The Jacobi Method

Produces all eigenvalues and eigenvectors simultaneously.

Consider the eigenvalue problem

$$A x_i = \lambda_i \underline{x}_i$$

Consider the coordinate transformation

$$\underline{x}_i = R \underline{y}_i$$

such that

$$A R \underline{y}_i = \lambda_i R \underline{y}_i$$

If R is a unitary transformation

$$R^T = R^{-1}$$

$$R^T A R \underline{y}_i = R^T \lambda_i R \underline{y}_i$$

$$R^T A R \underline{y}_i = \lambda_i \underline{y}_i$$

We would like to obtain R so

$$R^T A R = \Lambda, \text{ then } y_i = 1, \text{ all other values } 0$$

We can't get R by simple observation, but we can apply repeated coordinate rotations to try and eliminate the off-diagonal terms

Example

$$A_0^{(0)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -\sqrt{2} \\ 0 & -\sqrt{2} & 1 \end{bmatrix}$$

think Mohr's circle

First step is to eliminate a_{12} (a_{21}) by a coordinate transformation.

The angle of rotation for the coordinates 1+2 is

$$\tan 2\theta_1 = \frac{2\,a_{12}^{(0)}}{a_{11}^{(0)} - a_{22}^{(0)}} = \frac{2\,(-1)}{2-3} = 2$$

$$\theta_1 = .5536 \text{ rad}$$

A coordinate transformation matrix is formed

$$R_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

sin θ takes sign of tan θ

$$= \begin{bmatrix} .85 & -.52 & 0 \\ .52 & .85 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_1 = R_1^T A_0 R_1 = \begin{bmatrix} 1.38 & 0 & -.743 \\ 0 & 3.618 & -1.20 \\ -.743 & -1.203 & 1 \end{bmatrix}$$

Next, we want to remove $a_{13}^{(1)}$ (in the 1-3 plane)
We will rotate about the 2 axis

$$\tan 2\theta_2 = \frac{2(-.743)}{1.38-1} = -3.89$$

$$\theta_2 = -.66 \text{ rad}$$

The coordinate transformation matrix is

$$R_2 = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}$$

$$A_2 = R_2^T A_1 R_2 = \begin{bmatrix} 1.959 & .737 & 0 \\ & 3.618 & -.951 \\ \cancel{\text{diag}}\ \text{sym} & & .423 \end{bmatrix}$$

Note that the off diagonal terms are getting smaller each time.

Rotating about the x axis yields

$$\tan 2\theta_3 = \frac{2(-.951)}{3.618-.423} = -.268$$

$$A_3 = \begin{bmatrix} 1.959 & .711 & .1955 \\ & 3.880 & 0 \\ & & .1619 \end{bmatrix}$$

Repeatedly rotating about the "coordinate axes" yields

$$A = \begin{bmatrix} 1.7459 & & \text{—} \\ & 4.1149 & \\ \text{—} & & .1392 \end{bmatrix}$$

The ~~vecto~~ matrix of eigenvectors is

$$\prod_{i=1}^{\infty} R_i = R_1 R_2 R_3 \cdots = \begin{bmatrix} .878 & -.395 & .269 \\ .223 & .836 & .5008 \\ -.423 & -.380 & .823 \end{bmatrix}$$