transfer function is the amplitude ratio between the input and output, and its argument is the phase shift $\phi$. Both are functions of $\omega$ and usually plotted against $\log \omega$, with $M$ expressed in decibels as $m = 20 \log M$. The logarithmic plots can be sketched by using low- and high-frequency asymptotes intersecting at the corner frequencies.

Most transfer functions consist of a combination of the following terms: $K$, $s$, $\tau s + 1$, and $s^2 + 2\zeta\omega_n s + \omega_n^2$, so we analyzed the frequency response of each term. The composite frequency response plot consists of the sum of the plots of the numerator terms minus the sum of the plots of the denominator terms.

Important engineering applications of frequency response covered in this chapter are an understanding of resonance and bandwidth. The bandwidth measures the filtering property of the system. This concept can be used with the Fourier series representation of a periodic function, which is an infinite series, by eliminating those terms that lie outside the bandwidth. The truncated series can be used to compute the response to a general periodic input.

The form of the transfer function and an estimate of the numerical values of the parameters can often be obtained from experiments involving frequency response.

The MATLAB Control Systems toolbox provides the `bode`, `bodemag`, `evalfr`, and `freresp` functions, which are useful for frequency response analysis.

Now that you have finished this chapter, you should be able to

1. Sketch frequency response plots using asymptotes, and use the plots or the frequency transfer function to determine the steady-state output amplitude and phase that results from a sinusoidal input.
2. Compute resonance frequencies and bandwidth.
3. Analyze vibration isolation and absorber systems, and the effects of base motion.
4. Determine the steady-state response to a periodic input, given the Fourier series description of the input.
5. Estimate the form of a transfer function and its parameter values, given frequency response data.
6. Use MATLAB as an aid in the preceding tasks.

## PROBLEMS

### Section 9.1 Frequency Response of First-Order Systems

**9.1** Use the following transfer functions to find the steady-state response $y_{ss}(t)$ to the given input function $f(t)$.

a. $$T(s) = \frac{Y(s)}{F(s)} = \frac{25}{14s + 18}, \quad f(t) = 15 \sin 1.5t$$

b. $$T(s) = \frac{Y(s)}{F(s)} = \frac{15s}{3s + 4}, \quad f(t) = 5 \sin 2t$$

c. $$T(s) = \frac{Y(s)}{F(s)} = \frac{s + 50}{s + 150}, \quad f(t) = 3 \sin 100t$$

d. $$T(s) = \frac{Y(s)}{F(s)} = \frac{33\ s + 100}{200\ s + 33}, \quad f(t) = 8 \sin 50t$$



**9.2** Use asymptotic approximations to sketch the frequency response plots for the following transfer functions.

a. $$T(s) = \frac{15}{6s + 2}$$

b. $$T(s) = \frac{9s}{8s + 4}$$

c. $$T(s) = 6\frac{14s + 7}{10s + 2}$$

**9.3** Figure P9.3 is a representation of the effects of the tide on a small body of water connected to the ocean by a channel. Assume that the ocean height $h_i$ varies sinusoidally with a period of 12 hr with an amplitude of 3 ft about a mean height of 10 ft. If the observed amplitude of variation of $\hat{h}$ is 2 ft, determine the time constant of the system and the time lag between a peak in $h_i$ and a peak in $\hat{h}$.

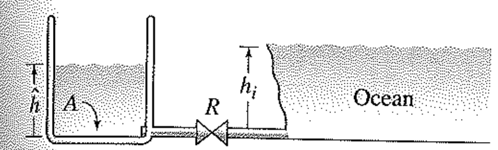



**Figure P9.3**

**9.4** A single-room building has four identical exterior walls, 5 m wide by 3 m high, with a perfectly insulated roof and floor. The thermal resistance of the walls is $R = 4.5 \times 10^{-3}$ K/W · m$^2$. Taking the only significant thermal capacitance to be the room air, obtain the expression for the steady-state room air temperature if the outside air temperature varies sinusoidally about 15°C with an amplitude of 5° and a period of 24 h. The specific heat and density of air at these conditions are $c_p = 1004$ J/kg · K and $\rho = 1.289$ kg/m$^3$.

