WI 2006 Final Exam Soln

Short Answer

1) With the reduction of one constraint, the eigenvalues will all shift down or remain the same, but not shift below the previous next lowest eigenvalue
i.e. $\lambda_1 \le \tilde{\lambda}_2 \le \lambda_2 \le \tilde{\lambda}_3 \le \lambda_4 \ldots$
Where $\tilde{\lambda}_i$ are eigenvalues of the constrained system.
This tells you that mesh refinement in FEA should shift estimates of nfs down, that your estimates using FEA is typically high. Further the effects of constraints imposed for modelling purposes (eg BCs) cause high estimates of natural frequencies.

2) $\frac{1}{2}$ the computational effort. Faster and easier to code

3) Stationarity of the Rayleigh Quotient $R = \frac{(u, Lu)}{(u, Mu)}$

4) Green's Method. It adjusts the subspace within which eigenvectors are presumed to exist via a MDOF variant of Green's Method.

5) Non self-adjoint
Damped
Forced
Nonlinear

$\Big\}$ 2 required

Short problems

1) $y_1 = \sqrt{2}\sin \pi x \qquad y_2 = \sqrt{3}\,x \qquad 0<x<1$

(Normalized by observation)

Using Gram Schmidt

$X_1 = y_1 = \sqrt{2}\sin\pi x$

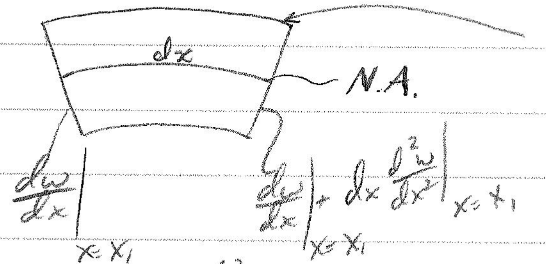
$\hat{X}_2 = y_1 - (y_1, y_2) = \sqrt{2}\sin\pi x - \left(\sqrt{6}\int_0^1 x\sin\pi x\,dx\right)\sqrt{3}\,x$

$= \sqrt{2}\left(\sin\pi x - \frac{3x}{\pi}\right)$

Normalized

$$X_2 = \frac{\hat{X}_2}{(\hat{X}_2, \hat{X}_2)^{1/2}} = \sqrt{\frac{2}{\pi^2-6}}\left(\pi\sin\pi x - 3x\right)$$

2)



change in length of top fiber is negative change in slope times dx (negative because as shown a negative change in slope positively stretches top fiber) times distance from NA (because plane sections remain plane)

Fiber stretch $= y\,dx\,\frac{d^2w}{dx^2}$

Fiber strain $= \epsilon = y\,\frac{d^2w}{dx^2}$

Fiber stress $= E\epsilon = Ey\,\frac{d^2w}{dx^2}$

Strain Energy $= \frac{1}{2}\sigma\epsilon = \frac{1}{2}Ey^2\frac{d^2w}{dx^2}$

Strain Energy at cross section

$$SE = \frac{1}{2}E\underbrace{\int_A y^2\,dA}_{I}\,\frac{d^2w}{dx^2} = \frac{1}{2}EI\frac{d^2w}{dx^2}$$

SE in beam $= \frac{1}{2}\int_0^{\ell} EI\,\frac{d^2w}{dx^2}\,dx$ (EI can change with x)

Long Problems

X_0 is zero strain of spring

1) $\vec{r} = (X_0 + X)\cos\omega t\,\hat{i} + (X_0 + X)\sin\omega t\,\hat{j}$

$\vec{V} = (\dot{X}\cos\omega t - (X_0+X)\omega\sin\omega t)\hat{i} + (\dot{X}\sin\omega t + (X_0+X)\omega\cos\omega t)\hat{j}$

$V^2 = \dot{X}^2 + ((X_0+X)\omega)^2$

$U = \frac{1}{2}KX^2 + mg(X+X_0)\sin\omega t$

$T = \frac{1}{2}m(\dot{x}^2 + ((X_0+X)\omega)^2)$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} = 0$$

$m\ddot{x} - m(X_0+X)\omega^2 + KX + mg\sin\omega t = 0$

$$\boxed{m\ddot{x} + (K - m\omega^2)X = mX_0\omega^2 - mg\sin\omega t}$$

The zero point (about which it oscillates) is

$$X = \frac{mX_0\omega^2}{K - m\omega^2}$$

Clearly it is unstable if $\omega > \sqrt{\frac{K}{m}}$ because this point goes to infinity. This is born out in the EOM.

