Matroids

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Start with a set of objects, for example: $E = \{ 1, 2, 3, 4, 5 \}$

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The power set of E is the set of all possible subsets of E:

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A matroid is a subset I of the power set of E such that

1. For all $I_1 \in I$, if $I_2 \subset I_1$ then $I_2 \in I$.
2. For all $I_1, I_2 \in I$ s.t. $|I_2| = |I_1|+1$, $\exists\ e \in I_2$ *s.t.* $I_1 \cup \{e\} \in I$

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Max cardinality subset

{1,2,3}, {1,2,4}, {1,2,5}, {1,3,4}, {1,3,5},
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{1,4}, {2,4} $\notin$ I

{1,2}, {1,3}, {1,4}, {1,5}, {2,3},
{2,4}, {2,5}, {3,4}, {3,5}, {4,5},

{1,2,3}, {1,2,4}, {1,2,5}, {1,3,4}, {1,3,5},
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Start with a set of objects, for example: E={ 1, 2, 3, 4, 5 }
Now put weights on elements, subset weight = sum of weights of its elements.

{},

{1}[], {2}[5], {3}[6], {4}[12], {5}[8],

{1,2}[], {1,3}[], {1,4}[], {1,5}[], {2,3}[11], {2,4}[17], {2,5}[13], {3,4}[18], {3,5}[], {4,5}[20],

{1,2,3}[], {1,2,4}[], {1,2,5}[], {1,3,4}[], {1,3,5}[], {1,4,5}[], {2,3,4}[23], {2,3,5}[], {2,4,5}[25], {3,4,5}[],

{1,2,3,4}[], {1,2,3,5}[], {1,2,4,5}[], {1,3,4,5}[], {2,3,4,5}[]

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Start with a set of objects, for example: E={ 1, 2, 3, 4, 5 }
Consider: Find max card subset of min (max) weight

{},

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Matroids

How to solve it?

Given: Matroid (E,I) with weights on elements of E
Find: Maximum Cardinality Independent Set of I with
minimum (maximum) weight

Set T=Φ;
Order all elements in E by increasing (decreasing) weight;
Repeat the following until E is empty:
Let e be the first element of E;
Remove e from E (pop it off);
If $T \cup \{e\} \in I$ then Set $T = T \cup \{e\}$;
Output T;

Matroids

{},

{1}[], {2}[5], {3}[6], {4}[12], {5}[8],

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Order: {2}[5], {3}[6], {5}[8], {4}[12]

Matroids

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{1,2,3,4,5}[]

Order: {2}[5], {3}[6], {5}[8], {4}[12]
Choose: {2}[5] :: T = {2} since {2} is in I

Matroids

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Order: {2}[5], {3}[6], {5}[8], {4}[12]
Choose: {2}[5] :: T = {2} since {2} is in I
Choose: {3}[6] :: T = {2,3} since {2,3} is in I

Matroids

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Choose: {5}[8] :: T = {2,3} since {2,3,5} is not in I
Choose: {4}[12] :: T = {2,3,4} since {2,3,4} is in I

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Depends on the particular problem!!!
With computer structures there is always some way to tell very easily.

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Depends on the particular problem!!!
With computer structures there is always some way to tell very easily.

Let's consider some examples:

1. Minimum Cost Network:
   Given: a graph (vertices,edges) and costs on the edges
   Find: a least cost subset of edges s.t. for all pairs of vertices <x,y> there is a path going solely through edges in the subset

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1. Is "For all $I_1 \in I$, if $I_2 \subset I_1$ then $I_2 \in I$." good?

