos \sigma_n l + \cosh \sigma_n l & \sin \sigma_n l + \sinh \sigma_n l \end{bmatrix}}_{A} \begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det(A) = (\cos \sigma_n l - \cosh \sigma_n l)(\sin \sigma_n l + \sinh \sigma_n l) - (\cos \sigma_n l + \cosh \sigma_n l)(\sin \sigma_n l - \sinh \sigma_n l) = 0$$

Multiplied out, many terms cancel. Writing those kept

$$2 \cos \sigma_n \sinh \sigma_n l - 2 \cosh \sin \sigma_n l = 0$$

$$\tanh \sigma_n l = \tan \sigma_n l$$

$\tan \sigma_n l$ repeats every $\pi$, so our answer will be similar. Plotting shows answers near $0, \pi, 2\pi, 3\pi \ldots$

Using calculator

$\sigma_n l = 3.92, 7.07, 10.21, 13.35, 16.49, 19.64$

We need the 1st. However $\sigma_n l = 0$ doesn't count/work.

Noting $X_n'''' = \sigma_n^4 X_n$

$$\left(\frac{EI}{\rho A}\sigma_n^4 - \omega_n^2\right) X_n = 0$$

$$\omega_n = \sigma_n^2 \sqrt{\frac{EI}{\rho A}}$$

$$\omega_1 = \frac{15.37}{l^2}\sqrt{\frac{EI}{\rho A}}$$

Presuming $A_n = 1$ (just to find $\frac{B_n}{A_n}$)

$$B_n = \frac{-(\cos\sigma_n l - \cosh\sigma_n l)}{\sin\sigma_n l - \sinh\sigma_n l} = \frac{+25.92}{-25.92} = \underline{-1.001}$$

$$X_1 = (\cos\sigma_1 x - \cosh\sigma_1 x) - 1.001(\sin\sigma_1 x - \sinh\sigma_1 x)$$

where $\sigma_1 = \frac{3.92}{l}$, $\omega_1 = \frac{15.37}{l^2}\sqrt{\frac{EI}{\rho A}}$

This is equivalent to the book, even though not in the same form.