Notes for Computational Methods in Structural Dynamics

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Contents

1 Concepts of Linear Algebra, Appendix B

1.1 Linear Vector Spaces

Definition 1 *(Field) Set of scalars possessing certain algebraic properties (F).*

Consider field of real numbers. To be a field it must satisfy:

1. Commutativity: $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$
2. Associativity:$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$
3. Distributivity: $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$
4. Identity: $\alpha + 0 = \alpha, \alpha \cdot 1 = \alpha$
5. Inverse: $\alpha + (-\alpha) = 0, \alpha\alpha^{-1} = 1$

$\therefore$ The set of real numbers is a field.

Definition 2 *(Linear Vector Space L) If vector addition is defined and scalar multiplication is defined, then the set of L and F are a* linear vector space *over a field F. They satisfy*

1. Commutativity: $(\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x})$
2. Associativity: $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$
3. Identity: There exists a vector $\mathbf{0}$ such that $\boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$
4. Inverse: There exists for each $\boldsymbol{x}$ a vector $(-\boldsymbol{x})$ so that $\boldsymbol{x} + (-\boldsymbol{x}) = \mathbf{0}$

For any $\boldsymbol{x}$, there also exists the vector $\alpha\boldsymbol{x}$ in L.
A vector space L possessing n elements of F is a vector space L^n. (consisting of vectors length n).
Let S be a subset of the vectors in L. Then S is a subspace if:

1. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are in S, then $\boldsymbol{x} + \boldsymbol{y}$ is in S
2. If $\boldsymbol{x}$ is in S and α is in F, than $\alpha\boldsymbol{x}$ is in S

1.2 Linear Dependence

A set of vectors

$$\boldsymbol{x}_i, \qquad i = 1, 2, 3, \ldots, n$$

in a linear space $\mathcal{C}^n$[1] are linearly independent iff

$$\sum \alpha_i \boldsymbol{x}_i = \mathbf{0}$$

cannot be satisfied without all $\alpha_i = 0 \qquad i = 1, \ldots, n$

[1] Complex Vector Space

1.2.1 Example

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 = \mathbf{0}$$

$$\alpha_1 1 + \alpha_2 0 = 0 \qquad \alpha_1 = 0$$

$$\alpha_1 2 + \alpha_2 2 = 0 \qquad \alpha_2 = 0$$

$\therefore \boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are independent

1.2.2 Example

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \qquad \boldsymbol{x}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \qquad \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 7 \\ 5 \end{bmatrix}$$

$$\alpha_1 = 2, \alpha_2 = 1, \alpha_3 = -1$$

satisfies

$$\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 + \alpha_3 \boldsymbol{x}_3 = \mathbf{0}$$

Thus $\boldsymbol{x}_1, \boldsymbol{x}_2$, and $\boldsymbol{x}_3$ are not independent.
The subspace S of L spanned by the vectors $\boldsymbol{x}_i$ is defined by

$$\sum_{i=1}^{n} \alpha_i \boldsymbol{x}_i$$

for all values of α_i.

1.3 Bases and Dimension of a Vector Space

- A vector space over a field of scalars is finite dimensional if there exists a finite set of vectors $\boldsymbol{x}_i$ such that $\boldsymbol{x}_i$ spans the space.
- The vectors $\boldsymbol{x}_i$ are called a *generating system.*
- If they are independent, then they are *basis vectors* for the space L.

The standard basis set is

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \ldots \qquad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Any vector $\boldsymbol{x}$ may be written

$$\boldsymbol{x} = x_1 \hat{\mathbf{e}}_1 + x_2 \hat{\mathbf{e}}_2 + x_3 \hat{\mathbf{e}}_3 + \ldots x_n \hat{\mathbf{e}}_n$$

1.3.1 Example

The vectors of examples 1.2.1 and 1.2.2 each are generating systems for spaces of dimension (n).
However, they *do not* span the same space.
The vectors of example 1.2.2 span a 2-d space that can be visualized as the plane $\perp$ to

$$\boldsymbol{x}_1 \times \boldsymbol{x}_2 = \begin{bmatrix} -4 \\ -3 \\ 5 \end{bmatrix}$$

The dimension of the space IS NOT the length of the vector!

1.4 Inner Products and Orthogonal Vectors

The *complex inner product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by

$$(\boldsymbol{x}, \boldsymbol{y}) = x_1\bar{y}_1 + x_2\bar{y}_2 + \ldots + x_n\bar{y}_n$$

where $\bar{y}_i$ is the complex conjugate of y_i. The inner product space L^n defined over the field of complex numbers is called *unitary space*.
When $\boldsymbol{x}$ and $\boldsymbol{y}$ are real

$$(\boldsymbol{x}, \boldsymbol{y}) = x_1y_1 + x_2y_2 + \ldots + x_ny_n$$

is the *real inner product*.
Note:

1. $(\boldsymbol{x}, \boldsymbol{x}) \geq 0$ for all $\boldsymbol{x}$ in L^n
2. $(\boldsymbol{x}, \boldsymbol{x}) = 0$ only for $\boldsymbol{x} = \boldsymbol{0}$
3. $(\boldsymbol{x}, \boldsymbol{y}) = (\overline{\boldsymbol{y}, \boldsymbol{x}})$
4. $(\lambda\boldsymbol{x}, \boldsymbol{y}) = \lambda(\boldsymbol{x}, \boldsymbol{y})$
   $(\boldsymbol{x}, \lambda\boldsymbol{y}) = \bar{\lambda}(\boldsymbol{x}, \boldsymbol{y})$ for all λ in F
5. Distributive law
   $(\boldsymbol{x}, \boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x}, \boldsymbol{y}) + (\boldsymbol{x}, \boldsymbol{z})$ in L^n

1.4.1 Vector norms

A measure of the length of a vector is the *norm*, $\|\boldsymbol{x}\|$

1. $\|\boldsymbol{x}\| \geq 0$ and $\|\boldsymbol{x}\| = 0$ only if $\boldsymbol{x} = \boldsymbol{0}$
2. $\|\lambda\boldsymbol{x}\| = |\lambda| \, \|\boldsymbol{x}\|$ for any λ in F
3. $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$

Definition 3 *(Quadratic norm)* $\|\boldsymbol{x}\| = (\boldsymbol{x}, \boldsymbol{x})^{1/2}$ *length of* $\boldsymbol{x}$

Definition 4 *(Euclidean norm (*$\boldsymbol{x}$ *real))*

$$\|\boldsymbol{x}\| = \left(\sum_{l=1}^{n} x_i^2 \right)^{\frac{1}{2}}$$

Definition 5 *(Unit Vector) A unit vector is a vector whose quadratic norm is 1.*

A vector can be *normalized* by

$$\hat{\boldsymbol{x}} = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$$

Two vectors are orthogonal if

$$(\boldsymbol{x}, \boldsymbol{y}) = 0$$

If all pairs of vectors in a set are orthogonal, the set is an *orthogonal set*.
If they have unit length, they are an *orthonormal set*.
Any set of mutually orthonormal vectors are linearly independent. The converse is not true.

1.5 Gram - Schmidt Orthogonalization

Takes a set of independent vectors and renders them orthogonal (orthonormal if we choose to normalize).
Independent vectors are given by $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \ldots$
Desired orthogonal vectors are $\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \ldots$
Desired orthonormal set is $\hat{\boldsymbol{y}}_1, \hat{\boldsymbol{y}}_2, \hat{\boldsymbol{y}}_3, \ldots$

1. $\hat{\boldsymbol{y}}_1 = \hat{\boldsymbol{x}}_1 = \boldsymbol{x}_1 / \|\boldsymbol{x}_1\|$
2. Note that we want $\boldsymbol{y}_2 \perp \hat{\boldsymbol{y}}_1$ (we can normalize later).

   $\therefore (\boldsymbol{y}_2, \hat{\boldsymbol{y}}_1) = 0$

   A vector $\boldsymbol{y}_2$ that satisfied this is

   $$\boldsymbol{y}_2 = \boldsymbol{x}_2 - (\boldsymbol{x}_2, \hat{\boldsymbol{y}}_1)\hat{\boldsymbol{y}}_1$$

   since

   $$\begin{aligned} (\boldsymbol{y}_2, \hat{\boldsymbol{y}}_1) &= (\boldsymbol{x}_2 - (\boldsymbol{x}_2, \hat{\boldsymbol{y}}_1)\hat{\boldsymbol{y}}_1, \hat{\boldsymbol{y}}_1) \\ &= (\boldsymbol{x}_2, \hat{\boldsymbol{y}}_1) - (\boldsymbol{x}_2, \hat{\boldsymbol{y}}_1)\overbrace{(\hat{\boldsymbol{y}}_1, \hat{\boldsymbol{y}}_1)}^{1} \\ &= 0 \end{aligned} \tag{1}$$

   $$\therefore \boldsymbol{y}_2 = \boldsymbol{x}_2 - (\boldsymbol{x}_2, \hat{\boldsymbol{y}}_1)\hat{\boldsymbol{y}}_1$$

3. We now want $(\boldsymbol{y}_3, \hat{\boldsymbol{y}}_1) = 0$ and $(\boldsymbol{y}_3, \hat{\boldsymbol{y}}_2) = 0$

   $$\boldsymbol{y}_3 = \boldsymbol{x}_3 - (\boldsymbol{x}_3, \hat{\boldsymbol{y}}_1)\hat{\boldsymbol{y}}_1 - (\boldsymbol{x}_3, \hat{\boldsymbol{y}}_2)\hat{\boldsymbol{y}}_2$$

   is a solution

   $$\begin{aligned} (\boldsymbol{y}_3, \hat{\boldsymbol{y}}_1) &= (\boldsymbol{x}_3, \hat{\boldsymbol{y}}_1) - (\boldsymbol{x}_3, \hat{\boldsymbol{y}}_1)\overbrace{(\hat{\boldsymbol{y}}_1, \hat{\boldsymbol{y}}_1)}^{1} - (\boldsymbol{x}_3, \boldsymbol{y}_2)\overbrace{(\hat{\boldsymbol{y}}_2, \hat{\boldsymbol{y}}_1)}^{0} \\ &= 0 \end{aligned} \tag{2}$$

   and

   $$\begin{aligned} (\boldsymbol{y}_3, \hat{\boldsymbol{y}}_2) &= (\boldsymbol{x}_3, \hat{\boldsymbol{y}}_2) - (\boldsymbol{x}_3, \hat{\boldsymbol{y}}_1)\overbrace{(\hat{\boldsymbol{y}}_1, \hat{\boldsymbol{y}}_2)}^{0} - (\boldsymbol{x}_3, \boldsymbol{y}_2)\overbrace{(\hat{\boldsymbol{y}}_2, \hat{\boldsymbol{y}}_2)}^{1} \\ &= 0 \end{aligned} \tag{3}$$

The modified Gram-Schmidt process described next in the book yields better results. It basically normalizes values and further iterates.

1.6 The Eigenvalue Problem

The eigenvalue problem is obtained in a very wide array of mathematical problems.[2] In vibration testing, it results from modal analysis and in calculation of the H_v frequency response function.
The general eigenvalue is one of finding λ and $\boldsymbol{x}$ when

$$A\boldsymbol{x} = \lambda B\boldsymbol{x} \tag{4}$$

and A and B are square $N \times N$ matrices. There are n linearly independent solutions $\boldsymbol{x}$ called the *eigenvectors*. Each eigenvector has a corresponding *eigenvalue* λ. The eigenvalues λ are *usually* unique, but often are not. The more general form of the eigenvalue problem is

$$AX = BX\Lambda \tag{5}$$

where $X = [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n]$, and Λ is a diagonal matrix of the eigenvalues corresponding to the eigenvectors assembled to form X.
Under special circumstances (for example, pre-multiplying by A^{-1}, the matrix B is the identity matrix and equation (5) becomes

$$AX = X\Lambda \tag{6}$$

which is more familiar to most. Eigen-solvers are more often capable of handling the eigenvalue problem in the form of equation (6) than the form of equation (5). Many eigensolvers that can solve the eigenvalue problem form of equation (5) do so by transformation to the form of equation (6). This case can be obtained by pre-multiplying (5) by B^{-1}. For large matrices, Gauss elimination should be used to obtain $B^{-1}A$. Better yet, Cholesky decomposition [**?**] should be applied to B such that

$$B = B_c^T B_c \tag{7}$$

where B_c is an upper triangular matrix[3]. The substitution

$$Y = B_c X \tag{8}$$

[2] The eigenvalue problem is also used for calculating principle stresses and strains, as well is principle axes and moments of inertias of rigid bodies

[3] All values below the diagonal are zero

can be made in equation (5), and then pre-multiplying by $B_c^{T^{-1}}$ gives

$$\left(B_c^{T^{-1}} A B_c\right) Y = Y\Lambda \tag{9}$$

which is an equivalent form to equation (6). Once the solution is obtained, the eigenvectors X can be obtained from $X = B_C^{-1} Y$.

This methodology has significant advantages over simply inverting B. One is that it is more numerically robust, resulting in reduced propagation of numerical errors. A second is that when A and B are *Hermitian*[4], the eigenvectors are orthogonal and the eigenvalues are real. These two pieces of information can be used by the algorithm programmer to improve computational efficiency, and can be used by the user to verify the quality of the results.

Equation (6) is often written as a matrix decomposition form of

$$X^{-1} A X = \Lambda \tag{10}$$

The rows of X^{-1} are referred to as the *left eigenvectors* because transposing equation (10) results in the alternate eigenvalue problem (using $(ABC)^T = C^T B^T A^T$)

$$X^T A^T X^{-1^T} = \Lambda \tag{11}$$

For the sake of substitution and solution of linear algebra problems, a matrix is often decomposed using equation (10) such that

$$A = X\Lambda X^{-1} \tag{12}$$

Further, if A is Hermitian, $X^{-1} = X^T$, and then

$$A = X\Lambda X^T \tag{13}$$

The eigenvalue problem can thus be used to identify whether or not a matrix is *non-singular* or invertible. If a matrix is invertible, it may not have any eigenvalues equal to zero because (using $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$)

$$A^{-1} = \left(X^T\right)^{-1} \Lambda^{-1} X^{-1} \tag{14}$$

Thus, since Λ is diagonal, and its inverse is

$$\Lambda^{-1} = \text{diag}\left[\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1}\right] \tag{15}$$

any zero eigenvalue will cause a singularity precluding an inverse.

[4] A Hermitian Matrix, A, is one where its conjugate transpose, represented by $(\cdot^H)$, is equal to itself.

1.6.1 Singular Value Decomposition - Principle Component Analysis

Singular value decomposition, or SVD, is an extension of the eigenvalue decomposition to non-square matrices. It is often used to identify, consolidate, and rank contributions of vectors to a large non-square matrix. The SVD of an $m \times n$ matrix A, where $m \geq n$ is defined by

$$\underbrace{A}_{m\times n} = \underbrace{U}_{m\times n}\underbrace{\Sigma}_{n\times n}\underbrace{V^T}_{n\times n} \tag{16}$$

where

$$U^T U = I \tag{17}$$

$$VV^T = I \tag{18}$$

and

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_n \end{bmatrix} \tag{19}$$

Some algorithms will alternatively return results satisfying

$$\underbrace{A}_{m\times n} = \underbrace{U}_{m\times m}\underbrace{\Sigma}_{m\times n}\underbrace{V^T}_{n\times n} \tag{20}$$

by appending zero rows to the matrix Σ and appending additional mutually orthogonal vectors to the matrix U. Since the nonzero appended columns of U correspond to zero-valued rows of Σ they have no impact on the matrix A when computed.

Just as in the eigenvalue problem, practical solution methodologies are extensive and sophisticated and will not be addressed here. The interested reader is referred to [**?**].

For illustration, consider obtaining the SVD of the matrix A defined by

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1.1 \end{bmatrix} \tag{21}$$

The problem of finding the singular values can quickly transformed into one of finding eigenvalues. Pre-multiplying (16) by itself transposed yields

$$A^T A = V\Sigma U^T U\Sigma V^T \tag{22}$$

Applying equation (17), this simplifies to

$$A^T A = V\Sigma^2 V^T \tag{23}$$

Comparing this to equation (12) one should recognize the Σ^2 are the eigenvalues of the matrix $A^T A$. Further, the eigenvalue problem is a symmetric one because $A^T A$ is symmetric, which is easily proven by taking its transpose (likewise this can be done with the right size of equation (23). The solution of the eigenvalue problem yields

$$\Sigma = \begin{bmatrix} 2.9 & 0.00 \\ 0.00 & 0.06 \end{bmatrix} \tag{24}$$

and

$$V^T = \begin{bmatrix} 0.70 & 0.72 \\ 0.72 & -0.70 \end{bmatrix} \tag{25}$$

The matrix U can then be found using (16) to be

$$U = AVS^{-1} = \begin{bmatrix} 0.49 & 0.30 \\ 0.49 & 0.30 \\ 0.49 & 0.30 \\ 0.52 & -0.85 \end{bmatrix} \tag{26}$$

where we have taken advantage of the fact that $V^T = V^{-1}$.

- Observing the original matrix A, ones first impression is that it is almost entirely constructed of values 1.
- The matrix U contains the *principal components*, or vectors, in the matrix A.
- The first column of A is a vector with nearly the same direction as a vector of ones but having been normalized.
- Notice then that the second column primarily shows a deviation of the fourth value. This is due to that small contribution of adding 0.1 to $A_{4,2}$. Importantly,

- Σ illustrates the degree of contribution of each of these vectors to the matrix and is almost always ordered from the highest to lowest value.
- Note that the first singular value is much greater than the second, indicating that the nearly constant vector is a much greater contributor to A.
- The matrix V can then be recognized as an organizer. It determines how much of each σ-weighted column of U is needed to produce the corresponding column of A.

In practice, for very large matrices with redundant data, the higher-index values of σ tend towards 0, and for all practical purposes can be treated as zero. When this happens, the SVD problem is written as

$$\underbrace{A}_{n\times m} = \underbrace{U}_{m\times n} \underbrace{\begin{bmatrix} \underbrace{\Sigma_p}_{p\times p} & [0] \\ [0] & [0] \end{bmatrix}}_{n\times n} \underbrace{V^T}_{n\times n} \tag{27}$$

where the near-zero values of Σ have been set to zero leaving only p non-zero values. As a result, columns $p+1$ through n of U and rows $p+1$ through n of V^T can be discarded along with all of Σ outside of Σ_p. The resulting SVD problem statement then appears as

$$\underbrace{A}_{n\times m} = \underbrace{U}_{m\times p} \underbrace{\Sigma_p}_{p\times p} \underbrace{V^T}_{p\times n} \tag{28}$$

where $m \geq n \geq p$.

1.7 Properties of Matrices

A matrix A for which

$$A^T = A$$

is symmetric
A matrix A for which

$$A^T = -A$$

is skew symmetric

Any arbitrary matrix can be written as the sum of a symmetric and a skew symmetric matrix
Proof: Suppose $A = B + C$ where B is symmetric and C is skew symmetric
Can we solve for B and C in the general case?

$$A^T = B^T + C^T = B - C$$

$$A + A^T = B + C + B - C = 2B$$

$$\therefore B = \frac{A + A^T}{2}$$

$$A - A^T = B + C - B + C = 2C$$

$$\therefore C = \frac{A - A^T}{2}$$

Yes.
Define $A^H = \bar{A}^T$ to be the *Hermitian adjoint* of A. This is what is returned by MATLAB when you take a transpose.
If A is such that

$$A^H = A,$$

then A is said to be Hermitian.
The real part is symmetric, the imaginary part is skew symmetric.
Because A is equal to its adjoint, Hermitian matrices are said to be self-adjoint.

4 Multi-Degree of Freedom Systems, Chapter 4 (see also page 88-)

4.1 Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial \mathcal{F}}{\partial \dot{q}_i} = Q_i$$

$L = T - V$
T is kinetic energy
V is potential energy

$$\mathcal{F} = \frac{1}{2}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(c_{ij}\dot{q}_{ij}^2\right)\right)$$

is Rayleigh's dissipation function.
$c_{ij} = c_{ji}$ are the damping coefficients and $\dot{q}_{ij} = \dot{q}_i - \dot{q}_j$ We can:

1. Linearize about equilibrium
2. Perform a coordinate transformation (shift coordinates) to set the equilibrium to be at coordinates zero.

The later is what is commonly done in linear vibrations when motion is in the direction of gravity.

4.2 Small Motions about Equilibrium Points

A dynamic system moves in what is known as state space. The trajectory through any point in state space is unique. The state space is defined by the generalized coordinates and their first time derivatives.
A constant solution $q_i = q_{io} = const$, $\dot{q}_i = \dot{q}_{io} = 0$ defines an equilibrium point. All accelerations and higher time derivatives are zero as well.
From Langrange's Equation, equilibrium points are found from

$$\frac{\partial U}{\partial q_i} = 0$$

where

$$\underbrace{U}_{\text{dynamic potential}} = \underbrace{V}_{\text{potential energy}} - \underbrace{T_0}_{\substack{\text{kinetic energy not dependent on velocity of} \\ \text{generalized coordinates}}}$$

This is derived by setting $\dot{q}_i = 0$ and $\ddot{q}_i = 0$.
T_0 is the kinetic energy that does not depend on the velocity of any generalized coordinates.[5]

[5]For example, the kinetic energy of us due to the spin of the Earth. Since the rotational speed of the Earth is presumed not to be affected by our individual motion, it is *not* a generalized coordinate, and thus its impact on kinetic energy is T_0.

4.3 Example

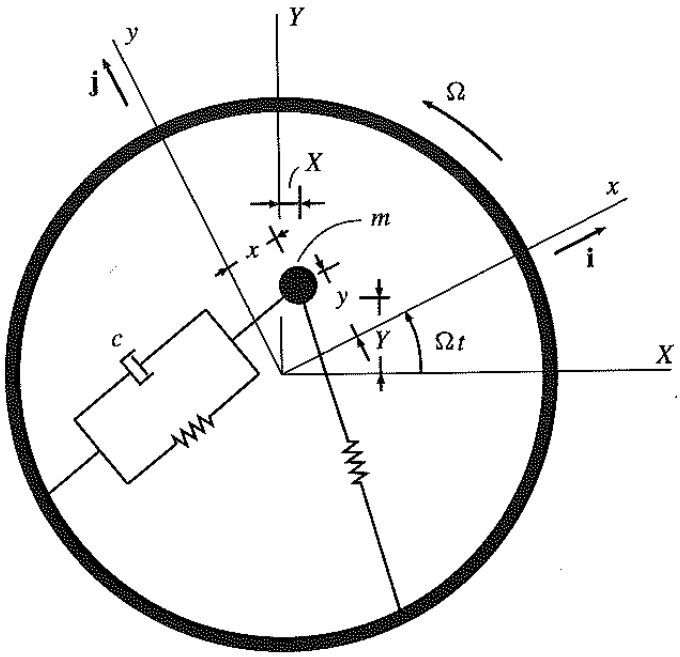


Figure 4.1 Mass connected to a rotating rigid ring through damper and springs

Derive equations of motion, linearize, derive matrices.
Attaching x and y to spinning hub

$$T = \frac{1}{2}m\left((\dot{x} - y\Omega)^2 + (\dot{y} + x\Omega)^2\right)$$

$\mathcal{F} = \frac{1}{2}c\dot{x}^2$ is the Rayleigh Dissipation Function
From sophomore dynamics, for conservative forces

$$F_i = -\frac{\partial V}{\partial q_i}$$

so

$$\begin{aligned} V &= -\int_0^x F_{s1}(\xi)\, d\xi - \int_0^y F_{s2}(\xi)\, d\xi \\ &= \frac{1}{2}\left(k_1\left(x^2 + \frac{\epsilon_1}{2}x^4\right) + \frac{1}{2}k_2\left(y^2 + \frac{\epsilon_2}{2}y^4\right)\right) \end{aligned}$$

substituting into Lagrange's equations
For x:

$$m\ddot{x} - 2\Omega m\dot{y} - m\Omega^2 x + k_1\left(x + \epsilon_1 x^3\right) + c\dot{x} = 0$$

For y:

$$m\ddot{y} + 2\Omega m\dot{x} - m\Omega^2 y + k_2\left(y + \epsilon_2 y^3\right) = 0$$

The equations of motion are

$$M\ddot{\boldsymbol{x}} + (C+G)\,\dot{\boldsymbol{x}} + (K+H)\,\boldsymbol{x} = 0$$

H and G are skew-symmetric matrices

$$M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$$C = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & -2\Omega m \\ 2\Omega m & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 - m\Omega^2 & 0 \\ 0 & k_2 - m\Omega^2 \end{bmatrix}$$

Observing T we can expand to

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + m\Omega\left(x\dot{y} - y\dot{x}\right) + \frac{1}{2}m\Omega^2\left(x^2 + y^2\right)$$

$$= T_2 + T_1 + T_0$$

$$T_2 = \underbrace{\frac{1}{2}\begin{bmatrix}\dot{x} & \dot{y}\end{bmatrix}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix}\dot{x} \\ \dot{y}\end{bmatrix}}_{\text{Quadratic in generalized velocities due to time rate change of coord system.}}$$

$$T_1 = \underbrace{\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 0 & m\Omega \\ -m\Omega & 0 \end{bmatrix}\begin{bmatrix}\dot{x} \\ \dot{y}\end{bmatrix}}_{\text{Linear in generalized velocities.}}$$

T_1 produces Coriolis type forces and is the gyroscopic term.

$$T_0 = \underbrace{\frac{1}{2}\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} m\Omega^2 & 0 \\ 0 & m\Omega^2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{No generalized velocities}}$$

T_0 behaves like a potential energy and causes centrifugal forces.
The potential energy is

$$V = \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

(linearized[6])

$$\mathcal{F} = \frac{1}{2}\begin{bmatrix} \dot{x} & \dot{y} \end{bmatrix}\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

Definitions:

$$T_2 = \frac{1}{2}\dot{\boldsymbol{q}}^T M \dot{\boldsymbol{q}}, \qquad T_1 = \boldsymbol{q}^T G \dot{\boldsymbol{q}}$$

$$\mathcal{F} = \frac{1}{2}\dot{\boldsymbol{q}}^T C \dot{\boldsymbol{q}}, \qquad U = \frac{1}{2}\boldsymbol{q}^T K \boldsymbol{q}$$

More definitions:

1. A matrix A is positive definite iff
2. $f = \boldsymbol{x}^T A \boldsymbol{x} > 0$ for all non-zero vectors $\boldsymbol{x}$
3. A matrix A is positive semi-definite iff
4. $f = \boldsymbol{x}^T A \boldsymbol{x} \geq 0$ for all non-zero vectors $\boldsymbol{x}$

In the previous example:

1. M is positive definite
2. C is positive semi-definite
3. K is positive definite only for small Ω

[6] Means curvature, and higher derivatives, of force set to zero

4.4 Energy Considerations

$$M\ddot{\boldsymbol{q}} + (C+G)\dot{\boldsymbol{q}} + (K+H)\boldsymbol{q} = \boldsymbol{0} \tag{29}$$

Consider the skew-symmetric matrix G pre-multiplied by $\dot{\boldsymbol{q}}^T$

$$\begin{aligned} \dot{\boldsymbol{q}}^T G\dot{\boldsymbol{q}} &= (\dot{\boldsymbol{q}}^T G\dot{\boldsymbol{q}})^T, && \text{scalar} \\ &= \dot{\boldsymbol{q}}^T(\dot{\boldsymbol{q}}^T G)^T && \text{(note: } (AB)^T = B^T A^T\text{)} \\ &= \dot{\boldsymbol{q}}^T G^T \dot{\boldsymbol{q}} \end{aligned}$$

By definition, since G is skew symmetric, $G^T = -G$

$$\dot{\boldsymbol{q}}^T G\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^T G^T \dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^T(-G)\dot{\boldsymbol{q}} = -(\dot{\boldsymbol{q}}^T G\dot{\boldsymbol{q}})$$

This is only true if $\dot{\boldsymbol{q}}^T G\dot{\boldsymbol{q}} = 0$
Pre-multiplying the equation of motion (29) by $\dot{\boldsymbol{q}}^T$

$$\dot{\boldsymbol{q}}^T M\ddot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^T C\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^T K\boldsymbol{q} + \dot{\boldsymbol{q}}^T H\boldsymbol{q} = \boldsymbol{0}$$

Integrating the first three terms and rearranging gives

$$\frac{1}{2}\frac{d}{dt}\underbrace{\left(\dot{\boldsymbol{q}}^T M\dot{\boldsymbol{q}} + \boldsymbol{q}^T K\boldsymbol{q}\right)}_{\text{Hamiltonian: } \mathcal{H}=T_2+U} = -\dot{\boldsymbol{q}}^T C\dot{\boldsymbol{q}} - \underbrace{\dot{\boldsymbol{q}}^T H\boldsymbol{q}}_{\text{circulatory forces constraint damping forces}}$$

Hamiltonian $\mathcal{H} = T_2 + U$

$$\frac{d}{dt}\mathcal{H} = -2\left(\mathcal{F}' + \mathcal{F}\right), \quad (\mathcal{F}' = \dot{\boldsymbol{q}}^T H\boldsymbol{q} \text{ is the circulatory dissipation function})$$

If there are no viscous damping or circulatory forces,

$$H = const$$

and the system is *conservative*. When $T_1 = T_0 = 0$

$$T = T_2 = \frac{1}{2}\dot{\boldsymbol{q}}^T M\dot{\boldsymbol{q}}$$

$$U = V$$

$$H = T + V = E = const$$

This is known as the principle of *conservation of energy*.

