Introduction to Basic Logarithms, Exponential Functions and Applications with Logarithms

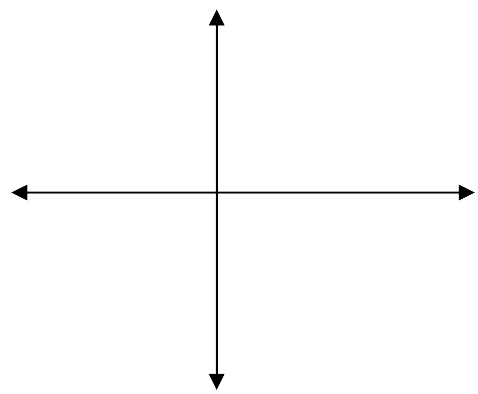
What is a logarithm?
This common question can only be answered by first understanding what an exponential function is and how exponential and logarithmic functions are related.

Well, then what is an exponential function? A good way to understand this type of function is through the following example.

Exponential function example: Say that your math teacher says he or she will pay you two cents per day to attend class, and each day you will receive double what you did the previous day. How much will you have earned in 30 days? (Note: this would never *really* happen!)

To figure this out, make a table of x and y values, where x is the day and y is the amount earned that day. Write next to the amount earned an equivalent amount in exponential form with a base of 2.

X (day)	Y (amount earned)
1	2 cents (2^1)
2	4 cents (2^2)
3	8 cents (2^3)
4	
5	
6	
…	…
X	2^x cents

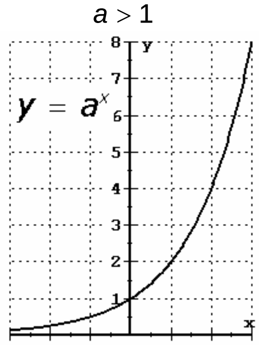


Now, plot these points on the right and connect the points to see the shape of the graph.

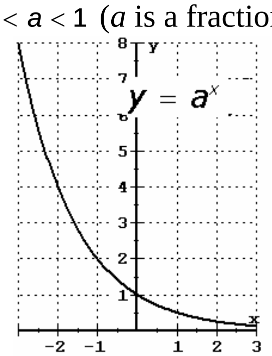
On the 30th day, we can use the *exponential function* $y = 2^x$ to find the value of y (amount earned) when x (the number of days) is 30. If you enter in your calculator $y = 2^{30}$ (keystrokes are 2 $\boxed{\wedge}$ 30, or 2 $\boxed{x^y}$ 30), you will see that you will earn over 1 billion cents, or over 10 million dollars! Imagine that!

Definition of an Exponential Function: For a > 0, where $a \neq 1$, the *exponential function with base a* is defined by $f(x) = a^x$ (or equivalently $y = a^x$), where "a" is the called the *base*, and "x" is called the "exponent". The exponential function has the following graphical representations:

$a > 1$



$0 < a < 1$ (*a* is a fraction)

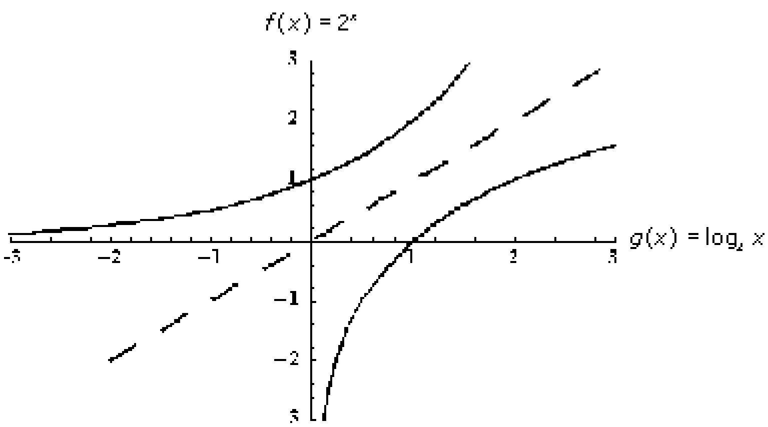


The Natural Exponential Function
There is an irrational number, denoted *e*, that arises in many logarithmic and exponential function problems as a base. The value of *e* is approximately 2.71828…. The *natural exponential function* is $f(x) = e^x$ (or equivalently $y = e^x$), and can be graphed on your calculator using the $\boxed{e^x}$ button.