**9.5** For the rotational system shown in Figure P9.5, $I = 2$ kg · m$^2$ and $c = 4$ N · m · s/rad. Obtain the transfer function $\Omega(s)/T(s)$, and derive the expression for the steady-state speed $\omega_{ss}(t)$ if the applied torque in N · m is given by

$$T(t) = 30 + 5 \sin 3t + 2 \cos 5t$$

**9.6** For the system shown in Figure P9.6, the bottom area is $A = 4\pi$ ft$^2$ and the linear resistance is $R = 1500$ sec$^{-1}$ft$^{-1}$. Suppose the volume inflow rate is

$$q_{vi}(t) = 0.2 + 0.1 \sin 0.002t \text{ ft}^3\text{/sec}$$

Obtain the expression for the steady-state height $\hat{h}_{ss}(t)$. Compute the lag in seconds between a peak in $q_{vi}(t)$ and a peak in $\hat{h}_{ss}(t)$.

**Figure P9.5**

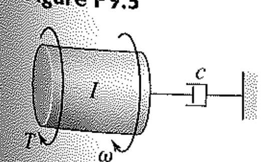
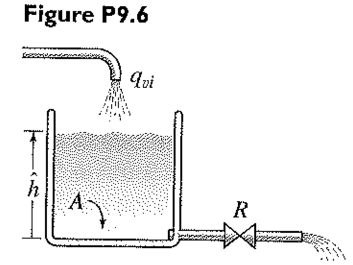



**Figure P9.6**



## Section 9.2 Frequency Response of Higher-Order Systems

**9.7** Use the following transfer functions to find the steady-state response $y_{ss}(t)$ to the given input function $f(t)$.

a. $$T(s) = \frac{Y(s)}{F(s)} = \frac{10}{(10s+1)(4s+1)}, \quad f(t) = 10\sin 0.2t$$

b. $$T(s) = \frac{Y(s)}{F(s)} = \frac{1}{2s^2+20s+200}, \quad f(t) = 16\sin 5t$$

**9.8** Use the following transfer functions to find the steady-state response $y_{ss}(t)$ to the given input function $f(t)$.

a. $$T(s) = \frac{Y(s)}{F(s)} = \frac{8}{s(s^2+10s+100)}, \quad f(t) = 6\sin 9t$$

b. $$T(s) = \frac{Y(s)}{F(s)} = \frac{10}{s^2(s+1)}, \quad f(t) = 9\sin 2t$$

c. $$T(s) = \frac{Y(s)}{F(s)} = \frac{s}{(2s+1)(5s+1)}, \quad f(t) = 9\sin 0.7t$$

d. $$T(s) = \frac{Y(s)}{F(s)} = \frac{s^2}{(2s+1)(5s+1)}, \quad f(t) = 9\sin 0.7t$$

**9.9** The model of a certain mass-spring-damper system is

$$10\ddot{x} + c\dot{x} + 20x = f(t)$$

Determine its resonant frequency $\omega_r$ and its peak magnitude $M_r$ if (a) $\zeta = 0.1$ and (b) $\zeta = 0.3$.

**9.10** The model of a certain mass-spring-damper system is

$$10\ddot{x} + c\dot{x} + 20x = f(t)$$

How large must the damping constant $c$ be so that the maximum steady-state amplitude of $x$ is no greater than 3, if the input is $f(t) = 11\sin\omega t$, for an arbitrary value of $\omega$?

**9.11** The model of a certain mass-spring-damper system is

$$13\ddot{x} + 2\dot{x} + kx = 10\sin\omega t$$

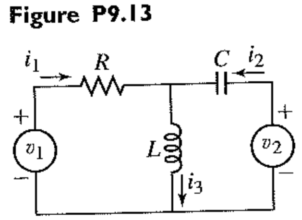
Determine the value of $k$ required so that the maximum response occurs at $\omega = 4$ rad/sec. Obtain the steady-state response at that frequency.