4.5 Lyapunov Stability

Let $\boldsymbol{x}(0)$ represent the vector of initial conditions (states at time zero) of a given system. The system is said to have a stable equilibrium if for any arbitrary positive number ϵ there exists a positive number $\delta(\epsilon)$ such that whenever
$\|\boldsymbol{x}(0)\| < \delta$, then $\|\boldsymbol{x}(t)\| < \epsilon$.

4.5.1 Example

$$m\ddot{x} + kx = 0$$

$$\|\boldsymbol{x}(t)\| = \left(\boldsymbol{x}^T\boldsymbol{x}\right)^{\frac{1}{2}} = \sqrt{x(t)^2 + \dot{x}(t)^2}$$

Consider i.c. of $x(0) = 0$, $\dot{x}(0) = \sqrt{\frac{k}{m}} = \omega$
The solution is $x(t) = \sin(\omega t)$
The system is Lyapunov stable because for

$$\|\boldsymbol{x}(0)\| = \left(x^2(0) + \dot{x}^2(0)\right)^{\frac{1}{2}} = \omega \qquad (\delta)$$

$$\|\boldsymbol{x}(t)\| = \left(\sin^2\omega t + \omega^2\cos^2\omega t\right)^{\frac{1}{2}} < \left(1 + \omega^2\right)^{\frac{1}{2}} \qquad (\epsilon)$$

We must then choose $\epsilon > (1 + \omega^2)^{\frac{1}{2}}$.
This worked only because we knew the solution.
The Lyapunov direct method or Liapunov second method does not require soln. of EOM.
The method consists of devising a suitable scalar testing function which can be used in conjunction with its total time derivative to determine the characteristics of equilibrium points.
Definition: A function $V(\boldsymbol{x})$ is said to be *positive definite* if it is positive for all values of $\boldsymbol{x} \neq \boldsymbol{0}$.
Definition: A function $V(\boldsymbol{x})$ is said to be *positive semi-definite* if it is ≥ 0 for all $\boldsymbol{x} \neq \boldsymbol{0}$.
Definition: A function $V(\boldsymbol{x})$ is said to be *indefinite* or *sign-variable* if the sign varies.
Negative semi-definite and *negative definite* can be defined likewise.

4.5.2 Lyapunov Stability Criteria

1. If $V(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$ and $\dot{V}(\boldsymbol{x}) \leq 0$, then the system is stable.
2. If $V(\boldsymbol{x}) > 0$ for all $\dot{\boldsymbol{x}} \neq \boldsymbol{0}$, and $\dot{V}(\boldsymbol{x}) < 0$, then the system is asymptotically stable
3. If $V(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$, and $\dot{V}(\boldsymbol{x})$ is indefinite, the stability is not known.

4.5.3 Conservative Systems

$$M\ddot{\boldsymbol{x}} + K\boldsymbol{x} = \boldsymbol{0}$$

Assume M and K are P.D. (Positive Definite) matrices

$$V(\boldsymbol{x}) = \boldsymbol{x}_1^T M \boldsymbol{x}_1 > 0, \qquad \boldsymbol{x} \neq \boldsymbol{0}$$

and

$$V(\boldsymbol{x}) = \boldsymbol{x}_2^T K \boldsymbol{x}_2 > 0, \qquad \boldsymbol{x} \neq \boldsymbol{0}$$

Let's pick as a Lyapunov function

$$V(\boldsymbol{x}) = \frac{1}{2}\left(\dot{\boldsymbol{x}}^T M \dot{\boldsymbol{x}} + \boldsymbol{x}^T K \boldsymbol{x}\right) \qquad \text{(mechanical energy)}$$

Since M and K are P.D., $V(\boldsymbol{x})$ is P.D.

$$\begin{aligned}\frac{d}{dt}\left(V(\boldsymbol{x})\right) &= \dot{\boldsymbol{x}}^T M \ddot{\boldsymbol{x}} + \dot{\boldsymbol{x}}^T K \boldsymbol{x} \\ &= \dot{\boldsymbol{x}}^T \left(M\ddot{\boldsymbol{x}} + K\boldsymbol{x}\right) \\ &= \dot{\boldsymbol{x}}^T \boldsymbol{0} = 0\end{aligned}$$

Since $\dot{V}(\boldsymbol{x}) = 0$ and $V(\boldsymbol{x}) > 0$, the system is stable.

4.5.4 Systems with Damping

$$M\ddot{\boldsymbol{x}} + C\dot{\boldsymbol{x}} + K\boldsymbol{x} = \boldsymbol{0}$$

Let

$$V(\boldsymbol{x}) > 0$$

still Then, using the equation of motion

$$\dot{V}(\boldsymbol{x}) = \left(\dot{\boldsymbol{x}}^T M \ddot{\boldsymbol{x}} + \dot{\boldsymbol{x}}^T K \boldsymbol{x}\right) = -\left(\dot{\boldsymbol{x}}^T C \dot{\boldsymbol{x}}\right)$$

If C is positive definite, then the system is asymptotically stable.
If C is positive semi-definite, the system is asymptotically stable iff:

$$\operatorname{rank}\begin{bmatrix} C \\ CK \\ CK^2 \\ \vdots \\ CK^{n-1} \end{bmatrix} = n$$

where n is the number DoF
(Inman, '89)
This is equivalent to proving that all of the modal damping ratios are greater than zero (of course, if all of the modal damping ratios are greater than zero, then the system is also asymptotically stable).
Alternatively: The system is asymptotically stable if none of the modes are in the null space of the damping matrix.

4.5.5 Gyroscopic Systems

$$M\ddot{\boldsymbol{x}} + G\dot{\boldsymbol{x}} + K\boldsymbol{x} = \boldsymbol{0}$$

Assume M and K are P.D. and G is skew-sym.
$\boldsymbol{x}^T G \boldsymbol{x} = 0$ for any $\boldsymbol{x}$, so previous Lyapunov function still works, and equilibrium is still stable. If K is P.D., system is stable.
If K is indefinite, semidefinite, or negative definite, the system may still be stable.

1. If K is negative definite, and $4K - GM^{-1}G$ is negative definite, the system is unstable.
2. Special cases for $n = 2$ have been examined (Inman, 1989)
3. If $4K - GM^{-1}G$ is P.D. and $(GM^{-1}K - KM^{-1}G)$ is Ps- D, then the system is stable.
4. If $GM^{-1}K = KM^{-1}G$, the system is stable iff $4K - GM^{-1}G$ is P.D.

4.5.6 Damped Gyroscopic Systems

$$M\ddot{\boldsymbol{x}} + (C + G)\,\dot{\boldsymbol{x}} + K\boldsymbol{x} = \mathbf{0}$$

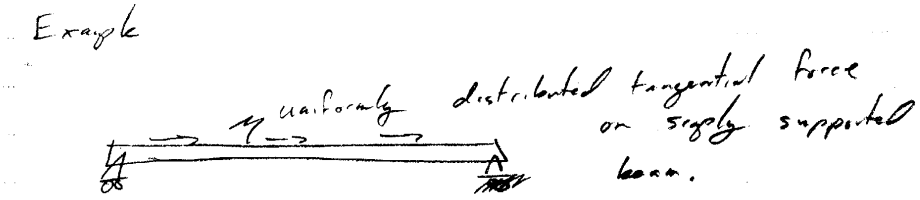
M is P.D.
From Inman '89.

1. If K and C are PD the system is asymptotically stable
2. If K is not PD and C is PD, the system is unstable. (Note, damping can destabilize a gyroscopic system with rigid body modes).
3. If K is PD and C is Ps-D, the system is:
   (a) Asymptotically stable is none of the eigenvectors of the undamped gyroscopic system are in the null space of C (Are an eigenvector of a zero eigenvalue of C)
   (b) Stable if proportionally damped.

4.5.7 Circulatory Systems

$$M\ddot{\boldsymbol{x}} + (K + H)\,\boldsymbol{x} = 0$$
$$H = -H^T$$

Example



The phenomenon occurs in aeroelasticity. Results for stability are not as well developed.

4.5.8 General Asymmetric Systems

$$\ddot{\boldsymbol{x}} + A_2\dot{\boldsymbol{x}} + A_3\boldsymbol{x} = \boldsymbol{0}$$

$$A_2 = M^{-1}(C+G), \qquad A_3 = M^{-1}(K+M)$$

Any real matrix can be written as the product of two symmetric matrices.

$$A_2 = T_1 T_2$$

$$A_3 = S_1 S_2$$

If $T_1 = S_1$ exists and is PD, the system is symmetrizable, Such systems are asymptotically stable if the eigenvalues of A_2 and A_3 are > 0. (S_2 and T_2 are PD).
See Inman '89 for more details.

4.6 Self-Adjoint (Symmetric) Systems

The majority of structures can be modeled as *self-adjoint* systems with n degrees of freedom. These systems are governed by equations of motion of the form

$$M\ddot{\boldsymbol{x}}(t) + C\dot{\boldsymbol{x}}(t) + K\boldsymbol{x}(t) = \boldsymbol{f}(t) \tag{30}$$

where M is a positive-definite $n \times n$ matrix, C and K are positive-semi-definite $n \times n$ matrices, and all are real and symmetric. The displacement vector

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{bmatrix} \tag{31}$$

represents displacements of the n degrees of freedom (generalized coordinates), which may be in any direction, x, y, z, or any combination thereof, or a rotation about any unit direction vector in three dimensional space. The force vector

$$\boldsymbol{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ \vdots \\ f_n(t) \end{bmatrix} \tag{32}$$

represents forces acting on each of the n generalized coordinates. Such equations of motion are typical of non-rotating structures without inclusion of aerodynamic loading.
Solution of equation (30) is most often performed in modal coordinates due to physical insight obtained and numerical advantages. In order to transform into modal coordinates, we first consider the un-forced, or homogeneous, and undamped, $C = 0$, equation of motion given by

$$M\ddot{\boldsymbol{x}} + K\boldsymbol{x} = \boldsymbol{0} \tag{33}$$

where for the sake of simplicity dependence on time is no longer explicitly shown. A solution for $\boldsymbol{x}(t)$ is then assumed to be

$$\boldsymbol{x} = \boldsymbol{\psi} e^{\lambda t} \tag{34}$$

Substituting into equation (33) gives

$$(M\lambda^2 + K)\boldsymbol{\psi} = \boldsymbol{0} \tag{35}$$

Equation (35) should be recognized as an eigenvalue problem with the eigenvalue λ^2 and the eigenvector $\boldsymbol{\psi}$. Since the vector is n elements long, and we also have the unknown λ^2 to solve for, we have only n equations but $n + 1$ unknowns. In addition, the equations are nonlinear in the combined unknowns λ^2 and ψ_i. A first attempt at solving these equations for $\boldsymbol{\psi}$ would be to premultiply equation (35) by $(M\lambda^2 + K)^{-1}$ giving

$$\begin{aligned} (M\lambda^2 + K)^{-1}(M\lambda^2 + K)\boldsymbol{\psi} &= (M\lambda^2 + K)^{-1}\boldsymbol{0} \\ I\boldsymbol{\psi} &= (M\lambda^2 + K)^{-1}\boldsymbol{0} \\ \boldsymbol{\psi} &= (M\lambda^2 + K)^{-1}\boldsymbol{0} \end{aligned} \tag{36}$$

At first observation, the only solution appears to be $\boldsymbol{\psi} = \boldsymbol{0}$. However, this is the so-called *trivial solution*. In fact observing equation (33), $\boldsymbol{x} = \boldsymbol{0}$ is a viable solution, if useless for understanding dynamic response. An alternative solution is the case where $A = (M\lambda^2 + K)^{-1}$ does not exist.
The inverse of a matrix A is defined as

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)} \tag{37}$$

where adj(A) is the *adjoint*[7][?, ?] or *classical adjoint*[?] matrix of A and det(A) is the determinant of A. Of importance is the denominator. If the det(A) $= 0$, then the inverse of A is undefined, and the solution in the form of equation (36) makes no sense. Thus, since we are expecting solutions other than the trivial solution, finding the value/s of λ^2 such that det($M\lambda^2+K$) $= 0$ is the only feasible solution.

Taking the determinant of $M\lambda^2 + K$ and setting it equal to zero results in an *nth* order polynomial in λ^2, the solution of which results in n real and negative values of λ^2. Solving for λ results in n imaginary conjugate pairs of values $\lambda = \pm j\omega$. For each value of λ_m^2, a unique (linearly independent from the rest) $\boldsymbol{\psi}_m$ can be found. Even when there are repeated values, i.e. $\lambda_m^2 = \lambda_{m+1}^2$, linearly independent vectors $\boldsymbol{\psi}_m$ and $\boldsymbol{\psi}_{m+1}$ can be found for each occurrence of a solution, λ^2. In the current case, the vectors $\boldsymbol{\psi}_m$ are real and linearly independent. The vectors $\boldsymbol{\psi}_m$ are the mode shape of the structure and the corresponding values ω_m are the natural frequencies in radians/sec[8]. The methods for obtaining λ_m and $\boldsymbol{\psi}_m$ from a given mass matrix, M, and stiffness matrix, K, pair vary greatly in practice from the elementary methods often learned in introductory courses. Most often methods such as subspace iteration, the QR method, inverse iteration, or other even more sophisticated algorithms are used[?, ?] and are beyond the scope of this class.

It is convenient to *mass normalize* the eigenvectors $\boldsymbol{\psi}_m$ such that

$$\boldsymbol{\psi}_m^T M \boldsymbol{\psi}_m = 1 \tag{38}$$

After doing so, the eigenvectors are *mass orthonormal* and *stiffness orthogonal*. Consider the two unique eigenvalue solutions ($l \neq m$), written in a slightly different form

$$K\boldsymbol{\psi}_l = -M\lambda_l^2\boldsymbol{\psi}_l \tag{39a}$$

$$K\boldsymbol{\psi}_m = -M\lambda_m^2\boldsymbol{\psi}_m \tag{39b}$$

Pre-multiplying each by the alternate eigenvector transposed gives

$$\boldsymbol{\psi}_m^T K\boldsymbol{\psi}_l = -\boldsymbol{\psi}_m^T M\lambda_l^2\boldsymbol{\psi}_l = -\lambda_l^2\boldsymbol{\psi}_m^T M\boldsymbol{\psi}_l \tag{40a}$$

$$\boldsymbol{\psi}_l^T K\boldsymbol{\psi}_m = -\boldsymbol{\psi}_l^T M\lambda_m^2\boldsymbol{\psi}_m = -\lambda_m^2\boldsymbol{\psi}_l^T M\boldsymbol{\psi}_m \tag{40b}$$

[7]The i, j element of the adjoint of a matrix is the determinant of the remaining matrix when the jth row and the ith column are removed, multiplied by -1^{i+j}. In practice, other methods are used to calculate the inverse when necessary.

[8]It is very rare for a unit of time other than seconds to be used for a structure

Subtracting (40a) from the transpose of (40b), and again noting that M and K are symmetric, gives

$$\begin{aligned} \boldsymbol{\psi}_m^T K \boldsymbol{\psi}_l - \boldsymbol{\psi}_m^T K \boldsymbol{\psi}_l &= -\lambda_m^2 \boldsymbol{\psi}_m^T M \boldsymbol{\psi}_l + \lambda_l^2 \boldsymbol{\psi}_m^T M \boldsymbol{\psi}_l \\ 0 &= \left(\lambda_l^2 - \lambda_m^2\right) \boldsymbol{\psi}_m^T M \boldsymbol{\psi}_l \end{aligned} \tag{41}$$

If $(\lambda_l^2 - \lambda_m^2) \neq 0$, then $\boldsymbol{\psi}_m^T M \boldsymbol{\psi}_l = 0$, and since $\boldsymbol{\psi}_m^T M \boldsymbol{\psi}_m = 1$,

$$\boldsymbol{\psi}_m^T M \boldsymbol{\psi}_l = \delta_{lm} = \begin{cases} 1, & l = m \\ 0, & l \neq m \end{cases} \tag{42}$$

where δ_{lm} is the *Kronecker delta function* and is as defined above. Considering now equation (40a)

$$\begin{aligned} \boldsymbol{\psi}_m^T K \boldsymbol{\psi}_l &= -\lambda_l^2 \boldsymbol{\psi}_m^T M \boldsymbol{\psi}_l \\ &= -\lambda_l^2 \delta_{lm} \\ &= \begin{cases} -\lambda_l^2, & l = m \\ 0, & l \neq m \end{cases} \end{aligned} \tag{43}$$

In the case where there are repeated solutions, i.e. $\lambda_m^2 = \lambda_{m+1}^2 = \ldots$, substituting the coordinate transformation

$$\boldsymbol{x} = M^{-1/2} \boldsymbol{q} \tag{44}$$

into equation (30) and pre-multiplying by $\left(M^{-1/2}\right)^T$ where $M^{-1/2}$ is defined such that

$$M = \left(M^{1/2}\right)^T M^{1/2} \tag{45}$$

gives

$$\begin{aligned} \left(M^{-1/2}\right)^T M M^{-1/2} \ddot{\boldsymbol{q}} + \left(M^{-1/2}\right)^T K M^{-1/2} \boldsymbol{q} &= \left(M^{-1/2}\right)^T \boldsymbol{f} \\ I \ddot{\boldsymbol{q}} + \tilde{K} \boldsymbol{q} &= \left(M^{-1/2}\right)^T \boldsymbol{f} \end{aligned} \tag{46}$$

where $\tilde{K}$ is the mass normalized stiffness matrix. Solving the homogeneous equation by assuming a solution of the form

$$\boldsymbol{q}(t) = \boldsymbol{v} e^{\lambda t} \tag{47}$$

yields the single matrix eigenvalue problem

$$(I\lambda^2 + \tilde{K})\boldsymbol{v} = \mathbf{0} \tag{48}$$

Since $\tilde{K}$ is symmetric, its eigenvectors, $\boldsymbol{v}_m$, are orthonormal[**?**]. Substituting (34) and (47) into (44) gives

$$\boldsymbol{\psi} = M^{-1/2}\boldsymbol{v} \tag{49}$$

Since the vectors $\boldsymbol{v}_m$ are orthonormal, and $M^{-1/2}$ must be non-singular, the vectors $\boldsymbol{\psi}_m$ must be linearly independent even in the case of repeated eigenvalues.
Given the multiplicity of solutions for ω^2 and $\boldsymbol{\psi}$, the solution for $\boldsymbol{x}$ must then be considered the linear summation

$$\boldsymbol{x} = \sum_{m=1}^{n} \left(a_m e^{+j\omega_m t} + \bar{a}_m e^{-j\omega_m t}\right) \boldsymbol{\psi}_m \tag{50}$$

Applying the Euler relation this can be written in the more physically intuitive form

$$\boldsymbol{x} = \sum_{m=1}^{n} R_m \sin(\omega_m t + \phi_m)\boldsymbol{\psi}_m \tag{51}$$

where R_m is a *modal amplitude*. If we further define

$$r_m(t) = R_m \sin(\omega_m t + \phi_m) \tag{52}$$

to be our *modal coordinates*, then equation (51) can be written in matrix form as

$$\boldsymbol{x}(t) = \Psi \boldsymbol{r}(t) \tag{53}$$

where

$$\Psi = \begin{bmatrix} \boldsymbol{\psi}_1 & \boldsymbol{\psi}_2 & \boldsymbol{\psi}_3 & \cdots & \boldsymbol{\psi}_n \end{bmatrix} = \begin{bmatrix} \psi_{1,1} & \psi_{1,2} & \cdots & \psi_{1,m} & \cdots & \psi_{1,n} \\ \psi_{2,1} & \psi_{2,2} & \cdots & \psi_{2,m} & \cdots & \psi_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ \psi_{l,1} & \psi_{l,2} & \cdots & \psi_{l,m} & \cdots & \psi_{l,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_{n,1} & \psi_{n,2} & \cdots & \psi_{n,m} & \cdots & \psi_{n,n} \end{bmatrix} \tag{54}$$

Substituting equation (53) into equation (30),

$$M\Psi\ddot{\boldsymbol{r}}(t) + K\Psi\boldsymbol{r}(t) = \boldsymbol{f}(t) \tag{55}$$

Pre-multiplying (55) by Ψ^T, and using equations (42) and (43) the equations are now transformed into individual uncoupled *modal equations*

$$I\ddot{\boldsymbol{r}}(t) + \Omega^2\boldsymbol{r}(t) = \tilde{\boldsymbol{f}}(t) \tag{56}$$

where I is the identity matrix,

$$\Omega = \begin{bmatrix} \omega_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \omega_2^2 & 0 & \cdots & 0 \\ 0 & 0 & \omega_3^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \omega_n^2 \end{bmatrix} \tag{57}$$

and

$$\tilde{\boldsymbol{f}}(t) = \Psi^T\boldsymbol{f}(t) \tag{58}$$

is the modal force vector. The resulting decoupled equations can each then be solved using SDOF methods. Applying equation (53), the solution can be transformed back into physical coordinates. Also, using equation (53), the initial conditions can be transformed into modal coordinates for use in solving equations (56) giving

$$\boldsymbol{r}(0) = \Psi^{-1}\boldsymbol{x}(0) \tag{59a}$$

$$\dot{\boldsymbol{r}}(0) = \Psi^{-1}\dot{\boldsymbol{x}}(0) \tag{59b}$$

This procedure can also be followed for damped systems under limited cases. In the case of viscous damping, where the governing equation of motion is

$$M\ddot{\boldsymbol{x}}(t) + C\dot{\boldsymbol{x}}(t) + K\boldsymbol{x}(t) = \boldsymbol{f}(t), \tag{60}$$

then if

$$CM^{-1}K = KM^{-1}C \tag{61}$$

the damping matrix, C, is diagonalized by $\Psi^T C\Psi = \text{diag}(2\zeta_i\omega_i)$[?]. Consider substituting

$$\boldsymbol{x}(t) = M^{-1/2}\boldsymbol{q}(t) \tag{62}$$

into equation (60), but with no forcing vector, where

$$(M^{-1/2})^T M M^{-1/2} = I \tag{63}$$

Pre-multiplying by $(M^{-1/2})^T$ yields

$$I\ddot{\boldsymbol{q}}(t) + \tilde{C}\dot{\boldsymbol{q}}(t) + \tilde{K}\boldsymbol{q}(t) = \boldsymbol{0} \tag{64}$$

Now let

$$\boldsymbol{q}(t) = \Upsilon \boldsymbol{r}(t) \tag{65}$$

where Υ is the orthonormal matrix of the eigenvectors of $\tilde{K}$, $\boldsymbol{\upsilon}$, such that

$$\Upsilon^T \Upsilon = I \tag{66}$$

$$\Upsilon^T \tilde{K} \Upsilon = \Omega^2 \tag{67}$$

substituting (65) into equation (64) and pre-multiplying by Υ^T yields

$$I\ddot{\boldsymbol{r}}(t) + \Upsilon^T \tilde{C} \Upsilon \dot{\boldsymbol{r}}(t) + \Omega^2 \boldsymbol{r}(t) = \boldsymbol{0} \tag{68}$$

In order for the equations of motion to be decoupled it is necessary that $\Upsilon^T \tilde{C} \Upsilon = 2\text{diag}(\zeta_i \Omega_i)$ be diagonal.
Two matrices are simultaneously diagonalized if and only if they commute, i.e.

$$\tilde{C}\tilde{K} = \tilde{K}\tilde{C} \tag{69}$$

The modal equations then are

$$I\ddot{\boldsymbol{r}}(t) + 2\text{diag}(\zeta_i \Omega_i)\dot{\boldsymbol{r}}(t) + \Omega^2 \boldsymbol{r}(t) = \tilde{\boldsymbol{f}}(t) \tag{70}$$

A special case of this is *Rayleigh damping*, often referred to as *proportional damping*, where $C = \alpha M + \beta K$. In this case the damping ratio is given by

$$\zeta_i = \frac{1}{2}\left(\frac{\alpha}{\Omega_i} + \beta \Omega_i\right) \tag{71}$$

Similarly, for a complex stiffness model of the form

$$M\ddot{\boldsymbol{x}}(t) + (K + K'j)\boldsymbol{x}(t) = \boldsymbol{f}(t), \tag{72}$$

if

$$KM^{-1}K' = K'M^{-1}K \tag{73}$$

then the imaginary part of the stiffness matrix, K', is diagonalized by $\Psi^T K' \Psi = \text{diag}(\eta_i \omega_i^2)$ yielding modal equations of motion

$$\begin{aligned} I\ddot{\boldsymbol{r}}(t) + (\Omega^2 + \text{diag}(\eta_i)\Omega^2 j)\boldsymbol{r}(t) &= \tilde{\boldsymbol{f}}(t) \\ I\ddot{\boldsymbol{r}}(t) + \Omega^2(1 + \text{diag}(\eta_i)j)\boldsymbol{r}(t) &= \tilde{\boldsymbol{f}}(t) \end{aligned} \tag{74}$$

The *modal frequency response functions* are then obtained by taking the Fourier transform of the appropriate modal equation of motion, either (56), (70), or (74), and solving for

$$\tilde{h}_i = \frac{R_i(j\omega)}{\tilde{F}_i(j\omega)} = \begin{cases} \frac{1}{\omega_i^2 - \omega^2}, & \text{undamped} \\ \frac{1}{\omega_i^2 + 2\zeta_i j\omega_i\omega - \omega^2}, & \text{viscous damping} \\ \frac{1}{\omega_i^2(1+\eta_i j) - \omega^2}, & \text{complex stiffness damping} \end{cases} \tag{75}$$

Consider now the full system modal equations in frequency response form

$$\boldsymbol{R}(j\omega) = \tilde{H}(j\omega)\tilde{\boldsymbol{F}}(j\omega) \tag{76}$$

where $\tilde{H}(j\omega) = \text{diag}(\tilde{h}_i)$. Premultiplying by Ψ and substituting $\boldsymbol{X}(j\omega) = \Psi\boldsymbol{R}(j\omega)$ and $\tilde{\boldsymbol{F}}(j\omega) = \Psi^T\boldsymbol{F}(j\omega)$ into equation (76) and premultiplying by Ψ^{-1} gives

$$\begin{aligned} \Psi\boldsymbol{R}(j\omega) &= \Psi\tilde{H}(j\omega)\Psi^T\boldsymbol{F}(j\omega) \\ \boldsymbol{X}(j\omega) &= \Psi\tilde{H}(j\omega)\Psi^T\boldsymbol{F}(j\omega) \\ &= H(j\omega)\boldsymbol{F}(j\omega) \end{aligned} \tag{77}$$

where $H(j\omega)$ is the matrix of transfer functions between forces and displacements in physical coordinates. Expressing Ψ using equation (54), and considering equation (77)

$$
\begin{aligned}
H(j\omega) &= \begin{bmatrix} \boldsymbol{\psi}_1 & \boldsymbol{\psi}_2 & \boldsymbol{\psi}_3 & \cdots & \boldsymbol{\psi}_n \end{bmatrix} \tilde{H}(j\omega) \begin{bmatrix} \boldsymbol{\psi}_1^T \\ \boldsymbol{\psi}_2^T \\ \boldsymbol{\psi}_3^T \\ \vdots \\ \boldsymbol{\psi}_n^T \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{\psi}_1 \tilde{h}_1(j\omega) & \boldsymbol{\psi}_2 \tilde{h}_2(j\omega) & \boldsymbol{\psi}_3 \tilde{h}_3(j\omega) & \cdots & \boldsymbol{\psi}_n \tilde{h}_n(j\omega) \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}_1^T \\ \boldsymbol{\psi}_2^T \\ \boldsymbol{\psi}_3^T \\ \vdots \\ \boldsymbol{\psi}_n^T \end{bmatrix} \qquad (78) \\
&= \sum_{i=1}^{n} \boldsymbol{\psi}_i \boldsymbol{\psi}_i^T \tilde{h}_i(j\omega) \\
&= \sum_{i=1}^{n} {}_iA \tilde{h}_i(j\omega)
\end{aligned}
$$

The variable, ${}_iA = \boldsymbol{\psi}_i \boldsymbol{\psi}_i^T$, represents the outer product of $\boldsymbol{\psi}_i$ with itself and is called the *modal constant* or *residue*. That is,

$$
{}_iA = \boldsymbol{\psi}_i \boldsymbol{\psi}_i^T = \begin{bmatrix} \psi_{1,i}\psi_{1,i} & \psi_{1,i}\psi_{2,i} & \cdots & \psi_{1,i}\psi_{m,i} & \cdots & \psi_{1,i}\psi_{n,i} \\ \psi_{2,i}\psi_{1,i} & \psi_{2,i}\psi_{2,i} & \cdots & \psi_{2,i}\psi_{m,i} & \cdots & \psi_{2,i}\psi_{n,i} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ \psi_{l,i}\psi_{1,i} & \psi_{l,i}\psi_{2,i} & \cdots & \psi_{l,i}\psi_{m,i} & \cdots & \psi_{l,i}\psi_{n,i} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_{n,i}\psi_{1,i} & \psi_{n,i}\psi_{2,i} & \cdots & \psi_{n,i}\psi_{m,i} & \cdots & \psi_{n,i}\psi_{n,i} \end{bmatrix} \qquad (79)
$$

Thus an individual frequency response function between an input l and output, m, is

$$
H_{l,m} = \sum_{i=1}^{n} \psi_{l,i}\psi_{m,i}\tilde{h}_i(j\omega) \qquad (80)
$$

An alternative representation of the frequency response function can be obtained directly from equations (72) or (60). Taking the Fourier transform of each equation,

$$
\left(-\omega^2 M + j\omega C + K\right) \boldsymbol{X}(j\omega) = \boldsymbol{F}(j\omega), \qquad \text{viscous} \qquad (81a)
$$

$$\left(-\omega^2 M + (K + K'j)\right) \boldsymbol{X}(j\omega) = \boldsymbol{F}(j\omega), \qquad \text{hysteretic} \tag{81b}$$

Since the analysis is the same for either form of damping, we show only the viscous case for the sake of clarity. The frequency response function matrix is then

$$\boldsymbol{X}(j\omega) = \left(-\omega^2 M + j\omega C + K\right)^{-1} \boldsymbol{F}(j\omega) = H(j\omega)\boldsymbol{F}(j\omega), \qquad \text{viscous} \tag{82}$$

Since

$$H(j\omega) = \left(-\omega^2 M + j\omega C + K\right)^{-1} \tag{83}$$

using the definition of the inverse of a matrix of

$$H_{l,m}(j\omega) = \frac{\text{adj}(-\omega^2 M + j\omega C + K)_{l,m}}{\det(-\omega^2 M + j\omega C + K)} \tag{84}$$

When

$$\text{adj}(-\omega^2 M + j\omega C + K)_{l,m} = 0 \tag{85}$$

the frequency response function will exhibit a *zero* or *anti-resonance*. Recall that the l, m element of the adjoint of a matrix is the determinant of the remaining matrix when the lth row and the mth column are removed, multiplied by -1^{l+m}, called the l, m *cofactor*. When $l = m$, this is equivalent to constraining the equations of motion such that $x_l = 0$.

