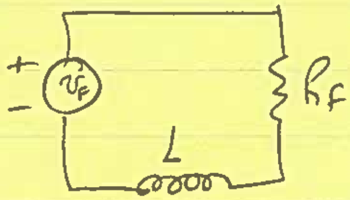
System Dynamics Fall 2015 Final Exam

1)

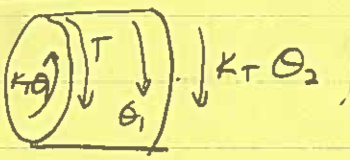


$$v_f - R_f i - L\dot{i} = 0$$

or

$$V_f(s) = (R + sL)I(s) \qquad (1)$$

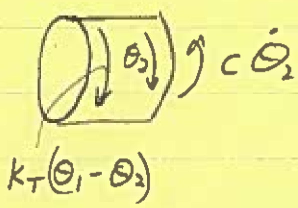
FBD 1:



$$\Sigma M = I_1\ddot{\theta}_1 = T + k_T\theta_2 - k_T\theta_1$$

$$(I_1 s^2 + k_T)\theta_1 - k_T\theta_2 = T(s) \qquad (2)$$

FBD 2:



$$\Sigma M = I_2\ddot{\theta}_2 = k_T(\theta_1 - \theta_2) - c\dot{\theta}_2$$

$$(I_2 s^2 + cs + k_T)\theta_2 = k_T\theta_1 \qquad (3)$$

Note: $T(s) = I k_t$ (4)

lower-case t for torque constant

Solving 3 for θ_1

$$\theta_1 = \frac{1}{K_T}\left(I_2 s^2 + cs + K_T\right)\theta_2 \qquad (5)$$

Substituting 5 into 2

$$\left(\left(I_1 s^2 + K_T\right)\frac{1}{K_T}\left(I_2 s^2 + cs + K_T\right) - K_T\right)\theta_2 = T$$

$$\left(\frac{I_1 I_2}{K_T} s^4 + \frac{I_1 c}{K_T} s^3 + (I_1 + I_2)s^2 + cs\right)\theta_2 = T \qquad (6)$$

Combining 4 and 1

$$T(s) = I K_t = \frac{K_t}{R + sL} V_f \qquad (7)$$

Substituting 7 into 6

$$\left(\frac{I_1 I_2}{K_T} s^4 + \frac{I_1 c}{K_T} s^3 + (I_1 + I_2)s^2 + cs\right)\theta_2 = \frac{K_t}{R + sL} V_f \qquad (8)$$

Simplifying

$$\boxed{\begin{aligned}\frac{\theta_2}{V_f} &= \frac{K_T K_t}{(R + sL)\left(I_1 I_2 s^4 + I_1 c s^3 + (I_1 + I_2) K_T s^2 + c K_T s\right)} \\ &= \frac{K_T K_t}{\begin{aligned}&I_1 I_2 s^5 + (R I_1 I_2 + L I_1 c)s^4 + [R I_1 c + L K_T (I_1 + I_2)]s^3 \\ &+ (R K_T (I_1 + I_2) + L c K_T)s^2 \cdots \\ &+ R c K_T s\end{aligned}}\end{aligned}}$$

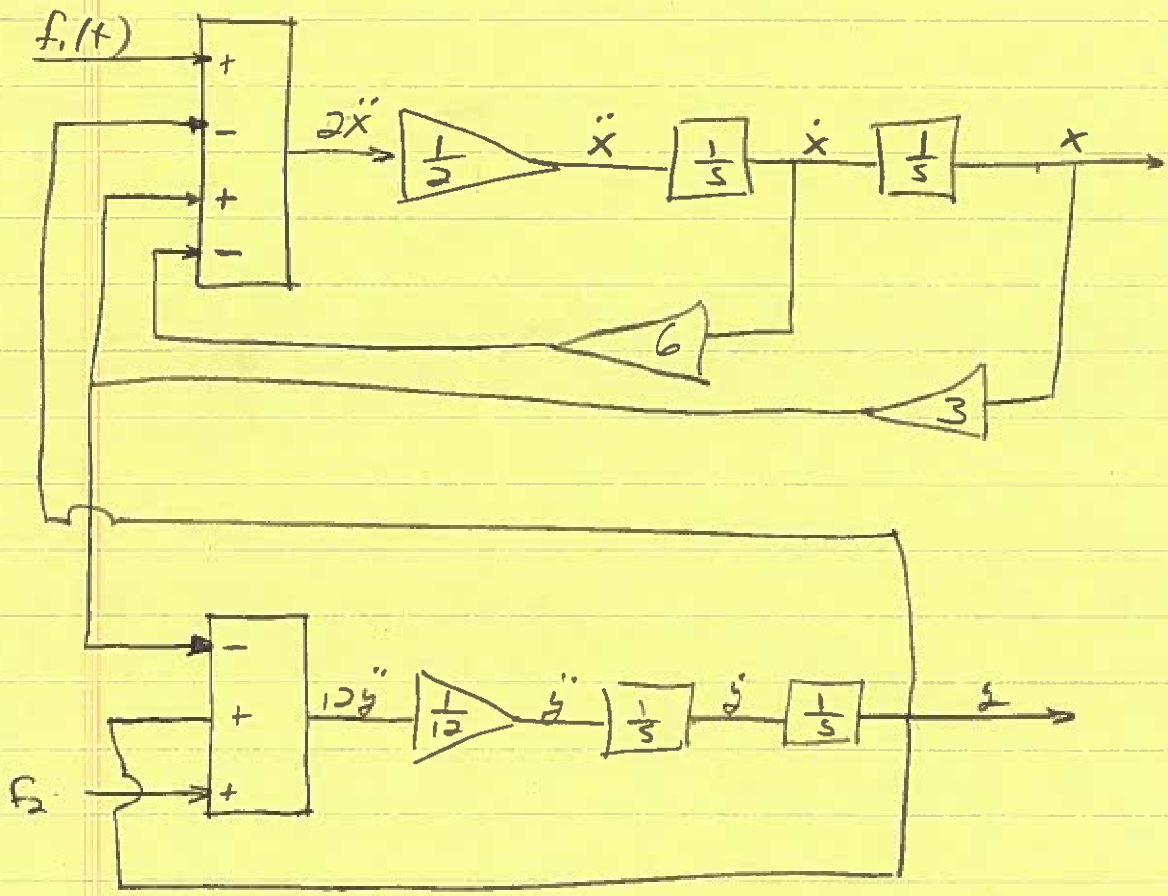
Getting the algebra correct is difficult. Full credit if your answer shows no obvious errors, or you identify them. Bonus if correct.

