Exam2

July 25, 2016

1 ME 7120, Finite Element Method Applications, Exam 2, Summer 2016

1.0.1 1: 10 points: A 3-noded rod element has nodes at $x = 0, 0.75, 1.0$. Determine the Jacobian presuming nodes at $\xi = -1, 0, 1$ in natural coordinates.

The shape functions can be written as quadratic polynomials with three unknown coefficients because we must have an equal number of nodal values and unknown terms in the polynomial.

So, $N_i(\xi) = a_0 + a_1\xi + a_2\xi^2$, with the unknown values a_j being different for each shape function.

For shape function 1, $N_1(-1) = 1$, $N_1(0) = 0$, $N_1(1) = 0$.

Evaluating at each nodal location

$$N_1(-1) = 1 = a_0 - a_1 + a_2$$
$$N_1(0) = 0 = a_0$$
$$N_1(1) = 0 = a_0 + a_1 + a_2$$

In matrix form, these equations can be written

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

These can be solved for coefficients of shape function 1 yielding $N_1 = \frac{1}{2}(-\xi + \xi^2)$ which can be observed to satisfy these equations. Likewise, $N_2 = 1 - \xi^2$ and $N_3 = \frac{1}{2}(\xi + \xi^2)$

By definition, $J = \frac{dx}{d\xi}$

Since $x = N_1 x_1 + N_1 x_2 + N_3 x_3$

$$J = \frac{1}{2}\begin{bmatrix} 2\xi - 1 & -4\xi & 2\xi + 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.75 \\ 1.0 \end{bmatrix} = \frac{1}{2} - \frac{1}{2}\xi$$

1.0.2 2: 10 points: Find the K_{12} for the preceding element presuming E is constant and $A = A_0(1 - x)$.

$$K = \int_{-1}^{1} B^T E A B J d\xi$$

From the previous problem

$$x(\xi) = \frac{1}{2}(-\xi + \xi^2)0 + (1 - \xi^2)0.75 + \frac{1}{2}(\xi + \xi^2)1 = 0.75 + 0.5\xi - 0.25\xi^2$$

so

$$A(x) = A_0(1-x) = 0.25 - 0.5\xi + 0.25\xi^2$$

B is defined as $B = \frac{d[N]}{dx} = \frac{d[N]}{d\xi}\frac{d\xi}{dx}$ so

$$B = \frac{1}{2}\begin{bmatrix}2\xi-1 & -4\xi & 2\xi+1\end{bmatrix}\frac{1}{J} = \frac{1}{2}\begin{bmatrix}2\xi-1 & -4\xi & 2\xi+1\end{bmatrix}\frac{1}{\frac{1}{2}-\frac{1}{2}\xi}$$

$$\begin{aligned}
K_{12} &= \frac{1}{4}\int_{-1}^{1} B_1 E A B_2 J d\xi \\
&= EA_0\frac{1}{4}\int_{-1}^{1} -4\xi(2\xi-1)(0.25-0.5\xi+0.25\xi^2)\frac{1}{\frac{1}{2}-\frac{1}{2}\xi}d\xi \\
&= EA_0\frac{1}{4}\int_{-1}^{1} 2\xi(1-2\xi)\frac{1-2\xi+\xi^2}{1-\xi}d\xi \\
&= EA_0\frac{1}{4}\int_{-1}^{1} 2\xi(1-2\xi)\frac{(1-\xi)^2}{1-\xi}d\xi \\
&= EA_0\frac{1}{4}\int_{-1}^{1} 2\xi(1-2\xi)1-\xi d\xi \\
&= EA_0\frac{1}{4}\int_{-1}^{1} 2\xi-6\xi^2+4\xi^3 d\xi \\
&= -EA_0
\end{aligned} \tag{1}$$

If one followed the intent of the problem, which was unstated, a 3rd order polynomial integration can be obtained with the 2 point rule yielding the same result much more simply.

$$\begin{aligned}
K_{12} &= \frac{1}{4}\int_{-1}^{1} B_1 E A B_2 J d\xi \\
&= EA_0\frac{1}{4}\int_{-1}^{1} -4\xi(2\xi-1)(0.25-0.5\xi+0.25\xi^2)\frac{1}{\frac{1}{2}-\frac{1}{2}\xi}d\xi \\
&= EA_0\frac{1}{4}\left(\frac{8\left(\frac{1}{3}+\frac{1}{2\sqrt{3}}\right)\left(1+\frac{2}{\sqrt{3}}\right)}{\sqrt{3}\left(1+\frac{1}{\sqrt{3}}\right)} + \frac{8\left(1-\frac{2}{\sqrt{3}}\right)\left(\frac{1}{3}-\frac{1}{2\sqrt{3}}\right)}{\sqrt{3}\left(1-\frac{1}{\sqrt{3}}\right)}\right) \\
&= EA_0\frac{1}{4}(-0.0754991027012 - 3.9245008973) \\
&= -EA_0
\end{aligned} \tag{2}$$

Note: Performing numerical evaluations instead of substitutions into expressions like shown above is much more efficient. These are only shown like this for illustration of the solution.

