Vibration Testing, With Modal Analysis and Health Monitoring

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Chapter 1

Review of Fundamentals: Single Degree of Freedom Systems

Vibration is the exchange of potential and kinetic energy, usually in some periodic, if complex, fashion, and usually without significant energy dissipation. The simplest representation of such a system with some energy dissipation and an external excitation is the single degree of freedom, or *SDOF*, system of Figure 1.1 represented by the differential equation

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) \tag{1.1}$$

where m is the mass, c is the linear damping coefficient, k is the linear spring stiffness, and $F(t)$ is

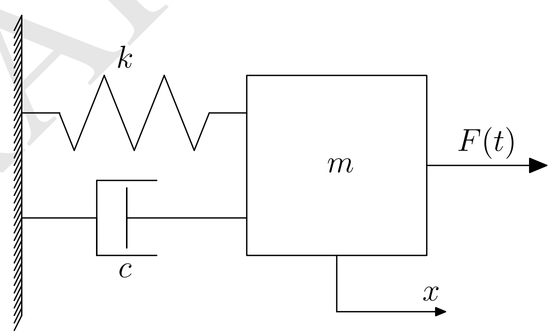


Figure 1.1: Linear forced single degree of freedom system.

the externally applied load. Equation (1.1) can be derived by Newton's law, Lagrange's equation, or a variety of other methods. Solution of this equation is standard fair in introductory differential equations, linear systems, and introductory vibrations courses. Often the emphasis is on solution assuming a solution in real form. In the following sections we focus on solution using complex exponentials as they lend themselves better to solutions of more complex equations.

1.1 Application of Complex Exponentials to Solving Linear Differential Equations

1.1.1 The Euler Relations

Euler's relation states that

$$e^{\beta j} = \cos(\beta) + j\sin(\beta) \tag{1.2}$$

where $j = \sqrt{-1}$. This is illustrated in Figure 1.2. The vector $e^{j\beta}$ is plotted in the complex plain. The vector has unit length. Thus from trigonometry, one can obtain all necessary relationships between the magnitude, phase (β), the real, and the imaginary parts. Multiplying the complex number by a real constant would make the vector longer without changing its direction (since both the imaginary and real parts would increase proportionally to the constant).

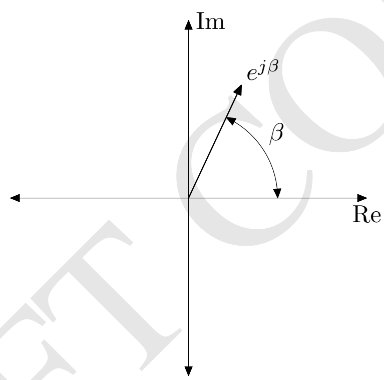


Figure 1.2: Phasor

For example, if we have a complex number, $A^* = A' + jA''$, it can also be represented as $A^* = Ae^{j\beta}$ where $A = \sqrt{A'^2 + A''^2}$ and $\beta = \arctan\left(\frac{A''}{A'}\right)$. Here care must be taken to use a two argument inverse tangent because a positive ratio $\frac{A'}{A''}$ could mean β is either in quadrant 1 or quadrant 3. Likewise, a negative ratio could mean β is in either quadrant 2 or quadrant 4. From basic trigonometry one can also see that $A' = A\cos(\beta)$ and $A'' = A\sin(\beta)$. The convenience of the complex exponential method is derived primarily from the identity $e^{j\alpha}e^{j\beta} = e^{j(\alpha+\beta)}$.

