ME 7120 Lecture Notes

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1 1-D elements

Material properties, geometry, and unknown variables (stress, strain, pressure, temperature) are described as a function of 1 coordinate. Example: Rod, beam, 1-D conduction, 1-D flow

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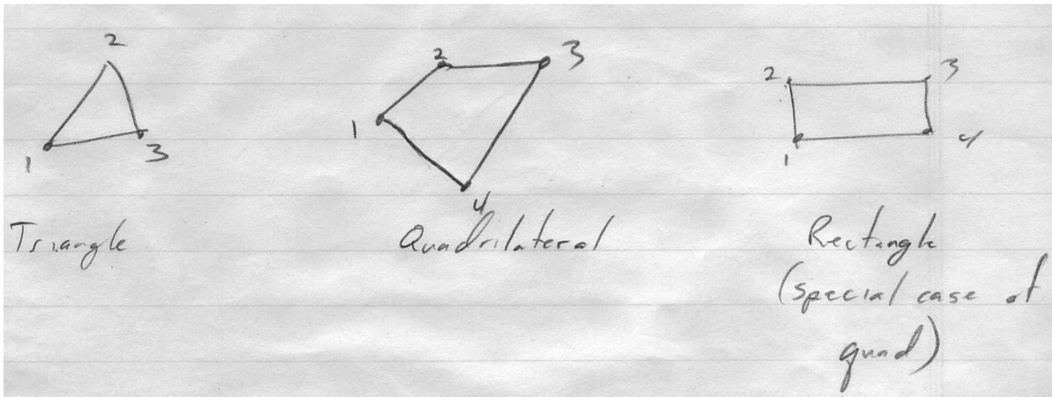
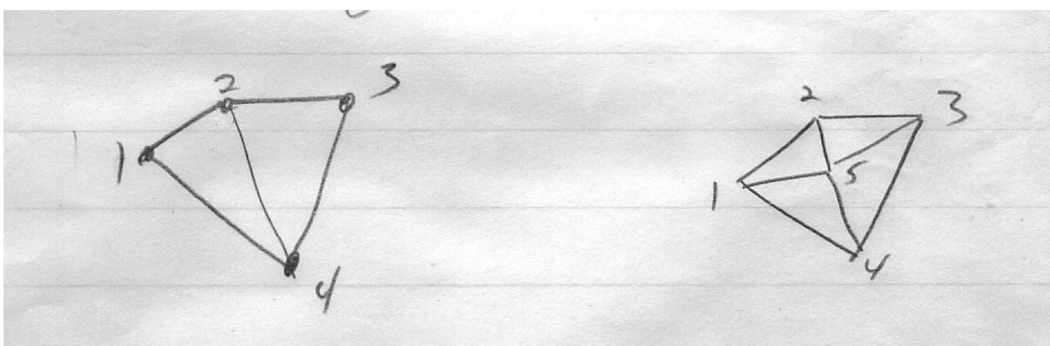
2 2-D Elements

Problems in 2-D require 2-D elements.



Quadrilateral elements (and therefore rectangular) can be formed using multiple triangle elements.

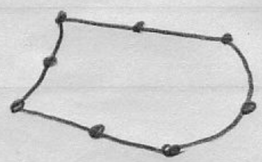


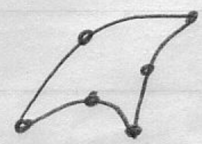
Use of quadrilateral formulation -

1. Reduces total # of elements
2. Reduces computer time
3. Increases accuracy in some cases

Higher Order Elements

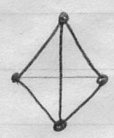
Higher Order Elements



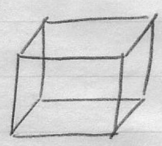


3 3-D elements

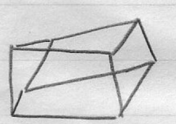
3-D Elements



Tetrahedron: Extension of triangle

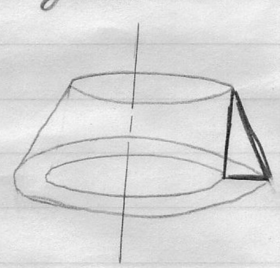


Rectangular prism: Extension of rectangle



Hexahedron: Extension of Quadrilateral

Axisymmetric Elements



4 Types of elements

Element type choice depends on physical problem. Element must be able to represent the essential phenomenon of the problem.

Element	Used for
Rod/truss	Extensional load/deformation
Beam	Bending/Transverse loads
Torsion Rod	Twisting of element
Plane Stress	Membrane problems
Plate	Thicker plates (with bending stress)
Brick/Hex	3-D objects

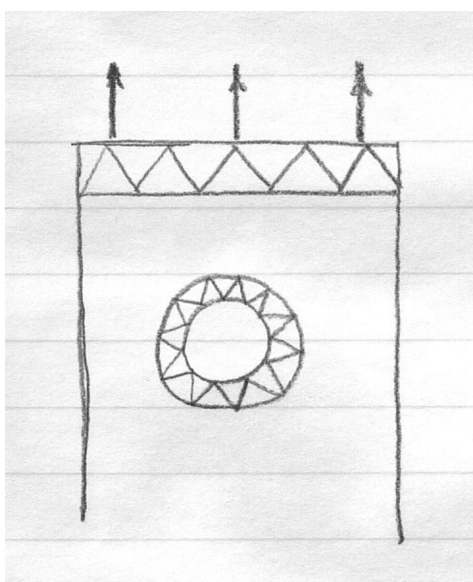
5 Size of elements

The size of the element changes the solution.(Remember that the solution is not what is really going on in the real world!)

Large elements: Fast computation, poor results

Small elements: Slow computation, better results

Elements must be smaller where things are "happening", i.e. notches, corners, discontinuities



6 Node Numbering Schemes

FEM matrices tend to be sparse (most values are zero). If the DOF's can be numbered appropriately, the matrices become banded. The bandwidth is the horizontal (or vertical) span of the matrix within which all non-zero elements reside.

$$\mathbf{K} = \begin{bmatrix} 10 & -1 & 0 & 0 \\ -1 & 9 & 2 & 0 \\ 0 & 2 & 8 & 1 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

The bandwidth is 3. The semi-bandwidth is $b = 2$ (total bandwidth is $2b - 1$). The storage requirement is $b * n$ ($n = \#$ DoF). In band form this matrix is

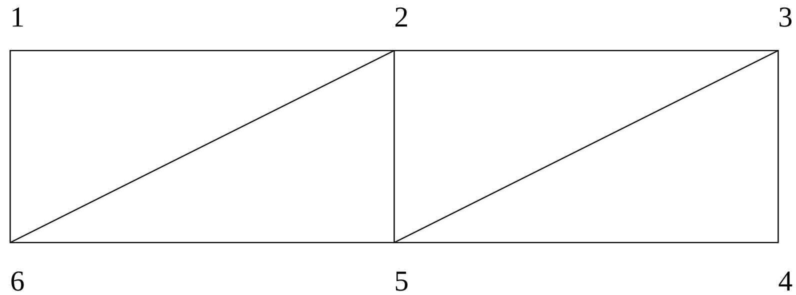
$$\begin{bmatrix} 10 & -1 \\ 9 & 2 \\ 8 & 1 \\ 7 & 0 \end{bmatrix}$$

which is half the size.

For a typical model the density is often 1%, meaning the actual storage required is 0.5%-0.75% of that for the full matrix.

Matrices become banded by properly selecting node #'s, or banding algorithms that automatically renumber the nodes to compress the matrix.

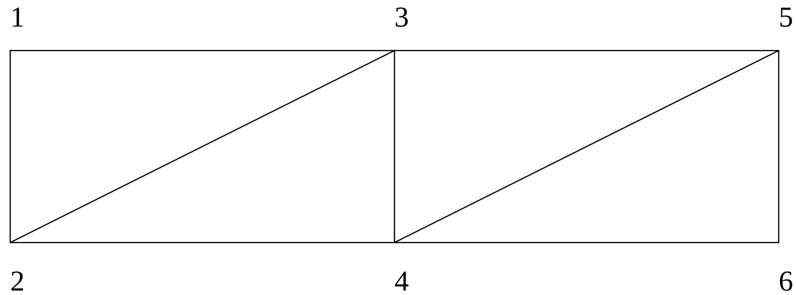
Example: Bad Numbering



$$\mathbf{K} = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} X & X & & & & X \\ & X & X & & X & X \\ & & X & X & X & \\ & & & X & X & \\ & & & & X & X \\ & & & & & X \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

b=6

Example: Good Numbering



$$\mathbf{K} = \begin{array}{cc} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\ \left[\begin{array}{cccccc} X & X & X & & & \\ & X & X & X & & \\ & & X & X & X & \\ & & & X & X & X \\ & & & & X & X \\ & & & & & X \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

b=3

Same example: lets reduce bandwidth by simply redefining DoF's of matrix (ignore geometry).

Swap 6 with 3

$$\mathbf{K} = \begin{array}{cc} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\ \left[\begin{array}{cccccc} X & X & X & & & \\ & X & X & & X & X \\ & & X & & X & \\ & & & X & X & X \\ & & & & X & X \\ & & & & & X \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

Swap new 6 with 4

$$
\mathbf{K} = \begin{array}{c c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\
\begin{bmatrix}
X & X & X & & & \\
 & X & X & X & X & \\
 & & X & & X & \\
 & & & X & X & X \\
 & & & & X & X \\
 & & & & & X
\end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}
$$

Swapping 4 with 5 and 2 with 3 results in banded matrix b=3. $n = [1\ 6\ 2\ 5\ 3\ 4\ ]$, new node 2 was old node 6

7 Basic Elements

The constitutive law for a linear material is

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = [E] \underbrace{\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}}_{\boldsymbol{\epsilon}} + \boldsymbol{\sigma_o} = [E](\boldsymbol{\epsilon} - \boldsymbol{\epsilon_o}) \tag{1}$$

where

$$\boldsymbol{\sigma_o} = -[E]\boldsymbol{\epsilon_o}$$

and

$$[E] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ \nu & (1-\nu) & \nu & 0 & 0 & 0 \\ \nu & \nu & (1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \tag{2}$$

in the 3-D case. (see equation 3.1-5)
$\boldsymbol{\epsilon_o}$ are initial strains due to:

- Temp change
- Crystal growth
- Piezo effects

$\boldsymbol{\sigma_o}$ are residual stresses from manufacture.
$\boldsymbol{\sigma_o}$ do not impact the overall equilibrium. They are localized effects that must be considered only when considering total stress, and thus are generally ignored at this stage (added to total stress later).

Alternatively, the constitutive equation can be written as

$$\boldsymbol{\epsilon} = [C](\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) = [C]\boldsymbol{\sigma} + \epsilon_0 \tag{3}$$

where (see 10.5-1)

$$[C] = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \tag{4}$$

8 General derivation of linear finite elements

sec 3.3

For each element

$$\boldsymbol{u}(x,y,z) = N\boldsymbol{d}$$

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \boldsymbol{u} = [\partial]\boldsymbol{u} = B\boldsymbol{d}$$

$$B = [\partial]N$$

The Constitutive law can then be written

$$\boldsymbol{\sigma} = [E]\ B\ \boldsymbol{d}^{(e)} - [E]\boldsymbol{\epsilon}_o$$

where as an example strain due to thermal expansion would be

$$\boldsymbol{\epsilon}_o = \begin{bmatrix} \alpha T \\ \alpha T \\ \alpha T \end{bmatrix}$$

The strain energy of the element is

$$\begin{aligned} \Pi^{(e)} &= \frac{1}{2} \iiint\limits_V \sigma_i\ \epsilon_i \,\mathrm{d}V \\ &= \frac{1}{2} \iiint\limits_V \boldsymbol{\epsilon}_T^T\ \boldsymbol{\sigma} \,\mathrm{d}V \end{aligned}$$

There $\boldsymbol{\epsilon}_T^T$ is the total strain, $\boldsymbol{\epsilon} - \boldsymbol{\epsilon_o}$, transposed

$$
\begin{aligned}
\Pi^{(e)} &= \frac{1}{2} \iiint\limits_V \ \sigma_i \ \epsilon_i \, \mathrm{d}V \\
&= \frac{1}{2} \iiint\limits_V \left(\boldsymbol{d}^T B^T - \boldsymbol{\epsilon_o}^T \right) [E] \left(B\boldsymbol{d} - \boldsymbol{\epsilon_o} \right) \mathrm{d}V \\
&= \frac{1}{2} \iiint\limits_V \boldsymbol{d}^T B^T [E] B \boldsymbol{d} \, \mathrm{d}V \\
&- \frac{1}{2} \iiint\limits_V \boldsymbol{d}^T B^T [E] \boldsymbol{\epsilon_o} \, \mathrm{d}V - \frac{1}{2} \iiint\limits_V \boldsymbol{\epsilon_o}^T [E] B \boldsymbol{d} \, \mathrm{d}V + \frac{1}{2} \iiint\limits_V \boldsymbol{\epsilon_o}^T [E] \boldsymbol{\epsilon_o} \, \mathrm{d}V
\end{aligned}
$$

9 Nodal Loads

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The work done by external loads is

$$W = \iint\limits_S \boldsymbol{u}^T \boldsymbol{\Phi} \, \mathrm{d}S + \iiint\limits_V \boldsymbol{u}^T \boldsymbol{F} \, \mathrm{d}V + \boldsymbol{d}^T \boldsymbol{P} \ (-\Omega \text{ in text 4.3-6, 4.4-7})$$

Since $\boldsymbol{u}^T = \boldsymbol{d}^T N^T$

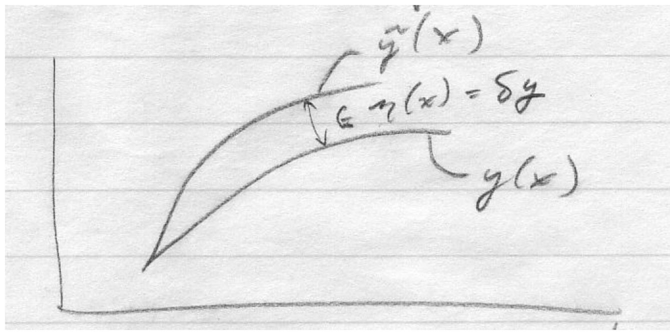
$$W = \boldsymbol{d}^T \left[\iint\limits_S N^T \boldsymbol{\Phi} \, \mathrm{d}S + \iiint\limits_V N^T \boldsymbol{F} \, \mathrm{d}V + \boldsymbol{P} \right]$$

The kinetic energy is

$$\begin{aligned} T &= \iiint\limits_V \frac{1}{2} \dot{\boldsymbol{u}}^T \rho \dot{\boldsymbol{u}} \, \mathrm{d}V \\ &= \frac{1}{2} \iiint\limits_V \dot{\boldsymbol{d}}^T N^T \rho N \dot{\boldsymbol{d}} \, \mathrm{d}V \end{aligned}$$

10 Variational Calculus

Use of the δ operator (variation of a function).



$$\tilde{y}(x) - y(x) = \epsilon\eta(x) = \delta\left(y(x)\right)$$

$$\delta\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \left(\frac{\mathrm{d}\tilde{y}}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\tilde{y} - y\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\delta y\right)$$

δ and derivatives commute
δ and integrals commute
From Taylor series

$$\tilde{F}(x, y, y') = F(x, y, y') + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \cdots$$

$$\delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'$$

10.1 Example: Beam

$$\Pi_p = \int_0^l \underbrace{\frac{EI}{2}\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 - pw}_{\text{Total potential/unit length.}} \; \mathrm{d}x$$

p is the distributed load.

$\delta\Pi_p = 0 \qquad$ (variation of potential energy is zero at solution)

$$\delta\Pi_p = \int\limits_0^l EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\delta\frac{\mathrm{d}^2w}{\mathrm{d}x^2} - p\delta w\,\mathrm{d}x = 0$$

$$= \int\limits_0^l EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\frac{\mathrm{d}^2\delta w}{\mathrm{d}x^2}\,\mathrm{d}x - \int\limits_0^l p\delta w\,\mathrm{d}x$$

Integrating first term by parts twice

$$= \int\limits_0^l \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right)\delta w\,\mathrm{d}x + EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\delta\frac{\mathrm{d}w}{\mathrm{d}x}\bigg|_0^l - \frac{\mathrm{d}}{\mathrm{d}x}EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\delta w\bigg|_0^l - \int_0^l p\delta w dx = 0$$

The DE is $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right) = p(x)$

The BC's are either

$$EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2} = 0 \text{ or } \delta\frac{\mathrm{d}w}{\mathrm{d}x} = 0$$

at each end

AND

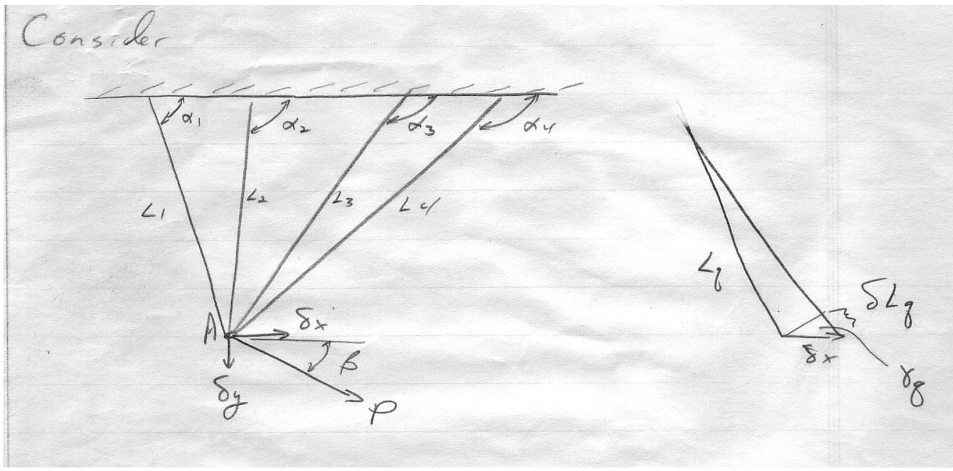
$$\frac{\mathrm{d}}{\mathrm{d}x}EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2} = 0 \text{ or } \delta w = 0$$

at each end

11 Example of virtual work

p116 Shames

Consider



Each rod q has a virtual change in length of

$$\delta L_q = \delta_x \cos \gamma_q \approx \delta_x \cos \alpha_q$$

as a result of δ_x and

$$\delta L_q \approx \delta_y \sin \alpha_q$$

as a result of δ_y.

The strain as a result of each are

$$(\delta \epsilon_q)_x = \frac{\delta_x \cos \alpha_q}{L_q}$$

and

$$(\delta \epsilon_q)_y = \frac{\delta_y \sin \alpha_q}{L_q}$$

The actual displacement of the pin A is $(\bar{u}, \bar{v})$. The strain in each rod as a result of these are

$$\bar{\epsilon}_q = \frac{\bar{u} \cos \alpha_q}{L_q} + \frac{\bar{v} \sin \alpha_q}{L_q}$$

Hence the stress $\bar{\sigma}_q$ in each rod is

$$\bar{\sigma}_q = E_q \left(\frac{\bar{u} \cos \alpha_q}{L_q} + \frac{\bar{v} \sin \alpha_q}{L_q} \right)$$

The principle of virtual work for the δ_x variation is

$$\begin{aligned}
P\cos\beta\delta_x &= \sum_{q=1}^{4} \bar{\sigma}_q \delta\epsilon_q L_q A_q \\
&= \sum_{q=1}^{4} E_q A_q \left[\bar{u}\cos\alpha_q \frac{\delta_x \cos\alpha_q}{L_q} + \bar{v}\sin\alpha_q \frac{\delta_x \cos\alpha_q}{L_q} \right] \\
&\text{Since } \delta_x \neq 0 \\
P\cos\beta &= \sum_{q=1}^{4} \frac{E_q A_q}{L_q} \left(\bar{u}\cos^2\alpha_q + \bar{v}\cos\alpha_q \sin\alpha_q \right) \\
&\text{Likewise in the y direction} \\
P\sin\beta &= \sum_{q=1}^{4} \frac{E_q A_q}{L_q} \left(\bar{u}\cos\alpha_q \sin\alpha_q + \bar{v}\sin^2\alpha_q \right)
\end{aligned}$$

This results in 2 Eqns and 2 unknowns. They can be solved for $\bar{u}$ and $\bar{v}$.

12 Hamilton's Principle

See section 4.2

$$\delta L = \delta \int\limits_{t_1}^{t_2} (T - \Pi_p)\ dt = 0$$

where

$$\Pi_p = U + \Omega$$

Here U is the potential energy and Ω is the *potential of the applied load.* Ω can also be considered to to be negative of the work done by the applied load (a la Fd), or $\Omega = -W$.

$$\begin{aligned} \int_{t_1}^{t_2} \delta T dt &= \delta \int\limits_{t_1}^{t_2} \frac{1}{2} \iiint\limits_V \dot{\boldsymbol{d}}^T N^T \rho N \dot{\boldsymbol{d}} \,\mathrm{d}V\,\mathrm{d}t \\ &= \int\limits_{t_1}^{t_2} \iiint\limits_V \delta\left(\dot{\boldsymbol{d}}^T\right) N^T \rho N \dot{\boldsymbol{d}} \,\mathrm{d}V\,\mathrm{d}t \end{aligned}$$

Integrating by parts in time, swapping order of integrals

$$\begin{aligned} u &= N^T \rho N \dot{\boldsymbol{d}} & \mathrm{d}v &= \delta \dot{\boldsymbol{d}}\,\mathrm{d}t \\ \mathrm{d}u &= N^T \rho N \ddot{\boldsymbol{d}}\,\mathrm{d}t & v &= \delta\left(\boldsymbol{d}^T\right) \end{aligned}$$

$$\iiint\limits_V \left[\delta\left(\boldsymbol{d}^T\right) N^T \rho N \dot{\boldsymbol{d}} \Big|_{t_1}^{t_2} - \int\limits_{t_1}^{t_2} \delta\left(\boldsymbol{d}^T\right) N^T \rho N \ddot{\boldsymbol{d}}\,\mathrm{d}t \right] \mathrm{d}V$$

because we presume to know $\boldsymbol{d}^T$ at t_1 and t_2

$$\delta\left(\boldsymbol{d}^T\right) N^T \rho N \dot{\boldsymbol{d}} \Big|_{t_1}^{t_2} = 0$$

$\therefore$

$$\int_{t_1}^{t_2} \delta T dt = -\int\limits_{t_1}^{t_2} \iiint\limits_V \delta\left(\boldsymbol{d}^T\right) N^T \rho N \ddot{\boldsymbol{d}} \, \mathrm{d}V \, \mathrm{d}t$$

Also

$$\delta\Pi^{(e)} = \iiint\limits_V \delta\boldsymbol{d}^T B^T E B \boldsymbol{d} \, \mathrm{d}V - \iiint\limits_V \delta\boldsymbol{d}^T B E \epsilon_0 \, \mathrm{d}V$$

$$\delta W = \delta\boldsymbol{d}^T \left[\iint\limits_s N^T \boldsymbol{\Phi} \, \mathrm{d}s + \iiint\limits_V N^T \boldsymbol{F} \, \mathrm{d}V \right] + \delta\boldsymbol{d}^T \boldsymbol{P}$$

The elemental eqn of motion is then

$$\underbrace{\iiint\limits_V B^T E B \, \mathrm{d}V}_{\text{Stiffness matrix } K^{(e)}} \boldsymbol{d} + \underbrace{\iiint\limits_V N^T \rho N \, \mathrm{d}V}_{\text{Mass matrix } M^{(e)}} \ddot{\boldsymbol{d}}$$

$$= \underbrace{\iiint\limits_V B^T E \epsilon_0 \, \mathrm{d}V}_{\substack{\text{Elemental node} \\ \text{“forces” to cause} \\ \text{initial strains}}} + \underbrace{\iint\limits_{D_1} N^T \boldsymbol{\Phi} \, \mathrm{d}s_1}_{\substack{\text{Vector of} \\ \text{elemental node} \\ \text{forces provided} \\ \text{by surface forces}}} + \underbrace{\iiint\limits_V N^T \boldsymbol{F} \, \mathrm{d}V}_{\substack{\text{Body force} \\ \text{vector}}} + \underbrace{\boldsymbol{P}}_{\substack{\text{Concentrated} \\ \text{loads}}}$$

The formulae remain the same in general, but the matrix definitions vary by case.