Something of the form

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8*(1 - 2/sqrt(3))*(1/3 - 1/(2.*sqrt(3))))/(sqrt(3)*(1 - 1/sqrt(3))) 1
-4/sqrt(3)*(2/sqrt(3)-1)*(0.25-0.5/sqrt(3)+0.25/(sqrt(3)^2))*1/(1/2-1/(2*sqrt(3)
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for $\xi = \frac{1}{\sqrt{3}}$ and similar for $\xi = \frac{-1}{\sqrt{3}}$ would be much more efficient.

1.0.3 3: Find the stress at (x,y)=(0,1) of a bilinear quadrilateral (Q4) element with nodes 1-4 at (0,0), (2,0), (2,2), and (0,1) in terms of u_3 and v_3 (presume all other nodal displacements are zero).

The location of interest is node 4, which is at $(\xi,\eta)=(-1,1)$

$$\begin{aligned}\begin{bmatrix}u\\v\end{bmatrix} &= \begin{bmatrix}N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0\\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4\end{bmatrix}\begin{bmatrix}u_1\\v_1\\u_2\\v_2\\u_3\\v_3\\u_4\\v_4\end{bmatrix} \\ &= \frac{1}{4}(1+\xi)(1+\eta)\begin{bmatrix}u_3\\v_3\end{bmatrix}\end{aligned} \tag{3}$$

$$\begin{aligned}[J] &= \frac{1}{4}\begin{bmatrix}-(1-\eta) & 1-\eta & 1+\eta & -(1+\eta)\\ -(1-\xi) & -(1+\xi) & 1+\xi & 1-\xi\end{bmatrix}\begin{bmatrix}x_1 & y_1\\x_2 & y_2\\x_3 & y_3\\x_4 & y_4\end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix}0 & 0 & 2 & -2\\ -2 & 0 & 0 & 2\end{bmatrix}\begin{bmatrix}0 & 0\\2 & 0\\2 & 2\\0 & 1\end{bmatrix} \\ &= \begin{bmatrix}1 & \frac{1}{2}\\ 0 & \frac{1}{2}\end{bmatrix}\end{aligned} \tag{4}$$

$$\begin{aligned}\begin{bmatrix}\frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \eta}\end{bmatrix} &= J\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{bmatrix} \\ \begin{bmatrix}\frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \eta}\end{bmatrix} &= \begin{bmatrix}1 & \frac{1}{2}\\ 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{bmatrix}\end{aligned} \tag{5}$$

$$\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{bmatrix} = \begin{bmatrix}1 & -1\\ 0 & 2\end{bmatrix}\begin{bmatrix}\frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \eta}\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{bmatrix} = \begin{bmatrix}\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\\ 2\frac{\partial u}{\partial \eta}\end{bmatrix} = \begin{bmatrix}\left(\frac{1}{4}(1+\eta) - \frac{1}{4}(1+\xi)\right)u_3\\ 2\frac{1}{4}(1+\xi)u_3\end{bmatrix} = \begin{bmatrix}\frac{1}{4}(\eta-\xi)u_3\\ \frac{1}{2}(1+\xi)u_3\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}u_3$$

$$\begin{bmatrix}\frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial y}\end{bmatrix} = \begin{bmatrix}\frac{\partial v}{\partial \xi} - \frac{\partial v}{\partial \eta}\\ 2\frac{\partial v}{\partial \eta}\end{bmatrix} = \begin{bmatrix}\left(\frac{1}{4}(1+\eta) - \frac{1}{4}(1+\xi)\right)v_3\\ 2\frac{1}{4}(1+\xi)v_3\end{bmatrix} = \begin{bmatrix}\frac{1}{4}(\eta-\xi)v_3\\ \frac{1}{2}(1+\xi)v_3\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}v_3$$

$$\begin{bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}u_3\\ 0\\ \frac{1}{2}v_3\end{bmatrix}$$

One should know at this point that the problem is underdefined. Is the situation plane strain or plane stress? Answering that

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [E] \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = [E] \begin{bmatrix} \frac{1}{2}u_3 \\ 0 \\ \frac{1}{2}v_3 \end{bmatrix}$$

For plane stress,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2}u_3 \\ 0 \\ \frac{1}{2}v_3 \end{bmatrix} = \begin{bmatrix} \frac{\frac{1}{2}u_3 E}{1-\nu^2} \\ \frac{\frac{1}{2}u_3 E\nu}{1-\nu^2} \\ \frac{\frac{1}{4}v_3 E}{(1+\nu)} \end{bmatrix}$$

While for plane strain

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2}u_3 \\ 0 \\ \frac{1}{2}v_3 \end{bmatrix} = \begin{bmatrix} \frac{E(1-\nu)\frac{1}{2}u_3}{(1+\nu)(1-2\nu)} \\ \frac{E\nu\frac{1}{2}u_3}{(1+\nu)(1-2\nu)} \\ \frac{\frac{1}{4}Ev_3}{(1+\nu)} \end{bmatrix}$$