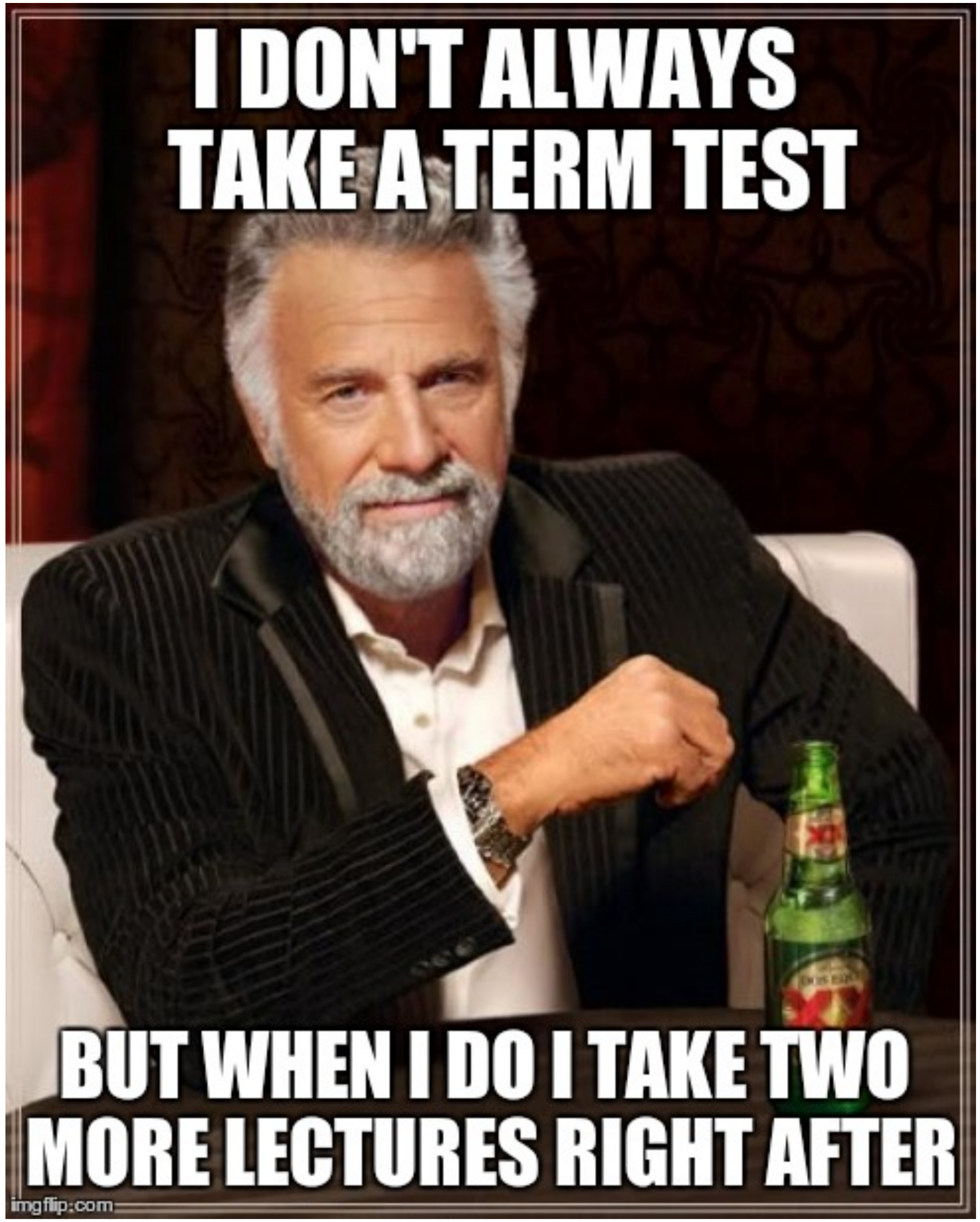
CSC165 Week 9

Larry Zhang, November 4, 2014


I DON'T ALWAYS
TAKE A TERM TEST
BUT WHEN I DO I TAKE TWO
MORE LECTURES RIGHT AFTER
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today's outline