13 Example: Rod Element

Displacement field is

$$u(x) = \left[\begin{array}{cc} \left(1 - \frac{x}{l}\right) & \frac{x}{l} \end{array} \right] \left[\begin{array}{c} d_1 \\ d_2 \end{array} \right]$$

The strain is

$$\epsilon = \frac{du}{dx} = \frac{d}{dx} \left[\begin{array}{cc} \left(1 - \frac{x}{l}\right) & \frac{x}{l} \end{array} \right] \left[\begin{array}{c} d_1 \\ d_2 \end{array} \right]$$

and stress is

$$\sigma = E\frac{du}{dx} = E\frac{d}{dx}N\boldsymbol{d}$$

The potential energy is

$$\begin{aligned}
\Pi_p &= \int\limits_0^l \frac{EA}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x - \boldsymbol{d}^T\boldsymbol{P} \\
\delta\Pi_p &= \int\limits_0^l EA\frac{\mathrm{d}u}{\mathrm{d}x}\delta\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x - \delta\boldsymbol{d}^T\boldsymbol{P} \\
&= \int\limits_0^l EA\frac{\mathrm{d}}{\mathrm{d}x}\left(N\boldsymbol{d}\right)\delta\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(N\boldsymbol{d}\right)\right)\mathrm{d}x - \delta\boldsymbol{d}^T\boldsymbol{P}
\end{aligned}$$

Factoring out $\delta\boldsymbol{d}^T$ and $\boldsymbol{d}$ and substituting for N

$$\begin{aligned}
\delta\Pi_p &= \delta\boldsymbol{d}^T\int_0^l EA\left[\begin{array}{cc} \frac{-1}{l} & \frac{1}{l} \end{array} \right]^T \left[\begin{array}{cc} \frac{-1}{l} & \frac{1}{l} \end{array} \right] dx\boldsymbol{d} - \delta\boldsymbol{d}^T\boldsymbol{P} \\
&= \delta\boldsymbol{d}^T\frac{EA}{l}\left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right]\boldsymbol{d} - \delta\boldsymbol{d}^T\boldsymbol{P}
\end{aligned}$$

The Kinetic energy is given by

$$T = \frac{1}{2}\int\limits_0^l \rho A\dot{u}^2\,\mathrm{d}x$$

The variation is

$$\delta T = \rho A \int_0^l \delta \dot{u} \dot{u} \, \mathrm{d}x = -\rho A \int_0^l \delta u \ddot{u} \, \mathrm{d}x$$

after integrating by parts in time. Substituting for u

$$\delta T = -\rho A \int_0^l \delta \boldsymbol{d}^T N^T N \ddot{\boldsymbol{d}} \, \mathrm{d}x = -\delta \boldsymbol{d}^T M \ddot{\boldsymbol{d}}$$

Using Hamilton's principle, this results in

$$\begin{aligned} -M\ddot{\boldsymbol{d}} + \ -K\boldsymbol{d} + \boldsymbol{P} &= 0 \\ M\ddot{\boldsymbol{d}} + K\boldsymbol{d} &= \boldsymbol{P} \end{aligned}$$

where

$$\begin{aligned} M &= \rho A \int_0^l N^T N \, \mathrm{d}x \\ &= \rho A \int_0^l \begin{bmatrix} 1 - \frac{x}{l} \\ \frac{x}{l} \end{bmatrix} \begin{bmatrix} 1 - \frac{x}{l} & \frac{x}{l} \end{bmatrix} \mathrm{d}x \\ &= \frac{\rho A l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{aligned}$$

14 Beam

Displacements are represented by

$$v(x) = N\boldsymbol{d}$$

$$N = \begin{bmatrix} 1 - 3\left(\frac{x}{l}\right)^2 + 2\left(\frac{x}{l}\right)^3 \\ x - 2\left(\frac{x^2}{l}\right) + \left(\frac{x^3}{l^2}\right) \\ 3\left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right)^3 \\ -\left(\frac{x^2}{l}\right) + \left(\frac{x^3}{l^2}\right) \end{bmatrix}^T \qquad \boldsymbol{d} = \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

The generalized strain for a beam is

$$\epsilon = -\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} N\boldsymbol{d}$$

The generalized stress is

$$\sigma = -EI\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = -EI\frac{\mathrm{d}^2}{\mathrm{d}x^2} N\boldsymbol{d}$$

$$\begin{aligned} \Pi_p &= \int_0^l \frac{\sigma\epsilon}{2}\,\mathrm{d}x - \boldsymbol{d}^T\boldsymbol{P} \\ &= \int_0^l \frac{1}{2}EI\left(\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right)^2\,\mathrm{d}x - \boldsymbol{d}^T\boldsymbol{P} \end{aligned}$$

$$\delta\Pi_p = \int_0^l EI\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\delta\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\,\mathrm{d}x - \delta\boldsymbol{d}^T\boldsymbol{P}$$

Substituting for the displacement field

$$\begin{aligned}
\delta\Pi_p &= \int_0^l EI \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} N\boldsymbol{d}\right)^T \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} N\delta\boldsymbol{d}\right) \mathrm{d}x - \delta\boldsymbol{d}^T\boldsymbol{P} \\
&= \int_0^l \delta\boldsymbol{d}^T \frac{\mathrm{d}^2}{\mathrm{d}x^2} N^T EI \frac{\mathrm{d}^2}{\mathrm{d}x^2} N\boldsymbol{d}\,\mathrm{d}x - \delta\boldsymbol{d}^T\boldsymbol{P}
\end{aligned}$$

Since $\delta\boldsymbol{d}^T \neq 0$

$$\int_0^l \frac{\mathrm{d}^2}{\mathrm{d}x^2} N^T EI \frac{\mathrm{d}^2}{\mathrm{d}x^2} N\,\mathrm{d}x\ \boldsymbol{d} = \boldsymbol{P}$$

Then

$$K = \int_0^l \frac{\mathrm{d}^2}{\mathrm{d}x^2} N^T EI \frac{\mathrm{d}^2}{\mathrm{d}x^2} N\,\mathrm{d}x$$

$$K = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ & & & 4l^2 \end{bmatrix}$$

Likewise the mass matrix can be found from Hamilton's principle

$$T = \int_0^l \frac{1}{2} \rho A \dot{v}^2 \, \mathrm{d}x$$

$$\delta T = \rho A \int_0^l \delta \dot{v} \dot{v} \, \mathrm{d}x$$

Integrating from t_1 to t_2 by parts (See Hamilton's principle)

$$= -\rho A \int_0^l \delta v \ddot{v} \, \mathrm{d}x$$

$$= -\rho A \int_0^l \delta \boldsymbol{d}^T N^T N \ddot{\boldsymbol{d}} \, \mathrm{d}x$$

$$= -\delta \boldsymbol{d}^T \frac{ml}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ & 4l^2 & 13l & -3l^2 \\ & & 156 & -22l \\ & & & 4l^2 \end{bmatrix} \ddot{\boldsymbol{d}}$$

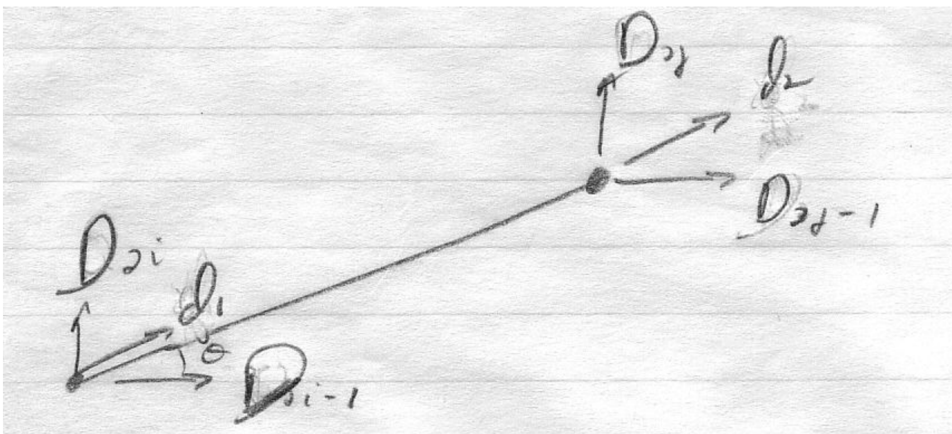
15 Coordinate transformation

15.1 Rotation of linear elements

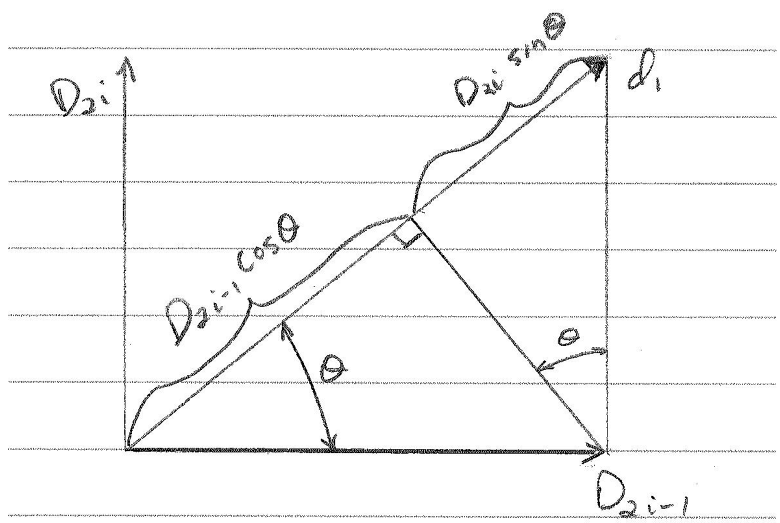
Rod: 1-D to 2-D

d_i: local DoF

D_i: global DoF



d_i: local DoF
D_i: global DoF



$$d_1 = D_{2i-1}\cos\theta + D_{2i}\sin\theta$$
$$d_2 = D_{2j-1}\cos\theta + D_{2j}\sin\theta$$

Define $\cos\theta = l$, $\sin\theta = m$

$$\boldsymbol{d}^e = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{bmatrix} D_{2i-1} \\ D_{2i} \\ D_{2j-1} \\ D_{2j} \end{bmatrix} = T\boldsymbol{D}$$

The elemental strain energy is

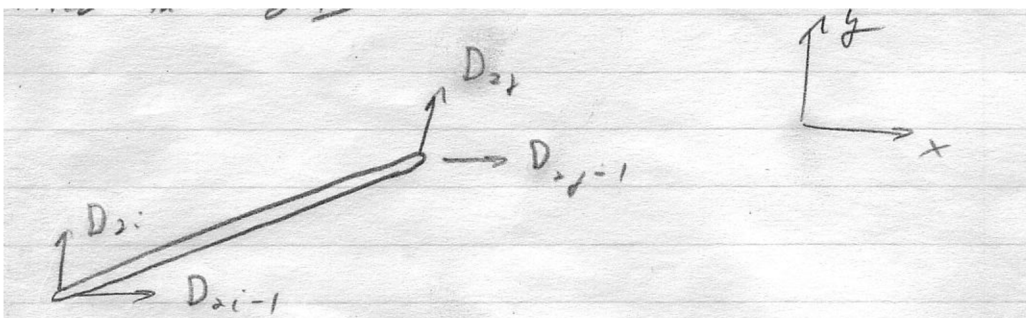
$$\begin{aligned} \Pi^{(e)} &= \frac{1}{2} \underbrace{\boldsymbol{d}^T}_{1\times 2} \underbrace{K}_{2\times 2} \underbrace{\boldsymbol{d}}_{2\times 1} \\ &= \frac{1}{2} \underbrace{\boldsymbol{D}^T}_{1\times 4} \underbrace{T^T}_{4\times 2} \underbrace{K}_{2\times 2} \underbrace{T}_{2\times 4} \underbrace{\boldsymbol{D}}_{4\times 1} \end{aligned}$$

So, in global coordinates,

$$K = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$= \frac{EA}{L} \left[\begin{array}{cc|cc} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ \hline -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{array} \right]$$

Transverse stiffness of a rod is zero. Transverse inertia of a rod is not. Therefor the rod to be used in 2-D must be derived in 2-D.



$$\boldsymbol{u}(x) = \begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = \begin{bmatrix} 1 - \frac{x}{l} & 0 & \frac{x}{l} & 0 \\ 0 & 1 - \frac{x}{l} & 0 & \frac{x}{l} \end{bmatrix} \begin{bmatrix} D_{2i-1} \\ D_{2i} \\ D_{2j-1} \\ D_{2j} \end{bmatrix}$$

Thus

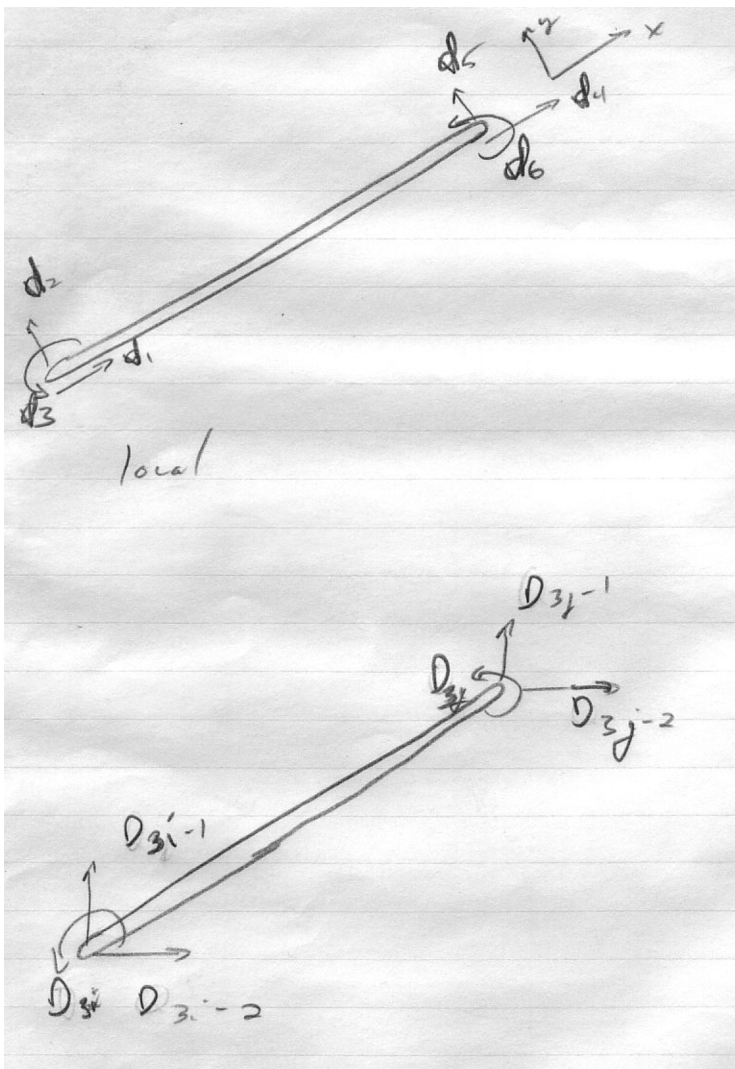
$$\boldsymbol{u}(x) = N\boldsymbol{D}$$

$$
\begin{aligned}
M &= \rho A \int\limits_0^l N^T N \,\mathrm{d}x \text{ for constant } \rho \text{ and } A \\
&= \rho A \int\limits_0^l \begin{bmatrix} \left(1-\frac{x}{l}\right)^2 & 0 & \left(\frac{x}{l}-\frac{x^2}{l^2}\right) & 0 \\ & \left(1-\frac{x}{l}\right)^2 & 0 & \left(\frac{x}{l}-\frac{x^2}{l^2}\right) \\ & & \frac{x^2}{l^2} & 0 \\ \text{sym} & & & \frac{x^2}{l^2} \end{bmatrix} \mathrm{d}x \\
&= \frac{\rho A l}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ & 2 & 0 & 1 \\ & & 2 & 0 \\ \text{sym} & & & 2 \end{bmatrix}
\end{aligned}
$$

15.2 Combined Beam/Rod Element

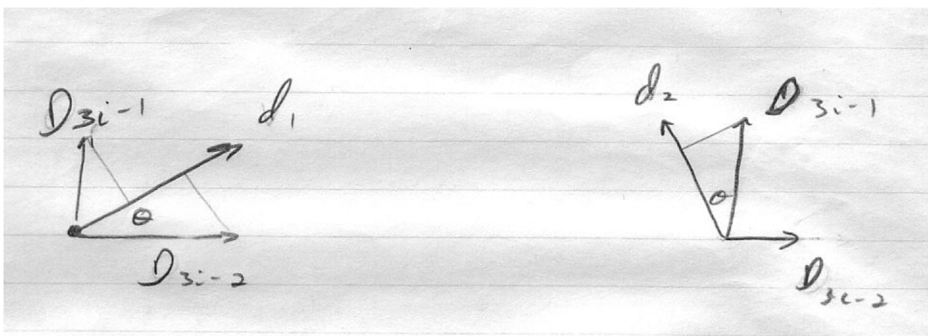
Stiffness:

$$K = \frac{EI}{l^3} \begin{bmatrix} \frac{Al^2}{I_{zz}} & 0 & 0 & \frac{-Al^2}{I_{zz}} & 0 & 0 \\ & 12 & 6l & 0 & -12 & 6l \\ & & 4l^2 & 0 & -6l & 2l^2 \\ & & & \frac{Al^2}{I_{zz}} & 0 & 0 \\ & & & & 12 & -6l \\ & & & & & 4l^2 \end{bmatrix}$$



$$K = T^T K^{(e)} T$$

$$T = \begin{bmatrix} l_{ox} & m_{ox} & 0 & & & \\ l_{oy} & m_{oy} & 0 & & \text{zeros} & \\ 0 & 0 & 1 & & & \\ & & & l_{ox} & m_{ox} & 0 \\ & \text{zeros} & & l_{oy} & m_{oy} & 0 \\ & & & 0 & 0 & 1 \end{bmatrix}$$



$$d_1 = D_{3i-2}\cos\theta + D_{3i-1}\sin\theta$$
$$d_2 = -D_{3i-2}\sin\theta + D_{3i-1}\cos\theta$$

$$l_{ox} = \cos\theta \quad m_{ox} = \sin\theta$$
$$l_{oy} = -\sin\theta \quad m_{oy} = \cos\theta$$

For the mass matrix

$$\boldsymbol{u}(x) = \begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = [N] \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_6 \end{bmatrix}$$

$$N = \begin{bmatrix} 1 - \frac{x}{l} & 0 & 0 & \frac{x}{l} & 0 & 0 \\ 0 & 1 - \frac{3x^2}{l} + \frac{2x^3}{l} & \dots & & & \end{bmatrix}$$

$$M = \iiint\limits_V N^T \rho N \, \mathrm{d}V$$

$$= \rho A L \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 & 0 \\ & \frac{13}{35} & \frac{11l}{210} & 0 & \frac{9}{70} & \frac{-13l}{420} \\ & & \frac{l^2}{105} & 0 & \frac{13l}{420} & \frac{-l^2}{140} \\ & & & \frac{1}{3} & 0 & 0 \\ & & & & \frac{13}{35} & \frac{-11l}{210} \\ & & & & & \frac{l^2}{105} \end{bmatrix}$$

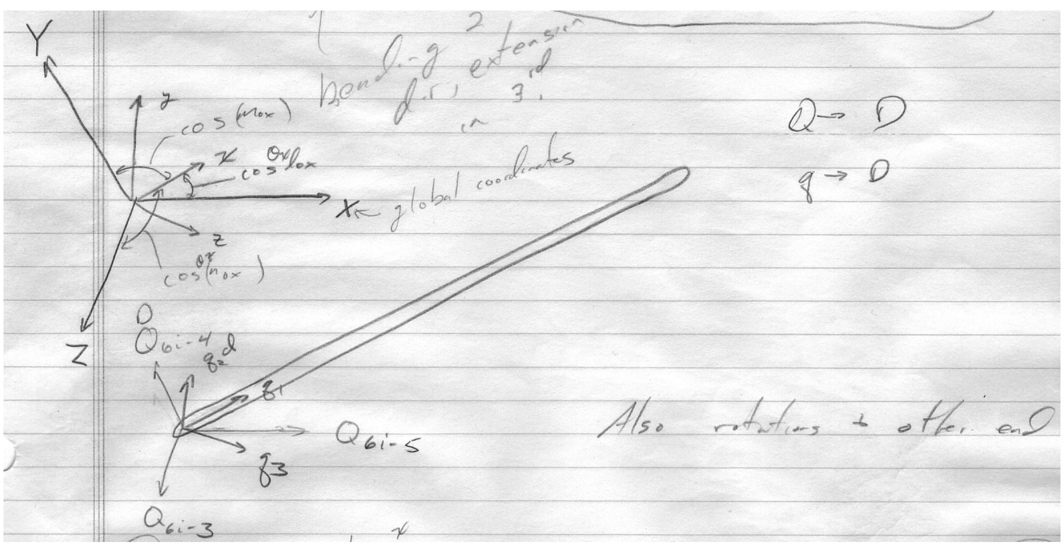
HW - Find mass matrix rotated by 45 degrees

15.3 Total beam element

Bending in 2 dir, extension in 3rd.

$$K^e = [12 \times 12]$$

HW: Write out full 3-D element in local coordinates



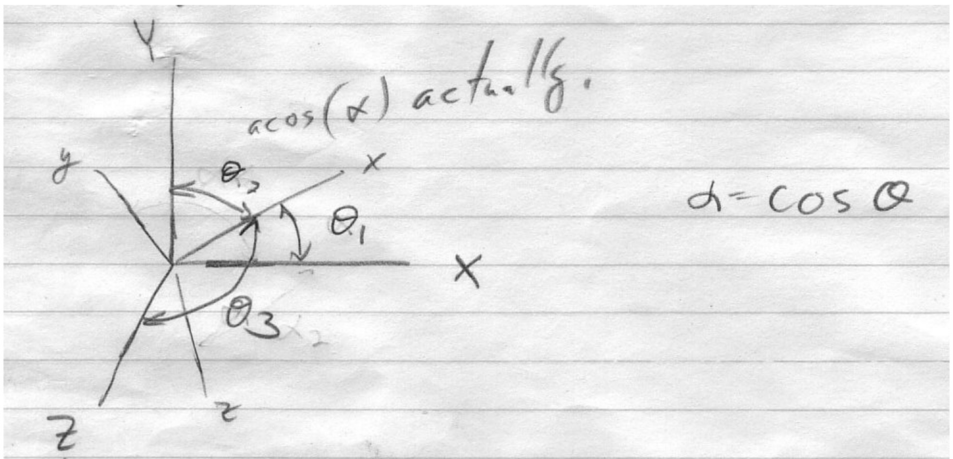
$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} l_{ox} & m_{ox} & n_{ox} & & & & & & & & & \\ l_{oy} & m_{oy} & n_{oy} & & & & & & & & Zeros & \\ l_{oz} & m_{oz} & n_{oz} & & & & & & & & & \\ & & & l_{ox} & m_{ox} & n_{ox} & & & & & & \\ & & & l_{oy} & m_{oy} & n_{oy} & & & & & & \\ & & & l_{oz} & m_{oz} & n_{oz} & & & & & & \\ & & & & & & l_{ox} & m_{ox} & n_{ox} & & & \\ & & & & & & l_{oy} & m_{oy} & n_{oy} & & & \\ & & & & & & l_{oz} & m_{oz} & n_{oz} & & & \\ & & & & & & & & & l_{ox} & m_{ox} & n_{ox} \\ & Zeros & & & & & & & & l_{oy} & m_{oy} & n_{oy} \\ & & & & & & & & & l_{oz} & m_{oz} & n_{oz} \end{bmatrix} \begin{bmatrix} D_{3i-5} \\ D_{3i-4} \\ D_{3i-3} \\ \vdots \\ D_{3j-5} \\ D_{3j-4} \\ \vdots \end{bmatrix}$$

The confusion in coding these elements is that the transformation is underdetermined. Although x clearly goes along the beam, the direction of the y and z axes relative to the global axes cannot be determined simply by knowing where the endpoints of the beam are.