Using Equation (1.2),

$$e^{\gamma j} + e^{-\gamma j} = \cos(\gamma) + j\sin(\gamma) + \cos(-\gamma) + j\sin(-\gamma) \tag{1.3}$$

where γ is some arbitrary angle. Noting that $\cos(-\gamma) = \cos(\gamma)$ and $\sin(-\gamma) = -\sin(\gamma)$, this can be simplified to

$$e^{\gamma j} + e^{-\gamma j} = 2\cos(\gamma) \tag{1.4}$$

Solving for $\cos(\gamma)$ yields the identity

$$\cos(\gamma) = \frac{e^{\gamma j} + e^{-\gamma j}}{2} \tag{1.5}$$

Alternatively, subtracting the two exponentials yields,

$$e^{\gamma j} - e^{-\gamma j} = \cos(\gamma) + j\sin(\gamma) - \cos(-\gamma) - j\sin(-\gamma) = 2j\sin(\gamma) \tag{1.6}$$

which results in the identity

$$\sin(\gamma) = \frac{e^{\gamma j} - e^{-\gamma j}}{2j} \tag{1.7}$$

1.1.2 The Phasor Representation of a Periodic Function

The advantage of using the Euler relations in vibration analysis is their use in representing periodic functions. Instead of writing a periodic function of time as

$$x(t) = A\cos(\omega t) + B\sin(\omega t) = X\cos(\omega t + \phi), \tag{1.8}$$

we can use equations (1.5) and (1.7) to write it as

$$x(t) = A\frac{e^{\omega tj} + e^{-\omega tj}}{2} + B\frac{e^{\omega tj} - e^{-\omega tj}}{2j} = X^* e^{j\omega t} + \bar{X}^* e^{-j\omega t} = x^*(t) + \bar{x}^*(t) \tag{1.9}$$

Here

$$X^* = \frac{A - Bj}{2} \tag{1.10}$$

and $\bar{X}^*$ means complex conjugate of X^*, e.i. $\bar{X}^* = \frac{A+Bj}{2}$. The term $x^*(t) = X^* e^{j\omega t}$ is often referred to as a *phasor*. It has a magnitude of $|X| = \sqrt{A^2 + B^2}$ and a phase of $\measuredangle(X^*) = \arctan(\frac{-B}{a}) + \omega t$. The derivative of a phasor is

$$\frac{d}{dt}x^*(t) = \frac{d}{dt}X^* e^{j\omega t} = j\omega X^* e^{j\omega t} \tag{1.11}$$

which illustrates the convenience of phasors: the time derivative of a phasor is simply j times the frequency of the phasor (ω for our phasor) times the phasor itself. What this does is turns a calculus operation (a tim derivative) into a simple algebraic operation. A derivative increases the amplitude by ω (or decreased the magnitude if $\omega < 1$) and adds a phase of 90°. Performing a time derivative on $x(t)$ of equation (1.8) in real form is a significantly more complex operation. For large complex systems, where many derivatives are involved, the savings from this approach in terms of understanding and simplicity becomes significant. The convenient properties of the exponential also make multiplying and dividing functions written as phasors easy. For example, consider $f^*(t) = F^* e^{j\Omega t}$ multiplied by $x^*(t)$.

$$f^*(t)x^*(t) = F^* e^{j\Omega t} X^* e^{j\omega t} = F^* X^* e^{j(\omega+\Omega)t} \tag{1.12}$$

or consider

$$\frac{x^*(t)}{f^*(t)} = \frac{X^* e^{j\omega t}}{F^* e^{j\Omega t}} = \frac{X^*}{F^*} e^{j(\omega-\Omega)t} \tag{1.13}$$

Both of these calculations are much easier using phasors than their real-functional form counterparts.