**9.12** Determine the resonant frequencies of the following models.

a. $$T(s) = \frac{7}{s(s^2+6s+58)}$$

b. $$T(s) = \frac{7}{(3s^2+18s+174)(2s^2+8s+58)}$$

**Figure P9.13**

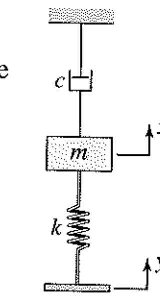



**9.13** For the circuit shown in Figure P9.13, $L = 0.1$ H, $C = 10^{-6}$ F, and $R = 100\ \Omega$. Obtain the transfer functions $I_3(s)/V_1(s)$ and $I_3(s)/V_2(s)$. Using asymptotic approximations, sketch the $m$ curves for each transfer function and discuss how the circuit acts on each input voltage (does it act like a low-pass filter, a high-pass filter, or other?).



**9.14** (a) For the system shown in Figure P9.14, $m = 1$ kg and $k = 600$ N/m. Derive the expression for the peak amplitude ratio $M_r$ and resonant frequency $\omega_r$, and discuss the effect of the damping $c$ on $M_r$ and on $\omega_r$. (b) Extend the derivation of the expressions for $M_r$ and $\omega_r$ to the case where the values of $m$, $c$, and $k$ are arbitrary.

**Figure P9.14**




**9.15** For the *RLC* circuit shown in Figure P9.15, $C = 10^{-5}$ F and $L = 5 \times 10^{-3}$ H. Consider two cases: (a) $R = 10\ \Omega$ and (b) $R = 1000\ \Omega$. Obtain the transfer function $V_o(s)/V_s(s)$ and the log magnitude plot for each case. Discuss how the value of $R$ affects the filtering characteristics of the system.

**9.16** A model of a fluid clutch is shown in Figure P9.16. Using the values $I_1 = I_2 = 0.02\ \text{kg}\cdot\text{m}^2$, $c_1 = 0.04\ \text{N}\cdot\text{m}\cdot\text{s/rad}$, and $c_2 = 0.02\ \text{N}\cdot\text{m}\cdot\text{s/rad}$, obtain the transfer function $\Omega_2(s)/T_1(s)$, and derive the expression for the steady-state speed $\omega_2(t)$ if the applied torque in N · m is given by

$$T_1(t) = 4 + 2\sin 1.5t + 0.9\sin 2t$$

**Figure P9.15**

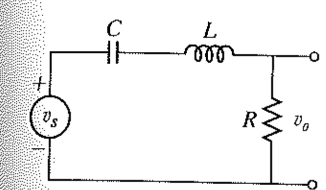
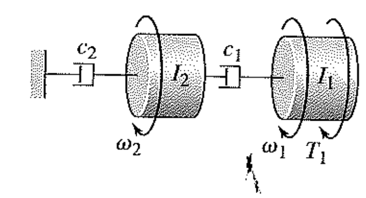



**Figure P9.16**




## Section 9.3 Frequency Response Applications

**9.17** Determine the beat period and the vibration period of the model

$$3\ddot{x} + 75x = 7\sin 5.2t$$

**9.18** Resonance will produce large vibration amplitudes, which can lead to system failure. For a system described by the model

$$\ddot{x} + 64x = 0.2\sin\omega t$$

where $x$ is in feet, how long will it take before $|x|$ exceeds 0.1 ft, if the forcing frequency $\omega$ is close to the natural frequency?

**9.19** The quarter-car weight of a certain vehicle is 625 lb and the weight of the associated wheel and axle is 190 lb. The suspension stiffness is 8000 lb/ft and the tire stiffness is 10,000 lb/ft. If the amplitude of variation of the road surface is 0.25 ft with a period of 20 ft, determine the critical (resonant) speeds of this vehicle.

**9.20** A certain factory contains a heavy rotating machine that causes the factory floor to vibrate. We want to operate another piece of equipment nearby and we measure the amplitude of the floor's motion at that point to be 0.01 m. The mass of the equipment is 1500 kg and its support has a stiffness of $k = 2 \times 10^4$ N/m and a damping ratio of $\zeta = 0.04$. Calculate the maximum force that will be transmitted to the equipment at resonance.