They must be known and defined in some way.

16 Orthonormal transformations

130 Goldstein



Denote $\hat{i}, \hat{j}, \hat{k}$ unit direction vectors X, Y, Z
and $\hat{i}', \hat{j}', \hat{k}'$ unit direction vectors x, y, z
The direction cosines are defined as

$$\left(\hat{i}', \hat{i}\right) = \hat{i}' \cdot \hat{i} = (\alpha_1) = \cos\theta_1$$
$$\left(\hat{i}', \hat{j}\right) = \hat{i}' \cdot \hat{j} = (\alpha_2) = \cos\theta_2$$
$$\left(\hat{i}', \hat{k}\right) = \hat{i}' \cdot \hat{k} = (\alpha_3) = \cos\theta_3$$

The vector $\hat{i}'$ can be written as

$$\hat{i}' = \alpha_1 \hat{i} + \alpha_2 \hat{j} + \alpha_3 \hat{k}$$

This is the unit direction vector along the beam.
The other vectors are

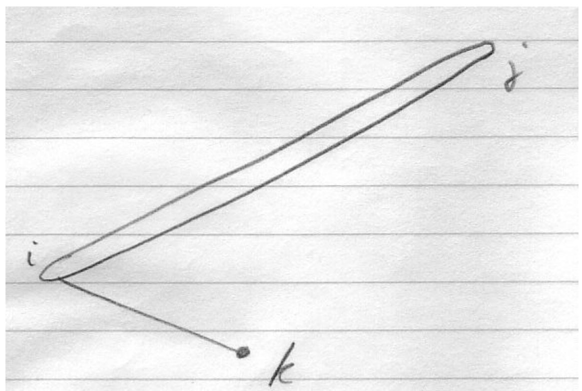
$$\hat{j}' = \beta_1 \hat{i} + \beta_2 \hat{j} + \beta_3 \hat{k}$$
$$\hat{k}' = \gamma_1 \hat{i} + \gamma_2 \hat{j} + \gamma_3 \hat{k}$$

The coordinate transform matrix T is then

$$T = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}$$

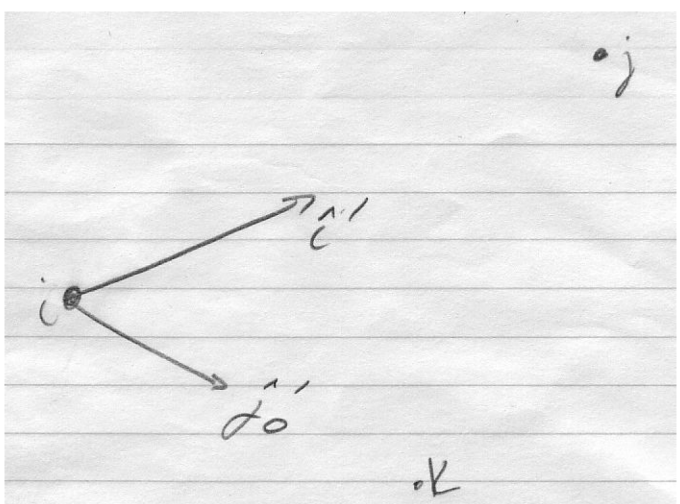
This requires that when we define a beam's location, we also define the direction of an axis, perhaps y'. The z' axis can be found via a cross product of x' with y'.

If you define a point K



The three points form a plane that can be defined as the direction y.

Practical aspect: it is unlikely that the data will be orthogonal. Data checking must occur. In addition, $\hat{k}$ can be moved in the plane to produce orthogonality.



First, obtain $\hat{\imath}'$ by normalizing the relative position of the points i and j by the length. Then, obtain $\hat{\jmath}'_o$ similarly.

$\hat{\imath}' \cdot \hat{\jmath}'_o$ should be zero. If not, it is the component of $\hat{\jmath}'_o$ in the $\hat{\imath}'$ direction. Define $\hat{\jmath}'$ to be the normalized value of $(\hat{\jmath}'_o - (\hat{\imath}' \dot{} \hat{\jmath}'_o)\hat{\imath}')$ then $\hat{k}' = \hat{\imath}' \times \hat{\jmath}'$ (into page).

Now the direction cosines α, β, and γ can be obtained via dot products with the vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

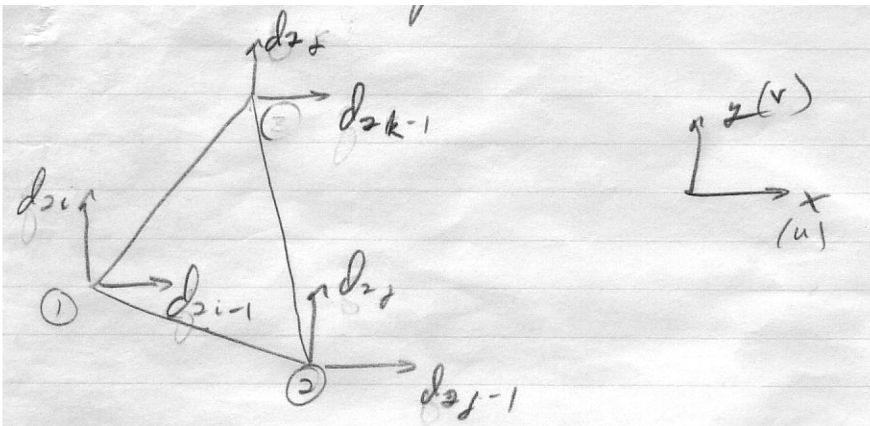
The matrix T must be an orthonormal matrix (just as in ME 460/660 for P). What are its properties?

$$T^T = T^{-1} \qquad \det(T) = 1$$

17 Triangular membrane element

Cooke p91

1st consider in-plane stress only.



Must # counterclockwise with this formulation. Simplest model for displacement field is

$$
\begin{aligned}
u(x,y) &= a_1 + a_2 x + a_3 y \\
v(x,y) &= a_4 + a_5 x + a_6 y
\end{aligned}
$$

BC's

$$
\begin{array}{lll}
u = d_{2i-1} & \text{and} & v = d_{2i} \text{ @ } (x_i, y_i) \\
u = d_{2j-1} & \text{and} & v = d_{2j} \text{ @ } (x_j, y_j) \\
u = d_{2k-1} & \text{and} & v = d_{2k} \text{ @ } (x_k, y_k)
\end{array}
$$

$$
\vec{u} = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = N\boldsymbol{d}
$$

$$
\begin{aligned}
N &= \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \\
N_1 &= \frac{1}{2A}\left[y_{32}(x - x_2) - x_{32}(y - y_2)\right] \\
N_2 &= \frac{1}{2A}\left[y_{13}(x - x_3) - x_{13}(y - y_3)\right] \\
N_3 &= \frac{1}{2A}\left[y_{21}(x - x_1) - x_{21}(y - y_1)\right] \\
A &= \frac{1}{2}(x_{32}y_{21} - x_{21}y_{32}) \\
\text{or } 2A &= \det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}
\end{aligned}
$$

$$
\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix}
$$

$$
\boldsymbol{\epsilon} = B\boldsymbol{d}
$$

where

$$
B = \frac{1}{2A}\begin{bmatrix} y_{32} & 0 & -y_{31} & 0 & y_{21} & 0 \\ 0 & -x_{32} & 0 & x_{31} & 0 & -x_{21} \\ -x_{32} & y_{32} & x_{31} & -y_{31} & -x_{21} & y_{21} \end{bmatrix}
$$

$$
\boldsymbol{\sigma} = [E]\boldsymbol{\epsilon}
$$

$$
[E] = \frac{E}{1 - \nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}
$$

$$
K^e = \int_V B^T [E] B \, \mathrm{d}V
$$

17.1 Plain stress and strain

Two possibilities exist for [E]

1. Plane Stress

   Dimension is small in one coordinate direction. Stresses are assumed to not vary through plate (membrane) (actual: stresses in through direction are tiny). We assume $\sigma_{zz} = \sigma_{zx} = \sigma_{yz} = 0$ where z is $\perp$ to object. Applying this to equation (3)

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ 0 \\ \tau_{xy} \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

   Which writing only the equations in ϵ_{zz}, ϵ_{zx}, and ϵ_{yz} gives

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} \tag{6}$$

   Which can also be written as

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \tag{7}$$

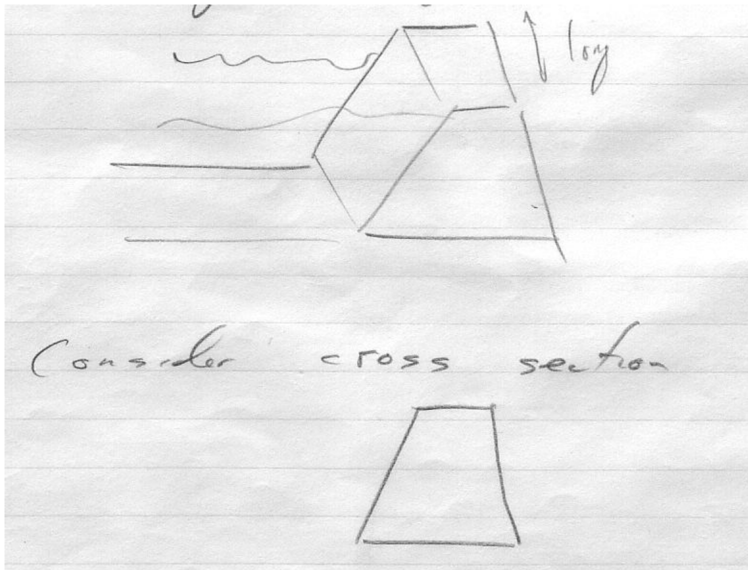
   It is important to note that Strain $\epsilon_{zz} \neq 0$ but that

$$\epsilon_{zz} = \frac{-\nu}{E}(\tau_{xx} + \tau_{yy})$$

$$\epsilon_{zy} = \epsilon_{zx} = 0$$

2. Plane strain (see p 244)

   Long bodies whose geometry and loading do not vary in longitudinal direction



constraint is that there is no out of plane motion

$$w = 0,\, \epsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$$

at each cross section. Applying these to equation (1) gives

$$\boldsymbol{\sigma} = [E](\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_o)$$

$$[E] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) - E\alpha T$$

$$\sigma_{yz} = \sigma_{xz} = 0$$

17.2 Problems

1. Called CST for Constant Strain Triangle. In bending problems it creates a spurious shear that doesn't exist.

2. Locks in bending

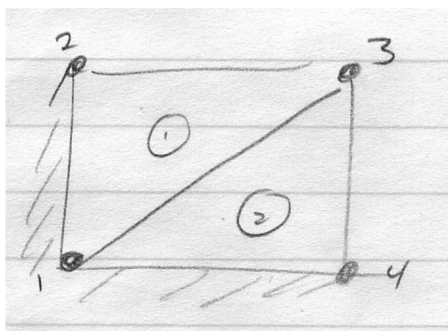
Consider the plain strain

$$[E] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Consider volumetric strain

$$\begin{aligned} \Delta &= \frac{(\mathrm{d}x + \mathrm{d}u)(\mathrm{d}y + \mathrm{d}v) - \mathrm{d}x\mathrm{d}y}{\mathrm{d}x\mathrm{d}y} \\ &= \frac{\mathrm{d}x\mathrm{d}v + \mathrm{d}u\mathrm{d}y + \mathrm{d}u\mathrm{d}v}{\mathrm{d}x\mathrm{d}y} \\ &= \epsilon_y + \epsilon_x + \text{ H.O.T.} \end{aligned}$$

under uniform pressure this is the strain form. If $\nu = 0.5$ as in rubber, the stress to cause such a strain $\rightarrow \infty$ $\boldsymbol{\sigma} = \underbrace{[E]}_{\infty} \boldsymbol{\epsilon}$



Any motion of 3 will cause volumetric strain in either element 1 or 2. With that not being allowed, the mesh has become "locked."

18 Isoparametric Elements

Isoparametric elements make it possible to generate elements that have curved sides.

An element is isoparametric if the shape functions are the same for the coordinates and the field variables.

Example 1-D

$$x = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ eq 3.62}$$

$$\phi = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \text{ eq 3.64}$$

2-D

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

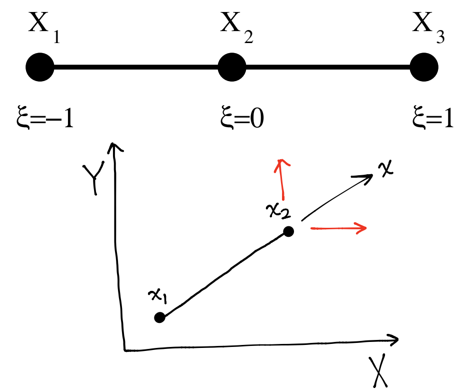
$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

where $N_i = L_i$

These are all Isoparametric

Higher order (quadratic, cubic) can have curved sides.

18.1 Isoparametric Bar element



$$(1)\ x = \begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad u = \begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix} \begin{bmatrix} a_4 \\ a_5 \\ a_6 \end{bmatrix}$$

Where a_n is a generalized degree of freedom. Evaluating (1) for each node gives:

$$(2) \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{array}{c} \begin{matrix} 1 & \xi & \xi^2 \end{matrix} \\ \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$(3) \quad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Substitute (3) into (1) gives:

$$x = \underbrace{\begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1}}_{N\text{, the shape matrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$N = \begin{bmatrix} \frac{1}{2}\left(-\xi + \xi^2\right) & 1 - \xi^2 & \frac{1}{2}\left(\xi + \xi^2\right) \end{bmatrix}$$

$$\epsilon_x = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} N \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Since N is a function of ξ, we need $\frac{\mathrm{d}}{\mathrm{d}x}$ in terms of ξ.

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}\xi}{\mathrm{d}x} \frac{\mathrm{d}}{\mathrm{d}\xi}$$

Define $J = \frac{\mathrm{d}x}{\mathrm{d}\xi}$ as the Jacobian

$$x = N \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\therefore$$

$$J = \frac{\mathrm{d}x}{\mathrm{d}\xi} = \frac{\mathrm{d}}{\mathrm{d}\xi} N \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(-1+2\xi) & -2\xi & \frac{1}{2}(1+2\xi) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The strain is then

$$\epsilon_x = \frac{1}{J} \frac{\mathrm{d}u}{\mathrm{d}\xi} = B \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$B = \frac{1}{J} \frac{\mathrm{d}}{\mathrm{d}\xi} N = \frac{1}{J} \begin{bmatrix} \frac{1}{2}(-1+2\xi) & -2\xi & \frac{1}{2}(1+2\xi) \end{bmatrix}$$

The stiffness is then

$$K = \int_0^l B^T E B A \, \mathrm{d}x$$

Since $dx = J \, \mathrm{d}\xi$

$$K = \int_{-1}^{1} B^T E B A J \, \mathrm{d}\xi$$

$$M = \int_{-1}^{1} N^T \rho N A J \, \mathrm{d}\xi$$

18.1.1 Finding shape functions for an isoparametric 3 noded beam element

For N_1

$$\begin{array}{rccl} N_1 = & 1 & @ & \xi = -1 \\ & 0 & @ & \xi = 0 \\ & 0 & @ & \xi = 1 \\ N_1' = & 0 & @ & \begin{bmatrix} \xi = -1 \\ \xi = 0 \\ \xi = 1 \end{bmatrix} \end{array}$$

Huh?

$$N_1 = a + b\xi + c\xi^2 + d\xi^3 + e\xi^4 + f\xi^5$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 3 & -4 & 5 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix}$$

See WFEM/belshfuncs.m for calculations of shape functions. For interpolating other properties, we could interpolate as

$$N_1 = \frac{1}{2}\left(-\xi + \xi^2\right), \qquad N_2 = \frac{1}{2}\left(\xi + \xi^2\right), \qquad N_3 = 1 - \xi^2$$

but this would err, not recognizing that a linear change in h results in a cubic change. Instead, if J, A, I_{xx} and I_{yy} are known, b, g, h and the shape constant can be calculated and re-interpolated.

18.2 Numerical Integration

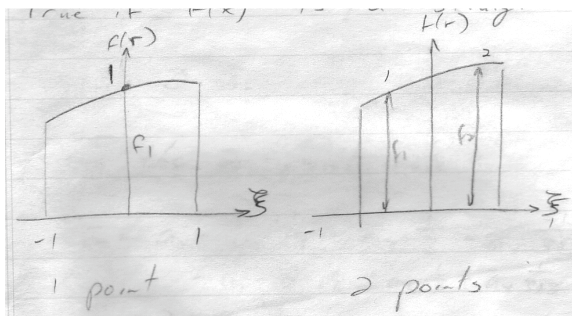
18.2.1 1-D Gauss Quadrature

$$I = \int_{-1}^{1} f(\xi)\,\mathrm{d}\xi$$

First approximation is

$$I = 2f(0)$$

True if $f(\xi)$ is a straight line.



For a 2 point estimate, we sample at $\xi = \pm\frac{1}{\sqrt{3}}$, and weight by 1

$$I = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

which is valid for a cubic polynomial.

Example: Consider

$$\phi = a_1 + a_2\xi + a_3\xi^2 + a_4\xi^3$$

$$I = \int_{-1}^{1} \phi\,\mathrm{d}\xi = 2a_1 + \frac{2}{3}a_3$$

using the 1 point rule

$$I = 2a_1$$

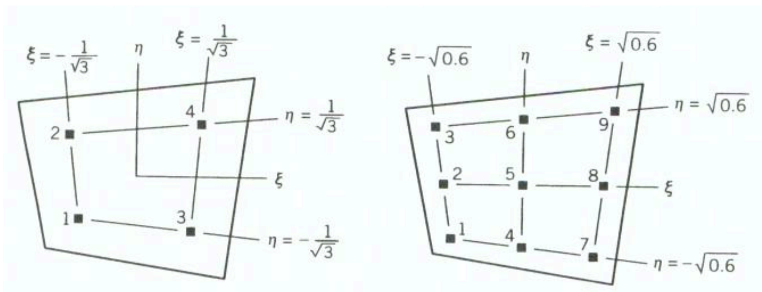
using the 2 point rule

$$\begin{aligned} I =& a_1 + a_2\frac{-1}{\sqrt{3}} + a_3\frac{1}{3} + a_4\frac{-1}{3\sqrt{3}} + a_1 + a_2\frac{1}{\sqrt{3}} + a_3\frac{1}{3} + a_4\frac{1}{3\sqrt{3}} \\ =& 2a_1 + \frac{2}{3}a_3 \end{aligned}$$

A polynomial of degree $2n-1$ is exactly integrated by n-point Gauss quadrature.

Gauss quadrature is not exact for other functions. Instead it is the exact solution for a polynomial fit to those points.

18.2.2 2-D Gauss quadrature of Rectangular regions: 1-direction @ a time



$$
\begin{aligned}
I &= \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)d\xi d\eta = \int_{-1}^{1}\sum_{i=1}^{2} w_i f(\xi_i,\eta)d\eta \\
&= \sum_{j=1}^{2} w_j \left(\sum_{i=1}^{2} w_i f(\xi_i,\eta)\right)\Bigg|_{\eta=\eta_j} \\
&= \sum_{i=1}^{2}\sum_{j=1}^{2} w_i w_j f(\xi_i,\eta_j) \\
&= w_1 w_1 f(\xi_1,\eta_1) + w_2 w_1 f(\xi_2,\eta_1) + w_1 w_2 f(\xi_1,\eta_2) + w_2 w_2 f(\xi_2,\eta_2) \\
&= 1\times 1 f\left(\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right) + 1\times 1 f\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right) \\
&+ 1\times 1 f\left(\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) + 1\times 1 f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)
\end{aligned}
$$

For 3 points, this becomes:

$$
\begin{aligned}
I &= \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)d\xi d\eta \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} w_i w_j f(\xi_i,\eta_j)
\end{aligned}
$$

Then

$$
\begin{aligned}
I = \frac{5}{9}\frac{5}{9}f(-\sqrt{0.6},\sqrt{0.6}) \qquad &+\frac{5}{9}\frac{8}{9}f(0,\sqrt{0.6}) \qquad &+\frac{5}{9}\frac{5}{9}f(\sqrt{0.6},\sqrt{0.6}) \\
+\frac{5}{9}\frac{8}{9}f(-\sqrt{0.6},0) \qquad &+\frac{8}{9}\frac{8}{9}f(0,0) \qquad &+\frac{5}{9}\frac{8}{9}f(\sqrt{0.6},0) \\
+\frac{5}{9}\frac{5}{9}f(-\sqrt{0.6},-\sqrt{0.6}) \qquad &+\frac{5}{9}\frac{8}{9}f(0,-\sqrt{0.6}) \qquad &+\frac{5}{9}\frac{5}{9}f(\sqrt{0.6},-\sqrt{0.6})
\end{aligned}
$$

Weights are 1 for 2-4-8 points (1-2-3 D). Weights are $\frac{5}{9}$, $\frac{8}{9}$, $\frac{5}{9}$ for 3-9-27 pints (1-2-3 D). See Table 6.3-1.

