1) $\delta \int_{t_1}^{t_2} -\frac{1}{2}\int_0^l EI(\alpha')^2\,dx - \frac{1}{2}\int_0^l KGA\left(\alpha - \frac{\partial v}{\partial x}\right)^2 dx$

$+ \frac{1}{2}\int_0^l \rho A \dot{v}^2\,dx + \frac{1}{2}\int_0^l \rho I \dot{\alpha}^2\,dx$

$+ \delta W_{nc}\; dt = 0$

① Integrating the $1^{st}$ term by parts and such

$$-\int_0^l EI\,\alpha'\,\delta\alpha'\,dx$$

$$* \quad = -EI\,\alpha'\,\delta\alpha\Big|_0^l + \int_0^l \frac{\partial}{\partial x} EI\,\alpha'\,\delta\alpha\,dx$$

② $\delta\left[-\frac{1}{2}\int_0^l \left(KGA(\alpha^2 - 2\alpha v' + v'^2\right)dx\right]$

$$= -\int_0^l KGA\,\alpha\,\delta\alpha\,dx + \int_0^l KGA(\alpha\,\delta v' + \delta\alpha\; v')\,dx$$

$$- \int_0^l KGA\,v'\,\delta v'\,dx$$

$$= -\int_0^l KGA\,\alpha\,\delta\alpha\,dx + \int_0^l KGA\,v'\,\delta\alpha\,dx$$

$$* \quad + KGA\,\alpha\,\delta v\Big|_0^l - \int_0^l \frac{\partial}{\partial x}(KGA\,\alpha)\,\delta v\,dx$$

$$- KGA\,v'\,\delta v\Big|_0^l + \int_0^l \frac{\partial}{\partial x}(KGA\,v')\,\delta v\,dx$$

③ $\int_0^l \rho A \dot{v}\,\delta v \Big|_{t_1}^{t_2} dx - \int_{t_1}^{t_2}\int_0^l \rho A \ddot{v}\,\delta v\,dx\,dt$

④ $\int_0^l \rho I \dot{\alpha}\,\delta\alpha \Big|_{t_1}^{t_2} dx - \int_{t_1}^{t_2}\int_0^l \rho I \ddot{\alpha}\,\delta\alpha\,dx\,dt$

Combining
$\therefore$ EOM

① $\frac{\partial}{\partial x} EI\alpha' - KGA(\alpha - v') - \rho I \ddot{\alpha} = 0$

② $\frac{-\partial}{\partial x}(KGA\,\alpha) + \frac{\partial}{\partial x}(KGA\,v') - \rho A \ddot{v} + p(x,t) = 0$

② $\frac{\partial}{\partial x}\left(KGA(\alpha - v')\right) + \rho A \ddot{v} = p(x,t)$

with B.C.

$\underbrace{EI\alpha'}_{①}\ \underbrace{\delta\alpha}_{②} = 0$ at $x=0,\ x=l$

$\underbrace{KGA\left(\alpha - \frac{\partial v}{\partial x}\right)}_{①}\ \underbrace{\delta v}_{②} = 0$ @ $x=0,\ x=l$

2)

$$m_{11} = \int_0^1 2\xi(1-\xi)^4 \, d\xi$$

$$= \int_0^1 2\xi(1-2\xi+\xi^2)^2 \, d\xi$$

$$= \int_0^1 2\xi(1-4\xi+6\xi^2-4\xi^3+\xi^4) \, d\xi$$

$$= \xi^2 - \frac{8}{3}\xi^3 + \frac{12}{4}\xi^4 - \frac{8}{5}\xi^5 + \frac{2}{6}\xi^6 \Big|_0^1$$

$$= 1 - \frac{8}{3} + 3 - \frac{8}{5} + \frac{2}{6} = .06666$$

$$m_{12} = \int_0^1 2\xi^2(1-\xi)^4 \, d\xi$$

$$= \int_0^1 2\xi^2(1-4\xi+6\xi^2-4\xi^3+\xi^4) \, d\xi$$

$$= \frac{2}{3}\xi^3 - 2\xi^4 + \frac{12}{5}\xi^5 - \frac{4}{3}\xi^6 + \frac{2}{7}\xi^7 \Big|_0^1$$

$$= \frac{2}{3} - 2 + \frac{12}{5} - \frac{4}{3} + \frac{2}{7} = .01905$$

$$m_{22} = \int_0^1 2\xi^3(1 - 4\xi + 6\xi^2 - 4\xi^3 + \xi^4)\, d\xi$$

$$= \frac{2}{4}\xi^4 - \frac{8}{5}\xi^5 + \frac{12}{6}\xi^6 - \frac{8}{7}\xi^7 + \frac{2}{8}\xi^8 \Big|_0^1$$

$$= \frac{1}{2} - \frac{8}{5} + 2 - \frac{8}{7} + \frac{1}{4} = .007143$$

$$K_{11} = \int_0^1 (1-\xi)^2 \frac{\partial^2}{\partial\xi^2} \frac{2}{3}\xi^3 \frac{\partial^2}{\partial\xi^2}(1-\xi)^2\, d\xi$$

$$= \int_0^1 (1-\xi)^2 \frac{\partial^2}{\partial\xi^2} \frac{2}{3}\xi^3 \; 2\, d\xi$$

$$= \int_0^1 (1-\xi)^2 \frac{\partial^2}{\partial\xi^2} \frac{4}{3}\xi^3\, d\xi$$