**9.21** An electronics module inside an aircraft must be mounted on an elastic pad to protect it from vibration of the airframe. The largest amplitude vibration produced by the airframe's motion has a frequency of 40 cycles per second. The module weighs 200 N, and its amplitude of motion is limited to 0.003 m because of space. Neglect damping and calculate the percent of the airframe's motion transmitted to the module.

**9.22** An electronics module used to control a large crane must be isolated from the crane's motion. The module weighs 2 lb. (a) Design an isolator so that no more than 10% of the crane's motion amplitude is transmitted to the module. The crane's vibration frequency is 3000 rpm. (b) What percentage of the crane's motion will be transmitted to the module if the crane's frequency can be anywhere between 2500 and 3500 rpm?

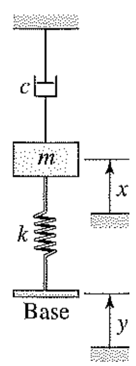
**9.23** Design a vibrometer having a mass of 0.1 kg, to measure displacements having a frequency near 200 Hz.

**9.24** A motor mounted on a beam vibrates too much when it runs at a speed of 6000 rpm. At that speed the measured force produced on the beam is 60 lb. Design a vibration absorber to attach to the beam. Because of space limitations, the absorber's mass cannot have an amplitude of motion greater than 0.08 in.

**9.25** The supporting table of a radial saw weighs 160 lb. When the saw operates at 200 rpm, it transmits a force of 4 lb to the table. Design a vibration absorber to be attached underneath the table. The absorber's mass cannot vibrate with an amplitude greater than 1 in.

## Section 9.4 Filtering Properties of Dynamic Systems

**Figure P9.26**




**9.26** A certain mass is driven by base excitation through a spring (see Figure P9.26). Its parameter values are $m = 200$ kg, $c = 2000$ N · s/m, and $k = 2 \times 10^4$ N/m. Determine its resonant frequency $\omega_r$, its resonance peak $M_r$, and its bandwidth.

**9.27** A certain series $RLC$ circuit has the following transfer function.

$$T(s) = \frac{I(s)}{V(s)} = \frac{Cs}{LCs^2 + RCs + 1}$$

Suppose that $L = 300$ H, $R = 10^4\ \Omega$, and $C = 10^{-6}$ F. Find the bandwidth of this system.

**9.28** Obtain the expressions for the bandwidths of the two circuits shown in Figure P9.28.

**Figure P9.28**

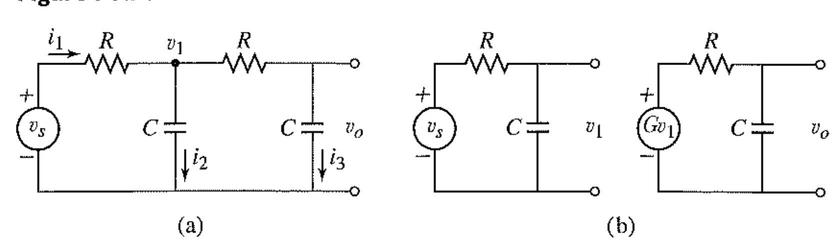
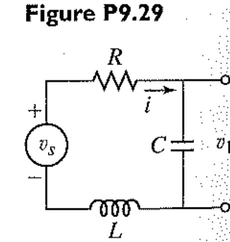



**Figure P9.29**




**9.29** For the circuit shown in Figure P9.29, $L = 0.1$ H and $C = 10^{-6}$ F. Investigate the effect of the resistance $R$ on the bandwidth, resonant frequency, and resonant peak over the range $100 \le R \le 1000\ \Omega$.

**9.30** For the circuit shown in Figure 9.1.7a, can values be found for $R_1$, $R_2$, and $C$ to make a low-pass filter? Prove your answer mathematically.