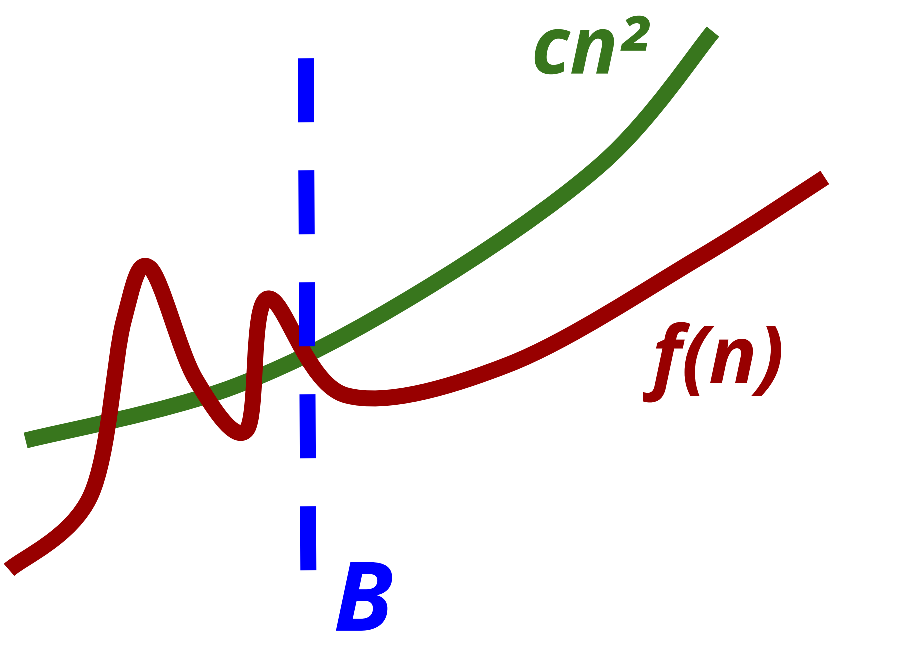
- ➔ exercises of big-Oh proofs
- ➔ prove big-Oh using limit techniques

Recap definition of $O(n^2)$

a function $f(n)$ is in $O(n^2)$ iff

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \text{such that } \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2$$

*Beyond **breakpoint B**, **f(n)** is upper-bounded by **cn^2**, where **c** is some wisely chosen constant multiplier.*



Notation of $O(n^2)$ being a set of functions

functions that take in a **natural number** and return a **non-negative real number**

$$\mathcal{O}(n^2) = \{\overline{f : \mathbb{N} \to \mathbb{R}^{\geq 0}} \mid \underline{c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2}\}$$

beyond breakpoint B, f(n) is upper-bounded by cn^2

set of all red functions which satisfy the green.

$$\mathcal{O}(n^2) = \{f : \mathbb{N} \to \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2\}$$

Prove $3n^2 + 2n \in \mathcal{O}(n^2)$

thoughts: it's all about picking c and B

- ➔ **tip 1: c** should probably be larger than 3 (the constant factor of the highest-order term)
- ➔ **tip 2**: see what happens when n = 1
- ➔ if n = 1
  - ◆ $3n^2 + 2n = 3 + 2 = 5 = 5n^2$
  - ◆ so pick **c = 5**, with **B = 1** is probably good
  - ◆ double check for n=2,3,4..., yeah it's all good

$$\mathcal{O}(n^2) = \{f : \mathbb{N} \to \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2\}$$

