Householder's Method (Reduces symmetric matrix to tri-diagonal form)

More efficient than Givens Method
(half as many multiplications)

1) Annihilates whole row and column at a time.
3) $n-2$ transformations are used to annihilate $n-2$ rows.
2) No successive transformation affects previous rows + columns already reduced.

Consider the following transformation

$$A_k = P_k A_{k-1} P_k \qquad (A_0 = A)$$

where

$$P_k = I - 2\,\underline{v}_k \underline{v}_k^T, \quad \underline{v}_k^T \underline{v}_k = 1$$

is called the Householder transformation matrix and is symmetric and orthonormal.
The matrix A_1, corresponding to $k=1$, must have the form

$$A_1 = P_1 A_0 P_1$$

(transform row and column 1)

$$= \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & 0 & 0 & 0 & \cdots & 0 \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & & & \cdots & a_{2n} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & & & & \vdots \\ 0 & a_{42} & & & & & \vdots \\ \vdots & & & & & & \\ 0 & a_{n2} & \cdots & & & & a_{nn}^{(1)} \end{bmatrix}$$

The requirement is then

$$a_{31}^{(1)} = a_{41}^{(1)} = \cdots = a_{n1}^{(1)} = 0 \quad \text{with } \underline{v}_1^T \underline{v}_1 = 1$$

This is $n-1$ constraints ~~(n-2+1)~~.
Hence, 1 element of $\underline{v}_1$ may be choosen

Let's pick $v_{1,1} = 0$ so

$$P_1 = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1-2v_{1,2}^2 & -2v_{1,2}v_{1,3} & \cdots & -2v_{1,2}v_{1,n} \\ \vdots & -2v_{1,2}v_{1,3} & 1-2v_{1,3}^2 & & \\ & & & \ddots & \\ 0 & -2v_{1,2}v_{1,n} & & & 1-2v_{1,n}^2 \end{bmatrix}$$

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Consider the product

$$A_1 = P_1 A_0 P_1 = P_1 A P_1$$

The first row of $P_1 A$ is simply the first row of A_0. Denote this row as $a_1^{(k)T}$.

The first row of A_1 is

$$\underline{a}_1^{(1)T} = \underline{a}_1^{T} P = \underline{a}_1^T - 2(\underline{a}_1^T \underline{v}_1)\,\underline{v}_1^T$$

where $\underline{a}_1^T = \underline{a}_1^{(0)T}$ is the 1^{st} row of A.

This implies

$$a_{11}^{(1)} = a_{11}^{(0)}, \quad a_{1j}^{(1)} = a_{1j}^{(0)} - 2(\underline{a}_1^T \underline{v}_1)\, v_{1,j} \qquad (1)$$
$$j = 2, \ldots n$$

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From this we conclude

$$a_{1j} - 2(\underline{a}_1^T \underline{v}_1)\, v_{1j} = 0 \qquad j = 3, \ldots \quad ②$$

and $v_{1,2}^2 + v_{1,3}^2 + \ldots v_{1,n}^2 = 1$

Note $\underline{a}_1^{(1)T} \underline{a}_1^{(1)} = \underline{a}_1^{(0)T} P_1 P_1 \underline{a}_1^{(0)}$

$$= \underline{a}_1^{(0)T} \underline{a}_1^{(0)}$$

Because $a_{11}^{(1)} = a_{11}$ we must have

$$\left(a_{12}^{(1)}\right)^2 = \sum_{j=2}^{n} a_{1j}^2 = \alpha_1^2 \quad ③$$

where $\alpha_1 = \left(\sum_{j=2}^{n} a_{1j}^2\right)^{1/2}$ is regarded as a positive constant.

From ① and ③

$$a_{12}^{(1)} = a_{12}^{(0)} - 2(\underline{a}_1^T \underline{v}_1)\, v_{1,2} = \pm \alpha_1 \quad ④$$

Multiply ③ by $v_{1,2}$ and ② by $v_{1,j}$ ($j = 3, \ldots n$) and add the results

$$\sum_{j=1}^{n} \left(\left[a_{1j} - 2(\underline{a}_1^T \underline{v}_1)\, v_{1,j} \right] v_{1,j} \right) = \pm \alpha_1 v_{1,2}$$

which can be reduced to

$$\underline{a}_1^T \underline{v}_1 - 2(\underline{a}_1^T \underline{v}_1)\underbrace{(\underline{v}_1^T \underline{v}_1)}_{1} = \pm \alpha_1 v_{1,2}$$