Example 4.1 *A system is defined by*

$$M\ddot{\boldsymbol{x}} + C\dot{\boldsymbol{x}} + K\boldsymbol{x} = \boldsymbol{0} \tag{86}$$

where

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 4 \end{bmatrix} \tag{87}$$

Find the poles and zeros of the frequency response function, as well as the FRF itself, between the first and third degrees of freedom.
Solution:
The poles (natural frequencies) of the system are given by the solution of

$$\begin{aligned} \det(-M\omega^2 + K) &= 0 \\ -\omega^6 + 9\omega^4 - 23\omega^2 + 13 &= 0 \end{aligned} \tag{88}$$

the solution of which is

$$\omega \approx \pm 0.8864, \pm 1.8813, \pm 2.1622 \tag{89}$$

The zeros of this specific frequency response function are given by

$$\begin{aligned} adj_{1,3}(K - \omega^2 M) &= 0 \\ \det\left(\begin{bmatrix} -1 & -1 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\omega^2\right) &= 0 \\ 4 - \omega^2 &= 0 \end{aligned} \tag{90}$$

Solving for ω gives zeros at

$$\omega = \pm 2 \tag{91}$$

The mass normalized mode shapes are

$$\boldsymbol{\psi}_1 = \begin{bmatrix} 0.7558 \\ 0.5207 \\ 0.3971 \end{bmatrix}, \quad \boldsymbol{\psi}_2 = \begin{bmatrix} 0.6318 \\ -0.7392 \\ -0.2332 \end{bmatrix}, \quad \textit{and } \boldsymbol{\psi}_3 = \begin{bmatrix} 0.1721 \\ 0.4271 \\ -0.8877 \end{bmatrix} \tag{92}$$

Using equation (80), the frequency response function is

$$\begin{aligned} H_{1,3}(j\omega) &= \sum_{i=1}^{3} \psi_{1,i}\psi_{3,i}\tilde{h}_i(j\omega) \\ &= \frac{0.3001}{0.7857 - \omega^2} + \frac{-0.1473}{3.5392 - \omega^2} + \frac{-0.1528}{4.6751 - \omega^2} \end{aligned} \tag{93}$$

Alternatively, applying equation (84), the FRF can also be represented as

$$H_{1,3}(j\omega) = \frac{-\omega^2 + 4}{-\omega^6 + 9\omega^4 - 23\omega^2 + 13} \tag{94}$$

5 Qualitative Aspects of the Algebraic Eigenvalue Problem (Chapter 5)

5.1 Geometric Properties of Eigenvalue Problem

5.2 Stationarity of Rayleigh Quotient

5.3 Maximum-Minimum Characteristics of Eigenvalues

Download and run minmaxplot3d.m in matlab

Consider

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{95}$$

The eigenvalues are $\lambda = 1, 3$. The first eigenvalue is the minimum value of

$$f(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$$

where $\boldsymbol{x}$ is normalized.
Trial 1:

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad f = 2$$

Trial 2:

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad f = 2$$

Trial 3:

$$\boldsymbol{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad f = 3$$

Trial 4:

$$\boldsymbol{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad f = 1$$

This could have been written as

$$\frac{df}{dx_1} = 0, \qquad ||\boldsymbol{x}|| = 1 \tag{96}$$

$$\begin{aligned} f(x_1) &= \begin{bmatrix} x_1 & \sqrt{1-x_1^2} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ \sqrt{1-x_1^2} \end{bmatrix} \\ &= 2x_1^2 - x_1\sqrt{1-x_1^2} - x_1\sqrt{1-x_1^2} + 2(1-x_1^2) \\ &= 2(x_1^1 - x_1\sqrt{1-x_1^2} + 1 - x_1^2) \\ &= 2(1 - x_1\sqrt{1-x_1^2}) \end{aligned} \tag{97}$$

Then, setting $f' = 0$

$$\begin{aligned} \frac{df}{dx_1} &= -\sqrt{1-x_1^2} + x_1^2\left(1-x_1^2\right)^{-\frac{1}{2}} = 0 \\ (-1+x_1^2) + x_1^2 &= 0 \\ 2x_1^2 &= 1 \\ x_1 &= \pm\frac{1}{\sqrt{2}}, \qquad \text{Note: by trial, } (+) \text{ is the correct sign} \end{aligned} \tag{98}$$

To find the second eigenvalue, we will pick values of $\boldsymbol{v}$ such that $\boldsymbol{x}$ is constrained by

$$\boldsymbol{x}^T\boldsymbol{v} = 0$$

For each vector $\boldsymbol{v}$ we minimize $f(\boldsymbol{x})$. The highest value of all of these minimums is our second eigenvalue.
In this case, because the system has only two dimensions ($\mathcal{R}^2$), choosing $\boldsymbol{v}$ is equivalent to choosing $\boldsymbol{x}$
Trial 1:

$$\boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad f = 2$$

Trial 2:

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad f = 2$$

Trial 3:

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad f = 1$$

Trial 4:

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \boldsymbol{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad f = 3$$

$f = 3$ is the second eigenvalue.
Extrapolating, the $(n+1)th$ eigenvalue is the highest minimum value of $\boldsymbol{x}^T A\boldsymbol{x}$ subject to n constraints.

5.4 The Inclusion Principle

What is the effect of using fewer degrees of freedom to represent a system? This is a vital concern when using finite elements and the more general continuum methods to be discussed later.
Consider a system described by the Hermitian matrix A $(n \times n)$. Consider another system defined by B $(n-1 \times n-1)$ formed from A by deleting the last row and column.
The eigenvalues of A are named $\lambda_1, \lambda_2, \lambda_3, \ldots$ and the eigenvalues of B are $\gamma_1, \gamma_2, \gamma_3, \ldots$.

$$\boldsymbol{y}^H B \boldsymbol{y} = \boldsymbol{x}^H A \boldsymbol{x} \qquad \text{if} \quad x_i = y_i, \text{ and } x_n = 0$$

$x_n = 0$ can be considered a constraint such that $\boldsymbol{x}^T \hat{\boldsymbol{e}}_n = 0$
From Rayleigh's principle,

$$\gamma_1 = \min \boldsymbol{y}^H B \boldsymbol{y}, \qquad ||\boldsymbol{y}|| = 1$$

This is equivalent to the constrained minimization

$$\gamma_1 = \tilde{\lambda}_1(\hat{\boldsymbol{e}}_n) = \min \boldsymbol{x}^H A \boldsymbol{x}, \qquad ||\boldsymbol{x}|| = 1, \, \boldsymbol{x}^H \hat{\boldsymbol{e}}_n = 0$$

From the min-max principle,

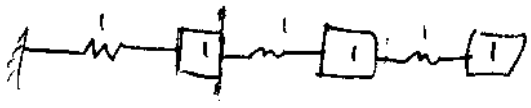
$$\lambda_1 \leq \gamma_1 \leq \lambda_2$$

Successively considering the remaining eigenvalues yields

$$\lambda_1 \leq \gamma_1 \leq \lambda_2 \leq \gamma_2 \leq \lambda_3 \leq \ldots \lambda_n$$

which is called *the inclusion principle*

5.4.1 Example: The inclusion principal



The original stiffness matrix is:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

If we constrain degree of freedom 1 to zero amplitude, the new stiffness matrix is

$$B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

The eigenvalues of A are $0.1981, 1.5550, 3.2470$. The eigenvalues of B are $0.3820, 2.6180$. The eigenvalues are interwoven with one another. This applies for more general constraints as well. For example, setting $x_1 = x_2$. Note that this also modifies the mass matrix. Verify that the inclusion principle works for this case for homework.

5.5 Perturbation of the Symmetric Eigenvalue Problem

We have already developed that the symmetric eigenvalue problem can be reduced to the eigenproblem of a single matrix. Define that system $A_0 = M_0^{-\frac{1}{2}} K_0 M_0^{-\frac{1}{2}}$ such that its eigensolution satisfies

$$A_0 \boldsymbol{x}_{0i} = \lambda_{0i} \boldsymbol{x}_{0i} \tag{99}$$

The eigenvectors satisfy orthogonality, i.e.

$$\boldsymbol{x}_{0i}^T \boldsymbol{x}_{0j} = \delta_{ij}, \qquad \boldsymbol{x}_{0i}^T A_0 \boldsymbol{x}_{0j} = \lambda_{0i} \delta_{ij}, \qquad i, j = 1, 2, \ldots, n \tag{100}$$

Presume that the system has been changed by some small amount that we define ϵA_1 so that the new system is defined by

$$A = A_0 + \epsilon A_1 \tag{101}$$

where A_0 is the unperturbed system matrix, ϵ is the *perturbation parameter*, and A_1 is the *perturbation matrix*. It is presumed that the magnitude of the significant non-zero quantities in A_1 are of the same order as corresponding quantities of A_0 and that $\epsilon < 1$. Agreement is much better for values $\epsilon \ll 1$. If the only stiffness matrix is changed, then $K = K_0 + K_1$ and $\epsilon A_1 = M_0^{-\frac{1}{2}} K_1 M_0^{-\frac{1}{2}}$. Alternatively, if both stiffness and mass are perturbed, then $M = M_0 + M_1$ as well, and $\epsilon A_1 = (M_0 + M_1)^{-\frac{1}{2}} (K_0 + K_1) (M_0 + M_1)^{-\frac{1}{2}} - A_0$. While this may be a rather computationally expensive, keep in mind that the eigensolution is much more difficult.
The solution to the perturbed eigenvalue problem is

$$A \boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i \tag{102}$$

Here we would like to estimate the values of $\boldsymbol{x}_i$ and λ_i without repeating the solution to the eigenvalue problem. Thus

$$\boldsymbol{x}_i^T \boldsymbol{x}_j = \delta_{ij}, \qquad \boldsymbol{x}_i^T A \boldsymbol{x}_j = \lambda_i \delta_{ij}, \qquad i, j = 1, 2, \ldots, n \tag{103}$$

The solution of the perturbed eigenvalue problem can then be presumed to be perturbations of the unperturbed solution. Thus

$$\begin{aligned} \lambda_i(\epsilon) &= \lambda_{0i} + \frac{1}{1!}\epsilon\frac{d\lambda_i}{d\epsilon} + \frac{1}{2!}\epsilon^2\frac{d^2\lambda_i}{d\epsilon^2} + \cdots \\ &= \lambda_{0i} + \epsilon\lambda_{1i} + \epsilon^2\lambda_{2i} + \cdots, \qquad i = 1, 2, \ldots, n \end{aligned} \tag{104a}$$

$$\begin{aligned} \boldsymbol{x}_i(\epsilon) &= \boldsymbol{x}_{0i} + \frac{1}{1!}\epsilon\frac{d\boldsymbol{x}_i}{d\epsilon} + \frac{1}{2!}\epsilon^2\frac{d^2\boldsymbol{x}_i}{d\epsilon^2} + \cdots \\ &= \boldsymbol{x}_{0i} + \epsilon\boldsymbol{x}_{1i} + \epsilon^2\boldsymbol{x}_{2i} + \cdots, \qquad i = 1, 2, \ldots, n \end{aligned} \tag{104b}$$

Substituting (104) and equation (101) into equation (102) gives

$$\begin{aligned} &A_0\boldsymbol{x}_{0i} + \epsilon A_0\boldsymbol{x}_{1i} + \epsilon A_1\boldsymbol{x}_{0i} + \epsilon^2 A_1\boldsymbol{x}_{1i} + \epsilon^2 A_0\boldsymbol{x}_{2i} + \epsilon^3 A_1\boldsymbol{x}_{2i} + \cdots \\ &\quad = \lambda_{0i}\boldsymbol{x}_{0i} + \epsilon\lambda_{0i}\boldsymbol{x}_{1i} + \epsilon\lambda_{1i}\boldsymbol{x}_{0i} + \epsilon^2\lambda_{2i}\boldsymbol{x}_{0i} + \epsilon^2\lambda_{1i}\boldsymbol{x}_{1i} + \epsilon^2\lambda_{0i}\boldsymbol{x}_{2i} + \cdots \end{aligned} \tag{105}$$

Acknowledging equation (99) this becomes

$$\begin{aligned} &\epsilon A_0\boldsymbol{x}_{1i} + \epsilon A_1\boldsymbol{x}_{0i} + \epsilon^2 A_1\boldsymbol{x}_{1i} + \epsilon^2 A_0\boldsymbol{x}_{2i} + \epsilon^3 A_1\boldsymbol{x}_{2i} + \cdots \\ &\quad = \epsilon\lambda_{0i}\boldsymbol{x}_{1i} + \epsilon\lambda_{1i}\boldsymbol{x}_{0i} + \epsilon^2\lambda_{0i}\boldsymbol{x}_{2i} + \epsilon^2\lambda_{1i}\boldsymbol{x}_{1i} + \epsilon^2\lambda_{2i}\boldsymbol{x}_{0i} + \cdots \end{aligned} \tag{106}$$

Considering only first-order terms in ϵ yields

$$A_0\boldsymbol{x}_{1i} + A_1\boldsymbol{x}_{0i} = \lambda_{0i}\boldsymbol{x}_{1i} + \lambda_{1i}\boldsymbol{x}_{0i} \tag{107}$$

Solution of this equation for $\boldsymbol{x}_{1i}$ and λ_{1i} is the solution of the first order perturbation problem.
To do this, we presume that $\boldsymbol{x}_{1i}$ can be written as a linear combination of the non-corresponding unperturbed system eigenvectors[9], i.e.

$$\boldsymbol{x}_{1i} = \sum_{j=1}^{n} \kappa_{ij}\boldsymbol{x}_{0j}\tilde{\delta}_{ij}, \qquad i = 1, 2, \ldots, n \tag{108}$$

[9]There is no benefit to perturbing an eigenvector by itself times a constant because, after normalization, it doesn't change.

where

$$\tilde{\delta}_{ij} = \begin{cases} 1 & i \neq j \\ 0 & i = j \end{cases} \tag{109}$$

Is the opposite of the Kronecker delta.
Substituting equation (108) into equation (107) yields

$$A_0 \sum_{j=1}^{n} \kappa_{ij} \boldsymbol{x}_{0j} \tilde{\delta}_{ij} + A_1 \boldsymbol{x}_{0i} = \lambda_{0i} \sum_{j=1}^{n} \kappa_{ij} \boldsymbol{x}_{0j} \tilde{\delta}_{ij} + \lambda_{1i} \boldsymbol{x}_{0i} \tag{110}$$

Applying equation (99) gives

$$\sum_{j=1}^{n} \kappa_{ij} \lambda_{0j} \boldsymbol{x}_{0j} \tilde{\delta}_{ij} + A_1 \boldsymbol{x}_{0i} = \lambda_{0i} \sum_{j=1}^{n} \kappa_{ij} \boldsymbol{x}_{0j} \tilde{\delta}_{ij} + \lambda_{1i} \boldsymbol{x}_{0i} \tag{111}$$

Pre-multiplying by $\boldsymbol{x}_{0k}^T$, and recognizing that

$$\boldsymbol{x}_{0k}^T \boldsymbol{x}_{0j} = \delta_{jk} \quad \text{and} \quad \boldsymbol{x}_{0k}^T A_0 \boldsymbol{x}_{0j} = \lambda_{0j} \delta_{jk} \tag{112}$$

for a self-adjoint system, gives

$$\kappa_{ik} \left(\lambda_{0k} - \lambda_{0i} \right) + \boldsymbol{x}_{0k}^T A_1 \boldsymbol{x}_{0i} = \lambda_{1i} \delta_{ik} \tag{113}$$

For $k = i$,

$$\lambda_{1i} = \boldsymbol{x}_{0i}^T A_1 \boldsymbol{x}_{0i} \tag{114}$$

The perturbed eigenvalue is then given by substituting equation (114) into equation (104a). Applying equation (113) for $k \neq i$

$$\kappa_{ik} = \frac{\boldsymbol{x}_{0k}^T A_1 \boldsymbol{x}_{0i}}{\lambda_{0i} - \lambda_{0k}} \tag{115}$$

and the perturbed eigenvector is obtained by substituting equation (115) into equation (108) then into equation (104b).

5.5.1 Example: Perturbation

Presume

$$A_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{116}$$

The eigensolution is given by

$$\boldsymbol{x}_{01} = \begin{bmatrix} -0.32799 \\ -0.59101 \\ -0.73698 \end{bmatrix}, \quad \boldsymbol{x}_{02} = \begin{bmatrix} 0.73698 \\ 0.32799 \\ -0.59101 \end{bmatrix}, \quad \boldsymbol{x}_{03} = \begin{bmatrix} -0.59101 \\ 0.73698 \\ -0.32799 \end{bmatrix} \tag{117}$$

with eigenvalues

$$\lambda_{01} = 0.19806, \quad \lambda_{02} = 1.55496, \quad \lambda_{03} = 3.24698 \tag{118}$$

We would like to obtain the eigenvalues and eigenvectors of the following matrix without resolving the eigenvalue problem:

$$A = \begin{bmatrix} 2.00000 & -1.00000 & 0.00000 \\ -1.00000 & 2.00000 & -1.00000 \\ 0.00000 & -1.00000 & 1.10000 \end{bmatrix} \tag{119}$$

Then

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{120}$$

with $\epsilon = 0.1$
Applying equation (114)

$$\lambda_{11} = \begin{bmatrix} -0.328 & -0.591 & -0.737 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.32799 \\ -0.59101 \\ -0.73698 \end{bmatrix} = 0.543 \tag{121}$$

thus, from equation (104a)

$$\lambda_1 = 0.19806 + 0.1 \times 0.54313 = 0.25238 \tag{122}$$

as compared to the true value of 0.25077 which is an error or -0.63843%.
Then, applying equation (115)

$$\kappa_{12} = \frac{\begin{bmatrix} -0.328 & -0.591 & -0.737 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.73698 \\ 0.32799 \\ -0.59101 \end{bmatrix}}{0.19806 - 1.55496} = -0.321 \tag{123}$$

and

$$\kappa_{13} = \frac{\begin{bmatrix}-0.328 & -0.591 & -0.737\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-0.591\\ 0.737\\ -0.328\end{bmatrix}}{0.19806 - 3.24698} = -0.079 \tag{124}$$

Thus, applying equation (108)

$$\boldsymbol{x}_{11} = \begin{bmatrix}-0.18971\\ -0.16371\\ 0.21571\end{bmatrix} \tag{125}$$

and using equation (104b)

$$\boldsymbol{x}_1 = \begin{bmatrix}-0.32799\\ -0.59101\\ -0.73698\end{bmatrix} + 0.1\begin{bmatrix}-0.18971\\ -0.16371\\ 0.21571\end{bmatrix} = \begin{bmatrix}-0.34696\\ -0.60738\\ -0.71540\end{bmatrix} \tag{126}$$

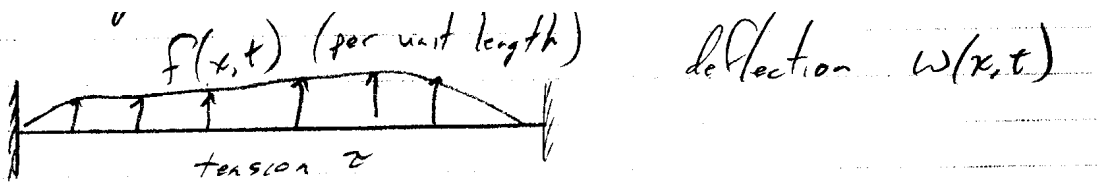
which are errors of

$$\begin{bmatrix}0.02762\\ -0.05061\\ -0.07694\end{bmatrix}\% \tag{127}$$

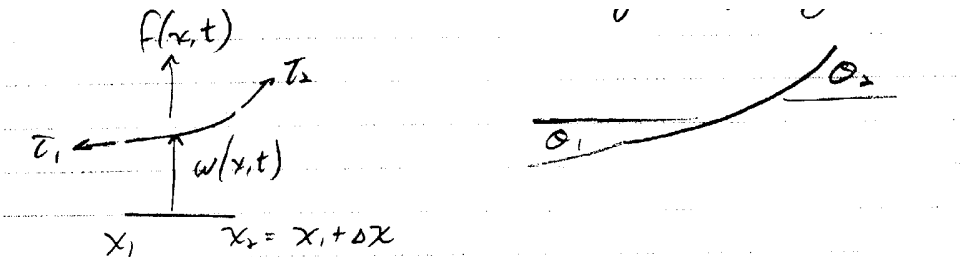
Thus a 10% change in one value of the matrix has caused less than a 1% change in the first eigenvalue and corresponding eigenvectors. This is consistent with the remainder of the eigenvalue and eigenvectors.

6 Vibration of Continuous Systems (Introduction)

6.1 Strings and Cables



Consider an infinitesimal element, neglect gravity



$$\sum F_y = \rho \Delta x \frac{\partial^2 w}{\partial t^2}$$

$$f(x,t)\Delta x + \tau_2 \frac{\partial w}{\partial x}\Big|_{x_2} - \tau_1 \frac{\partial w}{\partial x}\Big|_{x_1} = \rho \Delta x \frac{\partial^2 w}{\partial t^2}$$

$$\sum F_x = 0 = \tau_2 \cos\theta_2 - \tau_1 \cos\theta_1$$

For small θ_1, $\cos\theta_1 \approx 1 \approx \cos\theta_2$ $\tau_1 = \tau_2 = \tau$

$$\tau \left[\frac{\partial w}{\partial x}\Big|_{x_2} - \frac{\partial w}{\partial x}\Big|_{x_1} \right] + \Delta x f(x,t) = \rho \Delta x \frac{\partial^2 w}{\partial t^2}$$

From Taylor series expansion:

$$\frac{\partial w}{\partial x}\Big|_{x_2} = \frac{\partial w}{\partial x}\Big|_{x_1} + \Delta x \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)\Big|_{x_1} + \mathcal{O}(\Delta x^2)$$

substituting

$$\tau \Delta x \frac{\partial^2 w}{\partial x^2}\Big|_{x_1} + \Delta x f(x,t) = \Delta x \rho \frac{\partial^2 w}{\partial t^2}$$

Dividing by Δx, and since Δx is infinitesimal

$$\tau \frac{\partial^2 w}{\partial x^2} + f(x,t) = \rho \frac{\partial^2 w}{\partial t^2}$$

or

$$\tau w_{xx} + f(x,t) = \rho w_{tt}$$

where $c = \sqrt{\frac{\tau}{\rho}}$ is the wave speed.

6.1.1 Finding Mode Shapes and Natural Frequencies of a String

Presume fixed boundary conditions. The equation of motion for a string is

$$c^2 w_{xx} = w_{tt}$$

Let's assume $w(x,t) = X(x)T(t)$
Substituting

$$c^2 X'' T = X\ddot{T}$$
$$\frac{X''}{X} = \frac{\ddot{T}}{c^2 T}$$
$$f(x) = \frac{1}{c^2} g(t)$$

The only way this can be satisfied is if each side is equal to a constant.

$$\therefore \qquad \frac{X''}{X} = a = -\sigma^2 \tag{128}$$

also

$$\frac{\ddot{T}}{c^2 T} = -\sigma^2 \tag{129}$$

We call it $-\sigma^2$ because later in the solution we realize that it would have been convenient to do earlier. However, at this point in the solution process, there is no justification for it. Rearranging (128) yields

$$X'' + \sigma^2 X = 0$$

The solution to this is

$$X(x) = A\cos\sigma x + B\sin\sigma x$$

The boundary conditions are

$$X(0) = 0 = A, \qquad X(\ell) = 0 = B\sin\sigma\ell$$
$$\therefore \sigma_n = \frac{\pi n}{\ell}$$

and the mode shapes are

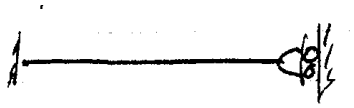
$$X(x) = A\sin\left(\frac{\pi n}{\ell} x\right)$$

Substituting into the temporal part:

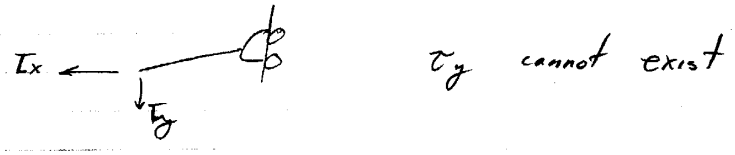
$$\ddot{T} + c^2\sigma_n^2 T = 0$$

$$\therefore \omega_n = \sqrt{\frac{\tau}{\rho}}\left(\frac{\pi n}{\ell}\right)$$

Next, consider:



The only difference is at the right end.



