Fall 2008 ME 460 Final Exam Solns

1) $$\frac{X(j\omega)}{Y(j\omega)} = \frac{k}{k - m\omega^2 + cj\omega}$$

n	ω	$\lvert H(j\omega)\rvert$	$\measuredangle H(j\omega)$	X_n
0	0	$\frac{1}{2}$	0	$\frac{1}{2}$
1	π	1.01	-0.018°	0.32
2	2π	1.04	-0.037°	0.17
3	3π	1.10	-0.059	0.12

$$x(t) = \frac{1}{2} - 0.32\sin(\pi t - 0.018^\circ) - 0.17\sin(2\pi t - 0.037^\circ) - 0.12\sin(3\pi t - 0.06^\circ)$$

phases in rad
-3×10^{-4}
-6.5×10^{-4}
-1×10^{-3}

Note: Excitations are far from resonance. However, this means they are all nearly quasi-static excitations. Such an assumption only leads to a 10% error in the 3rd term. A quasi static excitation solution would give

$$x(t) = \frac{1}{2} - 0.32\sin\pi t - 0.16\sin 2\pi t - 0.11\sin 3\pi t$$

2) See solution to problem 2, on Fall '08 Exam # 2. This uses convolution integral, 2 times. (but the simplification doesn't happen)

3) $\underline{r}(0) = S^{-1}\,\underline{x}(0)$

a) I gave formula for inverting matrix
b) Columns of S are orthogonal, so its inverse is its transpose

$$\underline{r}(0) = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\underline{v}(0) = \underline{0}$$

so $r_1(t) = \frac{1}{\sqrt{2}}\cos 3t$, $r_2(t) = \frac{1}{\sqrt{2}}\cos 10t$

$$x(t) = S\,\underline{r} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}\cos 3t\\ \cos 10t\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{2}\cos 3t + \frac{1}{2}\cos 10t\\ \frac{1}{2}\cos 3t - \frac{1}{2}\cos 10t\end{bmatrix}$$

4) $U = \frac{1}{2}k\,x_{spring}^2 - mgx$ $\qquad x_{spring} = \frac{1}{2}x$

$= \frac{1}{2}\frac{k}{4}x^2 - mgx$

$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2$ $\qquad J = \frac{1}{2}m_{disk}R^2$
subst for J and $\dot{\theta}$ $\qquad \dot{\theta} = \frac{\dot{x}}{R^2}$

$T = \frac{1}{2}\left(m + \frac{m_{disk}}{2}\right)\dot{x}^2$

Using Lagrange's Eqn $\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = \left(m + \frac{m_{disk}}{2}\right)\ddot{x}$

$\frac{\partial U}{\partial x} = \frac{k}{4}x - mg$, so

$$\boxed{\left(m + \frac{m_{disk}}{2}\right)\ddot{x} + \frac{k}{4}x - mg = f}$$

5) $w_{tt} - c^2 w_{xx} = 0$

If ends are fixed, we can solve for mode shapes as (see prev exams/book)

$$X(x) = \sin \frac{n\pi x}{\ell}$$

So $\quad w(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{\ell}$

Substituting

$$\Big(\ddot{T} + \underbrace{c^2 \left(\frac{n\pi}{\ell}\right)^2}_{\omega_n^2} T\Big) \sin \frac{n\pi x}{\ell} = 100\, \delta(x - \tfrac{\ell}{3})\, \delta(t)$$

Multiplying by $\sin \frac{m\pi x}{\ell}$ and integrating over $0 < x < \ell$

$$\ddot{T}_n + c^2 \left(\frac{n\pi}{\ell}\right)^2 T_n = \frac{2}{1}\, 100 \sin \frac{n\pi}{3}\, \delta(t)$$

The impulse response is (for undamped system)

$$T_n(t) = \frac{\hat{F}}{m\omega_n} \sin \omega_n t$$

$$= 200 \sin\left(\frac{n\pi}{3}\right) \frac{\ell}{n\pi} \sqrt{\frac{\rho}{\tau}} \sin\left(\frac{n\pi}{\ell} \sqrt{\frac{\tau}{\rho}}\, t\right)$$

$$w(x,t) = \sum_{n=1}^{\infty} 200 \frac{\ell}{n\pi} \sqrt{\frac{\rho}{\tau}} \sin \frac{n\pi}{3} \sin\left(\frac{n\pi}{\ell} \sqrt{\frac{\tau}{\rho}}\, t\right) \sin \frac{n\pi x}{\ell}$$