Sp '06 Exam 2 Solns

1) Shape functions are

$$N = \left[\tfrac{1}{2}\xi(\xi-1) \quad 1-\xi^2 \quad \tfrac{1}{2}\xi(\xi+1)\right]$$

For a rod element

$$K = \int_{x_1}^{x_3} \frac{\partial}{\partial x} N^T \, EA \, \frac{\partial}{\partial x} N \, dx$$

Since $X = N \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $x_2 = \dfrac{x_1 + x_3}{2}$

$$X = \cancel{\tfrac{1}{2}\xi^2 x_1} - \tfrac{1}{2}\xi x_1 + \frac{x_1+x_3}{2} - \cancel{\xi^2 \frac{x_1+x_3}{2}} + \cancel{\tfrac{1}{2}\xi^2 x_3} + \tfrac{1}{2}\xi x_3$$

$$= \frac{x_1+x_3}{2} + \left(\frac{x_3-x_1}{2}\right)\xi$$

$$\therefore \quad J = \frac{dx}{d\xi} = \frac{x_3-x_1}{2} = \frac{\ell}{2}$$

$$K = \int_{-1}^{1} \frac{\partial}{\partial \xi} N^T \, EA \, \frac{\partial}{\partial \xi} N \, \frac{\ell}{2} \, d\xi$$

$$B = \frac{1}{J}\frac{\partial}{\partial \xi} N = \left[\xi - \tfrac{1}{2} \quad -2\xi \quad \xi + \tfrac{1}{2}\right]\frac{2}{\ell}$$

see script

$$K = \frac{2EA}{\ell}\begin{bmatrix} 1.167 & -1.33 & .167 \\ & 2.67 & -1.33 \\ & & 1.67 \end{bmatrix}$$

```
% Problem 1
disp('Problem 1')
K=zeros(3,3);
Xi=[-1 1]/sqrt(3);
for i=1:2
xi=Xi(i)
B=[xi-1/2 -2*xi xi+1/2]
K=K+B'*B
end
disp('Problem 2')

evaledpoly= [-1 1  -1  1; %poly evaluated at -1
              3 -2  1  0; %poly derivative evaluated at -1
              1  1  1  1; %poly evaluated at 1
              3  2  1  0];%poly dervative evaluated at 1
disp('Shape function 1 coefficients')
N1=evaledpoly\[1 0 0 0]'


disp('Shape function 1 coefficients')
N2=evaledpoly\[0 1 0 0]'


disp('Shape function 1 coefficients')
N3=evaledpoly\[0 0 1 0]'

disp('Shape function 1 coefficients')
N4=evaledpoly\[0 0 0 1]'
gset key
xi=-1:.01:1;
grid on
plot(xi,polyval(N1,xi),"-;N1;",xi,polyval(N2,xi),'-;N2;',xi,polyval(N3,xi),'-;N3;',xi,polyval(N4,
xi),'-;N4;')

B1=polyderiv(polyderiv(N1))

B2=polyderiv(polyderiv(N2))

B3=polyderiv(polyderiv(N3))

B4=polyderiv(polyderiv(N4))

disp("Don't forget that terms need to be multiplied ordivided by J as shown in the written soluti
on.")
```

2) Shape functions are derived in script

$$K_{14} = \int_{-1}^{1} B^T \, E I(\xi) \, B \, J \, d\xi$$

$$N_1 = \frac{\xi^3}{4} - \frac{3\xi}{4} + \frac{1}{2}$$

$$N_2 = \frac{1}{4}\left(\xi^3 - \xi^2 - \xi + 1\right) J$$

$$N_3 = \frac{1}{4}\left(\xi^3 + 3\xi + 2\right)$$

$$N_4 = \frac{1}{4}\left(\xi^3 + \xi^2 - \xi - 1\right) J$$

Condition for N_2 is $\theta = 1 @ x_1$

$$\frac{d}{dx} v(x)\Big|_{x=x_1} = 1$$

Using shape functions in natural coordinates

$$\frac{d}{d\xi} V(\xi)\Big|_{\xi=-1} = \frac{dx}{d\xi}\frac{d}{dx} V(x)\Big|_{x=x_1}$$

$$= J \frac{d}{dx} V(x)\Big|_{x=x_1}$$

So since $\frac{d}{dx} v(x)\Big|_{x=x_1} = 1$, then $\frac{d}{d\xi} v(\xi)\Big|_{\xi=-1} = J$. For computational purposes, we set them to 1, then scale the coefficients by J

$$I = \frac{I_1}{2}(1-\xi) + \frac{I_2}{2}(1+\xi)$$

$$J = \frac{L}{2}$$

$$B = \frac{1}{J^2}\frac{d^2}{d\xi^2} N$$

$$= \frac{4}{L^2}\left[\frac{3}{2}\xi \quad \left(\frac{3}{2}\xi - \frac{1}{2}\right)\frac{L}{2} \quad \frac{-3}{2}\xi \quad \left(\frac{3}{2}\xi + \frac{1}{2}\right)\frac{L}{2}\right]$$