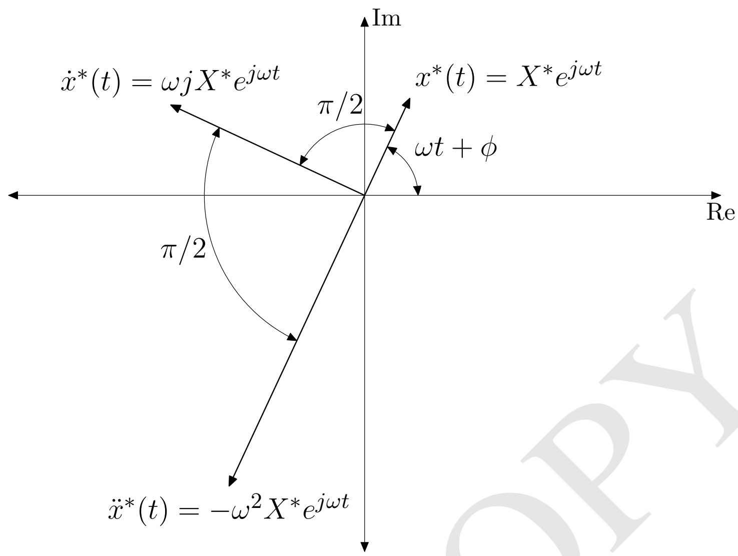


Figure 1.3: Representation of $x(t)$ and its derivatives as phasors.

1.1.3 Solution for the Homogeneous (free response) Solution

Consider the homogeneous equation of motion

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{1.14}$$

The solution can be written as

$$x(t) = X_h e^{\lambda t} \tag{1.15}$$

where the constants X_h and λ need to be determined. Substituting the solution into the equation of motion yields the result

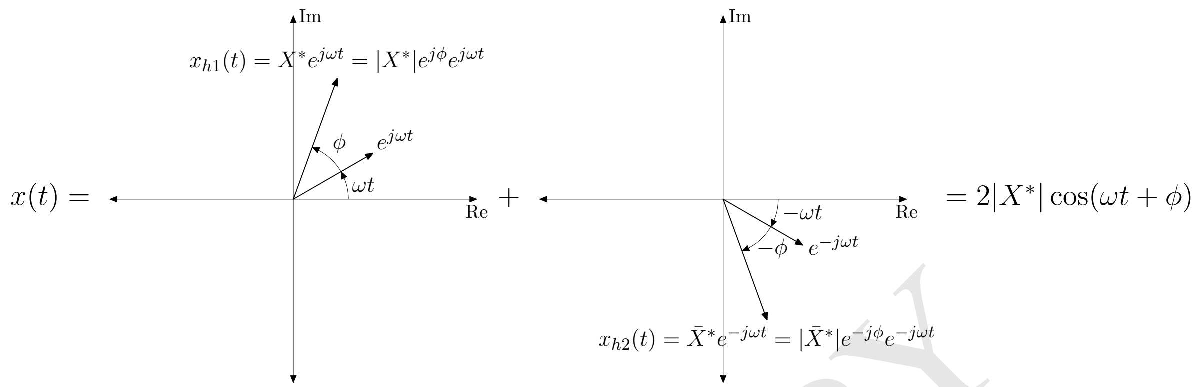
$$\lambda = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m} \tag{1.16}$$

In the case of no damping, i.e. $c = 0$, this yields $\lambda = \pm\sqrt{\frac{k}{m}}j$. Noting that two solutions exist

$$x(t) = X_{h1} e^{\sqrt{\frac{k}{m}}t} + X_{h2} e^{-\sqrt{\frac{k}{m}}t} = X_{h1} e^{\omega t} + X_{h2} e^{-\omega t} \tag{1.17}$$

where $\omega = \sqrt{k}m$.

This is illustrated in Figure 1.4. The unit phasors $e^{j\omega t}$ of Figure 1.4(a) drives the individual solution $x_{h1}(t)$ counter-clockwise in time. The coefficient X^* can be factored into its magnitude, $|X^*|$, and its own constant phasor, $e^{j\phi}$. The phasor results in a constant phase lead (as shown) or lag relative to the time-varying phasor, while the magnitude increases or decreases the apparent length of the phasor representing the solution, $x_{h1}(t)$. When we add the conjugate part of the



(a) Counter-clockwise rotating solution.

(b) Clockwise rotating solution.

Figure 1.4: Clockwise and counter-clockwise rotating phasor representation of $x(t)$.

solution as shown in Figure 1.4(b) the result is twice the real part of either solution, with a phase lead of ϕ.

