1) $T = T(\dot{q}_i) \qquad V = V(q_i, \dot{q}_i)$

$L = T - V = L(q_i, \dot{q}_i)$

Hamilton's principle

$$\delta \int_{t_1}^{t_2} L + W_{nc} \, dt = 0$$

$$= \int_{t_1}^{t_2} \delta L(q_i, \dot{q}_i) + \sum_{i=1}^{n} Q_i \delta q_i \, dt$$

$$= \int_{t_1}^{t_2} \sum_{i=1}^{n} \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + Q_i \delta q_i \right] dt$$

$$= \sum_{i=1}^{n} \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + Q_i \delta q_i \right) dt$$

Integrating the second term

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \, dt = \frac{\partial L}{\partial \dot{q}_i} \underbrace{\cancel{\delta q_i}}_{0} \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \delta q_i \, dt$$

since q_i are assumed to be known at t_1 and t_2

Substituting

$$= \sum_{i=1}^{n} \int_{t_1}^{t_2} \underbrace{\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + Q_i \right)}_{=0} \delta q_i \, dt$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$$

QED

2) $L = \frac{d^2}{dx^2}\left(EI \frac{d^2}{dx^2}\right)$

$w(0) = 0, \quad w'(0) = 0$

$\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=L} = 0, \quad \left.\frac{\partial}{\partial x}\left(EI \frac{\partial^2 w}{\partial x^2}\right)\right|_{x=L} = m \left.\frac{\partial^2 w}{\partial t^2}\right|_{x=L}$

$(u, Lv) = (v, Lu)$ for self adjointness where u and v are comparison functions.

$$\int_0^L u \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 v}{\partial x^2}\right) dx$$

$$= u \frac{\partial}{\partial x}\left(EI \frac{\partial^2 v}{\partial x^2}\right)\Big|_0^L - \int_0^L \frac{\partial u}{\partial x} \frac{\partial}{\partial x}\left(EI \frac{\partial^2 v}{\partial x^2}\right) dx$$

$$= \underbrace{\cancel{u}}_{=0 @ x=0} \frac{\partial}{\partial x}\left(EI \frac{\partial^2 v}{\partial x^2}\right)\Big|_0^L - \underbrace{\cancel{\frac{\partial u}{\partial x}}}_{=0 @ x=0} EI \underbrace{\cancel{\frac{\partial^2 v}{\partial x^2}}}_{=0 @ x=L}\Big|_0^L + \int_0^L \frac{\partial^2 u}{\partial x^2} EI \frac{\partial^2 v}{\partial x^2} dx$$

$$\frac{\partial}{\partial x}\left(EI \frac{\partial^2 v}{\partial x^2}\right)\Big|_L = m \frac{\partial^2 v}{\partial t^2}\Big|_L$$

Assuming a solution of the form $\frac{\partial^2 v}{\partial t^2} = \lambda v$

$$= u m \lambda v \Big|_L + \int_0^L \frac{\partial^2 u}{\partial x^2} EI \frac{\partial^2 v}{\partial x^2} dx$$

which is symmetrical in u and v.
Thus, the operator L is self-adjoint with respect to the given B.C.s.

To be positive definite, (u, Lu) must be positive for any non-zero comparison function.

$$(u, Lu) = m\lambda u^2\Big|_{x=L} + \int_0^L \left(\frac{\partial^2 u}{\partial x^2}\right)^2 EI\,dx$$

The first term is ≥ 0

The second term is > 0 for any non-zero u assuming EI is always positive.

Thus, the operator is also positive definite.

3) $V = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + GJ\left(\frac{\partial \theta}{\partial x}\right)^2 dx$

$T = \frac{1}{2}\int_0^L \left(\rho A \dot{w}^2 + \rho I_p \dot{\theta}^2\right) dx$

$+ \frac{1}{2} m v^2$

where v is the velocity of the mass m.