## Section 9.5 Response to Nonperiodic Inputs

**9.31** The voltage shown in Figure P9.31 is produced by applying a sinusoidal voltage to a *full wave rectifier*. The Fourier series approximation to this function is

$$v_s(t) = \frac{20}{\pi} - \frac{40}{\pi}\left[\frac{\cos 240\pi t}{1(3)} + \frac{\cos 480\pi t}{3(5)} + \frac{\cos 720\pi t}{5(7)} + \cdots\right]$$

Suppose this voltage is applied to a series $RC$ circuit whose transfer function is

$$\frac{V_o(s)}{V_s(s)} = \frac{1}{RCs + 1}$$

where $R = 600\ \Omega$ and $C = 10^{-6}$ F. Keeping only those terms in the Fourier series whose frequencies lie within the circuit's bandwidth, obtain the expression for the steady-state voltage $v_o(t)$.

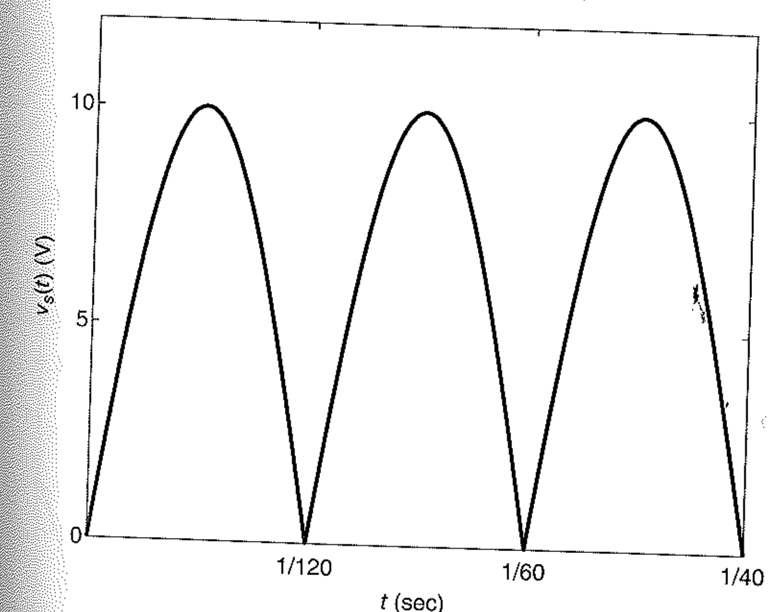



**Figure P9.31**

**9.32** The voltage shown in Figure P9.32 is called a square wave. The Fourier series approximation to this function is

$$v_s(t) = 5\left[1 + \frac{4}{\pi}\left(\frac{\sin 120\pi t}{1} + \frac{\sin 360\pi t}{3} + \frac{\sin 600\pi t}{5} + \cdots\right)\right]$$

Suppose this voltage is applied to a series $RC$ circuit whose transfer function is

$$\frac{V_o(s)}{V_s(s)} = \frac{1}{RCs + 1}$$

where $R = 10^3\ \Omega$ and $C = 10^{-6}$ F. Keeping only those terms in the Fourier series whose frequencies lie within the circuit's bandwidth, obtain the expression for the steady-state voltage $v_o(t)$.

**9.33** The displacement shown in Figure P9.33a is produced by the cam shown in part (b) of the figure. The Fourier series approximation to this function is

$$y(t) = \frac{1}{20\pi}\left[\pi - 2\left(\frac{\sin 10\pi t}{1} + \frac{\sin 20\pi t}{2} + \frac{\sin 30\pi t}{3} + \cdots\right)\right]$$