Because the vector $\underline{v}_1$ has unit length, this reduces to

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$$\underline{a}_1^T \underline{v}_1 = \mp \alpha_1 v_{1,2} \qquad (5)$$

Subst (5) into (4)

$$v_{1,2} = \left[\frac{1}{2}\left(1 \mp \frac{a_{12}}{\alpha_1}\right)\right]^{1/2} \qquad (6)$$

which permits us to obtain $v_{1,j}$

Subst (5) into (2) gives

$$v_{1,j} = \mp \frac{a_{1j}}{2\alpha_1 v_{1,2}} \qquad (7)$$

where $v_{1,2}$ is found from (6)

The sign of (6) should be chosen to be the same as a_{12} to avoid obtaining zero as a result for use in (7). The choice of signs in (6) and (7) must be the same. Thus $\underline{v}$ is now defined.

~~The procedure~~

The procedure can be generalized to

$$\underline{V}_k = [0 \quad 0 \quad \cdots \quad 0 \quad V_{k,k+1} \quad V_{k,k+2} \cdots V_{k,n}]^T$$

$$V_{k,k+1} = \left[\frac{1}{2}\left(1 \mp \frac{a_{k,k+1}^{(k-1)}}{\alpha_k}\right)\right]^{1/2}$$

$$V_{k,j} = \mp \frac{a_{kj}^{(k-1)}}{2\alpha_k V_{k,k+1}} \qquad j = k+2, k+3, \ldots n$$

where

$$\alpha_k = \left[\sum_{j=k+1}^{n} \left(a_{kj}^{(k-1)}\right)^2\right]^{1/2} \qquad k = 1, 2, \ldots n-2$$

This takes $n-2$ iterations. The eigenvectors of $A(\underline{u})$ can be related to the eigenvectors of $A_k(\underline{x})$ by

$$\underline{u} = P\,\underline{x} \quad \text{where} \quad P = \prod_{i=1}^{k} P_i$$

$$A_1 = \begin{bmatrix} 1 & -1.732 & 0 & 0 \\ & 7.667 & -.544 & -1.122 \\ \text{sym} & & .328 & .167 \\ & & & .955 \end{bmatrix}$$

$k=2$

$$\alpha_2 = \left[\sum_{i=3}^{4} a_{2i}^{(1)\,2}\right]^{1/2} = \left(.544^2 + 1.122^2\right)^{1/2} = 1.247$$

$$V_{2,3} = \left[\frac{1}{2}\left(1 \mp \frac{a_{23}^{(1)}}{\alpha_2}\right)\right]^{1/2} = \left(\frac{1}{2}\left(1 + \frac{.544}{1.247}\right)\right)^{1/2} = .847$$

$$V_{2,4} = \mp \frac{a_{24}}{2\alpha_2 V_{2,3}} = \frac{1.122}{2 \cdot .847 \cdot 1.247} = .531$$

$$\underline{V}_2 = \begin{bmatrix} 0 & 0 & .847 & .531 \end{bmatrix}^T$$

$$P_2 = I - 2\underline{V}_2\underline{V}_2^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & -.437 & -.900 \\ & & & .437 \end{bmatrix}$$

$$A_2 = P_2 A_1 P_2 = \begin{bmatrix} 1 & -1.732 & 0 & 0 \\ & 7.667 & 1.247 & 0 \\ & & .976 & -.124 \\ & & & .357 \end{bmatrix}$$

Example

Tri diagonalize the following.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$k=1$

$$\alpha_1 = \left[\sum_{j=2}^{4} \left(a_{1j}^{(0)}\right)^2 \right]^{1/2} = \left(1^2 + 1^2 + 1^2\right)^{1/2} = \sqrt{3}$$

$$V_{1,2} = \left[\frac{1}{2}\left(1 \pm \frac{a_{12}^{(0)}}{\alpha_1}\right)\right]^{1/2} = \left[\frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right)\right]^{1/2} = .888$$

$$V_{1,3} = \mp \frac{a_{13}^{(0)}}{2\alpha_1 V_{1,2}} = \frac{1}{2\sqrt{3}\ .888} = .325$$

$$V_{1,4} = \frac{a_{14}}{2\alpha_1 V_{1,2}} = .325$$

$$\underline{V_1} = \begin{bmatrix} 0 \\ .888 \\ .325 \\ .325 \end{bmatrix}$$

$$P_1 = I - 2\underline{V_1}^T \underline{V_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -.577 & -.5773 & -.577 \\ 0 & & .789 & -.211 \\ 0 & \text{sym} & & .789 \end{bmatrix}$$

$$A_1 = P_1 A_0 P_1$$