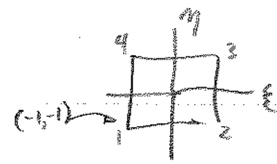
1) $(\xi, \eta) = (-1,-1) \longrightarrow$ node 1



$$N = \frac{1}{4}\left[(1-\xi)(1-\eta) \quad (1+\xi)(1-\eta) \quad (1+\xi)(1+\eta) \quad (1-\xi)(1+\eta)\right]$$

$$J = \frac{1}{4}\begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix} \begin{array}{c} \begin{array}{cc} x_i & y_i \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \end{array}$$

$$J_{(-1,-1)} = \frac{1}{4}\begin{bmatrix} -2 & 2 & 0 & 0 \\ -2 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 \\ -1/2 & -1/2 \end{bmatrix}$$

$$J^{-1}_{(-1,-1)} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

$\underline{u} = N_3 u_3$

$\underline{v} = N_3 v_3$

all other u's & v's = 0

$\frac{\partial u}{\partial \xi} = \frac{\partial N_3}{\partial \xi} u_3$

$\begin{bmatrix} \frac{\partial N_3}{\partial \xi} \\ \frac{\partial N_3}{\partial \eta} \end{bmatrix}$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}_{(-1,-1)} = J^{-1}_{(-1,-1)} \frac{1}{4}\begin{bmatrix} (1+\eta) \\ (1+\xi) \end{bmatrix}_{(-1,-1)} u_3 = J^{-1}_{(-1,-1)} \frac{1}{4}\begin{bmatrix} 0 \\ 0 \end{bmatrix} u_3$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} u_3$$

$$\begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix}_{(-1,-1)} = J^{-1}_{(-1,-1)} \frac{1}{4}\begin{bmatrix} (1+\eta) \\ (1+\xi) \end{bmatrix} v_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} v_3$$

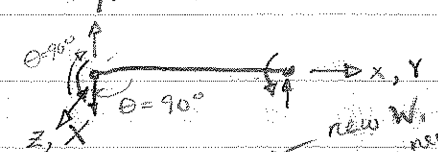
$\frac{\partial u}{\partial x} = \varepsilon_x$

$\frac{\partial v}{\partial y} = \varepsilon_y$

$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{xy}$

$$\boxed{\begin{array}{c} \varepsilon_x = \phi \\ \varepsilon_y = \phi \\ \gamma_{xy} = \phi \end{array}}$$

2)



new W_1 new θ_{x_1} new W_2 new θ_{x2}

$$K = \frac{EI_x}{L^3}\begin{array}{c} \begin{array}{cccc} V_1 & \theta_{z_1} & V_2 & \theta_{z_2} \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

$W_i \rightarrow$ disp. in global Z direction

$\theta_{x_i} \rightarrow$ rotation about global X axis

$$T^T = \begin{array}{cc} \begin{array}{cccc} V_1 & \theta_{z_1} & V_2 & \theta_{z_2} \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} u_1 \\ v_1 \\ w_1 \\ \theta_{x_1} \\ \theta_{y_1} \\ \theta_{z_1} \\ u_2 \\ v_2 \\ w_2 \\ \theta_{x2} \\ \theta_{y2} \\ \theta_{z2} \end{array} \end{array}$$

Transformation Matrix Transpose

$T^T K T = K_G$

gives 12×12

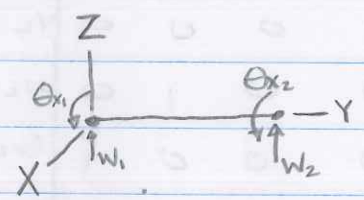
2 nodes $\times$ 6 dof each

$\longrightarrow$ $[K_G]$ on back

I_Y in Global = I_x in Local

$$[K_G] = \frac{EI_Y}{L^3}\;
\begin{array}{c}
\begin{array}{cccccccccccc} U_1 & V_1 & W_1 & \theta_{x_1} & \theta_{Y_1} & \theta_{Z_1} & U_2 & V_2 & W_2 & \theta_{x_2} & \theta_{Y_2} & \theta_{Z_2} \end{array} \\
\left[\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 12 & 6L & 0 & 0 & 0 & 0 & -12 & 6L & 0 & 0 \\
0 & 0 & 6L & 4L^2 & 0 & 0 & 0 & 0 & -6L & 2L^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -12 & -6L & 0 & 0 & 0 & 0 & 12 & -6L & 0 & 0 \\
0 & 0 & 6L & 2L^2 & 0 & 0 & 0 & 0 & -6L & 4L^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
\begin{array}{c} \\ U_1 \\ V_1 \\ W_1 \\ \theta_{x_1} \\ \theta_{Y_1} \\ \theta_{Z_1} \\ U_2 \\ V_2 \\ W_2 \\ \theta_{x_2} \\ \theta_{Y_2} \\ \theta_{Z_2} \end{array}$$

new beam orientation



⟹ Since the axes simply replaced one another the values in the stiffness matrix do not change. Simply the coordinate corresponding to each d.o.f changes.

(We are simply changing the labels)

3) 2 pt. Gauss $\longrightarrow$ need values of θ

$$\int_0^{l} f(x_i)\,dx \longrightarrow f(x_i)w_i + f(x_i)w_i$$

$x_i = \theta_i$, $l = \pi \longrightarrow J = \frac{l}{2} = \frac{\pi}{2}$

$J = \frac{\pi}{2} \leftarrow l = \pi$

wt.	ξ	θ	
1	$-\frac{1}{\sqrt{3}}$	0.211π	$\leftarrow \frac{\pi}{2} - \frac{\pi}{2}\left(\frac{1}{\sqrt{3}}\right)$
1	$\frac{1}{\sqrt{3}}$	0.789π	$\leftarrow \frac{\pi}{2} + \frac{\pi}{2}\left(\frac{1}{\sqrt{3}}\right)$

$$\int_0^{\pi} \sin\theta\,d\theta = \left[\sin(0.211\pi)(1) + \sin(0.789\pi)(1)\right]\left(\frac{\pi}{2}\right)$$

$$= (0.615386 + 0.615386)\left(\frac{\pi}{2}\right) = 1.23077\left(\frac{\pi}{2}\right)$$

$$= \boxed{1.93329}$$

actual $\int_0^{\pi} \sin(\theta)\,d\theta = 2$

$$\%\ \text{error} = \frac{2 - 1.93329}{2} \times 100\% = \boxed{3.33\%}$$

Note 3pt. Gauss gives $\int_0^{\pi} \sin(\theta)\,d\theta = 1.98747 \longrightarrow 0.626\%$ error

4pt. gives $\int_0^{\pi} \sin(\theta)\,d\theta = 1.99753 \longrightarrow 0.124\%$ error

* For a sine wave, the more Gauss points you use, the closer to the closed form solution you will be.