2) $2\ddot{x} + 6\dot{x} - 3x + y = f_1(t)$

$12\ddot{y} - y + 3x = f_2(t)$

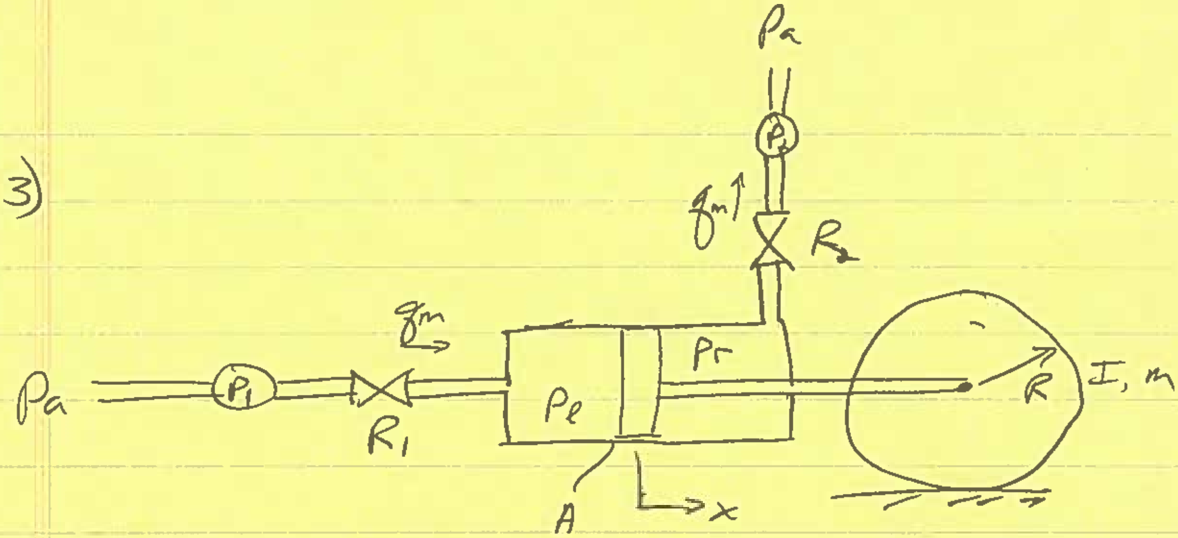
$$2\ddot{x} = f_1(t) - 6\dot{x} + 3x - y$$

$$12\ddot{y} = f_2(t) - 3x + y$$



It is clearly unstable because the coefficient to x in the 1st equation is negative. Further, the coefficient to y is negative in the second eqn.

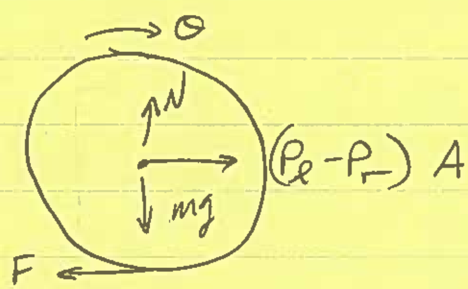
3)



Using all gauge pressures (all derived pressures are 0 at p_a)

$$(p_e - p_1) = -R_1 q_m \qquad (p_r - p_2) = R_2 q_m$$

$$\frac{1}{\rho} q_m = A\dot{x} \qquad \text{(conservation of volume)}$$



$$\Sigma F = m\ddot{x} = (p_e - p_r)A - F$$

$$\Sigma M = I\ddot{\alpha} = FR$$

$$F = \frac{I}{R}\ddot{\alpha}$$

so

$$m\ddot{x} + \frac{I}{R}\ddot{\alpha} = (p_e - p_r)A$$

since $\ddot{\alpha}R = \ddot{x}$

$$\left(m + \frac{I}{R^2}\right)\ddot{x} = (p_e - p_r)A$$

Since $p_\ell = p_1 - R_1 q_m$ and $p_r = p_2 + R_2 q_m$

$$\left(m + \frac{I}{R^2}\right)\ddot{x} = (p_1 - R_1 q_m - p_2 - R_2 q_m) A$$

$$= (p_1 - p_2)A - A(R_1 + R_2) q_m$$

Since $q_m = \rho A \dot{x}$

$$\left(m + \frac{I}{R^2}\right)\ddot{x} + (R_1 + R_2)\rho A^2 \dot{x} = (p_1 - p_2) A$$

b)
$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & \frac{-(R_1+R_2)\rho A^2}{m + \frac{I}{R^2}} \end{bmatrix}}_{A} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ \frac{A}{m+\frac{I}{R^2}} & \frac{-A}{m+\frac{I}{R^2}} \end{bmatrix}}_{B} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ p_\ell \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -R\rho A \end{bmatrix}}_{C} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{D} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

because $p_\ell = p_1 - R q_m = p_1 - R\rho A \dot{x}$