Exploration 1: Graph $y = 2^x$, $y = 3^x$, and $y = e^x$ on your graphing calculator. What can you say about the steepness of the curve with bases 2 and 3 compared to base *e*?

Exploration 2: Using the $\boxed{e^x}$ button on your calculator, find the values of the following (round to 4 decimal places):
a) e^3 b) $2e^{-0.53}$ c) $e^{4.8}$

Logarithmic Functions
Now that you know about exponential functions, understanding logarithmic functions will be a lot easier. Recall the graph of the exponential function $f(x) = a^x$, where a > 1. Consider the graph below, of $f(x) = 2^x$ (in this case, a = 2). If you reflect the graph across the line y = x, you get the *inverse function* g(x).



This inverse function g(x) is what is called the *logarithmic function,* $\log_a$, where in this case a = 2.

Definition of the Logarithmic Function: For a > 0 with $a \neq 1$, the *logarithmic function with base a,* denoted $\log_a$, is defined by $\log_a x = y$ if and only if $a^y = x$.

In words, this means $\log_a x$ is the *exponent* to which the base a must be raised to to give x.

You can easily switch between *logarithmic form* and *exponential form* as follows:

Logarithmic form

exponent
$$\log_a x = y$$
base

Exponential form

exponent
$$a^y = x$$
base

Examples: The chart below illustrates how to switch between logarithmic and exponential forms. Notice you are just changing around the exponent and the base according the above definitions.

Logarithmic form	**Exponential form**
$\log_{10} 100,000 = 5$	$10^5 = 100,000$
$\log_2 8 = 3$	$2^3 = 8$
$\log_2 \left(\frac{1}{8}\right) = -3$	$2^{-3} = \frac{1}{8}$
$\log_5 p = q$	$5^q = p$

You try!

1. Express the following in exponential form:

a) $\log_2 32 = 5$ b) $\log_4 2 = \frac{1}{2}$ c) $\log_2 \left(\frac{1}{16}\right) = -4$

2. Express the following in logarithmic form:

a) $10^4 = 10,000$ b) $10^{-3} = 0.001$ c) $81^{\frac{1}{2}} = 9$

3. **Make up your own!** Write an equation in exponential form. See if you can convert it into logarithmic form. Then, change the equation in logarithmic form back to the equation in exponential form.

   Now, write an equation in logarithmic form. See if you can convert it into exponential form. Then, convert it back into logarithmic form.

Special Bases for Logarithms

There are two special bases to consider when working with logarithms. One is the value *e*, which was introduced above in the discussion of the natural exponential function. The other special base is 10.

The logarithm with base *e* is called the **natural logarithm** and is denoted ***ln:*** $\ln x = \log_e x$, where $\ln x = y$ if and only if $e^y = x$. (Note that "ln x" is a *function of x*, not variables *l, n, and x* multiplied as $l \bullet n \bullet x$. This is similar to f(x), where the function or rule is called "f" and x is the input variable).

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base: $\log_{10} x = \log x$ (if you do not see a base in a log statement, the implied base is 10).

Properties of Logarithms
There are certain helpful properties to know about when working with logarithms. These properties will help you to simplify logarithmic expressions and solve logarithmic equations.

Log Property	Equivalent Exponential Form	Reason
1. $\log_a 1 = 0$	$a^0 = 1$	We must raise *a* to the power of 0 to get 1.
2. $\log_a a = 1$	$a^1 = a$	We must raise *a* to the power of 1 to get a.
3. $\log_a a^x = x$	$a^x = a^x$	We must raise *a* to the power of x to get a^x.
4. $a^{\log_a x} = x$	None	$\log_a x$ is the power to which *a* must be raised to get *x*.

