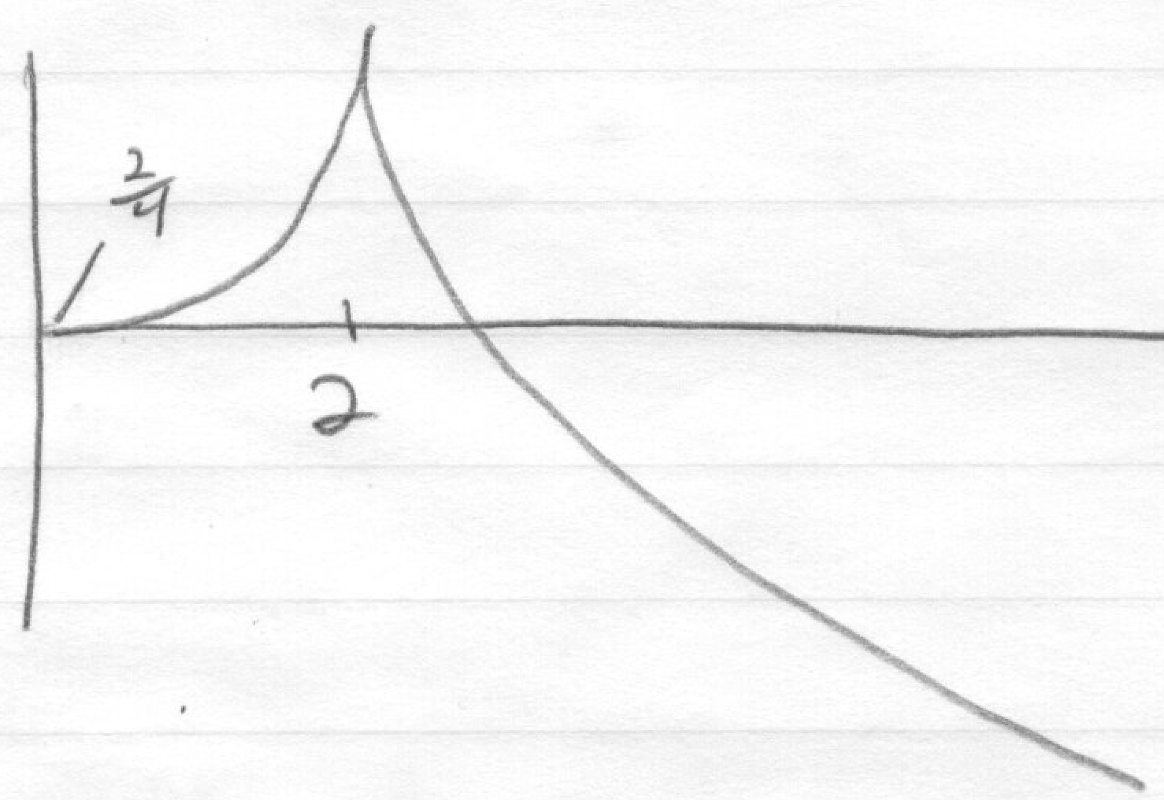
1) $$H(j\omega) = \frac{-\omega^2}{1000 - 10\omega^2 + .01 j\omega}$$

2) Natural frequencies are $\omega_1 = 2$ rad/s , $\omega_2 = 3$ rad/s
Damping ratios are $\zeta_1 = .01$, $\zeta_2 = .008$

