20170918(1)

Devoir 1

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1.(a) i) for n, $t_a = \frac{3^n}{100}$ (sec)

$\Rightarrow$ for $t_a = \frac{3^n}{100} = 30 \cdot 24 \cdot 3600$ (sec)

$\Rightarrow 3^n = 30 \cdot 24 \cdot 3600 \cdot 100$

log: $n \cdot \log 3 = \log(30 \cdot 24 \cdot 3600 \cdot 100)$

$n = \frac{\log(30 \cdot 24 \cdot 3600 \cdot 100)}{\log 3}$

$\therefore n = 17.6342$

ii) for n, $t_b = \frac{n^6}{100}$ (sec)

$\Rightarrow$ for $t_b = \frac{n^6}{100} = 30 \cdot 24 \cdot 3600$

$\Rightarrow n^6 = 30 \cdot 24 \cdot 3600 \cdot 100$

log: $6 \cdot \log n = \log 30 \cdot 24 \cdot 3600 \cdot 100$

$\log n = \frac{1}{6} \log(30 \cdot 24 \cdot 3600 \cdot 100)$

$n = \exp(\frac{1}{6} \log(30 \cdot 24 \cdot 3600 \cdot 100))$

$\therefore n = 25.2506$

1.(b) i) for n, $t_a' = \frac{3^n}{100} \cdot \frac{1}{10^6}$ (sec)

$\Rightarrow$ for $t_a' = \frac{3^n}{10^8} = 30 \cdot 24 \cdot 3600$

$\Rightarrow 3^n = 30 \cdot 24 \cdot 3600 \cdot 10^8$

log: $n \cdot \log 3 = \log(30 \cdot 24 \cdot 3600 \cdot 100)$

$n = \frac{1}{\log 3} \cdot \log(30 \cdot 24 \cdot 3600 \cdot 100)$

$\therefore n = 30.209585$

ii) for n, $t_b' = \frac{n^6}{100} \cdot \frac{1}{10^6}$ (sec)

$\Rightarrow$ for $t_b' = \frac{n^6}{10^8} = 30 \cdot 24 \cdot 3600$

$\Rightarrow n^6 = 30 \cdot 24 \cdot 3600 \cdot 10^8$

$6 \cdot \log n = \log(30 \cdot 24 \cdot 3600 \cdot 10^8)$

$\log n = \frac{1}{6} \log(30 \cdot 24 \cdot 3600 \cdot 10^8)$

$\therefore n = \exp(\frac{1}{6} \log(30 \cdot 24 \cdot 3600 \cdot 10^8)) = 252.5065$ (sec)

20170418(2)

1. (c) $\lim_{n\to\infty} \frac{t_a}{t_b} = \lim_{n\to\infty} \frac{\frac{3^n}{100}}{\frac{n^6}{100}} = \lim_{n\to\infty} \frac{3^n}{n^6} \overset{\text{l'Hopital}}{=} \lim_{n\to\infty} \frac{\ln 3 \cdot 3^n}{6 \cdot n^5}$

$\vdots$

$= \lim_{n\to\infty} \frac{(\ln 3)^6 \cdot 3^n}{6!} = \infty$

$\therefore \exists n_0 \in \mathbb{N}^+$ s.t $\forall n \geq n_0$, $\frac{3^n}{100} \geq \frac{n^6}{100}$

$\therefore$ Yes

2.(a) let $c = 1$,

then $n^2 \leq n^3$ for $\forall n \in \mathbb{N}$

$\therefore \exists c \in \mathbb{R}^+$ s.t $\forall n \in \mathbb{N}$ $n^2 \leq n^3$.

$\therefore \exists c \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}$ s.t $\forall n \in \mathbb{N}$ $n^2 \leq n^3$

$\therefore n^2 \in O(n^3)$

$\therefore$ True

2.(b) $\neg(\exists d \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0, n^2 \geq dn^3)$

$\Leftrightarrow \forall d \in \mathbb{R}^+, \forall n_0 \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq n_0 \wedge n^2 < dn^3$

assume $d \in \mathbb{R}^+$, $n_0 \in \mathbb{N}$

let $n = \max(\lceil \frac{1}{d} \rceil, n_0) + 1$, then $n \in \mathbb{N}$

$\Rightarrow n \geq n_0$

$n > \frac{1}{d}$ $\big\downarrow \times n^2$

$\Rightarrow n^2 < d \cdot n^3$

$\Rightarrow n \geq n_0 \wedge n^2 < dn^3$

$\Rightarrow \exists n \in \mathbb{N}, n \geq n_0 \wedge n^2 < dn^3$

$\Rightarrow \forall d \in \mathbb{R}^+, \forall n \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq n_0 \wedge n^2 < dn^3$

$\Leftrightarrow \neg(n^2 \in \Omega(n^3))$

$\therefore n^2 \notin \Omega(n^3)$

$\therefore$ False

2.(c) let $c = 1$, $n_0 = 1$

then $\forall\, n \geq n_0$, $2^n \leq c \cdot 2^{n+1}$

$\therefore\ 2^n \in O(2^{n+1})$

let $c = \frac{1}{4}$, $n_0 = 1$

then $\forall\, n \geq n_0$, $2^n \geq c \cdot 2^{n+1} = \frac{1}{2} \cdot 2^n$

$\therefore\ 2^n \in \Omega(2^{n+1})$

$\Rightarrow\ 2^n \in O(2^{n+1})$ and $2^n \in \Omega(2^{n+1})$

$\therefore\ 2^n \in \Theta(2^{n+1})$

$\therefore$ True

2.(d) Prove $n! \notin \Omega((n+1)!)$

assume $d \in \mathbb{R}^+$, $n_0 \in \mathbb{N}$

let $n = \max\left(\left\lceil \frac{1}{d} - 1 \right\rceil, n_0\right) + 1$, then $n \in \mathbb{N}$

