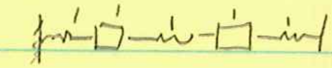
ME 710 Exam 2 Solns, Wi 2011

1) By observation, this system is 
Mode shapes are $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$, $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

$$\omega_1^2 = \frac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 1, \omega_1 = 1 \text{ rad/s}$$

$$\omega_2^2 = \frac{1}{2}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}3\\-3\end{bmatrix} = 3, \omega_2 = 1.73$$

For perturbation, $A_0 = \begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}$

$$A = \begin{bmatrix}1.1 & 0\\0 & 1\end{bmatrix}^{-1/2}\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\begin{bmatrix}1.1 & 0\\0 & 1\end{bmatrix}^{-1/2} = \begin{bmatrix}1.82 & -0.95\\-0.95 & 2\end{bmatrix}$$

While we are here, $\omega_1 = 0.98$ rad/s, $\omega_2 = 1.69$ rad/s

So, $\epsilon A_1 = \begin{bmatrix}-0.18 & 0.05\\0.05 & 0\end{bmatrix}$, If we choose $\epsilon = 0.1$,

$A_1 = \begin{bmatrix}-1.8 & 0.5\\0.5 & 0\end{bmatrix}$, so this is a small change

So, using perturbation, $\omega_1^2 = 1 + (0.1)\frac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}-1.8 & 0.5\\0.5 & 0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$

$$= 1 + 0.05(-0.8) = 0.96, \omega_1 = 0.98$$

$$\omega_2^2 = 2.86, \omega_2 = 1.69 \text{ rad/s}$$

There is no error to two places.

2) $T = \int_0^\ell \rho A \dot{w}^2 \, dx + \frac{1}{2} m \dot{w}^2 \Big|_{x=0}$

$U = \int_0^\ell EI \left(\frac{d^2 w}{dx^2} \right)^2 dx + \frac{1}{2} k w^2 \Big|_{x=0}$

$$\delta \int_{t_1}^{t_2} \int_0^\ell T - U \, dx \, dt = 0$$

Considering T, $\delta \int_{t_1}^{t_2} \int_0^\ell T \, dx \, dt = \delta \int_{t_1}^{t_2} \int_0^\ell - \rho A \ddot{w} \, \delta w \, dx + m \ddot{w} \, \delta w \Big|_{x=0} dt$

Considering δU, $\int_{t_1}^{t_2} \int_0^\ell EI \frac{d^4 w}{dx^4} \delta w \, dx - EI \frac{d^3 w}{dx^3} \delta w \Big|_0^\ell + EI \frac{d^2 w}{dx^2} \delta w' \Big|_0^\ell \; dt$

$+ k w \, \delta w \Big|_{x=0}$

Collecting coefs of δx inside integral over x

EOM: $+EI \frac{d^4 w}{dx^4} + \rho A \ddot{w} = 0$

At left end $(x=0)$: $m \ddot{w} + EI \frac{d^3 w}{dx^3} - k w = 0$

and $EI \frac{d^2 w}{dx^2} = 0$

At right end, $w = 0$, $w' = 0$.

3) $EI w'''' + \rho A \ddot{w} = 0$

Subst

$$w(x,t) = u(x,t) + a \sin \omega t$$

$$EI u'''' + \rho A \ddot{u} = -\rho A a \omega^2 \sin \omega t$$

Mass normalized modes are

$$\mathcal{U}(x) = \sqrt{\frac{2}{\rho A}} \sin \frac{n\pi x}{\ell}$$

$$\int_0^\ell \frac{1}{2\rho A} \cdot \rho A \sin^2 \frac{n\pi x}{\ell} dx$$

$$= \frac{1}{2} \int_0^\ell \sin^2 \frac{n\pi x}{\ell} dx = \frac{1}{4}$$

should have been $\sqrt{\frac{2}{\rho A}}$

linear undamped, sine in sine out

$$u_n(x,t) = A_n \sin \omega t \; \mathcal{U}_n(x)$$

Subst

$$\sum_{n=1}^{\infty} \left[EI \left(\frac{n\pi}{\ell}\right)^4 \sqrt{\frac{2}{\rho A}} \sin \frac{n\pi x}{\ell} \sin \omega t - \rho A \sqrt{\frac{2}{\rho A}} \, \omega^2 \sin \frac{n\pi x}{\ell} \sin \omega t \right] = -\rho A a \omega^2 \sin \omega t$$

Mult by $\sqrt{\frac{2}{\rho A}} \sin \frac{m\pi x}{\ell}$ and integrate over domain

$$A_n \left[EI \left(\frac{m\pi}{\ell}\right)^2 \frac{1}{\rho A} \cancel{\sin \omega t} - \omega^2 \cancel{\sin \omega t} \right] = -\rho A a \omega^2 \cancel{\sin \omega t} \sqrt{\frac{2}{\rho A}} (1 - \cos m\pi$$

$$A_n = \frac{-a\omega^2 \rho A \sqrt{\frac{2}{\rho A}} (1 - \cos m\pi)}{\left(\frac{m\pi}{\ell}\right)^2 \frac{EI}{\rho A} - \omega^2}$$

So, $$w(x,t) = \sum_{m=1}^{\infty} \frac{-a\omega^2 (1 - \cos m\pi)}{\left(\frac{m\pi}{\ell}\right)^2 \frac{EI}{\rho A} - \omega^2} \, 2 \sin \frac{m\pi x}{\ell} \sin \omega t - a \sin \omega t$$

4)

Shear strain at point r is

$$r \frac{d\theta}{dx}$$

Shear stress is

$$G r \frac{d\theta}{dx}$$

Moment of stress about center is

$$G r^2 \frac{d\theta}{dx}$$

Total moment is

$$\int_A G r^2 \frac{d\theta}{dx} \, dA$$

$$= G \frac{d\theta}{dx} \underbrace{\int_A r^2 \, dA}_{\text{definition of } J}$$

$$= G J \frac{d\theta}{dx}$$