Note: In property #3, the *log base a of a* "cancels out" and the "x" drops down. In property #4, the *a* raised to the *log base a* power also "cancels out" and the "x" drops down.

Practice: Identify the property being used.

a) $\log_3 3 = 1$ b) $\log_7 7^{(2x+1)} = 2x+1$ c) $\log_5 1 = 0$ d) $2^{\log_2(3x^2)} = 3x^2$

Using Logarithmic Properties to Evaluate Expressions
You can use the properties above to evaluate logarithmic expressions. When looking at an expression, try to decide which property would apply, and then convert the expression into the correct form.

Example: Evaluate $\log_2 32$.

> **Solution:** This expression looks most like Property 2 or Property 3, but in both cases, you need the base 2 and the value of 32 to be the same somehow. So, think of a way that you could perhaps convert 32 into a 2 raised to a power. Since $32 = 2^5$, the expression can be rewritten as $\log_2 2^5$, which resembles most Property 3. Therefore, the answer is 5.

Example: Evaluate $4^{\log 0.25}$.

> **Solution:** This expression looks most like Property 4, but the "4" and the "0.25" need to be the same somehow. How could you rewrite the 0.25 to be a "4" raised to a power? You could write it this way: $0.25 = \frac{1}{4} = 4^{-1}$. Then, the entire expression can be rewritten as follows: $4^{\log 4^{-1}}$, and by Property 4, the expression evaluates to –1.

You try! Evaluate the following expressions using the Log Properties.

a. $\log_3 1$ b. $\log_7 49$ c. $\log 10^{12345}$ d. $\log_4\left(\frac{1}{2}\right)$

Properties of Natural Logarithms
In addition to the regular logarithm properties, there are similar properties for natural logarithms which are very similar.

Log Property	Equivalent Exponential Form	Reason
1. $\ln 1 = 0$	$e^0 = 1$	We must raise *e* to the power of 0 to get 1.
2. $\ln e = 1$	$e^1 = e$	We must raise *e* to the power of 1 to get *e*.
3. $\ln e^x = x$	$e^x = e^x$	We must raise *e* to the power of x to get e^x.
4. $e^{\ln x} = x$	None	$\ln x$ is the power to which *e* must be raised to get *x*.

Actually, these properties are the same as above. Remember that $\ln x = \log_e x$.

Practice: Rewrite the log properties in the chart above in terms of "$\log_e$" instead of "$\ln$" and compare with the previous logarithm property chart.

For example, $\ln 1$ can be written as $\log_e 1$, and from the previous chart, it is clear that this expression evaluates to 0. Try the others and compare.

Laws of Logarithms
In addition to the logarithm properties, there are Laws of Logarithms that are derived from the Laws of Exponents that you may have studied in previous classes. Knowing about these laws will help you to condense or expand logarithmic expressions, which will be important when solving equations and applied problems involving logarithms.

Let *a* be a positive number, with $a \neq 1$. Let A > 0, B > 0, and C be any real number.

Law	**Description**
1. $\log_a(AB) = \log_a A + \log_a B$	The logarithm of a product of numbers is the ***sum*** of the logarithms of the numbers.
2. $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$	The logarithm of a quotient of numbers is the ***difference*** of the logarithms of the numbers.
3. $\log_a\left(A^C\right) = C\log_a A$	The logarithm of number raised to an exponent is the exponent times the logarithm of the number.

Note: Since $\ln x = \log_e x$, there are equivalent laws for the natural logarithm. Create your own chart with the natural logarithm. The first entry has been done for you:

Law	**Description**
1. $\ln(AB) = \ln A + \ln B$	The natural logarithm of a product of numbers is the ***sum*** of the natural logarithms of the numbers.
2.	
3.	