Summing moments in the vertical direction, it's clear that the slope must be zero at the right end.
The boundary conditions are

$$X(0) = 0 = A, \qquad X'\Big|_\ell = 0 = B\sigma\cos\sigma\ell$$

$$\therefore \sigma_n\ell = \frac{\pi n}{2}, \qquad n = 1, 3, 5, \ldots$$

or

$$\sigma_n = \frac{\pi(2n-1)}{2\ell}, \qquad n = 1, 2, 3, 4, \ldots$$

and the mode shapes are

$$X(x) = a_n \sin\left(\frac{\pi(2n-1)}{2\ell}x\right)$$

Substituting into the temporal part:

$$\ddot{T} + c^2\sigma_n^2 T = 0$$

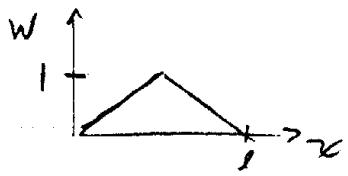
$$\therefore \omega_n = \sqrt{\frac{\tau}{\rho}}\frac{\pi(2n-1)}{2\ell}, \qquad n = 1, 2, 3, 4, \ldots$$

6.1.2 Free Response of a String

Find the response of a string plucked in the middle.

$$w(x,0) = \frac{2}{\ell}x, \qquad 0 < x < \frac{\ell}{2}$$

$$w(x,0) = \frac{2}{\ell}(\ell - x), \qquad \frac{\ell}{2} < x < \ell$$



$$w(x,t) = \sum_{n=1}^{\infty} \underbrace{\sin\frac{n\pi x}{\ell}}_{\text{Mode shapes}} \underbrace{(c_n \sin\omega_n t + d_n \cos\omega_n t)}_{\text{Temporal solution}} \tag{130}$$

Applying the initial conditions

$$w(x,0) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{\ell} d_n \tag{131}$$

where $w(x,0)$ was defined earlier, and

$$0 = \dot{w}(x,0) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{\ell} c_n \omega_n \tag{132}$$

c_n is clearly zero.
Multiplying (131) by $\sin\frac{m\pi x}{\ell}$ and integrating over ℓ

$$\int_0^\ell w(x,0)\sin\frac{m\pi x}{\ell}dx = \int_0^\ell \left(\sum_{n=1}^{\infty} \sin\frac{n\pi x}{\ell} d_n \sin\frac{m\pi x}{\ell}\right) dx \tag{133}$$

From orthogonality,

$$\int_0^\ell \sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell}dx = \begin{cases} 0 & m \neq n \\ \frac{\ell}{2} & m = n \end{cases}$$

Substituting, solving for d_n, and performing the integration

$$d_n = \frac{8}{\pi^2 n^2} \sin \frac{n\pi}{2}, \qquad n = 1, 2, 3, \ldots$$

Thus

$$w(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{\ell} \left(\frac{8}{\pi^2 n^2} \sin \frac{n\pi}{2} \cos \omega_n t \right)$$

where $\sigma_n = \frac{n\pi}{\ell}$.

6.1.3 Example, Forced response of a string

Solve for the steady-state (particular) response of the following system if the boundary conditions are presumed to be fixed-fixed where $c = \sqrt{\tau/\rho}$.

$$w_{tt}(x,t) - c^2 w_{xx}(x,t) = 100 \sin(t) \delta(x - l/2)$$

Recall that the integral of a Dirac delta function times another function is equal to the "another function" evaluated when the argument of the argument of the Dirac delta function is zero.

$$w(x,t) = T(t)X(t)$$
$$\ddot{T}X - c^2 T X'' = 0$$

Using separation of variables

$$\frac{\ddot{T}}{c^2 T} = \frac{X''}{X} = -\sigma^2$$

$$\therefore \qquad X(x) = A \cos \sigma x + B \sin \sigma x$$

The boundary conditions are

$$X(0) = 0 = A, \qquad X(\ell) = 0 = B \sin \sigma \ell$$

$$\therefore \sigma_n = \frac{\pi n}{\ell}$$

Substituting into the temporal part:

$$\ddot{T} + c^2 \sigma_n^2 T = 0$$

$$\therefore \omega_n = c\sigma_n$$

Assume a form of the solution[10]

$$w(x,t) = \sum_{n=1}^{\infty} a_n \sin t \sin n\pi \frac{x}{\ell}$$

Substituting into the equation of motion

$$\sum_{n=1}^{\infty} \left((-a_n \sin t \sin n\pi \frac{x}{\ell} + c^2 \left(\frac{n\pi}{\ell}\right)^2 a_n \sin t \sin n\pi \frac{x}{\ell} \right) = 100 \sin t \, \delta \left(x - \frac{\ell}{2} \right)$$

Multiplying by $\sin m\pi \frac{x}{\ell}$ and integrating over ℓ,

$$-a_n \frac{\ell}{2} + c^2 \left(\frac{n\pi}{\ell}\right)^2 a_n \frac{\ell}{2} = 100 \sin \frac{n\pi}{2}$$

Recall

$$\int_0^{\ell} \sin \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell} dx = \begin{cases} 0 & m \neq n \\ \frac{\ell}{2} & m = n \end{cases}$$

Solving gives

$$a_n = \frac{2}{\ell} \frac{100 \sin \frac{n\pi}{2}}{c^2 \left(\frac{n\pi}{\ell}\right)^2 - 1}$$

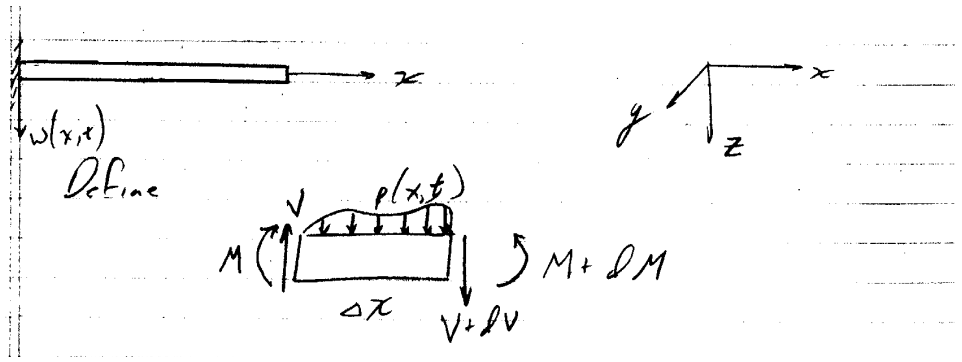
$$\sin \frac{n\pi}{2} = \begin{cases} -1 & n = 3, 7, 11, \ldots \\ 0 & n = 2, 4, 6, \ldots \\ 1 & n = 1, 5, 9, \ldots \end{cases}$$
$$= \left(\frac{1}{2} \left(-1^n - 1 \right) \right)^{\frac{n-1}{2}}$$

Then

$$w(x,t) = \sum_{n=1}^{\infty} \frac{200 \sin \frac{n\pi}{2}}{\ell \left(c^2 \left(\frac{n\pi}{\ell}\right)^2 - 1 \right)} \sin t \sin n\pi \frac{x}{\ell}$$

[10] We know that undamped systems always have a phase difference of 0° or 180°.

6.2 Bending Vibration of a Beam



$$M = -EI(x)\frac{\partial^2 w}{\partial x^2}$$

[11]

$$V = \frac{\partial M}{\partial x} = \frac{\partial}{\partial x}\left(-EI(x)\frac{\partial^2 w}{\partial x^2}\right) \tag{134}$$

$$= -E\frac{\partial I(x)}{\partial x}\frac{\partial^2 w}{\partial x^2} - EI(x)\frac{\partial^3 w}{\partial x^3}$$

$$\sum F_z = \Delta x \rho(x)\ddot{w} \qquad \text{(linear density)}$$

$$(V + \partial V) - V + p(x,t)\Delta x = \Delta x \rho(x)\ddot{w}$$

dividing by Δx and taking the limit as $\Delta x \to 0$

$$\frac{\partial V}{\partial x} + p(x,t) = \rho(x)\ddot{w}$$

substituting for V

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right) + \rho(x)\frac{\partial^2 w}{\partial t^2} = p(x,t)$$

Taking moments about the right end

$$-M + (M + \partial M) - V\Delta x + p(x,t)\Delta x\frac{\Delta x}{2} = 0$$

[11] Derive this by assuming plane sections remain plane, linear elasticity, and no Poisson effect.

dividing by Δx

$$\frac{\partial M}{\Delta x} - V + p\frac{\Delta x}{2} = 0$$

Taking the limit as $\Delta x \to 0$

$$\frac{\partial M}{\partial x} = V$$

as was stated in equation (134).
Allowable Boundary Conditions:
Clamped: w is known and w' is known.
Pinned: w is known and M is known.
Free: V is known and M is known.
Sliding end: w' is known and V is known.

6.2.1 Example: Mode shapes of cantilever beam

Find the mode shapes of a cantilever beam
By separation of variables

$$\ddot{T}X + \frac{EI}{\rho}TX'''' = 0 \tag{135}$$

$$\frac{\ddot{T}}{T} = \frac{-EI}{\rho}\frac{X''''}{X} = \frac{-EI}{\rho}\beta^4 = -\omega^2 \tag{136}$$

$$X'''' = \frac{\rho}{EI}\omega^2 X = \beta^4 X \tag{137}$$

or

$$X'''' - \beta^4 X = 0 \tag{138}$$

There must be 4 independent solutions to a 4th order homogeneous ODE (see Wronskian in an introductory DE text).
Four functions that satisfy this equation are sin, cos, sinh, and cosh. The total solution is then

$$X(x) = A\sin(\beta x) + B\cos(\beta x) + C\sinh(\beta x) + D\cosh(\beta x) \tag{139}$$

The boundary conditions for a cantilever beam are

$$X(x)\Big|_{x=0} = 0, \qquad \text{displacement is zero} \tag{140a}$$

$$X'(x)\Big|_{x=0} = 0, \qquad \text{slope is zero} \tag{140b}$$

$$X''(x)\Big|_{x=l} = 0, \qquad \text{moment is zero} \tag{140c}$$

$$X'''(x)\Big|_{x=l} = 0, \qquad \text{shear is zero} \tag{140d}$$

Applying equation (140a)

$$B + D = 0 \tag{141}$$

Applying equation (140b)

$$A\beta + C\beta = 0 \tag{142}$$

Applying equation (140c)

$$-A\sin(\beta l)\beta^2 - B\cos(\beta l)\beta^2 + C\sinh(\beta l)\beta^2 + D\cosh(\beta l)\beta^2 = 0 \tag{143}$$

Using equations (141) and (142) and simplifying (by substituting for C and D)

$$A\left(\sin(\beta l) + \sinh(\beta l)\right) + B\left(\cos(\beta l) + \cosh(\beta l)\right) = 0 \tag{144}$$

Then applying equation (140d)

$$-A\cos(\beta l)\beta^3 + B\sin(\beta l)\beta^3 + C\cosh(\beta l)\beta^3 + D\sinh(\beta l)\beta^3 = 0 \tag{145}$$

Using equations (141) and (142) and simplifying

$$A\left(\cos(\beta l) + \cosh(\beta l)\right) - B\left(\sin(\beta l) - \sinh(\beta l)\right) = 0 \tag{146}$$

Combining equations (144) and (146)

$$\begin{bmatrix} \left(\sin(\beta l) + \sinh(\beta l)\right) & \left(\cos(\beta l) + \cosh(\beta l)\right) \\ \left(\cos(\beta l) + \cosh(\beta l)\right) & -\left(\sin(\beta l) - \sinh(\beta l)\right) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{147}$$

Setting the determinant of the matrix equal to zero gives

$$-\cos^2(\beta l) - 2\cosh(\beta l)\cos(\beta l) - \cosh^2(\beta l) - \sin^2(\beta l) + \sinh^2(\beta l) = 0 \tag{148}$$

Since $\sin^2(\beta l) + \cos^2(\beta l) = 1$ and $\cosh^2(\beta l) - \sinh^2(\beta l) = 1$, this simplifies to

$$\cos(\beta l)\cosh(\beta l) = -1 \tag{149}$$

Having a value for βl we can use equation (146) to obtain the ratio between A and B as

$$\frac{A}{B} = \sigma_n = \frac{\sin(\beta l) - \sinh(\beta l)}{\cosh(\beta l) + \cos(\beta l)} \tag{150}$$

Not that this could also be solved for using equation (144)

$$\frac{A}{B} = \sigma_n = \frac{\cos(\beta l) + \cosh(\beta l)}{\sin(\beta l) + \sinh(\beta l)} \tag{151}$$

These are equivalent expressions for the values of βl that satisfy equation 149. It is left to the reader to prove this.
Thus the mode shape is

$$X_n(x) = A_n \left((\cos(\beta x) - \cosh(\beta x)) + \sigma_n \left(\sin(\beta x) - \sinh(\beta x) \right) \right) \tag{152}$$

6.2.2 Example: Forced response of cantilever beam

Find the response of a cantilever beam (uniform) to $F \sin \omega t$ at its free end
The E.O.M. is

$$\rho w_{tt} + EI w_{xxxx} = F_o \delta(x - \ell) \sin \omega t$$

Where $\delta(x - \ell)$ is the Dirac delta function
Assume a solution of the form

$$w(x,t) = \Sigma_{n=1}^{\infty} w_n(x,t) = \Sigma_{n=1}^{\infty} X_n(x) T_n(t) \tag{153}$$

The mode shapes for a fixed-free beam are given by equation (152), σ_n are given by equation (150) and $\beta_n l$ are obtained by solving equation (149). Natural frequencies are then given by

$$X_n'''' = \frac{\rho}{EI} \omega_n^2 X_n = \beta_n^4 X_n \tag{154}$$

Substituting equation (153) into the equation of motion

$$\sum_{n=1}^{\infty} \left(\ddot{T}_n + \omega_n^2 T_n \right) X_n = \frac{F_0}{\rho} \delta(x - \ell) \sin \omega t \tag{155}$$

The mode shapes can be normalized by

$$\int_o^{\ell} X_n^2 dx = (X_n, X_n) = 1$$

$$A_n^2 \int_o^{\ell} \left(\cosh \beta_n x - \cos \beta_n x - \sigma_n \left(\sinh \beta_n x - \sin \beta_n x \right) \right)^2 dx = 1$$

$$\begin{aligned}A_n = 4\beta_n(&4\beta_n\ell + 2\sigma_n\cos(2\beta_n\ell) - 2\sigma_n\cosh(2\beta_n\ell)\\ &- 4\cosh(\beta_n\ell)\sin(\beta_n\ell) - 4\sigma_n^2\cosh(\beta_n\ell)\\ &+ \sin 2\beta_n\ell - \sigma_n^2\sin 2\beta_n\ell - 4\cos\beta_n\ell\sinh\beta_n\ell\\ &+ 4\sigma_n^2\cos(\beta_n\ell)\sinh(\beta_n\ell) + 8\sigma_n\sin(\beta_n\ell)\sinh(\beta_n\ell)\\ &+ \sinh(2\beta_n\ell) + \sigma_n^2\sinh(2\beta_n\ell))^{-1}\end{aligned}$$

Multiply the EOM by X_m, integrating over x, and using

$$\int_0^\ell X_n X_m dx = \begin{cases}0 & m \neq n\\ 1 & m = n\end{cases}$$

$$\begin{aligned}\ddot{T}_n + \omega_n^2 T_n &= \frac{F_0}{\rho}\int_0^\ell X_n(x)\delta(x-\ell)dx\sin(\omega t)\\ &= \frac{F_o}{\rho}X_n(\ell)\sin(\omega t)\end{aligned}$$

Solving for T_n

$$T_n = \frac{F_o}{\rho A}\frac{X_n(\ell)}{\omega_n^2 - \omega^2}\sin(\omega t)$$

The total solution is

$$w(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x)$$

Recall ω_n was given as

$$\omega_n = \beta_n^2\sqrt{\frac{EI}{\rho}}$$

where β_n can be found from

$$\cos\beta_n\ell\cosh\beta_n\ell = -1$$

6.3 Nondimensionalizing E.O.M.

Consider the EOM of a rod (extension or torsion), or a string

$$c^2 w_{xx}(x,t) = w_{tt}(x,t)$$

$$c = \sqrt{\frac{\tau}{\ell}}$$

Let

$$\xi = \frac{x}{\ell}, \text{ then } \frac{dx}{d\xi} = \ell$$

$$\frac{\partial w}{\partial \xi} = \frac{\partial w}{\partial x}\frac{dx}{d\xi} = \frac{\partial w}{\partial x}\ell$$

or

$$\frac{\partial w}{\partial x} = \frac{1}{\ell}\frac{\partial w}{\partial \xi}$$

Also,

$$\begin{aligned}
\frac{\partial^2 w}{\partial \xi^2} &= \frac{\partial}{\partial \xi}\left(\frac{\partial w}{\partial \xi}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial \xi}\right)\frac{dx}{d\xi} \\
&= \ell\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial \xi}\right) \\
&= \ell\frac{\partial}{\partial x}\left(\ell\frac{\partial w}{\partial x}\right) \\
&= \ell^2\frac{\partial^2 w}{\partial x^2}
\end{aligned}$$

So

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{\ell^2}\frac{\partial^2 w}{\partial \xi^2}$$

Substituting into E.O.M.

$$\frac{c^2}{\ell^2} w_{\xi\xi}(\xi, t) = w_{tt}(\xi, t)$$

Similarly, let

$$\gamma = \frac{ct}{\ell} \qquad \text{(dimensionless time)}$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{c^2}{\ell^2}\frac{\partial^2 w}{\partial \gamma^2}$$

Substituting into the E.O.M.

$$w_{\xi\xi}(\xi, \gamma) = w_{\gamma\gamma}(\xi, \gamma)$$

define $y = \frac{w}{\ell}$

$$y_{\xi\xi}(\xi, \gamma) = y_{\gamma\gamma}(\xi, \gamma)$$

Which represents the nondimensionalized E.O.M.

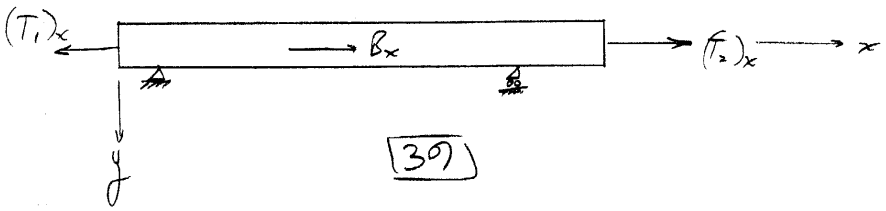
7 Energy Methods

7.1 Virtual Work (Shames Solid Mechanics) , p. 62 our book (discrete) and p. 369, eqn. 7.20 our book

Virtual work is the work done on a particle by all the forces acting on the particle as the particle is given a small hypothetical displacement, a "virtual displacement," which is consistent with the constraints present. Applied forces are constant during the virtual displacement.
The virtual work acting on a body is

$$\delta W_{virt} = \underbrace{\iiint_V \vec{B} \cdot \delta\vec{u} dV}_{\text{Body forces}} + \underbrace{\iint_S \vec{T} \cdot \delta\vec{u} dA}_{\text{Traction forces}}$$

$\delta\vec{u}$ is a virtual displacement field that satisfies the boundary conditions.
Consider the virtual work for a virtual displacement field ($\delta u_x(x)$) in the following:



$$\begin{aligned}\delta W &= \iiint_V B_x \delta u_x dV + \left(\iint_S T_x \delta u_x dA\right)_2 - \left(\iint_S T_x \delta u_x dA\right)_1 \\ &= \iiint_V B_x \delta u_x dV + \iint_S \left((T_x \delta u_x)_2 - (T_x \delta u_x)_1\right) dA\end{aligned}$$

Note:

$$(T_x \delta u_x)_2 - (T_x \delta u_x)_1 = \int_1^2 \frac{d}{dx}\left(T_x \delta u_x\right) dx$$

Note that T_x is force/unit area. Thus $T_x = \sigma_{xx}$ (Direct stress in the x direction). Thus

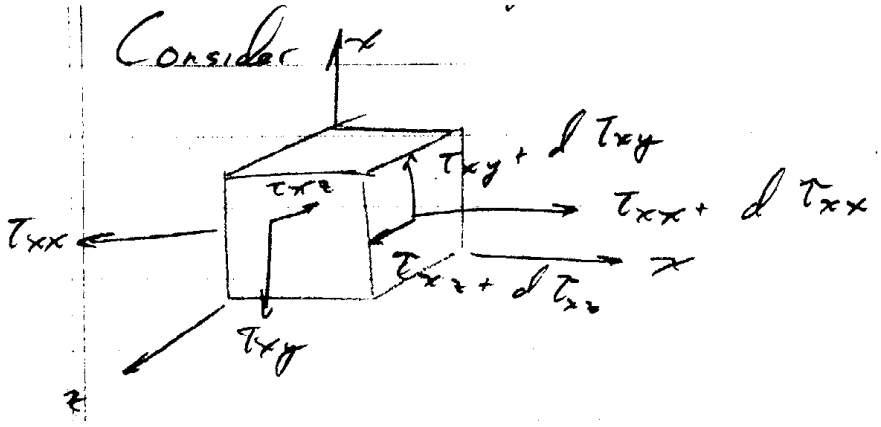
$$\delta W = \iiint_V B_x \delta u_x dV + \iiint_V \frac{d}{dx}\left(\sigma_{xx} \delta u_x\right) dV$$

Carrying out the differentiation and collecting terms

$$\delta W = \iiint_V \left(\left(B_x + \frac{d\sigma_{xx}}{dx} \right) \delta u_x + \sigma_{xx} \frac{d}{dx} \delta u_x \right) dV$$

7.1.1 Review from Strength of Materials

Consider



$$\begin{aligned}\sum F_x = &\left(\sigma_{xx} + d\sigma_{xx}\right) dydz - \sigma_{xx} dydz \\ &+ \left(\sigma_{yx} + d\sigma_{yx}\right) dxdz - \sigma_{yx} dxdz \\ &+ \left(\sigma_{zx} + d\sigma_{zx}\right) dxdy - \sigma_{zx} dxdy \\ &+ B_x dxdydz = 0\end{aligned}$$

Dividing by $dxdydz$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + B_x = 0$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + B_y = 0$$
$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + B_z = 0$$

(Continue derivation) The first term in the integrand is zero. It is the equation of equilibrium for an element.
Thus

$$\delta W = \iiint_V \left(\sigma_{xx} \left(\delta \underbrace{\frac{du_x}{dx}}_{\epsilon_x} \right) \right) dV$$

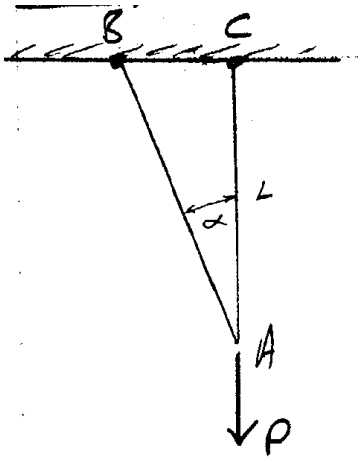
More generally

$$\iiint_V \vec{B} \cdot \delta\vec{u} dV + \iint_S \vec{T} \cdot \delta\vec{u} dA = \iiint_V \sigma_{ij} \delta\epsilon_{ij} dV$$

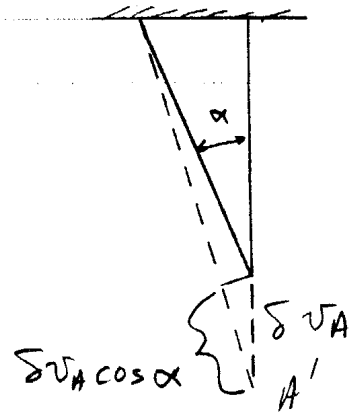
where

$$\sigma_{ij}\delta\epsilon_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_{ij}\delta\epsilon_{ij}$$

7.1.2 Example



Members AB and AC have the same modulus of elasticity and cross-sectional area. Find the forces in AB and AC and deflection of pin.
1st: Apply virtual displacement in y direction



$$\delta\epsilon_{AC} = \frac{\delta v_A}{L}$$

$$\delta\epsilon_{AB} = \frac{\delta v_A \cos\alpha}{\frac{L}{\cos\alpha}} = \frac{\delta v_A \cos^2\alpha}{L}$$

With no body forces, the principle of virtual work is

$$P\delta v_A = \int_0^{L/\cos\alpha} \sigma_{AB}\delta\epsilon_{AB} A d\ell + \int_0^{L} \sigma_{AC}\delta\epsilon_{AC} A d\ell$$

substituting for the strains

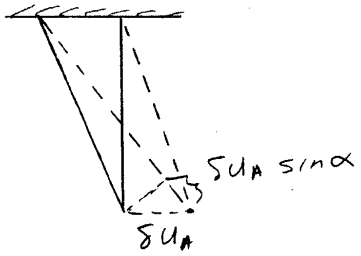
$$P\delta v_A = \int_0^{L/\cos\alpha} \sigma_{AB}\delta v_A \frac{\cos^2\alpha}{L} A d\ell + \int_0^{L} \sigma_{AC}\delta v_A \frac{A}{L} d\ell$$

Since δv_A is non-zero, integrating gives

$$P = \sigma_{AB}\frac{\cos^2\alpha}{L} A \frac{L}{\cos\alpha} + \sigma_{AC} A$$

$$P = \sigma_{AB}\cos\alpha A + \sigma_{AC} A \tag{156}$$

2nd: Apply virtual displacement in x direction.



$$\delta\epsilon_{AC} = 0$$

$$\delta\epsilon_{AB} = \frac{\delta u_A \sin\alpha}{L\cos\alpha} = \frac{\delta u_A}{L}\sin\alpha\cos\alpha$$

The principle of virtual work

$$0\delta u_A = \int_0^{L/\cos\alpha} \sigma_{AB}\frac{\delta u_A}{L}\sin\alpha\cos\alpha A d\ell + \int_0^{L} \sigma_{AC} 0 A d\ell$$

$$0 = \sigma_{AB}(\sin\alpha\cos\alpha)A \tag{157}$$

$$\sigma_{AB} = 0$$

$$\sigma_{AC} = \frac{P}{A}$$

Now let's find the movement of the pin. Consider actual displacements $\bar{u}_A$ and $\bar{v}_A$

$$\epsilon_{AC} = \frac{\bar{v}_A}{L} \qquad \epsilon_{AB} = \frac{\bar{v}_A \cos^2\alpha}{L} + \frac{\bar{u}_A}{L}\sin\alpha\cos\alpha$$

Using Hooke's Law

$$\sigma_{AB} = E\epsilon_{AB} = \frac{E}{L}\left(\bar{v}_A\cos^2\alpha + \bar{u}_A\sin\alpha\cos\alpha\right)$$

$$\sigma_{AC} = E\epsilon_{AC} = \frac{E}{L}\left(\bar{v}_A\right)$$

Substituting the stresses into the virtual work equations (156) and (157):

$$P = \frac{E}{L}\left(\bar{v}_A\cos^2\alpha + \bar{u}_A\sin\alpha\cos\alpha\right)\cos\alpha + \frac{E}{L}\bar{v}_A A \tag{158}$$

$$0 = \frac{E}{L}\underbrace{\left(\bar{v}_A\cos^2\alpha + \bar{u}_A\sin\alpha\cos\alpha\right)}_{\text{must be 0 because: } \sin\alpha\neq 0,\ \cos\alpha\neq 0,\ A\neq 0} A\sin\alpha\cos\alpha \tag{159}$$

