Exponential and Logarithmic Rules

Remember that we define a **logarithm** in terms of the behavior of an exponential function as follows. Note that $\log_b a$ is read **"the logarithm of *a* base *b*."**

Definition: $y = \log_b a$ **means that** $b^y = a$. (*) (We assume b>0.)

So when we raise b to the $\log_b a$ power, we get a as the answer!

For example, $\log_2 8 = 3$ since $2^3 = 8$.

Also, since $2^0 = 1$, $\log_2(1) = 0$. In fact, $\log_b 1 = 0$, for any b > 0, since $b^0 = 1$.

Similarly, since $2^{(-1)} = \frac{1}{2}$, we have $\log_2\left(\frac{1}{2}\right) = -1$.

The two most commonly used bases are 10 and e (which is approximately 2.7182818284590452354). In the former case we have $\log_{10}$, which is usually just denoted "**log**" and called the **common log**, and in the latter case we have $\log_e$, which is almost always denoted "**ln**" and called the **natural log**. These two log bases are the most often used bases in applications and the two that are built into most calculators; this fact will help us later as we solve equations involving logarithmic and exponential functions.

In addition, we can draw several important conclusions directly from the definition of logs, including:

$\log_b b^n = n$ **and** $b^{(\log_b a)} = a$ (**)

Notice that the *domain* of the log function base b ($f(x) = \log_b x$) is exactly the *range* of the exponential function base b ($g(x) = b^x$) and vice versa.

Since $b^y > 0$ for all real numbers y and the **range of the exponential function base b is the interval (0, ∞), the domain of the log function base b is also (0, ∞).**
Similarly, since we can raise b to any (real number) power, y in (*) above can be any real number, i.e., that **the *domain* of the exponential function base b is ($-\infty$, ∞)**; this means that **the *range* of each of the log functions is also ($-\infty$, ∞).**

Rules of Exponents and Logs

The following rules of exponents should be familiar:

$x^n x^m = x^{(n+m)}$

$\frac{x^n}{x^m} = x^{(n-m)}$

$(x^n)^m = x^{(n\,m)}$

From the definition of logs and the rules of exponents above we can derive the following "rules of logs" (which are really just logarithmic versions of the rules of exponents), where we assume x>0 and y>0.

$$\log_b xy = \log_b x + \log_b y$$

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

$$\log_b x^n = n \log_b x \quad (***)$$

For example, if one wanted to simplify **ln(125) - 2ln(5)**, one could to do the following:

ln(125) - 2ln(5) $= \ln 125 - \ln 5^2$ *(re-positioning the exponent!)* $\Rightarrow$

$= \ln\left(\frac{125}{5^2}\right)$ *(check your rules!)* $\Rightarrow$

$= \ln\left(\frac{125}{25}\right) \Rightarrow$

$=$ **ln 5**

Solving Simple Exponential and Log Equations

Solving the exponential equation $3^x = 9$ is simple since one must merely raise three to an *integral* power to generate 9; clearly x = 2 here.
However, the problem of solving $3^x = 11$ is more difficult since there is no integral power to which 3 can be raised to obtain 11. Consequently, we must implement another strategy to solve for x. The difficulty here is obviously the position of the variable x. Using (***) we see that when we apply a logarithm *of any base* to an exponential,one can shift the exponent to the front of the expression and retain equality; this would put x in a much more accessible location! We have no logarithm in this equation yet, but since logarithms of equivalent expressions are always equal, we can take the logarithm of both sides of this equation to help obtain a solution.

Thus, in the case of $3^x = 11$, one can solve via this process:

$\log 3^x = \log 11 \Rightarrow$

$x \log 3 = \log 11 \Rightarrow$

$x = \frac{\log 11}{\log 3}$, which is approximately 2.182658338

We can also solve simple equations involving logs.
For example, to solve equation $\ln x^3 = 4$ notice that this means $\log_e x^3 = 4$.
By definition of logs this is equivalent to $e^4 = x^3$.

$\ln x^3 = 4 \Rightarrow$

$x^3 = e^4 \Rightarrow$

$(x^3)^{\left(\frac{1}{3}\right)} = (e^4)^{\left(\frac{1}{3}\right)} \Rightarrow$

$x = e^{\left(\frac{4}{3}\right)}$, which is approximately 3.793667893

Check this answer to make sure it is indeed a solution to the original equation.