ME 710 2006 Midterm Solns

1) $EI\,W'''' + \rho A \omega^2 W = 0$

$W'''' + \beta^4 W = 0$

$W(x) = A\sin\beta x + B\cos\beta x + C\sinh\beta x + D\cosh\beta x$

Applying BC

$W(0)=0: \quad 0 = B + D \qquad D = -B$

$W'(0)=0: \quad 0 = A + C \qquad C = -A$

① $W(l)=0: 0 = A(\sin\beta l - \sinh\beta l) + B(\cos\beta l - \cosh\beta l)$ ①

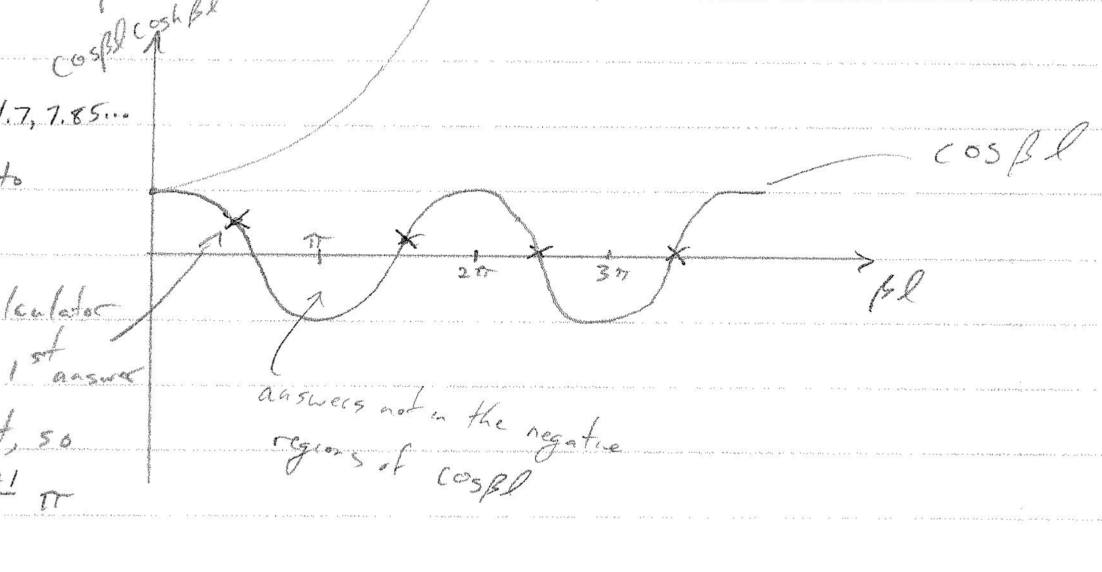
$W'(l)=0: 0 = A(\cos\beta l - \cosh\beta l) + B(-\sin\beta l - \sinh\beta l)$

$-(\sin\beta l - \sinh\beta l)(\sin\beta l + \sinh\beta l) - (\cos\beta l - \cosh\beta l)^2 = 0$

$\underbrace{-\sin^2\beta l - \cos^2\beta l}_{-1} + \underbrace{\sinh^2\beta l - \cosh^2\beta l}_{-1} + 2\cos\beta l\cosh\beta l = 0$

(-2)

$\cos\beta l\cosh\beta l = 1$, which must be solved numerically for βl



$\beta l \approx 1.3, 4.7, 7.85 \ldots$

converging to $\frac{2n-1}{2}\pi$

However, calculator plot shows 1st answer doesn't exist, so

$\beta_n l \approx \frac{2n+1}{2}\pi$

From 1: $B = A \dfrac{\sin\beta_n l - \sinh\beta_n l}{\cos\beta_n l - \cosh\beta_n l} = A\,\sigma_n$ (2)

so

$$W_n(x) = (\sin\beta_n x - \sinh\beta_n x) + \sigma_n(\cos\beta_n x - \cosh\beta_n x)$$

where σ_n is defined by (2), and β_n is defined by the solution of $\cos\beta l \cosh\beta l = 0$

$$\omega_n = \beta_n^2 \sqrt{\frac{EI}{\rho A}}$$

2) Mode shapes are

$W_n = \sin \frac{n\pi x}{l}$ or $\sin \sigma x$ where $\sigma = \frac{n\pi}{l}$

Total Soln is

$$w(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{l} \qquad (1)$$

Subst into EOM

$$\left(EI\, \sigma_n^4\, T_n(t) - c\, \sigma_n^2\, \dot{T}_n(t) + \rho A\, \ddot{T}_n(t) \right) \sin \sigma_n x = \delta(t)\, \delta(x - \tfrac{l}{2})$$

Multiplying by $\sin \sigma_m x$ and integrating over $0 < x < l$

$$\rho A\, \ddot{T}_n - c\, \sigma_n^2\, \dot{T}_n + EI\, \sigma_n^4\, T_n = \underbrace{\frac{2}{l} \sin \sigma_n \frac{l}{2}}_{F}\, \delta(t)$$

Soln to impulse is

$$T_n(t) = \frac{F}{m \omega_n} e^{-\zeta \omega_n t} \sin \omega_d t$$

where $\omega_n = \sigma_n^2 \sqrt{\frac{EI}{\rho A}}$, $\zeta_n = \frac{-c\, \sigma_n^2}{2 \rho A\, \omega_n}$

Note ζ_n is negative, system is unstable as defined.

