Fa 2009 Exam 1 Solns

1) $\frac{X(j\omega)}{F(j\omega)} = \frac{1}{1000 - \omega^2 + 0.1\omega j}$

f	ω	$\frac{X(j\omega)}{F(j\omega)}$	$\lvert X \rvert$	$\angle X$
1	6.23	$-1\times10^{-3} - 6.8\times10^{-7} j$	5.2×10^{-3}	$0°$
5	31.4	$7.2\times10^{-2} - 1.7\times10^{-3} j$	0.37	$-14°$
10	62.8	$-3.4\times10^{-4} - 7.2\times10^{-7} j$	1.7×10^{-3}	$180°$
100	628	$-2.5\times10^{-6} - 4.1\times10^{-10} j$	1.3×10^{-5}	$180°$

a) $5.2\times10^{-3} \sin 6.23t$

b) $0.37 \sin(31.4t - \underbrace{0.2}_{\text{rad}})$

c) $-1.7\times10^{-3} \sin 62.8t$

d) $-1.3\times10^{-5} \sin 628t$

2) At high amplitudes $\delta = \frac{1}{7} \ln\left(\frac{1}{.6}\right) = 0.073$

At low amplitude $\delta = \frac{1}{7} \ln\left(\frac{0.28}{0.21}\right) = 0.041$

For viscous damping, δ = const.
Since damping is higher at high amplitudes, there is some air (quadratic) damping involved.

3) $U = L_3(1-\cos\theta)mg + \frac{1}{2}k(L_1\theta - z(t))^2$

(This is obviously a grossly exagerated displacement in the figure)

$T = \frac{1}{2} L_3^2 m \dot{\theta}^2 + \frac{1}{2} I \dot{\theta}^2$

$= \frac{1}{2} I_{eff} \dot{\theta}^2$

where $I_{eff} = (L_3 - L_2)^2 m + I$

$R = \frac{1}{2} c (L_2 \dot{\theta})^2$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = I_{eff} \ddot{\theta}$

$\frac{\partial T}{\partial \theta} = 0$

$\frac{\partial U}{\partial \theta} = mg L_3 \sin\theta + L_1 k(L_1\theta - z)$

$\frac{\partial R}{\partial \dot{\theta}} = c L_2^2 \dot{\theta}$

Subst into Lagrange gives

$$(L_3^2 m + I)\ddot{\theta} + c L_2^2 \dot{\theta} + L_1^2 k\theta + mg L_3 \sin\theta = L_1 k z$$

4) $X(x) = A \sin \sigma x + B \cos \sigma x$ (for rod/string ...)

$X'(0) = \sigma A \cos \sigma x \Big|_{x=0} - B\sigma \sin \sigma x \Big|_{x=0} = 0$

$\sigma A = 0, \quad A = 0$

$X(\ell) = 0 = B \cos \sigma \ell$

$\cos \sigma \ell = 0$

$\sigma \ell = \frac{\pi}{2}, \frac{3\pi}{2} \ldots$

$\sigma = \frac{(2n-1)\pi}{2\ell}, \quad n = 1, 2, 3$

$X(x) = \cos \frac{(2n-1)\pi}{2\ell} x$

Subst into PDE

$-\frac{E}{\rho} \sigma_n^2 \cos(\sigma_n x) T_n(t) = -\omega_n^2 \cos(\sigma_n x) T_n(t)$

$$\boxed{\begin{aligned} \omega_n &= \sigma_n \sqrt{\frac{E}{\rho}} = \frac{(2n-1)\pi}{2\ell} \sqrt{\frac{E}{\rho}} \quad n = 1, 2, \ldots \\ X_n(x) &= \cos \frac{(2n-1)\pi}{2\ell} x \end{aligned}}$$