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ME 710 WI '96 Final Exam Solutions
Slater

1) Find the 1st natural frequency and mode shape of a string using a 2 term approximation for the mode shape.
The string has length 1 m. $T = 4\ N$, $m = 1\ kg/m$ and a point mass of 1 kg at the center.
Choose the best reasonable 2 term series.

$$W_r^{(2)} = \sum_{i=1}^{2} a_{ir} \underbrace{\sin(2i-1)\pi x}_{\phi_i}$$

1st and 3rd mode shape of string

$$m_{ij} = \int_0^1 m(x)\ \phi_i\ \phi_j\ dx$$

$$m(x) = 1 + \delta(x - \tfrac{1}{2})$$

$$m_{11} = \int_0^1 (\sin \pi x)^2 dx + \int_0^1 \delta(x - \tfrac{1}{2})(\sin \pi x)^2 dx$$

$$= \frac{1}{2} + 1 = 1.5$$

$$m_{12} = m_{21} = \int_0^1 \sin \pi x \sin 3\pi x\ dx + \int_0^1 \delta(x-\tfrac{1}{2}) \sin \pi x \sin 3\pi x\ dx$$

$$= 0 - 1 = -1$$

$$m_{22} = \int_0^1 (\sin 3\pi x)^2 dx + \int_0^1 \delta(x - \tfrac{1}{2})(\sin 3\pi x)^2 dx$$

$$= 1.5$$

$$K_{ij} = \int_0^1 \phi_i L \phi_j \, dx = -\int_0^1 T \phi_i \frac{d^2}{dx^2} \phi_j \, dx$$

$$K_{11} = T \pi^2 \int_0^1 \sin^2 \pi x \, dx = 2\pi^2$$

$$K_{12} = K_{21} = 9\pi^2 T \int_0^1 \sin \pi x \sin 3\pi x \, dx = 0$$

$$K_{22} = 9\pi^2 T \int_0^1 \sin^2 3\pi x \, dx = 18\pi^2$$

$$M = \begin{bmatrix} 1.5 & -1 \\ -1 & 1.5 \end{bmatrix} \qquad K = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} 2\pi^2$$

$$\det(M\omega^2 - K) = 0$$

$$\det \begin{bmatrix} 1.5\omega^2 - 2\pi^2 & -\omega^2 \\ -\omega^2 & 1.5\omega^2 - 18\pi^2 \end{bmatrix}$$

$$= 2.25\omega^4 - 3\omega^2\pi^2 - 27\omega^2\pi^2 + 36\pi^4 - \omega^4$$

$$= 1.25\omega^4 - 30\omega^2\pi^2 + 36\pi^4$$

$$\omega^2 = \pi^2 \left(\frac{30 \pm \sqrt{900 - 180}}{2.5} \right) = \frac{30 \pm 26.8}{2.5} \pi^2$$

$$= 1.267 \pi^2$$

$$\omega = 1.1256\pi \text{ rad/s}, \; 4.768\pi \text{ rad/s}$$

a) The equation of motion for a simply supported beam with an end load is

$$EI \frac{\partial^4}{\partial x^4} W(x,t) + P \frac{\partial^2}{\partial x^2} W(x,t) + m \frac{\partial^2}{\partial t^2} W(x,t) = 0$$

Determine for what positive values of P the system is stable.

$$L = EI \frac{d^4}{dx^4} + P \frac{d^2}{dx^2}$$

For stability

$(u, Lu) > 0$ for all $u \neq 0$

where u is a comparison function.

$$(u, Lu) = \int_0^L u EI u'''' + u P u'' \, dx$$

$$= \cancelto{0}{u u''' EI \Big|_0^L} - \int_0^L u' EI u''' \, dx + \cancelto{0}{u P u' \Big|_0^L} - \int_0^L u' P u' \, dx$$

$$= -\cancelto{0}{u' EI u'' \Big|_0^L} + \int_0^L u''^2 EI \, dx - \int_0^L P u'^2 \, dx$$

$$= \int_0^L u''^2 EI - P u'^2 \, dx$$

The solution to the eigenvalue problem is

$u(x) = a \sin\left(n\pi \frac{x}{L}\right) \qquad n = 1, 2, 3, \ldots$

substituting into (u, Lu) gives

$$(u, Lu) = \int_0^L a^2 \left(\frac{n\pi}{L}\right)^4 \sin^2 n\pi\frac{x}{L}\, EI - Pa^2\left(\frac{n\pi}{L}\right)^2 \cos^2 n\pi\frac{x}{L}\, dx$$