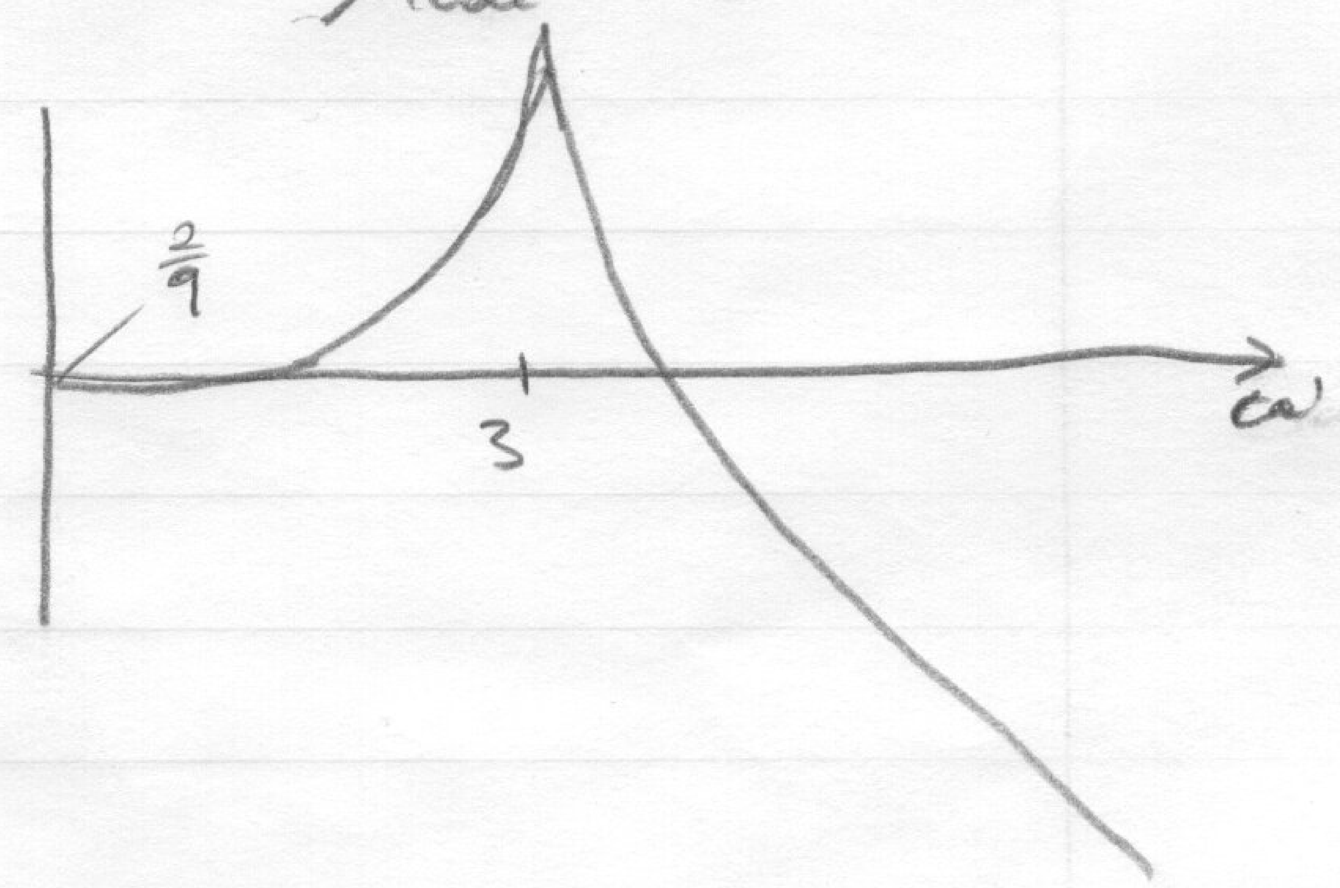
$$H_{12} = \frac{2}{4 + .04 j\omega - \omega^2} + \frac{-2}{9 + .05 j\omega - \omega^2}$$

$${}_1A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad {}_2A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

