Solutions

1) $w(x,t) = X(x)\,T(t)$

$\ddot{w} = -\omega_n^2 X(x)\,T(t)$

Subst into EOM

$$EI\,X''''(x) - \rho A \omega_n^2 X(x) = 0$$

$X(x) = A\sin\beta x + B\cos\beta x + C\sinh\beta x + D\cosh\beta x$

$X'''' = \beta_n^4 X(x)$

$\therefore\ \beta_n^4 = \dfrac{\rho A \omega_n^2}{EI}$

Applying B.C.s, $X(0) = 0,\ X''(0) = 0,\ X(l) = 0,\ X''(l) = 0$

$X(0) = 0 = B + D$

$X''(0) = 0 = (-B + D)\beta^2 \qquad B = D = 0$

$X(l) = 0 = A\sin\beta l + C\sinh\beta l$

$X''(l) = 0 = (-A\sin\beta l + C\sinh\beta l)\beta^2$

$2\sin\beta l\,\sinh\beta l = 0$

$\sin\beta l = 0 \qquad \beta_n l = \pi, 2\pi, 3\pi \ldots$

$X(l) = A\sin n\pi + C\sinh(n\pi) = 0 \qquad \beta_n = \dfrac{n\pi}{l}$

$= C\sinh(n\pi) = 0$

$\therefore\ C = 0$

Mode shape

$$\boxed{X(x) = A\sin\beta_n x \qquad \omega_n = \frac{(n\pi)^2}{l^2}\sqrt{\frac{EI}{\rho A}}}$$

2) Modes are the same. Subst into EOM

$$\sum_{n=1}^{\infty}\left(\rho A\omega_n^2 T + C\beta_n^2\dot{T} + \rho A\ddot{T}\right)\sin\beta_n x = \delta(t)\delta(x-\tfrac{l}{2})$$

Multiply by $\beta_m x$ and integrate over l

$$\rho A\omega_n^2 T_n + C\beta_n^2\dot{T}_n + \rho A\ddot{T}_n = \frac{2}{l}\sin(\beta_n\tfrac{l}{2})\,\delta(t)$$

Dividing by ρA

$$\ddot{T}_n + \frac{C\beta_n^2}{\rho A}\dot{T}_n + \omega_n^2 T_n = \frac{2}{l\rho A}\sin(\beta_n\tfrac{l}{2})\,\delta(t)$$

The solution is the IRF

$$T(t) = \frac{\hat{F}}{\omega_n}e^{-\zeta\omega_n t}\sin\omega_d t$$

where $\hat{F} = \frac{2}{l\rho A}\sin(\beta_n\tfrac{l}{2})$

$\zeta = \frac{C\beta_n^2}{2\rho A\omega_n}$

ω_n is given in problem 1, β_n is given in problem 1.

3) $$T = \frac{1}{2}\int_0^\ell \rho A \dot{v}^2 + J\rho\dot{\theta}^2\, dx + \frac{1}{2}m\left(r\dot{\theta}(\ell) + \dot{v}(\ell)\right)^2$$

$$\delta T = \frac{1}{2}\int_0^\ell 2\rho A \dot{v}\,\delta\dot{v} + 2J\rho\dot{\theta}\,\delta\dot{\theta}\, dx + \frac{1}{2}m\Big(r^2\dot{\theta}(\ell)\,\delta\dot{\theta}(\ell) + 2r\dot{\theta}(\ell)\,\delta\dot{v}(\ell) + 2r\dot{v}(\ell)\,\delta\dot{\theta}(\ell) + 2\dot{v}(\ell)\,\delta\dot{v}(\ell)\Big)$$

Integrating by parts

$$\int_{t_1}^{t_2}\delta T\, dt = \int_{t_1}^{t_2}\int_0^\ell -\rho A\ddot{v}\,\delta v - J\rho\ddot{\theta}\,\delta\theta\, dx - \left(mr^2\ddot{\theta}(\ell) + mr\ddot{v}(\ell)\right)\delta\theta\Big|_{x=\ell} - m\left(\ddot{v}(\ell) + r\ddot{\theta}(\ell)\right)\delta v\Big|_{x=\ell}$$

extra prime was missing

$$V = \int_0^\ell \frac{1}{2}GJ\theta'^2 + \frac{1}{2}EI v''^2\, dx$$

δV, after integrating by parts, is

$$\delta V = GJ\theta'\,\delta\theta\Big|_0^\ell - \int_0^\ell GJ\theta''\,\delta\theta\, dx$$

looks like previous line if error not corrected

$$+ EIv''\,\delta v'\Big|_0^\ell - \frac{d}{dx}(EIv'')\,\delta v\Big|_0^\ell + \int_0^\ell \frac{d^2}{dx^2}(EIv'')\,\delta v\, dx$$

Substituting δV and δT into Hamilton's principal, the EOMs are

$$J\rho\ddot{\theta} - GJ\theta'' = 0, \quad \rho A\ddot{v} + \frac{d^2}{dx^2}\left(EIv''\right) = 0$$

With BC $\theta = 0$, $v = 0$, $v' = 0$ @ $x = 0$

and

$$mr^2\ddot{\theta} + mr\ddot{v} + GJ\theta' = 0, \quad EIv'' = 0 \text{ and}$$

$$m\ddot{v} + mr\ddot{\theta} - \frac{d}{dx}\left(EIv''\right) = 0 \quad @\ x = \ell$$

4) define $\xi = \frac{x}{l}$,

$$\frac{d\xi}{dx} = \frac{1}{l}$$

$$\frac{d}{dx} = \frac{1}{l}\frac{d}{d\xi}$$

Likewise $\frac{d^2}{dx^2} = \frac{1}{l^2}\frac{d^2}{d\xi^2}$

so

$$\frac{EI}{l^4}\frac{d^4}{d\xi^4} w + \rho A \frac{d^2 w}{dt^2} = 0$$

define $\gamma = \sqrt{\frac{EI}{l^4 \rho A}}\, t$

$$\frac{d\gamma}{dt} = \sqrt{\frac{EI}{l^4 \rho A}} , \quad \frac{d^2\gamma}{dt^2} = \frac{EI}{l^4 \rho A}$$

$$\therefore \frac{d^2}{dt^2} = \frac{EI}{l^4 \rho A}\frac{d^2}{d\gamma^2}$$

so

$$\frac{d^4}{d\xi^4} w + \frac{d^2}{d\gamma^2} w = 0$$

define $w = y\,l$, then

$$\boxed{\frac{d^4}{d\xi^4} y + \frac{d^2}{d\gamma^2} y = 0}$$

All variables are non-dimensional