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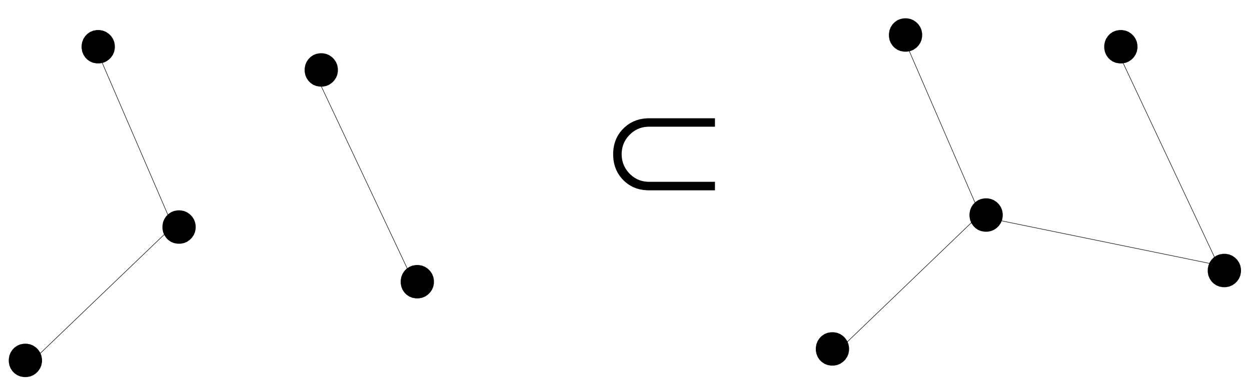
Let's see:

1. Is "For all $I_1 \in I$, if $I_2 \subset I_1$ then $I_2 \in I$." good?
   Each subset is a forest



I_2 I_1

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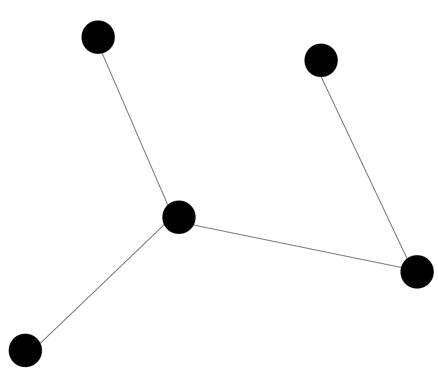
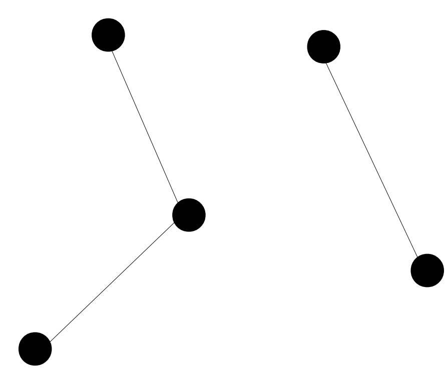


I_2

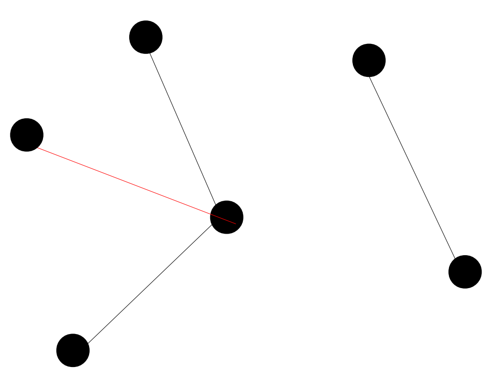


I_1

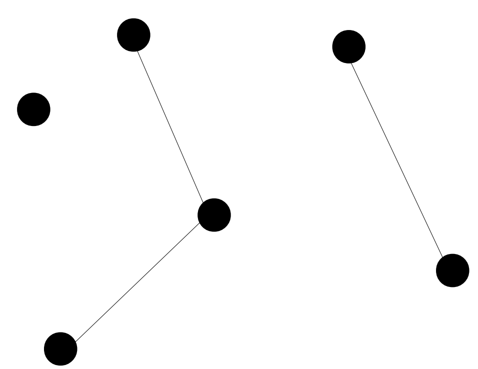
Matroids

Three cases:

1. There is an edge of the larger subset which has at most one vertex that is an endpoint of an edge of the smaller set.



I_2

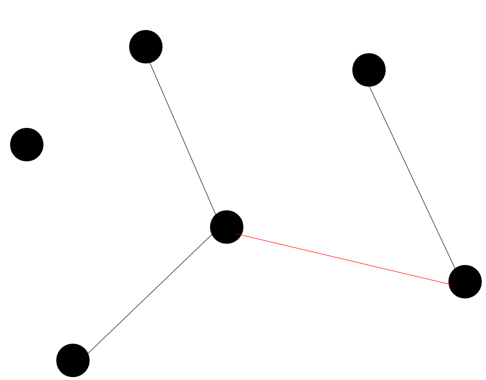


I_1

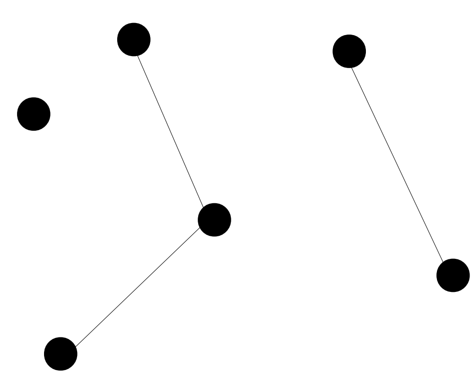
Matroids

Three cases:

2. There is an edge of the larger subset which has one endpoint in one connected component of the smaller subset and the other endpoint in another connected component of the smaller set.



I_2



I_1

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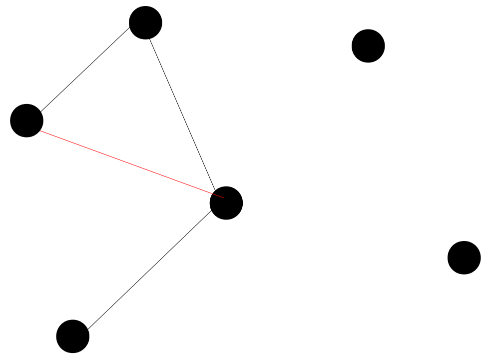
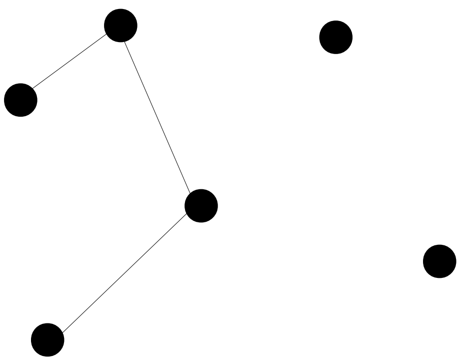
Three cases:

3. What's left: all edges of the larger subset are in the same connected component of the smaller subset



I_2



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But how to check that $T \cup \{e\} \in I$?

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But how to check that $T \cup \{e\} \in I$?

Need only check whether all the edges comprise a forest (that is, no cycles) – we discussed already ways to do this

Matroids

Here is another example:

Integer Deadline Scheduling Problem:

Given: Set of jobs, each with deadline and profit with unit processing time

Find: A schedule of lowest cost such that total profit is maximized (no profit for job completed after its deadline)

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This is a toughie!!!

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Right, the jobs!!

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This is a toughie!!!
Try this: Any set of jobs that <u>can</u> be scheduled so that all can be completed before their deadlines

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Is this a matroid???

1. Is "For all $I_1 \in I$, if $I_2 \subset I_1$ then $I_2 \in I$." good?

Job	Dead	Profit
8	2	501
3	3	623
0	4	532
5	4	441
1	5	321
4	6	278

I_1

Job	Dead	Profit
8	2	501
3	3	623
5	4	441
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2. Is "For all $I_1, I_2 \in I$ s.t. $|I_2|=|I_1|+1$,

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Job	Dead	Profit
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2	3	321
5	4	441
9	6	112

I_1

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So what weight should we use for elements?

Matroids

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So how do we determine whether $T \cup \{e\} \in I$?

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So how do we determine whether $T \cup \{e\} \in I$?
Right, keep 'em ordered by increasing deadline in T
and add 'em at the highest “open” deadline slot