Using the notation $X = X' + X''j$ to represent the real and imaginary parts of variables, substituting, and expanding, this can be written, as

$$\begin{aligned} x(t) &= X'_{h1}\cos(\sqrt{\frac{k}{m}}t) + X'_{h1}j\sin(\sqrt{\frac{k}{m}}t) + X''_{h1}j\cos(\sqrt{\frac{k}{m}}t) - X''_{h1}\sin(\sqrt{\frac{k}{m}}t) \\ &+ X'_{h2}\cos(\sqrt{\frac{k}{m}}t) - X'_{h2}j\sin(\sqrt{\frac{k}{m}}t) + X''_{h2}j\cos(\sqrt{\frac{k}{m}}t) + X''_{h2}\sin(\sqrt{\frac{k}{m}}t) \end{aligned} \tag{1.18}$$

after using Equation (1.2), $\cos(-\gamma) = \cos(\gamma)$, and $\sin(-\gamma) = -\sin(\gamma)$. Noting that $x(t)$ must be a real function, the imaginary part must be zero, and thus

$$\begin{aligned} 0 &= X'_{h1}j\sin(\sqrt{\frac{k}{m}}t) + X''_{h1}j\cos(\sqrt{\frac{k}{m}}t) - X'_{h2}j\sin(\sqrt{\frac{k}{m}}t) + X''_{h2}j\cos(\sqrt{\frac{k}{m}}t) \\ &= (X'_{h1} - X'_{h2})\sin(\sqrt{\frac{k}{m}}t) + (X''_{h1} + X''_{h2})\cos(\sqrt{\frac{k}{m}}t) \end{aligned} \tag{1.19}$$

For this expression to be satisfied for all time the coefficients of $\sin(\sqrt{\frac{k}{m}}t)$ and $\cos(\sqrt{\frac{k}{m}}t)$ must be zero, i.e.

$$X'_{h1} = X'_{h2} \tag{1.20}$$

and

$$X''_{h1} = -X''_{h2} \tag{1.21}$$

which means that X_{h1} and X_{h1} are complex conjugates ($X_{h1} = \bar{X}_{h1}$). Noting this, Equation (1.18) can be written as

$$\begin{aligned} x(t) &= 2X'_{h1}\cos(\sqrt{\frac{k}{m}}t) - 2X''_{h1}\sin(\sqrt{\frac{k}{m}}t) \\ &= A\cos(\sqrt{\frac{k}{m}}t) + B\sin(\sqrt{\frac{k}{m}}t) \\ &= C\cos(\sqrt{\frac{k}{m}}t + \phi) \end{aligned} \tag{1.22}$$

where the third form of the solution can be obtained using trigonometric identities. It is in this form that one notes that the period of oscillation is $\sqrt{\frac{k}{m}}$ which is usually represented at ω_n, the natural frequency. It is often referred to as the undamped natural frequency, since in the damped case, the frequency of oscillation is different.

In the case of damping, the solution is then given by

$$x(t) = X_{h1}e^{\lambda_1 t} + X_{h2}e^{\lambda_2 t} \tag{1.23}$$

where λ_i, $i = 1, 2$, are given by equation (1.16). Depending upon the value of c, one of the following three cases results: underdamped, overdamped, and critically damped. The level of damping is quantified by the non-dimensional value $\zeta = \frac{c}{2\sqrt{km}}$, the damping ratio.

Overdamped, $\zeta > 1$

If $c > 2\sqrt{km}$ then both roots, λ, are real. As long as $c > 0$, then they are also both negative. This is important as substituting into the solution, Equation (1.23), yields a purely exponential response. If either root is positive, then the corresponding term grows without bound in what is called an unstable response. In the overdamped case, it is not possible for the solution to cross $x(t) = 0$ more than once. No oscillation will be observed, and the coefficients X_{h1} and X_{h2} can be determined by evaluating $x(t)$ and its first time derivative at $t = 0$, and comparing to the initial displacement and velocity.