The Lyapunov function is indefinite for small values of x (see U)

$U = \frac{1}{2}Kx^2 + mg(X+X_0)\sin\omega t$

The lowest value of the second term is -1.

For that, we require

$$\frac{1}{2}Kx^2 - mgX - mgX_0 > 0$$

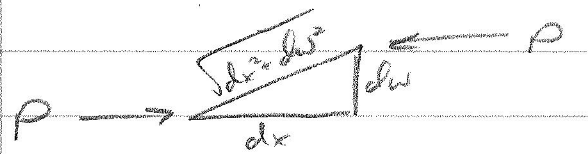
$$X = \frac{mg \pm \sqrt{(mg)^2 - 2KmgX_0}}{K}$$

For X close to zero, U is negative. For large X, U is positive. We've exhausted the course material

2) See midterm

3) See file

4)
$$T = \tfrac{1}{2} m \dot{w}^2$$
$$V = \int_0^{\ell} \tfrac{1}{2} EI \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx + V_{compression}$$



$$W_c = -V = P\,ds$$
$$ds = \sqrt{dx^2+dw^2} - dx \simeq dx + \tfrac{1}{2}\left(\frac{dw}{dx}\right)^2 dx - dx$$
$$\simeq \tfrac{1}{2}\left(\frac{dw}{dx}\right)^2 dx$$
$$\therefore\ V_{compression} = -\int_0^{\ell} \tfrac{1}{2} P (w')^2\,dx$$

Hamilton's Principle
$$\delta T - \delta V = 0$$

integrating by parts
$$\delta T = \delta \int_{t_1}^{t_2}\int_0^{\ell} \tfrac{1}{2} m \dot{w}^2\,dx\,dt = -\int_{t_1}^{t_2}\int_0^{\ell} m \ddot{w}\,\delta w\,dx\,dt$$

$$\delta V = \int_{t_1}^{t_2} \underbrace{\int_0^{\ell} \delta \tfrac{1}{2} EI w''^2\,dx}_{\delta V_1} + \underbrace{-\int_0^{\ell} \delta \tfrac{1}{2} P w'^2\,dx}_{\delta V_2}\,dt$$

Integrating the 1st term by parts twice

$$\delta V_1 = \int_{t_1}^{t_2} \int_0^l \frac{\partial^2}{\partial x^2}(EI\, w'')\, \delta w\, dx + \frac{\partial}{\partial w} EI\, w'' \delta w \Big|_0^l$$

$$- EI\, w'' \delta w' \Big|_0^l \, dt$$

Integrating the 2nd term once

$$\delta V_2 = -\int_{t_1}^{t_2} -\int_0^l P w'' \delta w\, dx + P w' \delta w \Big|_0^l \, dt$$

Combining terms inside the integrals, the EOM is

$$\boxed{\begin{gathered} -m\ddot{w} - \frac{\partial^2}{\partial x^2}(EI\, w'') - P w'' = 0 \\ w(x,t)\Big|_{x=0,l} = 0 \quad \text{and} \quad EI\, w''(x,t)\Big|_{x=0,l} = 0 \end{gathered}}$$

Bonus

The solution for the 1st mode is (for constant cross section)

$$w(x,t) = T(t) \sin \frac{\pi x}{l}$$

Substituting into the EOM gives

$$\left(-m\ddot{T}(t) - \left(\frac{\pi}{l}\right)^4 EI\, T(t) + P\left(\frac{\pi}{l}\right)^2 T(t)\right) \sin\frac{\pi x}{l} = 0$$

Since $\sin\frac{\pi x}{l} \neq 0$,

$$m\ddot{T} + \left(\frac{\pi}{l}\right)^2 \left(EI\left(\frac{\pi}{l}\right)^2 - P\right) T = 0$$

If $P > EI\left(\frac{\pi}{l}\right)^2$, the beam is unstable and buckles