Smoothness Rule

- Definition: $f : N \to \Re^{\geq 0}$ is eventually non-decreasing if there is an integer n_0 so that $f(n) \leq f(n+1)$ for $n \geq n_0$.
- Definition: Let $b \geq 2$ be an integer. f is b-smooth if it is eventually non-decreasing and $f(bn) \in O(f(n))$. That is, there are n_0 and c so that $f(bn) < cf(n)$ for $n \geq n_0$.
- Definition: A function is smooth if it is b-smooth for every integer $b \geq 2$. (different c's)

- Example: If $a, b > 0$ and $t(n)$ is defined by

$$t(n) = \begin{cases} a & \text{if } n = 1 \\ 4t(\lceil n/2 \rceil) + bn & \text{otherwise} \end{cases},$$

show $t(n)$ is $\Theta(n^2)$.

- Let $f : N \to \Re^{\geq 0}$ be a smooth function, let $t : N \to \Re^{\geq 0}$ be an eventually non-decreasing function, and let $b \geq 2$ be an integer. Then $t(n)$ is $\Theta(f(n))$ whenever $t(n) is$ $\Theta(f(n) \mid n$ is a power of $b)$

- $\lceil n \rceil$ is 'easy' if n is a power of 2.
- W.T.S.
  - $f(n) = n^2$ is smooth
    * $f(n)$ is eventually non-decreasing
    * $f(2n) \leq cf(n)$ for some c and all $n \geq n_0$
  - $t(n)$ is eventually non-decreasing
  - $t(n)$ is $\Theta(n^2)$ when n is a power of 2
- Then the smoothness rule yields the desired result.