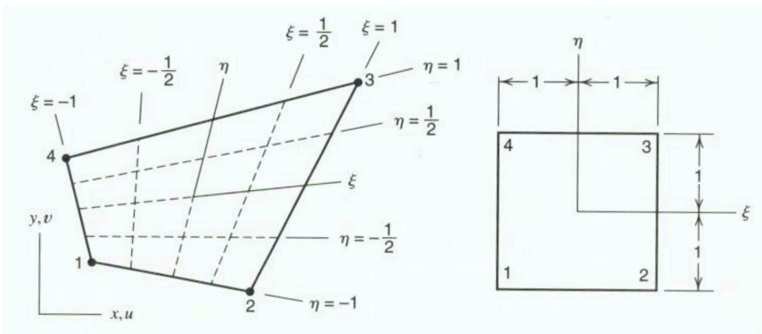
Using 4 points in 2-D exactly calculates *all* terms with up to $\xi^3\eta^3$

$$
\begin{array}{cccccccc}
 & & & 1 & & & 1pt & \\
 & & \xi & & \eta & & & \\
 & \xi^2 & & \xi\eta & & \eta^2 & & 2pt \\
\xi^3 & & \xi^2\eta & & \xi\eta^2 & & \eta^3 & \\
 & \xi^3\eta & & \xi^2\eta^2 & & \xi\eta^3 & & \\
 & & \xi^3\eta^2 & & \xi^2\eta^3 & & & \\
 & & & \xi^3\eta^3 & & & &
\end{array}
$$

18.3 2-D Quadrilateral Element (Q-4)

Pg 205 Cook

In natural coordinates:



The Cartesian and natural coordinates are related by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix}}_{\text{Coordinates of nodes}}$$

where $N_i = \frac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i)$. The shape functions are linear on side edge and quadratic along diagonal. If ϕ is a function of the natural coordinates, partials with respect to x and y can be obtained by

$$\frac{\partial \phi}{\partial \xi} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \xi}$$
$$\frac{\partial \phi}{\partial \eta} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\begin{bmatrix} \frac{\partial \phi}{\partial \xi} \\ \frac{\partial \phi}{\partial \eta} \end{bmatrix} = [J] \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{[J]} \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{bmatrix}$$

Thus

$$\begin{bmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \end{bmatrix} = [J]^{-1} \begin{bmatrix} \frac{\partial\phi}{\partial\xi} \\ \frac{\partial\phi}{\partial\eta} \end{bmatrix} \tag{8}$$

Recall

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sum N_i x_i \\ \sum N_i y_i \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \sum N_i u_i \\ \sum N_i v_i \end{bmatrix}$$

thus

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi} \\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta} \end{bmatrix} = \begin{bmatrix} \sum \frac{\partial N_i}{\partial\xi} x_i & \sum \frac{\partial N_i}{\partial\xi} y_i \\ \sum \frac{\partial N_i}{\partial\eta} x_i & \sum \frac{\partial N_i}{\partial\eta} y_i \end{bmatrix}$$

$$[J] = \frac{1}{4} \begin{bmatrix} -(1-\eta) & 1-\eta & 1+\eta & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & 1+\xi & 1-\xi \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

By definition

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \boldsymbol{\epsilon} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}}_{[\alpha]} \begin{bmatrix} u_{,x} \\ u_{,y} \\ v_{,x} \\ v_{,y} \end{bmatrix} \tag{9}$$

But, by the chain rule

$$\begin{bmatrix} u_{,\xi} \\ u_{,\eta} \end{bmatrix} = [J] \begin{bmatrix} u_{,x} \\ u_{,y} \end{bmatrix}, \begin{bmatrix} v_{,\xi} \\ v_{,\eta} \end{bmatrix} = [J] \begin{bmatrix} v_{,x} \\ v_{,y} \end{bmatrix}$$

so

$$\begin{bmatrix} u_{,x} \\ u_{,y} \\ v_{,x} \\ v_{,y} \end{bmatrix} = \overbrace{\begin{bmatrix} [J]^{-1} & 0 \\ 0 & [J]^{-1} \end{bmatrix}}^{[\beta]} \begin{bmatrix} u_{,\xi} \\ u_{,\eta} \\ v_{,\xi} \\ v_{,\eta} \end{bmatrix} \tag{10}$$

Since $\boldsymbol{u} = N\boldsymbol{d}$, $\boldsymbol{d}$ are displacements

$$\begin{bmatrix} u_{,\xi} \\ u_{,\eta} \\ v_{,\xi} \\ v_{,\eta} \end{bmatrix} = \underbrace{\begin{bmatrix} N_{1,\xi} & 0 & N_{2,\xi} & 0 & \\ N_{1,\eta} & 0 & N_{2,\eta} & 0 & \cdots \\ 0 & N_{1,\xi} & 0 & N_{2,\xi} & \\ 0 & N_{1,\eta} & 0 & N_{2,\eta} & \end{bmatrix}}_{[\gamma]} \boldsymbol{d} \tag{11}$$

From equations 9, 10, and 11

$$B = [\alpha][\beta][\gamma]$$

$$K = \iint_A B^T [E] B t \mathrm{d}x \mathrm{d}y$$

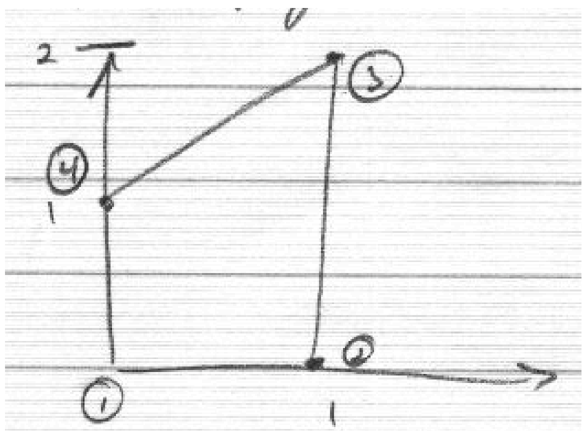
Note that

$$\mathrm{d}x\mathrm{d}y = \det(J)\mathrm{d}\xi\mathrm{d}\eta$$

Note: $\det(J) = \frac{A}{4}$ for a parallelogram

18.3.1 Example

Find the strain at (x, y)=(0,0) of a bilinear quadrilateral (Q4) element with nodes 1-4 at (0,0), (1,0), (1,2), and (0,1) in terms of u_2 and v_2 (presume all other nodal displacements are zero).



$$\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_2}{\partial \xi} u_2 \\ \frac{\partial N_2}{\partial \eta} u_2 \\ \frac{\partial N_2}{\partial \xi} v_2 \\ \frac{\partial N_2}{\partial \eta} v_2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} (1-\eta)u_2 \\ -(1+\xi)u_2 \\ (1-\eta)v_2 \\ -(1+\xi)v_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}u_2 \\ 0 \\ \frac{1}{2}v_2 \\ 0 \end{bmatrix}$$

We need the Jacobian at $\xi = -1$ and $\eta = -1$

$$J = \frac{1}{4} \begin{bmatrix} -2 & 2 & 0 & 0 \\ -2 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

We know then that

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = \begin{bmatrix} u_2 \\ 0 \end{bmatrix}$$

and also

$$\begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} = \begin{bmatrix} v_2 \\ 0 \end{bmatrix}$$

Thus

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} u_2 \\ 0 \\ v_2 \end{bmatrix}$$

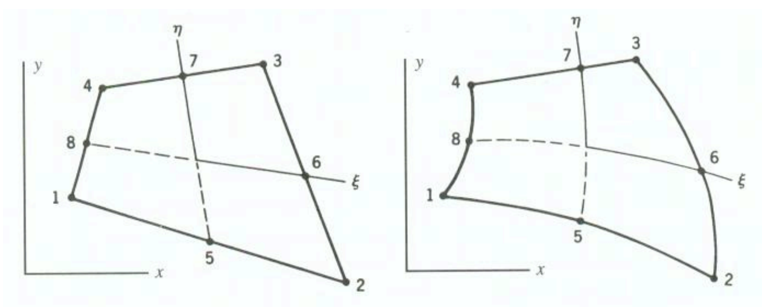
18.4 Quadratic Quadrilaterals (Q8 and Q9)

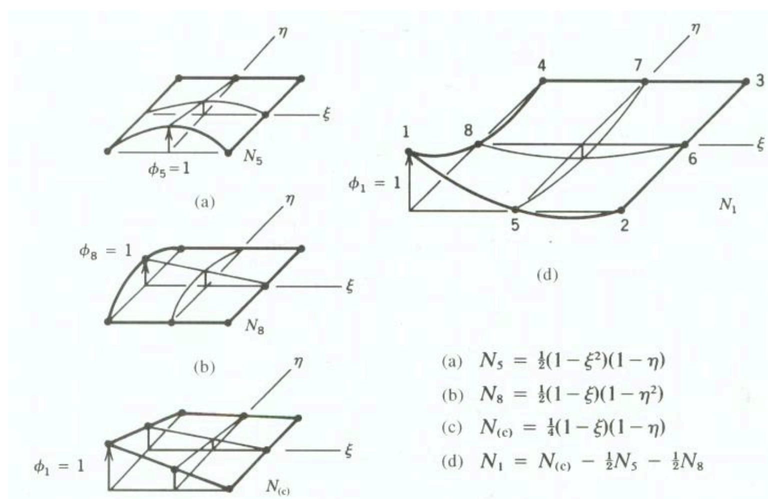
18.4.1 Q8

All new: Nodes on each side! *They don't need to be in the center!*

4 new shape functions

All are now quadratic in η or ξ. See book for derivation of shape functions.





Displacement fields are represented by

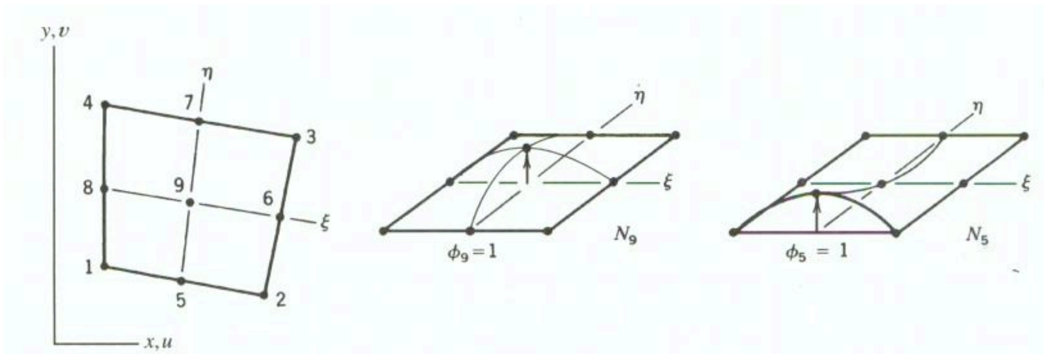
$$u = a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^2\eta + a_8\xi\eta^2$$

and

$$v = a_9 + a_{10}\xi + a_{11}\eta + a_{12}\xi^2 + a_{13}\xi\eta + a_{14}\eta^2 + a_{15}\xi^2\eta + a_{16}\xi\eta^2$$

Notice the missing $\xi^2\eta^2$ term (whether it's missing or not is subjective). Strain can vary linearly along an edge, but quadratically along a diagonal.

18.4.2 Q9



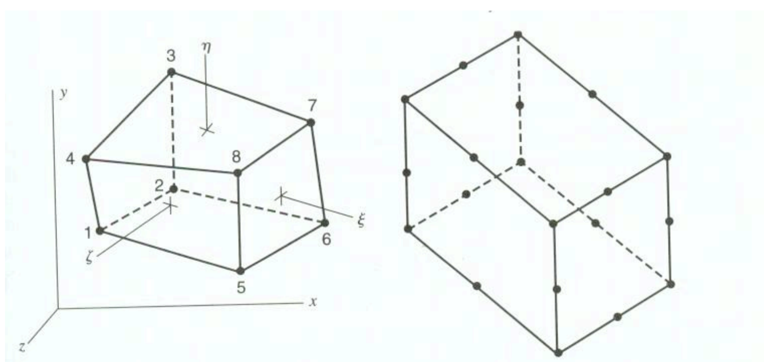
Displacement fields are represented by

$$u = a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^2\eta + a_8\xi\eta^2 + a_9\xi^2\eta^2$$

and

$$v = a_{10} + a_{11}\xi + a_{12}\eta + a_{13}\xi^2 + a_{14}\xi\eta + a_{15}\eta^2 + a_{16}\xi^2\eta + a_{17}\xi\eta^2 + a_{18}\xi^2\eta^2$$

18.5 Hexahedron (Brick) Element



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [N] \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ x_2 \\ \vdots \\ z_8 \end{bmatrix}$$

$$[N] = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & & N_8 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \cdots & 0 & N_8 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & & 0 & 0 & N_8 \end{bmatrix}$$

$$N_i = \frac{1}{8}(1+\xi\xi_i)(1+\eta\eta_i)(1+\zeta\zeta_i)$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = [N] \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ \vdots \\ w_8 \end{bmatrix}$$

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{bmatrix} = \underbrace{[B]}_{6x24} \underbrace{\boldsymbol{d}^{(e)}}_{24x1}$$

$$\underbrace{B}_{6x24} = [[B_1]\,[B_2]\,[B_3]\cdots[B_8]]$$

$$\underbrace{B_i}_{6x3} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial z} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \xi} + \frac{\partial N_i}{\partial z}\frac{\partial z}{\partial \xi} \\ \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \eta} + \frac{\partial N_i}{\partial z}\frac{\partial z}{\partial \eta} \\ \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \zeta} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \zeta} + \frac{\partial N_i}{\partial z}\frac{\partial z}{\partial \zeta} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ & \vdots & \\ & & \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{bmatrix} = [J] \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{bmatrix}$$

$$J = \begin{bmatrix} \sum_{i=1}^{8}\left(\frac{\partial N_i}{\partial \xi} x_i\right) & \sum_{i=1}^{8}\left(\frac{\partial N_i}{\partial \xi} y_i\right) \\ \sum_{i=1}^{8}\left(\frac{\partial N_i}{\partial \eta} x_i\right) & \ddots \\ & \end{bmatrix}$$

Where $\frac{\partial N_i}{\partial \xi} = \frac{1}{8}\xi_i(1+\eta\eta_i)(1+\zeta\zeta_i)$

Now we need to obtain $\frac{\partial N_i}{\partial x}$ ...

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{bmatrix} = [J]^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} \end{bmatrix}$$

Use these to plug into B. The element stiffness matrix is

$$K^e = \int_V B^T [E] B \, \mathrm{d}V$$

we need $\mathrm{d}V$ in natural coordinates.

$$K^e = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} B^T [E] B \det[J] \, \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}\zeta$$

Use Gauss integration to obtain element matrices.

18.5.1 The volume element

$$\mathbf{d\boldsymbol{\xi}} = \begin{bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \\ \frac{\partial z}{\partial \xi} \end{bmatrix} \mathrm{d}\xi \,, \; \mathbf{d\boldsymbol{\eta}} = \begin{bmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \\ \frac{\partial z}{\partial \eta} \end{bmatrix} \mathrm{d}\eta \,, \; \mathbf{d\boldsymbol{\zeta}} = \begin{bmatrix} \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \zeta} \end{bmatrix} \mathrm{d}\zeta$$

Since the triple product is

$$\det \begin{vmatrix} \mathbf{d\boldsymbol{\xi}} & \mathbf{d\boldsymbol{\eta}} & \mathbf{d\boldsymbol{\zeta}} \end{vmatrix}$$

and we can factor $\mathrm{d}\xi \; \mathrm{d}\eta \; \mathrm{d}\zeta$

$$\mathrm{d}V = \det \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{vmatrix} \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}\zeta = \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

18.5.2 Incompatible Modes

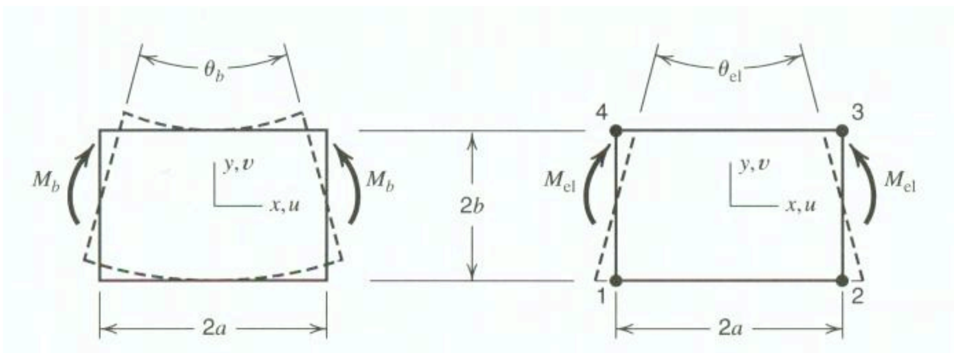
Both the Quad and Hex elements exhibit *shear locking*.

When bent they exhibit shear as well as bending strain. This shear absorbs strain energy so the element is stiffer then it should be.

Example: Q4 element

$$U = \frac{1}{2}\int \boldsymbol{\epsilon}^t [E] \boldsymbol{\epsilon} \, \mathrm{d}V$$

$$\mathrm{d}V = t \, \mathrm{d}x \, \mathrm{d}y$$



reality	element
$\epsilon_x = -\frac{\theta_b y}{2a}$	$\epsilon_x = -\frac{\theta_{el} y}{2a}$
$\epsilon_y = \nu\frac{\theta_b y}{2a}$	$\epsilon_y = 0$
$\gamma_{xy} = 0$	$\gamma_{xy} = -\frac{\theta_{el} x}{2a}$

If $M_b = M_{el}$, then

$$\frac{\Theta_{el}}{\Theta_b} = \frac{1-\nu^2}{1+\frac{1-\nu}{2}\left(\frac{a}{b}\right)^2}$$

This ratio $\to 0$ as $\frac{a}{b} \to \infty$ and is known as *locking*.

Higher order Q elements (8,9) don't have this issue, but are more expensive.

Incompatible modes can alternatively be added. This makes the element subparametric.

$$u = \sum_{i=1}^{4} N_i u_i + (1-\xi^2)a_1 + (1-\eta^2)a_2$$

$$v = \sum_{i=1}^{4} N_i v_i + (1-\xi^2)a_3 + (1-\eta^2)a_4$$

a_i are nodeless DOF's.

$$B = \begin{bmatrix} B_d & B_a \end{bmatrix}$$

B_d is the same, and B_a operates on the nodeless DOF

Jacobian calculations are based only on nodal DOFs because x and y are still functions of the original shape functions. This adds 4 columns to B. For Q4, γ is appended by

$$[\gamma_a] = \begin{bmatrix} -2\xi & 0 & 0 & 0 \\ 0 & -2\eta & 0 & 0 \\ 0 & 0 & -2\xi & 0 \\ 0 & 0 & 0 & -2\eta \end{bmatrix}$$

This is now the Q6 element

If it is non-rectangular, it now fails to be able to represent a constant stress state. This is bad.

Remedy:

Consider constant stress $\boldsymbol{\sigma_o}$. We require that this causes no displacements of a_i because they represent non-constant stress. This requires that load terms associated with a_i for a uniform stress be zero.

$$\boldsymbol{r_{e_{a_i}}} = \iint\limits_V B_a^T \boldsymbol{\sigma_o} \,\mathrm{d}V = \mathbf{0}$$

Since $\boldsymbol{\sigma_o}$ is constant non zero and arbitrary

$$= \iint\limits_V B_a^T \,\mathrm{d}V = [0]$$

$$= \int_{-1}^{1}\int_{-1}^{1} B_a^T t J \,\mathrm{d}\xi \,\mathrm{d}\eta$$

($J = det([J])$) When J constant, this is no problem (rectangular). To fix this, we fix the values of $[J]$ and J to be those at $\xi = \eta = \zeta = 0$, forcing this to integrate to zero. This is the QM6 element.

At all Gauss points during integration when B_a is involved, we always use $[J]$ at the origin.

The integration of B_d is *NOT* modified.

18.6 Guyan Reduction/Static Condensation

Quite often the mass matrix is singular or nearly singular. DOFs with low $\frac{m_{ii}}{k_{ii}}$ are candidates for removal from the model. This is also the case when we derive a subparametric stiffness matrix. Also necessary when applying BC using Lagrange Multipliers.

$\boldsymbol{d_1}$ are coords to be kept

$\boldsymbol{d_2}$ are coords to remove

$$T = \frac{1}{2}\begin{bmatrix} \dot{\boldsymbol{d}}_1 \\ \dot{\boldsymbol{d}}_2 \end{bmatrix}^T \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{d}}_1 \\ \dot{\boldsymbol{d}}_2 \end{bmatrix}$$

$$U = \frac{1}{2}\begin{bmatrix} \boldsymbol{d_1} \\ \boldsymbol{d_2} \end{bmatrix}^T \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{d_1} \\ \boldsymbol{d_2} \end{bmatrix}$$

$$\frac{\partial U}{\partial \boldsymbol{d_2}} = \frac{\partial}{\partial \boldsymbol{d_2}}\left(\boldsymbol{d_1}^T K_{11}\boldsymbol{d_1} + \boldsymbol{d_1}^T K_{12}\boldsymbol{d_2} + \boldsymbol{d_2}^T K_{21}\boldsymbol{d_1} + \boldsymbol{d_2}^T K_{22}\boldsymbol{d_2}\right) = 0$$

$$= \boldsymbol{d_1}^T K_{12} + \boldsymbol{d_1}^T K_{21} + 2\boldsymbol{d_2}^T K_{22} = 0$$

$$2\boldsymbol{d_1}^T K_{12} + 2\boldsymbol{d_2}^T K_{22} = 0$$

$$\boldsymbol{d_2} = -K_{22}^{-1}K_{21}\boldsymbol{d_1}$$

Assume then

$$\boldsymbol{d} = \begin{bmatrix} \boldsymbol{d_1} \\ \boldsymbol{d_2} \end{bmatrix} = Q\boldsymbol{d_1}$$

Where

$$Q = \begin{bmatrix} I \\ -K_{22}^{-1}K_{21} \end{bmatrix}$$

$$M\ddot{\boldsymbol{d}} + K\boldsymbol{d} = \boldsymbol{f}$$

$$\underbrace{Q^T M Q}_{M_{red}}\ddot{\boldsymbol{d}}_1 + \underbrace{Q^T K Q}_{K_{red}}\boldsymbol{d_1} = Q^T\boldsymbol{f}$$

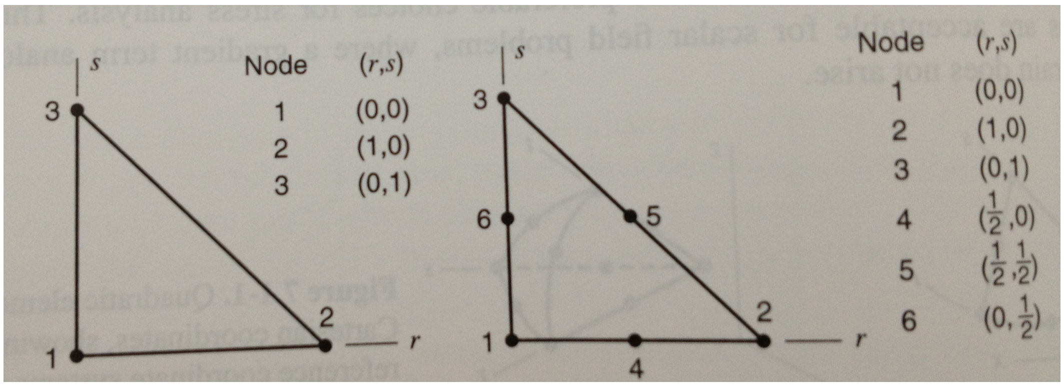
$$K_{red} = Q^T K Q = K_{11} - K_{12}K_{22}^{-1}K_{21}$$

$$M_{red} = Q^T M Q = M_{11} - K_{21}^T K_{22}^{-1}M_{21} - M_{12}K_{22}^{-1}K_{21} + K_{21}^T K_{22}^{-1}M_{22}K_{22}^{-1}K_{21}$$

This method is really a Rayleigh-Ritz Method, with good guesses for the basis vectors.

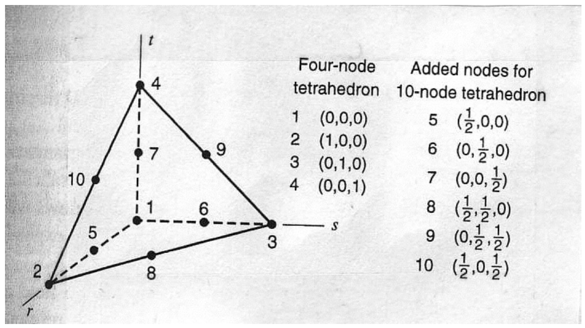
18.7 Isoparametric Triangles and Tetrahedra

Recall: an element is *isoparametric* if its geometry and its field quantity ϕ use the same set of shape functions.



$$\begin{array}{rcc} & \text{Linear} & \text{Quadratic} \\ N_1 = & 1-r-s & (1-r-s)(1-2r-2s) \\ N_2 = & r & r(2r-1) \\ N_3 = & s & s(2s-1) \\ N_4 = & & 4r(1-r-s) \\ N_5 = & & 4rs \\ N_6 = & & 4s(1-r-s) \end{array}$$

Tetrahedron



$$\begin{array}{rcc}
N_1 = & 1-r-s-t & (1-r-s-t)(1-2r-2s-2t) \\
N_2 = & r & r(2r-1) \\
N_3 = & s & s(2s-1) \\
N_4 = & t & t(2t-1) \\
N_5 = & & 4r(1-r-s-t) \\
N_6 = & & 4s(1-r-s-t) \\
N_7 = & & 4t(1-r-s-t) \\
N_8 = & & 4rs \\
N_9 = & & 4st \\
N_{10} = & & 4tr
\end{array}$$

$$x = \sum N_i x_i$$

$$y = \sum N_i y_i$$

When 3-D

$$z = \sum N_i z_i$$

and the field variable is

$$\phi = \sum N_i \phi_i$$

For elasticity we have 2 field variables in 2-D, 3 in 3-D.

18.8 CST and LST Matrices

For a single field problem,

$$\begin{bmatrix} \phi_{,x} \\ \phi_{,y} \end{bmatrix} = B \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$

Since the shape functions are in terms of r and s, we cannot take the partials of them w.r.t. x and y.

