ME 710, WI 96, Exam 1

1) Find the natural frequencies and mode shapes for a rod with the equation of motion $(E/\rho)w_{xx}(x,t) = w_{tt}(x,t)$ with boundary conditions $w(0,t) = 0$ and $w_x(l,t) = 0$. (30 points)

2) Derive the equation of motion for a simple pendulum (unforced) with an arm that is elastic in extension. Assume a point mass m, a stiffness k, and a stretched (under the influence of gravity) spring length l (unstretched length+mg/k). Don't forget to include gravity. (30 points)

Lagrange's equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

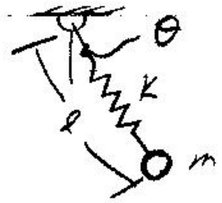
3) Find the forced response of a string of length l with tension τ and density per unit length ρ. The natural frequencies are $\omega_n = \sqrt{\tau/\rho}\,\sigma_n$ where $\sigma_n = n\pi/l$ with normalized mode shapes $\sqrt{2/l}\sin(n\pi x/l)$. Write out the first few terms of the solution. (40 points)

The equation of motion is

$$\rho w_{tt}(x,t) - \tau w_{xx}(x,t) = \delta(x - l/2)\sin\left(\frac{2\pi}{l}\sqrt{\frac{\tau}{\rho}}t\right)$$

ME 710 Exam 1 Solutions

2)



l is unstretched length

Let the stretched by gravity length be $l_1 = l + \frac{mg}{K}$, x be the stretch of the spring relative to l_1

The potential energy is

$$V = \frac{1}{2} K \left(x + \frac{mg}{K}\right)^2 + mg\left(l_1 - (l_1 + x)\cos\theta\right)$$

The Kinetic energy is

$$T = \frac{1}{2} m \left(\dot{x}^2 + \left[\dot{\theta}(l_1 + x)\right]^2 \right)$$

$$L = T - V$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}\left(m(l_1 + x)^2 \dot{\theta}\right) = 2m\dot{x}(l_1 + x)\dot{\theta} + m(l_1 + x)^2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = mg(l_1 + x)\sin\theta$$

$$m(l_1 + x)^2 \ddot{\theta} + 2m(l_1 + x)\dot{\theta}\dot{x} + mg(l_1 + x)\sin\theta = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{\partial L}{\partial x} = -K\left(x - \frac{mg}{K}\right) + mg\cos\Theta + m\dot{\theta}^2(x + l_1)$$

$$m\ddot{x} + K\left(x - \frac{mg}{K}\right) - mg\cos\theta + m\dot{\theta}^2(x + l_1) = 0$$

1) $\left(\frac{E}{\rho}\right) \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial t^2}$

Assume $w(x,t) = T(t)\, X(x)$

$\left(\frac{E}{\rho}\right) X'' T = \ddot{T} X$

$\frac{E}{\rho} \frac{X''}{X} = \frac{\ddot{T}}{T} = \text{const} = -\lambda$

$\frac{E}{\rho} \frac{X''}{X} = -\lambda$

$X'' + \lambda \frac{\rho}{E} X = 0$

Solutions are sin and cos

$X(x) = A \sin \alpha_n x + B \cos \alpha_n x$

$\alpha_n = \sqrt{\lambda \frac{\rho}{E}}$

$X(0) = 0 = B \cos \alpha_n 0 \quad , \quad B = 0$

$X'(\ell) = 0 = \alpha_n A \cos \alpha_n \ell - \alpha_n B \sin \alpha_n x$

$\cos \alpha_n \ell = 0$

$\alpha_n = \frac{(2n-1)\pi}{2\ell}$

$X(x) = A \sin\left(\frac{2n-1}{2\ell}\right) \pi x$ mode shapes

$$\lambda = \left(\frac{(2n-1)\pi}{2\ell}\right)^2 \frac{E}{\rho}$$

The temporal equation is then

$$\ddot{T} + \lambda T = 0$$

So $\omega_n = \sqrt{\lambda} = \dfrac{(2n-1)\pi}{2\ell}\sqrt{\dfrac{E}{\rho}}$

3) $\rho W_{tt} - \tau W_{xx} = \delta(x - \ell/2) \sin\left(\frac{2\pi}{\ell}\sqrt{\frac{\tau}{\rho}}\, t\right)$