From (158)

$$P = \frac{E}{L}\bar{v}_A A$$

$$\bar{v}_A = \frac{PL}{EA}$$

From (159)

$$\frac{PL}{EA}\cos\alpha + \bar{u}_A\sin\alpha = 0$$

$$\bar{u}_A = \frac{-PL\cos\alpha}{EA\sin\alpha}$$

7.2 Derivation of Hamilton's Principle from Virtual Work

Consider Newton's Law for a particle

$$\sum \vec{F} = m\vec{a}$$

Rearrange it as

$$\sum \vec{F} + -m\vec{a} = 0$$

The form is now that of a statics problem.
This is called D'Alembert's principle.
For a body, this force is

$$-m\vec{a} = -\iiint_V \rho \frac{d^2 u_i}{dt^2} dV$$

and the virtual work due to this force is

$$-\iiint_V \rho \frac{d^2 u_i}{dt^2} \delta u_i dV$$

The Principle of Virtual Work is now

$$\iiint_V B_i \delta u_i dV + \iint_S T_i \delta u_i dA - \iiint_V \rho \frac{d^2 u_i}{dt^2} \delta u_i dV = \iiint_V \sigma_{ij} \delta \epsilon_{ij} dV \quad (160)$$

Define the work of the external forces (the *Nonconservative Work*)

$$W_{nc} = \iiint_V B_i u_i dV + \iint_S T_i u_i dA$$

This is also the negative of the "Force Potential"[12]. The variation of the nonconservative work is

$$\delta W_{nc} = \iiint_V B_i \delta u_i dV + \iint_S T_i \delta u_i dA$$

Assume the existence of a strain energy density function $\mathcal{U}$ such that

$$\sigma_{ij} = \frac{\partial \mathcal{U}}{\partial \epsilon_{ij}}$$

[12] Potential energy of externally applied loads.

then

$$\iiint_V \sigma_{ij}\delta\epsilon_{ij}dV = \delta \iiint_V \mathcal{U}dV = \delta V$$

So, the principle of virtual work is

$$-\iiint_V \rho\frac{d^2u_i}{dt^2}\delta u_i dV + \delta W_{nc} - \delta V = 0 \tag{161}$$

To make this apply for a range of times we integrate with respect to time.

$$-\int_{t_1}^{t_2}\iiint_V \rho\frac{d^2u_i}{dt^2}\delta u_i dV dt + \int_{t_1}^{t_2}\delta W_{nc}dt - \int_{t_1}^{t_2}\delta V dt = 0 \tag{162}$$

Consider the first term. We can swap the order of the integrations, and thus

$$-\int_{t_1}^{t_2}\iiint_V \rho\frac{d^2u_i}{dt^2}\delta u_i dV dt = -\iiint_V \left(\int_{t_1}^{t_2}\rho\frac{d^2u_i}{dt^2}\delta u_i dt\right) dV$$

Integrating in time by parts

$$\int_{t_1}^{t_2}\frac{d^2u_i}{dt^2}\delta u_i dt = \frac{du_i}{dt}\delta u_i|_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{du_i}{dt}\frac{d}{dt}\left(\delta u_i\right)dt$$

Then

$$\int_{t_1}^{t_2}\frac{d^2u_i}{dt^2}\delta u_i dt = \frac{du_i}{dt}\delta u_i|_{t_1}^{t_2} - \int_{t_1}^{t_2}\delta\frac{1}{2}\left(\frac{du_i}{dt}\right)^2 dt$$

Substituting into (162)

$$\iiint_V \frac{\rho}{2}\int_{t_1}^{t_2}\delta\dot{u}_1^2 dt dV - \frac{du_i}{dt}\delta u_i|_{t_1}^{t_2} + \delta\int_{t_1}^{t_2}W_{nc}dt - \delta\int_{t_1}^{t_2}V dt = 0 \tag{163}$$

The first term can be written

$$\int_{t_1}^{t_2}\left(\delta\underbrace{\iiint_V \frac{1}{2}\rho\dot{u}_i^2 dV}_{T}\right)dt = \int_{t_1}^{t_2}\delta T dt = \delta\int_{t_1}^{t_2}T dt \tag{164}$$

substituting into (163)

$$\delta\int_{t_1}^{t_2}\left(T - V + W_{nc}\right)dt - \frac{du_i}{dt}\delta u_i|_{t_1}^{t_2} = 0$$

Where we typically adopt the rules that the variation $\delta u_i = 0$ at $t = t_1$, and $t = t_2$.
V is the total potential energy. Then

$$T - V = L \qquad (Lagrangian)$$

Thus

$$\delta \int_{t_1}^{t_2} L + W_{nc} dt = 0$$

This is *Hamilton's Principle*
Note

$$\delta V = -\delta W_c$$

That is, the potential energy of the system is negative of the conservative work that it has *done*. e.g. compressing a spring a distance x, the spring stores $V = \frac{1}{2}kx^2$, but the work done *by* the spring is $W_c = -\frac{1}{2}kx^2$.

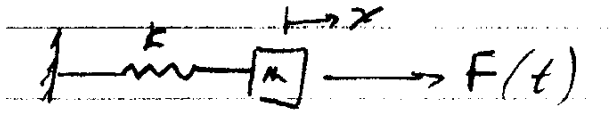
7.2.1 Example

Hamilton's Principle (Shames 112-114, 323-329: Energy and Finite Element Methods in Structural Mechanics)

$$\int_{t_1}^{t_2} (\delta L + \delta W_{nc})\, dt = 0$$

$$L = T - V = T + W_c$$

Example: SDOF system



$$T = \frac{1}{2} m \dot{x}^2, \qquad \delta T = m \dot{x} \delta \dot{x}$$

$$W_c = -\frac{1}{2} k x^2, \qquad \delta W_c = -kx\delta x$$

$$W_{nc} = Fx, \qquad \delta W_{nc} = F\delta x$$

$$\int_{t_1}^{t_2} (\delta T + \delta W_c + \delta W_{nc})\, dt = \int_{t_1}^{t_2} (m\dot{x}\delta\dot{x} - kx\delta x + F\delta x)\, dt$$

$$= \underbrace{\int_{t_1}^{t_2} m\dot{x}\delta\dot{x}dt}_{\text{integrate this one by parts}} + \int_{t_1}^{t_2} (F - kx)\, \delta x dt = 0$$

Integrating the first term by parts

$$\int_{t_1}^{t_2} m\dot{x}\delta\dot{x}dt$$

$$u = m\dot{x} \qquad dv = \delta\dot{x}dt$$

$$du = m\ddot{x}dt \qquad v = \delta x$$

$$= m\dot{x}\delta x\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} m\ddot{x}\delta x dt$$

δx is zero at t_1 and t_2 (x is known, we are looking for the equation describing the trajectory between these two times)

Substituting back

$$\int_{t_1}^{t_2} (-m\ddot{x} + F - kx)\, \delta x dt = 0$$

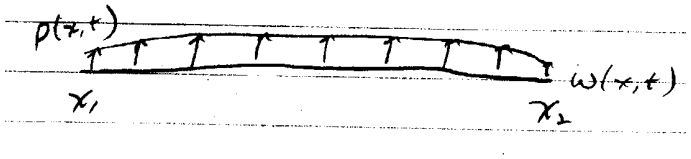
Since δx is arbitrary, this can be satisfied only if

$$-m\ddot{x} + F - kx = 0$$

for all time
Thus, the EOM is

$$m\ddot{x} + kx = F(t)$$

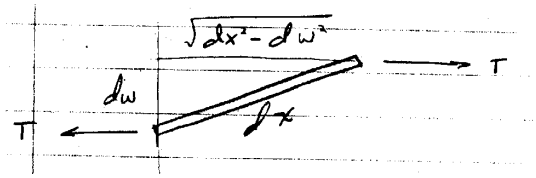
7.2.2 Example 2: String with tension T



The Kinetic energy is

$$T = \int_{x_1}^{x_2} \frac{1}{2} \rho \dot{w}^2 (x,t)\, dx$$

Conservative work



The shortening is $dx - \sqrt{dx^2 - dw^2}$

$$= dx - dx\sqrt{1 - \left(\frac{dw}{dx}\right)^2}$$

$$\approx \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx$$

Per unit length, $W_c = -\frac{1}{2}\left(\frac{dw}{dx}\right)^2 T dx$

Nonconservative work

$$W_{nc} = p\left(x,t\right) w\left(x,t\right)$$

Applying Hamilton's Principle

$$\int_{t_1}^{t_2} \delta T + \delta W_c + \delta W_{nc} dt = 0$$

$$\delta \left(\int_{t_1}^{t_2} \left(\int_{x_1}^{x_2} \frac{1}{2} \rho \dot{w}^2 \left(x,t\right) dx + \int_{x_1}^{x_2} -\frac{1}{2} \left(\frac{dw}{dx} \right)^2 T + p(x,t) w dx \right) dt \right) = 0$$

$$\int_{t_1}^{t_2} \left(\int_{x_1}^{x_2} \rho \dot{w} \delta \dot{w} dx + \int_{x_1}^{x_2} \left(-T \underbrace{w'}_{\frac{dw}{dx}} \delta w' + p(x,t) \right) \delta w dx \right) dt = 0$$

The order of the integrations is not important so the first term is

$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} \rho \dot{w} \delta \dot{w} dt dx$$

Integrating by parts with respect to time yields

$$\int_{x_1}^{x_2} \left(\rho \dot{w} \delta w \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \rho \ddot{w} \delta w dt \right) dx$$

Recall that we prescribe $\delta w = 0$ at $t = t_1$ and t_2
The total integral is then

$$\int_{t_1}^{t_2} \left(\int_{x_1}^{x_2} -\rho \ddot{w} \delta w dx + \int_{x_1}^{x_2} -T w' \delta w' dx + \int_{x_1}^{x_2} p(x,t) \delta w dx \right) dt = 0$$

Let's now consider the 2nd term

$$\int_{x_1}^{x_2} -T w' \delta w' dx$$

$$u = -T w' \qquad dv = \delta w' dx$$

$$du = -T w'' dx \qquad v = \delta w$$

$$= -T w' \delta w \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} -T w'' \delta w dx$$

Substituting and collecting terms

$$\int_{t_1}^{t_2} \left(-Tw'\delta w \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left(Tw'' - \rho\ddot{w} + p(x,t) \right) \delta w dx \right) dt = 0$$

The first term yields the boundary conditions:
At x_1:
Either $\delta w = 0$ (w is known) or $-Tw' = 0$
At x_2:
Either $\delta w = 0$ (w is known) or $-Tw' = 0$
Actually, the true boundary conditions are such that $-Tw'$ is known. How to incorporate this into Hamilton's principle follows.
The second term must be zero for any arbitrary δw for all time. Thus

$$T\frac{d^2w}{dx^2} + p\left(x,t\right) = \rho\frac{d^2w}{dt^2}$$

(see 6.158 Inman) The ability to have non fixed ends with non-zero forces is accomplished by adding nonconservative work terms at each end of the string to the total work. i.e.

$$W = \int_{x_1}^{x_2} \left(-\frac{1}{2}\left(\frac{dw}{dx}\right)^2 T + p(x,t)w \right) dx + p_1^* w_1 + p_2^* w_2$$

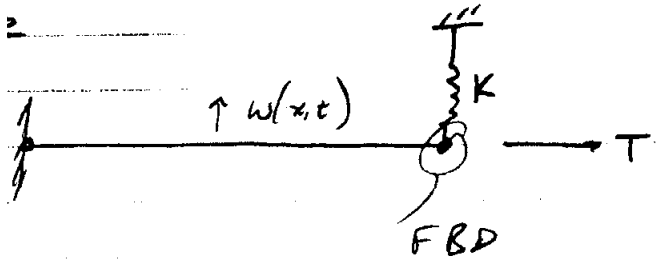
Thus the 1st term of the final form of Hamilton's equation becomes

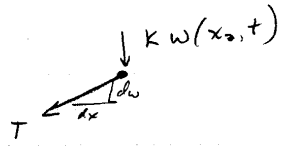
$$-Tw'\delta w \Big|_{x_1}^{x_2} + p_1^*\delta w_1 + p_2^*\delta w_2 = 0$$

$$\left(Tw_1' + p_1^*\right)\delta w_1 + \left(-Tw_2' + p_2^*\right)\delta w_2 = 0$$

So $Tw_1' = -p_1^*$ at x_1 is a valid B.C.
and $Tw_2' = p_2^*$ at x_2 is a valid B.C.

7.2.3 String Boundary Condition Example



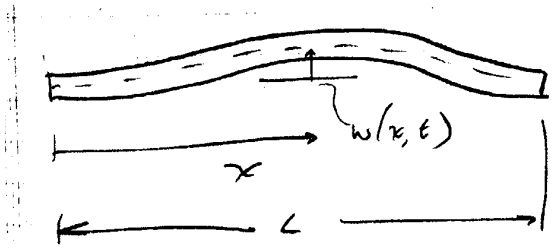


$$\sum F_y = 0 = -T\frac{dw}{dx} - kw \text{ at } x_2$$

Note: small angle approximation $\tan\theta \approx \sin\theta \approx \theta$
Using the result from Hamilton's principle

$$Tw_2' = -kw_2$$

7.3 Lagrange's Equation for a Continuous System



Kinetic Energy:

$$T(t) = \int_0^\ell \underbrace{\hat{T}(x,t)}_{\text{kinetic energy density}} \quad dx$$

$\hat{T}(x,t)$ will have the form $\hat{T}(\dot{w}, \dot{w}')$
$\hat{T}$ depends on x and t because $w\,(x,t)$
Virtual Work:

$$\delta W(t) = \int_0^\ell \delta\hat{W}(x,t)dx$$

$$\delta\hat{W}(x,t) = \delta\hat{W}_c(x,t) + \delta\hat{W}_{nc}(x,t)$$

$$\delta\hat{W}_c(x,t) = -\delta\hat{V}(x,t) \qquad \hat{V} : \text{potential energy density}$$

$$\hat{V}(x,t) = \hat{V}\,(w, w', w'')$$

$$\delta\hat{W}_{nc}(x,t) = p(x,t)\delta w(x,t)$$

Hamilton's Principle states

$$\int_{t_1}^{t_2} \delta T + \delta W dt = 0 \qquad \delta W = 0 \,@\, t = t_1, t_2 \tag{165}$$

or

$$\int_{t_1}^{t_2} \int_0^{\ell} \delta \hat{T} + \delta \hat{W} dx dt = 0 \qquad \delta W = 0 \,@\, t = t_1, t_2 \tag{166}$$

$$\int_{t_1}^{t_2} \int_0^{\ell} \delta \hat{L} + \delta W_{nc} dx dt = 0 \qquad \delta W = 0 \,@\, t = t_1, t_2 \tag{167}$$

where $\delta \hat{L} = \delta \hat{T} - \delta \hat{V}$ is the variation of the Lagrangian Density

$$\delta \hat{L} = \frac{\partial \hat{L}}{\partial w} \delta w + \frac{\partial \hat{L}}{\partial w'} \delta w' + \frac{\partial \hat{L}}{\delta w''} \delta w'' + \frac{\partial L}{\partial \dot{w}} \delta \dot{w} + \frac{\partial L}{\delta \dot{w}'} \delta \dot{w}' \tag{168}$$

substituting (168) into (167) and using the definition of δW_{nc}

$$\int_{t_1}^{t_2} \int_0^{\ell} \left(\frac{\partial \hat{L}}{\partial w} \delta w + \frac{\partial \hat{L}}{\partial w'} \delta w' - \frac{\partial \hat{L}}{\partial w''} \delta w'' + \frac{\partial \hat{L}}{\partial \dot{w}} \delta \dot{w} + \frac{\partial \hat{L}}{\partial \dot{w}'} \delta \dot{w}' + p(x,t) \delta w \right) dx = 0 \tag{169}$$

Like we did using Hamilton's Principle before, we want the integrand to contain only δw, so we integrate by parts.

a)

$$\int_0^{\ell} \frac{\partial \hat{L}}{\partial w'} \delta w' dx = \frac{\partial \hat{L}}{\partial w'} \delta w \Big|_0^{\ell} - \int_0^{\ell} \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w'} \right) \delta w dx$$

b)

$$\int_0^{\ell} \frac{\partial \hat{L}}{\partial w''} \delta w'' dx = \frac{\partial \hat{L}}{\partial w''} \delta w' \Big|_0^{\ell} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w''} \right) \delta w \Big|_0^{\ell} + \int_0^{\ell} \frac{\partial^2}{\partial x^2} \left(\frac{\partial \hat{L}}{\partial w''} \right) \delta w dx$$

c)

$$\int_{t_1}^{t_2} \frac{\partial \hat{L}}{\partial \dot{w}} \delta \dot{w} dx = \frac{\partial \hat{L}}{\partial \dot{w}} \delta w \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}} \delta w \right) dt$$

d)

$$\int_{t_1}^{t_2}\int_0^{\ell}\frac{\partial\hat{L}}{\partial\dot{w}'}\delta\dot{w}'dxdt = \int_{t_1}^{t_2}\left(\frac{\partial\hat{L}}{\partial\dot{w}'}\delta\dot{w}\Big|_0^{\ell} - \int_0^{\ell}\frac{\partial}{\partial x}\left(\frac{\partial\hat{L}}{\partial\dot{w}'}\right)\delta\dot{w}dx\right)dt$$
$$= \frac{\partial\hat{L}}{\partial\dot{w}'}\delta w\Big|_0^{\ell}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\left(\frac{\partial}{\partial t}\frac{\partial\hat{L}}{\partial\dot{w}'}\delta w\Big|_0^{\ell}\right)dt$$
$$-\left(\int_0^{\ell}\frac{\partial}{\partial x}\left(\frac{\partial\hat{L}}{\partial\dot{w}'}\right)\delta w dx\right)_{t_1}^{t_2}$$
$$+\int_{t_1}^{t_2}\int_0^{\ell}\frac{\partial^2}{\partial t\partial x}\left(\frac{\partial\hat{L}}{\partial\dot{w}'}\right)\delta w dxdt$$

Substituting into (169) and recalling $\delta w = 0$ at $t = t_1, t_2$

$$\int_{t_1}^{t_2}\left[\int_0^{\ell}\left(\frac{\partial\hat{L}}{\partial w} - \frac{\partial}{\partial x}\frac{\partial\hat{L}}{\partial w'} + \frac{\partial^2}{\partial x^2}\frac{\partial\hat{L}}{\partial w''} - \frac{\partial}{\partial t}\left(\frac{\partial\hat{L}}{\partial\dot{w}}\right) + \frac{\partial^2}{\partial x\partial t}\left(\frac{\partial\hat{L}}{\partial\dot{w}'}\right) + p(x,t)\right)\delta w dx\right.$$
$$\left. + \left(\frac{\partial\hat{L}}{\partial w'} - \frac{\partial}{\partial x}\left(\frac{\partial\hat{L}}{\partial w''}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\dot{w}'}\right)\right)\delta w\Big|_0^{\ell} + \frac{\partial\hat{L}}{\delta w''}\delta w'\Big|_0^{\ell}\right]dt = 0 \tag{170}$$

Since δw is arbitrary

$$\frac{\partial\hat{L}}{\partial w} - \frac{\partial}{\partial x}\frac{\partial\hat{L}}{\partial w'} + \frac{\partial^2}{\partial x^2}\frac{\partial\hat{L}}{\partial w''} - \frac{\partial}{\partial t}\frac{\partial\hat{L}}{\partial\dot{w}} + \frac{\partial^2}{\partial x\partial t}\left(\frac{\partial\hat{L}}{\partial\dot{w}'}\right) + p(x,t) = 0$$

with the boundary conditions

$$\frac{\partial\hat{L}}{\partial w'} - \frac{\partial}{\partial x}\left(\frac{\partial\hat{L}}{\partial w''}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\dot{w}'}\right) = 0$$

or

$$w \quad \text{is known}$$

AND

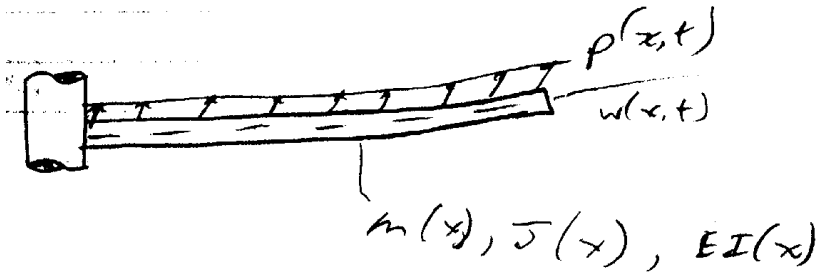
$$\frac{\partial\hat{L}}{\partial w''} = 0$$

or

$$w' \quad \text{is known at each end}$$

7.3.1 Example: Beam is bending on spinning shaft

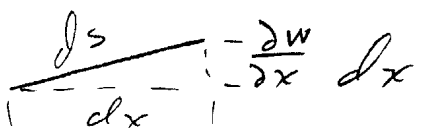
Shaft is spinning about vertical axis at angular velocity Ω. Derive the equations of motion.



$$T(t) = \underbrace{\frac{1}{2}\int_0^\ell m(x)\dot{w}^2 dx}_{\text{Kinetic energy in translation}} + \underbrace{\frac{1}{2}\int_0^\ell J(x)\dot{w}'^2 dx}_{\text{Kinetic energy in rotation}}$$

$$V(t) = \frac{1}{2}\int_0^\ell EI(x)w''^2 dx + \int_0^\ell P(x,t)(ds - dx)$$

$P(x,t)$ is axial force (centrifugal effects)



$ds - dx$ is the shortening of the beam

$$ds - dx = \left(\left(dx^2\right) + \left(\frac{\partial w}{\partial x}dx\right)^2\right)^{\frac{1}{2}} - dx \approx \frac{1}{2}w'^2 dx$$

The axial load is assumed to be

$$P(x,t) = P(x) = \int_x^\ell m\left(\xi\right)\Omega^2 \xi d\xi$$

The potential energy can now be written

$$V(t) = \frac{1}{2}\int_0^\ell EI(x)w''^2 dx + \frac{1}{2}\int_0^\ell \left(\int_x^\ell m\left(\xi\right)\Omega^2 \xi d\xi\right) w'^2 dx$$

Consider $p(x,t)$ (external transverse load) to be the weight of the blade

$$\delta W_{nc} = -\int_0^\ell m(x) g \delta w(x,t) dx$$

$\hat{L}$ is then

$$\hat{L} = \frac{1}{2} m \dot{w}^2 + \frac{1}{2} J \left(\dot{w}'\right)^2 - \frac{1}{2} EI(x) \left(w''\right)^2 - \frac{1}{2} \left(\int_x^\ell m(\xi) \Omega^2 \xi d\xi \right) \left(w'\right)^2$$

Evaluating terms of Lagrange's equation one at a time:

$$\frac{\partial \hat{L}}{\partial w} = 0$$

$$\frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w'} \right) = \frac{\partial}{\partial x} \left(\int_\ell^x m(\xi) \Omega^2 \xi d\xi \left(w'\right) \right)$$

$$= m(x) \Omega^2 x w' - \int_x^\ell m(\xi) \Omega^2 \xi d\xi w''$$

$$\frac{\partial^2}{\partial x^2} \left(\frac{\partial \hat{L}}{\partial w''} \right) = -\frac{\partial^2}{\partial x^2} \left(EI(x) w'' \right)$$

$$\frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}} \right) = m \ddot{w}$$

$$\frac{\partial^2}{\partial x \partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}'} \right) = \frac{\partial}{\partial x} \left(J \ddot{w}' \right)$$

So the E.O.M. is

$$-m\Omega^2 x w' + \int_x^\ell m(\xi) \Omega^2 \xi d\xi w'' - \frac{\partial^2}{\partial x^2} (EI(x) w'') - m\ddot{w} + \frac{\partial}{\partial x} (J \ddot{w}') - mg = 0 \qquad 0 < x < \ell$$

To get the B.C., let's again evaluate each term.

$$\frac{\partial \hat{L}}{\partial w'} = -\left(\int_x^\ell m \Omega^2 \xi d\xi \right) w'$$

$$\frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w''} \right) = -\frac{\partial}{\partial x} \left(EI(x) w'' \right)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{w}}\right) = J\ddot{w}'$$

$$\frac{\partial \hat{L}}{\partial w''} = -EIw''$$

So the B.C. are

$$-\left(\int_x^\ell m\Omega^2 \xi d\xi\right) w' + \frac{\partial}{\partial x}(EIw'') - J\ddot{w}' = 0 \quad \text{or} \quad w = 0$$

AND

$$-EIw'' = 0 \quad \text{or} \quad w' = 0$$

At the left end, $w = 0$ and $w' = 0$. At the right end $w \neq 0$ and $w' \neq 0$, so

$$\frac{\partial}{\partial x}(EIw'')\big|_{x=\ell} = J\ddot{w}'\big|_{x=\ell} \qquad \text{(Integral is zero)}$$

AND

$$-EIw''\big|_{x=\ell} = 0$$

Boundary conditions demanded by geometry ($w = 0, w' = 0$) are *geometric* or *essential* boundary conditions. These are typically BC through $\frac{\partial}{\partial x}$.
The other boundary conditions are *dynamic* boundary conditions. These are BC through $\frac{\partial^3}{\partial x^3}$

8 The Eigenvalue Problem

Assume a solution of

$$w(x,t) = W(x)F(t)$$

substituting into the E.O.M.

$$\begin{aligned}\left(-m\Omega^2 xW' + \left(\int_x^\ell m\Omega^2 \xi d\xi\right) W'' - \frac{d^2}{dx^2}(EIW'')\right)F \\ = \left(mW - \frac{d}{dx}(JW')\right)\ddot{F}, \qquad 0 < x < \ell\end{aligned} \tag{171}$$

The boundary conditions at the left end are $W(0)F = 0$, and $W'\,|_{x=0}\, F = 0$ and the BCs at the right end are

$\left(\frac{d}{dx}\left(EIW''\right)\right)_{x=\ell} F = JW' \mid_{x=\ell} F$, and $\left(EIW''\right) \mid_{x=\ell} F = 0$
Solving for $\frac{-\ddot{F}}{F}$

$$\frac{m\Omega^2 xW' - \left(\int_x^\ell m\Omega^2 \xi d\xi\right) W'' + \frac{d^2}{dx^2}\left(EIW''\right)}{mW - \frac{d(JW')}{dx}} = \frac{-\ddot{F}}{F} = \lambda$$

Thus, the temporal equation is

$$\ddot{F} + \lambda F = 0$$

$$f(t) = Ae^{st}$$

substituting

$$s^2 + \lambda = 0$$

Let's assume $\lambda = \omega^2, s = \pm i\omega$

$$F(t) = A_1 e^{iwt} + A_2 e^{-iwt}$$

The spacial equation is

$$m\Omega^2 xW' - \left(\int_x^\ell m\Omega^2 \xi d\xi\right) W'' + \frac{d}{dx^2}\left(EIW''\right) = \lambda\left(mW - \frac{d}{dx}\left(JW'\right)\right)$$

with the BCs
$W(0) = 0$, $W' \mid_{x=0} = 0$, $-\left(\frac{d}{dx} EIW''\right)_{x=\ell} = \lambda\left(JW'\right) \mid_{x=\ell}$, and $EIW'' \mid_{x=\ell} = 0$
Determining λ such that the spacial equation and the B.C. are satisfied is called the *eigenvalue problem.*
There is no closed form solution for this particular eigenvalue problem. A finite difference or finite element method must be employed to get the "complete" solusion.

8.1 Self-adjoint systems

Let W be a function of a spacial variable x.

$$LW = A_0(x)W + A_1(x)\frac{dW}{dx} + A_2(x)\frac{d^2W}{dx^2} + \ldots A_{2p}\frac{d^{2p}}{dx^{2p}}$$

LW is a linear homogeneous combination of W and its derivatives through order $2p$.

