short answers

1) It works. Piezoceramic based devices drain charge too quickly for low freq use.

2) a) real eigenvalues
b) real eigenvectors
c) orthogonal eigenvectors
d) 1/2 the storage space

3) $m\omega_n = .1$ $\qquad \omega_d = .9 = \omega_n\sqrt{1-\zeta^2}$ $\qquad \zeta\omega_n = .01$

$$\frac{.01}{.9} = \frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}} = \frac{\zeta}{\sqrt{1-\zeta^2}}$$

$$.9\zeta = .01\sqrt{1-\zeta^2}$$

$$.81\zeta^2 = .0001(1-\zeta^2)$$

$$.8101\zeta^2 = .0001$$

$$\zeta = \sqrt{\frac{.0001}{.8101}} = .011 \quad \left(\text{or } \zeta \approx \frac{.01}{.9}\right)$$

$$\omega_n = \frac{.01}{.011} = .9 \text{ rad/s}$$

$$m = \frac{.1}{\omega_n} = 0.11 \text{ units} \quad \text{(I didn't give them)}$$

$m = 0.11$, $c = 2\zeta\omega_n m$ $\qquad k = m\omega_n^2$

$$\boxed{0.11\ddot{x} + 0.002\dot{x} + 0.09\dot{x} = F(t)}$$

4) Energy Method doesn't allow inclusion of
a) energy dissipation
b) external forces

5) Easier to handle geometric nonlinearities

6) Yes, through the mass and stiffness matrices

7) K is symmetric and Positive semi-definite
(all eigenvalues ≥ 0)

M is symmetric and Positive definite.

8) In addition to orthogonality through K and M
$\underline{u}_i^T K \underline{u}_i = \omega_i^2 \qquad \underline{u}_i^T M \underline{u}_i = 1$
when they are mass normalized

9) Orthogonality of $\underline{v}_i$ (eigenvectors of $\tilde{K}$)

10) Numerical simulation

Long Problems

1) a) 1) $y_t = \frac{-mg}{K} \cos \sqrt{\frac{K}{m}}\, t$

2) Equation one has no transient response given the initial deflection.

b) 1) $y_{ss} = \frac{mg}{K}$. 2) $y_{ss} = 0$ for eqn 1

c) $y_2 = \frac{mg}{K}\left(1 - \cos \sqrt{\frac{K}{m}}\, t\right)$

d) Max is $\frac{2mg}{K}$. Min is zero. Max is twice static deflection.

2) $a_0 = 1$

Real form is easiest for this problem

$$a_n = \frac{2}{T}\left(\int_0^{T/2} 2\cos\frac{2\pi n t}{T}\,dt + \int_{T/2}^{T} -1\cos\frac{2\pi n t}{T}\,dt\right)$$

$$= \frac{2}{T}\left(\frac{2T}{2\pi n}\sin\frac{2\pi n t}{T}\Big|_0^{T/2} - \frac{T}{2\pi n}\sin\frac{2\pi n t}{T}\Big|_{T/2}^{T}\right)$$

$$= \frac{2}{T}\left(\frac{T}{\pi n}\left(\sin\cancel{\frac{2\pi n}{2}}^{0} - 0\right) - \frac{T}{2\pi n}\left(\sin\cancel{2\pi n}^{0} - \sin\cancel{\frac{2\pi n}{2}}^{0}\right)\right)$$

$$= 0$$

$$b_n = \frac{2}{T}\left(\int_0^{T/2} 2\sin\frac{2\pi n t}{T}\,dt + \int_{T/2}^{T} -1\cos\frac{2\pi n t}{T}\,dt\right)$$

$$= \frac{2}{T}\left(\frac{-2T}{2\pi n}\cos\frac{2\pi n t}{T}\Big|_0^{T/2} + \frac{T}{2\pi n}\cos\frac{2\pi n t}{T}\Big|_{T/2}^{T}\right)$$

$$= \frac{-1}{\pi n}\left[2\left(\underbrace{\cos\pi n}_{1/-1\text{ even/odd}} - \underbrace{\cos 0}_{1}\right) - \left(\underbrace{\cos 2\pi n}_{1} - \underbrace{\cos\pi n}_{1/-1\text{ even/odd}}\right)\right]$$

