Introduction to algorithms Fall 2017

IFT2125-6001 TA: Maëlle Zimmermann

Demonstration 4

1

Q. Let $f \in O$ (n $_{\log\ _b\ a\text{-}\varepsilon)}$ o`u $\varepsilon > 0$. Let $T: N \rightarrow R_{\geq 0}$ such that

$$T (b_{k)} = \begin{cases} c & \text{if } k = k_{0,} \\ aT (b_{k-1)} + f (b_{k)} & \text{otherwise} \end{cases}$$

Show that $T \in \Theta$ (n $_{\log\ _b\ a:}$ n is a power of b).

Solution: Let g $_{(k}$ b $_{0)}$ $= c / a$ $_{k\ 0,}$ and g (b $_{k)}$ $= f$ (b $_{k)}$ / a $_k$ for all $k > k_{0.}$ We have being shown by induction that T (b $_{k)}$ $= a$ $_k$ [g $_{(k}$ b $_{0)}$ + g (b $_{k\ 0\ +1)}$ + ... + g (b $_{k)],}$ which proves that $T \in \Omega$ (n $_{\log\ _b\ a}$ n is a power of b) as a $_k$ = (b $_{k)}$ $_{\log\ _b\ a.}$

It remains to prove that `$T \in O$ (n $_{\log\ _b\ a:}$ n is a power of b). Since $f \in O$ (n $_{\log\ _b\ a\text{-}\varepsilon),}$ there are $n \in N$ $_{0,}$ $d \in$ $_{R>0}$ such that $f(n) \leq dn$ $_{\log\ _b\ a\text{-}\varepsilon}$ for all $n \geq n_{0.}$

Let $i \geq \max$ (n $_{0,}$ k $_{0),}$ then:

$$\begin{aligned} g_{(i}\ b) &= f (b_{i)} / a_i \\ &\leq d (b_{i)}\ _{\log\ _b\ a\text{-}\varepsilon} / a_i \\ Db &= _{i (\log\ _b\ a\text{-}\varepsilon)} / b\ _{\log\ _b\ a\ i} \\ &= D / b\ _{\varepsilon i.} \end{aligned}$$

Thus for $k \geq \max$ (n $_{0,}$ k $_{0),}$ we have

$$T(b^k) = a^k\left[g(b^{k_0}) + g(b^{k_0+1}) + \dots + g(b^k)\right]$$

$$= A^k\left[\sum_{i=k_0}^{\max(n_0,k_0)-1} g(b^i) + \sum_{i=\max(n_0,k_0)}^{k} g(b^i)\right]$$

$$\le a^k\left[\sum_{i=k_0}^{\max(n_0,k_0)-1} g(b^i) + \sum_{i=\max(n_0,k_0)}^{k} d/b^{\varepsilon i}\right]$$

$$\le a^k\left[\underbrace{\sum_{i=k_0}^{\max(n_0,k_0)-1} g(b^i) + d/(1-(1/b^{\varepsilon}))}_{\text{a constant}}\right].$$

Since $a^k = (b^k)^{\log_b a}$, we conclude that $T \in O(n^{\log_b a} \mid n$ is a power of b) and so we have $T \in \Theta(n^{\log_b a} \mid n$ is a power of b).

2

Question: Show that all conditions in order to apply the harmony rule are required. Specifically, exhibit $b \geq 2$, and $f, t: N \rightarrow R_{\geq 0}$ such as $t(n) \in \Theta(f(n)$: n is a power of b), but $t \notin \Theta(f)$. Give three pairs of functions f, t subject with the following additional conditions:

1. f is harmonious, but t is not possibly non-decreasing
2. f and t are possibly non-decreasing but $f(bn) \notin O(f(n))$

(F (n)) and t is non-decreasing, but f is not possibly non-decreasing.

Solution:

1. Necessity of t .nd: Let $f(n) = n$ and

$$t(n) = \begin{cases} n & \text{if } n \text{ is a power of } b, \\ 1 & \text{otherwise} \end{cases}$$

When n is a power of b, we have $t(n) = f(n) = n$. Let us show that $t \notin \Theta(f)$. Suppose the opposite is absurd. Then there exists $n \in N_0$, $c \in R_{>0}$

such that $\forall n \geq n_0$ we have $t(n) \geq cf(n)$. Let $m > \max(d_0, 1/c)$ a number which is not a power of b. According to our assumption, we have $t(m) \geq cf(m) = cm > c\frac{1}{c} = 1$. Now by definition $t(m) = 1$, which is a contradiction with the above.