Assume a solution of the form

$$W(x,t) = \sum_{n=1}^{\infty} T_n(t)\, X_n(x)$$

where we know $X_n(x) = \sqrt{\frac{2}{\ell}} \sin\frac{n\pi x}{\ell}$

$$W_{xx} = -\sum_{n=1}^{\infty} T_n \left(\frac{n\pi}{\ell}\right)^2 \sqrt{\frac{2}{\ell}} \sin\frac{n\pi x}{\ell}$$

Substituting the assumed form of the solution into the equation of motion

$$\sum_{n=1}^{\infty} \left(\rho \ddot{T}_n + \left(\frac{n\pi}{\ell}\right)^2 \tau \; T_m\right)\sqrt{\frac{2}{\ell}} \sin\frac{n\pi x}{\ell} = \delta\left(x - \frac{\ell}{2}\right) \sin\left(\frac{2\pi}{\ell}\sqrt{\frac{\tau}{\rho}}\, t\right)$$

Multiplying both sides by $\sqrt{\frac{2}{\ell}} \sin\frac{m\pi x}{\ell}$ and integrating over the length gives

$$\rho \ddot{T}_m + \left(\frac{m\pi}{\ell}\right)^2 \tau\, T_m = \sqrt{\frac{2}{\ell}} \sin\frac{m\pi}{2} \sin\left(\frac{2\pi}{\ell}\sqrt{\frac{\tau}{\rho}}\, t\right)$$

For m even, $\sin\frac{m\pi}{2} = 0$. For m odd ± 1.

Assuming a solution $T_n(t) = A_n \sin\left(\overbrace{\frac{2\pi}{\ell}\sqrt{\frac{\tau}{\rho}}}^{\omega_{dr}}\, t\right)$

$$-A_n\left(\frac{2\pi}{\ell}\right)^2 \tau \sin\omega_{dr} t + A_n\left(\frac{n\pi}{\ell}\right)^2 \tau \sin\omega_{dr} t = \sqrt{\frac{2}{\ell}}\,\sin\frac{n\pi}{2}\,\sin\omega_{dr} t$$

$$A_n = \frac{\sqrt{\frac{2}{\ell}}\,\sin\frac{n\pi}{2}\,\ell^2}{-(2\pi)^2\tau + (n\pi)^2\tau}$$

The total solution is then

$$W(x,t) = \sum_{n=1}^{\infty} \frac{2\ell}{\tau\pi^2}\,\sin\frac{2\pi}{\ell}\sqrt{\frac{T}{\rho}}\,t\,\sin\frac{n\pi x}{\ell}\,\frac{\sin\frac{n\pi}{2}}{n^2-4}$$

$$= \frac{-2}{3}\,\frac{\ell}{\tau\pi^2}\,\sin\frac{2\pi}{\ell}\sqrt{\frac{T}{\rho}}\,t\,\sin\frac{\pi x}{\ell}$$

$$-\frac{2}{5}\,\frac{\ell}{\tau\pi^2}\,\sin\frac{2\pi}{\ell}\sqrt{\frac{T}{\rho}}\,t\,\sin\frac{3\pi x}{\ell}$$

$$+\frac{2}{21}\,\frac{\ell}{\tau\pi^2}\,\sin\frac{2\pi}{\ell}\sqrt{\frac{T}{\rho}}\,t\,\sin\frac{5\pi x}{\ell}$$

$$-\frac{2}{45}\,\frac{\ell}{\tau\pi^2}\,\sin\frac{2\pi}{\ell}\sqrt{\frac{T}{\rho}}\,t\,\sin\frac{7\pi x}{\ell}$$

$$+ \cdots$$