Su 2016 Exam 1 Solutions

1) $N_4 = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$N_4(0) = 0, \quad a_0 = 0$

$N_4'(0) = 0, \quad a_1 = 0$

$N_4(L) = 0, \quad 0 = a_2 L^2 + a_3 L^3 \implies a_2 = -a_3 L$

$N_4'(L) = -1, \quad -1 = -2a_2 L + 3a_3 L^2$

$-1 = -2a_3 L^2 + 3a_3 L^2$

$\frac{-1}{L^2} = a_3$

$a_2 = \frac{+1}{L}$

$N_4 = \frac{1}{L} x^2 - \frac{1}{L^2} x^3$

2) $x = \left(\frac{\xi + 1}{2}\right) L$

substituting into solution from 1

$N_4 = \left[\frac{\xi^2 + 2\xi + 1}{4} - \frac{(\xi+1)^3}{2}\right] L$

$= \frac{L}{8}\left(1 + \xi - \xi^2 - \xi^3\right)$

2) Alternatively $N_4 = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3$

$N_4(-1) = 0 = a_0 - a_1 + a_2 - a_3$ ①

$N_4'(-1) = 0 = a_1 - 2a_2 + 3a_3$ ②

$N_4(1) = 0 = a_0 + a_1 + a_2 + a_3$ ③

$N_4'(1) = -1$, however note that

$$N_4'(x) = \frac{dN_4(x)}{dx} = \frac{dN_4(x)}{dx}\frac{dx}{d\xi} = \frac{dN_4}{d\xi}$$

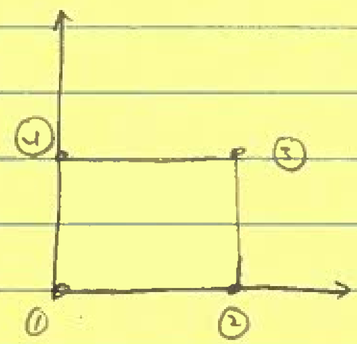
$$x = \left(\frac{1+\xi}{2}\right)L \quad , \text{ so } \frac{dx}{d\xi} = \frac{L}{2}$$

$$-1\,\frac{L}{2} = a_1 + 2a_2 + 3a_3 \qquad ④$$

Solving ①, ②, ③ and ④ gives

$$N_4 = \frac{L}{8}\left(1 + \xi - \xi^2 - \xi^3\right)$$

3)



$$P = \int_0^1 N \, \Phi \, dy$$

We just need x component of Node 3 force

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

with the pertinent part of $\Phi$ being $y$.

$J = \frac{1}{4}$, but you can calculate this.

$[J] = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, by inspection, but math is simple

$$[J] = \frac{1}{4}\begin{bmatrix} -(1-\eta) & (1-\eta) & 1+\eta & -1+\eta \\ -(1-\xi) & -(1+\xi) & 1+\xi & 1-\xi \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

Since it's uniform, set $\eta = 0$, $\xi = 0$

$$[J] = \frac{1}{4}\begin{bmatrix} -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

The load $y$ becomes $f(\eta) = (\eta+1)\frac{1}{2}$

$$\int_{-1}^{1} \frac{1}{2}(\eta+1) \cdot \frac{1}{4}(2)(1+\eta) \underset{\substack{\uparrow \\ \det J}}{\left(\frac{1}{4}\right)} d\eta)$$

$$= \int_{-1}^{1} \frac{1}{16}\left(1 + 2\eta + \eta^2\right) d\eta$$

No request for Gauss point was made, so

$\frac{1}{16}\,\frac{8}{3} = \boxed{\frac{1}{6}}$

4) $\int_0^{\pi} \sin\theta \, d\theta = -\cos\theta \Big|_0^{\pi} = 1 + 1 = 2$

$A = \frac{\pi}{2} \cdot \left( \sin \pi\left(\frac{1}{2} + \frac{1}{2\sqrt{3}}\right) + \sin \pi\left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\right)\right)$

$= \pi \left( \cos \frac{\pi}{2\sqrt{3}} \right) = 1.936$

$\% \text{ error} = \frac{-0.064}{2} = \underline{-3.2\% \text{ error}}$