ME 710 Exam 1 Solns Wi 2012

1) $\underline{x} = \sum \alpha_i \underline{v}_i$ where $\underline{v}_i$ are the eigenvectors of A (normalized) in order of increasing eigenvalues.

$$= \underbrace{\begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \dots \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\underline{\alpha}}$$

Since $\underline{x}^T \underline{x} = 1$

$$\underline{\alpha}^T P^T P \underline{\alpha} = 1$$

where P are eigenvectors of A.
Since P is an orthonormal matrix, $P^T P = I$,
So $\underline{\alpha}^T \underline{\alpha} = 1$, and $\underline{\alpha}$ is a unit vector

Evaluating $\underline{x}^T A \underline{x} = \underline{\alpha}^T P^T A P \underline{\alpha}$

$$= \underline{\alpha}^T \Lambda \underline{\alpha}$$

$$= \alpha_1^2 \lambda_1 + \alpha_2^2 \lambda_2 + \dots \alpha_n^2 \lambda_n$$

Since $|\underline{\alpha}| = 1$, we can write $\alpha_1^2 = 1 - \sum_{i=2}^{n} \alpha_i^2$
Substituting for α_1^2

$$\underline{x}^T A \underline{x} = \lambda_1 + \alpha_2^2(\lambda_2 - \lambda_1) + \dots + \alpha_n^2(\lambda_n - \lambda_1)$$

since $\lambda_i > \lambda_1$ for $i \neq 1$,

$\underline{x}^T A \underline{x} \geq \lambda_1$

So the problem statement is proven false.

Consider $\underline{x}^T A \underline{x} = \alpha_1^2 \lambda_1 + \cdots \alpha_n^2 \lambda_n$

Since $\alpha_n^2 = 1 - \sum_{i=1}^{n-1} \alpha_i^2$, substituting

$$\underline{x}^T A \underline{x} = \alpha_1^2 (\lambda_1 - \lambda_n) + \alpha_2^2 (\lambda_2 - \lambda_n) + \cdots \lambda_n$$

Since $\lambda_i < \lambda_n$ for $i \neq n$

$\underline{x}^T A \underline{x} \leq \lambda_n$, also violating the problem statement.

The true statement is

$$\boxed{\lambda_1 \leq \underline{x}^T A \underline{x} \leq \lambda_n}$$

2) $EI\,w'''' + \rho A \ddot{w} = 0$

Start with given substitution

$$w(x,t) = y(x,t) + a\sin(\omega t)\frac{x}{\ell}$$

$$w'''' = y'''' \qquad \ddot{w} = \ddot{y} - a\omega^2 \sin\omega t\left(\frac{x}{\ell}\right)$$

So

$$EI\,y'''' + \rho A \ddot{y} = \rho A a \omega^2 \sin\omega t\left(\frac{x}{\ell}\right)$$

Now we can use the pinned-pinned mode shapes and treat the motion as a load (it's a D'Alembert force)

$$y(x,t) = T(t)\,X(x)$$

$$X_n(x) = b_n \sin\frac{n\pi x}{\ell} \qquad n = 1,2,3$$

If we mass normalize $X(x)$

$$1 = \int_0^\ell \rho A\, X^2(x)\, dx = \rho A b^2 \frac{\ell}{2}$$

$$b_n = \sqrt{\frac{2}{\rho A \ell}}$$

Then we can substitute for $w(x,t)$

$$\left(EI\left(\frac{n\pi}{\ell}\right)^4 \sqrt{\frac{2}{\rho A\ell}}\, T(t) + \rho A \sqrt{\frac{2}{\rho A\ell}}\, \ddot{T}\right)\sin\frac{n\pi x}{\ell} = \rho A a \omega^2 \sin\omega t \frac{x}{\ell}$$

Multiplying by $\sqrt{\frac{2}{\rho A\ell}} \sin\frac{m\pi x}{\ell}$ and integrating over domain

$$\ddot{T} + \omega_n^2 T(t) = \sqrt{\frac{2\rho A}{\ell}}\; a\,\omega^2 \int_0^\ell \sin\frac{n\pi x}{\ell}\,\frac{x}{\ell}\,dx \;\sin\omega t$$

where $\omega_n = \frac{n^2\pi^2}{\ell^2}\sqrt{\frac{EI}{\rho A}}$

Performing the integral (integration by parts/Mathematica)

$$\ddot{T} + \omega_n^2 T(t) = \frac{\sqrt{2\rho A\ell}}{n\pi}\; a\omega^2 (-1)^n \sin\omega t$$

$$T_n(t) = \frac{1}{\omega_n^2 - \omega^2}\;\frac{\sqrt{2\rho A\ell}}{n\pi}\; a\omega^2 (-1)^n \sin\omega t$$

So

$$y(x,t) = \sum_{n=1}^{\infty} \frac{2a\omega^2(-1)^n}{n\pi(\omega_n^2 - \omega^2)} \sin\frac{n\pi x}{\ell}\sin\omega t$$

$$\boxed{w(x,t) = a\sin\omega t\left(\frac{x}{\ell}\right) + \sum_{n=1}^{\infty} \frac{2a\omega^2(-1)^n}{n\pi(\omega_n^2 - \omega^2)} \sin\frac{n\pi x}{\ell}\sin\omega t \quad \text{where } \omega_n = \left(\frac{n\pi}{\ell}\right)^2\sqrt{\frac{EI}{\rho A}}}$$

All terms are solved here explicitly (simply subst n)

3) $\epsilon = 0.01$ $\quad A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\lambda_1 \approx 1.2679 - 0.01 \cdot 0.6220 = 1.2617$$

$$\lambda_2 \approx 3.000 - 0.01 \cdot 0.333 = 2.9967$$

$$\lambda_3 \approx 4.7321 - 0.01 \cdot 0.0447 = 4.7316$$

Actual eigenvalues of B are identical to this many decimal places.

$$\int_0^{\ell} \sin\frac{n\pi x}{\ell}\ \frac{x}{\ell}\, dx$$

$$u = \frac{x}{\ell} \qquad dv = \sin\frac{n\pi x}{\ell}\, dx$$

$$du = \frac{1}{\ell}\, dx \qquad v = \frac{-\ell}{n\pi} \cos\frac{n\pi x}{\ell}$$

$$= \left(\frac{x}{\ell}\ \frac{-\ell}{n\pi} \cos\frac{n\pi x}{\ell}\right)\Bigg|_0^{\ell} - \int_0^{\ell} \frac{1}{\ell}\ \frac{-\ell}{n\pi} \cos\frac{n\pi x}{\ell}\, dx$$

$$= \underbrace{\left(\frac{\ell}{n\pi} \cos\frac{n\pi \ell}{\ell} - 0\right)}_{= \frac{\ell}{n\pi} \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}} + \underbrace{\frac{1}{n\pi}\ \frac{\ell}{n\pi} \sin\frac{n\pi x}{\ell}\Bigg|_0^{\ell}}_{0}$$

$$= \frac{\ell}{n\pi} (-1)^n$$