IFT 2125 **Geña Hahn**

Introduction à l'algorithmique

Second MIDTERM

11 November 2013

Duration: 50 minutes

Value: 20% of the the grade, 20% of the grade if part of comprehensives

Instructions:

- One side of letter-size paper of notes is allowed, nothing else.
- Reply to each question in the space provided. Use the backs of the sheets for notes and rough work. **The space provided is no indication of the length of an answer! It is usually much too big.**
- Unless otherwise indicated, no points will be given for answers - correct or not - with no justification or proof.
- Note the difference between *justify* (a quick short argument) and *prove* or *show* (detailed argument).
- You can use the results from class, tutorials, books, or the appendix, provided you quote them correctly and precisely (unless, of coursem, they are to be proven).
- To answer a question you can also use results from other questions on this exam even if you did not prove them.
- Recall that $\mathbb{N}$ is the set of non-negative integers, $\mathbb{R}$ the set of reals, $\mathbb{R}^{\geq 0}$ the set of non-negative reals, $\mathbb{R}^{>0}$ the set of positive reals. Also, $\lg n = \log_2 n$ and $\ln n = \log_e n$ (where e is the base of the natural logarithm). Finally, $\log n$ is a "generic" logarithm : the base can be any $b \in \mathbb{R}^{>1}$.

1.	________ /15	4.	________ /25
2.	________ /15	5.	________ /15
3.	________ /10	6.	________ /20

Total: ________+________=________ /100

Name: ______________________________ **Student code:** ______________

☐ **Mark here if you are taking the exam as part of your comprehensives.**

1. **(15 points)** Let $D = (V, A)$ be a simple directed graph weighted by $c : A \longrightarrow \mathbb{R}^{\geq 0}$, given by its distance matrix $C(G)$ ($C(i,j) = c(ij)$ if $ij \in A$, $C(i,j) = \infty$ otherwise). We assume that $V = \{1, \ldots, 8\}$. Let $s = 3$.

0	3	∞	∞	∞	∞	∞	1
∞	0	4	∞	∞	∞	∞	∞
∞	∞	0	2	∞	12	∞	8
∞	∞	∞	0	2	∞	7	∞
∞	6	∞	∞	0	1	2	∞
∞	∞	∞	∞	∞	0	1	∞
1	∞	∞	∞	∞	∞	0	3
3	4	∞	∞	∞	3	∞	0

In $D_i(j)$, the table D_i contains a triple $(d_i(j), x_j, y_j)$. It indicates that after the ith iteration, the distance from s to j is $d_i(j)$, that it can be realised by passing through the vertex $x_j \in F_i$ and that j is or is not in F_i — $y_j = 1$ if $j \in F_i$, $y_j = 0$ otherwise. If $d_i(j) = \infty$, we put $x_j = 0$.

The following table is D_4.

D_4:							
$(\infty,0,0)$	(10,5,0)	(0,3,1)	(2,3,1)	(4,4,1)	(5,5,1)	(6,5,0)	(8,3,0)

Fill in the table below so it becomes D_5.

D_5:

2. **(15 points)** Find the exact solution to the recurrence

$$t(n) = \begin{cases} 1 & n = 1 \\ 2 & n = 5 \\ 2t(\lceil \frac{n}{5} \rceil) - t(\lfloor \frac{n}{25} \rfloor) & \text{sinon} \end{cases}$$

for $P_5 = \{5^k : k \in \mathbb{N}\}$.

3. **(20 points)** Prove that $\lg n! \in \Theta(n \lg n)$.

4. **(15 points)** Let $t_i : i = 0, 1, \ldots, 9$ be a set of tasks to execute. Each needs a unit of time to be executed and the task i must be executed at the latest at moment d_i in order to bring a profit of g_i, as indicated in the following table.

	t_0	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9
g	21	15	37	2	6	15	19	10	30	7
d	1	4	2	2	4	3	4	4	3	9

(a) Using the fast (student) algorithm, what is the order of execution of the tasks that maximise the profit while keeping the execution time minimum and what is the profit?

Order:

Gain: ________

(b) What is the state of the table in which the slow (protestant) algorithme computes the order of the task after the 4th iteration?

5. **(25 points)** Find the exact solution of the following recurrence $n = 2^{2^k}$, $k \in \mathbb{N}$ and give the exact order of the solution. Is the order valid for all $n \in \mathbb{N}$?

$$T(n) = \begin{cases} 1 & n = 2 \\ 2T(\sqrt{n}) + \lg n & \text{otherwise} \end{cases}$$

Boni (20 points) Let $f, g : \mathbb{N} \longrightarrow \mathbb{R}^{\geq 0}$ be two smooth fucntions. En each of the following cases prove the claim true or false.

1. $\frac{1}{f}$ is smooth
2. $f + g$ is smooth
3. $f \cdot g$ is smooth
4. $f \circ g$ is smooth (here we assume that $g : \mathbb{N} \longrightarrow \mathbb{N}$ and that the constants needed for g to be smooth are integers).

GOOD LUCK

APPENDIX

(1) Let R be the homogeneous recurrence

$$\sum_{i=0}^{k} a_i t_{n-i} = 0$$

and let $p(x)$ be its characteristic polynomial with distinct roots $r_1, \ldots, r_\ell$ of multiplicity $m_1, \ldots, m_\ell$, respectively. The general solution of R, is then

$$\sum_{i=1}^{\ell} \sum_{j=0}^{m_i - 1} c_{ij} n^j r_i^n.$$

(2) Let R^* be the NO homogeneous recurrence

$$\sum_{i=0}^{k} a_i t_{n-i} = \sum_{i=1}^{s} b_i^n q_i(n)$$

with $b > 0$ and $q_i(n)$ a polynomial in n of degree d_i, $i = 1, \ldots, s$. Let $p^*(x)$ be the characteristic polynomial of R^*, with distinct roots $r_1, \ldots, r_\ell$ of multiplicity $m_1, \ldots, m_\ell$, respectively,

$$p^*(x) = p(x) \prod_{i=1}^{s} (x - b_i)^{d_i + 1}.$$

The general solution of R^* is

$$\sum_{i=1}^{\ell} \sum_{j=0}^{m_i - 1} c_{ij} n^j r_i^n.$$

(3) Consider the recurrence

$$t(n) = \ell t(\frac{n}{b}) + cn^k$$

when $n \geq n_0$, with $b, k, n_0 \in \mathbb{N}$, $b \geq 2$, $n_0 \geq 1$, $n > 0$, $c \in \mathbb{R}^{>0}$. Then

$$t(n) \in \begin{cases} \Theta(n^k) & \text{when } b^k > \ell \\ \Theta(n^{\log_b \ell}) & \text{when } b^k < \ell \\ \Theta(n^k \log n) & \text{when } b^k = \ell \end{cases}$$

provided t is eventually nondecreasing.
Remark. When a variable or the range is changed, the initial conditions do not disappear but are transformed to the new paradigm.