Introduction to algorithms Fall 2017

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Demonstration 3

1

Question: A demonstrator initially teaches students. `Every week, at least a quarter of its students drop out of the course. Estimate the number of weeks, at the most, that will elapse before there is nobody left over.

Solution: The maximum number of weeks will elapse if the minimum number of students drops the course each week. This minimum number is by definition a quarter, therefore $\lceil \frac{n}{4} \rceil$, because the number of students must be integer.

We define a function T: N → N which corresponds to a number of students the maximum number of weeks the course can last before being empty. We define T recursively thus:

$$T(n) = \begin{cases} 0 & \text{if } n = 0, \\ T(n - \lceil \frac{n}{4} \rceil) + 1 & \text{if } n > 0 \end{cases} \quad (1)$$

Since T is recursive, it must be analyzed to find the maximum number of weeks depending on the initial number of students. It first notes that $\lceil n - \frac{n}{4} \rceil = \lfloor \frac{3n}{4} \rfloor$.

Idea: show by induction that for all i ≥ 4 we have T (4 i) ≤ 5 (I-4) + T (4 4).

Base Case: i = 4: We have T (4 4) ≤ 5 (4-4) + T (4 4).

Induction step: Let i> 4. Suppose that the proposition is true for i - 1. We have

$T(4^i) = T(3 \cdot 4^{i-1}) + 1$ by def. of T

$T = (3^2 \cdot 4^{i-2}) + 2$ by def. of T

$= T(3^3 \cdot 4^{i-3}) + 3$ by def. of T

$T = (3 \cdot {}_4 4^{i-4}) + 4$ by def. of T

$T = (3 \cdot 4^5{}_{i-5}) + 5$ by def. of T

$= T((\frac{3}{4})$ April 5 $^{i)}$ 5

$\leq T(4^{i-1}) + 5$ because T non-decreasing and $(\frac{3}{4}) \leq {}_5 \frac{1}{4}$

$5 = (i - 1 - 4) + T({}_4 4) + 5$ by hyp. ind.

$5 = (i - 4) + T(4^4)$.

This concludes the proof by induction. Thus, $\forall n \in N$ such that $n = 4^i$ for some i, we have $T(n) \leq 5 (\log_4 n-4) + T(4^4)$. So we get that $T(n) \in O(\log n$ $_{4:}$ $n = 4^i)$. Since the function $\log_4 n$ is 4-harmonious and that T is not decreasing, we conclude that $T(n) \in O(\log_4 n) = O(\log n)$.

2

Question: Does the function $t: N \to R_{\geq 0}$ Next

$$t(n) = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1 \\ t(\lfloor \frac{n}{b} \rfloor) + f(n) & \text{if } n > 1 \end{cases}$$

is optionally nondecreasing for all $a_0, a_1 \in R_{\geq 0}$, for all $b \in N$ and any non-decreasing function f?

Solution: We can find a counterexample. Consider the function next:

$$t(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n = 1 \\ t(\lfloor \frac{n}{3} \rfloor) & \text{if } n > 1 \end{cases}$$

Show by induction on i that $t(3^i) = 0$ and $t(2 \cdot 3^i) = 1$.

Base case: $i = 0$:

$$t(3^0) = t(1) = 0$$

$$t(2 \cdot 3^0) = t(2) = t(\lfloor \frac{2}{3} \rfloor) = t(0) = 1$$

Induction step: Let $i > 0$. Suppose the proposition is true for $i-1$ and show that it is true for i.

$$t(3^i) = t(3^{i-1}) \quad \text{by def. of } t$$
$$= 0 \quad \text{by hyp. ind.}$$
$$t(2 \cdot 3^i) = t(2 \cdot 3^{i-1}) \quad \text{by def. of } t$$
$$= 1 \quad \text{by hyp. ind.}$$

This proves the proposition by induction. Suppose now by the absurd that t is possibly non-decreasing. By definition, $\exists n_0 \in N$ such that $\forall$ n, $n \geq n_0$, $n \geq n \rightarrow t(n) \geq t(n)$. Now take $n = 2 \cdot 3^{n_0}$ and $n = 3^{n_0+1}$. We have $n \geq n$. On the other hand, we obtain $0 = t(n) < t(n) = 1$, which is a contradiction.