Proof $3n^2 + 2n \in \mathcal{O}(n^2)$

first, $3n^2 + 2n : \mathbb{N} \to \mathbb{R}^{\geq 0}$ # $3n^2+2n \geq 0$ when n is natural number

pick $c = 5$, then $c \in \mathbb{R}^+$ # 5 is a positive real

pick $B = 1$, then $B \in \mathbb{N}$ # 1 is a natural number

assume $n \in \mathbb{N}$ # generic natural number

assume $n \geq 1$ # the antecedent

then $3n^2 + 2n \leq 3n^2 + 2n \times n$ # $2n\times 1 \leq 2n\times n$ since $1 \leq n$

$= 5n^2 = cn^2$ # $c = 5$

then $3n^2 + 2n \leq cn^2$ # transitivity of $\leq$

then $n \geq B \Rightarrow 3n^2 + 2n \leq cn^2$ # introduce antecedent

then $\forall n \in \mathbb{N}, n \geq B \Rightarrow 3n^2 + 2n \leq cn^2$ # introduce $\forall$

then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 3n^2 + 2n \leq cn^2$ # introduce $\exists$

then $3n^2 + 2n \in \mathcal{O}(n^2)$ # by definition


MAKE IT MORE COMPLEX, PLEASE
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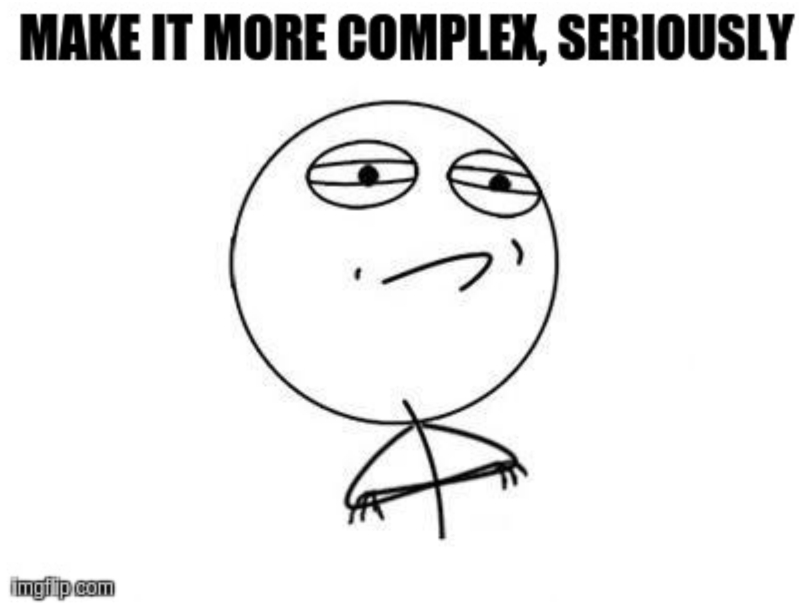
what if we add a constant?

Prove $3n^2 + 2n + 5 \in \mathcal{O}(n^2)$

thoughts: it's all about picking c and B

- **tip 1: c** should probably be larger than 3 (the constant factor of the highest-order term)
- **tip 2**: see what happens when n = 1
- if n = 1
  - $3n^2 + 2n + 5 = 3 + 2 + 5 = 10 = 10n^2$
  - so pick c = **10**, with B = 1 is probably good
  - double check for n=2,3,4…, yeah it's all good


MAKE IT MORE COMPLEX, SERIOUSLY
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$$\mathcal{O}(n^2) = \{f : \mathbb{N} \to \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2\}$$

Prove $7n^6 - 5n^4 + 2n^3 \in \mathcal{O}(6n^8 - 4n^5 + n^2)$

thoughts:

- ➔ *assume* $\boxed{n \geq 1}$
- ➔ ***upper-bound*** *the left side by* ***overestimating***
- ➔ ***lower-bound*** *the right side by* ***underestimating***
- ➔ *choose a* ***c*** *that connects the two bounds*