Figure P9.32

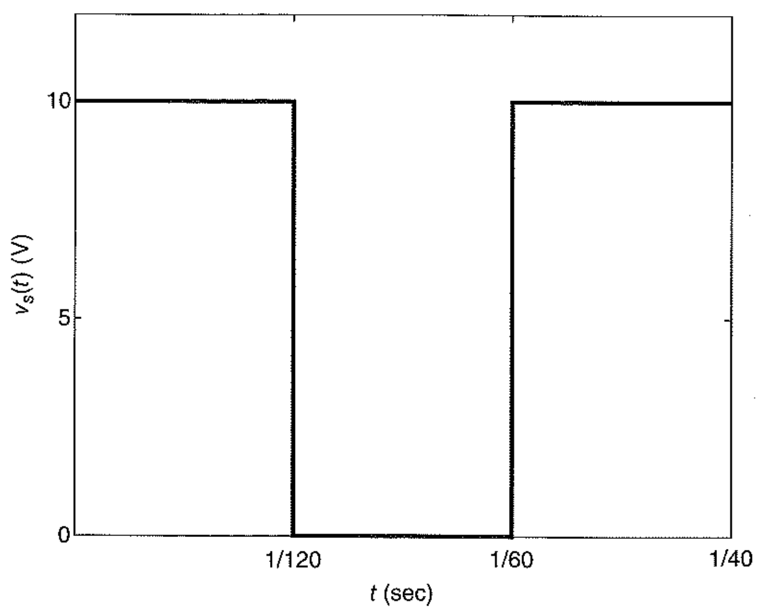
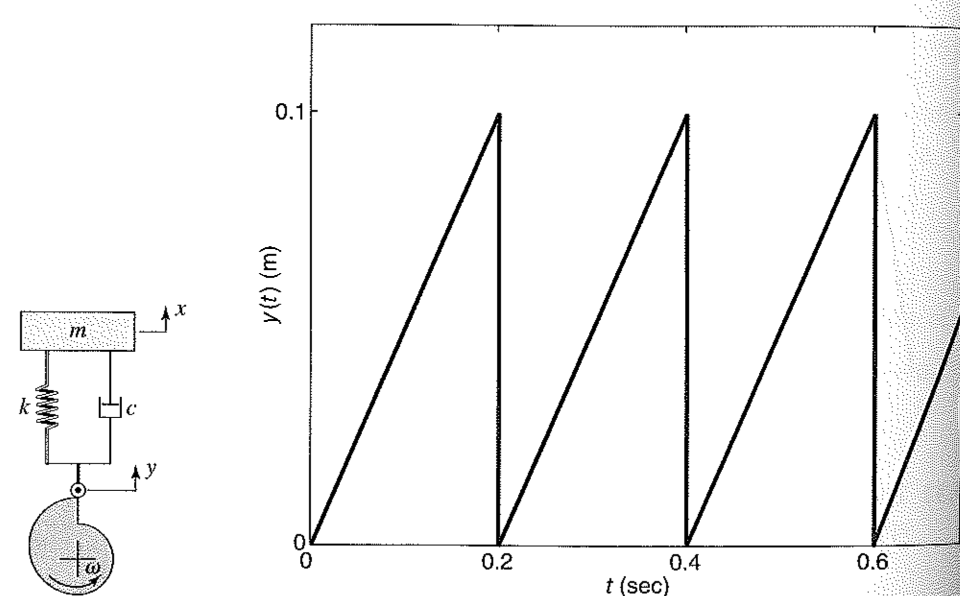



Figure P9.33




(a) (b)

For the values $m = 1$ kg, $c = 98$ N · s/m, and $k = 4900$ N/m, keeping only those terms in the Fourier series whose frequencies lie within the system's bandwidth, obtain the expression for the steady-state displacement $x(t)$.

**9.34** Given the model

$$0.5\dot{y} + 5y = f(t)$$

with the following Fourier series representation of the input

$$f(t) = \sin 4t + 4\sin 8t + 0.04\sin 12t + 0.06\sin 16t + \cdots$$

Find the steady-state response $y_{ss}(t)$ by considering only those components of the $f(t)$ expansion that lie within the bandwidth of the system.



**9.35** A mass-spring-damper system is described by the model

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

where $m = 0.25$ slug, $c = 2$ lb-sec/ft, $k = 25$ lb/ft, and $f(t)$ (lb) is the externally applied force shown in Figure P9.35. The forcing function can be expanded in a Fourier series as follows:

$$f(t) = -\left(\sin 3t + \frac{1}{3}\sin 9t + \frac{1}{5}\sin 15t + \frac{1}{7}\sin 21t + \cdots + \frac{1}{n}\sin 3nt \pm \cdots\right)$$

$n$ odd

Find an approximate description of the output $x_{ss}(t)$ at steady state, using only those input components that lie within the bandwidth.