2. Necessity of $f(bn) \in O(f(n))$) Let $b = 2$,

$$t(n) = 2^{n \lceil \log n \rceil},$$
$$f(n) = 2^{n \lfloor \log n \rfloor}.$$

By definition, t and f are not decreasing, and we have $t(2^i) = f(2^i) = 2^{2^i i}$. Suppose $f(2n) \in O(f(n))$, then $\exists n_0, c \in R_{>0}$ such that $n \geq_0 \forall n$ we $f(2n) \leq cf(n)$. Let $m = \max(d_0, \lceil c \rceil + 1)$. We have $f(2^{m+1}) / f(2^m) \leq c$, or

$$\frac{f(2^{m+1})}{f(2^m)} = \frac{2^{2^{m+1}(m+1)}}{2^{2^m m}} \geq \frac{2^{2^{m+1}(m+1)}}{2^{2^{m+1} m}} = 2^{2^{m+1}} \geq m > c,$$

which is a contradiction.

Let us now show that $t \notin \Theta(f)$. Suppose that $t \in \Theta(f)$, then $\exists n_0 \in N, c \in R_{>0}$ such that $\forall n \geq n_0$ we have $t(n) \leq cf(n)$. Let $m = \max(d_0, \lceil c \rceil + 1)$.
We $t(2^m + 1) / f(2^m + 1) \leq c$, or

$$\frac{t(2^m + 1)}{f(2^m + 1)} = \frac{2^{(2^m+1)(m+1)}}{2^{(2^m+1)m}} = 2^{2^m+1} \geq m > c,$$

which is a contradiction. Thus, $t \notin \Theta(f)$.

3. Necessity of f end: Let $t(n) = n$ and

$$f(n) = \begin{cases} n & \text{if } n \text{ is a power of } b, \\ 1 & \text{otherwise} \end{cases}$$

When n is a power of b, we have $t(n) = f(n) = n$. Moreover, $f(bn) \in O(f(n))$. Indeed, if $b\ n =^k$ we have

$$f(bn) = f(b^{k+1}) = b^{k+1} = bf(b^k) = bf(n),$$

and if n is not a power of b we have

$$f(bn) = 1 \leq b = bf(n).$$

Now, $t \notin \Theta(f)$, by an argument similar to that given in point 1.

Question: Consider a function $t: N \rightarrow R_{>0}$ eventually nondecreasing such than

$$\forall n \geq n_0 \qquad t(n) \leq t(\lfloor n/2 \rfloor) + t(\lceil n/2 \rceil) + t(1 + \lceil n/2 \rceil) + cn.$$

Borne t with O notation.

Solution: Let m_0 `the threshold from which t is nondecreasing. Let $n \geq \max(2m_0, n_0)$, then $\lfloor n/2 \rfloor \geq_0 m$. Thus $t(\lfloor n/2 \rfloor) \leq t(\lceil n/2 \rceil) \leq t(1 + \lceil n/2 \rceil)$ and of the blow

$$t(n) \leq 3t(1 + \lceil n/2 \rceil) + cn. \qquad (1)$$

Let $T(n) = t(n + 2)$. Let $n \geq \max(2m_o, n_0)$, then

$$\begin{aligned} T(n) &= t(n+2) \\ &\leq 3t(1 + \lceil (n+2)/2 \rceil) + c(n+2) \\ &= 3t(2 + \lceil n/2 \rceil) + c(n+2) \\ &\leq 3t(2 + \lceil n/2 \rceil) + c \cdot 2n \\ &= 3T(\lceil n/2 \rceil) + 2cn. \end{aligned}$$

Thus, $T(n) \leq 3T(\lceil n/2 \rceil) + (2c)$ for all $n \geq n \max(2m_0, n_0)$. By applying a theorem in class (theorem on asymptotic recurrences) with

$$a = 3, b = 2, f(n) = n, \varepsilon = 1/2 > 0$$

we get $T \in O(n^{\log_2 3})$. Since $t(n) = T(n - 2) \leq T(n)$, we conclude that $t \in O(n^{\log_2 3}) = O(n^{1.585})$.