3

Question: Prove that the function $t: N \rightarrow R_{\geq 0}$ is possibly not next decreasing:

$$t(n) = \begin{cases} d & \text{if } 0 \leq n \leq n_0, \\ a_1 t(\lceil \frac{n}{b} \rceil) + a_2 t(\lfloor \frac{n}{b} \rfloor) + cf(n) & \text{if } n >_0 n \end{cases}$$

o`un $_0 > 0$, $c, d \in R_{\geq 0}$, $a_1, a_2, b \in N$, $a_1 + a_2 \geq 1 \geq b$ 2 and $f: N \rightarrow R_{\geq 0}$ is not decreasing.

Solution show by induction on n that $t(n+1) \geq T(n)$ for all $n \geq n_0$.

Base case: $n = n_0$:

$$t(n_0+1) = a_1 t(\lceil \frac{n_0+1}{b} \rceil) + a_2 t(\lfloor \frac{n_0+1}{b} \rfloor) + cf(n_0+1) \quad \text{by def. of } t$$
$$= A_1 t(k_1) + a_2 t(k_2) + cf(n_0+1) \quad \text{o`uk}_1, k_2 \leq n_0 \text{ for } n_0 + 1 \geq 2 \geq 2 \text{ and b}$$
$$= (A_1 + a_2) d + cf(n_0+1) \quad \text{by def. of } t$$
$$\geq (a_1 + a_2) d \quad \text{because } c \geq 0 \text{ and f not neg.}$$
$$\geq d \quad \text{as } a_1 + a_2 \geq 1$$
$$= T(n_0) \quad \text{by def. of } t$$

Induction Step: Let $n > n_0$. It is assumed that the proposition is true for all

$n_0 \leq k < n$. Let us show that the proposition is true for n:

$$t(n+1) = a_1 t(\lceil \frac{n+1}{b} \rceil) + a_2 t(\lfloor \frac{n+1}{b} \rfloor) + cf(n+1) \quad \text{by def. of } t$$
$$\geq {}_1 t(\lceil \frac{n+1}{b} \rceil) + a_2 t(\lfloor \frac{n+1}{b} \rfloor) + cf(n) \quad \text{because f not decreasing}$$
$$\geq {}_1 t(\lceil \frac{not}{b} \rceil) + a_2 t(\lfloor \frac{not}{b} \rfloor) + cf(n) \quad \text{because } n, b \geq 2 \text{ and by hyp. ind.}$$
$$= t(n) \quad \text{by def. of } t$$

Question: Let T be the array resulting from the algorithm of belonging to a group of permutations. Knowing that a permutation belonging to this group can be expressed as

$$T [m, y_{m]} T * [m - 1, j_{m-1]} * \cdots * T [2, j_{2]} * T [1, j_{1]}$$

where T [i, j] comes from the ith line of the upper diagonal of the table, justify this notation is unique.

Solution: Suppose a permutation can be written in two distinct ways as products of elements above the diagonal of the table. We have:

$$* \; a_m \; a_{m-1} * \cdots *_2 *_1 = b_m * b_{m-1} * \cdots * b_2 * b_1$$

o`ua $_k$ and b $_k$ are permutations of the line k. Let i be the smallest index of line o`ua $_i$ = b $_i$. So

$$* \; a_m \; a_{m-1} * \cdots * a_i \; b =_m * b_{m-1} * \cdots * b_i$$

Note that as a $_i$ and b $_i$ are two different permutations of the row i of the table, they do not send i on the same point, that is `a say i $_{a_i}$ = i $_{b_i}$. Then multiply the equality by a $^{-1}_{i+1}$ * ··· * a $^{-1}_m$ on each side:

$$a_i = a^{-1}_{i+1} * \cdots * a^{-1} b \; m * m * b \; m\text{-}1 * \cdots * b \; i \; b \; i + 1 *$$

Since the left and right permutations are identical, they send i to the same point. By definition of the table, the permutations a $_{i+1}$, ..., a $_m$, b $_{i+1}$, ..., b $_m$ set i, and their inverses also. This implies that the image of i by a $^{-1}_{i+1}$ * ··· * a $^{-1}$ b m * m * b $_{m-1}$ * ··· * b $_i$ is i $_{b_i}$ (b $_i$ point which sends i). So i $_{a_i}$ = i $_{b_i}$.

This is a contradiction because a $_i$ and b $_i$ i do not send the same point.