18.8.1 Linear Triangle (CST)

Using the chain rule

$$\left[\begin{array}{c} \frac{\partial \phi}{\partial r} \\ \frac{\partial \phi}{\partial s} \end{array}\right] = \left[\begin{array}{c} \frac{\partial x}{\partial r}\frac{\partial \phi}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial \phi}{\partial y} \\ \frac{\partial x}{\partial r}\frac{\partial \phi}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial \phi}{\partial y} \end{array}\right] = \underbrace{\left[\begin{array}{cc} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{array}\right]}_{[J]} \left[\begin{array}{c} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{array}\right]$$

From expressions of x and y in terms of shape functions,

$$\frac{\partial x}{\partial r} = \sum \frac{\partial N_i}{\partial r} x_i$$

$$\frac{\partial y}{\partial r} = \sum \frac{\partial N_i}{\partial r} y_i$$

$$\frac{\partial x}{\partial s} = \sum \frac{\partial N_i}{\partial s} x_i$$

$$\frac{\partial y}{\partial s} = \sum \frac{\partial N_i}{\partial s} y_i$$

Substituting,

$$[J] = \left[\begin{array}{ccc} -1 & 1 & 0 \\ -1 & 0 & 1 \end{array}\right] \left[\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{array}\right] = \left[\begin{array}{cc} x_{21} & y_{21} \\ x_{31} & y_{31} \end{array}\right]$$

where $x_{ij} = x_i - x_j$

Then

$$\left[\begin{array}{c} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{array}\right] = [J]^{-1} \left[\begin{array}{c} \frac{\partial \phi}{\partial r} \\ \frac{\partial \phi}{\partial s} \end{array}\right]$$

where

$$[J]^{-1} = \frac{\left[\begin{array}{cc} y_{31} & -y_{21} \\ -x_{31} & x_{21} \end{array}\right]}{x_{21}y_{31} - x_{31}y_{21}}$$

Recalling

$$\phi = \sum N_i \phi_i$$

Then

$$\frac{\partial \phi}{\partial r} = \sum \frac{\partial N_i}{\partial r}\phi_i \,, \; \frac{\partial \phi}{\partial s} = \sum \frac{\partial N_i}{\partial s}\phi_i$$

Then

$$\begin{bmatrix} \frac{\partial \phi}{\partial r} \\ \frac{\partial \phi}{\partial s} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$

Thus

$$\begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{bmatrix} = \underbrace{[J]^{-1} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$

For elasticity

$$B = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{12} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{21} \end{bmatrix}$$

$[E]$ is given by (3.1-5a) (Note $E_{11} = E_{22} = E_{33} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}$)

18.8.2 Quadratic Triangle (LST)

Summation now has six terms. Jacobian matrix is still 2×2.

eg.

$$\frac{\partial x}{\partial s} = (4r + 4s - 3)x_1 + 0x_2 + (4s - 1)x_3 - 4rx_4 + 4rx_5 - 4(r + 2s - 1)x_6$$

Consider $x_4 = \frac{1}{2}(x_1 + x_2)$, $x_5 = \frac{1}{2}(x_2 + x_3)$, $x_6 = \frac{1}{2}(x_3 + x_1)$.

$$\frac{\partial x}{\partial s} = x_3 - x_1$$

- The Jacobian of a straight-sided quadratic triangle element is given by the Jacobian of the corresponding linear element.

Side nodes allow volumetric-strain-relief that allows it to work when $\nu = 0.5$.

18.8.3 Tetrahedra

Supplement with the coordinate t, and follow the same procedure for linear or quadratic tetrahedra

18.8.4 Integration of Triangles: Numerical

When sides are curved or nodes are not evenly spaced, we call the element "distorted." Closed-form integration isn't possible.

The equation for K is

$$K = \int_A B^T[E]Bt \, \mathrm{d}A$$

Consider only one term, say k_{11}

$$k_{11} = \int_A \left[B^T[E]Bt\right]_{11} \mathrm{d}A = \sum_{i=1}^{n} \left[B^T[E]Bt\right]_{11_i} J_i W_i$$

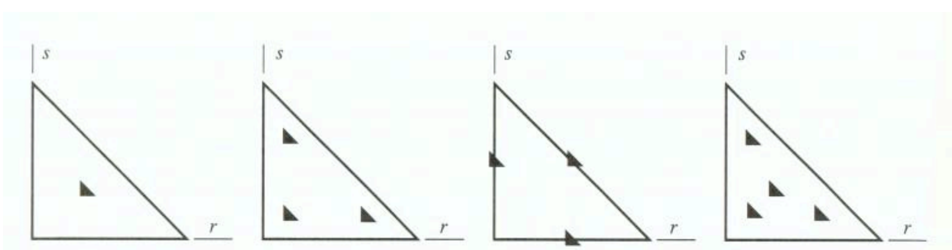
where $[B^T[E]Bt]_{11_i}$ is the value evaluated at the ith point

$J_i = \frac{1}{2}|J|_i$ is $\frac{1}{2}$ the determinant of the Jacobian evaluated at the ith point

W_i is the "weight" at the ith point.

See table 7.4-1 for the points and weights.

$$I = \sum_{i=1}^{n} w_i f(r,s)$$



For N=1

$$w_1 = 1, \qquad (r,s) = \left(\frac{1}{3}, \frac{1}{3}\right)$$

For N=3

$$w_1 = \frac{1}{3} \quad (r,s) = \left(\tfrac{2}{3}, \tfrac{1}{6}\right)$$
$$w_2 = \frac{1}{3} \quad (r,s) = \left(\tfrac{1}{6}, \tfrac{1}{6}\right)$$
$$w_3 = \frac{1}{3} \quad (r,s) = \left(\tfrac{1}{6}, \tfrac{2}{3}\right)$$

or

$$
\begin{aligned}
w_1 &= \frac{1}{3} \quad (r,s) = \left(\tfrac{1}{2}, 0\right) \\
w_2 &= \frac{1}{3} \quad (r,s) = \left(0, \tfrac{1}{2}\right) \\
w_3 &= \frac{1}{3} \quad (r,s) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)
\end{aligned}
$$

A rule of precision K exactly integrates all terms of the form $r^l s^m$, $l+m \leq K$ and is inexact for at least on term $l+m > K$ (see section on analytical integration.)

18.8.5 Integration of Tetrahedra: Numeric

$$K_{11} = \int_V [B^T[E]B]_{11} \, \mathrm{d}V = \sum_{i=1}^{n} [B^T[E]B]_{11_i} J_i W_i$$

$$J_i = \frac{1}{6} |[J_i]|$$

For an undistorted element, $|[J_i]| = 6V \therefore J_i = V$

See Table 7.4-2

18.9 The Patch Test

Consider solving a problem repeatedly with a finer and finer mesh. Will the sequence of solutions converge? The answer is yes only if the element can pass the patch test, which asks if an assembly of elements can display a constant strain.

Procedure:

- Build a "patch" of elements
- Element shapes must be irregular (not in their "design" local coordinate shape)
- The boundary can be (and is easiest to have) rectangular with evenly spaced nodes
- Apply loading and boundary conditions that must result in a uniform strain
- Evaluate the solution to assure that this is true for each uniform strain/stress (e.g. σ_{xx}, τ_{xy})
- If the elements all have uniform strain at each corner (and each element), then the element passes the patch test.

The patch test assures that as mesh size is refined, and thus the variation of strain/stress from one element to the next is minimized, that it can converge to zero variation between elements even when elements are skewed as is necessary in real problems.

Care must be taken that boundaries are allowed to slide appropriately so as not to cause undue Poisson effects.

18.9.1 The Weak Patch Test

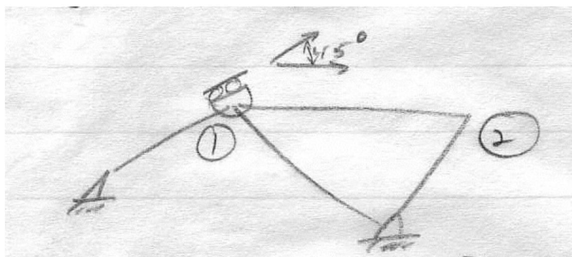
An element is said to pass the weak patch test if it can satisfy the requirements of the patch test, but only if the elements are shaped such that the Jacobian is constant (e.g. parallelograms for the Q4 element).

19 Constraints (Boundary Conditions) (chapter 13)

By Transformation

Example.

Say we want DOF 1 to be equal to DOF 2 for the following:



$$\boldsymbol{D} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_1 \\ u_2 \\ v_2 \end{bmatrix}$$

define

$$\boldsymbol{D}_r = \begin{bmatrix} u_1 \\ u_2 \\ v_2 \end{bmatrix}$$

Then

$$\boldsymbol{D} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{C} \boldsymbol{D}_r$$

$$\Pi = \frac{1}{2}\boldsymbol{D}^T K \boldsymbol{D} = \frac{1}{2}\boldsymbol{D}_r^T C^T K C \boldsymbol{D}_r$$

$$\therefore K_r = C^T K C$$

19.1 Constraints by transformation 2

Constraint equations are

$$\begin{bmatrix} C_r & C_c \end{bmatrix} \boldsymbol{D} - \boldsymbol{Q} = \boldsymbol{0}$$

$$\boldsymbol{D} = \begin{bmatrix} \boldsymbol{D}_r \\ \boldsymbol{D}_c \end{bmatrix}$$

C_c is square and nonsingular because

1. # of constraints is # condensed DOFs

2. constraints are independent

   $\boldsymbol{D}_c$ can be solved for

$$C_r\boldsymbol{D}_r + C_c\boldsymbol{D}_c = \boldsymbol{Q}$$

$$\boldsymbol{D}_c = C_c^{-1}(\boldsymbol{Q} - C_r\boldsymbol{D}_r)$$

$$\boldsymbol{D} = \begin{bmatrix} \boldsymbol{D}_r \\ \boldsymbol{D}_c \end{bmatrix} = \underbrace{\begin{bmatrix} I \\ -C_c^{-1}C_r \end{bmatrix}}_{T} \boldsymbol{D}_r + \underbrace{\begin{bmatrix} \boldsymbol{0} \\ C_c^{-1}\boldsymbol{Q} \end{bmatrix}}_{Q_o}$$

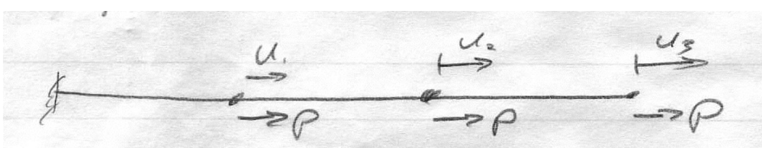
Original eqns are

$$M\ddot{\boldsymbol{D}} + K\boldsymbol{D} = \boldsymbol{R}$$

Subst for $\boldsymbol{D}$, premultiply by T^T

$$M_r = T^TMT,\ K_r = T^TKT,\ \boldsymbol{R_r} = T^T\left(\boldsymbol{R} - K\begin{bmatrix} \boldsymbol{0} \\ C_c^{-1}\boldsymbol{Q} \end{bmatrix}\right)$$

19.1.1 Example



$$\begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} P \\ P \\ P \end{bmatrix}$$

Say $u_3 = u_2$, $\therefore$

$$\begin{bmatrix} \underbrace{0 \quad 1}_{C_r} & \underbrace{-1}_{C_c} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \boldsymbol{0}$$

$$T = \begin{bmatrix} I \\ -C_c^{-1}C_r \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$K' = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}$$

$$R' = \begin{bmatrix} P \\ 2P \end{bmatrix}$$

19.2 Constraints By Lagrange Multipliers

Consider the constraints

$$C\boldsymbol{D} = \boldsymbol{Q}$$

Then we can introduce

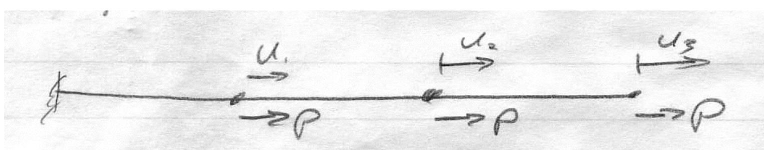
$$\Pi_p(\boldsymbol{D}, \boldsymbol{\lambda}) = \frac{1}{2}\boldsymbol{D}^T K \boldsymbol{D} - \boldsymbol{D}^T \boldsymbol{R} + \boldsymbol{\lambda}^T (C\boldsymbol{D} - \boldsymbol{Q})$$

$$\delta\Pi_p = \delta\boldsymbol{D}^T K \boldsymbol{D} - \delta\boldsymbol{D}^T \boldsymbol{R} + \delta\boldsymbol{\lambda}^T (C\boldsymbol{D} - \boldsymbol{Q}) + \delta\boldsymbol{D}^T C^T \boldsymbol{\lambda}$$

$\therefore K\boldsymbol{D} + C^T\boldsymbol{\lambda} = \boldsymbol{R}$ and $C\boldsymbol{D} = \boldsymbol{Q}$

$$\therefore \underbrace{\begin{bmatrix} K & C^T \\ C & 0 \end{bmatrix}}_{\tilde{K}} \begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} \\ \boldsymbol{Q} \end{bmatrix}$$

19.2.1 Previous Example



Say $u_3 = u_2$

$$C = [\ 0 \quad 1 \quad -1\ ]$$

$$\tilde{K} = \begin{bmatrix} 2k & -k & 0 & 0 \\ -k & 2k & -k & 1 \\ 0 & -k & k & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} P \\ P \\ P \\ 0 \end{bmatrix}$$

Note: This matrix results in a singular mass matrix. For a dynamic analysis, Guyan Reduction must be performed *before* eigensolution is performed.

20 Dynamic Analysis

The equation of motion is defined by Finite Elements is

$$M\ddot{\boldsymbol{D}} + K\boldsymbol{D} = \boldsymbol{R}$$

Finite Elements is generally not effective for deriving the damping matrix C which represents energy dissipation.

Assume

$$\boldsymbol{D}(t) = \overline{\boldsymbol{D}}e^{j\omega t}$$

ω is the natural frequency

$$(-M\omega^2 + K)\overline{\boldsymbol{D}}e^{j\omega t} = \mathbf{0}$$

$$e^{j\omega t} \neq \mathbf{0}$$

$$(K - M\omega^2)\overline{\boldsymbol{D}} = \mathbf{0}$$

let $\lambda = \omega^2$

$$K\overline{\boldsymbol{D}} = \lambda M\overline{\boldsymbol{D}}$$

This is the general eigenvalue problem (see eig in MATLAB)

20.1 Preparation for solving the eigenvalue problem

M and K are both real, symmetric matrices. Many algorithms work on a single matrix. Let's manipulate the equation so that only one matrix exists.

The mass matrix should always be positive definite. If it is not, Guyan reduction should be applied to the equations to remove massless DOFs.

20.1.1 Positive Definite

A matrix is positive definite if

1. all of its eigenvalues > 0
2. determinants of all of its principle minors are > 0.

The second is easy to calculate and is relatively robust computation compared to the first.

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$|m_{11}| > 0\,, \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} > 0\,, |M| > 0$$

20.1.2 Positive Semi-Definite

If the principle minor's determinants are ≥ 0 then the matrix is positive semi-definite.

A lumped-mass matrix is often positive semi-definite. eg for a beam the lumped mass matrix used is sometimes

$$M = \frac{m}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This requires use of Guyan reduction see HRZ Lumping

20.1.3 HRZ Lumping

1. Calculate only diagonal elements of a mass matrix (each element)

2. Scale them so total linear mass is preserved. Scale rotational diagonals by the same amount/ratio

Example: Beam element

$$M = \frac{m}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ & 4L^2 & 13L & -3L^2 \\ & sym & 156 & -22L \\ & & & 4L^2 \end{bmatrix}$$

Lumped, this becomes

$$M = m \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ & \frac{L^2}{78} & 0 & 0 \\ & sym & \frac{1}{2} & 0 \\ & & & \frac{L^2}{78} \end{bmatrix}$$

$$\frac{L^2}{78} = \left(\frac{210}{156}\right) \frac{4L^2}{420}$$

If M is positive definite, it can be decomposed into $M = L^T L$ where L is a nonsingular matrix (does not have a zero eigenvalue and is invertible)

The eigenvalue problem is then

$$K\overline{\boldsymbol{D}} = \lambda L^T L \overline{\boldsymbol{D}}$$

Consider the substitution

$$\boldsymbol{v} = L\overline{\boldsymbol{D}}$$

$$KL^{-1}\boldsymbol{v} = \lambda L^T \boldsymbol{v}$$

pre-multiply by $(L^T)^{-1}$

$$(L^T)^{-1}KL^{-1}\boldsymbol{v} = \lambda \boldsymbol{v}$$

is now the standard eigenvalue problem.

$$A\boldsymbol{v} = \lambda \boldsymbol{v}$$

$$A = (L^T)^{-1}KL^{-1}$$

Note that the values λ_i are the same, but the vectors $\overline{\boldsymbol{D}}$ and $\boldsymbol{v}$ are not. $\overline{\boldsymbol{D}}$ can be obtained from $\overline{\boldsymbol{D}} = L^{-1}\boldsymbol{v}$

Note, if M is diagonal, $L = \sqrt{M} = \sqrt{m_{ii}}$.

If A is $n \times n$, there will be n eigenvalues and n eigenvectors $(\lambda_i, \boldsymbol{u_i})$

Eigenvalues are sorted in increasing order

$$\lambda_1 \leq \lambda_2 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

If two eigenvalues are equal, then we say we have "repeated modes."

In this case, any linear combination of eigenvectors corresponding to the repeated eigenvalues is also a mode. We choose eigenvectors to be orthogonal for computational efficiency.

If A is positive definite $\lambda_i > 0$ otherwise $\lambda_i \geq 0$.

The zero eigenvalues correspond to rigid body modes. This will happen when K is positive semi-definite instead of positive definite. There are a variety of ways to take advantage of the special properties of M, K.

1. Symmetric
2. Banded
3. Real
4. Eigenvalues and vectors are real

Once obtained, the eigenvectors are normalized such that

$$\overline{\boldsymbol{D}}_i^T M \overline{\boldsymbol{D}}_i = 1$$

The eigenvectors satisfy the relation

$$\overline{\boldsymbol{D}}_i^T M \overline{\boldsymbol{D}}_j = \delta_{ij}$$

where δ_{ij} is the Kronecker Delta

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

The eigenvalue equation is

$$K\overline{\boldsymbol{D}}_i = \lambda_i M \overline{\boldsymbol{D}}_i$$

$$\overline{\boldsymbol{D}}_j^T K \overline{\boldsymbol{D}}_i = \lambda_i \overline{\boldsymbol{D}}_j^T M \overline{\boldsymbol{D}}_i$$

$$\overline{\boldsymbol{D}}_j^T K \overline{\boldsymbol{D}}_i = \lambda_i \delta_{ij}$$

gives the eigenvalues when $i = j$ and is the K- orthogonality condition.

Eigenvectors of repeated eigenvalues must satisfy the orthogonality condition for clarity.

If we write

$$S = [\overline{\boldsymbol{D}}_1, \overline{\boldsymbol{D}}_2, \overline{\boldsymbol{D}}_3, \ldots, \overline{\boldsymbol{D}}_i, \ldots, \overline{\boldsymbol{D}}_n]$$

$$S^T K S = \Lambda$$

$$S^T M S = I$$

Λ is a diagonal matrix I is the identity matrix Checking eigenvalue calculation

$$A\overline{\boldsymbol{D}}_i = \lambda_i \overline{\boldsymbol{D}}_i$$

1. $\operatorname{trace}(A) = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}$

2. $\det(A) = \prod_{i=1}^{n} \lambda_i$

20.1.4 SEREP: System Equivalent Reduction-Expansion Process

Assume the equation of motion of the form

$$M\ddot{\boldsymbol{D}} + C\dot{\boldsymbol{D}} + K\boldsymbol{D} = \boldsymbol{F}$$

We want to reduce the size of the system to a smaller number so that the model only includes p modes or the m degrees of freedom that are necessary to the model. Here p is defined based on the expected bandwidth of the excitation and m is defined based on the number of degrees of freedom that are acted upon by an external load, or are necessary as output states in the model. The number of retained coordinates is constrained to be equal to the number of retained modes, and is chosen to be the larger of p or m.

We then partition the coordinate vector into two parts, x_r and x_t, the retained and truncated coordinates, and reorganize the equations of motion in the form (considering only the undamped case for now)

$$\begin{bmatrix} M_{rr} & M_{rt} \\ M_{tr} & M_{tt} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{D}}_r \\ \ddot{\boldsymbol{D}}_t \end{bmatrix} + \begin{bmatrix} K_{rr} & K_{rt} \\ K_{tr} & K_{tt} \end{bmatrix} \begin{bmatrix} \boldsymbol{D}_r \\ \boldsymbol{D}_t \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_r \\ \boldsymbol{0} \end{bmatrix}$$

Consider the eigensolution for the mass normalized eigenvectors, $\Phi = \begin{bmatrix} \Phi_{ar} & \Phi_{at} \end{bmatrix}$ where r and t stand for retained and truncated, and a represents that all coordinates are retained in the vector. The modes to be retained, Φ_{ar}, is an $n \times m$ matrix, and the modes to be truncated, Φ_{at}, is an $n \times (n-m)$ matrix. Substituting

$$\boldsymbol{D} = \begin{bmatrix} \boldsymbol{D}_r \\ \boldsymbol{D}_t \end{bmatrix} = \Phi \boldsymbol{r} = \begin{bmatrix} \Phi_{ar} & \Phi_{at} \end{bmatrix} \begin{bmatrix} \boldsymbol{r}_r \\ \boldsymbol{r}_t \end{bmatrix} = \begin{bmatrix} \Phi_{rr} & \Phi_{rt} \\ \Phi_{tr} & \Phi_{tt} \end{bmatrix} \begin{bmatrix} \boldsymbol{r}_r \\ \boldsymbol{r}_t \end{bmatrix}, \tag{12}$$

and pre-multiplying by

$$\Phi^T = \begin{bmatrix} \Phi_{ar} & \Phi_{at} \end{bmatrix}^T$$

yields the equations of motion in modal coordinates

$$I\ddot{\boldsymbol{r}} + \Lambda \boldsymbol{r} = \Phi^T \boldsymbol{F} \tag{13}$$

with a transformed mass matrix of

$$\begin{bmatrix} \Phi_{ar}^T \\ \Phi_{at}^T \end{bmatrix} M \begin{bmatrix} \Phi_{ar} & \Phi_{at} \end{bmatrix} = \begin{bmatrix} \Phi_{ar}^T M \Phi_{ar} & \Phi_{ar}^T M \Phi_{at} \\ \Phi_{at}^T M \Phi_{ar} & \Phi_{at}^T M \Phi_{at} \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & I_t \end{bmatrix}$$

and a transformed stiffness matrix of

$$\begin{bmatrix}\Phi_{ar}^T \\ \Phi_{at}^T\end{bmatrix} K \begin{bmatrix}\Phi_{ar} & \Phi_{at}\end{bmatrix} = \begin{bmatrix}\Phi_{ar}^T K \Phi_{ar} & \Phi_{ar}^T K \Phi_{at} \\ \Phi_{at}^T K \Phi_{ar} & \Phi_{at}^T K \Phi_{at}\end{bmatrix} = \begin{bmatrix}\Lambda_r & 0 \\ 0 & \Lambda_t\end{bmatrix}$$

We then truncate the modal vector $\boldsymbol{r} = \begin{bmatrix}\boldsymbol{r}_r^T & \boldsymbol{r}_t^T\end{bmatrix}^T$ by assuming $\boldsymbol{r}_t = \boldsymbol{0}$. Equation (13) then reduces to

$$I_r \ddot{\boldsymbol{r}}_r + \Lambda_r \boldsymbol{r}_r = \Phi_{ar}^T \boldsymbol{F} \tag{14}$$

Since now $\boldsymbol{D}_r = \Phi_{rr} \boldsymbol{r}_r$ from equation (12), we can substitute $\boldsymbol{r}_r = \Phi_{rr}^{-1} \boldsymbol{D}_r$ into equation (14) yielding

$$\Phi_{rr}^{-1^T} \Phi_{ar}^T M \Phi_{ar} \Phi_{rr}^{-1} \ddot{\boldsymbol{D}}_r + \Phi_{rr}^{-1^T} \Phi_{ar}^T K \Phi_{ar} \Phi_{rr}^{-1} \boldsymbol{D}_r = \Phi_{rr}^{-1^T} \Phi_{ar}^T \boldsymbol{F}$$

We can then define a coordinate transformation matrix T where

$$T = \Phi_{ar} \Phi_{rr}^{-1} = \begin{bmatrix}\Phi_{rr} \\ \Phi_{tr}\end{bmatrix} \Phi_{rr}^{-1} = \begin{bmatrix} I \\ \Phi_{tr} \Phi_{rr}^{-1}\end{bmatrix}$$

so that

$$\tilde{M} \ddot{\boldsymbol{D}}_r + \tilde{C} \dot{\boldsymbol{D}}_r + \tilde{K} \boldsymbol{D}_r = \tilde{\boldsymbol{F}}$$

where $\tilde{M} = T^T M T$, $\tilde{C} = T^T C T$, $\tilde{K} = T^T K T$ and $\tilde{\boldsymbol{F}} = T^T \boldsymbol{F}$ yields our reduced order model.