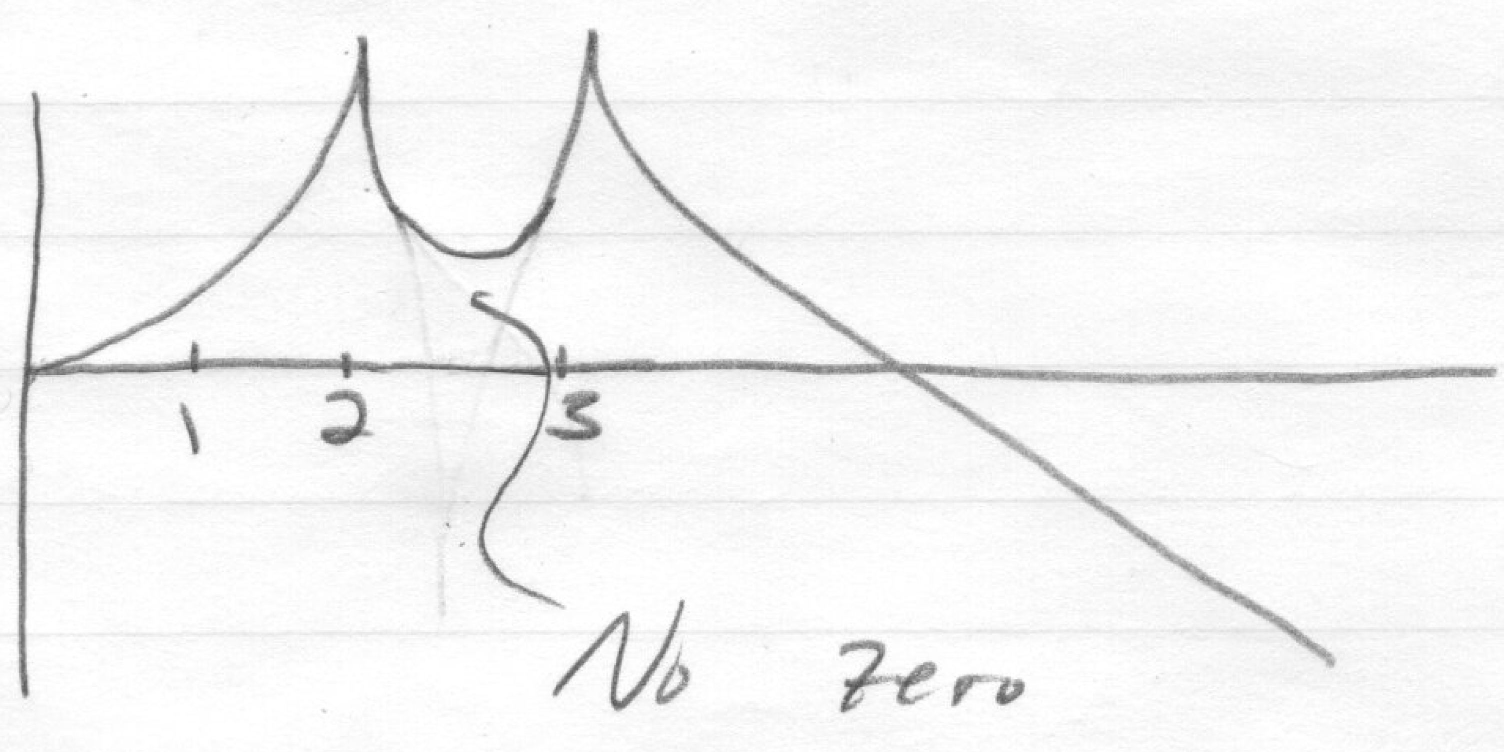
Mode 1



Mode 2



Net is difference



3) $f(t) = \sum_{n=1}^{\infty} b_n \sin \frac{2\pi n t}{T}$ ($a_n = 0$ because function is odd)

$$b_n = \frac{2}{T} \int_0^T f(t) \sin \frac{2\pi n t}{T} \, dt$$

$$= \frac{1}{2}\left[\int_0^2 \sin \frac{2\pi n t}{4} \, dt - \int_2^4 \sin \frac{2\pi n t}{4} \, dt\right]$$

$$= \frac{1}{2}\left[\frac{-2}{n\pi} \cos \frac{2\pi n}{4} t \Big|_0^2 - \frac{-2}{n\pi} \cos \frac{2\pi n}{4} t \Big|_2^4\right]$$

$$= \frac{-1}{2n\pi}\left[(\cos \pi n - \cos 0) - (\cos 2\pi n - \cos \pi n)\right]$$

$$= \frac{-1}{2n\pi}\left[-1 + 2\cos n\pi - \cos 2n\pi\right]$$

$$= \frac{2}{2n\pi}\left[1 - \cos n\pi\right]$$

$$= \frac{4}{n\pi} \qquad n = 1, 3, 5, 7 \ldots$$

$$f(t) = \sum_{n=1,3,5}^{\infty} \frac{4}{n\pi} \sin \frac{2\pi n t}{4}$$

4) a)

$$\dot{z} = -\frac{16}{9} z + \frac{1}{9} f(t)$$

$$y = z$$

b)

$$z(t) = z_0 e^{at}, \quad \dot{z} = a e^{at}$$

$$a e^{at} = -\frac{16}{9} e^{at}$$

$$\therefore z(t) = z_0 e^{-\frac{16}{9} t}$$

$$S(\tau) = e^{-\frac{16}{9} \tau}$$

5) Controllability matrix is

$$\mathcal{C} = [B \quad AB \quad A^2B \quad A^3B]$$

System is controllable if rank $\mathcal{C} = 4$

$$\mathcal{C} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \\ 1 & 0 & -2 \end{bmatrix} \cdots$$

system is already uncontrollable because rank < 4

Note

$$A^2 = \begin{bmatrix} [0] & I \\ A' & [0] \end{bmatrix} \begin{bmatrix} [0] & I \\ A' & [0] \end{bmatrix} = \begin{bmatrix} A' & [0] \\ [0] & A' \end{bmatrix}$$

$$\therefore A^2B = \begin{bmatrix} 0 \\ 0 \\ A'\begin{bmatrix}1\\1\end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ -2 \end{bmatrix}$$

Solution 2:

Recognize system could be

$$I\ddot{\underline{x}} + \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

Because of symmetry of system (or eigen solution)

$$\underline{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{2}}, \quad \underline{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\frac{1}{\sqrt{2}}$$

Transforming to modal coordinates

$$I\ddot{\underline{r}} + \overset{\text{NOT Needed}}{\Lambda}\,\underline{r} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} u(t)$$

2^{nd} mode is uncontrollable.