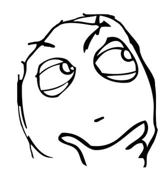
large

$$6n^8 - 4n^5 + n^2$$

$$6n^8 - 4n^5$$

$$6n^8 - 4n^8 = \underline{2n^8}$$

$$\boxed{9n^6 \leq \frac{9}{2} \cdot 2n^8}$$



$$7n^6 + 2n^6 = \underline{9n^6}$$

$$7n^6 + 2n^3$$

$$7n^6 - 5n^4 + 2n^3$$

small

$$\mathcal{O}(n^2) = \{f : \mathbb{N} \to \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2\}$$

Proof $7n^6 - 5n^4 + 2n^3 \in \mathcal{O}(6n^8 - 4n^5 + n^2)$

then $7n^6 - 5n^4 + 2n^3 : \mathbb{N} \to \mathbb{R}^{\geq 0}$ **# non-negative for n ≥ 0**

pick $c = 9/2$

pick $B = 1$

assume $n \in \mathbb{N}$

assume $n \geq 1$ **# n ≥ B, the antecedent**

then $7n^6 - 5n^4 + 2n^3 \leq 7n^6 + 2n^3$ **# $-5n^4 \leq 0$**

$\leq 7n^6 + 2n^6 = 9n^6$ **# $n^3 \leq n^6$ since n ≥ 1**

$= (9/2) \cdot 2n^6 = c \cdot 2n^6$ **# c = 9/2**

$\leq c \cdot 2n^8 = c \cdot (6n^8 - 4n^8) \leq c \cdot (6n^8 - 4n^5)$ **# $-n^8 \leq -n^5$**

$\leq c \cdot (6n^8 - 4n^5 + n^2)$ **# $0 \leq n^2$**

then $7n^6 - 5n^4 + 2n^3 \leq c \cdot (6n^8 - 4n^5 + n^2)$

. . . introduce antecedent and quantifiers (omitted) . . .

how about disproving?

$$\mathcal{O}(n^2) = \{f : \mathbb{N} \to \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2\}$$

Prove $n^3 \notin \mathcal{O}(3n^2)$

- ➔ first, negate it

$$\neg(\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow n^3 \leq c \cdot 3n^2)$$

$$\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge n^3 > c \cdot 3n^2$$

- ➔ then, prove the negation

Prove $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge n^3 > c \cdot 3n^2$

thoughts

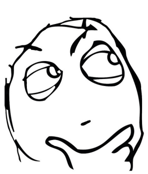
want to make $n^3 > c \cdot 3n^2$

that is $n > c \cdot 3 = 3c$

but be careful with B since we also need $n \geq B$

so, we want both $n > 3c$ and $n \geq B$

pick $n > \max(3c, B)$



e.g., $n = \max(\lceil 3c \rceil, B) + 1 \ \in \mathbb{N}$

$n^3 \notin \mathcal{O}(3n^2)$

Proof: $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \land n^3 > c \cdot 3n^2$

assume $c \in \mathbb{R}^+, B \in \mathbb{N}$

pick $n = \max(\lceil 3c \rceil, B) + 1$, then $n \in \mathbb{N}$

then $n \geq B$ **# definition of max**

also $n > 3c$ **# definition of max and ceiling**

then $n \cdot n^2 > 3c \cdot n^2$ **# multiply both sides by $n^2 > 0$**

then $n^3 > c \cdot 3n^2$

then $n \geq B \land n^3 > c \cdot 3n^2$ **# conjunction introduction**

then $\exists n \in \mathbb{N}, n \geq B \land n^3 > c \cdot 3n^2$ **# introduce ∃**

then $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \land n^3 > c \cdot 3n^2$ **# intro ∀**

then $n^3 \notin \mathcal{O}(3n^2)$ **# definition of big-Oh**

summary

- the ways we talked about of picking **c** and **B**, the main point of them is that we know how to show the **bounding** relation when picking in these ways.
- these are not the only ways, you can be more flexible when you are more familiar with these types of proofs.
- In general, larger **c** or larger **B** don’t hurt.

so far all functions we talked about are polynomials e.g., $2n^9 + 7n^6 + 44n^5 + 5n^2 + 11n + 6$

➔ between polynomials, it is fairly easy to figure out who is big-Oh of whom

➔ simply look at the highest-degree term

➔ ***f(n)*** is in ***O(g(n))*** exactly when the high-degree of ***f(n)*** is **no larger** than that of ***g(n)***

polynomials

$165n^{148} - 137n^{108} + 1130n^{11}$ is in $\mathcal{O}$ of...