21 Solution Techniques

$$\det(K - \lambda M) = 0$$

Obtain polynomial called characteristic equation. Does not work well for 1000s of DOF

21.1 Rayleigh Ritz method

Rayleigh's Quotient: $R = \frac{\bar{\boldsymbol{D}}^T K \bar{\boldsymbol{D}}}{\bar{\boldsymbol{D}}^T M \bar{\boldsymbol{D}}}$

Minimize R for all vectors $\bar{\boldsymbol{D}}$ will give approximation of first eigenvalue. System can be reduced to leave $n-1$ eigenvalues, repeat.

Works well when no zero eigenvalues and first mode shape can be guessed.

In Ritz method, problem is reduced by assuming

$$\bar{\boldsymbol{D}}_n = \sum a_{in} \boldsymbol{\Psi}_i$$

a_i are Ritz coordinates, $\boldsymbol{\Psi}_i$ are Ritz basis vectors.

$$\underbrace{S'}_{nxp} = \underbrace{\Psi}_{nxp} \underbrace{[a]}_{pxp}$$

Reduces size of eigenvalue problem to p.

21.1.1 Rayleigh Ritz Example

$$(A - \lambda I)\bar{\boldsymbol{D}} = 0$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2.5 & -1.5 \\ 0 & -1.5 & 3 \end{bmatrix}$$

Assume

$$\boldsymbol{\Psi}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}}, \ \boldsymbol{\Psi}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} (A - \lambda)\bar{\boldsymbol{D}} &= 0 \\ AS &= S\Lambda \\ S^T AS &= \Lambda \end{aligned}$$

because $S^T = S^{-1}$.

Assume (guess)

$$S' = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$(S')^T AS' = \Lambda'$$

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \Lambda'$$

$$[a]^T \underbrace{\begin{bmatrix} 2.5 & -1.7678 \\ -1.7678 & 2.5 \end{bmatrix}}_{A_p} [a] = \Lambda'$$

$$\Lambda' = \begin{bmatrix} 0.7322 & 0 \\ 0 & 4.2678 \end{bmatrix} \qquad \begin{bmatrix} 0.7258 & & \\ & 2.3198 & \\ & & 4.4544 \end{bmatrix}$$

$$[a] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$S' = \begin{bmatrix} 0.5 & -0.5 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0.5 & 0.5 \end{bmatrix} \qquad \begin{bmatrix} 0.548 & 0.7907 & 0.2729 \\ 0.6983 & -0.2528 & -0.6697 \\ 0.4606 & -0.5575 & 0.6907 \end{bmatrix}$$

21.2 Cholesky Decomposition

Decomposition of a matrix into simpler matrices.

$$A = L^T L$$

This must exist for positive definite A.

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & 0 \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix}$$

By inspection

$$l_{ii} = \left(a_{ii} - \sum_{j=1}^{i-1} l_{ij}^2 \right)^{\frac{1}{2}}$$

$$l_{ki} = \left(a_{ik} - \sum_{j=1}^{i-1} l_{ij} l_{kj} \right) \frac{1}{l_{ii}}$$

21.2.1 Example

$$A = \begin{bmatrix} 16 & -20 & -24 \\ -20 & 89 & -50 \\ -24 & -50 & 280 \end{bmatrix}$$

$$\begin{aligned}
l_{11} =& a_{11}^{\frac{1}{2}} = 4 \\
l_{21} =& \frac{1}{l_{11}}(a_{12}) = -5 \\
l_{22} =& (a_{22} - l_{21}^2)^{\frac{1}{2}} = (89 - (-5)^2)^{\frac{1}{2}} = 8 \\
l_{31} =& \frac{1}{l_{11}} a_{13} = \frac{-24}{4} = -6 \\
l_{32} =& \frac{1}{l_{22}}(a_{23} - l_{21} l_{31}) = \frac{1}{8}[-50 - (-5)(-6)] = -10 \\
l_{33} =& (a_{33} - l_{31}^2 - l_{32}^2)^{\frac{1}{2}} = (280 - (-6)^2 - (-10)^2)^{\frac{1}{2}} = 12
\end{aligned}$$

$$L = \begin{bmatrix} 4 & 0 & 0 \\ -5 & 8 & 0 \\ -6 & -10 & 12 \end{bmatrix}$$

Consider

$$M\ddot{\boldsymbol{D}} + K\boldsymbol{D} = \boldsymbol{0}$$

$$M = L^T L$$

set $\boldsymbol{D} = L^{-1}\boldsymbol{q}$

$$L^{T^{-1}} L^T L L^{-1} \ddot{\boldsymbol{q}} + L^{T^{-1}} K L^{-1} \boldsymbol{q} = \boldsymbol{0}$$

$$I\ddot{\boldsymbol{q}} + \tilde{K}\boldsymbol{q} = \boldsymbol{0}$$

This is a very robust way of mass normalizing discrete equations.

21.3 Matrix Iteration

Power Method,

Consider a matrix A size $n \times n$.

Consider a vector

$$\boldsymbol{x} = \sum c_i \bar{\boldsymbol{D}}_i$$

c_i and $\bar{\boldsymbol{D}}_i$ unknown, $\bar{\boldsymbol{D}}_i$ are unknown eigenvectors

$$\boldsymbol{x}_1 = A\boldsymbol{x} = c_1 A\bar{\boldsymbol{D}}_1 + c_2 A\bar{\boldsymbol{D}}_2 + \cdots$$

This can also be written

$$A\bar{\boldsymbol{D}} = \mu\bar{\boldsymbol{D}}$$

$$\boldsymbol{x}_1 = c_1\mu_1\bar{\boldsymbol{D}}_1 + c_2\mu_2\bar{\boldsymbol{D}}_2 + \cdots$$

Normalize $\boldsymbol{x}_1$

$$\boldsymbol{x}_r = c_1\mu_1^r\bar{\boldsymbol{D}}_1 + c_2\mu_2^r\bar{\boldsymbol{D}}_2 + \cdots$$

When this converges

$$\boldsymbol{x}_r \approx c_1\mu_1^r\bar{\boldsymbol{D}}_1$$

Assuming $\mu_m \gg \mu_{m+1}$, the m^{th} term of the summation dominates the $m-1$ term.

So $c_1\boldsymbol{x}_{r+1} \approx c_1\mu_1\boldsymbol{x}_r$

Then

$$\frac{|\boldsymbol{x}_{r+1}|}{|\boldsymbol{x}_r|} = \mu_1$$

21.3.1 Example

$$K^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{x}_1 = K^{-1}\boldsymbol{x}_0 / \left|K^{-1}\boldsymbol{x}_0\right| = \frac{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}{\left|\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right|} = \begin{bmatrix} -0.4472 \\ 0.8944 \end{bmatrix} \quad \text{norm} = \sqrt{5}$$

$$\boldsymbol{x}_2 = K^{-1}\boldsymbol{x}_1 / \left|K^{-1}\boldsymbol{x}_1\right| = \begin{bmatrix} -0.6247 \\ 0.7809 \end{bmatrix} \qquad 2.86$$

$$\boldsymbol{x}_3 = K^{-1}\boldsymbol{x}_2 / \left|K^{-1}\boldsymbol{x}_2\right| = \begin{bmatrix} -0.6805 \\ 0.7328 \end{bmatrix} \qquad 2.98$$

$$\boldsymbol{x}_4 = K^{-1}\boldsymbol{x}_3 / \left|K^{-1}\boldsymbol{x}_3\right| = \begin{bmatrix} -0.6983 \\ 0.7158 \end{bmatrix} \qquad 2.998$$

$$\boldsymbol{x}_5 = K^{-1}\boldsymbol{x}_4 / \left|K^{-1}\boldsymbol{x}_4\right| = \begin{bmatrix} -0.7042 \\ 0.7100 \end{bmatrix} \qquad 2.9998$$

$$\boldsymbol{x}_6 = K^{-1}\boldsymbol{x}_5 / \left|K^{-1}\boldsymbol{x}_5\right| = \begin{bmatrix} -0.7061 \\ 0.7081 \end{bmatrix} \qquad 3.000$$

21.4 Matrix Deflation

A new Matrix D1 is formed

$$D1 = K^{-1} - \mu_1 \bar{\boldsymbol{D}}_1 \bar{\boldsymbol{D}}_1^T$$

It will not have the eigenvalue μ_1

$$D1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

It has eigenvalues 0 and 1. The remaining eigenvalues can be determined by repeating this with the new matrix.

21.5 Subspace Iteration

Consider the eigenvalue problem

$$K\bar{\boldsymbol{D}} = \lambda M\bar{\boldsymbol{D}}$$

for the case where we only need the 1st p eigenvalues and eigenvectors

Start with an initial guess X_1 $(n \times q)$ like the Rayleigh Ritz method

$$\underbrace{\bar{X}_2}_{n\times q} = K^{-1}M\underbrace{X_1}_{n\times q}$$

which can be solved for using Gauss elimination

A reduced eigenvalue problem is obtained

$$\underbrace{K_2}_{q\times q} = \bar{X}_2^T K\bar{X}_2$$

$$M_2 = \bar{X}_2^T M\bar{X}_2$$

The reduced eigenvalue problem is

$$K_2Q_2 = M_2Q_2\Lambda_2$$

which can be solved for using other techniques better for smaller eigenvalue problems.

In general, $q \ll n$.

An improved approximation for the eigenvectors is

$$\underbrace{X_2}_{n\times q} = \underbrace{\bar{X}_2}_{n\times q}\underbrace{Q_2}_{q\times q}$$

Repeat this

$$\underbrace{\bar{X}_{k+1}}_{n\times q} = K^{-1}M\underbrace{X_k}_{n\times q}$$

$$\underbrace{K_{k+1}}_{q\times q} = \bar{X}_{k+1}^T K\bar{X}_{k+1}$$

$$M_{k+1} = \bar{X}_{k+1}^T M\bar{X}_{k+1}$$

Solve

$$K_{k+1}Q_{k+1} = M_{k+1}Q_{k+1}\Lambda_{k+1}$$

for Q_{k+1}, then

$$\underbrace{X_{k+1}}_{n\times q} = \underbrace{\bar{X}_{k+1}}_{n\times q}\underbrace{Q_{k+1}}_{q\times q}$$

Repeat until convergence.

Example;

```
K =

   2.00000  -1.00000   0.00000
  -1.00000   2.50000  -1.50000
   0.00000  -1.50000   3.00000
```

```
M =

        1        0        0
        0        1        0
        0        0        1


>> x1=[1 1;0 1;0 1]

x1 =

        1        1
        0        1
        0        1

>> x2bar=K\M*x1
x2bar =

  0.70000  1.30000
  0.40000  1.60000
  0.20000  1.13333

>> k2=x2bar'*K*x2bar
k2 =

  0.70000  1.30000
  1.30000  4.03333
>> m2=x2bar'*M*x2bar
m2 =

  0.69000  1.77667
  1.77667  5.53444
>> [q2,lam2]=eig(m2\k2);
>> q2=q2(:,[2,1]),lam2=diag(sort(diag(lam2)))% need to sort eigenvalues
q2 =

  -0.02669   0.95230
   0.99964  -0.30515
```

```
lam2 =

  0.72874  0.00000
  0.00000  2.34844

>> x2=x2bar*q2;x2=x2/[norm(x2(:,1)) 0;0 norm(x2(:,2))]% normalize vector lengths

x2 =

   0.54935   0.81937
   0.68141  -0.32580
   0.48362  -0.47169

>> x3bar=K\M*x2

x3bar =

   0.75384   0.34890
   0.95832  -0.12157
   0.64037  -0.21801

>> k3=x3bar'*K*x3bar
k3 =

  1.37683  0.00339
  0.00339  0.42832

>> m3=x3bar'*M*x3bar
m3 =

  1.89671  0.00690
  0.00690  0.18404

>> [q3,lam3]=eig(m3\k3);q3=q3(:,[2,1]),lam3=diag(sort(diag(lam3)))% need to sort e
q3 =

   0.00417  -0.99998
```

```
  -0.99999  -0.00548

lam3 =

  0.72590  0.00000
  0.00000  2.32760
```

```
>> x3=x3bar*q3;x3=x3/[norm(x3(:,1)) 0;0 norm(x3(:,2))] %normalize eigenvectors

x3 =

  -0.54874  -0.80601
  -0.69534   0.29272
  -0.46410   0.51445
```

After 9 iterations

$$\Lambda = \begin{bmatrix} 0.7258 & 0 \\ 0 & 2.3198 \end{bmatrix}$$

and

$$\bar{\boldsymbol{D}} = \begin{bmatrix} 0.5480 & 0.7909 \\ 0.6983 & -0.2533 \\ 0.4606 & -0.5571 \end{bmatrix}$$

Initial vectors are chosen to be:

1. Diagonal of mass matrix
2. Vectors of +1 where m_{ii}/k_{ii} are maximum.

Only the first p eigenvalues are useful. q is generally $\min(2p, p+8)$.

This will converge to the lowest p eigenvalues given that all of the vectors in X_1 are *not* orthogonal to any of the corresponding eigenvectors.

21.6 Shifting

Consider the eigenvalue problem

$$(K - \lambda M)\bar{\boldsymbol{D}} = \mathbf{0}$$

Subspace iteration does not work with a singular K. (rigid body motion capable system)

Consider redefining $\lambda = \mu - 1$. Then the eigenvalue problem is restated as

$$\begin{aligned}
(K - \lambda M)\bar{\boldsymbol{D}} &= \mathbf{0} \\
(K - (\mu - 1)M)\bar{\boldsymbol{D}} &= \mathbf{0} \\
((K + M) - \mu M)\,\bar{\boldsymbol{D}} &= \mathbf{0} \\
(K' - \mu M)\,\bar{\boldsymbol{D}} &= \mathbf{0}
\end{aligned} \tag{15}$$

K' is non-singular. The obtained eigenvectors are the same. The eigenvalues can be obtain by *unshifting* the eigenvalues using $\lambda = \mu - 1$.

22 Transient Response

$$M\ddot{\boldsymbol{D}} + K\boldsymbol{D} = \boldsymbol{F}(t)$$

Find $\boldsymbol{D}(t)$
From $\boldsymbol{D}(t)$, stresses and strains may be found.
Let Φ be a matrix of mode shapes, $\bar{\boldsymbol{D}}$.
Let $\boldsymbol{D} = \Phi\boldsymbol{r}$

$$\Phi^T\left(M\Phi\ddot{\boldsymbol{r}} + K\Phi\boldsymbol{r} = \boldsymbol{F}(t)\right)$$

$$\Phi^T M\Phi\ddot{\boldsymbol{r}} + \Phi^T K\Phi\boldsymbol{r} = \Phi^T\boldsymbol{F}(t)$$

$$I\ddot{\boldsymbol{r}} + \Lambda\boldsymbol{r} = \Phi^T\boldsymbol{F}(t) = \boldsymbol{f}(t)$$

$\boldsymbol{r}$ are modal coordinates
Λ is diag(ω_i^2)
$\Phi^T\boldsymbol{F}(t)$ are modal forces

Using SDOF techniques, we can solve for $r_i(t)$ for each r_i that is "significant." Then use $\boldsymbol{D}(t) = \Phi\boldsymbol{r}(t)$ to find total displacement.
If we have damping we cannot do this unless

$$CM^{-1}K = KM^{-1}C$$

where C is the damping matrix

$$M\ddot{\boldsymbol{D}} + C\dot{\boldsymbol{D}} + K\boldsymbol{D} = \boldsymbol{F}(t)$$

A special case is Rayleigh damping,

$$C = \alpha M + \beta K$$

α: Air damping
β: Material damping

$$\Phi^T M\Phi\ddot{\boldsymbol{r}} + \Phi^T(\alpha M + \beta K)\Phi\dot{\boldsymbol{r}} + \Phi^T K\Phi\boldsymbol{r} = \Phi^T\boldsymbol{F}(t)$$

$$I\ddot{\boldsymbol{r}} + (\underbrace{\alpha I + \beta\Lambda}_{diagonal})\dot{\boldsymbol{r}} + \Lambda\boldsymbol{r} = \Phi^T\boldsymbol{F}(t)$$

$$\zeta_i = \frac{\alpha + \beta\omega_i^2}{2\omega_i}$$

22.1 Numerical Integration

Finite Differences: 1st derivative

$\dot{\boldsymbol{D}}(t) = \frac{\boldsymbol{D}(t+\Delta t)-\boldsymbol{D}(t)}{\Delta t}$ Forward

$\dot{\boldsymbol{D}}(t) = \frac{\boldsymbol{D}(t+\Delta t)-\boldsymbol{D}(t-\Delta t)}{2\Delta t}$ Central

$\dot{\boldsymbol{D}}(t) = \frac{\boldsymbol{D}(t)-\boldsymbol{D}(t-\Delta t)}{\Delta t}$ Backward

2nd Derivative

$\ddot{\boldsymbol{D}}(t) = \frac{\dot{\boldsymbol{D}}(t+\Delta t)-\dot{\boldsymbol{D}}(t)}{\Delta t}$ Forward

$\ddot{\boldsymbol{D}}(t) = \frac{\dot{\boldsymbol{D}}(t+\Delta t)-\dot{\boldsymbol{D}}(t-\Delta t)}{2\Delta t}$ Central

$\ddot{\boldsymbol{D}}(t) = \frac{\dot{\boldsymbol{D}}(t)-\dot{\boldsymbol{D}}(t-\Delta t)}{\Delta t}$ Backward

22.1.1 Central Difference Method

$$\ddot{\boldsymbol{D}}(t) = \frac{\boldsymbol{D}(t+\Delta t) - 2\boldsymbol{D}(t) + \boldsymbol{D}(t-\Delta t)}{\Delta t^2}$$

$$\dot{\boldsymbol{D}} = \frac{\boldsymbol{D}(t+\Delta t) - \boldsymbol{D}(t-\Delta t)}{2\Delta t}$$

Substituting into EOM.

$$\begin{aligned}\left(\frac{1}{\Delta t^2}M + \frac{1}{2\Delta t}C\right)\boldsymbol{D}(t+\Delta t) =& \boldsymbol{F}(t) - \left(K - \frac{2}{\Delta t^2}M\right)\boldsymbol{D}(t) \\ &- \left(\frac{1}{\Delta t^2}M - \frac{1}{2\Delta t}C\right)\boldsymbol{D}(t-\Delta t)\end{aligned} \tag{16}$$

Solve for $\boldsymbol{D}(t+\Delta t)$

We presume to know $\boldsymbol{D}(0)$ and $\dot{\boldsymbol{D}}(0)$

Using Backwards differences

$$\dot{\boldsymbol{D}}(0)\Delta t = \boldsymbol{D}(0) - \boldsymbol{D}(-\Delta t)$$

$$\boldsymbol{D}(-\Delta t) = \boldsymbol{D}(0) - \dot{\boldsymbol{D}}(0)\Delta t$$

We can now apply Equation 16 repeatedly.

Problem,

Δt must be $< \frac{T_n}{\pi} = \Delta t_{crit}$ where T_n is smallest period of model.

Modal truncation must be performed

$$I\ddot{\boldsymbol{r}} + [2\zeta_i\omega_i]\dot{\boldsymbol{r}} + \Lambda\boldsymbol{r} = \boldsymbol{f}(t)$$

1. Perform Guyan reduction on modal model to remove higher order modes

2. Use SEREP

22.2 Runge-Kutta

Convert to State-Space form

$$M\ddot{\boldsymbol{D}} + C\dot{\boldsymbol{D}} + K\boldsymbol{D} = \boldsymbol{F}(t)$$

Define $\boldsymbol{z_1} = \boldsymbol{D}$, $\boldsymbol{z_2} = \dot{\boldsymbol{D}}$

$$\dot{\boldsymbol{z}} = \begin{bmatrix} \dot{\boldsymbol{z}}_\mathbf{1} \\ \dot{\boldsymbol{z}}_\mathbf{2} \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{D}} \\ \ddot{\boldsymbol{D}} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}}_{A} \begin{bmatrix} \boldsymbol{D} \\ \dot{\boldsymbol{D}} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ M^{-1}\boldsymbol{F}(t) \end{bmatrix}}_{\boldsymbol{u}}$$

$$\dot{\boldsymbol{z}} = A\boldsymbol{z} + \boldsymbol{u}$$

To go to next step

$$\begin{aligned} \boldsymbol{z}(t+\Delta t) &= \boldsymbol{z}(t) + \frac{\Delta t}{6}\left(\boldsymbol{k}_{n1} + 2\boldsymbol{k}_{n2} + 2\boldsymbol{k}_{n3} + \boldsymbol{k}_{n4}\right) \\ \boldsymbol{k}_{n1} &= A\boldsymbol{z}(t) + \boldsymbol{u}(t) \\ \boldsymbol{k}_{n2} &= A\left(\boldsymbol{z}(t) + \frac{\Delta t}{2}\boldsymbol{k}_{n1}\right) + \frac{1}{2}[\boldsymbol{u}(t) + \boldsymbol{u}(t+\Delta t)] \\ \boldsymbol{k}_{n3} &= A\left(\boldsymbol{z}(t) + \frac{\Delta t}{2}\boldsymbol{k}_{n2}\right) + \frac{1}{2}[\boldsymbol{u}(t) + \boldsymbol{u}(t+\Delta t)] \\ \boldsymbol{k}_{n4} &= A\left(\boldsymbol{z}(t) + \Delta t\boldsymbol{k}_{n3}\right) + \boldsymbol{u}(t+\Delta t) \end{aligned}$$

Inital conditions are used only in setting up vector $\boldsymbol{z}(0)$.
(See http://www.cs.wright.edu/~jslater/vtoolbox/vtb9_4.m)

Adaptive methods also exist (ode23 and ode45 in matlab).

22.3 Newmark Method

The Newmark integration method is based on the assumption that the acceleration varies linearly, or is constant (two methods) between two instants of time. The resulting expressions for the velocity and displacement vectors $\dot{\boldsymbol{D}}(t+\Delta t)$ and $\boldsymbol{D}(t+\Delta t)$ are written as

$$\dot{\boldsymbol{D}}(t+\Delta t) = \dot{\boldsymbol{D}}(t) + \Delta t \left[(1-\gamma)\ddot{\boldsymbol{D}}(t) + \gamma \ddot{\boldsymbol{D}}(t+\Delta t) \right] \tag{17}$$

$$\boldsymbol{D}(t+\Delta t) = \boldsymbol{D}(t) + \Delta t \dot{\boldsymbol{D}}(t) + \frac{1}{2}\Delta t^2 \left[(1-2\beta)\, \ddot{\boldsymbol{D}}(t) + 2\beta \ddot{\boldsymbol{D}}(t+\Delta t) \right] \tag{18}$$

The average acceleration(presumed constant between time steps) method is given by using the parameters $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$

When $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$, the previous equations correspond to the linear acceleration method.