4

Question: Solve the following recurrence exactly

$$t_n = \begin{cases} n + 1 & \text{if } n = 0 \text{ or if } n = 1, \\ 3t_{n-1} - 2t_{n-2} + 3 \cdot 2^{n-2} & \text{if not} \end{cases} \qquad (2)$$

Solution: For $n > 1$ we have $t_n - 3t_{n-1} + 2t_{n-2} = (3/4) \cdot 2^n$. Thus the polynomial Characteristic of recurrence is $p(x) = (x^2 - 3x + 2)(x-2) = (x-1)(x-2)(x-2)$. The roots are 1 and 2 (multiplicity 2). The root 2 generates

$$(c_1 + c_2 n) \cdot 2^n$$

and the root 1 generates

$$c_3 \cdot 1^n.$$

So,

$$t_n = c_1 2^n + c_2 2^n n + c_3.$$

Using the initial conditions given by (2) we get the following system

$$\begin{aligned} t_0 &= c_1 + c_3 = 1, \\ t_1 &= 2c_1 + 2c_2 + c_3 = 2, \\ t_2 &= 4c_1 + 8c_2 + c_3 = 7. \end{aligned}$$

As we have

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 8 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

So we get $t_n = -2 \cdot 2^n + (3/2) 2^n n + 3 = (3n - 4) 2^{n-1} + 3$.

5

Question: Solve the following recurrence exactly

$$T(n) = \begin{cases} at & \text{if } n = 0 \text{ or if } n = 1, \\ T(n-1) + T(n-2) + c & \text{otherwise} \end{cases}$$

Solution: For n> 1 we have $T(n) - T(n-1) - T(n-2) = c = 1^n c$. Thus the Characteristic polynomial recurrence is $p(x) = (x^2 - x - 1)(x - 1)$. Thus the roots of p are 1 and $\frac{1 \pm \sqrt{5}}{2}$, All of multiplicity 1. Denoting $\phi = \frac{1+\sqrt{5}}{2}$, We have

$$T(n) = c_1 + c_2 \phi^n + c_3 (1 - \phi)^n.$$

We obtain the system

$$\begin{array}{lllll} T(0) = c_1 & + & c_2 & + c_3 & = at, \\ T(1) = c_1 & + C_2 \phi & + c_3 (1-\phi) & = & at, \\ T(2) = c_1 & + C_2 \phi^2 & + C_3 (1-\phi)^2 & & = 2a + c. \end{array}$$

We can solve the system either by directly applying the Gauss-Jordan as previously on the matrix

$$\begin{bmatrix} 1 & 1 & 1 & at \\ 1 & \phi & 1-\phi & at \\ 1 & \phi^2 & (1-\phi)^2 & 2a+c \end{bmatrix}$$

or by proceeding first by substitution, which gives:

$$c_1 = a - c_2 - c_3$$

The system then becomes

$$(\phi - 1) c_2 - \phi c_3 = 0$$
$$(\phi^2 - 1) c_2 + (\phi^2 - 2\phi) c_3 = a + c$$

With Gauss-Jordan we get

$$\begin{bmatrix} \phi - 1 & -\phi & 0 \\ \phi^2 - 1 & \phi^2 - 2\phi & a + c \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{-\phi}{\phi - 1} & 0 \\ 0 & \phi(2\phi - 1) & a + c \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & \frac{a+c}{(\phi-1)(2\phi-1)} \\ 0 & 1 & \frac{a+c}{\phi(2\phi-1)} \end{bmatrix}$$

Since $2\phi - 1 = \sqrt{5}$ and $\phi(1 - \phi) = -1$, we obtain $c = -c_1$, $c_2 = \phi(a + c) / \sqrt{5}$, $c_3 = -(1 - \phi)(a + c) / \sqrt{5}$. We conclude that $T(n) = -c + \frac{a+c}{\sqrt{5}} \phi^{n+1} - \frac{a+c}{\sqrt{5}} (1 - \phi)^{n+1}$.

6

Question: Bound the following recurrence for n a power of 2

$$T(n) = \begin{cases} 1 & \text{if } n = 1, \\ 2T(n/2) + \log n & \text{otherwise} \end{cases}$$

Solution: We can use the lemma of the powers of b. If $n = 2^k$ we

$$T(2^k) = \begin{cases} 1 & \text{if } k = 0, \\ 2T(2^{k-1}) + k & \text{if} \end{cases}$$

We find the general case the lemma with $a = 2$, $b = 2$ $\log_b a = \log_2 2 = 1$ and $f(n) = \log(n)$. Since we can bound $f(n)$ by $\sqrt{n}$, we can apply query the case 1 of the lemma with $\varepsilon = 1/2$. Indeed, we have found $\varepsilon > 0$ such that $f(n) \in O(n^{\log_b a - \varepsilon})$ as $O(n^{\log_2 2 (1/2)}) = O(\sqrt{n}$ not). This implies that $T(n) \in \Theta(n^{\log_b a}$: n is a power of b) $= \Theta$(n: n is a power of 2).