Polynomial to be integrated is 3^{rd} order

$2n - 1 = 3$, $n = 2$. 2 points are sufficient

$$K_{14} = \frac{4}{L^2}\sum_{i=1}^{2} \frac{3}{2}\xi\left(\frac{3}{2}\xi + \frac{1}{2}\right)\left(\frac{I_1}{2}(1-\xi) + \frac{I_2}{2}(1+\xi)\right)$$

$$\xi_1 = \frac{-1}{\sqrt{3}}, \quad \xi_2 = \frac{1}{\sqrt{3}}$$

$$= \frac{2I_1 + 4I_2}{L^2} \, E$$

3) There is an error in the problem. The equation given for $\bar{x}$ actually gives $\bar{y}$. $\bar{x}$ is given by $\frac{\int x \, dA}{\int dA}$

Since the polynomial is 1^{st} order, one point should be sufficient. Let's see

$$y = N\,[y] \qquad\qquad x = N\,[x]$$

$$= 2s - rs - s^2 \qquad\qquad = r$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} = \begin{bmatrix} 1 & -s \\ 0 & 2-r-2s \end{bmatrix}$$

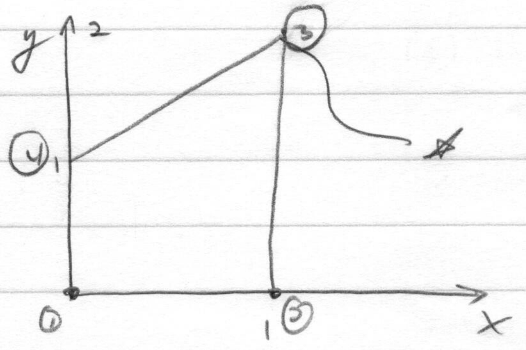
very important

$$\det[J] = \mathcal{J} = 2 - r - 2s$$

So $\int_0^1 \int_0^1 y \, \mathcal{J} \, dr \, ds = \int_0^1 \int_0^1 (2s - rs - s^2)(2 - r - 2s) \, dr \, ds$

is a 3^{rd} order polynomial and requires 4 points.
For exam, one loop makes the most sense
See script for solution.
$\bar{x} = \frac{1}{3}$, $\bar{y} = \frac{1}{3}$. You should be able to do these by hand (you should have them memorized.)

```
%Problem 3
A=0;
Qx=0;
Qy=0;
weights=[-27/48 25/48 25/48 25/48];
r=[1/3 3/5 1/5 1/5]
s=[1/3 1/5 1/5 3/5]
for i=1:4
    w=weights(i)
    x=r(i);
    y=2*s(i)-r(i)*s(i)-s(i)^2
    J=1/2*(2-r(i)-2*s(i))
    A=A+w*J
    Qx=Qx+w*y*J
    Qy=Qy+w*x*J
end
Qx/A
Qy/A
```

4) 

Find Stresses @ 3 presuming only $U_3, V_3 \neq 0$

We need $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$.

To get them, we need J and $\frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \frac{\partial v}{\partial \xi}, \frac{\partial v}{\partial \eta}$

$u = \sum N_i u_i \quad , \quad v = \sum N_i v_i$

We can evaluate these where we need them

$$\frac{\partial u}{\partial \xi} = \frac{\partial N_3}{\partial \xi} u_3 = \frac{1}{2} u_3 \ , \quad \frac{\partial u}{\partial \eta} = \frac{\partial N_3}{\partial \eta} u_3 = \frac{1}{2} u_3 \ , \quad \frac{\partial v}{\partial \xi} = \frac{\partial N_3}{\partial \xi} v_3 = \frac{1}{2} v_3 \ , \quad \frac{\partial v}{\partial \eta} = \frac{\partial N_3}{\partial \eta} v_3 = \frac{1}{2} v_3$$

$$J = \frac{1}{4}\begin{bmatrix} 0 & 0 & 2 & -2 \\ 0 & -2 & 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial v} \end{bmatrix} = J^{-1}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} u_3 = \frac{1}{2} U_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{likewise} \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \frac{1}{2} v_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So
$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{xy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} u_3 \\ \frac{1}{2} v_3 \\ \frac{1}{2}(u_3 + v_3) \end{bmatrix}.$$

We need more information to determine what [E] to use.