For n odd

$$b_n = \frac{-1}{\pi n}\left(2(-1-1) - (1+1)\right)$$

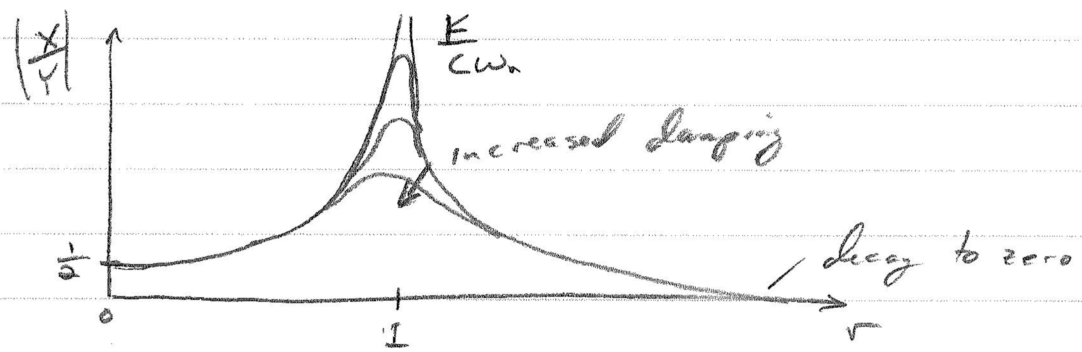
$$= \frac{6}{\pi n}$$

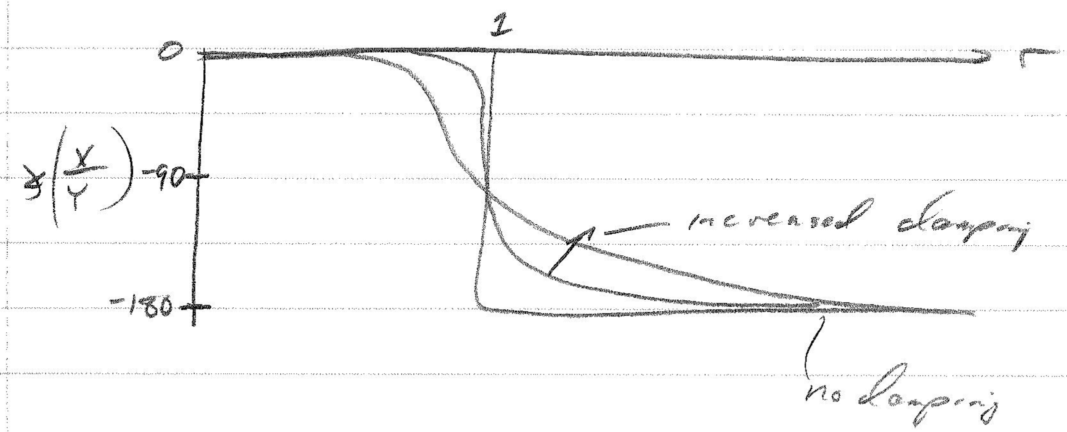
For n even

$$b_n = \frac{-1}{\pi n}(0) = \boxed{0}$$

$$F(t) = \frac{1}{2} + \sum_{n=1} \frac{6}{n\pi}\sin\frac{2\pi n t}{T}$$

3) $\frac{X}{Y} = \frac{K}{2K - m\omega^2 + cj\omega}$ $\qquad \omega_n = \sqrt{\frac{2K}{m}}$





4) $\underline{f} = S^T \underline{F} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \delta(t) \\ 0 \end{bmatrix}$

(note S is orthonormal, conveniently making $S^{-1} = S^T$)

$$\underline{f} = \frac{1}{\sqrt{2}} \begin{bmatrix} \delta(t) \\ \delta(t) \end{bmatrix}$$

$h(t) = \frac{1}{m\omega_n} e^{-\zeta\omega_n t} \sin \omega_d t \qquad \zeta = 0,\ \omega_d = \omega_n$

$r(t) = \hat{F}(t)\, h(t)$

$r_1(t) = \frac{1}{\sqrt{2}} \frac{1}{3} \sin 3t \qquad$ Modal mass is always 1

$r_2(t) = \frac{1}{\sqrt{2}} \frac{1}{10} \sin 10t$

$$x(t) = S\ r(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}}\frac{1}{3} \sin 3t \\ \frac{1}{\sqrt{2}}\frac{1}{10} \sin 10t \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6} \sin 3t + \frac{1}{20} \sin 10t \\ \frac{1}{6} \sin 3t - \frac{1}{20} \sin 10t \end{bmatrix}$$

5) $\det(K - M\omega^2) = 0$

$$(35 - 2\lambda)(35 - 3\lambda) - 35^2 = 0$$

$$35^2 - 5 \cdot 35\lambda + 6\lambda^2 - 35^2 = 0$$

$$6\lambda^2 = 175\lambda$$

$$\lambda = 0,\ \frac{175}{6}\ (29.27)$$

$$\omega = 0,\ 5.4\ \text{rad/s}$$

$\underline{\omega_1 = 0}$:

$$\begin{bmatrix} 35 & -35 \\ -35 & 35 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$u_1 = u_2 \qquad \underline{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\omega_2 = 5.4$

$$\begin{bmatrix} 35 - 58.3 & -35 \\ -35 & 35 - 87.5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-23.3u_1 + -35u_2 = 0$

$-35u_1 - 52.5u_2 = 0$

$\underline{u}_2 = \begin{bmatrix} 1 \\ -.666 \end{bmatrix}$

You should check using orthogonality if time remains.