$A.$ $0.0001n^{149} + 7n^{77} - 3n^{33} + 6n$ ✔

$B.$ $10000n^{147} + 99999n^{146} + 473736743$ ✘

$C.$ $n^{148} + 2$ ✔

how about non-polynomials?

$$2^n \;?\; \mathcal{O}(n^2)$$

$$\text{probably} \quad 2^n \notin \mathcal{O}(n^2)$$

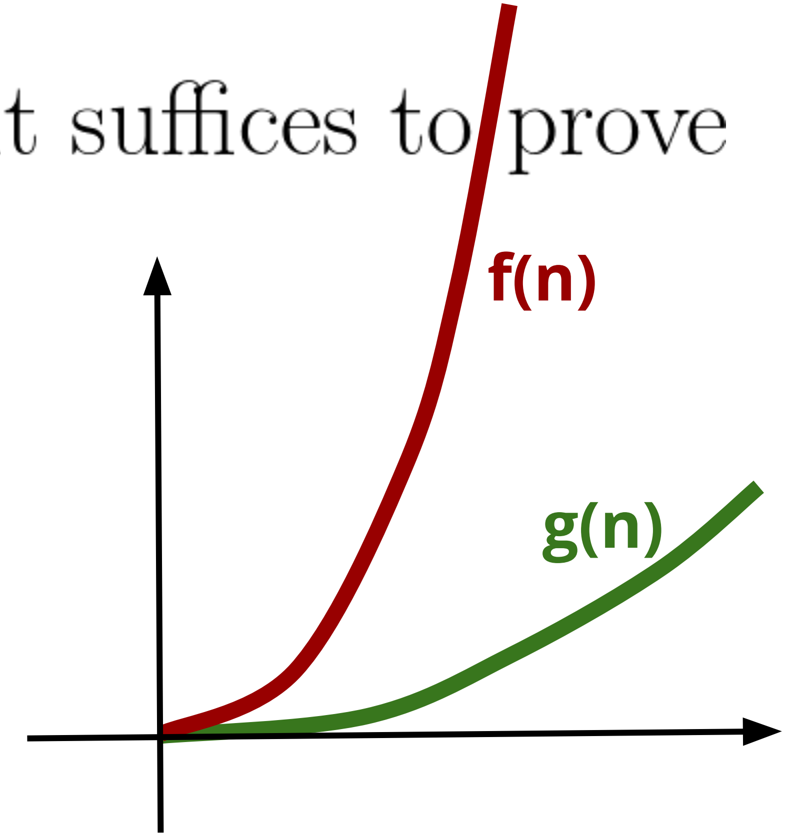
but how do we prove it?