Expanding Expressions using Laws of Logarithms

To "expand" something means to make it larger. When you are asked to expand a logarithmic expression, you want to use one or more of the laws to go from the lefthand side of the equations to the righthand side of the equations in the Logarithm Law chart as follows:

Expand $\log_2(3x)$**:**

$$\log_2(3x) = \log_2 3 + \log_2 x \qquad \text{(by Law 1)}$$

Sometimes, after expanding, you can evaluate part or all of each piece:

Expand $\log_2(8x)$**:**

$$\begin{aligned} \log_2(8x) &= \log_2 8 + \log_2 x && \text{(by Law 1)} \\ &= \log_2 2^3 + \log_2 x && \text{(rewrite 8 as } 2^3\text{)} \\ &= 3 + \log_2 x && \text{(evaluate } \log_2 2^3 \text{ using Property 3)} \end{aligned}$$

Also, many times, you can combine the laws to expand the expression as much as possible.

Expand $\ln\left(\frac{ab}{\sqrt[3]{c}}\right)$**:**

$$\begin{aligned} \ln\left(\frac{ab}{\sqrt[3]{c}}\right) &= \ln(ab) - \ln\left(\sqrt[3]{c}\right) && \text{(by Law 2)} \\ &= \ln a + \ln b - \ln\left(\sqrt[3]{c}\right) && \text{(by Law 1)} \\ &= \ln a + \ln b - \ln(c^{\frac{1}{3}}) && \text{(since } \sqrt[3]{c} = c^{\frac{1}{3}}\text{)} \\ &= \ln a + \ln b - \frac{1}{3}\ln c && \text{(by Law 3)} \end{aligned}$$

It is wise to check your final answer to make sure there is no other expanding you can do using the Laws of Logarithms and that there is no way to simplify any of the pieces. In this case, there is not. Double check to make sure.

You try! Expand the following logarithmic expressions:

1. $\log_2(x(x-1))$
2. $\log_6 \sqrt[4]{17}$
3. $\log_3(xy)^{10}$
4. $\log\left(\frac{x^2}{yz^3}\right)$

Condensing Logarithmic Expressions
To "condense" something is to make it smaller. When you are asked to condense a logarithmic expression, you want to use the Logarithm Laws in reverse, meaning you are going from the righthand side of the equations to the lefthand side of the equations in the Logarithm Law chart as follows:

Condense (or write as a single logarithm): $5\log x + \frac{1}{2}\log(x+1)$

$$5\log x + \frac{1}{2}\log(x+1) \quad = \log x^5 + \log(x+1)^{\frac{1}{2}} \quad \text{(by Law 3)}$$

$$= \log\left(x^5(x+1)^{\frac{1}{2}}\right) \quad \text{(by Law 1)}$$

$$= \log\left(x^5\sqrt{x+1}\right) \quad \text{(rewrite } (x+1)^{\frac{1}{2}} \text{ as } \sqrt{x+1}\text{)}$$

Note: It is best to only apply one Logarithm Law at a time. If you try to do more than one at a time, you may inadvertently make an algebraic error.

You try! Condense the following logarithmic expressions into a single logarithm.

1. $\log 12 + \frac{1}{3}\log 7 - \log 5$

2. $\log_2 A + \log_2 B - 2\log_2 C$

3. $2(\log_5 x + 2\log_5 y - 3\log_5 z)$

Common Errors to Avoid
It is a natural inclination to want to use other algebraic properties, such as distribution, with logarithms. However, be very careful to only use the laws and properties for logarithms when working with logs. Not all of the normal algebraic properties work with logarithms!

Common Error #1: Although the Law of Logarithms tells us how to compute the logarithm of a product or a quotient, there is no corresponding rule for the sum or difference of a logarithm!

$$\log_a(x+y) \neq \log_a x + \log_a y$$

In fact, you know that the right hand side is actually $\log_a(xy)$ by Law 1.

$$\log_a(x-y) \neq \log_a x - \log_a y$$

What is the righthand side in this case?

Common Error #2: Incorrect simplification of quotients or powers of logarithms can lead to incorrect answers.

$$\frac{\log 6}{\log 2} \neq \log\left(\frac{6}{2}\right) = \log 3$$

Use your calculator to verify that the lefthand and righthand sides are *not* equivalent.

$$(\log_2 x)^3 \neq 3\log_2 x$$

Put in any value for "x" on the left and use your calculator to compute. Put the same value for "x" on the right side and computer with your calculator. Verify that they're not equivalent.

