How to deal with linear recurrence that it's characteristic polynomial has multiple roots?

example ,

$$a_n = 6a_{n-1} - 9a_{n-2}, a_0 = 0, a_1 = 1$$

what is the a_n?

In fact, I want to know there are any way to deal with this situation.

(linear-algebra) (combinations)

edited Aug 16 '14 at 3:18
Thomas Andrews
119k 10 138 266

asked Aug 16 '14 at 3:11
lanse2pty
644 3 16

1 @lanse2pty, en.wikipedia.org/wiki/Recurrence_relation#Solving – lab bhattacharjee Aug 16 '14 at 3:34

1 Besides solving this recurrence, I advise you to learn how to write formal proofs and verify others' proofs so that you can understand perfectly why a given method works or does not work. Frankly I cannot emphasize this enough. – user21820 Aug 16 '14 at 3:43

3 Answers

We assume that you are dealing with linear homogeneous recurrences with constant coefficients.

When the characteristic equation has degree 2 and a double root r, the general solution is $Ar^n + Bnr^n$.

If the equation has degree 3, a double root r and a single root s, use $Ar^n + Bnr^n + Cs^n$.

If there is a triple root r, use $Ar^n + Bnr^n + Cn^2r^n$.

And so on.

Remark: The situation is analogous to what we do with linear homogeneous differential equations with constant coefficients when the characteristic equation has multiple roots.

edited Aug 16 '14 at 3:20

answered Aug 16 '14 at 3:14
André Nicolas
434k 35 380 755

Even though you have not proof your way,but I feel it is right. If I want to study the the linear recurrence theory,what book I should read? – lanse2pty Aug 16 '14 at 6:32

1 Sorry, I am not familiar with current texts. The book by Rosen on discrete mathematics is fairly comprehensive. As to proof, all we need to verify (for your case) is that r^n and nr^n **are** solutions (easy) and that they are linearly independent. That the space of solutions is 2-dimensional is linear algebra, and then we get that the space of solutions is given by linear combinations of r^n and nr^n. The proof for the general case is similar, but gets notationally messy. – André Nicolas Aug 16 '14 at 6:38

A bare-hands method would be as follows:

$$a_n - 3a_{n-1} = 3(a_{n-1} - 3a_{n-2})$$

$$a_n - 3a_{n-1} = 3^{n-1}(a_1 - 3a_0)$$

$$\sum_{k=1}^{n} 3^{n-k}(a_k - 3a_{k-1}) = \sum_{k=1}^{n} 3^{n-k}\left(3^{k-1}(a_1 - 3a_0)\right)$$

$$\sum_{k=1}^{n} 3^{n-k}a_k - 3^{n-(k-1)}a_{k-1} = \sum_{k=1}^{n} 3^{n-1}(a_1 - 3a_0)$$

$$a_n - 3^n a_0 = n3^{n-1}(a_1 - 3a_0)$$

The constants were obtained from the factorization of the quadratic.

Note that the above method works regardless of whether the roots are repeated, although the last step will differ.

edited Aug 24 '14 at 17:19
Did
231k 21 185 416

answered Aug 16 '14 at 3:25
user21820
30.3k 4 32 102

1 Nice trick. You might want to add a step or two between the second and third identities. – Did Aug 19 '14 at 19:14

@Did: Yea I thought of doing that but didn't know how much detail I should provide. Alright I'll add the summation in. =) – user21820 Aug 24 '14 at 8:20

Reading the expanded version, I think it is impossible to fail to get the idea... :-) – Did Aug 24 '14 at 17:19

Here is a fun way to get to the answer:

Suppose we have a recursion $A_n = c_1 A_{n-1} + c_2 A_{n-2} + \cdots + c_k A_{n-k}$. When you are doing the usual approach, you are looking for geometric sequences that satisfy this recursion, and you end up looking for nonzero roots r of:

$$x^n = c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_k x^{n-k}$$

Now remember that when we have a double root of a polynomial, then it is also a root of the derivative of that polynomial. So if we are in that case and r is a double root of this let's differentiate the above equation, and then multiply by x to keep it looking nice:

$$nx^n = c_1(n-1)x^{n-1} + c_2(n-2)x^{n-2} + \cdots + c_k(n-k)x^{n-k}$$

We know r must still must be a root of this, so we plug in r and look carefully this says that the sequence $A_n = nr^n$ satisfies our original recursion. Hooray!

If r is a root of multiplicity more than 2, then we can differentiate the equation more times to get more sequences of the form $A_n = n^k r^n$ for k less than the multiplicity of r.

I know another way to get at this using linear algebra that I think is a bit more enlightening as to why these are the answers, and why you never get anything else, but in my opinion this differentiation trick is pretty slick.

answered Aug 21 '14 at 19:39
Nate
5,887 1 9 20