Critically, $\zeta = 1$

If $c = 2\sqrt{km}$ then both roots, λ, are real and equal (i.e. $\lambda_1 = \lambda_2$). In this case, since both solutions are the same, but two are required[1]. The solution it then

$$x(t) = X_{h1}e^{\lambda t} + X_{h2}te^{\lambda t} \tag{1.24}$$

Again, it is not possible for oscillation to occur and the coefficients X_{h1} and X_{h2} can be determined by evaluating $x(t)$ and its first time derivative at $t = 0$, and comparing to the initial displacement and velocity.

[1] Discussion of why is left to a differential equations text. Pick your favorite, and look up Wronskian.

Underdamped, $\zeta < 1$

If $c < 2\sqrt{km}$ then both roots, λ, are complex conjugates. Noting that $\zeta = \frac{c}{2\sqrt{km}}$, and $\omega_n = \sqrt{\frac{k}{m}}$, the roots can be written

$$\lambda = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}j \tag{1.25}$$

Substituting for λ_i in Equation (1.23) gives

$$\begin{aligned} x(t) &= X_{h1}e^{\lambda_1 t} + X_{h2}e^{\lambda_2 t} \\ &= X_{h1}e^{(-\zeta\omega_n+\omega_n\sqrt{1-\zeta^2}j)t} + X_{h2}e^{(-\zeta\omega_n-\omega_n\sqrt{1-\zeta^2}j)t} \\ &= \left(X_{h1}e^{\omega_n\sqrt{1-\zeta^2}jt} + X_{h2}e^{\omega_n\sqrt{1-\zeta^2}jt}\right)e^{-\zeta\omega_n t} \end{aligned} \tag{1.26}$$

Comparing the coefficient of $e^{-\zeta\omega_n t}$ to Equation (1.22), the total solution can be written

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t - \phi) \tag{1.27}$$

Here the coefficient A and the phase angle ϕ are determined by evaluating $x(t)$ and its first time derivative at $t = 0$, and comparing to the initial displacement and velocity. The quantity $\omega_n\sqrt{1-\zeta^2}$ is generally referred to as the damped natural frequency and is written as ω_d.

Undamped, $\zeta = 0$

When damping is exceptionally light, expected to be very light, or where peak response at resonance is not a critical factor, the undamped solution

$$x(t) = A\sin(\omega_n t - \phi) \tag{1.28}$$

is often used.

1.1.4 Particular (Steady State) Solution to a Harmonic Excitation

Consider the equation

$$m\ddot{x} + c\dot{x} + kx = Y\cos(\omega_{dr}t) \tag{1.29}$$

The solution can be written as

$$x(t) = X_p\cos(\omega_{dr}t + \phi) \tag{1.30a}$$

or

$$x(t) = A\cos(\omega_{dr}) + B\sin(\omega_{dr}) \tag{1.30b}$$

where either the pair X_p and ω_{dr} of solution (1.30a) or the pair A and B of solution (1.30b) must be determined by substituting the assumed form of the solution $x(t)$ into equation (1.29). The solution can be particularly cumbersome, but can be obtained by substituting solution (1.30b) into equation

(1.29), solving for the unknown constants A and B, then using the equivalence of the solutions (1.30) along with trigonometric identities to obtain X_p and ϕ. Consider instead the equation

$$m\ddot{x} + c\dot{x} + kx = Y\sin(\omega_{dr}t) \tag{1.31}$$

for which the solution is similarly obtained. An alternative method of solution is to solve both equation (1.29) and equation (1.31) simultaneously. In order to keep track of the two solutions, the solution of equation (1.29) will be referred to as x_1, while the solution to equation (1.31) will be referred to as x_2. Multiplying equation (1.31) by $j = \sqrt{-1}$, and adding it to equation (1.29) yields

$$m(\ddot{x}_1 + \ddot{x}_2 j) + c(\dot{x}_1 + \dot{x}_2 j) + k(x_1 + x_2 j) = Y\left(\cos(\omega_{dr}t) + j\sin(\omega_{dr}t)\right) \tag{1.32}$$

or more simply as

$$m\ddot{x} + c\dot{x} + kx = Y\left(\cos(\omega_{dr}t) + j\sin(\omega_{dr}t)\right) \tag{1.33}$$

where

$$x = x_1 + jx_2 \tag{1.34}$$

Using the Euler relation, the right side of equation (1.33) can be written as

$$Y\left(\cos(\omega_{dr}t) + j\sin(\omega_{dr}t)\right) = Ye^{j\omega_{dr}t} \tag{1.35}$$