One Last Formula: The "Change of Base" Formula

So, what do we do with something like $\frac{\log 6}{\log 2}$? We can actually rewrite as $\frac{\log_{10} 6}{\log_{10} 2}$ since these are common logarithms. Notice that the base of each log is the same. To evaluate this expression, you can use the following "Change of Base" formula:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Notice, on the right side, the logs have the ***same base***. You can ONLY use this formula to evaluate a quotient of two logs with the same base.

We can now write $\frac{\log_{10} 6}{\log_{10} 2} = \log_2 6$.

This is helpful when you are asked to evaluate something like $\log_8 5$. Using the change of base formula, rewrite as $\log_8 5 = \frac{\log_{10} 5}{\log_{10} 8}$. Notice that it is convenient to use "10" as the base because now you can enter each quantity into your calculator using the $\boxed{\text{log}}$ button (which has 10 as a base). Evaluate $\frac{\log_{10} 5}{\log_{10} 8}$ using the $\boxed{\text{log}}$ button on your calculator. Did you get 0.77398?

You try! Use the Change of Base formula to evaluate the following expressions.

1. $\log_2 7$ 2. $\log_3 11$ 3. $\log_6 92$

Solving Exponential and Logarithmic Equations

You are now ready to solve equations involving exponential and logarithmic functions. You will need this skill to solve applied problems. There is a three-step process to follow when solving these types of equations:

Solving Exponential Equations

1. Isolate the exponential expression on one side of the equation.
2. Take the logarithm of each side, then use the Laws of Logarithms to "bring down the exponent".
3. Solve for the variable.

Example 1: Solve $3^{x+2} = 7$.

$3^{x+2} = 7$	
$\log(3^{x+2}) = \log 7$	(take the log of both sides)
$(x+2)\log 3 = \log 7$	(Law 3 – bring down the exponent)
$x + 2 = \dfrac{\log 7}{\log 3}$	(divide by log 3)
$x = \dfrac{\log 7}{\log 3} - 2$	(subtract a 2 from each side)
≈ -0.228756	(use a calculator)

You try! Use the same steps, but use the natural logarithm *ln* instead of "log". Remember to isolate the exponential expression first!

Solve $8e^{2x} = 20$.   (Did you get 0.458?)

Practice: Solve the following equations.

1. $3^{2x-1} = 5$ 2. $4 + 3^{5x} = 8$ 3. $\left(\dfrac{1}{4}\right)^x = 75$

Solving Logarithmic Equations

This process is very similar to the one for solving exponential equations. However, these equations have the log function of some variable or value already in them. So, we proceed a little differently:

1. Isolate the logarithmic term on one side of the equation; you may first need to condense logarithmic terms, writing them as one expression.
2. Write the equation in exponential form (or raise the base to each side of the equation).
3. Solve for the variable.

Example 1: Solve $\log_2(x+2)=5$.

$$\log_2(x+2)=5$$

$x+2=2^5$ (write in exponential form)

$x+2=32$ (evaluate 2^5=32)

$x=30$ (subtract 2 from each side)

Example 2: Solve $\ln x=8$.

$$\ln x=8$$

$e^{\ln x}=e^8$ (raise both sides of the equation up over *e*)

$x=e^8$ (by property 4)

Example 3: Solve $4+(\log 2+\log x)^3=16$

$$4+(\log 2+\log x)^3=16$$

$(\log 2+\log x)^3=12$ (subtract 4 from each side to isolate the log expression)

$3(\log 2+\log x)=12$ (bring down the 3 by Law 3)

$\log 2+\log x=4$ (divide both sides by 3)

$\log(2x)=4$ (condense into one expression by Law 1)

$10^{\log_{10}(2x)}=10^4$ (raise both sides over 10 since $\log(2x)=\log_{10}(2x)$)

$2x=10000$ (bring down the (2x) by Property 4)

$x=5000$ (divide both sides by 2)

You try! Solve the following logarithmic equations.