$\Rightarrow\ n \geq n_0$ and $n > \frac{1}{d} - 1$

$\times n!$

$\Rightarrow\ n! < (n+1)! \cdot d$

$\Rightarrow\ n \geq n_0 \land n! < d(n+1)!$

$\Rightarrow\ \exists\, n \in \mathbb{N},\ n \geq n_0 \land n! < d(n+1)!$

$\Rightarrow\ \forall\, d \in \mathbb{R}^+, \forall\, n \in \mathbb{N},\ \exists\, n \in \mathbb{N},\ n \geq n_0 \land n! < d(n+1)!$

$\Leftrightarrow\ \neg(n! \in \Omega((n+1)!))$

$\therefore\ n! \notin \Omega((n+1)!)$

$\therefore\ n! \notin \Theta((n+1)!)$

$\therefore$ False

20170918(4)

4. $S = O((2n)^3 \log n) \cup \Omega(n^5)$

$f(n) = n^4 (\log n)^2$.

$a(n) \equiv (2n)^3 \log n$. $b(n) \equiv n^5$

4(a) $f \notin S$?

$f \notin S \iff f \notin O(a(n)) \wedge f \notin \Omega(n^5)$

$\lim_{n\to\infty} \frac{a(n)}{f(n)} = \frac{(2n)^3 \log n}{n^4 (\log n)^2} = \lim_{n\to\infty} \frac{2^3}{n \cdot \log n} = 0$

$\therefore f \notin O((2n)^3 \log n)$

$\lim_{n\to\infty} \frac{f(n)}{b(n)} = \lim_{n\to\infty} \frac{n^4 (\log n)^2}{n^5} = \lim_{n\to\infty} \frac{(\log n)^2}{n} \overset{\text{l'Hopital}}{=} \lim_{n\to\infty} \frac{2 \cdot \log n \cdot \frac{1}{n}}{1} = \lim_{n\to\infty} \frac{2 \cdot \log n}{n}$

$\overset{\text{l'Hopital}}{=} \lim_{n\to\infty} \frac{2 \frac{1}{n}}{1} = 0$

$\therefore n^4 (\log n)^2 \notin \Omega(n^5)$

$\therefore f(n) \notin O(a(n)) \wedge f(n) \notin \Omega(n^5)$

$\therefore f(n) \notin S$

$\therefore$ True.

4.(b) $S \cap \Theta(f(n))$?

$S = O(a(n)) \cup \Omega(b(n))$

$\Rightarrow S \cap \Theta(f(n)) = (O(a(n)) \cap \Theta(f(n))) \cup (\Omega(b(n)) \cap \Theta(f(n)))$

i) prove $O(a(n)) \cap \Theta(f(n)) = \emptyset$

if $O(a(n)) \cap \Omega(f(n)) = \emptyset$, then $O(a(n)) \cap \Theta(f(n)) = \emptyset$

prove. $O(a(n)) \cap \Omega(f(n)) = \emptyset$,

~~let~~ $t(n) \in O(a(n))$

$\lim_{n\to\infty} \frac{t(n)}{f(n)} = \lim_{n\to\infty} \frac{t(n)}{f(n)} \frac{a(n)}{a(n)} = \lim_{n\to\infty} \underbrace{\frac{t(n)}{a(n)}}_{\mathbb{R}^+ \text{ or } 0} \cdot \underbrace{\frac{a(n)}{f(n)}}_{0} = 0$.

$\therefore$ for $\forall\, t(n) \in O(a(n))$, $t(n) \notin \Omega(f(n))$

$\therefore O(a(n)) \cap \Omega(f(n)) = \emptyset$

$\therefore O(a(n)) \cap \Theta(f(n)) = \emptyset$ cont.

4.(b) cont.

iii) $\Omega(b(n)) \cap \Theta(f(n))$?

if $\Omega(b(n)) \cap o(f(n)) = \phi$ then $\Omega(b(n)) \cap \Theta(f(n)) = \phi$

prove $\Omega(b(n)) \cap o(f(n)) = \phi$.

$\Rightarrow$ let $t(n) \in O(f(n))$

$$\lim_{n\to\infty} \frac{t(n)}{b(n)} = \lim_{n\to\infty} \frac{t(n)}{b(n)} \cdot \frac{f(n)}{f(n)} = \lim_{n\to\infty} \overbrace{\frac{t(n)}{f(n)}}^{\mathbb{R}^+ \text{ or } 0} \cdot \overbrace{\frac{f(n)}{b(n)}}^{0} = 0.$$

$\therefore$ for $\forall\, t(n) \in O(f(n))$, $t(n) \notin \Omega(b(n))$

$\therefore$ $\Omega(b(n)) \cap O(f(n)) = \phi$

$\therefore$ $\Omega(b(n)) \cap \Theta(f(n)) = \phi$

iiii) Since $O(a(n)) \cap \Theta(f(n)) = \phi$

$\Omega(b(n)) \cap \Theta(f(n)) = \phi$

$\Rightarrow S \cap \Theta(f(n)) = \phi$

$\therefore$ $S \cap \Theta(f(n)) = \phi$

4.(c) since $S \cap \Theta(f(n)) = \phi$

$S \cup \Theta(f(n)) = S \cup \Theta(f(n))$ ......