Winter 2002 Final

1.) The equation of motion of a fixed-fixed string is given by:

$$\mathcal{T} w'' - \rho(x) A \ddot{w} = 0$$

or think of this as

$$-\mathcal{T} w'' + \rho(x) A \ddot{w} = 0 \qquad \text{so that } L = -\mathcal{T}\frac{\partial^2}{\partial x^2}$$

and

$$m(x) = \rho(x) A = MA\,\delta(x - L/2)$$

Estimate the first natural fequency using a two term series using:

a.) Rayleigh Ritz method.

first assume your comparison functions. (Must satisfy B.C.'s $w(0) = 0 = w(L)$)

assume:

$$\phi_1 = \sin\left(\frac{\pi x}{L}\right) \qquad \phi_2 = \sin\left(\frac{3\pi x}{L}\right)$$

now discretize the system using RR

$$k_{11} = \int_0^L \phi_1 L \phi_1 \, dx = -\int_0^L \phi_1 \mathcal{T} \frac{\partial^2 \phi_1}{\partial x^2} dx = \int_0^L \frac{\mathcal{T}\pi^2}{L^2} \sin^2\left(\frac{\pi x}{L}\right) dx$$

$$k_{11} = \frac{\mathcal{T}\pi^2}{L^2}\left(\frac{L}{2}\right) = \frac{\mathcal{T}\pi^2}{2L}$$

$$k_{12} = \int_0^L \phi_1 L \phi_2 \, dx = -\int_0^L \phi_1 \mathcal{T} \frac{\partial^2 \phi_2}{\partial x^2} dx = \int_0^L \frac{9\mathcal{T}\pi^2}{L^2} \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{3\pi x}{L}\right) dx$$

$$k_{12} = 0$$

$$K_{21} = \int_0^L \phi_2 L \phi_1 \, dx = -\int_0^L \phi_2 \, \zeta \, \frac{\partial^2 \phi_1}{\partial x^2} \, dx = \int_0^L \frac{\zeta \pi^2}{L^2} \sin\left(\frac{3\pi x}{L}\right) \sin\left(\frac{\pi x}{L}\right) dx$$

$$K_{21} = 0$$

$$K_{22} = \int_0^L \phi_2 L \phi_2 \, dx = -\int_0^L \phi_2 \, \zeta \, \frac{\partial^2 \phi_2}{\partial x^2} \, dx = \int_0^L \frac{9\pi^2}{L^2} \zeta \sin^2\left(\frac{3\pi x}{L}\right) dx$$

$$K_{22} = \frac{9 \zeta \pi^2}{L^2}\left(\frac{L}{2}\right) = \frac{9 \zeta \pi^2}{2L}$$

so that

$$K = \frac{\zeta \pi^2}{L} \begin{bmatrix} 1/2 & 0 \\ 0 & 9/2 \end{bmatrix}$$

$$M_{11} = \int_0^L \phi_1 \, m(x) \, \phi_1 \, dx = \int_0^L \phi_1 \, MA \, \delta(x - L/2) \, \phi_1 \, dx = \int_0^L MA \sin^2\left(\frac{\pi x}{L}\right) \delta(x - L/2) \, dx$$

$$M_{11} = MA$$

$$M_{12} = \int_0^L \phi_1 \, m(x) \, \phi_2 \, dx = \int_0^L \phi_1 \, MA \, \delta(x - L/2) \, \phi_2 \, dx = \int_0^L MA \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{3\pi x}{L}\right) \delta(x - L/2) \, dx$$

$$m_{12} = -MA$$

$$M_{21} = \int_0^L \phi_2 \, m(x) \, \phi_1 \, dx = \int_0^L \phi_2 \, MA \, \delta(x - L/2) \, \phi_1 \, dx = \int_0^L MA \sin\left(\frac{3\pi x}{L}\right) \sin\left(\frac{\pi x}{L}\right) \delta(x - L/2) \, dx$$

$$M_{21} = -MA$$

$$M_{22} = \int_0^L \phi_2 \, m(x) \, \phi_2 \, dx = \int_0^L \phi_2 \, MA \, \delta(x - L/2) \, \phi_2 \, dx = \int_0^L \sin^2\left(\frac{3\pi x}{L}\right) \delta(x - L/2) \, dx =$$

$$M_{22} = MA$$

$$M = MA \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The eigen value problem is

$$\frac{\Im \pi^2}{LMA} (K - \lambda M) \underline{X} = \underline{0}$$

now you can solve this two ways:
by hand.