L is a "linear homogeneous differential operator"

$$L = A_0(x) + A_2(x)\frac{d}{dx} + \ldots + A_{2p}(x)\frac{d^{2p}}{dx^{2p}}$$

Let M be an operator similar to L, but of order $2q, q < p$, and write

$$LW = \lambda MW, \qquad 0 < x < \ell$$

Where λ comes from separation of variables.[13]
The boundary conditions can be written as

$$B_i W = \lambda C_i W \qquad i = 1, 2, \ldots, p \qquad x = 0, \ell$$

B_i and C_i are also linear homogenous differential operators.
The maximum order of B_i is $2p - 1$
The maximum order of C_i is $2q - 1$
For the previous problem:

$$L = m\Omega^2 x\frac{d}{dx} - \int_x^\ell m\Omega^2\xi d\xi \frac{d^2}{dx^2} + E\frac{d^2}{dx^2}I\frac{d^2}{dx^2} + EI\frac{d^4}{dx^4} \qquad (p = 2)$$

$$M = m - \frac{d}{dx}J\frac{d}{dx} - J\frac{d^2}{dx^2} \qquad q = 1$$

At $x = 0$:

$$B_1 = 1, \qquad C_1 = 0 \tag{172}$$

$$B_2 = \frac{d}{dx}, \qquad C_2 = 0 \tag{173}$$

and at $x = \ell$

$$B_1 = -E\frac{dI}{dx}\frac{d^2}{dx^2} - EI\frac{d^3}{dx^3}, \qquad C_1 = J\frac{d}{dx} \tag{174}$$

$$B_2 = EI\frac{d^2}{dx^2}, \qquad C_2 = 0 \tag{175}$$

Define the inner product of functions f and g where f and g are: real, piecewise smooth (continuous and in 0th and 1st derivative).

$$(f, g) = \int_0^\ell f(x)g(x)dx$$

[13] $M = -\tau^2$ for a string.

$f(x)$ and $g(x)$ are orthogonal if $(f, g) = 0$
$\| f \|$ is the norm of $f(x)$ defined by

$$\| f \|^2 = (f, f) = \int_0^\ell f^2(x)dx < \infty$$

The space of such functions is denoted K^0
Consider a set of functions ϕ_i, $i = 1, \ldots n$
The set is linearly dependent if

$$c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + \ldots c_n\phi_n = 0 \tag{176}$$

without all $c_r = 0, \qquad r = 1, \ldots n$
Assume (for the moment) that ϕ_r are orthogonal. i.e. $(\phi_r, \phi_s) = 0 \qquad r \neq s$
Multiply equation (176) by ϕ_s and integrate over $0 \leq x \leq \ell$

$$\sum_{r=1}^{n} c_r \int_0^\ell \phi_r\phi_s dx = 0$$

$$\sum_{r-1}^{n} c_r \| \phi_s \|^2 \delta_{rs} = 0$$

$$c_s \| \phi_s \|^2 = 0$$

Since $\| \phi_s \| \neq 0$, c_s, for any s, must be zero to satisfy (176).
Therefore, orthogonal functions are always linearly independent. The converse is not always true.
Consider $\phi_r(r)$, where $\| \phi_r(x)^2 \| = 1$, and an arbitrary function $f(x)$. Let c_r be defined by

$$c_r = (f, \phi_r) = \int_0^\ell f\phi_r dx \qquad r = 1, 2, \ldots$$

Consider the norm of $(f - \sum_{r=1}^{n} c_r\phi_r)$

$$\| f - \sum_{r=1}^{n} c_r\phi_r \|^2 = \int_0^\ell \left(f - \sum_{r=1}^{n} c_r\phi_r \right)^2 dx \geq 0$$

Expanding the integral

$$\begin{aligned}\int_0^\ell f^2 dx - 2\sum_{r=1}^n c_r \underbrace{\int_0^\ell f\phi_r dx}_{c_r} + \sum_{r=1}^n c_r^2 \\ =\| f \|^2 -2\sum_{r=1}^n c_r^2 + \sum_{r=1}^n c_r^2 \\ =\| f \|^2 - \sum_{r=1}^n c_r^2 \geq 0\end{aligned} \tag{177}$$

Therefore

$$\| f \| \geq \sum_{r=1}^n c_r^2$$

Bessel's Inequality: The sum of the squares of the coefficients c_r is always bounded.
This is significant when approximating $f(x)$ by $\sum_{r=1}^n d_r\phi_r$.
If we represent the system by $f(x) = \sum_{r=1}^n d_r\phi_r$ then the mean square error of the approximation is

$$M = \int_0^\ell \left(f - \sum_{r=1}^n d_r\phi_r \right)^2 dx$$

To best approximate f, we want to minimize this. First expand the integral

$$\begin{aligned}M =\| f \|^2 -2\sum_{r=1}^n d_r c_r + \sum_{r=1}^n d_r^2 \\ =\| f \|^2 - \sum_{r=1}^n c_r^2 + \sum_{r=1}^n d_r^2 - 2d_r c_r + c_r^2 \\ =\| f \|^2 - \sum_{r=1}^n c_r^2 + \sum_{r=1}^n (d_r - c_r)^2\end{aligned} \tag{178}$$

which is minimum for $d_r = c_r$. (recall we are minimizing with respect to d_r).
An approximation by this method is an "approximation in the mean." If n is large enough, the error can be smaller than some arbitrarily small ϵ. Such a set is considered to be *complete*.

Let $f_n = \sum_{r=1}^{n} c_r \phi_r$ be the series representation of f. Then

$$\lim_{n\to\infty} \parallel f - f_n \parallel = 0$$

The completeness of a set *does not* require the functions to be orthonormal, but only that they be independent.
Considering the original eigenvalue problem (in $w(x,t)$) The space is denoted K_B^{2p}. The $2p$ derivative must have finite energy and the boundary conditions must be satisfied.
Call functions in this space K_B^{2p} *comparison functions.* They need not satisfy the differential equation.
The space of eigenfunctions *is a subspace* of K_B^{2p}.

Consider the helicopter blade problem again.
If J is neglected (it is usually small) the eigenvalue problem is

$$LW = \lambda MW \qquad 0 < x < \ell$$

$$B_i W = 0 \qquad i = 1, 2, \ldots p \qquad x = 0, \ell$$

which is much simpler than before.
Consider two comparison functions, u and v. The system is considered to be *self-adjoint* iff

$$(u, Lv) = (v, Lu)$$

This is the symmetry we often obtain for discrete systems. Discrete systems with symmetric stiffness matrices (and symmetric damping matrices) are also self-adjoint. (this means no gyroscopic matrices, ...).
Recall L is of order $2p$
Define

$$[u, v] = \int_0^\ell \sum_{k=0}^{p} f(x) \frac{d^k u}{dx^k} \frac{d^k v}{dx^k} dx + \sum_{l=0}^{p-1} b_l \frac{d^l u}{dx^l} \frac{d^l v}{dx^l} \Big|_0^\ell$$

to be the energy inner product (obtained by integration by parts).
The energy inner product is symmetric in u and v and their derivatives. (Property of self-adjoint systems).
Note that (u, v) only has derivatives through p and boundary conditions only through $p - 1$ (characteristic of Geometric Boundary Conditions).
Questions: Can we allow functions outside of K_B^{2p}? Yes. We can enlarge this space to include functions with only p derivatives having finite energy.
Define the expanded space K_G^p. The G indicates that the members of the space satisfy the geometric boundary conditions. This is known as the *energy space*. Members are *energy functions*, or *admissible functions*.

8.1.1 Example: Self-adjointness of beam stiffness operator

$$L = \frac{d^4}{dx^4} \qquad p = 2$$

$$
\begin{aligned}
(u, Lv) &= \int_0^\ell u \frac{d^4 v}{dx^4} dx = -\int_0^\ell \frac{du}{dx}\frac{d^3 v}{dx^3} dx + u\frac{d^3 v}{dx^3}\,|_0^\ell \\
&= \int_0^\ell \frac{d^2 u}{dx^2}\frac{d^2 v}{dx^2} dx - \underbrace{\frac{du}{dx}\frac{d^2 v}{dx^2}\,|_0^\ell + u\frac{d^3 v}{dx^3}\,|_0^\ell}_{\text{If } u \text{ and } v \text{ satisfy the BCs, these terms are zero}} \qquad (179) \\
&= \underbrace{[u, v]}_{\text{energy inner product}}
\end{aligned}
$$

Likewise

$$(v, Lu) = \int_0^\ell \frac{d^2 u}{dx^2}\frac{d^2 v}{dx^2} dx$$

Thus, L is self-adjoint.
This indicates that only the derivative p of u and v is required to have finite energy and functions must satisfy only geometric B.C., so the space can be enlarged by to K_G^p.
Consider (u, Lu) If $(u, Lu) = \int_0^\ell uLudx \geq 0$ for all u, and zero only for $u = 0$, then the system is positive definite (Compare to $\boldsymbol{x}^T A\boldsymbol{x} \geq 0$ for all non-zero $\boldsymbol{x}$).
If the system is positive definite, all $\lambda_r > 0$.

$$Lu = \lambda M u$$

$$\int_0^\ell u\,(Lu)\,dx = \int_0^\ell u\lambda M u dx$$

$$(u, Lu) = \lambda \int_0^\ell u^2 M dx$$

If $(u, Lu) > 0$, since $M > 0$ and $\| u^2 \| > 0$, λ must be > 0.
Recall $\frac{-\ddot{F}}{F} = \lambda$ (separation of variables) $\lambda = \omega^2$
If $\lambda > 0$, ω has no real part, and the system is marginally stable.
$\therefore$ Positive definite systems are stable.
$\therefore$ Positive semi-definite systems are marginally stable (Rigid body rotation)

8.2 Proof that the eigenfunctions are orthogonal for self-adjoint systems.

Consider two solutions (λ_r, W_r and λ_s, W_s)

to the eigenproblem

$$LW_r = m\lambda_r W_r$$

$$LW_s = m\lambda_s W_s$$

Multiply and integrate

$$\int_0^\ell W_s L W_r = \lambda_r \int_0^\ell m W_s W_r dx$$

$$\int_0^\ell W_r L W_s = \lambda_s \int_0^\ell m W_r W_s dx$$

Subtracting

$$\int_0^\ell W_s L W_r dx - \int_0^\ell W_r L W_s dx = (\lambda_r - \lambda_s) \int_0^\ell m W_s W_r dx$$

If the system is self-adjoint, the left side is zero by definition. If we assume λ_r and λ_s are distinct.

$$0 = \int_0^\ell m W_s W_r dx, \qquad m \neq 0$$

$\therefore$ W_s and W_r are orthogonal *with respect to the function* m.
Considering the eigenproblem again

$$LW_r = \lambda_r m W_r$$

$$\int_0^\ell W_s L W_r dx = \lambda_r \underbrace{\int_0^\ell m W_s W_r dx}_{=0}$$

$$\int_0^\ell W_s L W_r dx = 0$$

For self-adjoint systems the eigenfunctions are orthogonal-also in the energy inner product.

8.3 Non self-adjoint systems

Some systems, such as these involving flutter or oil whip, are not self-adjoint. Consider the linear operator L. It can be shown that the operator L has and adjoint L^* such that

$$(v, Lu) = (u, L^* v)$$

Where u and v are in the domain of L and L^*.
Consider u_i and v_j, eigenfunctions of L and L^*.

$$L_i u_i = \lambda_i u_i, \qquad L_j^* v_j = \lambda_j^* v_j$$

The set of eigenfunctions v_j is said to be adjoint to the eigenfunctions u_i.
Multiply and integrating

$$(v_j, Lu_i) = \int_0^\ell v_j L u_i dx = \lambda_i \int_0^\ell v_j u_i dx$$

$$(u_i, L^* v_j) = \int_0^\ell u_i L^* v_j dx = \lambda_i^* \int_0^\ell u_i v_j dx$$

Subtracting

$$0 = \left(\lambda_i - \lambda_j^*\right) \int_0^\ell u_i v_j dx$$

If $\lambda_i \neq \lambda_j^*$ then

$$(u_i, v_j) = \int_0^\ell u_i v_j dx = 0, \qquad i, j = 1, 2, \ldots$$

$\therefore$ The eigenfunction of L corresponding to the eigenvalue λ_i is orthogonal to the eigenfunction of L^* corresponding to λ_j^*, where λ_j^* is distinct from λ_i. This means that there is a biorthogonality relation, (orthogonal to two sets).
$Lu_i = \lambda u_i$ can be written

$$(L - \lambda_i)\, u_i = 0$$

$$(v_i, (L - \lambda_i)\, u_i) = 0$$

For any eigenfunction function v_i of L^*

$$\begin{aligned}(v_i, Lu_i) - \lambda_i (v_i, u_i) &= (L^* v_i, u_i) - \lambda_i (v_i, u_i) \\ &= ((L^* - \lambda_i)\, v_i, u_i) = 0\end{aligned}$$

$\therefore u_i$ is orthogonal to every function $(L^* - \lambda_i)\, v_i$
Consider: Assume $(L^* - \lambda_i)\, v_i$ can represent any arbitrary function f. Then $(f, u_i) = 0$.
This is not true for any function, but only those orthogonal to u_i, or those equal to 0. Since that is a solution.

$$\therefore (L^* - \lambda_i)\, v_i = 0$$

Thus, the eigenvalues of L^* are the same as those for L. (The eigenfunctions are not the same.)
If u_i and v_i, $1 = 1, 2, \ldots \infty$, are normalized, then

$$(u_i, v_j) = \int_0^\ell u_i v_j dx = \delta_{ij}, \qquad i, j = 1, 2, 3, \ldots \infty \tag{180}$$

The sets of eigenfunctions are assumed to be complete, so that for any arbitrary function f,

$$f = \sum_{i=1}^{\infty} \alpha_i u_i$$

$$\alpha_i = (u_i, f)$$

from (180).
Likewise for

$$f = \sum_{i=1}^{\infty} \beta_i v_i, \qquad \beta_i = (v_i, f)$$

$$Lf = \sum_{i=1}^{\infty} \alpha_i L u_i = \sum_{i=1}^{\infty} \alpha_i \lambda_i u_i$$

$$L^* f = \sum_{i=1}^{\infty} \beta_i L^* v_i = \sum_{i=1}^{\infty} \beta_i \lambda_i v_i$$

If $L^* = L$, then the system is self-adjoint.
Non self-adjoint systems are not generally easy to solve closed-form.

8.4 Repeated eigenvalues

Consider λ_i has multiplicity m_i.

- There are m_i eigenfunctions corresponding to λ_i.

- Any linear combination of the eigenfunctions is also an eigenfunction.
- They can (and should) be taken so that they are mutually orthogonal.

They are normalized such that

$$\int_0^\ell m(x) w_r^2 dx = 1$$

Recall that the energy inner product is

$$\begin{aligned}
[w_r, w_s] &= \int_0^\ell w_r L w_s dx \\
&= \int_0^\ell \sum_{K=1}^{p} a_k \frac{d^k w_r}{dx^k} \frac{d^k w_s}{dx^k} dx + \sum_{\ell=1}^{p-1} b_\ell \frac{d^\ell w_r}{dx^\ell} \frac{d^\ell w_s}{dx^\ell} \Big|_0^\ell \qquad (181) \\
&= \lambda_r \delta_{rs}
\end{aligned}$$

and

$$\left(\sqrt{m} w_r, \sqrt{m} w_s\right) = \int_0^\ell m w_r w_s dx = \delta_{rs}$$

Every function w with continuous Lw and satisfying the boundary conditions can be expanded in a convergent series.

$$w = \sum_{r=1}^{\infty} c_r \underbrace{w_r}_{\text{eigenfunctions}}$$

where

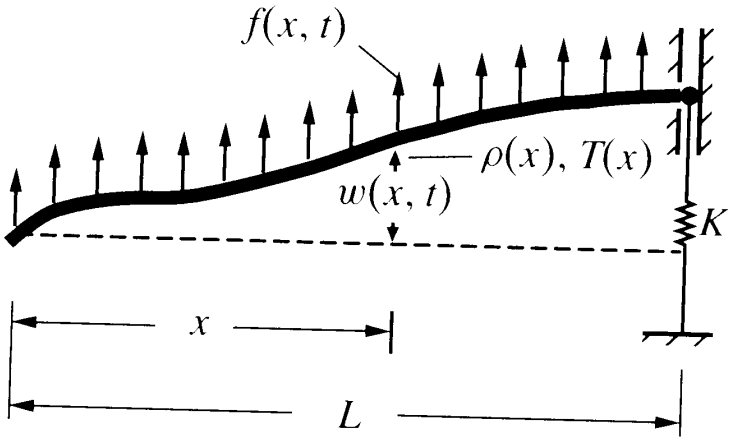
$$c_r = \int_0^\ell m w w_r dx$$

8.5 Conclusion- approximation of eigenfunctions

This solution form is written in terms of the eigenfunctions. Often they cannot be found closed-form.
When closed-form solutions do not exist, approximate methods must be used.
The approximations are linear combinations of comparison or admissible functions.
Admissible functions are preferred because there are more of them.

8.6 Vibration of Rods, Shafts and Strings

All are represented by a second-order PDE.
Consider



$$T(t) = \frac{1}{2}\int_0^\ell m(x)\dot{w}(x,t)^2 dx$$

$$V(t) = \frac{1}{2}\int_0^\ell \left(s(x)w'(x,t)^2 + k(x)w(x,t)^2\right) dx + \frac{1}{2}Kw\left(\ell,t\right)^2$$

$$\delta W_{nc}(t) = \int_0^\ell p(x,t)\delta w(x,t)dx$$

Hamilton's Principle:

$$\int_{t_1}^{t_2}\left(\int_0^\ell \left(\delta\hat{L} + \delta\hat{W}_{nc}\right)dx + \underbrace{\delta L_0}_{\text{discrete Lagrangian}}\right)dt = 0$$

$$L_0 = -\frac{1}{2}Kw\left(\ell,t\right)^2$$

$$\delta L = \delta\hat{T} - \delta\hat{V}$$

Applying Hamilton's principle or the Lagrange equation yields

$$-k(x)w + \frac{\partial}{\partial x}\left(sw'\right) - m\ddot{w} + p = 0 \qquad 0 < x < \ell$$

$$w(0,t) = 0, \qquad \text{Geometric B.C.}$$
$$sw' + Kw = 0 \qquad @x = \ell, \qquad \text{Dynamic B.C.}$$

Consider the unforced problem. Separation of variables yields

$$-\frac{d}{dx}(sW') + kW = \omega^2 mW \qquad 0 < x < \ell$$
$$W(0) = 0, \qquad sW' + KW = 0 \qquad @x = \ell$$

The operators are
$L = -\frac{d}{dx}\left(s\frac{d}{dx}\right) + k$
"L" is of order 2
Since $p = 1$

$$\beta_1 = 1 \qquad @x = 0$$
$$\beta_2 = s\frac{d}{dx} + K \qquad @x = \ell$$

First: Let's see if L is self-adjoint. Consider two comparison functions u and v.

$$\begin{aligned}\int_0^\ell uLvdx &= \int_0^\ell u\left(-\frac{d}{dx}\left(s\frac{dv}{dx}\right) + kv\right)dx \\ &= -\int_0^\ell u\frac{d}{dx}\left(s\frac{dv}{dx}\right)dx + \int_0^\ell ukvdx \\ &= -us\frac{dv}{dx}\,|_0^\ell + \int_0^\ell \frac{du}{dx}s\frac{dv}{dx}dx + \int_0^\ell ukvdx\end{aligned} \tag{182}$$

Recall the boundary conditions

$$v(0) = 0$$

(also true for u)
and

$$sv'\,|_{x=\ell} = -Kv\,|_\ell$$

Substituting

$$= uKv\,|_\ell + \int_0^\ell \frac{du}{dx}s\frac{dv}{dx}dx + \int_0^\ell ukvdx$$

is symmetric in u and v and their first derivatives.
Thus, L (and the system) is self-adjoint. Thus the eigenfunctions are orthogonal.

Letting $v = u$

$$\int_0^\ell uLudx = u^2 K \mid_\ell + \int_0^\ell u'^2 s + u^2 k dx > 0$$

So the system is positive definite, and all of the eigenvalues are positive.
Note: We did not need to solve the E.O.M. to reach these conclusions.
Let $s(x), k(x)$ and $m(x)$ all be constants.
Then

$$-\frac{d}{dx}\left(sW'\right) + kW = \omega^2 mW$$

becomes

$$W'' + \beta^2 W = 0$$
$$\beta^2 = \left(\omega^2 m - k\right)/s$$

The solution is

$$W(x) = C_1 \sin \beta x + C_2 \cos \beta x$$

From the boundary conditions

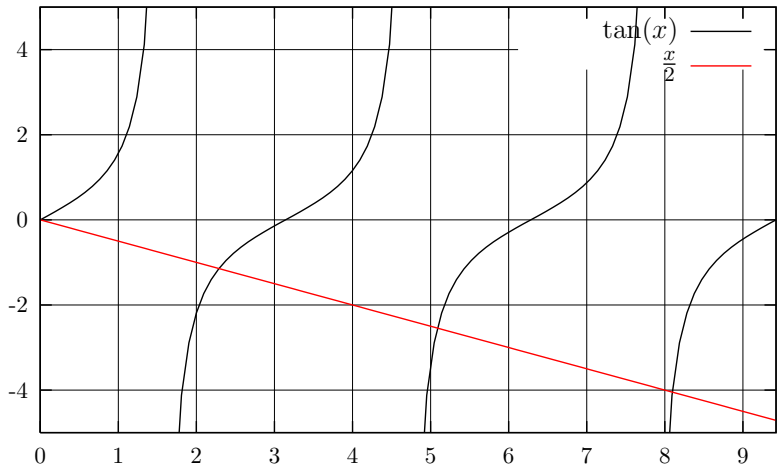
$$C_2 = 0$$

$$s\beta C_1 \cos \beta\ell + KC_1 \sin \beta\ell = 0$$

Simplifying

$$\tan \beta\ell = -\frac{s}{K\ell}\beta\ell$$

which must be solved numerically



The eigenfunctions are

$$W_r(x) = C_r \sin \beta_r x \qquad r = 1, 2, \ldots$$

$$\omega_r = \sqrt{\lambda_r} = \sqrt{\frac{(s\beta_r^2 + k)}{m}} \qquad r = 1, 2, \ldots$$

The eigenfunctions should now be normalized such that

$$1 = \int_0^\ell m W_r^2 dx$$

$$C_r = 2\sqrt{\frac{\beta_r}{m}\left(2\beta_r \ell - \sin 2\beta_r \ell\right)}$$

Note that the distributed spring does not effect the eigenvalues $\beta\ell$ or the eigenfunctions. (It does effect ω). (Note $\omega = \sqrt{\lambda} \neq \sqrt{\beta\ell}$)
The orthogonality of the modes must be checked.

$$\begin{aligned}
\int_0^\ell \sin \beta_r x \sin \beta_s x dx &\stackrel{?}{=} C\delta_{rs} \\
&= \frac{1}{2}\int_0^\ell \cos(\beta_r - \beta_s)x - \cos(\beta_r + \beta_s)x dx \\
&= \frac{1}{2(\beta_r^2 - \beta_s^2)} \begin{pmatrix} (\beta_r + \beta_s)\sin(\beta_r - \beta_s)x \\ \quad + (\beta_r - \beta_s)\sin(\beta_r + \beta_s)x \end{pmatrix}\Bigg|_0^\ell \qquad (183) \\
&= \frac{1}{\beta_r^2 - \beta_s^2} \underbrace{\left(\beta_s \sin \beta_r \ell \cos \beta_s \ell - \beta_r \sin \beta_s \ell \cos \beta_r \ell\right)}_{=0 \text{ for } s=r, \text{ but not for } s\neq r}
\end{aligned}$$

So, $\frac{0}{0}$ doesn't give us anything ($s = r$), and we don't have zero for $s \neq r$.
Recall the characteristic equation

$$\tan \beta\ell = -\frac{s}{K\ell}\beta\ell$$

thus

$$\sin \beta\ell = -\frac{s}{K\ell}\beta\ell \cos \beta\ell$$

Since $\beta_s l$ and $\beta_r \ell$ are both solutions we can substitute for the sine functions. Thus

$$\therefore \frac{1}{\beta_r^2 - \beta_s^2}\left(-\beta_s \cos(\beta_s \ell)\frac{s}{K\ell}\beta_r \ell \cos(\beta_r \ell) + \beta_r \cos(\beta_r l)\frac{s}{K\ell}\beta_s \ell \cos(\beta_s \ell)\right) = 0$$

for $r \neq s$

8.7 Bending Vibration of a Helicopter Blade

The differential equation with $J = 0$ is

$$\frac{d^2}{dt^2}\left(EIW''\right) - \frac{d}{dx}\left(\left(\int_x^\ell m\Omega^2\xi d\xi\right)W'\right) = \lambda mW$$

with B.C.

$$W(0) = 0, \qquad W'(0) = 0$$

$$EIW'' \mid_{x=\ell} = 0$$

$$-\frac{d}{dx}\left(EIW''\right) \mid_{x=\ell} = 0$$

There is no known closed-form solution.
Let's consider the self-adjointness and positive definiteness.
The operator L is

$$L = \frac{d^2}{dx^2}\left(EI\frac{d^2}{dx^2}\right) - \frac{d}{dx}\left(\left(\int_x^\ell m\Omega^2\xi d\xi\right)\frac{d}{dx}\right)$$

To consider the self-adjointness, consider two comparison functions u and v.

$$\begin{aligned}
(u, Lv) =& \int_0^\ell uLvdx \\
=& \int_0^\ell u\left((EIv'')'' - \left[\left(\int_x^\ell m\Omega^2\xi d\xi\right)v'\right]'\right)dx \\
=& u\left(EIv''\right)' \mid_0^\ell - u'\left(EIv''\right) \mid_0^\ell - u\left(\int_x^\ell m\Omega^2\xi d\xi\right)v' \mid_0^\ell \\
&+ \int_0^\ell u''EIv'' + u'\left(\int_x^\ell m\Omega^2\xi d\xi\right)v'dx
\end{aligned}$$

Since u and v satisfy all the B.C., this reduces to

$$(u, Lv) = \int_0^\ell u''EIv'' + u'\left(\int_x^\ell m\Omega^2\xi d\xi\right)v'dx$$

which means that L is self-adjoint

$\therefore$ The mass weighted eigenfunctions are orthogonal. Since $(u, Lv) = [u, v]$ we can check for positive definiteness by setting $v = u$ and observing the sign.

$$\begin{aligned}[u,u] &= \int_0^\ell uLudx \\ &= \int_0^\ell \underbrace{EIu''^2}_{\geq 0} dx + \int_0^\ell \underbrace{u'^2}_{\geq 0} \underbrace{\left(\int_x^\ell m\Omega^2 \xi d\xi\right)}_{\geq 0} dx > 0 \end{aligned} \tag{184}$$

So the eigenvalues are all positive (positive definite system)
Consider the hinged blade (no longer clamped at left end)... articulated blade. Then

$$W(0) = 0 \Longrightarrow EIW'' \,|_{x=0} = 0$$

(no moment at left end)
Consider: Does $W = x$ cause $\lambda = 0$?
Substituting into $[u, u]$ gives

$$[u,u] = \int_0^\ell \left(\int_x^\ell m\Omega^2 \xi d\xi\right) dx > 0$$

so the system is still positive definite.
For kicks try $W = x$ in the EOM

$$-\frac{d}{dx}\int_x^\ell m\Omega^2 \xi d\xi = \lambda m x$$

$$m\Omega^2 x = m\lambda x$$

$$\Omega^2 = \lambda$$

Since $\omega^2 = \lambda$, $\omega = \Omega$ is the frequency of the flapping mode.

