Exam 1, 2004 Solutions

1)

$$\delta\pi = \delta \underline{d}^T \underbrace{\int_0^l N^T \, k \, N \, dx}_{\Delta K} \underline{d}$$

For the foundation

$$\therefore \Delta K_{11} = \int_0^l N_1^2 \, k(x) \, dx \qquad \Delta K_{21} = \int_0^l N_1 N_2 \, k(x) \, dx$$

eg for constant $k(x) = k$

$$\Delta K_{11} = k \int_0^l 1 - \frac{3x^2}{l^2} + 2\frac{x^3}{l^3} - 3\frac{x^2}{l^2} + 9\frac{x^4}{l^2} - 6\frac{x^5}{l^5} + 2\frac{x^3}{l^3} - 6\frac{x^5}{l^5} + 4\frac{x^6}{l^6} \, dx$$

$$= k \int_0^l 1 - 6\frac{x^2}{l^2} + 4\frac{x^3}{l^3} + 9\frac{x^4}{l^4} - 12\frac{x^5}{l^5} + 4\frac{x^6}{l^6} \, dx$$

$$= k \left[l - 2l + l + \frac{9}{5} l - \frac{12}{6} l + \frac{4}{7} l \right]$$

$$= kl \frac{78}{210} = \boxed{kl \frac{26}{70}}$$

$$\Delta K_{12} = k \int_0^l x - 2\frac{x^2}{l} + \frac{x^3}{l^2} - 3\frac{x^3}{l^2} + 6\frac{x^4}{l^3} - 3\frac{x^5}{l^4} + 2\frac{x^4}{l^3} - 4\frac{x^5}{l^4} + 2\frac{x^6}{l^5} \, dx$$

$$= k \int_0^l x - 2\frac{x^2}{l} - 2\frac{x^3}{l^2} + 8\frac{x^4}{l^3} - 7\frac{x^5}{l^4} + 2\frac{x^6}{l^5} \, dx$$

$$= k\left(\frac{1}{2} l^2 - \frac{2}{3} l^2 - \frac{1}{2} l^2 + \frac{8}{5} l^2 - \frac{7}{6} l^2 + \frac{2}{7} l^2 \right).$$

$$= \boxed{kl^2 \frac{11}{210}}$$

2) $F = \int_0^l N^T f(x)\,dx$

On DOF 1 $\quad F_1 = a\int_0^l x - \frac{3x^3}{l^2} + 2\frac{x^4}{l^3}\,dx$

$= a\left(\frac{l^2}{2} - \frac{3}{4}l^2 + \frac{2}{5}l^2\right)$

$= \frac{3}{20}\,a\,l^2$

Likewise

$F_2 = a\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right)l^3$

$= \frac{1}{30}\,a\,l^3$

$F_3 = a\left(\frac{3}{4} - \frac{2}{5}\right)l^2$

$= \frac{7}{20}\,a\,l^2$

$F_4 = a\left(\frac{-1}{4} + \frac{1}{5}\right)l^3$

$= \frac{-1}{20}\,a\,l^3$

3) $M = \int_0^{\ell} N^T \rho A(x) N \, dx$

$N = \left[1 - \frac{x}{\ell} \quad \frac{x}{\ell}\right] \qquad A(x) = A_1 + (A_2 - A_1)\frac{x}{\ell}$

$$M_{22} = \rho \int_0^{\ell} \left(\frac{x}{\ell}\right)^2 A_1 + (A_2 - A_1)\left(\frac{x}{\ell}\right)^3 dx$$

$$= \rho \left(\frac{\ell^3}{3\ell^2} A_1 + (A_2 - A_1) \frac{\ell^4}{4\ell^3} \right)$$

$$= \rho \left(\frac{1}{3} A_1 \ell + (A_2 - A_1) \frac{1}{4} \ell \right)$$

$$= \underline{\rho \ell \left(\frac{1}{12} A_1 + \frac{1}{4} A_2 \right)}$$

By symmetry of the problem (renaming node 1 -> node 2 etc)

$$M_{11} = \underline{\rho \ell \left(\frac{1}{4} A_1 + \frac{1}{12} A_2 \right)}$$

$$M_{12} = \rho \int_0^{\ell} \frac{x}{\ell}\left(1 - \frac{x}{\ell}\right)\left(A_1 + (A_2 - A_1)\frac{x}{\ell}\right) dx$$

$$= \rho \left(\frac{1}{2} A_1 \ell + \frac{1}{3}(A_2 - A_1)\ell - \rho \ell \left(\frac{1}{12} A_1 + \frac{1}{4} A_2 \right) \right)$$

$$= \underline{\frac{1}{12} \rho (A_1 + A_2) \ell}$$

by symmetry $\underline{M_{12} = M_{21}}$

For ordering $[u_1, v_1, w_1, u_2, v_2, w_2]$

c) $M_{11} = M_{22} = M_{33} = (M_{11} \text{ in 2-D})$
$M_{14} = M_{25} = M_{36} = (M_{12} \text{ in 2-D})$
$M_{44} = M_{55} = M_{66} = (M_{22} \text{ in 2-D})$
All other elements are zero in the upper triangular part, matrix is symmetric.

Explanation: Mass matrix for a rod has equivalent derivation in each of the 3 directions because the shape functions are the same in the 3 directions, i.e. $N_1 = N_2 = N_3$, $N_4 = N_5 = N_6$.

5) $N_1 = \left(1 - \frac{x}{l}\right) \qquad N_2 = \frac{x}{l}$

$$u(x) = u_1 N_1 + u_2 N_2$$

$$u' = u_1 N_1' + u_2 N_2' = \underbrace{\begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix}}_{N'} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

There is a typo. Clearly the 1^{st} term should be $\frac{1}{2} E A \frac{du}{dx}^2$

Also, for the potential field

$$V' = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Integrating over the element, taking the variation of things that can vary.

$$\delta U = \delta \underline{u} \int_0^l N'^T E A N' \, dx \, \underline{u} - \delta \underline{V} \int_0^l N'^T eAN' dx \, \underline{u}$$

$$- \delta \underline{u} \int_0^l N'^T eAN' dx \, \underline{V} - \delta \underline{V} \int_0^l N'^T \epsilon A N' dx \, \underline{V}$$

Factoring coefficients and performing integrals

$$\frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \underline{u} - \frac{eA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \underline{V} = \underline{0}$$

and

$$\frac{eA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \underline{u} - \frac{\epsilon A}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \underline{V} = \underline{0}$$