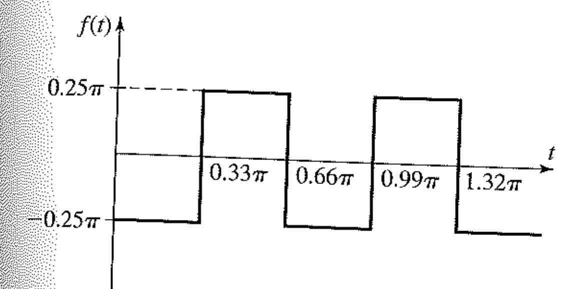



Figure P9.35

## Section 9.6 System Identification from Frequency Response

**9.36** An input $v_s(t) = 20\sin\omega t$ V was applied to a certain electrical system for various values of the frequency $\omega$, and the amplitude $|v_o|$ of the steady-state output was recorded. The data are shown in the following table. Determine the transfer function $V_o(s)/V_s(s)$.

| $\omega$ (rad/s) | $\lvert v_o\rvert$ (V) |
|---|---|
| 0.1 | 5.48 |
| 0.2 | 5.34 |
| 0.3 | 5.15 |
| 0.4 | 4.92 |
| 0.5 | 4.67 |
| 0.6 | 4.40 |
| 0.7 | 4.14 |
| 0.8 | 3.89 |
| 0.9 | 3.67 |
| 1 | 3.20 |
| 1.5 | 2.59 |
| 2 | 2.05 |
| 3 | 1.42 |
| 4 | 1.08 |
| 5 | 0.87 |
| 6 | 0.73 |
| 7 | 0.63 |

**9.37** An input $f(t) = 15\sin\omega t$ N was applied to a certain mechanical system for various values of the frequency $\omega$, and the amplitude $|x|$ of the steady-state output was recorded. The data are shown in the following table. Determine the transfer function $X(s)/F(s)$.

| $\omega$ (rad/s) | $\lvert x \rvert$ (mm) |
|---|---|
| 0.1 | 209 |
| 0.4 | 52 |
| 0.7 | 28 |
| 1 | 19 |
| 2 | 7 |
| 4 | 2 |
| 6 | 1 |

**9.38** The following data were taken by driving a machine on its support with a rotating unbalance force at various frequencies. The machine's mass is 50 kg, but the stiffness and damping in the support are unknown. The frequency of the driving force is $f$ Hz. The measured steady-state displacement of the machine is $|x|$ mm. Estimate the stiffness and damping in the support.

| $f$ (Hz) | $\lvert x \rvert$ (mm) | $f$ (Hz) | $\lvert x \rvert$ (mm) |
|---|---|---|---|
| 0.2 | 2 | 3.8 | 26 |
| 1 | 4 | 4 | 22 |
| 2 | 8 | 5 | 16 |
| 2.6 | 24 | 6 | 14 |
| 2.8 | 36 | 7 | 12 |
| 3 | 50 | 8 | 12 |
| 3.4 | 36 | 9 | 12 |
| 3.6 | 30 | 10 | 10 |

### Section 9.7 Frequency Response Analysis Using MATLAB

**9.39** For the system shown in Figure P9.39, $m_1 = m_2$, $k_1 = k_2$, and $k_1/m_1 = 64$ N/(m · kg). Obtain the transfer function $X_1(s)/Y(s)$ and its Bode plots. Identify the resonant frequencies and bandwidth.

**9.40** For the system shown in Figure P9.40, $I_1 = I_2$, $c_{T_1} = c_{T_2}$, $c_{T_1}/I_1 = 0.1$ rad$^{-1}$ · s$^{-1}$, and $k_T/I_1 = 1$ s$^{-2}$. Obtain the transfer function $\Theta_1(s)/\Phi(s)$ and its Bode plots. Identify the resonant frequencies and bandwidth.