$\omega_d = \sqrt{1 - \zeta^2}$

$$w(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \sigma_n x$$

3) $U = mgh = mg\frac{L}{2}(1-\cos\theta)$

$$T = \frac{1}{2}\underbrace{\left(\frac{1}{3}mL^2\right)}_{J}\dot{\theta}^2 + \int_0^L \frac{1}{2}\left(\frac{m}{L}\right)\underbrace{(\Omega x \sin\theta)}_{\substack{\text{Velocity in circular}\\ \text{motion in horizontal}\\ \text{plane}}}{}^2\,dx$$

Consider 2nd term

$$T_0 = \frac{1}{2}\frac{m}{L}\Omega^2 \sin^2\theta \int_0^L x^2\,dx$$

$$= \frac{1}{2}\frac{m}{L}\Omega^2 \sin^2\theta \frac{L^3}{3} = \frac{1}{2}\left(\frac{1}{3}mL^2\sin^2\theta\right)\Omega^2$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = \frac{1}{3}mL^2\ddot{\theta}$$

$$\frac{\partial T}{\partial \theta} = \frac{1}{3}mL^2 \sin\theta\cos\theta\,\Omega^2$$

$$\frac{\partial U}{\partial \theta} = mg\frac{L}{2}\sin\theta$$

a) $\longrightarrow$ $\frac{1}{3}mL^2\ddot{\theta} + m\sin\theta\left(\frac{gL}{2} - \frac{1}{3}L^2\cos\theta\,\Omega^2\right) = 0$

Eq points are at $\sin\theta\left(\frac{gL}{2} - \frac{1}{3}L^2\cos\theta\,\Omega^2\right) = 0$

b) $\longrightarrow$ $\theta = 0, \pi, \operatorname{acos}\left(\frac{3g}{2L\Omega^2}\right)$

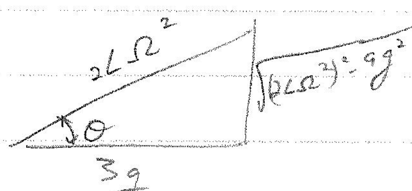
Linearize about $\theta = a\cos \frac{3g}{2L\Omega^2}$

$$f(\theta) \simeq f(\theta_0) + \left.\frac{\partial f}{\partial \theta}\right|_{\theta_0} \Delta\theta + HOT$$

$f(\theta_0) = 0$ (definition of equilibrium)

$$\frac{\partial f}{\partial \theta} = m\cos\theta\left(\cancelto{0}{5gme}\right) + m\sin\theta\,\frac{1}{3}L^2\sin\theta\,\Omega^2$$

$$\left.\frac{\partial f}{\partial \theta}\right|_{\theta = a\cos\frac{3g}{2L\Omega^2}} = \left.\frac{L^2 m}{3}\sin^2\theta\,\Omega^2\right|_{\theta = a\cos\frac{3g}{2L\Omega^2}}$$



$$\sin^2\theta = \frac{(2L\Omega^2)^2 - 9g^2}{(2L\Omega^2)^2}$$

$$\left.\frac{\partial f}{\partial \theta}\right|_{\theta=\theta_0} = \frac{m}{12}\left(4L^2\Omega^4 - 9g^2\right)$$

So linearized eqn is

$$\frac{1}{3}mL^2\ddot{\theta}' + \frac{m}{12}\left(4L^2\Omega^4 - 9g^2\right)\theta' = 0$$

This is stable if $4L^2\Omega^4 > 9g^2$

or

$$\Omega > \sqrt{\frac{3g}{2L}}$$

Note: Consider $\Omega = 0$. For small motion

$\omega_n = \sqrt{\frac{3g}{2L}}$

Prove

4) $\lambda_1 \leq \underline{x}^T A \underline{x} \leq \lambda_2$

Note $0 < a < 1$

$-1 < b < 1$

$a = \sqrt{1-b^2}$

$$\underline{X} = \frac{a}{\sqrt{a^2+b^2}} \underline{U}_1 + \frac{b}{\sqrt{a^2+b^2}} \underline{U}_2$$

$$\underline{X}^T A \underline{x} = \frac{a^2}{a^2+b^2} \lambda_1 + \frac{b^2}{a^2+b^2} \lambda_2$$

For $a = 1, b = 0$ $\quad \underline{X}^T A \underline{x} = \lambda_1$

$a = 0, b = 1$ or -1 $\quad \underline{x}^T A \underline{x} = \lambda_2$

Since $\frac{a^2}{a^2+b^2} < 1$ and $\frac{b^2}{a^2+b^2} < 1$

and

$\frac{a^2 + b^2}{a^2+b^2} = 1$, $\underline{X}^T A \underline{x}$ is bounded by λ_1 and λ_2 presuming $\underline{x}^T \underline{x} = 1$

5) $$w(x,t) = \sum_{n=1}^{\infty} \frac{F\, X_n(x_1)\, X_n(x_2)}{\omega_n^2 - \omega_{dr}^2} \sin \omega_{dr} t$$