1. $\ln(2+x)=1$
2. $4-\ln(3-x)^2=0$
3. $\log x+\log(x-1)=\log(4x)$

Applications of Exponential and Logarithmic Functions

Compound Interest Problems

Now we get to the really fun stuff! Let's talk money for a bit, shall we? Using logarithms, you can find out how much money you would earn if you invested a certain amount at a certain interest rate over a period of time. There are three main equations to know about when calculating compound interest:

If a principal P (the money you put in) is invested at an interest rate *r* for a period of *t* years, then the amount A (how much you make) of the investment can be calculated by:

1. $A=P(1+r)$ Simple interest for 1 year.
2. $A=P\left(1+\frac{r}{n}\right)^{nt}$ Interest compounded *n* times per year.
3. $A=Pe^{rt}$ Interest compounded continuously.

Example: A sum of $5000 is invested at an interest rate of 9% per year. Find the time required for the money to double if the interest is compounded according to the following methods:

a. Semiannually (meaning twice per year)
In this case, you would use the formula #2 for compound interest with P = $5000, A = $10,000 (since you want the investment to double), r = 0.09 (write all interest rates as a decimal), and n = 2 (since it is twice per year). Set up the equation and solve:

$5,000\left(1+\frac{0.09}{2}\right)^{2t}=10,000$	(write the equation)
$5,000(1.045)^{2t}=10,000$	(perform operations within the parentheses)
$(1.045)^{2t}=2$	(divide both sides by 5,000 to isolate the exponential expression)
$\log(1.045)^{2t}=\log 2$	(take the logarithm of both sides)
$2t\log 1.045=\log 2$	(bring down the "2t" exponent)
$t=\frac{\log 2}{\log 1.045}$	(divide both sides by 2 and log1.045)
$t\approx 7.9$ years	(use calculator to compute)

This means it will take around 7.9 years for your money to double at that rate.

b. Continuously
In this case, you would use the formula #3 for compound interest, with P = $5,000, A = $10,000, r = 0.09, and solve the resulting exponential equation.

$5,000e^{0.09t}=10,000$	(write the equation)
$e^{0.09t}=2$	(divide both sides by 5,000 to isolate the exponential expression)
$\ln e^{0.09t}=\ln 2$	(take the natural logarithm of both sides)
$0.09t=\ln 2$	(bring down the 0.09t)
$t=\frac{\ln 2}{0.09}$	(divide both sides by 0.09)
$t\approx 7.702$ years	(use calculator to compute)

This means it will take around 7.7 years for your money to double at that rate.

So, out of the two choices, which would you choose? Why?

You try! Compare how long will it take for an investment of $1000 to triple in value if the interest rate is 8.5% per year and it is compounded either four times per year or continuously. Which will take longer?

Exponential Growth Problems

To calculate the growth (or decay) of an animal or bacterium population, there is a certain formula you can use that requires the logarithmic manipulation you have just learned.

If n_0 is the initial size of the population, then the population $n(t)$ at a time t is given by: $$n(t) = n_0 e^{rt}$$ where r is the relative rate of growth expressed as a fraction of the population.

Example: World Population Projections
The population of the world in 1995 was 5.7 billion, and the estimated relative growth rate is 2% per year. If the population continues to grow at this rate, when will it reach 57 billion?

In this case, use $n_0 = 5.7$ billion, r = 0.02, and n(t) = 57 billion. Using the given formula:

$5.7e^{0.02t} = 57$	(write the equation)
$e^{0.02t} = 10$	(divide both sides by 5.7)
$\ln e^{0.02t} = \ln 10$	(take the natural logarithm of both sides)
$0.02t = \ln 10$	(bring down the 0.02t)
$t \approx 115.129$ years	(divide both sides by 0.02)

So, it will take a little over 115 years for the population to reach 57 billion.

You try! A culture starts with 10,000 bacteria, and the number doubles every 40 min.

a) Find a formula for the number of bacteria at time t.

b) Find the number of bacteria after one hour.

c) After how many minutes will there be 50,000 bacteria?