For small motion, $v = e\dot{\theta}\big|_{x=L} + \dot{w}\big|_{x=L}$

Hamilton's principle states

$\delta \int_{t_1}^{t_2} T - V + W_{nc} \, dt = 0$

$W_{nc} = 0$ here

Let's work by term

The first term of T yields

$\int_{t_1}^{t_2}\int_0^L \rho A \dot{w}\,\delta\dot{w} \, dx\, dt$

Integrating by parts with respect to time yields

$$\int_0^L \delta w \, \dot{w} \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \ddot{w} \, \rho A \delta w \, dt \, dx$$

Likewise, the second term of T yields

$$\int_0^L \delta \theta \, \dot{\theta} \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \ddot{\theta} \, \rho I_p \, \delta \theta \, dt \, dx$$

Taking the variation of the first term of the Potential energy yields

$$\int_{t_1}^{t_2} \int_0^L EI \frac{\partial^2 w}{\partial x^2} \delta \left(\frac{\partial^2 w}{\partial x^2} \right) dx \, dt$$

Integrating by parts twice yields

$$\int_{t_1}^{t_2} EI \frac{\partial^2 w}{\partial x^2} \delta \frac{\partial w}{\partial x} \Big|_0^L - \int_0^L EI \frac{\partial^3 w}{\partial x^3} \delta \frac{\partial w}{\partial x} \, dx \, dt$$

$$= \int_{t_1}^{t_2} EI \frac{\partial^2 w}{\partial x^2} \delta \frac{\partial w}{\partial x} \Big|_0^L - EI \frac{\partial^3 w}{\partial x^3} \delta w \Big|_0^L + \int_0^L EI \frac{\partial^4 w}{\partial x^4} \delta w \, dx \, dt$$

Taking the variation of the second term of the potential energy and integrating by parts yields

$$\int_{t_1}^{t_2} \frac{\partial \theta}{\partial x} \delta \theta \Big|_0^L - \int_0^L \frac{\partial^2 \theta}{\partial x^2} \delta \theta \, dx \, dt$$

The third term of the kinetic energy is

$$T_3 = \frac{1}{2} m v^2 = \frac{1}{2} m \left(e\dot{\theta}_L + \dot{w}_L \right)^2$$

$$\delta T_3 = m \left(e \delta \dot{\theta}_L + \delta \dot{w}_L \right) \left(e\dot{\theta}_L + \dot{w}_L \right)$$

$$= m e \delta \dot{\theta}_L \left(e \dot{\theta}_L + \dot{w}_L \right) + m \delta \dot{w}_L \left(e \dot{\theta}_L + \dot{w}_L \right)$$

Integrating T_3 by parts with respect to time yields

$$= m e \left(e \dot{\theta}_L + \dot{w}_L \right) \delta \theta_L \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} m e \left(e \ddot{\theta}_L + \ddot{w}_L \right) \delta \theta_L \, dt$$

$$+ m \left(e \dot{\theta}_L + \dot{w}_L \right) \delta w_L \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} m \left(e \ddot{\theta}_L + \ddot{w}_L \right) \delta w_L \, dt$$

Combining these 5 results yields

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = 0$$

$$GJ \frac{\partial^2 w}{\partial x^2} - \rho I_p \frac{\partial^2 w}{\partial t^2} = 0$$

with the 6 boundary conditions

$$w(0) = 0 \quad \text{or} \quad \left.\frac{\partial^3 w}{\partial x^3}\right|_{x=0} = 0$$

$$w'(0) = 0 \quad \text{or} \quad \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0} = 0$$

$$\theta(0) = 0 \quad \text{or} \quad \theta'(0) = 0$$

$$\theta(l) = 0 \quad \text{or} \quad +\left.\frac{\partial \theta}{\partial x}\right|_{x=L} = -me\left(e\ddot{\theta} + \ddot{w}\right)\Big|_{x=L}$$

$$w(l) = 0 \quad \text{or} \quad +EI\left.\frac{\partial^3 w}{\partial x^3}\right|_{x=L} = +m\left(e\ddot{\theta} + \ddot{w}\right)\Big|_{x=L}$$

$$w'(l) = 0 \quad \text{or} \quad \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=L} = 0$$