ME 460/660 Final Exam Solutions Spring 2009

1) $(K - M\omega_n^2)\,\underline{u}_n = \underline{0}$ (with a 20-year old calculator, this takes 3 min, but here it is old-school)

let $\begin{bmatrix} 3000-100\omega_n^2 & -1000 \\ -1000 & 4000-200\omega_n^2 \end{bmatrix} = 0$

$$1.2\times10^7 - 1\times10^6\,\omega_n^2 + 20000\,\omega_n^4 - 1\times10^6 = 0$$

$$2\omega_n^4 - 100\,\omega_n^2 + 1100 = 0$$

$$\omega_n^2 = \frac{100 \pm \sqrt{10{,}000 - 8800}}{4} = 25 \pm 8.66$$

$$= 16.34,\ 33.66$$

$$\omega_n = 4.04,\ 5.8 \text{ rad/s}$$

<u>For $\omega_1 = 4.04$ rad/s</u>

$$(3000-1634)\,u_1 - 1000\,u_2 = 0$$
$$1366\,u_1 - 1000\,u_2 = 0$$

$$\underline{u}_1 = \begin{bmatrix} 1 \\ 1.366 \end{bmatrix} \text{ or } \begin{bmatrix} 0.732 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0.591 \\ 0.807 \end{bmatrix}$$

Mass normalized also o.k.

Similarly, for $\omega_2 = 4.04$

$$\underline{u}_2 = \begin{bmatrix} 1 \\ -0.366 \end{bmatrix} \text{ or } \begin{bmatrix} 2.73 \\ -1 \end{bmatrix} \text{ or } \begin{bmatrix} 0.939 \\ -0.344 \end{bmatrix}$$

2) Step response is valid for $0 < t < \frac{1}{30}$

$$x_1(t) = \frac{F}{8.888 \times 10^4}\left(1 - \cos 94.2t\right)$$

Solution for $\frac{1}{30} < t$ is the sum of the response to 2 steps.
The "2nd Step" is $-F$ at $t = \frac{1}{30}$, so

$$x_{s2}(t) = \frac{-F}{8.888 \times 10^4}\left(1 - \cos 94.2\left(t - \tfrac{1}{30}\right)\right)$$

$94.2/30 = \pi$

$$= \frac{-F}{8.888 \times 10^4}\left(1 + \cos 94.2t\right)$$

-1.1×10^{-5}

$$x(t) = X_1(t) + X_{s2}(t)$$

$$= \frac{-2F}{8.888 \times 10^4}\left(\cos 94.2t\right)$$

-2.2×10^{-5}

See exam 2 on how to solve using convolution integral

3) First, find modal forces

$$\underline{f_m} = S^T \underline{f} = \begin{bmatrix} \frac{10}{\sqrt{2}} \sin 5t \\ \frac{10}{\sqrt{2}} \sin 5t \end{bmatrix}$$

Find modal responses.

$$r_1(t) = \frac{10}{\sqrt{2}} \frac{1}{\underbrace{\omega_1^2}_{3^2} - 5^2} \sin 5t$$

$$= -0.442 \sin 5t$$

$$r_2(t) = \frac{10}{\sqrt{2}} \frac{1}{10^2 - 5^2} \sin 5t = 0.094 \sin 5t$$

$$\underline{x}(t) = S \, \underline{r}(t)$$

$$= \begin{bmatrix} -0.313 \sin 5t + 0.067 \sin 5t \\ \\ -0.313 \sin 5t - 0.067 \sin 5t \end{bmatrix}$$

$$= \begin{bmatrix} -0.246 \sin 5t \\ -0.379 \sin 5t \end{bmatrix}$$

4) Clearly 2 DOF, given 2 DOF listed! Use Lagrange

$$T = \frac{1}{2} m \dot{y}^2 + \frac{1}{2}\left(\frac{1}{12} m \ell^2\right)\dot{\theta}^2$$

$$U = mgy + \frac{1}{2}(2k_2)\left(y + \theta\frac{\ell}{2}\right)^2 + \frac{1}{2}(2k_1)\left(y - \theta\frac{\ell}{4}\right)^2$$

<u>For y:</u>

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{y}} = m\ddot{y} \qquad \frac{\partial U}{\partial y} = mg + 2k_2\left(y + \theta\frac{\ell}{2}\right) + 2k_1\left(y - \theta\frac{\ell}{4}\right)$$

$+mg$

$$\underline{m\ddot{y} + 2(k_1 + k_2)y + \ell\left(k_2 - \frac{k_1}{2}\right)\theta + mg = 0}$$

<u>For θ:</u>

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = \frac{1}{12} m \ell^2 \ddot{\theta}$$

$$\frac{\partial U}{\partial \theta} = \ell k_2\left(y + \theta\frac{\ell}{2}\right) - \frac{1}{2}\ell k_1\left(y - \theta\frac{\ell}{4}\right)$$

$$\underline{\frac{1}{12} m \ell^2 \ddot{\theta} + \left(\ell k_2 - \frac{\ell k_1}{2}\right)y + \left(\frac{\ell^2 k_2}{2} + \frac{\ell^2 k_1}{8}\right)\theta = 0}$$

5) You can find mode shapes, or remember them

$$X\ddot{T} = c^2 X'' T$$

$$X\, T\omega_n^2 = c^2 X'' T$$

$$\frac{X''}{X} = -\sigma_n^2 = \frac{-\omega_n^2}{c^2} = \frac{\ddot{T}}{c^2 T}$$

$$X'' + \sigma_n^2 X = 0 \quad , \quad X(x) = A\cos\sigma_n l + B\sin\sigma_n l$$

$$X'(x)\Big|_{x=0} = 0 \quad , \quad X'(x)\Big|_{x=l} = 0 \qquad \therefore B = 0$$

$$\sigma_n \sin\sigma_n l = 0 \quad , \quad \sigma_n = 0, \frac{n\pi}{l} \quad , \quad \underline{\sigma_n = \frac{n\pi}{l}} \qquad n = 0,1,2,\ldots$$

$$w(x,t) = \sum_{n=0}^{\infty} \cos(\sigma_n x)\, a_n \underbrace{\sin 3t}_{\text{we know temporal form}}$$

Sub into eqn

$$-9\sum_{n=0}^{\infty} a_n \cos\sigma_n x \sin 3t + c^2\sigma_n^2 \sum_{n=0}^{\infty} \cos(\sigma_n x) a_n \sin 3t = 100\,\delta\!\left(x - \frac{l}{3}\right)\sin 3t$$

Multiply by $\cos\sigma_m x$, integrate from 0 to l.

$$-9\,\frac{l}{2}\,a_m + c^2\sigma_m^2\,\frac{l}{2}\,a_m = 100\cos\sigma_m \frac{l}{3}$$

$$\underline{a_m = \frac{200}{l\left(c^2\left(\frac{m\pi}{l}\right)^2 - 9\right)} \cos\frac{m\pi}{3}}$$

$$w(x,t) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{l} \sin 3t$$ where a_m is defined above.