The QR Method

The QR Method for obtaining eigenvalues is computationally expensive if the matrix is full. It is very efficient when Given's method is applied 1st.

Example

$$A_1^{(0)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2.5 & -1.5 \\ 0 & -1.5 & 3 \end{bmatrix}$$

1st: Annihilate subdiagonal elements using Given's Method

$$\sin\theta_1 = \frac{-1}{\sqrt{2^2+(-1)^2}} = -.4472$$

$$\cos\theta_1 = \frac{2}{\sqrt{2^2+(-1)^2}} = .8944$$

The rotation matrix Θ is

$$\Theta_1 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_1^{(1)} = \Theta_1 A_1^{(0)} = \begin{bmatrix} 2.2361 & -2.0125 & .6708 \\ 0 & 1.7889 & -1.3416 \\ 0 & -1.5 & 3.000 \end{bmatrix}$$

Now do a rotation to remove $a_{3,2}$

$$\sin\theta_2 = \frac{-1.5}{\sqrt{1.7889^2+1.5^2}} = -.6425$$

$$\cos\theta_2 = \frac{1.7889}{\sqrt{1.7889^2+1.5^2}} = .7663$$

$$\Theta_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .7663 & -.6425 \\ 0 & .6425 & .7663 \end{bmatrix}$$

$$A_1^{(2)} = \Theta_2 A_1^{(1)} = \begin{bmatrix} 2.2361 & -2.0125 & .6708 \\ 0 & 2.3345 & -2.9556 \\ 0 & 0 & 1.4367 \end{bmatrix}$$

Which is now upper triangular

Q_1 is now

$$Q_1 = \Theta_1^T \Theta_2^T = \begin{bmatrix} .8944 & .3427 & .2873 \\ -.4472 & .6854 & .5747 \\ 0 & -.6425 & .7663 \end{bmatrix}$$

Finally. The First iteration is

$$A_2 = R_1 Q_1 = A_1^{(2)} Q_1 = \Theta_2\Theta_1 A_1^{(0)} \Theta_1^T \Theta_2^T$$

$$= \begin{bmatrix} 2.9 & -1.044 & 0 \\ -1.044 & 3.4991 & -.9231 \\ 0 & -.9231 & 1.1009 \end{bmatrix}$$

recall $A_1^{(2)} = \Theta_2\Theta_1 A_1^{(0)}$, so

$$A_2 = \Theta_2 \Theta_1 A_1^0 \Theta_1^T \Theta_2^T$$

Now. Solve the two eigen values for the bottom right 2×2 matrix.

original matrix $\det \begin{vmatrix} 2.5-\lambda & -1.5 \\ -1.5 & 3-\lambda \end{vmatrix} = 0$

Eigenvalue closest to 3 is 4.27069. (λ_1)

And

$$\det \begin{vmatrix} 3.4991-\lambda & -.9231 \\ -.9231 & 1.1009-\lambda \end{vmatrix} = 0$$

Eigenvalue closest to 1.1009 is .7867 (λ_2)

$$\left| \frac{\lambda_2}{\lambda_1} - 1 \right| = .81579 > \frac{1}{2}$$

A shift is <u>not</u> in order
(see eqn j, p170, or 5.192)

$$A_2 = \begin{bmatrix} 2.9 & -1.044 & 0 \\ -1.044 & 3.4991 & -.9231 \\ 0 & -.9231 & 1.1009 \end{bmatrix}$$

Repeat cycle, annihilate a_{21}

$$\sin\theta_1 = \frac{-1.044}{\sqrt{1.044^2 + 2.9^2}} = -.3387$$

$$\cos\theta_1 = \frac{2.9}{\sqrt{1.044^2 + 2.9^2}} = .9407$$

$$\Theta_1 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2^{(1)} = \begin{bmatrix} 3.0822 & -2.1676 & .3127 \\ 0 & 2.9386 & -.8686 \\ 0 & -.9231 & 1.1009 \end{bmatrix}$$

$$\sin\theta_1 = \frac{-.9231}{\sqrt{.9231^2 + 2.9386^2}} = -.2997$$

$$\cos\theta_1 = \frac{2.9386}{\sqrt{.9231^2 + 2.9386^2}} = .9540$$

$$\Theta_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$A_2^{(2)} = \Theta_2 A_2^{(1)} = \begin{bmatrix} 3.0822 & -2.1676 & .3127 \\ 0 & 3.0802 & -1.1586 \\ 0 & 0 & .7900 \end{bmatrix}$$

Q_2 is now

$$Q_2 = \Theta_1^T \, \Theta_2^T = \begin{bmatrix} .9409 & .3232 & .1015 \\ -.3387 & .8976 & .2820 \\ 0 & -.2997 & .9540 \end{bmatrix}$$

The second iteration is then

$$A_3^{(0)} = \overset{R_2}{A_2^{(2)}} \, Q_2 = \begin{bmatrix} 3.6342 & -1.0433 & 0 \\ -1.0433 & 3.1121 & -.2368 \\ 0 & -.2368 & .7537 \end{bmatrix}$$

Recall $\lambda_2 = .7867$

$$\lambda_3 = \text{eig} \begin{bmatrix} 3.1121 & -.2368 \\ -.2368 & .7537 \end{bmatrix} \text{ closest to } .7537$$

$$= .7301$$

$$\left| \frac{\lambda_3}{\lambda_2} - 1 \right| = \left| \frac{.\cancel{7537}\,301}{.7867} - 1 \right| = \cancel{.0419}\ .0719 << \frac{1}{2}$$

A shift is advisable.

The next iteration begins using

$$A_3^{(0)} - .7301\, I = \begin{bmatrix} 2.9041 & -1.0433 & 0 \\ -1.0433 & 2.3820 & -.2368 \\ 0 & -.2368 & .0235 \end{bmatrix}$$

repeating the cycle using $A_3^{(0)}$

$$A_3^{(2)} \quad R_3 = \begin{bmatrix} 3.0858 & -1.7873 & .0801 \\ 0 & 1.9037 & -.2240 \\ 0 & 0 & -.0044 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} .9411 & .3355 & -.0421 \\ -.3381 & .9338 & .1170 \\ 0 & -.1244 & .9922 \end{bmatrix}$$

$$A_4^{(0)} = R_3 Q_3 + .7301 I = \begin{bmatrix} 4.2385 & -.6437 & 0 \\ -.6437 & 2.5356 & .0005 \\ 0 & .0005 & .7258 \end{bmatrix}$$

The 3-3 element is almost independent, so The 1st eigenvalue is ~ .7258. We can continue further for more accuracy, or truncate this eigenvalue and continue with finding the eigenvalues of

$$\begin{bmatrix} 4.2385 & -.6437 \\ -.6437 & 2.5356 \end{bmatrix}$$

The 1st true eigenvalue is .725817
We obtained .725773

31.37