$$\det \left(\begin{bmatrix} 1/2 & 0 \\ 0 & 9/2 \end{bmatrix} - \begin{bmatrix} \lambda & -\lambda \\ -\lambda & \lambda \end{bmatrix} \right) = 0$$

$$\det \begin{bmatrix} 1/2 - \lambda & \lambda \\ \lambda & 9/2 - \lambda \end{bmatrix} = 9/4 - 5\lambda + \lambda^2 - \lambda^2 = 0$$

$$9/4 - 5\lambda = 0$$

$$\lambda = \frac{9}{20} = 0.45$$

So $\omega_1 = \sqrt{\dfrac{9 \Im \pi^2}{20 LMA}} = \boxed{0.6708\pi \sqrt{\dfrac{\Im}{LMA}}} = 2.107 \sqrt{\dfrac{\Im}{MAL}}$

you can also solve the eigen value problem on the computer using this Little trick

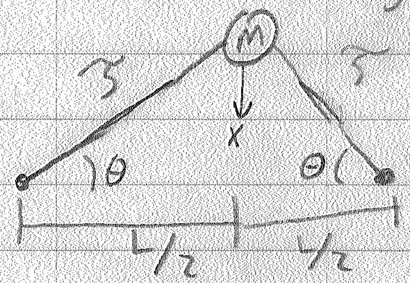
Let $(B - \alpha I)\underline{X} = \underline{0}$ be the new eigen value problem where $B = K^{-1}M$ then solve for α where $\alpha = \frac{1}{\lambda}$

λ being the eigen values you want.

this will also give you

$$\boxed{\omega_1 = 2.107\sqrt{\frac{\tau}{MAL}}}$$

this is a very good approximation. If you were to solve this analytically you would get:



$$M\ddot{x} = -2\tau\sin\theta = -2\tau\frac{x}{L/2} = -\frac{4\tau x}{L}$$

$$m\ddot{x} + \frac{4\tau x}{L} = 0 \qquad \ddot{x} + \underbrace{\frac{4\tau}{mL}}_{\omega^2} x = 0 \qquad m = \rho A = MA$$

so $\boxed{\omega = 2\sqrt{\frac{\tau}{MLA}}}$

b) use the collocation method.

assume weighting functions

$\psi_1 = \delta(x-a) \qquad \psi_2 = \delta(x-b)$

use same comparison functions as part (a.)

$\phi_1 = \sin\left(\frac{\pi x}{L}\right) \qquad \phi_2 = \sin\left(\frac{3\pi x}{L}\right)$

discretize the system where:

$$L = -\zeta \frac{\partial^2}{\partial x^2}$$

$$k_{11} = \int_0^L \psi_1 L \phi_1 dx = -\zeta \int_0^L \psi_1 \frac{\partial^2 \phi_1}{\partial x^2} dx = \frac{\zeta \pi^2}{L^2} \int_0^L \delta(x-a) \sin\left(\frac{\pi x}{L}\right) dx$$

$$k_{11} = \frac{\zeta \pi^2}{L^2} \sin\left(\frac{\pi a}{L}\right)$$

$$k_{12} = \int_0^L \psi_1 L \phi_2 dx = -\zeta \int_0^L \psi_1 \frac{\partial^2 \phi_2}{\partial x^2} dx = \frac{9\zeta \pi^2}{L^2} \int_0^L \delta(x-a) \sin\left(\frac{3\pi x}{L}\right) dx$$

$$k_{12} = \frac{9\zeta \pi^2}{L^2} \sin\left(\frac{3\pi a}{L}\right)$$

$$k_{21} = \int_0^L \psi_2 L \phi_1 dx = -\zeta \int_0^L \psi_2 \frac{\partial^2 \phi_1}{\partial x^2} dx = \frac{\zeta \pi^2}{L^2} \int_0^L \delta(x-b) \sin\left(\frac{\pi x}{L}\right) dx$$

$$k_{21} = \frac{\zeta \pi^2}{L^2} \sin\left(\frac{\pi b}{L}\right)$$

$$K_{22} = \int_0^L \Psi_2 L \Phi_2 \, dx = -\zeta \int_0^L \Psi_2 \frac{\partial^2 \Phi_2}{\partial x^2} dx = \frac{9\zeta\pi^2}{L^2} \int_0^L \delta(x-b) \sin\left(\frac{3\pi x}{L}\right) dx$$

$$K_{22} = \frac{9\zeta\pi^2}{L^2} \sin\left(\frac{3\pi b}{L}\right)$$

$$K = \frac{\zeta\pi^2}{L^2} \begin{bmatrix} \sin\left(\frac{\pi a}{L}\right) & 9\sin\left(\frac{3\pi a}{L}\right) \\ \sin\left(\frac{\pi b}{L}\right) & 9\sin\left(\frac{3\pi b}{L}\right) \end{bmatrix}$$

Now you will see why I did not plus in numbers for a and b.