6) $U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 + \frac{1}{2} k_3 x_2^2$

$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} J_2 \dot{\theta}_2^2$

These are disks, so $\dot{\theta} = \frac{\dot{x}}{r}$

$T = \frac{1}{2}\left(m_1 + \frac{J_1}{r_1^2}\right)\dot{x}_1^2 + \frac{1}{2}\left(m_2 + \frac{J_2}{r_2^2}\right)\dot{x}_2^2$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = \left(m_1 + \frac{J_1}{r_1^2}\right)\ddot{x}_1$

$\frac{\partial U}{\partial x_1} = k_1 x_1 + k_2(x_1 - x_2)$

Eqn 1 is

$$\boxed{\left(m_1 + \frac{J_1}{r_1^2}\right)\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0}$$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = \left(m_2 + \frac{J_2}{r_2^2}\right)\ddot{x}_2$

$\frac{\partial U}{\partial x_2} = k_2(x_2 - x_1) + k_2 x_3$

Eqn 2 is

$$\boxed{\left(m_2 + \frac{J_2}{r_2^2}\right)\ddot{x}_2 + (k_2 + k_3)x_2 - k_2 x_1 = 0}$$

7) (Grad Problem)

$$W_{tt} = c^2 \; W_{xx}$$

$$-\omega_n^2 \; X(x) = c^2 \; X''(x)$$

$$X''(x) + \overbrace{\frac{\omega_n^2}{c^2}}^{\sigma^2} X(x) = 0$$

$$X(x) = A \sin \sigma x + B \cos \sigma x$$

Free-Free conditions mean

$$X'|_{x=0} = 0 = A\sigma \cos(\sigma 0) - B\sigma \sin(\sigma 0) \quad ①$$

$$X'|_{x=\ell} = 0 = A\sigma \cos(\sigma \ell) - B\sigma \sin(\sigma \ell) \quad ②$$

From ①, $A = 0$

From ② $\sin \sigma_n \ell = 0$

$$\sigma_n = \frac{n\pi}{\ell} \qquad n = 0, 1, 2, \ldots \quad \text{(Yes, zero)}$$

Mode shapes are

$$X_n(x) = B_n \cos \frac{n\pi x}{\ell}$$

Then, $\omega_n = \sqrt{\sigma_n^2 c^2} = \frac{n\pi}{\ell}\sqrt{\frac{T}{\rho}}$

To normalize them $B_n = \sqrt{\frac{2}{\ell}}$, except $B_0 = \frac{1}{\sqrt{\ell}}$

$$w(t) = \sum_{n=1}^{\infty} T_n(t) \, X_n(t)$$

Subst into EOM, multiply over domain, gives

$$\left(\ddot{T}_n + \underbrace{c^2 \sigma_n^2}_{\omega_n^2 =} T \right) = B_n \, 100 \cos \frac{n\pi}{2} \; \delta(t)$$

$$\hat{F} = B_n\, 100 \cos\frac{n\pi}{2} \qquad \cos\frac{n\pi}{2} = \begin{cases} 0 & n \text{ odd} \\ 1 & n = 0, 4, 8, \ldots \\ -1 & n = 2, 6, 10, \ldots \end{cases}$$

$$h(t) = \frac{1}{\omega_n} \sin\omega_n t$$

$$T_n(t) = 100\, B_n \cos\frac{n\pi}{2} \frac{1}{\omega_n} \sin\omega_n t$$

Except for $n = 0$. For $n = 0$, impulse is initial velocity of a free system, so

$$\dot{T}_n = 100 \frac{1}{\sqrt{\ell}}, \qquad T_n = 100 \frac{1}{\sqrt{\ell}}\, t$$

$$w(x,t) = \frac{100}{\ell}\, t + \sum_{n=1}^{\infty} \frac{200}{\ell\omega_n} \underbrace{\cos\frac{n\pi}{2}}_{\frac{1}{B_n}\left(X_n\left(\frac{\ell}{2}\right)\right)} \cos\frac{n\pi x}{\ell} \sin\left(\frac{n\pi}{\ell}\sqrt{\frac{I}{\ell}}\, t\right)$$

where $\omega_n = \frac{n\pi}{\ell}\sqrt{\frac{I}{\ell}}$