The particular solution to this equation is

$$x(t) = X_p^* e^{j\omega_{dr}t} \tag{1.36}$$

where the unknown constant coefficient X_p^* is complex (as the solution will demonstrate). Using $\frac{d}{dt}e^{j\omega_{dr}t} = j\omega_{dr}e^{j\omega_{dr}t}$ and $\frac{d^2}{dt^2}e^{j\omega_{dr}t} = -\omega_{dr}^2 e^{j\omega_{dr}t}$, then substituting (1.36) into equation (1.33) yields

$$mX_p^*(-\omega_{dr}^2)e^{j\omega_{dr}t} + cX_p^* j\omega_{dr}e^{j\omega_{dr}t} + kX_p^* e^{j\omega_{dr}t} = Ye^{j\omega_{dr}t} \tag{1.37}$$

factoring out $e^{j\omega_{dr}t}$ and solving for X_p^* yields

$$X_p^* = \frac{1}{-\omega_{dr}^2 m + j\omega_{dr}c + k} = \frac{(k - m\omega_{dr}^2) - j\omega_{dr}c}{(k - m\omega_{dr}^2)^2 + (c\omega_{dr})^2} = X_p e^{j\theta} \tag{1.38}$$

where X_p^* has been expressed in complex form with a magnitude X_p and a phase $\theta = \arctan\left(\frac{-c\omega_{dr}}{k - m\omega_{dr}^2}\right)$ using Euler's equation. Note that θ found here is identical to ϕ found from solving equation (1.29). Substituting for X_p in equation (1.36) gives

$$\begin{aligned} x(t) &= X_p e^{j\theta} e^{j\omega_{dr}t} \\ &= X_p e^{j(\omega_{dr}t + \theta)} \\ &= X_p\left(\cos(\omega_{dr}t + \theta) + j\sin(\omega_{dr}t + \theta)\right) \end{aligned} \tag{1.39}$$

Recalling equation (1.34) yields

$$x_1 = X_p\cos(\omega_{dr}t + \theta) \tag{1.40a}$$

$$x_2 = X_p \sin(\omega_{dr} t + \theta) \tag{1.40b}$$

Thus the solution to equation (1.29), represented by equation (1.40a) can more easily be obtained by:

1. Assuming the excitation is in the form of a complex exponential.
2. Assume the response is in the form of equation (1.36).
3. Substitute into the equation of motion. For this example, the equation of motion is given by equation (1.29), (1.31), or (1.33). Your equation will likely be different.
4. Obtain the amplitude of the response, X_p, and the phase, θ by factoring out $e^{j\omega_{dr}t}$ from each term and solving for X_p^*.
5. The solution will have the same trigonometric function as the excitation, will have an amplitude X_p, and a relative phase of θ.

Alternatively, a solution to Equation (1.29) can be obtained by using Equation (1.5). Substituting $\cos(\omega_{dr}t) = \frac{e^{\omega_{dr}tj}+e^{-\omega_{dr}tj}}{2}$ into Equation (1.29) yields

$$m\ddot{x} + c\dot{x} + kx = Y\left(\frac{e^{\omega_{dr}tj} + e^{-\omega_{dr}tj}}{2}\right) = \frac{Y}{2}e^{\omega_{dr}tj} + \frac{Y}{2}e^{-\omega_{dr}tj} \tag{1.41}$$

for which the solution is

$$x_p(t) = \frac{1}{2}X_p^* e^{\omega_{dr}tj} + \frac{1}{2}\bar{X}_p^* e^{-\omega_{dr}tj} \tag{1.42}$$

