Subspace Iteration (Bathe + Wilson)

$$K\phi = \lambda M \phi$$

Consider the case where we only need the 1st p eigenvalues and eigenvectors

Start with an initial guess $\underset{n\times q}{X_1}$ (recall Rayleigh-Ritz method)

$$\underset{n\times q}{\bar{X}_2} = K^{-1} M \underset{n\times q}{X_1}$$

which can be solved using Gauss elimination.

A reduced eigenvalue problem is obtained

$$\underset{q\times q}{K_2} = \underset{q\times n}{\bar{X}_2^T}\ \underset{n\times n}{K}\ \underset{n\times q}{\bar{X}_2}$$

$$M_2 = \bar{X}_2^T M \bar{X}_2$$

$$K_2 Q_2 = M_2 \underset{q}{Q_2} \underset{q}{\Lambda_2}$$

and must be solved using previous techniques (matrix iteration, Jacobi method...). Note that in general, $q << n$ where q is the number of basis vectors chosen.

An improved approximation for the eigenvectors is

$$X_2 = \bar{X}_2 \, Q_2$$

Repeat

$$\bar{X}_{k+1} = K^{-1} M X_k$$

$$K_{k+1} = \bar{X}_{k+1}^T \, K \, \bar{X}_{k+1}$$

$$M_{k+1} = \bar{X}_{k+1}^T M \, \bar{X}_{k+1}$$

Solve

$$K_{k+1} \, Q_{k+1} = M_{k+1} \, Q_{k+1} \, \Lambda_{k+1}$$

for Q_{k+1}

$$X_{k+1} = \bar{X}_{k+1} \, Q_{k+1}$$

<u>Example</u>

$$K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2.5 & -1.5 \\ 0 & -1.5 & 3 \end{bmatrix} \qquad M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Assume $X_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

$$\bar{X}_2 = \begin{bmatrix} .7 & 1.3 \\ .4 & 1.6 \\ .2 & 1.133 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} .7 & 1.3 \\ 1.3 & 4.03 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} .69 & 1.777 \\ 1.777 & 5.534 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} -.0267 & .9523 \\ .9996 & -.3052 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} .5494 & .8194 \\ .6814 & -.3258 \\ .4836 & -.4717 \end{bmatrix}$$

$$\left(\Lambda = \begin{bmatrix} .7287 & \\ & 2.3484 \end{bmatrix} \right)$$

$$\bar{X}_3 = \begin{bmatrix} .7538 & +.3489 \\ .9583 & -.1216 \\ .6404 & -.2180 \end{bmatrix}$$

$$K_3 = \begin{bmatrix} 1.3768 & \text{sym} \\ .0034 & .4283 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 1.8967 & .0069 \\ \text{sym} & .1840 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 1 & -.0042 \\ .0055 & 1 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} .5487 & .8060 \\ .6953 & -.2927 \\ .4641 & -.5145 \end{bmatrix}$$

$$\vdots \qquad \left(\Lambda = \begin{bmatrix} .7259 & \\ & 2.3276 \end{bmatrix} \right)$$

$$\left.\begin{aligned} \Lambda &= \begin{bmatrix} .7258 & 0 \\ 0 & 2.3198 \end{bmatrix} \\ \phi &= \begin{bmatrix} .5480 & .7909 \\ .6983 & -.2533 \\ .4606 & -.5571 \end{bmatrix} \end{aligned}\right\} \text{After 9 iterations}$$

Initial vectors are chosen to be

1) The diagonal of the mass matrix

2) Vectors with entries of +1 where m_{ii}/k_{ii} are maximum. See prev. example.

<u>Note:</u>
Lowest eigenvalues are generally found to higher accuracy than higher eigenvalues

$$\left| \frac{\lambda_i^{(k+1)} - \lambda_i^{k}}{\lambda_i^{(k+1)}} \right| < tol$$

tol should be # of digits of accuracy.

Only the 1st p eigenvalues are used/checked.

q is generally min (2p, p+8)

Will converge to p lowest eigenvalues provided X_1 is not mass orthogonal to any of lowest p eigenvectors.