$$= \int_0^1 (1-\xi)^2 \; 8\xi\, d\xi$$

$$= \int_0^1 (1 - 2\xi + \xi^2)\, 8\xi\, d\xi$$

$$= \int_0^1 8\xi - 16\xi^2 + 8\xi^3\, d\xi$$

$$= 4\xi^2 - \frac{16}{3}\xi^3 + 2\xi^4 \Big|_0^1$$

$$= 4 - \frac{16}{3} + 2 = \frac{2}{3} = .6666$$

$$K_{12} = \int_0^1 \xi(1-\xi)^2 \; 8\xi \, d\xi$$

$$= \int_0^1 8\xi^2 - 16\xi^3 + 8\xi^4 \, d\xi$$

$$= \frac{8}{3}\xi^3 - 4\xi^4 + \frac{8}{5}\xi^5 \Big|_0^1$$

$$= \frac{8}{3} - 4 + \frac{8}{5} = .2666$$

$$K_{22} = \int_0^1 \xi(1-\xi)^2 \frac{\partial^2}{\partial\xi^2} \frac{2}{3}\xi^3 \frac{\partial^2}{\partial\xi^2} \xi(1-\xi)^2 \, d\xi$$

from scrap paper

$$= \int_0^1 \xi(1-\xi)^2 \frac{\partial^2}{\partial\xi^2} \left(4\xi^4 - \frac{8}{3}\xi^3\right) d\xi$$

$$= \int_0^1 \xi(1-\xi)^2 \left(48\xi^2 - 16\xi\right) d\xi$$

$$= \int_0^1 16\xi^2 \left(1 - 2\xi + \xi^2\right)\left(3\xi - 1\right) d\xi$$

$$= \int_0^1 16\xi^2 \left(3\xi - 6\xi^2 + 3\xi^3 - 1 + 2\xi - \xi^2\right) d\xi$$

$$= \int_0^1 16\xi^2 \left(-1 + 5\xi - 7\xi^2 + 3\xi^3\right) d\xi$$

$$= 16\left(\frac{-1}{3} + \frac{5}{4} - \frac{7}{5} + \frac{3}{6}\right)$$

$$= .2666$$

$$M = \begin{bmatrix} .06666 & .01905 \\ .01905 & .007143 \end{bmatrix} \qquad K = \begin{bmatrix} .6666 & .2666 \\ .2666 & .2666 \end{bmatrix}$$

1 term: $\omega = \sqrt{\frac{K_{11}}{m_{11}}} = \sqrt{10} = 3.16$

2 term:

$$det \begin{bmatrix} .6666 - .0666\lambda & .26666 - .01905\lambda \\ / & .26666 - .007143\lambda \end{bmatrix} \doteq 0$$

from scrap

$.106667 - .01238\lambda + .0001133\lambda^2 = 0$

$\lambda = 9.43 \quad , 99.84$

$\omega = 3.07 \quad , 9.99$

3) 1 term

$$m_{11} = 2\xi(1-\xi)^2 \Big|_{\xi=.5}$$

$$= .25$$

$$K_{11} = \frac{\partial^2}{\partial \xi^2} \frac{2}{3}\xi^3 \frac{\partial^2}{\partial \xi^2}(1-\xi)^2 \Big|_{\xi=\frac{1}{2}}$$

From 2

$$= 8\xi \Big|_{\xi=\frac{1}{2}} = 4$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{4}{.25}} = \sqrt{16} = 4 \text{ rad/s}$$

2 term

$$m_{11} = 2\xi(1-\xi)^2 \Big|_{\xi=\frac{1}{3}} = .2963$$

$$m_{21} = 2\xi(1-\xi)^2 \Big|_{\xi=\frac{2}{3}} = .148148$$

$$m_{12} = 2\xi^2(1-\xi)^2 \Big|_{\xi=\frac{1}{3}} = .098765$$

$$m_{22} = 2\xi^2(1-\xi)^2 \Big|_{\xi=\frac{2}{3}} = .098765$$

$$K_{11} = 8\xi\Big|_{1/3} = \frac{8}{3} = 2.666$$

$$K_{21} = 8\xi\Big|_{2/3} = \frac{16}{3} = 5.333$$

$$K_{12} = \frac{\partial^2}{\partial\xi^2}\left(\frac{2}{3}\xi^3 \ \frac{\partial^2}{\partial\xi^2}\ \xi(1-\xi)^2\right)\Big|_{1/3}$$

From ②

$$= 48\xi^2 - 16\xi\Big|_{1/3} = 0$$

$$K_{22} = 48\xi^2 - 16\xi\Big|_{2/3} = 10.6667$$

$$M = \begin{bmatrix} .2963 & .098765 \\ .148148 & .098765 \end{bmatrix}$$

$$K = \begin{bmatrix} 2.6666 & 0 \\ 5.3333 & 10.6667 \end{bmatrix}$$

$\lambda = 10.36,\ 187.6$

$\omega = 3.219,\ 13.70$ rad/s

4) $C$ is semi-definite

The only way for the zero eigenvalue to show up as zero damping for a mode is for it to have the same eigenvector as a system mode shape. Observing $M$ and $K$, $\begin{bmatrix}1\\1\end{bmatrix}\frac{1}{\sqrt{2}}$ is not a mode shape.
(It is the eigenvector of $C$ corresponding to the zero eigenvalue.)