Solving equation (18) for $\ddot{\boldsymbol{D}}(t+\Delta t)$, then substituting into (17) gives

$$\ddot{\boldsymbol{D}}(t+\Delta t) = \frac{1}{\beta \Delta t^2} \left(\boldsymbol{D}(t+\Delta t) - \boldsymbol{D}(t) - \Delta t \dot{\boldsymbol{D}}(t) \right) - \left(\frac{1}{2\beta} - 1 \right) \ddot{\boldsymbol{D}}(t) \tag{19}$$

$$\dot{\boldsymbol{D}}(t+\Delta t) = \frac{\gamma}{\beta \Delta t} \left(\boldsymbol{D}(t+\Delta t) - \boldsymbol{D}(t) \right) - \left(\frac{\gamma}{\beta} - 1 \right) \dot{\boldsymbol{D}}(t) - \Delta t \left(\frac{\gamma}{2\beta} - 1 \right) \ddot{\boldsymbol{D}}(t) \tag{20}$$

These two expressions are substituted into the equation of motion, then it is solved for $\boldsymbol{D}(t)$ giving

$$\begin{aligned} \boldsymbol{D}(t+\Delta t) = &\left[\frac{1}{\beta(\Delta t)^2} M + \frac{\gamma}{\beta \Delta t} C + K \right]^{-1} \\ \times &\left[\boldsymbol{R}(t+\Delta t) + M \left(\frac{1}{\beta(\Delta t)^2} \boldsymbol{D}(t) + \frac{1}{\beta \Delta t} \dot{\boldsymbol{D}}(t) + \left(\frac{1}{2\beta} - 1 \right) \ddot{\boldsymbol{D}}(t) \right) \right. \\ &\left. + C \left(\frac{\gamma}{\beta \Delta t} \boldsymbol{D}(t) + \left(\frac{\gamma}{\beta} - 1 \right) \dot{\boldsymbol{D}}(t) + \left(\frac{\gamma}{\beta} - 2 \right) \frac{\Delta t}{2} \ddot{\boldsymbol{D}}(t) \right) \right] \end{aligned}$$

The Newmark method can be summarized in the following steps:

1. From the known values of $\boldsymbol{D}_0$ and $\dot{\boldsymbol{D}}_0$, find $\ddot{\boldsymbol{D}}_0$.

2. Select suitable values of Δt, γ, and β.

3. Calculate the displacement vector $\boldsymbol{D}(t + \Delta t)$, using the expression shown above.

4. Find the acceleration and velocity vectors at time $t + \Delta t$:

5. Repeat

Similarly, if β is greater than $\frac{1}{2}$, a positive damping is introduced. This reduces the magnitude of response even without real damping in the problem. The method is unconditionally stable for $\gamma \geq \frac{1}{4}(\beta + \frac{1}{2})^2$ and $\beta \geq \frac{1}{2}$.

22.3.1 Additional notes on Newmark method

β and γ can have a wide variety of values resulting in different methods.

Unconditional stability when $2\beta \geq \gamma \geq \frac{1}{2}$. Conditionally stable for

$$\gamma \geq \frac{1}{2}, \qquad \beta < \frac{1}{2}\gamma \qquad \Delta t \leq \frac{\Omega_{\text{crit}}}{\omega_{\text{max}}} = \frac{\Omega_{\text{crit}} T_{\text{min}}}{2\pi}$$

$$\Omega_{\text{crit}} = \frac{\zeta\left(\gamma - \frac{1}{2}\right) + \sqrt{\frac{\gamma}{2} - \beta + \zeta^2\left(\gamma - \frac{1}{2}\right)^2}}{\frac{\gamma}{2} - \beta}$$

ζ is the damping ratio of ω_{max} frequency (highest natural frequency) and T_{min} is the period of the fastest natural frequency.

Unstable for $\gamma < \frac{1}{2}$

	β	γ	Ω_{crit}	Accuracy
Trapezoid Rule (Avg. accel)	$\frac{1}{4}$	$\frac{1}{2}$	∞	$O(\Delta t^2)$
Linear Acceleration	$\frac{1}{6}$	$\frac{1}{2}$	$2\sqrt{3}^u$, eqn^d	$O(\Delta t^2)$
Central difference method	$\frac{1}{2}$	0	?	$O(\Delta t^2)$
Backward difference method	1	$\frac{3}{2}$	?	$O(\Delta t^2)$
Galerkin method	$\frac{4}{5}$	$\frac{3}{2}$	?	$O(\Delta t^2)$
Fox-Goodwin	$\frac{1}{12}$	$\frac{1}{2}$	$\sqrt{6}^u$, eqn^d	$O(\Delta t^4)$
Artificially Damped	$\geq \frac{1}{4}(\gamma + \frac{1}{2})^2$	$> \frac{1}{2}$	∞	$O(\Delta t^2)$

u:undamped, d:damped, central difference and artificially damped case M and C must be diagonal.

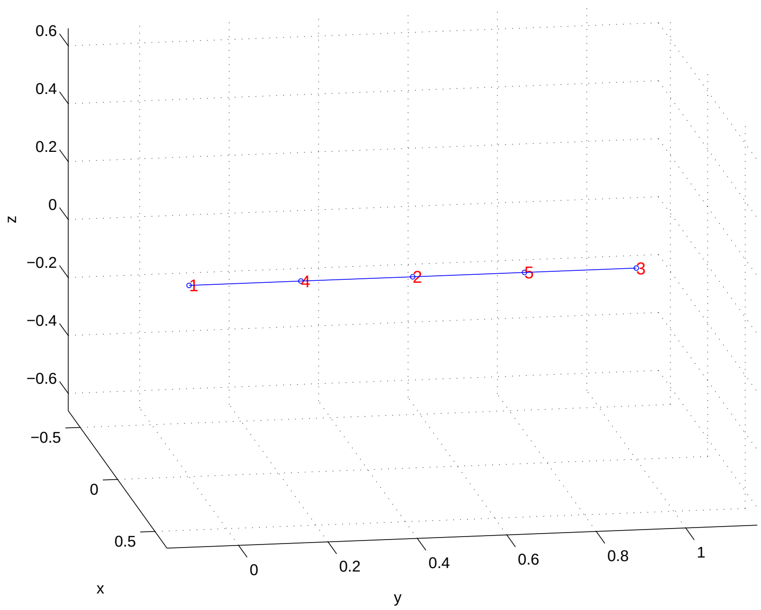
23 Model Correlation

23.1 Modal Assurance Criterion (MAC)

The *Modal Assurance Criteria* (*MAC*) or *Modal Shape Correlation Coefficient* (*MSCC*) is a measure of the correlation between two modes, whether they be two experimental modes, a physics derived mode and an experimental modes, or two physics derived modes and is given by

$$\mathrm{MAC}(\boldsymbol{\psi}_l, \boldsymbol{\psi}'_m) = \frac{\left(\boldsymbol{\psi}_l^H \boldsymbol{\psi}'_m\right)^2}{\left(\boldsymbol{\psi}_l^H \boldsymbol{\psi}_l\right)\left(\left(\boldsymbol{\psi}'_m\right)^H \boldsymbol{\psi}'_m\right)} \tag{21}$$

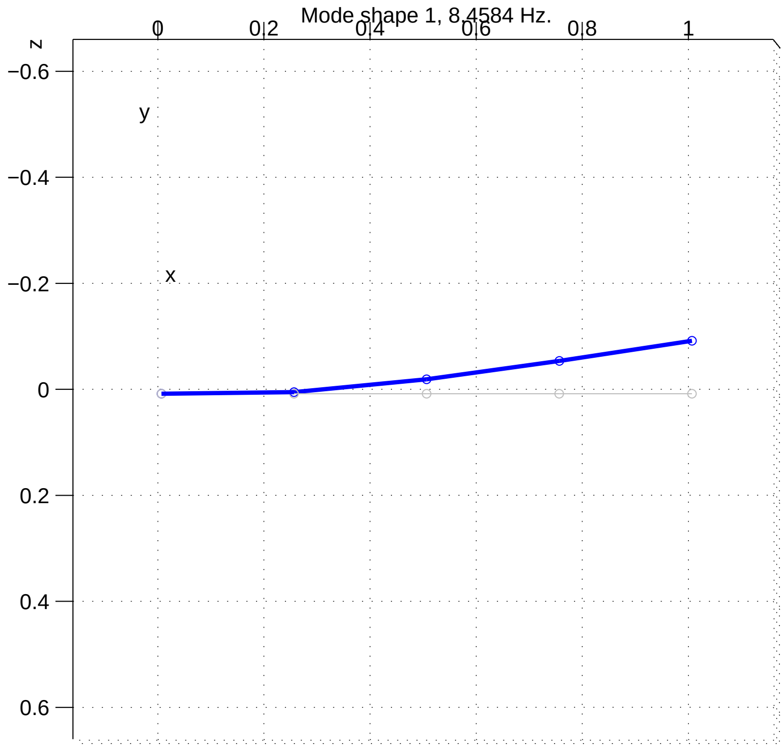
Consider an example of a cantilever beam in 3-D space before and after a point mass is added to the tip.



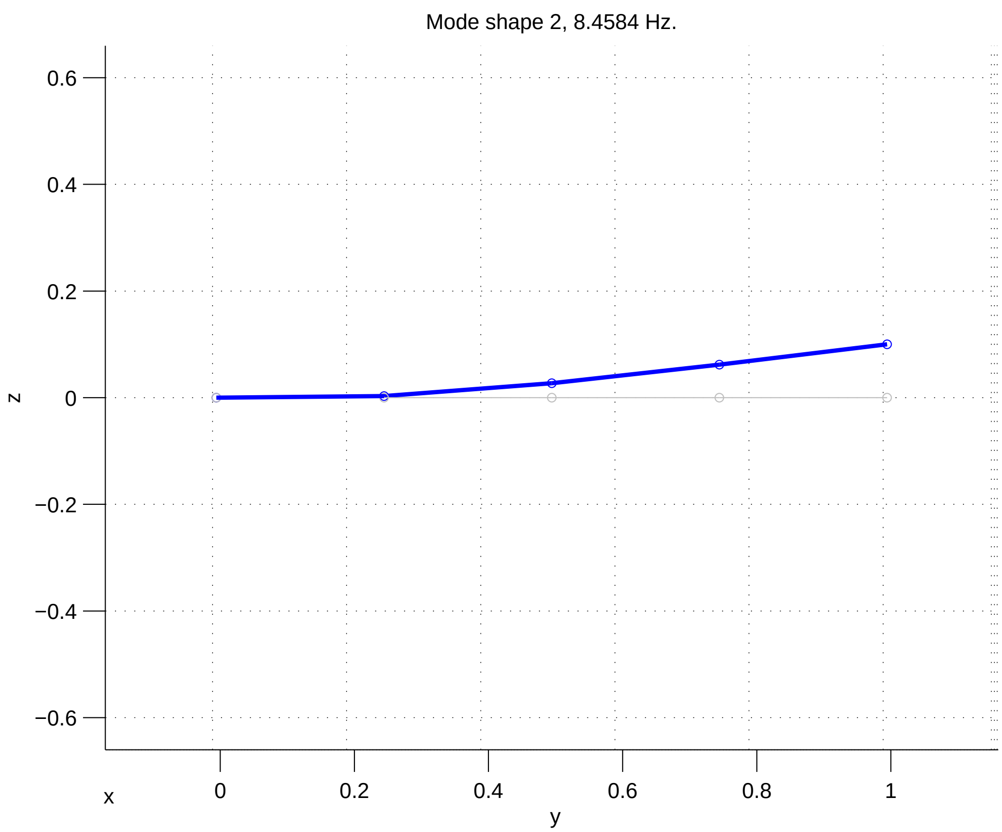
Mode number	Natural Frequencies (Hz)	Natural Frequencies with tip mass added (Hz)
1	8.458	7.474
2	8.458	7.474
3	64.950	60.382
4	64.950	60.382
5	165.155	160.974
6	165.155	160.974
7	422.206	416.581
8	422.206	416.581
9	523.326	523.326
10	764.570	756.918
11	764.570	756.918
12	1017.303	949.384

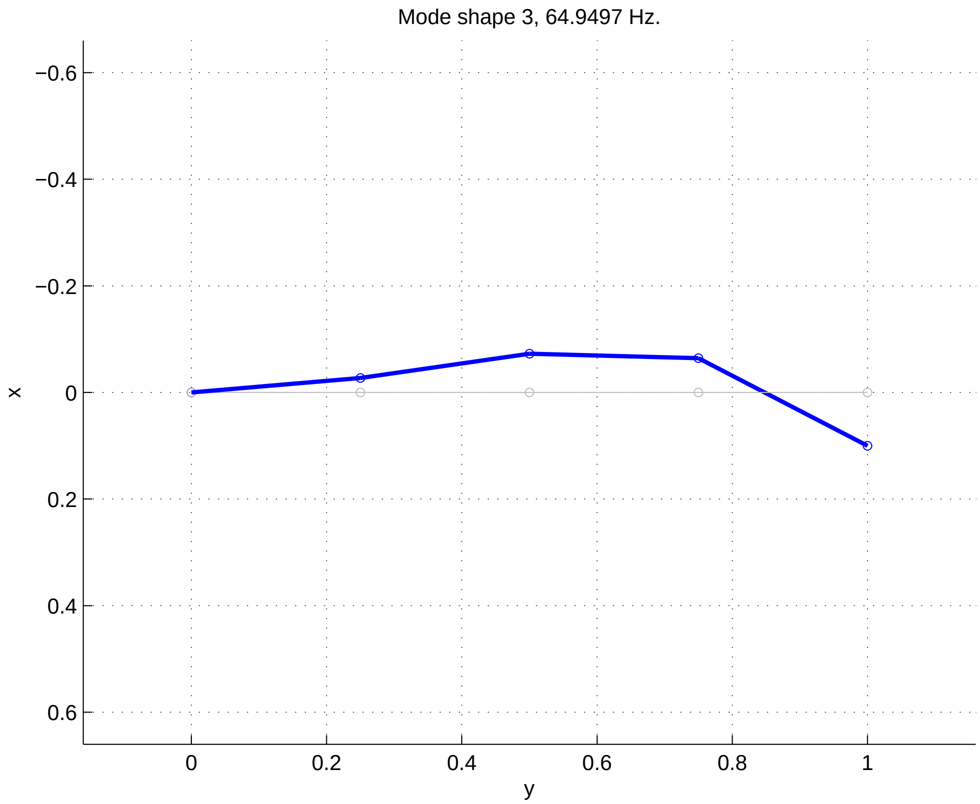
Modal Assurance Criteria

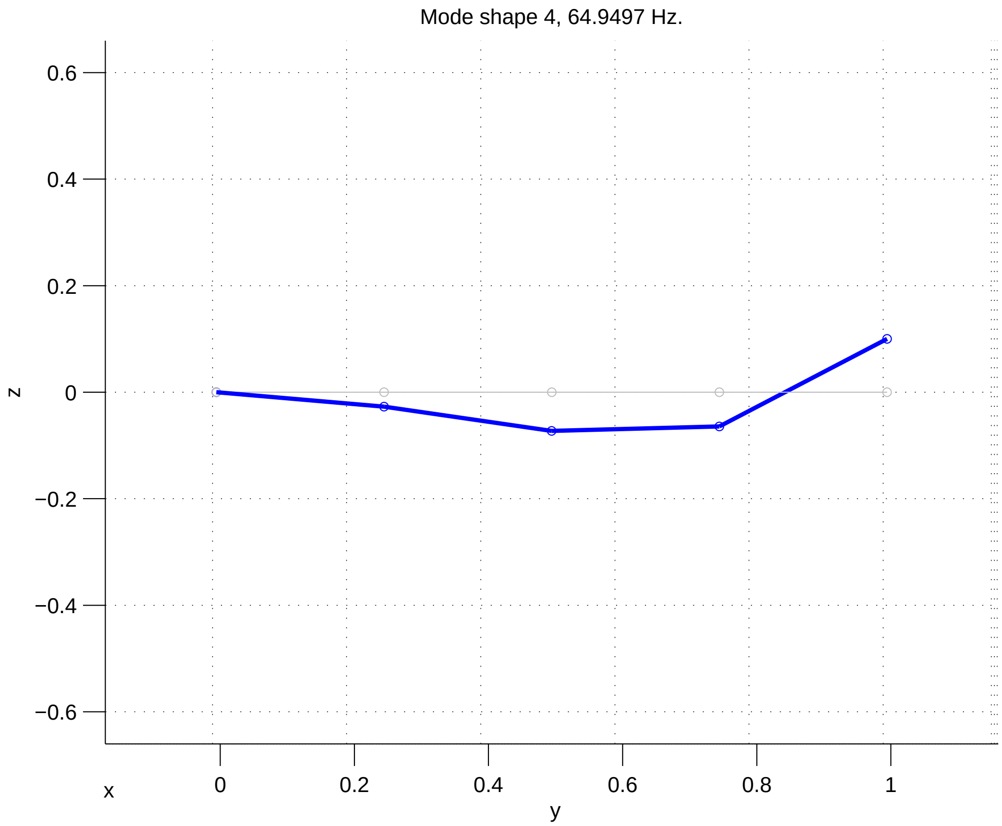
	1	2	3	4	5	6	7	8	9	10	11	12
1	0.99	0.01	0.00	0.30	0.02	0.00	0.07	0.00	0.00	0.00	0.12	0.00
2	0.01	0.99	0.30	0.00	0.00	0.02	0.00	0.07	0.00	0.12	0.00	0.00
3	0.00	0.37	0.99	0.00	0.00	0.38	0.00	0.38	0.00	0.29	0.00	0.00
4	0.37	0.00	0.00	0.99	0.38	0.00	0.38	0.00	0.00	0.00	0.29	0.00
5	0.04	0.00	0.00	0.47	1.00	0.00	0.36	0.00	0.00	0.00	0.12	0.00
6	0.00	0.04	0.47	0.00	0.00	1.00	0.00	0.36	0.00	0.12	0.00	0.00
7	0.00	0.09	0.43	0.00	0.00	0.39	0.00	0.99	0.00	0.87	0.00	0.00
8	0.09	0.00	0.00	0.43	0.39	0.00	0.99	0.00	0.00	0.00	0.87	0.00
9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00
10	0.13	0.00	0.00	0.31	0.12	0.00	0.90	0.00	0.00	0.00	1.00	0.00
11	0.00	0.13	0.31	0.00	0.00	0.12	0.00	0.90	0.00	1.00	0.00	0.00
12	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00

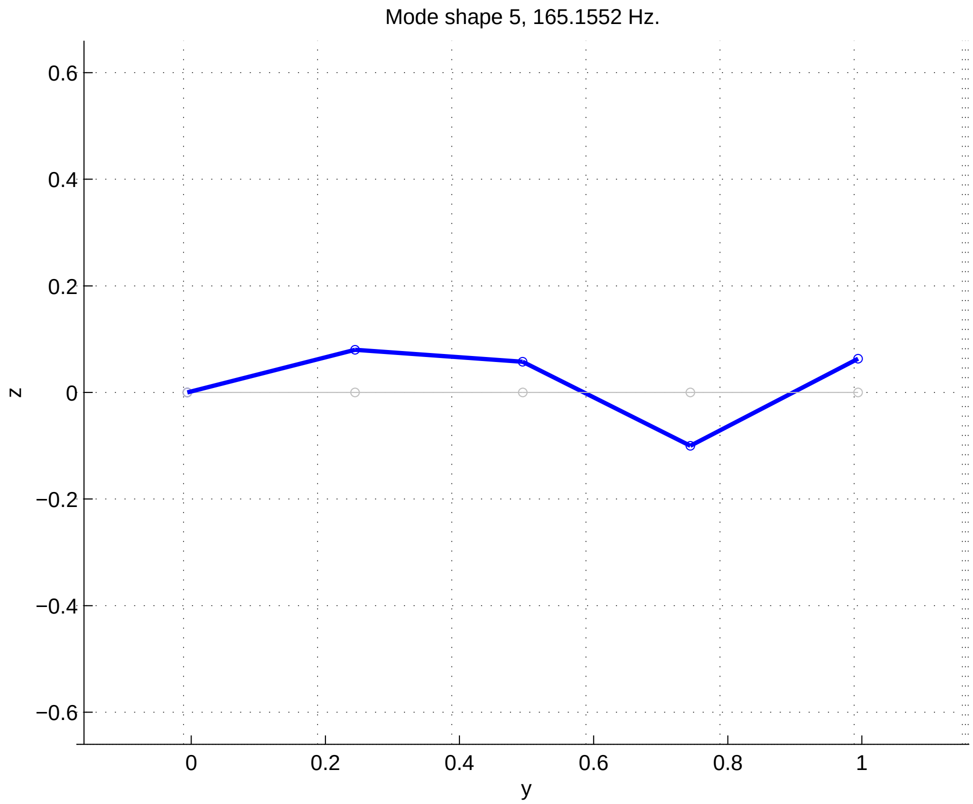


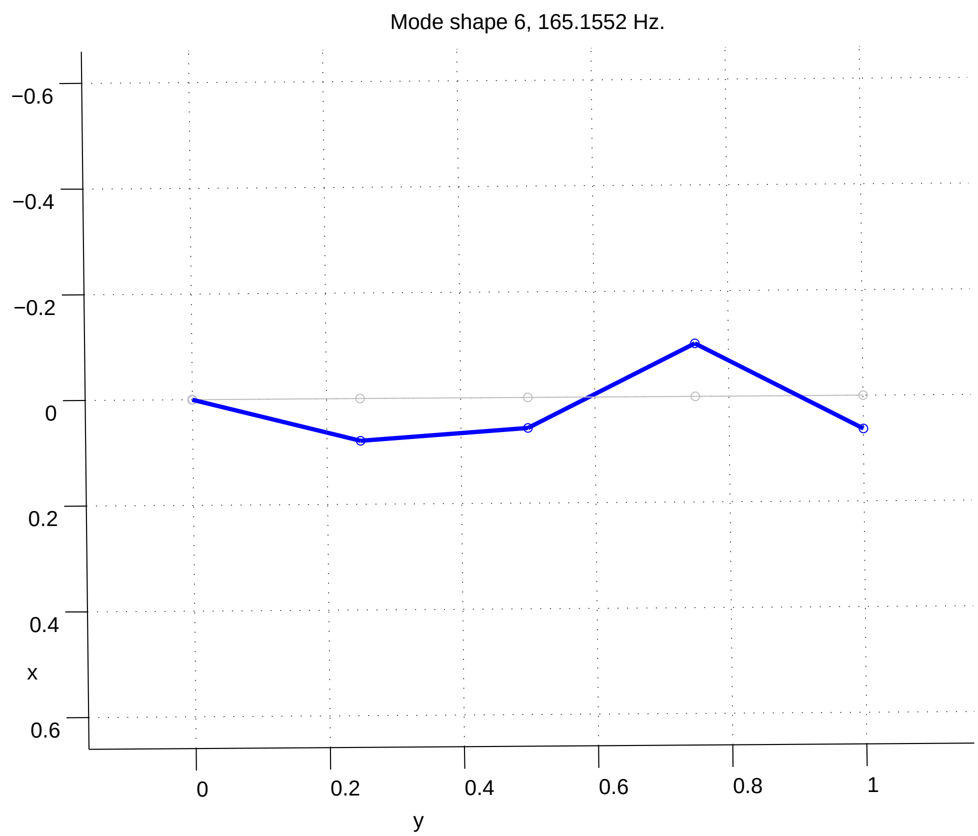
Mode shape 1, 8.4584 Hz.
0
0.2
0.4
0.6
0.8
1
z
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
y
x