➔ we need to use a math tool, which we didn't like a lot but studied anyway

LIMIT

to prove $2^n \notin \mathcal{O}(n^2)$, it suffices to prove

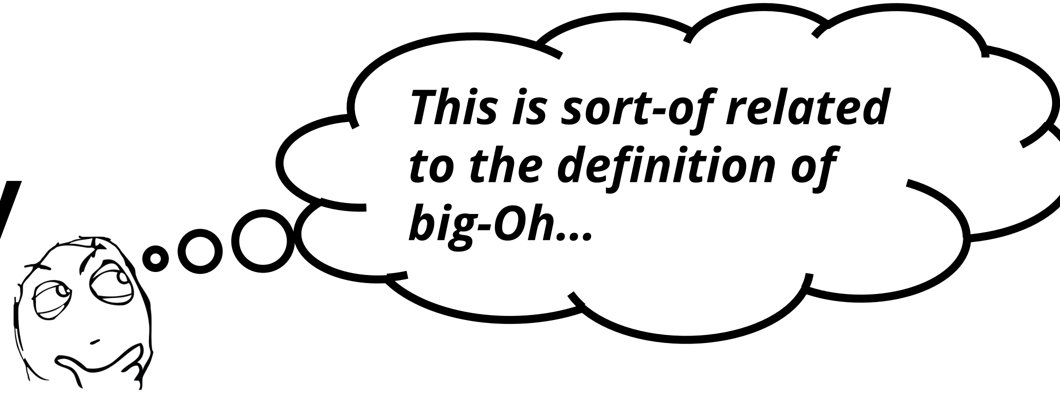
$$\lim_{n\to\infty} \frac{2^n}{n^2} = \infty$$



Intuition:

➔ *if the ratio* $\frac{f(n)}{g(n)}$ *approaches infinity when **n** becomes larger and larger, that means* ***f(n) grows faster*** *than* ***g(n)***

more precisely



$$\lim_{n\to\infty} \frac{2^n}{n^2} = \infty \quad \text{by definition of limit:}$$

$$\forall c \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow \frac{2^n}{n^2} > c$$

give me any big number

I can find a breakpoint

beyond which

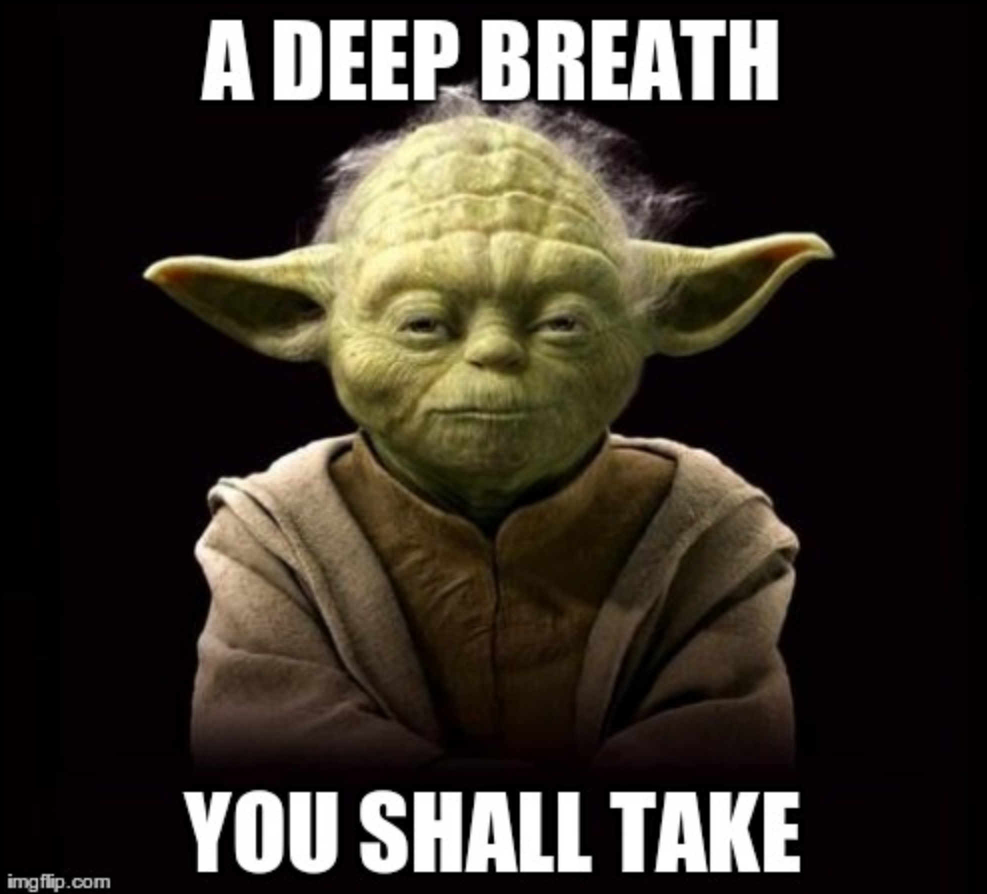
the ratio is bigger than that big number