where $\bar{X}_p^*$ means complex conjugate of X_p^*, and X_p^* is given by Equation (1.38). Introducing the angle $\phi = \arctan\left(\frac{X_p''}{X_p'}\right)$, and manipulating by pre- and post-multiplying both terms by $e^{\phi j}$ and $e^{-\phi j}$ yields

$$\begin{aligned} x_p(t) &= \frac{1}{2}X_p^* e^{-\phi j} e^{\phi j} e^{\omega_{dr}tj} + \frac{1}{2}\bar{X}_p^* e^{\phi j} e^{-\phi j} e^{-\omega_{dr}tj} \\ &= \frac{1}{2}X_p e^{(\omega_{dr}t+\phi)j} + \frac{1}{2}X_p e^{\phi j} e^{-(\omega_{dr}t+\phi)j} \\ &= X_p \cos(\omega_{dr}t + \phi) \end{aligned} \tag{1.43}$$

Example 1.1 *Consider a single degree of freedom system attached to a sinusoidally moving base via a spring and a dashpot. The governing equation of motion for the system is*

$$m\ddot{x} + c\dot{x} + kx = ky(t) + c\dot{y}(t)$$

where $y(t) = Y\cos(\omega_{dr}t)$ is the motion of the base. Find $x(t)$ as a function of m, c, k, and Y. If we instead use $y(t) = Ye^{\omega_{dr}t} = Y(\cos(\omega_{dr}t) + j\sin(\omega_{dr}t))$, we can write the equation of motion as

$$m\ddot{x} + c\dot{x} + kx = Ye^{\omega_{dr}t}(k + cj\omega_{dr})$$

Assuming a form of the solution $x(t) = Xe^{j\omega_{dr}t}$, and substituting into the equation of motion, gives

$$X = Y\frac{k + cj\omega_{dr}}{k - m\omega_{dr}^2 + cj\omega_{dr}}$$

The magnitude and phase of the response are $|X|$ and $\sphericalangle X$ respectively, and the solution for $x(t)$ can now be written as

$$x(t) = \left(\frac{k^2 + (c\omega_{dr})^2}{(k - m\omega_{dr}^2)^2 + (c\omega_{dr})^2}\right)\cos(\omega_{dr}t + \phi)$$

where

$$\phi = \arctan\left(\frac{-mc\omega_{dr}^3}{k^2 + c^2 - km\omega_{dr}^2}\right)$$

1.2 Laplace Transform

A convenient way of solving linear differential equations it the *Laplace transform*. It does this by transforming differential equations into an algebraic ones in the *s-domain*, which in turn can be solve using algebraic tools. The solution is then transformed back into the time domain. The Laplace transform of a function $x(t)$ is defined as

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^\infty x(t)e^{-st}dt \tag{1.44}$$

where s is a complex variable. Using this method, it is presumed that the current states, or initial conditions, are known, and thus integrating before $t = 0$ is unnecessary. To demonstrate how a differential equation is turned into an algebraic one, it is instructive to consider the Laplace transform of $\dot{x}(t)$. Applying the definition of the Laplace transform, and using integration by parts,

$$\begin{aligned}\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} &= \int_0^\infty \frac{dx(t)}{dt}e^{-st}dt \\ &= x(t)e^{-st}\Big|_{t=0}^{t=\infty} - \int_0^\infty x(t)(-s)e^{-st}dt \\ &= -x(0) + s\mathcal{L}\{x(t)\}\end{aligned} \tag{1.45}$$

Thus, applying the Laplace transform to the single degree of freedom forced system gives

$$\begin{aligned}\mathcal{L}\{m\ddot{x} + c\dot{x} + kx\} &= \mathcal{L}\{f(t)\} \\ m\left(s^2X(s) - sx(0) - \dot{x}(0)\right) + c\left(sX(s) - x(0)\right) + kX(s) &= F(s)\end{aligned} \tag{1.46}$$

The solution for $x(t)$ is then obtained by taking the inverse of $X(s)$, or

$$X(s) = \frac{F(s) + msx(0) + m\dot{x}(0) + cx(0)}{ms^2 + cs + k} \tag{1.47}$$