ME 712 Sp '12 Exam 1

1) $\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix}$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix}, \quad \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix}$$

$$J = \frac{1}{4} \begin{bmatrix} -(1-\eta) & 1-\eta & 1+\eta & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & 1+\xi & 1-\xi \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} -2 & 2 & 0 & 0 \\ -2 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} -2 & 4 \\ -2 & -2 \end{bmatrix}$$

$$\left.\frac{\partial u}{\partial \xi}\right|_{①} = \frac{\partial N_3}{\partial \xi} u_3 \bigg|_{①} = 0 \qquad \left.\frac{\partial u}{\partial \eta}\right|_{①} = \frac{\partial N_3}{\partial \eta} u_3 \bigg|_{①} = 0$$

$$\left.\frac{\partial v}{\partial \xi}\right|_{①} = \frac{\partial N_3}{\partial \xi} v_3 \bigg|_{①} = 0 \qquad \left.\frac{\partial v}{\partial \eta}\right|_{①} = \frac{\partial N_3}{\partial \eta} u_3 \bigg|_{①} = 0$$

There is no strain at node 1 if nodes 1, 2 and 4 are fixed.

2) a) J is the most "skewed" at node 2. Evaluating J for the given coordinates yields

$$J = \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{2} & -1 \end{bmatrix}$$

By comparison, at node 4, $J = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

Ideal is an orthogonal transformation. Norms of colums should be similar.

If J is an orthogonal transformation, then $J^T = \alpha J^{-1}$. The degree to which $J \times J^T$ is diagonal can be used as an indicator. For this case, $J \times J^T = \begin{bmatrix} 1.25 & -0.75 \\ -0.75 & 1.25 \end{bmatrix}$
If node 2 is moved to $y = 1$,
$J \times J^T = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$. Applying this metric for a wide range of values y_2
$0.75 < y_2 < 1.25$ result in off-diagonal terms under 25% of the diagonal terms.

b) Since this is a constant strain element, only uniformity is possible. Presuming the element passes the (yet to be discussed) patch test, simple "squareness," is all that can be used, so that the ability of strain to vary is consistent. Consistency of det(J) throughout the element is simplest metric across multiple elements in all directions

3)

$$K = \begin{bmatrix} \frac{12EI_x}{a^3} + \frac{12EI_x}{b^3} & \frac{-6EI_x}{a^2} & \frac{6EI}{b^2} \\ \frac{-6EI_x}{a^2} & \frac{4EI_x}{a} & 0 \\ \frac{6EI_x}{b^2} & 0 & \frac{4EI_x}{b} \end{bmatrix} \begin{bmatrix} v \\ \theta_1 \\ \theta_2 \end{bmatrix}$$

(I_z is actually correct, but I_x was given in exam)

4) Using coordinates v_1, $\theta_1 L$, v_2, $\theta_2 L$

$$K = \frac{EI_x}{L^3} \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & 6 & 2 \\ -12 & 6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

We can actually rotate 90° in either direction (but that's renaming a positive x deflection, adding another problem)

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{T'} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

Note that K must be expanded to include U_1, U_2 before rotation. Then

$$T = \begin{bmatrix} T' & 0 & 0 & 0 \\ 0 & 1 & & \\ 0 & & T' & 0 \\ & & 0 & 1 \end{bmatrix}$$

$$K = \frac{EI}{L^3}\begin{bmatrix} 12 & 0 & -6 & -12 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 0 & 4 & -6 & 0 & 2 \\ -12 & 0 & -6 & 12 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 0 & 2 & 6 & 0 & 4 \end{bmatrix}$$

or reduced to only traditional coordinates

$$K = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & 6L^2 \\ -6L & 2L^2 & 6L^2 & 4L^2 \end{bmatrix}$$

Terms that differ due to rotation are circled for illustration.