How to use limit to prove big-Oh

General procedure

1. prove $\lim_{n\to\infty} \frac{2^n}{n^2} = \infty$ using "some calculus"
2. translate the limit into its definition with c and n', i.e., $\forall c \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow \frac{2^n}{n^2} > c$
3. relate this definition to the definition of big-Oh, i.e., $2^n \notin \mathcal{O}(n^2)$ means

$$\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge 2^n > c \cdot n^2$$


A DEEP BREATH
YOU SHALL TAKE
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http://mathworld.wolfram.com/LHospitalsRule.html

Step 1. prove $\lim_{n\to\infty} \frac{2^n}{n^2} = \infty$ **using “some calculus”**

“some calculus”: **L’Hopital’s rule**

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

derivatives

l’hopital

$$\lim_{n\to\infty} \frac{2^n}{n^2} = \lim_{n\to\infty} \frac{(2^n)'}{(n^2)'} = \lim_{n\to\infty} \frac{\ln 2 \cdot 2^n}{2n}$$

l’hopital again!

$$= \lim_{n\to\infty} \frac{(\ln 2 \cdot 2^n)'}{(2n)'} = \lim_{n\to\infty} \frac{\ln 2 \cdot \ln 2 \cdot 2^n}{2} = \infty$$

Step 2. translate the limit into its definition

we have proven $\lim_{n\to\infty} \frac{2^n}{n^2} = \infty$

then by definition of limit

$$\forall c \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow \frac{2^n}{n^2} > c$$

give me any big number

I can find a breakpoint

beyond which

the ratio is bigger than that big number

Step 3. relate it to the definition of big-Oh

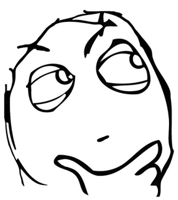
$$\lim_{n\to\infty}\frac{2^n}{n^2}=\infty$$

what we have

$$\forall c \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, \boxed{n \geq n'} \Rightarrow \frac{2^n}{n^2} > c$$

we have this awesome n', all n ≥ n' satisfy 2^n > cn²

$$2^n > c \cdot n^2$$



pick n = max(n', B)

$$\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, \boxed{n \geq B} \wedge 2^n > c \cdot n^2$$

$$2^n \notin \mathcal{O}(n^2)$$

wanna pick a wise n

to satisfy both

$$\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge 2^n > c \cdot n^2$$

Proof: $2^n \notin \mathcal{O}(n^2)$

assume $c \in \mathbb{R}^+, B \in \mathbb{N}$ **# generic c and B**

then $\lim\limits_{n\to\infty} \frac{2^n}{n^2} = \infty$ **# applying L'hopital's rule twice, (show the steps, omitted here)**

then $\forall c \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow \frac{2^n}{n^2} > c$ **# definition of limit**

pick $n = \max(n', B)$, then $n \in \mathbb{N}$

then $n \geq B$ **# definition of max**

and $n \geq n'$ **# definition of max**

then $2^n > c \cdot n^2$ **# definition the limit**

then $n \geq B \wedge 2^n > c \cdot n^2$ **# conjunction introduction**

then $\exists n \in \mathbb{N}, n \geq B \wedge 2^n > c \cdot n^2$ **# intro ∃**

then $\forall c \in \mathbb{R}^+, \forall, B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge 2^n > c \cdot n^2$ **# intro ∀**

then $2^n \notin \mathcal{O}(n^2)$ **# negation of definition of big-Oh**

next week

- ➔ more proofs on O, Ω, Θ
- ➔ (maybe) computability