8.8 Timoshenko Beams

8.9 Vibration of Membranes

8.10 Vibration of Plates

8.11 Distributed Gyroscopic systems

8.12 Variational Characterization of the Eigenvalues

Consider a self-adjoint system. Define the Rayleigh quotient as

$$R(u) = \frac{[u,u]}{(\sqrt{m}u, \sqrt{m}u)} \tag{185}$$

where u is some arbitrary function (satisfying B.C.)
Recall that we can represent u by $u = \sum_{r=1}^{\infty} c_r W_r$
where W_r are the eigenfunctions. Substituting gives

$$R(c_1, c_2, \ldots.) = \frac{\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} c_r c_s [W_r, W_s]}{\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} c_r c_s (\sqrt{m}W_r, \sqrt{m}W_s)}$$

Since the eigenfunctions are orthogonal, and

$$[W_r, W_r] = \lambda_r \text{ and } \left(\sqrt{m}W_r, \sqrt{m}W_r\right) = 1$$

$$R(c_1, c_2, \ldots) = \frac{\sum_{r=1}^{\infty} c_r^2 \lambda_r}{\sum_{r=1}^{\infty} c_r^2}$$

The first variation of Rayleigh's quotient is

$$\delta R = \frac{\partial R}{\partial c_1}\delta c_1 + \frac{\partial R}{\partial c_2}\delta c_2 + \ldots = \sum_{i=1}^{\infty} \frac{\partial R}{\partial c_i}\delta c_i$$

At a stationary point; δR must vanish
$\therefore \frac{\partial R}{\partial c_i} = 0$ (small variations of c don't change R)

Substituting for R gives

$$\begin{aligned}\frac{\partial R}{\partial c_i} =& \frac{\partial}{\partial c_i}\left(\left(\sum_{r=1}^{\infty} c_r^2\lambda_r\right)\left(\sum_{r=1}^{\infty} c_r^2\right)^{-1}\right)\\ =& \left(\sum_{r=1}^{\infty} 2c_r\frac{\partial c_r}{\partial c_i}\lambda_r\right)\left(\sum_{r=1}^{\infty} c_r^2\right)^{-1}\\ &+\left(\sum_{r=1}^{\infty} c_r^2\lambda_r\right)\left(-1\sum_{r=1}^{\infty} 2c_r\frac{\partial c_r}{\partial c_i}\right)\left(\sum_{r=1}^{\infty} c_r^2\right)^{-2} = 0\end{aligned} \tag{186}$$

Since c_r and c_i are independent

$$\frac{\partial c_r}{\partial c_i} = \delta_{ri}$$

and

$$\begin{aligned}\frac{\partial R}{\partial c_i} &= \frac{2c_i\lambda_i\sum_{r=1}^{\infty} c_r^2 - 2c_i\sum_{r=1}^{\infty} c_r^2\lambda_r}{\left(\sum_{r=1}^{\infty} c_r^2\right)^2}\\ &= \frac{2c_i\sum_{r=1}^{\infty} c_r^2\left(\lambda_i-\lambda_r\right)}{\left(\sum_{r=1}^{\infty} c_r^2\right)^2} = 0\end{aligned} \tag{187}$$

If u is an eigenfunction, say $u = W_i$, then $c_r = c_i\delta_{ir}$.

$$\frac{\partial R}{\partial c_i} = \frac{2c_i^3\left(\lambda_i-\lambda_i\right)}{c_i^4} = \frac{2\left(\lambda_i-\lambda_i\right)}{c_i} = 0$$

is satisfied.
Hence the Rayleigh quotient has stationary points at the system eigenfunctions.
Letting $u = c_iW_i$ in the Rayleigh Quotient,

$$R(W_i) = \lambda_i$$

$\therefore$ The stationary values of the Rayleigh quotient are the system eigenvalues.

Assume u has the form $W_i + \epsilon v$ where ϵ is a small value and v is an arbitrary function.

$$\begin{aligned} R\left(W_i + \epsilon v\right) &= \frac{\left[W_i + \epsilon v, W_i + \epsilon v\right]}{\left(\sqrt{m}\left(W_i + \epsilon v\right), \sqrt{m}\left(W_i + \epsilon v\right)\right)} \\ &= \frac{\left[W_i, W_i\right] + 2\epsilon\left[W_i, v\right] + \epsilon^2\left[v, v\right]}{\left(\sqrt{m}W_i, \sqrt{m}W_i, \right) + 2\epsilon\left(\sqrt{m}W_i, \sqrt{m}v\right) + \epsilon^2\left(\sqrt{m}v, \sqrt{m}v\right)} \end{aligned} \tag{188}$$

Using the Binomial Expansion Theorem

$$\begin{aligned} R\left(W_i + \epsilon v\right) &\approx R\left(W_i\right) + 2\epsilon\frac{\left[W_i, v\right]\left(\sqrt{m}W_i, \sqrt{m}W_i\right) - \left[W_i, W_i\right]\left(\sqrt{m}W_i, \sqrt{m}v\right)}{\left(\sqrt{m}W_i, \sqrt{m}W_i\right)^2} + \mathcal{O}\left(\epsilon^2\right) \\ &\approx \lambda_i + 2\epsilon\frac{\left[W_i, v\right] - \lambda_i\left(\sqrt{m}W_i, \sqrt{m}v\right)}{\left(\sqrt{m}W_i, \sqrt{m}W_i\right)} + \mathcal{O}\left(\epsilon^2\right) \end{aligned} \tag{189}$$

For a fixed v, R depends only on ϵ. If $R(W_i + \epsilon v)$ is stationary, then the second term must be zero for non-zero ϵ.

$$\int_0^\ell vLW_i dx - \lambda_i \int_0^\ell mvW_i dx = 0$$

$$\int_0^\ell v\left(LW_i - \lambda_i mW_i\right) dx = 0$$

For this to be true, v must be orthogonal to $LW_i - \lambda_i mW_i$. But since v is arbitrary

$$LW_i - \lambda_i mW_i = 0$$

So, the stationarity of Rayleigh's quotient is equivalent to the eigenvalue problem.
If λ_1 is the lowest eigenvalue, then $R(u) \geq \lambda_1$.
If u is orthogonal to the first s eigenvalues then

$$R(u) \geq \lambda_{s+1}$$

But this requires the eigenfunctions
A better characterization is

$$\lambda_{s+1} = \max \min\, R(u) \qquad \underbrace{(u, v_i) = 0 \qquad i = 1, 2, \ldots s}_{s \text{ constraints}}$$

v_i are arbitrary functions, they are in effect constraints on the function u.
So λ_{s+1} is the max value of ($\min R(u)$ with respect to u) with respect to v_i.
To apply this we can use admissible functions u since $[u, u]$ satisfies the dynamic B.C.s automatically and $(\sqrt{m}u, \sqrt{m}u)$ does not require their consideration.

8.13 Integral Formulation of the Eigenvalue Problem

For a discrete system with stiffness matrix A ($M = I$, $K = A$) with eigenvalues λ and eigenvectors $\boldsymbol{x}$, the eigenvectors of A^{-1} (assuming A is non-singular) are λ^{-1} and the eigenvectors are $\boldsymbol{x}$.

$$(A - \lambda I)\,\boldsymbol{x} = \mathbf{0}$$

$$A^{-1}(A - \lambda I)\,\boldsymbol{x} = \mathbf{0}$$

$$\left(I - \lambda A^{-1}\right)\boldsymbol{x} = \mathbf{0}$$

$$\left(A^{-1} - \lambda^{-1} I\right)\boldsymbol{x} = \mathbf{0}$$

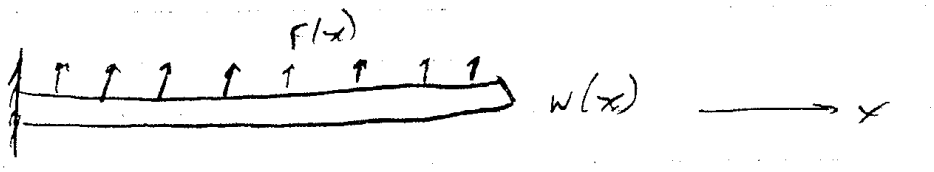
$\therefore$ the inverse of the eigenvalues of A^{-1} are the eigenvalues of A. So we actually had a choice of solving eigenvalue problem of A or A^{-1}!
A^{-1} is the compliance matrix or the inverse operator.
Consider the differential equation

$$LW(x) = f(x) \qquad 0 < x < \ell$$

L is P.D. and self-adjoint.
Does $Af(x) = W(x)$ where A is the inverse operator of L exist?
Yes, it is usually an integral operator.
Assume:

$$W = Af = \int_0^\ell a\,(x, \xi)\, f\,(\xi)\, d\xi$$

$a(x, \xi)$ is the kernel of the integrator operator.
$a(x, \xi)$ is known as the *Green's function* or *influence function.*
Consider a cantilever beam



Let $a(x,\xi)$ be the displacement at x due to a unit load at point ξ. The total displacement $W(x)$ is

$$W(x) = \int_0^\ell a(x,\xi) f(\xi) d\xi$$

Note that from Maxwell's reciprocity theorem

$$a(x,\xi) = a(\xi,x)$$

Consider the E.O.M.

$$LW = -m\ddot{W}$$

since

$$\ddot{W} = -\omega^2 W$$

$$LW = m\omega^2 W = \lambda m W \qquad \lambda = \omega^2$$

Comparing to the original equation

$$f(x) = \lambda m W$$

so

$$W(x) = \int_0^\ell a(x,\xi) \lambda m(\xi) W(\xi) d\xi$$

which is the eigenvalue problem in integral form.
Recall from discrete systems

$$M\ddot{\boldsymbol{x}} + K\boldsymbol{x} = 0$$

$$-\omega^2 M\boldsymbol{x} + K\boldsymbol{x} = 0$$

$$\boldsymbol{x} = \omega^2 K^{-1} M\boldsymbol{x} = \lambda A M\boldsymbol{x} =$$

If M has been diagonalized

$$x_i = \lambda \sum_{j=1}^{n} a_{ij} m_j x_j$$

The continuous integral form can be used in an iterative process.

$$W_{i+1}(x) = \lambda \int_0^\ell a(x,\xi) m(\xi) W_i(\xi) d\xi$$

If the first function W_0 is an admissible function, then the resulting function is a comparison function.

8.13.1 Example: Cantilever beam. EI=const.

Influence function (static response)

$$a(x,\xi) = \frac{1}{2EI}x^2\left(\xi - \frac{1}{3}x\right) \qquad x < \xi$$

$$a(x,\xi) = \frac{1}{2EI}\xi^2\left(x - \frac{1}{3}\xi\right) \qquad \xi < x$$

Pick an initial admissible function $W_0(x) = x^2$ (Picking an admissible function gets us closer to the answer quicker. However *any* initial displacement, even one that does not satisfy the boundary conditions, will also work). Substitution yields

$$W_1(x) = \frac{\lambda m}{2EI}\left(\int_0^x \underbrace{\xi^2}_{W(\xi)}\underbrace{\xi^2\left(x - \frac{1}{3}\xi\right)}_{a(x,\xi)}d\xi + \int_x^\ell \underbrace{\xi^2}_{W(\xi)}\underbrace{x^2\left(\xi - \frac{1}{3}x\right)}_{a(x,\xi)}d\xi\right)$$

New trial function

$$= \frac{\lambda m}{2EI}\left(\ell^3\left(\frac{1}{4}x^2\ell - \frac{1}{9}x^3\right) + \frac{1}{180}x^6\right)$$

Taking the second derivative:

$$W_1''(x) = \frac{\lambda m}{2EI}\left(\ell^3\left(\frac{1}{2}\ell - \frac{2}{3}x\right) + \frac{1}{6}x^4\right)$$

Evaluating at the right end:

$$W_1''(\ell) = 0$$

And the shear is proportional to the 3rd derivative:

$$W_1'''(x) = \frac{\lambda m}{2EI}\left(-\frac{2}{3}\ell^3 + \frac{2}{3}x^3\right)$$

$$W_1'''(\ell) = 0$$

So, the BCs are now satisfied with our current estimate of the first mode. HW: Apply Rayleigh's quotient and compare to ω true.

8.13.2 Example: String solution using Green's functions

Assume T and m const, fixed at each end (length 1)

$$a(x,\xi) = \begin{cases} x(1-\xi) & x < \xi \\ \xi(1-x) & \xi < x \end{cases}$$

For example, if a point load is applied at $\xi = 1/2$, the deflection shapes would be $x/2$ and $1/2 - x/2$. If a point load is applied at $\xi = 1/4$, the deflection shapes would be $3x/4$ and $3/4 - 3x/4$.
Pick an initial trial function that satisfies the boundary conditions

$$W_0(x) = .25 - (x - .5)^2$$

$$W_1(x) = \lambda_m \left(\int_0^x \xi\,(1-x)\left(.25 - (\xi - .5)^2\right) d\xi + \int_x^\ell x\,(1-\xi)\left(.25 - (\xi - .5)^2\right) d\xi \right)$$

$$W_1(x) = \frac{1}{12}x - \frac{1}{6}x^3 + \frac{1}{12}x^4$$

continuing

$$W_2(x) = \frac{1}{120}x - \frac{1}{72}x^3 + \frac{1}{120}x^5 - \frac{1}{360}x^6$$

The resulting Rayleigh quotients are

$$R_0 = \frac{T(-2)^2}{\rho(.25 - (x - .5)^2)^2} = 10.0000c^2$$

$$R_1 = 9.87097c^2$$

$$R_2 = 9.86962c^2$$

$$R = 9.86960c^2 \qquad True$$

9 Discretization of continuous systems

9.1 The Rayleigh-Ritz method

Consider again the eigenvalue problem

$$LW = m\lambda W \qquad W = W(x)$$

$$B_i W = 0 \qquad i = 1, 2, \dots p$$

L is self-adjoint of order p.
Instead of solving this, we will seek stationary values of the Rayleigh quotient, equation (185)

$$R(W) = \frac{[W, W]}{(\sqrt{m}W, \sqrt{m}W)}$$

where W is a trial function.
The function space considered will be the finite subspace S^n of K_G^p
Denote a function approximating W in $S^n : W^n$

$$R(W^n) = \frac{[W^n, W^n]}{(\sqrt{m}W^n, \sqrt{m}W^n)}$$

Select a sequence of functions $(\phi_1(x), \phi_2(x), \phi_3(x), \dots, \phi_n(x))$ of x that are linearly independent and are complete in energy.
That is to say for any $W(x)$ in K_G^p and any small ϵ, the are enough functions such that

$$\| W - \sum_{i=1}^{n} a_i \phi_i(x) \| < \epsilon$$

$\therefore \sum_{i=1}^{N} a_i \phi_i(x)$ can adequately represent $W(x)$

$$W^n = \sum_{i=1}^{n} a_i \phi_i$$

a_i need to be determined

$$\begin{aligned} R(a_i, a_2, \dots, a_n) &= \frac{[\sum_{i=1}^{n} a_i\phi_i, \sum_{i=1}^{n} a_i\phi_i]}{(\sqrt{m}\sum_{i=1}^{n} a_i\phi_i, \sqrt{m}\sum_{i=1}^{n} a_i\phi_i)} \\ &= \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j [\phi_i, \phi_j]}{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j (\sqrt{m}\phi_i, \sqrt{m}\phi_j)} \end{aligned} \tag{190}$$

Next, define

$$[\phi_i, \phi_j] = K_{ij}$$
$$\left(\sqrt{m}\phi_i, \sqrt{m}\phi_j\right) = M_{ij}$$

These are the mass and stiffness coefficients (elements of mass and stiffness matrices).
Note that they must be symmetric as a result of L being self-adjoint.

$$R(a_1, a_2, \ldots, a_n) = \frac{N(a_1, a_2, \ldots)}{D(a_1, a_2, \ldots)}$$

$$N(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} K_{ij} a_i a_j \qquad \text{compare to } \boldsymbol{x}^T K \boldsymbol{x}$$

$$D(a_i, a_2, \ldots, a_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij} a_i a_j \qquad \text{compare to } \boldsymbol{x}^T M \boldsymbol{x}$$

Recall that for stationarity

$$\begin{aligned} \frac{\partial R}{\partial a_r} &= 0 \qquad r = 1, 2, \ldots n \\ \frac{\partial R}{\partial a_r} &= \frac{\left(\frac{\partial N}{\partial a_r}\right) D - \left(\frac{\partial D}{\partial a_r}\right) N}{D^2} \\ &= \frac{\frac{\partial N}{\partial a_r} - \Lambda^n \frac{\partial D}{\partial a_r}}{D} \end{aligned} \tag{191}$$

Note that $\Lambda^n = \frac{N}{D}$ at the stationary point.
Further,

$$\begin{aligned} \frac{\partial N}{\partial a_r} &= \sum_{i=1}^{n}\sum_{j=1}^{n} K_{ij}\left(\frac{\partial a_i}{\partial a_r} a_j + a_i \frac{\partial a_j}{\partial a_r}\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} K_{ij}\left(\delta_{ir} a_j + \delta_{rj} a_i\right) \\ &= \sum_{j=1}^{n} a_j K_{rj} + \sum_{i=1}^{n} a_i K_{ir} \\ &= 2\sum_{i=1}^{n} a_i K_{ir} \end{aligned} \tag{192}$$

Likewise

$$\frac{\partial D}{\partial a_j} = 2\sum_{i=1}^{n} a_i M_{ir}$$

$$\frac{\partial R}{\partial a_r} = 0$$

gives

$$\left(\frac{\partial N}{\partial a_r}\right) - \Lambda^n \left(\frac{\partial D}{\partial a_r}\right) = 0$$

$$2\sum_{i=1}^{n} a_i K_{ir} - \Lambda^n 2\sum_{i=1}^{n} a_i M_{ir} = 0$$

$$\left(K - \Lambda^n M\right)\boldsymbol{a} = \mathbf{0}$$

where K_{ir} are the elements of K and M_{ir} are the elements of M
The eigenvalue problem is now algebraic. The n eigenvalues Λ^n are approximations of the first n eigenvalues of the actual model, the quality of which is determined by the ability of

$$W^n = \sum_{i=1}^{n} a_i \phi_i$$

to represent each mode.

$$W_r^n = \sum_{i=1}^{n} a_{ir} \phi_i$$

a_{ir} is the ith element of the rth vector
W_r^n are the *Ritz eigenfunctions*, Λ_r^n are the *Ritz eigenvalues*.
Space S^n is finite, we can consider W^n to be part of the solution for W, with

$$a_{n+1} = a_{n+2} = \ldots = 0$$

(we ignored higher terms)
Since λ_1 is the minimum value of Rayleigh's quotient over K_G^P, and Λ_1^n is the minimum in the subspace S^n of K_G^P, $\qquad \lambda_1 \leq \Lambda_1^n$

Since the higher eigenfunctions are

- orthogonal to the first eigenfunction

- presumably not completely represented by $\sum a_i \phi_i$.

Then

- The Ritz eigenvalues bound the true eigenvalues from above.
- Usually the lower eigenvalues are best represented.
- The Rayleigh-Ritz method requires trying $W^1 = a_1\phi_1, W^2 = a_1\phi_1 + a_2\phi_2, W^3 = a_1\phi_1 + a_2\phi_2 + a_3\phi_3, \ldots$ and observing convergence of the eigenvalues.
- Often (but not usually) the Rayleigh-Ritz method is applied to find only the first (fundamental) natural frequency. A function approximating the first natural frequency is tried. This is what was done in the section on Green's functions.
- Note that using additional functions, ϕ, is simple since for each additional term in the series, only 3 new calculations must be made. i.e.

$$K^{n+1} = \begin{bmatrix} K^n & | \\ - & + \end{bmatrix} \qquad M^{n+1} = \begin{bmatrix} M^n & | \\ - & + \end{bmatrix}$$

Let Λ_i^n be the i^{th} eigenvalue of the n-term Rayleigh-Ritz solution and Λ_i^{n+1} be the i^{th} eigenvalue of the $n+1$-term Rayleigh-Ritz solution. Then

$$\Lambda_1^{n+1} \leq \Lambda_1^n \leq \Lambda_2^{n+1} \leq \Lambda_2^n \leq \ldots \Lambda_n^n \leq \Lambda_{n+1}^{n+1}$$

This is the p-type FEA method.

Note:

- As n increases, the approximations of the eigenvalues improves monotonically
- As n increases, the approximations of the eigenvalues approach the true eigenvalues from above
- The rate of convergence depends on the nature of the admissible functions chosen

Important constraints:

1. The admissible functions must be linearly independent
2. They must be complete in the energy space K_G^p
3. Simple functions should be chosen to simplify calculations
4. Example sets: Power series, trig functions, Bessel functions, Legendre polynomials, Tchebycheff polynomials
5. Orthogonality is not required, but if

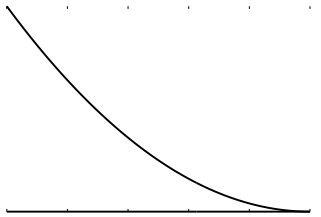
$$\left(\sqrt{m}\phi_r, \sqrt{m}\phi_s\right) = \delta_{rs}$$

   the effort will be reduced
6. The best method is often to solve a more simple similar problem and use the eigenfunctions as admissible functions for the more difficult problem.

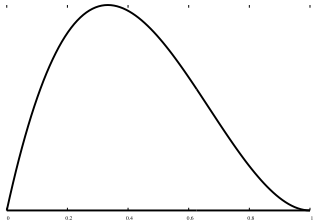
9.1.1 Example: Cantilever Beam, Shames, p 340

Find the first two natural frequencies of a beam free at the left end, and clamped at the right end.
Assume two admissible functions:

$\phi_1(x) = \left(1 - \frac{x}{\ell}\right)^2$



$\phi_2(x) = \frac{x}{\ell}\left(1 - \frac{x}{\ell}\right)^2$



$$W^2(x) = a_1\left(1 - \frac{x}{\ell}\right)^2 + a_2\frac{x}{\ell}\left(1 - \frac{x}{\ell}\right)^2$$

$$L = EI\frac{d^4}{dx^4}$$

$$K_{ij} = [\phi_i, \phi_j] = \int_0^\ell \phi_i EI\frac{d^4}{dx^4}\phi_j dx$$

$$= EI\int_0^\ell \left(\frac{d^2}{dx^2}\phi_i\right)\left(\frac{d^2}{dx^2}\phi_j\right)dx \qquad \text{other terms are zero}$$

$$K_{11} = EI\int_0^\ell \left(\phi''\right)^2 dx = EI\int_0^\ell \left(\frac{2}{\ell^2}\right)^2 dx = \frac{4EI}{\ell^3}$$

$$K_{12} = K_{21} = EI\int_0^\ell \left(\phi_1''\right)\left(\phi_2''\right)dx = EI\int_0^\ell \frac{2}{\ell^2}\frac{2}{\ell^2}\left(3\frac{x}{\ell} - 2\right)dx = -\frac{2EI}{\ell^3}$$

$$K_{22} = EI\int_0^\ell \left(\phi_2''\right)^2 dx = EI\int_0^\ell \left(\frac{2}{\ell^2}\right)^2\left(3\frac{x}{\ell} - 2\right)^2 dx = \frac{4EI}{\ell^3}$$

$$M_{ij} = \left(\sqrt{m}\phi_i, \sqrt{m}\phi_j\right)$$

$$M_{11} = m\int_0^\ell \left(1-\frac{x}{\ell}\right)^4 dx = \frac{m\ell}{5}$$

$$M_{12} = M_{21} = m\int_0^\ell \left(1-\frac{x}{\ell}\right)^2 \frac{x}{\ell}\left(1-\frac{x}{\ell}\right)^2 dx = \frac{m\ell}{30}$$

$$M_{22} = m\int_0^\ell \left(\frac{x}{\ell}\right)^2 \left(1-\frac{x}{\ell}\right)^2 dx = \frac{m\ell}{105}$$

$$K = \frac{EI}{\ell^3}\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$$

$$M = m\ell\begin{bmatrix} \frac{1}{5} & \frac{1}{30} \\ \frac{1}{30} & \frac{1}{105} \end{bmatrix}$$

$$(K-\Lambda M)\,\boldsymbol{a} = \mathbf{0}$$

$$\Lambda_1^2 = 12.60\frac{EI}{m\ell^4}$$

$$\Lambda_2^2 = 1212\frac{EI}{m\ell^4}$$

True values are

$$\Lambda_1 = 12.30\frac{EI}{m\ell^4}$$

$$\Lambda_2 = 483\frac{EI}{m\ell^4}$$

The second function chosen was a lousy choice, and thus our estimate of the second natural frequency is way too high.

9.2 The Assumed-Modes Method

Similar to the Rayleigh-Ritz method except the solution is assumed to be

$$w^n(x,t) = \sum_{\ell=1}^{n} \phi_i(x) q_i(t)$$

The kinetic energy can be written

$$T = \frac{1}{2}\int_0^\ell m\dot{w}^2 dx$$

and the potential energy can be written in general form

$$V = \frac{1}{2}[w, w]$$

substituting the series approximation into T yields

$$T = \frac{1}{2}\int_0^\ell m\left(\sum_{i=1}^n \phi_i \dot{q}_i\right)\left(\sum_{j=1}^n \phi_j \dot{q}_j\right) dx =$$

$$= \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \dot{q}_i \dot{q}_j \int_0^\ell m\phi_i\phi_j dx$$

$$= \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \dot{q}_i \dot{q}_j M_{ij}$$

where

$$M_{ij} = \int_0^\ell m\phi_i\phi_j dx, \qquad i, j = 1, 2, 3, \ldots n$$

Likewise for the potential energy

$$V = \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n q_i q_j K_{ij}$$

where

$$K_{ij} = [\phi_i, \phi_j], \qquad i, j = 1, 2, 3, \ldots n$$

Note that the mass and stiffness matrices are identical to those obtained in the Rayleigh-Ritz method.
Lagrange's equation for a discrete system (No damping, external loads, or T_0 or T_1 energy) are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0 \qquad r = 1, ..., n$$

$$L = T - V$$

Substituting for L yields

$$\sum_{j=1}^n (M_{rj}\ddot{q}_j + K_{rj}q_j) = 0 \qquad r = 1, 2, ..., n$$

Assuming $\ddot{q}_r = -\omega_r^2 q_r = -\lambda_r q_r$

$$\sum_{j=1}^{n} \left(K_{rj} - \Lambda M_{rj}\right) q_j = 0 \qquad r = 1, 2, ..., n$$

which is identical to the Rayleigh-Ritz method.

9.3 Weighted Residual Methods

Rayleigh-Ritz: Variation of Rayleigh's quotient is zero.
Weighted residual methods are applied directly using the DE and BCs.
Consider a *trial* function $W(x)$ in the space K_B^{2p} substituted into the differential equation.

$$LW(x) = \lambda m W(x)$$

The residual is the error

$$R\left(W, x\right) = LW - \lambda m W$$

and is a function of the trial function W and depends on the position x.
If the trial function is an eigenfunction,

$$R\left(W, x\right) = 0$$

Next consider a *test* function v (weighting function) from the space K^0
Define the weighted residual

$$vR = v\left(LW - \lambda m W\right)$$

If v is orthogonal to R, then $(v, R) = 0$
Restrictions on the space K^{2p} can be changed using integration by parts.
Again, assume a solution of the form

$$W^n = \sum_{i=1}^{n} a_i \phi_i(x)$$

$\phi_i\left(x\right)$ are a complete set of trial functions (comparison function)
$\phi_i(x)$ are a basis for the subspace S^n of K_B^{2p}
Choose n functions ψ_i and regard then as a basis for the subspace V^n
S^n is the trial space
V^n is the test space
The coefficients a_i are determined by placing the constraint that ψ_i be orthogonal to $R\left(W^n, x\right)$

$$\left(\psi_i, R\right) = \int_0^{\ell} \psi_i \left(LW^n - m\lambda^n W^n\right) dx = 0 \qquad \ell = 1, 2, \ldots, n$$

Why does this work?

As $n \to \infty, V^n$ fills the entire space,
Then the only way for $(\psi_i, R) = 0$ is for $R = 0$.

$$\lim_{n\to\infty} (LW^n - \lambda^n m W^n) = LW - \lambda m W = 0$$

Substituting the approximation for W yields

$$(\psi_i, R) = \int_0^\ell \psi_i \left(\sum_{j=1}^n a_j L\phi_i - \lambda^n \sum_{j=1}^n a_j m\phi_j \right) dx$$

$$(\psi_i, R) = \sum_{j=1}^n \left(K_{ij} - \lambda^n \sum M_{ij} \right) a_j = 0$$

$$K_{ij} = (\psi_i, L\phi_j) = \int_0^\ell \psi_i L\phi_j dx$$

$$M_{ij} = (\psi_i, m\phi_j) = \int_0^\ell \psi_i m\phi_j dx$$

K_{ij} is usually not symmetric, regardless of whether or not L is self-adjoint. (Depends on technique.)