Noting that $\int_0^L \sin^2 n\pi x\, dx = \int_0^L \cos^2 n\pi x\, dx \geq 0$

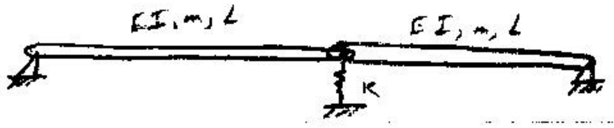
$$\cancel{a^2} \left(\frac{n\pi}{L}\right)^{\cancel{4}\,2} EI > P \cancel{a^2} \left(\frac{n\pi}{L}\right)^2$$

$$P < EI\,(n\pi)^2$$

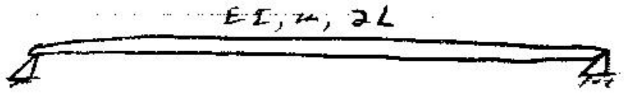
$$P < \frac{EI\pi^2}{L^2}$$

b) P is the critical buckling value.

3) Obtain the equation of motion for the system of beams shown.



Is there any similarity between the modes of the system above and the beam below?



For the left beam

$$EI \frac{\partial^4}{\partial x^4} W_1(x,t) + m \frac{\partial^2}{\partial t^2} W_1(x,t) = 0$$

$$W_1(0,t) = 0, \quad W_1''(0,t) = 0$$

$$W_1''(L,t) = 0, \quad EI \left.\frac{\partial^3 W_2}{\partial x^3}\right|_{x=L} = K x + EI \left.\frac{\partial^3 W_1}{\partial x^3}\right|_{x=0}$$

For the right beam

$$EI \frac{\partial^4}{\partial x^4} W_2(x,t) + m \frac{\partial^2}{\partial t^2} W_2(x,t) = 0$$

$$W_1(0,t) = W_2(0,t) \qquad W_2''(0,t) = 0$$

$$W_2(L,t) = 0, \qquad W_2''(L,t) = 0$$

x4 The modes $n = 2, 4, 6, \ldots$ of the second beam will also be modes of the first beam.

4a) Solve the following using Gaussian Elimination

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 4 \end{bmatrix} \underline{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

b) Complete one cycle of the Jacobi method (3 rotations) on the following matrix

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 6 & -2 \\ -1 & -2 & 4 \end{bmatrix}$$

4b) $A^0 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 6 & -2 \\ -1 & -2 & 4 \end{bmatrix}$

Annihilate (2,1)

$\theta_1 = \frac{1}{2} \text{atan} \frac{2(-1)}{2-6} = .2318$

$R_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} .9732 & -.2298 & 0 \\ +.2298 & .9732 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$A^1 = R_1^T A^0 R_1 = \begin{bmatrix} 1.7639 & 0 & -1.4328 \\ 0 & 6.2361 & -1.7167 \\ -1.4328 & -1.7167 & 4.00 \end{bmatrix}$

Annihilate 3,3

$\theta_2 = \frac{1}{2} \text{atan} \frac{2(-1.4328)}{1.7639-4} = .4541$

$R_2 = \begin{bmatrix} .8987 & 0 & -.4386 \\ 0 & 1 & 0 \\ .4386 & 0 & .8987 \end{bmatrix}$

$A^2 = R_2^T A^1 R_2 = \begin{bmatrix} 1.0646 & -.7530 & 0 \\ & 6.2361 & -1.5428 \\ \text{sym} & & 4.6993 \end{bmatrix}$

Annihilate 3,2, $\theta_3 = -.5544 \left(\frac{1}{2} \text{atan} \frac{2(-1.5428)}{6.236-4.699} \right)$

$R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .8502 & .5264 \\ 0 & -.5264 & .8502 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1.0646 & -.6403 & -.3964 \\ & 7.1912 & 0 \\ \text{sym} & & 3.744 \end{bmatrix}$

4a) $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 6 & -2 \\ -1 & -2 & 4 \end{bmatrix} \underline{X} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Subtract row 2 from row 3

$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 6 & -2 \\ 0 & -8 & 6 \end{bmatrix} \underline{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Add $\frac{1}{2}$ of row 1 to 2

$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 5.5 & -2.5 \\ 0 & -8 & 6 \end{bmatrix} \underline{x} = \begin{bmatrix} 1 \\ 1.5 \\ 0 \end{bmatrix}$

Add $\frac{8}{5.5}$ of row 2 to row 3.

$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 5.5 & -2.5 \\ 0 & 0 & 2.364 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.5 \\ 2.182 \end{bmatrix}$

$X_3 = \frac{2.182}{2.364} = .923$

$X_2 = (1.5 + 2.5 \cdot .923) \frac{1}{5.5} = .692$

$X_1 = (1 + .923 + .692) \frac{1}{2} = 1.308$