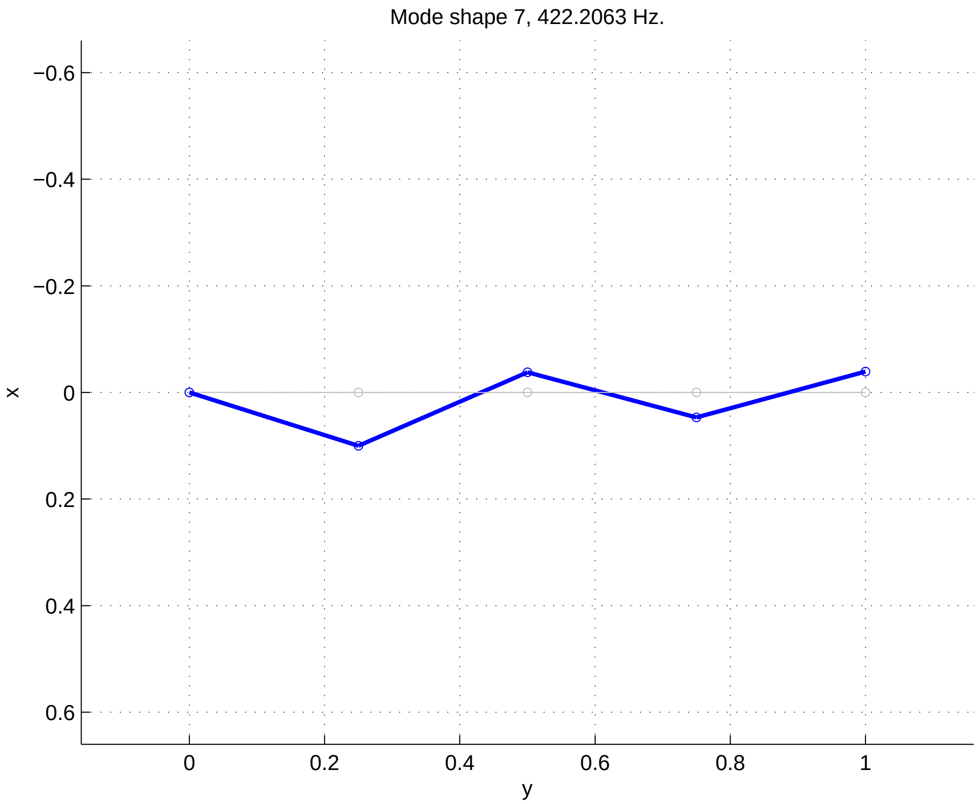

Mode shape 2, 8.4584 Hz.
0.6
0.4
0.2
z
0
−0.2
−0.4
−0.6
x
0
0.2
0.4
0.6
0.8
1
y

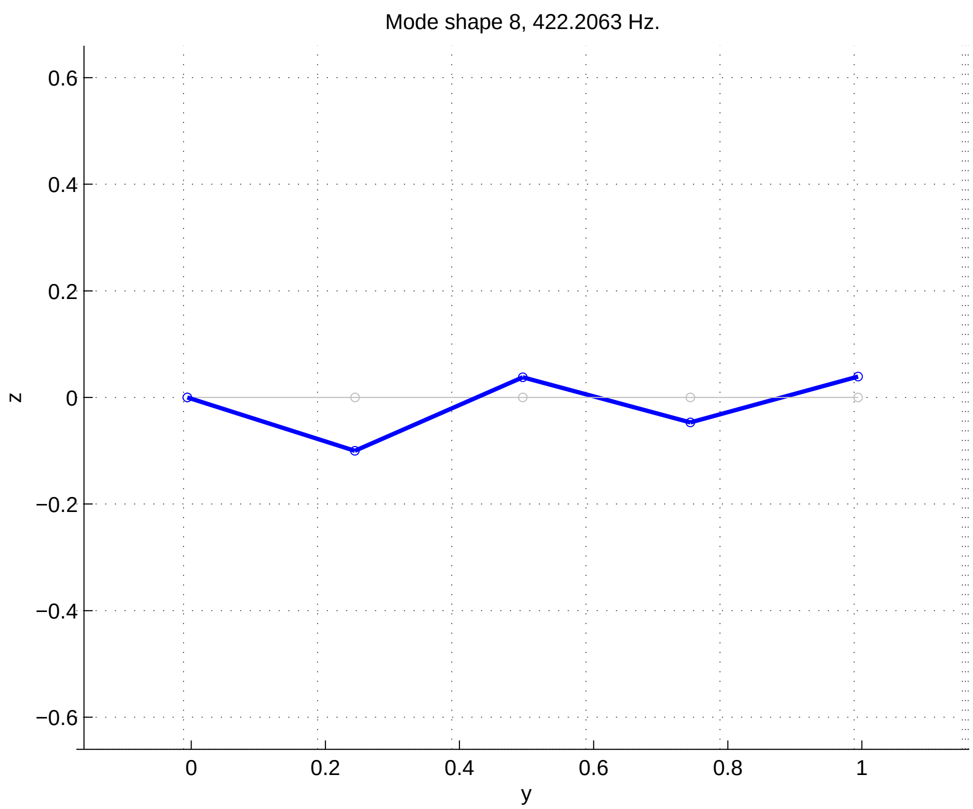

Mode shape 3, 64.9497 Hz.
x
y
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0
0.2
0.4
0.6
0.8
1


Mode shape 4, 64.9497 Hz.
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
z
x
0
0.2
0.4
0.6
0.8
1
y

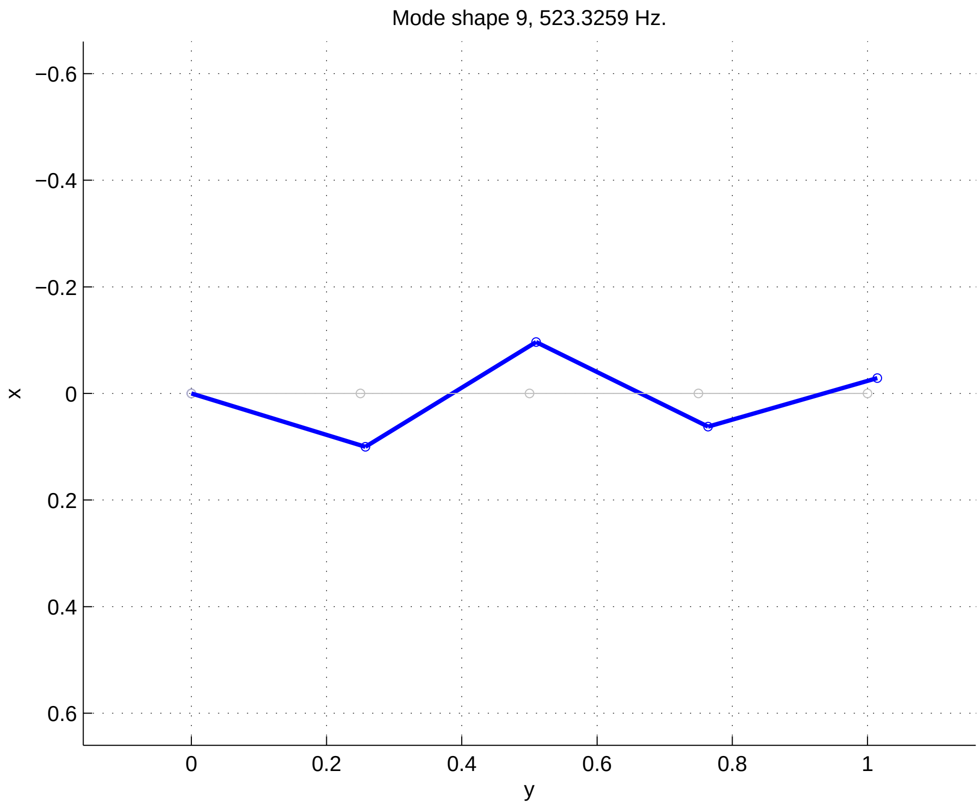

Mode shape 5, 165.1552 Hz.
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
z
0
0.2
0.4
0.6
0.8
1
y


Mode shape 6, 165.1552 Hz.
−0.6
−0.4
−0.2
0
0.2
0.4
x
0.6
0
0.2
0.4
0.6
0.8
1
y

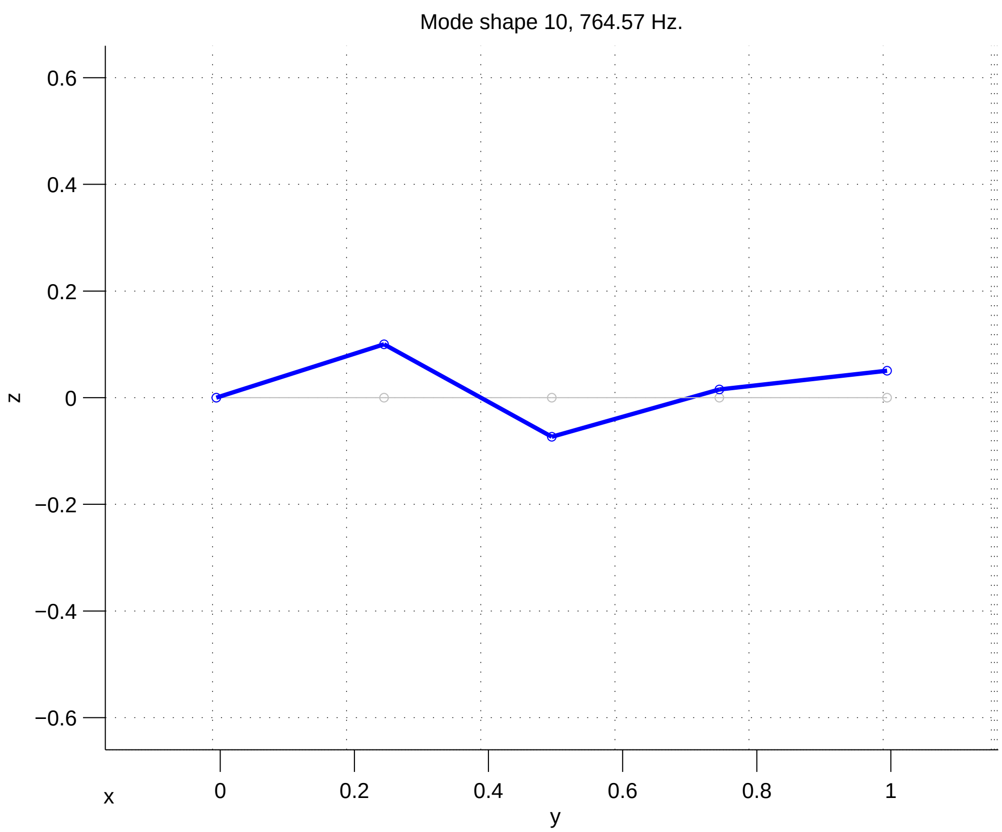

Mode shape 7, 422.2063 Hz.
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
x
0
0.2
0.4
0.6
0.8
1
y



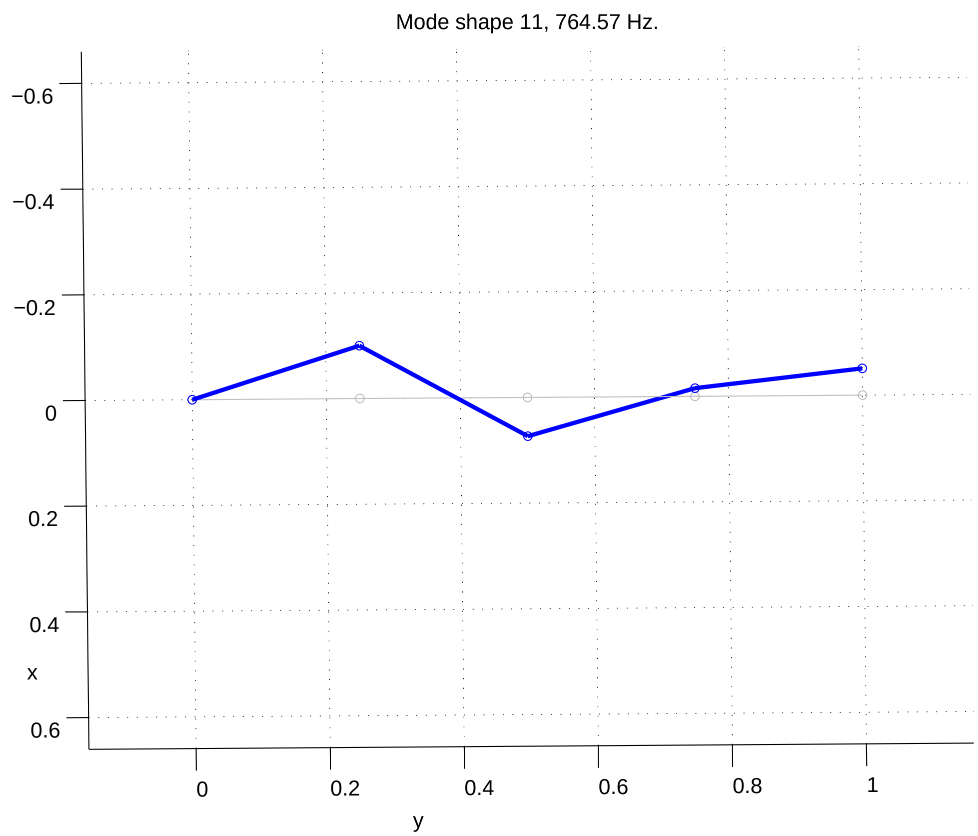
Mode shape 8, 422.2063 Hz.
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
z
0
0.2
0.4
0.6
0.8
1
y

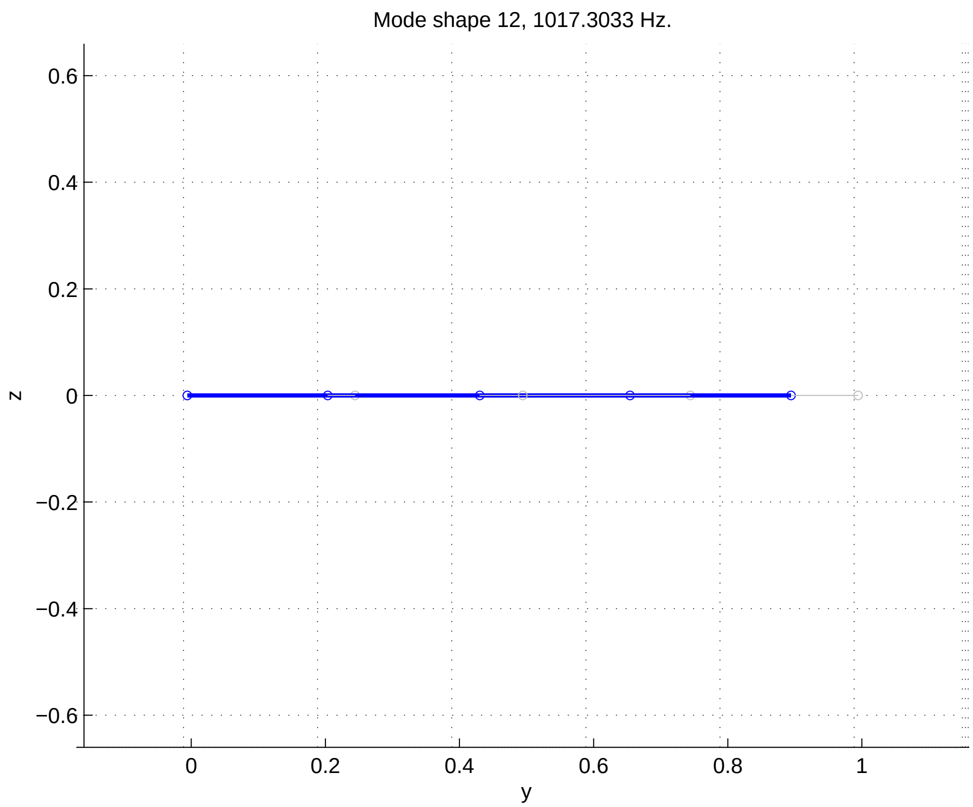


Mode shape 9, 523.3259 Hz.
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
x
0
0.2
0.4
0.6
0.8
1
y



Mode shape 10, 764.57 Hz.
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
z
x
0
0.2
0.4
0.6
0.8
1
y


Mode shape 11, 764.57 Hz.
−0.6
−0.4
−0.2
0
0.2
0.4
x
0.6
0
0.2
0.4
0.6
0.8
1
y



Mode shape 12, 1017.3033 Hz.
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
z
0
0.2
0.4
0.6
0.8
1
y

23.2 Coordinate Modal Assurance Criterion (COMAC)

The Coordinate Modal Assurance Criterion (COMAC) provides a spatial compliment to the MAC. Provided that modes from two disparate sources have been correlated well (e.g. the first mode of one set is also the first mode of the other set), it is a correlation factor of modal participation of a selected degree of freedom across all modes of interest. It can be recognized as the correlation coefficient between the rows of Ψ and Ψ' (see section **??**). The value of the COMAC for the lth degree of freedom is given by

$$COMAC_l = \frac{\left(\sum_{i=1}^{n} \psi_i(l)\psi_i'(l)\right)^2}{\left(\sum_{i=1}^{n} \psi_i(l)\psi_i(l)\right)\left(\sum_{i=1}^{n} \psi_i'(l)\psi_i'(l)\right)} \tag{22}$$

Conflicting versions of this expression are given by Maia and Silva (Maia, 1997) and Harris (1995). As there is no physics based derivation, neither those two or the one presented can be proven correct. However, each has a means by which to account for the fact that a correlation coefficient between two modes of -1 is by definition a perfect correlation, since the negative of a mode shape is also a mode shape. However, when performing the COMAC calculations, if a negative correlation between any two of the modes (contrary to the remainder) will cause a decrease in the correlation. The simple solution is to assure that corresponding modes have a positive correlation prior to calculation of the COMAC. By avoiding automatically using the positive of each independent term, as shown in Maia and Silva, sign discrepancies between a few elements of corresponding modes shapes will provide an appropriate decrease in the calculated COMAC.

DOF	COMAC value
1	0.992
2	0.992
3	0.987
4	0.987
5	0.997
6	0.997
7	0.988
8	0.988

24 Acoustics

Consider a compressible fluid having no viscosity and no net flow, whose density is uniform with the exception of small changes due to sound waves. e.g. sound propagation in buildings, inside cars.

The governing equation in 1-D in the x direction is

$$\frac{\partial p}{\partial x} = -\rho \ddot{u} \tag{23}$$

That is analogous to $F = \Delta p A = m\ddot{u} = \rho A \Delta x \ddot{u}$ for a discrete volume. Dividing by $A\Delta x$ and taking the limit at $\Delta x \to 0$ gives the 1-D equation.

Taking the partial derivative with respect to x, and adding the corresponding equations for the y and z directions:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = -\rho \left(\frac{\partial \ddot{u}}{\partial x} + \frac{\partial \ddot{v}}{\partial y} + \frac{\partial \ddot{w}}{\partial z} \right) \tag{24}$$

or

$$\nabla^2 p = -\rho \frac{\mathrm{d}^2}{\mathrm{d}t^2} (\epsilon_x + \epsilon_y + \epsilon_z) \tag{25}$$

The bulk modulus is defined as the ratio of pressure to the fractional volume change it produces:

$$B = -\frac{p}{dV/V} = -\frac{p}{\epsilon_x + \epsilon_y + \epsilon_z} \tag{26}$$

Equations (25) and (26) yield the non-dissipative wave equation

$$\nabla^2 p = \frac{\rho}{B} \ddot{p} = \frac{1}{c^2} \ddot{p} \tag{27}$$

where c is the speed of sound. The "essential" boundary condition is $p = 0$, which means a free surface exhibiting no surface waves.

The "nonessential" boundary condition is

$$\frac{\partial p}{\partial n} = -\rho \ddot{u}_n \tag{28}$$

where n represents the direction normal to a rigid surface. Provided the rigid surface is not moving, the boundary condition becomes

$$\frac{\partial p}{\partial n} = 0 \tag{29}$$

Both of these boundary conditions reflect waves without absorbing acoustic energy.

The functional used to derive the elements is given by

$$\Pi = \int \left(\frac{\frac{\partial p}{\partial x}^2 + \frac{\partial p}{\partial y}^2 + \frac{\partial p}{\partial z}^2}{2} + \frac{1}{c^2} p\ddot{p} \right) dV + \rho \int \ddot{u}_n p dS \tag{30}$$

which can be seen to be the negative of L from section 12 after integration by parts in time of the kinetic energy term. The reason for this strangeness is that the text doesn't use Hamilton's principle, so they don't have the Hamiltonian L defined, and thus can't figure out what to call it.

Nevertheless, we understand enough about finite elements that taking the variation of this and setting it equal to zero (thus the minus sign doesn't matter) with generate governing equations.

Further, in finite elements, we start by using shape functions, i.e.

$$p = NP_e \tag{31}$$

where P_e represent nodal pressures (or for some elements can contain the derivative of pressures just like the beam element DOFs include derivatives of the deflection).

Substituting we get

$$k_F = \int \left(\frac{\partial N_F}{\partial x}^T \frac{\partial N_F}{\partial x} + \frac{\partial N_F}{\partial y}^T \frac{\partial N_F}{\partial y} + \frac{\partial N_F}{\partial z}^T \frac{\partial N_F}{\partial z} \right) dV \tag{32}$$

$$m_F = \frac{1}{c^2} \int N_f^T N_f dV \tag{33}$$

$$r_F = \rho \int N_f^T \ddot{u}_n dS \tag{34}$$

for an element These must be assembled to generate K_F, M_F, and R_F.

Keep in mind that we are, in many cases, just going to use the same shape functions that we've used in similarly shaped elements because shape functions are not generated based on the physics. i.e. for a 1-D problem we use the rod element shape functions.

Applying stationarity (Hamilton's principle) and factoring out the non-variational components we get

$$M_F \ddot{P} + K_F P = -R_F \tag{35}$$

24.1 Acoustic modes

Consider a rigid cavity so that $\ddot{u}_n = 0$. Thus there is no forcing function.

To find the modes, we assume

$$p = \bar{p} \sin \omega t \tag{36}$$

In the finite element equation, this becomes

$$\left(K_F - \omega^2 M_F\right) \bar{P} = 0 \tag{37}$$

If there are no openings to the cavity, the boundary condition $p = 0$ is not imposed at any surface node. In that case, the first resonance will be $\omega = 0$ because K_F is singular, the trivial case. Meaningful answers start with the second frequency.

24.1.1 1-D case

Consider a piping system. Provided that the transverse dimension is small, we can neglect derivatives in the transverse direction.

To derive a single element, we use

$$k_F = \int \frac{\mathrm{d}N_F}{\mathrm{d}x}^T \frac{\mathrm{d}N_F}{\mathrm{d}x} A dx \tag{38}$$

$$m_F = \frac{1}{c^2} \int N_F^T N_F A dx \tag{39}$$

For a 2-noded isoparametric element, we use

$$N_F = \begin{bmatrix} \frac{-\xi+1}{2} & \frac{\xi+1}{2} \end{bmatrix} \tag{40}$$

Applying (38) yields

$$k_F = \frac{A}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{41}$$

and applying equation(39) yields

$$m_F = \frac{AL}{6c^2} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{42}$$

Using two linear elements to represent a cavity of length $2L$, the lowest mode has a frequency of $\omega_1 = \sqrt{3} = 1.7321c/L$. With 4 elements this becomes

$\omega_1 = 1.611c/L$. The solution with two 3-node elements is given by the text as $\omega_1 = 1.5767c/L$ while the closed form solution is $\omega_1 = \pi/2 = 1.5708c/L$. Note that finite elements forms an upper bound to the solution and converges as higher order elements or more elements are added. This also highlights that with the same number of nodes, the higher order elements outperform the lower order elements.

24.1.2 Boundary Absorption

In reality, boundaries typically absorb some energy. The functional used to derive the finite element equations is then augmented with a surface dissipative term like Rayleigh's dissipation function (covered in ME 710) so that it becomes.

$$\Pi = \int \left(\frac{\frac{\partial p}{\partial x}^2 + \frac{\partial p}{\partial y}^2 + \frac{\partial p}{\partial z}^2}{2} + \frac{1}{c^2} p\ddot{p} \right) dV + \rho \int \ddot{u}_n p dS + \frac{\beta}{c} \int p\dot{p} dS \tag{43}$$

the derivation of which is presented in the text. The important things to note are

- $0 < \beta < 1$ must be determined experimentally and represents the fraction of energy that is absorbed at the surface. ($\beta > 1$ would represent adding to the fully reflected pressure wave.

- Since the damping takes place only at surfaces, its spatial distribution if vastly different from structural damping.

The governing equation then becomes

$$M_F \ddot{P} + C_F \dot{P} + K_F P = -R_F \tag{44}$$

where

$$C_F = \frac{\beta}{c} \int N_F^T N_F dS \tag{45}$$