9.3.1 Galerkin's Method (Ritz's Second Method)

Assume the weighting functions are the same as the trial functions

$$\psi_i = \phi_i \qquad \ell = 1, 2, ...n$$

$$K_{ij} = (\phi_i, L\phi_j)$$
$$M_{ij} = (\phi_i, m\phi_j)$$

Which are the same as the Ritz results if L is self-adjoint.
Consider the non self-adjoint eigenvalue problem

$$-\frac{d}{dx}\left(s\frac{dW}{dx}\right) + r\frac{dW}{dx} = \lambda m W$$

$$K_{ij} = (\phi_i, L\phi_i) = \int_0^\ell \phi_i \left[-\frac{d}{dx}\left(s\frac{d\phi_j}{dx}\right) + r\frac{d\phi_j}{dx}\right] dx$$

Integrating by parts and considering the boundary conditions yields

$$K_{ij} = \int_0^\ell \left(s\phi_i'\phi_j' + r\phi_i\phi_j'\right) dx$$

which is not symmetric in ϕ_i and ϕ_j. What happened to the boundary condition terms when integrating by parts?

9.3.2 Example: Clamped-clamped beam (Dimarogonas)

Consider a clamped-clamped beam, length ℓ
Choose $\psi_1(x) = 1 - \cos\frac{2\pi x}{\ell}$, $\psi_2(x) = 1 - \cos\frac{4\pi x}{\ell}$

$$R_i = \int_0^\ell \psi_i(LW - m\omega^2 W)dx$$

For the two trial functions

$$R_1 = \int_0^\ell \psi_1(LW^2 - m\omega^2 W^2)dx = \ell(a_1\frac{8EI\pi^4}{\ell^4} - \frac{3}{2}a_1 m\omega^2 - a_2 m\omega^2)$$

$$R_2 = \int_0^\ell \psi_2(LW^2 - m\omega^2 W^2)dx = \ell(a_2\frac{128EI\pi^4}{\ell^4} - a_1 m\omega^2 - \frac{3}{2}a_2 m\omega^2)$$

Where we recall that W^2 is the two term representation of the solution. Thus

$$\begin{bmatrix} \frac{8EI\pi^4}{\ell^4} & 0 \\ 0 & \frac{128EI\pi^4}{\ell^4} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \omega^2 \begin{bmatrix} \frac{3}{2}m & m \\ m & \frac{3}{2}m \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{0}$$

or

$$\begin{bmatrix} 8\pi^4 & 0 \\ 0 & 128\pi^4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \lambda \begin{bmatrix} \frac{3}{2} & 1 \\ 1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{0}$$

where

$$\lambda = \frac{\omega^2 m\ell^4}{EI}$$

$\lambda^{1/2}$	22.47	-	124.06
True	22.373	61.673	120.90

The second mode was missed due to a poor selection of functions ψ_i.
For the first mode

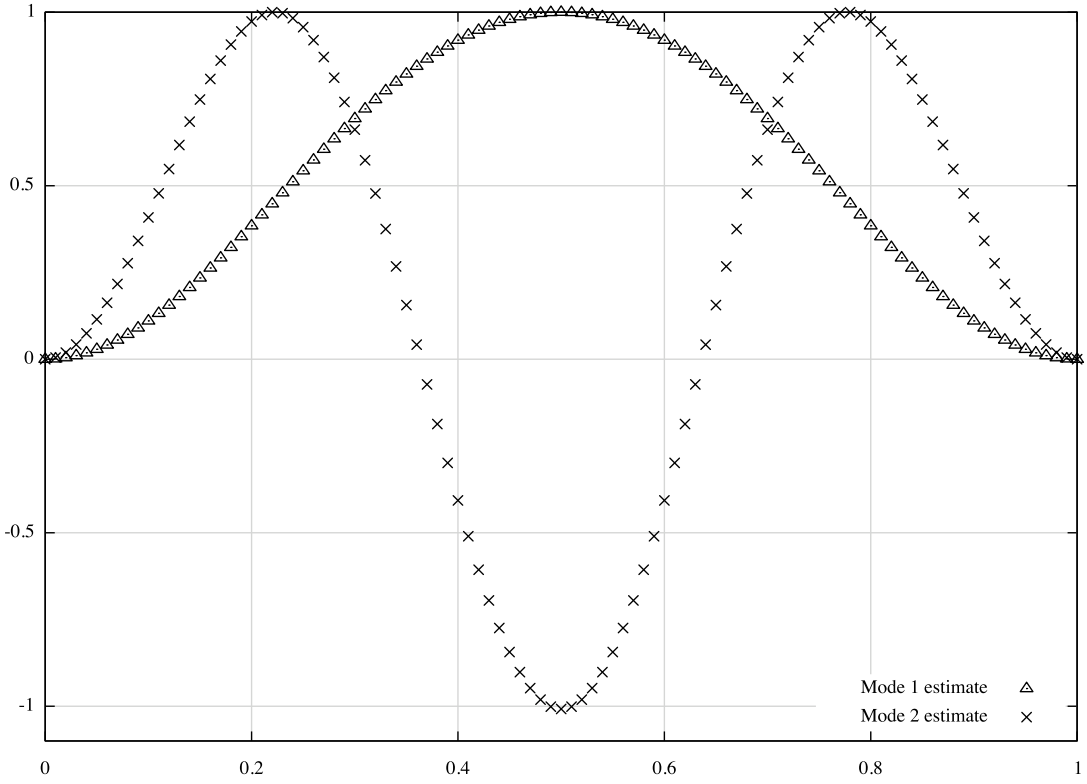
$$a_1 = 23.2, \qquad a_2 = 1$$

$$\phi_1 = 23.2\left(1 - \cos\frac{2\pi x}{\ell}\right) + 1 - \cos\frac{4\pi x}{\ell}$$

For the second mode

$$a_1 = -.69, \qquad a_2 = 1$$

$$\phi_2 = -.69\left(1 - \cos\frac{2\pi x}{\ell}\right) + 1 - \cos\frac{4\pi x}{\ell}$$


1
0.5
0
-0.5
-1
0
0.2
0.4
0.6
0.8
1
Mode 1 estimate
Mode 2 estimate

9.3.3 The Collocation Method

The weighting functions are spatial Dirac delta functions

$$\psi_i = \delta(x - x_i)$$

The points $x = x_i$ are chosen in advance

$$(\psi_i, R) = \int_0^\ell \delta(x - x_i)\left(LW^n - \lambda m W^n\right) dx = 0$$
$$= \left(LW^n - \lambda^n m W^n\right)|_{x=x_i} = 0$$

What this means is the D.E. is satisfied at n preselected locations.
As n increases, the equation is satisfied everywhere.
Now

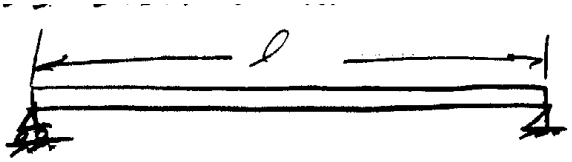
$$K_{ij} = \int_0^\ell \delta(x - x_i) L\phi_j dx = L\phi_j(x_i)$$

$$M_{ij} = \int_0^\ell \delta(x - x_i) m\phi_j dx = m\phi_j(x_i)$$

The coefficients here are not symmetric because δ is not a comparison function of L.
This is true even when L is self-adjoint.
Pro: Easy to find K_{ij}, M_{ij}
Con: Difficult to obtain solution to non-symmetric eigenvalue problem.

9.3.4 Collocation Method Example

(interior method)
Simply-supported beam with non-uniform mass distribution.



$$m(x) = m_0 \sin\frac{\pi x}{\ell} \qquad EI = const$$

weighting functions	trial functions
$\psi_1 = \delta(x - \frac{\ell}{4})$	$\phi_1 = \sin\frac{\pi x}{\ell}$
$\psi_2 = \delta(x - \frac{\ell}{2})$	$\phi_2 = \sin\frac{2\pi x}{\ell}$
$\psi_3 = \delta(x - \frac{3\ell}{4})$	$\phi_3 = \sin\frac{3\pi x}{\ell}$

$$K_{ij} = L\phi_j(x_i) = EI\frac{d^4}{dx^4}\phi_j(x_i)$$

$$= EI\left(\frac{j\pi}{\ell}\right)^4 \sin\frac{j\pi x_i}{\ell}$$

$$K_{11} = EI\left(\frac{\pi}{\ell}\right)^4 \frac{1}{\sqrt{2}}, \quad K_{12} = EI\left(\frac{2\pi}{\ell}\right)^4, \quad K_{13} = EI\left(\frac{3\pi}{\ell}\right)^4 \frac{1}{\sqrt{2}}$$

$$K_{21} = EI\left(\frac{\pi}{\ell}\right)^4, \quad K_{22} = 0, \quad K_{23} = -EI\left(\frac{3\pi}{\ell}\right)^4$$

$$K_{31} = EI\left(\frac{\pi}{\ell}\right)^4 \frac{1}{\sqrt{2}}, \quad K_{32} = -EI\left(\frac{2\pi}{\ell}\right)^4, \quad K_{33} = EI\left(\frac{3\pi}{\ell}\right)^4 \frac{1}{\sqrt{2}}$$

$$K = EI\left(\frac{\pi}{\ell}\right)^4 \begin{bmatrix} \frac{1}{\sqrt{2}} & 16 & 57.27 \\ 1 & 0 & -81 \\ \frac{1}{\sqrt{2}} & -16 & 57.27 \end{bmatrix}$$

$$M_{ij} = m\left(x_i\right)\phi_j\left(x_i\right)$$

$$M_{11} = \frac{1}{2}m_0, \quad M_{12} = \frac{1}{\sqrt{2}}m_0, \quad M_{13} = \frac{1}{2}m_0$$

$$M_{21} = m_0, \quad M_{22} = 0, \quad M_{23} = -m_0$$

$$M_{31} = \frac{1}{2}m_0, \quad M_{32} = -\frac{1}{\sqrt{2}}m_0, \quad M_{33} = \frac{1}{2}m_0$$

Assembling:

$$M = m_0 \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ 1 & 0 & -1 \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

$$\sqrt{eig\left(M^{-1}K\right)} = \left(\frac{1}{\ell^2}\right)\sqrt{\frac{EI}{m_0}}\left(11.74, 39.48, 105.6\right)$$

For comparison, a uniform simply supported beam:

$$\omega_n = \left(\frac{1}{\ell^2}\right)\sqrt{\frac{EI}{m_0}}(9.87, 39.48, 88.83)$$

9.4 System Response By Approximate Methods: Galerkin's Method - the foundation of Finite Elements

9.4.1 Damped (and undamped) Non-gyroscopic System

Consider

$$Lw(x,t) + C\frac{\partial w(x,t)}{\partial t} + M\frac{\partial^2 w(x,t)}{\partial t^2} = f(x,t)$$

Let

$$w(x,t)^n = \sum_{j=1}^{n} \phi_j(x) q_j(t)$$

where ϕ_j are comparison functions.
Substituting yields

$$\sum_{j=1}^{n} \left(L\phi_j q_j + C\phi_j \dot{q}_j + m\left(x\right)\phi_j \ddot{q}_j \right) = f(x,t)$$

Pre-multiplying by ϕ_i, $i = 1, 2, \ldots n$ and integrating over ℓ.

$$M\ddot{\boldsymbol{q}} + C\dot{\boldsymbol{q}} + K\boldsymbol{q} = \boldsymbol{F}(t)$$

$$M_{ij} = \int_0^\ell \phi_i m(x) \phi_j dx$$

$$K_{ij} = \int_0^\ell \phi_i L \phi_j dx$$

$$C_{ij} = \int_0^\ell \phi_i C \phi_j dx$$

$$F_i = \int_0^\ell \phi_i f dx$$

The discrete eqns can be solved as shown before.

10 The Computational Eigenvalue Problem

10.1 Householder's method

Reduced a symmetric matrix to tri-diagonal form

More efficient than Given's method, 1/2 as many multiplications. Requires n-2 transformations.

$$A_k = P_k A_{k-1} P_k, \qquad k = 1, 2, 3, \ldots, n-2$$

where

$$P_k = I - 2\boldsymbol{v}_k \boldsymbol{v}_k^T$$

A_1 must have the form

$$A_1 = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & 0 & \cdots & 0 \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & \cdots & a_{3n}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(1)} & a_{n3}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix}$$

the requirement is then

$$a_{31}^{(1)} = a_{41}^{(1)} = \cdots = a_{n1}^{(1)} = 0$$

with $\boldsymbol{v}_k^T \boldsymbol{v}_k = 1$
Let's pick $v_{1,1} = 0$, so

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 - 2v_{12}^2 & -2v_{12}v_{13} & \cdots & -2v_{1,2}v_{1n} \\ 0 & -2v_{12}v_{13} & 1 - 2v_{13}^2 & \cdots & -2v_{1,3}v_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -2v_{12}v_{1n} & \cdots & \cdots & 1 - 2v_{1n}^2 \end{bmatrix}$$

Then

$$v_{1,2} = \left(\frac{1}{2} \left(1 \mp \frac{a_{1,2}}{\alpha_1} \right) \right)^{\frac{1}{2}}$$

where

$$\alpha_1 = \left(\sum_{j=2}^{n} a_{1j}^2 \right)^{\frac{1}{2}}$$

then

$$v_{i,j} = \mp \frac{a_{1j}}{2\alpha_1 v_{12}}$$

Where the signs are chosen to be the same as that of a_{12}.
The procedure is generalized to

$$\boldsymbol{v}_k^T = \begin{bmatrix} 0 & 0 & 0 \cdots & v_{k,k+1} & v_{k,k+2} & \cdots & v_{k,n} \end{bmatrix}$$

where

$$v_{k,k+1} = \left(\frac{1}{2} \left(1 \mp \frac{a_{k,k+1}}{\alpha_k} \right) \right)^{\frac{1}{2}}$$

and

$$v_{k,j} = \mp \frac{a_{kj}}{2\alpha_k v_{k,k+1}}$$

where

$$\alpha_k = \left(\sum_{j=k+1}^{n} a_{kj}^2 \right)^{\frac{1}{2}}$$

The eigenvectors of the transformed matrix, $T = A_k$, are then related to those of the original matrix A, by $T = P^T AP$.
Example:

```
A =

     1     1     1     1
     1     2     2     2
     1     2     3     3
     1     2     3     4

k = 1
```

$$\alpha_k = \left(\sum_{j=k+1}^{n} a_{kj}^2 \right)^{\frac{1}{2}}$$

```
alpha = 1.7321
```

$$v_{k,k+1} = \left(\frac{1}{2} \left(1 \mp \frac{a_{k,k+1}}{\alpha_k} \right) \right)^{\frac{1}{2}}$$

and

$$v_{k,j} = \mp \frac{a_{kj}}{2\alpha_k v_{k,k+1}}$$

```
V =

  0.00000
  0.88807
  0.32506
  0.32506
```

$$P_k = I - 2\boldsymbol{v}_k\boldsymbol{v}_k^T$$

```
P1 =

   1.00000   0.00000   0.00000   0.00000
   0.00000  -0.57735  -0.57735  -0.57735
   0.00000  -0.57735   0.78868  -0.21132
   0.00000  -0.57735  -0.21132   0.78868

A =

   1.00000  -1.73205   0.00000   0.00000
  -1.73205   7.66667  -0.54466  -1.12201
   0.00000  -0.54466   0.37799   0.16667
   0.00000  -1.12201   0.16667   0.95534

k = 2
alpha = 1.2472
V =

  0.00000
  0.00000
  0.84755
  0.53071

Psub =

   1.00000   0.00000   0.00000   0.00000
   0.00000   1.00000   0.00000   0.00000
   0.00000   0.00000  -0.43670  -0.89961
```

```
   0.00000   0.00000  -0.89961   0.43670

A =

   1.00000  -1.73205   0.00000   0.00000
  -1.73205   7.66667   1.24722  -0.00000
   0.00000   1.24722   0.97619  -0.12372
   0.00000   0.00000  -0.12372   0.35714
```

The resulting matrix is a tridiagonal matrix

$$A_k = PA_0P \tag{193}$$

where

$$P = \prod_1^k P_i \tag{194}$$

10.2 The QR Method

The QR Method for obtaining eigenvalues is computationally expensive if the matrix is full. It is very efficient when Given's method (or the more efficient Householder's method) is applied first.

```
A1=[2 -1 0;-1 2.5 -1.5;0 -1.5 3]
A1 =
    2.0000   -1.0000         0
   -1.0000    2.5000   -1.5000
         0   -1.5000    3.0000
First, try to get rid of sub-diagonal elements
s1=-1/sqrt(2^2+1)
s1 =
   -0.4472
c1=2/sqrt(2^2+1)
c1 =
    0.8944
```

The rotation matrix Θ_1 is

$$\Theta_1 = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
t1=[c1 s1 0;-s1 c1 0;0 0 1]
t1 =
    0.8944   -0.4472         0
    0.4472    0.8944         0
         0         0    1.0000
```

We now do a partial rotation, $A_{1p1} = \Theta_1 A_1$

```
A1p1=t1*A1
A1p1 =
    2.2361   -2.0125    0.6708
         0    1.7889   -1.3416
         0   -1.5000    3.0000
```

Now do a rotation to remove $a_{3,2}$

```
s2=A1p1(3,2)/sqrt(A1p1(3,2)^2+A1p1(2,2)^2)
s2 =
   -0.6425
c2=A1p1(2,2)/sqrt(A1p1(3,2)^2+A1p1(2,2)^2)
c2 =
    0.7663
```

The rotation matrix Θ_2 is

$$\Theta_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

```
t2=[1 0 0;0 c2 s2;0 -s2 c2]
t2 =
    1.0000         0         0
         0    0.7663   -0.6425
         0    0.6425    0.7663
A1p1 =
    2.2361   -2.0125    0.6708
         0    1.7889   -1.3416
         0   -1.5000    3.0000
A1p2=t2*A1p1
A1p2 =
    2.2361   -2.0125    0.6708
         0    2.3345   -2.9556
         0   -0.0000    1.4367
```

Now we need to post-multiply by our rotations

```
Q1=t1'*t2'
Q1 =
    0.8944    0.3427    0.2873
   -0.4472    0.6854    0.5747
         0   -0.6425    0.7663
A2=A1p2*Q1
A2 =
    2.9000   -1.0440    0.0000
   -1.0440    3.4991   -0.9231
    0.0000   -0.9231    1.1009
```

This is the completion of one step Note that this is identical to

$$A_2 = \Theta_2 \Theta_1 A_1 \Theta_1^T \Theta_2^T \tag{195}$$

The beginning of subsequent steps begins with the solution of the eigenvalue problem for the lowest 2×2 principle minor of the *original* matrix. i.e.

$$\text{eig}\left(\begin{bmatrix} 2.5 & -1.5 \\ -1.5 & 3 \end{bmatrix}\right)$$

```
eig(A1(2:3,2:3))
ans =
  1.22931
  4.27069
```

The eigenvalue closest to 3 (bottom right value) is 4.27.
Next, solve the eigenvalue problem for the lowest 2×2 principle minor of the *new* matrix.

```
eig(A2(2:3,2:3))
ans =
    3.8133
    0.7867
```

The eigenvalue closest to 1.1009 (bottom right value) is 0.7867.

$$\left|\frac{0.7867}{4.27} - 1\right| = .81 > \frac{1}{2}$$

Thus, a shift is not in order (we will have one later). So, repeat the process again from the start.

```
s1=A2(2,1)/sqrt(A2(2,1)^2+A2(1,1)^2)
s1 =
   -0.3387
c1=A2(1,1)/sqrt(A2(2,1)^2+A2(1,1)^2)
c1 =
    0.9409
t1=[c1 s1 0;-s1 c1 0;0 0 1]
t1 =
    0.9409   -0.3387         0
    0.3387    0.9409         0
         0         0    1.0000
A2p1=t1*A2
A2p1 =
    3.0822   -2.1676    0.3127
   -0.0000    2.9386   -0.8686
    0.0000   -0.9231    1.1009
s2=A2p1(3,2)/sqrt(A2p1(3,2)^2+A2p1(2,2)^2)
s2 =
   -0.2997
c2=A2p1(2,2)/sqrt(A2p1(3,2)^2+A2p1(2,2)^2)
c2 =
    0.9540
t2=[1 0 0;0 c2 s2;0 -s2 c2]
```

```
t2 =
    1.0000         0         0
         0    0.9540   -0.2997
         0    0.2997    0.9540
A2p2=t2*A2p1
A2p2 =
    3.0822   -2.1676    0.3127
   -0.0000    3.0802   -1.1586
    0.0000    0.0000    0.7900
Q2=t1'*t2'
Q2 =
    0.9409    0.3232    0.1015
   -0.3387    0.8976    0.2820
         0   -0.2997    0.9540
A3=A2p2*Q2
A3 =
    3.6342   -1.0433    0.0000
   -1.0433    3.1121   -0.2368
    0.0000   -0.2368    0.7537
```

This is the end of a second rotation. We already have the eigensolution of the 2×2 minor of A_2. We need it for A_3.

```
eig(A3(2:3,2:3))
ans =
    3.1356
    0.7301
```

$$\left| \frac{0.7301}{0.7867} - 1 \right| = 0.0719 < \frac{1}{2}$$

A shift is thus advisable. The next iteration begins by using $A_3 - 0.7301I$.

```
A3p=A3-eye(3)*ans(2)
A3p =
    2.9041   -1.0433    0.0000
   -1.0433    2.3820   -0.2368
    0.0000   -0.2368    0.0235
s1=A3p(2,1)/sqrt(A3p(2,1)^2+A3p(1,1)^2)
s1 =
   -0.3381
c1=A3p(1,1)/sqrt(A3p(2,1)^2+A3p(1,1)^2)
c1 =
    0.9411
t1=[c1 s1 0;-s1 c1 0;0 0 1]
```

```
t1 =
    0.9411   -0.3381         0
    0.3381    0.9411         0
         0         0    1.0000
A3p1=t1*A3p
A3p1 =
    3.0858   -1.7873    0.0801
   -0.0000    1.8889   -0.2228
    0.0000   -0.2368    0.0235
s2=A3p1(3,2)/sqrt(A3p1(3,2)^2+A3p1(2,2)^2)
s2 =
   -0.1244
c2=A3p1(2,2)/sqrt(A3p1(3,2)^2+A3p1(2,2)^2)
c2 =
    0.9922
t2=[1 0 0;0 c2 s2;0 -s2 c2]
t2 =
    1.0000         0         0
         0    0.9922   -0.1244
         0    0.1244    0.9922
A3p2=t2*A3p1
A3p2 =
    3.0858   -1.7873    0.0801
   -0.0000    1.9037   -0.2240
    0.0000    0.0000   -0.0044
```

```
Q3=t1'*t2'
Q3 =
    0.9411    0.3355    0.0421
   -0.3381    0.9338    0.1170
         0   -0.1244    0.9922
```

We now complete the rotation and add back the subtracted approximation to the eigenvalue.

```
A4=A3p2*Q3+(eye(3)*.7301)
A4 =
    4.2385   -0.6437   -0.0000
   -0.6437    2.5356    0.0005
    0.0000    0.0005    0.7258
```

the 3×3 element is almost completely independent, so the first extracted eigenvalue is approximately 0.7258 (we could, or course, keep going to converge further). This compares to the true value of 0.725817
The entire process is then repeated again for the sub-matrix created from removing the last row and column.

10.3 Subspace Iteration

Consider the eigenvalue problem

$$K\Phi = \lambda M\Phi$$

for the case where we only need the 1st p eigenvalues and eigenvectors
Start with an initial guess X_1 $(n \times q)$ like the Rayleigh Ritz method

$$\underbrace{\bar{X}_2}_{n\times q} = K^{-1}M \underbrace{X_1}_{n\times q}$$

which can be solved for using Gauss elimination
A reduced eigenvalue problem is obtained

$$\underbrace{K_2}_{q\times q} = \bar{X}_2^T K \bar{X}_2$$

$$M_2 = \bar{X}_2^T M \bar{X}_2$$

The reduced eigenvalue problem is

$$K_2 Q_2 = M_2 Q_2 \Lambda_2$$

which can be solved for using other techniques better for smaller eigenvalue problems.
In general, $q \ll n$.
An improved approximation for the eigenvectors is

$$\underbrace{X_2}_{n\times q} = \underbrace{\bar{X}_2}_{n\times q}\underbrace{Q_2}_{q\times q}$$

Repeat this

$$\underbrace{\bar{X}_{k+1}}_{n\times q} = K^{-1}M \underbrace{X_k}_{n\times q}$$

$$\underbrace{K_{k+1}}_{q\times q} = \bar{X}_{k+1}^T K \bar{X}_{k+1}$$

$$M_{k+1} = \bar{X}_{k+1}^T M \bar{X}_{k+1}$$

Solve

$$K_{k+1} Q_{k+1} = M_{k+1} Q_{k+1} \Lambda_{k+1}$$

for Q_{k+1}, then

$$\underbrace{X_{k+1}}_{n\times q} = \underbrace{\bar{X}_{k+1}}_{n\times q}\underbrace{Q_{k+1}}_{q\times q}$$

Repeat until convergence.
Example;

```
K =

   2.00000  -1.00000   0.00000
  -1.00000   2.50000  -1.50000
   0.00000  -1.50000   3.00000

M =

        1         0         0
        0         1         0
        0         0         1


>> x1=[1 1;0 1;0 1]

x1 =

        1         1
        0         1
        0         1

>> x2bar=K\M*x1
x2bar =

  0.70000  1.30000
  0.40000  1.60000
  0.20000  1.13333

>> k2=x2bar'*K*x2bar
k2 =

  0.70000  1.30000
```

```
  1.30000  4.03333
>> m2=x2bar'*M*x2bar
m2 =

  0.69000  1.77667
  1.77667  5.53444
>> [q2,lam2]=eig(m2\k2);
>> q2=q2(:,[2,1]),lam2=diag(sort(diag(lam2)))% need to sort eigenvalues
q2 =

  -0.02669   0.95230
   0.99964  -0.30515

lam2 =

  0.72874  0.00000
  0.00000  2.34844

>> x2=x2bar*q2;x2=x2/[norm(x2(:,1)) 0;0 norm(x2(:,2))]% normalize vector lengths

x2 =

   0.54935   0.81937
   0.68141  -0.32580
   0.48362  -0.47169

>> x3bar=K\M*x2

x3bar =

   0.75384   0.34890
   0.95832  -0.12157
   0.64037  -0.21801

>> k3=x3bar'*K*x3bar
k3 =

  1.37683  0.00339
```

```
  0.00339  0.42832

>> m3=x3bar'*M*x3bar
m3 =

  1.89671  0.00690
  0.00690  0.18404

>> [q3,lam3]=eig(m3\k3);q3=q3(:,[2,1]),lam3=diag(sort(diag(lam3)))% need to sort e
q3 =

   0.00417  -0.99998
  -0.99999  -0.00548

lam3 =

  0.72590  0.00000
  0.00000  2.32760
```

```
>> x3=x3bar*q3;x3=x3/[norm(x3(:,1)) 0;0 norm(x3(:,2))] %normalize eigenvectors

x3 =

  -0.54874  -0.80601
  -0.69534   0.29272
  -0.46410   0.51445
```

After 9 iterations

$$\Lambda = \begin{bmatrix} 0.7258 & 0 \\ 0 & 2.3198 \end{bmatrix}$$

and

$$\Phi = \begin{bmatrix} 0.5480 & 0.7909 \\ 0.6983 & -0.2533 \\ 0.4606 & -0.5571 \end{bmatrix}$$

Initial vectors are chosen to b:

1. Diagonal of mass matrix
2. Vectors of $+1$ where m_{ii}/k_{ii} are maximum.

Only the first p eigenvalues are useful. q is generally $\min(2p, p+8)$.
This will converge to the lowest p eigenvalues given that all of the vectors in X_1 are *not* orthogonal to any of the corresponding eigenvectors.

10.4 Shifting

Consider the eigenvalue problem

$$(K - \lambda M)\boldsymbol{\psi} = \mathbf{0}$$

Subspace iteration does not work with a singular K. (rigid body motion capable system)
Consider redefining $\lambda = \mu - 1$. Then the eigenvalue problem is restated as

$$\begin{aligned} (K - \lambda M)\boldsymbol{\psi} &= \mathbf{0} \\ (K - (\mu - 1)M)\boldsymbol{\psi} &= \mathbf{0} \\ ((K + M) - \mu M)\,\boldsymbol{\psi} &= \mathbf{0} \\ (K' - \mu M)\,\boldsymbol{\psi} &= \mathbf{0} \end{aligned} \tag{196}$$

K' is non-singular. The obtained eigenvectors are the same. The eigenvalues can be obtain by *unshifting* the eigenvalues using $\lambda = \mu - 1$.

References