$$M_{11} = \int_0^L \Psi_1 m(x) \Phi_1 \, dx = \int_0^L \delta(x-a) \, MA \, \delta(x - L/2) \sin\left(\frac{\pi x}{L}\right) dx$$

$$M_{12} = \int_0^L \Psi_1 m(x) \Phi_2 \, dx = \int_0^L \delta(x-a) \, MA \, \delta(x - L/2) \sin\left(\frac{3\pi x}{L}\right) dx$$

$$M_{21} = \int_0^L \Psi_2 m(x) \Phi_1 \, dx = \int_0^L \delta(x-b) \, MA \, \delta(x - L/2) \sin\left(\frac{\pi x}{L}\right) dx$$

$$M_{22} = \int_0^L \Psi_2 m(x) \Phi_2 \, dx = \int_0^L \delta(x-b) \, MA \, \delta(x - L/2) \sin\left(\frac{3\pi x}{L}\right) dx$$

Lookins at M_{ij}'s you see somethins that looks like

$$\int_0^L \delta(x-a)\, MA\, \delta(x-L/2) \sin\left(\frac{n\pi x}{L}\right) dx \quad (1)$$

We know that $\int_0^L \delta(x-a) f(x)\, dx = f(a)$

So in equation (1) above

$$f(x) = MA\,\delta(x-L/2) \sin\left(\frac{n\pi x}{L}\right)$$

So the solution to the integral is

$$f(a) = MA\,\delta(a-L/2) \sin\left(\frac{n\pi a}{L}\right)$$

Unfortunately $\delta(x-L/2) = \begin{cases} \infty, & x = L/2 \\ 0, & x \neq L/2 \end{cases}$

so $M_{ij} = \begin{cases} 0, & a \neq L/2 \\ \infty, & a = L/2 \end{cases}$

this means that <u>you can not</u> solve

this problem using the collocation method because of the point mass. If you want to be creative you can say that point masses are not really real and say the mass exists over a very small finite distance β

then $m(x)$ is just $m(x) = \frac{MA}{\beta}$ a constant.

now you can find the m matrix as

$$M = \frac{MA}{\beta} \begin{bmatrix} \sin\left(\frac{\pi a}{L}\right) & \sin\left(\frac{3\pi a}{L}\right) \\ \sin\left(\frac{\pi b}{L}\right) & \sin\left(\frac{3\pi b}{L}\right) \end{bmatrix}$$

now the eigen value problem is

$$\frac{\tilde{S}\pi^2 \beta}{MAL^2} \left(K - \lambda M \right) \underline{X} = \underline{0}$$

$$\det \begin{bmatrix} \sin\left(\frac{\pi a}{L}\right) - \lambda \sin\left(\frac{\pi a}{L}\right) & 9\sin\left(\frac{3\pi a}{L}\right) - \lambda \sin\left(\frac{3\pi a}{L}\right) \\ \sin\left(\frac{\pi b}{L}\right) - \lambda \sin\left(\frac{\pi b}{L}\right) & 9\sin\left(\frac{3\pi b}{L}\right) - \lambda \sin\left(\frac{3\pi b}{L}\right) \end{bmatrix} = 0.$$

$$\det \begin{bmatrix} (1-\lambda)\sin\left(\frac{\pi a}{L}\right) & (9-\lambda)\sin\left(\frac{3\pi a}{L}\right) \\ (1-\lambda)\sin\left(\frac{\pi b}{L}\right) & (9-\lambda)\sin\left(\frac{3\pi b}{L}\right) \end{bmatrix} = 0$$

$$(9 - 10\lambda + \lambda^2)\sin\left(\frac{\pi a}{L}\right)\sin\left(\frac{3\pi b}{L}\right) - (9 - 10\lambda + \lambda^2)\sin\left(\frac{\pi b}{L}\right)\sin\left(\frac{3\pi a}{L}\right) = 0$$

Let $a = \frac{L}{2}$ and $b = \frac{3L}{4}$

$$0.7071(9 - 10\lambda + \lambda^2) + 0.7071(9 - 10\lambda + \lambda^2) = 0$$

$$\sqrt{2}(9 - 10\lambda + \lambda^2) = 0.$$

$$9 - 10\lambda + \lambda^2 = 0. \qquad \frac{10 \pm \sqrt{100-36}}{2} = \frac{10 \pm 8}{2}$$

$\lambda_1 = 1$ and $\lambda_2 = 9$

Since we want ω_1 look at λ_1

$$\lambda_1 = 1 \Rightarrow \omega_1^2 = \frac{3\pi^2 B}{mAL^2} \text{ so}$$

if $\frac{\beta\pi}{L} = 2$ the answer is the same as the analytical solution. This shows that if you pick β to be $\frac{2L}{\pi}$ for $a = \frac{L}{2}$ and $b = \frac{3L}{4}$ you will get the right answer. However this should not make you feel any better about using this method on this particular problem.