**Figure P9.39**

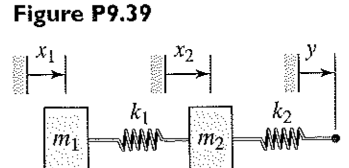
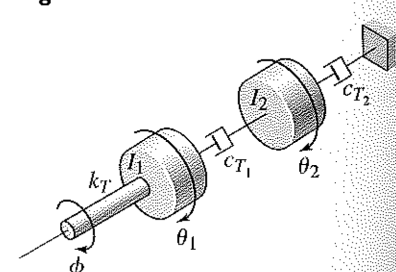



**Figure P9.40**




**9.41** A certain mass is driven by base excitation through a spring (see Figure P9.40). Its parameter values are $m = 50$ kg, $c = 200$ N · s/m, and $k = 5000$ N/m.



Determine its resonant frequency $\omega_r$, its resonance peak $M_r$, and the lower and upper bandwidth frequencies.

**9.42** The transfer functions for an armature-controlled dc motor with the speed as the output are

$$\frac{\Omega(s)}{V(s)} = \frac{K_T}{(Is + c)(Ls + R) + K_b K_T}$$

$$\frac{\Omega(s)}{T_d(s)} = -\frac{Ls + R}{(Is + c)(Ls + R) + K_b K_T}$$

A certain motor has the following parameter values:

$$K_T = 0.04 \text{ N} \cdot \text{m/A} \qquad K_b = 0.04 \text{ V} \cdot \text{s/rad}$$
$$c = 7 \times 10^{-5} \text{ N} \cdot \text{m} \cdot \text{s/rad} \qquad R = 0.6\ \Omega$$
$$L = 0.1 \text{ H} \qquad I_m = 2 \times 10^{-5} \text{ kg} \cdot \text{m}^2$$
$$I_L = 4 \times 10^{-5} \text{ kg} \cdot \text{m}^2$$

where $I_m$ is the motor's inertia and $I_L$ is the load inertia. Thus $I = I_m + I_L$. Obtain the frequency response plots for both transfer functions. Determine the bandwidth and resonant frequency, if any.

**9.43** The transfer function of the speaker model derived in Chapter 6 is, for $c = 0$,

$$\frac{X(s)}{V(s)} = \frac{K_f}{mLs^3 + mRs^2 + (kL + K_f K_b)s + kR}$$

where $x$ is the diaphragm's displacement and $v$ is the applied voltage. A certain speaker has the following parameter values:

$$m = 0.002 \text{ kg} \qquad k = 10^6 \text{ N/m}$$
$$K_f = 20 \text{ N/A} \qquad K_b = 15 \text{ V} \cdot \text{s/m}$$
$$R = 10\ \Omega \qquad L = 10^{-3} \text{ H}$$

Obtain the speaker's frequency response plots. Determine the speaker's bandwidth and resonant frequency, if any.

**9.44** The following is a two-mass model of a vehicle suspension.

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 - c_1\dot{x}_2 - k_1x_2 = 0$$

$$m_2\ddot{x}_2 + c_1\dot{x}_2 + (k_1 + k_2)x_2 - c_1\dot{x}_1 - k_1x_1 = k_2y$$

Mass $m_1$ is one-fourth the mass of the car body, and $m_2$ is the mass of the wheel-tire-axle assembly. Spring constant $k_1$ represents the suspension's elasticity, and $k_2$ represents the tire's elasticity. The input is the road surface displacement $y$. A certain vehicle has the following parameter values:

$$m_1 = 250 \text{ kg} \qquad k_1 = 1.5 \times 10^4 \text{ N/m}$$
$$m_2 = 40 \text{ kg} \qquad k_2 = 1.5 \times 10^5 \text{ N/m}$$
$$c_1 = 1000 \text{ N} \cdot \text{m/s} \qquad c_2 = 0$$

(a) Obtain the suspension's frequency response plots. Determine the bandwidth and resonant frequencies, if any. (b) If the road surface is approximately sinusoidal with a period of 10 m, at what speeds will the car mass experience the greatest oscillation amplitude?